

ON GENERALIZED SMOOTH GROUPS

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A. M. Mohamed
(A. M. Elkholy)

To my parents, my wife, and my children

" Gehad and Joseph "

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ON GENERALIZED SMOOTH GROUPS

Introduction

Following Schmidt [4], a group G is called smooth if G has a maximal chain of subgroups in which any two intervals of the same length are isomorphic (as lattices). We will call a group G totally smooth if all its maximal chains are smooth. The main concept in this work is that of a generalized smooth group. We will call G a generalized smooth group if $[G/H]$ is totally smooth for every subgroup H of G of prime order.

Throughout this thesis, G is a finite group and $L(G)$ its subgroup lattice.

The first aim is to determine the structure of totally smooth groups. Then we study the structure of groups all of whose proper subgroups are totally smooth. Finally, we will determine the structure of generalized smooth groups.

This thesis includes three Chapters:

The first chapter is concerned with the notation and the basic definitions that will be used throughout this thesis. Also we list a number of well-known results which will be used throughout this work, referring the reader to their proofs in the literature.

In the second chapter, we will prove that G is totally smooth if and only if one of the following holds: G is cyclic of prime power order, or G is a p -group, or G is cyclic of square free order. This result determines the structure of totally smooth groups and plays an important role in this work. We will prove also that all maximal subgroups of a group G are totally smooth if and only if one of the following holds: G is of order $p_1 p_2 p_3$ where $p_1, p_2,$ and p_3 are not necessarily distinct primes in $\pi(G)$, or G is a totally smooth group, or $G = PQ$ where P is an elementary abelian normal Sylow p -subgroup and Q is a cyclic group of order q operating irreducibly on P , or $G = PQ$ where P is an elementary abelian normal Sylow p -subgroup of order p^q , Q is a cyclic group of order q^2 which operates irreducibly on P and $q \mid p-1$.

The purpose of the third chapter is to determine the structure of a generalized smooth groups. As the structure of groups for which $n \leq 3$ is well-known where n is the longest length of a maximal chain of their subgroups, we prove the following result under the assumption that $n \geq 4$.

G is a solvable generalized smooth group if and only if one of the following holds:

- (i) G is totally smooth.

- (ii) G has a normal elementary abelian Sylow p - subgroup P of order p^2 , $G/P \cong Q$ is cyclic of order q^2 and Q and $\Phi(Q)$ are operating irreducibly on P .
- (iii) $G = PK$ where $|P| = p_1$, $P \triangleleft G$ and K is cyclic of order $p_2 p_3 \dots p_n$ operating faithfully on P with $p_i \neq p_j$ for $i \neq j$ and $i, j \in \{1, 2, \dots, n\}$, where p_i are primes.

Finally, we will show that the alternating group A_5 is the unique nonsolvable generalized smooth group.

CHAPTER I

Notation, Definitions and Known Results

1. 1. Introduction:

This chapter consists of two parts. In the first part, we present our notation and basic definitions that will be used throughout this thesis. And in the second part, we list a number of well- known results which will be used throughout this work.

1. 2. Notation and Definitions:

Groups and subgroups will be denoted by capital letters with or without indices and their elements will be denoted by small letters.

We will use the symbol 1 for both the identity subgroup and the identity element of a group.

We list below the other notation:

$x \in G$	x belongs to G .
$x \notin G$	x does not belong to G .

$\langle x, y \rangle$	group generated by x and y .
$ G $	the order of G .
$\pi(G)$	the set of distinct primes dividing $ G $.
$ \pi(G) $	the number of primes in $\pi(G)$.
$H \leq G$	H is a subgroup of G .
$H < G$	H is a proper subgroup of G .
$H \triangleleft G$	H is a normal subgroup of G .
G/H	the factor group of G by H .
$N_G(H)$	the normalizer of H in G .
$C_G(H)$	the centralizer of H in G .
$Z(G)$	the center of G .
$H \cong K$	H is isomorphic to K .
$p \mid G $	$ G $ is divisible by the prime p .
$[x, y]$	$= x^{-1}y^{-1}xy$, the commutator of x and y .
G'	the commutator subgroup of G .
$\Phi(G)$	the Frattini subgroup of G , that is the intersection of all maximal subgroups of G .
$L_2(q)$	the projective special linear group of degree 2 over the field $GF(q)$ with q elements.
A_n	the alternating group of degree n .
$L(G)$	the subgroup lattice of G .
$P_n(p)$	the subgroup lattice of an elementary abelian group of order p^n .
$Q_n(p)$	the subgroup lattice of PQ where P is elementary abelian of order p^n and Q is cyclic of prime order operating irreducibly on P .

The following example illustrates the pattern used to cite the references: [1; Theorem 8. 2, p. 310] refers to the result stated as Theorem 8. 2 on page 310 in the source numbered 1 in the references. Also numbers in the square brackets [] refer to the references.

Let $1 = G_0 < G_1 < \dots < G_n = G$ be a maximal chain of subgroups of a group G . This chain is called smooth if $[G_{i+j}/G_j] \cong [G_i/1]$ for all $i, j \in \mathbb{N}$ such that $i+j \leq n$. A group G is called smooth if its subgroup lattice $L(G)$ has a smooth chain. This concept was introduced by Schmidt (see [4]) in 1997. Also for $k \in \{2, 3, \dots, n-1\}$ the lattice $L(G_k)$ is called the k - type of G , or of the smooth chain; every interval of length k in the chain is isomorphic to this lattice (see [5]).

Let p be a prime and $n \geq 2$. We say that G belongs to the class $P(n,p)$ if G is either elementary abelian of order p^n or $G = PQ$ where P is an elementary abelian normal subgroup of G of order p^{n-1} and Q is of prime order $q \neq p$ and induces a nontrivial power automorphism on P . We call G a P - group if $G \in P(n, p)$ for some n and p . In 1939, Baer proved that all groups in the class $P(n, p)$ are lattice- isomorphic (see [3], p. 49).

We say that H operates nontrivially on P if P and H are contained in a group G and $1 \neq [P, H] \leq P$. And H operates irreducibly on P if 1 and P are the only H - invariant subgroups of P .

1. 3. Known results:

We will need the following well- known results:

1. 3. 1. Burnside's Theorem [2; Hauptsatz 2. 6, p. 419]

Let P be a Sylow p - subgroup of a group G . If $P \leq Z(N_G(P))$, then G has a normal p - complement.

1. 3. 2. Maschke's Theorem [1; Theorem 3. 2, p. 69]

Let G be a p' - group of automorphisms of an abelian p - group V and suppose that V_1 is a G - invariant direct factor of V . Then $V = V_1 \times V_2$, where V_2 is also G - invariant.

1. 3. 3. [2; Hilfssatz c, p. 258]

If $G = \langle x, y \rangle$, then $G' = \langle [x, y], G'' \rangle$.

1. 3. 4. [2; Hilfssatz 3. 3, p. 269]

If $H \triangleleft G$, then $\Phi(H) \leq \Phi(G)$.

1. 3. 5. [1; Theorem 2. 1, p. 421]

Let G be a group with a self- centralizing Sylow 2- subgroup of order 4. Then one of the following holds:

- (i) G has a normal 2- complement.
- (ii) $G/O(G)$ is isomorphic to $L_2(q)$, $q \equiv 3, 5 \pmod{8}$ where $O(G)$ is the largest normal subgroup of odd order in G . In particular, if G is simple, then $G \cong L_2(q)$, with $q \equiv 3, 5 \pmod{8}$ and $q > 3$.

1. 3. 6. [2, Satz 8.2, p 310]

A finite p - group G has exactly one subgroup of order p if and only if G is cyclic or a generalized quaternion group.

1. 3. 7. [1; Theorem 6. 4, p. 259]

A simple nonabelian group has order divisible either by the cube of its smallest prime or by 12.

Finally, we will need some more special results on subgroup lattices of groups.

1. 3. 8. [5; Lemma 1.3, p. 184]

Let $1 = G_0 < G_1 < G_2 = G$ be a maximal chain of subgroups of G . If it is smooth, then $|G| = p^2$ or $|G| = pq$ or $L(G) \cong Q_m(p)$ for $p, q \in \pi(G)$ and $m \geq 2$.

1. 3. 9. [5; Theorem A, p. 185]

Let K_2 be the chain of length two. Then G is a smooth group of 2-type K_2 if and only if G is cyclic of order p^n for some prime p and integer $n \geq 2$.

1. 3. 10. [5; Lemma 4. 1, p. 195]

Let $G = NK$ where N is a normal Hall subgroup of G with complement K . Let $H \leq K$ and $L = [[N/H]/1]_H$ be the lattice of H -invariant subgroups of $[N, H]$. Suppose that N and K are abelian and that every $X \in L$ is normal in G . Then

$$[G/H] \cong L \times [C_G(H)/H]$$

1. 3. 11. [5; Lemma 2. 2, p. 186]

Let $H \leq G$ such that $[G/H] \cong P_m(p)$. If there exists $X \leq G$ such that $H \leq X < G$ and $X \triangleleft G$, then $H \triangleleft G$.

1. 3. 12. [5; Proposition 2.4, p. 186]

If $H \leq G$ such that $[G/H] \cong P_m(p)$ and $|H| = q$ is a prime, then one of the following holds:

- (i) $H \triangleleft G$.
- (ii) $H = N_G(H)$ and G is a nonabelian P - group of order $p^m q$, $p > q$.
- (iii) $m = 2$, $H = C_G(G) < N_G(H) < G$, $p \mid q - 1$, and G is simple.

1. 3. 13. [5; Corollary 2.5, p. 187]

If $|H| = p$ and $[G/H] \cong P_m(p)$, then $H \triangleleft G$.

1. 3. 14. [6; Lemma 3.1]

Let p and q be different primes dividing the order of G such that $G = PQ$ where P is an elementary abelian normal subgroup of G of order p^n ($n \in \mathbb{N}$) and $Q = \langle x \rangle$ is a cyclic q - group. Then the following properties are equivalent:

- (i) Every subgroup Q_1 of Q either is irreducible on P or normalizes every subgroup of P .
- (ii) One of the following holds:
 - (a) $G = P \times Q$ or x induces a power automorphism in P .

- (b) $q \mid p-1$, $|P| = p^q$, and x induces an automorphism of order q^{k+1} (resp. at least $q^{k+1} = 4$ in case $q = 2$ and $k = 1$) in P where k is the largest integer such that $q^k \mid p-1$.
- (c) $n \geq 2$, q does not divide $p^r - 1$ for $1 \leq r < n$, $q^m \mid p^n - 1$, and x induces an automorphism of order q^m in P ($m \in \mathbb{N}$).

CHAPTER II

ON TOTALLY SMOOTH GROUPS

A group G is called smooth if G has a maximal chain of subgroups in which any two intervals of the same length are isomorphic (as lattices). This concept was introduced by Schmidt (see [4]) in 1997. In this chapter, we determine the structure of a group G for which all maximal chains of subgroups of G are smooth, we call groups with this property totally smooth.

2. 1. Definition:

A group G is called totally smooth if all maximal chains of G are smooth.

We give the structure of these groups in the following Theorem.

Theorem 2. 2.

A group G is totally smooth if and only if one of the following holds:

- (i) G is cyclic of prime power order.
- (ii) G is a P- group.
- (iii) G is cyclic of square free order.

Proof. It is clear that if G satisfies (i), (ii), or (iii), then every maximal chain of subgroups of G is smooth and hence G is totally smooth.

For the converse, suppose that the Theorem is false and let G be a counterexample of minimal order.

If $L(G)$ has a maximal chain of length $n \leq 2$, then by 1.3.8, $|G| = p^2$ or $|G| = pq$ or $L(G) \cong Q_m(p)$ for $p, q \in \pi(G)$ and $m \geq 2$. Since G is totally smooth, it follows that $L(G)$ is not isomorphic to $Q_m(p)$. Then G is a cyclic group of prime power order or a P- group or cyclic of square free order which contradicts our choice of G . So all maximal chains have length at least 3.

By induction, every maximal subgroup of G satisfies (i), (ii), or (iii). So we have to handle the following three cases:

Case 1. There exists a cyclic maximal subgroup of prime power order p^k where $k \geq 2$. By 1.3.9, G is a cyclic group of order p^n , a contradiction.

Case 2. There exists a maximal subgroup M which is a P -group.

If G is a p -group, then M is elementary abelian. Let H be any minimal subgroup of M . Since G is totally smooth, it follows that

$$[G/H] \cong L(M) \cong P_{n-1}(p)$$

and hence $\Phi(G) \leq H$. Since $|M| \geq p^2$, there exist at least two minimal subgroups of M , say, H_1 and H_2 . Therefore,

$$\Phi(G) \leq H_1 \cap H_2 = 1.$$

Hence G is elementary abelian, a contradiction. Thus G is not a p -group. Then M is either elementary abelian or a semi-direct product of an elementary abelian group L , say, of order p^{n-2} by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on L .

Suppose, first, that M is elementary abelian and let H be any minimal subgroup of M . Since $[G/H] \cong L(M) \cong P_{n-1}(p)$, it follows from 1.3.13 that

$$H \triangleleft G$$

and that G/H is a nonabelian P -group of order $p^{n-2}q$, $p > q$. So M is the unique Sylow p -subgroup of G . As shown above, every subgroup of M is normal in G which implies that Q induces a universal power automorphism since Q is not centralizing M . Then G is a P -group, a contradiction.

Therefore M is a nonabelian P - group of order $p^{n-2}q$, $p > q$. Let K be a subgroup of M such that $|K|=q$. Obviously, K is not normal in M . Since G is totally smooth, it follows that

$$[G/K] \cong P_{n-1}(p).$$

Hence by applying 1.3.12, we have $K = N_G(K)$ and G is a P - group, a contradiction.

Case 3. There is no maximal subgroup which is cyclic of prime power order or a P - group.

Then all maximal subgroups of G are cyclic of square free order. Therefore all Sylow subgroups of G are of prime order.

If all Sylow subgroups of G are normal in G , then we have that G is cyclic of square free order, a contradiction.

Thus we can assume that there exist a Sylow q - subgroup Q , say, of G such that Q is not normal in G , where $q \in \pi(G)$. Then $N_G(Q)$ is cyclic and hence $Q \leq Z(N_G(Q))$ which implies that G has a normal q - complement N , say, and $G/N \cong Q$. Clearly, N is cyclic. Let P_i be any Sylow p_i - subgroup of N . Since all maximal chains in G have length at least 3, $P_i Q < G$ and hence $P_i Q$ is cyclic. Then $Q \triangleleft P_i Q$. Therefore, $Q \triangleleft G$ which contradicts our choice of Q . This final contradiction completes the proof.

We study now groups for which all maximal subgroups are totally smooth. The following theorem gives the structure of these groups.

Theorem 2. 3.

All maximal subgroups of a group G are totally smooth if and only if one of the following holds:

- (i) G is of order $p_1 p_2 p_3$, where $p_1, p_2,$ and p_3 are not necessary distinct primes in $\pi(G)$.
- (ii) G is a totally smooth group.
- (iii) $G = PQ$, where P is an elementary abelian normal Sylow p -subgroup and Q is cyclic of order q which operates irreducibly on P .
- (iv) $G = PQ$, where P is an elementary abelian normal Sylow p -subgroup of order p^q , Q is cyclic of order q^2 which operates irreducibly on P and $q \mid p-1$.

Proof. Assume, first, that all maximal subgroups of G are totally smooth. Applying Theorem 2.2, all maximal subgroups of G are supersolvable and hence by Huppert's Theorem (see [2; Satz 9. 6, p. 718]) G is solvable.

If the maximal length n of a chain in $L(G)$ is at most 3, then G is of order $p_1 p_2 p_3$ or $p_1 p_2$ or p_1 , where $p_1, p_2,$ and p_3 are not necessary distinct primes in $\pi(G)$ and hence we are done. Thus we may assume that $n \geq 4$.

Now we have the following cases:

Case 1. $|\pi(G)|=1$. Hence G is a p -group of order p^n . By Theorem 2.2, every maximal subgroup of G is cyclic or elementary abelian. Suppose that M and N are maximal subgroups of G , where M is elementary abelian and N is cyclic. Then $M \cap N \neq 1$ and since $M \cap N$ is cyclic and elementary abelian, it follows that

$$|M \cap N|=p.$$

Then $n=3$, a contradiction since $n \geq 4$. Thus either all maximal subgroups of G are elementary abelian or all maximal subgroups of G are cyclic.

Assume, first, that all maximal subgroups of G are cyclic. Then G has exactly one subgroup of order p and hence G is cyclic since $n \geq 4$.

Now assume that all maximal subgroups of G are elementary abelian. Then

$$M_1 \cap M_2 \leq Z(G)$$

where M_i is a maximal subgroup of G for $i=1, 2$. Hence $G' \leq Z(G)$. We argue that G is abelian. If not, then there exist $x, y \in G$ such that $xy \neq yx$. Then

$$G = \langle x, y \rangle.$$

By 1.3.3, $G' = \langle [x, y] \rangle$ and G' is cyclic of order p since it is elementary abelian. Since $G/G' = \langle x, y \rangle G'/G'$ has order p^2 , it follows that $|G| = p^3$, a contradiction since $n \geq 4$. Thus G is abelian, and consequently G is elementary abelian. In particular, G is totally smooth.

Case 2. $|\pi(G)| = 2$. Since G is solvable, G has a minimal normal subgroup L , say, of order p^f for some prime $p \in \pi(G)$. Let P be a Sylow p -subgroup of G and Q be a Sylow q -subgroup of G .

Suppose, first, $|L| = p$. Let Q_1 be a maximal subgroup of Q . If $L = P$, $|L| = |P| = p$ and so LQ_1 is a proper subgroup of G . It follows, by hypothesis, that LQ_1 is totally smooth. Hence Theorem 2.2 shows that LQ_1 is a nonabelian P -group or cyclic of square free order. Then $|Q_1| = q$ and so $|Q| = q^2$, which implies that $n = 3$, a contradiction.

Thus L is a proper subgroup of P . Hence LQ is a proper subgroup of G . By hypothesis, LQ is totally smooth. Applying Theorem 2.2, LQ is a nonabelian P -group or cyclic of square free order and hence

$$|Q| = q.$$

Assume that $Q \triangleleft G$ and so HQ is a proper subgroup of G , where H is a maximal subgroup of P . Since HQ is totally smooth, Theorem 2.2 implies that $|H| = p$ as $Q \triangleleft G$. Since H is a maximal subgroup of P , we have that $|P| = p^2$ and hence $|G| = p^2q$, and $n = 3$, a contradiction as $n \geq 4$. Thus Q is not normal in G . By hypothesis, P is totally smooth. Applying Theorem 2.2, P is elementary abelian or cyclic.

We argue that $P \triangleleft G$. If not, $P = N_G(P)$ as P is a maximal subgroup of G . By 1.3.1, $Q \triangleleft G$, a contradiction. Thus

$$P \triangleleft G.$$

If P is cyclic, $H_1 \triangleleft G$ where H_1 is a maximal subgroup of P and so $H_1 Q < G$ which implies, by using Theorem 2.2, that $|H_1| = p$ since H_1 is cyclic. Then $|G| = p^2 q$, and $n = 3$, a contradiction as $n \geq 4$. Thus P is elementary abelian. By 1.3.2, P is completely reducible under Q and so L has a complement K , say, in P which is normal in G . Hence KQ is a proper subgroup of G . Once again, by hypothesis and Theorem 2.2, KQ is a nonabelian P -group or cyclic. Since $n \geq 4$ and $|L| = p$, it follows that $|K| > p$ and so KQ is a nonabelian P -group. Then all subgroups of K are normal in KQ and hence normal in G .

Let P_1 be any subgroup of P of order p such that $P_1 \neq L$. Since $n \geq 4$, $U = LP_1$ is a proper subgroup of P . By Dedekind's rule

$$U = LK \cap U = L(K \cap U).$$

Since L and $K \cap U$ are Q -invariant, we have that UQ is a proper subgroup of G and so totally smooth. Applying Theorem 2.2, UQ is a nonabelian P -group as $|U| = p^2$. It follows that Q does not centralize L and $P_1 \triangleleft UQ$. Since P is elementary abelian, $P_1 \triangleleft G$. Therefore, every subgroup of P is normal in G and Q induces a universal power automorphism. Then G is a nonabelian P -group. In particular, G is totally smooth.

Now suppose $|L| > p$. If L is a proper subgroup of P , LQ is a proper subgroup of G and by hypothesis, totally smooth. By Theorem 2.2, LQ is a nonabelian P -group. Then there exists a normal subgroup L_1 , say, of LQ of order p . Since $L_1 \triangleleft LQ$ and $L_1 \triangleleft P$, it follows that $L_1 \triangleleft G$ which contradicts the minimality of L . Thus $L = P$ and hence P is a minimal normal subgroup of G .

Assume that $|Q| = q$. Since P is a minimal normal subgroup of G , then Q operates irreducibly on P and we have case (iii).

Thus we can assume $|Q| \geq q^2$. Let Q_1 be a maximal subgroup of Q . So PQ_1 is a proper subgroup of G . By hypothesis, PQ_1 is totally smooth and by Theorem 2.2, PQ_1 is a P -group. Hence $|Q_1| = q$ and so $|Q| = q^2$. If Q would be elementary abelian, PQ_i would be a nonabelian P -group for every maximal subgroup Q_i of Q . Then G would have a normal subgroup of order p which would contradict the minimality of P . Thus Q is cyclic, operates irreducibly on P and Q_1 normalizes every subgroup of P . As Q operates irreducibly on P and $q \mid p-1$, it follows by 1.3.14 that $|P| = p^q$. Therefore, we have case (iv) and we are done.

Case 3. $|\pi(G)| \geq 3$. Let $p_j \in \pi(G)$; $j = 1, 2, \dots, m$ and $3 \leq m \leq n$. Suppose, for a contradiction, that $p_1^2 \mid |G|$. Since G is solvable, G has a Sylow basis $\{P_1, P_2, \dots, P_m\}$. Hence P_1P_i is a proper subgroup of G for $i = 2, 3, \dots, m$. By our hypothesis, P_1P_i is totally smooth and by Theorem 2.2, P_1P_i is a nonabelian P -group since $|P_1| > p_1$. Hence $|P_i| = p_i$ for $i = 2, 3, \dots, m$. Then P_1 has a normal subgroup H , say, of G of order p_1 . Consequently, $HP_2P_3 \dots P_m$ is a proper subgroup of G . Then by Theorem 2.2, $HP_2P_3 \dots P_m$

is cyclic of square free order which implies that P_i centralizes H for $i=2,3, \dots, m$, a contradiction as P_1P_i is a nonabelian P - group.

Thus every Sylow subgroup of G is cyclic of prime order. Hence for every $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, there exists $k \in \{1, 2, \dots, n\}$ such that $P_iP_jP_k$ is a proper subgroup of G where $i \neq k \neq j$. By Theorem 2.2, $P_iP_jP_k$ is cyclic for every i, j, k . Hence

$$[P_i, P_j] = 1$$

which implies that G is abelian and consequently G is cyclic of square free order; in particular, G is totally smooth.

For the converse, if G satisfies (i), (ii), or (iii) then, clearly, all maximal subgroups of G are totally smooth.

So assume that $G = PQ$, where P is an elementary abelian normal subgroup of order p^q , Q is a cyclic subgroup of order q^2 operating irreducibly on P and $q \mid p-1$. Let Q_1 be the subgroup of Q of order q . Since $q \mid p-1$, Q_1 is not irreducible on P . By 1.3.14, it follows that Q_1 normalizes every subgroup of P . Then all maximal subgroups of G are totally smooth.

CHAPTER III

ON GENERALIZED SMOOTH GROUPS

In this final chapter, we study the structure of generalized smooth groups.

3. 1. Definition:

A group G is called generalized smooth if $[G/H]$ is totally smooth for every subgroup H of G of prime order.

Since every lattice of length two is totally smooth, it is clear that every group G for which the maximal length n of a chain in $L(G)$ is at most 3 is generalized smooth. Furthermore, the structure of groups with this property is well- known. So we will usually assume that $n \geq 4$.

Theorem 3. 2.

Let G be a p - group of order p^n with $n \geq 4$. Then G is a generalized smooth group if and only if one of the following holds:

- (i) G is elementary abelian.
- (ii) G is cyclic.

Proof. It is clear that if G is totally smooth, then all maximal chains of G are smooth and hence $[G/H]$ is totally smooth for every minimal subgroup H of G . In particular, G is a generalized smooth group.

Thus assume that G is generalized smooth. Hence, for every minimal subgroup H of G , $[G/H]$ is totally smooth. Clearly, G has a normal subgroup H , say, of order p . By hypothesis, G/H is totally smooth and by Theorem 2.2, G/H is elementary abelian or cyclic.

If $\Phi(G) = 1$, G is elementary abelian and we are done. Thus assume that $\Phi(G) \neq 1$ and let H be a normal subgroup of G of order p such that G/H is elementary abelian. Then $\Phi(G) = H$.

Let R be a minimal subgroup of G of order p such that $R \neq H$. By hypothesis, $[G/R]$ is totally smooth. Let M be any maximal subgroup of G containing HR . Since $[G/R]$ is smooth, it follows that

$$[M/R] \cong [G/HR]$$

and since $[G/HR] \cong P_{n-2}(p)$, we obtain that $[M/R] \cong P_{n-2}(p)$ and hence, $R \triangleleft M$.

Since G/HR is elementary abelian and $|G/HR| > p$, we obtain $R \triangleleft G$. By hypothesis and Theorem 2.2, G/R is cyclic or elementary abelian. If G/R would be cyclic, then since G/HR is elementary abelian, it follows that $n = 3$ which contradicts our hypothesis that $n \geq 4$. Thus G/R is elementary abelian. Then $\Phi(G) \leq H \cap R = 1$, a contradiction.

Thus H is the only minimal subgroup of G . Then by 1.3.6, G is a generalized quaternion group as G/H is elementary abelian and $n \geq 4$. Hence $|\Phi(G)| = 2$ which implies that $n = 3$, a final contradiction.

It remains to consider the case where G/H is cyclic for every normal subgroup H of order p of G .

Since $\Phi(G) \neq 1$, it contains a normal subgroup H of order p of G . Since G/H is cyclic, $G/\Phi(G)$ is cyclic and hence G is cyclic.

Now we can assume that $|\pi(G)| \geq 2$ and first we prove the following result.

Theorem 3. 3.

Suppose that p and q are distinct primes in $\pi(G)$ such that $|G| = p^\alpha q^\beta$, where α and β are non-zero positive integers and let $n \geq 4$. Then G is a generalized smooth group if and only if one of the following holds:

- (i) G is a nonabelian P - group.
- (ii) G has an elementary abelian normal subgroup P of order p^2 , $G/P \cong Q$ is cyclic of order q^2 and Q and $\Phi(Q)$ are operating irreducibly on P .

Proof. Clearly, if G is a nonabelian P - group, then by Theorem 2.2, G is totally smooth and hence it is generalized smooth. So suppose that G satisfies (ii) and let H be a subgroup of G of prime order. If $|H| = q$,

$H = \Phi(Q)^g$ for some $g \in G$ and hence $\Phi(Q)$ is a maximal subgroup in Q and $P\Phi(Q)$ which implies that $[G/H]$ is of length two. Then $[G/H]$ is totally smooth. If $|H| = p$, then since Q and $\Phi(Q)$ are operating irreducibly on P , it follows that $[G/H]$ is a chain. Then $[G/H]$ is totally smooth and hence G is a generalized smooth group.

For the converse, suppose that G is generalized smooth. Since G is solvable, G has a minimal normal subgroup N , say, of prime power order which is elementary abelian. Suppose that N is a p -group, let P be a Sylow p -subgroup and let Q be a Sylow q -subgroup of G . By hypothesis, G/N is totally smooth. Applying Theorem 2.2, G/N is a P -group or cyclic of square free order or cyclic of prime power order. We handle the cases:

Case 1. Assume that G/N is a P -group.

Let N_1 be a maximal subgroup of N . Suppose, for a contradiction, that $N_1 \neq 1$. Then

$$[M/N_1] \cong [G/N]$$

for every maximal subgroup M of G containing N . Since G/N is a P -group, it follows that $[M/N_1]$ is complemented. If K is a complement to N in $[M/N_1]$, then $N_1 = N \cap K \triangleleft K$ and so $N_1 \triangleleft M$. Then N_1 is normal in every maximal subgroup of G containing N . Hence $N_1 \triangleleft G$, which contradicts the minimality of N . Thus

$$|N| = p.$$

Suppose, first, that G/N is elementary abelian. Then $N = P$ and P is a minimal normal subgroup of G of order p . Hence $|Q| \geq q^3$ since $n \geq 4$. As $\text{Aut } P$ is cyclic, there exists $Q_1 \leq C_Q(P)$ such that $|Q_1| = q$. Hence $Q_1 \triangleleft G$. Since $PQ_1/Q_1 \triangleleft G/Q_1$ and $|Q/Q_1| \geq q^2$, Theorem 2.2 shows that G/Q_1 is not totally smooth. This contradicts our hypothesis.

Thus G/N is a nonabelian P -group. Suppose, for a contradiction, that $q > p$. Then $NQ \triangleleft G$ and $NQ/N \cong Q$ is elementary abelian. Since $n \geq 4$, $|Q| \geq q^2$. Further, $NQ_1 \triangleleft G$ where Q_1 is any minimal subgroup of Q . As NQ_1 is nilpotent, we have $Q_1 \text{ char } NQ_1$. Hence $Q_1 \triangleleft G$.

Since $N \triangleleft G$, G/Q_1 is cyclic of square free order which implies that $|P| = p$, a contradiction since $|P| = p^2$.

Thus $p > q$. Hence $|Q| = q$ and P is normal in G of order p^e , $e \geq 3$. We claim that P is elementary abelian and every minimal subgroup of P is normal in G .

Let H be a minimal subgroup of P such that $H \neq N$. By hypothesis, $[G/H]$ is totally smooth and since $n \geq 4$, G/HN is a nonabelian P -group. Then

$$[G/HN] \cong [P/H].$$

This implies that $H \triangleleft P$ and P/H is elementary abelian. Then

$$\Phi(P) \leq H \cap N = 1.$$

Therefore P is elementary abelian. Let M be a maximal subgroup of G containing NH and Q . Since $[G/H]$ is smooth,

$$[M/H] \cong [G/NH] \cong P_{n-2}(p).$$

Since $|H|=p$, it follows by 1.3.13, that $H \triangleleft M$. Hence, $H \triangleleft G$.

Therefore, Q induces a power automorphism on P which is nontrivial since G/N is a nonabelian P -group. Then G is a nonabelian P -group.

Case 2. G/N is cyclic of order pq .

Hence $|Q|=q$. We argue that $|N|=p$. If not, then

$$[G/N] \cong [P/N_1]$$

where N_1 is a maximal subgroup of N . Since G/N is cyclic of order pq and $[P/N_1]$ is an interval in a p -group, we get a contradiction. Thus $|N|=p$. Therefore, $n=3$ which contradicts the fact that $n \geq 4$.

Case 3. G/N is cyclic of prime power order.

Then P is a minimal normal subgroup of G and Q is operating irreducibly on P . Hence the maximal subgroups of G are $P\Phi(Q)$ and the conjugates of Q . Let Q_1 be the minimal subgroup of Q .

Assume that $|P| \geq p^3$ and let H be a maximal subgroup of P . Hence,

$$[PQ_1/H] \cong P_2(p).$$

Then by 1.3.11, $H \triangleleft PQ_1$ and PQ_1/H is a nonabelian P -group. Since $G/P \cong Q$ is cyclic and $[G/H]$ is totally smooth, it follows that $|Q| = q$. Then $H \triangleleft G$, a contradiction. Thus $|P| \leq p^2$.

Assume that $|P| = p$. Then $|Q| \geq q^3$ and so $[Q/Q_1]$ is a chain of length at least two. But since $[G/\Phi(Q)]$ is not a chain, we get a contradiction. Therefore,

$$|P| = p^2.$$

Suppose that Q_1 operates reducibly on P . Then there exists a minimal subgroup H , say, of P such that $H \triangleleft PQ_1$. By hypothesis, $[G/H]$ is totally smooth and therefore,

$$[G/P] \cong [P\Phi(Q)/H].$$

Since G/P is cyclic of prime power order and $[P\Phi(Q)/H]$ is not a chain, we get a contradiction. Thus Q_1 is operating irreducibly on P and so Q_1 is a maximal subgroup of PQ_1 . Hence

$$[G/PQ_1] \cong [P\Phi(Q)/Q_1].$$

If Q_1 would be a proper subgroup of $\Phi(Q)$, then $[P\Phi(Q)/Q_1]$ would not be a chain and hence $[G/Q_1]$ would not be totally smooth which contradicts our hypothesis. Thus $|Q_1| = |\Phi(Q)| = q$ and hence $|Q| = q^2$.

Therefore, $|G| = p^2q^2$ and (ii) holds. This completes the proof.

Now we consider the case that G is solvable and $|\pi(G)| \geq 3$.

Theorem 3. 4.

Suppose that $n \geq 4$ and let p_1, p_2, \dots, p_m be different primes in $\pi(G)$ ($m \geq 3$). Then G is a solvable generalized smooth group if and only if one of the following holds:

- (i) G is cyclic of square free order.
- (ii) $G = PK$, where $|P| = p$, $P \triangleleft G$ and K is cyclic of order $p_1 p_2 \dots p_{n-1}$ such that $p_i \neq p$ ($i = 1, 2, \dots, n-1$) operating faithfully on P .

Proof. If G is cyclic of square free order, then, for every subgroup H of prime order of G , $[G/H]$ is totally smooth and so G is generalized smooth. And if (ii) holds, then G/P is cyclic of square free order and if H is a subgroup of K of order p_i , then by 1.3.10, $[G/H]$ is a direct product of chains of length one and hence $[G/H]$ is totally smooth. Since every minimal subgroup of G different from P is conjugate to a subgroup of K , it follows that $[G/H]$ is totally smooth for every minimal subgroup H of G . In particular, G is generalized smooth.

For the converse, let N be a minimal normal subgroup of G of order p^e which is elementary abelian and let $N \leq P$, where P is a Sylow p -subgroup of G . By hypothesis G/N is totally smooth and hence Theorem 2.2 shows that G/N is a nonabelian P -group or cyclic of square free order since $|\pi(G)| \geq 3$.

Case 1. Suppose that G/N is a nonabelian P - group.

Since $|\pi(G)| \geq 3$, it follows that P is a minimal normal subgroup of G and $|G/P| = q^s r$ where $q > r$, $s \geq 1$. Let Q be a Sylow q - subgroup of G . Clearly, Q is elementary abelian. Let Q_1 be any minimal subgroup of Q . If $|P| \geq p^2$, there exists a nontrivial maximal subgroup L of P . Since $[G/L]$ is totally smooth, we have that

$$[PQ_1/L] \cong P_2(q).$$

It follows from 1.3.11 that $L \triangleleft PQ_1$. Hence $PQ_1/L \in P(2, q)$ and P/L is a normal subgroup in PQ_1/L of order $p \neq q$, a contradiction. Thus

$$|P| = p.$$

Since $n \geq 4$, G/PQ_1 is a nonabelian P - group. Hence if $Q_1 \triangleleft G$, we have that G/Q_1 is nonabelian and by Theorem 2.2, G/Q_1 is a nonabelian P -group which implies that $|Q| = q$ and hence $n = 3$, a contradiction. Thus Q_1 is not normal in G . Since $[G/Q_1]$ is totally smooth,

$$[M/Q_1] \cong [G/PQ_1]$$

for every maximal subgroup M of G containing PQ_1 . Since

$$[G/PQ_1] \cong P_e(q) \text{ for } e \geq 2, \text{ we have}$$

$$[M/Q_1] \cong P_e(q).$$

Then by 1.3.13, $Q_1 \triangleleft M$ and consequently $Q_1 \triangleleft G$, which is not the case as shown above.

Case 2. Suppose that G/N is cyclic of square free order.

We have the following two cases:

Case 2a. Assume that $p \mid |G/N|$.

Hence $P \triangleleft G$ and N is a maximal subgroup of P . If P is not elementary abelian, $\Phi(P) \neq 1$ and hence $N \leq \Phi(P)$ since N is a minimal normal subgroup of G . Hence $G/\Phi(P)$ is cyclic. Since $P \triangleleft G$, we get by 1.3.4, that $\Phi(P) \leq \Phi(G)$. As $G/\Phi(P)$ is cyclic, it follows that $G/\Phi(G)$ is cyclic and so G is cyclic. Hence, for any prime $r \neq p$ in $\pi(G)$, $R \triangleleft G$ where R is a Sylow r -subgroup of G of prime order. Since G/R is totally smooth and G is cyclic, we get by Theorem 2.2, that G/R is cyclic of square free order. Hence $|P| = p$, a contradiction. Thus $\Phi(P) = 1$ and so P is elementary abelian.

Suppose that $|N| > p$ and let N_1 be a maximal subgroup of N . If R is any Sylow r -subgroup of G where $r \neq p$, then by hypothesis $[PR/N_1]$ is totally smooth and hence,

$$[PR/N] \cong [P/N_1] \cong P_2(p).$$

Then PR/N is a P -group, a contradiction since PR/N is cyclic. Thus

$$|N| = p.$$

Let K be a complement of P in G . Suppose further that there exists a subgroup Q , say, of K of order $q \neq p$ such that Q centralizes P . Since K is cyclic, $Q \triangleleft K$ and hence $Q \triangleleft G$. As G/NQ is cyclic of square free order, it follows by Theorem 2.2, that G/Q is cyclic of square free order which implies that $|P| = p$, a contradiction. Thus there is no subgroup of K which is normal in G . Since K is completely reducible on P , there exists a proper subgroup L of P such that $P = L \times N$ and $K \leq N_G(L)$. Hence, $L \triangleleft G$.

It follows, by Theorem 2.2, that G/L is cyclic of square free order. Then

$$G' \leq L \cap N = 1$$

and consequently G is abelian which implies that the subgroups of K are normal in G , a contradiction.

Case 2b. The order of G/N is not divisible by p .

So P is a minimal normal subgroup in G . We argue that $|P| = p$. If not, then P has a nontrivial maximal subgroup P_1 , say. Let R be any subgroup of prime order $r \neq p$. Then

$$[PR/P_1] \cong [A/P],$$

for a suitable subgroup A of G containing P . Since G/P is cyclic of square free order, A/P is cyclic of square free order and so $[PR/P_1]$ is

complemented. Hence, there exists a complement K_1 to P_1 in $[PR/P_1]$.

Then

$$P_1 = P \cap K_1 \triangleleft K_1.$$

Therefore, $P_1 \triangleleft PR$.

As R is an arbitrary subgroup of G of prime order ($\neq p$), we have $P_1 \triangleleft G$ which contradicts the minimality of P . Therefore,

$$|P| = p.$$

Next suppose that there exists a subgroup Q of K of order q which centralizes P where K is a complement to P in G . Hence $Q \triangleleft G$ as $Q \triangleleft K$. Since G/Q is totally smooth and $n \geq 4$, it follows by Theorem 2.2, that G/Q is a nonabelian P -group or cyclic of square free order. If G/Q is a nonabelian P -group, then $|\pi(G)| = 3$ and so $n = 3$, a contradiction. Thus G/Q is cyclic of square free order. Hence, $G' \leq P \cap Q = 1$. Then G is abelian and therefore G is cyclic of square free order.

So we may assume that no nontrivial subgroup of K centralizes P . This implies that K operates faithfully on P and (ii) holds. This completes the proof of Theorem 3.4.

Finally, we determine the nonsolvable groups which are generalized smooth. In fact, we show that A_5 is the only such group.

Theorem 3.5.

Suppose that G is a nonsolvable group. Then G is generalized smooth if and only if G is isomorphic to A_5 .

Proof. Suppose, first, that $G \cong A_5$ and let H be a subgroup of G of prime order. If $|H| \neq 2$, then $[G/H]$ is of length two and hence $[G/H]$ is totally smooth. So suppose that $|H| = 2$.

Suppose that $H = G_1 < G_2 < \dots < G_n = G$ is a maximal chain of subgroups of G . If $|G_2| = 6$, then the length of the chain is 2 and hence the chain is smooth. Similarly, if $|G_2| = 10$, the chain is of length 2 and hence it is smooth and we are done. Therefore we are in a position to assume the final case that $|G_2| = 4$. Then G_2 is a Sylow 2- subgroup of G and hence $G_3 = N_G(G_2)$, $[G/G_2]$ and $[N_G(G_2)/H]$ are chains of length 2. Hence $[G/G_2] \cong [N_G(G_2)/H]$. Therefore the chain $H = G_1 < G_2 < G_3 < G_4 = G$ is smooth. It follows that G is generalized smooth.

For the converse, suppose that G is generalized smooth. As G is a nonsolvable group, we have the following two cases:

Case 1. Suppose that G is a nonabelian simple group.

Let S be a Sylow 2- subgroup of G . First assume that $|S| \geq 8$. If S is a proper subgroup of $N_G(S)$, then since $N_G(S)$ is a solvable generalized smooth group, Theorem 3.3 and Theorem 3.4 show that $|S| \leq 4$, a contradiction as $|S| \geq 8$. Thus $S = N_G(S)$. Since G is simple, it follows, by Theorem 3.2 and 1.3.1, that $|S|$ would be of order 8 and S is a non-abelian Sylow 2- subgroup of G . As G has no normal 2- complement,

by Frobenius's Theorem there exists a subgroup S_1 , say, of S such that $N_G(S_1)/C_G(S_1)$ is not a 2- group (see [1; p. 253]). It is clear that $N_G(S_1)$ is containing S and S_1 is a Sylow 2-subgroup of $C_G(S_1)$ of order 4. Let us study $N_G(S_1)$. Obviously, $N_G(S_1)$ is solvable. If $|\pi(N_G(S_1))| \geq 3$, Theorem 3.4 shows that $|S| = 2$, a contradiction as $|S| = 8$. Thus $|\pi(N_G(S_1))| = 2$ and hence by Theorem 3.3, we get $|S| = 4$, a contradiction. Therefore

$$|S| = 4.$$

Further by 1.3.7, $|G|$ is divisible by 12. Suppose that $|N_G(S)| > 12$. If $|\pi(N_G(S))| \geq 3$, we get by Theorem 3.4 that $|S| = 2$, a contradiction. Thus $|\pi(N_G(S))| = 2$ and so by Theorem 3.3, $N_G(S)$ has a cyclic Sylow 3-subgroup R of order 3^2 and the subgroups of R are operating irreducibly on S . Since $|N_G(S)/C_G(S)| = 3$, R_1 centralizes S where R_1 is the minimal subgroup of R , which contradicts Theorem 3.3. Therefore, $|N_G(S)| = 12$ and $S = C_G(S)$. Hence $N_G(S) \cong A_4$. Then by 1.3.5,

$$G \cong L_2(q) \text{ with } q \equiv 3, 5 \pmod{8} \text{ and } q > 3.$$

Suppose, for a contradiction, that $N_G(S)$ is not a maximal subgroup of G and let M be a subgroup of G such that $N_G(S)$ is a second maximal subgroup of M . Then, there exists a subgroup A of M containing $N_G(S)$ as a maximal subgroup. If L is a minimal subgroup of S , we get

$$[M/N_G(S)] \cong [A/S] \cong [N_G(S)/L].$$

But $[N_G(S)/L]$ is a chain and since

$$[M/N_G(S)] \cong [A/R],$$

it follows that $R = N_A(R)$ where R is a Sylow 3- subgroup of $N_G(S)$. As R is cyclic, $N_A(R) = C_A(R)$ and by 1.3.1, A has a normal 3- complement K . Clearly, $S \leq K$. Since $[A/S]$ is a chain, we get $K = S$. This implies that $A = N_G(S)$ which contradicts our assumption that $N_G(S)$ is a maximal subgroup of A . Therefore $N_G(S)$ is a maximal subgroup of G .

As $G \cong L_2(q)$ and $q \equiv 3, 5 \pmod{8}$, it follows that G has a dihedral subgroup D of order $4m$ where m is odd, $m > 1$ if $q > 5$ (see [2; p. 213]). Then by Sylow's Theorem,

$$S \leq D^g \cap N_G(S),$$

for some $g \in G$. Hence $[G/S]$ is not a chain and therefore it is not isomorphic to $[N_G(S)/L]$ where L is a minimal subgroup of S . Hence $[G/L]$ is not totally smooth which contradicts our hypothesis. Thus $q = 5$ and hence

$$G \cong L_2(5) \cong A_5.$$

Case 2. Suppose that G is not simple.

Hence G has a nonsolvable chief factor H/K and $H < G$ since G is generalized smooth.

Let M be a maximal subgroup of G containing H . By induction, $M \cong A_5$. Therefore $H = M$ and $K = 1$, that is, $M \triangleleft G$ and $|G:M| = p$ for some prime. By the Frattini argument, $G = HN_G(S)$ where S is a Sylow 2- subgroup of H .

Let L be a minimal subgroup of S . Since $H \cong A_5$, $N_H(S) \cong A_4$ and hence $[N_H(S)/L]$ is a chain. Since $G = HN_G(S)$, clearly $N_H(S) < N_G(S)$ and so $[G/N_H(S)]$ is not a chain which implies that $[G/L]$ is not totally smooth which contradicts our hypothesis. Thus $N_H(S) = N_G(S)$ and hence $N_G(S) \leq H$. Then $G = H$, a contradiction. This final contradiction shows that G has no nontrivial normal subgroup which completes our proof.

We collect our results of this chapter in the following Main Theorem which gives the characterization of the generalized smooth groups.

Main Theorem.

A finite group G is generalized smooth if and only if one of the following holds:

- (i) $|G| = p_1 p_2 p_3$ where p_1, p_2 and p_3 are not necessarily distinct primes.
- (ii) G is totally smooth.
- (iii) G has an elementary abelian normal subgroup P of order p^2 , $G/P \cong Q$ is cyclic of order q^2 and Q and $\Phi(Q)$ are operating irreducibly on P where p and q are primes.
- (iv) $G = PK$ where $|P| = p_1$, $P \triangleleft G$ and K is cyclic of order $p_2 p_3 \dots p_n$ and operates faithfully on P where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in \{1, 2, \dots, n\}$.
- (v) $G \cong A_5$.

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