

L^p -estimates for the wave equation
associated to the Grušin operator

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Zusammenfassung

Mit $G := -(\partial_x^2 + x^2\partial_u^2)$ bezeichnen wir den *Grušin* Operator auf \mathbb{R}^2 . Das Cauchy-Problem der assoziierten Wellengleichung auf $\mathbb{R} \times \mathbb{R}^2$ ist gegeben durch

$$\left(\frac{\partial^2}{\partial t^2} + G\right)v = 0, \quad v|_{t=0} = f, \quad \frac{\partial v}{\partial t}|_{t=0} = g,$$

wobei sich t auf die Zeit bezieht und f, g geeignete Funktionen sind. Die Lösung dieses Problems ist formal gegeben durch

$$v(t, x, u) := [\cos(t\sqrt{G})f](x, u) + \left[\frac{\sin(t\sqrt{G})}{\sqrt{G}}g\right](x, u).$$

Das Thema dieser Dissertation sind Glattheitseigenschaften der Lösung v in Abhängigkeit von den Anfangsdaten. Wir betrachten dabei einen festen Zeitpunkt t . Die Glattheit einer Lösung messen wir bezüglich Sobolev-Normen $\|f\|_{L_p^\alpha} := \|(1 + G)^{\alpha/2}f\|_{L_p}$, definiert in Termen des Differentialoperators G . S_C bezeichne den Streifen $S_C := \{(x, u) \in \mathbb{R}^2; |x| \leq C\}$ im \mathbb{R}^2 . Wir beweisen, dass für $1 \leq p \leq \infty$ die Lösung v in $L_p^{-\alpha}$ liegt, falls unsere Anfangsdaten f und g in einem Streifen S_C , $C > 0$, getragene L_p -Funktionen sind und zudem $\alpha > |1/p - 1/2|$ gilt. Hierzu zeigen wir, dass sich für alle $C > 0$ der Operator $\exp(it\sqrt{G})(1 + G)^{-\alpha/2}$, definiert auf dem Schwartzraum \mathcal{S} , zu einem beschränkten Operator von $L_p(S_C)$ nach $L_p(\mathbb{R}^2)$ fortsetzen läßt, sofern $\alpha > |1/p - 1/2|$ gilt.

Abstract

Let $G := -(\partial_x^2 + x^2 \partial_u^2)$ denote the *Grušin* operator on \mathbb{R}^2 . Consider the Cauchy problem for the associated wave equation on $\mathbb{R} \times \mathbb{R}^2$, given by

$$\left(\frac{\partial^2}{\partial t^2} + G\right)v = 0, \quad v|_{t=0} = f, \quad \frac{\partial v}{\partial t}|_{t=0} = g,$$

where t denotes time and f, g are suitable functions. The solution to this problem is formally given by

$$v(t, x, u) := [\cos(t\sqrt{G})f](x, u) + \left[\frac{\sin(t\sqrt{G})}{\sqrt{G}}g\right](x, u).$$

The focus of this thesis lies on smoothness properties of the solution v for fixed time t with respect to the initial data. Smoothness can be measured in terms of Sobolev norms $\|f\|_{L_p^\alpha} := \|(1 + G)^{\alpha/2}f\|_{L_p}$, defined in terms of the differential operator G . Let S_C denote the strip $S_C := \{(x, u) \in \mathbb{R}^2; |x| \leq C\}$ in \mathbb{R}^2 . We prove that for $1 \leq p \leq \infty$ the solution v is in $L_p^{-\alpha}$ if our initial data f and g are L_p -functions supported in a fixed strip S_C , $C > 0$, and if $\alpha > |1/p - 1/2|$ holds. In fact, we show that for every $C > 0$ the operator $\exp(it\sqrt{G})(1 + G)^{-\alpha/2}$, defined for Schwartz functions, extends to a bounded operator from $L_p(S_C)$ to $L_p(\mathbb{R}^2)$ for all $\alpha > |1/p - 1/2|$.

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0 Introduction

0.1 Context and background

Let

$$L(x, \partial_x) = - \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha \quad (0.1)$$

be a linear partial differential operator of order 2 with smooth real coefficients in an open set $\Omega \subseteq \mathbb{R}^d$ with principal symbol $L_{pr}(x, \xi) := \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha$. We say L is *elliptic in x* , if

$$L_{pr}(x, \xi) \neq 0 \text{ for all } \xi \in T_x \Omega \setminus 0 := \{\xi \in T_x \Omega; \xi \neq 0\}.$$

We call L *elliptic*, if L is elliptic for all $x \in \Omega$. L is called *non-elliptic*, if L is not elliptic.

In addition, we assume that L is positive and essentially selfadjoint. We consider now the following Cauchy problem for the wave equation associated to L on Ω :

$$\frac{\partial^2 v}{\partial t^2} + Lv = 0, \quad v|_{t=0} = f, \quad \frac{\partial v}{\partial t}|_{t=0} = g,$$

where t denotes time and f, g are suitable functions. The solution to this problem is formally given by

$$v(t, x) := \cos(t\sqrt{L})f(x) + \frac{\sin(t\sqrt{L})}{\sqrt{L}}g(x), \quad (x, t) \in \Omega \times \mathbb{R}.$$

The functions of L are defined by the spectral theorem and the above expression for v makes sense at least for $f, g \in L^2(\mathbb{R}^d)$.

Smoothness properties of the solution v , for fixed time t , can be measured in terms of Sobolev norms $\|f\|_{L_p^\alpha} := \|(1+L)^{\alpha/2}f\|_{L_p}$ adapted to L . We are especially interested in estimates of the following kind.

For every $t > 0$, $1 < p < \infty$ and $\alpha > \alpha(d, p)$ there exists a constant $C_{p,t}^\alpha$ such that

$$\|\cos(t\sqrt{L})f\|_{L_p^{-\alpha}} \leq C_{p,t}^\alpha \|f\|_p \quad (0.2)$$

and

$$\left\| \frac{\sin(t\sqrt{L})}{\sqrt{L}}g \right\|_{L_p^{-\alpha+1}} \leq C_{p,t}^\alpha \|g\|_p \quad (0.3)$$

hold.

We call these estimates *wave estimates*.

Wave estimates for the Laplacian on Euclidean space

For $L = -\Delta$ and $\Omega = \mathbb{R}^d$, we have the usual Cauchy problem on the Euclidean space. For this case, estimates have been established by Sigrid Sjöstrand [25], Akihiko Miyachi [14] and Juan Peral [21]. In 1980, Peral and Miyachi independently showed the estimates (0.2), (0.3) for $\alpha(d, p) := (d-1)|1/p - 1/2|$. In fact, they showed that these estimates also hold true for the endpoint $\alpha = \alpha(d, p) = (d-1)|1/p - 1/2|$ if $1 < p < \infty$. Moreover, both operators are bounded from the Hardy space $H_1(\mathbb{R}^d)$ to $L_1(\mathbb{R}^d)$ for $\alpha = (d-1)/2$. These estimates are optimal. In the following we call $\alpha(d, p) := (d-1)|1/p - 1/2|$ the *critical index*.

The solution v can also be written as

$$\begin{aligned} v(x, t) &= \sum_{\epsilon=\pm 1} (2\pi)^{-d} \int \int e^{i((x-y) \cdot \xi + \epsilon|\xi|t)} \frac{1}{2} \left(f(y) + \epsilon \frac{g(y)}{i|\xi|} \right) dy d\xi \\ &=: \sum_{\epsilon=\pm 1} A_{\epsilon,0}^t f(x) + A_{\epsilon,1}^t g(x). \end{aligned}$$

The operators $A_{\epsilon,k}^t$ are Fourier integral operators. Andreas Seeger, Christopher D. Sogge and Elias M. Stein [23] showed that wave estimates (0.2), (0.3) hold true for a wide class of Fourier integral operators for the critical index $\alpha(d, p)$ and d the topological dimension of the underlying space.

In general let

$$P := \partial_t^2 + L(x, \partial_x),$$

with L defined as in (0.1), and put $p(x, \tau, \xi) := -\tau^2 + L_{pr}(x, \xi)$. If L is an elliptic operator, then P is *strictly hyperbolic*, which means that

$$\begin{aligned} p(x, \tau, \xi) &\text{ has two real distinct roots } \tau_1(x, \xi), \tau_2(x, \xi) \\ &\text{ for each } (x, \xi) \text{ with } \xi \neq 0. \end{aligned}$$

It is well known (see J. J. Duistermaat [4]) that in this case one can find elliptic Fourier integral operators $T_{j,k}^t$ such that for small t the solution $v(t, x)$ of the Cauchy problem with initial conditions $\partial_t^k v(0, \cdot) = f_k$ for $k = 0, 1$ is given by

$$v(t, x) = \sum_{j=1,2, k=0,1} T_{j,k}^t f_k(x),$$

modulo an infinitely smoothing operator (this technic is often called the *geometrical optics ansatz*). Therefore, the estimates (0.2) and (0.3) hold true for a wide class of elliptic operators L , provided t is small and $\alpha(d, p) = (d-1)|1/p - 1/2|$. In fact, Seeger, Sogge and Stein [23] showed these estimates for elliptic differential operators of order m on compact, smooth manifolds of dimension d . Locally such an operator of order 2 is of the form (0.1).

Hörmander type operators

Now we remove the requirement of ellipticity. To ensure that the question of smoothness of a solution v of the wave equation is reasonable, we should demand that $L(x, \partial_x)$ is *hypoelliptic*, which means that for all $v \in D'(\Omega)$

$$\text{singsupp } v \subseteq \text{singsupp } L(x, \partial_x)v$$

holds. An elliptic operator L is hypoelliptic.

In 1967, Lars Hörmander showed that operators of the form

$$L(x, \partial_x) := -\left(\sum_{\ell=1}^m X_\ell^2 + X_0\right),$$

where all X_ℓ are real vector fields on an open set Ω , are hypoelliptic under the following, rather weak condition:

For all $x \in \Omega$, the tangential space $T_x\Omega$ in x is spanned by $\{X_1, \dots, X_m\}$ and finitely many iterated commutators of X_1, \dots, X_m .

This condition is often called *Hörmander's condition* and an operator which fulfills it is called an *operator of Hörmander type*. If we write L as $\sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$, we can show that $-\sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \geq 0$. Operators fulfilling this inequality are often called *degenerate elliptic* operators.

The associated wave equation to a non-elliptic operator is not strictly hyperbolic, and that is why we cannot use a geometrical optic ansatz to write the solutions by using Fourier integral operators. Since we have no "straight forward" way of computing the solutions of the wave equation, we can presently only hope to get results for special operators. Furthermore, the underlying geometry is sub-Riemannian and, in general, substantially more complex than the geometry for wave equations associated to elliptic operators.

Nevertheless, we expect that for many Hörmander type operators (0.2) and (0.3) hold true for the critical index $(d-1)|1/p-1/2|$, where d is the topological dimension of the underlying space and that such a result is optimal, except for the endpoint. As far as we know, there is only one result of this type for a non-elliptic operator yet known. We define this operator in the following.

Wave estimates for the sub-Laplacian on the Heisenberg group.

Let \mathbb{H}_m denote the $2m+1$ -dimensional Heisenberg group. As a manifold \mathbb{H}_m is the \mathbb{R}^{2m+1} . The vector fields $X_j := \partial_{x_j} - \frac{1}{2}y_j\partial_u$, $Y_j := \partial_{y_j} + \frac{1}{2}x_j\partial_u$, $U := \partial_u$ form a natural basis for the Lie algebra of left-invariant vector fields. The *sub-Laplacian*

$$L := -\sum_{j=1}^m (X_j^2 + Y_j^2)$$

is non-elliptic. Nevertheless, L is a hypoelliptic operator, since $[X_j, Y_j] = U$ and hence the Hörmander condition is fulfilled.

In 1999, Detlef Müller and Elias M. Stein [19] showed that the estimates (0.2), (0.3) hold true for the critical index $(d-1)|1/p - 1/2|$, where $d := 2m + 1$ is the topological dimension of \mathbb{H}_m . Except for the endpoint $\alpha(d, p)$, this result is optimal. One can reduce the proof to showing that the operator $\exp(i\sqrt{L})(1+L)^{-\alpha/2}$ is bounded on L_p for $\alpha > \alpha(d, p)$. Furthermore, it can be restricted to the case $p = 1$. For this case, Müller and Stein showed that the corresponding convolution kernel of this operator lies in $L_1(\mathbb{H}_m)$.

Before we present our result, we want to mention a recent result by Michael Cowling and Adam Sikora. They studied a sub-Laplacian on the group $SU(2)$. This sub-Laplacian is also of Hörmander type. Since the $SU(2)$ is connected to the Heisenberg group (see Fulvio Ricci [22]), one can hope to get wave estimates for the wave equation associated to this sub-Laplacian for the critical index $\alpha(d, p) = (d-1)|1/p - 1/2| = 2|1/p - 1/2|$. We know, by oral communication, that Cowling and Müller are working on this topic and have developed some new technics, which yield, in principle, the wave estimates on the group $SU(2)$ for the critical index. But they have not worked out all details yet.

Instead of wave equations, Cowling and Sikora studied multipliers and proved a spectral multiplier theorem for this sub-Laplacian. By general functional calculus one can deduce multiplier theorems from wave estimates (see Müller [16]).

Spectral multiplier theorems

We say that for an operator $L(x, \partial_x)$ a *Mikhlin-Hörmander multiplier theorem* holds if for all bounded Borel functions $m : \mathbb{R}^+ \rightarrow \mathbb{C}$ with

$$\sup_{t \in [1, \infty[} \|\eta(\cdot)m(t\cdot)\|_{H_s} < \infty \quad (0.4)$$

the operator $m(L)$ is a bounded operator on L_p for $1 < p < \infty$ and $m(L)$ is of weak type $(1, 1)$, provided s is bigger than an index s_0 . η is here a non trivial cut-off function on \mathbb{R}^+ . Müller and Stein [20] and independently Waldemar Hebisch [10] proved a Mikhlin-Hörmander multiplier theorem for the sub-Laplacian L on \mathbb{H}_m for the index $s_0 = d/2 = (2m + 1)/2$, which is half of the topological dimension of \mathbb{H}_m . This result is optimal, except for the endpoint. It follows also from the mentioned estimates for the wave equation by Müller and Stein by the method of subordination. In fact, Hebisch [10], and also Ricci, Müller and Stein [17] showed multiplier theorems for generalized Heisenberg groups and not only for \mathbb{H}_m .

In 2001, Cowling and Sikora showed that a Mikhlin-Hörmander multiplier theorem for a sub-Laplacian on the group $SU(2)$ (see [3]) holds true for $s_0 = 3/2$,

which is half of the topological dimension of $SU(2)$ and therefore, this result is the analogue for $SU(2)$ of the result obtained by Müller and Stein and as well of the result by Hebisch. This result is optimal, except for the endpoint.

By using the methods of Müller in [16], it should be possible to prove a Mihlin-Hörmander multiplier theorem for the Grušin operator. We conjecture the following theorem.

Conjecture. *Let $m : \mathbb{R}^+ \rightarrow \mathbb{C}$ a bounded Borel function that fulfills (0.4). Then $m(G)$ is a bounded operator on L_p for $1 < p < \infty$ and $m(G)$ is of weak type $(1, 1)$, provided $s > 1$.*

0.2 The main result

Let S_C denote the strip $S_C := \{(x', u') \in \mathbb{R}^2; |x'| \leq C\}$ in \mathbb{R}^2 . In this thesis, we show that for the most basic and historically first studied non-elliptic Hörmander type operator the estimates (0.2) and (0.3) hold true with critical index $\alpha(d, p)$ and d the topological dimension of the underlying space, provided the initial data f and g are supported in a fixed strip S_C . This operator is the Grušin operator.

The *Grušin operator* G is defined by

$$G := -(\partial_x^2 + x^2 \partial_u^2)$$

on \mathbb{R}^2 . Though this operator is non-elliptic, it is still a hypoelliptic operator, since it fulfills the Hörmander condition. Since G is one of the easiest Hörmander type operators, it is predestinated as a starting point for a systematic study of wave equations for non-elliptic operators of this type.

G possesses less invariance properties than the sub-Laplacian L on \mathbb{H}_m . That is why the study of waves associated to this operator is more difficult than the study of waves associated to L . In contrast to L , the Grušin operator is not translation invariant. Waves associated to G that start near the axis $x' = 0$ exhibit a behavior similar to the behavior of waves on \mathbb{H}_1 . Waves associated to G that start far away from the axis $x' = 0$ behave like waves associated to an elliptic operator. Especially the transition area, $0 < x' \lesssim 1$, is very interesting and gives new insights in the general theory of wave estimates for operators of Hörmander type.

G is connected to the sub-Laplacian L on \mathbb{H}_1 , since it can be written as an image of L under a certain representation of \mathbb{H}_1 .

Our result reads as follows.

Theorem 1. *For every $C > 0$, $t > 0$, $1 \leq p \leq \infty$ and $\alpha > |1/p - 1/2|$ there exists a constant $C_{p,t,C}^\alpha$ such that for all f and g in \mathcal{S} and supported in S_C the estimates*

$$\left\| \frac{\cos(t\sqrt{G})}{(1+G)^{\alpha/2}} f \right\|_{L_p(\mathbb{R}^2)} \leq C_{p,t}^\alpha \|f\|_{L_p(\mathbb{R}^2)},$$

and

$$\left\| \frac{\sin(t\sqrt{G})}{\sqrt{G}(1+G)^{(\alpha-1)/2}} g \right\|_{L_p(\mathbb{R}^2)} \leq C_{p,t,C}^\alpha \|g\|_{L_p(\mathbb{R}^2)},$$

hold.

Since the topological dimension is $d = 2$, and hence $d - 1 = 1$, this theorem is a localized analogue for G of the result by Müller and Stein for the sub-Laplacian on the Heisenberg group.

We can restrict to $t = 1$, since G is homogenous with respect to the dilation $\delta_r : (x, u) \mapsto (rx, r^2u)$, $r > 0$. Instead of $\cos(\sqrt{G})(1+G)^{-\alpha/2}$ and $\sin(\sqrt{G})G^{-1/2}(1+G)^{-\alpha/2}$ we study the operator $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$. The assertion for $\cos(\sqrt{G})(1+G)^{-\alpha/2}$ follows immediately. For the operator $\sin(\sqrt{G})G^{-1/2}(1+G)^{-\alpha/2}$, we use that it suffices to prove the assertion for $\eta(G)\sin(t\sqrt{G})G^{-1/2}G^{-\alpha/2}$, where η is a smooth function supported away from the origin. Thus our theorem can be reduced to the following.

Theorem 2. *For every $C > 0$, $1 \leq p \leq \infty$, the operator $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ extends to a bounded operator from $L_p(S_C)$ to $L_p(\mathbb{R}^2)$, provided $\alpha > |1/p - 1/2|$.*

By standard interpolation arguments, it suffices to show the case $p = 1$. Since G is not translation invariant, $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ has no convolution kernel. To prove the case $p = 1$ we show that this operator has an integral kernel K such that $\exp(i\sqrt{G})(1+G)^{-\alpha/2}f(x, u) = \int K(x', u', x, u) f(x', u') d(x', u')$ for every $f \in \mathcal{S}$ and that

$$\|K(x', u', \cdot, \cdot)\|_{L_1(\mathbb{R}^2)} \tag{0.5}$$

is uniformly bounded for $|x'| \leq C$ and $u' \in \mathbb{R}$.

As we have mentioned before, due to the lack of translation invariance, the behavior of a wave highly depends on its starting point. For waves starting near the axis $x' = 0$, we use ideas of Müller and Stein and adapt them to our situation. For waves starting far away from the axis $x' = 0$, it should be possible to reduce by scaling arguments to the results by Seeger, Sogge and Stein for elliptic operators. For this case we only present the general idea and do not go into the details.

It turns out that the most crucial part, but also the most interesting part, of the proof is the case when waves start near, but not exactly on the axis $x' = 0$. For waves starting at $x' = 0$ the methods of Müller and Stein work very well, but as soon as the starting point is a little bit away from $x' = 0$ matters become

a lot more difficult. A sketch of the proof of Theorem 2 will be given in Chapter 2.

Before we start to prove Theorem 2, we have to calculate the integral kernel of $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ in a very explicit way.

Peter C. Greiner, David Holcman and Yakar Kannai derived in [8] formulas for the distribution kernel of $\sin(t\sqrt{G})G^{-1/2}$. We do not use their formulas for two reasons. First, we are interested in the "smoothed" wave propagators $\cos(t\sqrt{G})(1+G)^{-\alpha/2}$ and $\sin(t\sqrt{G})G^{-1/2}(1+G)^{-\alpha/2}$ resp. $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ instead of $\sin(t\sqrt{G})G^{-1/2}$.

The second reason is that Greiner, Holcman and Kannai identify \mathbb{R}^2 with \mathbb{C} and their formulas involve contour integrals. This representation seems not to be explicitly enough to show wave estimates, or, it is not clearly evident how to use them. Moreover, we do not know how to get similar formulas for higher dimensional Grušin operators by this approach. Though we prove wave estimates only for the two dimensional case, our method of calculating the integral kernel of $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ can also be used, in principle, for the higher dimensional case.

We use a trick that was used beforehand by Müller and Stein for the sub-Laplacian L on \mathbb{H}_m . G can be written as $(iU)(-iGU^{-1})$ and hence one can derive formulas for the integral kernel of $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ by using the functional calculus of iU and the functional calculus of $-iGU^{-1}$. The calculus for $-iLU^{-1}$, L instead of G , has been studied by Robert S. Strichartz in [28]. We adapt his methods to our situation and calculate $m(-iGU^{-1})$, for a bounded Borel function m . This gives us a representation of the integral kernel of $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ that can be handled by oscillatory integral methods.

0.3 Organization of this thesis

In CHAPTER 1, Section 1.1, we define the Grušin operator and the sub-Laplacian L on the Heisenberg group \mathbb{H}_m .

The Grušin operator is the image of L under a certain representation of the polarized Heisenberg group of dimension 3. Therefore, transference methods are applicable. We show that for G a weak multiplier theorem holds. This will be done in Section 2.

The next section, Section 1.3, is devoted to the study of the underlying geometry of our problem. We give explicit formulas for geodesics belonging to optimal control metric associated to G . This, together with a result by Richard Melrose [13], allows us to estimate the speed of propagation of our waves. At the end of this section, we present figures of geodesics and balls belonging to the optimal control metric associated to G , and a figure of the sphere in the optimal control metric associated to L .

In CHAPTER 2, we state our main theorem and a conjecture for higher dimensional Grušin operators. Moreover, we give a short sketch of the proof of Theorem 2.

In CHAPTER 3, we study the joint functional calculus for $iU = i\partial_u$ and G . Since G can be written as $(iU)(-iGU^{-1})$, we are especially interested in $m(-iGU^{-1})$, where m is a bounded Borel function.

The functional calculus for $-iLU^{-1}$, L instead of G , has been studied by Strichartz [28]. Strichartz derived explicit formulas for joint eigenfunctions $\phi_{\lambda,n}$ associated to the eigenvalues λ of L and the eigenvalues $\epsilon\lambda/(1+2n)$ of iU . He showed that the operators $m(-iLU^{-1})$, m a bounded Borel function, can be written as a sum over certain generalized projection operators $\mathcal{P}_{n,\epsilon}^{\text{H}}$ associated to rays of the Heisenberg fan. Formally $\mathcal{P}_{n,\epsilon}^{\text{H}}$ is the convolution operator with kernel $\int \phi_{\lambda,n} d\lambda$. We adapt these ideas to our situation. Since G is not translation invariant, the corresponding projection operators $\mathcal{P}_{n,\epsilon}$ are no longer convolution operators. But they are still L_2 -bounded singular integral operators. For a bounded Borel function m , we get the spectral decomposition $m(-iGU^{-1})f = \sum_{\epsilon=\pm 1} \sum_{n=0}^{\infty} m(\epsilon(2n+1)) \mathcal{P}_{n,\epsilon}(f)$.

In CHAPTER 4, we show that it suffices to prove the theorem for $p = 1$. Let K denote the integral kernel of $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$. To prove the theorem, we show that

$$\|K(x', u', \cdot, \cdot)\|_{L^1(\mathbb{R}^2)} \quad (0.6)$$

is uniformly bounded for $|x'| \leq C$ and $u' \in \mathbb{R}$. Since G is translation invariant with respect to u , we only have to consider the case $u' = 0$. We formally show how one can use scaling arguments and the result by Seeger, Sogge and Stein for elliptic operators, to proof that (0.6) is also true for large x' .

Furthermore, instead of estimating $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$, we are allowed to estimate $h^\alpha(G) := \eta(G) G^{-\alpha/2} \exp(i\sqrt{G})$, where η is a smooth function supported away from the origin. In addition we show, due to the finite speed of propagation of our waves, that it suffices to show that the integral kernel $K_{h^\alpha(G)}(x', 0, \cdot, \cdot)$ of $h^\alpha(G)$ is in $L_1(B_G((x', 0), C))$, where $B_G((x', 0), C)$ is a ball with respect to the optimal control metric associated to G centered in $(x', 0)$ with radius C and C a constant.

In CHAPTER 5, we use a dyadic decomposition of the joint spectrum of G and iU to decompose the integral kernel $K_{h^\alpha(G)}$ in dyadic parts $K_{k,j}$. The proof of the theorem is then reduced to showing that $\sup_{|x'| \lesssim 1} \|K_{k,j}(x', 0, \cdot, \cdot)\|_{L_1(B_G((x', 0), C))}$ is summable in j and k . We derive explicit formulas for these dyadic parts by using the projection operators $\mathcal{P}_{n,\epsilon}$ we have defined in Chapter 3. Furthermore, we introduce new coordinates.

In CHAPTER 6, we show the desired L_1 -estimates for the integral kernels $K_{k,j}$.

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1 Preliminaries

1.1 Remarks about integral kernels

We know that if $A : L_p(\mathbb{R}^2) \rightarrow L_q(\mathbb{R}^2)$, $1 \leq p, q \leq \infty$ is a translation invariant operator, then there exists a distribution $u \in \mathcal{S}'(\mathbb{R}^2)$ with $Af = f * u$ for all $f \in \mathcal{S}(\mathbb{R}^2)$. Furthermore, if u is in $L_1(\mathbb{R}^2)$ then the operator $\mathcal{S} \rightarrow C^\infty$, $f \mapsto f * u$ extends to a bounded operator on L_1 . (In fact, we know that this operator extends to a bounded operator on L_1 if and only if u is a finite Borel measure.)

Since the operator of interest $G = -(\partial_x^2 + x^2 \partial_u^2)$ is not translation invariant, most of the operators we study in this work do not have a convolution kernel. Though by the Schwartz kernel theorem, we know that for every continuous linear map $A : C_0^\infty(\mathbb{R}^2) \rightarrow D'(\mathbb{R}^2)$ there exists a distribution K such that

$$\langle A\phi, \psi \rangle = K(\psi \otimes \phi), \quad (1.1)$$

where $\psi \otimes \phi$ denotes the tensor product of ψ and ϕ . We designate K as the *distribution kernel* of A .

If K is a measurable function such that for every $f \in \mathcal{S}$

$$Af(x) = \int K(x', x) f(x') dx',$$

we say that K is the *integral kernel* of A . If we consider more than one operator, we usually denote the integral kernel of an operator A by K_A . We also write

$$K_A(x', \cdot) = A\delta_{x'},$$

where $\delta_{x'}$ denotes the Dirac measure at the point x' .

Given such an A and K we assume now that

$$\sup_{x'} \int |K(x', x)| dx \quad \text{and} \quad \sup_x \int |K(x', x)| dx'$$

are bounded. By Schur's test, A is bounded on L_p for $1 \leq p \leq \infty$. For the L_1 -boundedness, we only need that $\sup_{x'} \int |K(x', x)| dx$ is bounded.

Definition. Let A be an operator with measurable integral kernel $K(x', x)$ such that

$$A(f)(x) = \int K(x', x) f(x') dx', \quad (1.2)$$

for all $f \in \mathcal{S}$. We define the *Schur norm* of K , denoted by $\|K\|_{Schur}$, by

$$\|K\|_{Schur} := \sup_{x'} \int |K(x', x)| dx.$$

In the following, we only use the phrases " K has bounded Schur norm" or "the kernels K_t have bounded Schur norms, uniformly for $t \in I$ " if $(K_t)_t$ is a family of kernels and there exists a constant C such that $\|K_t\|_{Schur} \leq C$ for all t in an interval I .

Thus an operator A with integral kernel K such that K has bounded Schur norm is bounded on L_1 .

1.2 The Grušin operator and the sub-Laplacian on \mathbb{H}_m

The Grušin operator

Let $n \in \mathbb{N}$. We define the *Grušin operator* G_n on \mathbb{R}^{n+1} by

$$G_n := -(\Delta_x + |x|^2 \partial_u^2)$$

G_n is positive and essentially selfadjoint on $C_0^\infty(\mathbb{R}^{n+1})$. Though this operator is not elliptic for $x = 0$, it is still a hypoelliptic operator, since it fulfills the Hörmander condition. G is one of the easiest Hörmander type operators.

V. V. Grušin studied in 1970 (see [9]) a class of operators that is not contained in the class of Hörmander type operators. He gave sufficient and necessary condition for operators in this class to be hypoelliptic. The Grušin operator is a prototype of these operators.

This work will be restricted to the case $n = 1$. Therefore, we define

$$G := G_1.$$

For every $r > 0$, we define the dilation δ_r on \mathbb{R}^2 by

$$\delta_r(x, u) := (rx, r^2u). \quad (1.3)$$

Then for every suitable f and $r > 0$,

$$G(f \circ \delta_r)(x, u) = r^2(Gf) \circ \delta_r(x, u)$$

holds. Hence G is homogenous of degree 2 with respect to δ_r .

The Heisenberg group and the sub-Laplacian

Let \mathbb{H}_m denote the Heisenberg group, which is $\mathbb{R}^{2m} \times \mathbb{R}$ endowed with the group law

$$(x, y, u) \cdot (x', y', u') := \left(x + x', y + y', u + u' + \frac{1}{2}\omega((x, y), (x', y'))\right)$$

for $x, y, x', y' \in \mathbb{R}^n$, $u, u' \in \mathbb{R}$, where ω is the canonical symplectic form

$$\omega((x, y), (x', y')) := x \cdot y' - x' \cdot y, \quad x, y, x', y' \in \mathbb{R}^n$$

on \mathbb{R}^{2n} . \mathbb{H}_m is a connected, simply connected and nilpotent Lie group.

The Lebesgue measure on the Euclidean space \mathbb{R}^{2n+1} is a bi-invariant Haar measure on \mathbb{H}_m . The convolution of two functions $f, g \in L_1(\mathbb{H}_m)$ is defined by

$$(f * g)(x, y, u) = \int f(x', y', u') g((x', y', u')^{-1}(x, y, u)) d(x', y', u').$$

The dilation $\delta_r^{\mathbb{H}} : (x, y, u) \mapsto (rx, ry, r^2u)$ is an automorphism of \mathbb{H}_m for every $r > 0$. The vector fields

$$X_j := \partial_{x_j} - \frac{1}{2}y_j\partial_u, \quad Y_j := \partial_{y_j} + \frac{1}{2}x_j\partial_u, \quad U := \partial_u$$

form a natural basis for the Lie algebra \mathfrak{h}_m of left-invariant vector fields with commutator relations

$$\begin{aligned} [X_j, Y_k] &= \delta_{k,j}U_j, \\ [X_j, X_k] &= [Y_j, Y_k] = [X_j, U] = [Y_j, U] = 0. \end{aligned}$$

We define now the *sub-Laplacian* on \mathbb{H}_m by

$$L := - \sum_{j=1}^m (X_j^2 + Y_j^2). \quad (1.4)$$

Explicitly L is given by

$$L = -\Delta_{x,y} + x \cdot \partial_y - y \cdot \partial_x + \frac{1}{4}|(x, y)|^2 \partial_u^2.$$

L is positive, essentially selfadjoint and homogenous with respect to $\delta_r^{\mathbb{H}}$. Moreover, L is non-elliptic, but hypoelliptic, since the Hörmander condition is fulfilled.

Another common way of writing \mathbb{H}_m is as a set of matrices

$$\{A(p, q, v); p, q \in \mathbb{R}^n, v \in \mathbb{R}\}$$

with

$$A(p, q, v) := \begin{pmatrix} 1 & p_1 & \cdots & p_n & v \\ & 1 & & 0 & q_1 \\ & & \ddots & & \vdots \\ & 0 & & 1 & q_n \\ & & & & 1 \end{pmatrix}$$

and endowed with the usual matrix product. By identifying $A(p, q, v)$ with (p, q, v) , we get a group on $\mathbb{R}^{2n} \times \mathbb{R}$ with group law

$$(p, q, v) \cdot (p', q', v') = (p + p', q + q', v + v' + p \cdot q').$$

This group is called the polarized Heisenberg group and we denote it by $\widetilde{\mathbb{H}}_m$. \mathbb{H}_m is isomorphic to $\widetilde{\mathbb{H}}_m$ by

$$\Psi : \mathbb{H}_1 \rightarrow \widetilde{\mathbb{H}}_1, (x, y, u) \mapsto (p, q, v) = (x, y, u + x \cdot y/2)$$

with inverse transformation

$$\Psi^{-1} : \widetilde{\mathbb{H}}_1 \rightarrow \mathbb{H}_1, (p, q, v) \mapsto (x, y, u) = (p, q, v - p \cdot q/2).$$

A basis of the Lie algebra is given by

$$P_j := \partial_{p_j}, \quad Q_j := \partial_{q_j} + p_j \partial_v, \quad V := \partial_v$$

and the sub-Laplacian on this group is

$$\widetilde{L} := - \sum_{j=1}^m (\partial_{p_j}^2 + (\partial_{q_j} + p_j \partial_v)^2).$$

For more information about the Heisenberg group we refer to Stein [27] and Taylor [30].

L^p -estimates for the wave equation on the Heisenberg group

In analogy to the Cauchy problem for the wave equation on Euclidean space, we consider the following Cauchy problem on $\mathbb{H}_m \times \mathbb{R}$.

$$\frac{\partial^2 v}{\partial t^2} + Lv = 0, \quad v|_{t=0} = f, \quad \frac{\partial v}{\partial t}|_{t=0} = g.$$

Since the work of Müller and Stein about L_p -estimates for solutions of this Cauchy problem is the starting point of this work, we want to state its main result once again.

D. Müller and E. M. Stein established in [19] the following estimates.

Theorem. (D. Müller, E. M. Stein (1999)) *Let $d := 2m + 1$ denote the topological dimension of \mathbb{H}_m . For every $t > 0$, $1 \leq p \leq \infty$ and $\alpha > (d - 1)|1/p - 1/2|$ there exists a constant $C_{p,t}^\alpha$ such that for all f and g in \mathcal{S} the estimates*

$$\left\| \frac{\cos(t\sqrt{L})}{(1 + L)^{\alpha/2}} f \right\|_{L_p(\mathbb{H}_m)} \leq C_{p,t}^\alpha \|f\|_{L_p(\mathbb{H}_m)},$$

and

$$\left\| \frac{\sin(t\sqrt{L})}{\sqrt{L}(1 + L)^{(\alpha-1)/2}} g \right\|_{L_p(\mathbb{H}_m)} \leq C_{p,t}^\alpha \|g\|_{L_p(\mathbb{H}_m)},$$

hold.

By a standard interpolation argument and since L is homogeneous with respect to $\delta_r^{\mathbb{H}}$, it suffices to prove the theorem for $p = 1$ and $t = 1$. In fact, Müller and Stein showed that the operator $\exp(i\sqrt{L})(1 + L)^{-\alpha/2}$ extends to a bounded operator on $L_1(\mathbb{H}_m)$, when $\alpha > m$. For this purpose, they showed that the corresponding convolution kernel belongs to $L_1(\mathbb{H}_m)$.

The proof of this theorem is strongly involved in the proof of our Theorem 2. One reason for this is that G is the image of L under a certain representation of the polarized Heisenberg group $\tilde{\mathbb{H}}_1$.

1.3 Transference

In this section, we denote the polarized Heisenberg group $\tilde{\mathbb{H}}_1$ by \mathcal{G} with elements $g \in \mathcal{G}$, $g = (p, q, v)$. The Lie algebra of \mathcal{G} we denote by \mathfrak{g} . Define

$$P := \partial_p, \quad Q := \partial_q + p\partial_v, \quad V := \partial_v. \quad (1.5)$$

The sub-Laplacian is now given by $L = -(P^2 + Q^2)$.

Let \mathfrak{h} denote the smallest subalgebra of \mathfrak{g} with $Q \in \mathfrak{h}$. With $\exp_G : \mathfrak{g} \rightarrow \mathcal{G}$ denoting the exponential function, we define now

$$\mathcal{H} := \exp(\mathfrak{h}) \subseteq \mathcal{G}.$$

For every $f : \mathcal{G} \rightarrow \mathbb{C}$ we put

$$\|f\|_{\mathcal{H}} := \left(\int |f(p, 0, v)|^2 dp dv \right)^{1/2}$$

and we define

$$\mathcal{H} := \{f; \|f\|_{\mathcal{H}} < \infty \text{ and } f(hg) = f(g) \forall (g, h) \in \mathcal{G} \times \mathcal{H}\}.$$

\mathcal{H} is a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$. We denote the set of unitary operators on \mathcal{H} by $\mathcal{U}(\mathcal{H})$.

$$\begin{aligned} \pi : \mathcal{G} &\rightarrow \mathcal{U}(\mathcal{H}), \\ [\pi(g)f](h) &:= f(hg) \end{aligned}$$

defines an unitary representation π of \mathcal{G} with representation space \mathcal{H} . We denote the associative algebra of left-invariant differential operators with C^∞ -coefficients by $\mathcal{D}_\ell(\mathcal{G})$. $d\pi$ denotes the representation of $\mathcal{D}_\ell(\mathcal{G})$ derived from π . Then

$$d\pi(P) = \partial_p, \quad d\pi(Q) = p\partial_v,$$

and hence

$$d\pi(L) = -(\partial_p^2 + p^2\partial_v^2). \quad (1.6)$$

Thus the Grušin operator G is the image of L under $d\pi$.

For $m \in C_\infty(\mathbb{R}^+)$, the operator $m(L)$, defined by the functional calculus for L , is contained in the C^* -algebra $C^*(\mathcal{G})$. We denote the $*$ -representation of $C^*(\mathcal{G})$ which corresponds to π again by π . If $m \in \mathcal{S}$, then we know by a result of Hulanicki [12] that

$$m(L)f = f * M,$$

with $M \in \mathcal{S}(G)$. Now, let $m \in C_\infty(\mathbb{R}^+)$. If we assume that $m(L)$ is given by $m(L)f = f * M$ with $M \in L_1(G)$, we have

$$\pi(m(L)) = \int_{\mathcal{G}} M(x) \pi(x) dx.$$

Furthermore, we have that the functional calculus commutes with the representation π , i.e.

$$\pi(m(L)) = m(d\pi(L)).$$

For a proof see e.g. Proposition 1.1 in [15].

These facts allow us to compute the integral kernel M_G of $m(G)$ by using the convolution kernel M_L of $m(L)$, provided M_L is in $L_1(\mathcal{G})$.

Proposition 1.1. *Let $m \in C_\infty(\mathbb{R}^+)$ such that the operator $m(L)$ has a convolution kernel $M_L \in L_1(\mathcal{G})$. Then for every $f \in \mathcal{S}(\mathbb{R}^2)$*

$$[m(G)f](p, v) = \int M_L(p' - p, q', v' - v - pq') dq' f(p', v') dp' dv'$$

holds.

Proof. Let $\tilde{f} \in \mathcal{H} \cap \mathcal{S}$, $g := (p, q, v) \in \mathcal{G}$. Then

$$\begin{aligned} [\pi(m(L))\tilde{f}](g) &= \int M_L(x) \tilde{f}(gx) dx \\ &= \int M_L((p, q, v)^{-1}(p', q', v')) dq' \tilde{f}(p', 0, v') dp' dv' \\ &= \int M_L(p' - p, q' - q, v' - v + p(q - q')) dq' \tilde{f}(p', 0, v') dp' dv' \\ &= \int M_L(p' - p, q', v' - v - pq') dq' \tilde{f}(p', 0, v') dp' dv'. \end{aligned}$$

Since $m(G) = m(\pi(L)) = \pi(m(L))$, we finally get for $f \in \mathcal{S}(\mathbb{R}^2)$

$$[m(G)f](p, v) = \int M_L(p' - p, q', v' - v - pq') dq' f(p', v') dp' dv'.$$

□

By this observation and the Fubini theorem, we deduce the following corollary.

Corollary 1.2. *Let $m \in C_\infty(\mathbb{R}_{\geq 0})$ such that the operator $m(L)$ has a convolution kernel $M_L \in L_1(\mathcal{G})$. Then $m(G)$ is bounded on $L_1(\mathbb{R}^2)$ and has an integral kernel M_G with bounded Schur norm. Furthermore, $\|M_G\|_{Schur} = \|M_L\|_{L_1}$.*

REMARK. In the coordinates (x, y, u) of the Heisenberg group \mathbb{H}_1 , and with $L = -(X^2 + Y^2)$ and $G = -(\partial_x^2 + x^2 \partial_u^2)$, we get

$$[m(G)f](x, u) = \int M_L(x' - x, y', u' - u - (x + x')y'/2) dy' f(x', u') dx' du'.$$

◇

From the wave estimates for the Heisenberg group by Müller and Stein (see the last part of Section 1.2), we know that the convolution kernel M_L^α of the operator $m^\alpha(L) := \exp(i\sqrt{L})(1 + L)^{-\alpha/2}$ lies in $L_1(\mathbb{H}_1)$, for $\alpha > (d - 1)/2$ and with $d = 3$ the topological dimension of \mathbb{H}_1 . Hence we obtain that the operator $\exp(i\sqrt{G})(1 + G)^{-\alpha/2}$ is bounded in $L_1(\mathbb{R}^2)$ for $\alpha > 1$.

Comparing this result with our result in Theorem 2, for $p = 1$, we see that by this approach we miss half a derivative. One can show that the q' -integral

$$\int M_L^\alpha(p' - p, q', v' - v - pq') dq'$$

is an oscillatory integral and hence one can hope to get this missing half a derivative by using the method of stationary phase. Müller and Stein derived an explicit formula for M_L^α as a one dimensional oscillatory integral. So, by using Proposition 1.1 we end up with a two dimensional oscillatory integral. Unfortunately, this integral turned out to be very complicated. Hence we do not use this representation of the integral kernel of $\exp(i\sqrt{G})(1 + G)^{-\alpha/2}$ for the proof of Theorem 2.

Nevertheless, Corollary 1.2 is very useful to show that a weak multiplier theorem for G holds. We use this multiplier theorem in the proof of Theorem 2.

A weak multiplier theorem for G

Proposition 1.3. (Multipliers for G) *Suppose $\psi \in C^k(\mathbb{R}^+)$, with k assumed to be sufficiently large. If ψ satisfies the inequalities*

$$\begin{cases} |\xi^\ell \partial_\xi^\ell \psi(\xi)| \lesssim \xi^{1/2} & , \quad \text{for all } 0 < \xi \leq 1, \\ |\xi^\ell \partial_\xi^\ell \psi(\xi)| \lesssim \xi^{-1/2} & , \quad \text{for all } 1 \leq \xi < \infty, \end{cases}$$

for $0 \leq \ell \leq k$, then the integral kernel of $\psi(G)$ has bounded Schur norm.

Proof. Müller and Stein showed in [19], Section 1.1 that under these conditions for ψ the convolution kernel of $\psi(L)$ is in $L_1(\mathbb{H}_1)$. By Corollary 1.2, the integrability of the kernel of $\psi(L)$ implies the boundedness of the Schur norm of the integral kernel of $\psi(G)$. \square

The fact that G is an image of L under a representation of \mathbb{H}_1 implies much more than we have stated here. In general, let A be an operator on a well-behaved group and π a representation of this group. The process of getting information for the operator $\pi(A)$ from properties of A is often called "transference method". Several people have worked on this subject. We refer here to a paper by Ronald R. Coifman and Guido Weiss [2].

1.4 The optimal control metric associated to G

In this section we study the optimal control metric of G . The definition for optimal control metrics resp. Carnot-Carathéodory metrics we have taken from the paper [29] by Strichartz.

Let M be a connected C^∞ manifold, X_1, \dots, X_m smooth real vector fields on M . Let $x \in M$ and $v \in T_x M$. If v is in the linear span of the vector fields $X_1(x), \dots, X_m(x)$ we define

$$\|v\|_x^2 := \inf\{\xi_1^2 + \dots + \xi_m^2; \xi_1 X_1(x) + \dots + \xi_m X_m(x) = v\}.$$

If v is not in the linear span of $X_1(x), \dots, X_m(x)$ we define

$$\|v\|_x^2 := \infty.$$

Let I be an interval and $\gamma : I \rightarrow M$ be a piecewise C^1 -curve. We call γ *admissible*, if $\|\dot{\gamma}(t)\|_{\gamma(t)} < \infty$ for all $t \in I$.

We assume now that γ is admissible and $I = [0, 1]$. Set

$$L(\gamma) := \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt.$$

The *Carnot-Carathéodory metric* associated to the vector fields X_1, \dots, X_m on M we define by

$$d_{cc}(x, y) := \inf\{L(\gamma); \gamma(0) = x, \gamma(1) = y, \gamma \text{ regular}\}.$$

Now let $A := -(\sum_{\ell=1}^m X_\ell^2)$. If A is a Hörmander type operator, then we define the *optimal control metric* d_A associated to A by $d_A := d_{cc}$, where d_{cc} is the Carnot-Carathéodory metric associated to the vector fields X_1, \dots, X_m . In [13] this metric is called *A-distance*.

Now let $M := \mathbb{R}^2$ and define vector fields X_1 and X_2 by

$$X_1 := \partial_{x_1} = (1, 0) \quad X_2 := x_1 \partial_{x_2} = (0, x_1).$$

Then $[X_1, X_2] = \partial_{x_2} = (0, 1)$. The Carnot-Carathéodory distance d_{cc} to these vector fields is the optimal control metric of our operator $G = -(X_1^2 + X_2^2)$.

Let $g^{11}(x) = 1$, $g^{22}(x) = x_1^2$ and $g^{12}(x) = g^{21}(x) = 0$. The Riemannian metric g_{jk} , if there were one, ought to be the inverse of the metric g^{jk} , which does not exist. g^{jk} is often called sub-Riemannian metric.

Here we are interested in balls $B_G((x_1, x_2), R)$, belonging to this metric, of radius R and centered at (x_1, x_2) . Especially we like to show that

$$B_G((x_1, x_2), R) \subseteq B(x_1, cR) \times B(x_2, cR(R + |x_1|)),$$

with some constant c and $B(x, R)$ the ball with respect to the Euclidean metric on \mathbb{R} of radius R and centered at x . To show this we study geodesics.

Let $x, y \in M$. We want to find an admissible curve $\gamma = (\gamma^1, \gamma^2)$ with $\gamma(0) = x$, $\gamma(1) = y$ and $d_G(x, y) = L(\gamma)$. We can assume that $\|\dot{\gamma}(t)\|_{\gamma(t)}$ is constant. To find this curve we have to minimize $\int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt$ with $\gamma(0) = x$ and $\gamma(1) = y$. Instead of minimizing this integral, we are allowed to minimize

$$\int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)}^2 dt = \int_0^1 (\dot{\gamma}^1)^2 + \frac{(\dot{\gamma}^2)^2}{(\gamma^1)^2} dt,$$

with $\gamma(0) = x$, $\gamma(1) = y$. The minimizer is a solution of the Euler-Lagrange equations

$$\begin{aligned} \ddot{\gamma}^1 + \frac{(\dot{\gamma}^2)^2}{(\gamma^1)^3} &= 0 \\ \frac{\dot{\gamma}^2}{(\gamma^1)^2} &= \text{constant}. \end{aligned} \tag{1.7}$$

For $\gamma(0) = (0, 0)$ these equations have the solutions

$$\begin{aligned} \gamma_{b,c}(t) &:= \left(\frac{c}{b} \sin(bt), \frac{c^2}{b} \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) \right), \quad b, c \in \mathbb{R} \\ \beta_c(t) &:= (ct, 0), \quad c \in \mathbb{R}. \end{aligned}$$

For d_G -Balls centered in the origin we obtain essentially

$$B_G(0, R) \sim \{(x_1, x_2); |x_1| < R, |x_2| < R^2\}.$$

These calculations have also been done by Greiner, Holcman and Kannai in [8].

With some more calculus, we get also formulas for $\gamma(0) = (x_1, 0)$. Define

$$\begin{aligned}\gamma^1 &:= c_1 \sin(bt)/b + c_2 \cos(bt)/b, \\ \gamma^2 &:= \frac{1}{b} \left(c_1^2 \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) + \frac{c_1 c_2}{b} \sin^2(bt) + c_2^2 \left(\frac{t}{2} + \frac{\sin(2bt)}{4b} \right) \right) + d,\end{aligned}$$

with $c_1, c_2, b, d \in \mathbb{R}$. All solutions of (1.7) are of one of these forms, or given by β_c .

Since $\gamma(0)$ should be $(x_1, 0)$ we choose $c_2 = x_1 b$ and $d = 0$. Thus by defining

$$\begin{aligned}\gamma_{b,c_1}^1 &:= c_1 \sin(bt)/b + x_1 \cos(bt), \\ \gamma_{b,c_1}^2 &:= \frac{c_1^2}{b} \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) + \frac{x_1 c_1}{b} \sin^2(bt) + x_1^2 b \left(\frac{t}{2} + \frac{\sin(2bt)}{4b} \right),\end{aligned}$$

the function $\gamma_{b,c_1}(t) : [0, 1] \rightarrow \mathbb{R}^2$, $t \mapsto (\gamma_{b,c_1}^1(t), \gamma_{b,c_1}^2(t))$ is part of a geodesic starting in $(x_1, 0)$ of length $\sqrt{c_1^2 + x_1^2 b^2}$. For every $c_1 \in \mathbb{R}$, there exists a $c \geq |x_1 b|$ and an $\epsilon \in \{+1, 1\}$ such that $c_1 = \epsilon \sqrt{c^2 - x_1^2 b^2}$. Define

$$\begin{aligned}\gamma_{b,c,\epsilon}^1 &:= \epsilon \sqrt{c^2 - x_1^2 b^2} \sin(bt)/b + x_1 \cos(bt), \\ \gamma_{b,c,\epsilon}^2 &:= \frac{c^2}{b} \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) + \frac{\epsilon x_1}{b} \sqrt{c^2 - x_1^2 b^2} \sin^2(bt) + x_1^2 \frac{\sin(2bt)}{2}.\end{aligned}$$

Now the function $\gamma_{b,c,\epsilon}(t) : [0, 1] \rightarrow \mathbb{R}^2$, $t \mapsto (\gamma_{b,c,\epsilon}^x(t), \gamma_{b,c,\epsilon}^u(t))$ is part of a geodesic starting in $(x_1, 0)$ of length c .

An easy calculation shows, that we can write the components of $\gamma_{b,c,\epsilon}$ in the following form

$$\begin{aligned}\gamma_{b,c,\epsilon}^1 &= \epsilon |x_1 b| \sqrt{c^2/(x_1^2 b^2) - 1} \sin(bt)/b + x_1 \cos(bt), \\ \gamma_{b,c,\epsilon}^2 &= \frac{c^2 t}{2b} - c^2 \frac{\sin(2bt)}{4b^2} + \epsilon x_1 |x_1| \sqrt{c^2/(x_1^2 b^2) - 1} \sin^2(bt) + x_1^2 \frac{\sin(2bt)}{2}.\end{aligned}$$

With this information it is not difficult to show that there exists a constant c with

$$B_G((x_1, 0), 1) \subseteq B(x_1, c) \times B(0, c(1 + |x_1|)).$$

To prove this we have to study the functions $\gamma_{b,1,\epsilon}$. We can restrict to $|x_1|^2 \geq C$, with $C > 1$ a sufficiently large constant and hence $|b| \leq 1/|x_1| \leq 1/C$ is small. Therefore, $\sin(2bt)/(2b^2) - t/b \lesssim 1$ and $x_1^2 \sin(2bt) \lesssim |x_1|$. Now, we get for $1/(b^2 x_1^2) \geq 2$ by the Taylor expansion of the sine function

$$|\sin(bt)|x_1| \sqrt{b^{-2} x_1^{-2} - 1} \lesssim 1, \quad |\sin^2(bt) x_1^2 \sqrt{b^{-2} x_1^{-2} - 1}| \lesssim 1.$$

And for $1/(b^2x_1^2) \leq 2$ we get the same estimates. Hence

$$|\gamma_{b,1,\epsilon}^1(t) - x_1| \lesssim 1 \text{ and } |\gamma_{b,1,\epsilon}^2(t)| \lesssim 1 + |x_1|,$$

for all $t \leq 1$. Since $G = -X_1^2 - X_2^2$ is homogeneous with respect to the automorphic dilation δ_r and translation invariant with respect to the variable x_2 we get

$$B_G((x_1, x_2), R) \subseteq B(x_1, cR) \times B(x_2, cR(R + |x_1|)).$$

We now want to use our previous notation. We denote the first variable by x and the second by u . Then $G = -(\partial_x^2 + x^2\partial_u^2)$ and we have proven the following proposition.

Proposition 1.4. *Let $B_G((x, u), R)$ denote the ball with respect to the optimal control metric associated to G , centered in (x, u) and with radius $R \in \mathbb{R}^+$. There exists a constant c such that for all $x, u \in \mathbb{R}$ and $R \in \mathbb{R}^+$*

$$B_G((x, u), R) \subseteq B(x, cR) \times B(u, cR(R + |x|))$$

holds.

This proposition allows us to make a statement about the speed of propagation of our wave.

Proposition 1.5. (Finite wave propagation speed) *Let K_t denote the distribution kernel of $\cos(t\sqrt{G})$. There exists a constant \mathcal{C}_0 such that*

$$\text{supp } K_t \subseteq \{(x', u', x, u); (x, u) \in B(x', \mathcal{C}_0 t) \times B(u', \mathcal{C}_0 t(t + |x'|))\}. \quad (1.8)$$

Proof. This assertion follows by Proposition 1.4 and since

$$\text{supp } K_t \subseteq \{(x', u', x, u); (x, u) \in B_G((x', u'), t)\},$$

which was shown by Melrose in [13]. In fact, Melrose showed that this is true for an arbitrary positive selfadjoint differential operator of second order on a compact manifold. The compactness is not essential here, since the operator G is homogeneous with respect to δ_r and hence we have to show (1.8) only for small t . \square

REMARK. A formal proof of (1.8) can also be obtained in the following way. The support of the distribution $\cos(t\sqrt{L})\delta_0$ is contained in $B_t^{\mathbb{H}}$, where $B_t^{\mathbb{H}}$ denotes the ball with respect to the optimal control metric associated to L on \mathbb{H}_1 , centered in 0 and with radius t . There exists a constant $C \geq 1$ such that

$$B_{|t|}^{\mathbb{H}_1} \subseteq \{(z, t) \in \mathbb{H}_1; |z| \leq Ct, |u| \leq C^2 t^2\}.$$

Let $K_{\cos(t\sqrt{L})}$ denote the distribution kernel of $\cos(t\sqrt{L})$. Formally,

$$K_t(x', u', x, u) = \int K_{\cos(t\sqrt{L})}(x' - x, y', u' - u - (x + x')y'/2) dy'.$$

Observe that $|x' - x|, |y'| \leq Ct$ and $|u' - u| \geq 2C^2t(t + |x'|)$ implies

$$|u' - u - (x + x')y'/2| \geq |u' - u| - |(x + x')y'/2| \geq 3C^2t^2/2 > C^2t^2.$$

Hence, if $K_{\cos(t\sqrt{L})}$ would be integrable, the assertion (1.8) would follow immediately. Unfortunately, $K_{\cos(t\sqrt{L})}$ is not integrable and so this second proof is only formally true.

◇

To get an impression how the geometry looks like, we now want to show some figures. Define

$$S_G(x', u') := \{(x, u); d_G((x', u'), (x, u)) = 1\}.$$

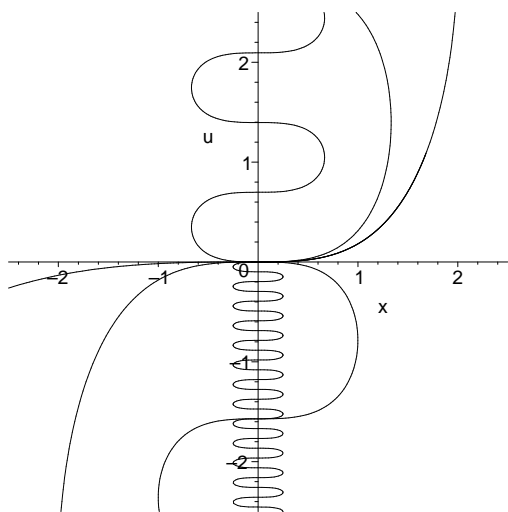


Figure 1.

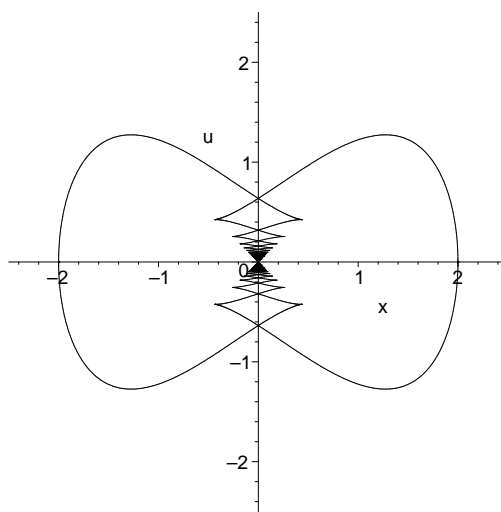


Figure 2.

Figure 1 shows geodesics starting in the origin. Figure 2 shows the sphere $S_G(0,0)$. It has a highly complicated structure. Remarkable is that it has an inner structure consisting of infinitely many edges tending to the origin. This is a tribute to the non-ellipticity of G in 0. The sphere is symmetric with respect to the axis $x = 0$.

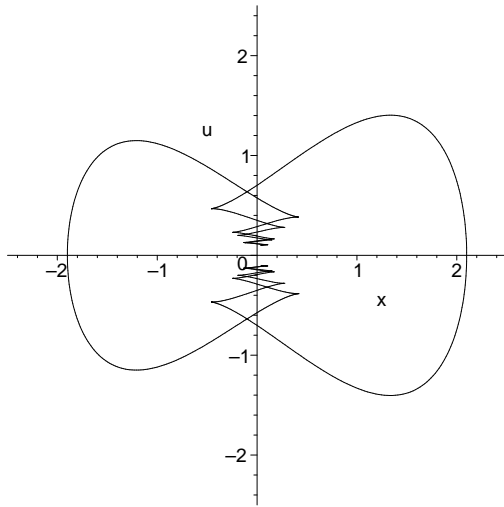


Figure 3.

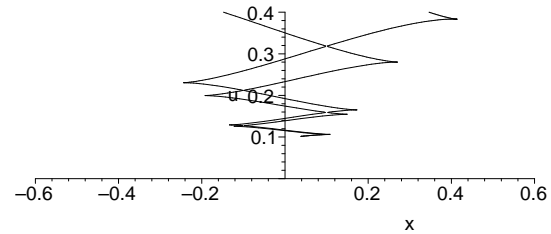


Figure 4.

Figure 3 shows the sphere $S_G(x', 0)$, where $x' = 0.1$. Figure 4 is an enlargement of Figure 3 near the point $(0.1, 0)$. The inner structure is given by only finitely many edges. $S_G(0.1, 0)$ is, in contrast to $S_G(0, 0)$, not symmetric with respect to any axis $x = c$.

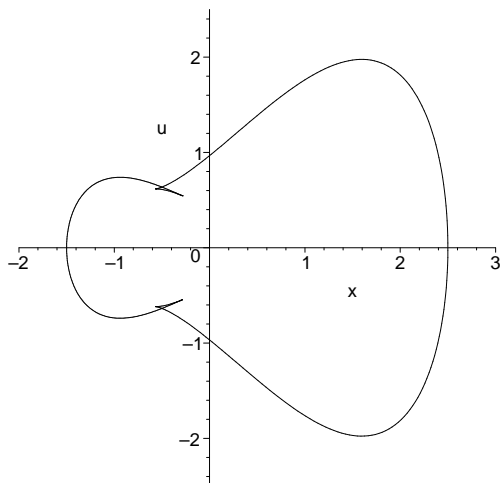


Figure 5.

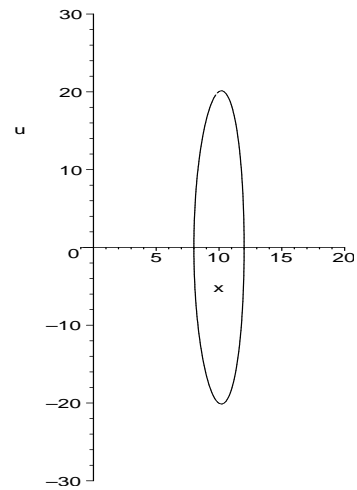


Figure 6.

Figure 5 shows the sphere $S_G(0.5, 0)$. The sphere $S_G(x, 0)$ in figure 6 is given for $x = 10$ and is nearly an ellipsoid with elongation comparable to $|x|$ in the u -direction and elongation comparable to 1 in the x -direction. Near this sphere the operator G is elliptic.

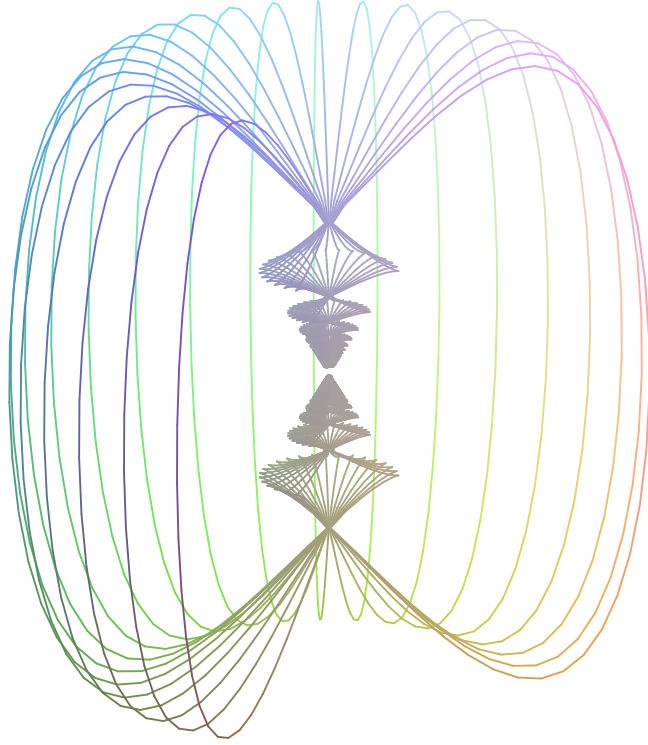


Figure 7.

Figure 7 shows the sphere belonging to the optimal control metric of the sub-Laplacian L on the Heisenberg group \mathbb{H}_1 . We get this picture by rotating the sphere in figure 2 around the axis $x = 0$. This reflects the fact that G is the image of L under the representation π given in the previous section.

If we consider the Cauchy problem

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = 0, \quad v|_{t=0} = f, \quad \frac{\partial v}{\partial t}|_{t=0} = g,$$

for the usual wave equation, then the solution v is given by

$$v(t, x) = f * Q_t(x) + g * P_t(x),$$

where the wave propagators Q_t, P_t are distributions with

$$\text{singsupp } Q_t = \text{singsupp } P_t = S^t(0) = \{x \in \mathbb{R}^d; |x| = t\}.$$

For our wave equation we get a similar result. If we denote the distribution kernel of $\cos(t\sqrt{G})$ and $\sin(t\sqrt{G})/\sqrt{G}$ by Q_t and P_t respectively, we have

$$\text{singsupp } Q_t = \text{singsupp } P_t \subseteq \{(x', u', x, u) \in \mathbb{R}^2; d_G((x', u'), (x, u)) = t\}.$$

This has been proven by Melrose in [13].

Therefore, for given x' , the set $S_{x'} := S_G(x', 0)$ in \mathbb{R}^2 , shown in the figures 2 to 6 has the property that $(\text{singsupp } Q_1) \cap ((x', 0) \times \mathbb{R}^2) \subseteq (x', 0) \times S_{x'}$.

In the introduction, we mentioned that the most crucial part of the proof of Theorem 2 is the case when waves start near, but not exactly on the axis $x = 0$. Now, since we know the underlying geometry better, we want to pick up this subject once more.

First we thought that estimating waves starting in the origin should be most difficult, since G is not elliptic in $(0, 0)$. Though, by comparing figure 2 and figure 7 one can see that this situation is very similar to the situation on the Heisenberg group. By a slightly modification of the methods of Müller and Stein we can prove that the integral kernel of $\exp(i\sqrt{G})(1+G)^{-\alpha/2}\delta_{(x', 0)}$ lies in $L_1(\mathbb{R}^2)$.

Waves starting in $(x', 0)$ with x' very far away from 0 behave like waves for an elliptic operator, as one can see in figure 6. Therefore, this case should also be not very difficult. We do not consider this case in detail.

The most difficult case is when $0 < |x'| < c$, where c is a small constant. A reason for this is that in this case the set $S_{x'}$ has a highly complex structure, as one can see in figures 3 and 4. Especially the lack of symmetry, which we have for S_0 , causes that explicit formulas for $\exp(i\sqrt{G})(1+G)^{-\alpha/2}\delta_{(x', 0)}$ are very complicated.

2 The main theorem and a conjecture

Since we now have the fundamentals, we want to present the main theorem one more time. Furthermore, we state a conjecture for higher dimensional Grušin operators, and give a short sketch of the proof of our theorem.

The main theorem of this thesis reads as follows. Put $S_{\mathcal{C}_1} := \{(x, u) \in \mathbb{R}^2; |x| \leq \mathcal{C}_1\}$.

Theorem 1. *For every $\mathcal{C}_1 > 0$, $t > 0$, $1 \leq p \leq \infty$ and $\alpha > |1/p - 1/2|$ there exists a constant $C_{p,t,\mathcal{C}_1}^\alpha$ such that for all f in \mathcal{S} with $\text{supp } f \subseteq S_{\mathcal{C}_1}$ the estimates*

$$\left\| \frac{\cos(t\sqrt{G})}{(1+G)^{\alpha/2}} f \right\|_{L_p(\mathbb{R}^2)} \leq C_{p,t,\mathcal{C}_1}^\alpha \|f\|_{L_p(\mathbb{R}^2)},$$

and

$$\left\| \frac{\sin(t\sqrt{G})}{\sqrt{G}(1+G)^{(\alpha-1)/2}} f \right\|_{L_p(\mathbb{R}^2)} \leq C_{p,t,\mathcal{C}_1}^\alpha \|f\|_{L_p(\mathbb{R}^2)},$$

hold.

For higher dimensional Grušin operators G_n we conjecture that the following holds.

Conjecture. *Let $d := n + 1$ denote the topological dimension. For every $\mathcal{C}_1 > 0$, $t > 0$, $1 \leq p \leq \infty$ and $\alpha > (d - 1)|1/p - 1/2|$ there exists a constant $C_{p,t,\mathcal{C}_1}^\alpha$ such that for all f in \mathcal{S} with $\text{supp } f \subseteq S_{\mathcal{C}_1}$ the estimates*

$$\left\| \frac{\cos(t\sqrt{G_n})}{(1+G_n)^{\alpha/2}} f \right\|_{L_p(\mathbb{R}^d)} \leq C_{p,t,\mathcal{C}_1}^\alpha \|f\|_{L_p(\mathbb{R}^d)},$$

and

$$\left\| \frac{\sin(t\sqrt{G_n})}{\sqrt{G_n}(1+G_n)^{(\alpha-1)/2}} f \right\|_{L_p(\mathbb{R}^d)} \leq C_{p,t,\mathcal{C}_1}^\alpha \|f\|_{L_p(\mathbb{R}^d)},$$

hold.

Many our technics we use for the proof of Theorem 1 are also applicable in the higher dimensional case. Our computations for this case so far give rise to the hope that the conjecture is really true, but we have not gone into the details yet.

Instead of Theorem 1 we show the following.

Theorem 2. *Let $\mathcal{C}_1 > 0$, $1 \leq p \leq \infty$. The operator $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ extends to a bounded operator from $L_p(S_{\mathcal{C}_1})$ to $L_p(\mathbb{R}^2)$ for $\alpha > |1/p - 1/2|$.*

The restriction to time $t = 1$ is not essential, since G is homogenous with respect to the dilation $\delta_r : (x, u) \mapsto (rx, r^2u)$, $r > 0$, and hence one can deduce the case $t \neq 1$ by the case $t = 1$. As we mentioned before, the assertion for $\cos(\sqrt{G})(1 + G)^{-\alpha/2}$ follows immediately. For the operator $\sin(t\sqrt{G})G^{-1/2}(1 + G)^{-(\alpha-1)/2}$, we use that it suffices to show the assertion for $\eta(G) \sin(t\sqrt{G})G^{-1/2}G^{-(\alpha-1)/2}$, where η is a smooth function supported away from the origin (see Proposition 4.1 in Section 4.2).

We simplify our notation by defining

$$m^\alpha(G) := \exp(i\sqrt{G})(1 + G)^{-\alpha/2}.$$

Furthermore, we fix a constant $\mathcal{C}_1 > 0$.

Throughout our calculations it turned out that it is very useful to compare our results and formulas with the results and formulas that have been derived by Müller and Stein. The topological dimension of the Heisenberg group \mathbb{H}_m is $2m + 1$ and the critical index $\alpha(d, p) = 2m|1/p - 1/2|$ is m , for $p = 1$. In our work the dimension of \mathbb{R}^2 can be written as $2 \times 1/2 + 1$ and the critical index $\alpha(d, p) = |1/p - 1/2|$ is $1/2$, for $p = 1$. Therefore, we set m to be equal to $1/2$,

$$m := 1/2.$$

In REMARKS we consider formulas for L on \mathbb{H}_m and only there we take m to be in \mathbb{N} . If A is some term with respect to G , we denote the corresponding term for L on \mathbb{H}_m by $A^{\mathbb{H}}$.

A short sketch of the proof of Theorem 2

By standard interpolation arguments, it suffices to prove the case $p = 1$, hence we show that for all $\alpha > 1/2$ the integral kernel of $m^\alpha(G)$ has bounded Schur norm. We can restrict to high frequencies in the spectrum of G , and instead of $m^\alpha(G)$ we are allowed to study the operator $h^\alpha(G)$, with

$$h^\alpha(\xi) := \eta_N(\xi) \xi^{-\alpha/2} e^{i\sqrt{\xi}}$$

and η_N smooth and supported in $\{\xi \in \mathbb{R}^+; \xi \geq N\}$.

Let $\mathcal{X}_B(x', u', x, u) := 1$ for all $(x, u) \in B(x', 2\mathcal{C}_0) \times B(u', 2\mathcal{C}_0(2 + |x'|))$ and 0 otherwise. By using the finite speed of propagation (see Proposition 1.8), it suffices to show that the integral kernel $\mathcal{X}_B K_{h^\alpha(G)}$ has bounded Schur norm. This is Proposition 4.1.

In the next chapter we present an idea how one can proof our result also in the case that f is not supported in a strip S_C . This idea reads as follows. By scaling in u , we can transform G into the operator $\tilde{G} := -\partial_x^2 - x^2/x'^2 \partial_u^2$. Now

let $x' \gg 1$. On the set $|x - x'| \leq 1$, $u \in \mathbb{R}$ the operator \tilde{G} is elliptic and just a smooth perturbation of the Laplacian. Hence it should be possible to use Fourier integral operator methods to obtain estimates for solutions to the wave equation $\partial_t^2 + \tilde{G}$ (see Seeger, Sogge, Stein [23]). From these estimates one would get that

$$\sup_{|x'| \geq C, u' \in \mathbb{R}} \|K_{h^\alpha(G)}(x', u', \cdot, \cdot)\|_{L_1}$$

is bounded, where C is a constant.

We exchange \mathcal{X}_B by a smooth variant $\tilde{\mathcal{X}}_B$ of \mathcal{X}_B . Let $\Omega := S_{\mathcal{E}_1}$. The proof of the theorem is then reduced to showing that the operator

$$f \mapsto \int \tilde{\mathcal{X}}_B K_{h^\alpha(G)}(x', u', \cdot, \cdot) f(x', u') dx' du'$$

is bounded from $L_p(\Omega)$ to $L_p(\mathbb{R}^2)$, for every $1 < p < \infty$. This is Proposition 4.5.

G and iU are strongly commuting operators. Their joint spectrum is the closure of the union of rays

$$\mathcal{R}_{n,\epsilon} := \left\{ (\epsilon\lambda, \tau); \tau = (2n+1)\lambda, \lambda > 0 \right\}, \quad \epsilon := \pm 1, n \in \mathbb{N}_0.$$

By a dyadic decomposition of the joint spectrum, we write $h^\alpha(G)$ as a sum of operators $H_{k,j}^\epsilon$, $k \in \mathbb{Z}$, $j \in \mathbb{N}_0$, $\epsilon = \pm 1$, where $H_{k,j}^\epsilon$ is given by

$$H_{k,j}^\epsilon f = 2^{-\alpha k} \sum_n \mathcal{X}_j(2n+1) \gamma_n^\epsilon(iU) (\mathcal{P}_n^\epsilon f),$$

with $\gamma_n^\epsilon(\lambda) := \tilde{\mathcal{X}}_{2k-j}(\epsilon\lambda) e^{i\sqrt{(2n+1)|\lambda|}}$, $\mathcal{X}_j := \mathcal{X}(2^{-j} \cdot)$ and $\tilde{\mathcal{X}}_{2k-j} := \tilde{\mathcal{X}}(2^{-2k+j} \cdot)$ cut off functions and \mathcal{P}_n^ϵ the spectral projection operator that corresponds to the ray $\mathcal{R}_{n,\epsilon}$ in the joint spectrum. These projection operators and especially their integral kernels we study in Chapter 3. If we denote the integral kernel of the operator $H_{k,j}^\epsilon$ by $K_{k,j}^\epsilon$, then away from the diagonal $K_{k,j}^\epsilon$ is given by

$$K_{k,j}^\epsilon(x', 0, x, u) = \frac{2^{-\alpha k}}{2\pi} \sum_{n=0}^{\infty} \mathcal{X}_j(2n+1) \int_{-\infty}^{\infty} P_n^\epsilon(x', 0, x, u-s) \Phi_{k,j,n}^\epsilon(s) ds, \quad (2.1)$$

where P_n^ϵ is the integral kernel of the projection operator \mathcal{P}_n^ϵ , and $\Phi_{\ell,n}^\epsilon$ is given by an oscillatory integral. We can reduce to the case $\epsilon = 1$. By the method of stationary phase, $\Phi_{k,j,n}^1$ is roughly given by $2^{3k/2-j} f(\sqrt{(m+n)/(2^{2k-j})}s^{-1}) e^{i(m+n)/s}$, with $f \in C_0^\infty(\mathbb{R})$, supported away from the origin. Now we choose α to be the critical index $m = 1/2$. To prove the theorem, it suffices to show that for every $\epsilon > 0$ there exists a constant C_ϵ with

$$\sup_{|x'| \leq 2^{\mathcal{E}_1}} \sum_{j \in \mathbb{N}_0} \|\tilde{\mathcal{X}}_B K_{k,j}^1(x', 0, \cdot, \cdot)\|_{L_1} \leq C_\epsilon 2^{\epsilon k}. \quad (2.2)$$

Because, having this assertion allows us to sum over all k , which gives us the desired result for $h^\alpha(G)$. That the assertion (2.2) is true is stated in Proposition 5.3.

For the proof of Proposition 5.3, we use many technical lemmata. Though, there are two main observations that we want to mention. The first thing we need is an explicit formula for P_n^c . In Chapter 3 we derive

$$P_n(x', 0, x, u/2) = C [Q_n - Q_{n-2}],$$

with

$$Q_n = \sum_{\ell=0}^n \frac{\Gamma(\ell + m + 1) \Gamma(n - \ell + m + 1)}{\Gamma(\ell + 1) \Gamma(n - \ell + 1)} e^{-i2\ell\sigma} e^{i\ell\sigma} \frac{(x^2 + x'^2 - iu)^{n/2}}{(x^2 + x'^2 + iu)^{n/2+m+1}}$$

and σ depending on x , x' and u . Careful examinations reveals that Q_n behaves like the sum of two terms of the form

$$Q_n^\pm = \frac{n^m}{|1 - e^{-i2\sigma}|^{3/2}} e^{\pm i\sigma} \frac{(x^2 + x'^2 - iu)^{n/2}}{(x^2 + x'^2 + iu)^{n/2+m+1}}.$$

REMARK. On the Heisenberg group \mathbb{H}_m , Müller, Stein and Strichartz derived similar formulas. The corresponding convolution kernel $P_n^{\mathbb{H}}$ is given by $P_n^{\mathbb{H}} = C_m [Q_n^{\mathbb{H}_m} - Q_{n-1}^{\mathbb{H}_m}]$ with

$$Q_n^{\mathbb{H}} = i^{m+1} (-1)^n \frac{(n+m)!}{n!} \frac{(x^2 + y^2 - 4iu)^n}{(x^2 + y^2 + 4iu)^{n+m+1}}.$$

Observe that $\frac{(n+m)!}{n!} \sim n^m$.

◇

For $x' = 0$, the factor $e^{i\sigma}$ is equal to i and hence $|1 - e^{-i2\sigma}|$ is just 2. Roughly we get

$$Q_n^\pm(0, 0, \cdot, \cdot) = C_m n^m i^{\pm n} \frac{(x^2 - iu)^{n/2}}{(x^2 + iu)^{n/2+m+1}}.$$

This coincidence allows us to use many methods that were used by Müller and Stein in their proof. Unfortunately, matters become very difficult when we choose x' away from 0.

The second important observation we use is the following. By using our formulas we derived for Q_n , and after some scaling, we see that the integral in (2.1) is an oscillatory integral with phase function $s \mapsto \varphi(s) + 2^{k-j}/s$, where φ is roughly given by

$$\varphi(s) = \arctan \left(\frac{Rw - 2xx_1(u - s)}{2Rxx_1 + (u - s)w} \right),$$

with $R := x^2 + x_1^2$ and $w := ((x^2 - x_1^2)^2 + (u - s)^2)^{1/2}$. By

$$X := \frac{Rw - 2xx_1(u - s)}{2Rxx_1 + (u - s)w}, \quad Y := \frac{(R^2 + (u - s)^2)w}{2Rxx_1 + (u - s)w},$$

$$s := s, \quad \psi(x, u, s) := (X, Y, s).$$

we define new coordinates such that $\varphi(s) = \arctan(X)$ and $\partial_s \varphi(s) = X/Y$. For $x_1 = 0$, these coordinates coincides with coordinates which were used by Müller and Stein. On first sight, it is not clear that using them is really possible. The problem is that the functional determinant of ψ^{-1} can not be computed directly, or, to be more exact, in a straight forward way.

Nevertheless, it is possible to find an expression for $\det(\psi^{-1})$. We can estimate the integral in (2.1) by using these new coordinates and a refined partial integration. We prove Proposition 5.3, which completes the proof of Theorem 2.

3 The joint functional calculus for G and iU

In [28] Strichartz regarded harmonic analysis on the Heisenberg group \mathbb{H}_m as the *joint spectral theory of the operator L and the operator iU* , where $U = \partial_u$ is the partial derivative with respect to the central variable u of the group.

The joint spectrum of these operators is the *Heisenberg fan*, which is the closure of the union of rays $\mathcal{R}_{n,\epsilon} := \{(\lambda, \tau); \tau = \epsilon\lambda/(m + 2n), \lambda > 0\}$. The spectral decomposition of a function f in L_2 can be given as

$$f = \sum_{k,\epsilon} \int_0^\infty f * \phi_{\lambda,k,\epsilon} d\lambda,$$

where the functions $f * \phi_{\lambda,k,\epsilon}$ are joint eigenfunctions of iU and L . The functions $\phi_{\lambda,k,\epsilon}$ can be explicitly calculated in terms of Laguerre polynomials.

In this chapter we adapt the methods of Strichartz to our situation. Instead of the joint spectrum of the sub-Laplacian iU and L we study the joint spectrum of iU and G , which also consists of rays in $\mathbb{R} \times \mathbb{R}^+$. Our main concern here is to derive simple formulas for certain spectral projection operators belonging to these rays and, in the end, to derive a simple formula for the integral kernel $m^\alpha(G)$.

3.1 Spectral projection operators to rays

Let $U := \partial_u$. Since G and iU are essentially self-adjoint and strongly commuting operators, they have a well defined joint spectrum. This spectrum consists of the union of rays

$$\mathcal{R}_{n,\epsilon} := \{(\epsilon\lambda, \tau); \tau = (2n + 1)\lambda, \lambda > 0\}$$

for $\epsilon := \pm 1$, $n \in \mathbb{N}_0$ together with the limit ray $\mathcal{R}_\infty = \{(0, \tau); \tau \geq 0\}$. Here $\epsilon\lambda$ refers to the spectrum iU and τ to the spectrum of G . In analogy to the Heisenberg group we will call the closure of the union of rays $\mathcal{R}_{n,\epsilon}$ the *Grušin fan*.

We write

$$G = (iU)(-iGU^{-1}).$$

The operator iU is easy. The operator $-iGU^{-1}$ can be written as a sum over spectral projection operators to rays.

Proposition 3.1. *Let m be a bounded measurable function on the Grušin fan. Then for every $f \in \mathcal{S}$*

$$[m(G, iU)f](x, u) = \sum_{\epsilon=\pm 1} \sum_{n=0}^{\infty} \int_0^{\infty} m(\epsilon\lambda, (2n+1)\lambda) [\mathcal{P}_{\lambda, n, \epsilon} f](x, u) d\lambda$$

with

$$\begin{aligned} [\mathcal{P}_{\lambda, n, \epsilon} f](x, u) &:= \int \phi_{\lambda, n, \epsilon}(x', u', x, u) f(x', u') dx' du', \\ \phi_{\lambda, n, \epsilon}(x', u', x, u) &:= \frac{\lambda^{1/2}}{2\pi} e^{-i\epsilon\lambda(u-u')} h_n(\lambda^{1/2}x') h_n(\lambda^{1/2}x). \end{aligned}$$

and

$$h_n(x) := (\sqrt{\pi} n! 2^n)^{-1/2} (\partial_x - x)^n e^{-x^2/2}$$

the n -th Hermite function.

Proof. Observe that under the Fourier transform with respect to the variable u the operator G is the Hermite operator $H_\lambda := -(\partial_x^2 - \lambda x^2)$. By using the spectral decomposition of the Hermite operator and the Fourier inversion formula with respect to the variable u , we get for any Schwartz function f on \mathbb{R}^2

$$f(x, u) = \sum_{\epsilon=\pm 1} \sum_{n=0}^{\infty} \int_0^{\infty} \int \phi_{\lambda, n, \epsilon}(x', u', x, u) f(x', u') dx' du' d\lambda,$$

with

$$\phi_{\lambda, n, \epsilon}(x', u', x, u) := \frac{\lambda^{1/2}}{2\pi} e^{-i\epsilon\lambda(u-u')} h_n(\lambda^{1/2}x') h_n(\lambda^{1/2}x).$$

The function $\Phi_{\lambda, n, \epsilon}(x, u) := e^{-i\epsilon\lambda u} h_n(\lambda^{1/2}x)$ is a joint eigenfunction of G and iU with

$$\begin{aligned} G\Phi_{\lambda, n, \epsilon} &= (2n+1)\lambda \Phi_{\lambda, n, \epsilon} \\ iU\Phi_{\lambda, n, \epsilon} &= \epsilon\lambda \Phi_{\lambda, n, \epsilon}. \end{aligned}$$

Hence

$$\frac{G}{iU}\Phi_{\lambda, n, \epsilon} = \epsilon(2n+1) \Phi_{\lambda, n, \epsilon}$$

the proposition follows by the spectral theorem. \square

This proposition implies the next corollary.

Corollary 3.2. *Let m be a bounded measurable function then for every $f \in \mathcal{S}$*

$$[m(-iGU^{-1})f](x, u) = \sum_{\epsilon=\pm 1} \sum_{n=0}^{\infty} m(\epsilon(2n+1)) [\mathcal{P}_{n,\epsilon}f](x, u)$$

with

$$[\mathcal{P}_{n,\epsilon}f](x, u) = \int_0^{\infty} \int \frac{\lambda^{1/2}}{2\pi} e^{-i\epsilon\lambda(u-u')} h_n(\lambda^{1/2}x') h_n(\lambda^{1/2}x) f(x', u') dx' du' d\lambda.$$

The operator $\mathcal{P}_{n,\epsilon}$ will be called the *spectral projection operator to the ray* $\mathcal{R}_{n,\epsilon}$.

Observe that with $S(x, u) := (x, -u)$

$$[\mathcal{P}_{n,-1}f](x, u) = -[\mathcal{P}_{n,1}(f \circ S)](x, -u).$$

Hence it suffices to study $\mathcal{P}_{n,1}$. We assume now $\epsilon = 1$ and define $\mathcal{P}_n := \mathcal{P}_{n,1}$ to make the notation simpler.

For $x, u \in \mathbb{R}$ and $f \in \mathcal{S}$ supported away from u , we see by using partial integration that $[\mathcal{P}_n f](x, u)$ can be given by an absolutely convergent integral. By Fubini's theorem we get

$$[\mathcal{P}_n f](x, u) = \int \int_0^{\infty} \frac{\lambda^{1/2}}{2\pi} e^{-i\lambda(u-u')} h_n(\lambda^{1/2}x') h_n(\lambda^{1/2}x) d\lambda f(x', u') dx' du'.$$

We define now

$$P_n(x', u', x, u) := \int_0^{\infty} \frac{\lambda^{1/2}}{2\pi} e^{-i\lambda(u-u')} h_n(\lambda^{1/2}x') h_n(\lambda^{1/2}x) d\lambda \quad (3.1)$$

and

$$\phi_{\lambda,n}(x', u', x, u) := \frac{\lambda^{1/2}}{2\pi} e^{-i\lambda(u-u')} h_n(\lambda^{1/2}x') h_n(\lambda^{1/2}x). \quad (3.2)$$

Then

$$[\mathcal{P}_n f](x, u) = \int P_n(x', u', x, u) f(x', u') dx' du',$$

for $f \in \mathcal{S}$, supported away from u .

REMARK. Strichartz showed that the projection operators \mathcal{P}_n^{H} for iU and L are Calderon-Zygmund operators and that

$$\mathcal{P}_n^{\text{H}} = c_n \delta_0 + p.v.P_n^{\text{H}}.$$

$P_n^{\mathbb{H}}$ is a kernel of similar type than P_n and $p.v.P_n^{\mathbb{H}}$ is the operator with kernel $P_n^{\mathbb{H}}$ in the principal value sense.

The same should also be true here, but we do not need this additional information. ◇

We compute P_n explicitly. Recall the Mehler formula, according to which for all $x, y \in \mathbb{R}$ and $z \in \mathbb{C}$, $|z| < 1$

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(x) h_n(y) z^n &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{H_n(x) H_n(y) z^n}{2^n n!} e^{-\frac{x^2+y^2}{2}} \\ &= \frac{1}{\sqrt{\pi} \sqrt{1-z^2}} e^{\frac{2xyz - z^2(x^2+y^2)}{1-z^2}} e^{-\frac{x^2+y^2}{2}} \end{aligned}$$

holds. Thus

$$\begin{aligned} &\sum_{n=0}^{\infty} r^n \int_0^{\infty} \phi_{\lambda,n}(x', u', x, u) d\lambda \\ &= \sum_{n=0}^{\infty} r^n \int_0^{\infty} \frac{\lambda^{1/2}}{2\pi} e^{-i\lambda(u-u')} h_n(\lambda^{1/2}x) h_n(\lambda^{1/2}x') d\lambda \\ &= \frac{1}{2\pi^{3/2}} \int_0^{\infty} \frac{\sqrt{\lambda}}{\sqrt{1-r^2}} e^{\frac{2r\lambda x x' - r^2(\lambda x^2 + \lambda x'^2)}{1-r^2}} e^{-\lambda \frac{x^2+x'^2}{2}} e^{-i\lambda(u-u')} d\lambda \\ &= \frac{1}{2\pi^{3/2}} \int_0^{\infty} \frac{\sqrt{\lambda}}{\sqrt{1-r^2}} e^{-\lambda \left[(x^2+x'^2) \frac{1+r^2}{2(1-r^2)} - \frac{2rxx'}{1-r^2} + i(u-u') \right]} d\lambda. \end{aligned}$$

For small r this integral converges since, $x^2 + x'^2 \geq 2xx'$.

REMARK. For $x' = 0 = u'$ we obtain

$$\frac{1}{2\pi^{3/2}} \int_0^{\infty} \frac{\sqrt{\lambda}}{\sqrt{1-r^2}} e^{-\lambda \left[x^2 \frac{1+r^2}{2(1-r^2)} + iu \right]} d\lambda.$$

On the Heisenberg group \mathbb{H}_1 , Strichartz established the following formula

$$\sum_{n=0}^{\infty} r^n \int_0^{\infty} \phi_{\mathbb{H},\lambda,n}(z, u) d\lambda = \frac{1}{4\pi^2} \int_0^{\infty} \frac{\lambda}{1-r} e^{-\lambda \left[|z|^2 \frac{1+r}{4(1-r)} + iu \right]} d\lambda,$$

with $|z|^2 = x^2 + y^2$. ◇

The λ -integration can be computed easily, since the Gamma function has the integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

for $\operatorname{Re}(z) > 0$. We get

$$\begin{aligned} \sum_{n=0}^{\infty} r^n \int_0^\infty \phi_{\lambda,n}(x', u', x, u) d\lambda &= \frac{\Gamma(3/2)2^{1/2}}{\pi^{3/2}} (1 - r^2) \\ &\times ((x^2 + x'^2) + 2i(u - u') + r^2((x^2 + x'^2) - 2i(u - u')) - 4rxx')^{-3/2}. \end{aligned}$$

Put

$$\begin{aligned} \phi(x', u', x, u) &:= \frac{\Gamma(3/2)2^{1/2}}{\pi^{3/2}} (1 - r^2) \\ &\times ((x^2 + x'^2) + 2i(u - u') + r^2((x^2 + x'^2) - 2i(u - u')) - 4rxx')^{-3/2} \end{aligned} \quad (3.3)$$

Since our operator is translation invariant with respect to the u -direction, we only have to consider the case $u' = 0$. In the following, we are interested in $P_n(x', 0, x, u/2)$, thus we replace $2u$ by u .

We use the abbreviations

$$R := x^2 + x'^2, \quad z := R + iu, \quad p := 4xx', \quad (3.4)$$

and

$$\begin{aligned} w &:= \sqrt{x^4 + x'^4 - 2(xx')^2 + u^2} = \sqrt{R^2 - 4(xx')^2 + u^2}, \\ a &:= \sqrt{(x^2 + x'^2)^2 + u^2} = \sqrt{R^2 + u^2}. \end{aligned} \quad (3.5)$$

With these definitions we get $\phi = C_m (1 - r^2)(z + r^2\bar{z} - rp)^{-m-1}$, with C_m a constant. Recall that m is always $1/2$, except when we speak about the Heisenberg group \mathbb{H}_m .

Lemma 3.3. *Let $\alpha \in \mathbb{N}/2$ then*

$$\frac{\partial_r^n}{n!} \Big|_{r=0} (z + r^2\bar{z} - rp)^{-\alpha} = C_\alpha b_{n,\alpha} \frac{\bar{z}^{n/2}}{z^{\alpha+n/2}} e^{in\sigma}$$

holds with

$$\begin{aligned} b_{n,\alpha} &= \sum_{\ell=0}^n \frac{\Gamma(\ell + \alpha) \Gamma(n - \ell + \alpha)}{\Gamma(\ell + 1) \Gamma(n - \ell + 1)} e^{-i2\ell\sigma}, \\ e^{i\sigma} &= \left(\frac{p}{2|z|} + i\sqrt{1 - \frac{p^2}{4|z|^2}} \right) \\ &= \frac{2xx' + i\sqrt{x^4 + x'^4 - 2(xx')^2 + u^2}}{\sqrt{(x^2 + x'^2)^2 + u^2}} = \frac{2xx' + iw}{\sqrt{R^2 + u^2}} \end{aligned}$$

and C_α a constant only depending on α .

Proof. Put $e^{i\vartheta} := \frac{z}{|z|}$ and $\tilde{r} := re^{-i\vartheta}$. Then

$$\begin{aligned} (z + r^2\bar{z} - rp)^{-\alpha} &= z^{-\alpha} \left(1 + \tilde{r}^2 - \tilde{r}\frac{p}{|z|}\right)^{-\alpha} \\ &= z^{-\alpha} [(\tilde{r} - \beta_1)(\tilde{r} - \beta_2)]^{-\alpha}. \end{aligned}$$

The roots β_1 and β_2 are given by $\beta_1 = \frac{p}{2|z|} + i\sqrt{1 - \frac{p^2}{4|z|^2}} = e^{i\sigma}$ and $\beta_2 = \overline{\beta_1}$. We have $|\beta_1| = 1 = |\beta_2|$ and

$$(z + r^2\bar{z} - rp)^{-\alpha} = z^{-\alpha} e^{i2\vartheta\alpha} [(r - e^{i(\vartheta+\sigma)})(r - e^{i(\vartheta-\sigma)})]^{-\alpha}.$$

Thus

$$\begin{aligned} &\frac{\partial_r^n}{n!} \Big|_{r=0} (z + r^2\bar{z} - rp)^{-\alpha} \\ &= \frac{z^{-\alpha} e^{i2\vartheta\alpha}}{n!} \left[\sum_{\ell=0}^n \frac{n!}{\ell!(n-\ell)!} (-1)^\ell \alpha \cdot \dots \cdot (\alpha + \ell - 1) (-e^{i(\vartheta+\sigma)})^{-\alpha-\ell} \right. \\ &\quad \left. \times (-1)^{n-\ell} \alpha \cdot \dots \cdot (\alpha + (n-\ell) - 1) (-e^{i(\vartheta-\sigma)})^{-\alpha-(n-\ell)} \right] \\ &= (-1)^{2\alpha} \Gamma(\alpha)^{-2} z^{-\alpha} e^{-i(\vartheta-\sigma)n} \sum_{\ell=0}^n \frac{\Gamma(\ell + \alpha) \Gamma(n - \ell + \alpha)}{\Gamma(\ell + 1) \Gamma(n - \ell + 1)} e^{-i2\ell\sigma} \\ &= C_\alpha \sum_{\ell=0}^n \frac{\Gamma(\ell + \alpha) \Gamma(n - \ell + \alpha)}{\Gamma(\ell + 1) \Gamma(n - \ell + 1)} e^{-i2\ell\sigma} z^{-\alpha} e^{-i(\vartheta-\sigma)n} \\ &= C_\alpha b_{n,\alpha} \frac{\bar{z}^{n/2}}{z^{\alpha+n/2}} e^{i\sigma n}. \end{aligned}$$

□

We are only interested in $\alpha = 3/2$ and $\alpha = 1/2$. Thus we put

$$q_{n,m} := b_{n,3/2} \text{ and } q_{n,m-1} := b_{n,1/2}. \quad (3.6)$$

Furthermore, we observe that

$$1 - e^{i2\sigma} = \frac{R^2 + u^2 - (4(xx')^2 + i4xx'w - w^2)}{R^2 + u^2} = 2 \frac{w^2 - i2xx'w}{R^2 + u^2},$$

and thus

$$|1 - e^{i2\sigma}| = 2 \frac{\sqrt{w^4 + 4(xx')^2 w^2}}{R^2 + u^2} = 2w \frac{\sqrt{w^2 + 4(xx')^2}}{R^2 + u^2} = 2 \frac{w}{a}.$$

For $z = R + iu$, we get

$$\begin{aligned}
\left(\frac{R - iu}{R + iu}\right)^{1/2} e^{i\sigma} &= \left(\frac{\bar{z}}{z}\right)^{1/2} e^{i\sigma} = \left(\frac{\bar{z}^2}{|z|^2}\right)^{1/2} e^{i\sigma} = \frac{\bar{z}}{|z|} e^{i\sigma} = \frac{(R - iu)(2xx' + iw)}{|z|^2} \\
&= \frac{R2xx' + uw + i(Rw - 2xx'u)}{|z|^2} = \frac{|z|^2}{|z|^2} \exp\left(i \arctan\left(\frac{Rw - 2xx'u}{R2xx' + uw}\right)\right) \\
&= \exp\left(i \arctan\left(\frac{Rw - 2xx'u}{R2xx' + uw}\right)\right).
\end{aligned} \tag{3.7}$$

\arctan denotes the branch of \tan^{-1} taking values in $[0, \pi]$, since $|Rw| \geq |2xx'u|$ and thus the imaginary part is always positive. We also get

$$\left(\frac{R - iu}{R + iu}\right)^{1/2} e^{-i\sigma} = \exp\left(-i \arctan\left(\frac{Rw + 2xx'u}{R2xx' - uw}\right)\right), \tag{3.8}$$

where \arctan denotes the same branch of \tan^{-1} taking values in $[0, \pi]$.

For the integral kernel P_n of \mathcal{P}_n we get

$$P_n(x', 0, x, u/2) = \int_0^\infty \phi_{\lambda, n}(x', 0, x, u/2) d\lambda = C [Q_n - Q_{n-2}]. \tag{3.9}$$

For Q_n and with $m = 1/2$ we have the following formula

$$\begin{aligned}
Q_n = Q_n(x', 0, x, u) &= \sum_{\ell=0}^n \frac{\Gamma(\ell + m + 1) \Gamma(n - \ell + m + 1)}{\Gamma(\ell + 1) \Gamma(n - \ell + 1)} e^{-i2\ell\sigma} \\
&\times e^{i\sigma} \frac{(x^2 + x'^2 - iu)^{n/2}}{(x^2 + x'^2 + iu)^{n/2+m+1}},
\end{aligned} \tag{3.10}$$

with

$$e^{i\sigma} = \frac{2xx' + i\sqrt{x^4 + x'^4 - 2(xx')^2 + u^2}}{\sqrt{(x^2 + x'^2)^2 + u^2}} = \frac{2xx_1 + iw}{a}. \tag{3.11}$$

For completeness we also want to state another formula for the projection operators. It is based on the following observation

Lemma 3.4. *Let $f, g \in C^\infty$ with $g^{(n)}(r)|_{r=0} = 0$ for all $n > 2$. Then*

$$\frac{1}{n!} (f \circ g)^{(n)}|_{r=0} = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \frac{1}{2^\ell \ell! (n - 2\ell)!} (g')^{n-2\ell} (g'')^\ell f^{(n-\ell)} \circ g|_{r=0}.$$

By this lemma we get easily

$$\begin{aligned}
& \frac{\partial_r^n}{n!} \Big|_{r=0} (z + r^2 \bar{z} - rp)^{-\alpha} \\
&= \sum_{\ell=0}^{\lfloor n/2 \rfloor} \frac{1}{2^\ell \ell! (n-2\ell)!} (-p)^{n-2\ell} (2\bar{z})^\ell (-1)^{(n-\ell)} \frac{\Gamma(n-\ell+\alpha)}{\Gamma(\alpha)} z^{-\alpha-(n-\ell)} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \frac{\Gamma(n-\ell+\alpha)}{\ell! (n-2\ell)!} p^{n-2\ell} \frac{\bar{z}^\ell}{z^{\alpha+(n-\ell)}} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \frac{\Gamma(n-\ell+\alpha)}{\ell! (n-2\ell)!} (4xx_1)^{n-2\ell} \frac{(x^2 + x'^2 - iu)^\ell}{(x^2 + x'^2 + iu)^{\alpha+(n-\ell)}}.
\end{aligned}$$

Hence

$$\int_0^\infty \phi_{\lambda,n}(x', t', x, 0) d\lambda = C_m [Q_n - Q_{n-2}].$$

with Q_n given by

$$\begin{aligned}
Q_n &= \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \frac{\Gamma(n-\ell+m+1)}{\ell! (n-2\ell)!} (4xx')^{n-2\ell} \\
&\quad \times \frac{(x^2 + x'^2 - iu)^\ell}{(x^2 + x'^2 + iu)^{m+1+(n-\ell)}}.
\end{aligned} \tag{3.10a}$$

It turns out, that this formula is not very useful. In the following we only use formula (3.10).

The region where w is small

For the region where w is small we establish a second formula for the P_n . Let

$$\begin{aligned}
\Phi &:= -\frac{C_m}{im} (z + r^2 \bar{z} - rp)^{-m} \\
&= -\frac{C_m}{im} (x^2 + x_1^2 + iu + r^2(x^2 + x_1^2 - iu) - 4rxx_1)^{-m}.
\end{aligned}$$

Then $\phi = \partial_u \Phi$ and by Lemma 3.3 we obtain

$$\begin{aligned}
\frac{\partial_r^n}{n!} \Big|_{r=0} \Phi_\epsilon &= \frac{\partial_r^n}{n!} \Big|_{r=0} \frac{C_m}{i(\alpha+1)} (z + r^2 \bar{z} - rp)^{-\alpha+1} \\
&= C_m q_{n,m-1} \frac{\bar{z}^{n/2}}{z^{m+n/2}} e^{in\sigma} =: C_m q_{n,m-1} \frac{(R-iu)^{n/2}}{(R+iu)^{m+n/2}} e^{in\sigma}.
\end{aligned}$$

Define

$$R_n := q_{n,m-1} \frac{(R - iu)^{n/2}}{(R + iu)^{m+n/2}} e^{in\sigma}. \quad (3.12)$$

For P_n we get

$$\begin{aligned} P_n &= C_m [Q_n - Q_{n-2}] = \frac{1}{n!} \partial_r^n |_{r=0} \phi = \frac{1}{n!} \partial_r^n |_{r=0} (\partial_t \Phi) = \frac{1}{n!} \partial_t \partial_r^n |_{r=0} \Phi \\ &= C_m \partial_t R_n. \end{aligned} \quad (3.13)$$

REMARK. In the Heisenberg situation, the corresponding function $\phi^{\mathbb{H}}$ is given by

$$\begin{aligned} \phi^{\mathbb{H}}(x, y, u) &= 2^{m+1} \pi^{-m-1} m! (1 - r) \\ &\quad \times ((x^2 + y^2) + 4iu + r((x^2 + y^2) - 4iu))^{-m-1}. \end{aligned}$$

For the convolution kernel $P_n^{\mathbb{H}}$ we have a very similar expressions as for P_n .

$$P_n^{\mathbb{H}} = C_m [Q_n^{\mathbb{H}m} - Q_{n-1}^{\mathbb{H}m}]$$

and

$$P_n^{\mathbb{H}} = C_m \partial_t R_n^{\mathbb{H}m},$$

with

$$Q_n^{\mathbb{H}} = i^{m+1} (-1)^n \frac{(n+m)!}{n!} \frac{(x^2 + y^2 - 4iu)^n}{(x^2 + y^2 + 4iu)^{n+m+1}}$$

and

$$R_n^{\mathbb{H}} = i^m (-1)^n \frac{(m+n-1)!}{n!} \frac{((x^2 + y^2) - 4iu)^n}{((x^2 + y^2) + 4iu)^{m+n}}.$$

◇

Careful examination rends that Q_n , given by (3.10), can be roughly written as a sum of two functions Q_n^+ and Q_n^- , where each behaves like

$$\frac{n^m}{|\sigma|^{m+1}} e^{\pm in\sigma} \frac{(x^2 + x'^2 - iu)^{n/2}}{(x^2 + x'^2 + iu)^{n/2+m+1}}.$$

For R_n we get a similar expression. These calculations will be done in the next section.

3.2 Properties of the function Q_n

In the last chapter we obtained the following formulas

$$P_n(x', 0, x, u/2) = \int_0^\infty \phi_{\lambda,n}(x', 0, x, u/2) d\lambda = C [Q_n - Q_{n-2}]$$

with

$$Q_n = Q_n(x', 0, x, u) = \sum_{\ell=0}^n \frac{\Gamma(\ell + m + 1)}{\Gamma(\ell + 1)} \frac{\Gamma(n - \ell + m + 1)}{\Gamma(n - \ell + 1)} e^{-i2\ell\sigma} \\ \times e^{i\sigma} \frac{(x^2 + x'^2 - iu)^{n/2}}{(x^2 + x'^2 + iu)^{n/2+m+1}}.$$

where $m = 1/2$.

Let $\mathcal{X}^+, \mathcal{X}^- \in C_0^\infty$ with $\mathcal{X}^+(x) + \mathcal{X}^-(x) = 1$ for all $x \in]0, 1$. Furthermore $\mathcal{X}^+(x) = 0$ for $x \geq 3/4$ and $\mathcal{X}^-(x) = 0$ for $x \leq 1/4$.

$$q_{n,m}^\pm := \sum_{\ell=0}^n \mathcal{X}^\pm(\ell/n) \frac{\Gamma(\ell + m + 1)}{\Gamma(\ell + 1)} \frac{\Gamma(n - \ell + m + 1)}{\Gamma(n - \ell + 1)} e^{-i2\ell\sigma}.$$

Then

$$Q_n = (q_{n,m}^+ + q_{n,m}^-) e^{i\sigma} \frac{(x^2 + x'^2 - iu)^{n/2}}{(x^2 + x'^2 + iu)^{n/2+m+1}}. \quad (3.14)$$

By the transformation $\ell \rightarrow n - \ell$, we get easily

$$q_{n,m}^- = \sum_{\ell=0}^n \mathcal{X}^-((n - \ell)/n) \frac{\Gamma(\ell + m + 1)}{\Gamma(\ell + 1)} \frac{\Gamma(n - \ell + m + 1)}{\Gamma(n - \ell + 1)} e^{i2\ell\sigma} e^{-i2n\sigma} \\ =: \tilde{q}_n^- e^{-i2n\sigma}$$

Thus

$$Q_n = (q_{n,m}^+ e^{i\sigma} + q_{n,m}^- e^{i\sigma}) \frac{(x^2 + x_1^2 - iu)^{n/2}}{(x^2 + x_1^2 + iu)^{n/2+m+1}} \\ = (q_{n,m}^+ e^{i\sigma} + \tilde{q}_n^- e^{-i\sigma}) \frac{(x^2 + x_1^2 - iu)^{n/2}}{(x^2 + x_1^2 + iu)^{n/2+m+1}} \quad (3.15)$$

$\mathcal{X}^-((n - \cdot)/n)$ is of the same type as \mathcal{X}^+ and $q_{n,m}^+$ and \tilde{q}_n^- behave in the same way. So, we can exchange \tilde{q}_n^- in (3.15) by $q_{n,m}^+$.

Partial summation and the beta function

We define for a sequence $a := a_n$ the difference Operator Δ by

$$\Delta(a)_n = \Delta a_n := a_n - a_{n-1}.$$

The product rule reads as follows

$$\Delta(a_n b_n) = a_n b_n - a_{n-1} b_{n-1} = (a_n - a_{n-1}) b_n + a_{n-1} (b_n - b_{n-1}) \\ = \Delta(a)_n b_n + a_{n-1} \Delta(b)_n. \quad (3.16)$$

Define

$$\Gamma_\ell := \frac{\Gamma(\ell + m + 1)}{\Gamma(\ell + 1)}.$$

Recall the definition of the beta function. For $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$ the function $B(z, w)$ is defined by

$$B(z, w) := \int_0^1 t^{z-1} (1-t)^{w-1} dt.$$

and $B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$. So we have

$$\Gamma_\ell = \frac{(\ell + 1)B(\ell + 3/2, 1/2)}{\Gamma(1/2)}. \quad (3.17)$$

Lemma 3.5. *Let $k, N \in \mathbb{N}$, $z, w \in \mathbb{C}$ and $\beta \in \mathbb{N}/2$.*

(a)

$$\Delta_k^N B(z + k, w) = (-1)^N B(z + k - N, w + N)$$

holds if $\operatorname{Re} z + k - N > 0$ and $\operatorname{Re} w > 0$.

(b)

$$B(k - \beta, \beta) \leq c \Gamma(\beta) k^{-\beta}$$

holds if $k - \beta > 0$.

(c)

$$\Delta_\ell^N \Gamma_\ell \leq c_N \ell^{1/2-N}$$

holds if $\ell - N > 0$.

Proof. The first assertion follows by induction. For the main step we use the equality

$$\begin{aligned} \Delta_k B(z + k, w) &= B(z + k, w) - B(z + k - 1, w) \\ &= \int_0^1 t^{z+k-1} (1-t)^{w-1} dt - \int_0^1 t^{z+k-2} (1-t)^{w-1} dt \\ &= - \int_0^1 t^{z+k-2} (1-t) (1-t)^{w-1} dt = - \int_0^1 t^{z+k-2} (1-t)^w dt \\ &= -B(z + k - 1, w + 1). \end{aligned}$$

For the second assertion we study the case $\beta \in \mathbb{N}$ first. In this case we have

$$B(k - \beta, \beta) = \frac{\Gamma(k - \beta)\Gamma(\beta)}{\Gamma(k)} \leq \Gamma(\beta) k^{-\beta}.$$

For $\beta \in \mathbb{N} - 1/2$ we use the Wallis product formula. This formula can be derived by using the product development of the sine function and reads as follows

$$\lim_{n \rightarrow \infty} \left(\frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} \right)^2 \frac{1}{2n} = \frac{\pi}{2}.$$

Thus $\Gamma(k + 1/2)/\Gamma(k) \leq c k^{1/2}$ and

$$\begin{aligned} B(k - \beta, \beta) &= \frac{\Gamma(k - \beta)\Gamma(\beta)}{\Gamma(k)} \\ &= \Gamma(\beta) \frac{\Gamma(k - \beta)}{((k - 1) \cdot (k - 2) \cdot \dots \cdot (k - \beta - 1/2))\Gamma(k - \beta - 1/2)} \\ &\leq c \Gamma(\beta) \frac{k^{1/2}}{k^{\beta+1/2}} = c \Gamma(\beta) k^{-\beta}. \end{aligned}$$

By using (3.17) the third assertion is now an easy consequence. \square

We also need some estimates for derivatives of the beta function. Put

$$\psi := (\log \Gamma)' = \Gamma'/\Gamma.$$

This function is sometimes called the *digamma function*. The first derivative is connected with the half series

$$\zeta(x, s) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \text{ for } x > 0,$$

called the *Hurwitz zeta function*. In fact we have

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} = \zeta(x, 2)$$

and the derivatives $\psi^{(N)}(x)$ of n -th order in the point x is bounded by $c_N |x|^{-N}$.

REMARK. For $x = 1$, the Hurwitz zeta function is the well known *Riemannian zeta function* $\zeta(s)$. \diamond

By these facts we obtain the following lemma.

Lemma 3.6. For all $x \gtrsim 1$, $w > 0$

$$\partial_x^N B(x, w) \leq C_w |x|^{-w-N}$$

holds.

Proof. First we want to study the case $N = 1$. Observe that

$$\begin{aligned}\partial_x B(x, w) &= \partial_x \frac{\Gamma(x)\Gamma(w)}{\Gamma(x+w)} = \frac{\Gamma'(x)}{\Gamma(x+w)}\Gamma(w) - \frac{\Gamma'(x+w)}{\Gamma(x+w)}\frac{\Gamma(x)\Gamma(w)}{\Gamma(x+w)} \\ &= [\psi(x) - \psi(x+w)]B(x, w).\end{aligned}$$

The estimate follows by the mean value theorem. The higher cases $N > 1$ follow in the same way by using the product formula (3.16). \square

By these observations it is obvious to regard the functions Γ_ℓ and $\Gamma_{n-\ell}$ as “symbols” of order $1/2$ with respect to the operators Δ and ∂ . By the previous lemmata, one gets

$$|\partial_\ell^{N_1} \Delta_\ell^{N_2} \Gamma_\ell| \leq c_{N_1, N_2} \ell^{1/2 - N_1 - N_2} \quad (3.18)$$

for all $N_2 \leq \ell$. In fact

$$\begin{aligned}\partial_\ell^{N_1} \Delta_\ell^{N_2} \Gamma_\ell &= \frac{1}{\Gamma(1/2)} \partial_\ell^{N_1} \Delta_\ell^{N_2} (\ell + 1) B(\ell + 3/2, 1/2) \\ &= \frac{1}{\Gamma(1/2)} \partial_\ell^{N_1} (\Delta_\ell^{N_2-1} B(\ell + 3/2, 1/2) + \ell \Delta_\ell^{N_2} B(\ell + 3/2, 1/2)) \\ &= \frac{1}{\Gamma(1/2)} \partial_\ell^{N_1} ((-1)^{N_2-1} B(\ell + 5/2 - N_2, -1/2 + N_2) \\ &\quad + (-1)^{N_2} \ell B(\ell + 3/2 - N_2, 1/2 + N_2))\end{aligned}$$

and the estimate (3.18) follows. Similarly

$$|\partial_\ell^{N_1} \Delta_\ell^{N_2} \Delta_n^{N_3} \Gamma_{n-\ell}| \leq c_{N_1, N_2, N_3} n^{1/2 - N_1 - N_2 - N_3} \quad (3.19)$$

for all $N_2 + N_3 \leq n/4$ with $\ell \leq 3n/4$. The function $(\ell, n) \mapsto \mathcal{X}(\ell/n)$ behaves even better. An easy calculation shows that

$$|\partial_\ell^{N_1} \Delta_n^{N_2} \Delta_\ell^{N_3} \mathcal{X}(\ell/n)| \leq c_{N_1, N_2, N_3} n^{-(N_1 + N_2 + N_3)} \quad (3.20)$$

holds, for all $\ell \leq 3n/4$. In fact

$$\Delta_\ell \mathcal{X}(\ell/n) = \mathcal{X}(\ell/n) - \mathcal{X}((\ell-1)/n) = n^{-1} \tilde{\mathcal{X}}_n(\ell)$$

with $\tilde{\mathcal{X}}_n := n(\mathcal{X}(x/n) - \mathcal{X}((x-1)/n))$ and $\tilde{\mathcal{X}}_n$ is of the same type than \mathcal{X}_n . Furthermore,

$$\Delta_n \mathcal{X}(\ell/n) = \mathcal{X}(\ell(n-1)/(n^2-n)) - \mathcal{X}(\ell n/(n^2-n)) = n^{-1} \tilde{\tilde{\mathcal{X}}}_n(\ell)$$

with $\tilde{\tilde{\mathcal{X}}}_n := n(\mathcal{X}(x/n) - \mathcal{X}(x/(n-1)))$ and $\tilde{\tilde{\mathcal{X}}}_n$ is of the same type than \mathcal{X}_n .

For the estimates to come we also need “half derivatives”. In some sense, we get the next lemma by applying the operator “ $\Delta^{1/2}$ ”. We do our calculation in the Fourier space, where we estimate integrals. Finally, we use the Poisson summation formula.

Lemma 3.7. *Let $\beta \in \mathbb{R}$ and $c > 1$. Let $\mathcal{X}_n \in C_0^\infty$ with $\mathcal{X}_n(x) = 0$ for $x \leq 1/2$ or $x \geq \lfloor n/c \rfloor + 1$ and $\mathcal{X}_n(x) = 1$ for $1 \leq x \leq \lfloor n/c \rfloor$. For all σ with $|1 - e^{i\sigma}| \leq 1/2$*

$$\left| \sum_{\ell=1}^{\lfloor n/c \rfloor} \ell^{-1/2} (n - \ell)^\beta e^{i\sigma\ell} \right| = \left| \sum \mathcal{X}_n(\ell) \ell^{-1/2} (n - \ell)^\beta e^{i\sigma\ell} \right| \lesssim \frac{n^\beta}{|1 - e^{i\sigma}|^{1/2}}$$

holds.

Proof. Since our function is periodic we can assume that $0 \leq \sigma \leq 1/2$ holds. Put

$$\begin{aligned} I_n(\sigma - x) &= I := \int \mathcal{X}_n(t) t^{-1/2} (n - t)^\beta e^{i(\sigma - x)t} dt \\ &= n^{1/2 + \beta} \int \mathcal{X}_n(nt) t^{-1/2} (1 - t)^\beta e^{in(\sigma - x)t} dt. \end{aligned}$$

Then we have

$$\sum \mathcal{X}_n(\ell) \ell^{-1/2} (n - \ell)^\beta e^{i\sigma\ell} = \sum_{x \in \mathbb{Z}} I_n(\sigma - x).$$

Put $y := n(\sigma - x)$. For $|y| \leq C$ we have $|I_n| \lesssim n^{1/2 + \beta}$.

For $|y| \geq C$ we get easily by partial integration and since $\mathcal{X}_n(n \cdot)'$ is zero outside a set of measure $\sim 1/n$.

$$\begin{aligned} |I| &\lesssim \frac{n}{|y|^N} \int_0^1 |\partial_t^N [\mathcal{X}_n(nt) t^{-1/2} (1 - t)^\beta]| dt \\ &= \frac{n}{|y|^N} \sum_{M=0}^N c_M \int_0^1 |\partial_t^M [\mathcal{X}_n(nt)] [\partial_t^{N-M} t^{-1/2} (1 - t)^\beta]| dt \\ &\lesssim \frac{n}{|y|^N} \sum_{M=1}^N c_M \int_0^1 |\partial_t^M [\mathcal{X}_n(nt)] [\partial_t^{N-M} t^{-1/2} (1 - t)^\beta]| dt + c_0 \frac{n}{|y|^N} (1/n)^{1/2 - N} \\ &\lesssim \frac{n}{|y|^N} n^M 1/n (1/n)^{-1/2 - N + M} + n^{1/2} \left(\frac{n}{|y|}\right)^N = n^{1/2} \left(\frac{n}{|y|}\right)^N \\ &= n^{1/2} |\sigma - x|^{-N}. \end{aligned}$$

For $|y| \geq C$ we also have a second estimate. Observe that

$$\begin{aligned} I &= \frac{n}{y^{1/2}} \int_0^y \Psi_n(t/y) t^{-1/2} (1 - t/y)^\beta e^{it} dt = \frac{n}{y^{1/2}} \int_0^1 \dots dt + \frac{n}{y^{1/2}} \int_1^y \dots dt \\ &=: \frac{n}{y^{1/2}} (I_1 + I_2), \end{aligned}$$

with $\Psi_n(\cdot) = \mathcal{X}_n(n \cdot)$, holds. The first Integral can be estimated by a constant. For the second integral we get by integration by parts

$$|I_2| \lesssim |\Psi_n(1/y)| + \left| \int_1^y (\partial_t \Psi_n(t/y)) t^{-1/2} (1-t/y)^\beta e^{it} dt \right| \\ + \left| \int_1^y \Psi_n(t/y) (\partial_t t^{1/2} (1-t/y)^\beta) e^{it} dt \right|.$$

But this is bounded by a constant, since Ψ_n is a bounded function, the support of $\partial_t \Psi_n(\cdot/y)$ has measure $\sim y/n$, $\partial_t \Psi_n(t/y) \lesssim n/y$, $\partial_t t^{1/2}$ is integrable and $\partial_t (1-t/y)^\beta \lesssim 1/y$. For $x = 0$, we get the following estimates

$$|I_n(\sigma - x)| \lesssim n^{1/2+\beta} \lesssim \frac{n^\beta}{\sigma^{1/2}}, \quad \text{if } n\sigma \leq C \text{ and} \\ |I_n(\sigma - x)| \lesssim \frac{n^{1/2+\beta}}{y^{1/2}} = \frac{n^\beta}{\sigma^{1/2}}, \quad \text{if } n\sigma \geq C.$$

Hence $|I_n(\sigma)| \lesssim n^\beta \sigma^{-1/2}$. For $|x| \geq 1$ we have $y \geq C$ and

$$\sum_{|x| \geq 1} |I_n(\sigma - x)| \lesssim n^\beta \sum_{|x| \geq 1} |x|^{-2} \lesssim n^\beta \lesssim n^\beta \sigma^{-1/2}.$$

The lemma follows by the Poisson summation formula. \square

Corollary 3.8.

$$\sum_{\ell=1}^{n-1} \Delta_n[\mathcal{X}((\ell-1)/n) \Gamma_{n-\ell+1}] \Delta_\ell \Gamma_\ell e^{-i2\ell\sigma} e^{i\omega n} \lesssim \frac{n^{-1/2}}{|1 - e^{-i2\sigma}|^{1/2}} \quad (3.21)$$

holds for all σ with $|1 - e^{i\sigma}| \leq 1/2$.

Proof. (3.19) and (3.20) together with the product formula (3.16) imply that the function $(n, \ell) \mapsto \Delta_n[\mathcal{X}((\ell-1)/n) \Gamma_{n-\ell+1}]$ behaves like $(n-\ell)^{-1/2}$. (3.18) implies that the function $\ell \mapsto \Delta_\ell \Gamma_\ell$ behaves like $\ell^{-1/2}$. Furthermore, $\mathcal{X}((\ell-1)/n)$ is supported in $[0, \lfloor n/c \rfloor]$ with a constant $c > 1$. Hence this corollary follows by the proof of Lemma 3.7 with slightly modifications. \square

Proposition 3.9. (a) For every ϵ there exists a constant C_ϵ such that

$$\left| \sum_{n \sim 2^j} \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\Gamma(\ell+m+1)}{\Gamma(\ell+1)} \frac{\Gamma(n-\ell+m+1)}{\Gamma(n-\ell+1)} e^{-i2\ell\sigma} e^{i\omega n} \right| \\ \leq C_\epsilon \frac{2^{j/2+j\epsilon}}{|1 - e^{-i2\sigma}|^{3/2}} \frac{2^j}{(1 + 2^j |1 - e^{i\omega}|)^{1+\epsilon}}, \quad \text{for all } \sigma \text{ and } \omega. \quad (3.22)$$

(b) For every ϵ there exists a constant C_ϵ such that

$$\begin{aligned} & \left| \sum_{n \sim 2^j} \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\ell \Gamma(\ell+m+1) \Gamma(n-\ell+m+1)}{\Gamma(\ell+1) \Gamma(n-\ell+1)} e^{-i2\ell\sigma} e^{i\omega n} \right| \\ & \leq C_\epsilon \frac{2^{j/2+j\epsilon}}{|1-e^{-i2\sigma}|^{5/2}} \frac{2^j}{(1+2^j|1-e^{i\omega}|)^{1+\epsilon}}, \text{ for all } \sigma \text{ and } \omega. \end{aligned} \quad (3.23)$$

The proof is mainly based on the fact that Γ_ℓ behaves like $\ell^{1/2}$ and that $\mathcal{X}(\ell/n) \Gamma_{n-\ell}$ behaves like $n^{1/2}$, since $\ell \leq 3n/4$. Observe that the double sum in (3.22) is bounded by 2^{3j} . By partial summation with respect to n we win $2^{-j}/|1-e^{i\omega}|$. By partial summation with respect to ℓ we get roughly two terms $\ell^{-1/2} (n-\ell)^{1/2}$ and $\ell^{1/2} (n-\ell)^{-1/2}$. We win $2^{-j}/|1-e^{-i2\sigma}|$. Hence the double sum is bounded by $2^{2j}/|1-e^{-i2\sigma}|$. To get an additional factor $2^{-j/2}/|1-e^{-i2\sigma}|^{1/2}$ we use Corollary 3.8. Hence, together with the partial summation in n , we get our result.

Just one more partial summation in ℓ would not give us the desired result, since we would have to sum over $\ell^{-3/2} (n-\ell)^{1/2}$ which gives, after summing all up, $2^{3/2j}/|1-e^{-i2\sigma}|^2$ by interpolating this with our previous result we would get $2^{3/2j+1/4j}/|1-e^{-i2\sigma}|^{3/2}$, which is not sufficient.

Proof. For $j \leq 10$ the estimate (3.22) is obvious. So we can restrict to $j \geq 10$. Now $n \geq c2^{10}$. We assume that $\omega \in [-\pi, \pi]$, then $|1-e^{i\omega}| \sim |\omega|$. Furthermore, we assume that $\sigma \in [-\pi/2, \pi/2]$, then $|1-e^{-i2\sigma}| \sim |\sigma|$.

Since

$$\sum_{\ell=\ell_0}^{\ell_1} (a_\ell - a_{\ell-1}) e^{i\omega\ell} = \sum_{\ell=\ell_0}^{\ell_1-1} a_\ell e^{i\omega\ell} (1 - e^{i\omega}) + a_{\ell_1} e^{i\omega\ell_1} - a_{\ell_0-1} e^{i\omega\ell_0}$$

for an arbitrary sequence a_ℓ holds, we get

$$\begin{aligned} & \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\Gamma(\ell+m+1) \Gamma(n-\ell+m+1)}{\Gamma(\ell+1) \Gamma(n-\ell+1)} e^{-i2\ell\sigma} \\ & = \sum_{\ell=0}^1 \mathcal{X}(\ell/n) \Gamma_\ell \Gamma_{n-\ell} e^{-i2\ell\sigma} + \sum_{\ell=2}^n \mathcal{X}(\ell/n) \Gamma_\ell \Gamma_{n-\ell} e^{-i2\ell\sigma} \\ & = \sum_{\ell=0}^1 \mathcal{X}(\ell/n) \Gamma_\ell \Gamma_{n-\ell} e^{-i2\ell\sigma} + \Gamma_1 \Gamma_{n-1} e^{-i2\sigma} (1 - e^{-i2\sigma})^{-1} \\ & \quad + \sum_{\ell=1}^n \Delta[\mathcal{X}(\ell/n) \Gamma_\ell \Gamma_{n-\ell}] e^{-i2\ell\sigma} (1 - e^{-i2\sigma})^{-1} =: q_n^1 + q_n^2 + q_n^3, \end{aligned}$$

since $\mathcal{X}(x) = 0$ for $x \geq 1$. By partial summation with respect to n the function $\sum_{n \sim 2^j} q_n^1 e^{i\omega n}$ can be written as a sum of two terms of the form

$$(1 - e^{i\omega})^{-1} \sum_{n \sim 2^j} \sum_{\ell=0}^1 \Delta_n[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}] e^{-i2\ell\sigma} e^{i\omega n}$$

and

$$(1 - e^{i\omega})^{-1} \sum_{\ell=0}^1 \mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell} e^{-i2\ell\sigma} e^{i\omega n}|_{n \sim 2^j}.$$

By the symbol properties (3.19)–(3.20) of the functions $\Gamma_{n-\ell}$ and $\mathcal{X}(\ell/n)$ both terms are bounded by $2^{j/2}|\omega|^{-1}$.

By the same arguments we obtain that $\sum_{n \sim 2^j} q_n^2 e^{i\omega n}$ is bounded by $2^{j/2}|\sigma|^{-1}|\omega|^{-1}$. We are left with the sum over q_n^3 . Once more we use partial summation with respect to n . Observe that

$$\sum_{n \sim 2^j} q_n^3 e^{i\omega n} = (1 - e^{-i2\sigma})^{-1} \sum_{n \sim 2^j} \sum_{\ell=1}^n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}] e^{-i2\ell\sigma} e^{i\omega n}$$

can be written as

$$\begin{aligned} & ((1 - e^{i\omega})(1 - e^{-i2\sigma}))^{-1} \sum_{n \sim 2^j} \Delta_n \left[\sum_{\ell=1}^n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}] e^{-i2\ell\sigma} \right] e^{i\omega n} \\ &= ((1 - e^{i\omega})(1 - e^{-i2\sigma}))^{-1} \left(\sum_{n \sim 2^j} \sum_{\ell=1}^{n-1} \Delta_n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}] e^{-i2\ell\sigma} e^{i\omega n} \right. \\ & \quad \left. + \sum_{n \sim 2^j} \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}]|_{\ell=n} e^{-i2n\sigma} e^{i\omega n} \right), \end{aligned} \quad (A)$$

together with boundary terms of the form

$$(1 - e^{i\omega})^{-1} \sum_{\ell=1}^n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}] e^{-i2\ell\sigma} (1 - e^{-i2\sigma})^{-1} e^{i\omega n}|_{n \sim 2^j}. \quad (B)$$

Since $\mathcal{X}((n-1)/n) = 0 = \Delta\mathcal{X}(1)$ we just have to study the first summand in (A). Using (3.18)–(3.20) and the product rule it is easy to see that (A) is bounded by $c 2^j|\sigma|^{-1}|\omega|^{-1}$, where c is a constant independent of σ , ω and j . Unfortunately we need a slightly better estimate. Observe that

$$\begin{aligned} \Delta_n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell\Gamma_{n-\ell}] &= \Delta_n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] \Gamma_\ell \\ & \quad + \Delta_n[\mathcal{X}((\ell-1)/n)\Gamma_{n-\ell+1}] \Delta_\ell \Gamma_\ell \end{aligned}$$

Applying one more time the summation operator Δ_ℓ to the first term and summing over all ℓ gives us

$$\sum_{\ell=1}^n \Delta_\ell[\Delta_n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] \Gamma_\ell] \lesssim n^{-1}$$

Thus, we get

$$((1 - e^{i\omega})(1 - e^{-i2\sigma}))^{-1} \sum_{n \sim 2^j} \sum_{\ell=1}^{n-1} \Delta_n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] \Gamma_\ell e^{-i2\ell\sigma} e^{i\omega n} \lesssim \frac{1}{|\sigma|^2 |\omega|}.$$

By interpolating this result with our previous one, we get the bound $2^{j/2} |\sigma|^{-3/2} |\omega|^{-1}$. For the second term, involving $\Delta_n[\mathcal{X}((\ell-1)/n)\Gamma_{n-\ell+1}] \Delta_\ell \Gamma_\ell$, we need a more refined method than just partial summation. For $|1 - e^{-i2\sigma}| \geq 1/2$ we use partial summation with respect to ℓ as we did for the first term. We get the estimate $2^{j/2} |\sigma|^{-2} |\omega|^{-1}$. This is weaker than it should be, but since $|1 - e^{-i2\sigma}| \geq 1/2$ this is bounded by $2^{j/2} |\sigma|^{-3/2} |\omega|^{-1}$. For $|1 - e^{-i2\sigma}| \leq 1/2$ we use Corollary 3.8 and get the expected bound directly. In fact

$$((1 - e^{i\omega})(1 - e^{-i2\sigma}))^{-1} \sum_{n \sim 2^j} \sum_{\ell=1}^{n-1} \Delta_n[\mathcal{X}((\ell-1)/n)\Gamma_{n-\ell+1}] \Delta_\ell \Gamma_\ell e^{-i2\ell\sigma} e^{i\omega n} \lesssim \frac{2^{j/2}}{|\sigma|^{3/2} |\omega|}.$$

For (B) all the calculations are very similar. The first estimate we get is that (B) is bounded by $c 2^j |\sigma|^{-1} |\omega|^{-1}$. We write

$$\begin{aligned} & \sum_{\ell=1}^n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_\ell \Gamma_{n-\ell}] e^{-i2\ell\sigma} \\ &= \sum_{\ell=1}^n ((\Delta \mathcal{X}(\ell/n))\Gamma_\ell \Gamma_{n-\ell} + \mathcal{X}((\ell-1)/n)(\Delta \Gamma_{n-\ell})\Gamma_\ell) e^{-i2\ell\sigma} \\ &+ \sum_{\ell=1}^n \mathcal{X}((\ell-1)/n)\Gamma_{n-\ell+1}(\Delta \Gamma_\ell) e^{-i2\ell\sigma}. \end{aligned}$$

For the first sum we use one more time partial summation which gives us an additional factor $(n|\sigma|)^{-1}$. Now by interpolation we get

$$((1 - e^{i\omega})(1 - e^{-i2\sigma}))^{-1} \sum_{\ell=1}^n \Delta_\ell[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] \Gamma_\ell e^{-i2\ell\sigma} e^{i\omega n} \Big|_{n \sim 2^j} \lesssim \frac{2^{j/2}}{|\sigma|^{3/2} |\omega|}.$$

As before we use Corollary 3.8 for the second sum and obtain

$$((1 - e^{i\omega})(1 - e^{-i2\sigma}))^{-1} \sum_{\ell=1}^n \mathcal{X}((\ell-1)/n)\Gamma_{n-\ell+1} (\Delta \Gamma_\ell) e^{-i2\ell\sigma} e^{i\omega n} \Big|_{n \sim 2^j} \lesssim \frac{2^{j/2}}{|\sigma|^{3/2} |\omega|}.$$

Of course, if we omit the partial summation in n and do similar calculations, we also get an estimate of the form

$$\left| \sum_{n \sim 2^j} \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\Gamma(\ell+m+1)}{\Gamma(\ell+1)} \frac{\Gamma(n-\ell+m+1)}{\Gamma(n-\ell+1)} e^{-i2\ell\sigma} e^{i\omega n} \right| \leq C \frac{2^{j/2}}{|\sigma|^{3/2}} 2^j.$$

Observe that $\frac{2^{j/2}}{|\sigma|^{3/2}|\omega|} \leq \frac{2^{j/2}}{|\sigma|^{3/2}|\omega|^{1+\epsilon}}$ and

$$\min\left\{ \frac{2^{j/2}}{|\sigma|^{3/2}|\omega|^{1+\epsilon}}, \frac{2^{j/2}}{|\sigma|^{3/2}} 2^j \right\} \lesssim \frac{2^{j/2+j\epsilon}}{|1-e^{-i2\sigma}|^{3/2}} \frac{2^j}{(1+2^j|1-e^{i\omega}|)^{1+\epsilon}}$$

which completes the proof (a).

(b) we get in a similar way. We write

$$\begin{aligned} \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\ell \Gamma(\ell+m+1)}{\Gamma(\ell+1)} \frac{\Gamma(n-\ell+m+1)}{\Gamma(n-\ell+1)} e^{-i2\ell\sigma} \\ = \sum_{\ell=0}^2 \mathcal{X}(\ell/n) \ell \Gamma_\ell \Gamma_{n-\ell} e^{-i2\ell\sigma} + 2 \Gamma_2 \Gamma_{n-2} e^{-i2\sigma} (1-e^{-i2\sigma})^{-1} \\ + \sum_{\ell=2}^n \Delta[\mathcal{X}(\ell/n) \ell \Gamma_\ell \Gamma_{n-\ell}] e^{-i2\ell\sigma} (1-e^{-i2\sigma})^{-1} =: \tilde{q}_n^1 + \tilde{q}_n^2 + \tilde{q}_n^3. \end{aligned}$$

For \tilde{q}_n^1 and \tilde{q}_n^2 we get with the same methods as in (a)

$$\sum_{n \sim 2^j} \tilde{q}_n^1 e^{i\omega n}, \sum_{n \sim 2^j} \tilde{q}_n^2 e^{i\omega n} \lesssim \frac{2^{j/2}}{|\sigma||\omega|}.$$

For \tilde{q}_n^3 we have

$$\begin{aligned} \sum_{\ell=2}^n \Delta[\mathcal{X}(\ell/n) \ell \Gamma_\ell \Gamma_{n-\ell}] e^{-i2\ell\sigma} (1-e^{-i2\sigma})^{-1} \\ = \sum_{\ell=2}^n \Delta[\ell \Gamma_\ell] \mathcal{X}(\ell/n) \Gamma_{n-\ell} e^{-i2\ell\sigma} (1-e^{-i2\sigma})^{-1} \\ + \sum_{\ell=2}^n (\ell-1) \Gamma_{\ell-1} \Delta[\mathcal{X}(\ell/n) \Gamma_{n-\ell}] e^{-i2\ell\sigma} (1-e^{-i2\sigma})^{-1}. \end{aligned}$$

Since $\Delta[\ell \Gamma_\ell]$ behaves like Γ_ℓ we get by the proof of (a)

$$(1-e^{-i2\sigma})^{-1} \sum_{n \sim 2^j} \sum_{\ell=2}^n \Delta[\ell \Gamma_\ell] \mathcal{X}(\ell/n) \Gamma_{n-\ell} e^{-i2\ell\sigma} (1-e^{-i2\sigma})^{-1} e^{i\omega n} \lesssim \frac{2^{j/2}}{|\sigma|^{5/2}|\omega|}.$$

For the second sum we use one time more partial summation in ℓ and get two terms, that behave like

$$(1 - e^{-i2\sigma})^{-2} \sum_{\ell} \Gamma_{\ell} \Delta[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] e^{-i2\ell\sigma} \quad (\text{A})$$

and

$$(1 - e^{-i2\sigma})^{-2} \sum_{\ell} \ell \Gamma_{\ell} \Delta^2[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] e^{-i2\ell\sigma} \quad (\text{B})$$

By partial summation with respect to n we get

$$\left| \sum_{n \sim 2^j} \sum_{\ell} \Gamma_{\ell} \Delta[\mathcal{X}(\ell/n)\Gamma_{n-\ell}] e^{-i2\ell\sigma} e^{i\omega n} \right| \lesssim 2^j |\omega|^{-1}.$$

Now, one more partial summation in ℓ yields that this sum is bounded by $|\sigma|^{-1} |\omega|^{-1}$. Interpolating these two results gives us our expected result. For (B) we get the same results. We omit the details. \square

For $q_{n,m-1}^+$ we get similar results.

Proposition 3.10. (a) For every ϵ there exists a constant C_{ϵ} such that

$$\begin{aligned} & \left| \sum_{n \sim 2^j} \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\Gamma(\ell+m)\Gamma(n-\ell+m)}{\Gamma(\ell)\Gamma(n-\ell)} e^{-i2\ell\sigma} e^{i\omega n} \right| \\ & \leq C_{\epsilon} \frac{2^{-j/2+j\epsilon}}{|1 - e^{-i2\sigma}|^{1/2}} \frac{2^j}{(1 + 2^j |1 - e^{i\omega}|)^{1+\epsilon}}, \text{ for all } \sigma \text{ and } \omega. \end{aligned} \quad (3.24)$$

(b) For every ϵ there exists a constant C_{ϵ} such that

$$\begin{aligned} & \left| \sum_{n \sim 2^j} \sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\ell \Gamma(\ell+m)\Gamma(n-\ell+m)}{\Gamma(\ell)\Gamma(n-\ell)} e^{-i2\ell\sigma} e^{i\omega n} \right| \\ & \leq C_{\epsilon} \frac{2^{-j/2+j\epsilon}}{|1 - e^{-i2\sigma}|^{3/2}} \frac{2^j}{(1 + 2^j |1 - e^{i\omega}|)^{1+\epsilon}}, \text{ for all } \sigma \text{ and } \omega. \end{aligned} \quad (3.25)$$

Proof. The proof is similar to the proof of the previous proposition. Hence we omit it. \square

4 Reductions

4.1 Interpolation, reduction to $p = 1$

Define the analytic family of operators $T_\alpha := e^{i\sqrt{G}}/(1+G)^{\alpha/2}$. If we assume now that the case $p = 1$ is true, we have

$$\begin{aligned} \|T_\alpha f\|_2 &\leq C_\alpha \|f\|_2, & \text{if } \operatorname{Re} \alpha = 0, \\ \|T_\alpha f\|_1 &\leq C_\alpha \|f\|_1, & \text{if } \operatorname{Re} \alpha > 1/2. \end{aligned}$$

Müller and Stein showed that the convolution kernel of the operator $(1+L)^{-\epsilon+i\gamma}$ has bounded L_1 -norm, for $\gamma \in \mathbb{R}$ and $\epsilon > 0$. The L_1 -norms grow polynomially in γ . By transference, Corollary 1.2, the operators $(1+G)^{-\epsilon+i\gamma}$ have integral kernels with bounded Schur norms and the Schur norms grow polynomially in γ . Hence we can use the analytic interpolation theorem in [26] and a standard duality argument to deduce the theorem, for arbitrary $1 \leq p \leq \infty$.

4.2 Reduction to an estimate for the local part of the kernel, part 1

From now on, we take α to be always bigger than $1/2$, let

$$\alpha > \frac{1}{2}.$$

Let η be an even $C_0^\infty(\mathbb{R})$ function, such that $\eta(\xi) = 1$ for small ξ , and $\eta(\xi) = 0$, if $|\xi| \geq 1$. For some large constant $N > 1$, to be chosen later, put $\eta_N(\xi) := \eta(\xi/N)$. Define

$$h_t^\alpha(\xi) := (1 - \eta_N)(\xi) \xi^{-\alpha/2} e^{it\sqrt{\xi}}, \text{ for all } \xi \in \mathbb{R}^+. \quad (4.1)$$

and $h^\alpha := h_1^\alpha$. Then $h_t^\alpha(\xi) = 0$ for all $\xi < N$. Furthermore, we define

$$B_r(x', u') := B(x', \mathcal{C}_0 r) \times B(u', \mathcal{C}_0 r(r + |x'|)).$$

Proposition 4.1. *Put*

$$\mathcal{X}_B(x', u', x, u) := \begin{cases} 1, & \text{if } (x, u) \in B_2(x', u') \\ 0, & \text{otherwise.} \end{cases}$$

Then the integral kernel $(1 - \mathcal{X}_B)K_{h^\alpha(G)}$ has bounded Schur norm. Furthermore, to prove the theorem, it suffices to show that $\mathcal{X}_B K_{h^\alpha(G)}$ has bounded Schur norm.

Proof. Define

$$\begin{aligned} f_{\alpha,t}(\xi) &:= (1 - \eta_N)(\xi) |\xi|^{-\alpha/2} \cos(t\sqrt{\xi}), \\ g_{\alpha,t}(\xi) &:= (1 - \eta_N)(\xi) |\xi|^{-\alpha/2} \sin(t\sqrt{\xi}) \end{aligned}$$

Then $h^\alpha(\xi) = f_{\alpha,1}(\xi) + ig_{\alpha,1}(\xi)$. Let

$$\tilde{g}_{\alpha,t}(\xi) := (1 - \eta_N)(\xi) |\xi|^{-\alpha/2} \sin(t\sqrt{|\xi|}) \operatorname{sgn}(\xi).$$

Then $\tilde{g}_{\alpha,t}$ is an odd function and

$$\begin{aligned} h^\alpha(\xi) &= f_{\alpha,1}(\xi) + i\tilde{g}_{\alpha,1}(\xi), \quad \text{for all } \xi \geq 0, \\ h^\alpha(G) &= f_{\alpha,1}(G) + i\tilde{g}_{\alpha,1}(G). \end{aligned}$$

It is easily seen that for $\xi > 0$

$$(1 - \eta_N)(\xi) |\xi|^{-\alpha/2} = \int \varphi(\tau) \cos(\tau\sqrt{\xi}) d\tau,$$

where φ is such that $\eta\varphi \in L_1$ and $(1 - \eta)\varphi \in \mathcal{S}$. Hence

$$f_{\alpha,t}(\xi) = \int (\eta\varphi)(\tau) \cos(\tau\sqrt{\xi}) \cos(t\sqrt{\xi}) d\tau + \Phi(\sqrt{\xi}) \cos(t\sqrt{\xi}),$$

with $\Phi \in \mathcal{S}$. By Proposition 1.8 the support of the distribution

$$\int (\eta\varphi)(\tau) \cos(\tau\sqrt{G}) \cos(t\sqrt{G}) d\tau$$

lies in the support of \mathcal{X}_B for all $t \leq 1$. Define

$$\psi(\xi) := \Phi(\sqrt{\xi}) \cos(t\sqrt{\xi}) - \Phi(0)e^{-\xi}.$$

Then

$$\begin{aligned} |\xi^\ell \partial_\xi^\ell \psi(\xi)| &\lesssim \xi^{1/2} \text{ for all } 0 < \xi \leq 1, \\ |\xi^\ell \partial_\xi^\ell \psi(\xi)| &\lesssim \xi^{-1/2} \text{ for all } 1 \leq \xi < \infty. \end{aligned}$$

And thus the integral kernel of $\Phi(\sqrt{G}) \cos(t\sqrt{G})$ has bounded Schur norm by Proposition 1.3 and the fact that e^{-G} is the heat-kernel, which has bounded Schur norm. Thus, if we denote the integral kernel of $f_{\alpha,t}(G)$ by K_f , then $(1 - \mathcal{X}_B) K_f$ has bounded Schur norm.

Since $\tilde{g}_{\alpha,t}$ is an odd function we get by the Fourier transform and the summation formula for the cos function

$$\begin{aligned} \tilde{g}_{\alpha,t} &= \int \varphi(\tau) \sin(\tau\sqrt{\xi}) \sin(t\sqrt{\xi}) d\tau \\ &= \int \varphi(\tau) \cos(\tau\sqrt{\xi}) \cos(t\sqrt{\xi}) d\tau - \int \varphi(\tau) \cos((\tau + t)\sqrt{\xi}) d\tau \end{aligned}$$

with some appropriate function φ . This can be written as

$$\begin{aligned} & \int (\eta\varphi)(\tau) \cos(\tau\sqrt{\xi}) \cos(t\sqrt{\xi}) d\tau + \Phi(\sqrt{\xi}) \cos(t\sqrt{\xi}) \\ & - \int (\eta\varphi)(\tau) \cos((\tau+t)\sqrt{\xi}) d\tau - \int ((1-\eta)\varphi)(\tau) \cos((\tau+t)\sqrt{\xi}) d\tau, \end{aligned}$$

with a smooth function $\Phi \in \mathcal{S}$. As for $f_{\alpha,t}$, we know that for all $\tau \leq 1$ the supports of the distributions

$$\begin{aligned} & \int (\eta\varphi)(\tau) \cos(\tau\sqrt{G}) \cos(t\sqrt{G}) d\tau \text{ and} \\ & \int (\eta\varphi)(\tau) \cos((\tau+t)\sqrt{G}) d\tau \end{aligned}$$

lie in the support of \mathcal{X}_B . Furthermore, the kernel of $\Phi(\sqrt{G}) \cos(t\sqrt{G})$ has bounded Schur norm for all $|t| \leq 1$, as we have seen before.

We are left with the operator $\int ((1-\eta)\varphi)(\tau) \cos((\tau+t)\sqrt{G}) d\tau$ which can be written as

$$\begin{aligned} & \int ((1-\eta)\varphi)(\tau) [\cos(\tau\sqrt{G}) \cos(t\sqrt{G}) - \sin(\tau\sqrt{G}) \sin(t\sqrt{G})] d\tau = \\ & \Phi_1(\sqrt{G}) \cos(t\sqrt{G}) + \Phi_2(\sqrt{G}) \sin(t\sqrt{G}) \end{aligned}$$

with appropriate functions Φ_1 and Φ_2 supported away from the origin. Once more we can show that the functions

$$\Phi_1(\sqrt{\xi}) \cos(t\sqrt{\xi}) - \Phi_1(0)e^{-\xi}, \quad \Phi_2(\sqrt{\xi}) \sin(t\sqrt{\xi}) - \Phi_2(0)e^{-\xi}$$

fulfill the conditions of Proposition 1.3 and thus the integral kernels of the operators $\Phi_1(\sqrt{G}) \cos(t\sqrt{G})$ and $\Phi_2(\sqrt{G}) \sin(t\sqrt{G})$ have bounded Schur norms, since the heat-kernel has bounded Schur norm. Thus, if we denote the integral kernel of $g_{\alpha,t}(G)$ by K_g , then $(1 - \mathcal{X}_B) K_g$ has bounded Schur norm. Hence, if we know that $\mathcal{X}_B K_{h^\alpha(G)}$ has bounded Schur norm, we could conclude that $K_{h^\alpha(G)}$ has bounded Schur norm.

The last thing we have to prove is that if the kernel of $h^\alpha(G)$ has bounded Schur norm, this is also true for the kernel of $(1+G)^{-\alpha/2} e^{i\sqrt{G}}$. We write

$$(1+\xi)^{-\alpha/2} e^{i\sqrt{\xi}} = \eta_N(\xi)(1+\xi)^{-\alpha/2} e^{i\sqrt{\xi}} + \xi^{\alpha/2}(1+\xi)^{-\alpha/2}(1-\eta_N)(\xi)\xi^{-\alpha/2} e^{i\sqrt{\xi}}.$$

The functions $\eta_N(\xi)(1+\xi)^{-\alpha/2} e^{i\sqrt{\xi}} - e^{i\xi}$, $\xi^{\alpha/2}(1+\xi)^{-\alpha/2} - 1 + e^{-\xi}$ satisfy the hypothesis of Proposition 1.3 and thus the operators $\eta_N(G)(1+G)^{-\alpha/2} e^{i\sqrt{G}}$ and $G^{\alpha/2}(1+G)^{\alpha/2}$ have bounded Schur norms. Since $(1-\eta_N)(G)G^{-\alpha/2} e^{i\sqrt{G}} = h^\alpha(G)$ the proposition has been proven. \square

A further reduction allows us to exchange the indicator function \mathcal{X}_B by a smooth variant, this will be shown in part 2 of this section.

4.3 The case of large x'

Before we go on with the proof of the theorem, we want to mention how one can get uniform estimates for $\|\mathcal{X}_B K_{h^\alpha(G)}(x', 0, \cdot, \cdot)\|_{L_1}$ for $x' \gg 1$. Since we have not done all calculations in this section in detail, we formulate the key estimate as a conjecture and show how this conjecture implies the result for $x' \gg 1$.

Let us define operators G_ϵ for $\epsilon \leq 1/4$ in the following way

$$G_\epsilon := -\partial_x^2 - (1 + 2\epsilon x + \epsilon^2 x^2)\partial_u^2.$$

These operators are uniformly elliptic operators on the set $\{(x, u); |(x, u)| \leq c\}$, for every $c > 0$. By Fourier integral methods, one can show that solutions to corresponding wave equations fulfill estimates in the sense of (0.2) and (0.3), uniformly in $\epsilon \leq 1/4$.

Proposition 4.2. *Let $\alpha \geq 1/2$, $1 < p < \infty$ and $c > 0$. There exists a constant $C_{\alpha,p,c}$ such that for all $\epsilon \leq 1/4$, t sufficiently small and f supported in $\{(x, u); |(x, u)| \leq c\}$*

$$\left\| \exp(it\sqrt{G_\epsilon})f \right\|_{L_p} \leq C_{\alpha,p,c} \|f\|_{W_p^\alpha} \quad (4.2)$$

holds.

Such estimates are well known. At the end of the chapter we will give a sketch of the proof. For details see [23] and [24]. In fact, Seeger, Sogge and Stein showed that Proposition 4.2 is true for an elliptic operator defined on a compact manifold instead of G_ϵ . The compactness is here not necessary, since we have finite speed of wave propagation and since the functions f are supported in a compact set.

Since the operator G_ϵ is elliptic on the support of f it should be possible to exchange the usual Sobolev norm $\|\cdot\|_{W_p^\alpha}$ by norms $\|\cdot\|_{L_p^\alpha} := \|(1 + G_\epsilon)^{\alpha/2} \cdot\|_{L_p}$. In addition, the assertion (4.3) should also hold true for $p = 1$ and with a slightly worse exponent $\alpha > 1/2$. We conjecture the following.

Conjecture 4.3. *Let $\alpha > 1/2$ and $c > 0$. There exists a constant $C_{\alpha,c}$ such that for all $\epsilon \leq 1/4$, t sufficiently small and f supported in $\{(x, u); |(x, u)| \leq c\}$*

$$\left\| \frac{\exp(it\sqrt{G_\epsilon})}{(1 + G_\epsilon)^{\alpha/2}} f \right\|_{L_1} \leq C_{\alpha,c} \|f\|_{L_1} \quad (4.3)$$

holds.

This conjecture implies now the following proposition.

Proposition 4.4. *Conjecture 4.3 implies that for every $\alpha > 1/2$ there exist constants C, C_α , such that*

$$\sup_{|x'| \geq C} \|K_{h^\alpha(G)}(x', 0, x, u)\|_{L^1(x, u)} \leq C_\alpha \quad (4.4)$$

holds.

Hence, if the conjecture was true, this proposition together with our theorem would imply that the operator $\exp(i\sqrt{G})(1+G)^{-\alpha/2}$ extends to a bounded operator on $L_p(\mathbb{R}^2)$ for $\alpha > |1/p - 1/2|$ and $1 \leq p \leq \infty$.

Proof. Since

$$\left(-\partial_x^2 - \left(1 + \frac{x^2}{x_1^2} + \frac{2x}{x_1}\right)\partial_u^2\right)f(\cdot + x_1, \cdot x_1)|_{(x-x_1, u/x_1)} = (-\partial_x^2 - x^2\partial_u^2)f|_{(x, u)},$$

we can express the kernel of $m(G)$ by the kernel of $m(G_{1/x_1})$ for every bounded function m . We have

$$\begin{aligned} & \int K_{m(G_{1/x_1})}(x', u', x - x_1, u/x_1) f(x' + x_1, u'x_1) dx' du' \\ &= \int K_{m(G)}(x', u', x, u) f(x', u') dx' du' \end{aligned}$$

and thus

$$x_1^{-1} K_{m(G_{1/x_1})}(x' - x_1, u'/x_1, x - x_1, u/x_1) = K_{m(G)}(x', u', x, u).$$

By Proposition 4.2 the operator $\exp(t\sqrt{G_\epsilon})(1+G_\epsilon)^{-\alpha/2}$, with kernel $M_{\epsilon, t}^\alpha$, is bounded from $L_1(\Omega_1)$ to L_1 for small $t < t_0$, with t_0 a sufficiently small constant. Hence

$$\|M_{\epsilon, t}^\alpha(x', 0, x, u)\|_{L^1(x, u)} \leq C_\alpha$$

for all $|x'| \leq 2$ and C_α independent of ϵ . Since the operator

$$(1 - \eta_N)(G_\epsilon)(1 + G_\epsilon)^{\alpha/2} G_\epsilon^{-\alpha/2 - \delta},$$

for small δ , is bounded on L_1 we get that the kernel $\widetilde{M}_{\epsilon, t}^{\alpha - \delta}$ of the operator $h_t^{\alpha - \delta}(G_\epsilon)$ has bounded Schur norm. Hence

$$x_1^{-1} \int |\widetilde{M}_{\epsilon, t}^{\alpha - \delta}(x' - x_1, 0, x - x_1, u/x_1)| dx du \leq C_\alpha,$$

for all $|x' - x_1| \leq 1$. With $\epsilon = 1/x_1$ this leads to

$$\int K_{h_t^{\alpha - \delta}(G)}(x', 0, x, u) dx du \leq C_\alpha,$$

for all $|x' - x_1| \leq 1$. With $x' = x_1$ we end up with

$$\int |K_{h_t^{\alpha-\delta}(G)}(x', 0, x, u)| dx du \leq C_\alpha,$$

uniformly in x' , provided $x' \geq 4$. By the homogeneity of G with respect to the dilation $(x, u) \mapsto (rx, r^2u)$, it follows that

$$\sup_{|x'| \geq 4/t_0} \int K_{h_t^{\alpha-\delta}(G)}(x', 0, x, u) dx du \leq C_\alpha.$$

By setting $C := 4/t_0$ the proposition has been proven. \square

Fourier integral operators

In this section we deal with operators of the form

$$Af(x) = \int a(x, \xi) e^{i\varphi(x, \xi)} \widehat{f}(\xi) d\xi, \quad (4.5)$$

where the amplitude a is a real valued function in a symbol class $S_{\rho, \delta}^m$, $\rho > 0$, $\delta < 1$, and the phase function ϕ fulfills

- (a) φ is smooth, real valued and homogeneous of degree 1 in ξ .
- (b) the gradient $\nabla_x \varphi$ is nowhere vanishing on the support of a , for all $\xi \neq 0$.

These operators are a special cases of *Fourier integral operators*. We refer to Sogge [24] and Duistermaat [4] for a general definition.

The aspect of interest for us is that the operator defined in (4.5) is essentially the solution operator to a strictly hyperbolic differential equation.

Sketch of the proof of Proposition 4.2

We use the abbreviations $z := (x, u)$, $\zeta := (\xi, \eta)$. Let $P_\epsilon := \partial_t^2 + G_\epsilon$. Since P_ϵ is strictly hyperbolic we can factor its principal symbol,

$$p(x, u, \xi, \eta, \tau) = (\tau - \lambda_+(x, u, \xi, \eta))(\tau - \lambda_-(x, u, \xi, \eta))$$

with $\lambda_\pm = \pm(\xi^2 + (1 + x\epsilon)^2\eta^2)^{1/2}$. The *eikonal equation*

$$\begin{cases} \partial_t \varphi^\pm = \lambda_\pm(x, \nabla_x \varphi^\pm) = \pm((\partial_x \varphi^\pm)^2 + (1 + x\epsilon)^2(\partial_u \varphi^\pm)^2)^{1/2}, \\ \varphi^\pm|_{t=0} = (z|\zeta) \end{cases},$$

is a system of two first order nonlinear differential equation for φ^+ and φ^- , and it can be solved at least for small t . By the initial condition $\varphi^\pm(0, z, \zeta) = (z|\zeta)$

the solutions φ^+ , φ^- will satisfy the requirements (a) and (b) for small t , $|t| \leq \delta$, δ sufficiently small and independent of ϵ .

We suppose that an approximate solution \tilde{v} to the Cauchy problem

$$(\partial_t^2 + G_\epsilon)v = P_\epsilon v = 0, \quad v|_{t=0} = f_0, \quad \partial_t v|_{t=0} = f_1, \quad (4.6)$$

for $f_0, f_1 \in \mathcal{S}$, can be written as

$$\tilde{v}(t, x, u) = \sum_{\iota \in \{+, -\}, k \in \{0, 1\}} a^{\iota, k}(t, z, \zeta) e^{i\varphi^\iota(t, z, \zeta)} \widehat{f}_k(\zeta) d\zeta,$$

where $a^{\iota, k}$ is a symbol of order $-k$. For simplicity we only consider the case $f_0 \in \mathcal{S}$ and $f_1 = 0$. The case $f_0 = 0$ and $f_1 \in \mathcal{S}$ can be obtained similarly. Then, the general case follows since our wave equation is linear.

Put $f_1 = 0$. We suppose that \tilde{v} is given as $\tilde{v} = \tilde{v}^+ + \tilde{v}^-$, with

$$\tilde{v}^\iota(t, x, u) = \int a^\iota(t, z, \zeta) e^{i\varphi^\iota(t, z, \zeta)} \widehat{f}_0(\zeta) d\zeta, \quad (4.7)$$

for $\iota \in \{+, -\}$ and a^ι is a symbol of order 0 and given as a sum over symbols a_k^ι in S^{-k}

$$a^\iota(t, z, \zeta) = \sum_{k \leq 0} a_k^\iota(t, z, \zeta). \quad (4.8)$$

Now we compute the symbols a_k^ι . Since $\varphi^\pm|_{t=0} = (z|\zeta)$ and u should be a solution with $u|_{t=0} = f_0$ and $\partial_t u|_{t=0} = 0$ we have the following equations for a^+ and a^- .

$$\begin{aligned} (a^+ + a^-)|_{t=0} &= 1 \\ [i\partial_t \varphi^+ a^+ + i\partial_t \varphi^- a^- + \partial_t(a^+ + a^-)]|_{t=0} &= 0 \end{aligned}$$

Since φ^\pm fulfills the eikonal equation (4.3) the second equation can be written as

$$[i\sigma(a^+ - a^-) + \partial_t(a^+ + a^-)]|_{t=0} = 0, \quad (4.9)$$

with $\sigma := (\xi^2 + (1 + x\epsilon)^2 \eta^2)^{1/2}$. Since σ is of order 1 and $\partial_t(a^+ + a^-)$ is a symbol of order 0, the highest order term in a has to fulfill

$$\sigma(a_0^+ - a_0^-)|_{t=0} = 0.$$

Put

$$d_k := a_k^+ + a_k^-, \quad b_k := a_k^+ - a_k^- \quad (4.10)$$

In order to fulfill (4.9) and with respect to the symbol orders of σ , b_k and d_k we choose b_k and d_k so that

$$i\sigma b_k|_{t=0} = -(\partial_t d_k)|_{t=0}. \quad (4.11)$$

Furthermore, to approximate a solution of the wave equation, i.e. to ensure that $P_\epsilon \tilde{v}$ is small, a_k^+ and a_k^- have to fulfill a transport equation. We get these equation in the following way. By Applying our operator P_ϵ to \tilde{v}^ι , for $\iota \in \{+, -\}$, we get

$$P_\epsilon \tilde{v}^\iota = \int c^\iota(t, z, \zeta) e^{i\varphi^\iota(t, z, \zeta)} \widehat{f}_0(\zeta) d\zeta$$

with

$$\begin{aligned} c^\iota(t, z, \zeta) &= e^{-i\varphi^\iota} P_\epsilon(e^{i\varphi^\iota} a^\iota) \\ &= i(2(\partial_t \varphi^\iota)(\partial_t a^\iota) - 2(\partial_x \varphi^\iota)(\partial_x a^\iota) - 2(1 + 2\epsilon x + \epsilon^2 x^2)(\partial_u \varphi^\iota)(\partial_u a^\iota)) \\ &\quad + i(P_\epsilon \varphi^\iota) a^\iota + (P_\epsilon a^\iota). \end{aligned} \quad (4.12)$$

Because \tilde{v} should be an approximate solution we have to show that c^\pm has order $-N$, with N sufficiently large. Thus we set c^\pm equal to zero and solve for the symbols a_k^\pm . The leading term is

$$i(2(\partial_t \varphi^\iota)(\partial_t a_0^\iota) - 2(\partial_x \varphi^\iota)(\partial_x a_0^\iota) - 2(1 + 2\epsilon x + \epsilon^2 x^2)(\partial_u \varphi^\iota)(\partial_u a_0^\iota)) + i(P_\epsilon \varphi^\iota) a_0^\iota$$

which is of order 1. Define

$$V^\iota := (\partial_t \varphi^\iota) \partial_t - (\partial_x \varphi^\iota) \partial_x - (1 + 2\epsilon x + \epsilon^2 x^2) (\partial_u \varphi^\iota) \partial_u.$$

We study now the *transport equation*

$$(V^\iota a_0^\iota)(t, z, \zeta) + (P_\epsilon \varphi) a_0^\iota(t, z, \zeta) = 0,$$

for $\iota \in \{+, -\}$ and with initial conditions

$$(a_0^+ - a_0^-)|_{t=0} = b_0|_{t=0} = 0, \quad (a_0^+ + a_0^-)|_{t=0} = d_0|_{t=0} = 1. \quad (4.13)$$

Since V^ι is a real vector field, we can solve these equations, on the same t -interval and get solutions a_0^ι in the symbol class S^0 . Furthermore, we have $a_0^+ - a_0^- = 0$ and $a_0^+ + a_0^- = 1$. By Rewriting (4.12) and setting equal to zero we get

$$V^\iota \left(\sum_{k \leq -1} a_k^\iota \right) + (P_\epsilon \varphi^\iota) \left(\sum_{k \leq -1} a_k^\iota \right) - i \left(P_\epsilon \sum_{k \leq 0} a_k^\iota \right) = 0,$$

for $\iota \in \{+, -\}$. This gives us new transport equations for a_{-1}^+ and a_{-1}^- .

$$(V a_{-1}^\iota)(t, z, \zeta) + (P_\epsilon \varphi^\iota) a_{-1}^\iota(t, z, \zeta) - i(P_\epsilon a_0^\iota) = 0.$$

Since we have already calculated d_0 , we get for these transport equations with initial conditions, chosen with respect to (4.11),

$$i\sigma(a_{-1}^+ - a_{-1}^-)|_{t=0} = i\sigma b_{-1}|_{t=0} = -\partial_t d_0|_{t=0}, \quad (a_{-1}^+ + a_{-1}^-)|_{t=0} = d_{-1}|_{t=0} = 0$$

unique solutions a_{-1}^+ and a_{-1}^- . Iteratively, we can solve the transport equations

$$(V^\iota a_k^\iota)(t, z, \zeta) + (P_\epsilon \varphi^\iota) a_k^\iota(t, z, \zeta) - i(P_\epsilon a_{k+1}^\iota) = 0.$$

with initial conditions

$$i\sigma(a_k^+ - a_k^-)|_{t=0} = i\sigma b_k|_{t=0} = -\partial_t d_{k+1}|_{t=0}, \quad (a_k^+ + a_k^-)|_{t=0} = d_k|_{t=0} = 0$$

for all $k \leq -1$ and $\iota \in \{+, -\}$.

We choose now a sufficiently large $N \in \mathbb{N}$. Put $\tilde{a}^\iota := \sum_{-N \leq k \leq 0} a_k^\iota$, for $\iota \in \{+, -\}$, and

$$\tilde{v}(t, z) := \sum_{\iota \in \{+, -\}} \int \tilde{a}^\iota(t, z, \zeta) e^{i\varphi^\iota(t, z, \zeta)} \widehat{f}_0(\zeta) d\zeta.$$

Then

$$\begin{aligned} (P_\epsilon \tilde{v})(t, z) &= F(t, z), \\ \tilde{v}|_{t=0} &= f_0, \end{aligned}$$

where

$$F(t, z) := \sum_{\iota \in \{+, -\}} \int \tilde{c}^\iota(t, z, \zeta) e^{i\varphi^\iota(t, z, \zeta)} \widehat{f}_0(\zeta) d\zeta$$

and \tilde{c}^ι is a symbol of order $-N$, for $\iota \in \{+, -\}$. Furthermore, $\partial_t \tilde{v}|_{t=0}$ is given by

$$\begin{aligned} \partial_t \tilde{v}(0, z) &= \sum_{k=-N}^0 \int [i\sigma b_k + (\partial_t d_k)](0, z, \zeta) e^{i(z|\zeta)} \widehat{f}_0(\zeta) d\zeta \\ &= \int (\partial_t d_{-N})(0, z, \zeta) e^{i(z|\zeta)} \widehat{f}_0(\zeta) d\zeta. \end{aligned}$$

Put $g(z) := \int (\partial_t d_{-N})(0, z, \zeta) e^{i(z|\zeta)} \widehat{f}_0(\zeta) d\zeta$.

Now, if a exact solution v for the Cauchy problem (4.6) is given, the function $w := \tilde{v} - v$ fulfills the inhomogeneous wave equation

$$\begin{cases} (\partial_t^2 + G_\epsilon)w = F, \\ w|_{t=0} = 0, \\ \partial_t w|_{t=0} = g. \end{cases}$$

Since our operator G_ϵ is just a smooth perturbation of the Laplacian, we have finite speed of wave propagation. Since f is compactly supported we can find an open set Ω_2 with $\overline{\Omega_2}$ compact and independent of ϵ such that $\text{supp } v \subseteq \Omega_2$.

By general theory (see [5]), in fact by using the energy inequality, we obtain the estimate

$$\|w(t, \cdot)\|_{L_2(\Omega_2)} \leq C_t \left(\|g\|_{L_2(\Omega_2)} + \int_0^t \|F(\tau, \cdot)\|_{L_2(\Omega_2)} d\tau \right), \quad (4.14)$$

with C_t independent of ϵ . Since \tilde{c}^\pm and $\partial_t d_{-N}$ are both symbols of order $-N$, we get

$$\|w(t, \cdot)\|_{L_2(\Omega_2)} \leq C_t \|f_0\|_{L_1}.$$

By the Cauchy-Schwarz inequality, the L_1 -norm of $w(t, \cdot)$ on Ω_2 is bounded by the L_2 -norm of $w(t, \cdot)$ on Ω_2 .

We are left with the estimation of Fourier integral operators of the form

$$f \mapsto Af(t, x, u) = \int a(t, z, \zeta) e^{i\varphi(t, z, \zeta)} \widehat{f}(\zeta) d\zeta,$$

where a is a symbol of order 0 and ϕ fulfills the requirements (a) and (b). Furthermore, a and φ depend smoothly on ϵ . By well known regularity properties of Fourier integral operators, we obtain that the operator A is bounded from W_p^α to L_p for $\alpha \geq |1/2 - 1/p|$ and $1 < p < \infty$. We refer here to [23] and [24]. This completes the sketch of the proof.

4.4 Reduction to an estimate for the local part of the kernel, part 2

We now exchange the indicator function \mathcal{X}_B by a smooth variant of it. Let $\tilde{\mathcal{X}}_B \in C^\infty(\mathbb{R}^4)$, with

$$\tilde{\mathcal{X}}_B(x', u', x, u) = 1 \text{ for all } (x', u', x, u) \in \text{supp } \mathcal{X}_B$$

and

$$\tilde{\mathcal{X}}_B(x', u', x, u) = 0 \text{ for all } (x', u', x, u) \text{ with } (x, u) \notin B_4(x', u').$$

$\tilde{\mathcal{X}}_B$ is a smooth function supported near the diagonal $(x, u) = (x', u')$.

Put

$$\Omega := \{(x, u) \in \mathbb{R}^2; |x| < 2\mathcal{C}_1\}.$$

Proposition 4.5. *To prove the theorem, it suffices to show that for all $\alpha > 1/2$ the operator*

$$f \mapsto \int \tilde{\mathcal{X}}_B K_{h^\alpha(G)}(x', u', \cdot, \cdot) f(x', u') dx' du' \quad (4.15)$$

is bounded from $L_p(\Omega)$ to $L_p(\mathbb{R}^2)$ for every $1 < p < \infty$.

Proof. Let $\alpha > 1/2$. We assume now that the operator defined in (4.15) is bounded from $L_p(\Omega)$ to $L_p(\mathbb{R}^2)$ for every $1 < p < \infty$. By Proposition 4.1 we have to show that there exists a constant C with

$$\sup_{x' \leq C_0, u'} \int |\mathcal{X}_B K_{h^\alpha(G)}(x', u', x, u)| dx du \leq C. \quad (4.16)$$

Since $\alpha > 1/2$ we can choose an $\epsilon > 0$ such that $\alpha - \epsilon > 1/2$. Put $\alpha' := \alpha - \epsilon$. Since we know from Proposition 4.1 that the kernel $(1 - \tilde{\mathcal{X}}_B)K_{h^{\alpha'}}$ has bounded Schur norm, we get that the operator $h^{\alpha'}(G)$ is bounded from $L_p(\Omega)$ to $L_p(\mathbb{R}^2)$ for $1 < p < \infty$.

In [19] it was shown that $(1 + L)^{-\epsilon}\delta_0$ is in L_p for some $p > 1$ and that $(1 + L)^{-\epsilon}\delta_0$ is rapidly decreasing away from the origin. By the transfer principle, Proposition 1.1, we get that the function

$$f_{x', u'}^1 : (x, u) \mapsto K_{(1+G)^{-\epsilon}}(x', u', x, u)\mathcal{X}_\Omega(x, u),$$

lies in L_p and that the function

$$f_{x', u'}^2 : (x, u) \mapsto K_{(1+G)^{-\epsilon}}(x', u', x, u)(1 - \mathcal{X}_\Omega)(x, u),$$

lies in L_2 . Now, by applying the operator $h^{\alpha'}(G)$ to the functions $f_{x', u'}^1$ and $f_{x', u'}^2$ for $|x'| \leq C_0$, C_0 a constant, we get

$$\|h^{\alpha'}(G)f_{x', u'}^1\|_p \leq C, \quad \|h^{\alpha'}(G)f_{x', u'}^2\|_2 \leq C$$

with a constant C only depending on C_0 , p and α . Since

$$K_{h^\alpha(G)}(x', u', x, u) = h^{\alpha'}(G)f_{x', u'}(x, u),$$

and since the function $(x, u) \mapsto \mathcal{X}_B(x', u', x, u)$ is in L_1 , with norm bounded by $(1 + |x'|)$ for every (x', u') , there is for every $C_0 > 0$ a constant $C_1 > 0$ with

$$\sup_{|x'| \leq C_0, u'} \int |\mathcal{X}_B K_{h^\alpha(G)}(x', u', x, u)| dx du \leq C_1.$$

This gives us the assertion (4.16) and proves the proposition. \square

With slight modifications of the proof of Proposition 4.5 we can also show that the Conjecture 4.3 together with the assumption (4.15) implies that for every $1 \leq p \leq \infty$ the operator $\exp(i\sqrt{G})(1 + G)^{-\alpha/2}$ extends to a bounded operator on $L_p(\mathbb{R}^2)$, provided $\alpha > |1/p - 1/2|$.

5 Preparations

5.1 Dyadic decomposition.

We have seen in Section 3.1 that the spectrum of the Grušin operator lies in the union of the rays

$$\begin{aligned}\mathcal{R}_{n,\epsilon} &:= \left\{ (\epsilon\lambda, \tau); \tau = (2n+1)\lambda, \lambda > 0 \right\}, \quad \epsilon := \pm 1, n \in \mathbb{N}_0 \\ \mathcal{R}_\infty &:= \{(0, \tau); \tau \geq 0\}.\end{aligned}$$

This is the joined spectrum of the operators G and iU . Since G is a positive operator, the spectrum of G is contained in $\bigcup_{n \in \mathbb{N}_0} \mathcal{R}_{n,1} \cup \mathcal{R}_\infty$.

In Section 1.4 we have studied spheres belonging to the optimal control metric associated to G . This gave us a description of the singularities of the distribution kernel of $\cos(\sqrt{G})$. The singularities lie in a rather complicated curve, that contains many, for $x' = 0$ infinitely many, edges.

In this section, we decompose the integral kernel of $h^\alpha(G)$ in a sum of integral kernels $K_{k,j}^\epsilon$ such that these parts of the integral kernel coincides, in some way, with the edges in the singular support of $\cos(\sqrt{G})$.

Roughly, for every (k, j, ϵ) , we choose a rectangle in the joint spectrum of iU and G of length 2^j in the n -direction and length 2^{2k-j} in the λ direction. $\epsilon(2n+1)$ corresponds to the spectrum of $-iGU^{-1}$ and λ to the spectrum of iU . The dyadic operators with integral kernels $K_{k,j}^\epsilon$ are of the form $h^\alpha(G)\mathcal{X}_{2^{k-j}j}(iU)\mathcal{X}_j(-iGU^{-1})$ where \mathcal{X} is a cut-off function. Since $G = (iU)(-iGU^{-1})$ we can compute the integral kernels of these dyadic parts by the functional calculus of iU and $-iGU^{-1}$.

The Fourier transform gives us a spectral decomposition of iU and in Chapter 3 we derived the spectral decomposition of the operator $-iGU^{-1}$ as a sum over singular integral operators $\mathcal{P}_{n,\epsilon}$. We get an explicit formula for the integral kernel $K_{k,j}^\epsilon(x', u', x, u)$ away from the diagonal $u' = u$.

By Proposition 3.1 the operator $h^\alpha(G)$ can be decomposed in the following way

$$h^\alpha(G)f = \sum_{\epsilon=\pm 1} \sum_{n=0}^{\infty} \int_0^{\infty} h^\alpha((2n+1)\lambda) [\mathcal{P}_{\lambda,n,\epsilon}f] d\lambda.$$

Let \mathcal{X}_j , $j \in \mathbb{Z}$ denote a dyadic decomposition of unity on \mathbb{R}^+ . Define now

$$H_{k,j}^\epsilon f := \sum_{n=0}^{\infty} \int_0^{\infty} h^\alpha((2n+1)\lambda) \mathcal{X}_{2^{k-j}j}(\lambda) \mathcal{X}_j(2n+1) [\mathcal{P}_{\lambda,n,\epsilon}f] d\lambda,$$

for $j \in \mathbb{N}_0$, $k \in \mathbb{Z}$, $\epsilon \in \{-1, 1\}$, $\lambda \geq 0$. Since

$$2^{k-1} \leq \sqrt{(2n+1)\lambda} \leq 2^{k+1}$$

on the support of $h_{k,j}^\epsilon$, we have $h_{k,j}^\epsilon = 0$, unless $2^k \geq N/4$. So, if we fix any $k_0 \gg 1$, we may choose N sufficiently large so that

$$h^\alpha(G) = \sum_{\substack{\epsilon=\pm 1 \\ k \geq k_0, j \in \mathbb{N}_0}} H_{k,j}^\epsilon. \quad (5.1)$$

Furthermore, we choose k_0 sufficiently large so that we can delete the factor $(1-\eta)(\cdot/N)$ in h^α . In the following, we use the abbreviation $\ell = 2k - j$. Define

$$\tilde{\mathcal{X}}(x) := |x|^{-\alpha/2} \mathcal{X}(x).$$

Then

$$\tilde{\mathcal{X}}_\ell(\lambda) \tilde{\mathcal{X}}_j(2n+1) = 2^{k\alpha} ((2n+1)|\lambda|)^{-\alpha/2} \mathcal{X}_\ell(\lambda) \mathcal{X}_j(2n+1)$$

and

$$\begin{aligned} h^\alpha((2n+1)\lambda) \mathcal{X}_\ell(\lambda) \mathcal{X}_j(2n+1) &= ((2n+1)\lambda)^{-\alpha/2} e^{i\sqrt{(2n+1)\lambda}} \mathcal{X}_\ell(\lambda) \mathcal{X}_j(2n+1) \\ &= 2^{-\alpha k} e^{i\sqrt{(2n+1)\lambda}} \tilde{\mathcal{X}}_\ell(\lambda) \tilde{\mathcal{X}}_j(2n+1). \end{aligned}$$

Since $\tilde{\mathcal{X}}$ is of similar type as \mathcal{X} , we shall again write \mathcal{X} in place of $\tilde{\mathcal{X}}$. Observe that

$$H_{k,j}^\epsilon f := \sum_{\tilde{\epsilon}=\pm 1} \sum_{n=0}^{\infty} \int_0^{\infty} 2^{-\alpha k} e^{i\sqrt{(2n+1)\lambda}} \mathcal{X}_\ell(\tilde{\epsilon}\lambda) \mathcal{X}_j(2n+1) [\mathcal{P}_{\lambda,n,\tilde{\epsilon}} f] d\lambda,$$

and hence we get with Corollary 3.2

$$\begin{aligned} H_{k,j}^\epsilon f &= 2^{-\alpha k} \sum_{\tilde{\epsilon}=\pm 1} \sum_n \mathcal{X}_j(\tilde{\epsilon}(2n+1)) \gamma_n^\epsilon(iU) (\mathcal{P}_n^{\tilde{\epsilon}} f) \\ &= 2^{-\alpha k} \sum_n \mathcal{X}_j(2n+1) \gamma_n^\epsilon(iU) (\mathcal{P}_n^\epsilon f), \end{aligned}$$

with

$$\gamma_n^\epsilon(\lambda) := \tilde{\mathcal{X}}_\ell(\epsilon\lambda) e^{i\sqrt{(2n+1)|\lambda|}}.$$

We denote the integral kernel of the operator $H_{k,j}^\epsilon$ by $K_{k,j}^\epsilon$. Away from the diagonal, $K_{k,j}^\epsilon$ is given by

$$\begin{aligned} K_{k,j}^\epsilon(x', 0, x, u) &= \frac{2^{-\alpha k}}{2\pi} \\ &\times \sum_{n=0}^{\infty} \mathcal{X}_j(2n+1) \int_{-\infty}^{\infty} P_n^\epsilon(x', 0, x, u-s) e^{i\sqrt{(2n+1)|\lambda|}} \mathcal{X}_\ell(\epsilon\lambda) e^{-i\lambda s} d\lambda ds. \end{aligned}$$

We put

$$\Phi_{\ell,n}^\epsilon(s) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sqrt{(2n+1)|\lambda|}} \mathcal{X}_\ell(\epsilon\lambda) e^{-i\lambda s} d\lambda. \quad (5.2)$$

Then $\Phi_{\ell,n}^{-1}(s) = \Phi_{\ell,n}^1(-s)$ and

$$K_{k,j}^\epsilon(x', 0, x, u) = \frac{2^{-\alpha k}}{2\pi} \sum_{n=0}^{\infty} \mathcal{X}_j(2n+1) \int_{-\infty}^{\infty} P_n^\epsilon(x', 0, x, u-s) \Phi_{\ell,n}^\epsilon(s) ds.$$

Since $P_n^{-1}(x', 0, x, u) = P_n^1(x', 0, x, -u)$ and $\Phi_{\ell,n}^{-1}(s) = \Phi_{\ell,n}^1(-s)$ we get

$$K_{k,j}^{-1}(x', 0, x, u) = K_{k,j}^1(x', 0, x, u).$$

Hence we can restrict to the case $\epsilon = 1$. Put

$$K_{k,j} := K_{k,j}^1.$$

Thus, to prove the theorem it suffices to show the following proposition. Recall that in the previous section we have defined

$$\Omega := \{(x, u) \in \mathbb{R}^2; |x| < 2\mathcal{C}_1\}.$$

Proposition 5.1. *If $\alpha > 1/2$, then*

$$\sum_{\substack{\epsilon=\pm 1 \\ k \geq k_0, j \geq 0}} \|\tilde{\mathcal{X}}_B K_{k,j}\|_{(L_p(\Omega), L_p)} < \infty.$$

for every p , $1 < p < \infty$, where $\|K\|_{(L_p(\Omega), L_p)}$ denotes the operator norm of the integral operator $f \mapsto \int K(x', u', \cdot, \cdot) f(x', u') dx' du'$ from $L_p(\Omega)$ to L_p .

5.2 Integral formulas for $K_{k,j}$

In this section we completely follow the proof of Müller and Stein. More exactly, we reproduce the Sections 1.2 and 2 of [19]. The only difference here is that we do not have translation invariant operators. Of course, we use our formulas for the spectral projection operators \mathcal{P}_n we derived in Section 3.1 and Section 3.2.

Recall the definition of $\Phi_{\ell,n}^1$,

$$\Phi_{\ell,n}^1(s) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sqrt{(2n+1)|\lambda|}} \mathcal{X}_\ell(\lambda) e^{-i\lambda s} d\lambda. \quad (5.3)$$

We define $\Phi_{\ell,n} := \Phi_{\ell,n}^1$.

Lemma 5.2. *Let $f \in C_0^\infty(\mathbb{R})$ be supported in $[1/2, 2]$. For every $N \in \mathbb{N}$ there exist functions $f_0, \dots, f_N \in C_0^\infty(\mathbb{R})$ supported in $[1/4, 4]$ and $E_N \in C^\infty(\mathbb{R}^2)$, such that for $(a, b) \in \mathbb{R}^2$ with $|(a, b)| > 1$*

$$\int_{-\infty}^{\infty} e^{i(ax-bx^2/2)} f(x) dx = e^{ia^2/(2b)} \sum_{\nu=0}^N b^{-1/2-\nu} f_\nu(a/b) + E_N(a, b),$$

where E_N satisfies

$$E_N^{(\alpha)} = O(|(a, b)|^{-N/2-1}), \text{ for every } \alpha \in \mathbb{N}^2.$$

Proof. Müller and Stein [19], Lemma 1.4. The proof bases upon the method of stationary phase. \square

We may apply the lemma to $\Phi_{\ell, n}$, since $\sqrt{(2n+1)2^\ell} \sim 2^k \gg 1$, and obtain

$$\begin{aligned} \Phi_{\ell, n}(u) = & e^{i(2n+1)/(4u)} 2^\ell \sum_{\nu=0}^N (2^{\ell+1}u)^{-1/2-\nu} f_\nu\left(\sqrt{\frac{2n+1}{2^\ell}} \frac{1}{2u}\right) \\ & + 2^\ell E_N(\sqrt{(2n+1)2^\ell}, 2^{\ell+1}u), \end{aligned} \quad (5.4)$$

with f_ν and E_N as in the lemma. Put $a_{n, \ell} := \sqrt{(2n+1)2^\ell}$. Since $a_{n, \ell} \sim 2^k$ and since

$$(2^{\ell+1}u)^{-1/2-\nu} = a_{n, \ell}^{-1/2-\nu} a_{n, \ell}^{1/2+\nu} (2^{\ell+1}u)^{-1/2-\nu} = a_{n, \ell}^{-1/2-\nu} \left(\sqrt{\frac{M+2n}{2^\ell}} \frac{1}{2u}\right)^{1/2+\nu},$$

the ν -th term in (5.4) is given by

$$a_{n, \ell}^{-1/2-\nu} \tilde{f}_\nu\left(\sqrt{\frac{M+2n}{2^\ell}} \frac{1}{2u}\right),$$

with $\tilde{f}_\nu u(x) = x^{1/2+\nu} f_\nu(x)$. Since \tilde{f}_ν is of the same type as f_ν and we only have to sum over finitely many ν we may reduce to the case where $\Phi_{\ell, n}$ is either of the form

$$(a) \quad 2^\ell a_{n, \ell}^{-1/2-\nu} f\left(\sqrt{\frac{2n+1}{2^\ell}} \frac{1}{4u}\right) e^{i(2n+1)/(4u)},$$

with $f \in C_0^\infty(\mathbb{R})$ supported in $[1/8, 2]$ and $\nu \in N_0$, or of the form

$$(b) \quad \Phi_{\ell, n}(u) := 2^\ell E_N(a_{n, \ell}, 2^{\ell+1}u).$$

First we study the case (b).

Müller and Stein showed that the corresponding operators with convolution kernels

$$\tilde{K}_{k,j} := 2^{-\alpha k} \sum_n \mathcal{X}_j(1+2n) \int P_n^{\mathbb{H}}(x, y, u-s) \Phi_{2k-j,n}(s) ds,$$

where $P_n^{\mathbb{H}}$ is the kernel of the projection operator $\mathcal{P}_n^{\mathbb{H}}$ on the Heisenberg group and $\Phi_{2k-j,n}$ is given by (b), are L_p bounded for $1 < p < \infty$. Furthermore, they showed that one can sum up all L_p -operator norms for $\alpha > 0$. Observe that $\alpha > 1/2$. Hence the operator with convolution kernel $\tilde{K} := \sum_{k,j} K_{k,j}$ is bounded on L_p for $1 < p < \infty$.

Let π be the representation of \mathbb{H}_1 with $\pi(L) = G$. We mentioned in Section 1.3 that one can transfer estimates for operators on \mathbb{H}_1 to operators on \mathbb{R}^2 . By transfer methods the operator with integral kernel $K := \sum_{k,j} K_{k,j}$, where $K_{k,j}$ is given by

$$K_{k,j}(x', u', x, u) = \frac{2^{-\alpha k}}{2\pi} \sum_{n=0}^{\infty} \mathcal{X}_j(2n+1) \int_{-\infty}^{\infty} P_n(x', u', x, u-s) \Phi_{\ell,n}(s) ds$$

and $\Phi_{\ell,n}$ is given by (b), is bounded on L_p . Furthermore, also the operator with truncated kernel $\tilde{\mathcal{X}}_B K$ is bounded on L_p . We do not want to go in the details here. There is one difficulty. One can use transference, usually, only for bounded measures and the convolution kernel \tilde{K} is not bounded. We just want to mention an argument that one has to use, to make the transference principle work.

We take a dyadic decomposition of unity ψ_r^ϵ in the following way. Put

$$\psi_r^\epsilon := \sum_{|j| \leq r, |k| \leq r} \mathcal{X}_{2k-j}(\epsilon\lambda) \mathcal{X}_j(2n+1),$$

where \mathcal{X} is chosen as in section 5.1. We define $\Psi_r^\epsilon := \mathcal{G}^{-1}(\psi_r^\epsilon)$, where \mathcal{G} is the Gelfand transform for the algebra of radial functions on \mathbb{H}_1 . Furthermore, we put $\Phi_r^\epsilon := \pi(\Psi_r^\epsilon)$. Then the set

$$X := \{\Phi_r^\epsilon(f); f \in C_0^\infty(\mathbb{R}^2), r \in \mathbb{N}_0, \epsilon = \pm 1\}$$

is dense in $L_p(\mathbb{R}^2)$. Now we define the operator A on $L_p(\mathbb{R}^2)$ by

$$A(\Phi_r^\epsilon f) := \sum_{k,j} \pi(\tilde{K}_{k,j} * \Psi_r^\epsilon) f. \quad (5.5)$$

By transfer methods we can deduce that $\pi(\phi_r^\epsilon * \tilde{K}_{k,j})$ is bounded on L_p , since $\phi_r^\epsilon * \tilde{K}_{k,j}$ is a bounded measure. The convolution kernel of the original operator $H_{k,j}^1$ on the Heisenberg group is given by $\mathcal{G}^{-1}(\phi_{k,j})$ where $\phi_{k,j} = (\lambda, n) \mapsto \phi_{k,j}(\lambda, n)$ has compact support in λ with $\lambda \sim 2^{2k-j}$ (compare [19], section 1.1). Thus we can localize the Fourier transform of $\Phi_{\ell,n}$, given by (b), to the same region. Hence for every (r, ϵ) there are only finitely many k, j we have to

sum up in (5.5). Therefore, the operator A is bounded on L_p . But A is equal to the operator with integral kernel K .

The case that $\Phi_{\ell,n}$ is given by (a) is left.

Since

$$a_{n,\ell}^{-1/2-\nu} \chi_j(2n+1) = 2^{-k/2-\nu k} \tilde{\chi}_j(2n+1),$$

with $\tilde{\chi}$ of similar type than \mathcal{X}_j , we only have to prove the following proposition instead of Proposition 5.1. We have exchanged α by the critical index m .

Proposition 5.3. *Suppose that $K_{k,j}$ is given by*

$$K_{k,j}(x', 0, x, u) := 2^{-mk} \sum_n \mathcal{X}_j(m+n) \int P_n(x', 0, x, u/2 - s/2) \Phi_{k,j,n}(s) ds$$

with

$$\Phi_{k,j,n}(s) := 2^{3k/2-j} f\left(\sqrt{\frac{m+n}{2^{2k-j}} \frac{1}{s}}\right) e^{i(m+n)/s},$$

$f \in C_0^\infty(\mathbb{R})$ supported in $[1/8, 2]$, $\chi \in C_0^\infty(\mathbb{R})$ supported in $[1/2, 2]$, $\chi_j = \chi(2^{-j} \cdot)$ and $k \geq 0$ sufficiently large. Then for every $\epsilon > 0$ there exists a constant C_ϵ with

$$\sup_{|x'| \leq 2^{\mathcal{C}_1}} \sum_{j \in \mathbb{N}_0} \|\tilde{\mathcal{X}}_B K_{k,j}(x', 0, \cdot, \cdot)\|_{L^1} \leq C_\epsilon 2^{\epsilon k}.$$

Before we prove Proposition 5.3, we do some more reductions. Let $\tilde{\chi} \in C_0^\infty$ with $\tilde{\chi}(x) = 1$ for $1/4 \leq x \leq 16$ and supported in $[1/8, 32]$. Since $\chi_j(m+n)\Phi_{k,j,n}(s) = 0$, unless $1/4 \leq 2^{k-j}s \leq 16$

$$\chi_j(m+n)\Phi_{k,j,n}(s) = \chi_j(m+n)\tilde{\chi}(2^{k-j}s)\Phi_{k,j,n}(s).$$

Moreover, writing

$$f\left(\sqrt{\frac{m+n}{2^{2k-j}} \frac{1}{s}}\right) = g\left(\log\left(\frac{m+n}{2^{2k-j}} s^{-2}\right)\right),$$

with g smooth and supported in $[\log 1/16, \log 2] \subseteq [-\pi, \pi]$, we see that

$$\Phi_{k,j,n}(s) = \sum_{\nu \in \mathbb{Z}} a_\nu \left(\frac{m+n}{2^{2k-j}s^2}\right)^{i\nu} 2^{3k/2-j} \tilde{\chi}(2^{k-j}s) e^{i(m+n)/s} =: \sum_{\nu \in \mathbb{Z}} a_\nu \Phi_{k,j,n,\nu}(s),$$

where

$$a_\nu = O(|\nu|^{-N}), \tag{5.6}$$

for every $N \in \mathbb{N}$. By the definition

$$K_{k,j,\nu}(x', 0, x, u) := 2^{-mk} \sum_n \mathcal{X}_j(m+n) \int P_n(x', 0, x, u/2 - s/2) \Phi_{k,j,n,\nu}(s) ds$$

we get

$$K_{k,j} = \sum_{\nu} a_{\nu} K_{k,j,\nu}.$$

By defining $\chi_{(\nu)}(s) := s^{-i2\nu} \tilde{\chi}(s)$, we still have $\chi_{(\nu)} \in C_0^{\infty}$ and supported in $[1/8, 32]$. Furthermore,

$$\|\chi_{(\nu)}^{(\alpha)}\|_{\infty} = O((1 + |\nu|)^{\alpha}), \quad (5.7)$$

for all $\alpha \in \mathbb{N}$ and

$$\begin{aligned} \Phi_{k,j,n,\nu}(s) &= 2^{3k/2-j} \left(\frac{m+n}{2^{2k-j}s^2} \right)^{i\nu} \tilde{\chi}(2^{k-j}s) e^{i(m+n)/s} \\ &= 2^{3k/2-j} \chi_{(\nu)}(2^{k-j}s) 2^{-ij\nu} (m+n)^{i\nu} e^{i(m+n)/s}. \end{aligned}$$

We define now

$$\chi_{(\nu),j}(x) := \chi_{(\nu)}(2^{-j}x) = (2^{-j}x)^{i\nu} \chi(2^{-j}x)$$

and get $\chi_{(\nu),j}(m+n) = 2^{-ij\nu} (m+n)^{i\nu} \chi_j(m+n)$. Thus,

$$\chi_j(m+n) \Phi_{k,j,n,\nu}(s) = 2^{3k/2-j} \chi_{(\nu),j}(m+n) \chi_{(\nu)}(2^{k-j}s) e^{i(m+n)/s}.$$

By inserting the formulas for $\Phi_{k,j,n,\nu}$, we obtain

$$\begin{aligned} K_{k,j,\nu}(x', 0, x, u) &:= 2^{-mk} \sum_n \mathcal{X}_j(m+n) \\ &\times \int P_n(x', 0, x, (u-s)/2) \Phi_{k,j,n,\nu}(s) ds \\ &= 2^{k/2} 2^{(m-1)(j-k)} \sum_n \chi_{(\nu),j}(m+n) \\ &\times \int \chi_{(\nu)}(2^{k-j}s) (2^{-mj} P_n)(x', 0, x, (u-s)/2) e^{i(m+n)/s} ds. \end{aligned} \quad (5.8)$$

Now we need our explicit formulas for P_n , which we derived in Chapter 3. We have

$$P_n(x', 0, x, (u-s)/2) = C [Q_n - Q_{n-2}](x', 0, x, (u-s)/2), \quad (5.9)$$

where

$$Q_n(x', 0, x, (u-s)/2) = (q_{n,m}^+ e^{in\sigma} + \tilde{q}_{n,m}^- e^{-in\sigma}) \frac{(x^2 + x_1^2 - i(u-s))^{n/2}}{(x^2 + x_1^2 + i(u-s))^{n/2+m+1}}.$$

$q_{n,m}^+$ is given by

$$\sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\Gamma(\ell+m+1) \Gamma(n-\ell+m+1)}{\Gamma(\ell+1) \Gamma(n-\ell+1)} e^{-i2\ell\sigma},$$

where $\mathcal{X} \in C_0^\infty$ with $\mathcal{X}(x) = 0$ for $x \geq 3/4$. Furthermore, $\tilde{q}_{n,m}^-$ is similar to the function $q_{n,m}^- := \overline{q_{n,m}^+}$. σ is given by

$$e^{i\sigma} = \frac{2xx' + iw}{\sqrt{R^2 + (u-s)^2}}, \quad (5.10)$$

with $w = \sqrt{(x^2 - x'^2)^2 + (u-s)^2}$ and $R = x^2 + x'^2$. Define now

$$\omega^+(x', x, u, s) = \arctan\left(\frac{Rw - 2xx'(u-s)}{R2xx' + (u-s)w}\right) + \frac{1}{s}$$

and

$$\omega^-(x', x, u, s) = -\arctan\left(\frac{Rw + 2xx'(u-s)}{R2xx' - (u-s)w}\right) + \frac{1}{s}.$$

\arctan denotes the branch of \tan^{-1} taking values in $[0, \pi]$, since $|Rw| \geq |2xx'u|$ and thus the imaginary part is always positive (compare (3.7) and (3.8)). Observe that

$$\begin{aligned} \omega^-(x', x, -u, -s) &= -\left(\arctan\left(\frac{Rw - 2xx'(u-s)}{R2xx' + (u-s)w}\right) + \frac{1}{s}\right) \\ &= -\omega^+(x', x, u, s) \end{aligned} \quad (5.11)$$

holds. We define

$$Q_n^\pm(x', 0, x, (u-s)/2) := q_n^\pm e^{\pm i n \sigma} \frac{(x^2 + x_1^2 - i(u-s))^{n/2}}{(x^2 + x_1^2 + i(u-s))^{n/2+m+1}}$$

and $P_n^\pm := Q_n^\pm + Q_{n-2}^\pm$. By these definitions we get

$$\begin{aligned} Q_n^\pm e^{i(m+n)/s} &= Q_n^\pm e^{i(m+n+1)/s} e^{-i/s} \\ &= q_{n,m}^\pm e^{\pm i n \sigma} \frac{(x^2 + x_1^2 - i(u-s))^{n/2}}{(x^2 + x_1^2 + i(u-s))^{n/2+m+1}} e^{i(m+n+1)/s} e^{-i/s} \\ &= q_{n,m}^\pm e^{i(n+m+1)\omega^\pm} e^{-i/s} e^{\mp i(m+1)\sigma} |R + i(u-s)|^{-m-1}. \end{aligned}$$

By (5.10) and (5.11), we find out that the term $Q_n^-(x', 0, x, (-u+s)) e^{i(m+n)/s}$ is similar to $\overline{Q_n^+(x', 0, x, (u-s)) e^{i(m+n)/s}}$. Since we are interested in L_1 -norms of $K_{k,j,\nu}$ and by (5.8), we only have to consider Q_n^+ in our estimates to come. In fact, if we define $K_{k,j,\nu}^+$ in such a way that it only involves $+$ -terms and $K_{k,j,\nu}^-$ so that it only involves $-$ -terms, we deduce from the L_1 -boundedness of $K_{k,j,\nu}^+$ the L_1 -boundedness of $\overline{K_{k,j,\nu}^+}$. If we now exchange u by $-u$ and s by $-s$ we get the L_1 -boundedness of $K_{k,j,\nu}^-$.

Furthermore, define

$$\zeta_{\nu,j,m}^{\pm}(x', x, u, s) := \sum_n \chi_{(\nu),j}(m+n) q_n^{\pm} 2^{-mj} e^{i(n+m+1)\omega^{\pm}}. \quad (5.12)$$

Thus,

$$\begin{aligned} & 2^{k/2} 2^{(m-1)(j-k)} \sum_n \chi_{(\nu),j}(m+n) \\ & \times \int \chi_{(\nu)}(2^{k-j}s) (2^{-mj} Q_n^{\pm})(x', 0, x, (u-s)/2) e^{i(m+n)/s} ds \\ & = 2^{k/2} 2^{(m-1)(j-k)} \\ & \times \int \frac{\zeta_{\nu,j,m}^{\pm}(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} e^{-i/s} e^{\mp i(m+1)\sigma} \chi_{(\nu)}(2^{k-j}s) ds. \end{aligned} \quad (5.13)$$

In order to simplify the notation, we only consider $\nu = 0$ from now on. Our estimates for $K_{k,j,\nu}$ are depending on estimates we get for $\zeta_{\nu,j,m}^{\pm}$ and $\zeta_{\nu,j,m-1}^{\pm}$. For $\zeta_{\nu,j,m}^{\pm}$ resp. $\zeta_{\nu,j,m-1}^{\pm}$ we use our estimates in Proposition 3.9 and Proposition 3.10. These estimates only depend on 4 derivatives of \mathcal{X}_{ν} . And hence, by (5.7), these estimates are bounded by $O((1+|\nu|)^4)$. Since a_{ν} descends as fast, as we want (see (5.6)), it will be clear, at the end of our proof, that it suffices to show estimates for $\nu = 0$. Put

$$\zeta_{j,m} := \zeta_{0,j,m}.$$

Hence we get, with this new definition and by (5.13), that

$$\begin{aligned} & 2^{k/2} 2^{(m-1)(j-k)} \\ & \times \sum_n \chi_{(0),j}(m+n) \int \chi_{(0)}(2^{k-j}s) (2^{-mj} Q_n^{\pm})(x', 0, x, (u-s)/2) e^{i(m+n)/s} ds \\ & = 2^{k/2} 2^{(m-1)(j-k)} \int \frac{\zeta_{j,m}^{\pm}(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} e^{-i/s} e^{\mp i(m+1)\sigma} \chi(2^{k-j}s) ds. \end{aligned}$$

For Q_{n-2}^+ we get

$$Q_{n-2}^+ e^{i(m+n)/s} = q_{n-2}^+ e^{i((n-2)+m+1)\omega} e^{i/s} e^{-i(m-1)\sigma} |R + iu|^{-m-1}.$$

Observe that

$$\mathcal{X}_j(m+n) = \mathcal{X}_j(m+n-2) + 2^{-j}(2^j \mathcal{X}_j(m+n) - 2^j \mathcal{X}_j(m+n-2)).$$

Now let $\tilde{\mathcal{X}}_j(x) := 2^j \mathcal{X}_j(x+2) - 2^j \mathcal{X}_j(x) = 2 \int_0^1 \mathcal{X}'((x+2-2t)/2^j) dt$. Since this

function has similar properties as \mathcal{X}_j , we obtain

$$\begin{aligned} & 2^{k/2+(m-1)(j-k)} \sum_n \chi_j(m+n) \int \chi_{(0)}(2^{k-j}s) (2^{-mj} Q_{n-2}^\pm) e^{i(m+n)/s} ds \\ &= 2^{k/2} 2^{(m-1)(j-k)} \left(\int \frac{\zeta_{j,m}^\pm(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} e^{i/s} e^{\mp i(m+1)\sigma} \chi(2^{k-j}s) ds \right. \\ & \quad \left. + 2^{-j} \int \frac{\tilde{\zeta}_{j,m}^\pm(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} e^{i/s} e^{\mp i(m+1)\sigma} \chi(2^{k-j}s) ds \right), \end{aligned}$$

with $\tilde{\zeta}_{j,m}$ of the same type as $\zeta_{j,m}$. From now on we denote $K_{k,j,0}$ again by $K_{k,j}$. Thus,

$$\begin{aligned} K_{k,j}(x', 0, x, u) &= \sum_{\epsilon \in \{+, -\}} C 2^{k/2} 2^{(m-1)(j-k)} \\ & \times \left(\int \frac{\zeta_{j,m}^\epsilon(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} (e^{-i/s} - e^{i/s}) e^{-\epsilon i(m+1)\sigma} \chi(2^{k-j}s) ds \right. \\ & \quad \left. - 2^{-j} \int \frac{\tilde{\zeta}_{j,m}^\epsilon(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} e^{i/s} e^{-\epsilon i(m+1)\sigma} \chi(2^{k-j}s) ds \right), \end{aligned} \quad (5.14)$$

For the region where $w(s)$ is small we use (3.13) and obtain

$$R_n = (q_{n,m-1}^+ e^{in\sigma} + \tilde{q}_{n,m-1}^- e^{-in\sigma}) \frac{(x^2 + x_1^2 - i(u-s))^{n/2}}{(x^2 + x_1^2 + i(u-s))^{n/2+m}}$$

with $q_{n,m-1}^+$ given by

$$\sum_{\ell=0}^n \mathcal{X}(\ell/n) \frac{\Gamma(\ell+m)}{\Gamma(\ell+1)} \frac{\Gamma(n-\ell+m)}{\Gamma(n-\ell+1)} e^{-i2\ell\sigma}.$$

and $\tilde{q}_{n,m-1}^-$ similar to $q_{n,m-1} := \overline{q_{n,m-1}^+}$. Put

$$R_n^+ := q_{n,m-1}^+ e^{in\sigma} \frac{(x^2 + x_1^2 - iu)^{n/2}}{(x^2 + x_1^2 + iu)^{n/2+m}} e^{i(m+n)/s} e^{-i/s} e^{-i(m+n)/s}.$$

Then

$$R_n^+ e^{i(m+n)/s} = q_{n,m-1}^+ e^{i(n+m)\omega^+} e^{-im\sigma} |R + iu|^{-m}.$$

Thus we obtain a second formula for $K_{k,j}$. $K_{k,j}$ is given by

$$\begin{aligned} K_{k,j}(x', 0, x, u) &= \sum_{\epsilon \in \{+, -\}} C 2^{k/2} 2^{(m-3)(j-k)} \\ & \times \int \frac{\zeta_{k,j,m-1}^\epsilon(x', x, u, s)}{(R^2 + (u-s)^2)^{m/2}} e^{-\epsilon im\sigma} \mathcal{X}(2^{k-j}s) ds, \end{aligned} \quad (5.15)$$

with

$$\zeta_{j,m-1}^{\pm}(x, x_1, u, s) := \sum_n \chi_j(m+n) q_{n,m-1}^{\pm} 2^{mj} e^{i(n+m)\omega^{\pm}}$$

and some constant C .

We now distinguish the cases when $j - k$ is small and when $k - j$ is small. We define the constant \mathcal{C}_2 by

$$\mathcal{C}_2 := \frac{1}{16^2} \frac{1}{(\mathcal{C}_0 + \mathcal{C}_1)^2}. \quad (5.16)$$

Furthermore, recall that we have to integrate $K_{k,j}$ over the set

$$\tilde{O}(x') := \{(x, u); |x - x'| \leq 2\mathcal{C}_0, |u| \leq 2\mathcal{C}_0(2 + |x'|)\} \quad (5.17)$$

and that $|x'| \leq 2\mathcal{C}_1$.

$K_{k,j}$ for the case $2^{j-k} \leq 1/\mathcal{C}_2$

Let $\rho \in C_0^\infty$ such that $\rho(x) = 1$ for $|x| \leq 2^{12}$ and $\rho(x) = 0$ for $|x| \geq 2^{13}$. We split $K_{k,j}$ into

$$\begin{aligned} K_{k,j}(x', 0, x, u) &:= 2^{-mk} \sum_n \mathcal{X}_j(m+n) \int P_n(x', 0, x, (u-s)/2) \Phi_{k,j,n}(s) ds \\ &= 2^{-mk} \sum_n \mathcal{X}_j(m+n) \int P_n(x', 0, x, (u-s)/2) \Phi_{k,j,n}(s) (1-\rho)(2^{4(k-j)}w(s)^2) ds \\ &\quad + 2^{-mk} \sum_n \mathcal{X}_j(m+n) \int P_n(x', 0, x, (u-s)/2) \Phi_{k,j,n}(s) \rho(2^{4(k-j)}w(s)^2) ds. \end{aligned}$$

We use integration by parts in the second term. We find that $K_{k,j}$ is a sum of terms of the following types

$$\begin{aligned} 2^{-mk} \sum_n \mathcal{X}_j(m+n) \int P_n(x', 0, x, (u-s)/2) \Phi_{k,j,n}(s) \\ \times (1-\rho)(2^{4(k-j)}w(s)^2) ds, \end{aligned} \quad (5.18a)$$

$$\begin{aligned} 2^{-mk+2k-j} \sum_n \mathcal{X}_j(m+n) \int R_n(x', 0, x, (u-s)/2) \tilde{\Phi}_{k,j,n}(s) \\ \times \rho(2^{4(k-j)}w(s)^2) ds \end{aligned} \quad (5.18b)$$

and

$$\begin{aligned} 2^{-mk} 2^{4(k-j)} \sum_n \mathcal{X}_j(m+n) \int R_n(x', 0, x, (u-s)/2) \Phi_{k,j,n}(s) (u-s) \\ \times \rho'(2^{4(k-j)}w(s)^2) ds. \end{aligned} \quad (5.18c)$$

We have seen before that it suffices to consider only $+$ -terms, since we get the L_1 -estimates for the $-$ -terms by exchanging u with $-u$ and s with $-s$. To make our notation simpler, we put $\omega := \omega^+$ and $\zeta_{j,m} := \zeta_{j,m}^+$. Observe that

$$\begin{aligned} \zeta_{j,m} e^{\pm i/s} &= \sum_{n=0}^{\infty} \mathcal{X}_j(m+n) q_{n,m}^+(\sigma) 2^{-mj} e^{i(n+m+1\pm 1)\omega} \\ &\quad \times \frac{R2xx' + (u-s)w \mp i(Rw - 2xx'(u-s))}{R^2 + (t-s)^2} \\ &= \tilde{\zeta}_j \frac{R2xx' + (u-s)w \mp i(Rw - 2xx'(u-s))}{R^2 + (t-s)^2} \end{aligned}$$

with $\tilde{\zeta}_{j,m}^{\pm} := \zeta_{j,m} e^{\pm i\omega}$ of the same type as ζ_j . Since $2^{-j} \leq 1$ we can write (5.18a), only considering the $+$ -terms, as a finite sum of terms of the following types (compare (5.14))

$$\begin{aligned} &2^{k/2} 2^{(m-1)(j-k)} \int \frac{\zeta_{j,m}(x', x, u, s)}{(R^2 + (u-s)^2)^{(m+1)/2}} \\ &\quad \times \frac{R2xx' + (u-s)w + \epsilon i(Rw - 2xx'(u-s))}{R^2 + (u-s)^2} e^{-i(m+1)\sigma} \\ &\quad \times (1-\rho)(2^{4(k-j)}w(s)^2) \mathcal{X}(2^{k-j}s) ds \end{aligned}$$

with $\epsilon = \pm 1$. We can restrict to the case $\epsilon = 1$, since the second case is similar. For (5.18b), only considering the $+$ -terms, we get (compare (5.15))

$$\begin{aligned} &2^{-mk+2k-j} \sum_n \mathcal{X}_j(m+n) \int R_n^+(x', 0, x, (u-s)/2) \tilde{\Phi}_{k,j,n}(s) \rho(2^{4(k-j)}w(s)^2) ds \\ &= 2^{k/2} 2^{(m-3)(j-k)} \int \frac{\zeta_{j,m-1}(x', x, u, s)}{(R^2 + (u-s)^2)^{m/2}} e^{-im\sigma} \\ &\quad \rho(2^{4(k-j)}w(s)^2) \mathcal{X}(2^{k-j}s) ds. \end{aligned}$$

And for the last term (5.18c), only considering the $+$ -terms, we get (compare (5.15))

$$\begin{aligned} &2^{-mk+4(k-j)} \sum_n \mathcal{X}_j(m+n) \\ &\quad \times \int R_n^+(x', 0, x, (u-s)/2) \tilde{\Phi}_{k,j,n}(s) (u-s) \rho'(2^{4(k-j)}w(s)^2) ds \\ &= 2^{k/2} 2^{(m-5)(j-k)-j} \int \frac{\zeta_{j,m-1}(x', x, u, s)}{(R^2 + (u-s)^2)^{m/2}} (u-s) e^{-im\sigma} \\ &\quad \rho'(2^{4(k-j)}w(s)^2) \mathcal{X}(2^{k-j}s) ds. \end{aligned}$$

We now replace x by $2^{(j-k)/2}x$, u by $2^{j-k}u$ and s by $2^{j-k}s$. Furthermore, we set $x_1 := 2^{(k-j)/2}x'$ and define

$$\zeta_{k,j,m}(x_1, x, u, s) := \sum_n \chi_j(m+n) q_{n,m}^+ 2^{mj} e^{i(n+m+1)\varphi(s)}$$

and

$$\zeta_{k,j,m-1}(x_1, x, u, s) := \sum_n \chi_j(m+n) q_{n,m-1}^+ 2^{mj} e^{i(n+m)\varphi(s)},$$

with

$$\varphi(s) := \arctan\left(\frac{Rw - 2xx_1(u-s)}{R2xx_1 + (u-s)w}\right) + \frac{2^{k-j}}{s}. \quad (5.19)$$

Hence we have to study the following terms.

$$\begin{aligned} F_{k,j}(x_1, x, u) &:= 2^{k/2} 2^{m(j-k)} \int \frac{\zeta_{k,j,m}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{(m+3)/2}} e^{-i(m+1)\sigma} \\ &\quad \times (R2xx_1 + (u-s)w - i(Rw - 2xx_1(u-s))) \\ &\quad \times (1-\rho)(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds \end{aligned} \quad (5.20a)$$

for (5.18a)).

$$\begin{aligned} G_{k,j}(x_1, x, u) &:= 2^{k/2} 2^{(m-1)(j-k)} \\ &\quad \times \int \frac{\zeta_{k,j,m-1}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{m/2}} e^{-im\sigma} \rho(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds \end{aligned} \quad (5.20b)$$

for (5.18b). And at last

$$\begin{aligned} H_{k,j}(x_1, x, u) &:= 2^{k/2} 2^{(m-2)(j-k)-j} \\ &\quad \times \int \frac{\zeta_{k,j,m-1}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{m/2}} (u-s) e^{-im\sigma} \rho'(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds \end{aligned} \quad (5.20c)$$

for (5.18c).

$K_{k,j}$ for the case $2^{j-k} \geq \mathcal{C}_2$

In this case we only use formula (5.15). Thus we have to study

$$G_{k,j}(x, x_1, u) := 2^{k/2} 2^{(m-1)(j-k)} \int \frac{\zeta_{k,j,m-1}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{m/2}} e^{-im\sigma} \mathcal{X}(s) ds. \quad (5.21)$$

5.3 Calculations for the phase function φ

In this section, we study the phase function φ in $\zeta_{k,j,m}$ resp. $\zeta_{k,j,m-1}$. This function was defined in (5.19). In the next chapter, we estimate $F_{k,j}$, $G_{k,j}$ and $H_{k,j}$ by partial integration, hence we have to study the first and second derivative of φ .

We use the abbreviations

$$\begin{aligned} R &:= x^2 + x_1^2, & w &:= \sqrt{R^2 - 4(xx_1)^2 + (u-s)^2}, \\ a &:= \sqrt{R^2 + (u-s)^2}. \end{aligned}$$

Let

$$\begin{aligned} \varphi_0(s) &:= \arctan\left(\frac{x^2 + x_1^2}{u-s}\right) + \frac{2^{k-j}}{s}, \\ \varphi(s) &:= \arctan\left(\frac{Rw - 2xx_1(u-s)}{2Rxx_1 + (u-s)w}\right) + \frac{2^{k-j}}{s}. \end{aligned}$$

Some easy computations yield

$$\begin{aligned} \partial_s w(s) &= \frac{1}{w}(u-s) = \frac{u-s}{((x^2 - x_1^2) + (u-s)^2)^{1/2}}, \\ \partial_s a(s) &= \frac{1}{a}(u-s) = \frac{u-s}{((x^2 + x_1^2) + (u-s)^2)^{1/2}}, \\ \partial_s \varphi_0(s) &= \frac{R}{R^2 + (u-s)^2} - \frac{2^{k-j}}{s^2}, \\ \partial_s \varphi(s) &= \frac{Rw - 2xx_1(u-s)}{(R^2 + (u-s)^2)w} - \frac{2^{k-j}}{s^2}, \\ \partial_s^2 \varphi_0(s) &= \frac{R}{(R^2 + (t-s)^2)^2} 2(t-s) + 2\frac{2^{k-j}}{s^3}, \\ \partial_s^2 \varphi(s) &= \frac{Rw^{-1}(u-s) + 2xx_1}{a^2w} \\ &\quad - \frac{Rw - 2xx_1(u-s)}{a^4w^2} ((2(u-s)w + a^2w^{-1}(u-s)) + 2\frac{2^{k-j}}{s^3}). \end{aligned}$$

REMARK. For $x_1 = 0$, the phase function is the same than in the Heisenberg case. We have

$$\begin{aligned} \varphi(s) &= \arctan\left(\frac{R}{t-s}\right) + \frac{2^{k-j}}{s} \\ \partial_s \varphi(s) &= \frac{R}{R^2 + (t-s)^2} - \frac{2^{k-j}}{s^2}. \end{aligned}$$

◇

Lemma 5.4. *If $s \sim 1$ then*

$$\begin{aligned} |\partial_s \varphi(s)| &\leq c2^{k-j} + 2a^{-1} \\ |\partial_s^2 \varphi(s)| &\leq c2^{k-j} + 6(aw)^{-1} \lesssim (2^{(k-j)/2} + (aw)^{-1/2})^2 \end{aligned}$$

and hence

$$\left| 2^{-j} \frac{\partial_s^2 \varphi}{(\partial_s \varphi)^2}(s) \right| \lesssim 2^{-j} \left(\frac{2^{(k-j)/2} + (aw)^{-1/2}}{\partial_s \varphi} \right)^2.$$

Proof. Follows easily since $|u - s| \leq w$ and $w \leq a$ and thus

$$|\partial_s^2 \varphi(s)| \leq 2a^{-1}w^{-1} + 2a^{-3}w + 2a^{-1}w^{-1} + c2^{k-j} \leq 6(aw)^{-1} + c2^{k-j}.$$

□

5.4 The change of coordinates

Let

$$\tilde{\varphi}(s) := \arctan \left(\frac{Rw - 2xx_1(u - s)}{2Rxx_1 + (u - s)w} \right).$$

Then

$$\partial_s \tilde{\varphi}(s) = \frac{Rw - 2xx_1(u - s)}{(R^2 + (u - s)^2)w}.$$

The following change of coordinates turns out to be useful

$$\begin{aligned} X &:= \frac{Rw - 2xx_1(u - s)}{2Rxx_1 + (u - s)w}, & Y &:= \frac{(R^2 + (u - s)^2)w}{2Rxx_1 + (u - s)w}, \\ s &:= s, & \psi(x, u, s) &:= (X, Y, s). \end{aligned}$$

Observe that $\text{sgn } X = \text{sgn } Y$. Some easy computations show

$$\begin{aligned} \frac{a^2}{|2Rxx_1 + (u - s)w|} &= \langle X \rangle, & w &= \frac{|Y|}{\langle X \rangle}, \\ \frac{Rw - 2xx_1(u - s)}{(R^2 + (u - s)^2)w} &= \frac{X}{Y} = \partial_s \tilde{\varphi}(s). \end{aligned} \tag{5.22}$$

One can compute the inverse of this transformation by solving a third order equation in x^2 and a second order equation in u . But unfortunately we have no simple solutions for these equations.

Nevertheless, we have a simple expression for the functional determinant in the new coordinates.

$$\begin{aligned} |\det(\psi')|^{-1} &= \frac{w|2xx_1R + (u-s)w|^3}{a^4|2x^5 - 2xx_1^4 + 2x(u-s)^2 - 2x_1(u-s)w|} \\ &= \frac{|Y|}{2\langle X \rangle^3} \frac{1}{\sqrt{|XY - x_1^2X^2|}}. \end{aligned} \quad (5.23)$$

This calculation will be done in the following two lemmata.

Lemma 5.5. *The following equation holds*

$$\begin{aligned} |\det(\psi')|^{-1} &= \frac{w|2xx_1R + (u-s)w|^3}{a^4} \frac{1}{|2x^5 - 2xx_1^4 + 2x(u-s)^2 - 2x_1(u-s)w|} \\ &= \frac{|Y|}{\langle X \rangle^4} \frac{a^2}{|2x^5 - 2xx_1^4 + 2x(u-s)^2 - 2x_1(u-s)w|}. \end{aligned}$$

Proof. The proof is elementary but complex. For these computations an algebra system like Maple is very useful. \square

Lemma 5.6. *The following holds*

$$XY - x_1^2X^2 = \frac{(2x^5 - 2xx_1^4 + 2x(u-s)^2 - 2x_1(u-s)w)^2}{4(2xx_1R + (u-s)w)^2}.$$

REMARK. For $x_1 = 0$ we get $XY = \frac{R(R^2 + (u-s)^2)}{(u-s)^2}$. \diamond

Proof.

$$\begin{aligned} XY - x_1^2X^2 &= \frac{(Rw - 2xx_1(u-s))((u-s)^2 + R^2)w}{(2xx_1R + (u-s)w)^2} - x_1^2 \frac{(Rw - 2xx_1(u-s))^2}{(2xx_1R + (u-s)w)^2} \\ &= \frac{(2x^5 - 2xx_1^4 + 2x(u-s)^2 - 2x_1(u-s)w)^2}{4(2xx_1R + (u-s)w)^2} \end{aligned}$$

As before, an algebra system is very useful for these calculations. \square

Thus we get

$$\begin{aligned} |\det(\psi')|^{-1} &= \frac{w|2xx_1R + (u-s)w|^3}{a^4} \frac{1}{|2x^5 - 2xx_1^4 + 2x(u-s)^2 - 2x_1(u-s)w|} \\ &= \frac{w|2xx_1R + (u-s)w|^2}{2a^4} \frac{1}{\sqrt{|XY - x_1^2X^2|}} \\ &= \frac{w}{2\langle X \rangle^2} \frac{1}{\sqrt{|XY - x_1^2X^2|}} = \frac{|Y|}{2\langle X \rangle^3} \frac{1}{\sqrt{|XY - x_1^2X^2|}} \end{aligned}$$

and we have proven equation (5.23).

REMARK. For $x_1 = 0$ we use the same coordinates as in [19], i.e.

$$X = \frac{R}{u-s}, \quad Y = \frac{R^2 + (u-s)^2}{u-s}, \quad s = s.$$

The functional determinant

$$|\det(\psi')|^{-1} = \frac{|Y|}{2\langle X \rangle^4} \frac{\langle X \rangle}{\sqrt{|XY|}},$$

is, of course, also the same, except for the extra term $\langle X \rangle / \sqrt{|XY|} = R^{-1/2}$, which reflects the fact that our manifold \mathbb{R}^2 has one dimension less than the Heisenberg group \mathbb{H}_1 .

◇

In Section 1.4 we mentioned that the case $x_1 \neq 0$ is much harder to understand than the case $x_1 = 0$. Here is one reason for this fact. Our coordinate transform we have to use is much more complicated for $x_1 \neq 0$ as for $x_1 = 0$.

As far as we know, there is no easier transform that allows us to estimate the L_1 -norm of the kernel $K_{k,j}$ properly.

To see that we really can use this transformation, we have to take a closer look at the inverse of it. Though we can not hope to get an explicit and useful formula for ψ^{-1} , we are able to show that ψ can be restricted to a finite number of sets Ω_n such that $\psi|_{\Omega_n}$ is invertible. Hence we are allowed to use the transformation formula.

Observe that by (5.22) we get the following equations for $\tilde{u} = u - s$

$$\begin{aligned} \frac{|X|}{\langle X \rangle} &= \frac{Rw - 2xx_1\tilde{u}}{R^2 + \tilde{u}^2} \\ \tilde{u}^2 &= \frac{Y^2}{\langle X \rangle^2} - R^2 + 4x^2x_1^2 \end{aligned} \tag{5.24}$$

From the first equation we deduce

$$\tilde{u} = \pm \sqrt{\frac{R|Y|}{|X|} - R^2 + x^2x_1^2 \frac{\langle X \rangle^2}{X^2} - xx_1 \frac{\langle X \rangle}{|X|}}.$$

This together with the second equation from (5.24) gives us for $\epsilon \in \{-1, 1\}$

$$\frac{Y^2}{\langle X \rangle^2} - R^2 + 4x^2x_1^2 = \left(\epsilon \sqrt{\frac{R|Y|}{|X|} - R^2 + x^2x_1^2 \frac{\langle X \rangle^2}{X^2} - xx_1 \frac{\langle X \rangle}{|X|}} \right)^2,$$

which implies

$$\left(\frac{Y^2}{\langle X \rangle^2} + 4x^2x_1^2 - 2x^2x_1^2\frac{\langle X \rangle^2}{X^2} - R\frac{Y}{X}\right)^2 - 4\left(R\frac{Y}{X} - R^2 + x^2x_1^2\frac{\langle X \rangle^2}{X^2}\right)x^2x_1^2\frac{\langle X \rangle^2}{X^2} = 0.$$

But this an equation of third order (due to $R^2x^2x_1^2 = (x^2 + x_1^2)^2x^2x_1^2$) in x^2 and hence can be solved.

Now, for a given (x, \tilde{u}) , one can find a locally defined function ψ_{-1} with

$$\psi_{-1}(X, Y) = \psi_{-1}\left(\frac{Rw - 2xx_1\tilde{u}}{2Rxx_1 + \tilde{u}w}, \frac{(R^2 + \tilde{u}^2)w}{2Rxx_1 + \tilde{u}w}\right) = x^2$$

and hence $x = \text{sgn}(x)\sqrt{\psi_{-1}(X, Y)}$. In addition,

$$Y^2/\langle X \rangle^2 - (\psi_{-1}(X, Y) + x_1^2)^2 + 4\psi_{-1}(X, Y)x_1^2 = \tilde{u}^2$$

and $\tilde{u} = \text{sgn}(u)\sqrt{Y^2/\langle X \rangle^2 - (\psi_{-1}(X, Y) + x_1^2)^2 + 4\psi_{-1}(X, Y)x_1^2}$.

Furthermore, there is a finite number N , a measurable set Ω_0 of measure 0 and open sets $\Omega_1, \dots, \Omega_N$ such that $\bigcup_{n=0}^N \Omega_n = \mathbb{R}^2$ and $\psi|_{\Omega_n}$ is a diffeomorphism for every $n \in \{1, \dots, N\}$. On all these sets we may apply the transformation formula. We obtain the following proposition.

Proposition 5.7. *There exists a constant C such that for every measurable $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ with $(X, Y, s) \mapsto f(X, Y, s)|Y| \langle X \rangle^{-3} |XY - x_1^2X^2|^{-1/2} \in L_1(\mathbb{R}^3)$*

$$\begin{aligned} \int |f(X(x, u, s), Y(x, u, s), s)| dx du ds &\leq C \int |f(X, Y, s)| |\det(\psi')|^{-1} dX dY ds \\ &= C \int |f(X, Y, s)| \frac{|Y|}{2\langle X \rangle^3} \frac{1}{\sqrt{|XY - x_1^2X^2|}} dX dY ds \end{aligned}$$

holds.

6 The proof of Proposition 5.3

Define

$$O(x_1) := \{(x, u); |x - x_1| \leq 2^{(k-j)/2} 2\mathcal{C}_0, |u| \leq 2^{k-j} 2\mathcal{C}_0(2 + 2\mathcal{C}_1)\}. \quad (6.1)$$

To prove Proposition 5.3, we now show that the following two estimates hold (compare (5.17)).

(1) For every $\epsilon > 0$ and for all $A \in \{F_{k,j}, G_{k,j}, H_{k,j}\}$, defined by (5.20a), (5.20b) and (5.20c), there exists a constant C_ϵ such that the estimate

$$\sum_{j \in \mathbb{N}_0, 2^{j-k} \leq 1/\mathcal{C}_2} \sup_{|x_1| \leq 2\mathcal{C}_1 2^{(k-j)/2}} \|A\|_{L_1(O(x_1))} \leq C_\epsilon 2^{k\epsilon} \quad (6.2)$$

holds.

(2) For every $\epsilon > 0$ and for $G_{k,j}$, defined by (5.21), there exists a constant C_ϵ such that the estimate

$$\sum_{j \in \mathbb{N}_0, 2^{k-j} \leq \mathcal{C}_2} \sup_{|x_1| \leq 2\mathcal{C}_1 2^{(k-j)/2}} \|G_{k,j}\|_{L_1(O(x_1))} \leq C_\epsilon 2^{k\epsilon} \quad (6.3)$$

holds.

Unfortunately, we cannot express $O(x_1)$ in the new coordinates. We only have to integrate over such X and Y so that for given s and x_1 the corresponding x and u , with $X = X(x_1, x, u, s)$ and $Y = Y(x_1, x, u, s)$, are in $O(x_1)$. For this we write

$$\int_{O(x_1)} \dots dX dY.$$

6.1 Some integrations in the new coordinates

Before we prove the assertions (6.2) and (6.3), we show a few technical lemmata concerning integration with respect to new coordinates X and Y .

Lemma 6.1. *The following estimates hold true*

(a)

$$\int_{c_0 \leq |Y| \leq c_1} \frac{\sqrt{|Y|}}{\sqrt{|Y - x_1^2 X|}} dY \lesssim c_1,$$

(b)

$$\int_{c_0 \leq |Y| \leq c_1} \frac{1}{\sqrt{|Y|}} \frac{1}{\sqrt{|Y - x_1^2 X|}} dY \lesssim 1 + \log\left(\frac{c_1}{c_0}\right).$$

Proof. This estimates follows easily by the transformation $Y \rightarrow Y/(x_1^2|X|)$. In fact

$$\int_{c_0 \leq |Y| \leq c_1} \frac{\sqrt{|Y|}}{\sqrt{|Y - x_1^2 X|}} dY = x_1^2 |X| \int_{c_0/(x_1^2 |X|) \leq |Y| \leq c_1/(x_1^2 |X|)} \frac{\sqrt{|Y|}}{\sqrt{|Y - 1|}} dY. \lesssim c_1.$$

For $x_1^2 |X| \lesssim 1$ we get for $|Y| \leq 2$ the bound $x_1^2 |X| c \lesssim c_1$ and for $|Y| \geq 2$ we get $x_1^2 |X| c_1/(x_1^2 |X|) = c_1$. For $x_1^2 |X| \gtrsim 1$ we get for $|Y| \leq 2$ the bound $c_1 \int_{|Y| \leq 2} (\sqrt{|Y|} \sqrt{|Y - 1|})^{-1} dY \lesssim c_1$. and for $|Y| \geq 2$ we get $x_1^2 |X| c_1/(x_1^2 |X|) = c_1$. This gives us estimate (a). For (b) observe that

$$\begin{aligned} \int_{c_0 \leq |Y| \leq c_1} \frac{1}{\sqrt{|Y|}} \frac{1}{\sqrt{|Y - x_1^2 X|}} dY &= \int_{\frac{c_0}{x_1^2 |X|} \leq |Y| \leq \frac{c_1}{x_1^2 |X|}} \frac{1}{\sqrt{|Y|}} \frac{1}{\sqrt{|Y - 1|}} dY \\ &\lesssim \int_{0 \leq |Y| \leq 1/2} \frac{1}{\sqrt{|Y|}} dY + \int_{\frac{c_0}{x_1^2 |X|} \leq |Y| \leq \frac{c_1}{x_1^2 |X|}, |Y| \geq 2} \frac{1}{|Y|} dY + \int_{Y \sim 1} \frac{1}{\sqrt{|Y - 1|}} dY \\ &\lesssim 1 + \log \left(\frac{c_1}{c_0} \right) \end{aligned}$$

holds. □

Lemma 6.2. (a) For all $\epsilon \geq 0$

$$\int_{|Y - a_2| \leq c} \frac{1}{\sqrt{|Y - a_1|}} dY \lesssim c^{1-\epsilon} |a_1 - a_2|^{-1/2+\epsilon}$$

holds with a constant not depending on a_1, a_2 and c .

(b) For all $\epsilon > 0$

$$\int_{|Y - a_2| \geq c} \frac{c}{|Y - a_2|} \frac{1}{\sqrt{|Y - a_1|}} dY \lesssim c^{1-\epsilon} |a_1 - a_2|^{-1/2+\epsilon}$$

holds with a constant not depending on a_1, a_2 and c .

Proof.

$$\int_{|Y - a_2| \leq c} \frac{1}{\sqrt{|Y - a_1|}} dY = \int_{|Y| \leq c|a_1 - a_2|^{-1}} \sqrt{|a_1 - a_2|} \frac{1}{\sqrt{|Y - 1|}} dY$$

For $Y \ll 1$ the integral is bounded by $c^{1-\epsilon} |a_1 - a_2|^{-1/2+\epsilon}$.

For $Y \sim 1$ we have $1 \lesssim c^{1-\epsilon}|a_1 - a_2|^{-1+\epsilon}$. Since the integral converges we get $c^{1-\epsilon}|a_1 - a_2|^{-1/2+\epsilon}$.

For $Y \gg 1$ we have $\sqrt{|Y-1|} \geq \sqrt{|Y|} \geq |Y|^\epsilon$. The integration gives us the bound $c^{1-\epsilon}|a_1 - a_2|^{-1+\epsilon} \sqrt{a_1 - a_2}$. Thus (a) holds.

$$\int_{|Y-a_2| \geq c} \frac{c}{|Y-a_2|} \frac{1}{\sqrt{|Y-a_1|}} dY \lesssim \int \left(\frac{c}{|Y|}\right)^{1-\epsilon} \frac{1}{|a_1-a_2|^{1/2-\epsilon}} \frac{1}{\sqrt{Y-1}} dY$$

For $Y \ll 1$ and $Y \sim 1$ we get by integration $c^{1-\epsilon}|a_1 - a_2|^{-1/2+\epsilon}$.

For $Y \gg 1$ we have $\sqrt{Y-1} \gtrsim \sqrt{Y}$ and the integral converges. We get $c^{1-\epsilon}|a_1 - a_2|^{-1/2+\epsilon}$. Thus (b) holds. \square

Lemma 6.3. *Let $\epsilon > 0$ and C_1 an arbitrary constant. We define*

$$\psi_{k,j}(X, s) := \frac{2^j}{(1 + 2^j|1 - e^{i(\arctan(X)+2^{k-j}/s)})|^{1+\epsilon}}.$$

Then there exists a constant C such that for all j and k , with $2^{j-k} \leq C_1$, and all $X \in \mathbb{R}$ the estimate

$$\left| \int \psi_{k,j}(X, s) \mathcal{X}(s) ds \right| \leq C$$

holds.

Proof. Put $z := \arctan(X) \in [0, \pi]$. Since $\text{supp } \mathcal{X} \subseteq [1/8, 32]$, we have

$$\begin{aligned} \int |\psi_{k,j}(X, s) \mathcal{X}(s)| ds &\leq 2^{j-k} \int_{2^{k-j-4}}^{2^{k-j+3}} \frac{2^j}{(1 + 2^j|1 - e^{i(z+s)})|^{1+\epsilon}} ds \\ &\leq 2^{j-k} (1 + 2^{k-j}) \int \frac{2^j}{(1 + 2^j|s|)^{1+\epsilon}} ds \lesssim 1 \end{aligned}$$

(compare Lemma 3.1 in [19]). \square

Lemma 6.4. *Let $k, j \in \mathbb{N}_0$ with $2^{k-j} \leq \mathcal{C}_2$. Then*

$$I := \int_{O(x_1)} \psi_{k,j}(X, s) \frac{|Y|^{1/2}}{\sqrt{|XY - x_1^2 X^2|}} \mathcal{X}(s) dX dY ds \lesssim 1$$

holds.

Proof. In this case we have $Y \sim 1$, $\langle X \rangle \sim 1$ and $|X| \lesssim 2^{k-j}$. We consider the case $|Y| \leq 2x_1^2|X|$ first. Since $x_1^2 \lesssim 2^{k-j}$, we get

$$\begin{aligned} &2^{(k-j)/2} \int_{Y \sim 1, |Y| \leq 2x_1^2|X|} \psi_{k,j}(X, s) \frac{1}{\sqrt{|X|}\sqrt{|Y - x_1^2 X^2|}} \mathcal{X}(s) dX dY ds \\ &\lesssim 2^{(k-j)/2} \int_{|X| \leq 1} \psi_{k,j}(X, s) \frac{1}{\sqrt{|X|}} \mathcal{X}(s) dX ds. \end{aligned}$$

For $|X| \leq 2^{k-j}$, we do the s -integration first and obtain

$$2^{(k-j)/2} 2^{j-k} \int_{|X| \leq 2^{k-j}} \frac{1}{\sqrt{|X|}} dX \lesssim 1.$$

For $|X| \geq 2^{k-j}$, we do the X -integration first and obtain

$$\begin{aligned} & 2^{(k-j)/2} \int_{s \sim 1, |X| \geq 1} \psi_{k,j}(X, s) \frac{1}{\sqrt{|X|}} dX ds \\ & \lesssim 2^{(k-j)/2} 2^{(j-k)/2} \int_{s \sim 1, |X| \geq 1} \psi_{k,j}(X, s) dX ds \lesssim \int_{s \sim 1} ds \lesssim 1. \end{aligned}$$

Now, the case $|Y| \geq 2x_1^2|X|$ is left. It is not difficult to see that for every $x' \leq 2\mathcal{C}_1$ and s in the support of \mathcal{X} there exists a set $\Omega(x', s)$ such that

$$Y(x_1, x, u, s) \in \Omega(x', s), \quad (6.4)$$

for $x_1 = 2^{(k-j)/2}x'$ and all $x, u \in O(x_1)$. In addition, there exists a constant C such that for all x' and s the estimate

$$|\Omega(x', s)| \lesssim 2^{k-j}$$

holds. To prove this we write

$$\frac{Y}{\langle X \rangle^2} = \frac{(R^2 + (u-s)^2)w}{a^4} (2Rxx_1 + (u-s)w). \quad (6.5)$$

Now, we have that $\langle X \rangle$, $(u-s)$ and w are $1 + O(2^{k-j})$. Furthermore, u, R, xx_1 are bounded by 2^{k-j} and s is similar to 1. Since $2^{k-j} \leq \mathcal{C}_2$ is small enough we get the result. We do not want to go into the details here. For $x' = 0$ this is evident, since in this case we have

$$\frac{Y}{\langle X \rangle^2} = u - s \quad (6.6)$$

and $|u| \lesssim 2^{k-j}$. With this result we get

$$\begin{aligned} & \int_{O(x_1), |Y| \geq 2x_1^2|X|} \psi_{k,j}(X, s) \frac{|Y|^{1/2}}{\sqrt{|XY - x_1^2X^2|}} \mathcal{X}(s) dX dY ds \\ & \lesssim \int_{O(x_1), |Y| \geq 2x_1^2|X|} \psi_{k,j}(X, s) \frac{1}{\sqrt{|X|}} \mathcal{X}(s) dX dY ds \\ & \lesssim 2^{k-j} \int_{|X| \lesssim 1} \psi_{k,j}(X, s) \frac{1}{\sqrt{|X|}} \mathcal{X}(s) dX ds \lesssim 2^{(k-j)/2} \lesssim 1. \end{aligned} \quad (6.7)$$

□

Lemma 6.5. *Let $\epsilon > 0$ and C an arbitrary constant. We define*

$$\psi_{k,j}(X, s) := \frac{2^j}{(1 + 2^j |1 - e^{i(\arctan(X) + 2^{k-j}/s)}|)^{1+\epsilon}}$$

and $\Sigma := \frac{2^{-j/2+j-k}\langle X \rangle}{|Y - 2^{j-k}s^2X|}$.

(a) *For all j and k with $2^{j-k} \leq C$, the estimate*

$$I := 2^{k/2+(k-j)/2} \times \int_{\Sigma \gtrsim 1} \psi_{k,j}(X, s) \mathcal{X}(s) \frac{\sqrt{Y}}{\sqrt{X} \sqrt{|Y - x_1^2 X|}} \frac{1}{\langle X \rangle^{5/2}} dX dY ds \lesssim 2^{k\epsilon}.$$

holds.

(b) *And similarly, for all j and k with $2^{j-k} \leq C$, the estimate*

$$I := 2^{k/2+(k-j)/2} \times \int_{\substack{\Sigma \lesssim 1, \\ |Y| \lesssim 2^{j-k}\langle X \rangle}} \psi_{k,j}(X, s) \mathcal{X}(s) \frac{(\Sigma + (2^{-j}\Sigma)^{1/2})\sqrt{Y}}{\sqrt{X} \sqrt{|Y - x_1^2 X|}} \frac{1}{\langle X \rangle^{5/2}} dX dY ds \lesssim 2^{k\epsilon}$$

holds.

Proof. Define $\rho_j(x) := 2^j (1 + 2^j |1 - e^{ix}|)^{-1-\epsilon}$. Then

$$\begin{aligned} \int \psi_{k,j}(X, s) \mathcal{X}(s) ds &= \int \rho_j(\arctan(X) + 2^{k-j}/s) \mathcal{X}(s) ds \\ &\lesssim 2^{j-k} \int_{s \sim 2^{k-j}} \rho_j(\arctan(X) + s) ds \lesssim \int_{s \sim 1} \frac{2^j}{(1 + 2^j |\arctan(X) + s|)^{1+\epsilon}} ds \lesssim 1 \end{aligned}$$

since ρ_j is a periodic function.

(a)

First we study the case $Y \geq 2x_1^2 X$ or $Y \leq x_1^2 X/2$. Here we have the estimate

$$I \lesssim 2^{k/2+(k-j)/2} 2^{-j/2+j-k} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{\sqrt{X}} \frac{1}{\langle X \rangle^{3/2}} dX ds \lesssim 1$$

Now $x_1^2 |X|/2 \leq |Y| \leq 2x_1^2 |X|$ and we suppose that $|Y| \geq 4 \cdot 2^{j-k} |X|$ or $|Y| \leq 2^{j-k} |X|/4$. We get

$$I \lesssim 2^{k/2+(k-j)/2} \int_{|Y| \lesssim 2^{-j/2+j-k}\langle X \rangle} \psi_{k,j}(X, s) \mathcal{X}(s) \frac{\sqrt{Y}}{\sqrt{X} \sqrt{|Y - x_1^2 X|}} \frac{1}{\langle X \rangle^{5/2}} dX dY ds$$

Since we only integrate where $Y \sim x_1^2 X$ and since

$$|x_1||X|^{1/2} \lesssim |Y|^{1/2} \lesssim 2^{-j/4+(j-k)/2} \langle X \rangle$$

we get from the Y -integration

$$\begin{aligned} I &\lesssim 2^{k/2+(k-j)/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) 2^{-j/4+(j-k)/2} \langle X \rangle^{1/2} \frac{|x_1|\sqrt{X}}{\sqrt{X}} \frac{1}{\langle X \rangle^{5/2}} dX ds \\ &\lesssim 2^{k/2-j/4} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{2^{-j/4+(j-k)/2} \langle X \rangle^{1/2}}{\sqrt{X}} \frac{1}{\langle X \rangle^2} dX ds \lesssim 1. \end{aligned}$$

We are left with the case $Y \sim 2^{j-k} X$ and $Y \sim x_1^2 X$. First we estimate $|\sqrt{Y}|$ by $2^{(j-k)/2} \sqrt{|X|}$. We get

$$I \lesssim 2^{k/2+(k-j)/2} 2^{(j-k)/2} \int_{\Sigma \gtrsim 1} \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{|Y - x_1^2 X|} \frac{1}{\langle X \rangle^{5/2}} dX dY ds.$$

By Lemma 6.2 (a) we obtain

$$I \lesssim 2^{(j-k)/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{2^{(-j/2+j-k)(-\tilde{\epsilon})}}{|(2^{j-k} s^2 - x_1^2) X|^{1/2-\tilde{\epsilon}}} \frac{1}{\langle X \rangle^{3/2+\tilde{\epsilon}}} dX ds$$

for every $\tilde{\epsilon} \geq 0$. For $X \geq 1$ we get by the transformation $\mu := \arctan X \in [0, \pi]$ and with $\tilde{\epsilon} = 0$

$$\begin{aligned} I &\lesssim 2^{(j-k)/2} \int_0^\pi \int \rho_j(\mu + 2^{k-j}/s) \mathcal{X}(s) \frac{1}{|2^{j-k} s^2 - x_1^2|^{1/2}} d\mu ds \\ &\lesssim 2^{(j-k)+(j-k)/2} \int_0^\pi \int_{s \sim 2^{k-j}} \rho_j(\mu + s) \frac{1}{|2^{k-j} s^{-2} - x_1^2|^{1/2}} d\mu ds \\ &\lesssim 2^{(j-k)+(j-k)/2} \int_0^\pi \int_{s \sim 2^{k-j}} \rho_j(\mu + s) \frac{2^{k-j}}{|2^{k-j} - x_1^2 s^2|^{1/2}} d\mu ds \end{aligned}$$

Since x_1^2 is comparable to 2^{j-k} we get

$$\begin{aligned} &2^{(j-k)/2} \int_0^\pi \int_{s \sim 2^{k-j}} \rho_j(\mu + s) \frac{2^{(k-j)/2}}{|s^2 - 2^{k-j} x_1^{-2}|^{1/2}} d\mu ds \\ &= 2^{(j-k)/2} \int_0^\pi \int_{s \sim 2^{k-j}} \rho_j(\mu + s) \frac{2^{(k-j)/2}}{|s - 2^{(k-j)/2} x_1^{-1}|^{1/2} |s + 2^{(k-j)/2} x_1^{-1}|^{1/2}} d\mu ds \end{aligned}$$

We know that $s + \operatorname{sgn}(x_1)2^{(k-j)/2}x_1^{-1} \geq s \gtrsim 2^{k-j}$. We can assume that $\operatorname{sgn}(x_1) = 1$. The case $\operatorname{sgn}(x_1) = -1$ is similar. We obtain

$$\begin{aligned} I &\lesssim 2^{(j-k)/2} \int_0^\pi \int_{s \sim 2^{k-j}} \rho_j(\mu + s) \frac{1}{|s - 2^{(k-j)/2}x_1^{-1}|^{1/2}} d\mu ds \\ &\lesssim 2^{(j-k)/2} \sum_{\substack{n \sim 2^{k-j} \\ n \sim 2^{k-j}, n \in \mathbb{N}}} \int_{s \in [-\pi, \pi]} \int \rho_j(\mu + n + s) \frac{1}{|s + n - 2^{(k-j)/2}x_1^{-1}|^{1/2}} d\mu ds \\ &= 2^{(j-k)/2} \sum_{\substack{n \sim 2^{k-j} \\ n \sim 2^{k-j}, n \in \mathbb{N}}} \int_{s \in [-\pi, 2\pi]} \int \rho_j(n + s) \frac{1}{|s + n - \mu - 2^{(k-j)/2}x_1^{-1}|^{1/2}} d\mu ds \end{aligned}$$

For $|n - 2^{(k-j)/2}x_1^{-1}| \leq 4\pi$ we get by integration with respect to μ

$$2^{(j-k)/2} \sum_{\substack{|n - 2^{(k-j)/2}x_1^{-1}| \leq 4\pi \\ n \in \mathbb{N}}} \int_{s \in [-\pi, 2\pi]} \rho_j(n + s) |s + n - 2^{(k-j)/2}x_1^{-1}|^{1/2} ds.$$

$|s + n - 2^{(k-j)/2}x_1^{-1}|^{1/2}$ is bounded by a constant. The last integration gives us a one more constant. Hence we get $I \lesssim 2^{(j-k)/2}$.

For $|n - 2^{(k-j)/2}x_1^{-1}| \geq 4\pi$ we get $|s + n - \mu - 2^{(k-j)/2}x_1^{-1}| \geq |n - 2^{(k-j)/2}x_1^{-1}|/2$ and hence

$$\begin{aligned} I &\lesssim 2^{(j-k)/2} \sum_{\substack{|n - 2^{(k-j)/2}x_1^{-1}| \geq 4\pi \\ n \sim 2^{k-j}, n \in \mathbb{N}}} \int_{s \in [-\pi, 2\pi]} \rho_j(n + s) \frac{1}{|n - 2^{(k-j)/2}x_1^{-1}|^{1/2}} ds \\ &\lesssim 2^{(j-k)/2} \sum_{\substack{|n - 2^{(k-j)/2}x_1^{-1}| \geq 4\pi \\ n \sim 2^{k-j}, n \in \mathbb{N}}} \frac{1}{|n - 2^{(k-j)/2}x_1^{-1}|^{1/2}}. \end{aligned}$$

It is easy to observe that this last sum is bounded by $2^{(k-j)/2}$ and hence $I \lesssim 1$.

For $X \leq 1$ and for $\operatorname{sgn}(x_1) = 1$, which we assume, we get with similar arguments and $\arctan X \sim X$

$$\begin{aligned} I &\lesssim 2^{j\epsilon/2+(j-k)/2} \int_{\substack{s \sim 2^{k-j} \\ |X| \leq 1}} \rho_j(X + s) \frac{1}{|s - 2^{(k-j)/2}x_1^{-1}| |X|^{1/2-\epsilon}} dX ds \\ &\lesssim 2^{j\epsilon/2+(j-k)/2} \sum_{\substack{n \sim 2^{k-j} \\ n \in \mathbb{N}}} \int_{\substack{s \in [-\pi, \pi] \\ |X| \leq 1}} \rho_j(X + n + s) \frac{|X|^{-1/2+\epsilon}}{|s + n - 2^{(k-j)/2}x_1^{-1}|^{1/2-\epsilon}} dX ds \\ &= 2^{j\epsilon/2+(j-k)/2} \sum_{\substack{n \sim 2^{k-j} \\ n \in \mathbb{N}}} \int_{\substack{s \in [-\pi, 2\pi] \\ |X| \leq 1}} \rho_j(n + s) \frac{|X|^{-1/2+\epsilon}}{|s + n - X - 2^{(k-j)/2}x_1^{-1}|^{1/2-\epsilon}} dX ds \end{aligned}$$

As before we get, for $|n - 2^{(k-j)/2}x_1^{-1}| \leq 4\pi$,

$$2^{j\epsilon/2} 2^{(j-k)/2} \sum_{|n-2^{(k-j)/2}x_1^{-1}| \leq 4\pi} \int_{s \in [-\pi, 2\pi]} \rho_j(n+s) ds \lesssim 2^{(j-k)/2} 2^{k\epsilon/2}$$

and, for $|n - 2^{(k-j)/2}x_1^{-1}| \geq 4\pi$, the estimate

$$\begin{aligned} 2^{j\epsilon/2} 2^{(j-k)/2} \sum_{\substack{|n-2^{(k-j)/2}x_1^{-1}| \geq 4\pi \\ n \sim 2^{k-j}, n \in \mathbb{N}}} \int_{s \in [-\pi, 2\pi]} \rho_j(n+s) |n - 2^{(k-j)/2}x_1^{-1}|^{-1/2+\epsilon} ds \\ \lesssim 2^{k\epsilon} \end{aligned}$$

holds.

(b)

The proof of (b) is very similar to the proof of (a).

For $Y \geq 2x_1^2X$ or $Y \leq 1/2x_1^2X$ we use

$$\Sigma^{1/2} \leq \Sigma^{1/2-\epsilon/2} \leq 2^{j\epsilon/4+(k-j)\epsilon/2} 2^{-j/2+j-k} |Y - 2^{j-k}s^2X|^{-1+\epsilon/2}$$

and get by integrating with respect to Y .

$$I \lesssim 2^{k\epsilon} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{\sqrt{X}} \frac{1}{\langle X \rangle^{3/2-\epsilon/2}} dX ds \lesssim 2^{k\epsilon}.$$

The case $Y \geq 4 \cdot 2^{j-k}X$ or $Y \leq 1/4 \cdot 2^{j-k}X$. For this part we have the estimate

$$I \lesssim \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{\sqrt{X} \sqrt{|Y - x_1^2X|}} \frac{1}{\langle X \rangle^{3/2}} dX dY ds$$

Since we only integrate where $Y \sim x_1^2X$ and since $|Y| \lesssim 2^{j-k}\langle X \rangle$ we get from the Y -integration

$$I \lesssim 2^{(j-k)/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{\sqrt{X}} \frac{1}{\langle X \rangle} dX ds \lesssim 1$$

The case $Y \sim 2^{j-k}X$ and $Y \sim x_1^2X$ is left. By Lemma 6.2 (b) we get

$$I \lesssim 2^{(j-k)/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{2^{(-j/2+j-k)(-\epsilon)}}{|(2^{j-k}s^2 - x_1^2)X|^{1/2-\epsilon}} \frac{1}{\langle X \rangle^{3/2+\epsilon}} dX ds$$

By similar estimates as in (a) we get $I \lesssim 2^{k\epsilon}$. \square

We get a similar result for $2^{k-j} \lesssim 1$.

Lemma 6.6. *Let $\epsilon > 0$ and $C \leq 1$ a constant. We define*

$$\psi_{k,j}(X, s) := \frac{2^j}{(1 + 2^j |1 - e^{i(\arctan(X) + 2^{k-j}/s)}|)^{1+\epsilon}}$$

and $\Sigma := \frac{2^{-k/2} \langle X \rangle}{|Y - 2^{j-k} s^2 X|}$.

(a) *For all j and k with $2^{k-j} \leq C$, the estimate*

$$I := 2^{k/2+(k-j)/2} \times \int_{\substack{\Sigma \gtrsim 1, X \lesssim 1, \\ Y \sim 1}} \psi_{k,j}(X, s) \mathcal{X}(s) \frac{\sqrt{Y}}{\sqrt{X} \sqrt{|Y - x_1^2 X|}} \frac{1}{\langle X \rangle^{5/2}} dX dY ds \lesssim 2^{j\epsilon}$$

holds.

(b) *And similarly, for all j and k with $2^{k-j} \leq C$, the estimate*

$$I := 2^{k/2+(k-j)/2} \times \int_{\substack{\Sigma \lesssim 1, X \lesssim 1, \\ Y \sim 1}} \psi_{k,j}(X, s) \mathcal{X}(s) \frac{\Sigma \sqrt{Y}}{\sqrt{X} \sqrt{|Y - x_1^2 X|}} \frac{1}{\langle X \rangle^{5/2}} dX dY ds \lesssim 2^{j\epsilon}$$

holds.

Proof. Define $\rho_j(x) := 2^j (1 + 2^j |1 - e^{ix}|)^{-1-\epsilon}$.

(a)

The case $Y \geq 2x_1^2 X$ or $Y \leq 1/2x_1^2 X$. For this part we have the estimate

$$I \lesssim 2^{k/2+(k-j)/2} 2^{-k/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{\sqrt{X}} \frac{1}{\langle X \rangle^{3/2}} dX ds$$

Now, for $|X| \geq 2^{k-j}$, we obtain $I \lesssim 2^{(k-j)/2} 2^{(j-k)/2} = 1$, by integration over X . For $|X| \leq 2^{k-j}$, we first do the s -integration and obtain

$$2^{(k-j)/2} 2^{j-k} \int_{|X| \leq 2^{k-j}} \frac{1}{\sqrt{X}} dX,$$

but this is bounded by a constant.

The case $Y \geq 4 \cdot 2^{j-k} X$ or $Y \leq 1/4 \cdot 2^{j-k} X$. Since we only integrate where $Y \sim x_1^2 X$ we get from the Y -integration

$$\begin{aligned} I &\lesssim 2^{k/2+(k-j)/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) 2^{-k/4} \langle X \rangle^{1/2} \frac{|x_1| \sqrt{X}}{\sqrt{X}} \frac{1}{\langle X \rangle^{5/2}} dX ds \\ &\lesssim 2^{k/2+(k-j)/2} \int \psi_{k,j}(X, s) \mathcal{X}(s) 2^{-k/4} \langle X \rangle^{1/2} \frac{2^{-k/4} \langle X \rangle^{1/2}}{\sqrt{X}} \frac{1}{\langle X \rangle^{5/2}} dX ds \lesssim 1 \end{aligned}$$

The case $Y \sim 2^{j-k}X$ and $Y \sim x_1^2X$

Since $X \lesssim 1$ and $Y \sim 1$ we get by Lemma 6.2 (a)

$$I \lesssim 2^{j\epsilon/2+(k-j)\epsilon} \int \psi_{k,j}(X, s) \mathcal{X}(s) \frac{1}{|(2^{j-k}s^2 - x_1^2)X|^{1/2-\epsilon}} X^{-1+\epsilon} dX ds$$

Now, with $\arctan X \sim X$ and by the transformation $s := 2^{k-j}/s$, we obtain

$$\begin{aligned} & 2^{k\epsilon/2+(k-j)/2} 2^{j-k} \int_{X \lesssim 1} \int_{s \sim 2^{k-j}} \rho_j(X+s) \frac{1}{|x_1^2 - 2^{k-j}/s^2|^{1/2-\epsilon}} X^{-1+\epsilon} dX ds \\ & \lesssim 2^{k\epsilon/2+(k-j)/2} 2^{(j-k)2\epsilon} \int_{X \lesssim 1} \int_{s \sim 2^{k-j}} \rho_j(X+s) \frac{x_1^{-1+2\epsilon}}{|s^2 - 2^{k-j}/x_1^2|^{1/2-\epsilon}} X^{-1+\epsilon} dX ds \end{aligned}$$

Since $x_1^2X \sim Y \sim 1$, the factor $x_1^{-1+2\epsilon}$ is bounded by $|X|^{1/2-\epsilon}$. We assume that $\text{sgn}(x_1) = 1$ and hence $|s + 2^{(k-j)/2}x_1^{-1}| \gtrsim 2^{k-j}$. We obtain

$$\begin{aligned} I & \lesssim 2^{k\epsilon/2} 2^{(j-k)\epsilon} \int_{X \lesssim 1} \int_{s \sim 2^{k-j}} \rho_j(X+s) \frac{1}{|s - 2^{(k-j)/2}/x_1|^{1/2-\epsilon}} X^{-1/2} dX ds \\ & = 2^{k\epsilon/2} 2^{(j-k)\epsilon} \\ & \quad \times \sum_{n \sim 2^{k-j}} \int_{X \lesssim 1} \int_{s \sim 1} \rho_j(n+s) \frac{1}{|s+n-X-2^{(k-j)/2}/x_1|^{1/2-\epsilon}} X^{-1/2} dX ds \end{aligned}$$

Since $2^{k-j} \leq 1$ there is only one summand. For $|n - 2^{(k-j)/2}x_1^{-1}| \leq 64$, we compute the x -integration and get

$$2^{k\epsilon/2} 2^{(j-k)\epsilon} \sum_{|n-2^{(k-j)/2}x_1^{-1}| \leq 64} \int_{s \sim 1} \rho_j(n+s) ds \lesssim 2^{j\epsilon}$$

For $|n - 2^{(k-j)/2}x_1^{-1}| \geq 64$, we get

$$2^{k\epsilon/2} 2^{(j-k)\epsilon} \sum_{\substack{n \sim 2^{k-j}, \\ |n-2^{(k-j)/2}x_1^{-1}| \geq 64}} \int_{s \sim 1} \rho_j(n+s) |n - 2^{(k-j)/2}x_1^{-1}|^{-1/2+\epsilon} ds \lesssim 2^{j\epsilon}$$

with a new ϵ .

Since the proof for (b) is very similar, we omit it. \square

6.2 Estimation for $2^{j-k} \leq 1/\mathcal{C}_2$

Estimation of $H_{k,j}$

Recall the definition of $H_{k,j}$ given by (5.20c). According to this

$$H_{k,j}(x_1, x, u) = 2^{k/2} 2^{(m-2)(j-k)-j} \\ \times \int \frac{\zeta_{k,j,m-1}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{m/2}} (u-s) \frac{(2xx_1 - iw)^m}{a^m} \rho'(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds.$$

Let

$$\kappa(s) := (u-s) \frac{(2xx_1 - iw)^m}{a^{2m}} \rho'(2^{2(k-j)}w(s)^2).$$

Then $a^m w^{-m} |\kappa| \lesssim (2xx_1 - iw)^m a^{-m} w^{-m+1} = w^{1-m}$.

REMARK. On the Heisenberg group \mathbb{H}_m , $\kappa^{\mathbb{H}}$ is defined as

$$\kappa^{\mathbb{H}} := (u-s)(R^2 + (t-s)^2)^{-m/2} \rho'(2^{2(k-j)}a^2)$$

and we have the estimate $\kappa^{\mathbb{H}} \lesssim a^{1-m}$.

◇

We know, by Proposition 3.10, that

$$\zeta_{k,j,m-1} \lesssim 2^{-mj} 2^{j/2+j\epsilon} \frac{a^m}{w^m} \psi_{k,j}(X, s) = 2^{j\epsilon} \frac{a^m}{w^m} \psi_{k,j}(X, s)$$

with

$$\psi_{k,j}(X, s) := \frac{2^j}{(1 + 2^j |1 - e^{i \arctan(X) + 2^{k-j}/s}|)^{1+\epsilon}}.$$

Thus

$$|H_{k,j}| \leq 2^{k/2} 2^{(m-2)(j-k)-j+\epsilon j} \int \psi_{k,j} w^{1-m} \rho'(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds.$$

Observe that for $\rho'(s) \neq 0$ we have

$$\frac{|Y|}{\langle X \rangle} = w(s) \sim 2^{j-k}.$$

Lemma 6.3 implies that the s -integration just gives us a constant. In combination with Lemma 6.1 (a) we find that

$$\|H_{k,j}\|_{L_1(O(x_1))} \lesssim 2^{k/2} 2^{(m-2)(j-k)-j+j\epsilon} 2^{j-k} \\ \times \int_{|Y| \sim 2^{j-k} \langle X \rangle} \psi_{k,j}(X, s) \frac{\langle X \rangle^{1/2}}{|Y|^{1/2}} \frac{|Y|}{2 \langle X \rangle^3} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} ds dX dY \\ \lesssim 2^{k/2} 2^{(m-2)(j-k)-j+\epsilon j} 2^{j-k} \int \frac{2^{j-k} \langle X \rangle}{\langle X \rangle^{2+1/2} |X|^{1/2}} dX \lesssim 2^{-j/2+\epsilon j}$$

holds.

Definition of $\Sigma(s)$

This was the easy part of the proof. For $F_{k,j}$ and $G_{k,j}$ we use partial summation in the s -variable to get the desired estimates. For this, we use a "control quantity", denoted by $\Sigma(s)$. If $\Sigma(s) \lesssim 1$ we "win" by partial integration a factor of size $2^{-j/2}$. For $\Sigma(s) \gtrsim 1$ we use support estimates and also "win" a factor of size $2^{-j/2}$. We define

$$\Sigma(s) := 2^{-j} \left(\frac{2^{(k-j)/2} + w^{-1}}{\partial_s \varphi} \right)^2 \quad (6.8)$$

Then

$$\begin{aligned} \Sigma(s) &= 2^{-j} \left(\frac{2^{(k-j)/2} + w^{-1}}{\frac{X}{Y} - \frac{2^{k-j}}{s^2}} \right)^2 \\ &\sim 2^{-j} \left(\frac{(2^{(j-k)/2} + 2^{j-k} w^{-1})Y}{Y - 2^{j-k} s^2 X} \right)^2 = 2^{-j} \left(\frac{2^{(j-k)/2} |Y| + 2^{j-k} \langle X \rangle}{Y - 2^{j-k} s^2 X} \right)^2. \end{aligned}$$

Estimation of $F_{k,j}$

Recall the definition of $F_{k,j}$. $F_{k,j}$ is given by

$$\begin{aligned} F_{k,j}(x_1, x, u) &= 2^{k/2} 2^{m(j-k)} \int \frac{\zeta_{k,j,m}(x, x_1, u, s)}{(R^2 + (u-s)^2)^{(m+3)/2}} e^{-i(m+1)\sigma} \\ &\quad \times (R2xx_1 + (u-s)w - i(Rw - 2xx_1(u-s))) \\ &\quad \times (1-\rho)(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds. \end{aligned} \quad (6.9)$$

Let

$$\begin{aligned} \kappa(s) &:= \frac{(2xx_1 - iw)^{m+1}}{a^{m+1}} \frac{R2xx_1 + (u-s)w - i(Rw - 2xx_1(u-s))}{a^{m+3}} \\ &\quad \times (1-\rho)(2^{2(k-j)}w(s)^2). \end{aligned}$$

Then we have

$$a^{m+1} w^{-m-1} |\kappa| \lesssim |2xx_1 - iw|^{m+1} a^{-m-1} w^{-m-1} = w^{-m-1}. \quad (6.10)$$

REMARK. On the Heisenberg group \mathbb{H}_m , $\kappa^{\mathbb{H}}$ is defined by

$$\kappa^{\mathbb{H}} := (u-s-iR)(R^2 + (u-s)^2)^{-(m+2)/2} (1-\rho)(2^{2(k-j)}a^2)$$

and we have the estimate $\kappa^{\mathbb{H}} \lesssim a^{-m-1}$. Once more we see that the role of a on the Heisenberg group coincides with the role of w in our proof. \diamond

Since

$$|R2xx_1 + (u-s)w - i(Rw - 2xx_1(u-s))|^2 = R^2a^2 + (u-s)^2a^2 = a^4$$

we get

$$|\partial_s \kappa| \lesssim |\kappa| a^{-1}. \quad (6.11)$$

For s in the support of $(1-\rho)$ and by (6.1) we have

$$c_0 2^{j-k} \leq w(s) \leq c_1 2^{k-j},$$

for suitable constants c_0 and c_1 and $c_0 \geq 64$. Thus

$$c_0 2^{j-k} \langle X \rangle \leq |Y| \leq c_1 2^{k-j} \langle X \rangle. \quad (6.12)$$

Hence $|Y - s^2 2^{j-k} X| \geq |Y|$ for all $s \in \text{supp } \mathcal{X}$ and

$$\begin{aligned} \Sigma(s) &= 2^{-j} \left(\frac{2^{(j-k)/2} |Y| + 2^{j-k} \langle X \rangle}{Y - 2^{j-k} s^2 X} \right)^2 \lesssim 2^{-j} \left(\frac{2^{(j-k)/2} |Y| + 2^{j-k} \langle X \rangle}{|Y|} \right)^2 \\ &\lesssim 2^{-j} (2^{(j-k)/2} + 1)^2 \sim 2^{-j}. \end{aligned}$$

Observe that

$$2^{-j} \left| \frac{\kappa'}{\varphi' \kappa} \right| \lesssim 2^{-j} \frac{w^{-1}}{|\varphi'|} \lesssim (2^{-j} \Sigma)^{1/2} \leq 2^{-j} (2^{(j-k)/2} + 1)^2 \sim 2^{-j} (2^{j-k} + 1)$$

and

$$2^{-j} \frac{|\mathcal{X}'|}{|\varphi' \mathcal{X}|} \lesssim \frac{1}{|\varphi|} \lesssim 2^{-j} \frac{|(a^{-1} + 2^{k-j})|}{|\partial_s \varphi|^2} \lesssim \Sigma(s).$$

hold true. By Lemma 5.4 we have

$$2^{-j} \left| \frac{\partial_s^2 \varphi}{\varphi'(s)^2} \right| \lesssim \Sigma(s)$$

and hence

$$2^{-j} \partial_s \left(\frac{\kappa \mathcal{X}}{\partial_s \varphi} \right) \lesssim (\Sigma(s) + (2^{-j} \Sigma(s))^{1/2}) |\kappa \mathcal{X}|(s).$$

Furthermore,

$$\left| \frac{2^{-j} w^{-1}}{\partial_s \varphi} \right| \lesssim (2^{-j} \Sigma(s))^{1/2}.$$

By the definition of κ we have

$$F_{k,j} = 2^{k/2} 2^{m(j-k)} \int \zeta_{k,j,m}(x_1, x, u, s) \kappa(s) \mathcal{X}(s) ds$$

and

$$\begin{aligned} \zeta_{k,j,m}(x_1, x, u, s) &= \sum_{n=0}^{\infty} \mathcal{X}_j(m+n) q_{n,m}(\sigma) 2^{-mj} \\ &\quad \times \frac{1}{i(n+m+1)\partial_s\varphi} [\partial_s e^{i(n+m+1)\varphi(s)}] \end{aligned}$$

with $\varphi(s) := \arctan(X) + 2^{k-j}/s$. Thus

$$\begin{aligned} F_{k,j} &= 2^{k/2} 2^{m(j-k)} \int \tilde{\zeta}_{j,m}(x_1, x, u-s, 2^{k-j}/s) 2^{-j} \partial_s \left(\frac{\kappa \mathcal{X}}{\partial_s \varphi} \right) ds \\ &\quad + 2^{k/2} 2^{m(j-k)} \int \tilde{\tilde{\zeta}}_{j,m}(x_1, x, u-s, 2^{k-j}/s) \frac{2^{-j} w^{-1}}{\partial_s \varphi} \kappa \mathcal{X} ds \end{aligned}$$

with

$$\begin{aligned} \tilde{\zeta}_{k,j,m} &:= \sum_{n=0}^{\infty} \mathcal{X}_j(m+n) q_{n,m}(\sigma) \frac{2^{(1-m)j}}{i(n+m+1)} e^{i(n+m+1)\varphi} \\ \tilde{\tilde{\zeta}}_{k,j,m} &:= \sum_{n=0}^{\infty} \mathcal{X}_j(m+n) [\partial_s q_{n,m}(\sigma)] \frac{2^{(1-m)j} w}{i(n+m+1)} e^{i(n+m+1)\varphi}. \end{aligned}$$

We know, by Proposition 3.9, that the following two inequalities

$$\begin{aligned} \tilde{\zeta}_{k,j,m} &\lesssim 2^{-mj} 2^{j/2+j\epsilon} \frac{a^{m+1}}{w^{m+1}} \frac{2^j}{(1+2^j|\varphi|)^{1+\epsilon}} \sim 2^{j\epsilon} \frac{a^{m+1}}{w^{m+1}} \psi_{k,j}(X, s), \\ \tilde{\tilde{\zeta}}_{k,j,m} &\lesssim 2^{-mj} 2^{j/2+j\epsilon} \frac{a^{m+2}}{w^{m+2}} a^{-1} w \frac{2^j}{(1+2^j|\varphi|)^{1+\epsilon}} \sim 2^{j\epsilon} \frac{a^{m+1}}{w^{m+1}} \psi_{k,j}(X, s) \end{aligned} \tag{6.13}$$

hold, with

$$\psi_{k,j}(X, s) := \frac{2^j}{(1+2^j|1 - e^{i \arctan(X) + 2^{k-j}/s}|)^{1+\epsilon}}.$$

Let

$$J := [c_0 2^{j-k} \langle X \rangle, c_1 2^{k-j} \langle X \rangle].$$

By Lemma 6.3 the s -integration $\int \psi_{k,j}(X, s) \mathcal{X}(s) ds$ yields a constant. In com-

bination with Lemma 6.1 (b) we thus find that

$$\begin{aligned}
& \left\| 2^{k/2} 2^{m(j-k)} \int \tilde{\zeta}_{k,j,m}(x_1, x, u, s) 2^{-j} \partial_s \left(\frac{\kappa \mathcal{X}}{\partial_s \varphi} \right) ds \right\|_{L_1(O(x_1))} \\
& \leq 2^{k/2} 2^{m(j-k)} \left\| \int |\tilde{\zeta}_{k,j,m}| (\Sigma(s) + (2^{-j}\Sigma(s))^{1/2}) |\kappa \mathcal{X}| ds \right\|_{L_1(O(x_1))} \\
& \lesssim 2^{-j/2+\epsilon j} \int_{|Y| \in J} \psi_{k,j}(X, s) \frac{\langle X \rangle^{3/2}}{|Y|^{3/2}} \frac{|Y|}{2\langle X \rangle^3} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} dX dY ds \\
& \lesssim 2^{-j/2+\epsilon j} \int_{|Y| \in J} \frac{1}{|Y|^{1/2} \langle X \rangle^{3/2}} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} dX dY \\
& \lesssim 2^{-j/2+\epsilon j} \int (1 + \log \left(\frac{C_1}{C_0} 2^{2(k-j)} \right)) \frac{1}{|X|^{1/2} \langle X \rangle^{3/2}} dX \lesssim 2^{-j/2+\epsilon j} k.
\end{aligned}$$

holds. For the term involving $\tilde{\zeta}_{k,j,m}$ we get the same result. This implies

$$\sum_{j-k \lesssim 1} \|F_{k,j}\|_{L_1(O(x_1))} \lesssim k^2.$$

Estimation of $G_{k,j}$

The proof for $G_{k,j}$ is similar to the proof for $F_{k,j}$, except for that we have here a set of points for which the partial integration does not yields 2^{-j} . We use estimates for the measure of this set, to get also an additional $2^{-j/2}$. $G_{k,j}$ is defined by

$$\begin{aligned}
G_{k,j}(x_1, x, u) &= 2^{k/2} 2^{(m-1)(j-k)} \\
&\times \int \frac{\zeta_{j,m-1}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{m/2}} \frac{(2xx_1 - iw)^m}{a^m} \rho(2^{2(k-j)}w(s)^2) \mathcal{X}(s) ds.
\end{aligned}$$

with

$$\zeta_{k,j,m-1}(x_1, x, u, s) = \sum_n \mathcal{X}_j(m+n) q_{n,m-1} 2^{mj} e^{i(n+m)\varphi}.$$

Let

$$\kappa(s) := \frac{(2xx_1 - iw)^m}{a^{2m}} \rho(2^{2(k-j)}w(s)^2).$$

Then $a^m w^{-m} |\kappa| \lesssim |2xx_1 - iw|^m a^{-m} w^{-m} = w^{-m}$.

REMARK. On the Heisenberg group \mathbb{H}_m , $\kappa^{\mathbb{H}}$ is defined as

$$\kappa^{\mathbb{H}} := (R^2 + (t-s)^2)^{-m/2} \rho(2^{2(k-j)}a^2)$$

and we have the estimate $\kappa^{\mathbb{H}} \lesssim a^{-m}$.

◇

For s in the support of ρ we have

$$w(s) \lesssim 2^{j-k}$$

and thus

$$|Y| \lesssim 2^{j-k} \langle X \rangle. \quad (6.14)$$

Let

$$\begin{aligned} \Sigma(s) &:= 2^{-j} \left(\frac{2^{(k-j)/2} + w^{-1}}{\partial_s \varphi} \right)^2 = 2^{-j} \left(\frac{2^{(k-j)/2} + w^{-1}}{\frac{X}{Y} - \frac{2^{k-j}}{s^2}} \right)^2 \\ &\sim 2^{-j} \left(\frac{(2^{(j-k)/2} + 2^{j-k} w^{-1}) Y}{Y - 2^{j-k} s^2 X} \right)^2 = 2^{-j} \left(\frac{2^{(j-k)/2} |Y| + 2^{j-k} \langle X \rangle}{Y - 2^{j-k} s^2 X} \right)^2. \end{aligned}$$

Since $|Y| \lesssim 2^{j-k} \langle X \rangle$ we see that

$$\Sigma(s) \sim 2^{-j} \left(\frac{2^{j-k} \langle X \rangle}{Y - 2^{j-k} s^2 X} \right)^2. \quad (6.15)$$

holds. This implies that

$$|Y - 2^{j-k} s^2 X| \lesssim 2^{-j/2+j-k} \langle X \rangle, \quad (6.16)$$

if $\Sigma(s) \geq 1$. Now, fix a cut-off function ρ_1 supported in $|x| \leq 2$ with $\rho_1(x) = 1$ for $|x| \leq 1$. We write

$$\begin{aligned} |G_{k,j}|(x, x_1, u) &= 2^{k/2} 2^{(m-1)(j-k)} \\ &\quad \times \left| \int \zeta_{k,j,m-1}(x, x_1, u, s) \kappa(s) (\rho_1(\Sigma(s)) + (1 - \rho_1)(\Sigma(s))) \mathcal{X}(s) ds \right| \\ &\lesssim G_{k,j}^1 + G_{k,j}^2 + G_{k,j}^3 + G_{k,j}^4 \end{aligned}$$

with

$$\begin{aligned} G_{k,j}^1 &:= 2^{k/2+(m-1)(j-k)} \int_{1 \leq \Sigma \leq 2} \left| \tilde{\zeta}_{k,j,m-1}(x_1, x, u, s) \left(\frac{\kappa \mathcal{X}}{\varphi'} \right)(s) 2^{-j} \Sigma'(s) \right| ds \\ G_{k,j}^2 &:= 2^{k/2+(m-1)(j-k)} \int_{\Sigma \leq 2} \left| \tilde{\zeta}_{k,j,m-1}(x_1, x, u, s) 2^{-j} \partial_s \left(\frac{\kappa \mathcal{X}}{\varphi'} \right)(s) \right| ds \\ G_{k,j}^3 &:= 2^{k/2+(m-1)(j-k)} \int_{\Sigma \geq 1} \left| \zeta_{k,j,m-1}(x_1, x, u, s) (\kappa \mathcal{X})(s) \right| ds \\ G_{k,j}^4 &:= 2^{k/2+(m-1)(j-k)} \int_{\Sigma \leq 2} \left| \tilde{\zeta}_{k,j,m-1}(x_1, x, u, s) 2^{-j} \left(\frac{\kappa \mathcal{X}}{w \varphi'} \right)(s) \right| ds \end{aligned}$$

and

$$\begin{aligned}\tilde{\zeta}_{k,j,m-1}(x, x_1, u, s) &= \sum_n \mathcal{X}_j(m+n) q_{n,m-1} \frac{2^{(1+m)j}}{i(n+m)} e^{i(n+m)\varphi} \\ \tilde{\tilde{\zeta}}_{k,j,m-1}(x, x_1, u, s) &= \sum_n \mathcal{X}_j(m+n) (\partial_s q_{n,m-1}) \frac{2^{(1+m)j}}{i(n+m)} e^{i(n+m)\varphi}\end{aligned}$$

Notice that

$$\begin{aligned}|\Sigma'(s)| &\lesssim 2^{-j/2} \Sigma^{1/2} \left(\left| \frac{w^{-2}}{\varphi'} \right| + 2^{j/2} \Sigma^{1/2} \left| \frac{\varphi''}{\varphi'} \right| \right) \\ &\lesssim (2^{j/2} \Sigma^{3/2} + 2^j \Sigma^2) |\varphi'| \lesssim 2^j |\varphi'|,\end{aligned}\tag{6.17}$$

if $\Sigma(s) \lesssim 1$, and hence $|G_{k,j}^1| \lesssim |G_{k,j}^3|$.

Estimates for $G_{k,j}^1$ and $G_{k,j}^3$.

We get by Proposition 3.9

$$\tilde{\zeta}_{k,j,m-1} \lesssim 2^{-mj} 2^{j/2+j\epsilon} \frac{a^m}{w^m} \frac{2^j}{(1+2^j|\varphi|)^{1+\epsilon}} \sim 2^{j\epsilon} \frac{a^m}{w^m} \psi_{k,j}(X, s),\tag{6.18}$$

with

$$\psi_{k,j} := \frac{2^j}{(1+2^j|1-e^{i\varphi}|)^{1+\epsilon}}.$$

By Lemma 6.5(a) we deduce

$$\begin{aligned}&\left\| 2^{k/2} 2^{(m-1)(j-k)} \int_{\Sigma(s) \geq 1} \tilde{\zeta}_{j,m-1}(x_1, x, u, s) \kappa(s) \mathcal{X}(s) ds \right\| \\ &\lesssim 2^{k/2+\epsilon j} 2^{(m-1)(j-k)} \\ &\quad \times \int_{\Sigma(s) \geq 1} \psi_{k,j}(X, s) \frac{\langle X \rangle^{1/2}}{|Y|^{1/2}} \frac{|Y|}{2 \langle X \rangle^3} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} dX dY ds \\ &= 2^{k/2-1+\epsilon j} 2^{(m-1)(j-k)} \\ &\quad \times \int_{\Sigma(s) \geq 1} \psi_{k,j}(X, s) \frac{|Y|^{1/2}}{\langle X \rangle^{5/2}} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} dX dY ds \\ &\lesssim 2^{2\epsilon k}.\end{aligned}$$

Estimates for $G_{k,j}^2$ and $G_{k,j}^4$.

We know by Proposition 3.9

$$\begin{aligned}\tilde{\zeta}_{k,j,m-1} &\lesssim 2^{mj} 2^{-j/2+j\epsilon} \frac{a^m}{w^m} \frac{2^j}{(1+2^j|\varphi|)^{1+\epsilon}} \sim 2^{j\epsilon} \frac{a^m}{w^m} \psi_{k,j}(X, s) \\ \tilde{\zeta}_{k,j,m-1} &\lesssim 2^{mj} 2^{-j/2+j\epsilon} \frac{a^{m+1}}{w^{m+1}} a^{-1} w \frac{2^j}{(1+2^j|\varphi|)^{1+\epsilon}} \sim 2^{j\epsilon} \frac{a^m}{w^m} \psi_{k,j}(X, s),\end{aligned}\tag{6.19}$$

with

$$\psi_{k,j} := \frac{2^j}{(1+2^j(1-e^{i\omega}))^{1+\epsilon}}.$$

Easy calculations show

$$2^{-j} \left(\frac{\kappa \mathcal{X}}{w\varphi'} \right) (s) \lesssim (2^{-j}\Sigma(s))^{1/2} (\kappa \mathcal{X})(s)\tag{6.20}$$

and

$$2^{-j} \partial_s \left(\frac{\kappa \mathcal{X}}{\varphi'} \right) (s) \leq (\Sigma + (2^{-j}\Sigma)^{1/2})(s) (\kappa \mathcal{X})(s).$$

In this case we gain by the integration by parts. We have the following estimate

$$2^{-j/2+j-k} \langle X \rangle \lesssim |Y - 2^{j-k} s^2 X| \lesssim 2^{j-k} \langle X \rangle$$

and hence

$$2^{k-j} \langle X \rangle^{-1} \lesssim \Sigma(s) \leq 1.$$

This implies, by Lemma 6.5(b),

$$\begin{aligned}\max\{\|G_{k,j}^2\|_{L_1(O(x_1))}, \|G_{k,j}^4\|_{L_1(O(x_1))}\} &\lesssim 2^{k/2+\epsilon j} 2^{(m-1)(j-k)} \\ \int_{\Sigma \leq 2} \psi_{k,j}(X, s) (\Sigma + (2^{-j}\Sigma)^{1/2}) \mathcal{X}(s) &\frac{|Y|^{1/2}}{\langle X \rangle^{5/2} |X|^{1/2}} \frac{1}{\sqrt{|Y - x_1^2 X|}} dX dY ds \\ &\leq 2^{2\epsilon k}.\end{aligned}$$

6.3 Estimation for $2^{k-j} \leq \mathcal{C}_2$

Definition of $\Sigma(s)$

We define the control quantity Σ in this case by

$$\Sigma(s) := 2^{-j} \left(\frac{2^{(k-j)/2}}{\partial_s \varphi} \right)^2 = 2^{-j} \left(\frac{2^{(k-j)/2}}{\frac{X}{Y} - \frac{2^{k-j}}{s^2}} \right)^2 \sim 2^{-j} \left(\frac{2^{(j-k)/2} Y}{Y - 2^{j-k} s^2 X} \right)^2.$$

Estimation of $G_{k,j}$

Recall that in this case $G_{k,j}$ is given by

$$G_{k,j} := 2^{k/2} 2^{(m-1)(j-k)} \int \frac{\zeta_{k,j,m-1}(x_1, x, u, s)}{(R^2 + (u-s)^2)^{m/2}} \frac{(2xx_1 - iw)^m}{a^m} \mathcal{X}(s) ds. \quad (6.21)$$

Let

$$\kappa(s) := \frac{(2xx_1 - iw)^m}{a^{2m}}.$$

Then $a^m w^{-m} |\kappa| = |2xx_1 - iw|^m a^{-m} w^{-m} = w^{-m}$.

REMARK. On the Heisenberg group \mathbb{H}_m , $\kappa^{\mathbb{H}}$ is defined as

$$\kappa^{\mathbb{H}} := (R^2 + (u-s)^2)^{-m/2}$$

and we have the estimate $\kappa^{\mathbb{H}} \lesssim a^{-m}$. ◇

The support of \mathcal{X} is contained in $[1/8, 32]$. By (5.16) and (6.1) we get for s in the support of \mathcal{X} the following estimates

$$\begin{aligned} |x^2 - x_1^2| &= |(x+x_1)(x-x_1)| \leq (2\mathcal{C}_0 2^{(k-j)/2} + 4\mathcal{C}_1 2^{(k-j)/2}) 2^{(k-j)/2+1} \mathcal{C}_0 \\ &\leq 2^{k-j+3} (\mathcal{C}_0 + \mathcal{C}_1)^2 \leq 8\mathcal{C}_2 (\mathcal{C}_0 + \mathcal{C}_1)^2 \leq 1/32, \\ R = x^2 + x_1^2 &\leq 16\mathcal{C}_2 (\mathcal{C}_0 + \mathcal{C}_1)^2 \leq 1/16, \\ u &\leq 2^{k-j+3} (\mathcal{C}_0 + \mathcal{C}_1)^2 \leq 8\mathcal{C}_2 (\mathcal{C}_0 + \mathcal{C}_1)^2 \leq 1/32, \\ (u-s) &\sim 1, \quad w(s) = \sqrt{(x^2 - x_1^2)^2 + (u-s)^2} \sim 1. \end{aligned}$$

We get

$$\begin{aligned} |X| &= \left| \frac{Rw + 2xx_1(u-s)}{(u-s)w - 2Rxx_1} \right| \lesssim \frac{Rw}{w} \lesssim 2^{k-j} < 1 \\ |Y| &= \left| \frac{(R^2 + (u-s)^2)w}{(u-s)w - 2Rxx_1} \right| \sim 1, \end{aligned}$$

which implies

$$|Y| \sim \langle X \rangle \sim 1. \quad (6.22)$$

Furthermore, $|\partial_s^2 \varphi| \lesssim 2^{k-j}$ and $|\partial_s \varphi| \lesssim 2^{k-j}$. Hence easy calculations show

$$2^{-j} \left(\frac{\kappa \mathcal{X}}{w \varphi'} \right) (s) \lesssim \frac{2^{k-2j}}{|\varphi'|^2} (\kappa \mathcal{X})(s) = \Sigma(s) (\kappa \mathcal{X})(s) \quad (6.23)$$

and

$$2^{-j} \partial_s \left(\frac{\kappa \mathcal{X}}{\varphi'} \right) (s) \lesssim \frac{2^{k-2j}}{|\varphi'|^2} (\kappa \mathcal{X})(s) = \Sigma(s) (\kappa \mathcal{X})(s).$$

Since $|Y| \sim 1$ we see that

$$\Sigma(s) \sim \frac{2^{-k}}{|Y - 2^{j-k} s^2 X|^2} \quad (6.24)$$

holds. This implies

$$|Y - 2^{j-k} s^2 X| \lesssim 2^{-k/2}, \quad (6.25)$$

if $\Sigma(s) \geq 1$. Fix a cut-off function ρ_1 supported in $|x| \leq 2$ with $\rho_1(x) = 1$ for $|x| \leq 1$. We can write $G_{k,j}$ as

$$\begin{aligned} |G_{k,j}| &= 2^{k/2} 2^{(m-1)(j-k)} \\ &\quad \left| \int \zeta_{k,j,m-1}(x_1, x, u, s) \kappa(s) \rho_1(\Sigma(s) + (1 - \rho_1(\Sigma(s))) \mathcal{X}(s)) ds \right| \\ &\lesssim G_{k,j}^1 + G_{k,j}^2 + G_{k,j}^3 + G_{k,j}^4 \end{aligned}$$

with

$$\begin{aligned} G_{k,j}^1 &:= 2^{k/2+(m-1)(j-k)-j} \int_{1 \leq \Sigma \leq 2} \left| \tilde{\zeta}_{k,j,m-1}(x_1, x, u, s) \left(\frac{\kappa \mathcal{X}}{\varphi'} \right)(s) (\partial_s \Sigma)(s) \right| ds \\ G_{k,j}^2 &:= 2^{k/2+(m-1)(j-k)} \int_{\Sigma \leq 2} \left| \tilde{\zeta}_{k,j,m-1}(x_1, x, u, s) (\Sigma + (2^{-j} \Sigma)^{1/2})(s) (\kappa \mathcal{X})(s) \right| ds \\ G_{k,j}^3 &:= 2^{k/2+(m-1)(j-k)} \int_{\Sigma \geq 1} \left| \zeta_{k,j,m-1}(x_1, x, u, s) (\kappa \mathcal{X})(s) \right| ds \\ G_{k,j}^4 &:= 2^{k/2+(m-1)(j-k)} \int_{\Sigma \leq 2} \left| \tilde{\zeta}_{k,j,m-1}(x_1, x, u, s) (2^{-j} \Sigma)^{1/2}(s) (\kappa \mathcal{X})(s) \right| ds, \end{aligned}$$

and $\tilde{\zeta}_{k,j,m-1}$ and $\tilde{\zeta}_{k,j,m-1}$ defined as before. We have $|G_{k,j}^1| \lesssim |G_{k,j}^3|$. Notice that

$$|\partial_s \Sigma|(s) \lesssim |2^{-j} \frac{2^{k-j}}{(\partial_s \varphi)^3} \partial_s^2 \varphi| \lesssim 2^j |\varphi'|, \quad (6.26)$$

if $\Sigma(s) \lesssim 1$. And hence $|G_{k,j}^1| \lesssim |G_{k,j}^3|$.

Estimates for $G_{k,j}^1$ and $G_{k,j}^3$.

In this case we have by Lemma 6.6

$$\begin{aligned} \|G_{k,j}^3\|_{L_1(O(x_1))} &= \left\| 2^{k/2} 2^{(m-1)(j-k)} \int_{\substack{O(x_1) \\ \Sigma(s) \geq 1}} \zeta_{k,j,m}(x_1, x, u, s) \kappa(s) \mathcal{X}(s) ds \right\|_{L_1(O(x_1))} \\ &\lesssim 2^{k/2+\epsilon j} 2^{(m-1)(j-k)} \\ &\quad \times \int_{\substack{O(x_1) \\ \Sigma(s) \geq 1}} \psi_{k,j}(X, s) \frac{|Y|^{1/2}}{2 < X >^{5/2}} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} dX dY ds \\ &\lesssim 2^{2\epsilon j}. \end{aligned}$$

In addition with the trivial estimate in Lemma 6.4,

$$\begin{aligned} \|G_{k,j}^3\|_{L_1(O(x_1))} &\lesssim 2^{k/2+\epsilon j} 2^{(m-1)(j-k)} \\ &\quad \times \int_{O(x_1)} \psi_{k,j}(X, s) \frac{|Y|^{1/2}}{2 < X >^{5/2}} \frac{1}{\sqrt{|XY - x_1^2 X^2|}} dX dY ds \\ &\lesssim 2^{k-j/2+\epsilon j}, \end{aligned}$$

we get

$$\|G_{k,j}^1\|_{L_1(O(x_1))} + \|G_{k,j}^3\|_{L_1(O(x_1))} \lesssim \min\{2^{2j\epsilon}, 2^{k-j/2+\epsilon j}\}.$$

Hence, with $2^M \sim 1/\mathcal{C}_2$,

$$\begin{aligned} \sum_{j>k+M} \|G_{k,j}^1\|_{L_1(O(x_1))} + \|G_{k,j}^3\|_{L_1(O(x_1))} \\ &\lesssim \sum_{j>k+M, j \leq 2k} 2^{2\epsilon j} + \sum_{j>k+M, j > 2k} 2^{k-j/2+\epsilon j} \\ &\lesssim k 2^{2\epsilon k} + \sum_{j=0}^{\infty} 2^{-j/2+\epsilon j} \lesssim k 2^{k\epsilon}. \end{aligned}$$

Estimates for $G_{k,j}^2$ and $G_{k,j}^4$.

In this case we gain by the integration by parts. We deduce by Lemma 6.6

$$\begin{aligned} \|G_{k,j}^2\|_{L_1(O(x_1))} + \|G_{k,j}^4\|_{L_1(O(x_1))} \\ &\lesssim 2^{k/2+\epsilon j+(m-1)(j-k)} \\ &\quad \times \int_{\substack{O(x_1) \\ \Sigma \leq 2}} \psi_{k,j}(X, s) \Sigma(s) \frac{|Y|^{1/2}}{\sqrt{|XY - x_1^2 X^2|}} dX dY ds \\ &\lesssim 2^{2\epsilon j}. \end{aligned}$$

Together with the trivial estimate, Lemma 6.4,

$$2^{k/2+\epsilon j+(m-1)(j-k)} \int_{\substack{O(x_1) \\ \Sigma \leq 1, Y \sim 1}} |\Sigma|^{1/2} \frac{|Y|^{1/2}}{\sqrt{|XY - x_1^2 X^2|}} dY \lesssim 2^{k-j/2+\epsilon j}$$

we get, with $2^M \sim 1/\mathcal{C}_2$,

$$\begin{aligned} \sum_{j>k+M} \|G_{k,j}^2\|_{L_1(O(x_1))} + \|G_{k,j}^4\|_{L_1(O(x_1))} &= \sum_{j>k+M, j<2k} 2^{2\epsilon j} + \sum_{j>k+M, j>2k} 2^{k-j/2+\epsilon j} \\ &\lesssim k 2^{2\epsilon k}. \end{aligned}$$

This completes the proof of Proposition 5.3 and hence completes the proof of Theorem 2.

Index of notation

\mathbb{C}	set of complex numbers
C_∞	set of continuous, in ∞ vanishing functions
C^∞	set of smooth functions
C_0^∞	set of compactly supported smooth function
D'	set of distributions
\mathbb{H}_1	Heisenberg group of dimension 3, page 11
\mathbb{H}_m	Heisenberg group of dimension $2m + 1$, page 11
L_1	set of integrable functions
L_p	set of measurable functions with $ f ^p \in L_1$
L_p^α	set of measurable function f with $\ f\ _{L_p^\alpha} < \infty$
\mathbb{N}	set of natural numbers
\mathbb{N}_0	$\mathbb{N} \cup \{0\}$
\mathbb{R}	set of real numbers
\mathbb{R}^+	set of real numbers x with $x \geq 0$
\mathcal{S}	set of Schwartz functions
W_p^s	L_p -Sobolev space of order s
\mathbb{Z}	set of integers
$\ \cdot\ _{L_p^\alpha}$	for a differential operator A , $\ f\ _{L_p^\alpha} = \ (1 + A)^{\alpha/2}f\ _p$, page 1
$\ \cdot\ _{Schur}$	Schur norm, page 10
a	$((x^2 + x'^2)^2 + u^2)^{1/2}$ resp. $((x^2 + x'^2)^2 + (u - s)^2)^{1/2}$, pages 34, 74
α	$> 1/2$, page 50
$\alpha(d, p)$	$(d - 1) 1/p - 1/2 $, page 2
B_A	ball with respect to the metric d_A , page 18
$\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$	general constants, pages 20, 26, 71
d_A	optimal control metric of the differential operator A , page 17
δ_r	$(x, u) \mapsto (rx, r^2u)$, automorphic dilation, page 11
G	Grušin operator, page 11
h^α	page 50
L	sub-Laplacian on \mathbb{H}_m , page 12
m	$1/2$, page 26
$m^\alpha(G)$	$\exp(i\sqrt{G})(1 + G)^{-\alpha/2}$, page 26
$O(x_1)$	page 79
$\mathcal{P}_{n,\epsilon}$	spectral projection operator belonging to the ray $\mathcal{R}_{n,\epsilon}$, page 31
R	$x^2 + x'^2$, pages 34, 74
$\mathcal{R}_{n,\epsilon}$	ray in the joint spectrum of G and iU , page 30

σ	page 34
Σ	pages 91, 97
$S_C, S_{\mathcal{C}_1}$	pages 5, 25
U	∂_u , page 30
w	$((x^2 - x'^2)^2 + u^2)^{1/2}$ resp. $((x^2 - x'^2)^2 + (u - s)^2)^{1/2}$, pages 34, 74
X, Y	new coordinates, page 75
\mathcal{X}_A	indicator function of the set A
$\tilde{\mathcal{X}}_B$	"smooth indicator function" associated to the ball B , page 59

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch meinen akademischen Lehrer, in Inhalt und Form selbst angefertigt habe. Ein Artikel mit Resultaten dieser Arbeit wurde im Oberwolfach Report No. 30/2005 des Mathematischen Forschungsinstituts Oberwolfach veröffentlicht. Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Des Weiteren versichere ich, noch keinen Promotionsversuch unternommen zu haben.

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