

The Additivity of the two-dimensional Miller ideal

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Introduction

For several of the classical tree forcings, such as Sacks forcing \mathbb{S} , Laver forcing \mathbb{L} and Miller forcing \mathbb{M} , there are corresponding σ -ideals. Already in 1935 Marcewski [Ma] studied the σ -ideal $\mathcal{J}(\mathbb{S}) := \{X \subseteq {}^\omega 2 \mid \forall p \in \mathbb{S} \exists q \in \mathbb{S} (q \leq p \wedge [q] \cap X = \emptyset)\}$, later, for example, Veličković, Judah, Miller and Shelah [Ve, JuMiSh] continued this investigation of $\mathcal{J}(\mathbb{S})$. Brendle, Goldstern, Johnson, Judah, Miller, Repický, Shelah and Spinas [Br, JuMiSh, GoReShSp, GoJSp, Sp1] studied the σ -ideals $J(Q) := \{X \subseteq {}^\omega \omega \mid \forall p \in Q \exists q \in Q (q \leq p \wedge [q] \cap X = \emptyset)\}$ for $Q \in \{\mathbb{M}, \mathbb{L}\}$. For every ideal \mathcal{I} one can define the additivity $\text{add}(\mathcal{I})$ and the covering number $\text{cov}(\mathcal{I})$ of \mathcal{I} . The additivity is the minimal cardinality of a subset of the ideal whose union is not in the ideal, and the covering number is the least cardinality of a subset of the ideal whose union is the whole set, on which the ideal is defined. Clearly, the following inequalities hold for every σ -ideal \mathcal{I} on ${}^\omega 2$ or ${}^\omega \omega$:

$$\omega_1 \leq \text{add}(\mathcal{I}) \leq \text{cov}(\mathcal{I}) \leq 2^\omega$$

and a typical question is whether any of the above inequalities could consistently be strict. Judah, Miller and Shelah built a model for $\text{add}(\mathcal{J}(\mathbb{S})) < \text{cov}(\mathcal{J}(\mathbb{S}))$ and Goldstern, Repický, Shelah and Spinas built models for $\text{add}(J(Q)) < \text{cov}(J(Q))$ for $Q \in \{\mathbb{M}, \mathbb{L}\}$. In [Sp1, Sp2, Sp3] Spinas started to develop a combinatorial theory for the two-dimensional Miller forcing \mathbb{M}^2 , where \mathbb{M}^n for $n \in \omega \setminus \{0\}$ consists of all n -tuples of superperfect trees and carries the coordinatewise ordering. It turned out that \mathbb{M}^2 is much closer to \mathbb{M} as it is to \mathbb{M}^3 . On the one hand, for example, both, \mathbb{M} and \mathbb{M}^2 do not add a Cohen-real (see [Mi, Sp2]), while on the other hand, by an unpublished result of Shelah, \mathbb{M}^3 does add a Cohen-real.

Jossen and Spinas [JoSp] started the investigation of the n -dimensional ideals

$$\begin{aligned} \mathcal{J}(Q^n) := \{X \subseteq ({}^\omega \omega)^n \mid \forall (p_0, \dots, p_{n-1}) \in Q^n \exists (q_0, \dots, q_{n-1}) \in Q^n \\ ((q_0, \dots, q_{n-1}) \leq (p_0, \dots, p_{n-1}) \wedge ([q_0] \times \dots \times [q_{n-1}]) \cap X = \emptyset)\} \end{aligned}$$

for $Q \in \{\mathbb{L}, \mathbb{M}\}$. They showed that only the two-dimensional Miller ideal $J(\mathbb{M}^2)$ is a σ -ideal, but neither the higher dimensional Miller ideals $J(\mathbb{M}^n)$

for $n \geq 3$ nor the Laver ideals $J(\mathbb{L}^m)$ for $m \geq 2$ are σ -ideals, thus the additivity of the higher dimensional ideals is ω . Therefore, it is interesting to pay more attention to $\mathcal{J}(\mathbb{M}^2)$. Analogously to the one-dimensional case, Jossen and Spinas have built a model for $\text{add}(J(\mathbb{M}^2)) < \text{cov}(J(\mathbb{M}^2))$. It is obtained – starting with a model of ZFC and the continuum hypothesis – by a countable support iteration of \mathbb{M}^2 of length ω_2 . Since no Cohen-reals are added, $\text{cov}(\mathcal{M}) < \text{cov}(J(\mathbb{M}^2))$ holds in their model, hence, since the continuum equals ω_2 , we have $\text{cov}(\mathcal{M}) = \text{add}(J(\mathbb{M}^2)) = \omega_1$ there, where \mathcal{M} is the ideal of all meager subsets of ${}^\omega\omega$.

Here we prove the consistency of $\text{cov}(\mathcal{M}) < \text{add}(J(\mathbb{M}^2))$ (Theorem 1.1.2). The natural forcings to increase the additivity numbers are called amoeba forcings. In general, given any definable tree forcing Q , an amoeba forcing $\mathbb{A}(Q^n)$ for Q^n is a forcing adding some $(p_0, \dots, p_{n-1}) \in Q^n$ such that every n -tuple of branches $(x_0, \dots, x_{n-1}) \in [p_0] \times \dots \times [p_{n-1}]$ is Q^n -generic. We construct an amoeba forcing $\mathbb{A}(\mathbb{M}^2)$ for \mathbb{M}^2 that does not add Cohen-reals. Our construction is inspired by the work of Spinas [Sp1], where, in the one-dimensional situation, amoeba forcings for \mathbb{L} and \mathbb{M} have been constructed, which have the Laver property. The Laver property is a combinatorial property ruling out that Cohen-reals are added. By a result of Shelah, the Laver property is preserved under countable support forcing iterations. Our $\mathbb{A}(\mathbb{M}^2)$ will also have the Laver property, hence we can increase $\text{add}(\mathcal{J}(\mathbb{M}^2))$ without adding Cohen-reals. Let us remark, that the natural amoeba forcings for \mathbb{L} , \mathbb{M} and \mathbb{M}^2 that come to one's mind are not suitable for our purpose, as they add lots of Cohen-reals (see [Sp1]).

Similar ideas as in [Sp1] have been used independently in [LoShVe], where implicitly an amoeba forcing for Sacks forcing has been constructed, which does not add Cohen-reals.

Judah, Miller, Shelah and Veličković [JuMiSh, Ve] have independently shown that Martin's Axiom does not imply that $\text{add}(\mathcal{J}(\mathbb{S})) = 2^\omega$. In contrast to this, it is possible to blow up $\text{add}(\mathcal{J}(\mathbb{L}))$ and $\text{add}(\mathcal{J}(\mathbb{M}))$ by a forcing fulfilling the countable chain condition. In fact, in [JuMiSh] it was shown that Martin's Axiom ($\mathfrak{t} = 2^\omega$ is enough, where \mathfrak{t} is the tower number) implies $\text{add}(\mathcal{J}(\mathbb{L})) = 2^\omega$. The analogous result for \mathbb{M} was claimed as well, but the proof was faulty. It was later corrected in [GoJSp]. Here we show that Martin's Axiom for σ -centered forcings ($\text{MA}(\sigma\text{-centered})$) implies that $\text{add}(\mathcal{J}(\mathbb{M}^2)) = 2^\omega$ hold (Theorem 2.0.5) by combining the ideas of these two papers with the combinatorial properties of \mathbb{M}^2 . As a corollary it turns out that under the assumption that $\text{MA}(\sigma\text{-centered})$ is true forcing with \mathbb{M}^2 does not collapse cardinals. By using the same methods we can prove that $\text{MA}(\sigma\text{-centered})$ implies Martin's Axiom for the forcing \mathbb{M}^2 (Theorem 2.3.5).

In chapter 3 we will treat a completely different problem with similar methods, in the sense that carefully chosen finite products of Mathias forcing are used that have the Laver property. In [Ma], Mathias introduced his famous Math-

ias forcing $M(U)$ restricted to a Ramsey ultrafilter U , i.e. $M(U) := \{(s, S) \in [\omega]^{<\omega} \times [\omega]^\omega \mid (\max(s) < \min(S) \text{ or } s = \emptyset) \text{ and } S \in U\}$ and the ordering is defined by $(s, S) \leq (t, T)$ if and only if $s \supseteq t$ and $S \cup (s \setminus t) \subseteq T$. The analysis of this forcing led him, among many other things, to the result that there are no analytical infinite maximal almost disjoint families in $\mathfrak{P}(\omega)$. Note that maximal almost disjoint families in $\mathfrak{P}(\omega)$ correspond to maximal antichains in the Boolean algebra $\mathfrak{P}(\omega)/\text{fin}$. Here we investigate infinite maximal antichains in finite products of $\mathfrak{P}(\omega)/\text{fin}$. Using similar but simpler ideas as in [LoShVe] and [Sp1], Shelah and Spinas [ShSp2] introduced an n -dimensional version of Mathias forcing with the Laver property. Combining their analysis of this forcing with some combinatorial facts about infinite maximal antichains in $(\mathfrak{P}(\omega)/\text{fin})^n$ we prove that for every $n \in \omega \setminus \{0, 1\}$ there are no analytical infinite maximal antichains in $(\mathfrak{P}(\omega)/\text{fin})^n$ (Theorem 3.1.1). Curiously, in dimension ω there exists a perfect (thus closed) partition, as the following example shows: choose $a \in [\omega]^\omega$ with $\omega \setminus a \in [\omega]^\omega$, too, and consider $\mathcal{A} := \{(a_i)_{i \in \omega} \mid \forall i \in \omega (a_i \in \{a, \omega \setminus a\})\}$.

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Chapter 0

Preliminaries

We fix our notation: For a set A let $[A]^{<\omega}$ denote the collection of all finite subsets of A and let $[A]^\omega$ denote the collection of all countably infinite subsets of A . ${}^{<\omega}A$ denotes the set of all functions $s : n \rightarrow A$ for some $n \in \omega$, ${}^\omega A$ is the set of all functions $f : \omega \rightarrow A$. For $s \in {}^{<\omega}A$ for any set A we write $|s| = n$ (the length of s) if $s : n \rightarrow A$. For $a \in A$ let $s \frown \langle a \rangle$ be the function $s \cup \{(|s|, a)\}$.

Trees: A set $p \subseteq {}^{<\omega}\omega$ is called a tree if for every $\sigma \in p$ and $\tau \subseteq \sigma$ we have $\tau \in p$. Given a tree $p \subseteq {}^{<\omega}\omega$, the set of all infinite branches through p is denoted by $[p]$. By $\text{st}(p)$ we denote the stem of p , this is the shortest splitnode in p . For $\sigma \in p$ let $(p)_\sigma$ be the subtree of p consisting of all $\nu \in p$ which are comparable with σ . For $\sigma \in p$ let $\text{succ}_p(\sigma)$ designates the set of all extensions of the form $\sigma \frown \langle n \rangle$, for $n \in \omega$, with $\sigma \frown \langle n \rangle \in p$. Call $\sigma \in p$ a splitnode, if $|\text{succ}_p(\sigma)| > 1$, an infinite splitnode, if $|\text{succ}_p(\sigma)|$ is infinite. The set of all infinite splitnodes of a tree p is denoted by $\text{split}(p)$. For $\sigma \in \text{split}(p)$ define $\text{Succ}_p(\sigma) := \{\tau \in \text{split}(p) \mid \sigma \subsetneq \tau \wedge \forall \rho (\sigma \subsetneq \rho \subsetneq \tau \Rightarrow \rho \notin \text{split}(p))\}$, the set of all direct successors of σ in the tree sense in $\text{split}(p)$. A tree $p \subseteq {}^{<\omega}\omega$ is called a *Laver tree* if it has a stem $\text{st}(p)$ and for every $\sigma \in p$ with $\sigma \supseteq \text{st}(p)$ we have $\sigma \in \text{split}(p)$. *Laver forcing* is the set of all Laver trees \mathbb{L} ordered by inclusion. A tree $p \subseteq {}^{<\omega}\omega$ is called a *Miller* or *superperfect tree*, if it has a stem and for every $\sigma \in p$ there exists an extension $\tau \supsetneq \sigma$ in p which is an infinite splitnode. Let \mathbb{M} denote the set of all superperfect trees p with the additional property, that every splitnode of p is an infinite one. *Miller forcing* is \mathbb{M} ordered by inclusion, i.e. $p \leq q$ if and only if $p \subseteq q$. Notice that \mathbb{M} is dense in the forcing sense in the set of all superperfect trees. During the whole paper, if we write p is “superperfect” or “Miller” we mean $p \in \mathbb{M}$. Let \mathbb{M}^2 and more generally \mathbb{M}^n denote the set of all n -tuples of superperfect trees carrying the coordinatewise ordering. We write $p \leq^0 q$ if $p \leq q$ and additionally we have $\text{st}(p) = \text{st}(q)$ and $(p', q') \leq^0 (p, q)$ if $(p', q') \leq (p, q)$ and we have that $\text{st}(p') = \text{st}(p)$ and $\text{st}(q') = \text{st}(q)$ hold.

A set $S \subseteq {}^\omega\omega$ is called superperfect, if there exists a superperfect tree $p \in$

\mathbb{M} with $S = [p]$. For $(p, q) \in \mathbb{M}^2$ by $[p] \times^+ [q]$ we denote the upper half of the superperfect rectangle $[p] \times [q]$, i.e. the set of all $(x, y) \in [p] \times [q]$ with $x(|\text{st}(p)|) < y(|\text{st}(q)|)$. Similar, $[p] \times^- [q]$ denotes the lower half of $[p] \times [q]$.

We use the following well-ordering $<$ of ${}^{<\omega}\omega$:

$$\begin{aligned} \sigma < \tau &: \Leftrightarrow \max\{|\sigma|, \max \text{ran}(\sigma)\} < \max\{|\tau|, \max \text{ran}(\tau)\} \\ &\quad \vee (\max\{|\sigma|, \max \text{ran}(\sigma)\} = \max\{|\tau|, \max \text{ran}(\tau)\} \wedge |\sigma| < |\tau|) \\ &\quad \vee (\max\{|\sigma|, \max \text{ran}(\sigma)\} = \max\{|\tau|, \max \text{ran}(\tau)\} \wedge |\sigma| = |\tau| \\ &\quad \quad \wedge \sigma \text{ precedes } \tau \text{ lexicographically}). \end{aligned}$$

Let $T : \omega \rightarrow {}^{<\omega}\omega$ be the order preserving enumeration of $({}^{<\omega}\omega, <)$ and write $\#\rho = n$ if $T(n) = \rho$, so we have $\#\rho \geq \max\{|\rho|, \max \text{ran}(\rho)\}$.

We repeat the following definition and fact of [Sp2]:

Definition 0.0.1 Let $(\sigma, \tau) \in ({}^{<\omega}\omega)^2$ and $(x, y) \in ({}^\omega\omega)^2$ with $\sigma \subseteq x$ and $\tau \subseteq y$. We say that (x, y) *oscillates infinitely often above* (σ, τ) if there exists a strictly increasing sequence $(k_i)_{i \in \omega}$ in ω such that the following hold for all $n \in \omega$:

$$\begin{aligned} k_0 &= |\tau|, k_1 > |\sigma|; \\ k_{2n} &= \min\{i \in \omega \mid y(i) > \#x \upharpoonright k_{2n+1}\}; \\ k_{2n+1} &= \min\{i \in \omega \mid x(i) > \#y \upharpoonright k_{2n+2}\}; \\ k_{2n+1} &< y(k_{2n}) < k_{2n+2} < x(k_{2n+1}) < k_{2n+3}. \end{aligned}$$

The sequence

$$(\sigma, \tau, x \upharpoonright k_1, y \upharpoonright k_2, x \upharpoonright k_3, y \upharpoonright k_4, \dots)$$

is called the $\text{type}_{\sigma, \tau}$ -sequence of the pair (x, y) .

Fact 0.0.2 [Sp2] *For every $(p, q) \in \mathbb{M}^2$ there exists $(p', q') \leq^0 (p, q)$ such that for every $(x, y) \in [p'] \times^+ [q']$, (x, y) oscillates infinitely often above $(\text{st}(p), \text{st}(q))$. Moreover, if $(\mu_0, \nu_0, \mu_1, \nu_1, \dots)$ is the type sequence of (x, y) , we have $\{\mu_n \mid n \in \omega\} \subseteq \text{split}(p')$ and $\{\nu_n \mid n \in \omega\} \subseteq \text{split}(q')$. \square*

Hence for every $(x, y) \in [p'] \times [q']$, (p', q') as in fact 0.0.2, there is a unique associated sequence $(k_i)_{i \in \omega}$ in ω , which is determined solely by (x, y) and $(\text{st}(p), \text{st}(q)) =: (\sigma, \tau)$. Define the 0-type pair of the pair of branches (x, y) by $\text{tp}_{\sigma, \tau}\text{-0-pair}(x, y) := (\sigma, \tau)$ and for every $n \in \omega$ define $\text{tp}_{\sigma, \tau}\text{-(}2n+1\text{)-pair}(x, y) = (x \upharpoonright k_{2n+1}, y \upharpoonright k_{2n})$ and $\text{tp}_{\sigma, \tau}\text{-(}2n+2\text{)-pair}(x, y) = (x \upharpoonright k_{2n+1}, y \upharpoonright k_{2n+2})$. Using this we can define a partial function

$$\text{tp}_{\sigma, \tau}^{p', q'} : ({}^{<\omega}\omega)^2 \rightarrow \omega$$

by letting $\text{tp}_{\sigma,\tau}^{p',q'}(\mu, \nu) = n$ if and only if there exists $(x, y) \in [p'] \times^+ [q']$ such that $(\mu, \nu) = \text{tp}_{\sigma,\tau}\text{-}n\text{-pair}(x, y)$.

For $(\mu, \nu) \in p' \times q'$ with $\text{tp}_{\sigma,\tau}^{p',q'}(\mu, \nu) = 2n$ there exists a unique sequence $(\mu_0, \nu_0, \dots, \mu_n, \nu_n)$, which is the initial sequence of length $2n$ of the type $_{\sigma,\tau}$ -sequence of (x, y) for some infinitely oscillating $(x, y) \in [p'] \times^+ [q']$ over (σ, τ) . We call this sequence the type $_{\sigma,\tau}$ -sequence of the pair (μ, ν) and define $\text{tp}_{\sigma,\tau}\text{-}i\text{-pair}(\mu, \nu) = \text{tp}_{\sigma,\tau}\text{-}i\text{-pair}(x, y)$ for all $i \leq 2n$. And similar for $(\mu, \nu) \in p' \times q'$ with $\text{tp}_{\sigma,\tau}^{p',q'}(\mu, \nu)$ is odd.

If (σ, τ) or (p', q') are clear from context, we omit them in the above notation.

For $(p, q) \in \mathbb{M}^2$ and $(\sigma, \tau) \in p \times q$ let

$$\text{TP}_{\sigma,\tau}(p, q) := \text{dom}(\text{tp}_{\sigma,\tau}^{p,q}) \cap (\text{split}((p)_\sigma) \times^+ \text{split}((q)_\tau)),$$

the set of all type pairs of (p, q) , and for $n \in \omega$ let $\text{TP}_{\sigma,\tau}^n(p, q)$ denote the set of all type pairs of (p, q) of type n . For $(\mu, \nu) \in \text{TP}_{\sigma,\tau}^n$ with n even let

$$\begin{aligned} \text{Sop}_{\sigma,\tau}^{p,q}(\mu, \nu) := \{ \mu' \in \text{split}(p_\mu) \mid \text{tp}_{\sigma,\tau}^{p,q}(\mu', \nu) = n + 1 \wedge \\ \text{tp}_{\sigma,\tau}\text{-}n\text{-pair}(\mu', \nu) = (\mu, \nu) \}, \end{aligned}$$

the set of all possible successive oscillation points, and similar, we define $\text{Sop}_{\sigma,\tau}^{p,q}(\mu, \nu)$, if n is odd.

Note that for (p', q') as in fact 0.0.2 and $(u, v) \leq (p', q')$ and $(\sigma, \tau) \in u \times v$ we in general have $\text{TP}_{\sigma,\tau}(p', q') \cap (u \times v) \neq \text{TP}_{\sigma,\tau}(u, v)$, but there exists $(u', v') \leq^0 (u, v)$ such that for almost all $n \in \omega$ and every $(x, y) \in [u'] \times^+ [v']$ we have $\text{tp}_{\sigma,\tau}\text{-}n\text{-pair}(x, y) \in \text{TP}_{\sigma,\tau}(u', v')$ (and therefore $\text{tp}_{\sigma,\tau}\text{-}n\text{-pair}(x, y) \in \text{split}(u') \times \text{split}(v')$).

For the rest of the paper we always assume that we work with elements $(p, q) \in \mathbb{M}^2$, which have the property of (p', q') of fact 0.0.2. By the fact, the set of all such (p, q) is dense in \mathbb{M}^2 , so with this partial ordering we get forcing extensions isomorphic to those of \mathbb{M}^2 .

We will need the following result of Miller, which is not very difficult to prove:

Fact 0.0.3 [Mi] *For every colouring of the splitnodes of a Miller tree p in finitely many colours, there exists a $q \in \mathbb{M}$ with $q \leq p$ and such that all splitnodes of q have the same colour.* \square

Ideals: For every ideal \mathcal{I} on ${}^\omega\omega$, its additivity and its covering number is defined by

$$\begin{aligned} \text{add}(\mathcal{I}) &:= \min\{|F| \mid F \subseteq \mathcal{I} \text{ and } \bigcup F \notin \mathcal{I}\}, \\ \text{cov}(\mathcal{I}) &:= \min\{|F| \mid F \subseteq \mathcal{I} \text{ and } \bigcup F = {}^\omega\omega\}. \end{aligned}$$

Let \mathcal{M} denote the ideal of the meager subsets of ${}^\omega\omega$.

Forcing: For a forcing P , an M -generic filter G for P and a P -name \dot{a} we write

$\text{val}_G(\dot{a})$ or \dot{a}_G for the value of \dot{a} . For elements x of the ground model we do not distinguish between the canonical name \check{x} and x itself.

Recall that a forcing P has the Laver property if there exists $F \in {}^\omega\omega \cap V$ such that for every P -name \dot{f} for an element of ${}^\omega\omega$, every $p \in P$ and every $g \in {}^\omega\omega \cap V$ with $p \Vdash_P \forall n \in \omega (\dot{f}(n) \leq g(n))$, there exist $q \leq p$ and $(H_n)_{n \in \omega}$ in V such that $|H_n| \leq F(n)$ and $q \Vdash_P \forall n (\dot{f}(n) \in H_n)$.

Chapter 1

The consistency of $\text{cov}(\mathcal{M}) < \text{add}(\mathcal{J}(\mathbb{M}^2))$

1.1 An amoeba forcing for \mathbb{M}^2

We repeat the definition of the introduction:

Definition 1.1.1 Let

$$J(\mathbb{M}^n) := \{X \subseteq (\omega^\omega)^n \mid \forall (p_0, \dots, p_{n-1}) \in \mathbb{M}^n \exists (q_0, \dots, q_{n-1}) \in \mathbb{M}^n \\ ((q_0, \dots, q_{n-1}) \leq (p_0, \dots, p_{n-1}) \wedge ([q_0] \times \dots \times [q_{n-1}] \cap X = \emptyset)\}.$$

It is easy to see that $J(\mathbb{M})$ is a σ -ideal. Jossen and Spinas [JoSp] have shown that $J(\mathbb{M}^2)$ is a σ -ideal but $J(\mathbb{M}^n)$ is not for $n > 2$. And they pointed out parallelisms between the behaviour of $\mathcal{J}(\mathbb{M})$ and $\mathcal{J}(\mathbb{M}^2)$. Here, we underpin this by showing that, analogous to the one-dimensional case, we have the following:

Theorem 1.1.2 *It is relatively consistent with ZFC that $\text{add}(J(\mathbb{M}^2)) = \omega_2$ and $\text{cov}(\mathcal{M}) = \omega_1$.*

For this define an amoeba forcing $\mathbb{A}(\mathbb{M}^2)$ for \mathbb{M}^2 as follows:

Definition 1.1.3 Let $\mathbb{A}(\mathbb{M}^2)$ be the set of all pairs $((s, p), (t, q))$ such that

- (i) $(p, q) \in \mathbb{M}^2$, $s : n \rightarrow \text{split}(p)$ and $t : m \rightarrow \text{split}(q)$ for some $n, m \in \omega \setminus \{0\}$ with $n = m$ or $n = m + 1$;
- (ii) the downward closure S of $\text{ran}(s)$ is a finite subtree of p and $\text{split}(S) \subseteq \text{ran}(s)$, the downward closure T of $\text{ran}(t)$ is a finite subtree of q with $\text{split}(T) \subseteq \text{ran}(t)$;

- (iii) the mappings $\Phi : (\text{ran}(s), \subseteq) \rightarrow (\text{ran}(T \upharpoonright n), \subseteq)$ and $\Psi : (\text{ran}(t), \subseteq) \rightarrow (\text{ran}(T \upharpoonright m), \subseteq)$ defined by $s(k) \mapsto T(k)$ and $t(k) \mapsto T(k)$ are both isomorphisms, where T is the order preserving enumeration of our well-ordering \prec of ${}^{<\omega}\omega$ fixed in the preliminaries;
- (iv) if $s(j)$ is the immediate predecessor in the tree sense of $s(k+1)$ in $\text{split}(p)$ then $s(k+1)(|s(j)|) > \max(\{\#s(i) \mid i \leq k\} \cup \{\#t(i) \mid i \leq k\})$ and $|s(k+1)| > s(k+1)(|s(j)|)$;
 if $t(j)$ is the immediate predecessor in the tree sense of $t(k+1)$ in $\text{split}(q)$ then $t(k+1)(|t(j)|) > \max(\{\#s(i) \mid i \leq k+1\} \cup \{\#t(i) \mid i \leq k\})$ and $|t(k+1)| > t(k+1)(|t(j)|)$.

Define a partial ordering \leq on $\mathbb{A}(\mathbb{M}^2)$ by

$$((s, p), (t, q)) \leq ((s', p'), (t', q')) \iff s \supseteq s', t \supseteq t', p \subseteq p' \text{ and } q \subseteq q'.$$

Oftenly we will write s and t as tuples.

1.2 $\mathbb{A}(\mathbb{M}^2)$ has the pure decision property

To rule out that $\mathbb{A}(\mathbb{M}^2)$ adds Cohen-reals, we need the following decision property of $\mathbb{A}(\mathbb{M}^2)$:

Lemma 1.2.1 $\mathbb{A}(\mathbb{M}^2)$ has the pure decision property.

Here, the *pure decision property* means that for every $\mathbb{A}(\mathbb{M}^2)$ -name Θ and every $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ such that

$$((s, p), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta \in \{0, 1\}$$

there exists $((s', p'), (t', q')) \in \mathbb{A}(\mathbb{M}^2)$ with $((s', p'), (t', q')) \leq ((s, p), (t, q))$ and $((s', p'), (t', q'))$ decides Θ (this means there exists $j < 2$ such that $((s', p'), (t', q')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = j$).

Proof of lemma 1.2.1: Let Θ be an $\mathbb{A}(\mathbb{M}^2)$ -name and $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ such that $((s, p), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta \in \{0, 1\}$. Our goal is to construct by a fusion a pair (p', q') of superperfect trees with $(p', q') \leq^0 (p, q)$, $((s, p'), (t, q')) \in \mathbb{A}(\mathbb{M}^2)$ and with the following property:

- (*) For all $(s', t') \in {}^{<\omega}(\text{split}(p')) \times {}^{<\omega}(\text{split}(q'))$, if there is some $(u, v) \leq (p', q')$ with $((s', u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$, $((s', u), (t', v)) \leq ((s, p'), (t, q'))$ and $((s', u), (t', v))$ decides Θ , then already $((s', p'), (t', q'))$ decides Θ .

Suppose $|s| = |t|$, so we have to extend s first. The case $|s| = |t| + 1$ is very similar and is not written down. We want to construct sequences $(\sigma_n)_{n \in \omega}$, $(\tau_n)_{n \in \omega}$ in ${}^{<\omega}\omega$ and $((u'_n, v'_n))_{n \in \omega \setminus \{0\}}$, $((u_n, v_n))_{n \in \omega}$ in \mathbb{M}^2 such that, letting $s_n := s \frown \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ and $t_n := t \frown \langle \tau_0, \dots, \tau_{n-1} \rangle$, for every $n \in \omega$ the following hold:

- (a) $(p, q) \geq (u_0, v_0) \geq (u'_{n+1}, v'_{n+1}) \geq (u_{n+1}, v_{n+1}) \geq (u'_{n+2}, v'_{n+2})$;
- (b) $((s_n, u_n), (t_n, v_n)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s_{n+1}, u'_{n+1}), (t_n, v'_{n+1})) \in \mathbb{A}(\mathbb{M}^2)$;
- (c) for every $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_n))$ with $s'(|s'| - 1) = \sigma_n$, if there exists $(u, v) \in \mathbb{M}^2$ such that $((s', u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$, $((s', u), (t', v)) \leq ((s, u'_{n+1}), (t, v'_{n+1}))$ and $((s', u), (t', v))$ decides Θ , then $((s', u'_{n+1}), (t', v'_{n+1}))$ decides Θ as well;
and for every $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_{n+1}))$ with $t'(|t'| - 1) = \tau_n$, if there exists $(u, v) \in \mathbb{M}^2$ with $((s', u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$, $((s', u), (t', v)) \leq ((s, u_{n+1}), (t, v_{n+1}))$ and $((s', u), (t', v))$ decides Θ , then $((s', u_{n+1}), (t', v_{n+1}))$ decides Θ as well.

For beginning the construction define $s_0 := s$, $t_0 := t$, $u_0 := p$ and $v_0 := q$.

Suppose $s_n = s \frown \langle \sigma_0, \dots, \sigma_{n-1} \rangle$, $t_n = t \frown \langle \tau_0, \dots, \tau_{n-1} \rangle$, u_n and v_n are already defined. Fix $\mu \in \text{ran}(s_n)$ maximal such that whenever the element $((s_n \frown \langle \sigma \rangle, u_n), (t_n, v_n)) \in \mathbb{A}(\mathbb{M}^2)$ extends $((s_n, u_n), (t_n, v_n))$ we have $\mu \subsetneq \sigma$. Choose $\sigma_n \in \text{Succ}_{u_n}(\mu)$ such that $((s_n \frown \langle \sigma_n \rangle, u_n), (t_n, v_n)) \in \mathbb{A}(\mathbb{M}^2)$ and define $s_{n+1} := s_n \frown \langle \sigma_n \rangle$.

Claim 1: There exists $(u'_{n+1}, v'_{n+1}) \in \mathbb{M}^2$ such that $(u'_{n+1}, v'_{n+1}) \leq (u_n, v_n)$, $((s_{n+1}, u'_{n+1}), (t_n, v'_{n+1}))$ is an element of $\mathbb{A}(\mathbb{M}^2)$ and the first part of (c) holds.

Proof of claim 1: Let $((\bar{s}_i, \bar{t}_i))_{i < N}$ for some $N \in \omega$ enumerate all pairs $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_n))$ with $s'(|s'| - 1) = \sigma_n$. We want to consider successively all pairs (\bar{s}_i, \bar{t}_i) and thin out (u_n, v_n) to get (u'_{n+1}, v'_{n+1}) . Let $p_0 := u_n$ and $q_0 := v_n$ and suppose we have already constructed $(p_i, q_i) \in \mathbb{M}^2$ for some $i < N$.

Consider (\bar{s}_i, \bar{t}_i) and suppose there exists a pair $(u, v) \leq (p_i, q_i)$ such that $((\bar{s}_i, u), (\bar{t}_i, v)) \in \mathbb{A}(\mathbb{M}^2)$, $((\bar{s}_i, u), (\bar{t}_i, v)) \leq ((s, p_i), (t, q_i))$ and $((\bar{s}_i, u), (\bar{t}_i, v))$ decides Θ (if there does not exist such a pair (u, v) we let $p_{i+1} := p_i$ and $q_{i+1} := q_i$). Let S_i be the downward closure of $\text{ran}(\bar{s}_i)$ in the tree sense and define

$$r_p := \bigcup \{(p_i)_\sigma \mid \sigma \in \text{ran}(s_{n+1}) \setminus S_i\},$$

$$r'_p := \bigcup \{(p_i)_{\sigma \frown \langle k \rangle} \mid \sigma \in (\text{ran}(s_{n+1}) \setminus \text{ran}(\bar{s}_i)) \cap S_i \wedge \sigma \frown \langle k \rangle \notin S_i\}$$

and let

$$p_{i+1} := u \cup r_p \cup r'_p.$$

Then $p_{i+1} \in \mathbb{M}$ and note that every $\sigma \in \text{ran}(s_{n+1}) \setminus \text{ran}(\bar{s}_i)$ remains an infinite splitnode of p_{i+1} . Analogous let T_i be the downward closure of $\text{ran}(\bar{t}_i)$ and define

$$r_q := \bigcup \{(q_i)_\tau \mid \tau \in \text{ran}(t_{n+1}) \setminus T_i\},$$

$$r'_q := \bigcup \{(q_i)_{\tau \frown \langle k \rangle} \mid \tau \in (\text{ran}(t_{n+1}) \setminus \text{ran}(\bar{t}_i)) \cap T_i \wedge \tau \frown \langle k \rangle \notin T_i\}$$

and let

$$q_{i+1} := v \cup r_q \cup r'_q.$$

Again, every $\tau \in \text{ran}(t_n) \setminus \text{ran}(\bar{t}_i)$ remains an infinite splitnode of q_{i+1} . Easily, $((s_{n+1}, p_{i+1}), (t_n, q_{i+1})) \in \mathbb{A}(\mathbb{M}^2)$ and $((s_{n+1}, p_{i+1}), (t_n, q_{i+1})) \leq ((s_{n+1}, p_i), (t_n, q_i))$. And $((\bar{s}_i, p_{i+1}), (\bar{t}_i, q_{i+1}))$ decides Θ as well, because for $((\bar{s}, \bar{u}), (\bar{t}, \bar{v})) \leq ((\bar{s}_i, p_{i+1}), (\bar{t}_i, q_{i+1}))$ our construction and (iv) of the definition of $\mathbb{A}(\mathbb{M}^2)$ guarantees that we have $\bar{s} \setminus \bar{s}_i \subseteq u$ and $\bar{t} \setminus \bar{t}_i \subseteq v$ (remember that $\bar{s}_i(|\bar{s}_i - 1|) = \sigma_n$). So for $\bar{u} := \bar{u} \setminus (r_p \cup r'_p)$ and $\bar{v} := \bar{v} \setminus (r_q \cup r'_q)$ we have $((\bar{s}, \bar{u}), (\bar{t}, \bar{v})) \in \mathbb{A}(\mathbb{M}^2)$ and $((\bar{s}, \bar{u}), (\bar{t}, \bar{v})) \leq ((\bar{s}_i, u), (\bar{t}_i, v))$, hence $((\bar{s}, \bar{u}), (\bar{t}, \bar{v}))$ and $((\bar{s}_i, u), (\bar{t}_i, v))$ are compatible, as required.

Finally let $u'_{n+1} := p_N$ and $v'_{n+1} := q_N$. Then (u'_{n+1}, v'_{n+1}) is as desired. \square (claim1)

Fix now $\nu \in \text{ran}(t_n)$, the point where we have to extend the finite tree generated by $\text{ran}(t_n)$. Choose an element $\tau_n \in \text{Succ}_{v'_{n+1}}(\nu)$ such that we have $((s_{n+1}, u'_{n+1}), (t_n \frown \langle \tau \rangle)) \in \mathbb{A}(\mathbb{M}^2)$ and define $t_{n+1} := t_n \frown \langle \tau_n \rangle$. In the same way as we got claim 1, we get

Claim 2: There exists $(u_{n+1}, v_{n+1}) \in \mathbb{M}^2$ such that $(u_{n+1}, v_{n+1}) \leq (u'_{n+1}, v'_{n+1})$, $((s_{n+1}, u_{n+1}), (t_{n+1}, v_{n+1})) \in \mathbb{A}(\mathbb{M}^2)$ and for every $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_{n+1}))$ with $t'(|t' - 1|) = \tau_n$, if there exists $(u, v) \in \mathbb{M}^2$ such that $((s', u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$, $((s', u), (t', v)) \leq ((s, u_{n+1}), (t, v_{n+1}))$ and the pair $((s', u), (t', v))$ decides Θ , then $((s', u_{n+1}), (t', v_{n+1}))$ decides Θ as well.

\square (claim 2)

This finishes our construction.

Let p' be the unique superperfect tree determined by

$$\text{split}(p') = \bigcup_{n \in \omega} \text{ran}(s_n)$$

and q' the unique superperfect tree determined by

$$\text{split}(q') = \bigcup_{n \in \omega} \text{ran}(t_n).$$

Claim 3: (p', q') satisfies property $(*)$ (see at the beginning of this proof).

Proof of claim 3: Suppose $(s', t') \in {}^{<\omega}(\text{split}(p')) \times {}^{<\omega}(\text{split}(q'))$ and there exists some $(u, v) \leq (p', q')$ with $((s', u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$, $((s', u), (t', v)) \leq ((s, p'), (t, q'))$ and $((s', u), (t', v))$ decides Θ . If $|s'| = |t'|$, there exists $n \in \omega$ with $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_{n+1}))$ and $t'(|t'| - 1) = \tau_n$. And if $|s'| = |t'| + 1$ there exists $n \in \omega$ such that $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_n))$ and $s'(|s'| - 1) = \sigma_n$. Since

$$\begin{aligned} ((s', u), (t', v)) &\leq ((s', p'), (t', q')) \leq \\ &((s', u_{n+1}), (t', v_{n+1})) \leq ((s', u'_{n+1}), (t', v'_{n+1})), \end{aligned}$$

property (c) gives us that $((s', u'_{n+1}), (t', v'_{n+1}))$ or $((s', u_{n+1}), (t', v_{n+1}))$ decides Θ , so $((s', p'), (t', q'))$ decides Θ as well. \square (claim 3)

Claim 4: $((s, p'), (t, q'))$ decides Θ .

Proof of claim 4: Suppose not. And suppose again that we have $|s| = |t|$; the case $|s| = |t| + 1$ is once more similar. We will construct sequences $((u_n, v_n))_{n \in \omega}$, $((u'_n, v'_n))_{n \in \omega}$ in \mathbb{M}^2 and $(s_n)_{n \in \omega}$ and $(t_n)_{n \in \omega}$ in ${}^{<\omega}({}^{<\omega}\omega)$ with $s_n = s \frown \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ and $t_n = t \frown \langle \tau_0, \dots, \tau_{n-1} \rangle$ such that for every $n \in \omega$ the following hold:

- (a') $s \subseteq s_n \subseteq s_{n+1}$ and $t \subseteq t_n \subseteq t_{n+1}$;
- (b') $(p', q') \geq (u'_n, v'_n) \geq (u_n, v_n) \geq (u'_{n+1}, v'_{n+1})$;
- (c') $((s_{n+1}, p'), (t_{n+1}, q')) \in \mathbb{A}(\mathbb{M}^2)$ and $((s_{n+1}, p'), (t_n, q')) \in \mathbb{A}(\mathbb{M}^2)$;
- (d') if $(s', t') \in {}^{<\omega}(\text{ran}(s_n)) \times {}^{<\omega}(\text{ran}(t_n))$ such that $|s'| = |t'|$, $t'(|t'| - 1) = \tau_{n-1}$ for $n > 0$, $((s', u'_n), (t', v'_n)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s', u'_n), (t', v'_n)) \leq ((s, u'_n), (t, v'_n))$, then there is no $\sigma \in \text{split}(u'_n)$ such that we have $((s' \frown \langle \sigma \rangle, u'_n), (t', v'_n)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s' \frown \langle \sigma \rangle, u'_n), (t', v'_n))$ decides Θ ;
- (e') if $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_n))$ such that $|s'| = |t'| + 1$, $s'(|s'| - 1) = \sigma_n$, $((s', u_n), (t', v_n)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s', u_n), (t', v_n)) \leq ((s, u_n), (t, v_n))$, then there is no $\tau \in \text{split}(v_n)$ such that we have $((s', u_n), (t' \frown \langle \tau \rangle, v_n)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s', u_n), (t' \frown \langle \tau \rangle, v_n))$ decides Θ .

Define $s := s_0$ and $t := t_0$. For beginning the induction fix $\mu \in \text{ran}(s)$, the unique point where we can extend the finite subtree of p generated by $\text{ran}(s)$, and prune p' by defining

$$\bar{p} := \bigcup \{(p')_{\mu \frown \langle k \rangle} \mid \forall \sigma \in \text{ran}(s) \setminus \{\mu\} (\mu \frown \langle k \rangle \not\subseteq \sigma)\}.$$

Then \bar{p} is a superperfect tree with stem μ and for $\rho \in \text{Succ}_{\bar{p}}(\mu)$ we have $(\bar{p})_\rho = (p')_\rho$. For every $\rho \in \text{Succ}_{\bar{p}}(\mu)$ we want to colour the set

$$S_\rho := \{\sigma \in \text{split}(p')_\rho \mid ((s \frown \langle \sigma \rangle, p'), (t, q')) \in \mathbb{A}(\mathbb{M}^2)\}$$

in three colours by defining

$$\begin{aligned} \varphi_\rho : S_\rho &\longrightarrow \{0, 1, 2\} \\ \sigma &\longmapsto \begin{cases} 0, & \text{if } ((s \frown \langle \sigma \rangle, p'), (t, q')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = 0, \\ 1, & \text{if } ((s \frown \langle \sigma \rangle, p'), (t, q')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = 1, \\ 2, & \text{if } ((s \frown \langle \sigma \rangle, p'), (t, q')) \text{ does not decide } \Theta. \end{cases} \end{aligned}$$

Fact 0.0.3 gives us for every $\rho \in \text{Succ}_{\bar{p}}(\mu)$ a superperfect tree $p^\rho \subseteq (p')_\rho$ whose splitnodes all have colour ε_ρ .

If there is an infinite set $A \subseteq \text{Succ}_{\bar{p}}(\mu)$ and an $\varepsilon \in \{0, 1\}$ such that for every $\rho \in A$ we have $\varepsilon_\rho = \varepsilon$, then we can define an element of $\mathbb{A}(\mathbb{M}^2)$ which forces Θ to be this ε as follows: Define

$$\bar{p}' := \bigcup \{p^\rho \mid \rho \in A\} \cup (p' \setminus \bar{p})$$

then clearly $((s, \bar{p}'), (t, q')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = \varepsilon$.

By the property (*) of our constructed pair (p', q') this implies that $((s, p'), (t, q'))$ decides Θ , a contradiction to our assumption.

So there must exist a finite set $F \subseteq \text{Succ}_{\bar{p}}(\mu)$ such that for every $\rho \in \text{Succ}_{\bar{p}}(\mu) \setminus F$ we have $\varepsilon_\rho = 2$. Define

$$u'_0 := \bigcup \{p^\rho \mid \rho \in \text{Succ}_{\bar{p}}(\mu) \setminus F\} \cup (p' \setminus \bar{p})$$

and $v'_0 := q'$. Then for no $\sigma \in \text{split}(u'_0)$ with $((s \frown \langle \sigma \rangle, u'_0), (t, v'_0)) \in \mathbb{A}(\mathbb{M}^2)$ the element $((s \frown \langle \sigma \rangle, u'_0), (t, v'_0))$ decides Θ . Choose $\sigma_0 \in \text{Succ}_{u'_0}(\mu)$ such that $((s \frown \langle \sigma_0 \rangle, u'_0), (t, v'_0)) \in \mathbb{A}(\mathbb{M}^2)$ and define $s_1 := s \frown \langle \sigma_0 \rangle$.

Now fix $\nu \in \text{ran}(t)$, the unique point where we can extend the finite subtree generated by $\text{ran}(t)$. We want to do the same as before at the right side: Prune v'_0 by defining

$$\bar{v} := \bigcup \{(v'_0)_{\nu \frown \langle k \rangle} \mid \forall \tau \in \text{ran}(t) \setminus \{\nu\} (\nu \frown \langle k \rangle \not\subseteq \tau)\}$$

and colour for every $\rho \in \text{Succ}_{\bar{v}}(\nu)$ the set

$$T_\rho := \{\tau \in \text{split}(v'_0)_\rho \mid ((s_1, u'_0), (t \frown \langle \tau \rangle, v'_0)) \in \mathbb{A}(\mathbb{M}^2)\}$$

by defining

$$\begin{aligned} \psi_\rho : T_\rho &\longrightarrow \{0, 1, 2\} \\ \tau &\longmapsto \begin{cases} 0, & \text{if } ((s_1, u'_0), (t \frown \langle \tau \rangle, v'_0)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = 0, \\ 1, & \text{if } ((s_1, u'_0), (t \frown \langle \tau \rangle, v'_0)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = 1, \\ 2, & \text{if } ((s_1, u'_0), (t \frown \langle \tau \rangle, v'_0)) \text{ does not decide } \Theta. \end{cases} \end{aligned}$$

Again we get for every $\rho \in \text{Succ}_{\bar{v}}(\nu)$ a superperfect tree $q^\rho \subseteq (v'_0)_\rho$, whose splitnodes all have colour ε_ρ . If there exists an infinite set $A \subseteq \text{Succ}_{\bar{v}}(\nu)$ and $\varepsilon \in \{0, 1\}$, such that for every $\rho \in A$ we have $\varepsilon_\rho = \varepsilon$, we can define

\bar{v}' such that $((s_1, u'_0), (t, \bar{v}')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = \varepsilon$ and therefore – again by property (*) – $((s_1, p'), (t, q'))$ decides Θ . But $((s_1, u'_0), (t, v'_0)) \leq ((s_1, p'), (t, q'))$, hence $((s_1, u'_0), (t, v'_0))$ decides Θ , this is a contradiction to the choice of (u'_0, v'_0) (property (d') for $n = 0$). Hence we get a finite set $F \subseteq \text{Succ}_{\bar{v}}(\nu)$ with $\varepsilon_\rho = 2$ for every $\rho \in \text{Succ}_{\bar{v}}(\nu) \setminus F$. Define

$$v_0 := \bigcup \{q^\rho \mid \rho \in \text{Succ}_{\bar{v}}(\nu) \setminus F\} \cup (v'_0 \setminus \bar{v})$$

and $u_0 := u'_0$. Then for no $\tau \in \text{split}(v_0)$ with $((s_1, u_0), (t \frown \langle \tau \rangle, v_0)) \in \mathbb{A}(\mathbb{M}^2)$ the element $((s_1, u_0), (t \frown \langle \tau \rangle, v_0))$ decides Θ . Choose $\tau_0 \in \text{Succ}_{v_0}(\nu)$ with $((s_1, u_0), (t \frown \langle \tau_0 \rangle, v_0)) \in \mathbb{A}(\mathbb{M}^2)$ and define $t_1 := t \frown \langle \tau_0 \rangle$.

Suppose we have constructed $((u_i, v_i))_{i \leq n}$, $((u'_i, v'_i))_{i \leq n}$, $(s_i)_{i \leq n+1}$ and $(t_i)_{i \leq n+1}$ with $s_i = s \frown \langle \sigma_0, \dots, \sigma_{i-1} \rangle$ and $t_i = t \frown \langle \tau_0, \dots, \tau_{i-1} \rangle$ for some $n \in \omega$. First we will describe how to get (u'_{n+1}, v'_{n+1}) : Let $((\bar{s}_i, \bar{t}_i))_{i < N}$ be an enumeration of all $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_{n+1}))$ with $|s'| = |t'|$ and $t'(|t'| - 1) = \tau_n$. By considering successively all pairs (\bar{s}_i, \bar{t}_i) , we will thin out (u_n, v_n) step by step to make (d') true. Let $p_0 := u_n$ and $q_0 := v_n$. Suppose (p_i, q_i) is already constructed for some $i < N$. Consider (\bar{s}_i, \bar{t}_i) and fix $\mu \in \text{ran}(\bar{s}_i)$, the unique point where the finite subtree of u_n generated by \bar{s}_i can be extended. Prune p_i by defining

$$\bar{p}_i := \bigcup \{(p_i)_{\mu \frown \langle k \rangle} \mid \forall \sigma \in \text{ran}(s_n) \setminus \{\mu\} (\mu \frown \langle k \rangle \not\leq \sigma)\}.$$

For every $\rho \in \text{Succ}_{\bar{p}}(\mu)$ we colour the set

$$S_\rho^i := \{\sigma \in \text{split}(p_i)_\rho \mid ((\bar{s}_i \frown \langle \sigma \rangle, p_i), (\bar{t}_i, q_i)) \in \mathbb{A}(\mathbb{M}^2)\}$$

with the three colours 0, 1 and 2 as in the first step of the induction. We get superperfect trees $p^\rho \subseteq (p_i)_\rho$ whose splitnodes all have colour ε_ρ .

If there is an $\varepsilon \in \{0, 1\}$ such that for infinitely many $\rho \in \text{Succ}_{\bar{p}_i}(\mu)$ we have $\varepsilon_\rho = \varepsilon$, we can easily define (\bar{u}, \bar{v}) such that $((\bar{s}_i, \bar{u}), (\bar{t}_i, \bar{v})) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta = \varepsilon$.

By property (*) we have that $((\bar{s}_i, p'), (\bar{t}_i, q'))$ decides Θ . Fix $k \leq n$ such that $\bar{s}_i(|\bar{s}_i| - 1) = \sigma_k$, then $((\bar{s}_i, u_k), (\bar{t}_i, v_k)) \leq ((\bar{s}_i, p'), (\bar{t}_i, q'))$ and therefore $((\bar{s}_i, u_k), (\bar{t}_i, v_k))$ decides Θ as well. But $(\bar{s}_i, \bar{t}_i \upharpoonright (|\bar{t}_i| - 1)) \in {}^{<\omega}(\text{ran}(s_{k+1})) \times {}^{<\omega}(\text{ran}(t_k))$ and $|\bar{s}_i| = |\bar{t}_i \upharpoonright (|\bar{t}_i| - 1)| + 1$, hence the element $((\bar{s}_i, u_k), (\bar{t}_i, v_k))$ does not decide Θ by property (e') for k , a contradiction.

So for all but finitely many $\rho \in \text{Succ}_{p_i}(\mu)$ we have $\varepsilon_\rho = 2$. Define

$$p_{i+1} := \bigcup \{p^\rho \mid \rho \in \text{Succ}_{p_i}(\mu) \wedge \varepsilon_\rho = 2\} \cup (p_i \setminus \bar{p}_i)$$

and $q_{i+1} := q_i$.

In the end, we define $u'_{n+1} := p_N$ and $v'_{n+1} := q_N$. Then (u'_{n+1}, v'_{n+1}) is as desired. Choose an element $\sigma_{n+1} \in \text{split}(u'_{n+1})$ such that we have $((s_{n+1} \frown \langle \sigma_0 \rangle, u'_{n+1}), (t_{n+1}, v'_{n+1})) \in \mathbb{A}(\mathbb{M}^2)$ and let $s_{n+2} := s_{n+1} \frown \langle \sigma_{n+1} \rangle$.

Analogously we get $(u_{n+1}, v_{n+1}) \leq (u'_{n+1}, v'_{n+1})$ in \mathbb{M}^2 such that property (e') holds. Choose an element $\tau_{n+1} \in \text{split}(v_{n+1})$ such that we have $((s_{n+2}, u_{n+1}), (t_{n+1} \frown \langle \tau_{n+1} \rangle, v_{n+1})) \in \mathbb{A}(\mathbb{M}^2)$ and let $t_{n+2} := t_{n+1} \frown \langle \tau_{n+1} \rangle$.

This finishes the construction. Now we can define u and v to be the unique superperfect trees with

$$\text{split}(u) = \bigcup \{ \text{ran}(s_n) \mid n \in \omega \}$$

and

$$\text{split}(v) = \bigcup \{ \text{ran}(t_n) \mid n \in \omega \}.$$

Then $((s, u), (t, v)) \in \mathbb{A}(\mathbb{M}^2)$ and the construction and property $(*)$ guarantee that no extension of $((s, u), (t, v))$ decides Θ , which is a contradiction. So claim 4 is proven and we are done. \square (claim 4) \square

Corollary 1.2.2 $\mathbb{A}(\mathbb{M}^2)$ has the following decision property:

If Θ is an $\mathbb{A}(\mathbb{M}^2)$ -name and $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ such that for some $n \in \omega$

$$((s, p), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta < n,$$

then there exists an element $((s, p'), (t, q')) \in \mathbb{A}(\mathbb{M}^2)$ with $((s, p'), (t, q')) \leq ((s, p), (t, q))$ and $((s, p'), (t, q'))$ decides Θ .

Proof: Suppose $((s, t), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta < n$. Choose an $\mathbb{A}(\mathbb{M}^2)$ -name \dot{a}_0 such that for every $\mathbb{A}(\mathbb{M}^2)$ -generic filter G we have $(\dot{a}_0)_G = 0$ if and only if $\Theta_G = 0$ and $(\dot{a}_0)_G = 1$ if and only if $\Theta_G \neq 0$. As $\Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{a}_0 \in \{0, 1\}$, therefore $((s, p), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{a}_0 \in \{0, 1\}$, the pure decision property of $\mathbb{A}(\mathbb{M}^2)$ gives us a pair $(p_0, q_0) \in \mathbb{M}^2$ such that $(p_0, q_0) \leq (p, q)$, $((s, p_0), (t, q_0)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s, p_0), (t, q_0))$ decides \dot{a}_0 . If $((s, p_0), (t, q_0)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{a}_0 = 0$ define $p' := p_0$ and $q' := q_0$; if $((s, p_0), (t, q_0)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{a}_0 = 1$ choose an $\mathbb{A}(\mathbb{M}^2)$ -name \dot{a}_1 such that for every $\mathbb{A}(\mathbb{M}^2)$ -generic filter G we have $(\dot{a}_1)_G = 0$ if and only if $\Theta_G = 1$ and $(\dot{a}_1)_G = 1$ if and only if $\Theta_G \neq 1$. Again the pure decision property of $\mathbb{A}(\mathbb{M}^2)$ gives us $(p_1, q_1) \leq (p_0, q_0)$ such that $((s, p_1), (t, q_1))$ decides \dot{a}_1 . If $((s, p_1), (t, q_1)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{a}_1 = 0$ let $p' := p_1$ and $q' := q_1$; if $((s, p_1), (t, q_1)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{a}_1 = 1$ choose an $\mathbb{A}(\mathbb{M}^2)$ -name \dot{a}_2 with $(\dot{a}_2)_G = 0$ if and only if $\Theta_G = 2$ and $(\dot{a}_2)_G = 1$ if and only if $\Theta_G \neq 2$ and so on. Since $((s, p), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \Theta < n$, this procedure stops after finitely many steps and we get (p', q') as desired. \square

1.3 $\mathbb{A}(\mathbb{M}^2)$ does not add Cohen-reals

Lemma 1.3.1 $\mathbb{A}(\mathbb{M}^2)$ has the Laver property.

As an abbreviation we will use the following:

Definition 1.3.2 For $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ and an element $s' \in {}^{<\omega}(\text{ran}(s))$ with $((s', p), (t', q)) \in \mathbb{A}(\mathbb{M}^2)$ for some t' define

$$p(s, s') := \bigcup \{(p)_{\sigma \smallfrown \langle k \rangle} \mid \sigma \in \text{ran}(s') \wedge \forall \sigma' \in \text{ran}(s) \setminus \text{ran}(s') (\sigma \smallfrown \langle k \rangle \not\subseteq \sigma')\}$$

and analogous define a superperfect tree $q(t, t')$ for $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ and $t' \in {}^{<\omega}(\text{ran}(t))$ with $((s', p), (t', q)) \in \mathbb{A}(\mathbb{M}^2)$ for some s' .

Proof of lemma 1.3.1: Suppose $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$, $g \in {}^\omega\omega \cap V$ and \dot{f} is an $\mathbb{A}(\mathbb{M}^2)$ -name for an element of ${}^\omega\omega$ such that

$$((s, p), (t, q)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \forall n (\dot{f}(n) < g(n)).$$

We want to construct by a fusion $(p', q') \in \mathbb{A}(\mathbb{M}^2)$ and a sequence $(H_n)_{n \in \omega}$ in V such that $((s, p'), (t, q')) \in \mathbb{A}(\mathbb{M}^2)$, $((s, p'), (t, q')) \leq ((s, p), (t, q))$, $|H_n| \leq 2^{m_n}$ for a strictly increasing sequence $(m_n)_{n \in \omega}$ of natural numbers not depending on $((s, p'), (t, q'))$, g and \dot{f} and $((s, p'), (t, q')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \forall n (\dot{f}(n) \in H_n)$. Oftenly m_n equals n , here we have to choose it a little bit larger, but as the specific value of m_n is insignificant, we avoid the finitary combinatorics.

Suppose $|s| = |t|$; the case $|s| = |t| + 1$ is similar. We construct sequences $(s_n)_{n \in \omega}$, $(t_n)_{n \in \omega}$ in ${}^{<\omega}({}^{<\omega}\omega)$, $((u_n, v_n))_{n \in \omega}$ in \mathbb{M}^2 , $(m_n)_{n \in \omega}$ in ω and a sequence $(H_n)_{n \in \omega}$ of subsets of ω in V by induction such that for every $n \in \omega$ the following hold:

- (i) $(p, q) \geq (u_n, v_n) \geq (u_{n+1}, v_{n+1})$;
- (ii) $((s_n, u_n), (t_n, v_n)) \in \mathbb{A}(\mathbb{M}^2)$;
- (iii) $s \subseteq s_n \subseteq s_{n+1}$ and $t \subseteq t_n \subseteq t_{n+1}$; and if n is even we have $s_{n+1} = s_n$ and if n is odd we have $t_{n+1} = t_n$;
- (iv) $((s, u_n), (t, v_n)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n) \in H_n$;
- (v) the sequence $(m_n)_{n \in \omega}$ is strictly increasing and only depends on n and is independent of $((s, p'), (t, q'))$, g and \dot{f} and $|H_n| \leq 2^{m_n}$.

For beginning the induction we find a natural number K and $(u, v) \in \mathbb{M}^2$ with $((s, u), (t, v)) \leq ((s, p), (t, q))$ and $((s, u), (t, v)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(0) = K$ by corollary 1.2.2. Define $(u_0, v_0) := (u, v)$, $s_0 := s$, $t_0 := t$, $m_0 := 0$ and $H_0 := \{K\}$.

Suppose we have already constructed $((s_i, u_i), (t_i, v_i)) \in \mathbb{A}(\mathbb{M}^2)$, m_i and H_i for $i \leq n$ for some $n \in \omega$ and suppose first that n is odd. Fix $\mu \in \text{ran}(s_n)$, the node where we have to extend the finite tree generated by $\text{ran}(s_n)$. Choose $\sigma \in \text{Succ}_{u_n}(\mu)$ such that $((s_n \smallfrown \langle \sigma \rangle, u_n), (t_n, v_n)) \in \mathbb{A}(\mathbb{M}^2)$ and define $s_{n+1} :=$

$s_n \frown \langle \sigma \rangle$ and $t_{n+1} := t_n$.

Claim 1: There exists a pair $(u, v) \in \mathbb{M}^2$ such that $(u, v) \leq (u_n, v_n)$ and $((s_{n+1}, u), (t_{n+1}, v)) \in \mathbb{A}(\mathbb{M}^2)$ and for every pair $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_{n+1}))$ with $((s', u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$ and $((s', u), (t', v)) \leq ((s, u_n), (t, v_n))$ there exists some K such that

$$((s', u(s_{n+1}, s')), (t', v(t_{n+1}, t'))) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K.$$

Proof of claim 1: Let $((\bar{s}_i, \bar{t}_i))_{i < N}$ for some $N \in \omega$ enumerate all pairs (s', t') as in claim 1. We want to consider successively all pairs (\bar{s}_i, \bar{t}_i) and thin out (u_n, v_n) step by step to get (u, v) : Define $p_0 := u_n$ and $q_0 := v_n$ and suppose we have constructed (p_i, q_i) for some $i < N$. Consider (\bar{s}_i, \bar{t}_i) . Suppose there exists $(\bar{u}, \bar{v}) \in \mathbb{M}^2$ with $((\bar{s}_i, \bar{u}), (\bar{t}_i, \bar{v})) \leq ((s, p_i), (t, q_i))$ (if not, we let $p_{i+1} := p_i$ and $q_{i+1} := q_i$). It is $((s, p_i), (t, q_i)) \leq ((s, p), (t, q))$, hence $((\bar{s}_i, \bar{u}), (\bar{t}_i, \bar{v})) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) < g(n+1)$, and the pure decision property of $\mathbb{A}(\mathbb{M}^2)$, corollary 1.2.2 to be precise, gives us a pair $(\bar{p}, \bar{q}) \in \mathbb{M}^2$ with $((\bar{s}_i, \bar{p}), (\bar{t}_i, \bar{q})) \in \mathbb{A}(\mathbb{M}^2)$, $((\bar{s}_i, \bar{p}), (\bar{t}_i, \bar{q})) \leq ((\bar{s}_i, \bar{u}), (\bar{t}_i, \bar{v}))$ and $((\bar{s}_i, \bar{p}), (\bar{t}_i, \bar{q})) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K$ for some K . Clearly, we can thin out \bar{p} and \bar{q} to $\bar{\bar{p}}$ and $\bar{\bar{q}}$ such that $(\bar{\bar{p}}, \bar{\bar{q}}) \leq (p_i(s_{n+1}, \bar{s}_i), q_i(t_{n+1}, \bar{t}_i))$ and $((\bar{s}_i, \bar{\bar{p}}), (\bar{t}_i, \bar{\bar{q}})) \in \mathbb{A}(\mathbb{M}^2)$. Define

$$p_{i+1} := \bar{\bar{p}} \cup (p_i \setminus p_i(s_{n+1}, \bar{s}_i))$$

and

$$q_{i+1} := \bar{\bar{q}} \cup (q_i \setminus q_i(t_{n+1}, \bar{t}_i)).$$

Then clearly we have

$$((\bar{s}_i, p_{i+1}(s_{n+1}, \bar{s}_i)), (\bar{t}_i, q_{i+1}(t_{n+1}, \bar{t}_i))) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K,$$

as well.

In the end, define $u := p_N$ and $v := q_N$.

□(Claim 1)

Fix $(u, v) \in \mathbb{M}^2$ as in claim 1 and define H' to be the collection of all K with $((s', u(s_{n+1}, s')), (t', v(t_{n+1}, t'))) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K$ for some (s', t') as in claim 1, then H' is finite and the cardinality of H' only depends on the number of σ_i 's and τ_i 's, which we have already chosen, hence on n .

Now we thin out u and v such that for every pair (s', t') as in claim 1 with $|s'| = |t'|$ and every $\rho \in u(s_{n+1}, s')$ such that $((s' \frown \langle \rho \rangle, u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$ and for every (s', t') as in claim 1 with $|s'| = |t'| + 1$ and every $\rho \in v(t_{n+1}, t')$ such that $((s', u), (t' \frown \langle \rho \rangle, v)) \in \mathbb{A}(\mathbb{M}^2)$, respectively, we have

$$\min\{|\rho|, \rho(|\mu|)\} > \max\{\#\sigma \mid \sigma \in \text{ran}(s_{n+1})\} \cup \{\#\tau \mid \tau \in \text{ran}(t_{n+1})\},$$

where μ is the direct predecessor of ρ in the tree sense in $\text{ran}(s')$, $\text{ran}(t')$, respectively. For this, we have to consider all pairs (s', t') and cut off finitely often finitely many superperfect trees, so there is no problem for the remaining tree to be a superperfect one. This implies the following property for (s', t') as in claim 1:

- (*) Suppose $\rho \in u(s_{n+1}, s')$ such that $((s' \frown \langle \rho \rangle, u), (t', v)) \in \mathbb{A}(\mathbb{M}^2)$. If $\bar{s} \supseteq s' \frown \langle \rho \rangle$ and $\bar{t} \supseteq t'$ with $((\bar{s}, u), (\bar{t}, v)) \in \mathbb{A}(\mathbb{M}^2)$ and with $((\bar{s}, u), (\bar{t}, v)) \leq ((s' \frown \langle \rho \rangle, u), (t', v))$, then we have $\bar{s} \subseteq u(s_{n+1}, s')$ and $\bar{t} \subseteq v(t_{n+1}, t')$.
 And similar suppose $\rho \in v(t_{n+1}, t')$ such that $((s', u), (t' \frown \langle \rho \rangle, v)) \in \mathbb{A}(\mathbb{M}^2)$. If $\bar{t} \supseteq t' \frown \langle \rho \rangle$ and $\bar{s} \supseteq s'$ with $((\bar{s}, u), (\bar{t}, v)) \in \mathbb{A}(\mathbb{M}^2)$ and $((\bar{s}, u), (\bar{t}, v)) \leq ((s', u), (t' \frown \langle \rho \rangle, v))$, then we have $\bar{s} \subseteq u(s_{n+1}, s')$ and $\bar{t} \subseteq v(t_{n+1}, t')$.

By a similar fusion as by the construction of the pair (p', q') in the proof of lemma 1.2.1, using the pure decision property of $\mathbb{A}(\mathbb{M}^2)$, we get $(u', v') \in \mathbb{M}^2$ such that $((s_{n+1}, u'), (t_{n+1}, v')) \in \mathbb{A}(\mathbb{M}^2)$, $((s_{n+1}, u'), (t_{n+1}, v')) \leq ((s_{n+1}, u), (t_{n+1}, v))$ and such that for every $(s', t') \in {}^{<\omega}(\text{split}(u')) \times {}^{<\omega}(\text{split}(v'))$ with $\text{ran}(s') \not\subseteq \text{ran}(s_n)$, $((s', u'), (t', v')) \in \mathbb{A}(\mathbb{M}^2)$ and $((s', u'), (t', v')) \leq ((s, u'), (t, v'))$, the element $((s', u'), (t', v'))$ decides $\dot{f}(n+1)$.

Claim 2: There exists a pair $(u'', v'') \in \mathbb{M}^2$ with $(u'', v'') \leq (u', v')$ such that $((s_{n+1}, u''), (t_{n+1}, v'')) \in \mathbb{A}(\mathbb{M}^2)$ and for every pair $(s', t') \in {}^{<\omega}(\text{ran}(s_{n+1})) \times {}^{<\omega}(\text{ran}(t_{n+1}))$ with $((s', u''), (t', v'')) \in \mathbb{A}(\mathbb{M}^2)$ and $((s', u''), (t', v'')) \leq ((s, u'), (t, v'))$ we have the following:

If $\tau \in \text{ran}(s_{n+1})$ with $((s' \frown \langle \tau \rangle, u''), (t', v'')) \in \mathbb{A}(\mathbb{M}^2)$ then there exists K such that for every $\rho \in \text{split}(u''(s_{n+1}, s' \frown \langle \tau \rangle))$ which properly extends τ we have $((s' \frown \langle \rho \rangle, u''), (t', v'')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K$;

if $\tau \in \text{ran}(t_{n+1})$ with $((s', u''), (t' \frown \langle \tau \rangle, v'')) \in \mathbb{A}(\mathbb{M}^2)$ then there exists K such that for every $\rho \in \text{split}(v''(t_{n+1}, t' \frown \langle \tau \rangle))$ which properly extends τ we have $((s', u''), (t' \frown \langle \rho \rangle, v'')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K$.

Proof of claim 2: Let $((\bar{s}_i, \bar{t}_i, \bar{\tau}_i))_{i < N}$ for some $N \in \omega$ enumerate all triplets (s', t', τ) as in the claim. Define $p_0 := u'$ and $q_0 := v'$ and suppose we have constructed (p_i, q_i) for some $i < N$. Consider $(\bar{s}_i, \bar{t}_i, \bar{\tau}_i)$ and suppose $\bar{\tau}_i \in \text{ran}(s_{n+1})$, so $|\bar{s}_i| = |\bar{t}_i|$; the case $\bar{\tau}_i \in \text{ran}(t_{n+1})$ is similar.

For all $\rho \in \text{split}(p_i(s_{n+1}, \bar{s}_i \frown \langle \bar{\tau}_i \rangle))$ which properly extends $\bar{\tau}_i$ and with $((\bar{s}_i \frown \langle \rho \rangle, p_i), (\bar{t}_i, q_i)) \in \mathbb{A}(\mathbb{M}^2)$ we have $\text{ran}(\bar{s}_i \frown \langle \rho \rangle) \not\subseteq \text{ran}(s_n)$ (by property (*)), $((\bar{s}_i \frown \langle \rho \rangle, u'), (\bar{t}_i, v')) \in \mathbb{A}(\mathbb{M}^2)$ and $((\bar{s}_i \frown \langle \rho \rangle, u'), (\bar{t}_i, v')) \leq ((s, u'), (t, v'))$.

The choice of (u', v') gives us that $((\bar{s}_i \frown \langle \rho \rangle, p_i), (\bar{t}_i, q_i))$ decides $\dot{f}(n+1)$, say $((\bar{s}_i \frown \langle \rho \rangle, p_i), (\bar{t}_i, q_i)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K_\rho$.

By colouring splitnodes as in the proof of claim 4 in lemma 1.2.1, we can find for every above considered $\rho \in \text{Succ}_{p_i(s_{n+1}, \bar{s}_i \frown \langle \bar{\tau}_i \rangle)}(\bar{\tau}_i)$ a K_ρ and a tree $p^\rho \in \mathbb{M}$ such that $p^\rho \subseteq (p_i)_\rho$ and for every $\nu \in \text{split}(p^\rho)$ we have $((\bar{s}_i \frown \langle \nu \rangle, p_i), (\bar{t}_i, q_i)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K_\rho$. There are infinitely many ρ but finitely many K_ρ , so there is an infinite set A of ρ 's and a K_i with $K_\rho = K_i$ for every $\rho \in A$. Fix $\mu \in \text{ran}(\bar{s}_i)$, the direct predecessor in the tree sense of $\bar{\tau}_i$ and define

$$p_{i+1} := \bigcup \{p^\rho \mid \rho \in A\} \cup \{\sigma \in p_i \mid \mu \not\subseteq \sigma \vee \exists \nu \in \text{ran}(s_{n+1}) (\mu \subsetneq \nu \subseteq \sigma \vee \mu \subseteq \sigma \subsetneq \nu)\}$$

and $q_{i+1} := q_i$.

In the end, define $u'' := p_N$ and $v'' := q_N$.

□(Claim 2)

Define $u_{n+1} := u''$ and $v_{n+1} := v''$ and let H'' be the collection of all K as in claim 2. Clearly, H'' is finite and the cardinality only depends on n . Define $H_{n+1} := H' \cup H''$ then we can choose $m_{n+1} \in \omega$ with $m_{n+1} > m_n$ and $|H''| \leq 2^{m_{n+1}}$.

Claim 3: $((s, u_{n+1}), (t, v_{n+1})) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) \in H_{n+1}$.

Proof of claim 3: Let $((s', u'), (t', v')) \in \mathbb{A}(\mathbb{M}^2)$ with $((s', u'), (t', v')) \leq ((s, u_{n+1}), (t, v_{n+1}))$ and $((s', u'), (t', v')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K$ for some K . Let \bar{s} be the maximal initial segment of s' with $\text{ran}(\bar{s}) \subseteq \text{ran}(s_{n+1})$ and \bar{t} the maximal initial segment of t' with $\text{ran}(\bar{t}) \subseteq \text{ran}(t_{n+1})$. Without loss of generality, the length of s' is longer than the length of \bar{s} and the length of t' is longer than the length of \bar{t} . Define $\mu := s'(|\bar{s}|)$ and $\nu := t'(|\bar{t}|)$. Distinguish the following two cases:

First case: $\mu \in u_{n+1}(s_{n+1}, \bar{s})$ and $\nu \in v_{n+1}(t_{n+1}, \bar{t})$, so by property (*) we have $\text{ran}(s') \subseteq u_{n+1}(s_{n+1}, \bar{s})$ and $\text{ran}(t') \subseteq v_{n+1}(t_{n+1}, \bar{t})$ and hence $((s', u'), (t', v'))$ and $((\bar{s}, u_{n+1}(s_{n+1}, \bar{s})), (\bar{t}, v_{n+1}(t_{n+1}, \bar{t})))$ are compatible. Since

$$((\bar{s}, u_{n+1}(s_{n+1}, \bar{s})), (\bar{t}, v_{n+1}(t_{n+1}, \bar{t}))) \leq ((\bar{s}, u(s_{n+1}, \bar{s})), (\bar{t}, v(t_{n+1}, \bar{t})))$$

and $((\bar{s}, u(s_{n+1}, \bar{s})), (\bar{t}, v(t_{n+1}, \bar{t}))) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) \in H'$ we conclude that $((s', u'), (t', v')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) \in H'$.

Second case: $\mu \notin u_{n+1}(s_{n+1}, \bar{s})$ or $\nu \notin v_{n+1}(t_{n+1}, \bar{t})$. Suppose $\mu \notin u_{n+1}(s_{n+1}, \bar{s})$, therefore μ extends an element of $\text{ran}(s_{n+1}) \setminus \text{ran}(\bar{s})$; the case $\nu \notin v_{n+1}(t_{n+1}, \bar{t})$ is similar. Choose $\sigma \in \text{ran}(s_{n+1})$ with maximal length such that $\mu \not\subseteq \sigma$. It is easy to see that we have $((\bar{s} \frown \langle \sigma \rangle, u_{n+1}), (\bar{t}, v_{n+1})) \in \mathbb{A}(\mathbb{M}^2)$, hence $(\bar{s}, \bar{t}, \sigma)$ is among the enumerated tripels of claim 2 above, say $(\bar{s}, \bar{t}, \sigma) = (\bar{s}_i, \bar{t}_i, \bar{\tau}_i)$. Then μ is an

element of $\text{split}(p_i(s_{n+1}, \bar{s}_i \hat{\ } \langle \bar{\tau}_i \rangle))$ with $\mu \not\subseteq \bar{\tau}_i$ and $((\bar{s}_i \hat{\ } \langle \mu \rangle, p_i), (\bar{t}_i, q_i)) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) = K_i \in H''$. Since $((s', u'), (t', v')) \leq ((\bar{s}_i \hat{\ } \langle \mu \rangle, p_i), (\bar{t}_i, q_i))$, also $((s', u'), (t', v')) \Vdash_{\mathbb{A}(\mathbb{M}^2)} \dot{f}(n+1) \in H''$. \square (claim 3)

If n is even, we have to do the analogous step at the right side by choosing $\tau \in \text{Succ}_{v_n}(\nu)$, where $\nu \in \text{ran}(t_n)$ is the node, where we have to extend $\text{ran}(t_n)$. Then we let $t_{n+1} := t_n \hat{\ } \langle \tau \rangle$ and $s_{n+1} := s_n$. This step is very similar to the even case, we do not carry it out. This finishes the construction.

In the end, define p' and q' to be the superperfect trees determined by

$$\text{split}(p') = \bigcup \{ \text{ran}(s_n) \mid n \in \omega \}$$

and

$$\text{split}(q') = \bigcup \{ \text{ran}(t_n) \mid n \in \omega \}.$$

Then (p', q') and $(H_n)_{n \in \omega}$ are as desired. \square

It is well known, that the Laver property of a forcing P implies, that forcing with P does not add Cohen-reals.

1.4 The model

Definition 1.4.1 Let G be an $\mathbb{A}(\mathbb{M}^2)$ -generic filter over V . Define (p_G, q_G) to be the unique pair $(p', q') \in \mathbb{M}^2$ such that

$$\text{split}(p') = \bigcup \{ \text{ran}(s) \mid \exists p, q \in \mathbb{M} \exists t \in {}^{<\omega}({}^{<\omega}\omega) ((s, p), (t, q)) \in G \}$$

and

$$\text{split}(q') = \bigcup \{ \text{ran}(t) \mid \exists p, q \in \mathbb{M} \exists s \in {}^{<\omega}({}^{<\omega}\omega) ((s, p), (t, q)) \in G \}.$$

Clearly, p_G and q_G are welldefined.

For justifying the name *amoeba* for our forcing $\mathbb{A}(\mathbb{M}^2)$ we want to prove the following lemma:

Lemma 1.4.2 *Suppose that $D \subseteq \mathbb{M}^2$ is open and dense and in V and G is $\mathbb{A}(\mathbb{M}^2)$ -generic over V .*

Then there exists a countable $F \subseteq D$ in V such that in $V[G]$ we have $[p_G] \times [q_G] \subseteq \bigcup \{ [p] \times [q] \mid (p, q) \in F \}$.

Moreover, every pair of branches of (p_G, q_G) is \mathbb{M}^2 -generic over V .

For the proof we use the following:

Fact 1.4.3 [Sp2] *Suppose $D \subseteq \mathbb{M}^2$ is open and dense and $(p, q) \in \mathbb{M}^2$. Then there exists $(p', q') \in \mathbb{M}^2$ with $(p', q') \leq^0 (p, q)$ and there exists some countable $F \subseteq D$ such that for every $(x, y) \in [p'] \times [q']$ there is $(u, v) \in F$ with $(x, y) \in [u] \times [v]$. \square*

This fact is the result of a difficult induction on types invented by Spinas.

Proof of lemma 1.4.2: So suppose G is $\mathbb{A}(\mathbb{M}^2)$ -generic over V and $D \subseteq \mathbb{M}^2$ is open and dense. Define

$$\bar{D} := \{(p', q') \in \mathbb{M}^2 \mid \text{there exists a countable } F \subseteq D \text{ such that} \\ [p'] \times [q'] \subseteq \bigcup \{[u] \times [v] \mid (u, v) \in F\}\},$$

then \bar{D} is \leq^0 -dense in \mathbb{M}^2 by fact 1.4.3; recall that this means that for every $(p, q) \in \mathbb{M}^2$ there exists $(p', q') \leq (p, q)$ with $(p', q') \in \bar{D}$ and $\text{st}(p') = \text{st}(p)$ and $\text{st}(q') = \text{st}(q)$. So, given $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ it is easy to find $(p', q') \leq^0 (p, q)$ such that $((s, p'), (t, q')) \in \mathbb{A}(\mathbb{M}^2)$ and there exists a countable $F \subseteq D$ such that $[p'] \times [q'] \subseteq \bigcup \{[u] \times [v] \mid (u, v) \in F\}$. Hence the set

$$D' := \{((s, p'), (t, q')) \in \mathbb{A}(\mathbb{M}^2) \mid \exists F \subseteq D \text{ countable} \\ ([p'] \times [q'] \subseteq \bigcup \{[u] \times [v] \mid (u, v) \in F\})\}$$

is \leq^0 -dense in $\mathbb{A}(\mathbb{M}^2)$, i.e. for every $((s, p), (t, q)) \in \mathbb{A}(\mathbb{M}^2)$ exists $(p', q') \leq^0 (p, q)$ such that $((s, p'), (t, q')) \in D'$. Choose $((s, p'), (t, q')) \in D' \cap G$ with $((s, p'), (t, q')) \leq^0 ((s, p), (t, q))$ and let $F \in V$ be the countable witness for this. Notice that the property of F is a Π_1^1 one and is therefore absolute for transitive models of ZFC, hence F has the same property in $V[G]$. For every $(x, y) \in [p_G] \times [q_G]$ and every $n \in \omega$ there is an element $((s_n, p_n), (t_n, q_n)) \in G$ such that for every $n \in \omega$ we have $((s_{n+1}, p_{n+1}), (t_{n+1}, q_{n+1})) \leq ((s_n, p_n), (t_n, q_n)) \leq ((s, p'), (t, q'))$ and $x \upharpoonright n \in S_n$ and $y \upharpoonright n \in T_n$, where S_n and T_n are the downward closures of $\text{ran}(s_n)$, $\text{ran}(t_n)$, respectively. Hence $(x, y) \in [p'] \times [q']$ and therefore $[p_G] \times [q_G] \subseteq \bigcup \{[p] \times [q] \mid (p, q) \in F\}$.

It remains to prove that every pair of branches in (p_G, q_G) is \mathbb{M}^2 -generic. For this we have to prove that every pair $(x, y) \in {}^\omega\omega \times {}^\omega\omega$ which has the following property:

(*) for every dense $D \subseteq \mathbb{M}^2$ in V there exists $(p, q) \in D$ with $(x, y) \in [p] \times [q]$

is a pair of Miller-reals, i.e. $H := \{(p, q) \in (\mathbb{M}^2)^V \mid (x, y) \in [p] \times [q]\}$ is \mathbb{M}^2 -generic over V .

Clearly, H is upwards closed. Suppose there are incompatible $(p, q), (p', q')$ in H . By Shoenfields absoluteness theorem there are incompatible in V , i.e. $[p] \cap [p']$ or $[q] \cap [q']$ (or both) does not contain the branches of a Miller tree, without

loss of generality, $[p] \cap [p']$ does not. Since $[p] \cap [p']$ is a closed subset of ${}^\omega\omega$, we can apply a fact of Kechris:

Fact 1.4.4 [Ke] *Every closed set $A \subseteq {}^\omega\omega$ is bounded with respect to \leq^* or contains the branches of a Miller tree.*

Choose by this fact an upper bound $f \in {}^\omega\omega$ of $[p] \cap [p']$. It is easy to see that the set $E := \{(\bar{p}, \bar{q}) \in \mathbb{M}^2 \mid \forall h \in [\bar{p}] (h \not\leq^* f)\}$ is dense in \mathbb{M}^2 , so by property (*) of (x, y) there exists $(\bar{p}, \bar{q}) \in E$ with $(x, y) \in [\bar{p}] \times [\bar{q}]$, hence we have $x \not\leq^* f$. But by choice $x \in [p] \cap [p']$, therefore $x \leq^* f$, this is a contradiction. Hence H is a filter.

The genericity of H is clear by property (*). \square

Now we are ready to prove the maintheorem of this chapter:

Proof of theorem 1.1.2: Suppose that $((P_\alpha)_{\alpha < \omega_2}, (\dot{Q}_\alpha)_{\alpha < \omega_2})$ is a countable support iteration of $\mathbb{A}(\mathbb{M}^2)$, i.e. for every $\alpha < \omega_2$, \dot{Q}_α is a P_α -name for $\mathbb{A}(\mathbb{M}^2)$ defined in the model V^{P_α} . Let G be P_{ω_2} -generic over V .

By lemma 1.3.1, every iterand of our iteration has the Laver property and by a result of Shelah ([Sh], see also [Go]) the Laver property is preserved under countable support iterations. So P_{ω_2} has the Laver property, hence $V[G] \models \text{cov}(\mathcal{M}) = \omega_1$. It remains to prove, that, in $V[G]$, we have $\text{add}(J(\mathbb{M}^2)) = \omega_2$. So suppose, in $V[G]$, $(X_\alpha)_{\alpha < \omega_1}$ is a family of elements of $J(\mathbb{M}^2)$ and let $X := \bigcup \{X_\alpha \mid \alpha < \omega_1\}$. Suppose $(p, q) \in \mathbb{M}^2$. We have to find $(p', q') \leq (p, q)$ in \mathbb{M}^2 such that $([p'] \times [q']) \cap X = \emptyset$.

For $\alpha < \omega_1$ define

$$D_\alpha := \{(u, v) \in \mathbb{M}^2 \mid ([u] \times [v]) \cap X_\alpha = \emptyset\}.$$

Clearly, D_α is open and dense in \mathbb{M}^2 . Our iteration has countable support, hence for every $\alpha < \omega_2$ with $\text{cf}(\alpha) = \omega_1$ we have for every $(u, v) \in (\mathbb{M}^2)^{V[G_\alpha]}$ that (u, v) already belongs to $V[G_\beta]$ for some $\beta < \alpha$ (see for example [Ku], ch. VIII, lemma 5.14). With this fact, by a Löwenheim-Skolem-argument, the set

$$C_\alpha := \{\beta < \omega_2 \mid D_\alpha \cap V[G_\beta] \in V[G_\beta] \text{ and}$$

$$D_\alpha \cap V[G_\beta] \text{ is open and dense in } (\mathbb{M}^2)^{V[G_\beta]}\}$$

is ω_1 -club in ω_2 , i.e. it is unbounded and closed under increasing sequences of length ω_1 . Hence $C := \bigcap_{\alpha < \omega_1} C_\alpha$ is ω_1 -club in ω_2 , too. But G is generic, hence by the above mentioned lemma of [Ku] we find $\gamma \in C$ and some $(s, t) \in {}^{<\omega}\text{split}(p) \times {}^{<\omega}\text{split}(q)$ such that $((s, p), (t, q)) \in G(\gamma)$, where $G(\gamma)$ is the $\dot{Q}_\gamma[G_\gamma]$ -generic filter induced by G . By lemma 1.4.2 in $V[G_{\gamma+1}]$ we have for every $\alpha < \omega_1$

$$[p_{G(\gamma)}] \times [q_{G(\gamma)}] \subseteq \bigcup \{[u] \times [v] \mid (u, v) \in D'_\alpha \cap V[G_\gamma]\}$$

for some countable $D'_\alpha \subseteq D_\alpha$. By absolutness this is also true in $V[G]$, hence $V[G] \models ([p_{G(\gamma)}] \times [q_{G(\gamma)}]) \cap X = \emptyset$ and as $(p_{G(\gamma)}, q_{G(\gamma)}) \leq (p, q)$ we are done. \square

Chapter 2

Martin's Axiom and the additivity of $\mathcal{J}(\mathbb{M}^2)$

In this chapter, we want to prove the following theorem:

Theorem 2.0.5 *MA(σ -centered) implies that $\text{add}(\mathcal{J}(\mathbb{M}^2))$ equals 2^ω .*

Where σ -centered denotes the class of all forcings which are σ -centered, where a forcing P is σ -centered, if it is a countable union of centered sets P_n , i.e. for $p, q \in P_n$ there exists $r \in P_n$ with $r \leq p, q$.

We will use the following:

Fact 2.0.6 [JoSp] *For $(p, q) \in \mathbb{M}^2$ and $X \in \mathcal{J}(\mathbb{M}^2)$ there exists $(p', q') \leq^0 (p, q)$ such that $([p'] \times [q']) \cap X = \emptyset$. \square*

In other words, the set $\{(p', q') \in \mathbb{M}^2 \mid ([p'] \times [q']) \cap X = \emptyset\}$ is \leq^0 -dense in \mathbb{M}^2 for fixed $X \in \mathcal{J}(\mathbb{M}^2)$. This fact was the mainstep in proving that $\mathcal{J}(\mathbb{M}^2)$ is a σ -ideal.

2.1 Good sequences

For the proof of theorem 2.0.5 we need the concept of good sequences and the corresponding sequences of superperfect trees defined in the next definitions:

Definition 2.1.1 Suppose $(p, q) \in \mathbb{M}^2$ and (p, q) have the property of (p', q') formulated in fact 0.0.2 (recall that we always assume this).

Let $I_{(p,q)} := \text{TP}_{\text{st}(p), \text{st}(q)}(p, q)$, then we have $I_{(p,q)} \subseteq \text{split}(p) \times^+ \text{split}(q)$. A sequence $\mathcal{A} = (A_{(\sigma, \tau)})_{(\sigma, \tau) \in I_{(p,q)}}$ is called *good for (p, q)* or a *good (p, q) -sequence* if the following hold for every $(\sigma, \tau) \in I_{(p,q)}$:

- (i) $A_{(\sigma,\tau)} \subseteq {}^{<\omega}\omega$ is infinite;
- (ii) if $\text{tp}(\sigma, \tau)$ is even, then $\forall \rho \in A_{(\sigma,\tau)} (\rho \supseteq \sigma \wedge |\rho| > \rho(|\sigma|))$ and $\forall \rho, \rho' \in A_{(\sigma,\tau)} (\rho \neq \rho' \Rightarrow \rho(|\sigma|) \neq \rho'(|\sigma|))$;
if $\text{tp}(\sigma, \tau)$ is odd, then $\forall \rho \in A_{(\sigma,\tau)} (\rho \supseteq \tau \wedge |\rho| > \rho(|\tau|))$ and $\forall \rho, \rho' \in A_{(\sigma,\tau)} (\rho \neq \rho' \Rightarrow \rho(|\tau|) \neq \rho'(|\tau|))$;
- (iii) $A_{(\sigma,\tau)} \subseteq \text{Sop}_{\text{st}(p), \text{st}(q)}^{p,q}(\sigma, \tau)$;
- (iv) if $\text{tp}(\sigma, \tau)$ is even and $\sigma', \sigma'' \in A_{(\sigma,\tau)}$ are such that $\sigma'(|\sigma|) < \sigma''(|\sigma|)$, then we have $A_{(\sigma',\tau)} \supseteq A_{(\sigma'',\tau)}$;
if $\text{tp}(\sigma, \tau)$ is odd and $\tau', \tau'' \in A_{(\sigma,\tau)}$ are such that $\tau'(|\tau|) < \tau''(|\tau|)$, then we have $A_{(\sigma,\tau')} \supseteq A_{(\sigma,\tau'')}$;
- (v) if $\text{tp}(\sigma, \tau)$ is even, then $\forall \sigma' \in A_{(\sigma,\tau)} \exists \sigma'' \in A_{(\sigma,\tau)} (\sigma''(|\sigma|) > \sigma'(|\sigma|) \wedge A_{(\sigma',\tau)} \supseteq A_{(\sigma'',\tau)})$;
if $\text{tp}(\sigma, \tau)$ is odd, then $\forall \tau' \in A_{(\sigma,\tau)} \exists \tau'' \in A_{(\sigma,\tau)} (\tau''(|\tau|) > \tau'(|\tau|) \wedge A_{(\sigma,\tau')} \supseteq A_{(\sigma,\tau'')})$.

If (p, q) is clear from context, we sometimes omit the mention of (p, q) .

For every good sequence $\mathcal{A} = (A_{(\sigma,\tau)})_{(\sigma,\tau) \in I_{(p,q)}}$ there is a corresponding sequence $((p_{(\sigma,\tau)}^{\mathcal{A}}, q_{(\sigma,\tau)}^{\mathcal{A}}))_{(\sigma,\tau) \in I_{(p,q)}}$ in \mathbb{M}^2 :

We just describe how to get $(p_{(\sigma,\tau)}^{\mathcal{A}}, q_{(\sigma,\tau)}^{\mathcal{A}})$ in the case where $\text{tp}(\sigma, \tau)$ is even, the odd case is similar. So suppose $\text{tp}(\sigma, \tau)$ is even. Define $p_{(\sigma,\tau)}^{\mathcal{A}}$ and $q_{(\sigma,\tau)}^{\mathcal{A}}$ by defining

$$\text{split}_n(p_{(\sigma,\tau)}^{\mathcal{A}}) := \{\bar{\sigma} \in \text{split}(p_{(\sigma,\tau)}^{\mathcal{A}}) \mid \bar{\sigma} \text{ has exactly } n \text{ different predecessors in the tree sense in } \text{split}(p_{(\sigma,\tau)}^{\mathcal{A}})\}$$

and

$$\text{split}_n(q_{(\sigma,\tau)}^{\mathcal{A}}) := \{\bar{\tau} \in \text{split}(q_{(\sigma,\tau)}^{\mathcal{A}}) \mid \bar{\tau} \text{ has exactly } n \text{ different predecessors in the tree sense in } \text{split}(q_{(\sigma,\tau)}^{\mathcal{A}})\}$$

by induction on n .

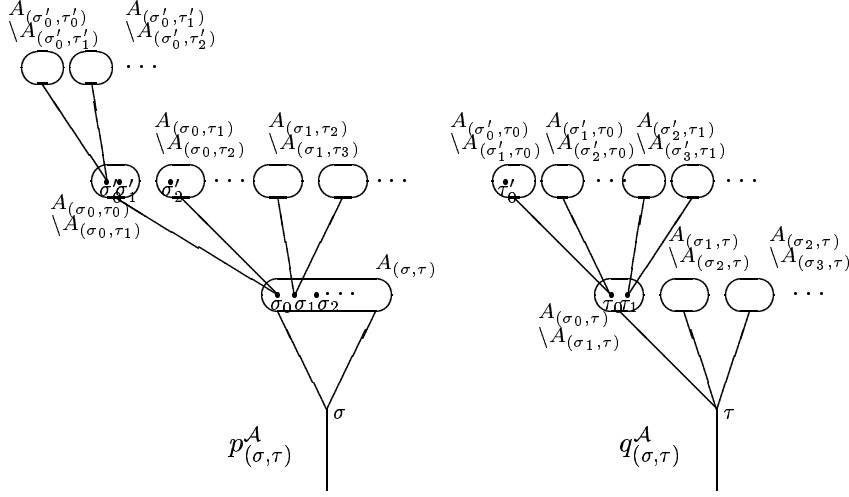
For $n = 0$ let $\text{split}_0(p_{(\sigma,\tau)}^{\mathcal{A}}) = \{\text{st}(p_{(\sigma,\tau)}^{\mathcal{A}})\} := \{\sigma\}$ and $\text{split}_0(q_{(\sigma,\tau)}^{\mathcal{A}}) = \{\text{st}(q_{(\sigma,\tau)}^{\mathcal{A}})\} := \{\tau\}$. Suppose for $k \leq n$ we have already defined $\text{split}_k(p_{(\sigma,\tau)}^{\mathcal{A}})$ and $\text{split}_k(q_{(\sigma,\tau)}^{\mathcal{A}})$. For defining $\text{split}_{n+1}(p_{(\sigma,\tau)}^{\mathcal{A}})$ we have to define what is in $\text{Succ}_{p_{(\sigma,\tau)}^{\mathcal{A}}}(\bar{\sigma})$ for every $\bar{\sigma} \in \text{split}_n(p_{(\sigma,\tau)}^{\mathcal{A}})$: Let

$$\text{Succ}_{p_{(\sigma,\tau)}^{\mathcal{A}}}(\bar{\sigma}) := \bigcup \{A_{(\bar{\sigma}, t)} \mid t \in \bigcup_{k \leq n} \text{split}_k(q_{(\sigma,\tau)}^{\mathcal{A}}) \wedge \text{tp}_{(\sigma,\tau)}(\bar{\sigma}, t) = 2n\}$$

for every $\bar{\sigma} \in \text{split}_n(p_{(\sigma,\tau)}^{\mathcal{A}})$. And then define

$$\text{Succ}_{q_{(\sigma,\tau)}^{\mathcal{A}}}(\bar{\tau}) := \bigcup \{A_{(s, \bar{\tau})} \mid s \in \bigcup_{k \leq n+1} \text{split}_k(p_{(\sigma,\tau)}^{\mathcal{A}}) \wedge \text{tp}_{(\sigma,\tau)}(s, \bar{\tau}) = 2n + 1\}$$

for every $\bar{\tau} \in \text{split}_n(q^A_{(\sigma,\tau)})$.



Notice that $(p^A_{(\sigma,\tau)}, q^A_{(\sigma,\tau)}) \leq (p, q)$, even $(p^A_{(\sigma,\tau)}, q^A_{(\sigma,\tau)}) \leq^0 ((p)_\sigma, (q)_\tau)$, for every $(\sigma, \tau) \in I_{(p,q)}$.

Remark 2.1.2 Fix $(p, q) \in \mathbb{M}^2$. Let $(p', q') \in \mathbb{M}^2$ with $(p', q') \leq (p, q)$ and such that for almost all $n \in \omega$ and every $(x, y) \in [p'] \times^+ [q']$ we have $\text{tp-}n\text{-pair}(x, y) \in \text{split}(p') \times \text{split}(q')$ (recall that for every $(u, v) \in \mathbb{M}^2$, $(u, v) \leq (p, q)$, there exists $(p', q') \leq^0 (u, v)$ with this property (compare the preliminaries)). Then we can define a partial good sequence $\mathcal{A} = (A_{(\sigma,\tau)})_{(\sigma,\tau) \in I'}$ for (p, q) , where $I' := I_{(p,q)} \cap (\text{split}(p') \times \text{split}(q'))$: For $(\sigma, \tau) \in I'$ with $\text{tp}(\sigma, \tau)$ even let $A_{(\sigma,\tau)}$ be the subset of $\text{Succ}_{p'}(\sigma) \cap \{\rho \mid \rho(|\sigma|) > \#\sigma\}$ thinned out such that for every $\rho \in A_{(\sigma,\tau)}$ we have $|\rho| > \rho(|\sigma|)$, $\min\{\rho(|\sigma|), |\rho|\} > \#\tau$ and $\rho(|\sigma|) > \#\rho'$ for every $\rho' \in A_{(\sigma,\tau)}$ with $\rho'(|\sigma|) < \rho(|\sigma|)$. Analogously, if $\text{tp}(\sigma, \tau)$ is odd let $A_{(\sigma,\tau)}$ be the subset of $\text{Succ}_{q'}(\tau) \cap \{\rho \mid \rho(|\tau|) > \#\tau\}$ thinned out such that for every $\rho \in A_{(\sigma,\tau)}$ we have $|\rho| > \rho(|\tau|)$, $\min\{\rho(|\tau|), |\rho|\} > \#\sigma$ and $\rho(|\tau|) > \#\rho'$ for every $\rho' \in A_{(\sigma,\tau)}$ with $\rho'(|\tau|) < \rho(|\tau|)$. It is easy to see that (i)-(v) of definition 2.1.1 hold for $\mathcal{A} = (A_{(\sigma,\tau)})_{(\sigma,\tau) \in I'}$.

Definition 2.1.3 Fix $(p, q) \in \mathbb{M}^2$. We define the following relations for any good (p, q) -sequences $\mathcal{A} = (A_{(\sigma,\tau)})_{(\sigma,\tau) \in I_{(p,q)}}$ and $\mathcal{B} = (B_{(\sigma,\tau)})_{(\sigma,\tau) \in I_{(p,q)}}$:

$$\mathcal{A} \leq \mathcal{B} \quad :\Leftrightarrow \quad (p^{\mathcal{A}}_{(\sigma,\tau)}, q^{\mathcal{A}}_{(\sigma,\tau)}) \leq (p^{\mathcal{B}}_{(\sigma,\tau)}, q^{\mathcal{B}}_{(\sigma,\tau)}) \text{ for every } (\sigma, \tau) \in I_{(p,q)}.$$

$$\begin{aligned} \mathcal{A} \approx \mathcal{B} \quad :\Leftrightarrow \quad & A_{(\sigma,\tau)} =^* B_{(\sigma,\tau)} \text{ for every } (\sigma, \tau) \in I_{(p,q)} \text{ and} \\ & A_{(\sigma,\tau)} = B_{(\sigma,\tau)} \text{ for all but finitely many } (\sigma, \tau) \in I_{(p,q)}. \end{aligned}$$

$$\mathcal{A} \leq^* \mathcal{B} \quad :\Leftrightarrow \quad \text{there exists a good sequence } \mathcal{C} \text{ such that } \mathcal{A} \approx \mathcal{C} \leq \mathcal{B}.$$

The good sequence \mathcal{C} in the definition of $\mathcal{A} \leq^* \mathcal{B}$ we often call a witness for $\mathcal{A} \leq^* \mathcal{B}$. Notice that $\mathcal{A} \leq^* \mathcal{B}$ implies $A_{(\sigma,\tau)} \subseteq^* B_{(\sigma,\tau)}$ for every $(\sigma, \tau) \in I_{(p,q)}$. Clearly, \approx is an equivalence relation on the set of good sequences for (p, q) . We have the following long but elementary lemma:

Lemma 2.1.4 *Suppose $(p, q) \in \mathbb{M}^2$, \mathcal{C} is good for (p, q) and $(\sigma, \tau) \in I_{(p,q)}$.*

- (a) *Suppose $\mu \in \text{split}(p_{(\sigma,\tau)}^{\mathcal{C}})$ with $\mu \not\supseteq \sigma$. Then we have the following:
If $\text{tp}(\sigma, \tau)$ is even, there exist sequences*

$$\mu = s_0 \not\supseteq \dots \not\supseteq s_k = \sigma \quad \text{and} \quad t_1 \not\supseteq \dots \not\supseteq t_k = \tau$$

such that $s_i \in C_{(s_{i+1}, t_{i+1})}$ for $i \in \{0, \dots, k-1\}$ and $t_i \in C_{(s_i, t_{i+1})}$ for $i \in \{1, \dots, k-1\}$. If we choose $t_1 \prec$ -maximal with $\mu \in C_{(s_1, t_1)}$ (notice there exists only finitely many t with $\mu \in C_{(s_1, t)}$), where s_1 is the direct predecessor in the tree sense of μ in $\text{split}(p_{(\sigma,\tau)}^{\mathcal{C}})$, then the sequences are unique.

If $\text{tp}(\sigma, \tau)$ is odd, there exist sequences

$$\mu = s_0 \not\supseteq \dots \not\supseteq s_k = \sigma \quad \text{and} \quad t_1 \not\supseteq \dots \not\supseteq t_{k+1} = \tau$$

such that $s_i \in C_{(s_{i+1}, t_{i+1})}$ for $i \in \{0, \dots, k-1\}$ and $t_i \in C_{(s_i, t_{i+1})}$ for $i \in \{1, \dots, k\}$. Again, if we choose $t_1 \prec$ -maximal with $\mu \in C_{(s_1, t_1)}$, where s_1 is the direct predecessor in the tree sense of μ in $\text{split}(p_{(\sigma,\tau)}^{\mathcal{C}})$, then the sequences are unique.

- (b) *Suppose $\nu \in \text{split}(q_{(\sigma,\tau)}^{\mathcal{C}})$ with $\nu \not\supseteq \tau$. Then we have the following:
If $\text{tp}(\sigma, \tau)$ is even, there exist sequences*

$$s_0 \not\supseteq \dots \not\supseteq s_k = \sigma \quad \text{and} \quad \nu = t_0 \not\supseteq \dots \not\supseteq t_k = \tau$$

such that $t_i \in C_{(s_i, t_{i+1})}$ and $s_i \in C_{(s_{i+1}, t_{i+1})}$ for $i \in \{0, \dots, k-1\}$. If we choose $s_0 \prec$ -maximal with $\nu \in C_{(s_0, t_1)}$, where t_1 is the direct predecessor in the tree sense of ν in $\text{split}(q_{(\sigma,\tau)}^{\mathcal{C}})$, then the sequences are unique.

If $\text{tp}(\sigma, \tau)$ is odd, there exist sequences

$$s_0 \not\supseteq \dots \not\supseteq s_k = \sigma \quad \text{and} \quad \nu = t_0 \not\supseteq \dots \not\supseteq t_{k+1} = \tau$$

such that $t_i \in C_{(s_i, t_{i+1})}$ for $i \in \{0, \dots, k\}$ and $s_i \in C_{(s_{i+1}, t_{i+1})}$ for $i \in \{0, \dots, k-1\}$. Again, if we choose $s_0 \prec$ -maximal with $\nu \in C_{(s_0, t_1)}$, where t_1 is the direct predecessor in the tree sense of ν in $\text{split}(q_{(\sigma,\tau)}^{\mathcal{C}})$, then the sequences are unique.

Proof: We only prove the first case of (a), the rest is similar. So suppose that $\text{tp}(\sigma, \tau)$ is even. Let $s_0 := \mu$ and choose $n \in \omega$ such that $s_0 \upharpoonright n$ is the direct predecessor in the tree sense of s_0 in $\text{split}(p_{(\sigma,\tau)}^{\mathcal{C}})$. Then there exists $t \in \text{split}(q_{(\sigma,\tau)}^{\mathcal{C}})$

with $\text{tp}(s_0 \upharpoonright n, t)$ is even and $s_0 \in C_{(s_0 \upharpoonright n, t)}$. Choose the \prec -maximal t with that property (property (v) of the definition of good sequences ensures that there are only finitely many such t) and define $s_1 := s_0 \upharpoonright n$ and $t_1 := t$.

Suppose now, s_j and t_j are defined for some $j \in \omega$. If $s_j = \sigma$ we have $t_j = \tau$ and $k = j$ and we are done. Otherwise, there exists $n \in \omega$ and $m \in \omega$ such that $t_j \in C_{(s_j, t_j \upharpoonright m)}$ and $s_j \in C_{(s_j \upharpoonright n, t_j \upharpoonright m)}$. Define $s_{j+1} := s_j \upharpoonright n$ and $t_{j+1} := t_j \upharpoonright m$. After finitely many steps the construction stops. \square

Lemma 2.1.5 *Suppose $(p, q) \in \mathbb{M}^2$ and \mathcal{A} and \mathcal{B} are good (p, q) -sequences.*

If there exists a good (p, q) -sequence \mathcal{C} with $\mathcal{A} \leq \mathcal{C} \approx \mathcal{B}$, then there exists a good (p, q) -sequence \mathcal{D} with $\mathcal{A} \approx \mathcal{D} \leq \mathcal{B}$, hence we have $\mathcal{A} \leq^ \mathcal{B}$. Moreover, \mathcal{D} can be chosen such that $\mathcal{D} \leq \mathcal{A}$.*

Proof: Suppose \mathcal{A} , \mathcal{B} and \mathcal{C} are good (p, q) -sequences with $\mathcal{A} \leq \mathcal{C} \approx \mathcal{B}$. Define

$$D_{(\sigma, \tau)} := \begin{cases} A_{(\sigma, \tau)} \cap \text{split}(p_{(\sigma, \tau)}^{\mathcal{B}}), & \text{if } \text{tp}(\sigma, \tau) \text{ is even,} \\ A_{(\sigma, \tau)} \cap \text{split}(q_{(\sigma, \tau)}^{\mathcal{B}}), & \text{if } \text{tp}(\sigma, \tau) \text{ is odd.} \end{cases}$$

Claim: $\mathcal{D} := (D_{(\sigma, \tau)})_{(\sigma, \tau) \in I_{(p, q)}}$ is a good (p, q) -sequence such that $\mathcal{A} \approx \mathcal{D} \leq \mathcal{B}$.

Proof of the claim: For proving that \mathcal{D} is good it is sufficient to show $D_{(\sigma, \tau)} =^* A_{(\sigma, \tau)}$ for every $(\sigma, \tau) \in I_{(p, q)}$.

If $\text{tp}(\sigma, \tau)$ is even, we prove $|A_{(\sigma, \tau)} \setminus \text{split}(p_{(\sigma, \tau)}^{\mathcal{B}})| < \omega$: For fixed $\mu \in A_{(\sigma, \tau)} \setminus \text{split}(p_{(\sigma, \tau)}^{\mathcal{B}})$, therefore $\mu \in \text{split}(p_{(\sigma, \tau)}^{\mathcal{C}})$, we have the unique sequences $\mu = s_0 \supsetneq \dots \supsetneq s_k = \sigma$ and $t_1 \supsetneq \dots \supsetneq t_k = \tau$ as in lemma 2.1.4. By choice, there must exist $i \in \{0, \dots, k\}$ such that $s_i \in C_{(s_{i+1}, t_{i+1})} \setminus B_{(s_{i+1}, t_{i+1})}$. Define $\Phi_{(\sigma, \tau)}(\mu)$ to be the shortest such s_i . Then $\Phi_{(\sigma, \tau)} \upharpoonright (A_{(\sigma, \tau)} \setminus \text{split}(p_{(\sigma, \tau)}^{\mathcal{B}}))$ is an injective function, because for $\mu, \mu' \in A_{(\sigma, \tau)}$ we have $\mu(|\sigma|) \neq \mu'(|\sigma|)$. So we have $|A_{(\sigma, \tau)} \setminus \text{split}(p_{(\sigma, \tau)}^{\mathcal{B}})| \leq \Sigma_{(s, t) \in I_{(p, q)}} |C_{(s, t)} \setminus B_{(s, t)}| < \omega$, the last inequality holds because we have $\mathcal{B} \approx \mathcal{C}$, and therefore $B_{(\sigma, \tau)} = C_{(\sigma, \tau)}$ for almost every $(\sigma, \tau) \in I_{(p, q)}$.

If $\text{tp}(\sigma, \tau)$ is odd, we get $|A_{(\sigma, \tau)} \setminus \text{split}(q_{(\sigma, \tau)}^{\mathcal{B}})| < \omega$ by a similar proof.

It remains to prove that we have $A_{(\sigma, \tau)} = D_{(\sigma, \tau)}$ for all but finitely many $(\sigma, \tau) \in I_{(p, q)}$. Define

$$E := \{(s, t) \in (<^\omega \omega)^2 \mid \exists (\sigma, \tau) \in I_{(p, q)} ((\sigma, \tau) \geq (s, t) \wedge B_{(\sigma, \tau)} \neq C_{(\sigma, \tau)})\}.$$

E is finite and for all (s, t) in $I_{(p, q)} \setminus E$ we have $(p_{(s, t)}^{\mathcal{B}}, q_{(s, t)}^{\mathcal{B}}) = (p_{(s, t)}^{\mathcal{C}}, q_{(s, t)}^{\mathcal{C}})$, hence $A_{(s, t)} = D_{(s, t)}$. \square

Corollary 2.1.6 *Fix $(p, q) \in \mathbb{M}^2$. Then our relation \leq^* is transitive on the set of good (p, q) -sequences.*

Proof: Suppose $\mathcal{A} \leq^* \mathcal{B} \leq^* \mathcal{C}$, so there exist good sequences \mathcal{A}' and \mathcal{B}' witnessing these. Hence we have the following relationships:

$$\begin{array}{c} \mathcal{C} \\ \vee | \\ \mathcal{B} \approx \mathcal{B}' \\ \vee | \quad \vee | \\ \mathcal{A} \approx \mathcal{A}' \approx \mathcal{D} \end{array}$$

\mathcal{D} exists by lemma 2.1.5 and is a witness for $\mathcal{A} \leq^* \mathcal{C}$. \square

Lemma 2.1.7 *Suppose $(p, q) \in \mathbb{M}^2$, \mathcal{A} is a good (p, q) -sequence and $X \in \mathcal{J}(\mathbb{M}^2)$. Then there exists a good (p, q) -sequence $\mathcal{B} \leq \mathcal{A}$ such that $([p_{(\sigma, \tau)}^{\mathcal{B}}] \times [q_{(\sigma, \tau)}^{\mathcal{B}}]) \cap X = \emptyset$ for every $(\sigma, \tau) \in I_{(p, q)}$.*

Proof: Let $((\sigma_n, \tau_n))_{n \in \omega}$ enumerate all pairs in $I_{(p, q)}$ such that $(\sigma_0, \tau_0) = (\text{st}(p), \text{st}(q))$. Let $n_0 := 0$ and choose by fact 2.0.6 $(p_{(\sigma_{n_0}, \tau_{n_0})}, q_{(\sigma_{n_0}, \tau_{n_0})})$ in \mathbb{M}^2 such that $(p_{(\sigma_{n_0}, \tau_{n_0})}, q_{(\sigma_{n_0}, \tau_{n_0})}) \leq^0 (p_{(\sigma_{n_0}, \tau_{n_0})}^{\mathcal{A}}, q_{(\sigma_{n_0}, \tau_{n_0})}^{\mathcal{A}})$ and

$$([p_{(\sigma_{n_0}, \tau_{n_0})}] \times [q_{(\sigma_{n_0}, \tau_{n_0})}]) \cap X = \emptyset.$$

By a remark in the preliminaries, there exists $(p'_{(\sigma_{n_0}, \tau_{n_0})}, q'_{(\sigma_{n_0}, \tau_{n_0})}) \in \mathbb{M}^2$ such that $(p'_{(\sigma_{n_0}, \tau_{n_0})}, q'_{(\sigma_{n_0}, \tau_{n_0})}) \leq^0 (p_{(\sigma_{n_0}, \tau_{n_0})}, q_{(\sigma_{n_0}, \tau_{n_0})})$ and for almost all $n \in \omega$ and every $(x, y) \in [p'_{(\sigma_{n_0}, \tau_{n_0})}] \times^+ [q'_{(\sigma_{n_0}, \tau_{n_0})}]$ we have $\text{tp}_{(\sigma_{n_0}, \tau_{n_0})}$ - n -pair $(x, y) \in \text{split}(p'_{(\sigma_{n_0}, \tau_{n_0})}) \times \text{split}(q'_{(\sigma_{n_0}, \tau_{n_0})})$. By remark 2.1.2 we get a partial good sequence $\mathcal{B}_0 = (B_{(\sigma, \tau)}^0)_{(\sigma, \tau) \in I_0}$ with $I_0 := I_{(p, q)} \cap (\text{split}(p'_{(\sigma_{n_0}, \tau_{n_0})}) \times \text{split}(q'_{(\sigma_{n_0}, \tau_{n_0})}))$. Suppose we have already constructed a strictly ascending sequence $(n_i)_{i \leq k}$ of natural numbers and a sequence $(\mathcal{B}_i)_{i \leq k}$ of partial good sequences $\mathcal{B}_i = (B_{(\sigma, \tau)}^i)_{(\sigma, \tau) \in I_i}$ such that $I_0 \subseteq \dots \subseteq I_{k-1} \subseteq I_k$ and \mathcal{B}_{i+1} extends the sequence \mathcal{B}_i for every $i < k$, this means $B_{(\sigma, \tau)}^{i+1} = B_{(\sigma, \tau)}^i$ for every $(\sigma, \tau) \in I_i$. Choose $n_{k+1} \in \omega$ minimal such that $(\sigma_{n_{k+1}}, \tau_{n_{k+1}}) \notin I_k$. Choose, by remark 2.0.6, $(p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}, q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})})$ in \mathbb{M}^2 with $(p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}, q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}) \leq^0 (p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}^{\mathcal{A}}, q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}^{\mathcal{A}})$ and

$$([p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}] \times [q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}]) \cap X = \emptyset.$$

Thin out $p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}$ and $q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}$ such that whenever $\sigma_{n_{k+1}} \in p_{(\sigma_{n_i}, \tau_{n_i})} \setminus \text{split}(p_{(\sigma_{n_i}, \tau_{n_i})})$ for some $i \leq k$ we have $p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})} \cap p_{(\sigma_{n_i}, \tau_{n_i})} = \emptyset$ and whenever $\tau_{n_{k+1}} \in q_{(\sigma_{n_i}, \tau_{n_i})} \setminus \text{split}(q_{(\sigma_{n_i}, \tau_{n_i})})$ for some $i \leq k$ we have $q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})} \cap q_{(\sigma_{n_i}, \tau_{n_i})} = \emptyset$ (Notice that we only have to cut out one single Miller tree for every relevant i). Again by the remark in the preliminaries there exists

$(p'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}, q'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}) \leq^0 (p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}, q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})})$ such that for almost all $n \in \omega$ and every $(x, y) \in [(p'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}) \times^+ [q'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}]]$ we have $\text{tp}_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}\text{-}n\text{-pair}(x, y) \in \text{split}(p'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}) \times \text{split}(q'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})})$. By remark 2.1.2 there exists a partial good sequence $\mathcal{C} = (C_{(\sigma, \tau)})_{(\sigma, \tau) \in J}$ for the pair $(p_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}, q_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})})$, where $J := I_{(p, q)} \cap (\text{split}(p'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}) \times \text{split}(q'_{(\sigma_{n_{k+1}}, \tau_{n_{k+1}})}))$.

Define $I_{k+1} := I_k \cup J$ and \mathcal{B}_{k+1} by

$$B_{(\sigma, \tau)}^{k+1} := \begin{cases} B_{(\sigma, \tau)}^k, & \text{if } (\sigma, \tau) \in I_k, \\ C_{(\sigma, \tau)}, & \text{if } (\sigma, \tau) \in I_{k+1} \setminus I_k. \end{cases}$$

It is easy to see that \mathcal{B}_{k+1} is a partial good sequence which extends \mathcal{B}_k .

In the end, we get a good sequence $\mathcal{B} = (B_{(\sigma, \tau)})_{(\sigma, \tau) \in \bigcup_{n \in \omega} I_n}$. It is $\bigcup_{n \in \omega} I_n = I_{(p, q)}$, because for every $(\sigma, \tau) \in I_{(p, q)}$ there exists $n_k \in \omega$ with $(\sigma, \tau) \in \text{split}(p_{(\sigma_{n_k}, \tau_{n_k})}) \times \text{split}(q_{(\sigma_{n_k}, \tau_{n_k})})$, hence $B_{(\sigma, \tau)}$ is defined. Clearly, we have $([p_{(\sigma, \tau)}^{\mathcal{B}}] \times [q_{(\sigma, \tau)}^{\mathcal{B}}]) \cap X = \emptyset$ for every $(\sigma, \tau) \in I_{(p, q)}$. \square

Lemma 2.1.8 *Suppose $\text{MA}_\kappa(\sigma\text{-centered})$ is true and fix $(p, q) \in \mathbb{M}^2$.*

If $(\mathcal{A}^\alpha)_{\alpha < \kappa}$ is a \leq^ -descending sequence of good (p, q) -sequences, then there exists a good (p, q) -sequence \mathcal{B} such that $\mathcal{B} \leq^* \mathcal{A}^\alpha$ for all $\alpha < \kappa$.*

Proof: Define a forcing P as follows:

Conditions in P are of the form $(F, \{T_{(\sigma, \tau)} \mid (\sigma, \tau) \in S\})$ such that

- (a) F is a finite subset of κ ;
- (b) S is a finite subset of $I_{(p, q)}$;
- (c) $T_{(\sigma, \tau)}$ is a finite subset of ${}^{<\omega}\omega$;
- (d) if $\text{tp}(\sigma, \tau)$ is even, then $\forall s \in T_{(\sigma, \tau)}(s \supseteq \sigma \wedge |s| > s(|\sigma|))$ and $\forall s, s' \in T_{(\sigma, \tau)}(s \neq s' \Rightarrow s(|\sigma|) \neq s'(|\sigma|))$,
if $\text{tp}(\sigma, \tau)$ is odd, then $\forall t \in T_{(\sigma, \tau)}(t \supseteq \tau \wedge |t| > t(|\tau|))$ and $\forall t, t' \in T_{(\sigma, \tau)}(t \neq t' \Rightarrow t(|\tau|) \neq t'(|\tau|))$;
- (e) $T_{(\sigma, \tau)} \subseteq \text{Sop}_{\text{st}(p), \text{st}(q)}^{(p, q)}(\sigma, \tau)$;
- (f) if $\text{tp}(\sigma, \tau)$ is even and $(\sigma, \tau), (\rho, \tau), (\rho', \tau)$ are in S such that $\rho, \rho' \in T_{(\sigma, \tau)}$ and $\rho(|\sigma|) < \rho'(|\sigma|)$, then we have $T_{(\rho, \tau)} \supseteq T_{(\rho', \tau)}$;
if $\text{tp}(\sigma, \tau)$ is odd and $(\sigma, \tau), (\sigma, \rho), (\sigma, \rho')$ are in S such that $\rho, \rho' \in T_{(\sigma, \tau)}$ and $\rho(|\tau|) < \rho'(|\tau|)$, then we have $T_{(\sigma, \rho)} \supseteq T_{(\sigma, \rho')}$.

Define $(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \leq (F', \{T'_{(\sigma,\tau)} \mid (\sigma,\tau) \in S'\}) :\Leftrightarrow$

$$\begin{aligned} F &\supseteq F', \\ S &\supseteq S', \\ \forall (\sigma,\tau) \in S' & (T_{(\sigma,\tau)} \supseteq T'_{(\sigma,\tau)}) \text{ and} \\ \forall (\sigma,\tau) \in S \forall \alpha \in F' & (T_{(\sigma,\tau)} \setminus T'_{(\sigma,\tau)} \subseteq \text{split}(r_{(\sigma,\tau)}^{A_\alpha})), \end{aligned}$$

where $r = \begin{cases} p, & \text{if } \text{tp}(\sigma,\tau) \text{ is even,} \\ q, & \text{if } \text{tp}(\sigma,\tau) \text{ is odd;} \end{cases}$ and $T'_{(\sigma,\tau)} = \emptyset$ if $(\sigma,\tau) \notin S'$.

Then P is σ -centered because

$$P = \bigcup \{P_{\{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}} \mid \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\} \text{ is as in the definition of } P\},$$

where $P_{\{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}} := \{(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \in P \mid F \subseteq \kappa \text{ finite}\}$. There are only countable many sets $\{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}$ and obviously each set $P_{\{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}}$ is centered.

In the following we define suitable dense subsets of P : For $\alpha < \kappa$ let

$$D_\alpha := \{(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \in P \mid \alpha \in F\}.$$

Clearly, D_α is dense for every $\alpha < \kappa$.

We define

$$E_{s,t,n} := \{(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \mid (s,t) \in S \wedge \exists \mu \in T_{(s,t)} (\mu(|s|) > n)\}$$

for $n \in \omega$ and $s, t \in {}^{<\omega}\omega$ such that $\text{tp}(s, t)$ is even and

$$F_{s,t,m} := \{(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \mid (s,t) \in S \wedge \exists \nu \in T_{(s,t)} (\nu(|t|) > m)\}$$

for $m \in \omega$ and $s, t \in {}^{<\omega}\omega$ such that $\text{tp}(s, t)$ is odd.

Claim 1: The set $E_{s,t,n}$ is dense for every $n \in \omega$ and every $s, t \in {}^{<\omega}\omega$ such that $\text{tp}(s, t)$ is even and $F_{s,t,m}$ is dense for every $m \in \omega$ and every $s, t \in {}^{<\omega}\omega$ such that $\text{tp}(s, t)$ is odd.

Proof of claim 1: We only prove the even case, the odd case is similar. Suppose $n \in \omega$ and s, t are in ${}^{<\omega}\omega$ such that $\text{tp}(s, t)$ even and $(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \in P \setminus E_{s,t,n}$. Define $S' := S \cup \{(s, t)\}$ (maybe $S = S'$). Let $(\alpha_0, \dots, \alpha_{k-1})$ be an ascending enumeration of F . Then we have $A_{\alpha_0} \geq^* \dots \geq^* A_{\alpha_{k-1}}$ and this implies that the set $\bigcap_{i < k} A_{(s,t)}^{\alpha_i}$ is infinite. We have to care about property (f) in the definition of P : Suppose there exists τ with $(s, \tau) \in S$ and $t \in T_{(s,\tau)}$ (τ is unique); else we have nothing to do. Let $(t_j)_{j < N}$ be an enumeration of all t' with $t' \in T_{(s,\tau)}$, $(s, t') \in S$ and $t'(|\tau|) < t(|\tau|)$ such that $t_j(|\tau|) < t_{j+1}(|\tau|)$ for every $j < N - 1$. Then we have $A_{(s,t_0)}^{\alpha_i} \supseteq^* A_{(s,t_1)}^{\alpha_i} \supseteq^* \dots \supseteq^* A_{(s,t)}^{\alpha_i}$ for every

$i < k$ and therefore $\bigcap_{i < k} A_{(s,t)}^{\alpha_i} \subseteq^* \bigcap_{i < k} A_{(s,t_{N-1})}^{\alpha_i} \subseteq^* \dots \subseteq^* \bigcap_{i < k} A_{(s,t_0)}^{\alpha_i}$. So the set

$$\bigcap_{i < k} A_{(s,t)}^{\alpha_i} \cap \bigcap_{i < k} A_{(s,t_{N-1})}^{\alpha_i} \cap \dots \cap \bigcap_{i < k} A_{(s,t_0)}^{\alpha_i}$$

is infinite. Choose $\mu \in \bigcap_{i < k} A_{(s,t)}^{\alpha_i} \cap \bigcap_{i < k} A_{(s,t_{N-1})}^{\alpha_i} \cap \dots \cap \bigcap_{i < k} A_{(s,t_0)}^{\alpha_i}$ with $\mu(|s|) > n$ and $\mu \notin \pi_1(S) = \{s \mid \exists t ((s,t) \in S)\}$. Define $T'_{(s,t)} := T_{(s,t)} \cup \{\mu\}$ ($T_{(s,t)} = \emptyset$ if $(s,t) \notin S$), $T'_{(s,t_j)} := T_{(s,t_j)} \cup \{\mu\}$ for every $j < N$ and $T'_{(\sigma,\tau)} := T_{(\sigma,\tau)}$ for every $(\sigma,\tau) \in S \setminus (\{(s,t)\} \cup \{(s,t_j) \mid j < N\})$. We have $T'_{(s,t')} \setminus T_{(s,t')} = \{\mu\} \subseteq \text{split}(p_{(s,t')})$ for every $t' \in \{t_j \mid j < N-1\} \cup \{t\}$, hence $(F, \{T'_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\} \cup \{T'_{(s,t)}\}) \leq (F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\})$ and $(F, \{T'_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\} \cup \{T'_{(s,t)}\})$ is in D_α . \square (claim1)

By MA(σ -centered) let G be a generic filter for the following dense sets: $E_{s,t,n}$ for $s, t \in {}^{<\omega}\omega$ such that $\text{tp}(s,t)$ is even and $n \in \omega$, $F_{s,t,m}$ for $s, t \in {}^{<\omega}\omega$ such that $\text{tp}(s,t)$ is odd and $m \in \omega$ and D_α for $\alpha < \kappa$. For $(s,t) \in I_{(p,q)}$ define

$$B_{(s,t)} := \{\rho \in {}^{<\omega}\omega \mid \exists (F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \in G ((s,t) \in S \wedge \rho \in T_{(s,t)})\}.$$

Claim 2: $\mathcal{B} := (B_{(s,t)})_{(s,t) \in I_{(p,q)}}$ is a good (p,q) -sequence.

Proof of claim 2: First suppose $\text{tp}(s,t)$ is even and $\rho, \rho' \in B_{(s,t)}$ with $\rho \neq \rho'$. So there are $(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\})$ and $(F', \{T'_{(\sigma,\tau)} \mid (\sigma,\tau) \in S'\})$ in G with $\rho \in T_{(s,t)}$ and $\rho' \in T'_{(s,t)}$, and there exists $(F'', \{T''_{(\sigma,\tau)} \mid (\sigma,\tau) \in S''\}) \leq (F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}), (F', \{T'_{(\sigma,\tau)} \mid (\sigma,\tau) \in S'\})$ in G . So we have $T''_{(s,t)} \supseteq T_{(s,t)}, T'_{(s,t)}$ and hence $\rho(|s|) \neq \rho'(|s|)$. The case that $\text{tp}(s,t)$ is odd is analogous, so (ii) of the definition of a good sequence (definition 2.1.1) holds.

For (i) we have to prove that $|B_{(s,t)}|$ is infinite. If $\text{tp}(s,t)$ is even, for every $n \in \omega$ there exists $(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \in G \cap E_{s,t,n}$, hence $(s,t) \in S$ and there is a $\mu \in T_{(s,t)}$ with $\mu(|s|) > n$. If $\text{tp}(s,t)$ is odd we can find for every $n \in \omega$ a $\nu \in T_{(s,t)}$ with $\nu(|t|) > n$.

(iii) and (iv) are easy to verify. For (v) suppose $\text{tp}(s,t)$ is even and $\rho \in B_{(s,t)}$. Fix $\nu \in B_{(\rho,\tau)}$. Since $G \cap E_{s,t,n} \neq \emptyset$ for every $n \in \omega$, we get $\rho' \in B_{(s,t)}$ such that $\rho'(|s|) > \#\nu$, hence $\nu \notin B_{(\rho',t)}$, hence we have $B_{(\rho,t)} \supsetneq B_{(\rho',t)}$. \square (claim 2)

Claim 3: $\mathcal{B} \leq^* \mathcal{A}^\alpha$ for every $\alpha < \kappa$.

Proof of claim 3: Suppose $\alpha < \kappa$ and fix $(F, \{T_{(\sigma,\tau)} \mid (\sigma,\tau) \in S\}) \in G \cap D_\alpha$.

Subclaim 1: For every $(s,t) \in I_{(p,q)} \setminus S$ we have $B_{(s,t)} \subseteq \text{split}(r_{(s,t)}^\alpha)$ where $r = p$ if $\text{tp}(s,t)$ is even and $r = q$ if $\text{tp}(s,t)$ is odd.

Proof of subclaim 1: Suppose $(s, t) \in I_{(p,q)} \setminus S$ and $\text{tp}(s, t)$ is even. Let $\mu \in B_{(s,t)}$, then there exists $(F', \{T'_{(\sigma,\tau)} \mid (\sigma, \tau) \in S'\}) \in G$ with $(s, t) \in S'$ and $\mu \in T'_{(s,t)}$. Choose $(F'', \{T''_{(\sigma,\tau)} \mid (\sigma, \tau) \in S''\}) \leq (F, \{T_{(\sigma,\tau)} \mid (\sigma, \tau) \in S\})$, $(F', \{T'_{(\sigma,\tau)} \mid (\sigma, \tau) \in S'\})$ in G . Then we have $(s, t) \in S''$, $\mu \in T''_{(s,t)}$, $(s, t) \notin S$ and $\alpha \in F$ and these imply $\mu \in T''_{(s,t)} \setminus T_{(s,t)} = T''_{(s,t)} \subseteq \text{split}(p_{(s,t)}^{A^\alpha})$, as desired. The odd case is once more analogous. \square (subclaim 1)

Subclaim 2: For the finitely many $(s, t) \in S$ we have $B_{(s,t)} \subseteq^* \text{split}(r_{(s,t)}^{A^\alpha})$ for $r = p$ if $\text{tp}(s, t)$ is even and $r = q$ if $\text{tp}(s, t)$ is odd.

Proof of subclaim 2: Suppose $(s, t) \in S$ and $\text{tp}(s, t)$ is even. We prove that we have $B_{(s,t)} \setminus T_{(s,t)} \subseteq \text{split}(p_{(s,t)}^{A^\alpha})$. Suppose $\mu \in B_{(s,t)} \setminus T_{(s,t)}$, so there exists $(F', \{T'_{(\sigma,\tau)} \mid (\sigma, \tau) \in S'\}) \in G$ with $(s, t) \in S'$ and $\mu \in T'_{(s,t)}$. Choose $(F'', \{T''_{(\sigma,\tau)} \mid (\sigma, \tau) \in S''\}) \leq (F, \{T_{(\sigma,\tau)} \mid (\sigma, \tau) \in S\})$, $(F', \{T'_{(\sigma,\tau)} \mid (\sigma, \tau) \in S'\})$ in G . Then we have $\mu \in T''_{(s,t)}$, $\mu \notin T_{(s,t)}$ and $\alpha \in F$ and these imply $\mu \in T''_{(s,t)} \setminus T_{(s,t)} \subseteq \text{split}(p_{(s,t)}^{A^\alpha})$. The odd case is analogous. \square (subclaim 2)
Subclaim 1 and 2 together easily imply $\mathcal{B} \leq^* \mathcal{A}^\alpha$. \square (claim 3)

\square

2.2 The proof of the maintheorem

Now we are able to prove the main theorem of this chapter, but we first write down a generalized version:

Theorem 2.2.1 $\text{MA}_\kappa(\sigma\text{-centered})$ implies $\text{add}(\mathcal{J}(\mathbb{M}^2)) > \kappa$.

Proof of theorem 2.2.1 and therefore of theorem 2.0.5: Suppose $\text{MA}_\kappa(\sigma\text{-centered})$ is true and $X_\alpha \in J(\mathbb{M}^2)$ for $\alpha < \kappa$ and $(p, q) \in \mathbb{M}^2$. We search for a pair $(p', q') \leq (p, q)$ in \mathbb{M}^2 such that $([p'] \times [q']) \cap \bigcup_{\alpha < \kappa} X_\alpha = \emptyset$. Define by induction a \leq^* -descending sequence $(\mathcal{A}^\alpha)_{\alpha \leq \kappa}$ of good (p, q) -sequences: For $\alpha = 0$ define \mathcal{A}^0 to be the good sequence for (p, q) defined in remark 2.1.2. Suppose \mathcal{A}^β for $\beta < \alpha$ is already defined and $\alpha < \kappa$ is a limit ordinal. Choose by lemma 2.1.8 (we use $\text{MA}_\kappa(\sigma\text{-centered})$ here) a good (p, q) -sequence \mathcal{A}^α with $\mathcal{A}^\alpha \leq^* \mathcal{A}^\beta$ for every $\beta < \alpha$.

For the successor step, suppose \mathcal{A}^α is defined. Choose a good (p, q) -sequence $\mathcal{A}^{\alpha+1} \leq^* \mathcal{A}^\alpha$ with $([p_{(\sigma,\tau)}^{\mathcal{A}^{\alpha+1}}] \times [q_{(\sigma,\tau)}^{\mathcal{A}^{\alpha+1}}]) \cap X_\alpha = \emptyset$ for every $(\sigma, \tau) \in I_{(p,q)}$ by lemma 2.1.7.

Notice that the pair $(p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}, q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa})$ of Miller trees does not have the desired property, because for $\alpha < \kappa$ we only have $\mathcal{A}^\kappa \leq^* \mathcal{A}^{\alpha+1}$, hence there

could exist a pair $(f, g) \in [p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}] \times [q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}]$ with $(f, g) \notin [p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^{\alpha+1}}] \times [q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^{\alpha+1}}]$, therefore we do not know whether (f, g) is in X or not. We will define a forcing Q to get $(p', q') \leq (p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}, q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa})$ such that for every $(f, g) \in [p'] \times [q']$ and every $\alpha < \kappa$ we have $(f, g) \in [p_{(f \upharpoonright n, g \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}] \times [q_{(f \upharpoonright n, g \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}]$ for suitable $n, m \in \omega$. In this situation the choice of $\mathcal{A}^{\alpha+1}$ guarantees $(f, g) \notin X$.

Define Q in the following way: Q is the set of all (u, v, X) with

- (a') $X \subseteq \kappa$ is finite;
- (b') $u : m \rightarrow \text{split}(p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa})$ and $v : m' \rightarrow \text{split}(q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa})$, $m, m' \in \omega$ with $m = m'$ or $m = m' + 1$;
- (c') the downward closures U and V of $\text{ran}(u)$ and $\text{ran}(v)$ are finite subtrees of $p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}, q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}$, respectively, with $\text{split}(U) \subseteq \text{ran}(u)$ and $\text{split}(V) \subseteq \text{ran}(v)$;
- (d') the functions $\Phi : (\text{ran}(u), \leq) \rightarrow (T \upharpoonright m, \leq)$ defined by $u(k) \mapsto T(k)$ and $\Psi : (\text{ran}(v), \leq) \rightarrow (T \upharpoonright m', \leq)$ defined by $v(k) \mapsto T(k)$ are both isomorphisms;
- (e') if $u(j)$ is the immediate predecessor in the tree sense of $u(k+1)$ in $\text{ran}(u)$, then we have $u(k+1)(|u(j)|) > \max(\{\#u(i) \mid i \leq k\} \cup \{\#v(i) \mid i \leq k\})$ and $|u(k+1)| > u(k+1)(|u(j)|)$,
if $v(j)$ is the immediate predecessor in the tree sense of $v(k+1)$ in $\text{ran}(v)$, then we have $v(k+1)(|v(j)|) > \max(\{\#u(i) \mid i \leq k+1\} \cup \{\#v(i) \mid i \leq k\})$ and $|v(k+1)| > v(k+1)(|v(j)|)$.

Define $(u, v, X) \leq (u', v', X') :\Leftrightarrow$

$$u \supseteq u', v \supseteq v', X \supseteq X' \text{ and}$$

for every $\alpha \in X'$ and every $(\sigma, \tau) \in (\text{ran}(u) \times \text{ran}(v)) \setminus (\text{ran}(u') \times \text{ran}(v'))$ we have $(\sigma, \tau) \in [p_{(\sigma \upharpoonright n, \tau \upharpoonright m)}^{\mathcal{A}^\alpha}] \times [q_{(\sigma \upharpoonright n, \tau \upharpoonright m)}^{\mathcal{A}^\alpha}]$
for every $n, m \in \omega$ such that $(\sigma \upharpoonright n, \tau \upharpoonright m) \in I_{(p, q)}$.

Q is σ -centered, because $Q = \bigcup \{Q_{(u, v)} \mid (u, v) \text{ is as in the definition of } Q\}$, where $Q_{(u, v)} := \{(u, v, X) \in Q \mid X \subseteq \kappa \text{ is finite}\}$ is obviously centered and there are only countable many (u, v) .

In the following, we will define suitable dense subsets of Q : For $\alpha < \kappa$ define

$$D_\alpha := \{(u, v, X) \in Q \mid \alpha \in X\}.$$

Clearly, D_α is dense for every $\alpha < \kappa$.

We define

$$E_{s,t,n} := \{(u, v, X) \in Q \mid (s, t) \in \text{ran}(u) \times \text{ran}(v) \wedge \\ \exists \mu \in \text{ran}(u) (s \not\subseteq \mu \wedge \mu(|s|) > n)\}$$

for $(s, t) \in I_{(p,q)}$ with $\text{tp}(s, t)$ even and $n \in \omega$ and

$$F_{s,t,m} := \{(u, v, X) \in Q \mid (s, t) \in \text{ran}(u) \times \text{ran}(v) \wedge \\ \exists \nu \in \text{ran}(v) (t \not\subseteq \nu \wedge \nu(|t|) > m)\}$$

for $(s, t) \in I_{(p,q)}$ with $\text{tp}(s, t)$ odd and $m \in \omega$. For $n \in \omega$ and $(s, t) \in I_{(p,q)}$ with $\text{tp}(s, t)$ even the set $E_{s,t,n}$ is dense below every $(\tilde{u}, \tilde{v}, \tilde{X})$ with $(s, t) \in \text{ran}(\tilde{u}) \times \text{ran}(\tilde{v})$. And for $m \in \omega$ and $(s, t) \in I_{(p,q)}$ with $\text{tp}(s, t)$ odd the set $F_{s,t,m}$ is dense below every $(\tilde{u}, \tilde{v}, \tilde{X})$ with $(s, t) \in \text{ran}(\tilde{u}) \times \text{ran}(\tilde{v})$. The proof of this equals the proof of claim 1 in lemma 2.1.8 and is skipped. Letting

$$\tilde{E}_{s,t,n} := \{(u, v, X) \in Q \mid \exists (u', v', X') \in E_{s,t,n} ((u, v, X) \leq (u', v', X'))\} \\ \cup \{(u, v, X) \in Q \mid (u, v, X) \text{ is incompatible with every element of } E_{s,t,n}\}$$

and

$$\tilde{F}_{s,t,m} := \{(u, v, X) \in Q \mid \exists (u', v', X') \in F_{s,t,m} ((u, v, X) \leq (u', v', X'))\} \\ \cup \{(u, v, X) \in Q \mid (u, v, X) \text{ is incompatible with every element of } F_{s,t,m}\}$$

we get dense subsets of Q . By $\text{MA}_\kappa(\sigma\text{-centered})$ let G be a generic filter for the dense sets D_α for $\alpha < \kappa$, $\tilde{E}_{s,t,n}$ for $(s, t) \in I_{(p,q)}$ with $\text{tp}(s, t)$ even and $n \in \omega$ and $\tilde{F}_{s,t,m}$ with $(s, t) \in I_{(p,q)}$ with $\text{tp}(s, t)$ odd and $m \in \omega$. Define p' and q' to be the superperfect trees determined by

$$\text{split}(p') = \bigcup \{\text{ran}(u) \mid \exists v \exists X ((u, v, X) \in G)\}$$

and by

$$\text{split}(q') = \bigcup \{\text{ran}(v) \mid \exists u \exists X ((u, v, X) \in G)\}.$$

Claim 1: $(p', q') \in \mathbb{M}^2$.

Proof of claim 1: Suppose $(s, t) \in p' \times q'$. Then there exists $(u, v, X) \in G$ and $s' \in \text{ran}(u)$ and $t' \in \text{ran}(v)$ with $s' \supseteq s$, $t' \supseteq t$, $(s', t') \in \text{ran}(u) \times \text{ran}(v)$ and $\text{tp}(s', t')$ exists. If $\text{tp}(s', t')$ is even, we have $G \cap E_{s',t',n} \neq \emptyset$ for every $n \in \omega$, therefore s' is an infinite splitnode above s , hence $p' \in \mathbb{M}$. If $\text{tp}(s', t')$ is odd we have $G \cap F_{s',t',m} \neq \emptyset$ for some $m \in \omega$ and it is possible to choose extensions s'' of s and t'' of t' with $\text{tp}(s'', t'')$ is even. Again, we have $G \cap E_{s'',t'',n} \neq \emptyset$ for every $n \in \omega$ and therefore s'' is an infinite splitnode above s . Analogously, we

get $q' \in \mathbb{M}$.

□(claim 1)

Obviously, we have $(p', q') \leq (p, q)$.

Claim 2: $([p'] \times [q']) \cap \bigcup_{\alpha < \kappa} X_\alpha = \emptyset$.

Proof of claim 2: It suffices to prove $([p'] \times [q']) \cap X_\alpha = \emptyset$ for every $\alpha < \kappa$. Fix $\alpha < \kappa$ and $(u, v, X) \in G \cap D_{\alpha+1}$, so $\alpha + 1 \in X$ and let U and V be the downward closures of $\text{ran}(u)$, $\text{ran}(v)$, respectively.

Subclaim: For every $(s, t) \in (p' \setminus U) \times (q' \setminus V)$ we have $(s, t) \in p_{(s \upharpoonright n, t \upharpoonright m)}^{\mathcal{A}^{\alpha+1}} \times q_{(s \upharpoonright n, t \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}$ for every $n, m \in \omega$ such that $(s \upharpoonright n, t \upharpoonright m) \in I_{(p, q)}$.

Proof of the subclaim: Suppose $(s, t) \in (p' \setminus U) \times (q' \setminus V)$ and choose $(s', t') \geq (s, t)$ with $(s', t') \in ((p' \setminus U) \times (q' \setminus V)) \cap (\text{split}(p_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}) \times \text{split}(q_{(\text{st}(p), \text{st}(q))}^{\mathcal{A}^\kappa}))$. There are $(u', v', X'), (u'', v'', X'') \in G$ such that $s' \in \text{ran}(u')$ and $t' \in \text{ran}(v'')$. Choose $(\bar{u}, \bar{v}, \bar{X}) \leq (u', v', X'), (u'', v'', X'')$ in G and then $(\tilde{u}, \tilde{v}, \tilde{X}) \leq (\bar{u}, \bar{v}, \bar{X}), (u, v, X)$ in G , as well. Then we have $s' \in \text{ran}(\tilde{u}) \setminus \text{ran}(u)$, $t' \in \text{ran}(\tilde{v}) \setminus \text{ran}(v)$ and $\alpha + 1 \in X$ and these imply $(s', t') \in (p_{(s' \upharpoonright n, t' \upharpoonright m)}^{\mathcal{A}^{\alpha+1}} \times q_{(s' \upharpoonright n, t' \upharpoonright m)}^{\mathcal{A}^{\alpha+1}})$ for every $n, m \in \omega$ such that $(s' \upharpoonright n, t' \upharpoonright m) \in I_{(p, q)}$. Therefore, (s, t) is in $p_{(s' \upharpoonright n, t' \upharpoonright m)}^{\mathcal{A}^{\alpha+1}} \times q_{(s' \upharpoonright n, t' \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}$, as well. □(subclaim)

Now we are able to prove claim 2: For $(f, g) \in [p'] \times [q']$ we get $(f, g) \in [p_{(f \upharpoonright n, g \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}] \times [q_{(f \upharpoonright n, g \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}]$ for suitable $n, m \in \omega$ and we have chosen $\mathcal{A}^{\alpha+1}$ in such a way that $([p_{(f \upharpoonright n, g \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}] \times [q_{(f \upharpoonright n, g \upharpoonright m)}^{\mathcal{A}^{\alpha+1}}]) \cap X_\alpha = \emptyset$ hold. □(claim 2)

□(theorem 2.2.1 and 2.0.5)

Since $\text{MA}_\kappa(\sigma\text{-centered})$ is true for every $\kappa < \mathfrak{p}$, where \mathfrak{p} is the least cardinality of a filter base on $([\omega]^\omega, \subseteq^*)$ without any lower bound, theorem 2.2.1 implies the following:

Corollary 2.2.2 $\text{add}(\mathcal{J}(\mathbb{M}^2)) \geq \mathfrak{p}$. □

Notice also that we have $\text{cov}(\mathcal{J}(\mathbb{M}^2)) \leq \mathfrak{d}$ like in the one-dimensional case, where \mathfrak{d} is the dominating number, i.e. the least cardinality of a dominating family in $({}^\omega\omega, \leq^*)$. For this, let $(f_\alpha)_{\alpha < \mathfrak{d}}$ be a dominating family in ${}^\omega\omega$ and define $X_\alpha := \{(f, g) \in {}^\omega\omega \times {}^\omega\omega \mid f, g \leq^* f_\alpha\}$. Then it is easy to see that $\bigcup_{\alpha < \mathfrak{d}} X_\alpha = {}^\omega\omega \times {}^\omega\omega$ and clearly, each X_α is in $\mathcal{J}(\mathbb{M}^2)$. Summarized, we have

$$\omega_1 \leq \mathfrak{p} \leq \text{add}(\mathcal{J}(\mathbb{M}^2)) \leq \text{cov}(\mathcal{J}(\mathbb{M}^2)) \leq \mathfrak{d} \leq 2^\omega.$$

Let us remark, that in the model, built for proving the main theorem 1.1.2 in the previous chapter, we have $\text{add}(\mathcal{J}(\mathbb{M}^2)) > \text{cov}(\mathcal{M}) \geq \mathfrak{t} \geq \mathfrak{p}$, where \mathfrak{t} is the tower number, that is the least cardinality of a decreasing chain in $([\omega]^\omega, \subseteq^*)$ without any lower bound. The last inequality is trivial, the second one is a result of Piotrowski and Szymański [PiSz].

2.3 MA(σ -centered) implies Martin's Axiom for \mathbb{M}^2

The following definitions are due to Veličković [Ve].

Definition 2.3.1 For a forcing P and a dense set $D \subseteq P$ we say that a sub-ordering $Q \subseteq P$ captures the density of D if $D \cap Q$ is dense in Q .

If \mathcal{D} is a family of dense sets of P , we say that Q captures the density of \mathcal{D} if Q captures the density of each $D \in \mathcal{D}$.

Definition 2.3.2 For a forcing P and a class \mathcal{Z} of forcings let $\mathcal{Z}(P)$ be the following statement:

Whenever \mathcal{D} is a family of at most 2^ω dense subsets of P and $p \in P$, then there exists a $Q \in \mathcal{Z}$ such that $p \in Q \subseteq P$ and Q captures the density of \mathcal{D} .

Definition 2.3.3 For a forcing P define $\text{MA}(P)$ to be the statement:

For any collection of less than 2^ω dense subsets of P there exists a filter on P intersecting all of them.

For a class \mathcal{Z} of forcings $\text{MA}(\mathcal{Z})$ means that $\text{MA}(P)$ is true for every P in \mathcal{Z} .

Thus, the usual MA is $\text{MA}(\text{ccc})$, where ccc denotes the class of all forcings satisfying the countable chain condition.

It is easy to prove the following:

Fact 2.3.4 For a class \mathcal{Z} of forcings and a single forcing P , $\text{MA}(\mathcal{Z})$ and $\mathcal{Z}(P)$ together imply $\text{MA}(P)$. □

We want to prove the following result:

Theorem 2.3.5 $\text{MA}(\sigma\text{-centered})$ implies $\text{MA}(\mathbb{M}^2)$.

Since MA implies $\text{MA}(\sigma\text{-centered})$ we get that MA implies $\text{MA}(\mathbb{M}^2)$.

Because of fact 2.3.4, the following theorem implies theorem 2.3.5.

Theorem 2.3.6 MA(σ -centered) implies σ -centered(\mathbb{M}^2).

For the proof we need the following definition and lemmata:

Definition 2.3.7 Suppose $(p, q) \in \mathbb{M}^2$. For a good (p, q) -sequence \mathcal{A} define

$$\begin{aligned} \mathbb{M}_{\mathcal{A}}^2 := \{ & (u, v) \in \mathbb{M}^2 \mid (\text{st}(u), \text{st}(v)) \in I_{(p,q)} \wedge \\ & \exists \mathcal{A}' \approx \mathcal{A} \forall (\sigma, \tau) \in I_{(p,q)} \cap (\text{split}(u) \times \text{split}(v)) \\ & ((p_{(\sigma,\tau)}^{\mathcal{A}'}, q_{(\sigma,\tau)}^{\mathcal{A}'}) \leq (u, v)) \}. \end{aligned}$$

Lemma 2.3.8 Suppose $(p, q) \in \mathbb{M}^2$ and suppose \mathcal{A} and \mathcal{B} are good (p, q) -sequences. Then the following hold:

- (a) $\mathbb{M}_{\mathcal{A}}^2$ is σ -centered.
- (b) $\mathcal{A} \approx \mathcal{B}$ implies $\mathbb{M}_{\mathcal{A}}^2 = \mathbb{M}_{\mathcal{B}}^2$.
- (c) $\mathcal{A} \leq^* \mathcal{B}$ implies $\mathbb{M}_{\mathcal{A}}^2 \supseteq \mathbb{M}_{\mathcal{B}}^2$.

Proof: For (a): For $(s, t) \in I_{(p,q)}$ define

$$M_{(s,t)}^{\mathcal{A}} := \{(u, v) \in \mathbb{M}_{\mathcal{A}}^2 \mid \text{st}(u) = s \wedge \text{st}(v) = t\}.$$

Then we have $\mathbb{M}_{\mathcal{A}}^2 = \bigcup \{M_{(s,t)}^{\mathcal{A}} \mid (s, t) \in I_{(p,q)}\}$ and since $I_{(p,q)}$ is countable, it suffices to prove that each $M_{(s,t)}^{\mathcal{A}}$ is centered. Fix (s, t) and assume $(u, v), (u', v') \in M_{(s,t)}^{\mathcal{A}}$. Then there are good sequences \mathcal{B} and \mathcal{B}' witnessing this. Suppose $\text{tp}(s, t)$ is even, the odd case is similar. It is easy to find a set $C_{(s,t)}$ such that $C_{(s,t)} =^* A_{(s,t)}$, $C_{(s,t)} =^* B_{(s,t)}$ and

$$\forall \sigma \in C_{(s,t)} \forall \sigma' \supseteq \sigma \forall \tau \supseteq t (\text{tp}(\sigma', \tau) \text{ exists} \Rightarrow B_{(\sigma', \tau)} = B'_{(\sigma', \tau)}).$$

Define $C_{(\sigma,\tau)} := B_{(\sigma,\tau)}$ for $(\sigma, \tau) \neq (s, t)$. Then $\mathcal{C} = (C_{(\sigma,\tau)})_{(\sigma,\tau) \in I_{(p,q)}}$ is a good sequence with $\mathcal{B} \approx \mathcal{C} \approx \mathcal{B}'$ and $\mathcal{C} \approx \mathcal{A}$. And $(p_{(s,t)}^{\mathcal{C}}, q_{(s,t)}^{\mathcal{C}})$ is an element of $M_{(s,t)}^{\mathcal{A}}$ smaller than (u, v) and (u', v') .

(b) is clear.

For (c): Suppose we have $\mathcal{A} \leq^* \mathcal{B}$, so there exists a good (p, q) -sequence \mathcal{A}' with $\mathcal{A} \approx \mathcal{A}' \leq \mathcal{B}$, and suppose $(u, v) \in \mathbb{M}_{\mathcal{B}}^2$, witnessed by \mathcal{B}' . By lemma 2.1.5 there exists a good (p, q) -sequence \mathcal{A}'' such that $\mathcal{A}' \approx \mathcal{A}'' \leq \mathcal{B}'$. Then for every $\forall (\sigma, \tau) \in I_{(p,q)} \cap (\text{split}(u) \times \text{split}(v))$ we have $(p_{(\sigma,\tau)}^{\mathcal{A}''}, q_{(\sigma,\tau)}^{\mathcal{A}''}) \leq (p_{(\sigma,\tau)}^{\mathcal{B}'}, q_{(\sigma,\tau)}^{\mathcal{B}'}) \leq (u, v)$, so \mathcal{A}'' witnesses that $(u, v) \in \mathbb{M}_{\mathcal{A}}^2$. \square

Lemma 2.3.9 Suppose $(p, q) \in \mathbb{M}^2$ and \mathcal{A} is good for (p, q) , $(u, v) \in \mathbb{M}^2$, $D \subseteq \mathbb{M}^2$ is dense and there exists a good (p, q) -sequence $\mathcal{B} \leq^* \mathcal{A}$ with $(u, v) \in \mathbb{M}_{\mathcal{B}}^2$. Then there exist $\mathcal{C} \leq^* \mathcal{A}$ and a pair $(u', v') \leq (u, v)$ such that $(u, v) \in \mathbb{M}_{\mathcal{C}}^2$ and $(u', v') \in D \cap \mathbb{M}_{\mathcal{C}}^2$. If D is \leq^0 -dense, we can choose $(u', v') \leq^0 (u, v)$.

Proof: Suppose $\mathcal{B} \leq^* \mathcal{A}$ and $(u, v) \in \mathbb{M}_{\mathcal{B}}^2$, witnessed by \mathcal{B}' . Since D is dense we find $(u', v') \leq (p_{(\text{st}(u), \text{st}(v))}^{\mathcal{B}'}, q_{(\text{st}(u), \text{st}(v))}^{\mathcal{B}'}) \leq (u, v)$ with $(u', v') \in D$ (If D is \leq^0 -dense, we choose (u', v') such that $\text{st}(u') = \text{st}(u)$ and $\text{st}(v') = \text{st}(v)$.) We define \mathcal{C} in the following way: If $\text{tp}(\sigma, \tau)$ is even and $(\sigma, \tau) \in I_{(p,q)} \cap (\text{split}(u) \times \text{split}(v))$ let $C_{(\sigma,\tau)}$ be a subset of $\text{Succ}_{u'}(\sigma) \cap \{\rho \mid \rho(|\sigma|) > \#\sigma\}$ such that for every $\rho \in C_{(\sigma,\tau)}$ we have $|\rho| > \rho(|\sigma|)$, $\min\{\rho(|\sigma|), |\rho|\} > \#\tau$ and $\rho(|\sigma|) > \#\rho'$ for every $\rho' \in C_{(\sigma,\tau)}$ with $\rho'(|\sigma|) < \rho(|\sigma|)$. Analogously for $(\sigma, \tau) \in I_{(p,q)} \cap (\text{split}(u) \times \text{split}(v))$ and $\text{tp}(\sigma, \tau)$ is odd. For $(\sigma, \tau) \in I_{(p,q)} \setminus (\text{split}(u) \times \text{split}(v))$ let $C_{(\sigma,\tau)} := \mathcal{B}'_{(\sigma,\tau)}$. If $\text{tp}(\sigma, \tau)$ is even we have $C_{(\sigma,\tau)} \subseteq \text{split}(p_{(\sigma,\tau)}^{\mathcal{B}'})$, if $\text{tp}(\sigma, \tau)$ is odd we have $C_{(\sigma,\tau)} \subseteq \text{split}(q_{(\sigma,\tau)}^{\mathcal{B}'})$, so $\mathcal{C} \leq \mathcal{B}' \leq^* \mathcal{A}$ and therefore $\mathcal{C} \leq^* \mathcal{A}$, and we have $(u, v) \in \mathbb{M}_{\mathcal{B}'}^2 = \mathbb{M}_{\mathcal{C}}^2 \subseteq \mathbb{M}_{\mathcal{C}}^2$ by lemma 2.3.8 (c). It is easy to prove that (u', v') is in $\mathbb{M}_{\mathcal{C}}^2$. \square

Now we are able to prove theorem 2.3.6:

Proof of theorem 2.3.6: Suppose $\text{MA}(\sigma\text{-centered})$ is true. Suppose also $\mathcal{D} = \{D_\alpha \mid \alpha < 2^\omega\}$ is a family of dense subsets of \mathbb{M}^2 and $(p, q) \in \mathbb{M}^2$. Let

$$\{(p_\alpha, q_\alpha), D_\alpha \mid \alpha < 2^\omega\}$$

be an enumeration of $\mathbb{M}^2 \times \mathcal{D}$ with $(p_0, q_0) = (p, q)$. We have to find a σ -centered forcing Q such that $(p, q) \in Q \subseteq \mathbb{M}^2$ and Q captures the density of \mathcal{D} .

Define a \leq^* -descending sequence $(\mathcal{A}^\alpha)_{\alpha < 2^\omega}$ of good (p, q) -sequences as follows: For $\alpha = 0$ define \mathcal{A}^0 to be the good sequence for $(p_0, q_0) = (p, q)$ defined in remark 2.1.2. Then clearly $(p, q) \in \mathbb{M}_{\mathcal{A}^0}^2$.

If $(\mathcal{A}^\beta)_{\beta < \alpha}$ is already defined and α is a limit ordinal, we choose by $\text{MA}(\sigma\text{-centered})$ and lemma 2.1.8 a good (p, q) -sequence \mathcal{A}^α such that $\mathcal{A}^\alpha \leq^* \mathcal{A}^\beta$ for every $\beta < \alpha$.

For the successor step suppose \mathcal{A}^α is already defined. If there is no good (p, q) -sequence $\mathcal{A} \leq^* \mathcal{A}^\alpha$ with $(p_\alpha, q_\alpha) \in \mathbb{M}_{\mathcal{A}}^2$, let $\mathcal{A}^{\alpha+1} := \mathcal{A}^\alpha$.

If there exists $\mathcal{A} \leq^* \mathcal{A}^\alpha$ with $(p_\alpha, q_\alpha) \in \mathbb{M}_{\mathcal{A}}^2$, lemma 2.3.9 gives us a good (p, q) -sequence $\mathcal{B} \leq^* \mathcal{A}^\alpha$ such that $(p_\alpha, q_\alpha) \in \mathbb{M}_{\mathcal{B}}^2$ and there is a pair $(p'_\alpha, q'_\alpha) \leq (p_\alpha, q_\alpha)$ with $(p'_\alpha, q'_\alpha) \in D_\alpha \cap \mathbb{M}_{\mathcal{B}}^2$. Let $\mathcal{A}^{\alpha+1} := \mathcal{B}$.

In the end, define

$$Q := \bigcup_{\alpha < 2^\omega} \mathbb{M}_{\mathcal{A}^\alpha}^2.$$

Clearly, we have $Q \subseteq \mathbb{M}^2$.

Claim 1: Q is σ -centered.

Proof of claim 1: For $(\sigma, \tau) \in I_{(p,q)}$ define $Q_{(\sigma,\tau)} := \{(u, v) \in Q \mid \text{st}(u) = \sigma \wedge \text{st}(v) = \tau\}$, then we have $Q = \bigcup \{Q_{(\sigma,\tau)} \mid (\sigma, \tau) \in I_{(p,q)}\}$. $I_{(p,q)}$ is countable and like in the proof of lemma 2.3.8 (a) using that $(A^\alpha)_{\alpha < 2^\omega}$ is a \leq^* -descending chain, we see that every $Q_{(\sigma,\tau)}$ is centered. \square (claim 1)

Claim 2: Q captures the density of \mathcal{D} .

Proof of claim 2: Suppose $D \in \mathcal{D}$ and $(\bar{p}, \bar{q}) \in Q$, so there exists $\alpha < 2^\omega$ such that $((\bar{p}, \bar{q}), D) = ((p_\alpha, q_\alpha), D_\alpha)$. We have to find $(p', q') \leq (\bar{p}, \bar{q})$ such that $(p', q') \in D \cap Q$.

Since $(\bar{p}, \bar{q}) \in Q$ we have $(\bar{p}, \bar{q}) \in \mathbb{M}_{\mathcal{A}^\beta}^2$ for some β , by lemma 2.3.8 we have $(\bar{p}, \bar{q}) \in \mathbb{M}_{\mathcal{A}^\gamma}^2$ for every $\gamma \geq \beta$. Hence, at step α in the construction of the sequence $(A^\alpha)_{\alpha < 2^\omega}$ arises the second case. There we have got $(p'_\alpha, q'_\alpha) \leq (\bar{p}, \bar{q})$ such that $(p'_\alpha, q'_\alpha) \in D \cap \mathbb{M}_{\mathcal{A}^{\alpha+1}}^2 \subseteq D \cap Q$. Let $(p', q') := (p'_\alpha, q'_\alpha)$ and we are done. \square (claim 2) \square

Corollary 2.3.10 MA(σ -centered) implies that \mathbb{M}^2 does not collapse cardinals.

Proof: Since \mathbb{M}^2 clearly satisfies the $(2^\omega)^+$ -chain-condition, it could only collapse cardinals $\leq 2^\omega$. Suppose there exist cardinals κ and λ such that $\kappa < \lambda \leq 2^\omega$ and there is a \mathbb{M}^2 -name τ with

$$\Vdash_{\mathbb{M}^2} \tau : \kappa \rightarrow \lambda \text{ is surjective.}$$

Define for each $\alpha < \lambda$

$$D_\alpha := \{(p, q) \in \mathbb{M}^2 \mid \exists \xi < \kappa ((p, q) \Vdash_{\mathbb{M}^2} \tau(\xi) = \alpha)\}.$$

Then D_α is open and dense for every $\alpha < \lambda$. As MA(σ -centered) is true, we have σ -centered(\mathbb{M}^2) and since $\lambda \leq 2^\omega$ there exists a forcing $P \subseteq \mathbb{M}^2$ such that each $D_\alpha \cap P$ is dense in P and P is σ -centered. But then P collapses κ to λ , this is a contradiction to the fact that P fulfills the countable chain condition. \square

Chapter 3

Infinite maximal antichains in $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$

3.1 Definitions and combinatorics

In the sequel, we always identify the elements of $\mathfrak{P}(\omega)/\text{fin}$ with their representatives in $[\omega]^\omega$. By $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$, for $n^* > 0$ in ω , we mean the full product carrying the coordinatewise ordering. Hence elements of $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$ are n^* -tupels of subsets of ω , elements of $(\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ are n^* -tupels of infinite subsets of ω , or we can view the elements as functions from n^* to $\mathfrak{P}(\omega)/\text{fin}$, $\mathfrak{P}(\omega)/\text{fin} \setminus \{0\}$, respectively. For $\bar{a} \in (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ and $F \subseteq n^*$ we write $\bar{a} \upharpoonright F$ for the restricted function $\bar{a} : F \rightarrow \mathfrak{P}(\omega)/\text{fin} \setminus \{0\}$. If F have just one element i , we write a_i instead of $\bar{a} \upharpoonright \{i\}$, so $\bar{a} = (a_i)_{i < n^*}$.

Two elements $\bar{a}, \bar{b} \in (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ are called *compatible* if there exists an element $\bar{c} \in (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ with $\bar{c} \leq \bar{a}$ and $\bar{c} \leq \bar{b}$, i.e. for $\bar{a} = (a_i)_{i < n^*}$ and $\bar{b} = (b_i)_{i < n^*}$ that $a_i \cap b_i$ is infinite for every $i < n^*$. As usual, if \bar{a} and \bar{b} are not compatible, they are called *incompatible*. A subset $A \subseteq (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ of pairwise incompatible elements is called an *antichain* of $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$.

Here, we are interested in infinite maximal antichains of $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$. Since $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$ can be densely embedded in the complete Boolean algebra of the regular open subsets of $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$ (see [Ko], for example), people often use the terminology of Boolean algebras and call maximal antichains in $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$ partitions.

Of course there are finite analytical maximal antichains of $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$ for every $n^* \in \omega \setminus \{0\}$: choose $a \in [\omega]^\omega$ with $\omega \setminus a \in [\omega]^\omega$, too, then $\{\bar{a} = (a_i)_{i < n^*} \mid a_i \in \{a, \omega \setminus a\} \text{ for every } i < n^*\}$ is a maximal antichain in $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$.

Our goal is to proof, analogously to Mathias' one-dimensional result, the following theorem:

Theorem 3.1.1 Fix $n^* \in \omega \setminus \{0\}$.

There are no analytical infinite maximal antichains in $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$.

Recall that a filter on ω is called *Ramsey* if for every descending sequence $(X_n)_{n \in \omega}$ with $X_n \in U$ for every $n \in \omega$ there exists $\{x_n \mid n \in \omega\} \in U$ such that $x_{n+1} \in X_{x_n}$ for every $n \in \omega$. In this case, we say that $\{x_n \mid n \in \omega\}$ *diagonalizes* the sequence $(X_n)_{n \in \omega}$. If $f : \omega \rightarrow \omega$ is an enumerating function for the set $\{x_n \mid n \in \omega\}$, hence $f(n+1) \in X_{f(n)}$ for every $n \in \omega$, we say that f diagonalizes $(X_n)_{n \in \omega}$.

The *Rudin-Keisler ordering* \leq_{RK} is defined by the following: For two ultrafilters U and U' we have $U \leq_{RK} U'$ if there exists a function $h : \omega \rightarrow \omega$ such that $U = \{x \subseteq \omega \mid h^{-1}[x] \in U'\}$. Two ultrafilters U, U' are called *Rudin-Keisler-equivalent* if and only if $U \leq_{RK} U'$ and $U' \leq_{RK} U$. Equivalently (see for example [Je]), U and U' are Rudin-Keisler-equivalent if there exists a bijection $h : \omega \rightarrow \omega$ such that $U = \{h[x] \mid x \in U'\}$.

It is well known that Martin's Axiom implies the existence of a Ramsey ultrafilter and the continuum hypothesis (CH) implies the existence as well (this is for example a consequence of lemma 3.3.1). And also a $\mathfrak{P}(\omega)/\text{fin}$ -generic filter over V is a Ramsey ultrafilter. Kunen got, starting with a model of CH and adding \aleph_2 random reals, a model in which there are no Ramsey ultrafilters (see for example [Je]).

For $n^* \in \omega \setminus \{0\}$ and an infinite maximal antichain $\mathcal{A} \subseteq (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ define

$$\mathcal{J}(\mathcal{A}) := \{\bar{x} \in (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*} \mid \exists k \in \omega \exists \bar{a}_0, \dots, \bar{a}_k \in \mathcal{A} \\ (\bar{x} \setminus (\bar{a}_0 \cup \dots \cup \bar{a}_k) \notin (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*})\}$$

and

$$\mathcal{J}(\mathcal{A})^+ := (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*} \setminus \mathcal{J}(\mathcal{A}) \\ = \{\bar{x} \in (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*} \mid \exists^\infty \bar{a} \in \mathcal{A} (\bar{a} \text{ and } \bar{x} \text{ are compatible})\}.$$

Here, \exists^∞ means “there are infinitely many”. For $\bar{x} \in (\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ define

$$\mathcal{A} \upharpoonright \bar{x} := \{\bar{a} \in \mathcal{A} \mid \bar{a} \text{ and } \bar{x} \text{ are compatible}\},$$

the set of all elements of \mathcal{A} compatible with \bar{x} .

We will use the following lemma of [Sp4]:

Lemma 3.1.2 [lemma 1.1 of [Sp4]]

Suppose $n^* \in \omega \setminus \{0\}$ and $\mathcal{A} \subseteq (\mathfrak{P}(\omega)/\text{fin})^{n^*}$ is an infinite maximal antichain. If $F \subseteq n^*$ is (with respect to \subseteq) maximal such that there exists $\mathcal{F} \in [\mathcal{A}]^\omega$ with the property that $\{\bar{a} \upharpoonright F \mid \bar{a} \in \mathcal{F}\}$ has the finite intersection property in $(\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$, then $|\mathcal{F}| = n^* - 1$.

Here, the finite intersection property in $(\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$ means, that for every finite subset there is a lower bound in $(\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$. In other words, every finite coordinatewise intersection of elements of \mathcal{F} is in every coordinate infinite. Of course, this lemma also holds for the restriction $\mathcal{A} \upharpoonright \bar{x}$ for every $\bar{x} = (x_0, \dots, x_{n^*-1}) \in (\mathfrak{P}(\omega)/\text{fin})^{n^*}$.

Corollary 3.1.3 *Suppose n^* and \mathcal{A} are as in lemma 3.1.2 and \bar{x} is an element of $(\mathfrak{P}(\omega)/\text{fin} \setminus \{0\})^{n^*}$. Then there exist $\mathcal{F} \in [\mathcal{A} \upharpoonright \bar{x}]^\omega$ and $i^* \in n^*$ such that for every $i \in n^* \setminus \{i^*\}$ there exists $b_i \in [\omega]^\omega$ with $b_i \subseteq^* a_i$ for every $\bar{a} \in \mathcal{F}$. Clearly, the set $\{a_{i^*} \mid \bar{a} \in \mathcal{F}\}$ must be an antichain.*

Lemma 3.1.4 *Suppose $n^* \in \omega \setminus \{0\}$ and $\mathcal{A} \subseteq (\mathfrak{P}(\omega)/\text{fin})^{n^*}$ is an infinite maximal antichain. Suppose also $\bar{b} = (b_0, \dots, b_{n^*-1}) \in \mathcal{J}(\mathcal{A})^+$ and $h : \omega \rightarrow \omega$ is a bijection.*

Then for every pair $k, l \in n^$ with $k < l$ there exists $b'_k \subseteq b_k$ and $b'_l \subseteq b_l$ such that we still have $(b_0, \dots, b'_k, \dots, b'_l, \dots, b_{n^*-1}) \in \mathcal{J}(\mathcal{A})^+$ and additionally $h[b'_k] \cap b'_l = \emptyset$ or $h[b'_l] \cap b'_k = \emptyset$ holds.*

Proof: So suppose $\bar{b} \in \mathcal{J}(\mathcal{A})^+$, $h : \omega \rightarrow \omega$ is bijective and $k < l < n^*$ are given. By corollary 3.1.3 there exist $i^* < n^*$ and $\mathcal{F} \in [\mathcal{A} \upharpoonright \bar{b}]^\omega$ such that for every $i \in n^* \setminus \{i^*\}$ there exists $b_i^* \in [\omega]^\omega$ with $b_i^* \subseteq^* c_i$ for every $\bar{c} \in \mathcal{F}$. Let $(\bar{c}^n)_{n \in \omega}$ be an enumeration of \mathcal{F} . Notice that the set $\{c_{i^*}^n \mid n \in \omega\}$ consists of pairwise almost disjoint subsets of ω , since \mathcal{F} is countable without loss of generality we have $c_{i^*}^n \cap c_{i^*}^m = \emptyset$ for $n \neq m$.

First case: $k, l \in n^* \setminus \{i^*\}$.

Consider the set $(b_l^* \cap b_l) \setminus h[b_k^* \cap b_k]$. If this set is infinite, let $b'_l := (b_l^* \cap b_l) \setminus h[b_k^* \cap b_k]$ and $b'_k := b_k^* \cap b_k$. Then clearly we have $h[b'_k] \cap b'_l = \emptyset$ and because of $b'_k \subseteq b_k^*$, $b'_l \subseteq b_l^*$ and the choice of b_k^* and b_l^* we have $(b_0, \dots, b'_k, \dots, b'_l, \dots, b_{n^*-1})$ is compatible with every $\bar{c} \in \mathcal{F}$, hence it is an element of $\mathcal{J}(\mathcal{A})^+$.

If $(b_l^* \cap b_l) \setminus h[b_k^* \cap b_k]$ is finite, divide the infinite set $b_l^* \cap b_l \cap h[b_k^* \cap b_k]$ in two infinite pieces $b' \cup b''$ and let $b'_l := b'$ and $b'_k := h^{-1}[b'']$. Again, we have $(b_0, \dots, b'_k, \dots, b'_l, \dots, b_{n^*-1}) \in \mathcal{J}(\mathcal{A})^+$.

Second case: $l = i^*$.

If there is an infinite subsequence $(c_{i^*}^{n_m})_{m \in \omega}$ of $(c_{i^*}^n)_{n \in \omega}$ such that $c_{i^*}^{n_m} \cap h[b_k \cap b_k^*] \cap b_l$ is infinite for every $m \in \omega$ then let

$$\begin{aligned} b'_k &:= h^{-1}[\bigcup_{m \in \omega} c_{i^*}^{n_{2m}}] \cap (b_k \cap b_k^*) & \text{and} \\ b'_l &:= \bigcup_{m \in \omega} c_{i^*}^{n_{2m+1}} \cap b_l. \end{aligned}$$

If there are only finitely many $c_{i^*}^{n_0}, \dots, c_{i^*}^{n_u}$ such that $c_{i^*}^{n_m} \cap h[b_k \cap b_k^*] \cap b_l$ is infinite for every $m \leq u$ then let

$$\begin{aligned} b'_k &:= b_k \cap b_k^* & \text{and} \\ b'_l &:= (\bigcup \{c_{i^*}^n \mid n \in \omega \setminus \{n_0, \dots, n_u\}\} \cap b_l) \setminus h[b_k \cap b_k^*]. \end{aligned}$$

Easily, in both subcases b'_k and b'_l are as claimed.

Third case: $k = i^*$.

This case is analogous to the second one, we just have to change the roles of k and l . \square

3.2 A finite-dimensional version of Mathias forcing

Now we introduce an n^* -dimensional version of Mathias forcing:

Definition 3.2.1 [ShSp2] Fix $n^* \in \omega \setminus \{0\}$. Let

$$Q := \{(s, S) \in [\omega]^{<\omega} \times [\omega]^\omega \mid \max(s) < \min(S) \vee s = \emptyset\}.$$

For defining the ordering on Q we need the following: let $(l_j)_{j < |s|}$ be the increasing enumeration of $s \in [\omega]^{<\omega}$ and define $s_i := \{l_j \mid j \equiv i \pmod{n^*}\}$ for $i < n^*$ and let $(k_j)_{j < \omega}$ be the increasing enumeration of $S \in [\omega]^\omega$ and define $S_i := \{k_j \mid j \equiv i \pmod{n^*}\}$ for $i < n^*$. Then we define

$$\begin{aligned} (s, S) \leq (t, T) \quad &\Leftrightarrow \quad s \cap (\max t + 1) = t \text{ or } t = \emptyset, \\ & s_i \setminus t_i \subseteq T_i \text{ for every } i < n^* \text{ and} \\ & S_i \subseteq T_i \text{ for every } i < n^*. \end{aligned}$$

We write $(s, S) \leq^0 (t, T)$ if $(s, S) \leq (t, T)$ and additionally $s = t$.

Notice that $(s, S), (t, T) \in Q$ are compatible if and only if

$$(\forall i < n^* (s_i \subseteq t_i \wedge t_i \setminus s_i \subseteq S_i)) \vee (\forall i < n^* (t_i \subseteq s_i \wedge s_i \setminus t_i \subseteq T_i))$$

and for every $i < n^*$ we have $|S_i \cap T_i| = \omega$.

As for the classical Mathias forcing we have the following relationship between Q -reals and generic filters: If G is Q -generic over some model M define $x := \bigcup \{s \mid \exists S ((s, S) \in G)\}$ and $x_i := \{k_j \mid j \equiv i \pmod{n^*}\}$, where $(k_i)_{i < \omega}$ is the increasing enumeration of x (notice that we have $x_i = \bigcup \{s_i \mid \exists S' ((s', S') \in G \wedge s'_i = s_i)\}$). Then $G = \{(s, S) \in Q^M \mid \forall i < n^* (s_i \subseteq x_i \subseteq S_i \cup s_i)\}$. The proof of this fact equals the classical proof. Hence a set x is a Q -real if the set $G_x := \{(s, S) \in Q^M \mid \forall i < n^* (s_i \subseteq x_i \subseteq S_i \cup s_i)\}$ is a Q -generic filter over M , where s_i is defined as above.

Suppose U_0, \dots, U_{n^*-1} are ultrafilters on ω . Then we define the subordering $Q(U_0, \dots, U_{n^*-1})$ of Q like follows: in $Q(U_0, \dots, U_{n^*-1})$ are only those pairs $(s, S) \in Q$ with $S_i \in U_i$ for every $i < n^*$.

In the following we list some of the results of the investigation of Q by Shelah and Spinas [ShSp2] which we will need later. All this results generalize well known facts about the one-dimensional Mathias forcing (see [Ma], for example). Fix $n^* \in \omega \setminus \{0\}$.

Fact 3.2.2 [ShSp2] *The forcing Q is equivalent to the forcing $(\mathfrak{P}(\omega)/\text{fin})^{n^*} * Q(\dot{G}_0, \dots, \dot{G}_{n^*-1})$, where $(\dot{G}_0, \dots, \dot{G}_{n^*-1})$ is the canonical name for the generic object added by the forcing $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$ consisting of pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters. \square*

Shelah and Spinas introduced a helpful game:

Definition 3.2.3 Let U_0, \dots, U_{n^*-1} be ultrafilters on ω . Then the game $\Gamma(U_0, \dots, U_{n^*-1})$ for two players I and II is defined as follows: player I chooses in the m th-move $(A_0^m, \dots, A_{n^*-1}^m) \in U_0 \times \dots \times U_{n^*-1}$ and player II responds playing $k_m \in A_{m \bmod n^*}^m$. Player II wins the game if and only if for every $i < n^*$ the set $\{k_n \mid n \equiv i \pmod{n^*}\}$ is an element of U_i .

Fact 3.2.4 [ShSp2] *Suppose U_0, \dots, U_{n^*-1} are pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters. Then player I does not have a winning strategy in the game $\Gamma(U_0, \dots, U_{n^*-1})$. \square*

Here, it is easy to see that it is necessary that the ultrafilters are pairwise not Rudin-Keisler-equivalent.

Fact 3.2.5 [ShSp2] *Let U_0, \dots, U_{n^*-1} are pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters on ω . Suppose $S \in [\omega]^\omega$ such that for every $i < n^*$ and for every $X \in U_i$ we have $S_i \subseteq^* X$. Then S is $Q(U_0, \dots, U_{n^*-1})$ -generic over V . \square*

As a corollary we get:

Fact 3.2.6 [ShSp2] *Let U_0, \dots, U_{n^*-1} are pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters. If $S \in [\omega]^\omega$ is $Q(U_0, \dots, U_{n^*-1})$ -generic over V and $T \in [\omega]^\omega$ with $T_i \subseteq S_i$ for every $i < n^*$, then T is $Q(U_0, \dots, U_{n^*-1})$ -generic over V as well. \square*

Fact 3.2.7 [ShSp2] *If U_0, \dots, U_{n^*-1} are pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters then the forcing $Q(U_0, \dots, U_{n^*-1})$ has the pure decision property.*

This means that for a given $Q(U_0, \dots, U_{n^-1})$ -name Θ such that $\Vdash_{Q(U_0, \dots, U_{n^*-1})} \Theta \in \{0, 1\}$ and given $(s, S) \in Q(U_0, \dots, U_{n^*-1})$, there exist $(t, T) \in Q(U_0, \dots, U_{n^*-1})$ and $\varepsilon \in \{0, 1\}$ such that $(t, T) \leq^0 (s, S)$ and $(t, T) \Vdash_{Q(U_0, \dots, U_{n^*-1})} \Theta = \varepsilon$. \square*

3.3 Finding Ramsey ultrafilters

Suppose \mathcal{A} is an infinite maximal antichain in $\mathfrak{P}(\omega)/\text{fin}$. Mathias [Ma] proved that CH implies that there exists a Ramsey ultrafilter U such that $U \subseteq \mathcal{J}(\mathcal{A})^+$. For the proof of Theorem 3.1.1 we need the following n^* -dimensional generalization:

Lemma 3.3.1 (CH) *Suppose $n^* \in \omega \setminus \{0\}$ and \mathcal{A} is an infinite maximal antichain in $(\mathfrak{P}(\omega)/\text{fin})^{n^*}$. Then there exist pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters U_0, \dots, U_{n^*-1} such that $U_0 \times \dots \times U_{n^*-1} \subseteq \mathcal{J}(\mathcal{A})^+$.*

Proof: Assume CH and fix the following enumerations: Let $(x_\alpha)_{\alpha < \omega_1}$ enumerate $[\omega]^\omega$; $(h_\alpha)_{\alpha < \omega_1}$ enumerate all bijections from ω to ω and let $({}^\alpha \bar{X})_{\alpha \in \text{Lim}(\omega_1)}$ enumerate all n^* -tuples of decreasing sequences (of length ω) in $[\omega]^\omega$, such that every ${}^\alpha \bar{X}$ occurs cofinal often.

We want to construct sequences $(a_0^\alpha)_{\alpha < \omega_1}, \dots, (a_{n^*-1}^\alpha)_{\alpha < \omega_1}$ in $[\omega]^\omega$ by induction such that we get U_i as the filter generated by $\{a_i^\alpha \mid \alpha < \omega_1\}$ for $i < n^*$.

For beginning the induction choose $(a_0^0, \dots, a_{n^*-1}^0) \in \mathcal{J}(\mathcal{A})^+$ arbitrarily.

Suppose $(a_0^\nu)_{\nu < \alpha}, \dots, (a_{n^*-1}^\nu)_{\nu < \alpha}$ are chosen such that $\{a_i^\beta \mid \beta < \nu\}$ generates a filter called F_i^ν for every $i < n^*$ and every $\nu < \alpha$ and we have $F_0^\nu \times \dots \times F_{n^*-1}^\nu \subseteq \mathcal{J}(\mathcal{A})^+$ for every $\nu < \alpha$.

First, suppose that α is a successor ordinal, say $\alpha = \beta + 1$. Consider the infinite subset x_β of ω . Let

$${}^0 b_i := a_i^\beta \cap x_\beta \quad \text{and} \quad {}^1 b_i := a_i^\beta \setminus x_\beta$$

for $i < n^*$. Because of $(a_0^\beta, \dots, a_{n^*-1}^\beta) \in \mathcal{J}(\mathcal{A})^+$ there exists at least one sequence $(\varepsilon_0, \dots, \varepsilon_{n^*-1}) \in \{0, 1\}^{n^*}$ such that $({}^{\varepsilon_0} b_0, \dots, {}^{\varepsilon_{n^*-1}} b_{n^*-1}) \in \mathcal{J}(\mathcal{A})^+$. Now consider the function h_β . By applying lemma 3.1.4 finitely many times we get the following: For every $i < n^*$ there is an infinite $a_i \subseteq {}^{\varepsilon_i} b_i$ such that $(a_0, \dots, a_{n^*-1}) \in \mathcal{J}(\mathcal{A})^+$ and for every pair $k, l < n^*$ with $k \neq l$ we have $h_\beta[a_k] \cap a_l = \emptyset$ or $h_\beta[a_l] \cap a_k = \emptyset$. Let $a_i^\alpha := a_i$ for every $i < n^*$.

Second, suppose that α is a limit ordinal. Consider the tuple ${}^\alpha \bar{X} = (({}^\alpha X_0^n)_{n \in \omega}, \dots, ({}^\alpha X_{n^*-1}^n)_{n \in \omega})$ of decreasing sequences in $[\omega]^\omega$.

First case: $\Pi_{i < n^*} \{{}^\alpha X_i^n \mid n \in \omega\} \subseteq \Pi_{i < n^*} \bigcup_{\nu < \alpha} F_i^\nu$. We want to construct for every $i < n^*$ strictly increasing functions $f_i : \omega \rightarrow \omega$ such that $f_i(n+1) \in {}^\alpha X_i^{f_i(n)}$ for every $i < n^*$ and every $n \in \omega$, $(f_0[\omega], \dots, f_{n^*-1}[\omega]) \in \mathcal{J}(\mathcal{A})^+$ and $\Pi_{i < n^*} (\bigcup_{\alpha < \nu} F_i^\nu \cup \{f_i[\omega]\})$ has the finite intersection property in $\mathcal{J}(\mathcal{A})^+$.

Fix for every $i < n^*$ an enumeration $(a_i^n)_{n \in \omega}$ of $\{a_i^\nu \mid \nu < \alpha\}$.

Choose $g_i^0(0) \in {}^\alpha X_i^0 \cap a_i^{f_0}$ for every $i < n^*$. If $g_0^0(k), \dots, g_{n^*-1}^0(k)$ are already chosen for some $k \in \omega$, take for every $i < n^*$

$$g_i^0(k+1) \in {}^\alpha X_i^0 \cap \dots \cap {}^\alpha X_i^{g_i^0(k)} \cap a_i^{f_0} \cap \dots \cap a_i^{f_{k+1}}$$

with $g_i^0(k+1) > g_i^0(k)$ (notice that in this first case the set ${}^\alpha X_i^0 \cap \dots \cap {}^\alpha X_i^{k+1} \cap a_i^{l_0} \cap \dots \cap a_i^{l_{k+1}}$ is infinite for every $i < n^*$). Hence we get strictly increasing functions $g_i^0 : \omega \rightarrow \omega$ for $i < n^*$ with $(g_0^0[\omega], \dots, g_{n^*-1}^0[\omega]) \in ([\omega]^\omega)^{n^*}$. As \mathcal{A} is a maximal antichain there is a $\bar{c}^0 = (c_0^0, \dots, c_{n^*-1}^0) \in \mathcal{A}$ such that $(g_0^0[\omega], \dots, g_{n^*-1}^0[\omega])$ and \bar{c}^0 are compatible in $(\mathfrak{B}(\omega)/\text{fin})^{n^*}$. Define

$$\varepsilon Y_i^n := \begin{cases} {}^\alpha X_i^n \setminus c_i^0, & \text{if } \varepsilon = 0; \\ {}^\alpha X_i^n, & \text{if } \varepsilon = 1. \end{cases}$$

Since we have $({}^\alpha X_0^n, \dots, {}^\alpha X_{n^*-1}^n) \in \mathcal{J}(\mathcal{A})^+$ for every $n \in \omega$ and of course $(c_0^0, \dots, c_{n^*-1}^0)$ is not in $\mathcal{J}(\mathcal{A})^+$, for every $n \in \omega$ there exists a sequence $(\varepsilon_0^n, \dots, \varepsilon_{n^*-1}^n) \in \{0, 1\}^{n^*}$ such that $\varepsilon_i^n = 0$ for at least one $i < n^*$ and we have $(\varepsilon_0^n Y_0^n, \dots, \varepsilon_{n^*-1}^n Y_{n^*-1}^n) \in \mathcal{J}(\mathcal{A})^+$. Because the sequences $({}^\alpha X_i^n)_{n \in \omega}$ are decreasing we have the following: If $(\varepsilon_0^n Y_0^n, \dots, \varepsilon_{n^*-1}^n Y_{n^*-1}^n) \notin \mathcal{J}(\mathcal{A})^+$ for some sequence $(\varepsilon_0, \dots, \varepsilon_{n^*-1}) \in \{0, 1\}^{n^*}$ then for every $m > n$ we have $(\varepsilon_0^m Y_0^m, \dots, \varepsilon_{n^*-1}^m Y_{n^*-1}^m) \notin \mathcal{J}(\mathcal{A})^+$. So we can choose an $\bar{n} \in \omega$ large enough such that for every $m > \bar{n}$ we have $\varepsilon_i^m = 1$ whenever $\varepsilon_i^{\bar{n}} = 1$. Let $\varepsilon_i := \varepsilon_i^{\bar{n}}$ for every $i < n^*$, then $\varepsilon_i = 0$ for at least one $i < n^*$ and $(\varepsilon_0 Y_0^{\bar{n}}, \dots, \varepsilon_{n^*-1} Y_{n^*-1}^{\bar{n}}) \in \mathcal{J}(\mathcal{A})^+$ for every $n \in \omega$.

Let ${}^1 Z_i^n := \varepsilon_i Y_i^n$ for every $i < n^*$ and every $n \in \omega$. So we have an n^* -tuple ${}^1 \bar{Z} = (({}^1 Z_0^n)_{n \in \omega}, \dots, ({}^1 Z_{n^*-1}^n)_{n \in \omega})$ of descending sequences in $[\omega]^\omega$ such that $({}^1 Z_0^n, \dots, {}^1 Z_{n^*-1}^n) \in \mathcal{J}(\mathcal{A})^+$ for every $n \in \omega$. Notice that we have $\Pi_{i < n^*} \{ {}^1 Z_i^n \mid n \in \omega \} \subseteq \Pi_{i < n^*} \bigcup_{\nu < \alpha} F_i^\nu$.

Now we construct functions $g_0^1, \dots, g_{n^*-1}^1 : \omega \rightarrow \omega$ in the same way as before but for ${}^1 \bar{Z}$ instead of ${}^\alpha \bar{X}$. Choose $g_i^1(0) \in {}^1 X_i^0 \cap a_i^{l_0}$ for every $i < n^*$ and if $g_0^1(k), \dots, g_{n^*-1}^1(k)$ are already chosen take for every $i < n^*$

$$g_i^1(k+1) \in {}^1 Z_i^0 \cap \dots \cap {}^1 Z_i^{g_i^1(k)} \cap a_i^{l_0} \cap \dots \cap a_i^{l_{k+1}}$$

with $g_i^1(k+1) > g_i^1(k)$. Then there exists an element $\bar{c}^1 \in \mathcal{A}$ such that $(g_0^1[\omega], \dots, g_{n^*-1}^1[\omega])$ and \bar{c}^1 are compatible in $(\mathfrak{B}(\omega)/\text{fin})^{n^*}$. As there is at least one index $i < n^*$ with ${}^1 Z_i^n = {}^\alpha X_i^n \setminus c_i^0$ for every $n \in \omega$ the elements \bar{c}^0 and \bar{c}^1 are different. As before we let ${}^0 Y_i^n := {}^1 Z_i^n \setminus c_i^1$ and ${}^1 Y_i^n := {}^1 Z_i^n$ and we find a sequence $(\varepsilon_0, \dots, \varepsilon_{n^*-1}) \in \{0, 1\}^{n^*}$ with $\varepsilon_i = 0$ for at least one $i < n^*$ such that $(\varepsilon_0 Y_0^n, \dots, \varepsilon_{n^*-1} Y_{n^*-1}^n) \in \mathcal{J}(\mathcal{A})^+$ for every $n \in \omega$. For every $i < n^*$ and every $n \in \omega$ let ${}^2 Z_i^n := \varepsilon_i Y_i^n$.

Hence we can find a sequence $({}^l \bar{Z})_{l \in \omega} = (({}^l Z_0^n)_{n \in \omega}, \dots, ({}^l Z_{n^*-1}^n)_{n \in \omega})$ of n^* -tuples of descending chains in $[\omega]^\omega$ with $({}^l Z_0^n, \dots, {}^l Z_{n^*-1}^n) \in \mathcal{J}(\mathcal{A})^+$ for every $n \in \omega$, a sequence $((g_0^l, \dots, g_{n^*-1}^l))_{l \in \omega}$ of n^* -tuples of increasing functions from ω to ω with $g_i^l(k+1) \in {}^l Z_i^{g_i^l(k)}$ and $(g_0^l[\omega], \dots, g_{n^*-1}^l[\omega]) \in ([\omega]^\omega)^{n^*}$ and a third sequence $(\bar{c}^l)_{l \in \omega}$ of pairwise different elements of \mathcal{A} such that $|c_i^l \cap g_i^l[\omega]| = \omega$ for every $i < n^*$ and every $l \in \omega$.

Now we are ready to define diagonalizing functions $f_0, \dots, f_{n^*-1} : \omega \rightarrow \omega$ for $({}^\alpha X_0^n)_{n \in \omega}, \dots, ({}^\alpha X_{n^*-1}^n)_{n \in \omega}$ such that $(f_0[\omega], \dots, f_{n^*-1}[\omega]) \in \mathcal{J}(\mathcal{A})^+$: Choose

$f_i(0) \in {}^\alpha X_i^0$ for $i < n^*$. For $k > 0$ there exists $j, l \in \omega$ such that $k = 2^j(2l + 1)$. Choose for every $i < n^*$

$$f_i(k) \in c_i^j \cap g_i^j[\omega] \cap {}^\alpha X_i^0 \cap \dots \cap {}^\alpha X_i^{f_i(k-1)} \cap a_i'^0 \cap \dots \cap a_i'^{k-1}$$

with $f_i(k) > f_i(k-1)$.

Now we define $a_i^\alpha := f_i[\omega]$ for every $i < n^*$. Hence for every $i < n^*$ we have that a_i^α diagonalizes $({}^\alpha X_i^n)_{n \in \omega}$; and as $(a_0^\alpha, \dots, a_{n^*-1}^\alpha)$ cuts infinitely many \bar{c}^j we have $(a_0^\alpha, \dots, a_{n^*-1}^\alpha) \in \mathcal{J}(\mathcal{A})^+$ and $\Pi_{i < n^*} \{a_i^\nu \mid \nu \leq \alpha\}$ has the finite intersection property in $\mathcal{J}(\mathcal{A})^+$.

Second case: $\Pi_{i < n^*} \{{}^\alpha X_i^n \mid n \in \omega\}$ is not a subset of $\Pi_{i < n^*} \bigcup_{\nu < \alpha} F_i^\nu$. Then choose $a_i^\alpha \in [\omega]^\omega$ arbitrarily such that $a_i^\alpha \subseteq^* a_i^\nu$ for every $\nu < \alpha$ and every $i < n^*$.

This finishes our construction. Our construction guarantees that for every $i < n^*$ the set $\{a_i^\alpha \mid \alpha < \omega_1\}$ has the finite intersection property, thus we can define U_i to be the filter generated by $\{a_i^\alpha \mid \alpha < \omega_1\}$. We have to check that U_i is an ultrafilter, that this ultrafilter is Ramsey and that U_i and U_j for $i \neq j$ are not Rudin-Keisler-equivalent.

First we check that each U_i is an ultrafilter: Suppose $a \in [\omega]^\omega$, so there exists $\alpha < \omega_1$ with $a = x_\alpha$. If $x_\alpha \notin U_i$ we have $a_i^{\alpha+1} \not\subseteq x_\alpha$, hence in the successor step we have chosen $a_i^{\alpha+1} \subseteq a_i^\alpha \setminus x_\alpha \subseteq \omega \setminus x_\alpha$, hence $\omega \setminus x_\alpha \in U_i$.

Check that each U_i is Ramsey: Suppose $(X_n)_{n \in \omega}$ is a descending chain in U_i . Then there exists $\alpha \in \text{Lim}(\omega_1)$ such that $(X_n)_{n \in \omega} = ({}^\alpha X_i^n)_{n \in \omega}$ and $\Pi_{i < n^*} \{{}^\alpha X_i^n \mid n \in \omega\} \subseteq \Pi_{i < n^*} \bigcup_{\nu < \alpha} F_i^\nu$, hence we are in the first case of the limit step of the construction and $a_i^\alpha \in U_i$ diagonalizes $({}^\alpha X_i^n)_{n \in \omega}$.

It remains to check that for $i, j < n^*$ with $i \neq j$ that the ultrafilters U_i and U_j are not Rudin-Keisler-equivalent: Suppose there exists a bijection $h : \omega \rightarrow \omega$ such that $U_i = \{h[x] \mid x \in U_j\}$. But there is an $\alpha < \omega_1$ with $h = h_\alpha$ and in the successor step of our construction we have chosen $a_i^{\alpha+1}$ and $a_j^{\alpha+1}$ such that $h_\alpha[a_i^{\alpha+1}] \cap a_j^{\alpha+1} = \emptyset$ or $h_\alpha[a_j^{\alpha+1}] \cap a_i^{\alpha+1} = \emptyset$, hence we get a contradiction. \square

3.4 The proof of the maintheorem

Proof of theorem 3.1.1: Suppose $n^* \in \omega \setminus \{0\}$ and $\mathcal{A} \subseteq (\mathfrak{P}(\omega)/\text{fin})^{n^*}$ is an infinite maximal antichain. Since we can collapse 2^ω to ω_1 by a σ -closed forcing, hence without changing the set $\mathfrak{P}(\omega)$, we can assume without loss of generality that CH is true. By lemma 3.3.1 there are pairwise not Rudin-Keisler-equivalent Ramsey ultrafilters U_0, \dots, U_{n^*-1} such that $U_0 \times \dots \times U_{n^*-1} \subseteq \mathcal{J}(\mathcal{A})^+$.

Suppose \mathcal{A} is analytical. Notice that in this case $\mathcal{J}(\mathcal{A})$ is analytical, too. Choose

a countable elementary submodel \mathcal{N} of H_Θ for a sufficiently large Θ such that $\mathcal{J}(\mathcal{A}), U_0, \dots, U_{n^*-1}, \text{code}(\mathcal{J}(\mathcal{A})) \in \mathcal{N}$, where $\text{code}(\mathcal{J}(\mathcal{A}))$ is the Σ_1^1 -code of the analytical set $\mathcal{J}(\mathcal{A})$. Consider the image of the Mostowski collapse $\varphi(\mathcal{N}) =: \tilde{\mathcal{N}}$ of \mathcal{N} , hence $\tilde{\mathcal{N}}$ is transitive, $\mathcal{J}(\mathcal{A})$ is analytical in $\tilde{\mathcal{N}}$ by the same code as in V , $\tilde{U}_i := \varphi(U_i) = U_i \cap \mathcal{N}$, hence $\tilde{U}_i \in \tilde{\mathcal{N}}$ for every $i < n^*$ and $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1}) = Q(U_0, \dots, U_{n^*-1}) \cap \mathcal{N} = Q(U_0, \dots, U_{n^*-1}) \cap \tilde{\mathcal{N}}$, hence $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1}) \in \tilde{\mathcal{N}}$. Let $(\dot{g}_0, \dots, \dot{g}_{n^*-1})$ be a name for a $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ -generic real over $\tilde{\mathcal{N}}$. By the pure decision property of $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ (fact 3.2.7) there exists $(\emptyset, S) \in Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ such that

$$(i) \quad (\emptyset, S) \Vdash_{Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})} (\dot{g}_0, \dots, \dot{g}_{n^*-1}) \in \mathcal{J}(\mathcal{A})$$

or

$$(ii) \quad (\emptyset, S) \Vdash_{Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})} (\dot{g}_0, \dots, \dot{g}_{n^*-1}) \notin \mathcal{J}(\mathcal{A}).$$

Claim: There exists $\bar{a} \in [\omega]^\omega$, which is $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ -generic over $\tilde{\mathcal{N}}$, such that $a_i \subseteq S_i$ and $a_i \in U_i$ for every $i < n^*$.

Proof of the claim: Let $(O_n)_{n \in \omega}$ be an enumeration of all open and dense subsets of $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ in $\tilde{\mathcal{N}}$. Consider the following strategy for player I in the game $G(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ (defined in definition 3.2.3): In the beginning, player I plays $(s^0, S^0) \in O_0$ with $(s^0, S^0) \leq (\emptyset, S)$. Player II answers with $k_0 \in S_0^0$. In the $(m+1)$ th-move, player I plays $(s^{m+1}, S^{m+1}) \in O_{m+1}$ with $(s^{m+1}, S^{m+1}) \leq (s^m, S^m)$ and additionally $k_m \in s^{m+1}$ (this is possible, since O_{m+1} is open and dense). Player II responds with $k_{m+1} \in S_{(m+1) \bmod n^*}^{m+1}$. By lemma 3.2.4 player I has no winning strategy, so the just defined one is not a winning one. Fix a play $((s^n, S^n)_{n \in \omega}, \{k_n \mid n \in \omega\})$ such that player I plays (s^n, S^n) as his above defined strategy orders, player II plays $\{k_n \mid n \in \omega\}$ and player I does not win. Hence we have $\{k_n \mid n \equiv i \pmod{n^*}\} \in U_i$ for every $i < n^*$. Define $\bar{a} := \bigcup_{i \in \omega} s^i$ and $a_i := \{k_j \mid j \equiv i \pmod{n^*}\}$, where $(k_i)_{i < \omega}$ is the increasing enumeration of \bar{a} . Clearly, we have $a_i \subseteq S_i$ and since $\{k_n \mid n \equiv i \pmod{n^*}\} \subseteq a_i$ we have $a_i \in U_i$ for every $i < n^*$. It remains to prove that \bar{a} is $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ -generic over $\tilde{\mathcal{N}}$. For this, by the remarks after the definition 3.2.1 of Q , we have to prove that for every open and dense set $O \subseteq Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$ with $O \in \tilde{\mathcal{N}}$ there exists $(t, T) \in O$ such that $t_i \subseteq a_i$ and $a_i \setminus t_i \subseteq T_i$ for every $i < n^*$. So suppose $O \in \tilde{\mathcal{N}}$ is open and dense in $Q(\tilde{U}_0, \dots, \tilde{U}_{n^*-1})$, hence $O = O_n$ for some $n \in \omega$. Consider (s^n, S^n) , the element of O_n played by player I . Easily, we have $s_i^n \subseteq a_i$ and $a_i \setminus s_i^n \subseteq S_i^n$, as required.

□(claim)

Now, if (i) is true we have

$$(i)' \quad \tilde{\mathcal{N}}[\bar{a}] \models (a_0, \dots, a_{n^*-1}) \in \mathcal{J}(\mathcal{A})$$

and if (ii) is true we have

$$(ii)' \quad \tilde{\mathcal{N}}[\bar{a}] \models (a_0, \dots, a_{n^*-1}) \notin \mathcal{J}(\mathcal{A}).$$

But by fact 3.2.6 this implies

$$(i)'' \quad \tilde{\mathcal{N}}[\bar{c}] \models (c_0, \dots, c_{n^*-1}) \in \mathcal{J}(\mathcal{A})$$

for every \bar{c} with $c_i \in [a_i]^\omega$ for every $i < n^*$ or

$$(ii)'' \quad \tilde{\mathcal{N}}[\bar{c}] \models (c_0, \dots, c_{n^*-1}) \notin \mathcal{J}(\mathcal{A})$$

for every \bar{c} with $c_i \in [a_i]^\omega$ for every $i < n^*$. But Σ_1^1 -definitions are absolute for transitive models, hence

$$(i)''' \quad V \models (c_0, \dots, c_{n^*-1}) \in \mathcal{J}(\mathcal{A})$$

for every \bar{c} with $c_i \in [a_i]^\omega$ for every $i < n^*$ or

$$(ii)''' \quad V \models (c_0, \dots, c_{n^*-1}) \notin \mathcal{J}(\mathcal{A})$$

for every \bar{c} with $c_i \in [a_i]^\omega$ for every $i < n^*$. But both is impossible. \square

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