
SUB-RIEMANNIAN GEOMETRY AND HEAT KERNEL ESTIMATES

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DISSERTATION

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität
zu Kiel



Kiel 2008

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Tag der mündlichen Prüfung: 25.4. / 2.5. / 7.5. 2008
Zum Druck genehmigt:

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Abstract

In this thesis we study the sub-Riemannian geometry associated with the Grušin operator and on 2-step nilpotent Lie groups to obtain heat kernel estimates.

In particular let \mathfrak{n} be a nilpotent Lie algebra of step 2, that is a finite dimensional real Lie algebra such that $\mathfrak{z} := [\mathfrak{n}, \mathfrak{n}] \neq \{0\}$ and $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{0\}$. The corresponding connected, simply connected Lie group N can always be realized as $\mathbb{R}^n \times \mathbb{R}^m$ with group product

$$(x, z) \cdot (y, u) = \left(x + y, z + u - \frac{1}{2} \left(\left\langle J^j x, y \right\rangle \right)_{j=1}^m \right), (x, z), (y, u) \in N$$

where J^j , $j = 1, \dots, m$ are skew-symmetric matrices acting on \mathbb{R}^n . Let X_1, \dots, X_n denote left invariant vector fields such that $X_j(0) = \partial_{x_j}$, $j = 1, \dots, n$, which generate the Lie algebra. We study the sub-Riemannian geometry associated with these vector fields and give a complete description of the Carnot-Carathéodory distance $|\cdot|_{CC}$ in the case, where $N = \mathbb{H}_{\nu, m} \approx \mathbb{R}^{2\nu} \times \mathbb{R}^m$ is of Heisenberg type. This allows us to give pointwise estimates for the heat convolution kernel k_t associated with the sub-Laplacian $\mathcal{L} = \sum_{j=1}^n X_j^2$. In particular we prove that

$$|k_t(x, z)| \leq C t^{-\frac{q}{2}} \min \left\{ B \left(\frac{|(x, z)|_{CC}}{|x|} \right), B \left(1 + \frac{|(x, z)|_{CC}^2}{4t} \right) \right\} e^{-\frac{1}{4t}|(x, z)|_{CC}^2},$$

where $B(\xi) = \begin{cases} 1 + \log \xi & , \text{ if } (\nu, m) \in \{(2, 3), (4, 7)\} \\ \xi^{\frac{q-3m-1}{2}} & , \text{ otherwise} \end{cases}$, $\xi > 0$,

for all $(x, z) \in \mathbb{H}_{\nu, m}$. Here $q = 2\nu + 2m$ is the homogeneous dimension of $\mathbb{H}_{\nu, m}$. This extends the results obtained by Beals, Gaveau and Greiner in [3] on the Heisenberg group to Heisenberg type groups.

Furthermore we study the singular sub-Riemannian geometry associated to the Grušin operator

$$G = \Delta_x + |x|^2 \partial_u^2 \text{ on } \mathbb{R}_x^n \times \mathbb{R}_u.$$

Again we are able to give explicit formulas for the Carnot-Carathéodory distance d_{CC} . Combined with the explicit formulas for the heat kernel K_t , we establish the following pointwise estimate

$$K_t(\zeta, \eta) \leq C t^{-\frac{q}{2}} \min \left\{ 1 + \frac{d_{CC}(\zeta, \eta)}{|x + \xi|}, 1 + \frac{d_{CC}(\zeta, \eta)}{4t} \right\}^\alpha e^{-\frac{1}{4t}d_{CC}(\zeta, \eta)^2},$$

for all $\zeta = (x, u_1), \eta = (\xi, u) \in \mathbb{R}^{n+1}$, where $\alpha = \max \left\{ \frac{q-4}{2}, 0 \right\}$. Here the homogeneous dimension is $q = n + 2$. Hence this result corresponds for $n \geq 2$ to that obtained by Beals, Gaveau and Greiner on the Heisenberg group.

Zusammenfassung

Wir behandeln in dieser Dissertation sowohl die Sub-Riemannsche Geometrie, die vom Grušin Operator erzeugt wird, als auch die auf 2-stufig nilpotenten Lie Gruppen. Ziel sind gewisse Abschätzungen von Wärmeleitungskernen.

Sei \mathfrak{n} eine nilpotente Lie Algebra der Stufe 2, d.h. eine endlich dimensionale reelle Lie Algebra mit der Eigenschaft, dass $\mathfrak{z} := [\mathfrak{n}, \mathfrak{n}] \neq \{0\}$ und $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{0\}$ ist. Wir können die zugehörige zusammenhängende, einfach zusammenhängende Lie Gruppe N als $\mathbb{R}^n \times \mathbb{R}^m$ realisieren, mit Gruppen Multiplikation

$$(x, z) \cdot (y, u) = \left(x + y, z + u - \frac{1}{2} \left(\left\langle J^j x, y \right\rangle \right)_{j=1}^m \right), \quad (x, z), (y, u) \in N,$$

wobei $J^j \in \mathbb{R}^{n \times n}$, $j = 1, \dots, m$ schiefsymmetrische Matrizen sind. Weiterhin seien mit X_1, \dots, X_n die linksinvarianten Vektorfelder mit $X_j(0) = \partial_{x_j}$ bezeichnet, welche die Lie Algebra erzeugen. Wir untersuchen die bezüglich dieser Vektorfelder erzeugte Sub-Riemannsche Geometrie und können im Falle von Heisenberg Typ Gruppen $\mathbb{H}_{\nu, m} \approx \mathbb{R}^{2\nu} \times \mathbb{R}^m$ den Carnot-Carathéodory Abstand $|\cdot|_{CC}$ vollständig beschreiben. Mithilfe der gewonnen Erkenntnisse können wir den Wärmeleitungs-Faltungskern k_t bezüglich des Sub-Laplace Operators $\mathcal{L} = \sum_{j=1}^n X_j^2$ abschätzen. Im Detail erhalten wir die folgende Abschätzung

$$|k_t(x, z)| \leq C t^{-\frac{q}{2}} \min \left\{ B \left(\frac{|(x, z)|_{CC}}{|x|} \right), B \left(1 + \frac{|(x, z)|_{CC}^2}{4t} \right) \right\} e^{-\frac{1}{4t} |(x, z)|_{CC}^2},$$

$$\text{wobei } B(\xi) = \begin{cases} 1 + \log \xi & , \text{ falls } (\nu, m) \in \{(2, 3), (4, 7)\} \\ \xi^{\frac{q-3m-1}{2}} & , \text{ sonst} \end{cases}, \quad \xi > 0,$$

für alle $(x, z) \in \mathbb{H}_{\nu, m}$. Die homogene Dimension ist in diesem Fall $q = 2\nu + 2m$. Somit können wir die Resultate von Beals, Gaveau und Greiner [3] auf der Heisenberg Gruppe auf Heisenberg Typ Gruppen verallgemeinern.

Desweiteren untersuchen wir die singuläre Sub-Riemannsche Geometrie, welche vom Grušin Operator

$$G = \Delta_x + |x|^2 \partial_u^2 \quad \text{on } \mathbb{R}_x^n \times \mathbb{R}_u$$

erzeugt wird. Auch hier können wir explizite Formeln für den Carnot-Carathéodory Abstand d_{CC} angeben. Kombiniert mit expliziten Formeln für den zugehörigen Wärmeleitungskern K_t erhalten wir die punktweise Abschätzung

$$K_t(\zeta, \eta) \leq C t^{-\frac{q}{2}} \min \left\{ 1 + \frac{d_{CC}(\zeta, \eta)}{|x + \xi|}, 1 + \frac{d_{CC}(\zeta, \eta)}{4t} \right\}^\alpha e^{-\frac{1}{4t} d_{CC}(\zeta, \eta)^2},$$

für alle $\zeta = (x, u_1), \eta = (\xi, u) \in \mathbb{R}^{n+1}$, wobei $\alpha = \max \left\{ \frac{q-4}{2}, 0 \right\}$. In diesem Fall beträgt die homogene Dimension $q = n + 2$. Somit erhalten wir ein Resultat, das für $n \geq 2$ mit dem auf der Heisenberg Gruppe von Beals, Gaveau und Greiner korrespondiert.

Introduction

Consider the (Euclidean) heat equation

$$(\partial_t - \Delta_x) u = 0 \text{ on } \mathbb{R}_{>0} \times \mathbb{R}^n, \quad (1)$$

where $\Delta_x = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian on \mathbb{R}^n . Then it is well known that the heat (convolution-) kernel is given by

$$k_t(x) = c_n t^{-\frac{n}{2}} e^{-\frac{1}{4t}|x|^2}, \quad (2)$$

with some constant $c_n > 0$. A natural question to ask is whether a corresponding result or at least a pointwise estimate holds in some “non-Euclidean setting”. In particular we are interested in the heat kernel on 2-step nilpotent Lie groups and the heat kernel for the Grušin operator.

On nilpotent Lie groups, Coulhon, Saloff-Coste, and Varopoulos show in their book [37] that the heat (convolution-) kernel corresponding to a sub-Laplacian satisfies the following estimate

$$|k_t(x)| \leq C_\epsilon V(\sqrt{t})^{-1} e^{-\frac{1}{4(1+\epsilon)t}|x|_{CC}^2}, \quad (3)$$

for every $\epsilon > 0$ where $|\cdot|_{CC}$ is the so-called Carnot-Carathéodory metric and $V(r)$ is the volume of the ball of radius $r > 0$ with respect to the Carnot-Carathéodory metric. We also find a lower bound in this book

$$|k_t(x)| \geq C' V(\sqrt{t})^{-1} e^{-\frac{c}{t}|x|_{CC}^2}, \quad (4)$$

with constants $C', c > 0$.

Given a collection of vector fields $\{X_1, \dots, X_k\}$, which fulfills Hörmander’s condition, i.e. X_j , $j = 1, \dots, k$ together with their iterated brackets span the tangent space at every point, the sub-Laplacian associated to these vector fields is $\mathcal{L} = \sum_{j=1}^k X_j^2$. Then Hörmander’s condition ensures that \mathcal{L} is hypoelliptic.

Heuristically the diffusion of \mathcal{L} “follows” the vector fields X_j , $j = 1, \dots, k$. This idea leads to the notion of “horizontal” curves that stay tangent to the span of the X_j , $j = 1, \dots, k$. Again Hörmander’s condition ensures that any two points can be connected by such a horizontal curve. Hence, as a natural generalization of the Riemannian metric, the Carnot-

Carathéodory distance, which is the infimum of the length over all horizontal curves connecting two given points, might be the “correct” geometrically object.

The estimate (3) can be improved (see the book [37] itself, [15, 9, 13]). The strongest estimate in this more general setting known to date seems to be in Sikora [31]. Actually he shows that

$$|k_t(x, y)| \leq C \min \left\{ t^{-\frac{d}{2}} \left(1 + \frac{d_{CC}(x, y)}{\sqrt{t}} \right)^{d-1}, t^{-\frac{D}{2}} \left(1 + \frac{d_{CC}(x, y)}{\sqrt{t}} \right)^{D-1} \right\} e^{-\frac{d_{CC}(x, y)^2}{4t}} \quad (5)$$

for the heat (integral-) kernel associated to the Laplace-Beltrami operator on a connected and complete Riemannian manifold M . This estimate holds under the assumption that

$$\sup_{x \in M} k_t(x, x) \leq \begin{cases} C t^{-d/2} & \text{if } t \leq 1 \\ C t^{-D/2} & \text{if } t > 1 \end{cases}. \quad (6)$$

As remarked in [31] this estimate (5) also holds true for sub-Laplacians on Lie groups as long as the diagonal estimate (6) remains true. In particular on stratified, connected, simply connected, nilpotent Lie groups the diagonal estimate (6) holds true for $d = D = q$ where q is the homogeneous dimension.

Sub-Riemannian geometry

Given a distribution $\mathcal{D} \subset TM$ on a smooth manifold M , an absolutely continuous curve $\gamma : [0, T] \rightarrow M$ is called horizontal, if $\dot{\gamma}(t) \in \mathcal{D}(\gamma(t))$ for almost all $t \in [0, T]$. Let g be a Riemannian metric on \mathcal{D} . Then the length of a horizontal curve is defined by

$$L(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_g dt, \quad (7)$$

and the Carnot-Carathéodory distance (or sub-Riemannian distance) is

$$d_{CC}(x, y) = \inf \{L(\gamma) : \gamma \text{ horizontal}, \gamma(0) = x, \gamma(T) = y\}. \quad (8)$$

In Riemannian geometry locally length minimizing arcs are called geodesics, and they are given by the geodesic equation. An obvious analogue to the geodesic equation is the equation of non-null bicharacteristics of the Hamiltonian of the sub-Riemannian metric. In contrast to the Riemannian case, the “space” projection of these bicharacteristics - we call them normal extremals - do not characterize all locally length-minimizing arcs. Thus in the sub-Riemannian setting there are two “candidates” for the definition of “geodesic”: either

the length-minimizing arcs or the normal extremals. Since there is no agreement in the literature on the definition of “geodesic” in this setting, we avoid the word “geodesic”.

Although not all length-minimizing arcs are normal extremals, every normal extremal is locally length minimizing. In order to characterize the necessary conditions to be length minimizing in terms of some differential equations, we use Control Theory to introduce “abnormal extremals”. Pontryagin’s Maximum Principle states that every length minimizing arc is either a normal extremal or an abnormal extremal. These two classes of extremals are not mutually exclusive and the abnormal extremals need not be length-minimizing. Abnormal extremals that are not normal are called strictly abnormal extremals and they even need not to be smooth.

Heat kernels on 2-step nilpotent Lie groups

Let N , realized as $\mathbb{R}^n \times \mathbb{R}^m$, be a 2-step nilpotent Lie group. Consider the heat equation

$$(\partial_t - \mathcal{L})u = 0 \quad \text{on } \mathbb{R}_{>0} \times N, \quad (9)$$

where

$$\mathcal{L} = \sum_{j=1}^n X_j^2 \quad (10)$$

is a sub-Laplacian on N . Then from Sikora [31] we know that the heat (convolution-) kernel admits the following bound

$$|k_t(x)| \lesssim t^{-\frac{q}{2}} \left(1 + \frac{|x|_{CC}^2}{4t}\right)^{\frac{q-1}{2}} e^{-\frac{1}{4t}|x|_{CC}^2}. \quad (11)$$

Here $q = n + 2m$ is the homogeneous dimension, derived from the non isotropic dilations $\delta_r(x, z) = (rx, r^2z)$, $(x, z) \in N$, $r > 0$. With respect to these dilations \mathcal{L} is homogeneous of degree 2.

On the Heisenberg group \mathbb{H}_ν , realized as $\mathbb{R}^{2\nu} \times \mathbb{R}$, which is one of the simplest examples and the prototype of a 2-step nilpotent Lie group, Beals, Gaveau and Greiner [3] improve this estimate (11). Basically they prove that

$$|k_t(x)| \lesssim t^{-\frac{q}{2}} \left(1 + \frac{|x|_{CC}^2}{4t}\right)^{\frac{q-4}{2}} e^{-\frac{1}{4t}|x|_{CC}^2}. \quad (12)$$

Note that $q = 2\nu + 2$, hence $\frac{q-4}{2} = \nu - 1$. The main ingredients for this proof are the knowledge of “explicit” formulas for the heat kernel as well as for the Carnot-Carathéodory distance.

Due to the seminal work of Gaveau in [19], there are as well explicit formulas for the heat kernel on 2-step nilpotent Lie groups (see also Cygan [14]). This gives hope to transfer the results (12) on the Heisenberg group to general 2-step nilpotent Lie groups. Provided that one has adequate formulas for the Carnot-Carathéodory distance.

But it turns out that the Carnot-Carathéodory distance on 2-step nilpotent Lie groups in general is not easy to understand. In contrast to the Heisenberg group the existence of abnormal extremals is not excluded (see example 1.8.1 on page 32). One may first think of the free 2-step nilpotent Lie group, since this group is somehow “easier”. Although this group admits no strictly abnormal extremals, the normal extremal equations have a very complicated structure.

The Heisenberg type groups do not admit abnormal extremals and are closer related to the Heisenberg group. It then turns out that the Carnot-Carathéodory distance is indeed easier to understand. Furthermore, we are able to generalize the result of (12) to Heisenberg type groups. Note that any Heisenberg group is also of Heisenberg type.

The Grušin operator

Now consider the Grušin operator

$$G = \Delta_x + |x|^2 \partial_u^2 \quad \text{on } \mathbb{R}_x^n \times \mathbb{R}_u \quad (13)$$

and the corresponding heat equation

$$(\partial_t - G) u = 0 \quad \text{on } \mathbb{R}_{>0} \times \mathbb{R}^{n+1}. \quad (14)$$

Then we can write G also as a “sum of squares” operator

$$G = \sum_{j=1}^n (X_j^2 + U_j^2), \quad (15)$$

where $X_j = \partial_{x_j}$, $U_j = x_j \partial_u$, $j = 1, \dots, n$ are smooth vector fields on \mathbb{R}^{n+1} . This collection $\mathcal{H} = \{X_j, U_j : j = 1, \dots, n\}$ of vector fields spans the tangent space everywhere except on the line $x = 0$. But since $[X_j, U_j] = \partial_u$, the Hörmander condition is fulfilled. Notice that in contrast to a regular distribution the dimension of

$$\mathcal{D}(x, u) := \text{span} \{X_j(x, u), U_j(x, u) : j = 1, \dots, n\} \subseteq T_{(x,u)} \mathbb{R}^{n+1}$$

changes from $n + 1$ to n whenever $x = 0$. This in turn means that \mathcal{D} does not lead to a sub-Riemannian geometry in the regular sense, but after all one can define the Carnot-Carathéodory distance, which, due to the lack of translation invariance of G , itself is not

translation invariant. Furthermore, the length-minimizing problem does not lead to abnormal curves.

The main results

In view of the above discussion the main results are certainly Corollary 2.4.15 in Section 2.4 on page 94 and Theorem 3.4.6 in Section 3.4 on page 136.

Let $\mathfrak{h}_{\nu,m}$ be a 2-step nilpotent Lie algebra spanned by the orthonormal basis $X_1, \dots, X_{2\nu}, Z_1, \dots, Z_m$ such that $[\mathfrak{h}_{\nu,m}, \mathfrak{h}_{\nu,m}] = \mathfrak{z} := \text{span}\{Z_1, \dots, Z_m\}$, and let $\mathbb{H}_{\nu,m}$ be the corresponding connected, simply connected Lie group realized as $\mathbb{R}^{2\nu} \times \mathbb{R}^m$ with group multiplication

$$(x, z) \cdot (y, u) = \left(x + y, z + u - \frac{1}{2} \left(\left\langle J^j x, y \right\rangle \right)_{j=1}^m \right), \quad (x, z), (y, u) \in \mathbb{H}_{\nu,m}$$

where J^j , $j = 1, \dots, m$ are skew-symmetric matrices acting on $\mathbb{R}^{2\nu}$. For $z \in \mathbb{R}^m$ we define the skew-symmetric matrices $J^z := \sum_{j=1}^m z_j J^j$. Then $\mathbb{H}_{\nu,m}$ is a Heisenberg type group if $(J^z)^2 = -|z|^2 I$ for every $z \in \mathbb{R}^m$. So let $\mathbb{H}_{\nu,m}$ be a Heisenberg type group and let

$$\mathcal{L} = \sum_{j=1}^{2\nu} X_j^2 \quad (16)$$

denote the sub-Laplacian corresponding to the left-invariant vector fields $X_1, \dots, X_{2\nu}$ which satisfy Hörmander's condition. Then $-\mathcal{L}$ is an essentially self-adjoint operator on $L^2(\mathbb{H}_{\nu,m})$, hypoelliptic and non-negative. Let $\{e^{t\mathcal{L}}\}_{t>0}$ denote the heat semigroup generated by \mathcal{L} . Since \mathcal{L} is left-invariant $e^{t\mathcal{L}}$ admits a convolution kernel k_t such that

$$e^{t\mathcal{L}} f = f * k_t \quad (17)$$

for every $t > 0$ and $f \in L^1(\mathbb{H}_{\nu,m})$. Then k_t is called heat kernel for \mathcal{L} and since $\partial_t - \mathcal{L}$ is hypoelliptic, the function $(t, g) \mapsto k_t(g)$ is smooth on $\mathbb{R}_{>0} \times \mathbb{H}_{\nu,m}$.

Furthermore let d_{CC} denote the Carnot-Carathéodory distance associated to the horizontal distribution $\mathcal{D} = \text{span}\{X_1, \dots, X_{2\nu}\}$. Then d_{CC} is left-invariant and we write $|g|_{CC} = d_{CC}(e, g)$, $g \in \mathbb{H}_{\nu,m}$. We can prove the following result:

Theorem. For $(x, z) \in \mathbb{H}_{\nu,m}$ we have

$$|k_t(x, z)| \leq C t^{-\frac{q}{2}} \min \left\{ B \left(\frac{|(x, z)|_{CC}}{|x|} \right), B \left(1 + \frac{|(x, z)|_{CC}^2}{4t} \right) \right\} e^{-\frac{1}{4t} |(x, z)|_{CC}^2}, \quad (18)$$

$$\text{where } B(\xi) = \begin{cases} \log(1 + \xi) & , \text{ if } (\nu, m) \in \{(2, 3), (4, 7)\} \\ \xi^{\frac{q-3m-1}{2}} & , \text{ otherwise} \end{cases}, \quad \xi > 0,$$

where $q = 2\nu + 2m$ is the homogeneous dimension.

Note that since $\mathbb{H}_{\nu,m}$ is a Heisenberg type group the integers $\nu, m \in \mathbb{N}$ has to satisfy specific properties. We always have $m \leq 2\nu - 1$ and therefore $\frac{q-3m-1}{2} \geq 0$, where $m = 2\nu - 1$ is only possible if $\nu \in \{1, 2, 4\}$ (see Lemma 2.4.10). This means that B is always increasing, except for $\nu = 1$ and $m = 1$. For $m = 1$ $\mathbb{H}_{\nu,1}$ is a Heisenberg group and this result exactly coincides with the result of Beals, Greiner, and Gaveau (12), [3]. With respect to the general result of Sikora (11) in [31] this is an improvement in the exponent of the second factor by $\frac{3m}{2}$ at least if $(\nu, m) \notin \{(2, 3), (4, 7)\}$.

We may note here that in the region $|z| \lesssim |x|^2$

$$\frac{|(x, z)|_{CC}}{|x|} \lesssim 1 \quad (19)$$

stays bounded, hence also the second factor in (18). This means that in this region we get an even greater improvement. This is also part of the theorem of Beals, Gaveau, and Greiner.

A similar result is obtained for the heat kernel with respect to the Grušin operator. Let $K_t(\zeta, \eta)$, $t > 0$, $\zeta, \eta \in \mathbb{R}^{n+1}$ denote the heat kernel in the sense that

$$e^{tG} f(\zeta) = \int_{\mathbb{R}^{n+1}} K_t(\zeta, \eta) f(\eta) d\eta, \quad t > 0 \quad (20)$$

for $f \in L^2(\mathbb{R}^{n+1})$. Thus $u(t, \zeta) = e^{tG} f(\zeta)$ is the solution of the heat equation

$$(\partial_t - G)u = 0, \quad t > 0 \quad (21)$$

with with the boundary condition

$$u(t, \cdot) \rightarrow f \text{ in } L^2(\mathbb{R}^{n+1}) \text{ as } t \rightarrow 0^+. \quad (22)$$

Theorem. For $\zeta = (x, 0)$, $\eta = (\xi, u) \in \mathbb{R}^{n+1}$ we have

$$|K_t(x, \xi, u)| \lesssim t^{-\frac{q}{2}} \min \left\{ 1 + \frac{d_{CC}(\zeta, \eta)}{|x + \xi|}, 1 + \frac{d_{CC}(\zeta, \eta)^2}{4t} \right\}^\alpha e^{-\frac{1}{4t} d_{CC}(\zeta, \eta)^2}, \quad (23)$$

with $\alpha = \max\left(\frac{q-4}{2}, 0\right)$ and $q = n + 2$ denotes the homogeneous dimension.

Again we produced a result which corresponds to that obtained by Beals, Gaveau, and Greiner on the Heisenberg group (12). At least for $n \geq 2$, but for $n = 1$ we might not expect an additional decay of the heat kernel.

Although we primarily targeted pointwise heat kernel estimates, from a differential geometric point of view the results in Section 2.3.1 and 3.2 are interesting on their own. We give a complete description of the sub-Riemannian geometry associated to $X_1, \dots, X_{2\nu}$ on Heisenberg type groups and a complete description of the (singular) sub-Riemannian geometry associated to the Grušin operator. Besides these complete descriptions we present some possibly useful formulas on free 2-step nilpotent Lie groups.

Organization of this thesis

This thesis consists of three parts which are more or less independent of each other.

CHAPTER 1 gives an overview of sub-Riemannian geometry. Due to the new phenomena that occur when passing from Riemannian geometry over to sub-Riemannian geometry, we give an introduction to the main terms and definitions in sub-Riemannian geometry and the length minimizing problem. Furthermore, we give a brief overview of Control Theory (*Section 1.7*). In *Section 1.8* we discuss the situation on 2-step nilpotent Lie groups and in *Section 1.9* on the singular sub-Riemannian geometry associated to the Grušin operator.

In CHAPTER 2 we recall some basic facts on 2-step nilpotent Lie groups and give the most important examples of the free group, Heisenberg type groups and Heisenberg groups (*Section 2.1*). In *Section 2.2* and *2.3* we discuss the sub-Riemannian geometry on these groups, and give a complete description of the sub-Riemannian geometry on Heisenberg type groups in *Section 2.3.1*.

The explicit formulas for the Carnot-Carathéodory distance and the heat kernel are then used in *Section 2.4* to establish the heat kernel upper bound (18) on Heisenberg type groups.

CHAPTER 3 on the Grušin operator has a similar structure as Chapter 2. First we give the preliminary definitions in *Section 3.1* to begin the discussion of the (singular) sub-Riemannian geometry in *Section 3.2*. In *Section 3.3* we derive the formula for the heat kernel by reducing it via a partial Fourier transform to heat kernels of some rescaled Hermite operators. This allows us to establish the heat kernel upper bound (23) for the Grušin operator in *Section 3.4*.

Acknowledgments

First of all, I thank my thesis advisor Prof. Dr. Detlef Müller for expert guidance and mentorship, his invaluable support and encouragement.

I feel deeply grateful to my parents for constantly supporting my studies all the years. Last but always first, for her encouragement and endless patience especially during the frustrating phases, I would like to express my sincere gratitude to Leona.

1 Some notes on sub-Riemannian geometry

There are some major differences when passing from Riemannian geometry over to the sub-Riemannian geometry as we already stated in the introduction. New phenomena occur and in view of some misleading literature, we want to give a short overview of sub-Riemannian geometry following in particular [23], but also a summary of [20, 29, 30, 26, 1].

When studying horizontal length minimizing arcs, two different classes of “extremals” have to be considered. The “normal extremals” which are the characteristics of the associated Hamiltonian generalize somehow the Riemannian geodesics. But in contrast to the Riemannian case the existence of “abnormal extremals” is not excluded. These two types of minimizers are not mutually exclusive.

The normal extremals have the property that they locally minimize length and are also differentiable. Since these two types of extremals are not mutually exclusive, the question arises whether strictly abnormal extremals, i.e. abnormal extremals that are not normal, do exist at all. Furthermore, one could ask if strictly abnormal extremals could minimize length and if they are differentiable.

It is not difficult to give examples for the existence of strictly abnormal extremals. The first example of a length minimizer that is not a normal extremal (strictly abnormal minimizer) was first given 1994 by Montgomery in [25]. For rank-2 distributions Liu and Sussmann [23] point to a generic class of abnormal extremals that also minimizes length.

Also in nilpotent Lie groups the analysis of abnormal extremals can not be avoided. In [20] we find an example of a graded nilpotent Lie group that admits “many” strictly abnormal minimizers.

Since we are mostly interested in 2-step nilpotent Lie groups, we focus our attention on these groups in Section 1.8. The free 2-step nilpotent Lie group (also known as the Gaveau-Brockett system) allows “many” abnormal extremals, but all of them are also normal. This minimizing problem was first considered by Gaveau in 1977 in [19] and then by Brockett in 1982 in [6] (see Remark 1.0.1).

Adapting the methods of [23], it is easy to prove that every non-constant abnormal minimizer on a general 2-step nilpotent Lie group is normal in some subgroup and thus smooth. This was done by Golé and Karidi in [20]. As another consequence we see that the abnormal extremals on nonsingular 2-step nilpotent Lie groups (Métivier groups - see Definition 1.8.5) are just the constant curves. Hence we need not care about abnormal

extremals if we consider the Carnot-Carathéodory distance on Heisenberg type groups which are nonsingular.

Furthermore we give an example (see Section 1.8.1) of a 2-step nilpotent Lie group with strictly abnormal extremals. In [20] we can find an example of a nilpotent Lie group with strictly abnormal minimizers. The Lie group constructed in [20] is 4-step nilpotent whereas the group we want to discuss in Section 1.8.1 is a 2-step nilpotent Lie group. It turns out that the strictly abnormal extremals, we are considering in this example, are not length-minimizing.

In Section 1.9 we study the length minimizing problem associated to the Grušin operator. Again we can prove that the locally length minimizing curves are fully characterized by normal extremals, in particular we show that the abnormal extremals are just constant curves.

Remark 1.0.1. Gaveau falsely stated 1977 in [19] that on free 2-step nilpotent Lie groups (with $n > 3$) there are points which can not be connected by a characteristic of the Hamiltonian. This would have implied the existence of “strictly abnormal minimizers”, since by Chow’s theorem there is always a horizontal curve with finite length connecting any given two points. The points considered by Gaveau can in fact be joined by a characteristic curve - as has been shown in Brockett [6] and Lemma 2.2.13 on page 70. An alternate proof (which was given in [23]) is given later in this section, which also shows that all “minimizers” are “normal” in this case (Theorem 1.8.3).

1.1 Preliminary definitions and notation

Let M be a smooth manifold. A subset $\mathcal{D} \subset TM$ which assigns to each $x \in M$ a linear subspace $\mathcal{D}(x) \subset T_x M$ is called **singular distribution**. If \mathcal{D} is actually a smooth subbundle of TM , then \mathcal{D} is called a **smooth distribution** or simply a distribution. Note that \mathcal{D} being a smooth distribution implies that $\dim \mathcal{D}(x)$ is constant. A **sub-Riemannian structure** (\mathcal{D}, g) on a smooth manifold M is given by a smooth distribution \mathcal{D} and a Riemannian metric g on \mathcal{D} . For $p \in M$ and $v, w \in T_p M$ we write $\langle v, w \rangle_g = g_p(v, w)$ and $\|v\|_g = \langle v, v \rangle_g^{1/2}$. A **sub-Riemannian manifold** (M, \mathcal{D}, g) is a smooth manifold M equipped with a sub-Riemannian structure (\mathcal{D}, g) .

Let \mathcal{D} be a smooth distribution on M , and $U \subset M$ open, then $\text{Sec}(\mathcal{D}, U)$ denotes the set of all smooth section of \mathcal{D} defined on U . Further let $\text{Sec}(\mathcal{D}) := \bigcup_U \text{Sec}(\mathcal{D}, U)$.

For $k \in \{0, 1, \dots, \infty\}$ let $V^k(M)$ denote the set of all vector fields of class C^k on M .

1.2 Horizontal curves and the Carnot-Carathéodory distance

Definition 1.2.1. Let (M, \mathcal{D}, g) be a sub-Riemannian manifold. An absolutely continuous curve $\gamma : [0, T] \rightarrow M$, $T > 0$ on M is called **horizontal** (or **\mathcal{D} -admissible**) if

$$\dot{\gamma}(t) = \frac{d}{dt}\gamma(t) \in \mathcal{D}(\gamma(t)) \quad (1.1)$$

for almost all $t \in [0, T]$. As in Riemannian geometry the **length of a horizontal curve** γ is defined by

$$L(\gamma) := \int_0^T \|\dot{\gamma}(t)\|_g dt. \quad (1.2)$$

γ is **parametrized by arc-length** if $\|\dot{\gamma}(t)\|_g = 1$ for almost all $t \in [0, T]$. We call $\gamma : [0, 1] \rightarrow M$ **parametrized with constant speed** if $\|\dot{\gamma}(t)\|_g = c$ for almost all $t \in [0, 1]$ where $c > 0$ is a constant.

The **Carnot-Carathéodory distance** (or **sub-Riemannian distance**) on (M, \mathcal{D}, g) is defined by

$$d_{CC}(p, q) := \inf \{L(\gamma) : \gamma \text{ is a horizontal curve and } \gamma \text{ joins } p, q\}, \quad p, q \in M, \quad (1.3)$$

where we use the convention that $\inf \emptyset = \infty$.

Definition 1.2.2. We call a distribution \mathcal{D} **bracket-generating** or **nonholonomic** if the vector fields in $\text{Sec}(\mathcal{D})$ and their iterated brackets span the tangent space $T_x M$ at every point $x \in M$. If \mathcal{D} is spanned by a global basis of smooth vector fields, this property is also called **Hörmander's condition**.

If M is connected and \mathcal{D} is bracket-generating, then by the Chow-Rashevskii theorem every two points $p, q \in M$ can be connected by a horizontal curve. Furthermore $d_{CC} : M \times M \rightarrow \mathbb{R}$ is a metric that induces the topology of M .

1.3 Hamiltonians and bicharacteristics

Let N be a symplectic manifold with symplectic 2-form Ω . For $H \in C^1(N)$ there is a unique vector field \vec{H} on N such that

$$\Omega(X, \vec{H}) = \langle dH, X \rangle \quad (1.4)$$

for every vector field X on N . We call this vector field \vec{H} the **Hamiltonian vector field** associated to H .

We will recall some properties:

- If $H, K \in C^k(N)$, $k \geq 1$, then the vector field \vec{H} is of class C^{k-1} .
- For $H, K \in C^1(N)$ the Poisson bracket is $\{H, K\} = \langle dK, \vec{H} \rangle = \Omega(\vec{H}, \vec{K})$.
- Furthermore we have $\{H, KL\} = \{H, K\}L + \{H, L\}K$,
- the Jacobi identity $\{H, \{K, L\}\} + \{K, \{L, H\}\} + \{L, \{H, K\}\} = 0$,
- and $\overrightarrow{HK} = H\vec{K} + K\vec{H}$.

For any subset $\mathcal{H} \subset C^1(N)$ we define $\vec{\mathcal{H}}(x) = \{\vec{H}(x) : H \in \mathcal{H}\} \subset T_x M$, for $x \in M$. An absolutely continuous curve $\Gamma : [0, T] \rightarrow T^*M$ is called a **bicharacteristic** of \mathcal{H} , if $\dot{\Gamma}(t) \in \vec{\mathcal{H}}(\Gamma(t))$ for almost every t . A **null bicharacteristic** of \mathcal{H} is a bicharacteristic of \mathcal{H} along which all the functions $H \in \mathcal{H}$ vanish. A bicharacteristic of a single function $H \in C^k$ is a curve of class C^k along which H is constant.

The cotangent bundle T^*M of a manifold M has a natural symplectic structure determined by the 2-form $\Omega_M = d\omega_M$, where ω_M is the 1-form given by $\omega_M(x, \xi)(v) = \langle \xi, d\pi_M^*(v) \rangle$ for $v \in T_{(x, \xi)}(T^*M)$ and π_M^* denotes the canonical projection of T^*M to M . Relative to a coordinate chart $T^*\kappa = (x^1, \dots, x^n, \xi_1, \dots, \xi_n)$ induced by a chart $\kappa = (x^1, \dots, x^n)$ on M , we have the formulas

$$\omega_M = \sum_j \xi_j dx^j, \quad \Omega_M = \sum_j d\xi_j \wedge dx^j, \quad (1.5)$$

$$\vec{H} = \sum_j \left(\frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial x^j} - \frac{\partial H}{\partial x^j} \frac{\partial}{\partial \xi_j} \right), \quad \text{and} \quad \{H, K\} = \sum_j \left(\frac{\partial H}{\partial \xi_j} \frac{\partial K}{\partial x^j} - \frac{\partial H}{\partial x^j} \frac{\partial K}{\partial \xi_j} \right). \quad (1.6)$$

Let M be a smooth manifold and $\mathcal{H} \subset C^1(T^*M)$, then a **characteristic (null characteristic)** of \mathcal{H} is a curve γ in M , which is the projection of a bicharacteristic (null bicharacteristic) Γ of \mathcal{H} to M , i.e. $\gamma = \pi_M^* \circ \Gamma$. A characteristic γ is called a **nontrivial characteristic (nontrivial null characteristic)** if we can choose Γ to be entirely contained in $T^\#M$. For $p \in M$ we denote $T_p^\#M := T_p^*M \setminus \{0\}$.

For each vector field X on M set

$$H_X : T^*M \rightarrow \mathbb{R}, \quad (x, \xi) \mapsto \langle \xi, X(x) \rangle.$$

(We will often use brackets to denote the value of the linear form $\xi \in T_p^*M$, $p \in M$ evaluate at $v \in T_pM$: $\langle \xi, v \rangle = \xi(v)$.) If $X \in V^1(M)$, then the vector field \vec{H}_X is called the **Hamiltonian lift** of X and $\{H_X, H_Y\} = H_{[X, Y]}$, $X, Y \in V^1(M)$. Moreover the characteristics of H_X are exactly the integral curves of X .

1.4 Abnormal extremals

There are several equivalent ways to define normal and abnormal extremals. The original definition came from Control Theory, but we will use the definition of [23], since this is more “differential geometric”.

Definition 1.4.1. Let \mathcal{D} be a smooth distribution on a smooth manifold M . An **abnormal biextremal** (**abnormal extremal**) of \mathcal{D} is an absolutely continuous curve in $T^\#M$ (in M) which is a null bicharacteristic (nontrivial null characteristic) of $\mathcal{H}_{\mathcal{D}} = \{H_X : X \text{ is a smooth section in } \mathcal{D}\}$.

Remark 1.4.2. Since $d\pi_M^* \left(\overrightarrow{H_X}(x, \xi) \right) = X(x)$, it follows that every abnormal extremal is a horizontal curve.

Remark 1.4.3. We have the following equivalent characterizations of abnormal extremals:

1. γ is an abnormal extremal iff γ is an abnormal extremal in the sense of Pontryagin’s Maximum Principle of Control Theory. Later we will give a short overview of Control Theory and the Maximum Principle. We give a proof of this equivalence in Proposition 1.7.3.
2. Let $p \in M$ and $H_{\mathcal{D}}(p)$ denote the set of all horizontal curves $\gamma : [0, 1] \rightarrow M$ that have finite energy, i.e. $\int_0^1 \|\dot{\gamma}(t)\|_g^2 dt < \infty$, and satisfy $\gamma(0) = p$. Then this set is independent of the choice of the metric g and one can define a differentiable structure on $H_{\mathcal{D}}(p)$ [5]. We define the endpoint map

$$\text{end}_p : H_{\mathcal{D}}(p) \rightarrow M, \gamma \mapsto \gamma(1). \quad (1.7)$$

Then end_p is a smooth map and $\gamma : [0, 1] \rightarrow M$ is an abnormal extremal iff γ is a singular point of $\text{end}_{\gamma(0)}$. In this context abnormal extremals are also called singular curves. For more details see for example [29, 30]

3. The abnormal biextremals are exactly the characteristic curves of \mathcal{D}^\perp that lie in $T^\#M$. For what this means exactly see [23].

1.5 Normal extremals

Definition 1.5.1. Let $(x, \xi) \in T^*M$, then the restriction $\xi|_{\mathcal{D}(x)}$ of ξ to $\mathcal{D}(x) \subset T_xM$ has a well-defined norm denoted again by $\|\xi\|_g$, since $\mathcal{D}(x)$ is an inner product space. The function $H : T^*M \rightarrow \mathbb{R}$ given by

$$H_2(x, \xi) = \frac{1}{2} \|\xi\|_g^2 \quad (1.8)$$

is called the **Hamiltonian** (function) of the sub-Riemannian structure (\mathcal{D}, g) . A **normal biextremal** is a curve Γ in T^*M such that Γ is a bicharacteristic of the Hamiltonian H and H does not vanish along Γ . A **normal extremal** is a curve in M which is the projection of a normal biextremal.

Remark 1.5.2. Given a local frame $\{X_1, \dots, X_n\}$ of \mathcal{D} , we define the matrix $g_{i,j}(p) = \langle X_i(p), X_j(p) \rangle_g$. Let $g^{i,j}(p)$ denotes its inverse, then

$$H_2(x, \xi) = \frac{1}{2} \sum_{i,j} g^{i,j}(x) \langle \xi, X_i(x) \rangle \langle \xi, X_j(x) \rangle \quad (1.9)$$

$$= \frac{1}{2} \sum_{i,j} g^{i,j} H_{X_i} H_{X_j}. \quad (1.10)$$

If $\{X_1, \dots, X_n\}$ is a orthonormal frame of \mathcal{D} (with respect to the sub-Riemannian inner product), then

$$H_2(x, \xi) = \frac{1}{2} \sum_{j=1}^n \langle \xi, X_j(x) \rangle^2. \quad (1.11)$$

Definition 1.5.3. A **biextremal (extremal)** is either an abnormal biextremal (abnormal extremal) or a normal biextremal (normal extremal).

Remark 1.5.4. A biextremal is either normal or abnormal, and the two possibilities are mutually exclusive. In general this is not true for extremals, i.e. an abnormal extremal can also be a normal extremal.

Definition 1.5.5. A **strictly abnormal extremal** of (\mathcal{D}, g) is an abnormal extremal which is not a normal extremal.

Remark 1.5.6. The smoothness of H_2 implies that normal (bi)extremals are smooth but an analogous statement for abnormal extremals is false (in general).

1.6 The length minimizing problem

Definition 1.6.1. Let (M, \mathcal{D}, g) be a sub-Riemannian manifold. A horizontal curve $\gamma : [0, T] \rightarrow M$ is called a **minimizer** if $d_{CC}(\gamma(0), \gamma(T)) = L(\gamma)$. An **abnormal minimizer** is an abnormal extremal which is a minimizer. A **strictly abnormal minimizer** is a strictly abnormal extremal which is a minimizer.

First note that the class of all non-constant minimizers is equivalent modulo reparametrization to the class of those minimizers that are parametrized by arc-length. Moreover there is a one-to-one correspondence of curves parametrized by arc-length and curves parametrized with constant speed. In particular $\gamma : [0, T] \rightarrow M$ is parametrized by arc-length iff $\gamma' : [0, 1] \rightarrow M$, $t \mapsto \gamma(t/T)$ is parametrized with constant speed.

Remark 1.6.2. As a consequence of Filipov's theorem we have the following existence result for minimizers. Let $p \in M$. Then for every q in a sufficiently small neighborhood of p there exists a curve of minimal length connecting p and q . We do not want to go into details of the various existence results (see for example [1, 10]). We just want to note that in the situations we are going to study in Section 1.8 and 1.9 the existence of a minimizers connecting any two points p, q is always ensured. This is another consequence of Filippov's theorem (see [1]).

We will now introduce the class of (locally) time optimal arcs.

Definition 1.6.3. Let $\gamma : [0, T] \rightarrow M$ be a non-constant horizontal curve with $\|\dot{\gamma}(t)\|_g \leq 1$ for almost all $t \in [0, T]$. Then γ is **time-optimal** (or **time-minimizing**) if $T \leq T'$ for all curves $\gamma' : [0, T'] \rightarrow M$ with $\|\dot{\gamma}'(t)\|_g \leq 1$ and $\gamma(0) = \gamma'(0)$, $\gamma(T) = \gamma'(T')$.

γ is called **locally time-optimal**, if there exists $\delta > 0$ such that the restriction of γ to every subinterval of $[0, T]$ of length δ is time-optimal.

This definition gives another characterization of the class of all minimizers: The class of those minimizers that are parametrized by arc-length is equal to the class of all time-optimal arcs.

We now give the main results for the relation between minimizers and extremals. Both theorems 1.6.4, 1.6.5 can be found in [23]. Later in Section 1.7 we will repeat the proof of Theorem 1.6.4 found in [23] by reducing it to a corollary of Pontryagin's Maximum Principle.

Theorem 1.6.4. *Let (M, \mathcal{D}, g) be a sub-Riemannian manifold and $\gamma : [0, T] \rightarrow M$ be a length minimizer parametrized by arc-length. Then γ is an extremal.*

This theorem gives us candidates for the minimizing problem. The next theorem gives a partial converse of Theorem 1.6.4:

Theorem 1.6.5. *Let (M, \mathcal{D}, g) be a sub-Riemannian manifold. Then every normal extremal is locally time-optimal.*

Remark 1.6.6. Theorem 1.6.5 tells us that the class of normal extremals is very useful when characterizing the minimizers, since this tells us that every normal extremal is at least locally optimal. The class of abnormal extremals which might be very large does not behave so nicely, because it turns out that they fail to be (locally) optimal in general. They fail to be even smooth in general.

1.6.1 Some examples

1. If $\mathcal{D} = TM$, then there are no abnormal extremals. Thus all extremals are locally optimal. (This is the Riemannian case.)

2. If \mathcal{D} is strongly bracket-generating but $\mathcal{D} \neq TM$, then the abnormal extremals are exactly the constant curves. A smooth distribution \mathcal{D} is called **strongly bracket-generating** if for all $p \in M$ and $X \in \text{Sec}(\mathcal{D})$, with $X(p) \neq 0$ we have $\mathcal{D} + [X, \text{Sec}(\mathcal{D})](p) = T_pM$.

1.7 Control systems

Definition 1.7.1. An **autonomous control system** F of class C^1 on a smooth manifold M is a parametrized family $F = \{X_a : a \in \mathcal{A}\}$ of vector fields of class C^1 on M , where \mathcal{A} is an arbitrary set. A **control** for an autonomous control system is a function $\alpha : [0, T] \rightarrow \mathcal{A}$. An **F -trajectory** generated by a control α is a curve $\gamma : [0, T] \rightarrow M$ such that $\dot{\gamma}(t) = X_{\alpha(t)}(\gamma(t))$ for almost all $t \in [0, T]$. A control α is called **admissible** if $(x, t) \mapsto X_\alpha(x, t) := X_{\alpha(t)}(x)$ satisfies the Carathéodory conditions:

1. $t \mapsto X_\alpha(x, t)$ is Lebesgue measurable as a function of t for fixed x ,
2. for every chart $\kappa : V \subset M \rightarrow \mathbb{R}^n$ and every compact subset $K \subset \kappa(V)$ there exists an integrable function $\varphi : [0, T] \rightarrow \mathbb{R}$ such that

$$\|X_\alpha^\kappa(x, t)\| + \left\| \frac{\partial X_\alpha^\kappa}{\partial x}(x, t) \right\| \leq \varphi(t), \text{ for all } (x, t) \in K \times [0, T], \quad (1.12)$$

where $X_\alpha^\kappa : \kappa(V) \times [0, T] \rightarrow \mathbb{R}^n$ is the representation of X_α with respect to the chart κ .

An **admissible trajectory** of F is an F -trajectory that is generated by an admissible control α . A non-constant, admissible trajectory $\gamma : [0, T] \rightarrow M$ is **time-optimal** if $T \leq T'$ for all admissible trajectories $\gamma' : [0, T'] \rightarrow M$ such that $\gamma(0) = \gamma'(0)$ and $\gamma(T) = \gamma'(T')$.

Notation. In Control Theory one uses the following notation and abbreviations:

- The manifold M is known as the state space of the system F .
- The set \mathcal{A} is known as the control space, or the set of control values.
- The word “control” is used a little vague; it either means the \mathcal{A} -valued “variable” a or the \mathcal{A} -valued function $\alpha : [0, T] \rightarrow \mathcal{A}$.
- A common way to introduce a control system is to say “the system $\dot{x} = X_a(x)$, $x \in M$, $a \in \mathcal{A}$ ”. This is to be interpreted as a control system $F = \{X_a : a \in \mathcal{A}\}$. The formal expression $\dot{x} = X_a(x)$ is an abbreviation for the fact that each control function $t \mapsto \alpha(t)$ gives rise for the ordinary differential equation $\dot{x}(t) = X_{\alpha(t)}(x(t))$, whose solutions are the trajectories of the system.

We define the **Hamiltonian of the control system** F by

$$H(x, \zeta, a) = H_{X_a}(x, \zeta) = \langle \zeta, X_a(x) \rangle, \quad (x, \zeta) \in T^*M, \quad a \in \mathcal{A}.$$

Let $\Gamma = (\gamma, \zeta) : [0, T] \rightarrow T^*M$ be an absolutely continuous curve, such that $\gamma : [0, T] \rightarrow M$ is a trajectory corresponding to an admissible control $\alpha : [0, T] \rightarrow \mathcal{A}$ and Γ satisfies the **adjoint equation**

$$\dot{\Gamma}(t) = \overrightarrow{H_{X_{\alpha(t)}}}(\Gamma(t)), \quad \text{for almost all } t \in [0, T], \quad (1.13)$$

where $\overrightarrow{H_{X_{\alpha(t)}}}$ denotes the Hamiltonian lift of $X_{\alpha(t)}$. Then ζ is called an **adjoint vector** along (γ, α) . An adjoint vector ζ along (γ, α) is called **maximizing** if there exists a constant $h_0 \geq 0$ such that

$$h_0 = H(\gamma(t), \zeta(t), \alpha(t)) = \max \{H(\gamma(t), \zeta(t), a) : a \in \mathcal{A}\} \quad (1.14)$$

for almost all $t \in [0, T]$. If $\zeta(t) \neq 0$ for every t , then ζ is called **nontrivial**.

With these notations the Pontryagin's Maximum Principle states:

Theorem 1.7.2. *If F is an autonomous control system of class C^1 on a manifold M , and γ is a time-optimal trajectory corresponding to an admissible control α , then there exists a nontrivial maximizing adjoint vector along (γ, α) .*

A proof can be found in many books of Control Theory for example in [1].

The language of Control Theory gives us now an equivalent formulation of our length minimizing problem and another characterization of normal and abnormal extremals. Let (M, \mathcal{D}, g) be a sub-Riemannian manifold. We will assume that \mathcal{D} has a global orthonormal basis of sections X_1, \dots, X_n . This assumption makes the proofs slightly simpler without losing much of the basic ideas. Furthermore, the situations we will consider later will always admit this property. In Remark 1.7.4 we give some hints for the general situation.

As control space we choose the closed unit ball in \mathbb{R}^n , i.e. $\mathcal{A} = \{a \in \mathbb{R}^n : |a| \leq 1\}$ and we define $X_a = \sum_{j=1}^n a_j X_j$ for $a \in \mathcal{A}$. Let $F = \{X_a : a \in \mathcal{A}\}$ be the control system. Then the Carathéodory conditions ensure that the F -admissible trajectories are precisely the horizontal curves $\gamma : [0, T] \rightarrow M$ with $\|\dot{\gamma}(t)\|_g \leq 1$ for almost all $t \in [0, T]$. Thus the time-optimal trajectories for F are exactly the time-optimal horizontal curves.

We now prove the following equivalent characterization for abnormal and normal extremals such that Theorem 1.6.4 just becomes a corollary of Pontryagin's Maximum Principle.

Proposition 1.7.3. *Let $\gamma : [0, T] \rightarrow M$ be a horizontal curve parametrized by arc-length. Then γ is an extremal if and only if there exists an admissible control $\alpha : [0, T] \rightarrow \mathcal{A}$*

corresponding to γ and a nontrivial maximizing adjoint vector ζ along (γ, α) , i.e. there exists $h_0 \geq 0$ such that

$$\dot{\zeta}(t) = -\frac{\partial H}{\partial x}(\gamma(t), \zeta(t), \alpha(t)), \quad (1.15)$$

$$h_0 = H(\gamma(t), \zeta(t), \alpha(t)) = \max_{a \in \mathcal{A}} H(\gamma(t), \zeta(t), a), \quad (1.16)$$

for almost all $t \in [0, T]$. Moreover γ is a normal extremal if and only if one can choose $h_0 \neq 0$ and γ is an abnormal extremal if and only if one can choose $h_0 = 0$.

Proof.

- First assume that γ is an abnormal extremal. Then there exists ζ such that $\Gamma = (\gamma, \zeta) : [0, T] \rightarrow T^{\#}M$ is an abnormal biextremal. i.e. $\dot{\Gamma}(t) \in \overrightarrow{\mathcal{H}_{\mathcal{D}}}(\Gamma(t))$ for almost all t and $H_X(\Gamma(t)) = 0$ for all smooth sections X in \mathcal{D} . Since $X_j, j = 1, \dots, n$ is an orthonormal basis of $\mathcal{D}(x)$ for every $x \in M$ there are functions $\tilde{\alpha}_j : [0, T] \times M \rightarrow \mathbb{R}$, where $x \mapsto \tilde{\alpha}_j(t, x)$ is smooth for each $j = 1, \dots, n$ and all $t \in [0, T]$ such that

$$\dot{\Gamma}(t) = \overrightarrow{H_{\tilde{\alpha}(t, \cdot)} X}(\Gamma(t)) \quad \text{for almost all } t. \quad (1.17)$$

Then

$$\dot{\gamma}(t) = \sum_{j=1}^n \tilde{\alpha}_j(t, \gamma(t)) X_j(\gamma(t)) \quad (1.18)$$

for almost all $t \in [0, T]$. Let $\alpha(t) := \tilde{\alpha}(t, \gamma(t))$ then $|\alpha(t)| = 1$ since $\|\dot{\gamma}(t)\|_g = 1$ for almost all $t \in [0, T]$. Then $\alpha : [0, T] \rightarrow \mathcal{A}$ is an admissible control for γ . Since $H_X(\Gamma(t)) = 0$ for every smooth section X we have

$$H(\gamma(t), \zeta(t), \alpha(t)) = 0 = \max_{a \in \mathcal{A}} H(\gamma(t), \zeta(t), a). \quad (1.19)$$

Furthermore we have by (1.17)

$$\begin{aligned} \dot{\zeta}(t) &= -\frac{\partial}{\partial x} \sum_{j=1}^n \tilde{\alpha}_j(t, x) \langle \zeta(t), X_j(x) \rangle \Big|_{x=\gamma(t)} \\ &= -\sum_{j=1}^n \langle \zeta(t), X_j(\gamma(t)) \rangle \frac{\partial}{\partial x} \tilde{\alpha}_j(t, \gamma(t)) \\ &\quad - \frac{\partial}{\partial x} \sum_{j=1}^n \langle \zeta(t), \alpha_j(t) X_j(x) \rangle \Big|_{x=\gamma(t)}. \end{aligned} \quad (1.20)$$

But since $\langle \zeta(t), X_j(\gamma(t)) \rangle = H_{X_j}(\Gamma(t)) = 0$ it follows that

$$\dot{\zeta}(t) = -\frac{\partial}{\partial x} H(\gamma(t), \zeta(t), \alpha(t))$$

almost everywhere and thus ζ is a nontrivial maximizing adjoint vector along (γ, α) .

- Now assume that γ is a normal extremal. Then there exists ζ such that $\Gamma = (\gamma, \zeta)$ is a normal biextremal, i.e. $\dot{\Gamma}(t) = \overrightarrow{H_2}(\Gamma(t))$ almost everywhere. In particular

$$\dot{\gamma}(t) = \frac{\partial}{\partial \zeta} H_2(\gamma(t), \zeta(t)) = \sum_{j=1}^n \langle \zeta(t), X_j(\gamma(t)) \rangle X_j(\gamma(t)) \quad (1.21)$$

almost everywhere. If we let $\alpha_j(t) := \langle \zeta(t), X_j(\gamma(t)) \rangle$ for $j = 1, \dots, n$, then $|\alpha(t)| = 1$ and $\alpha : [0, T] \rightarrow \mathcal{A}$ is an admissible control for γ and

$$\begin{aligned} \dot{\zeta}(t) &= -\frac{\partial}{\partial x} \frac{1}{2} \sum_{j=1}^n \langle \zeta(t), X_j(x) \rangle^2 \Big|_{x=\gamma(t)} \\ &= -\frac{\partial}{\partial x} \sum_{j=1}^n \langle \zeta(t), X_j(x) \rangle \Big|_{x=\gamma(t)} \langle \zeta(t), X_j(\gamma(t)) \rangle \\ &= -\frac{\partial}{\partial x} H(\gamma(t), \zeta(t), \alpha(t)) \end{aligned} \quad (1.22)$$

almost everywhere. Furthermore the maximum of $a \mapsto \sum_{j=1}^n a_j \langle \zeta(t), X_j(\gamma(t)) \rangle$ on the closed unit ball in \mathbb{R}^n is attained for the vector $(\langle \zeta(t), X_j(\gamma(t)) \rangle)_{j=1}^n = \alpha(t)$. Since $H_2(\Gamma(t))$ is constant in t and nonzero it follows that there exists $h_0 > 0$ such that

$$h_0 = H(\gamma(t), \zeta(t), \alpha(t)) = \max_{a \in \mathcal{A}} H(\gamma(t), \zeta(t), a).$$

Thus ζ is a nontrivial maximizing adjoint vector along (γ, α) .

Now let $\alpha : [0, T] \rightarrow \mathcal{A}$ be an admissible control for γ and suppose that there exists a nontrivial maximizing adjoint vector ζ along (γ, α) . We will show that γ is an abnormal extremal if the maximizing condition (1.16) holds with $h_0 = 0$ and γ is a normal extremal if (1.16) holds with some $h_0 > 0$.

- Assume that the maximizing condition (1.16) holds with $h_0 = 0$, i.e. the maximum of $a \mapsto \sum_{j=1}^n a_j \langle \zeta(t), X_j(\gamma(t)) \rangle$ on the closed unit ball is zero. This means that

$\langle \zeta(t), X_j(\gamma(t)) \rangle = 0$ for all $j = 1, \dots, n$ and thus $H_X(\Gamma(t)) = 0$ for all smooth sections X of \mathcal{D} . Since $\alpha(t)X = \sum_{j=1}^n \alpha_j(t)X_j$ is a smooth section of \mathcal{D} and

$$\dot{\gamma}(t) = \sum_{j=1}^n \alpha_j(t) X_j(\gamma(t)) = \frac{\partial}{\partial \zeta} H_{\alpha(t)X}(\gamma(t), \zeta(t)), \quad (1.23)$$

$$\dot{\zeta}(t) = -\frac{\partial}{\partial x} \sum_{j=1}^n \langle \zeta(t), \alpha_j(t) X_j(x) \rangle \Big|_{x=\gamma(t)} \quad (1.24)$$

almost everywhere we have $\dot{\Gamma}(t) \in \overrightarrow{\mathcal{H}}_{\mathcal{D}}(\Gamma(t))$ and thus γ is an abnormal extremal.

- Now assume that the maximizing condition (1.16) holds with $h_0 > 0$. By multiplying ζ by h_0^{-1} we obtain a new nontrivial maximizing adjoint vector such that

$$1 = H(\gamma(t), \zeta(t), \alpha(t)) = \max_{a \in \mathcal{A}} H(\gamma(t), \zeta(t), a) \quad (1.25)$$

holds almost everywhere. Since the maximum of the functional

$$a \mapsto \sum_{j=1}^n a_j \langle \zeta(t), X_j(\gamma(t)) \rangle \quad (1.26)$$

on the closed unit ball is 1 it follows that

$$\sum_{j=1}^n \langle \zeta(t), X_j(\gamma(t)) \rangle^2 = 1$$

so that $\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t))$ with

$$\alpha_j(t) = \langle \zeta(t), X_j(\gamma(t)) \rangle, \quad j = 1, \dots, n. \quad (1.27)$$

has length 1, and the maximum in (1.26) is attained for $a = \alpha(t)$. We conclude that

$$\begin{aligned} \dot{\gamma}(t) &= \sum_{j=1}^n \alpha_j(t) X_j(\gamma(t)) = \sum_{j=1}^n \langle \zeta(t), X_j(\gamma(t)) \rangle X_j(\gamma(t)) \\ &= \frac{\partial}{\partial \zeta} H_2(\gamma(t), \zeta(t)), \end{aligned}$$

and by the adjoint equation (1.15)

$$\begin{aligned}\dot{\zeta}(t) &= -\frac{\partial}{\partial x} \sum_{j=1}^n \alpha_j(t) \langle \zeta(t), X_j(x) \rangle \Big|_{x=\gamma(t)} \\ &= -\frac{\partial}{\partial x} \frac{1}{2} \sum_{j=1}^n \langle \zeta(t), X_j(x) \rangle^2 \Big|_{x=\gamma(t)} \\ &= -\frac{\partial}{\partial x} H_2(\gamma(t), \zeta(t))\end{aligned}$$

for almost all $t \in [0, T]$. Thus γ is a normal extremal. □

Remark 1.7.4. If we do not have a global orthonormal basis of smooth sections, we have to choose a different control system. In this case let the control space \mathcal{A} be the set of all smooth sections X of \mathcal{D} such that $\|X(x)\|_g \leq 1$ for all $x \in M$ and define $X_a := a$ for all $a \in \mathcal{A}$. Let $F = \{X_a : a \in \mathcal{A}\}$ be the control system. Then again the admissible trajectories of this control system are precisely the horizontal curves $\gamma : [0, T] \rightarrow M$ with $\|\dot{\gamma}(t)\|_g \leq 1$ almost everywhere.

Proposition 1.7.3 is still valid and can be proven with similar arguments. Essentially we proved that $\Gamma = (\gamma, \zeta)$ is a biextremal iff ζ is a nontrivial adjoint vector along (γ, α) with $h_0 \in \{0, 1\}$ where γ is an admissible trajectory corresponding to α . But these two statements are local properties. Indeed we can cover the image of γ by open sets $\{U_j\}_j$ such that on each U_j we have an orthonormal basis X_1^j, \dots, X_n^j of smooth sections for $\mathcal{D}(x)$, $x \in U_j$. Furthermore we can restrict $\Gamma = (\gamma, \zeta)$ to $\Gamma_j = (\gamma_j, \zeta_j) : I_j \rightarrow TM$ where the intervals $\{I_j\}_j$ cover $[0, T]$. Now we can repeat the above proof for each Γ_j .

As opposed to the Riemannian case being an extremal is not a local property in the sub-Riemannian case. Indeed, suppose $\gamma : [0, T] \rightarrow M$ is “locally extremal”, i.e. such that the restrictions $\gamma_j : I_j \rightarrow M$ of γ to intervals I_j that cover $[0, T]$ are extremals. Thus for each j there is ζ_j such that (γ_j, ζ_j) is a biextremal. In the Riemannian case the covectors ζ_j are uniquely determined and therefore they coincide, making it possible to define a global ζ . But in the sub-Riemannian case there is no reason why the ζ_j should match and thus it is impossible to construct a global ζ .

Remark 1.7.5. Proposition 1.7.3 shows that Theorem 1.6.4 is a corollary of Pontryagin’s Maximum Principle 1.7.2. If $\gamma : [0, T] \rightarrow M$ parametrized by arc-length is time-optimal for the length-minimizing problem then γ is also a time-optimal trajectory for the control system described above. By Pontryagin’s Maximum Principle and Proposition 1.7.3 we have that γ is either a normal or an abnormal extremal and this proves Theorem 1.6.4.

1.8 Normal and abnormal extremals on nilpotent Lie groups of step 2

We give first a short introduction of the notation we will use here. A more detailed overview of the basic facts on 2-step nilpotent Lie groups can be found in Chapter 2.

Let \mathfrak{n} denote a 2-step nilpotent real Lie algebra, i.e. $[\mathfrak{n}, \mathfrak{n}] \neq \{0\}$ and $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = \{0\}$, endowed with a scalar product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, $\mathfrak{z} := [\mathfrak{n}, \mathfrak{n}]$, $\mathfrak{v} := \mathfrak{z}^\perp$, and $\dim \mathfrak{v} = n$, $\dim \mathfrak{z} = m$. The complete structure of \mathfrak{n} is captured by the maps $\mathfrak{z} \ni Z \mapsto j(Z) \in \mathfrak{so}(\mathfrak{v})$ defined by

$$\langle j(Z)X, Y \rangle = -\langle [X, Y], Z \rangle, \quad X, Y \in \mathfrak{v}, \quad Z \in \mathfrak{z}. \quad (1.28)$$

By fixing an orthonormal basis X_1, \dots, X_n of \mathfrak{v} and an orthonormal basis Z_1, \dots, Z_m of \mathfrak{z} we can define the skew-symmetric matrices J^j , $j = 1, \dots, m$, by

$$\left(J^j \right)_{k,l} := \langle [X_k, X_l], Z_j \rangle = \langle j(Z_j)X_l, X_k \rangle, \quad 1 \leq k, l \leq n, \quad j = 1, \dots, m.$$

Let N denote the corresponding connected, simply connected 2-step nilpotent Lie group with exponential map

$$\exp(xX + zZ) = (x, z) \in N = \mathbb{R}^n \times \mathbb{R}^m$$

and group product

$$(x, z) \cdot (y, u) = \left(x + y, z + u - \frac{1}{2} \left(\left\langle J^j x, y \right\rangle \right)_{j=1}^m \right), \quad (x, z), (y, u) \in N$$

The left invariant vector fields which we again denote by $X_1, \dots, X_n, Z_1, \dots, Z_m$ are then given by

$$X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^m \left(J^j x \right)_i \partial_{z_j}, \quad i = 1, \dots, n \quad Z_j = \partial_{z_j}, \quad j = 1, \dots, m. \quad (1.29)$$

Since obviously the linear span \mathcal{D} of the vector fields X_1, \dots, X_n is a bracket-generating distribution on N , we have a sub-Riemannian structure together with the Riemannian metric defined by letting the left-invariant vector fields X_1, \dots, X_n being a global orthonormal basis.

Now we consider the control system $(\dot{x}, \dot{z}) = \sum_{j=1}^n a_j X_j$, i.e.

$$\dot{x} = a, \quad \dot{z}_j = -\frac{1}{2} \left\langle J^j x, a \right\rangle, \quad j = 1, \dots, m, \quad a \in \mathcal{A} \quad (1.30)$$

on N with control set $\mathcal{A} = \{a \in \mathbb{R}^n : |a| \leq 1\}$. Then the length minimizers parametrized by arc-length are precisely the time-optimal trajectories of this control system. The Hamiltonian of this control system is

$$H(x, z, \xi, \eta, a) = \left\langle \xi - \frac{1}{2} J^\eta x, a \right\rangle, \quad x, \xi \in \mathbb{R}^n, z, \eta \in \mathbb{R}^m, a \in \mathcal{A}. \quad (1.31)$$

Thus a trajectory $\gamma = (x, z) : [0, T] \rightarrow N$ corresponding to an admissible control $\alpha : [0, T] \rightarrow \mathcal{A}$ is an extremal if there exists a nowhere vanishing absolutely continuous function $(\xi(t), \eta(t))$ and a constant $h_0 \geq 0$ such that (ξ, η) satisfies the adjoint equations

$$\dot{\xi}(t) = -\frac{1}{2} J^\eta \alpha(t), \quad \dot{\eta} = 0, \quad (1.32)$$

and the maximizing condition

$$h_0 = H(x(t), z(t), \xi(t), \eta(t), \alpha(t)) = \max_{a \in \mathcal{A}} H(x(t), z(t), \xi(t), \eta(t), a) \quad (1.33)$$

almost everywhere. Notice that η is constant. Furthermore, γ is a normal extremal iff we can choose h_0 to be nonzero. In this case we can assume $h_0 = 1$. If h_0 can be chosen equal to 0, then γ is abnormal.

Lemma 1.8.1. *A trajectory $\gamma = (x, z) : [0, T] \rightarrow N$ corresponding to a control $\alpha : [0, T] \rightarrow \mathcal{A}$ is a normal extremal parametrized by arc-length if and only if there exists $\tilde{\eta} \in \mathbb{R}^m$ such that $\dot{\alpha}(t) = J^{\tilde{\eta}} \alpha(t)$ and $|\alpha(0)| = 1$, i.e. $\alpha(t) = e^{tJ^{\tilde{\eta}}} \alpha(0)$.*

Proof. This proof can also be found in [23].

Let $\gamma = (x, z) : [0, T] \rightarrow N$ corresponding to a control $\alpha : [0, T] \rightarrow \mathcal{A}$ be a normal extremal parametrized by arc-length. Then there exists a nontrivial adjoint vector (ξ, η) as above such that the adjoint equations (1.32) and the maximizing condition (1.33) are satisfied. Since the maximum of

$$a \mapsto \left\langle \xi(t) - \frac{1}{2} J^\eta x(t), a \right\rangle, a \in \mathcal{A} \quad (1.34)$$

on the closed unit ball is assumed to be 1 and this maximum is attained for $\alpha(t)$ we conclude that

$$\left\| \xi(t) - \frac{1}{2} J^\eta x(t) \right\| = 1$$

and

$$\alpha(t) = \xi(t) - \frac{1}{2} J^\eta x(t). \quad (1.35)$$

But this means that α is absolutely continuous. And by (1.32) we get

$$\dot{\alpha}(t) = \dot{\xi}(t) - \frac{1}{2}J^\eta \dot{x}(t) = -J^\eta \alpha(t).$$

Conversely suppose that $\gamma = (x, z)$ is a trajectory that corresponds to a control α , which satisfies $\dot{\alpha}(t) = J^{\tilde{\eta}}\alpha(t)$ for some $\tilde{\eta} \in \mathbb{R}^m$ and $|\alpha(0)| = 1$. Then $|\alpha(t)| = 1$ for all t and thus γ is parametrized by arc-length. Now let $\eta(t) = -\tilde{\eta}$ and $\xi(t) = \alpha(t) + \frac{1}{2}J^\eta x(t)$. Then (ξ, η) satisfies the adjoint equation (1.32) and the maximizing condition (1.33) holds with $h_0 = 1$. Thus γ is a normal extremal. \square

Lemma 1.8.2. *Let $\gamma = (x, z) : [0, T] \rightarrow N$ be a horizontal curve parametrized by arc-length. Then γ is an abnormal extremal if and only if there exists $\eta \in \mathbb{R}^m \setminus \{0\}$ such that $J^\eta \dot{x}(t) \equiv 0$.*

Proof. First suppose that γ is an abnormal extremal corresponding to a control $\alpha : [0, T] \rightarrow \mathcal{A}$. Then there exists a nontrivial adjoint vector (ξ, η) such that

$$\dot{\xi}(t) = -\frac{1}{2}J^{\eta(t)}\alpha(t), \quad \dot{\eta}(t) = 0 \text{ and} \tag{1.36}$$

$$0 = \left\| \xi(t) - \frac{1}{2}J^{\eta(t)}x(t) \right\|, \tag{1.37}$$

for almost all t . This means that $\eta(t) = \eta$ is constant and by (1.37) $\xi \equiv \frac{1}{2}J^\eta x$. Hence $0 = \dot{\xi}(t) - \frac{1}{2}J^\eta \dot{x}(t) = -J^\eta \alpha(t)$ and thus $\frac{1}{2}J^\eta \dot{x} = \frac{1}{2}J^\eta \alpha = 0$. $\eta = 0$ would imply that $\xi \equiv 0$ but since (ξ, η) is a nontrivial adjoint vector we conclude that $\eta \neq 0$.

Now suppose that $J^\eta \dot{x}(t) = 0$ for a nontrivial $\eta \in \mathbb{R}^m \setminus \{0\}$. Choose $\xi(t) = \frac{1}{2}J^\eta x(t)$ and let $\alpha(t) = \dot{x}(t)$. Then $\alpha : [0, T] \rightarrow \mathcal{A}$ is an admissible control corresponding to γ . Furthermore,

$$\dot{\xi}(t) = 0 = -\frac{1}{2}J^\eta \alpha(t), \quad \dot{\eta}(t) = 0 \text{ and} \tag{1.38}$$

$$0 = \left\| \xi(t) - \frac{1}{2}J^\eta x(t) \right\|. \tag{1.39}$$

Hence γ is an abnormal extremal. \square

Theorem 1.8.3. *Let F_n be a free 2-step nilpotent Lie group in n generators, so that $\dim F_n = n + \binom{n}{2}$. Then every length minimizer in F_n parametrized by arc-length is a normal extremal.*

Proof. See also [23]. Since the vector fields $X_j, j = 1, \dots, n$, are left invariant we can always assume that $\gamma(0) = 0$. Let $\gamma = (x, z) : [0, T] \rightarrow F_n$ be a non-constant length minimizing trajectory such that $\gamma(0, 0) = 0$ with corresponding control α . By the Maximum Principle

1.7.2 γ is a normal or an abnormal extremal. We prove that γ is a normal extremal by induction on n .

If $n = 2$, then $J^\eta \in \mathfrak{so}(2)$ has no nontrivial kernel for $\eta \in \mathbb{R} \setminus \{0\}$. By Lemma 1.8.2 this means that γ is an abnormal extremal iff $\dot{x} \equiv 0$. But then γ would be a constant curve.

Now let $n > 2$ and assume that every abnormal extremal in a free nilpotent Lie group $F_{\tilde{n}}$ with $\tilde{n} < n$ is normal. Let us assume that γ is an abnormal extremal in F_n , then by Lemma 1.8.2 there exists $\eta \in \mathbb{R}^m \setminus \{0\}$ such that $J^\eta \dot{x}(t) = J^\eta \alpha(t) = 0$. Since $x(0) = 0$ we have $J^\eta x \equiv 0$. Then $S = \ker J^\eta$ is a proper nontrivial subspace of \mathbb{R}^n . Let $\mathfrak{s} := \{sX : s \in S\}$ and $\tilde{\mathfrak{f}} := \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{s}]$. Then $\tilde{\mathfrak{f}}$ is a Lie subalgebra of \mathfrak{f} and according to the notation above we let $\tilde{F} = \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{m}}$ denote the corresponding Lie group, where $\tilde{n} = \dim \mathfrak{s} < n$ and $\tilde{m} = \dim [\mathfrak{s}, \mathfrak{s}]$. Without loss of generality we may assume that $S = \mathbb{R}^{\tilde{n}}$. Since $x(t), \alpha(t) \in S$ and $\dot{z}(t) = -\frac{1}{2} \langle J^j x(t), \alpha(t) \rangle_{j=1}^m = \exp\left(-\frac{1}{2} [xX, \alpha X]\right) \in \exp([\mathfrak{s}, \mathfrak{s}])$ we have that γ is a trajectory in \tilde{F} corresponding to the S -valued control $\alpha : [0, T] \rightarrow \tilde{\mathcal{A}} := \{a \in \mathbb{R}^{\tilde{n}} : |a| \leq 1\}$. Furthermore, γ minimizes length and is parametrized by arc length when regarded as an admissible arc for the sub-Riemannian manifold \tilde{F} . Now we consider two cases:

If $[\mathfrak{s}, \mathfrak{s}] = \{0\}$, then \tilde{F} is abelian and thus γ is a straight line which is obviously also normal in F_n .

If $[\mathfrak{s}, \mathfrak{s}] \neq \{0\}$, then \tilde{F} is a free 2-step nilpotent Lie group $F_{\tilde{n}}$ and by the induction hypothesis γ is a normal extremal in \tilde{F} . By Lemma 1.8.1 there exists $\tilde{\eta} \in \mathbb{R}^{\tilde{m}}$ such that $\dot{\alpha}(t) = J^{\tilde{\eta}} \alpha(t)$, $|\alpha(0)| = 1$ where $J^{\tilde{\eta}} \in \mathfrak{so}(\tilde{n})$. Then $J^{\tilde{\eta}} \in \mathfrak{so}(\tilde{n})$ extends to a skew-symmetric linear map $J^\eta \in \mathfrak{so}(n)$ with some $\eta \in \mathbb{R}^m$. Again by Lemma 1.8.1 this implies that γ is a normal extremal in F . \square

Theorem 1.8.4. *Let N be a 2-step nilpotent Lie group. Then every non-constant abnormal minimizer parametrized by arc-length in N is normal in some subgroup of N and therefore smooth.*

Proof. See [20]. Again we can assume that $\gamma(0) = 0$. Again by induction on $\dim N$, we can repeat the arguments of Theorem 1.8.3 up to the point where we extended $J^{\tilde{\eta}} \in \mathfrak{so}(\tilde{n})$, $\tilde{\eta} \in \mathbb{R}^{\tilde{m}}$ to a matrix $J^\eta \in \mathfrak{so}(n)$, $\eta \in \mathbb{R}^m$. Actually one can extend $J^{\tilde{\eta}} \in \mathfrak{so}(\tilde{n})$ to a skew-symmetric linear map $\Lambda \in \mathfrak{so}(n)$ but in general it is not clear that there exists $\eta \in \mathbb{R}^m$ such that $\Lambda = J^\eta$. For a general 2-step nilpotent Lie group $\{J^\eta : \eta \in \mathbb{R}^m\}$ is just a subspace of $\mathfrak{so}(n)$. For a free 2-step nilpotent Lie group the map $\eta \mapsto J^\eta$ is a vector space isomorphism $\mathbb{R}^m \rightarrow \mathfrak{so}(n)$. Anyway this proves that every non-constant abnormal minimizer is after all a normal extremal in some subgroup. \square

Definition 1.8.5. A 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, $\mathfrak{v} = \mathfrak{z}^\perp$ is called **nonsingular** if $\text{ad } X : \mathfrak{n} \rightarrow \mathfrak{z}$ is surjective for all $X \in \mathfrak{v} \setminus \{0\}$. A nonsingular 2-step nilpotent Lie group is also called **Métivier group**.

Remark 1.8.6. A 2-step nilpotent Lie algebra is nonsingular if and only if $J^\eta \in \mathfrak{so}(n)$ is nonsingular for every $\eta \in \mathbb{R}^m \setminus \{0\}$.

The Heisenberg type groups (note that this includes Heisenberg groups) are important examples of nonsingular 2-step nilpotent Lie groups. But not every nonsingular 2-step nilpotent Lie group is isomorphic to a Heisenberg type group. An example for a nonsingular 2-step nilpotent Lie group which is not isomorphic to a group of Heisenberg type can be found for instance in [28].

Theorem 1.8.7. *Let N be a nonsingular 2-step nilpotent Lie group. Then the abnormal extremals parametrized by arc-length are exactly the constant curves.*

Proof. Let $\gamma : [0, T] \rightarrow N$ be an abnormal extremal parametrized by arc-length. Then by Lemma 1.8.2 there exists $\eta \in \mathbb{R}^m \setminus \{0\}$ such that $0 = J^\eta \dot{x}(t)$. Since J^η is nonsingular, this implies that $\dot{x} \equiv 0$ and thus $\dot{z} \equiv 0$ which gives the claim. \square

Summary 1.8.8. This section shows that on 2-step nilpotent Lie groups the existence of abnormal extremals in general can not be avoided. But whether these abnormal extremals play a role in the minimizing process does depend on the particular structure of the Lie group. As we have seen in Theorem 1.8.7 there are no abnormal extremals to consider on nonsingular 2-step nilpotent Lie groups. And in Section 2.3.1 we are able to give very explicit formulas for the Carnot-Carathéodory distance on Heisenberg type groups.

The free 2-step nilpotent Lie groups do not admit any strictly abnormal minimizers but many abnormal extremals which are always normal. Thus the length minimizing problem of the Carnot-Carathéodory distance is completely described by the normal extremals. As in the Riemannian case the Hamiltonian differential equations gives us necessary conditions for a curve being minimizing.

1.8.1 An Example for strictly abnormal minimizer in 2-step nilpotent Lie groups

We now want to give an example for the existence of strictly abnormal extremals in 2-step nilpotent Lie groups. It turns out that the strictly abnormal extremals we are considering in Theorem 1.8.10 are not length minimizing. The first example for the existence of strictly abnormal minimizers was given by Montgomery in [25]. Liu and Sussmann [23] study a generic class of abnormal minimizers on manifolds with two-dimensional distribution, called regular abnormal extremals. An example of strictly abnormal minimizers on a nilpotent Lie group (of step 4) can be found in [20]. Here we want to present an example on a 2-step nilpotent Lie group.

Let \mathfrak{g} be the 6-dimensional real Lie algebra spanned by the orthonormal basis

$$\{e_1, e_2, e_3, e_4, f_1, f_2\}$$

with nontrivial commutator relations

$$[e_1, e_2] = f_1, [e_3, e_4] = f_2, [e_1, e_3] = [e_2, e_4] = f_1. \quad (1.40)$$

Then $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ where $\mathfrak{v} = \text{span}\{e_1, e_2, e_3, e_4\}$ and $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}] = \text{span}\{f_1, f_2\}$ is the center of \mathfrak{g} . Let G denote the corresponding Lie group which we realize as $\mathbb{R}^4 \times \mathbb{R}^2$. The structure matrices are given by

$$J^{f_1} = J^1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad J^{f_2} = J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (1.41)$$

The group product on G reads

$$(x, z) \cdot (y, u) = \left(x + y, z + u - \frac{1}{2} \left(\left\langle J^j x, y \right\rangle \right)_{j=1}^2 \right), \quad (x, z), (y, u) \in G,$$

and the left invariant vector fields are

$$X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^2 \left(J^j x \right)_i \partial_{z_j}, \quad i = 1, 2, 3, 4, \quad Z_1 = \partial_{z_1}, \quad Z_2 = \partial_{z_2}.$$

Proposition 1.8.9. *Let*

$$H_1 := \{(x, z) \in G : x_3 = x_4 = 0, z_2 = 0\}, \\ H_2 := \{(x, z) \in G : x_1 = x_4, x_2 = -x_3, z_1 = -z_2\}.$$

Then H_1, H_2 are subgroups of G and isomorphic to the Heisenberg group \mathbb{H}_1 . The abnormal extremals parametrized by arc-length of G starting in 0 are exactly the horizontal curves parametrized by arc-length in H_1 or H_2 . Furthermore an abnormal extremal $\gamma = (x, z) : [0, T] \rightarrow G$ parametrized by arc-length starting in 0 is strictly abnormal if $z(T) \neq 0$.

Proof. The subgroup properties are immediately clear if one observes that

$\log H_1 = \text{span}\{e_1, e_2, f_1\} =: \mathfrak{h}_1$ and $\log H_2 = \text{span}\{e_1 + e_4, e_2 - e_3, f_1 - f_2\} =: \mathfrak{h}_2$ and $\mathfrak{h}_1, \mathfrak{h}_2$ are Lie subalgebras of \mathfrak{g} since

$$[e_1, e_2] = f_1 \quad \text{and} \quad [e_1 + e_4, e_2 - e_3] = f_1 - f_1 - f_1 + f_2 = f_2 - f_1. \quad (1.43)$$

Thus H_1, H_2 are subgroups of G which are obviously isomorphic to the Heisenberg group \mathbb{H}_1 .

By Lemma 1.8.2 $\gamma = (x, z)$ is an abnormal extremal iff there exists $\eta \in \mathbb{R}^2 \setminus \{0\}$ such that $0 = J^\eta \dot{x}$ almost everywhere. Since we assumed that $x(0) = 0$ we have that $0 \equiv J^\eta x$. Now it is easy to see that J^η has a nontrivial kernel iff $\eta_1 = 0$ or $\eta_2 = \eta_1$. In case of $\eta_1 = 0$ the kernel of J^η is $\mathbb{R}^2 \times \{0\}$ and since $\eta_2 \dot{z}_2 = \langle \eta, \dot{z} \rangle = -\frac{1}{2} \langle J^\eta x, \dot{x} \rangle = 0$ we conclude that $z_2 \equiv 0$ and γ is a horizontal curve in H_1 . If now $\eta_1 = \eta_2$ then the kernel of J^η is the linear span of $(1, 0, 0, 1)$ and $(0, 1, -1, 0)$ and $\eta_1 (\dot{z}_1 + \dot{z}_2) = \langle \eta, \dot{z} \rangle = -\frac{1}{2} \langle J^\eta x, \dot{x} \rangle = 0$. Thus $z_1 \equiv -z_2$ and γ is a horizontal curve in H_2 . Furthermore every horizontal curve parametrized by arc-length in H_1 respectively H_2 is obviously an abnormal extremal in G .

Now let $\gamma = (x, z) : [0, T] \rightarrow G$ with control $\alpha : [0, T] \rightarrow \mathcal{A}$ be an abnormal extremal in G contained in H_1 . Then by Lemma 1.8.1 γ is a normal extremal iff there exists a $\eta \in \mathbb{R}^2$ such that $\dot{\alpha} = J^\eta \alpha$. Since $\alpha = (\alpha_1, \alpha_2, 0, 0)$ we have

$$\dot{\alpha} = \eta_1 (\alpha_2, -\alpha_1, -\alpha_1, -\alpha_2).$$

$\eta_1 \neq 0$ implies that $\alpha \equiv 0$ and therefore γ is constant equal to 0. If we assume that $\eta_1 = 0$, then it follows that $\alpha(t) = c$ is constant with $c \in \mathbb{R}^2 \times \{0\}$. Thus $x(t) = ct$, $\dot{z}_j(t) = -\frac{1}{2} \langle J^j x(t), \alpha(t) \rangle = 0$, $j = 1, 2$ and $z(t) = 0$.

If $\gamma = (x, z) : [0, T] \rightarrow G$ with control α is an abnormal extremal in G contained in H_2 and normal in H_2 , then by Lemma 1.8.1 there exists $\eta \in \mathbb{R}^2$ such that $\dot{\alpha} = J^\eta \alpha$. Since $\alpha = (\alpha_1, \alpha_2, -\alpha_2, \alpha_1)$ we have

$$\dot{\alpha} = \eta_1 (0, 0, -\alpha_1, -\alpha_2) + \eta_2 (0, 0, \alpha_1, \alpha_2).$$

This implies that $\dot{\alpha} \equiv 0$. Thus $x(t) = ct$ and $z(t) = 0$ with some constant $c \in \text{kern}(J^1 + J^2)$.

This means that an abnormal extremal $\gamma = (x, z)$ is strictly abnormal if $z(T) \neq 0$. \square

Proposition 1.8.9 gives us a complete description of the abnormal extremals in G . Now we want to ask the question whether the abnormal extremals minimize the length. It turns out that determining whether one of the strictly abnormal extremals is also minimizing is a very difficult task even in this “simple” example. Therefore we will restrict our further analysis to curves that connect $0 \in G$ and some point $(0, z) \in G$, $z \neq 0$.

Theorem 1.8.10. *Let $p = (s, 0), q = (s, -s) \in \mathbb{R}^2 \setminus \{0\}$. Then every length-minimizer connecting $0 \in G$ and $(0, p) \in G$ or $(0, q) \in G$ is normal and not abnormal. In particular*

$$d_{CC}(0, (0, p))^2 = 2\sqrt{2}\pi |s| < 4\pi |s| = \inf \left\{ L(\gamma)^2 : \gamma \text{ abnormal and connects } 0, (0, p) \right\}, \quad (1.44)$$

and

$$d_{CC}(0, (0, q))^2 = 4\sqrt{2}\pi |s| < 8\pi |s| = \inf \left\{ L(\gamma)^2 : \gamma \text{ abnormal and connects } 0, (0, q) \right\}. \quad (1.45)$$

Proof. By the previous Proposition 1.8.9 we have seen that every abnormal extremal connecting $0 \in G$ and $(0, p) \in G$ (or $(0, q) \in G$) is actually strictly abnormal and contained in H_1 or H_2 . Both subgroups H_1, H_2 are isomorphic to the Heisenberg group \mathbb{H}_1 and the length-minimizing problem on Heisenberg groups is well understood (see Section 2.3.1). Thus from Theorem 2.3.6 on page 77 we know that a length minimizing curve $\gamma_{\mathbb{H}}$ in \mathbb{H}_1 (which is a normal extremal in \mathbb{H}_1) connecting $0 \in \mathbb{H}_1$ and $(0, s) \in \mathbb{H}_1$ has length

$$L(\gamma_{\mathbb{H}})^2 = 4\pi |s|. \quad (1.46)$$

Via the Lie group isomorphism

$$H_1 \ni (x_1, x_2, 0, 0, z_1, 0) \mapsto (x_1, x_2, z_1) \in \mathbb{H}_1,$$

which obviously preserves the length of horizontal curves we can conclude that

$$4\pi |s| = \inf \left\{ L(\gamma)^2 : \gamma \text{ abnormal in } G \text{ and connects } 0, (0, p) \right\}.$$

Now observe that

$$\tilde{e}_1 := \frac{1}{\sqrt{2}}(e_1 + e_4), \quad \tilde{e}_2 := \frac{1}{\sqrt{2}}(e_2 - e_3), \quad \tilde{f}_1 := \frac{1}{2}(f_2 - f_1)$$

is a basis of the Lie algebra \mathfrak{h}_2 with the property that $\{\tilde{e}_1, \tilde{e}_2\}$ is orthonormal and $[\tilde{e}_1, \tilde{e}_2] = \tilde{f}_1$. Then the Lie group isomorphisms

$$\varphi : H_2 \rightarrow \mathbb{H}_1, (x_1, x_2, -x_2, x_4, z_1, -z_1) \mapsto (\sqrt{2}x_1, \sqrt{2}x_2, -2z_1)$$

has the property that γ is horizontal in G and contained in H_2 iff $\varphi \circ \gamma$ is horizontal in \mathbb{H}_1 , where $L(\gamma) = L(\varphi \circ \gamma)$. We conclude that

$$8\pi |s| = \inf \left\{ L(\gamma)^2 : \gamma \text{ abnormal and connects } 0, (0, q) \right\}.$$

Now we have to find the shortest normal extremals in G . This is a long proof and consists of several steps. After reparametrization we will assume here that our curves are parametrized with constant speed.

- Let $\gamma = (x, z) : [0, 1] \rightarrow G$ be a normal extremal parametrized with constant speed starting in $0 \in G$ with $x(1) = 0$. By the definition of normal extremals there exists a (ξ, η) such that $\Gamma = (x, z, \xi, \eta) : [0, 1] \rightarrow G$ is a bicharacteristic of the Hamiltonian

$$\mathbb{H}_2(x, z, \xi, \eta) = \frac{1}{2} \left| \xi - \frac{1}{2} J^\eta x \right|^2 = \frac{1}{2} |\zeta|^2, \quad \zeta := \xi - \frac{1}{2} J^\eta x \quad (1.47)$$

and H does not vanish along Γ . This means that Γ satisfies the Hamiltonian differential equations:

$$\dot{x}(t) = \zeta(t), \quad \dot{z}_j = -\frac{1}{2} \langle J^j x, \zeta \rangle, \quad \dot{\xi}(t) = -\frac{1}{2} J^{\eta(t)} \zeta(t), \quad \dot{\eta}(t) = 0. \quad (1.48)$$

Thus η is constant $\eta(t) = \eta$ and furthermore

$$\dot{\zeta}(t) = \dot{\xi}(t) - \frac{1}{2} J^\eta \dot{x}(t) = -J^\eta \zeta(t), \quad \text{where } \zeta(0) = \xi(0) =: \xi_0. \quad (1.49)$$

Solving equation (1.49) gives

$$\zeta(t) = e^{-tJ^\eta} \xi_0 \quad (1.50)$$

and taking into account that $x(0) = 0$ we get

$$x(t) = \frac{I - e^{-tJ^\eta}}{J^\eta} \xi_0, \quad \xi(t) = \frac{1}{2} (e^{-tJ^\eta} + I) \xi_0. \quad (1.51)$$

We define the term $\frac{I - e^{-A}}{A}$ by its power series $\sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} A^j$ which is convergent for every $A \in M(n)$.

Now let us denote the eigenvalues of J^η by $\pm i\lambda_1, \pm i\lambda_2$, $\lambda_1 \geq 0, \lambda_2 \geq 0$. Furthermore, $\text{ES}(A, \mu)$ will always denote the eigenspace of a matrix A corresponding to the eigenvalue μ . Then we can write $\xi_0 = \xi_1 + \xi_2$, where $\xi_j \in \text{ES}((J^\eta)^2, -\lambda_j^2)$, $j = 1, 2$ are uniquely determined. Thus the condition $x(1) = 0$ is equivalent to $\lambda_j \in 2\pi\mathbb{Z}^*$ or $\xi_j = 0$ for $j = 1, 2$. Now observe that

$$L(\gamma) = \int_0^1 |\dot{x}(t)| dt = |\xi_0|, \quad (1.52)$$

this means that at least for one $j = 1, 2$ we have $\lambda_j \in 2\pi\mathbb{Z}^*$ since otherwise it would follow that $L(\gamma) = 0$.

- Before we integrate the equation $\dot{z}_j = -\frac{1}{2} \langle J^j x, \zeta \rangle$ we want to compute the eigenvalues and eigenvectors of $(J^\eta)^2$. For this let $\eta = (a, b) \in \mathbb{R}^2$, then

$$(J^\eta)^2 = \begin{pmatrix} -2a^2 & 0 & 0 & a^2 + ab \\ 0 & -2a^2 & -(a^2 + ab) & 0 \\ 0 & -(a^2 + ab) & -(a^2 + b^2) & 0 \\ a^2 + ab & 0 & 0 & -(a^2 + b^2) \end{pmatrix}.$$

Since calculating the eigensystem in case $a = 0$, i.e.

$$(J^\eta)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -b^2 & 0 \\ 0 & 0 & 0 & -b^2 \end{pmatrix}$$

is easy we will assume here that $a \neq 0$. Let

$$\alpha_\pm := \frac{1}{2a} (a - b \pm \sqrt{R}) \text{ with } R := 5a^2 - 2ab + b^2 \text{ and}$$

$$\begin{aligned} \lambda_1 &= \frac{1}{\sqrt{2}} \sqrt{3a^2 + b^2 + (a + b) \sqrt{R}}, \\ \lambda_2 &= \frac{1}{\sqrt{2}} \sqrt{3a^2 + b^2 - (a + b) \sqrt{R}}. \end{aligned}$$

Then $\lambda_1, \lambda_2 \in \mathbb{R}$ are well-defined, since

$$\begin{aligned} (a + b)^2 R &= (a + b)^2 (4a^2 + (a - b)^2) \\ &= 4a^4 + 8a^3b + 4a^2b^2 + a^4 - 2a^2b^2 + b^4 \\ &= 9a^4 + 6a^2b^2 + b^4 - 4a^2(a - b)^2 \\ &\leq (3a^2 + b^2)^2. \end{aligned}$$

Furthermore, let

$$\begin{aligned} v_1 &:= (-\alpha_+, 0, 0, 1), & v_2 &:= (0, \alpha_+, 1, 0), \\ v_3 &:= (-\alpha_-, 0, 0, 1), & v_4 &:= (0, \alpha_-, 1, 0). \end{aligned}$$

We claim that v_1, v_2 are eigenvectors of $(J^\eta)^2$ corresponding to the eigenvalue $-\lambda_1^2$ and v_3, v_4 are eigenvectors of $(J^\eta)^2$ corresponding to the eigenvalue $-\lambda_2^2$. Indeed we have

$$\begin{aligned} (J^\eta)^2 v_1 &= \left(2a^2\alpha_+ + a^2 + ab, 0, 0, -\left(a^2 + ab\right)\alpha_+ - a^2 - b^2 \right) \\ &= \left(2a^2 + a\sqrt{R}, 0, 0, -\frac{1}{2}\left((a+b)(a-b) + 2a^2 + 2b^2 + (a+b)\sqrt{R}\right) \right) \\ &= \left(2a^2 + a\sqrt{R}, 0, 0, -\lambda_1^2 \right), \end{aligned}$$

$$\begin{aligned} (J^\eta)^2 v_3 &= \left(2a^2\alpha_- + a^2 + ab, 0, 0, -\left(a^2 + ab\right)\alpha_- - a^2 - b^2 \right) \\ &= \left(2a^2 - a\sqrt{R}, 0, 0, -\lambda_2^2 \right), \end{aligned}$$

$$\begin{aligned} (J^\eta)^2 v_2 &= \left(0, -2a^2\alpha_+ - a^2 - ab, -\left(a^2 + ab\right)\alpha_+ - a^2 - b^2, 0 \right) \\ &= \left(0, -2a^2 - a\sqrt{R}, -\lambda_1^2, 0 \right), \end{aligned}$$

$$\begin{aligned} (J^\eta)^2 v_4 &= \left(0, -2a^2\alpha_- - a^2 - ab, -\left(a^2 + ab\right)\alpha_- - a^2 - b^2, 0 \right) \\ &= \left(0, -2a^2 + a\sqrt{R}, -\lambda_2^2, 0 \right). \end{aligned}$$

But since

$$\begin{aligned} \lambda_1^2\alpha_+ &= \frac{1}{4a} \left(a - b + \sqrt{R} \right) \left(3a^2 + b^2 + (a+b)\sqrt{R} \right) \\ &= \frac{1}{4a} \left(3a^3 + b^2a - 3a^2b - b^3 + \left(a^2 - b^2 + 3a^2 + b^2 \right) \sqrt{R} + (a+b)R \right) \\ &= \frac{1}{4a} \left(8a^3 + 4a^2\sqrt{R} \right) \\ &= 2a^2 + a\sqrt{R} \quad \text{and} \\ \lambda_2^2\alpha_- &= 2a^2 - a\sqrt{R}, \end{aligned}$$

we have

$$(J^\eta)^2 v_1 = -\lambda_1^2 v_1, \quad (J^\eta)^2 v_2 = -\lambda_1^2 v_2, \quad (1.53a)$$

$$(J^\eta)^2 v_3 = -\lambda_2^2 v_3, \quad (J^\eta)^2 v_4 = -\lambda_2^2 v_4, \quad (1.53b)$$

and this shows that our claim was true.

- Now we distinguish the two cases of endpoints to be either $(0, p)$ or $(0, q)$.
- **First let $z(1) = p$ and assume that $L(\gamma)^2 \leq 2\sqrt{2}\pi |s|$.** We will show that then $L(\gamma)^2 = 2\sqrt{2}\pi |s|$ and actually that there exists a normal extremal with this length and endpoints.

First observe that

$$\begin{aligned} \langle \eta, \dot{z}(t) \rangle &= -\frac{1}{2} \langle J^\eta x(t), \zeta(t) \rangle = \frac{1}{2} (\langle \zeta(t), \zeta(t) \rangle - \langle \xi_0, \zeta(t) \rangle) \\ &= \frac{1}{2} (\langle \xi_0, \xi_0 \rangle - \langle \xi_0, \zeta(t) \rangle) \end{aligned}$$

which gives

$$\langle \eta, z(1) \rangle = \frac{1}{2} |\xi_0|^2 - \frac{1}{2} \langle \xi_0, x(1) \rangle = \frac{1}{2} |\xi_0|^2 = \frac{1}{2} L(\gamma)^2. \quad (1.54)$$

Since we assumed that $L(\gamma)^2 \leq 2\sqrt{2}\pi |s|$ and $z(1) = (s, 0)$ we conclude that

$$as = \langle \eta, z(1) \rangle = \frac{1}{2} L(\gamma)^2 \leq \sqrt{2}\pi |s|. \quad (1.55)$$

Thus we have that a has the same sign as s and $0 \neq |a| \leq \sqrt{2}\pi$. **First we will show that $\lambda_1, \lambda_2 \in 2\pi\mathbb{Z}^*$.** For this assume that $\lambda' \in \{\lambda_1, \lambda_2\}$ is not contained in $2\pi\mathbb{Z}^*$. Then for $\lambda \in \{\lambda_1, \lambda_2\}$, $\lambda' \neq \lambda$ we have $\lambda \in 2\pi\mathbb{Z}^*$ and ξ_0 is an eigenvector of $(J^\eta)^2$ corresponding to the eigenvalue $-\lambda^2$. Then

$$x(t) = \frac{I - e^{-tJ^\eta}}{J^\eta} \xi_0 = -\frac{1}{\lambda^2} (I - e^{-tJ^\eta}) J^\eta \xi_0 = \frac{1}{\lambda^2} (J^\eta \zeta(t) - J^\eta \xi_0)$$

and thus

$$\dot{z}_j(t) = -\frac{1}{2} \langle J^j x(t), \zeta(t) \rangle = \frac{1}{2\lambda^2} \langle J^2 J^\eta \xi_0, \dot{x}(t) \rangle - \frac{1}{2\lambda^2} \langle J^j J^\eta \zeta(t), \zeta(t) \rangle. \quad (1.56)$$

Now, for any skew-symmetric matrix $S \in \mathfrak{so}(n)$, we claim that

$$\langle S J^\eta \zeta(t), \zeta(t) \rangle = \langle S J^\eta \xi_0, \xi_0 \rangle \text{ for every } t. \quad (1.57)$$

To see this, observe that

$$\zeta(t) = e^{-tJ^\eta} \xi_0 = \cos(t\lambda) \xi_0 - \frac{\sin(t\lambda)}{\lambda} J^\eta \xi_0,$$

since ξ_0 is an eigenvector of $(J^\eta)^2$ corresponding to the eigenvalue $-\lambda^2$. Then

$$\begin{aligned} \langle SJ^\eta \zeta(t), \zeta(t) \rangle &= \cos^2(t\lambda) \langle SJ^\eta \xi_0, \xi_0 \rangle + \frac{\sin^2(t\lambda)}{\lambda^2} \langle S(J^\eta)^2 \xi_0, J^\eta \xi_0 \rangle \\ &\quad - \cos(t\lambda) \frac{\sin(t\lambda)}{\lambda} (\langle SJ^\eta \xi_0, J^\eta \xi_0 \rangle + \langle S \xi_0, \xi_0 \rangle) \\ &= \cos^2(t\lambda) \langle SJ^\eta \xi_0, \xi_0 \rangle + \sin^2(t\lambda) \langle SJ^\eta \xi_0, \xi_0 \rangle \\ &= \langle SJ^\eta \xi_0, \xi_0 \rangle \end{aligned}$$

and this gives the claim (1.57).

Using equation (1.57) we can integrate equation (1.56) to conclude that

$$z_j(1) = -\frac{1}{2\lambda^2} \langle J^j J^\eta \xi_0, \xi_0 \rangle. \quad (1.58)$$

Since ξ_0 is an eigenvector of $(J^\eta)^2$ we have $\xi_0 = \mu_1 v_1 + \mu_2 v_2$ or $\xi_0 = \mu_1 v_3 + \mu_2 v_4$ for some $0 \neq (\mu_1, \mu_2) \in \mathbb{R}^2$. Since both cases lead to the same calculations we may assume for simplicity that $\xi_0 = \mu_1 v_1 + \mu_2 v_2$. Then $J^\eta \xi_0 = \sigma \lambda (\mu_1 v_2 - \mu_2 v_1)$ for some $\sigma \in \{1, -1\}$ and

$$z_2(1) = \frac{1}{2\lambda^2} \langle J^\eta \xi_0, J^2 \xi_0 \rangle = \frac{\sigma}{2\lambda} (-\mu_1^2 - \mu_2^2) \neq 0.$$

Since $z_2(1)$ should be zero this is a contradiction and therefore we have that $\lambda_1, \lambda_2 \in 2\pi\mathbb{Z}^*$.

Now we will show that $a = -b$. Observe that

$$R = 5a^2 - 2ab + b^2 = 4a^2 + (a - b)^2 > (a - b)^2$$

and

$$\begin{aligned} 2 \min \{ \lambda_1^2, \lambda_2^2 \} &= 3a^2 + b^2 - |a + b| \sqrt{R} \\ &\leq 3a^2 + b^2 - |a^2 - b^2| \\ &\leq 4a^2 \leq 8\pi^2. \end{aligned} \quad (1.59)$$

Thus $\min \{ \lambda_1^2, \lambda_2^2 \} = 4\pi^2$. One may observe that the inequality (1.59) is strict if $a \neq -b$. Thus the assumption $a \neq -b$ would lead to a contradiction and we conclude that $a = -b$. Then

$$\lambda_1^2 = 2a^2 = \lambda_2^2 = 4\pi^2 \quad (1.60)$$

and thus $|a| = \sqrt{2}\pi$, $L(\gamma)^2 = 2\sqrt{2}\pi |s|$.

- To prove that there exists a normal extremal with length $L(\gamma)^2 = 2\sqrt{2}\pi|s|$ let $\eta = (a, -a)$, $a = \sqrt{2}\pi\frac{s}{|s|}$ and $\xi_0 \in \mathbb{R}^4$ such that $|\xi_0|^2 = 2\sqrt{2}\pi|s|$ and $\xi_0 = (\xi_1, \xi_2, \xi_2, -\xi_1)$. Let $\gamma = (x, z) : [0, 1] \rightarrow G$ be the normal extremal as above, i.e. the solution of the Hamiltonian equations with data $\gamma(0) = 0, \eta, \xi_0$. Then J^η has the eigenvalues $\pm i2\pi$ and thus $x(1) = 0$. Since $(J^\eta)^2 = -(2\pi)^2 I$, ξ_0 is an eigenvector of $(J^\eta)^2$ and by (1.58) we have

$$z_j(1) = -\frac{1}{2\lambda^2} \langle J^j J^\eta \xi_0, \xi_0 \rangle = -\frac{a}{8\pi^2} \langle J^j (J^1 - J^2) \xi_0, \xi_0 \rangle.$$

But $(J^1 - J^2) \xi_0 = (2\xi_2, -2\xi_1, 0, 0)$, $J^1 (J^1 - J^2) \xi_0 = (-2\xi_1, -2\xi_2, -2\xi_2, 2\xi_1)$ and therefore $z_2(1) = 0$ and

$$\begin{aligned} z_1(1) &= -\frac{a}{8\pi^2} \langle J^1 (J^1 - J^2) \xi_0, \xi_0 \rangle = \frac{a}{4\pi^2} |\xi_0|^2 \\ &= s. \end{aligned}$$

- **Now let $z(1) = q$ and assume that $L(\gamma)^2 \leq 4\sqrt{2}\pi|s|$.** We will show that then $L(\gamma)^2 = 4\sqrt{2}\pi|s|$ and actually that there exists a normal extremal with this length and endpoints.

In this case we have by (1.54)

$$(a - b)s = \langle \eta, z(1) \rangle = \frac{1}{2} |\xi_0|^2 = \frac{1}{2} L(\gamma)^2 \leq 2\sqrt{2}\pi|s|,$$

so that $a - b$ has the same sign as s and

$$0 \neq |a - b| \leq 2\sqrt{2}\pi. \quad (1.61)$$

Again we will first assume that there is $\lambda' \in \{\lambda_1, \lambda_2\}$ such that $\lambda' \notin 2\pi\mathbb{Z}^*$ and show that this is a contradiction. Let $\lambda \in \{\lambda_1, \lambda_2\}$ with $\lambda \in 2\pi\mathbb{Z}^*$ (this λ exists since we already mentioned that $\lambda_1, \lambda_2 \notin 2\pi\mathbb{Z}^*$ leads to a contradiction). Since ξ_0 is an eigenvector of $(J^\eta)^2$ with eigenvalue $-\lambda^2$ we have by (1.58)

$$z_j(1) = -\frac{1}{2\lambda^2} \langle J^j J^\eta \xi_0, \xi_0 \rangle,$$

and since $z(1) = (s, -s)$ we conclude

$$0 = z_1(1) + z_2(1) = -\frac{1}{\lambda^2} \langle (J^1 + J^2) J^\eta \xi_0, \xi_0 \rangle. \quad (1.62)$$

But this is a contradiction as we will show now. First consider the case $a = 0$; then $\xi_0 = (0, 0, \xi_3, \xi_4)$ and ξ_0 is an eigenvector of $(J^\eta)^2$ with eigenvalue $-\lambda^2$. Then

$$\langle (J^1 + J^2) J^\eta \xi_0, \xi_0 \rangle = b \langle (J^1 + J^2) J^2 \xi_0, \xi_0 \rangle = -b (\xi_3^2 + \xi_4^2)$$

and since $b \neq 0$ and $\xi_0 \neq 0$ this contradicts (1.62).

So we will assume that $a \neq 0$. Then $\xi_0 = \mu_1 v_1 + \mu_2 v_2$ or $\xi_0 = \mu_1 v_3 + \mu_2 v_4$ for some $0 \neq (\mu_1, \mu_2) \in \mathbb{R}^2$ since ξ_0 is an eigenvector of $(J^\eta)^2$ with eigenvalue $-\lambda^2$. Then

$$0 = \langle (J^1 + J^2) J^\eta \xi_0, \xi_0 \rangle = \sigma \lambda (\mu_1^2 + \mu_2^2) \langle (J^1 + J^2) v_2, v_1 \rangle$$

or

$$0 = \langle (J^1 + J^2) J^\eta \xi_0, \xi_0 \rangle = \sigma \lambda (\mu_1^2 + \mu_2^2) \langle (J^1 + J^2) v_4, v_3 \rangle$$

respectively. But this is equivalent to $(1 + \alpha_+)^2 = 0$ respectively $(1 + \alpha_-)^2 = 0$. But $\alpha_+ = -1$ respectively $\alpha_- = -1$ is only possible if $R = (3a - b)^2$. It is easy to see that $R = (3a - b)^2$ is equivalent to $a = b$. Indeed we have

$$(3a - b)^2 - R = 4a^2 - 4ab = 4a(a - b),$$

and $R = (3a - b)^2$ is equivalent to $a = b$ since $a \neq 0$. But we assumed by (1.61) that $a \neq b$ and thus the assumption $\lambda' \notin 2\pi\mathbb{Z}^*$ leads to a contradiction.

Thus we have $\lambda_1, \lambda_2 \in 2\pi\mathbb{Z}^*$. **Now we will show that $a = -b$.** For this observe that

$$\begin{aligned} 2 \min \{ \lambda_1^2, \lambda_2^2 \} &= 3a^2 + b^2 - \sqrt{(a+b)^2 (5a^2 - 2ab + b^2)} \\ &= 2a(a+b) - \sqrt{4a^2(a+b)^2 + (a-b)^2(a+b)^2} + (a-b)^2 \\ &\leq (a-b)^2 \leq 2 \cdot 4\pi^2. \end{aligned} \tag{1.63}$$

Thus $\min \{ \lambda_1^2, \lambda_2^2 \} = (2\pi)^2$. The inequality (1.63) is strict if $a+b \neq 0$ since $a \neq b$. But then $\min \{ \lambda_1^2, \lambda_2^2 \} \neq (2\pi)^2$ and this is a contradiction. So we conclude that $a = -b$ and then

$$\lambda_1^2 = 2a^2 = \lambda_2^2 = 4\pi^2. \tag{1.64}$$

Thus $|a| = \sqrt{2}\pi$, $|a - b| = 2|a| = 2\sqrt{2}\pi$ and $L(\gamma)^2 = 4\sqrt{2}\pi|s|$.

- To prove the existence of a normal extremal with length $L(\gamma)^2 = 4\sqrt{2}\pi|s|$ let $\eta = (a, -a)$ with $a = \sqrt{2}\pi \frac{s}{|s|}$ and $\xi_0 \in \mathbb{R}^4$ such that $|\xi_0|^2 = 4\sqrt{2}\pi|s|$ and $\xi_0 = (\xi_1, \xi_2, -\xi_2, \xi_1)$. Solving the Hamiltonian equations with data $\gamma(0) = 0$,

ξ_0, η leads to a normal extremal $\gamma = (x, z) : [0, 1] \rightarrow G$ with the following properties. Since $\pm i2\pi$ are the eigenvalues of J^η we have $x(1) = 0$ and

$$z_j(1) = -\frac{1}{2(2\pi)^2} \langle J^j J^\eta \xi_0, \xi_0 \rangle = -\frac{a}{8\pi^2} \langle J^j (J^1 - J^2) \xi_0, \xi_0 \rangle. \quad (1.65)$$

Now we have

$$(J^1 - J^2) \xi_0 = (0, 0, -2\xi_1, -2\xi_2)$$

and

$$\begin{aligned} J^1 (J^1 - J^2) \xi_0 &= (-2\xi_1, -2\xi_2, 0, 0), \\ J^2 (J^1 - J^2) \xi_0 &= (0, 0, -2\xi_2, 2\xi_1). \end{aligned}$$

Thus we have by (1.65):

$$\begin{aligned} z_1(1) &= \frac{a}{4\pi^2} (\xi_1^2 + \xi_2^2) = \frac{a}{8\pi^2} |\xi_0|^2 = s \text{ and} \\ z_2(1) &= -\frac{a}{4\pi^2} (\xi_1^2 + \xi_2^2) = -s. \end{aligned}$$

This completes the proof. \square

1.9 Singular sub-Riemannian manifolds and the Grušin operator

We consider the manifold $\mathbb{R}_x^n \times \mathbb{R}_u$ and the vector fields

$$X_j = \partial_{x_j}, \quad U_j = x_j \partial_u, \quad j = 1, \dots, n. \quad (1.66)$$

Then $\mathcal{D} = \text{span} \{X_j, U_j : j = 1, \dots, n\}$ is a singular distribution, which is bracket-generating, in particular $[X_j, U_j] = \partial_u$, $j = 1, \dots, n$. By requiring that X_1, \dots, X_n together with $|x| \partial_u$ is an orthonormal basis of $\mathcal{D}(x, u)$ for every $(x, u) \in \mathbb{R}^{n+1}$ we define a metric $\|\cdot\|_{(x,u)}$ on $\mathcal{D}(x, u)$. In particular we have for

$$v = \beta \partial_u + \sum_{j=1}^n \alpha_j \partial_{x_j} \in \mathcal{D}(x, u), \quad \alpha \in \mathbb{R}^n, \beta \in \mathbb{R} \quad (1.67)$$

that

$$\|v\|_{(x,u)}^2 = \begin{cases} |\alpha|^2 + \frac{\beta^2}{|x|^2} & , \text{ if } x \neq 0 \\ |\alpha|^2 & , \text{ if } x = 0 \end{cases}. \quad (1.68)$$

Note that this is not a sub-Riemannian structure in the sense of Section 1.1. Nevertheless we can call an absolutely continuous curve $\gamma : [0, T] \rightarrow \mathbb{R}^{n+1}$ **horizontal** if $\dot{\gamma}(t) \in \mathcal{D}(\gamma(t))$ for almost all $t \in [0, T]$ and define its length by

$$L(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_{\gamma(t)} dt. \quad (1.69)$$

Furthermore, we can define the Carnot-Carathéodory distance for $p, q \in \mathbb{R}^{n+1}$ as in the sub-Riemannian case by

$$d_{CC}(p, q) := \inf \left\{ L(\gamma) : \gamma : [0, T] \rightarrow \mathbb{R}^{n+1} \text{ is horizontal and } \gamma(0) = p, \gamma(T) = q \right\}. \quad (1.70)$$

By the Chow-Rashevskii theorem it follows that $d_{CC}(p, q) < \infty$ for $p, q \in \mathbb{R}^{n+1}$ since \mathcal{D} is bracket-generating.

We now want to translate the minimizing problem into the language of Control Theory.

Let

$$\mathcal{A} = \left\{ (a, b) \in \mathbb{R}^{2n} : |a|^2 + |b|^2 \leq 1 \right\} \quad (1.71)$$

be the control space and

$$F = \{ aX + bU : (a, b) \in \mathcal{A} \} \quad (1.72)$$

the control system. Observe that $aX + bU = \sum_{j=1}^n a_j \partial_{x_j} + \langle b, x \rangle \partial_u$, thus

$$\|aX + bU\|_{(x,u)}^2 = |a|^2 + \frac{\langle b, x \rangle^2}{|x|^2} \leq |a|^2 + |b|^2. \quad (1.73)$$

So the the admissible trajectories of the control system F are precisely the horizontal curves $\gamma : [0, T] \rightarrow \mathbb{R}^{n+1}$ with $\|\dot{\gamma}(t)\|_{\gamma(t)} \leq 1$. The Hamiltonian of this control system is given by

$$\begin{aligned} H(x, u, \xi, \eta, a, b) &= \langle (\xi, \eta), aX(x, u) + bU(x, u) \rangle \\ &= \langle a, \xi \rangle + \eta \langle b, x \rangle, \quad (x, u), (\xi, \eta) \in \mathbb{R}^{n+1}, (a, b) \in \mathcal{A}. \end{aligned} \quad (1.74)$$

Pontryagin's Maximum Principle gives us necessary conditions for a curve being length minimizing and we call γ an extremal if the necessary conditions are satisfied. In particular let $\gamma : [0, T] \rightarrow \mathbb{R}^{n+1}$ be an admissible trajectory with control $(\alpha, \beta) : [0, T] \rightarrow \mathcal{A}$. Then γ is an **extremal** if there exists a nontrivial maximizing adjoint vector ζ along $(\gamma, (\alpha, \beta))$, i.e. if there exists $h_0 \geq 0$ such that

$$\dot{\zeta}(t) = -\frac{\partial H}{\partial(x, u)}(\gamma(t), \zeta(t), (\alpha(t), \beta(t))), \text{ and} \quad (1.75)$$

$$h_0 = H(\gamma(t), \zeta(t), (\alpha(t), \beta(t))) = \max_{(a,b) \in \mathcal{A}} H(\gamma(t), \zeta(t), (a, b)) \quad (1.76)$$

for almost all $t \in [0, T]$. Moreover γ is called a **normal extremal** if we can choose $h_0 \neq 0$ and γ is called an **abnormal extremal** if we can choose $h_0 = 0$.

Theorem 1.9.1. *Let $\gamma = (x, u) : [0, T] \rightarrow \mathbb{R}^{n+1}$ be a length minimizer parametrized by arc-length. Then γ is a normal extremal. Furthermore a non constant curve γ parametrized by arc-length is a normal extremal if and only if there exists $\eta \in \mathbb{R}$ such that the following differential equations are satisfied*

$$\ddot{x}(t) = -\eta^2 x(t), \quad \dot{u}(t) = \eta |x(t)|^2, \quad \text{for all } t \in [0, T]. \quad (1.77)$$

Proof. Let $\gamma = (x, u) : [0, T] \rightarrow \mathbb{R}^{n+1}$ be a length minimizer parametrized by arc-length corresponding to the control $(\alpha, \beta) : [0, T] \rightarrow \mathcal{A}$. Then Pontryagin's Maximum Principle tells us that there exists a nontrivial maximizing adjoint vector $\zeta = (\xi, \eta) : [0, T] \rightarrow \mathbb{R}^{n+1}$ along $(\gamma, (\alpha, \beta))$, i.e.

$$\dot{\xi}(t) = -\frac{\partial H}{\partial x} = -\eta(t)\beta(t), \quad \dot{\eta}(t) = -\frac{\partial H}{\partial u} = 0, \quad (1.78)$$

and there exists $h_0 \geq 0$ such that

$$h_0 = H(x, u, \xi, \eta, \alpha, \beta) = \max_{(\tilde{\alpha}, \tilde{\beta}) \in \mathcal{A}} H(x, u, \xi, \eta, \tilde{\alpha}, \tilde{\beta}) \quad (1.79)$$

almost everywhere. Thus we have that $\eta(t) = \eta$ is constant.

First assume that $h_0 = 0$, i.e. γ is an abnormal extremal; then $(\xi(t), \eta x(t)) = 0$ since otherwise the maximum of $(a, b) \mapsto \langle a, \xi(t) \rangle + \eta \langle b, x(t) \rangle$ on the closed unit ball would not be zero. This means that $\xi \equiv 0$ and since (ξ, η) is nontrivial we have $\eta \neq 0$ and thus $x \equiv 0$. Since γ is an admissible trajectory we further have that $\dot{u}(t) = \langle \beta(t), x(t) \rangle = 0$ and thus γ is constant.

Now assume that $h_0 \neq 0$; then we can assume that $h_0 = 1$. Since the maximum of the functional

$$(a, b) \mapsto \langle a, \xi(t) \rangle + \eta \langle b, x(t) \rangle$$

on the closed unit ball is 1 and is attained for $(\alpha(t), \beta(t))$ we have that

$$|\xi(t)|^2 + \eta^2 |x(t)|^2 = 1$$

and we can conclude that

$$\alpha(t) = \xi(t), \quad \beta(t) = \eta x(t). \quad (1.80)$$

Then the adjoint equation (1.78) tells us that $\dot{\xi}(t) = -\eta\beta(t) = -\eta^2x(t)$. Since γ is an admissible trajectory for (α, β) we have by (1.80) that

$$\dot{x}(t) = \alpha(t) = \xi(t), \text{ and } \dot{u}(t) = \langle \beta, x(t) \rangle = \eta|x(t)|^2. \quad (1.81)$$

Thus γ and ξ are smooth and $\ddot{x}(t) = \dot{\xi}(t) = -\eta^2x(t)$.

Conversely suppose that γ is non constant and parametrized by arc-length and that there exists $\eta \in \mathbb{R}$ such that

$$\ddot{x}(t) = -\eta^2x(t), \text{ and } \dot{u}(t) = \eta|x(t)|^2.$$

Let $\alpha(t) := \dot{x}(t)$, $\beta(t) := \eta x(t)$; then $\dot{u}(t) = \eta|x(t)|^2 = \langle \beta(t), x(t) \rangle$. Since γ is parametrized by arc-length it follows that

$$1 = |\dot{x}(t)|^2 + \frac{\dot{u}(t)^2}{|x(t)|^2} = |\alpha(t)|^2 + |\beta(t)|^2$$

and thus (α, β) is an admissible control for γ . If we further let $\xi(t) := \alpha(t)$, and $\eta(t) = \eta$ to be constant, then

$$-\frac{\partial H}{\partial x}(\gamma(t), \xi(t), \eta, \alpha(t), \beta(t)) = -\eta\beta(t) = -\eta^2x(t) = \ddot{x}(t) = \dot{\alpha}(t) = \dot{\xi}(t),$$

and $-\frac{\partial H}{\partial u} = 0 = \dot{\eta}$. Furthermore

$$H(\gamma(t), \xi(t), \eta, a, b) \leq \sqrt{|\xi(t)|^2 + \eta^2|x(t)|^2} = \sqrt{|\alpha(t)|^2 + |\beta(t)|^2} = 1$$

for all $(a, b) \in \mathcal{A}$. This means that

$$1 = H(\gamma(t), \xi(t), \eta, \alpha(t), \beta(t)) = \max_{(a,b) \in \mathcal{A}} H(\gamma(t), \xi(t), \eta, a, b)$$

and thus (ξ, η) is a nontrivial maximizing adjoint vector. □

2 Nilpotent Lie groups of step 2

2.1 Preliminary definitions and notation

A real Lie algebra \mathfrak{n} is called a **2-step nilpotent Lie algebra** if the ideal $\mathfrak{z} := [\mathfrak{n}, \mathfrak{n}]$ is not trivial and contained in the center, i.e. $[\mathfrak{n}, \mathfrak{z}] = \{0\}$. Let \mathfrak{n} denote a nilpotent Lie algebra of step 2. We equip \mathfrak{n} with a positive definite inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Furthermore, we choose an orthonormal basis X_1, \dots, X_n of \mathfrak{v} and an orthonormal basis Z_1, \dots, Z_m of \mathfrak{z} .

The complete structure of \mathfrak{n} is captured by the maps $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ defined by the equation

$$\langle j(Z)X, Y \rangle = -\langle [X, Y], Z \rangle, \quad X, Y \in \mathfrak{v}, Z \in \mathfrak{z}. \quad (2.1)$$

Remark 2.1.1. Equivalently one may define $j(Z) : \mathfrak{v} \rightarrow \mathfrak{v}$ by

$$j(Z)X = \text{ad}^*(X)(Z) = -(\text{ad}(X))^*Z$$

where $(\text{ad}(X))^*$ denotes the transpose of the adjoint representation $\text{ad}(X) : \mathfrak{n} \rightarrow \mathfrak{z}$, $\text{ad}(X)(Y) = [X, Y]$, $Y \in \mathfrak{n}$ determined by $\langle \cdot, \cdot \rangle$.

Since we already fixed a basis of $\mathfrak{v}, \mathfrak{z}$ we may define the skew-symmetric matrices $J^j \in \mathfrak{so}(n)$, $j = 1, \dots, m$ by

$$(J^j)_{k,l} := \langle [X_k, X_l], Z_j \rangle, \quad j = 1, \dots, m, \quad 1 \leq k, l \leq n. \quad (2.2)$$

Then

$$J^z := \sum_{j=1}^m z_j J^j \in \mathfrak{so}(n) \quad (2.3)$$

is the matrix representation of $j(zZ)$ in the basis X_1, \dots, X_n .

A connected, simply connected Lie group is called a **2-step nilpotent Lie group** if the corresponding Lie algebra is a 2-step nilpotent Lie algebra. Using exponential coordinates, a 2-step nilpotent Lie group N can always be viewed as $N = \mathbb{R}^n \times \mathbb{R}^m$ with exponential map

$$\exp(xX + zZ) = (x, z) \in N = \mathbb{R}^n \times \mathbb{R}^m. \quad (2.4)$$

By the Campbell-Baker-Hausdorff formula the group product is then given by

$$\begin{aligned}
 (x, z) \cdot (y, u) &= \exp \left((x + y) X + (z + u) Z + \frac{1}{2} [xX, yX] \right) \\
 &= \exp \left((x + y) X + (z + u) Z + \frac{1}{2} \sum_{j=1}^m \sum_{k,l=1}^n x_k y_l J_{k,l}^j Z^j \right) \\
 &= \left(x + y, z + u - \frac{1}{2} \left(\langle J^j x, y \rangle \right)_{j=1}^m \right). \tag{2.5}
 \end{aligned}$$

The left invariant vector fields on N which we again denote by $X_i, i = 1, \dots, n$, $Z_j, j = 1, \dots, m$ are given by

$$Z_j = \partial_{z_j}, \quad j = 1, \dots, m \tag{2.6}$$

and for $f \in C^\infty(N)$

$$\begin{aligned}
 X_i f(x, z) &= \left. \frac{d}{dt} \right|_0 f((x, z) \cdot (te_i, 0)) \\
 &= \partial_{x_i} f(x, z) - \frac{1}{2} \sum_{j=1}^m \langle J^j x, e_i \rangle \partial_{z_j} f(x, z), \quad i = 1, \dots, n.
 \end{aligned}$$

Here $e_j \in \mathbb{R}^n, j \in \{1, \dots, n\}$ denotes the j -th standard unit vector in \mathbb{R}^n , i.e.

$$(e_j)_k = \delta_{j,k} = \begin{cases} 1 & , k = j \\ 0 & , k \neq j \end{cases}, \quad j, k = 1, \dots, n.$$

Then

$$X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^m (J^j x)_i \partial_{z_j}, \quad i = 1, \dots, n. \tag{2.7}$$

Remark 2.1.2. Recall the definition of the Carnot-Carathéodory distance:

$$d_{CC}(p, q) = \inf \{l(\gamma) : \gamma \text{ is horizontal and } \gamma(0) = p, \gamma(T) = q\}, \quad p, q \in N.$$

Since the vector fields X_1, \dots, X_n are left invariant the Carnot-Carathéodory distance is also left invariant, i.e.

$$d_{CC}(g \cdot p, g \cdot q) = d_{CC}(p, q), \quad p, q, g \in N.$$

We will therefore restrict our further analysis to the case $p = 0$ and write

$$|q|_{CC} := d_{CC}(0, q), \quad q \in N. \tag{2.8}$$

Remark 2.1.3. If we consider the non-isotropic dilations

$$\delta_r(x, z) = (rx, r^2z), (x, z) \in N, r > 0, \quad (2.9)$$

then the Carnot-Carathéodory metric is homogeneous of degree 1 with respect to these dilations, i.e.

$$|\delta_r(g)|_{CC} = r|g|_{CC}, g \in N, r > 0.$$

Now we want to give some important examples which will be revisited in following sections.

Example 2.1.4. The **Heisenberg algebra** \mathfrak{h}_ν with orthonormal basis $X_1, Y_1, \dots, X_\nu, Y_\nu, Z$ is characterized by the commutator relations

$$[X_i, Y_j] = \delta_{i,j}Z, i, j = 1, \dots, \nu \quad (2.10)$$

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_j, Z] = 0, i, j = 1, \dots, \nu. \quad (2.11)$$

Thus we have $\mathfrak{h}_\nu = \mathfrak{v} \oplus \mathfrak{z}$ with 1-dimensional center $\mathfrak{z} = \mathbb{R}Z = [\mathfrak{h}_\nu, \mathfrak{h}_\nu]$ and

$$\mathfrak{v} = \text{span} \{X_i, Y_i : i = 1, \dots, \nu\} = \mathfrak{z}^\perp.$$

The structure of \mathfrak{h}_ν is determined by the matrix

$$J = J^1 = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \end{pmatrix}.$$

Note that $J^2 = J \cdot J = -I$.

The corresponding Lie group, the Heisenberg group, is $\mathbb{H}_\nu = \mathbb{R}^{2\nu} \times \mathbb{R}$ with group product

$$(x, z) \cdot (y, u) = \left(x + y, z + u - \frac{1}{2} \langle Jx, y \rangle\right), (x, z), (y, u) \in \mathbb{R}^{2\nu} \times \mathbb{R}.$$

The next example is a generalization of the Heisenberg algebra with larger center.

Example 2.1.5. A **Heisenberg type algebra** $\mathfrak{h}_{\nu,m}$ with m -dimensional center $\mathfrak{z} = \text{span} \{Z_1, \dots, Z_m\}$ and 2ν -dimensional $\mathfrak{v} = \text{span} \{X_1, Y_1, \dots, X_\nu, Y_\nu\} = \mathfrak{z}^\perp$ is characterized by the condition that

$$j(Z)^2 = -\langle Z, Z \rangle Id_{\mathfrak{v}}, Z \in \mathfrak{z}. \quad (2.12)$$

With this property we necessarily need that the dimension of \mathfrak{v} has to be $n := 2\nu$ even.

The corresponding Lie group, a Heisenberg type group, is $\mathbb{H}_{\nu,m} = \mathbb{R}^{2\nu} \times \mathbb{R}^m$, endowed with the Campbell-Baker-Hausdorff product. Note that if $m = 1$, then a Heisenberg type group $\mathbb{H}_{\nu,1}$ is isomorphic to the Heisenberg group \mathbb{H}_{ν} .

Example 2.1.6. The free 2-step nilpotent Lie algebra $\mathfrak{f} = \mathfrak{v} \oplus \mathfrak{z}$ in n generators X_1, \dots, X_n is given by $\mathfrak{v} = \text{span}\{X_1, \dots, X_n\}$ and $\binom{n}{2}$ -dimensional center \mathfrak{z} spanned by $\{Z_{k,l} : 1 \leq k < l \leq n\}$, with commutation relations

$$[X_k, X_l] = Z_{k,l}, \quad 1 \leq k < l \leq n, \quad (2.13)$$

and all other commutators being zero. Thus the structure of \mathfrak{f} is determined by the matrices

$$J^{k,l} = 2e_k \wedge e_l, \quad 1 \leq k < l \leq n,$$

where the exterior product $x \wedge y \in \mathfrak{so}(n)$ for $x, y \in \mathbb{R}^n$ is given by

$$(x \wedge y)_{k,l} = \frac{1}{2}(x_k y_l - x_l y_k), \quad 1 \leq k, l \leq n$$

(see Section 2.1.1 on the next page).

The corresponding Lie group, the free 2-step nilpotent Lie group, is $F = \mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}}$ with group product

$$\begin{aligned} (x, z) \cdot (y, u) &= \left(x + y, z + u - \frac{1}{2} \left(\left\langle J^{k,l} x, y \right\rangle \right)_{1 \leq k < l \leq n} \right) \\ &= \left(x + y, z + u + \frac{1}{2} (x_k y_l - x_l y_k)_{1 \leq k < l \leq n} \right) \\ &= \left(x + y, z + u + \left((x \wedge y)_{k,l} \right)_{1 \leq k < l \leq n} \right), \quad (x, z), (y, u) \in F. \end{aligned} \quad (2.14)$$

The left invariant vector fields are given by

$$\begin{aligned} Z_{k,l} &= \partial_{z_{k,l}}, \quad 1 \leq k < l \leq n, \\ X_j &= \partial_{x_j} + \frac{1}{2} \left(\sum_{k=1}^{j-1} x_k \partial_{z_{k,j}} - \sum_{k=j+1}^n x_k \partial_{z_{j,k}} \right). \end{aligned} \quad (2.15)$$

We want to introduce some basic notation and properties on skew-symmetric matrices.

2.1.1 Some facts on $\mathfrak{so}(n)$ and the adjoint representation

Before we begin to study the sub-Riemannian geometry of 2-step nilpotent Lie groups we want to present some basic facts on $\mathfrak{so}(n)$ and the adjoint representation ad on $\mathfrak{so}(n)$ to establish the notation which we are going to use.

$\mathfrak{so}(n)$, the vector space of all $n \times n$ skew-symmetric matrices, is a Lie algebra with bracket $[A, B] = AB - BA$, $A, B \in \mathfrak{so}(n)$. The corresponding Lie group $SO(n)$ is the special orthogonal group. We equip $M(n)$ with the scalar product $\langle A, B \rangle = \text{tr}(AB^t)$ and denote $|A| = \langle A, A \rangle^{\frac{1}{2}} = (\text{tr}(AA^t))^{\frac{1}{2}}$.

The tensor product $x \otimes y \in M(n)$ of two vectors $x, y \in \mathbb{R}^n$ is the matrix with entries

$$(x \otimes y)_{k,l} = x_k y_l, \quad k, l = 1, \dots, n; \quad (2.16)$$

the exterior product (or wedge product) $x \wedge y \in M(n)$ is then given by

$$x \wedge y = \frac{1}{2}(x \otimes y - y \otimes x). \quad (2.17)$$

Obviously the exterior product defines a skew-symmetric matrix, i.e. $x \wedge y \in \mathfrak{so}(n)$ for $x, y \in \mathbb{R}^n$.

Next we give some very basic properties of the tensor product.

Lemma 2.1.7. *For $x, y, z, u \in \mathbb{R}^n$ and $A \in M(n)$ we have*

1. $(x \otimes y)^t = y \otimes x$,
2. $A(x \otimes y) = (Ax) \otimes y$,
3. $(x \otimes y)A = x \otimes (A^t y)$,
4. $(x \otimes y)z = \langle y, z \rangle x$,
5. $\langle x \otimes y, z \otimes u \rangle = \langle y, u \rangle \langle x, z \rangle$,
6. $|x \otimes y| = |x| |y|$.

And some basic properties for the exterior product:

Lemma 2.1.8. *For $x, y, z, u \in \mathbb{R}^n$, $A \in M(n)$, and $S \in \mathfrak{so}(n)$ we have*

1. $(x \wedge y)^t = y \wedge x = -x \wedge y$,
2. $A(x \wedge y)A^t = (Ax) \wedge (Ay)$,
3. $[S, x \wedge y] = Sx \wedge y + x \wedge Sy$,

4. $\langle S, x \wedge y \rangle = \langle Sy, x \rangle$
5. $\langle x \wedge y, z \wedge u \rangle = \frac{1}{2} (\langle x, z \rangle \langle y, u \rangle - \langle x, u \rangle \langle y, z \rangle),$
6. $|x \wedge y|^2 = \frac{1}{2} (|x|^2 |y|^2 - \langle x, y \rangle^2),$
7. Let $v_1, \dots, v_n \in \mathbb{R}^n$ be an orthonormal basis of \mathbb{R}^n , then $\sqrt{2}(v_k \wedge v_l), 1 \leq k < l \leq n$ is an orthonormal basis of $\mathfrak{so}(n)$.

Notation. In the sequel we use the convention that \wedge, \otimes have a weaker binding than other operations on vectors. Thus, for a matrix $A \in M(n)$ and vectors $x, y \in \mathbb{R}^n$, $Ax \wedge y$ will always denote $(Ax) \wedge y$.

Proof. Very simple calculations but also a source of errors.

1. $(x \wedge y)^t = \frac{1}{2} (y \otimes x - x \otimes y) = y \wedge x = -x \wedge y.$
2. $A(x \wedge y)A^t = \frac{1}{2} ((Ax) \otimes (Ay) - (Ay) \otimes (Ax)) = (Ax) \wedge (Ay).$
3. $[S, x \wedge y] = \frac{1}{2} (Sx \otimes y - Sy \otimes x - x \otimes (S^t y) + y \otimes (S^t x)) = (Sx) \wedge y + x \wedge (Sy).$
4. $\langle S, x \wedge y \rangle = \frac{1}{2} \text{tr}(Sy \otimes x - Sx \otimes y) = \frac{1}{2} (\langle Sy, x \rangle - \langle Sx, y \rangle) = \langle Sy, x \rangle.$
- 5.

$$\begin{aligned} \langle x \wedge y, z \wedge u \rangle &= \frac{1}{4} (\langle x \otimes y, z \otimes u \rangle - \langle y \otimes x, z \otimes u \rangle - \langle x \otimes y, u \otimes z \rangle + \langle y \otimes x, u \otimes z \rangle) \\ &= \frac{1}{2} (\langle x, z \rangle \langle y, u \rangle - \langle x, u \rangle \langle y, z \rangle). \end{aligned}$$

6. $|x \wedge y|^2 = \frac{1}{2} (\langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle \langle y, x \rangle) = \frac{1}{2} (|x|^2 |y|^2 - \langle x, y \rangle^2).$
7. By 5. we have $2 \langle v_k \wedge v_l, v_i \wedge v_j \rangle = \delta_{k,i} \delta_{l,j} - \delta_{k,j} \delta_{l,i} = \delta_{(k,l),(i,j)} - \delta_{(k,l),(j,i)} = \delta_{(k,l),(i,j)}$ (since $k < l, i < j$). Thus $\{\sqrt{2}(v_k \wedge v_l)\}_{1 \leq k < l \leq n}$ are orthogonal and by 6. they are also normalized. Since $\dim \mathfrak{so}(n) = \binom{n}{2}$, this is indeed an orthonormal basis.

□

As mentioned before $\mathfrak{so}(n)$ denotes the Lie algebra of all $n \times n$ skew-symmetric matrices. The corresponding Lie group $SO(n)$, the special orthogonal group, is the group of all orthogonal matrices with determinant 1. The exponential map is given by the usual power series

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}, \quad A \in \mathfrak{so}(n). \quad (2.18)$$

We will now study the map $\text{ad} : \mathfrak{so}(n) \rightarrow \mathfrak{gl}(\mathfrak{so}(n))$, the adjoint representation of $\mathfrak{so}(n)$, given by

$$\text{ad}(A)B := [A, B] = AB - BA, \quad A, B \in \mathfrak{so}(n). \quad (2.19)$$

By exponentiating this map one gets the adjoint representation $\text{Ad} : SO(n) \rightarrow GL(\mathfrak{so}(n))$ of $SO(n)$, i.e.

$$\text{Ad}(\exp(A)) := \exp(\text{ad}(A)) = \sum_{j=0}^{\infty} \frac{(\text{ad } A)^j}{j!}, \quad A \in \mathfrak{so}(n). \quad (2.20)$$

Furthermore, $\text{Ad}(Q) = \text{Int}(Q)_*$, $Q \in SO(n)$ where $\text{Int}(Q) : SO(n) \rightarrow SO(n)$ is given by $\text{Int}(Q)P := QPQ^{-1}$ and $\text{Int}(Q)_*$ is the differential of the map $\text{Int}(Q)$ at the identity $I \in SO(n)$, we have

$$\text{Ad}(e^A)X = e^A X (e^A)^{-1} = e^A X e^{-A}, \quad A, X \in \mathfrak{so}(n). \quad (2.21)$$

Lemma 2.1.9. *Let $A \in \mathfrak{so}(n)$, then $\text{ad}(A) \in \mathfrak{gl}(\mathfrak{so}(n))$ is skew-symmetric, i.e.*

$$\langle \text{ad}(A)B, C \rangle = -\langle B, \text{ad}(A)C \rangle, \quad B, C \in \mathfrak{so}(n). \quad (2.22)$$

Proof. Let $B, C \in \mathfrak{so}(n)$. Then

$$\begin{aligned} \langle \text{ad}(A)B, C \rangle &= \text{tr}((AB - BA)C^t) \\ &= \text{tr}(BC^tA - BAC^t) \\ &= \langle B, A^tC - CA^t \rangle \\ &= \langle B, \text{ad}(A^t)C \rangle = -\langle B, \text{ad}(A)C \rangle. \end{aligned}$$

□

Next we want to give some characterization of the eigenvalues and eigenvectors of $\text{ad}(\Lambda)$ for a given matrix $\Lambda \in \mathfrak{so}(n)$. Let $n = 2\nu$ if n is even or $n = 2\nu + 1$ if n is odd. As a skew-symmetric matrix Λ has eigenvalues $i\lambda_1, -i\lambda_1, \dots, i\lambda_\nu, -i\lambda_\nu$ for even n , and $\lambda_0 = 0, i\lambda_1, -i\lambda_1, \dots, i\lambda_\nu, -i\lambda_\nu$ for odd n , where $0 \leq \lambda_1 \leq \dots \leq \lambda_\nu$. So Λ^2 , which is symmetric, has only real eigenvalues $-\lambda_1^2, \dots, -\lambda_\nu^2$ for even n and $0, -\lambda_1^2, \dots, -\lambda_\nu^2$ for odd n .

Recall the following fundamental result on skew-symmetric matrices:

Lemma 2.1.10. *Let $\Lambda \in \mathfrak{so}(n)$ with eigenvalues as above. Then there is a orthonormal basis $v_1, \tilde{v}_1, \dots, v_\nu, \tilde{v}_\nu$ for even n (respectively $v_0, v_1, \tilde{v}_1, \dots, v_\nu, \tilde{v}_\nu$ for odd n) of \mathbb{R}^n such that*

$$\Lambda v_j = \lambda_j \tilde{v}_j \quad \Lambda \tilde{v}_j = -\lambda_j v_j, \quad j = 1, \dots, \nu, \quad (2.23)$$

and if n is odd $\Lambda v_0 = 0$.

Remark. The above lemma shows that Λ has the shape of a “block-diagonal” matrix in the basis $\{v_j, \tilde{v}_j\}_j$, i.e.

$$Q^t \Lambda Q = \begin{pmatrix} 0 & -\lambda_1 & & & & \\ \lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -\lambda_\nu & \\ & & & \lambda_\nu & 0 & \end{pmatrix},$$

if $Q = (v_1 \tilde{v}_1 \dots v_\nu \tilde{v}_\nu)$ and n is even. For odd n we write $Q = (v_0 v_1 \tilde{v}_1 \dots v_\nu \tilde{v}_\nu)$ to get

$$Q^t \Lambda Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_1 & & & \\ 0 & \lambda_1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 0 & -\lambda_\nu \\ 0 & & & & \lambda_\nu & 0 \end{pmatrix}.$$

Remark 2.1.11. Let $\Lambda \in \mathfrak{so}(n)$. Given a orthonormal basis $\{v_1, \dots, v_n\}$ of eigenvectors of Λ^2 such that $\Lambda^2 v_j = -\lambda_j^2 v_j$, $j = 1, \dots, n$ we define $\Lambda^{-1} \in M(n)$ by

$$\Lambda^{-1} v_j = \begin{cases} -\frac{1}{\lambda_j^2} \Lambda v_j & , \text{ if } \lambda_j \neq 0 \\ 0 & , \text{ if } \lambda_j = 0 \end{cases},$$

for $j = 1, \dots, n$. Then $\Lambda^{-1} \in \mathfrak{so}(n)$ and $\Lambda \Lambda^{-1} x = x = \Lambda^{-1} \Lambda x$ for all $x \in \text{im } \Lambda$. By identifying $\mathfrak{so}(n)$ with \mathbb{R}^m , $m = \binom{n}{2}$ we can also define $(\text{ad } \Lambda)^{-1}$.

Proposition 2.1.12. *Let $n = 2\nu$ even, $\Lambda \in \mathfrak{so}(n)$ and $\{v_1, \tilde{v}_1, \dots, v_\nu, \tilde{v}_\nu\}$ as in Lemma 2.1.10. For $1 \leq i < j \leq \nu$ let*

$$w_{i,j}^\pm := v_i \wedge v_j \pm \tilde{v}_j \wedge \tilde{v}_i, \quad (2.24a)$$

$$\tilde{w}_{i,j}^\pm := \tilde{v}_i \wedge v_j \pm v_i \wedge \tilde{v}_j, \quad (2.24b)$$

and for $1 \leq j \leq \nu$ let

$$w_{j,j} := \sqrt{2} v_j \wedge \tilde{v}_j. \quad (2.25)$$

Then $(\text{ad } \Lambda)^2$ has eigenvalues $0, -(\lambda_i \pm \lambda_j)^2, 1 \leq i < j \leq \nu$ and

$$\left\{ w_{i,j}^{\pm}, \tilde{w}_{i,j}^{\pm} : 1 \leq i < j \leq \nu \right\} \cup \left\{ w_{j,j} : 1 \leq j \leq \nu \right\}$$

is an orthonormal basis of eigenvectors of $(\text{ad } \Lambda)^2$. Furthermore,

$$\text{ad } \Lambda w_{i,j}^{\pm} = (\lambda_i \pm \lambda_j) \tilde{w}_{i,j}^{\pm}, \quad (2.26a)$$

$$\text{ad } \Lambda \tilde{w}_{i,j}^{\pm} = -(\lambda_i \pm \lambda_j) w_{i,j}^{\pm}, \quad 1 \leq i < j \leq \nu \quad (2.26b)$$

$$\text{ad } \Lambda w_{j,j} = 0, \quad 1 \leq j \leq \nu. \quad (2.26c)$$

Remark 2.1.13. This proposition is also known as the Cartan decomposition or root space decomposition of $\mathfrak{so}(n)$.

Proof. Since $\{v_1, \tilde{v}_1, \dots, v_\nu, \tilde{v}_\nu\}$ is an orthonormal basis for \mathbb{R}^n we know that

$$\left\{ \sqrt{2}v_i \wedge v_j, \sqrt{2}\tilde{v}_i \wedge v_j, \sqrt{2}v_i \wedge \tilde{v}_j, \sqrt{2}\tilde{v}_i \wedge \tilde{v}_j : 1 \leq i < j \leq \nu \right\} \cup \left\{ \sqrt{2}v_j \wedge \tilde{v}_j : 1 \leq j \leq \nu \right\}$$

is an orthonormal basis for $\mathfrak{so}(n)$. Now it is easy to see that

$$\left\{ w_{i,j}^{\pm}, \tilde{w}_{i,j}^{\pm} : 1 \leq i < j \leq \nu \right\} \cup \left\{ w_{j,j} : 1 \leq j \leq \nu \right\}$$

is as well an orthonormal basis for $\mathfrak{so}(n)$. To complete the proof we have to verify (2.26).

Let $1 \leq j \leq \nu$, then

$$\begin{aligned} \text{ad } \Lambda w_{j,j} &= \sqrt{2}(\Lambda v_j \wedge \tilde{v}_j + v_j \wedge \Lambda \tilde{v}_j) \\ &= \sqrt{2}(\lambda_j \tilde{v}_j \wedge \tilde{v}_j - \lambda_j v_j \wedge v_j) \\ &= 0. \end{aligned}$$

Now let $1 \leq i < j \leq \nu$, then

$$\begin{aligned} \text{ad } \Lambda w_{i,j}^{\pm} &= \Lambda v_i \wedge v_j + v_i \wedge \Lambda v_j \pm (\Lambda \tilde{v}_j \wedge \tilde{v}_i + \tilde{v}_j \wedge \Lambda \tilde{v}_i) \\ &= \lambda_i (\tilde{v}_i \wedge v_j \mp \tilde{v}_j \wedge v_i) + \lambda_j (v_i \wedge \tilde{v}_j \mp v_j \wedge \tilde{v}_i) \\ &= (\lambda_i \pm \lambda_j) (\tilde{v}_i \wedge v_j \pm v_i \wedge \tilde{v}_j) \\ &= (\lambda_i \pm \lambda_j) \tilde{w}_{i,j}^{\pm} \end{aligned}$$

and

$$\begin{aligned}
 \text{ad } \Lambda \tilde{w}_{i,j}^{\pm} &= \Lambda \tilde{v}_i \wedge v_j + \tilde{v}_i \wedge \Lambda v_j \pm (\Lambda v_i \wedge \tilde{v}_j + v_i \wedge \Lambda \tilde{v}_j) \\
 &= \lambda_i (-v_i \wedge v_j \pm \tilde{v}_i \wedge \tilde{v}_j) + \lambda_j (\tilde{v}_i \wedge \tilde{v}_j \mp v_i \wedge v_j) \\
 &= -(\lambda_i \pm \lambda_j) (v_i \wedge v_j \pm \tilde{v}_j \wedge \tilde{v}_i) \\
 &= -(\lambda_i \pm \lambda_j) w_{i,j}^{\pm}.
 \end{aligned}$$

□

The case for odd n is very similar:

Proposition 2.1.14. *Let $n = 2\nu + 1$ odd, $\Lambda \in \mathfrak{so}(n)$ and $\{v_0, v_1, \tilde{v}_1, \dots, v_\nu, \tilde{v}_\nu\}$ as in Lemma 2.1.10 on page 53. For $1 \leq i < j \leq \nu$ let*

$$w_{i,j}^{\pm} := v_i \wedge v_j \pm \tilde{v}_j \wedge \tilde{v}_i, \quad (2.27a)$$

$$\tilde{w}_{i,j}^{\pm} := \tilde{v}_i \wedge v_j \pm v_i \wedge \tilde{v}_j, \quad (2.27b)$$

and for $1 \leq j \leq \nu$ let

$$w_{j,j} := \sqrt{2}v_j \wedge \tilde{v}_j. \quad (2.28)$$

Further for $1 \leq j \leq \nu$ let

$$w_{0,j} := \sqrt{2}v_0 \wedge v_j \quad (2.29a)$$

$$\tilde{w}_{0,j} := \sqrt{2}v_0 \wedge \tilde{v}_j. \quad (2.29b)$$

Then $(\text{ad } \Lambda)^2$ has eigenvalues $0, -(\lambda_i \pm \lambda_j)^2$, $0 \leq i < j \leq \nu$ and

$$\{w_{i,j}^{\pm}, \tilde{w}_{i,j}^{\pm} : 1 \leq i < j \leq \nu\} \cup \{w_{j,j} : 1 \leq j \leq \nu\} \cup \{w_{0,j}, \tilde{w}_{0,j} : 1 \leq j \leq \nu\}$$

is an orthonormal basis of eigenvalues of $(\text{ad } \Lambda)^2$. Furthermore,

$$\begin{aligned}
 \text{ad } \Lambda w_{i,j}^{\pm} &= (\lambda_i \pm \lambda_j) \tilde{w}_{i,j}^{\pm}, \\
 \text{ad } \Lambda \tilde{w}_{i,j}^{\pm} &= -(\lambda_i \pm \lambda_j) w_{i,j}^{\pm}, \quad 1 \leq i < j \leq \nu, \\
 \text{ad } \Lambda w_{0,j} &= \lambda_j \tilde{w}_{0,j}, \\
 \text{ad } \Lambda \tilde{w}_{0,j} &= -\lambda_j w_{0,j}, \\
 \text{ad } \Lambda w_{j,j} &= 0, \quad 1 \leq j \leq \nu.
 \end{aligned}$$

Proof. The proof is analog to 2.1.12, if one takes into account that the additional $w_{0,j}, \tilde{w}_{0,j}$ completes the orthonormal basis in this case. □

2.2 Sub-Riemannian geometry on free 2-step nilpotent Lie groups

Let \mathfrak{f} denote the free nilpotent Lie algebra of step 2 with Lie group F as presented in Example 2.1.6.

In this section we want to study the sub-Riemannian geometry associated to the horizontal distribution $\mathcal{D} = \text{span}\{X_1, \dots, X_n\} \subset TF$. In particular we want to give some more “explicit” formulas for the Carnot-Carathéodory distance d_{CC} introduced in Section 1. Thus we need to find length minimizing curves connecting two given point $p, q \in F$, $p \neq q$. In Section 1.8 Theorem 1.8.3 we already discussed the minimizing problem on free 2-step nilpotent Lie groups and we proved that the abnormal minimizers are actually normal. This means that the Hamiltonian differential equations gives us necessary conditions for a curve to be minimizing.

Proposition 2.2.1. *Let $\gamma = (x, z) : [0, T] \rightarrow F$ be an absolute continuous curve in F . Then γ is horizontal if and only if*

$$J^{\dot{z}(t)} = x(t) \wedge \dot{x}(t) \tag{2.30}$$

for almost all $t \in [0, T]$. If γ is horizontal, its length is

$$L(\gamma) = \int_0^T |\dot{x}(t)| dt. \tag{2.31}$$

Proof. For almost all t we have

$$\begin{aligned} \dot{\gamma} &= \sum_{j=1}^n \dot{x}_j \partial_{x_j} + \sum_{1 \leq k < l \leq n} \dot{z}_{k,l} \partial_{z_{k,l}} \\ &= \dot{x}X - \frac{1}{2} \sum_{j=1}^n \dot{x}_j \left(\sum_{k=1}^{j-1} x_k \partial_{z_{k,j}} - \sum_{k=j+1}^n x_k \partial_{z_{j,k}} \right) + \sum_{1 \leq k < l \leq n} \dot{z}_{k,l} \partial_{z_{k,l}} \\ &= \dot{x}X + \sum_{1 \leq k < l \leq n} \left(\dot{z}_{k,l} - \frac{1}{2} (\dot{x}_l x_k - \dot{x}_k x_l) \right) \partial_{z_{k,l}}. \end{aligned} \tag{2.32}$$

So γ is horizontal, iff $\dot{z}_{k,l} = \frac{1}{2} (x_k \dot{x}_l - x_l \dot{x}_k)$, $1 \leq k < l \leq n$ for almost all t . This means that

$$J^{\dot{z}} = x \wedge \dot{x}.$$

If γ is horizontal, then its length is given by

$$L(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_{\gamma(t)} dt = \int_0^T |\dot{x}(t)| dt.$$

□

If we identify $p \in T_q^*F$, $q \in F$ with $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}}$, the Hamiltonian reads

$$\begin{aligned} \mathbf{H}_2(x, z, \xi, \eta) &= \frac{1}{2} \sum_{j=1}^n \left(\xi_j - \frac{1}{2} (J^\eta x)_j \right)^2 \\ &= \frac{1}{2} \left| \xi - \frac{1}{2} J^\eta x \right|^2 \end{aligned} \tag{2.33}$$

$$= \frac{1}{2} |\zeta|^2, \tag{2.34}$$

with $\zeta := \xi - \frac{1}{2} J^\eta x$.

This Hamiltonian function \mathbf{H}_2 generates a system of Hamiltonian differential equations in particular we have

$$\dot{x} = \frac{\partial \mathbf{H}_2}{\partial \xi}, \quad \dot{z} = \frac{\partial \mathbf{H}_2}{\partial \eta}, \quad \dot{\xi} = -\frac{\partial \mathbf{H}_2}{\partial x}, \quad \dot{\eta} = -\frac{\partial \mathbf{H}_2}{\partial z}. \tag{2.35}$$

For every solution $\Gamma = (x, z, \xi, \eta) : [0, T] \rightarrow T^*F$ to the Hamiltonian equations, i.e. bicharacteristic, such that \mathbf{H}_2 does not vanish along Γ , the projection $\gamma = (x, z) : [0, T] \rightarrow F$ is a normal extremal. Furthermore, by Theorem 1.6.4 every length minimizer parametrized by arc-length is a normal extremal.

Obviously the last equation is $\dot{\eta}(t) = 0$ and thus η is constant. Let $\Lambda = J^{\eta(0)} \in \mathfrak{so}(n)$. In the sequel we will often use the following notation: for $x \in \mathbb{R}^n$ we will write

$$x = x_K + x_I \text{ where } x_K \in \ker \Lambda \text{ and } x_I \in \operatorname{im} \Lambda$$

are uniquely determined.

Theorem 2.2.2. *The solution to the Hamiltonian equations with initial condition*

$$(x, z)(0) = (0, 0), \xi(0) = \alpha, J^{\eta(0)} = \Lambda \in \mathfrak{so}(n)$$

is given by

$$x(t) = \frac{1 - e^{-t\Lambda}}{\Lambda} \alpha, \quad (2.36)$$

$$\xi(t) = \frac{1}{2} (e^{-t\Lambda} + I) \alpha, \quad (2.37)$$

$$J^{\eta(t)} = \Lambda, \quad (2.38)$$

and

$$\begin{aligned} J^{z(t)} &= \Lambda^{-1} \xi_I(t) \wedge x_I(t) - \frac{1 - e^{-t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda} (\Lambda^{-1} \alpha_I \wedge \alpha_I) \\ &\quad + 2t \Lambda^{-1} \xi_I(t) \wedge \alpha_K - 2\Lambda^{-1} x_I(t) \wedge \alpha_K. \end{aligned} \quad (2.39)$$

Proof. (As mentioned before, we use the convention that the binding of \wedge is weaker than other operations on vectors, i.e. $Ax \wedge y = (Ax) \wedge y$, $A \in M(n)$, $x, y \in \mathbb{R}^n$.) We start with the easiest equation, i.e.

$$\dot{\eta}(t) = -\frac{\partial H_2}{\partial z} = 0, \quad J^{\eta(0)} = \Lambda.$$

This just means that η is constant, i.e. $J^{\eta(t)} = \Lambda$. Next we write the other equations more explicitly:

$$\dot{x} = \frac{\partial H_2}{\partial \xi} = \zeta, \quad \dot{\xi} = -\frac{\partial H_2}{\partial x} = \frac{1}{2} (\langle \Lambda e_i, \zeta \rangle)_{i=1}^n = -\frac{1}{2} \Lambda \zeta, \text{ and}$$

$$\dot{z}_{k,l} = \frac{\partial H_2}{\partial \eta_{k,l}} = -\frac{1}{2} \langle J^{k,l} x, \zeta \rangle = -\frac{1}{2} (x_l \zeta_k - x_k \zeta_l), \quad 1 \leq k < l \leq n, \quad \text{i.e.}$$

$$J^{\dot{z}} = x \wedge \zeta.$$

Solving these equations, we will first notice that

$$\dot{\zeta} = \dot{\xi} - \frac{1}{2} \Lambda \dot{x} = -\Lambda \zeta, \quad \zeta(0) = \xi(0) - \frac{1}{2} \Lambda x(0) = \alpha.$$

This gives

$$\zeta(t) = e^{-t\Lambda} \alpha, \quad (2.40)$$

and the remaining equations become

$$x(t) = \int_0^t \zeta(s) ds = \frac{I - e^{-t\Lambda}}{\Lambda} \alpha, \quad (2.41)$$

$$\xi(t) = -\frac{1}{2} \int_0^t \Lambda \zeta(s) ds + \xi(0) = \frac{1}{2} (I + e^{-t\Lambda}) \alpha, \quad (2.42)$$

where $\frac{I - e^{-t\Lambda}}{\Lambda}$ is defined by its power series. This proves (2.36) and (2.37). To integrate the last equation we write $\alpha = \alpha_K + \alpha_I$ as described above. Then $x_K = t\alpha_K$, $\xi_K = \alpha_K$ and

$$\begin{aligned} J^z &= x_I \wedge \zeta_I + x_I \wedge \zeta_K + x_K \wedge \zeta_I + x_K \wedge \zeta_K \\ &= \Lambda^{-1} \alpha_I \wedge \zeta_I - e^{-t \operatorname{ad} \Lambda} (\Lambda^{-1} \alpha_I \wedge \alpha_I) + x_I \wedge \alpha_K + t\alpha_K \wedge \zeta_I. \end{aligned} \quad (2.43)$$

Integrating the third summand of (2.43) gives

$$\begin{aligned} \int_0^t x_I(s) \wedge \alpha_K ds &= \int_0^t (I - e^{-s\Lambda}) \Lambda^{-1} \alpha_I \wedge \alpha_K ds \\ &= t\Lambda^{-1} \alpha_I \wedge \alpha_K - \Lambda^{-1} x_I(t) \wedge \alpha_K, \end{aligned} \quad (2.44)$$

and by integrating the last summand of (2.43) by parts

$$\begin{aligned} \int_0^t s\alpha_K \wedge \zeta_I(s) ds &= t\alpha_K \wedge x_I(t) - \int_0^t \alpha_K \wedge x_I(s) ds \\ &= t\alpha_K \wedge x_I(t) + t\Lambda^{-1} \alpha_I \wedge \alpha_K - \Lambda^{-1} x_I(t) \wedge \alpha_K. \end{aligned} \quad (2.45)$$

Combining (2.44) and (2.45) with (2.43) gives

$$\begin{aligned} J^{z(t)} &= \Lambda^{-1} \alpha_I \wedge x_I(t) - \frac{1 - e^{-t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda} (\Lambda^{-1} \alpha_I \wedge \alpha_I) \\ &\quad + t (2\Lambda^{-1} \alpha_I - x_I(t)) \wedge \alpha_K - 2\Lambda^{-1} x_I(t) \wedge \alpha_K \\ &= \Lambda^{-1} \alpha_I \wedge x_I(t) - \frac{1 - e^{-t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda} (\Lambda^{-1} \alpha_I \wedge \alpha_I) \\ &\quad + 2t\Lambda^{-1} \xi_I(t) \wedge \alpha_K - 2\Lambda^{-1} x_I(t) \wedge \alpha_K, \end{aligned}$$

where again $\frac{1 - e^{-t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda}$ is defined by its power series. \square

There is another useful equation which can be derived from the Hamiltonian equations:

Lemma 2.2.3. *Let $(x, z, \xi, \eta) : [0, T] \rightarrow F$ be the solution to the Hamiltonian equations with initial condition $(x, z)(0) = 0$, $\xi(0) = \alpha$ and $J^{\eta(0)} = \Lambda$. Then*

$$[\Lambda, J^{z(t)}] = 2\xi(t) \wedge x(t). \quad (2.46)$$

Proof. From the Hamiltonian equations we have $J^{\dot{z}} = x \wedge \zeta$. Thus

$$\begin{aligned} [\Lambda, J^{\dot{z}}] &= \Lambda x \wedge \zeta + x \wedge \Lambda \zeta \\ &= \alpha \wedge \zeta - \Lambda \zeta \wedge x \\ &= 2\xi \wedge \zeta - \Lambda \zeta \wedge x \\ &= 2\xi \wedge \dot{x} + 2\dot{\xi} \wedge x. \end{aligned}$$

Integrating this equation gives the desired result

$$[\Lambda, J^z] = 2\xi \wedge x.$$

□

Proposition 2.2.4. *All normal extremals parametrized with constant speed on the free 2-step nilpotent Lie group F starting in $0 \in F$ are given by $\gamma^{\alpha, \Lambda} = (x, z) : [0, 1] \rightarrow F$, where*

$$x(t) = \frac{1 - e^{-t\Lambda}}{\Lambda} \alpha \quad (2.47)$$

$$\begin{aligned} J^{z(t)} &= \Lambda^{-1} \xi_I(t) \wedge x_I(t) - \frac{1 - e^{-t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda} \left(\Lambda^{-1} \alpha_I \wedge \alpha_I \right) \\ &\quad + 2t \Lambda^{-1} \xi_I(t) \wedge \alpha_K - 2\Lambda^{-1} x_I(t) \wedge \alpha_K \end{aligned} \quad (2.48)$$

with parameters $\alpha \in \mathbb{R}^n \setminus \{0\}$, $\Lambda \in \mathfrak{so}(n)$. Their lengths are given by

$$L(\gamma^{\alpha, \Lambda}) = |\alpha|. \quad (2.49)$$

Proof. Since the normal extremals are the projections of the bicharacteristics given by Theorem 2.2.2 the first statement of the proposition is immediately clear.

Since $|\dot{x}| = |\zeta| = |\alpha|$ the length of $\gamma^{\alpha, \Lambda}$ is $L(\gamma^{\alpha, \Lambda}) = |\alpha|$. □

Now we come to the most difficult part. In view of the Carnot-Carathéodory distance we need to find those normal extremals connecting $0 \in F$ and any given point $(x, z) \in F$ and then compare their lengths to find the shortest. Thus we have to find those parameters $\alpha \in \mathbb{R}^n$, $\Lambda \in \mathfrak{so}(n)$ such that

$$\gamma^{\alpha, \Lambda}(0) = 0 \text{ and } \gamma^{\alpha, \Lambda}(1) = (x, z).$$

This turns out to be a very difficult problem, and we are not able to give a complete answer to this problem. Only for some nilpotent Lie groups, i.e. the Heisenberg type groups (and in particular the Heisenberg group - see [3]), we can solve these boundary conditions. These results are presented later when we discuss general nilpotent Lie groups in section 2.3.

From now on $(x, z) \in F$ will always denote a fixed point in F and not, as before, denote a curve. Since we are now only interested in the endpoint this is not a mismatch in notation.

The next lemma is an explicit formulation of the boundary condition $\gamma(0) = 0$, $\gamma(1) = (x, z) \in F$.

Lemma 2.2.5. *Let $(x, z) \in F$. Then $\gamma^{\alpha, \Lambda}$, $\alpha \in \mathbb{R}^n \setminus \{0\}$, $\Lambda \in \mathfrak{so}(n)$ is a normal extremal parametrized with constant speed connecting $0 \in F$ and $(x, z) \in F$ if and only if*

$$x = \frac{1 - e^{-\Lambda}}{\Lambda} \alpha, \tag{2.50}$$

$$J^z = \Lambda^{-1} \xi_I \wedge x_I - \frac{1 - e^{-\text{ad } \Lambda}}{\text{ad } \Lambda} \left(\Lambda^{-1} \alpha_I \wedge \alpha_I \right) + 2\Lambda^{-1} \xi_I \wedge \alpha_K - 2\Lambda^{-1} x_I \wedge \alpha_K. \tag{2.51}$$

For a further understanding of these equations we decompose \mathbb{R}^n into the eigenspaces of Λ^2 and $\mathfrak{so}(n)$ into the eigenspaces of $(\text{ad } \Lambda)^2$. In doing so, we need some more notation.

Let $\Lambda \in \mathfrak{so}(n)$ with unique nonzero eigenvalues $\pm i\lambda_1, \dots, \pm i\lambda_b$ with $b \in \mathbb{N}_0$, $\lambda_j > 0$, $j = 1, \dots, b$. We may sort the λ_j , $j = 1, \dots, b$ such that

$$\lambda_0 = 0 < \lambda_1 < \dots < \lambda_a \text{ and } \lambda_{a+1} < \dots < \lambda_b,$$

where

$$\lambda_1, \dots, \lambda_a \notin 2\pi\mathbb{Z}^*, \quad \lambda_{a+1}, \dots, \lambda_b \in 2\pi\mathbb{Z}^*.$$

For any vector $v \in \mathbb{R}^n$ we can uniquely write $v = \sum_{j=0}^b v_j$, where $v_j \in \text{ES}(\Lambda^2, -\lambda_j^2)$; and for a vector $v \in \text{ES}(\Lambda^2, -\lambda_j^2)$, $1 \leq j \leq b$ we denote $\tilde{v} = \frac{\Lambda}{\lambda_j} v$. As mentioned before $\text{ES}(A, \mu)$ denotes the eigenspace of a matrix A corresponding to the eigenvalue μ . Observe that if n is even $\text{ES}(\Lambda^2, 0) = \text{kern}(\Lambda)$ might be $\{0\}$ but if n is odd $\dim \text{kern } \Lambda \geq 1$.

Furthermore, let

$$V_\pi := \bigoplus_{j=a+1}^b \text{ES}(\Lambda^2, -\lambda_j^2), \quad V_K := \text{kern } \Lambda, \quad V_R := (V_\pi \oplus V_K)^\perp. \tag{2.52}$$

Then $\mathbb{R}^n = V_K \oplus V_\pi \oplus V_R$ and every vector $v \in \mathbb{R}^n$ can be uniquely written as

$$v = v_K + v_\pi + v_R \text{ with } v_K \in V_K, v_\pi \in V_\pi, v_R \in V_R.$$

Lemma 2.2.6. *Let $\Lambda \in \mathfrak{so}(n)$ and $v \in \mathbb{R}^n$. Then*

$$\begin{aligned} \tilde{v}_j \wedge v_j &\in \ker(\operatorname{ad} \Lambda) \text{ for all } j = 1, \dots, b, \\ v_j \wedge v_k, \tilde{v}_j \wedge \tilde{v}_k, \tilde{v}_j \wedge v_k, v_j \wedge \tilde{v}_k &\in \operatorname{im}(\operatorname{ad} \Lambda) \text{ for all } 1 \leq j < k \leq b, \\ v_0 \wedge v_k, v_0 \wedge \tilde{v}_k &\in \operatorname{im}(\operatorname{ad} \Lambda) \text{ for all } 1 \leq k \leq b. \end{aligned}$$

Proof. For $j = 1, \dots, b$ we have that

$$\begin{aligned} \operatorname{ad} \Lambda(\tilde{v}_j \wedge v_j) &= \Lambda \tilde{v}_j \wedge v_j + \tilde{v}_j \wedge \Lambda v_j \\ &= -\lambda_j v_j \wedge v_j + \lambda_j \tilde{v}_j \wedge \tilde{v}_j \\ &= 0. \end{aligned}$$

Hence $\tilde{v}_j \wedge v_j \in \ker \operatorname{ad} \Lambda$.

For $0 \leq j < k \leq b$ we can write

$$\begin{aligned} \tilde{v}_j \wedge v_k &= \frac{1}{2}(\tilde{v}_j \wedge v_k + v_j \wedge \tilde{v}_k) + \frac{1}{2}(\tilde{v}_j \wedge v_k - v_j \wedge \tilde{v}_k), \\ v_j \wedge \tilde{v}_k &= \frac{1}{2}(\tilde{v}_j \wedge v_k + v_j \wedge \tilde{v}_k) - \frac{1}{2}(\tilde{v}_j \wedge v_k - v_j \wedge \tilde{v}_k). \end{aligned}$$

In view of Proposition 2.1.12 and 2.1.14 we can conclude that

$$\begin{aligned} (\tilde{v}_j \wedge v_k + v_j \wedge \tilde{v}_k) &\in \operatorname{ES} \left((\operatorname{ad} \Lambda)^2, -(\lambda_j + \lambda_k)^2 \right) \text{ and} \\ (\tilde{v}_j \wedge v_k - v_j \wedge \tilde{v}_k) &\in \operatorname{ES} \left((\operatorname{ad} \Lambda)^2, -(\lambda_j - \lambda_k)^2 \right). \end{aligned}$$

Since $\lambda_j \neq \lambda_k$ and $\lambda_j, \lambda_k \neq 0$ we conclude that

$$\tilde{v}_j \wedge v_k, v_j \wedge \tilde{v}_k \in \operatorname{im}(\operatorname{ad} \Lambda).$$

Similarly we write

$$\begin{aligned} v_j \wedge v_k &= \frac{1}{2}(v_j \wedge v_k + \tilde{v}_j \wedge \tilde{v}_k) + \frac{1}{2}(v_j \wedge v_k - \tilde{v}_j \wedge \tilde{v}_k), \\ \tilde{v}_j \wedge \tilde{v}_k &= \frac{1}{2}(v_j \wedge v_k + \tilde{v}_j \wedge \tilde{v}_k) - \frac{1}{2}(v_j \wedge v_k - \tilde{v}_j \wedge \tilde{v}_k) \end{aligned}$$

to see that

$$v_j \wedge v_k, \tilde{v}_j \wedge \tilde{v}_k \in \operatorname{im}(\operatorname{ad} \Lambda).$$

Since

$$(\operatorname{ad} \Lambda)^2(v_0 \wedge v_k) = -\lambda_k^2 v_0 \wedge v_k, \quad (\operatorname{ad} \Lambda)^2(v_0 \wedge \tilde{v}_k) = -\lambda_k^2 v_0 \wedge \tilde{v}_k$$

we have that $v_0 \wedge v_k, v_0 \wedge \tilde{v}_k \in \text{im}(\text{ad } \Lambda)$. \square

Lemma 2.2.7. *Let $\Gamma = (\gamma, \vartheta) : [0, 1] \rightarrow T^*F$ be a normal biextremal with $\Gamma(0) = (0, 0, \alpha, \eta)$, $\alpha \in \mathbb{R}^n \setminus \{0\}$, $\Lambda = J^\eta \in \mathfrak{so}(n)$. Moreover, let $(x, z) = \gamma(1)$ and $(\xi, \eta) = \vartheta(1)$. Then*

$$\alpha_K = x_K, \quad \alpha_R = \frac{\frac{1}{2}\Lambda}{\sinh \frac{1}{2}\Lambda} e^{\frac{1}{2}\Lambda} x_R, \quad x_\pi = 0, \quad (2.53a)$$

$$\xi_K = x_K, \quad \xi_R = \frac{1}{2}\Lambda \coth \frac{1}{2}\Lambda x_R, \quad \xi_\pi = \alpha_\pi, \quad (2.53b)$$

and $J^z = Z_K + Z_I$, where $Z_K \in \text{kern}(\text{ad } \Lambda)$, $Z_I \in \text{im}(\text{ad } \Lambda)$ and

$$Z_K = \frac{1}{4} \sum_{j=1}^a \left(\frac{\lambda_j}{\sin^2 \frac{\lambda_j}{2}} - 2 \cot \frac{\lambda_j}{2} \right) \tilde{x}_j \wedge x_j + \sum_{j=a+1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j \quad (2.54)$$

$$\begin{aligned} Z_I &= \sum_{j=1}^a \left(\cot \frac{\lambda_j}{2} - \frac{2}{\lambda_j} \right) x_0 \wedge \tilde{x}_j + \sum_{j=a+1}^b \frac{2}{\lambda_j} x_0 \wedge \tilde{\alpha}_j \\ &+ \sum_{1 \leq j < k \leq a} \frac{\lambda_j \cot \frac{\lambda_j}{2} - \lambda_k \cot \frac{\lambda_k}{2}}{\lambda_j^2 - \lambda_k^2} (\lambda_k x_j \wedge \tilde{x}_k - \lambda_j \tilde{x}_j \wedge x_k) \\ &+ \sum_{1 \leq j \leq a < k \leq b} \frac{2}{\lambda_j^2 - \lambda_k^2} (\lambda_j \tilde{x}_j \wedge \alpha_k - \lambda_k x_j \wedge \tilde{\alpha}_k). \end{aligned} \quad (2.55)$$

Remark 2.2.8. Note that the decomposition $x = x_K + x_R + x_\pi$ is depending on Λ . Thus the vectors x_K, x_R, x_π are actually depending Λ , which is suppressed in this notation.

Proof. As a normal biextremal Γ is given as a solution to the Hamiltonian differential equation. Thus we can apply Theorem 2.2.2 on page 58 to calculate the endpoint $\Gamma(1)$. The first equations (2.53) follow easily by diagonalizing equation (2.36) and (2.37) in Theorem 2.2.2 on page 58. In particular

$$x = \frac{I - e^{-\Lambda}}{\Lambda} \alpha = \alpha_K + \frac{e^{\frac{1}{2}\Lambda} - e^{-\frac{1}{2}\Lambda}}{\Lambda} e^{-\frac{1}{2}\Lambda} \alpha_R + 0, \quad (2.56)$$

thus $x_K = \alpha_K$, $\alpha_R = \frac{\Lambda}{2 \sinh \frac{1}{2}\Lambda} e^{\frac{1}{2}\Lambda} x_R$, $x_\pi = 0$. Furthermore,

$$\xi = \frac{1}{2} (I + e^{-\Lambda}) \alpha = \alpha_K + \Lambda \frac{e^{\frac{1}{2}\Lambda} + e^{-\frac{1}{2}\Lambda}}{4 \sinh \frac{1}{2}\Lambda} x_R + \alpha_\pi, \quad (2.57)$$

thus $\xi_K = \alpha_K$, $\xi_\pi = \alpha_\pi$ and $\xi_R = \frac{1}{2}\Lambda \coth \frac{1}{2}\Lambda x_R$.

Turning our attention to the last equation we have by Lemma 2.2.3 that

$$J^z = 2(\operatorname{ad} \Lambda)^{-1}(\xi \wedge x) + P_K(J^z), \quad (2.58)$$

where $P_K : \mathfrak{so}(n) \rightarrow \ker(\operatorname{ad} \Lambda)$ denotes the orthogonal projection to the kernel of $\operatorname{ad} \Lambda$ and $(\operatorname{ad} \Lambda)^{-1}$ is defined as in Remark 2.1.11. Using equation (2.51) we have

$$P_K(J^z) = P_K(\Lambda^{-1}\xi_I \wedge x_I) - P_K(\Lambda^{-1}\alpha_I \wedge \alpha_I) + 2P_K(\Lambda^{-1}\xi_I \wedge \alpha_K - \Lambda^{-1}x_I \wedge \alpha_K).$$

The calculation of $P_K(J^z)$ takes several steps:

- Since $\alpha_K = x_K$ and $\xi_j = \lambda_j \cot \lambda_j x_j$, $j = 1, \dots, b$, we have

$$\begin{aligned} \Lambda^{-1}\xi_I \wedge \alpha_K &= -\sum_{j=1}^b \frac{1}{\lambda_j} \tilde{\xi}_j \wedge x_0 \\ &= -\sum_{j=1}^a \cot \lambda_j \tilde{x}_j \wedge x_0 - \sum_{j=a+1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_0, \\ \Lambda^{-1}x_I \wedge \alpha_K &= -\sum_{j=1}^b \frac{1}{\lambda_j} \tilde{x}_j \wedge x_0. \end{aligned}$$

By applying Lemma 2.2.6 we can conclude that $\Lambda^{-1}\xi_I \wedge \alpha_K, \Lambda^{-1}x_I \wedge \alpha_K \in \operatorname{im}(\operatorname{ad} \Lambda)$ and thus

$$P_K(\Lambda^{-1}\xi_I \wedge \alpha_K - \Lambda^{-1}x_I \wedge \alpha_K) = 0. \quad (2.59)$$

- Now write $\Lambda^{-1}\alpha_I \wedge \alpha_I$ as

$$\begin{aligned} \Lambda^{-1}\alpha_I \wedge \alpha_I &= -\sum_{j,k=1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_k \\ &= -\sum_{j=1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j + \sum_{1 \leq j < k \leq b} \left(\frac{1}{\lambda_k} \alpha_j \wedge \tilde{\alpha}_k - \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_k \right). \end{aligned}$$

Hence by Lemma 2.2.6

$$P_K(\Lambda^{-1}\alpha_I \wedge \alpha_I) = -\sum_{j=1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j. \quad (2.60)$$

- Furthermore, since $x_\pi = 0$ we have $\Lambda^{-1}\xi_I \wedge x_I = \Lambda^{-1}\xi_R \wedge x_R + \Lambda^{-1}\xi_\pi \wedge x_R$. To calculate $P_K(\Lambda^{-1}\xi_\pi \wedge x_R)$ we write

$$\Lambda^{-1}\xi_\pi \wedge x_R = - \sum_{j=a+1}^b \sum_{k=1}^a \frac{1}{\lambda_j} \xi_j \wedge x_k.$$

Since by (2.56)

$$x_k = \frac{2 \sinh \frac{1}{2}\Lambda}{\Lambda} e^{-\frac{1}{2}\Lambda} \alpha_k = \frac{2 \sin \frac{\lambda_k}{2}}{\lambda_k} \left(\cos \frac{\lambda_k}{2} \alpha_k - \sin \frac{\lambda_k}{2} \tilde{\alpha}_k \right), k = 1, \dots, a$$

and $\xi_j = \alpha_j$, $j = a+1, \dots, b$ we can conclude with Lemma 2.2.6 that $P_K(\Lambda^{-1}\xi_\pi \wedge x_R) = 0$. Now write $\Lambda^{-1}\xi_R \wedge x_R$ as

$$\begin{aligned} \Lambda^{-1}\xi_R \wedge x_R &= - \sum_{j,k=1}^a \frac{1}{\lambda_j} \tilde{\xi}_j \wedge x_k \\ &= - \sum_{j=1}^a \frac{1}{\lambda_j} \tilde{\xi}_j \wedge x_j - \sum_{1 \leq j < k \leq a} \left(\frac{1}{\lambda_j} \tilde{\xi}_j \wedge x_k + \frac{1}{\lambda_k} \tilde{\xi}_k \wedge x_j \right). \end{aligned}$$

Since $\tilde{\xi}_j = \frac{\lambda_j}{2} \cot \frac{\lambda_j}{2} \tilde{x}_j$, $j = 1, \dots, a$ we can apply Lemma 2.2.6 to see that

$$P_K(\Lambda^{-1}\xi_I \wedge x_I) = - \sum_{j=1}^a \frac{1}{\lambda_j} \tilde{\xi}_j \wedge x_j, \quad (2.61)$$

- Summarizing (2.59), (2.60), and (2.61) yields

$$\begin{aligned} P_K(J^z) &= \sum_{j=1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j - \sum_{j=1}^a \tilde{\xi}_j \wedge x_j \\ &= \frac{1}{4} \sum_{j=1}^a \left(\frac{\lambda_j}{\sin^2 \frac{\lambda_j}{2}} e^{\frac{1}{2} \text{ad} \Lambda} (\tilde{x}_j \wedge x_j) - 2 \cot \frac{\lambda_j}{2} \tilde{x}_j \wedge x_j \right) + \sum_{j=a+1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j \\ &= \frac{1}{4} \sum_{j=1}^a \left(\frac{\lambda_j}{\sin^2 \frac{\lambda_j}{2}} - 2 \cot \frac{\lambda_j}{2} \right) \tilde{x}_j \wedge x_j + \sum_{j=a+1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j. \end{aligned} \quad (2.62)$$

To calculate $(\text{ad} \Lambda)^{-1}(\xi \wedge x)$, we can write

$$\xi \wedge x = \sum_{j,k=0}^b \xi_j \wedge x_k = \sum_{j,k=1}^b \xi_j \wedge x_k, \quad (2.63)$$

because

$$\xi_0 \wedge x_0 = x_0 \wedge x_0 = 0.$$

Furthermore, we have

$$\xi_j \wedge x_j = \frac{1}{2} \lambda_j \cot \frac{\lambda_j}{2} x_j \wedge x_j = 0, \quad j = 1, \dots, a,$$

and by (2.56), (2.57)

$$\xi_j \wedge x_j = \vartheta_\xi \alpha_j \wedge (\vartheta_x \alpha_j + \vartheta_{\tilde{x}} \tilde{\alpha}_j) = \vartheta_\xi \vartheta_{\tilde{x}} \alpha_j \wedge \tilde{\alpha}_j \in \ker(\operatorname{ad} \Lambda), \quad j = a+1, \dots, b,$$

with some $\vartheta_\xi, \vartheta_x, \vartheta_{\tilde{x}} \in \mathbb{R}$. Hence, $\xi_j \wedge x_j \in \ker(\operatorname{ad} \Lambda)$ and $(\operatorname{ad} \Lambda)^{-1}(\xi_j \wedge x_j) = 0$. Thus by (2.63)

$$(\operatorname{ad} \Lambda)^{-1}(\xi \wedge x) = (\operatorname{ad} \Lambda)^{-1} \left(\sum_{j,k=1, k \neq j}^b \xi_j \wedge x_k \right).$$

Now we decompose the sum (2.63) into eigenvectors of $(\operatorname{ad} \Lambda)^2$, i.e for $j, k = 1, \dots, b, k \neq j$ write

$$\xi_j \wedge x_k = \frac{1}{2} (w_{j,k}^+ + w_{j,k}^-), \quad (2.64)$$

where

$$w_{j,k}^\pm = \xi_j \wedge x_k \pm \tilde{x}_k \wedge \tilde{\xi}_j. \quad (2.65)$$

By Lemma 2.1.12 (2.1.14) we can conclude that $w_{j,k}^\pm \in \operatorname{ES} \left((\operatorname{ad} \Lambda)^2, -(\lambda_j \pm \lambda_k)^2 \right)$, $j, k = 1, \dots, b, k \neq j$. Thus for $j \neq 0, k \neq 0$ we have

$$\begin{aligned} (\operatorname{ad} \Lambda)^{-1}(\xi_j \wedge x_k) &= \frac{-1}{2(\lambda_j + \lambda_k)^2} \operatorname{ad} \Lambda(w_{j,k}^+) + \frac{-1}{2(\lambda_j - \lambda_k)^2} \operatorname{ad} \Lambda(w_{j,k}^-) \\ &= -\frac{1}{2} \left(\frac{1}{\lambda_j + \lambda_k} (\tilde{\xi}_j \wedge x_k + \xi_j \wedge \tilde{x}_k) + \frac{1}{\lambda_j - \lambda_k} (\tilde{\xi}_j \wedge x_k - \xi_j \wedge \tilde{x}_k) \right) \\ &= -\frac{\lambda_j}{\lambda_j^2 - \lambda_k^2} \tilde{\xi}_j \wedge x_k + \frac{\lambda_k}{\lambda_j^2 - \lambda_k^2} \xi_j \wedge \tilde{x}_k. \end{aligned} \quad (2.66)$$

Further observe that for $k = 1, \dots, b$

$$\xi_0 \wedge x_k, \xi_k \wedge x_0 \in \operatorname{ES} \left((\operatorname{ad} \Lambda)^2, -\lambda_k^2 \right),$$

thus

$$\begin{aligned} (\operatorname{ad} \Lambda)^{-1} (\xi_0 \wedge x_k) &= -\frac{1}{\lambda_k^2} \operatorname{ad} \Lambda (\xi_0 \wedge x_k) \\ &= -\frac{1}{\lambda_k} \xi_0 \wedge \tilde{x}_k, \end{aligned} \quad (2.67)$$

and

$$(\operatorname{ad} \Lambda)^{-1} (\xi_k \wedge x_0) = -\frac{1}{\lambda_k} \tilde{\xi}_k \wedge x_0. \quad (2.68)$$

Hence

$$\begin{aligned} (\operatorname{ad} \Lambda)^{-1} (\xi \wedge x) &= \sum_{j=1}^b (\operatorname{ad} \Lambda)^{-1} (\xi_0 \wedge x_j + \xi_j \wedge x_0) + \sum_{j,k=1, j \neq k}^b (\operatorname{ad} \Lambda)^{-1} (\xi_j \wedge x_k) \\ &= -\sum_{j=1}^b \frac{1}{\lambda_j} (\xi_0 \wedge \tilde{x}_j + \tilde{\xi}_j \wedge x_0) + \sum_{j,k=1, j \neq k}^b \frac{1}{\lambda_j^2 - \lambda_k^2} (\lambda_k \xi_j \wedge \tilde{x}_k - \lambda_j \tilde{\xi}_j \wedge x_k) \\ &= -\sum_{j=1}^b \frac{1}{\lambda_j} (\xi_0 \wedge \tilde{x}_j + \tilde{\xi}_j \wedge x_0) \\ &\quad + \sum_{1 \leq j < k \leq b} \frac{1}{\lambda_j^2 - \lambda_k^2} (\lambda_k \xi_j \wedge \tilde{x}_k - \lambda_j \xi_k \wedge \tilde{x}_j - \lambda_j \tilde{\xi}_j \wedge x_k + \lambda_k \tilde{\xi}_k \wedge x_j) \\ &= \sum_{j=1}^a \left(\frac{1}{2} \cot \frac{\lambda_j}{2} - \frac{1}{\lambda_j} \right) x_0 \wedge \tilde{x}_j + \sum_{j=a+1}^b \frac{1}{\lambda_j} x_0 \wedge \tilde{\alpha}_j \\ &\quad + \frac{1}{2} \sum_{1 \leq j < k \leq a} \frac{\lambda_j \cot \frac{\lambda_j}{2} - \lambda_k \cot \frac{\lambda_k}{2}}{\lambda_j^2 - \lambda_k^2} (\lambda_k x_j \wedge \tilde{x}_k - \lambda_j \tilde{x}_j \wedge x_k) \\ &\quad + \sum_{1 \leq j \leq a < k \leq b} \frac{1}{\lambda_j^2 - \lambda_k^2} (\lambda_j \tilde{x}_j \wedge \alpha_k - \lambda_k x_j \wedge \tilde{\alpha}_k). \end{aligned}$$

This together with (2.62) and (2.58) completes the proof. \square

An immediate consequence is the following formula for the length:

Lemma 2.2.9. *Let $\gamma^{\alpha, \Lambda} : [0, 1] \rightarrow F$, $\alpha \in \mathbb{R}^n \setminus \{0\}$, $\Lambda \in \mathfrak{so}(n)$ be, as above, a normal extremal parametrized with constant speed such that $\gamma^{\alpha, \Lambda}(1) = (x, z) \in F$. Then its length is*

$$\begin{aligned} L(\gamma^{\alpha, \Lambda}) &= \sqrt{\left| \frac{\frac{1}{2}\Lambda}{\sinh \frac{1}{2}\Lambda} (x_R + x_K) \right|^2 + |\alpha_\pi|^2} \\ &= \sqrt{\frac{1}{4} \sum_{j=0}^a \frac{\lambda_j^2}{\sin^2 \frac{\lambda_j}{2}} |x_j|^2 + \sum_{j=a+1}^b |\alpha_j|^2}. \end{aligned}$$

Proof. Using Lemma 2.2.7 gives

$$\begin{aligned} L(\gamma^{\alpha, \Lambda})^2 &= |\alpha_K|^2 + |\alpha_R|^2 + |\alpha_\pi|^2 \\ &= |x_K|^2 + \left| \frac{\frac{1}{2}\Lambda}{\sinh \frac{1}{2}\Lambda} x_R \right|^2 + |\alpha_\pi|^2 \\ &= \left| \frac{\frac{1}{2}\Lambda}{\sinh \frac{1}{2}\Lambda} (x_R + x_K) \right|^2 + |\alpha_\pi|^2 \\ &= \frac{1}{4} \sum_{j=0}^a \frac{\lambda_j^2}{\sin^2 \frac{\lambda_j}{2}} |x_j|^2 + \sum_{j=a+1}^b |\alpha_j|^2. \end{aligned}$$

□

Lemma 2.2.7 gives us the structure of the skew-symmetric matrix J^z in terms of a basis of eigenvectors of $(\text{ad } \Lambda)^2$. This gives immediately the following formulas on the diagonal of J^z relative to an orthogonal basis of eigenvectors of Λ^2 :

Lemma 2.2.10. *Let $(x, z) \in F$, and $(\alpha, \Lambda) \in \mathbb{R}^n \times \mathfrak{so}(n)$ such that $\gamma^{\alpha, \Lambda}(1) = (x, z)$. Then*

$$\langle J^z x_j, \tilde{x}_j \rangle = \frac{1}{8} \left(\frac{\lambda_j}{\sin^2 \frac{\lambda_j}{2}} - 2 \cot \frac{\lambda_j}{2} \right) |x_j|^4, \quad j = 0, \dots, a \quad (2.69)$$

$$2\lambda_j \langle J^z \alpha_j, \tilde{\alpha}_j \rangle = |\alpha_j|^4, \quad j = a+1, \dots, b$$

Remark 2.2.11. Let

$$\mu(x) := \frac{x}{\sin^2 x} - \cot x, \quad x \in \mathbb{R} \setminus \pi\mathbb{Z}^*.$$

Then equation (2.69) is equivalent to

$$\langle J^z x_j, \tilde{x}_j \rangle = \frac{1}{4} \mu\left(\frac{\lambda_j}{2}\right) |x_j|^4. \quad (2.70)$$

The function μ also appears when studying the sub-Riemannian geometry on Heisenberg type groups - see Section 2.3.1 and Figure 2.1 on page 80. Hence we understand equation (2.70) and therefore the diagonal of J^z with respect to the orthogonal basis of the eigenvectors of Λ^2 very well. In particular if $|x_j| \neq 0$ we have finitely many solutions λ_j^k , $k = 1, \dots, n_k$ of equation (2.70). Moreover there is always exactly one solution in the interval $(-2\pi, 2\pi)$ and at most two solutions in an interval $(2m\pi, 2(m+1)\pi)$, $m \in \mathbb{N}$ if $\langle J^z x_j, \tilde{x}_j \rangle > 0$ respectively $(-2(m+1)\pi, -2m\pi)$, $m \in \mathbb{N}$ if $\langle J^z x_j, \tilde{x}_j \rangle < 0$.

But when looking for normal extremals $\gamma^{\alpha, \Lambda}$ which connect $0 \in F$ and a given point $(x, z) \in F$ we also need to solve the “off-diagonal equations” given in Lemma 2.2.7. From this it should follow which eigenvectors of Λ^2 one has to choose. But this seems to be highly complicated. It is not even clear that given one solution α, Λ one gets other solutions by altering only the eigenvalues with respect to equation (2.55). This may show that the “diagonal” and “off-diagonal equations” are highly intertwined.

Nevertheless we give some results for some special cases.

Lemma 2.2.12. *Let $(x, 0) \in F$, $x \neq 0$. Then $\gamma^{\alpha, \Lambda}$ connects $0 \in F$ with $(x, 0) \in F$ iff $x \in \ker \Lambda$ and $\alpha = x$. The length of $\gamma^{\alpha, \Lambda}$ and therefore the Carnot-Carathéodory distance in this case is*

$$|(x, 0)|_{CC} = L(\gamma^{\alpha, \Lambda}) = |x|. \quad (2.71)$$

Proof. If we choose any matrix Λ such that $x \in \ker \Lambda$ and $\alpha := x$, then by Lemma 2.2.7 we have $\gamma^{\alpha, \Lambda}(1) = (x, 0)$. Conversely suppose that $\gamma^{\alpha, \Lambda}$ with $\alpha \in \mathbb{R}^n$ and $\Lambda \in \mathfrak{so}(n)$ connects $0 \in F$ and $(x, 0) \in F$. Then, using Lemma 2.2.7, we have $Z_K = 0$, i.e. $x_i = 0$, $i = 1, \dots, a$ and $\alpha_j = 0$, $j = a + 1, \dots, b$. Thus

$$x = x_K \in \ker \Lambda \text{ and } \alpha = \alpha_K + \alpha_R = \alpha_K = x_K = x.$$

The length of such curves is always

$$L(\gamma^{\alpha, \Lambda}) = |\alpha| = |x|.$$

□

Lemma 2.2.13. *Let $(0, z) \in F$. Let $\pm i\theta_j$, $j = 1, \dots, c$ denote the nonzero eigenvalues of J^z counted with multiplicities, such that $\theta_1 \geq \dots \geq \theta_c > 0$. Then*

$$|(0, z)|_{CC}^2 = 2\pi \sum_{j=1}^c j\theta_j. \quad (2.72)$$

Proof. Let $\alpha \in \mathbb{R}^n, \Lambda \in \mathfrak{so}(n)$ such that $\gamma^{\alpha, \Lambda}$ connects $0 \in F$ with $(0, z) \in F$. Then $\alpha_K = 0, \alpha_R = 0$ and

$$J^z = \sum_{j=a+1}^b \frac{1}{\lambda_j} \tilde{\alpha}_j \wedge \alpha_j. \quad (2.73)$$

This means that Λ^2 has c different eigenvalues

$$-\lambda_j^2, \quad j = 1, \dots, c \text{ with } \lambda_j \in 2\pi\mathbb{Z}^*, \quad j = 1, \dots, c.$$

Since $(J^z)^2$ has eigenvalues $-\theta_j^2, j = 1, \dots, c$ and the square of the right side of (2.73) has eigenvalues $-\left(\frac{1}{\lambda_j} |\alpha_j|^2\right)^2$, we have after re-sorting the $\lambda_j, j = 1, \dots, c$

$$\theta_j = \frac{1}{\lambda_j} |\alpha_j|^2.$$

Moreover $\alpha_1, \tilde{\alpha}_1, \dots, \alpha_c, \tilde{\alpha}_c$ is a orthogonal basis of eigenvectors of $(J^z)^2$ and Λ^2 . The length of $\gamma^{\alpha, \Lambda}$ in this case is

$$\begin{aligned} L(\gamma^{\alpha, \Lambda})^2 &= \sum_{j=1}^c |\alpha_j|^2 \\ &= \sum_{j=1}^c \theta_j \lambda_j. \end{aligned}$$

Thus the shortest normal extremal in this case is given by $\gamma^{\alpha, \Lambda}$ if we choose $\lambda_j = 2j\pi$. \square

Another consequence of Lemma 2.2.5 is the following Lemma

Lemma 2.2.14. *Let $\gamma^{\alpha, \Lambda}, \alpha \in \mathbb{R}^n \setminus \{0\}, \Lambda \in \mathfrak{so}(n)$ be, as above, a normal extremal parametrized with constant speed. Then for $X \in \mathfrak{so}(n)$ with $[X, \Lambda] = 0$*

$$\langle J^z, X \rangle = -\langle X\Lambda^{-1}\xi_I, (x_I + x_K) \rangle + \langle X\Lambda^{-1}\alpha_I, \alpha_I \rangle + \langle X\Lambda^{-1}x_I, x_K \rangle,$$

in particular

$$\langle J^z, \Lambda \rangle = \left| \frac{\frac{1}{2}\Lambda}{\sinh \frac{1}{2}\Lambda} x_R \right|^2 + |\alpha_\pi|^2 - \frac{1}{2} \left\langle \Lambda \coth \frac{1}{2}\Lambda x_R, x_R \right\rangle, \quad (2.74)$$

or equivalently

$$L(\gamma^{\alpha, \Lambda}) = \frac{1}{2} \left\langle \Lambda \coth \frac{\Lambda}{2} (x_R + x_K), (x_R + x_K) \right\rangle + \langle J^z, \Lambda \rangle. \quad (2.75)$$

Proof. Use equation (2.51) in Lemma 2.2.5 to see that

$$\begin{aligned}
 \langle J^z, X \rangle &= -\langle X\Lambda^{-1}\xi_I, x_I \rangle - \left\langle X, \frac{I - e^{-\text{ad } \Lambda}}{\text{ad } \Lambda} (\Lambda^{-1}\alpha_I \wedge \alpha_I) \right\rangle - \langle X\Lambda^{-1}(\xi_I - x_I), \alpha_K \rangle \\
 &= -\langle X\Lambda^{-1}\xi_I, (x_I + \alpha_K) \rangle + \left\langle \frac{I - e^{\text{ad } \Lambda}}{\text{ad } \Lambda} (X), \Lambda^{-1}\alpha_I \wedge \alpha_I \right\rangle + \langle X\Lambda^{-1}x_I, \alpha_K \rangle \\
 &= -\langle X\Lambda^{-1}\xi_I, (x_I + \alpha_K) \rangle - \langle X, \Lambda^{-1}\alpha_I \wedge \alpha_I \rangle + \langle X\Lambda^{-1}x_I, \alpha_K \rangle \\
 &= -\langle X\Lambda^{-1}\xi_I, (x_I + \alpha_K) \rangle + \langle X\Lambda^{-1}\alpha_I, \alpha_I \rangle + \langle X\Lambda^{-1}x_I, \alpha_K \rangle.
 \end{aligned}$$

Since by Lemma 2.2.7 $\alpha_K = x_K$ this proves the first part of this Lemma. Setting $X = \Lambda$ gives

$$\begin{aligned}
 \langle J^z, \Lambda \rangle &= \langle \alpha_I, \alpha_I \rangle - \langle \xi_I, x_I \rangle \\
 &= \left| \frac{\frac{1}{2}\Lambda}{\sinh \frac{1}{2}\Lambda} x_R \right|^2 + |\alpha_\pi|^2 - \left\langle \frac{1}{2}\Lambda \coth \Lambda x_R, x_R \right\rangle.
 \end{aligned}$$

□

2.3 Sub-Riemannian geometry on general 2-step nilpotent Lie groups

In this section we want to revisit the Carnot-Carathéodory distance on a general 2-step nilpotent Lie group N . Contrary to free 2-step nilpotent Lie groups the existence of strictly abnormal extremals is not excluded here (see Section 1.8 and the example in Section 1.8.1). This fact makes the analysis of the Carnot-Carathéodory distance more difficult. Using the results on free 2-step nilpotent Lie groups in Section 2.2 we are able to solve the Hamiltonian differential equations. Even though the results on free 2-step nilpotent Lie groups carry over, the boundary conditions and the existence of strictly abnormal extremals are a lot more difficult. For this reason we will quickly pass to Heisenberg type groups in Section 2.3.1.

As in case of the free 2-step nilpotent Lie group we define the horizontal distribution $\mathcal{D} = \text{span}\{X_1, \dots, X_n\}$ which is again bracket-generating. By requiring that X_1, \dots, X_n is an orthonormal basis of \mathcal{H} we define a Riemannian metric on \mathcal{D} .

Proposition 2.3.1. *Let $\gamma = (x, z) : [0, 1] \rightarrow N$ be an absolute continuous curve in N . Then γ is horizontal if and only if*

$$\dot{z}_j(t) = \frac{1}{2} \langle J^j \dot{x}(t), x(t) \rangle \tag{2.76}$$

for almost all $t \in [0, 1]$. If γ is horizontal then its length is

$$L(\gamma) = \int_0^1 |\dot{x}(t)| dt.$$

Proof. For almost all t we have

$$\begin{aligned} \dot{\gamma} &= \sum_{i=1}^n \dot{x}_i \partial_{x_i} + \sum_{j=1}^m \dot{z}_j \partial_{z_j} \\ &= \dot{x}X + \sum_{j=1}^m \left(\dot{z}_j + \frac{1}{2} \sum_{i=1}^n \dot{x}_i (J^j x)_i \right) \partial_{z_j} \\ &= \dot{x}X + \sum_{j=1}^m \left(\dot{z}_j + \frac{1}{2} \langle J^j x, \dot{x} \rangle \right) \partial_{z_j}. \end{aligned}$$

Thus γ is horizontal, iff $\dot{z}_j(t) = \frac{1}{2} \langle J^j \dot{x}, x \rangle$, $j = 1, \dots, m$ for almost all t , and the length of a horizontal curve γ is

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt = \int_0^1 |\dot{x}(t)| dt.$$

□

The Hamiltonian is given by

$$\begin{aligned} \mathbb{H}_2(x, z, \xi, \eta) &= \frac{1}{2} \sum_{i=1}^n \left(\xi_i - \frac{1}{2} (J^\eta x)_i \right)^2 \\ &= \frac{1}{2} \left| \xi - \frac{1}{2} J^\eta x \right|^2 \\ &= \frac{1}{2} |\zeta|^2, \end{aligned} \tag{2.77}$$

if we write $\zeta := \xi - \frac{1}{2} J^\eta x$. This gives the Hamiltonian differential equations

$$\dot{x} = \frac{\partial \mathbb{H}_2}{\partial \xi}, \quad \dot{z} = \frac{\partial \mathbb{H}_2}{\partial \eta}, \quad \dot{\xi} = -\frac{\partial \mathbb{H}_2}{\partial x}, \quad \dot{\eta} = -\frac{\partial \mathbb{H}_2}{\partial z}, \tag{2.78}$$

and we have the following result.

Proposition 2.3.2. *The solution to the Hamiltonian equations with initial condition*

$$(x, z)(0) = (0, 0), \quad \xi(0) = \alpha, \quad \eta(0) = \beta, \quad (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^m$$

is given by

$$x(t) = \frac{1 - e^{-t\Lambda}}{\Lambda} \alpha, \quad (2.79)$$

$$\xi(t) = \frac{1}{2} (e^{-t\Lambda} + I) \alpha, \quad (2.80)$$

$$\eta(t) = \beta, \quad (2.81)$$

and

$$\begin{aligned} z_j(t) &= \frac{1}{2} \langle \Lambda^{-1} \xi_I(t), J^j x_I(t) \rangle + \frac{1}{2} \left\langle \frac{1 - e^{t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda} (J^j), \Lambda^{-1} \alpha_I \wedge \alpha_I \right\rangle \\ &\quad + t \langle \Lambda^{-1} (\xi_I(t) - x_I(t)), J^j \alpha_K \rangle, \end{aligned} \quad (2.82)$$

where $\Lambda := J^\beta$.

Proof. First notice that J^n is constant equal to Λ , since $\dot{\eta} = 0$. The other Hamiltonian equations yield

$$\dot{x}(t) = \zeta(t), \quad \dot{\xi}(t) = -\frac{1}{2} \Lambda \zeta(t),$$

$$\dot{z}_j = -\frac{1}{2} \langle J^j x(t), \zeta(t) \rangle = \frac{1}{2} \langle J^j, x(t) \wedge \zeta(t) \rangle.$$

Using the results on the free 2-step nilpotent Lie group in Section 2.2 Theorem 2.2.2 we conclude that

$$x(t) = \frac{I - e^{-t\Lambda}}{\Lambda} \alpha, \quad \xi(t) = \frac{1}{2} (I + e^{-t\Lambda}) \alpha,$$

$$\begin{aligned} z_j(t) &= \frac{1}{2} \langle \Lambda^{-1} \xi_I(t), J^j x_I(t) \rangle + \frac{1}{2} \left\langle \frac{I - e^{t \operatorname{ad} \Lambda}}{\operatorname{ad} \Lambda} (J^j), \Lambda^{-1} \alpha_I \wedge \alpha_I \right\rangle \\ &\quad + t \langle \Lambda^{-1} (\xi_I(t) - x_I(t)), J^j \alpha_K \rangle. \end{aligned}$$

□

2.3.1 Sub-Riemannian geometry on Heisenberg type groups

In this section we want to study the sub-Riemannian geometry on Heisenberg type groups $\mathbb{H}_{\nu, m} = \mathbb{R}^n \times \mathbb{R}^m$, $n = 2\nu$ introduced in example 2.1.5. The specific properties of these group makes it possible to give a complete answer to the length minimizing problem. As a nonsingular nilpotent Lie group of step 2 the only abnormal extremals are constant curves by Theorem 1.8.7. Thus, to calculate the Carnot-Carathéodory distance, we have to

compare the lengths of the normal extremals which are given as solutions to the Hamiltonian differential equations.

Immediate consequences of the defining property $(J^z)^2 = -|z|^2 I$, $z \in \mathbb{R}^m$, where I denotes the identity matrix on $\mathbb{R}^{2\nu}$, are the following facts:

Lemma 2.3.3. *Let $x, y \in \mathbb{R}^n$, $z, u \in \mathbb{R}^m$. Then*

$$\langle J^z x, J^u x \rangle = \langle z, u \rangle |x|^2, \quad x \in \mathbb{R}^n, \quad z, u \in \mathbb{R}^m, \quad (2.83)$$

$$\langle J^z x, J^z y \rangle = \langle x, y \rangle |z|^2, \quad x, y \in \mathbb{R}^n, \quad z \in \mathbb{R}^m, \quad (2.84)$$

$$J^z J^u + J^u J^z = -2 \langle z, u \rangle I, \quad z, u \in \mathbb{R}^m. \quad (2.85)$$

Proof. We start with equation (2.84)

$$\langle J^z x, J^z y \rangle = - \langle (J^z)^2 x, y \rangle = |z|^2 \langle x, y \rangle.$$

To see (2.83) and (2.85) notice that

$$\begin{aligned} -|z+u|^2 I &= (J^{z+u})^2 = (J^z + J^u)^2 \\ &= -|z|^2 I - |u|^2 I + J^z J^u + J^u J^z. \end{aligned}$$

This gives (2.85) since $|z+u|^2 = |z|^2 + |u|^2 + 2 \langle z, u \rangle$. Then equation (2.83) is a consequence of (2.85) since

$$\begin{aligned} \langle J^z x, J^u x \rangle &= -\frac{1}{2} (\langle J^u J^z x, x \rangle + \langle x, J^z J^u x \rangle) \\ &= \langle z, u \rangle |x|^2. \end{aligned}$$

□

Before we begin discussing the sub-Riemannian geometry we need some information on $(\text{ad } J^z)^k$, $k \in \mathbb{N}$ which is subject to the following lemma.

Lemma 2.3.4. *Let $z, u \in \mathbb{R}^m$. Then for every $k \in \mathbb{N}$ we have*

$$(\text{ad } J^z)^{2k} J^u = (2i)^{2k} |z|^{2k-2} (|z|^2 J^u - \langle z, u \rangle J^z) \quad \text{and} \quad (2.86a)$$

$$(\text{ad } J^z)^{2k+1} J^u = (2i)^{2k} |z|^{2k} \text{ad } (J^z) J^u. \quad (2.86b)$$

Furthermore, for $x \in \mathbb{R}^n$ we have

$$\langle J^z x, ((\text{ad } J^z)^k J^u) x \rangle = 0, \quad k \in \mathbb{N}. \quad (2.87)$$

Proof. Since $\text{ad}(J^z)J^z = 0$ equation (2.86b) is an immediate consequence of (2.86a). We prove (2.86a) by induction. For $k = 1$ we have by (2.85)

$$\begin{aligned} (\text{ad } J^z)^2 J^u &= J^z J^z J^u - 2J^z J^u J^z + J^u J^z J^z \\ &= -2|z|^2 J^u + 2(J^u J^z J^z + 2\langle z, u \rangle J^z) \\ &= -4(|z|^2 J^u - \langle z, u \rangle J^z). \end{aligned}$$

Now let $k \in \mathbb{N}$, then by the induction hypothesis we have

$$\begin{aligned} (\text{ad } J^z)^{2(k+1)} J^u &= (2i)^{2k} |z|^{2k-2} (\text{ad } J^z)^2 (|z|^2 J^u - \langle z, u \rangle J^z) \\ &= -4(2i)^{2k} |z|^{2k} (|z|^2 J^u - \langle z, u \rangle J^z) \\ &= (2i)^{2k+2} |z|^{2k} (|z|^2 J^u - \langle z, u \rangle J^z). \end{aligned}$$

Hence we have proven (2.86a) for all $k \in \mathbb{N}$.

To prove the remaining claim (2.87) of this lemma in view of (2.86) it suffices to prove (2.87) for $k = 1, 2$. First for $k = 1$ we have

$$\begin{aligned} \langle J^z x, (\text{ad}(J^z)J^u)x \rangle &= \langle J^z x, J^z J^u x \rangle - \langle J^z x, J^u J^z x \rangle \\ &= |z|^2 \langle x, J^u x \rangle - \langle J^z x, J^u J^z x \rangle \\ &= 0, \end{aligned}$$

since J^z and J^u are skew-symmetric. For $k = 2$ we have by (2.86a)

$$\begin{aligned} \langle J^z, ((\text{ad } J^z)^2 J^u)x \rangle &= -4|z|^2 \langle J^z x, J^u x \rangle + 4\langle z, u \rangle \langle J^z x, J^z x \rangle \\ &= -4|z|^2 \langle z, u \rangle |x|^2 + 4\langle z, u \rangle |z|^2 |x|^2 \\ &= 0. \end{aligned}$$

□

Theorem 2.3.5. *The normal extremals parametrized with constant speed are given by $\gamma^{\alpha, \eta} = (x, z) : [0, 1] \rightarrow \mathbb{H}_{\nu, m}$, $\alpha \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{R}^m$ where*

$$x(t) = 2 \frac{\sin(\frac{t}{2}|\eta|)}{|\eta|} e^{-\frac{t}{2}J^\eta} \alpha \tag{2.88}$$

$$z(t) = \begin{cases} \left(\frac{1}{2}t - \frac{\sin(\frac{t}{2}|\eta|)\cos(\frac{t}{2}|\eta|)}{|\eta|} \right) |\alpha|^2 \frac{\eta}{|\eta|^2} & , \text{ if } \eta \neq 0 \\ 0 & , \text{ if } \eta = 0 \end{cases}. \tag{2.89}$$

Proof. The normal extremals are given as the projections of the bicharacteristics of the Hamiltonian. Thus using Proposition 2.3.2 on page 73 and the fact that the kernel of J^η is trivial if $\eta \neq 0$ yields

$$\begin{aligned} x(t) &= \frac{I - e^{-tJ^\eta}}{J^\eta} \alpha = 2 \frac{\sinh\left(\frac{t}{2} J^\eta\right)}{J^\eta} e^{-\frac{t}{2} J^\eta} \alpha, \text{ and} \\ z_j(t) &= -\frac{1}{2} \left\langle J^j (J^\eta)^{-1} \xi(t), x(t) \right\rangle + \frac{1}{2} \left\langle \frac{I - e^{t \operatorname{ad} J^\eta}}{\operatorname{ad} J^\eta} (J^j), (J^\eta)^{-1} \alpha \wedge \alpha \right\rangle, \end{aligned}$$

for $j = 1, \dots, m$ and $\eta \neq 0$. If $\eta = 0$, then the kernel of J^η is \mathbb{R}^n and Proposition 2.3.2 on page 73 gives $z(t) = 0$. Furthermore,

$$\xi(t) = \frac{1}{2} \left(I + e^{-tJ^\eta} \right) \alpha = \cosh\left(\frac{t}{2} J^\eta\right) e^{-\frac{t}{2} J^\eta} \alpha.$$

Since $(J^\eta)^2 = -|\eta|^2 I$ has the only eigenvalue $-|\eta|^2$ we conclude that

$$\begin{aligned} x(t) &= 2 \frac{\sin\left(\frac{t}{2} |\eta|\right)}{|\eta|} e^{-\frac{t}{2} J^\eta} \alpha, \\ \xi(t) &= \cos\left(\frac{t}{2} |\eta|\right) e^{-\frac{t}{2} J^\eta} \alpha. \end{aligned}$$

Now assume that $\eta \neq 0$. Since then $(J^\eta)^{-1} = -\frac{1}{|\eta|^2} J^\eta$, by Lemma 2.3.4 and Lemma 2.3.3 we have for $j = 1, \dots, m$

$$\begin{aligned} z_j(t) &= -\frac{\sin\left(\frac{t}{2} |\eta|\right) \cos\left(\frac{t}{2} |\eta|\right)}{|\eta|} \left\langle J^j (J^\eta)^{-1} e^{-\frac{t}{2} J^\eta} \alpha, e^{-\frac{t}{2} J^\eta} \alpha \right\rangle \\ &\quad - \frac{t}{2|\eta|^2} \left\langle J^\eta \alpha, \frac{I - e^{t \operatorname{ad} J^\eta}}{t \operatorname{ad} J^\eta} (J^j) \alpha \right\rangle \\ &= \frac{\sin\left(\frac{t}{2} |\eta|\right) \cos\left(\frac{t}{2} |\eta|\right)}{|\eta|^3} \left\langle J^j J^\eta e^{-\frac{t}{2} J^\eta} \alpha, e^{-\frac{t}{2} J^\eta} \alpha \right\rangle + \frac{t}{2|\eta|^2} \left\langle J^\eta \alpha, J^j \alpha \right\rangle \\ &= \left(\frac{t}{2} - \frac{\sin\left(\frac{t}{2} |\eta|\right) \cos\left(\frac{t}{2} |\eta|\right)}{|\eta|} \right) |\alpha|^2 \frac{\eta_j}{|\eta|^2}. \end{aligned}$$

This proves the claim. \square

Theorem 2.3.6. *Let $(0, z) \in \mathbb{H}_{\nu, m}$, $z \neq 0$. Then $\gamma^{\alpha, \eta}$, $\alpha \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{R}^m$ is a normal extremal connecting $0 \in \mathbb{H}_{\nu, m}$ and $(0, z)$ if and only if $|\eta| \in 2\pi\mathbb{Z}^*$ such that*

$$\frac{z}{|z|} = \frac{\eta}{|\eta|} \text{ and } |\alpha| = \sqrt{2|\eta||z|}. \quad (2.90)$$

In this case the length of $\gamma^{\alpha,\eta}$ is

$$L(\gamma^{\alpha,\eta}) = \sqrt{2|\eta||z|}.$$

Moreover

$$|(0, z)|_{CC} = \sqrt{4\pi|z|}. \quad (2.91)$$

Proof. By Theorem 2.3.5 $\gamma^{\alpha,\eta}(1) = (0, z)$ is equivalent to

$$0 = 2 \frac{\sin\left(\frac{1}{2}|\eta|\right)}{|\eta|} e^{-\frac{1}{2}J\eta} \alpha \text{ and} \quad (2.92)$$

$$z = \left(\frac{1}{2} - \frac{\sin\left(\frac{1}{2}|\eta|\right) \cos\left(\frac{1}{2}|\eta|\right)}{|\eta|} \right) |\alpha|^2 \frac{\eta}{|\eta|^2}. \quad (2.93)$$

Equation (2.92) is in turn equivalent to $|\eta| \in 2\pi\mathbb{Z}^*$. Thus (2.93) is

$$2z|\eta|^2 = |\alpha|^2 \eta.$$

This proves the first part of the lemma. The length is given by

$$L(\gamma^{\alpha,\eta}) = |\alpha| = \sqrt{2|\eta||z|},$$

and the shortest normal extremal is obviously given by a parameter $|\eta| = 2\pi$ and thus

$$|(0, z)|_{CC} = \sqrt{4\pi|z|}.$$

□

Lemma 2.3.7. *Let $(x, z) \in \mathbb{H}_{\nu,m}$, $x \neq 0$. Then $\gamma^{\alpha,\eta}$, $\alpha \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$ is a normal extremal connecting $0 \in \mathbb{H}_{\nu,m}$ and $(x, z) \in \mathbb{H}_{\nu,m}$ if and only if $|\eta| \notin 2\pi\mathbb{N}$ and*

$$\alpha = \frac{|\eta|}{2 \sin \frac{|\eta|}{2}} e^{\frac{1}{2}J\eta} x \quad (2.94)$$

$$4 \frac{z}{|x|^2} = \left(\frac{1}{2} \frac{|\eta|}{\sin^2 \frac{|\eta|}{2}} - \cot \frac{|\eta|}{2} \right) \frac{\eta}{|\eta|}. \quad (2.95)$$

In this case the length is given by

$$L(\gamma^{\alpha,\eta}) = \frac{|\eta|}{2 \left| \sin \frac{|\eta|}{2} \right|} |x|. \quad (2.96)$$

Remark 2.3.8. The function $\lambda \mapsto \frac{\lambda}{\sin^2 \lambda} - \cot \lambda$ first defined $\mathbb{R} \setminus \pi\mathbb{Z}$ has a continuous extension on $\mathbb{R} \setminus \pi\mathbb{Z}^*$. Furthermore, $\lim_{\lambda \rightarrow 0} \left(\frac{\lambda}{\sin^2 \lambda} - \cot \lambda \right) = 0$. Thus the right side of (2.95) makes sense for $\eta = 0$ if we identify the term

$$\left(\frac{1}{2} \frac{|\eta|}{\sin^2 \frac{|\eta|}{2}} - \cot \frac{|\eta|}{2} \right) \frac{\eta}{|\eta|}$$

with its continuous extension, i.e. we define the right side of (2.95) to be $0 \in \mathbb{R}^n$ if $\eta = 0$.

Proof. By Theorem 2.3.5 $x \neq 0$ is equivalent to $|\eta| \notin 2\pi\mathbb{N}$. And by Theorem 2.3.5 we conclude that $\gamma^{\alpha, \eta}$ is connecting 0 and (x, z) iff

$$\alpha = \frac{|\eta|}{2 \sin \frac{|\eta|}{2}} e^{\frac{1}{2} J^\eta} x$$

and

$$z = \begin{cases} \left(\frac{1}{2} - \frac{\sin \frac{|\eta|}{2} \cos \frac{|\eta|}{2}}{|\eta|} \right) \frac{|\eta|^2}{4 \sin^2 \frac{|\eta|}{2}} |x|^2 \frac{\eta}{|\eta|^2} & , \text{ if } \eta \neq 0 \\ 0 & , \text{ if } \eta = 0 \end{cases}$$

$$= \frac{1}{4} \left(\frac{|\eta|}{2 \sin^2 \frac{|\eta|}{2}} - \cot \frac{|\eta|}{2} \right) |x|^2 \frac{\eta}{|\eta|}.$$

The length in this case is

$$L(\gamma^{\alpha, \Lambda}) = |\alpha| = \frac{|\eta|}{2 \left| \sin \frac{|\eta|}{2} \right|} |x|.$$

□

Definition 2.3.9. For $\lambda \notin \pi\mathbb{Z}^*$ let

$$\mu(\lambda) := \frac{\lambda}{\sin^2 \lambda} - \cot \lambda, \quad \text{and} \quad (2.97)$$

$$l(\lambda) := \frac{\lambda^2}{\sin^2 \lambda}. \quad (2.98)$$

As stated in Remark 2.3.8 we define μ, l in 0 by their continuous continuation, i.e. $\mu(0) = 0$ and $l(0) = 1$. Moreover $\mu, l \in C^\infty(\mathbb{R} \setminus \pi\mathbb{Z}^*)$.

This function μ is well understood since it also appeared in [3] when studying the sub-Riemannian geometry on Heisenberg groups. We will repeat here some of the properties found in [3]. These properties are also used in Section 3.2 when studying the sub-Riemannian geometry associated with the Grušin operator.

First we want to observe that μ is an odd function, i.e $\mu(-\lambda) = -\mu(\lambda)$ and l is even. We therefore may restrict our further analysis to $\mu|_{[0,\infty)\setminus\pi\mathbb{N}}$ respectively $l|_{[0,\infty)\setminus\pi\mathbb{N}}$. In figure 2.1 we plotted the graph of μ .

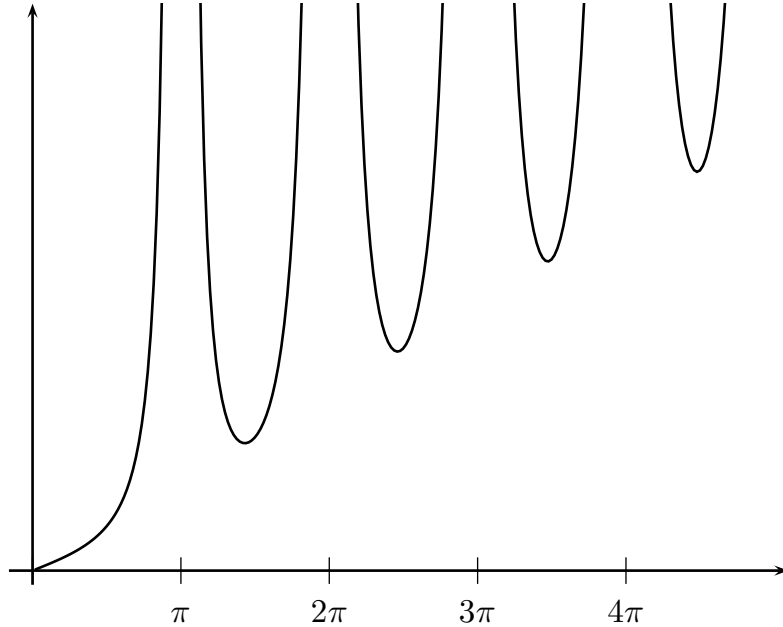


Figure 2.1: The graph of the function μ .

Lemma 2.3.10. *The function μ is a monotone increasing diffeomorphism of the interval $(-\pi, \pi)$ onto \mathbb{R} . On each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, μ has a unique critical point λ_m , which is a minimum. λ_m is given as the solution of the equation $1 - \lambda \cot \lambda = 0$ in the corresponding interval. On this interval μ decreases strictly from $+\infty$ to $\mu(\lambda_m)$ and then increases strictly to $+\infty$. Moreover*

$$\mu(\lambda_m) > m\pi, \tag{2.99}$$

for every $m \in \mathbb{N}$.

Proof. Although this proof can be found in [3] we want to give the proof here. Observe that

$$\mu'(\lambda) = 2 \frac{1 - \lambda \cot \lambda}{\sin^2 \lambda}, \quad \lambda \notin \pi\mathbb{Z}^*, \tag{2.100}$$

thus the critical point $\lambda_m \in (m\pi, (m+1)\pi)$, $m \in \mathbb{N}$ are exactly the solutions of $1 - \lambda \cot \lambda = 0$, $\lambda \notin \pi\mathbb{Z}$. Since μ' has the same sign as $1 - \lambda \cot \lambda$ it follows that $\mu|_{(m\pi, \lambda_m)}$ is monotone decreasing and $\mu|_{(\lambda_m, (m+1)\pi)}$ is monotone increasing. To prove that $\mu|_{(-\pi, \pi)}$

is a monotone increasing diffeomorphism onto \mathbb{R} we claim that $\mu'(\lambda) \geq \frac{2}{3}$ for $\lambda \in (-\pi, \pi)$. Since $\mu'(0) = \frac{2}{3}$ we show that $\mu''(\lambda) > 0$ for $\lambda \in (0, \pi)$. The second derivative of μ is

$$\begin{aligned}\mu''(\lambda) &= 2 \frac{\lambda - \sin \lambda \cos \lambda - 2(1 - \lambda \cot \lambda) \sin \lambda \cos \lambda}{\sin^4 \lambda} \\ &= 2 \frac{\lambda - 3 \sin \lambda \cos \lambda - 2\lambda \cos^2 \lambda}{\sin^4 \lambda}, \quad \lambda \notin \pi \mathbb{Z}^*.\end{aligned}\tag{2.101}$$

Let $f(\lambda) := \lambda - 3 \sin \lambda \cos \lambda - 2\lambda \cos^2 \lambda$. Then $f(0) = 0$ and

$$\begin{aligned}f'(\lambda) &= 1 - 3(\cos^2 \lambda - \sin^2 \lambda) + 2 \cos^2 \lambda - 4\lambda \sin \lambda \cos \lambda \\ &= 4 \sin \lambda (\sin \lambda - \lambda \cos \lambda) \\ &> 0,\end{aligned}$$

for $\lambda \in (0, \pi)$. Hence $\mu''(\lambda) > 0$ for $\lambda \in (0, \pi)$.

Furthermore, we have

$$\mu(\lambda + \pi) = \mu(\lambda) + \frac{\pi}{\sin^2 \lambda} \geq \mu(\lambda) + \pi.\tag{2.102}$$

Hence $\mu(\lambda_m) > m\pi$ since $\mu|_{(0,\pi)} > 0$. □

Theorem 2.3.11. *Given $(x, 0) \in \mathbb{H}_{\nu, m}$, $x \neq 0$, then there is exactly one normal extremal parametrized with constant speed connecting $0 \in \mathbb{H}_{\nu, m}$ and $(x, 0)$, namely $\gamma^{x, 0}$. Moreover*

$$|(x, 0)|_{CC} = |x|.\tag{2.103}$$

Proof. By Lemma 2.3.7 the normal extremals connecting 0 and $(x, 0)$ are given by $\gamma^{\alpha, \eta}$ where η is equal to zero or a solution of

$$0 = \left(\frac{1}{2} \frac{|\eta|}{\sin^2 \frac{|\eta|}{2}} - \cot \frac{|\eta|}{2} \right).\tag{2.104}$$

By Lemma 2.3.10 there is only one solution of (2.104), namely $\eta = 0$. Thus

$$\alpha = \frac{|\eta|}{2 \sin \frac{|\eta|}{2}} x = x$$

and

$$|(x, 0)|_{CC} = L(\gamma^{x, 0}) = |\alpha| = |x|.$$

□

Theorem 2.3.12. *Given $(x, z) \in \mathbb{H}_{\nu, m}$, $x \neq 0$, $z \neq 0$, then there are finitely many normal extremals connecting 0 and (x, z) . These are given by $\gamma^{\alpha, \eta}$ where η is a solution of*

$$\frac{4|z|}{|x|^2} = \mu\left(\frac{|\eta|}{2}\right), \quad \frac{z}{|z|} = \frac{\eta}{|\eta|} \quad (2.105)$$

and

$$\alpha = \frac{\eta}{2 \sin \frac{|\eta|}{2}} e^{\frac{1}{2}J^n} x.$$

Their length is strictly increasing with $|\eta|$. Moreover

$$|(x, z)|_{CC}^2 = \frac{|\eta_0|^2}{4 \sin^2 \frac{|\eta_0|}{2}} |x|^2 \quad (2.106a)$$

$$= \frac{|\eta_0|}{2} \cot \frac{|\eta_0|}{2} |x|^2 + 2|\eta_0| |z|, \quad (2.106b)$$

where $|\eta_0| \in [0, 2\pi)$ is the unique solution of (2.105) in the interval $[0, 2\pi)$.

Proof. First observe that if $|\eta|$ is a solution of (2.105) then

$$\begin{aligned} 2|\eta| |z| &= \frac{|\eta|}{2} \mu\left(\frac{|\eta|}{2}\right) |x|^2 \\ &= \frac{|\eta|^2}{4 \sin^2 \frac{|\eta|}{2}} |x|^2 - \frac{|\eta|}{2} \cot \frac{|\eta|}{2} |x|^2, \end{aligned}$$

which proves the identity (2.106). To prove the rest of the theorem we prove that for every $a > 0$ and any two solutions $s < t$ of the equation

$$a = \mu(\lambda), \lambda \in \mathbb{R} \setminus \pi\mathbb{Z}^* \quad (2.107)$$

it follows that

$$l(s) < l(t).$$

Before we prove this note that for $\lambda \notin \pi\mathbb{Z}^*$

$$\frac{d}{d\lambda} \lambda \cot \lambda = \cot \lambda - \frac{\lambda}{\sin^2 \lambda} = -\mu(\lambda), \quad (2.108)$$

$$\lambda \mu(\lambda) = l(\lambda) - \lambda \cot \lambda \quad (2.109)$$

and therefore

$$l'(\lambda) = -\mu(\lambda) + \mu(\lambda) + \lambda \mu'(\lambda) = \lambda \mu'(\lambda). \quad (2.110)$$

Lemma 2.3.10 on page 80 tells us that the equation (2.107) has at most two solutions in each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}_0$ and exactly one solution in the interval $(0, \pi)$.

First let $\mu(s) = a$ be a solution of (2.107) in the interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, then

$$\begin{aligned} l(s) &= l(\lambda_m) + \int_{\lambda_m}^s l'(\lambda) d\lambda \\ &= l(\lambda_m) + \int_{\lambda_m}^s \lambda \mu'(\lambda) d\lambda \\ &= l(\lambda_m) + \lambda \mu(\lambda) \Big|_{\lambda_m}^s - \int_{\lambda_m}^s \mu(\lambda) d\lambda. \end{aligned} \quad (2.111)$$

Since by (2.109)

$$l(\lambda_m) - \lambda_m \mu(\lambda_m) = \lambda_m \cot \lambda_m = 1, \quad (2.112)$$

we conclude that

$$l(s) = s\mu(s) - \int_{\lambda_m}^s \mu(\lambda) d\lambda. \quad (2.113)$$

Observe that equation (2.113) also holds true for a solution $s \in [0, \pi)$ of (2.107) if we let $\lambda_0 := 0$. Now let $\mu(s) = a = \mu(t)$, $s < t$ be two solutions of (2.107) in the same interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, then $s < \lambda_m < t$ and

$$l(s) - l(t) = -(t-s)a + \int_s^t \mu(\lambda) d\lambda < 0.$$

The last inequality follows since the maximum of μ on the interval $[s, t]$ is a and thus $\int_s^t \mu(\lambda) d\lambda < (t-s)a$.

If we have two solutions $s \in (m\pi, (m+1)\pi)$, $m \in \mathbb{N}_0$ and $t \in ((m+1)\pi, (m+2)\pi)$ of (2.107) with $\lambda_m \leq s < (m+1)\pi < t \leq \lambda_{m+1}$, then

$$l(s) - l(t) = -(t-s)a - \int_{\lambda_m}^s \mu(\lambda) d\lambda - \int_t^{\lambda_{m+1}} \mu(\lambda) d\lambda < 0,$$

since $\mu \geq 0$ on $[0, \infty) \setminus \pi\mathbb{N}$. □

2.4 Heat kernel estimates on Heisenberg type groups

Let G be a nilpotent Lie group of step 2 with left-invariant vector fields X_1, \dots, X_n as in Section 2.1. Consider the sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^n X_j^2 \quad (2.114)$$

and the corresponding heat equation

$$(\partial_t - \mathcal{L})u = 0 \text{ on } \mathbb{R}_{>0} \times G. \quad (2.115)$$

$-\mathcal{L}$ is a non-negative, hypoelliptic and essentially self-adjoint operator on $L^2(G)$. Then \mathcal{L} is the infinitesimal generator of the heat semigroup $\{e^{t\mathcal{L}}\}_{t>0}$. Since \mathcal{L} is also left-invariant for every $t > 0$ the operator $e^{t\mathcal{L}}$ is given by a convolution kernel, i.e.

$$e^{t\mathcal{L}}f = f * k_t.$$

k_t is called heat kernel for \mathcal{L} and since the differential operator $\partial_t - \mathcal{L}$ is hypoelliptic k_t has the additional feature that $(t, g) \mapsto k_t(g)$ is a smooth function on $(0, \infty) \times G$. This means that $e^{t\mathcal{L}}f$ is the unique solution of the heat equation (2.115) with boundary condition

$$e^{t\mathcal{L}}f \rightarrow f \text{ in } L^1(G) \text{ as } t \rightarrow 0^+. \quad (2.116)$$

Now we want to turn our attention to Heisenberg type groups. The heat kernel for a Heisenberg type group is given by the following formula:

Theorem 2.4.1. *The heat kernel on $\mathbb{H}_{\nu, m}$ is given by*

$$k_t(x, z) = c_{\nu, m} t^{-(m+\nu)} \int_{\mathbb{R}^m} \left(\frac{|\lambda|}{\sinh |\lambda|} \right)^\nu e^{i\langle \lambda, z \rangle} \exp\left(-\frac{|x|^2}{4t} |\lambda| \coth |\lambda| \right) d\lambda, \quad (2.117)$$

where $c_{\nu, m} = 2^{-\nu} (2\pi)^{-(m+\nu)}$.

There are different techniques to derive formulas of this type for 2-step nilpotent Lie groups. Cygan [14] uses irreducible unitary representations of free nilpotent Lie groups to derive formulas for the heat kernel on free 2-step nilpotent Lie groups. These formulas are then used to provide formulas for the heat kernel on general 2-step nilpotent Lie groups. But also the techniques of twisted convolutions given in [27] can be used to give this expression of the heat kernel.

We are now going to use this formula (2.117) for the heat kernel to produce pointwise estimates of the heat kernel. In particular we give estimates of the form

$$|k_t(x, z)| \lesssim t^{-(m+\nu)} f(x, z, t) e^{-\frac{1}{4t}|(x,z)|_{CC}^2},$$

where f is a function of polynomial growth. This can be derived by using the formulas of the Carnot-Carathéodory metric of Section 2.3.1.

Let us recall the result of Section 2.3.1 in a slightly different manner.

Remark 2.4.2. Let $(x, z) \in \mathbb{H}_{\nu, m}$. If $x \neq 0$, then

$$|(x, z)|_{CC}^2 = \frac{y^2}{\sin^2 y} |x|^2 = y \cot y |x|^2 + 4y |z|,$$

where $y \in [0, \pi)$ is the unique solution of the equation

$$\frac{4|z|}{|x|^2} = \frac{y}{\sin^2 y} - \cot y \tag{2.118}$$

in the interval $[0, \pi)$. If $x = 0$, then

$$|(0, z)|_{CC}^2 = 4\pi |z|.$$

For the further calculations it will be more convenient to deal with

$$h(x, z) := \int_{\mathbb{R}^m} \left(\frac{|\lambda|}{\sinh |\lambda|} \right)^\nu e^{-|x|^2 |\lambda| \coth |\lambda| + 4i \langle \lambda, z \rangle} d\lambda. \tag{2.119}$$

Then

$$k_t(x, z) = c_{\nu, m} t^{-(m+\nu)} h\left(\frac{x}{2\sqrt{t}}, \frac{z}{4t}\right). \tag{2.120}$$

Remark 2.4.3. Note that for every orthogonal matrix $A \in O(m)$ we have

$$h(x, z) = h(x, Az). \tag{2.121}$$

Hence we may always assume that $z = (z_1, 0, \dots, 0)$, $z_1 = |z|$.

Remark 2.4.4. The functions

$$V(\xi) := \frac{\xi}{\sinh \xi}, \quad \psi(\xi) := \xi \coth \xi, \quad \xi \in \mathbb{C} \setminus i\pi\mathbb{Z}^*, \tag{2.122}$$

are holomorphic with simple poles in $k\pi i$, $k \in \mathbb{Z}^*$.

Remark 2.4.5. Moreover we will assume that $m \geq 2$. The case $m = 1$ is the Heisenberg group \mathbb{H}_1 which has been intensively studied by Beals, Gaveau and Greiner in [3]. With

some modifications the below calculations also works for the case $m = 1$. In particular the “outer integration” does not occur which actually makes the estimates easier.

Lemma 2.4.6. *Let*

$$V_R(\lambda) := V\left(\sqrt{\lambda^2 + R^2}\right), \quad \psi_R(\lambda) := \psi\left(\sqrt{\lambda^2 + R^2}\right), \lambda \in \mathbb{R}, R \geq 0. \quad (2.123)$$

Then for $m \geq 2$, $(x, z) \in \mathbb{H}_{\nu, m}$ with $z = (z_1, 0, \dots, 0)$

$$h(x, z) = v_m \int_0^\infty R^{m-2} \int_{-\infty}^\infty V_R(\lambda)^\nu e^{-\psi_R(\lambda)|x|^2 + 4i\lambda z_1} d\lambda dR, \quad (2.124)$$

where $v_m = \text{vol}_{m-2}(S^{m-2}) = \frac{2\pi^{(m-1)/2}}{\Gamma((m-1)/2)}$ is the surface area of the $(m-2)$ -dimensional sphere in \mathbb{R}^{m-1} .

Proof. If $m = 2$ we have

$$\begin{aligned} h(x, z) &= \int_{\mathbb{R}} \int_{\mathbb{R}} V\left(\sqrt{\lambda^2 + \eta^2}\right)^\nu \exp\left(-\psi\left(\sqrt{\lambda^2 + \eta^2}\right)|x|^2 + 4i\lambda z_1\right) d\lambda d\eta \\ &= 2 \int_0^\infty \int_{-\infty}^\infty V_R(\lambda)^\nu \exp\left(-\psi_R(\lambda)|x|^2 + 4i\lambda z_1\right) d\lambda dR. \end{aligned}$$

Now if $m > 2$ we take polar coordinates on \mathbb{R}^{m-1} to see that

$$\begin{aligned} h(x, z) &= \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} V\left(\sqrt{\lambda^2 + |\eta|^2}\right)^\nu \exp\left(-\psi\left(\sqrt{\lambda^2 + |\eta|^2}\right)|x|^2 + 4i\lambda z_1\right) d\lambda d\eta \\ &= \text{vol}_{m-2}(S^{m-2}) \\ &\quad \cdot \int_0^\infty R^{m-2} \int_{-\infty}^\infty V\left(\sqrt{\lambda^2 + R^2}\right)^\nu \exp\left(-\psi\left(\sqrt{\lambda^2 + R^2}\right)|x|^2 + 4i\lambda z_1\right) d\lambda dR \\ &= \frac{2\pi^{m/2-1}}{\Gamma(m/2-1)} \int_0^\infty R^{m-2} \int_{-\infty}^\infty V_R(\lambda)^\nu e^{-\psi_R(\lambda)|x|^2 + 4i\lambda z_1} d\lambda dR. \end{aligned}$$

□

Remark 2.4.7. Since both functions V, ψ are even the definitions (2.123) of V_R, ψ_R , $R \geq 0$ do not depend on the choice of the square root. Thus we can extend the functions V_R and ψ_R to holomorphic functions

$$V_R, \psi_R : \mathbb{C} \setminus \left\{ \pm i\sqrt{\pi^2 k^2 + R^2} : k \in \mathbb{N} \right\} \rightarrow \mathbb{C}.$$

The next step is to move the line of integration of the “ λ -integral” to $\mathbb{R} + iy$ with some $-\pi < y < \pi$, but first we need some preliminary estimates for V_R and ψ_R .

Lemma 2.4.8. . *There is a constant $C > 0$ such that for all $R > 0$, $\lambda \in \mathbb{R}$, $0 \leq y < \pi$*

$$|V_R(\lambda + iy)| \leq C \frac{1}{R^2 + \pi^2 - y^2} e^{-\frac{1}{2}\sqrt{\lambda^2 + R^2}}, \quad (2.125)$$

$$|V_R(\lambda + iy)| \leq C \left(1 + \frac{1}{|\lambda|}\right). \quad (2.126)$$

Furthermore, for all $R > 0$, $\lambda \in \mathbb{R}$, $0 \leq y < \pi$ we have

$$\Re(\psi_R(\lambda + iy)) \geq y \cot y. \quad (2.127)$$

Proof. First let $R > 0$, $\lambda \in \mathbb{R}$, $0 \leq y < \pi$ and let $a + ib \in \mathbb{C}$ be a square root of $(\lambda + iy)^2 + R^2$, i.e.

$$(a + ib)^2 = (\lambda + iy)^2 + R^2.$$

Then

$$a^2 - b^2 = \lambda^2 - y^2 + R^2 \text{ and } ab = \lambda y. \quad (2.128)$$

From this we get

$$0 \leq b^2 \leq y^2, \quad (2.129)$$

because, if $b^2 > y^2$ it would follow by the second equality of (2.128) that $a^2 = \frac{\lambda^2 y^2}{b^2} \leq \lambda^2$ and by the first equality of (2.128)

$$b^2 = a^2 - \lambda^2 + y^2 - R^2 \leq y^2 - R^2 < y^2,$$

since $R > 0$. But this is a contradiction to the assumption $b^2 > y^2$ and (2.129) is indeed true. Furthermore, we can conclude from the first equation in (2.128) and (2.129) that

$$\lambda^2 + R^2 \geq a^2 = b^2 + \lambda^2 + R^2 - y^2 \geq \lambda^2 + R^2 - y^2. \quad (2.130)$$

Now we want to prove that

$$a^2 \geq \lambda^2. \quad (2.131)$$

By (2.128) we have for $b \neq 0$

$$b^2 = a^2 - \lambda^2 + y^2 - R^2 = \lambda^2 \left(\frac{y^2}{b^2} - 1 \right) + y^2 - R^2 \geq y^2 - R^2,$$

since by (2.129) $y^2 \geq b^2$. Thus by (2.128) we have

$$a^2 = b^2 + \lambda^2 + R^2 - y^2 \geq \lambda^2,$$

for $b \neq 0$. But if $b = 0$, then by the second equation of (2.128) $\lambda = 0$ or $y = 0$. If $\lambda = 0$, then obviously $a^2 \geq \lambda^2$; and if $y = 0$, then $a^2 = \lambda^2 + R^2 \geq \lambda^2$.

With these preliminary observations we are able to prove the estimates of this lemma.

- To show (2.127) observe that $0 \leq |b| \leq y$ and we write

$$\begin{aligned} \Re(\psi_R(\lambda + iy)) &= \Re(\psi(a + bi)) \\ &= \frac{1}{2} \frac{a \sinh(2a) + b \sin(2b)}{\sinh^2 a + \sin^2 b} \\ &= \frac{a \coth a \sinh^2 a + b \cot b \sin^2 b}{\sinh^2 a + \sin^2 b} \\ &= b \cot b + \sinh^2 a \frac{a \coth a - b \cot b}{\sinh^2 a + \sin^2 b} \\ &\geq b \cot b \\ &\geq y \cot y, \end{aligned}$$

since $b \mapsto b \cot b$ is even and monotone decreasing on $[0, \pi)$ and $a \coth a \geq 1 \geq b \cot b$.

- Now observe that

$$\begin{aligned} |V_R(\lambda + yi)|^2 &= |V(a + bi)|^2 \\ &= \frac{a^2 + b^2}{\sinh^2 a + \sin^2 b} \\ &\leq \left(1 + \frac{b^2}{a^2}\right) \frac{a^2}{\sinh^2 a} \\ &\lesssim \left(1 + \frac{1}{\lambda^2}\right), \end{aligned} \tag{2.132}$$

since $\frac{a}{\sinh a} \leq 1$ and $|b| \leq \pi$. So, this proves (2.126).

- Since V has simple poles in $k\pi i$, $k \in \mathbb{Z}^*$ we have that $f(z) := (z + \pi i)(z - \pi i)V(z)$ is holomorphic in $\left\{z \in \mathbb{C} : -\frac{3}{2}\pi < \Im z < \frac{3}{2}\pi\right\}$. In view of (2.132) we have for sufficiently

large $|\Re z|$, say $|\Re z| > s$, that $|f(z)| \lesssim e^{-\frac{1}{2}|\Re z|}$. For $|\Re z| \leq s$, $|\Im z| \leq \pi$ we have $|f(z)| \lesssim 1$. Hence, $|V(z)| \lesssim |z^2 + \pi^2| e^{-\frac{1}{2}|\Re z|}$ for all $|\Im z| < \pi$ and thus

$$\begin{aligned} |V_R(\lambda + iy)| &\lesssim \frac{1}{|R^2 + (\lambda + iy)^2 + \pi^2|} e^{-\frac{1}{2}\sqrt{\lambda^2 + R^2}} \\ &\leq \frac{1}{R^2 + \lambda^2 + \pi^2 - y^2} e^{-\frac{1}{2}\sqrt{\lambda^2 + R^2}} \\ &\leq \frac{1}{R^2 + \pi^2 - y^2} e^{-\frac{1}{2}\sqrt{\lambda^2 + R^2}}, \end{aligned}$$

for all $R > 0$, $\lambda \in \mathbb{R}$, $0 \leq y < \pi$.

□

Remark 2.4.9. Let $\mathbb{H}_{\nu, m}$ be a Heisenberg type group. Then the integers ν, m has to satisfy specific properties, i.e.

$$m < 8p + 2^q \tag{2.133}$$

if $2\nu = 2^{4p+q}\tilde{n}$ for $\tilde{n} \in \mathbb{N}$ odd and $p \in \mathbb{N}_0$, $q \in \{0, 1, 2, 3\}$. This is a result of Kaplan [22]. As a first consequence we have since $2^q \leq 2q + 2$, $q \in \{0, 1, 2, 3\}$ that

$$\begin{aligned} m &\leq 8p + 2^q - 1 \leq 2(4p + q - 1) + 3 \\ &= 2 \log_2(2^{4p+q-1}) + 3 \\ &\leq 2 \log_2(\nu) + 3. \end{aligned} \tag{2.134}$$

This observation was also given in [2].

Lemma 2.4.10. *Let $\mathbb{H}_{\nu, m}$ be a Heisenberg type group with $m \geq 2$. Then $\nu \geq 2$ and $m \leq 2\nu - 1$. Moreover, if $\nu \notin \{2, 4\}$, then $m \leq 2\nu - 2$.*

Proof. Let $2\nu = 2^{4p+q}\tilde{n}$ with $\tilde{n} \in \mathbb{N}$ odd and $p \in \mathbb{N}_0$, $q \in \{0, 1, 2, 3\}$ as above.

If $\nu \leq 4$, then by (2.133) $p = 0$ and thus $m \leq 2^q - 1 \leq 2\nu - 1$. Therefore, we can conclude that $\nu \geq 2$ since otherwise $m \leq 1$ which contradicts the assumption $m \geq 2$. If $\nu = 3$, then $6 = 2\nu = 2^{4p+q}\tilde{n}$ is only possible for $\tilde{n} = 3$, $p = 0$, and $q = 1$. Hence $m \leq 2^q - 1 = 1$ and this contradicts the assumption $m \geq 2$.

Now let $\nu > 4$. Then $\nu < 2^{\nu-2}$ and by (2.134) we conclude

$$m \leq 2 \log_2(\nu) + 3 < 2\nu - 1.$$

Hence $m \leq 2\nu - 2$ and this completes the proof.

□

Lemma 2.4.11. *Let $m \geq 2$. Then there is a constant $C > 0$ such that for $0 \leq y < \pi$ and $(x, z) \in \mathbb{H}_{\nu, m}$ with $z = (z_1, 0, \dots, 0)$:*

$$|h(x, z)| \leq C \tilde{B}_{\nu, m} \left(\frac{\pi}{\pi - y} \right) e^{-|x|^2 y \cot y - 4yz_1}. \quad (2.135)$$

for all $0 \leq y < \pi$ and $(x, z) \in \mathbb{H}_{\nu, m}$ with $z = (z_1, 0, \dots, 0)$, where

$$\tilde{B}_{\nu, m}(\xi) := \begin{cases} 1 + \log \xi & , \text{ if } m = 2\nu - 1 \\ \xi^{\frac{1}{2}(2\nu - m - 1)} & , \text{ if } m \neq 2\nu - 1 \end{cases}, \quad \xi > 0.$$

Remark 2.4.12. As a consequence of Lemma 2.4.10 B is strictly increasing.

Proof. As a consequence of the exponential decay of $|V_R(\lambda + iy)|$ in R and λ , shown in Lemma 2.4.8, we can move the line of integration to $\mathbb{R} + iy$ in the inner integral in (2.124). Thus

$$\begin{aligned} |h(x, z)| &= v_m \left| \int_0^\infty R^{m-2} \int_{-\infty}^\infty V_R(\lambda + iy)^\nu \exp\left(-\psi_R(\lambda + iy) |x|^2 + 4i(\lambda + iy)z_1\right) d\lambda dR \right| \\ &\leq v_m \int_0^\infty R^{m-2} \int_{-\infty}^\infty |V_R(\lambda + iy)|^\nu \exp\left(-\Re(\psi_R(\lambda + iy)) |x|^2 - 4yz_1\right) d\lambda dR \\ &\leq 2v_m e^{-|x|^2 y \cot y - 4yz_1} \int_0^\infty R^{m-2} \int_0^\infty |V_R(\lambda + iy)|^\nu d\lambda dR, \end{aligned}$$

where the last inequality follows by (2.127). To estimate the remaining integral

$$I := \int_0^\infty R^{m-2} \int_0^\infty |V_R(\lambda + iy)|^\nu d\lambda dR$$

we first split the outer integral:

$$I = \int_0^\pi R^{m-2} \int_0^\infty |V_R(\lambda + iy)|^\nu d\lambda dR + \int_\pi^\infty R^{m-2} \int_0^\infty |V_R(\lambda + iy)|^\nu d\lambda dR. \quad (2.136)$$

If we now use inequality (2.125) of Lemma 2.4.8 on page 87, we can estimate the second integral in (2.136) by

$$\int_{\pi}^{\infty} R^{m-2} \int_0^{\infty} |V_R(\lambda + iy)|^{\nu} d\lambda dR \lesssim \int_{\pi}^{\infty} R^{m-2} \int_0^{\infty} e^{-\nu \frac{1}{2} \sqrt{\lambda^2 + R^2}} d\lambda dR \lesssim 1.$$

For the first integral in (2.136) we split the inner integral into three parts:

$$\begin{aligned} \int_0^{\infty} |V_R(\lambda + iy)|^{\nu} d\lambda &= \int_0^r |V_R(\lambda + iy)|^{\nu} d\lambda + \int_r^{2\pi^2} |V_R(\lambda + iy)|^{\nu} d\lambda + \int_{2\pi^2}^{\infty} |V_R(\lambda + iy)|^{\nu} d\lambda \\ &= I_1 + I_2 + I_3, \quad \text{for } 0 \leq R \leq \pi, \end{aligned}$$

with $r := \pi^2 + R^2 - y^2$.

Inequality (2.125) gives

$$\begin{aligned} I_1 &= \int_0^r |V_R(\lambda + iy)|^{\nu} d\lambda \leq r \left(\frac{1}{\pi^2 + R^2 - y^2} \right)^{\nu} \\ &= \frac{1}{(\pi^2 + R^2 - y^2)^{\nu-1}}. \end{aligned} \tag{2.137}$$

The use of inequality (2.126) gives

$$\begin{aligned} I_2 &= \int_r^{2\pi^2} |V_R(\lambda + iy)|^{\nu} d\lambda \lesssim \int_r^{\infty} \frac{1}{\lambda^{\nu}} d\lambda \lesssim \frac{1}{r^{\nu-1}} \\ &= \frac{1}{(\pi^2 + R^2 - y^2)^{\nu-1}}, \end{aligned} \tag{2.138}$$

since $m \geq 2$ and by Lemma 2.4.10 $\nu \geq 2$. By (2.125) we have for I_3

$$I_3 = \int_{2\pi^2}^{\infty} |V_R(\lambda + iy)|^{\nu} d\lambda \lesssim \int_0^{\infty} e^{-\nu \frac{1}{2} \sqrt{R^2 + \lambda^2}} d\lambda \lesssim 1. \tag{2.139}$$

Putting (2.137), (2.138), and (2.139) together yields

$$\begin{aligned}
 \int_0^\pi R^{m-2} \int_0^\infty |V_R(\lambda + iy)|^\nu d\lambda dR &\lesssim \int_0^\pi \frac{R^{m-2}}{(\pi^2 + R^2 - y^2)^{\nu-1}} dR \\
 &= (\pi^2 - y^2)^{\frac{m-2}{2} - (\nu-1)} \int_0^{\pi/\sqrt{\pi^2 - y^2}} \frac{R^{m-2}}{(1 + R^2)^{\nu-1}} \sqrt{\pi^2 - y^2} dR \\
 &\lesssim \frac{1}{(\pi - y)^{\nu - \frac{m+1}{2}}} \int_0^{\pi/\sqrt{\pi^2 - y^2}} \frac{R^{m-2}}{(1 + R)^{2\nu-2}} dR.
 \end{aligned}$$

If $m \neq 2\nu - 1$, then by Lemma 2.4.10 $m \leq 2\nu - 2$ and thus

$$\int_0^{\pi/\sqrt{\pi^2 - y^2}} \frac{R^{m-2}}{(1 + R)^{2\nu-2}} dR \lesssim \int_0^\infty \frac{R^{m-2}}{(1 + R)^{2\nu-2}} dR \lesssim 1.$$

If now $m = 2\nu - 1$, then $\nu - \frac{m+1}{2} = 0$ and

$$\begin{aligned}
 \int_0^{\pi/\sqrt{\pi^2 - y^2}} \frac{R^{m-2}}{(1 + R)^{2\nu-2}} dR &\leq 1 + \int_1^{\pi/\sqrt{\pi^2 - y^2}} R^{-1} dR \\
 &\leq 1 + \log \frac{\pi}{\sqrt{\pi^2 - y^2}} \\
 &\leq 1 + \log \frac{\pi}{\pi - y}
 \end{aligned}$$

and this completes the proof. \square

Now if we choose $y = y_0$ in Lemma 2.4.11, where $y_0 \in [0, \pi)$ is the solution of (2.118), the exponential term in 2.4.11 is $e^{-|(x,z)|_{CC}^2}$. But $\frac{1}{\pi - y}$ tends to infinity if y approaches π . Therefore we choose $y = y_0 - \epsilon$ in Lemma 2.4.11 for an appropriate $\epsilon \geq 0$.

Theorem 2.4.13. *There is a constant $C > 0$, such that for all $(x, z) \in \mathbb{H}_{\nu, m}$*

$$|h(x, z)| \leq C \cdot \min \left\{ B_{\nu, m} \left(1 + |(x, z)|_{CC}^2 \right), B_{\nu, m} \left(\frac{|(x, z)|_{CC}}{|x|} \right) \right\} e^{-|(x, z)|_{CC}^2}, \quad (2.140)$$

where

$$B_{\nu, m}(\xi) = \begin{cases} 1 + \log \xi & , \text{ if } (\nu, m) \in \{(2, 3), (4, 7)\} \\ \xi^{\nu - \frac{m+1}{2}} & , \text{ otherwise} \end{cases}, \quad \xi > 0.$$

Remark 2.4.14. Note that if $\nu \notin \{2, 4\}$, then $m \leq 2\nu - 2$ and $\nu - \frac{m+1}{2} \geq \frac{1}{2}$.

Proof. If $m = 1$, then we can use the results on the Heisenberg group obtained by Beals, Gaveau, and Greiner in [3] to prove this Lemma. If $m \geq 2$, then $m = 2\nu - 1$ is only possible if $\nu \in \{2, 4\}$, i.e. $(\nu, m) \in \{(2, 3), (4, 7)\}$. Hence $\tilde{B}_{\nu, m} = B_{\nu, m}$ for all $m > 2$. So, assume that $m > 2$ and as remarked in 2.4.3 we will further assume that $z = (z_1, 0, \dots, 0)$, $z_1 = |z|$.

First assume that $x \neq 0$ and let $y_0 \in [0, \pi)$ be the solution of (2.118). Then the Carnot-Carathéodory distance is given by

$$\begin{aligned} |(x, z)|_{CC}^2 &= \frac{y_0^2}{\sin^2 y_0} |x|^2 \\ &= y_0 \cot y_0 |x|^2 + 4y_0 z_1. \end{aligned}$$

Observe that for $y \in [0, \frac{\pi}{2}]$ we have $\frac{y(\pi-y)}{\sin y} \geq (\pi - y) \geq \frac{\pi}{2}$; and for $y \in [\frac{\pi}{2}, \pi]$ we have $\frac{y(\pi-y)}{\sin y} \geq y \geq \frac{\pi}{2}$. We can conclude that

$$\frac{\pi}{\pi - y_0} \leq 2 \frac{y_0}{\sin y_0}.$$

Then we can use Lemma 2.4.11 on page 89 with $y = y_0$ to get

$$\begin{aligned} |h(x, z)| &\lesssim B_{\nu, m} \left(\frac{y_0}{\sin y_0} \right) e^{-y_0 \cot y_0 |x|^2 - 4y_0 z_1} \\ &= B_{\nu, m} \left(\frac{|(x, z)|_{CC}}{|x|} \right) e^{-|(x, z)|_{CC}^2}, \end{aligned}$$

but this estimates remains true if $x = 0$, since the right side becomes $+\infty$. To prove the remaining estimate

$$|h(x, z)| \lesssim B_{\nu, m} \left(1 + |(x, z)|_{CC}^2 \right) e^{-|(x, z)|_{CC}^2},$$

we again use Lemma 2.4.11. Let $y_0 \in [0, \pi]$ be the solution of (2.118) if $x \neq 0$ and $y_0 = \pi$ if $x = 0$. If $y_0 \leq \frac{\pi}{2}$, then $\frac{y_0}{\sin y_0} \leq \frac{\pi}{2}$, and one can immediately use Lemma 2.4.11 with $y = y_0$. So assume that $y_0 > \frac{\pi}{2}$ and let $y = y_0 - \epsilon$, with $\epsilon < \frac{\pi}{2}$ which will be specified later.

If $y_0 \neq \pi$, we have

$$\begin{aligned} y \cot y &= y_0 \cot y_0 + y(\cot y - \cot y_0) - \epsilon \cot y_0 \\ &\geq y_0 \cot y_0 - \epsilon \cot y_0, \end{aligned} \tag{2.141}$$

since \cot is monotone decreasing on $(0, \pi)$. Now the use of Lemma 2.4.11 with this y yields:

$$\begin{aligned} |h(x, z)| &\lesssim B_{\nu, m} \left(\frac{\pi}{\pi - y} \right) e^{-y_0 \cot y_0 |x|^2 - 4y_0 z_1 + \frac{\epsilon}{y_0} (y_0 \cot y_0 + 4y_0 z_1)} \\ &\lesssim B_{\nu, m} \left(\frac{\pi}{\epsilon} \right) e^{-\left(1 - \frac{\epsilon}{y_0}\right) |(x, z)|_{CC}^2}, \end{aligned} \quad (2.142)$$

first for $y_0 \neq \pi$. But, if $y_0 = \pi$, which means that $x = 0$, we claim that this estimate (2.142) remains true. Since in this case

$$\begin{aligned} |h(0, z)| &\lesssim B_{\nu, m} \left(\frac{\pi}{\pi - y} \right) e^{-4y_0 z_1 + \frac{\epsilon}{y_0} y_0 z_1} \\ &\lesssim B_{\nu, m} \left(\frac{\pi}{\epsilon} \right) e^{-\left(1 - \frac{\epsilon}{y_0}\right) |(x, z)|_{CC}^2}, \end{aligned}$$

the claim is indeed true.

By setting $\epsilon := \frac{1}{1 + |(x, z)|_{CC}^2}$, we get our desired result:

$$|h(x, z)| \lesssim B_{\nu, m} \left(1 + |(x, z)|_{CC}^2 \right) e^{-|(x, z)|_{CC}^2}.$$

□

Corollary 2.4.15. *There is a constant $C > 0$ such that for all $(x, z) \in \mathbb{H}_{\nu, m}$ the following estimate holds*

$$|k_t(x, z)| \leq C t^{-(\nu+m)} \min \left\{ B_{\nu, m} \left(\frac{|(x, z)|_{CC}}{|x|} \right), B_{\nu, m} \left(1 + \frac{|(x, z)|_{CC}^2}{4t} \right) \right\} e^{-\frac{1}{4t} |(x, z)|_{CC}^2}. \quad (2.143)$$

Proof. The proof is almost trivial if one uses Theorem 2.4.13 and the homogeneity of $|\cdot|_{CC}$.

□

Corollary 2.4.16. *Let $\alpha > 0$ and $\Omega_\alpha = \{(x, z) \in \mathbb{H}_{\nu, m} : |z| \leq \alpha |x|^2\}$. Then there is a constant $C_\alpha > 0$ depending on α such that*

$$|k_t(x, z)| \leq C_\alpha t^{-(\nu+m)} e^{-\frac{1}{4t} |(x, z)|_{CC}^2}, \quad (x, z) \in \Omega_\alpha. \quad (2.144)$$

Proof. This is immediately clear by 2.4.15 since $|(x, z)|_{CC} \sim |x| + \sqrt{|z|}$.

□

3 The Grušin operator

3.1 Introduction

In this chapter we study the geometric properties of the Grušin operator

$$G = \Delta_x + |x|^2 \partial_u^2 \quad \text{on } \mathbb{R}_x^n \times \mathbb{R}_u$$

to give estimates for the heat kernel of this operator. One may write G as

$$G = \sum_{j=1}^n (X_j^2 + U_j^2),$$

with the smooth vector fields

$$X_j := \partial_{x_j}, \quad U_j := x_j \partial_u, \quad j = 1, \dots, n.$$

Since G satisfies the Hörmander condition G is hypoelliptic.

We are interested in the Carnot-Carathéodory distance associated to these vector fields, i.e. the length of a minimizing horizontal curve. As we have seen in Section 1.9 on page 43, the only abnormal extremals are the constant curves. Depending on the start and end point of the curve, there are a different number of normal extremals. In addition to the results of Calin, Chang, Greiner and Kannai in [8], we can compare their lengths to give explicit formulas for the Carnot-Carathéodory distance. The normal extremals starting in $0 \in \mathbb{R}^{n+1}$ are very similar to those on the Heisenberg group (see Section 2.3.1 on page 74 and [3]). But comparing the lengths of normal extremals, starting in a generic point $(x, 0) \in \mathbb{R}^n \times \mathbb{R}$ gets more complicated. Another difference occurs when studying curves connecting $(x, 0)$ and $(\pm x, u)$, where one gets two different types of normal extremals (see Theorem (3.2.12) and (3.2.13)).

These results on the Carnot-Carathéodory distance allow us to give some pointwise estimates of the heat kernel.

Due to the fact that the partial Fourier transform of G in u leads to the rescaled Hermite operators, it is easy to calculate the heat kernel using Mehler's formula, up to the partial Fourier transform. One may also observe that G is translation invariant in the u -variable.

This allows us to write the heat kernel in the form $K_t(x, \xi, u)$, $x, \xi \in \mathbb{R}^n$, $u \in \mathbb{R}$. So the solution to the heat equation

$$\begin{cases} (\partial_t - G) u = 0 \\ u(0, \cdot) = f \end{cases}$$

is given by

$$u(t, x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} K_t(x, \xi, u - \lambda) f(\xi, \lambda) d\lambda d\xi.$$

Now consider the dilations

$$\delta_r(x, u) := (rx, r^2u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}, \quad (3.1)$$

for $r > 0$. Then G is homogeneous with degree 2 and in this setting the homogeneous dimension is $q = n + 2$.

3.2 The sub-Riemannian geometry associated to the Grušin operator

In this section we want to continue the analysis of the sub-Riemannian geometry associated with the singular horizontal distribution \mathcal{D} spanned by the vector fields $X_j, U_j, j = 1, \dots, n$.

Remark 3.2.1. With respect to the dilations (3.1) the Carnot-Carathéodory distance is homogeneous with degree 1, i.e.

$$d_{CC}(\delta_r(\zeta), \delta_r(\eta)) = r d_{CC}(\zeta, \eta), \quad \zeta, \eta \in \mathbb{R}^{n+1}.$$

In Theorem 1.9.1 on page 45 we already proved that the abnormal extremals are just constant curves. Thus every length minimizer parametrized with constant speed for $d_{CC}(p, q)$, $p \neq q$ is a normal extremal.

Theorem 3.2.2. *All normal extremals γ parametrized with constant speed starting in $(x_1, u_1) \in \mathbb{R}^n \times \mathbb{R}$ are given by $\gamma^{b,c} = (\gamma_{(1)}^{b,c}, \gamma_{(2)}^{b,c}) : [0, 1] \rightarrow \mathbb{R}^n \times \mathbb{R}$, where*

$$\gamma_{(1)}^{b,c}(t) = \frac{c}{b} \sin(bt) + x_1 \cos(bt) \quad (3.2a)$$

$$\begin{aligned} \gamma_{(2)}^{b,c}(t) &= \frac{|c|^2}{b} \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) + \frac{x_1 \cdot c}{b} \sin^2(bt) \\ &\quad + |x_1|^2 b \left(\frac{t}{2} + \frac{\sin(2bt)}{4b} \right) + u_1 \end{aligned} \quad (3.2b)$$

with parameters $b \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}^n$, and

$$\gamma^{0,c}(t) = (ct + x_1, u_1) \quad (3.3)$$

which is the limiting case $b \rightarrow 0$. The length is given by

$$L(\gamma^{b,c}) = \sqrt{|c|^2 + |x_1|^2 b^2}. \quad (3.4)$$

Remark. In this chapter we will denote the standard scalar product on \mathbb{R}^n by $x \cdot y$, $x, y \in \mathbb{R}^n$.

Proof. By Theorem 1.9.1 $\gamma = (x, u) : [0, 1] \rightarrow \mathbb{R}^{n+1}$ is a normal extremal iff there exists $b \in \mathbb{R}$ such that

$$\begin{aligned} \ddot{x}(t) &= -b^2 x(t), \\ \dot{u}(t) &= b |x(t)|^2, \end{aligned}$$

for all $t \in [0, 1]$. For $b \neq 0$ the solutions are given by

$$\begin{aligned} x(t) &= \alpha \sin(bt) + \beta \cos(bt), \\ u(t) &= b \left(|\alpha|^2 \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) + \alpha \cdot \beta \sin^2(bt) + |\beta|^2 \left(\frac{t}{2} + \frac{\sin(2bt)}{4b} \right) \right) + \mu \end{aligned}$$

with constants $\alpha, \beta \in \mathbb{R}^n$, $\mu \in \mathbb{R}$. For $b = 0$ the differential equations have the solutions

$$\begin{aligned} x(t) &= ct + \beta \\ u(t) &= \mu, \end{aligned}$$

with $c, \beta \in \mathbb{R}^n$, $\mu \in \mathbb{R}$. Using the initial condition $\gamma(0) = (x_1, u_1)$ leads to $\beta = x_1$ and $\mu = u_1$. Setting $\alpha := \frac{c}{b}$, we see that the normal extremals are indeed given by $\gamma^{b,c}$, $b \in \mathbb{R}$, $c \in \mathbb{R}^n$. The length is given by

$$\begin{aligned} L(\gamma^{b,c}) &= \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt \\ &= \int_0^1 \sqrt{|\dot{x}(t)|^2 + \frac{\dot{u}(t)^2}{|x(t)|^2}} dt \\ &= \int_0^1 \sqrt{|\dot{x}(t)|^2 + b^2 |x(t)|^2} dt. \end{aligned}$$

Since for $b \neq 0$ we have

$$\begin{aligned} |\dot{x}(t)|^2 + b^2 |x(t)|^2 &= b^2 \left| \frac{c}{b} \cos(bt) - x_1 \sin(bt) \right|^2 + b^2 \left| \frac{c}{b} \sin(bt) + x_1 \cos(bt) \right|^2 \\ &= |c|^2 + b^2 |x_1|^2, \end{aligned}$$

and for $b = 0$

$$|\dot{x}(t)|^2 + b^2 |x(t)|^2 = |c|^2$$

we conclude that

$$L(\gamma^{b,c}) = \sqrt{|c|^2 + b^2 |x_1|^2}.$$

□

To calculate the Carnot-Carathéodory distance, we are interested in the normal extremals joining two given points $(x_1, u_1), (x, u) \in \mathbb{R}^n \times \mathbb{R}$.

In our further analysis we will assume that $u_1 = 0$, since γ is a normal extremal connecting $(x_1, 0), (x, u)$, if and only if $\gamma + (0, u_1)$ is a normal extremal connecting $(x_1, u_1), (x, u + u_1)$; with the same length.

We will also assume that $u \geq 0$, since $\gamma_{(1)}^{b,c} = \gamma_{(1)}^{-b,c}$ and $\gamma_{(2)}^{b,c} = -\gamma_{(2)}^{-b,c}$.

Furthermore, observe that given $b \in \mathbb{R} \setminus \pi\mathbb{Z}^*$ the parameter $c \in \mathbb{R}^n$ is uniquely determined by $c = \frac{b}{\sin b} (x - x_1 \cos b)$, due to the boundary conditions. We will denote this extremal by γ^b for short.

Given $b \in \pi\mathbb{Z}^*$ (note that this means $x = \pm x_1$), the parameter $c \in \mathbb{R}^n$ is determined by the whole sphere $|c| = \sqrt{2bu - |x_1|^2 b^2}$. But any $c \in \mathbb{R}^n$ with $|c| = \sqrt{2bu - |x_1|^2 b^2}$ will give the same length.

Theorem 3.2.3. *Given $(x_1, 0), (x, 0) \in \mathbb{R}^n \times \mathbb{R}$, the only normal extremal parametrized with constant speed connecting them is given by*

$$\gamma^0(t) = (x_1 + t(x - x_1), 0) \tag{3.5}$$

and its length, which is equal to the Carnot-Carathéodory distance, is

$$L(\gamma^0) = d_{CC}((x_1, 0), (x, 0)) = |x - x_1|. \tag{3.6}$$

Proof. We have to prove that $b = 0$ is the only parameter that allows connecting $(x_1, 0)$ and $(x, 0)$. But from the differential equations in Theorem 1.9.1, we know that $\dot{\gamma}_{(2)} = b \left| \gamma_{(1)} \right|^2$. That means $\dot{\gamma}_{(2)}(t) \geq 0$ or $\dot{\gamma}_{(2)}(t) \leq 0$ for all $t \geq 0$. But since we want $\gamma_{(2)}(0) = 0$ and $\gamma_{(2)}(1) = 0$, we conclude that $\dot{\gamma}_{(2)} \equiv 0$, so that $b = 0$, hence $c = x - x_1$. □

Theorem 3.2.4. *Given $u > 0$, there are infinitely many normal extremals $\gamma^{m\pi, c_m}$, $m \in \mathbb{N}$, connecting $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ and $(0, u) \in \mathbb{R}^n \times \mathbb{R}$ given by (3.2) with $c_m \in \mathbb{R}^n$, $|c_m| = \sqrt{2m\pi u}$. Their lengths are*

$$L(\gamma^{m\pi, c_m}) = \sqrt{2m\pi u}. \tag{3.7}$$

In particular the shortest normal extremals in this case, which give the Carnot-Carathéodory distance, are γ^{π, c_1} , $|c_1| = \sqrt{2\pi u}$; so

$$d_{CC}((0, 0), (0, u)) = \sqrt{2\pi u}. \tag{3.8}$$

Proof. The boundary conditions $\gamma_{(1)}^{b,c}(1) = 0$ and $\gamma_{(2)}^{b,c}(1) = u$ are equivalent to

$$0 = \frac{c}{b} \sin b \quad \text{and} \quad u = \frac{|c|^2}{2b} \left(1 - \frac{\sin(2b)}{2b} \right).$$

Since $u \neq 0$, we have $c \neq 0$. But then there exists $m \in \mathbb{N}$ with $b = m\pi$, so that $|c|^2 = 2m\pi u$. This gives the claim. \square

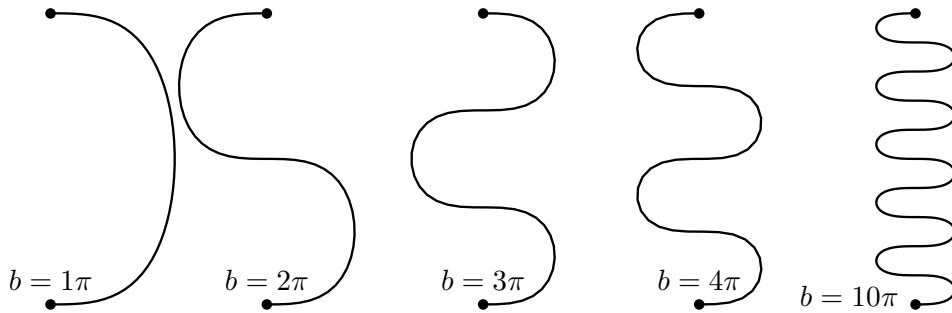


Figure 3.1: Normal extremals starting at $(0, 0)$ and ending in $(0, u)$ with different parameters.

Now we consider the case where we want to connect $(x_1, 0)$ and (x, u) with $x \neq \pm x_1$, $u > 0$. The first lemma is just a rewritten formulation of the boundary conditions $\gamma^{b,c}(0) = (x_1, 0)$ and $\gamma^{b,c}(1) = (x, u)$.

Lemma 3.2.5. *Suppose $(x_1, 0), (x, u) \in \mathbb{R}^n \times \mathbb{R}$, $x \neq \pm x_1$, $u > 0$. Then $\gamma^{b,c}$ given by (3.2) is a normal extremal connecting these two points, iff*

$$c = \frac{b}{\sin b} x - x_1 b \cot b$$

and b is any solution of

$$\frac{2u}{|x_1|^2 + |x|^2} = \frac{b}{\sin^2 b} - \cot b + \frac{2x \cdot x_1}{|x_1|^2 + |x|^2} \frac{1 - b \cot b}{\sin b}. \quad (3.9)$$

The square of the length of $\gamma^b = \gamma^{b,c}$ is then given by

$$L^2(\gamma^{b,c}) = \frac{b^2}{\sin^2 b} (|x_1|^2 + |x|^2 - 2x \cdot x_1 \cos b) \quad (3.10a)$$

$$= 2bu + (|x_1|^2 + |x|^2) b \cot b - 2x \cdot x_1 \frac{b}{\sin b}. \quad (3.10b)$$

Proof. To find those normal extremals connecting two points $(x_1, 0)$ and (x, u) , we have to find $b \neq 0$ and $c \in \mathbb{R}^n$, such that $\gamma_{(1)}^{b,c}(1) = x$ and $\gamma_{(2)}^{b,c}(1) = u$. Since $b \in \pi\mathbb{Z}$ means $x = \pm x_1$, we have that $b \notin \pi\mathbb{Z}$. Now $\gamma_{(2)}^{b,c}(1) = u$ gives:

$$\begin{aligned} 2u &= \frac{|c|^2}{b} \left(1 - \frac{\sin(2b)}{2b}\right) + 2\frac{x_1 \cdot c}{b} \sin^2(b) + |x_1|^2 b \left(1 + \frac{\sin(2b)}{2b}\right) \\ &= \frac{1}{b} (|c|^2 + |x_1|^2 b^2) - \frac{\sin(2b)}{2b^2} (|c|^2 - |x_1|^2 b^2) + 2\frac{x_1 \cdot c}{b} \sin^2 b \end{aligned} \quad (3.11)$$

And $\gamma_{(1)}^{b,c}(1) = x$ gives

$$\frac{b}{\sin b} x = c + x_1 b \cot b, \quad (3.12)$$

so that

$$\begin{aligned} \frac{b^2}{\sin^2 b} |x|^2 &= |c|^2 + 2c \cdot x_1 b \cot b + |x_1|^2 b^2 \cot^2 b \\ &= |c|^2 + 2c \cdot x_1 b \cot b - |x_1|^2 b^2 + \frac{|x_1|^2 b^2}{\sin^2 b} \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} |c|^2 &= \frac{b^2}{\sin^2 b} |x|^2 - 2x_1 \cdot x \frac{b^2 \cot b}{\sin b} + |x_1|^2 b^2 \cot^2 b \\ &= \frac{b^2}{\sin^2 b} (|x|^2 + |x_1|^2) - 2x_1 \cdot x \frac{b^2 \cot b}{\sin b} - |x_1|^2 b^2 \\ &= \frac{b^2}{\sin^2 b} (|x|^2 + |x_1|^2 - 2x \cdot x_1 \cos b) - |x_1|^2 b^2. \end{aligned} \quad (3.14)$$

We can continue with (3.11) by using equation (3.13)

$$\begin{aligned}
 2u &= \frac{1}{b} \left(|c|^2 + |x_1|^2 b^2 \right) - \cot b \left(|x|^2 - |x_1|^2 \right) + 2 \frac{c \cdot x_1}{b} \cos^2 b + 2 \frac{c \cdot x_1}{b} \sin^2 b \\
 &= \frac{1}{b} \left(|c|^2 + |x_1|^2 b^2 \right) - \cot b \left(|x|^2 - |x_1|^2 \right) + 2 \frac{c \cdot x_1}{b} \\
 &= \frac{1}{b} \left(|c|^2 + |x_1|^2 b^2 \right) - \cot b \left(|x|^2 + |x_1|^2 \right) + 2 \frac{c \cdot x_1 + |x_1|^2 b \cot b}{b}.
 \end{aligned}$$

Now use equation (3.12) and (3.14)

$$\begin{aligned}
 2u &= \frac{1}{b} \left(|c|^2 + |x_1|^2 b^2 \right) - \cot b \left(|x|^2 + |x_1|^2 \right) + 2 \frac{x \cdot x_1}{\sin b} \\
 &= \left(\frac{b}{\sin^2 b} - \cot b \right) \left(|x|^2 + |x_1|^2 \right) + 2x \cdot x_1 \frac{1 - b \cot b}{\sin b}.
 \end{aligned}$$

And the square of the length is given by

$$\begin{aligned}
 L^2(\gamma^{b,c}) &= \frac{b^2}{\sin^2 b} \left(|x|^2 + |x_1|^2 - 2x \cdot x_1 \cos b \right) \\
 &= 2bu + b \cot b \left(|x|^2 + |x_1|^2 \right) - 2 \frac{b}{\sin b} x \cdot x_1.
 \end{aligned}$$

□

Definition. For $b \in \mathbb{R} \setminus \pi\mathbb{Z}^*$, $-1 \leq a \leq 1$ define:

$$\mu(b) := \mu(b, a) := \frac{b}{\sin^2 b} - \cot b + a \frac{1 - b \cot b}{\sin b} \quad (3.15)$$

$$l(b) := l(b, a) := \frac{b^2}{\sin^2 b} (1 - a \cos b). \quad (3.16)$$

In the following we will often use the abbreviation

$$R = \sqrt{|x|^2 + |x_1|^2} \text{ and } a = \frac{2x_1 \cdot x}{|x_1|^2 + |x|^2}.$$

Next we will study the functions μ and $\tilde{\mu}$, $\hat{\mu}$, which will be introduced in the next lemma. (Note that $\hat{\mu}$ in this section does not mean Fourier transform!)

Lemma 3.2.6. For $b \in \mathbb{R} \setminus \pi\mathbb{Z}^*$, $0 \leq a \leq 1$, μ is a convex combination, i.e.

$$\mu(b) = (1 - a)\tilde{\mu} + a\hat{\mu} \left(\frac{b}{2} \right), \quad (3.17)$$

and for $-1 \leq a < 0$:

$$\mu(b) = (1 + a)\tilde{\mu}(b) - a\tilde{\mu}\left(\frac{b}{2}\right), \quad (3.18)$$

where

$$\tilde{\mu}(b) := \frac{b}{\sin^2 b} - \cot b, \quad b \in \mathbb{R} \setminus \pi\mathbb{Z}^*, \quad (3.19)$$

$$\hat{\mu}(b) := \frac{b}{\cos^2 b} + \tan b, \quad b \in \mathbb{R} \setminus \left\{ \left(k + \frac{1}{2}\right)\pi : k \in \mathbb{Z} \right\}. \quad (3.20)$$

Proof. For $b \in \mathbb{R} \setminus \pi\mathbb{Z}^*$:

$$\begin{aligned} \mu(b) &= (1 - a)\tilde{\mu}(b) + a \frac{1 - b \cot b + \frac{b}{\sin b} - \cos b}{\sin b} \\ &= (1 - a)\tilde{\mu}(b) + 2a \frac{\sin^2 \frac{b}{2} \left(1 + \frac{b}{\sin b}\right)}{\sin b} \\ &= (1 - a)\tilde{\mu}(b) + a \left(\frac{\frac{b}{2}}{\cos^2 \frac{b}{2}} + \tan \frac{b}{2} \right) \\ &= (1 - a)\tilde{\mu}(b) + a\hat{\mu}\left(\frac{b}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \mu(b) &= (1 + a)\tilde{\mu}(b) - a \frac{b \cot b - 1 + \frac{b}{\sin b} - \cos b}{\sin b} \\ &= (1 + a)\tilde{\mu}(b) - 2a \frac{\cos^2 \frac{b}{2} \left(\frac{b}{\sin b} - 1\right)}{\sin b} \\ &= (1 + a)\tilde{\mu}(b) - a \left(\frac{\frac{b}{2}}{\sin^2 \frac{b}{2}} - \cot \frac{b}{2} \right) \\ &= (1 + a)\tilde{\mu}(b) - a\tilde{\mu}\left(\frac{b}{2}\right) \end{aligned}$$

□

Remark. Observe that all these functions $\mu, \tilde{\mu}, \hat{\mu}$ are odd. The following lemmata will show that $\tilde{\mu}|_{[0, \infty)} \geq 0$, $\hat{\mu}|_{[0, \infty)} \geq 0$ and therefore $\mu|_{[0, \infty)} \geq 0$. Since we assumed that $u \geq 0$, we only need information of these functions for $b \geq 0$.

The function $\tilde{\mu}$, which also appears in the study of normal extremals on the Heisenberg group, where intensively studied by Beals, Gaveau, Greiner in [3]. Furthermore this function already appeared in Section 2.3.1. Recall the following result stated in Lemma 2.3.10:

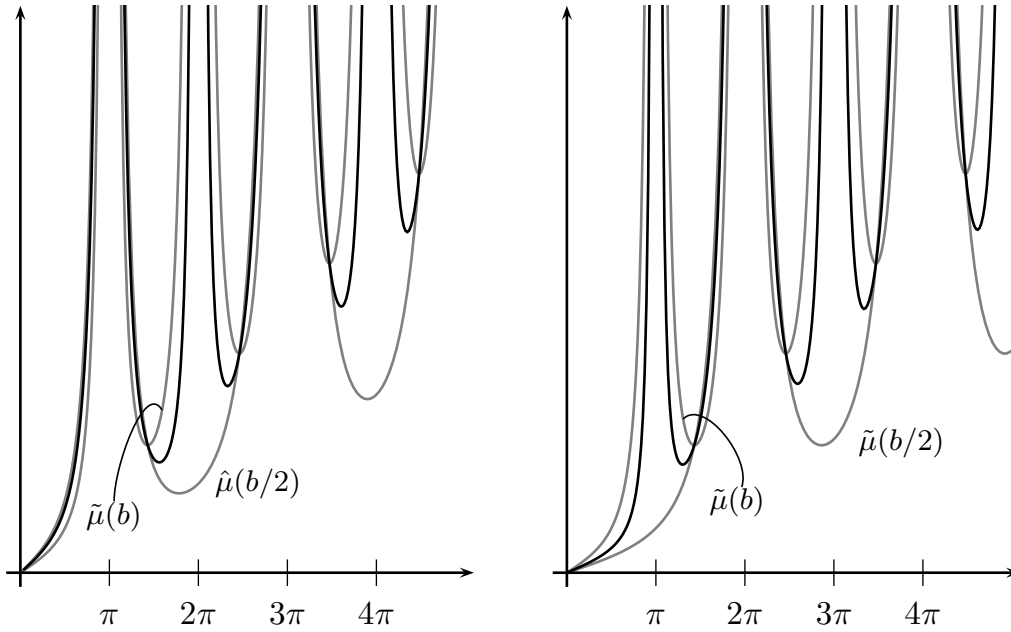


Figure 3.2: left: Plotting of $\tilde{\mu}(b)$ (gray), $\hat{\mu}(b/2)$ (gray) and $\mu(b)$ (black) with parameter $a = 0.7$.
 right: Plotting of $\tilde{\mu}(b)$ (gray), $\tilde{\mu}(b/2)$ (gray) and $\mu(b)$ (black) with parameter $a = -0.7$.

Lemma 3.2.7. *The function $\tilde{\mu}$ is a monotone increasing diffeomorphism of the interval $[0, \pi)$ onto $[0, \infty)$. On each interval $(m\pi, (m + 1)\pi)$, $m \in \mathbb{N}$, $\tilde{\mu}$ has a unique critical point \tilde{b}_m , which is a minimum. \tilde{b}_m is implicitly given as the solution of the equation $1 - b \cot b = 0$ in the corresponding interval. On this interval, $\tilde{\mu}$ decreases strictly from $+\infty$ to $\tilde{\mu}(\tilde{b}_m)$ and then increases strictly to $+\infty$. Moreover,*

$$\tilde{\mu}(\tilde{b}_m) + \pi < \tilde{\mu}(\tilde{b}_{m+1}), \tag{3.21}$$

and

$$\tilde{\mu}(\tilde{b}_m) > m\pi. \tag{3.22}$$

The function $\hat{\mu}$ has been studied by Calin, Chang, Greiner, Kannai in [8]:

Lemma 3.2.8. *The function $\hat{\mu}$ is a monotone increasing diffeomorphism of $[0, \frac{\pi}{2})$ onto $[0, \infty)$. On each interval $(m\pi + \frac{\pi}{2}, (m + 1)\pi + \frac{\pi}{2})$, $m \in \mathbb{N}_0$, $\hat{\mu}$ has an unique critical point \hat{b}_m , which is a minimum. \hat{b}_m is implicitly given as the solution of the equation $1 + b \tan b = 0$*

in the corresponding interval. On this interval, $\hat{\mu}$ decreases strictly from $+\infty$ to $\hat{\mu}(\hat{b}_m)$ and then increases strictly to $+\infty$. Moreover,

$$\hat{\mu}(\hat{b}_m) \geq \pi \left(m + \frac{1}{2} \right). \quad (3.23)$$

Remark. Moreover $\tilde{\mu}''(b) > 0$, $b \notin \pi\mathbb{N}$ and $\hat{\mu}''(b) > 0$, $b - \frac{\pi}{2} \notin \mathbb{N}_0$, i.e. $\tilde{\mu}$ is strictly convex on each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$ and $\hat{\mu}$ is strictly convex on each interval $(m\pi + \frac{\pi}{2}, (m+1)\pi + \frac{\pi}{2})$, $m \in \mathbb{N}_0$. This is another result of [3, 8]. Hence μ , as a convex combination, is also strictly convex on each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$.

Combining the previous two lemmata we get the following result for the function μ :

Lemma 3.2.9. *Let $-1 < a < 1$. The function μ is monotone increasing on $[0, \pi)$ onto $[0, \infty)$. On each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, μ has an unique critical point b_m , which is a minimum. For $a \geq 0$ and $m \in \mathbb{N}_0$*

$$\begin{aligned} (2m+1)\pi &< \tilde{b}_{2m+1} \leq b_{2m+1} \leq 2\hat{b}_m < (2m+2)\pi, \\ (2m+2)\pi &< b_{2m+2} \leq \tilde{b}_{2m+2} < (2m+3)\pi; \end{aligned} \quad (3.24)$$

and for $a < 0$ and $m \in \mathbb{N}$

$$\begin{aligned} 2m\pi &< \tilde{b}_{2m} \leq b_{2m} \leq 2\tilde{b}_m < (2m+1)\pi, \\ (2m+1)\pi &< b_{2m+1} \leq \tilde{b}_{2m+1} < (2m+2)\pi. \end{aligned} \quad (3.25)$$

On each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, μ decreases strictly from $+\infty$ to $\mu(b_m)$ and then increases strictly to $+\infty$. Moreover for all $m \in \mathbb{N}$,

$$\mu(b_m) \geq \frac{m-1}{2}\pi, \quad (3.26)$$

$$l(b_m) - b_m\mu(b_m) = 1 - a\delta(b_m), \quad (3.27)$$

where

$$\delta(b) := \cos b + \frac{b}{2} \sin b. \quad (3.28)$$

Proof. It is easy to see that $m\pi < \tilde{b}_m < m\pi + \frac{\pi}{2} < \hat{b}_m < (m+1)\pi$, see figure 3.3.

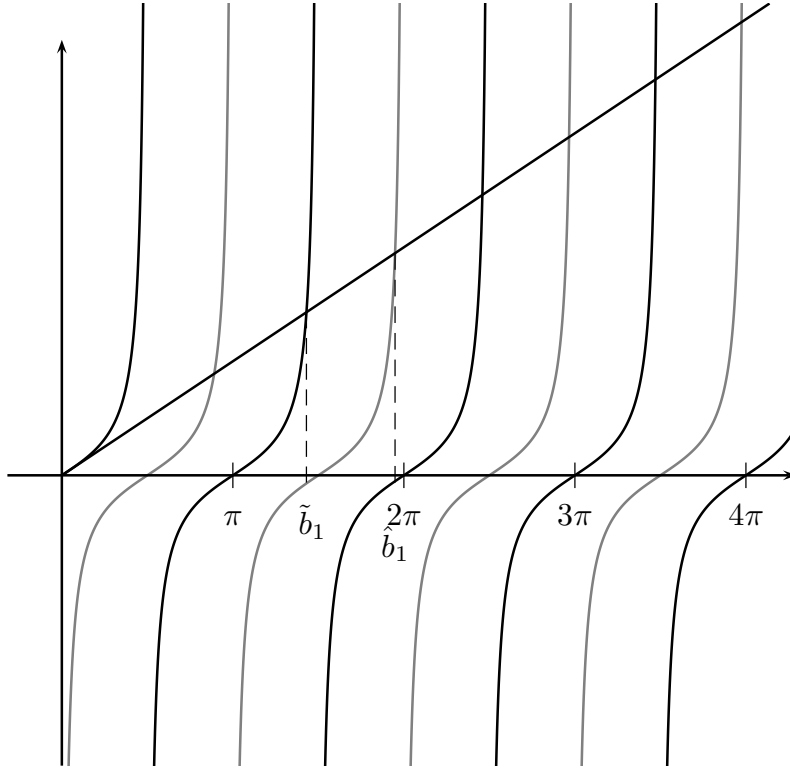


Figure 3.3: The graphs of the functions \tan (black) and $-\cot$ (gray) and the line $x \mapsto x$.

Since

$$\begin{aligned}
 1 - b \cot b &= 1 + \frac{b}{2} \tan \frac{b}{2} - \frac{b}{2} \cot \frac{b}{2} \\
 &\geq \begin{cases} 1 + \frac{b}{2} \tan \frac{b}{2} & b \in ((2m+1)\pi, (2m+2)\pi) \\ 1 - \frac{b}{2} \cot \frac{b}{2} & b \in (2m\pi, (2m+1)\pi) \end{cases}
 \end{aligned}$$

and $\tilde{b}_{2m+1}, 2\hat{b}_m \in ((2m+1)\pi, (2m+2)\pi)$ it follows that $\tilde{b}_{2m+1} < 2\hat{b}_m$; also, since $\tilde{b}_{2m}, 2\tilde{b}_m \in (2m\pi, (2m+1)\pi)$, it follows that $\tilde{b}_{2m} < 2\tilde{b}_m$.

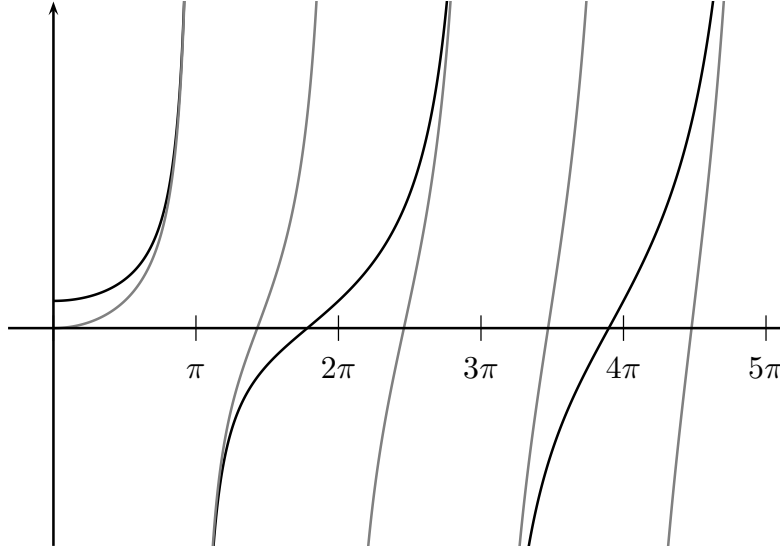


Figure 3.4: The graphs of the functions $x \mapsto 1 + \frac{x}{2} \tan \frac{x}{2}$ (black) and $x \mapsto 1 - x \cot x$ (gray).

If $a \geq 0$, μ' is a convex combination of $\tilde{\mu}'$ and $\frac{1}{2}\hat{\mu}'\left(\frac{b}{2}\right)$. On the interval $((2m+1)\pi, (2m+2)\pi)$ both functions are monotone increasing with zero \tilde{b}_{2m+1} of $\tilde{\mu}'$ lying left of $2\hat{b}_m$, so we have

$$\mu'(\tilde{b}_{2m+1}) = \frac{a}{2}\hat{\mu}'\left(\frac{\tilde{b}_{2m+1}}{2}\right) \leq 0 \text{ and } \mu'(2\hat{b}_m) = (1-a)\tilde{\mu}'(2\hat{b}_m) > 0.$$

Hence μ' has a zero b_m in between: $\tilde{b}_{2m+1} \leq b_m < 2\hat{b}_m$. Due to the fact that μ is strictly convex, this critical point is unique in the interval $((2m+1)\pi, (2m+2)\pi)$, and also a minimum.

In the interval $((2m+2)\pi, (2m+3)\pi)$ still both functions are monotone increasing, but only $\tilde{\mu}'$ has a zero at \tilde{b}_{2m+2} . $\frac{1}{2}\hat{\mu}'\left(\frac{b}{2}\right)$ is positive on this interval and since $\tilde{\mu}'(b)$ tends to $-\infty$ as $b \xrightarrow{b > (2m+2)\pi} (2m+2)\pi$, there is a zero b_{2m+2} of μ' in this interval $((2m+2)\pi, (2m+3)\pi)$ with $(2m+2)\pi \leq b_{2m+2} < \tilde{b}_{2m+2}$. This critical point is unique and a minimum, since μ is convex.

The case $a < 0$ is similar, if we use $\frac{1}{2}\tilde{\mu}\left(\frac{b}{2}\right)$ instead of $\frac{1}{2}\hat{\mu}\left(\frac{b}{2}\right)$ and the fact that in the interval $(2m\pi, (2m+1)\pi)$: $\tilde{b}_{2m} < 2\hat{b}_m$ are zeros of these two functions and $\frac{1}{2}\tilde{\mu}\left(\frac{b}{2}\right)$ has no zero in $((2m+1)\pi, (2m+2)\pi)$.

To obtain the lower bound (3.26) on μ , we consider the following cases:

1. case $a \geq 0$ and m even:

Since $\frac{b_m}{2} \in \left(\left(\frac{m-2}{2} + \frac{1}{2} \right) \pi, \left(\frac{m}{2} + \frac{1}{2} \right) \pi \right)$, we have

$$\begin{aligned} \mu(b_m) &= (1-a)\tilde{\mu}(b_m) + a\hat{\mu}\left(\frac{b_m}{2}\right) \\ &\geq (1-a)m\pi + a\frac{m-1}{2}\pi \\ &\geq \frac{m-1}{2}\pi. \end{aligned}$$

2. case $a \geq 0$ and m odd:

Since $\frac{b_m}{2} \in \left(\left(\frac{m-1}{2} + \frac{1}{2} \right) \pi, \frac{m+1}{2}\pi \right)$, we have

$$\begin{aligned} \mu(b_m) &\geq (1-a)m\pi + a\frac{m}{2}\pi \\ &\geq \frac{m}{2}\pi > \frac{m-1}{2}\pi. \end{aligned}$$

3. case $a < 0$ and m even:

Since $\frac{b_m}{2} \in \left(\frac{m}{2}\pi, \frac{m+1}{2}\pi \right)$, we have

$$\begin{aligned} \mu(b_m) &= (1+a)\tilde{\mu}(b_m) - a\tilde{\mu}\left(\frac{b_m}{2}\right) \\ &\geq (1+a)m\pi - a\frac{m}{2}\pi \\ &\geq \frac{m}{2}\pi > \frac{m-1}{2}\pi. \end{aligned}$$

4. case $a < 0$ and m odd:

Since $\frac{b_m}{2} \in \left(\frac{m-1}{2}\pi, \frac{m+1}{2}\pi \right)$, we have

$$\begin{aligned} \mu(b_m) &\geq (1+a)m\pi - a\frac{m-1}{2}\pi \\ &\geq \frac{m-1}{2}\pi. \end{aligned}$$

At last, to establish the equation (3.27), we need some more information of b_m , i.e. the critical points of μ . First observe, that for any $b \in \mathbb{R} \setminus \pi\mathbb{Z}^*$:

$$b\mu(b) + b \cot b - \frac{b}{\sin b}a = l(b) \tag{3.29}$$

and

$$\begin{aligned} \frac{d}{db} \left(b \cot b - \frac{b}{\sin b} a \right) &= \cot b - \frac{b}{\sin^2 b} - a \frac{1 - b \cot b}{\sin b} \\ &= -\mu(b). \end{aligned}$$

So

$$\begin{aligned} l'(b) &= \mu(b) + b\mu'(b) - \mu(b) \\ &= b\mu'(b). \end{aligned} \tag{3.30}$$

To find the critical points of μ , we differentiate:

$$\begin{aligned} \mu'(b) &= 2 \frac{1 - b \cot b}{\sin^2 b} + a \frac{\frac{b}{\sin b} - \cos b - \cos b(1 - b \cot b)}{\sin^2 b} \\ &= 2 \frac{1 - b \cot b}{\sin^2 b} + a \frac{b \sin b - 2 \cos b(1 - b \cot b)}{\sin^2 b}. \end{aligned}$$

So

$$\mu'(b) = 0 \Leftrightarrow a = \frac{1 - b \cot b}{\cos b(1 - b \cot b) - \frac{b}{2} \sin b}. \tag{3.31}$$

From this and (3.29) we get:

$$\begin{aligned} l(b_m) - b_m \mu(b_m) &= b_m \cot b_m - \frac{b_m}{\sin b_m} a \\ &= 1 - (1 - b_m \cot b_m) - \frac{b_m}{\sin b_m} a \\ &= 1 - a \left(\cos b_m (1 - b_m \cot b_m) - \frac{b_m}{2} \sin b_m + \frac{b_m}{\sin b_m} \right) \\ &= 1 - a \left(\cos b_m (1 - b_m \cot b_m) + \frac{b_m}{2} \sin b_m + b_m \cos b_m \cot b_m \right) \\ &= 1 - a \left(\cos b_m + \frac{b_m}{2} \sin b_m \right) \\ &= 1 - a\delta(b_m). \end{aligned}$$

□

For the proof of the next theorem we need another lemma:

Lemma 3.2.10. *On each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$ δ has a unique zero, and a unique critical point at \tilde{b}_m , which is a maximum with $\delta(\tilde{b}_m) > 0$, if m is even and a minimum with $\delta(\tilde{b}_m) < 0$, if m is odd .*

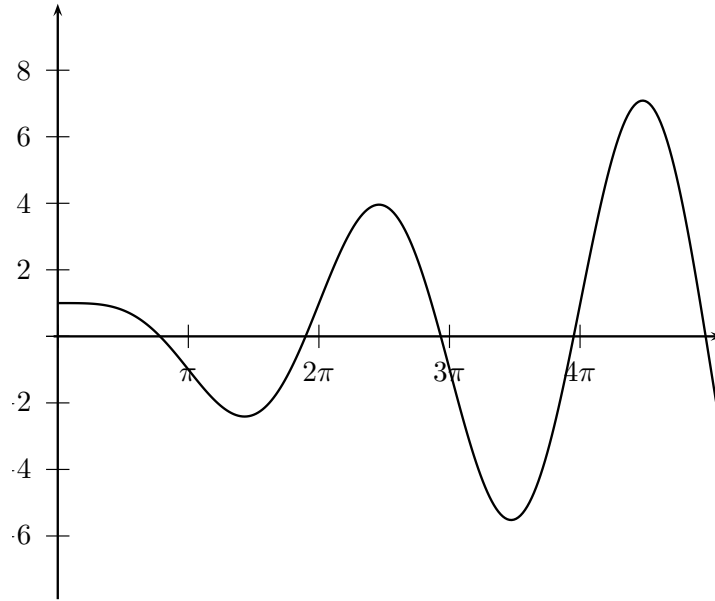


Figure 3.5: The function δ .

Proof. Since

$$\begin{aligned}\delta(b) = 0 &\Leftrightarrow \cos b + \frac{b}{2} \sin b = 0 \\ &\Leftrightarrow 2 + b \tan b = 0,\end{aligned}$$

there is a unique zero in each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}_0$.

The critical points of δ are given by

$$\begin{aligned}\delta'(b) = 0 &\Leftrightarrow -\sin b + \frac{1}{2} \sin b + \frac{b}{2} \cos b = 0 \\ &\Leftrightarrow b \cot b - 1 = 0 \\ &\Leftrightarrow b = \tilde{b}_m.\end{aligned}$$

Since

$$\delta''(b) = -\frac{b}{2} \sin b, \tag{3.32}$$

we see that \tilde{b}_m is a maximum, if m is even and a minimum, if m is odd. \square

Theorem 3.2.11. *Given $(x_1, 0), (x, u) \in \mathbb{R}^n \times \mathbb{R}$ with $x_1 \neq \pm x$ and $u > 0$, there are finitely many normal extremals parametrized with constant speed $\gamma^{b,c}$ joining these two points. These normal extremals are given by (3.2), where b is a solution of*

$$\frac{2u}{|x_1|^2 + |x|^2} = \mu(b), \tag{3.33}$$

and their lengths are strictly increasing with b . Moreover the shortest normal extremal joining $(x_1, 0)$ and (x, u) is given by the unique solution $b \in (0, \pi)$ of (3.33) in the interval $(0, \pi)$. With this solution b the Carnot-Carathéodory distance is

$$\begin{aligned} d_{CC}((x_1, 0), (x, u)) &= L(\gamma^b) \\ &= \frac{b}{\sin b} \sqrt{|x_1|^2 + |x|^2 - 2x_1 \cdot x \cos b} \\ &= \sqrt{2bu + (|x_1|^2 + |x|^2) b \cot b - 2\frac{b}{\sin b} x_1 \cdot x}. \end{aligned} \tag{3.34}$$

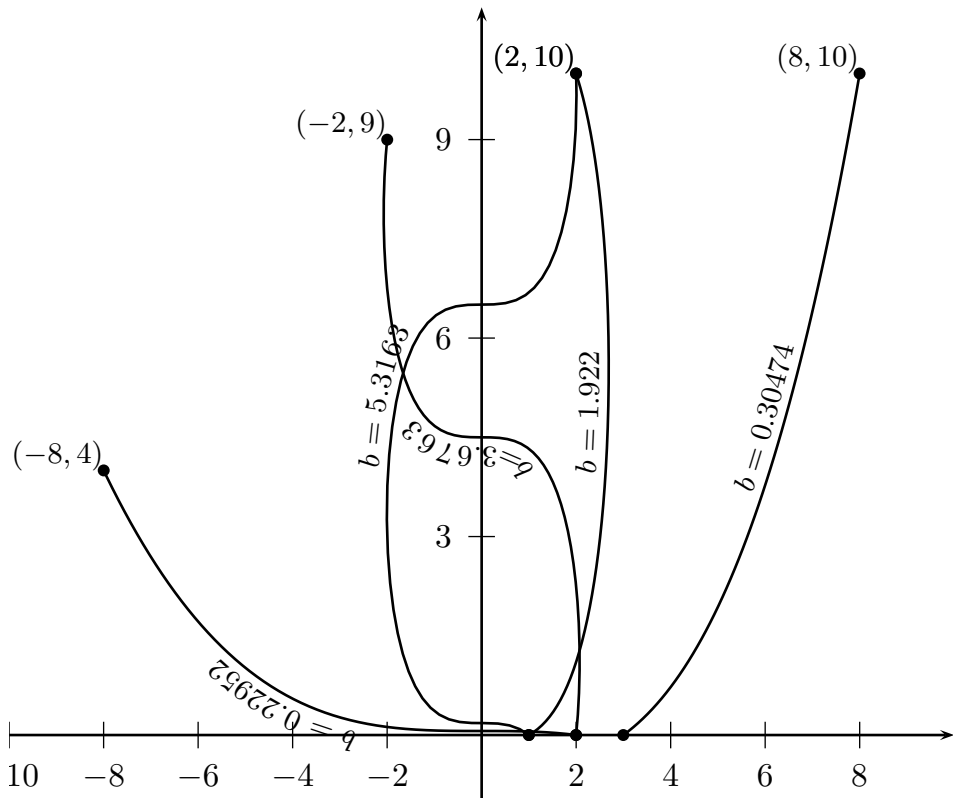


Figure 3.6: Normal extremals joining different points.

Proof. From Lemma 3.2.5 we have that the normal extremals in this case are implicitly given by solutions of the equation $\frac{2u}{|x_1|^2+|x|^2} = \mu(b)$. The properties of the function μ show us that there is exactly one solution in the interval $(0, \pi)$ and at most two solutions in each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$. So it remains to prove, that the values of l at these solutions are strictly increasing, i.e. if $s < t$ are two solutions, then $l(s) < l(t)$.

Let $s \in (m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, then we have by (3.30)

$$\begin{aligned} l(s) &= \int_{b_m}^s l'(b)db + l(b_m) \\ &= \int_{b_m}^s b\mu'(b)db + l(b_m) \\ &= b\mu(b)|_{b_m}^s - \int_{b_m}^s \mu(b)db + l(b_m) \\ &= s\mu(s) - \int_{b_m}^s \mu(b)db + l(b_m) - b_m\mu(b_m) \\ &= s\mu(s) - \int_{b_m}^s \mu(b)db + 1 - a\delta(b_m). \end{aligned}$$

So, if we have two solutions $s \leq t$ of (3.33) in the same interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, we see that

$$\begin{aligned} l(s) - l(t) &= s\mu(s) - \int_{b_m}^s \mu(b)db - t\mu(t) + \int_{b_m}^t \mu(b)db \\ &= (s-t)\mu(s) + \int_s^t \mu(b)db \\ &\leq 0. \end{aligned}$$

Now let $s_m, t_m \in (m\pi, (m+1)\pi)$, $s_m \leq t_m$, $m \in \mathbb{N}$ be the solutions of (3.33). To compare the corresponding “lengths” l at t_m and s_{m+1} , notice that

$$\begin{aligned} l(t_m) - l(s_{m+1}) &= -(s_{m+1} - t_m)\mu(t_m) - \int_{b_m}^{t_m} \mu(b)db - \int_{s_{m+1}}^{b_{m+1}} \mu(b)db - (\delta(b_m) - \delta(b_{m+1}))a. \quad (3.35) \end{aligned}$$

Now consider the following cases:

1. case $a \geq 0$ and m even:

Then, since $m\pi < b_m \leq \tilde{b}_m$, $\delta(m\pi) = 1$ and \tilde{b}_m is a maximum of δ , it follows that $\delta(b_m) > 1$. And, since $\tilde{b}_{m+1} \leq b_{m+1} < (m+2)\pi$, $\delta((m+2)\pi) = 1$ and \tilde{b}_{m+1} is a

minimum of δ , we have $\delta(b_m) < 1$. So $(\delta(b_m) - \delta(b_{m+1}))a \geq 0$, which gives the claim, since $\mu \geq 0$.

2. case $a < 0$ and m odd:

Then, since $m\pi < b_m \leq \tilde{b}_m$, $\delta(m\pi) = -1$ and \tilde{b}_m is a minimum of δ , it follows that $\delta(b_m) < -1$. And, since $\tilde{b}_{m+1} \leq b_{m+1} < (m+2)\pi$, $\delta((m+2)\pi) = -1$ and \tilde{b}_{m+1} is a maximum of δ , we have $\delta(b_m) > -1$. So $(\delta(b_m) - \delta(b_{m+1}))a \geq 0$, which gives the claim, since $\mu \geq 0$.

3. case $a \geq 0$ and m odd:

Here $b_m \leq 2\hat{b}_{\frac{m-1}{2}} < s_{m+1} \leq b_{m+1}$. Let

$$s := \begin{cases} t_m & , \text{if } \hat{\mu}\left(\frac{t_m}{2}\right) > \hat{\mu}\left(\frac{s_{m+1}}{2}\right) \\ s_{m+1} & , \text{if } \hat{\mu}\left(\frac{t_m}{2}\right) \leq \hat{\mu}\left(\frac{s_{m+1}}{2}\right) \end{cases}.$$

Then, since $2\hat{b}_{\frac{m-1}{2}}$ is a minimum of $\hat{\mu}\left(\frac{b}{2}\right)$ on $(m\pi, (m+2)\pi)$ and monotone on the left and right of $2\hat{b}_{\frac{m-1}{2}}$: $\max_{b \in [t_m, s_{m+1}]} \hat{\mu}\left(\frac{b}{2}\right) = \hat{\mu}\left(\frac{s}{2}\right)$ and therefore

$$\begin{aligned} -(s_{m+1} - t_m)\mu(t_m) &= -(s_{m+1} - t_m)\mu(s) \\ &= -(s_{m+1} - t_m) \left((1-a)\tilde{\mu}(s) + a\hat{\mu}\left(\frac{s}{2}\right) \right) \\ &\leq -(s_{m+1} - t_m) a\hat{\mu}\left(\frac{s}{2}\right) \\ &\leq -a \int_{t_m}^{s_{m+1}} \hat{\mu}\left(\frac{b}{2}\right) db. \end{aligned}$$

Notice that $\hat{\mu}\left(\frac{b}{2}\right)$ has no singularity at $(m+1)\pi$. Now (3.35) becomes, since $\mu(b) \geq a\hat{\mu}\left(\frac{b}{2}\right)$:

$$\begin{aligned} l(t_m) - l(s_{m+1}) &\leq -a \int_{b_m}^{b_{m+1}} \hat{\mu}\left(\frac{b}{2}\right) db - (\delta(b_m) - \delta(b_{m+1}))a \\ &= -a \int_{b_m}^{b_{m+1}} \left(\frac{b/2}{\cos^2 \frac{b}{2}} + \tan \frac{b}{2} \right) db - (\delta(b_m) - \delta(b_{m+1}))a \\ &= -ab \tan \frac{b}{2} \Big|_{b_m}^{b_{m+1}} - (\delta(b_m) - \delta(b_{m+1}))a \\ &= a \left(\delta(b_{m+1}) - b_{m+1} \tan \frac{b_{m+1}}{2} - \left(\delta(b_m) - b_m \tan \frac{b_m}{2} \right) \right). \end{aligned}$$

Further

$$\begin{aligned}
 \delta(b) - b \tan \frac{b}{2} &= \cos b + \frac{b}{2} \sin b - b \tan \frac{b}{2} \\
 &= b \sin \frac{b}{2} \left(\cos \frac{b}{2} - \frac{1}{\cos \frac{b}{2}} \right) - 2 \sin^2 \frac{b}{2} + 1 \\
 &= - \left(b \sin^2 \frac{b}{2} \tan \frac{b}{2} + 2 \sin^2 \frac{b}{2} \right) + 1 \\
 &= -2 \sin^2 \frac{b}{2} \left(1 + \frac{b}{2} \tan \frac{b}{2} \right) + 1.
 \end{aligned}$$

Since $b_m \leq 2\hat{b}_{\frac{m-1}{2}} \leq b_{m+1}$ and $1 + \hat{b}_{\frac{m-1}{2}} \tan \hat{b}_{\frac{m-1}{2}} = 0$ and $b \mapsto 1 + \frac{b}{2} \tan \frac{b}{2}$ is increasing on $(m\pi, (m+2)\pi)$, we conclude:

$$\begin{aligned}
 &l(t_m) - l(s_{m+1}) \\
 &\leq 2a \left(\sin^2 \frac{b_m}{2} \left(1 + \frac{b_m}{2} \tan \frac{b_m}{2} \right) - \sin^2 \frac{b_{m+1}}{2} \left(1 + \frac{b_{m+1}}{2} \tan \frac{b_{m+1}}{2} \right) \right) \\
 &\leq 0.
 \end{aligned}$$

4. case $a < 0$ and m even:

This is very similar to the third case, if we use $\tilde{\mu}$. Here $b_m \leq 2\tilde{b}_{\frac{m}{2}} < s_{m+1} \leq b_{m+1}$. Let

$$s := \begin{cases} t_m & , \text{ if } \tilde{\mu} \left(\frac{t_m}{2} \right) > \tilde{\mu} \left(\frac{s_{m+1}}{2} \right) \\ s_{m+1} & , \text{ if } \tilde{\mu} \left(\frac{t_m}{2} \right) \leq \tilde{\mu} \left(\frac{s_{m+1}}{2} \right) \end{cases}.$$

Then, since $2\tilde{b}_{\frac{m}{2}}$ is a minimum of $\tilde{\mu} \left(\frac{b}{2} \right)$ on $(m\pi, (m+2)\pi)$ and monotone on the left and right of $2\tilde{b}_{\frac{m}{2}}$: $\max_{b \in [t_m, s_{m+1}]} \tilde{\mu} \left(\frac{b}{2} \right) = \tilde{\mu} \left(\frac{s}{2} \right)$ and therefore

$$\begin{aligned}
 -(s_{m+1} - t_m)\mu(t_m) &= -(s_{m+1} - t_m)\mu(s) \\
 &= -(s_{m+1} - t_m) \left((1+a)\tilde{\mu}(s) - a\tilde{\mu} \left(\frac{s}{2} \right) \right) \\
 &\leq (s_{m+1} - t_m) a\tilde{\mu} \left(\frac{s}{2} \right) \\
 &\leq a \int_{t_m}^{s_{m+1}} \tilde{\mu} \left(\frac{b}{2} \right) db.
 \end{aligned}$$

Notice that $\tilde{\mu}(\frac{b}{2})$ has no singularity at $(m+1)\pi$. Now (3.35) becomes, since $\mu(b) \geq -a\tilde{\mu}(b/2)$:

$$\begin{aligned} l(t_m) - l(s_{m+1}) &\leq a \left(\int_{b_m}^{b_{m+1}} \tilde{\mu} \left(\frac{b}{2} \right) db - \delta(b_m) + \delta(b_{m+1}) \right) \\ &= a \left(\int_{b_m}^{b_{m+1}} \left(\frac{b/2}{\sin^2 \frac{b}{2}} - \cot \frac{b}{2} \right) db - \delta(b_m) + \delta(b_{m+1}) \right) \\ &= a \left(-b \cot \frac{b}{2} \Big|_{b_m}^{b_{m+1}} - \delta(b_m) + \delta(b_{m+1}) \right) \\ &= a \left(b_m \cot \frac{b_m}{2} - \delta(b_m) - (b_{m+1} \cot \frac{b_{m+1}}{2} - \delta(b_{m+1})) \right). \end{aligned}$$

Further

$$\begin{aligned} b \cot \frac{b}{2} - \delta(b) &= -\cos b - \frac{b}{2} \sin b + b \cot \frac{b}{2} \\ &= -b \cos \frac{b}{2} \left(\sin \frac{b}{2} - \frac{1}{\sin \frac{b}{2}} \right) - 2 \cos^2 \frac{b}{2} + 1 \\ &= -2 \cos^2 \frac{b}{2} \left(1 - \frac{b}{2} \cot \frac{b}{2} \right) + 1. \end{aligned}$$

Since $b_m \leq 2\tilde{b}_{\frac{m}{2}} \leq b_{m+1}$ and $1 - \tilde{b}_{\frac{m}{2}} \cot \tilde{b}_{\frac{m}{2}} = 0$ and $b \mapsto 1 - \frac{b}{2} \cot \frac{b}{2}$ is increasing on $(m\pi, (m+2)\pi)$, we conclude:

$$\begin{aligned} l(t_m) - l(s_{m+1}) &\leq -2a \left(\cos^2 \frac{b_m}{2} \left(1 - \frac{b_m}{2} \cot \frac{b_m}{2} \right) - \cos^2 \frac{b_{m+1}}{2} \left(1 - \frac{b_{m+1}}{2} \cot \frac{b_{m+1}}{2} \right) \right) \\ &\leq 0. \end{aligned}$$

If $t_0 \in (0, \pi)$ denotes the solution of (3.33), it remains to show that $l(t_0) < l(s_1)$. The above calculations can be extended to this, if one sets $b_0 = 0$. In detail we have

$$\begin{aligned} l(t_0) &= \int_0^{t_0} l'(b) db + l(0) \\ &= t_0 \mu(t_0) - \int_0^{t_0} \mu(b) db + 1 - a, \end{aligned}$$

so that

$$l(t_0) - l(s_1) = -(s_1 - t_0)\mu(s_1) - \int_0^{t_0} \mu(b) db - \int_{s_1}^{b_1} \mu(b) db - (1 - \delta(b_1)) a.$$

Again from the previous lemma one gets $l(t_0) < l(s_1)$ if $a \geq 0$, since $1 - \delta(b_1) > 0$ and $\mu > 0$. If $a < 0$, then since $\tilde{\mu}$ is increasing on $[0, \pi)$:

$$\begin{aligned} -(s_1 - t_0)\mu(s_1) &= -(s_1 - t_0) \left((1 + a)\tilde{\mu}(s_1) - a\tilde{\mu}\left(\frac{s_1}{2}\right) \right) \\ &\leq (s_1 - t_0)a\tilde{\mu}\left(\frac{s_1}{2}\right) \\ &\leq a \int_{t_0}^{s_1} \tilde{\mu}\left(\frac{b}{2}\right) db. \end{aligned}$$

So we get

$$\begin{aligned} l(t_0) - l(s_1) &\leq a \int_0^{b_1} \tilde{\mu}\left(\frac{b}{2}\right) db - (1 - \delta(b_1))a \\ &= a \left(-b \cot \frac{b}{2} \Big|_0^{b_1} - 1 + \delta(b_1) \right) \\ &= a \left(\delta(b_1) - b_1 \cot \frac{b_1}{2} + 1 \right) \\ &= 2a \cos^2 \frac{b_1}{2} \left(1 - \frac{b_1}{2} \cot \frac{b_1}{2} \right) \\ &\leq 0, \end{aligned}$$

since $1 - \frac{b}{2} \cot \frac{b}{2} \geq 0$ on $[0, 2\pi)$.

□

Theorem 3.2.12. *Given $(x, 0), (x, u) \in \mathbb{R}^n \times \mathbb{R}$, $x \neq 0$, $u > 0$, there are finitely many normal extremals parametrized with constant speed connecting $(x, 0)$, (x, u) . Namely:*

1. For any solution b of

$$\frac{u}{|x|^2} = \hat{\mu}\left(\frac{b}{2}\right) \tag{3.36}$$

there is a normal extremal γ^b given by (3.2) with square of length

$$L(\gamma^b)^2 = \frac{b^2}{\cos^2 \frac{b}{2}} |x|^2 = 2bu - 2|x|^2 b \tan \frac{b}{2}, \tag{3.37}$$

which strictly increases with b .

2. If $u \geq |x|^2 \pi$, then for each $m \in \mathbb{N}$ with $u \geq m\pi |x|^2$ there are normal extremals $\gamma^{b,c}$ given by (3.2) with $b = 2m\pi$, $|c| = 2\sqrt{m\pi u - (m\pi)^2 |x|^2}$, with lengths

$$L(\gamma^{b,c}) = 2\sqrt{m\pi u}. \tag{3.38}$$

Moreover the shortest normal extremal and therefore the Carnot-Carathéodory distance is given by the unique solution b of (3.36) in the interval $(0, \pi)$, with this b we have

$$d_{CC}((x, 0), (x, u)) = \frac{b}{\cos \frac{b}{2}} |x| = \sqrt{2bu - 2|x|^2 b \tan \frac{b}{2}}. \quad (3.39)$$

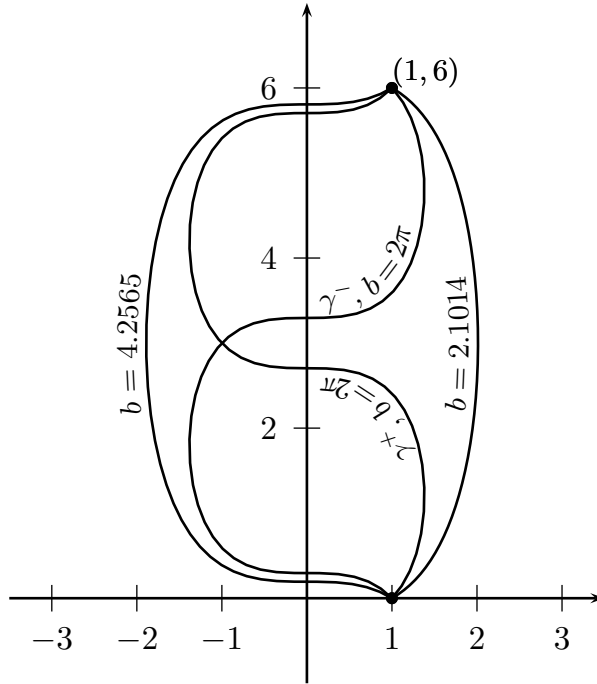


Figure 3.7: Normal extremals joining $(1, 0)$ and $(1, 6)$ with different $b \notin \pi\mathbb{Z}^*$ vs. the two normal extremals γ^\pm with $b = 2\pi$ and $c = \pm\sqrt{2bu - x_1^2 b^2}$.

Proof. Notice that in this case $a = 1$. We will consider two cases:

1. case $b \in \pi\mathbb{N}$:

Then since $\gamma_{(1)}^{b,c}(1) = x$, there exists $m \in \mathbb{N}$ with $b = 2m\pi$. Then $\gamma_{(2)}^{b,c}(1) = u$ is equivalent to

$$u = (|c|^2 + (2m\pi)^2 |x|^2) \frac{1}{4m\pi}.$$

So if $u \geq m\pi |x|^2$, there is a normal extremal given by (3.2) with

$$b = 2m\pi, \quad |c| = 2\sqrt{m\pi u - (m\pi)^2 |x|^2},$$

with length:

$$L^2(\gamma^{b,c}) = |c|^2 + |x|^2 b^2 = 4m\pi u$$

This gives the second part of the theorem.

2. case $b \notin \pi\mathbb{N}$:

In this case we have

$$\mu(b) = \hat{\mu}\left(\frac{b}{2}\right).$$

This is very similar to the situation in Theorem 3.2.11, but much easier, since we are only dealing with $\hat{\mu}$. From Lemma 3.2.8 we know that the equation

$$\frac{u}{|x|^2} = \hat{\mu}\left(\frac{b}{2}\right)$$

has at most two solutions in an interval $((2m+1)\pi, (2m+3)\pi)$, $m \in \mathbb{N}_0$, each defining a normal extremal. If s is a solution of this equation, the square of the length of the corresponding normal extremal γ is given by

$$\begin{aligned} L^2(\gamma) &= 2l(s)|x|^2 \\ &= 2\frac{s^2}{\sin^2 s}(1 - \cos s)|x|^2 \\ &= \frac{s^2}{\cos^2 \frac{s}{2}}|x|^2. \end{aligned}$$

If $s \in ((2m+1)\pi, (2m+3)\pi)$, $m \in \mathbb{N}_0$, we have

$$\begin{aligned} l(s) &= \int_{2\hat{b}_m}^s l'(b)db + l(2\hat{b}_m) \\ &= \int_{2\hat{b}_m}^s b\mu'(b)db + l(2\hat{b}_m) \\ &= \int_{2\hat{b}_m}^s \frac{b}{2}\hat{\mu}'\left(\frac{b}{2}\right)db + l(2\hat{b}_m) \\ &= s\hat{\mu}\left(\frac{s}{2}\right) - 2\hat{b}_m\hat{\mu}(\hat{b}_m) - \int_{2\hat{b}_m}^s \hat{\mu}\left(\frac{b}{2}\right)db + l(2\hat{b}_m) \end{aligned} \quad (3.40)$$

But now

$$\begin{aligned}
 l(2\hat{b}_m) - 2\hat{b}_m\hat{\mu}(\hat{b}_m) &= \frac{2\hat{b}_m^2}{\cos^2\hat{b}_m} - 2\hat{b}_m \left(\frac{\hat{b}_m}{\cos^2\hat{b}_m} + \tan\hat{b}_m \right) \\
 &= -2\hat{b}_m \tan\hat{b}_m \\
 &= 2.
 \end{aligned} \tag{3.41}$$

First let $s \leq t$ be two solutions in an interval $((2m+1)\pi, (2m+3)\pi)$, $m \in \mathbb{N}_0$. Then by (3.40) and (3.41)

$$\begin{aligned}
 l(s) - l(t) &= s\hat{\mu}\left(\frac{s}{2}\right) - \int_{2\hat{b}_m}^s \hat{\mu}\left(\frac{b}{2}\right) db - t\hat{\mu}\left(\frac{t}{2}\right) + \int_{2\hat{b}_m}^t \hat{\mu}\left(\frac{b}{2}\right) db \\
 &= -(t-s)\hat{\mu}\left(\frac{s}{2}\right) + \int_s^t \hat{\mu}\left(\frac{b}{2}\right) db \\
 &\leq 0.
 \end{aligned}$$

Now let $t_m \in ((2m+1)\pi, (2m+3)\pi)$ and $s_{m+1} \in ((2m+3)\pi, (2m+5)\pi)$ be two solutions, where t_m is the rightmost and s_{m+1} is the leftmost solution in the corresponding interval. Then by (3.40) and (3.41)

$$\begin{aligned}
 l(t_m) - l(s_{m+1}) &= t_m\hat{\mu}\left(\frac{t_m}{2}\right) - \int_{2\hat{b}_m}^{t_m} \hat{\mu}\left(\frac{b}{2}\right) db - s_{m+1}\hat{\mu}\left(\frac{s_{m+1}}{2}\right) + \int_{2\hat{b}_{m+1}}^{s_{m+1}} \hat{\mu}\left(\frac{b}{2}\right) db \\
 &= -(s_{m+1} - t_m)\hat{\mu}\left(\frac{t_m}{2}\right) - \int_{2\hat{b}_m}^{t_m} \hat{\mu}\left(\frac{b}{2}\right) db - \int_{s_{m+1}}^{2\hat{b}_{m+1}} \hat{\mu}\left(\frac{b}{2}\right) db \\
 &\leq 0,
 \end{aligned}$$

since $2\hat{b}_m \leq t_m$ and $s_{m+1} \leq 2\hat{b}_{m+1}$.

If $t \in (0, \pi)$ is the solution of (3.36), we have to check, that $l(t) < l(s_0)$. As above, we have

$$\begin{aligned}
 l(t) &= \int_0^t l'(b)db + l(0) \\
 &= t\hat{\mu}\left(\frac{t}{2}\right) - \int_0^t \hat{\mu}\left(\frac{b}{2}\right) db + 2,
 \end{aligned}$$

so that

$$\begin{aligned}
 l(t) - l(s_0) &= t\hat{\mu}\left(\frac{t}{2}\right) - \int_0^t \hat{\mu}\left(\frac{b}{2}\right) db - s_0\hat{\mu}\left(\frac{s_0}{2}\right) + \int_{2\hat{b}_0}^{s_0} \hat{\mu}\left(\frac{b}{2}\right) db \\
 &= -(s_0 - t)\hat{\mu}\left(\frac{t}{2}\right) - \int_0^t \hat{\mu}\left(\frac{b}{2}\right) db - \int_{s_0}^{2\hat{b}_0} \hat{\mu}\left(\frac{b}{2}\right) db \\
 &\leq 0.
 \end{aligned}$$

At last we have to compare the length of the shortest normal extremal of case 1 (if there is one) with that of case 2. Let $b \in (0, \pi)$ be the solution of (3.36), then since $\tan \frac{b}{2} \geq 0$:

$$\begin{aligned}
 L^2(\gamma^b) &= 2bu - 2|x|^2 b \tan \frac{b}{2} \\
 &\leq 2\pi u \\
 &\leq L^2(\gamma^{2\pi, c}).
 \end{aligned}$$

□

Theorem 3.2.13. *Given $(x, 0), (-x, u) \in \mathbb{R}^n \times \mathbb{R}$, $x \neq 0$, $u > 0$, there are finitely many normal extremals parametrized with constant speed connecting $(x, 0)$, $(-x, u)$. Namely:*

1. For any solution b of

$$\frac{u}{|x|^2} = \tilde{\mu}\left(\frac{b}{2}\right) \quad (3.42)$$

there is a normal extremal γ^b given by (3.2) with square of length

$$L(\gamma^b)^2 = \frac{b^2}{\sin^2 \frac{b}{2}} |x|^2 = 2bu + |x|^2 b \cot \frac{b}{2}, \quad (3.43)$$

which strictly increases with b .

2. If $2u \geq |x|^2 \pi$, then for each $m \in \mathbb{N}_0$ with $2u \geq (2m+1)\pi |x|^2$ there are normal extremals $\gamma^{b,c}$ given by (3.2) with $b = (2m+1)\pi$, $|c| = \sqrt{2(2m+1)\pi u - (2m+1)^2 \pi^2 |x|^2}$ and lengths

$$L(\gamma^{b,c}) = \sqrt{2(2m+1)\pi u}. \quad (3.44)$$

Moreover the shortest normal extremal and therefore the Carnot-Carathéodory distance is given by the unique solution b of (3.42) in the interval $(0, \pi)$, if $2u < |x|^2 \pi$. With this b we have:

$$d_{CC}((x, 0), (-x, u)) = \frac{b}{\sin \frac{b}{2}} |x| = \sqrt{2bu + |x|^2 b \cot \frac{b}{2}}. \quad (3.45)$$

And, if $2u \geq |x|^2 \pi$, then the Carnot-Carathéodory distance is given by

$$d_{CC}((x, 0), (-x, u)) = \sqrt{2\pi u}.$$

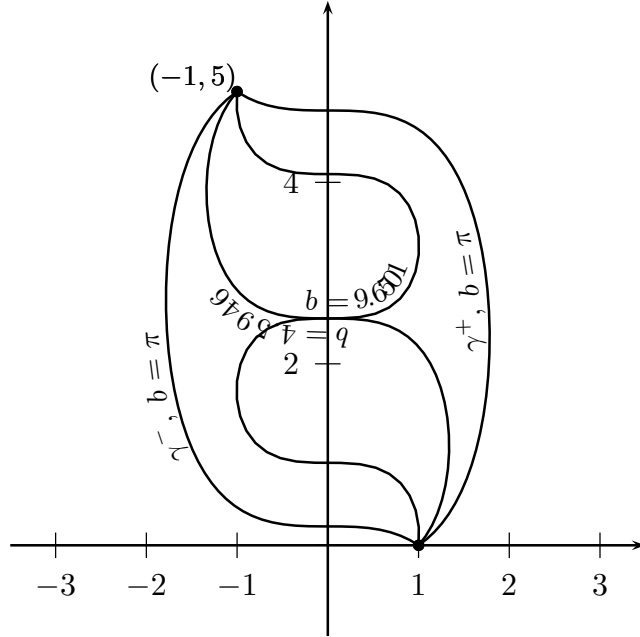


Figure 3.8: Normal extremals joining $(1, 0)$ and $(-1, 5)$ with different $b \notin \pi\mathbb{Z}^*$ vs. two normal extremals γ^\pm with $b = \pi$ and $c = \pm\sqrt{2bu - x_1^2 b^2}$.

Proof. Notice that in this case $a = -1$. The first part is the same proof as above, if one replaces $\hat{\mu}, \hat{b}$ with $\tilde{\mu}, \tilde{b}$. In particular:

1. case $b \in \pi\mathbb{N}$:

Then since $\gamma_{(1)}(1) = x$, there exists $m \in \mathbb{N}_0$ with $b = (2m + 1)\pi$. Then $\gamma_{(2)}(1) = u$ is equivalent to

$$u = \left(|c|^2 + (2m + 1)^2 \pi^2 |x|^2 \right) \frac{1}{2(2m + 1)\pi}.$$

So if $2u \geq (2m + 1)\pi |x|^2$, there are normal extremals $\gamma^{b,c}$ given by (3.2) with $b = (2m + 1)\pi$, $|c| = \sqrt{2(2m + 1)\pi u - (2m + 1)^2 \pi^2 |x|^2}$, with length

$$L^2(\gamma^{b,c}) = |c|^2 + |x|^2 b^2 = 2(2m + 1)\pi u$$

This gives the second part of the theorem.

2. case $b \notin \pi\mathbb{N}$:

In this case we have:

$$\mu(b) = \tilde{\mu}\left(\frac{b}{2}\right).$$

This is again very similar to the situation in Theorem 3.2.11, but a lot simpler, since we are only dealing with $\tilde{\mu}$. From Lemma 3.2.7 we know that the equation

$$\frac{u}{|x|^2} = \tilde{\mu}\left(\frac{b}{2}\right) \tag{3.46}$$

has at most two solutions in an interval $(2m\pi, (2m+2)\pi)$, $m \in \mathbb{N}$, each defining a normal extremal. If s is a solution of this equation, the square of the length of the corresponding normal extremal γ is given by

$$\begin{aligned} L^2(\gamma) &= 2l(s)|x|^2 \\ &= 2\frac{s^2}{\sin^2 s}(1 + \cos s)|x|^2 \\ &= \frac{s^2}{\sin^2 \frac{s}{2}}|x|^2. \end{aligned}$$

If $s \in (2m\pi, (2m+2)\pi)$, $m \in \mathbb{N}$, we have:

$$\begin{aligned} l(s) &= \int_{2\tilde{b}_m}^s l'(b)db + l(2\tilde{b}_m) \\ &= \int_{2\tilde{b}_m}^s b\mu'(b)db + l(2\tilde{b}_m) \\ &= \int_{2\tilde{b}_m}^s \frac{b}{2}\tilde{\mu}'\left(\frac{b}{2}\right)db + l(2\tilde{b}_m) \\ &= s\tilde{\mu}\left(\frac{s}{2}\right) - 2\tilde{b}_m\tilde{\mu}(\tilde{b}_m) - \int_{2\tilde{b}_m}^s \tilde{\mu}\left(\frac{b}{2}\right)db + l(2\tilde{b}_m) \end{aligned}$$

But now

$$\begin{aligned} l(2\tilde{b}_m) - 2\tilde{b}_m\tilde{\mu}(\tilde{b}_m) &= \frac{2\tilde{b}_m^2}{\sin^2 \tilde{b}_m} - 2\tilde{b}_m\left(\frac{\tilde{b}_m}{\sin^2 \tilde{b}_m} - \cot \tilde{b}_m\right) \\ &= 2\tilde{b}_m \cot \tilde{b}_m \\ &= 2. \end{aligned}$$

First let $s \leq t$ be two solutions of (3.46) in an interval $(2m\pi, (2m+2)\pi)$, $m \in \mathbb{N}$; then

$$\begin{aligned} l(s) - l(t) &= s\tilde{\mu}\left(\frac{s}{2}\right) - \int_{2\tilde{b}_m}^s \tilde{\mu}\left(\frac{b}{2}\right) db - t\tilde{\mu}\left(\frac{t}{2}\right) + \int_{2\tilde{b}_m}^t \tilde{\mu}\left(\frac{b}{2}\right) db \\ &= -(t-s)\tilde{\mu}\left(\frac{s}{2}\right) + \int_s^t \tilde{\mu}\left(\frac{b}{2}\right) db \\ &\leq 0. \end{aligned}$$

Now let $t_m \in (2m\pi, (2m+2)\pi)$ and $s_{m+1} \in ((2m+2)\pi, (2m+4)\pi)$ be two solutions, then

$$\begin{aligned} l(t_m) - l(s_{m+1}) &= t_m\tilde{\mu}\left(\frac{t_m}{2}\right) - \int_{2\tilde{b}_m}^{t_m} \tilde{\mu}\left(\frac{b}{2}\right) db - s_{m+1}\tilde{\mu}\left(\frac{s_{m+1}}{2}\right) + \int_{2\tilde{b}_{m+1}}^{s_{m+1}} \tilde{\mu}\left(\frac{b}{2}\right) db \\ &= -(s_{m+1} - t_m)\tilde{\mu}\left(\frac{t_m}{2}\right) - \int_{2\tilde{b}_m}^{t_m} \tilde{\mu}\left(\frac{b}{2}\right) db - \int_{s_{m+1}}^{2\tilde{b}_{m+1}} \tilde{\mu}\left(\frac{b}{2}\right) db \\ &\leq 0, \end{aligned}$$

since $2\tilde{b}_m \leq t_m$ and $s_{m+1} \leq 2\tilde{b}_m$.

If $t_0 \in (0, 2\pi)$ is the solution in the interval $(0, 2\pi)$, we have to show that $l(t_0) < l(s_1)$.

But the above calculations show that

$$\begin{aligned} l(t_0) &= \int_0^{t_0} l'(b) db + l(0) \\ &= t_0\tilde{\mu}\left(\frac{t_0}{2}\right) - \int_0^{t_0} \tilde{\mu}\left(\frac{b}{2}\right) db + 2. \end{aligned}$$

So we get

$$\begin{aligned} l(t_0) - l(s_1) &= t_0\tilde{\mu}\left(\frac{t_0}{2}\right) - \int_0^{t_0} \tilde{\mu}\left(\frac{b}{2}\right) db - s_1\tilde{\mu}\left(\frac{s_1}{2}\right) + \int_{2\tilde{b}_1}^{s_1} \tilde{\mu}\left(\frac{b}{2}\right) db \\ &= -(s_1 - t_0)\tilde{\mu}\left(\frac{t_0}{2}\right) - \int_0^{t_0} \tilde{\mu}\left(\frac{b}{2}\right) db - \int_{s_1}^{2\tilde{b}_1} \tilde{\mu}\left(\frac{b}{2}\right) db \\ &< 0. \end{aligned}$$

At last we have to compare the length of the shortest normal extremal $\gamma^{\pi,c}$, $|c| = \sqrt{2\pi u - \pi^2 |x|^2}$ of case 1 (if there is one) with that of case 2. Let $b \in (0, 2\pi)$ be the solution of (3.42). If $b < \pi$, then, since $\tilde{\mu}$ is monotone increasing on $(0, \pi)$, we get:

$$\frac{u}{|x|^2} = \tilde{\mu}\left(\frac{b}{2}\right) < \tilde{\mu}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

But this means, that $2u < \pi |x|^2$ and therefore there is no extra normal extremal. If now $b \geq \pi$, we have to show that $L^2(\gamma^b) \geq 2\pi u$. We have

$$\begin{aligned} \frac{u}{|x|^2} &= \tilde{\mu}\left(\frac{b}{2}\right) \\ &= \frac{\frac{b}{2}}{\sin^2 \frac{b}{2}} - \cot \frac{b}{2} \\ &= \cot \frac{b}{2} \left(\frac{b}{2 \sin \frac{b}{2} \cos \frac{b}{2}} - 1 \right) \\ &= \frac{b - \sin b}{\sin b} \cot \frac{b}{2}, \end{aligned}$$

so that

$$\begin{aligned} L^2(\gamma^b) &= 2bu + 2|x|^2 b \cot \frac{b}{2} \\ &= 2bu \left(1 + \frac{\sin b}{b - \sin b} \right) \\ &= 2u \frac{b^2}{b - \sin b} \\ &\geq 2\pi u. \end{aligned}$$

The last inequality holds since $f(b) := \frac{b^2}{b - \sin b} \geq \pi$, for all $b \in [\pi, 2\pi]$. To see this, observe, that $f(\pi) = \pi$ and f is monotone increasing:

$$\begin{aligned} f'(b) &= \frac{2b(b - \sin b) - b^2(1 - \cos b)}{(b - \sin b)^2} \\ &= \frac{b^2(1 + \cos b) - 2b \sin b}{(b - \sin b)^2} \\ &\geq 0. \end{aligned}$$

This completes the proof, since the normal extremals $\gamma^{b,c}$ in case 2 (with parameter $b = \pi$, $|c| = \sqrt{2\pi u - \pi^2 x^2}$) are shorter. \square

To close this section, we give a summarizing theorem about our results of this section:

Theorem 3.2.14. *Let $(x_1, u_1), (x, u) \in \mathbb{R}^{n+1}$. If $x_1 = -x$ and $2|u - u_1| \geq \pi|x|^2$, then the Carnot-Carathéodory distance is given by*

$$d_{CC}((x_1, u_1), (x, u)) = \sqrt{2\pi|u - u_1|}. \quad (3.47)$$

Otherwise the Carnot-Carathéodory distance is given by

$$d_{CC}((x_1, u_1), (x, u)) = \frac{b}{\sin b} \sqrt{|x_1|^2 + |x|^2 - 2x_1 \cdot x \cos b} \quad (3.48a)$$

$$= \sqrt{2b(u - u_1) + (|x_1|^2 + |x|^2)b \cot b - 2x_1 \cdot x \frac{b}{\sin b}} \quad (3.48b)$$

where $b \in (-\pi, \pi)$ is the unique solution of

$$\frac{2(u - u_1)}{|x_1|^2 + |x|^2} = \frac{b}{\sin^2 b} - \cot b + 2 \frac{x_1 \cdot x}{|x_1|^2 + |x|^2} \cdot \frac{1 - b \cot b}{\sin b} \quad (3.49)$$

in the interval $(-\pi, \pi)$.

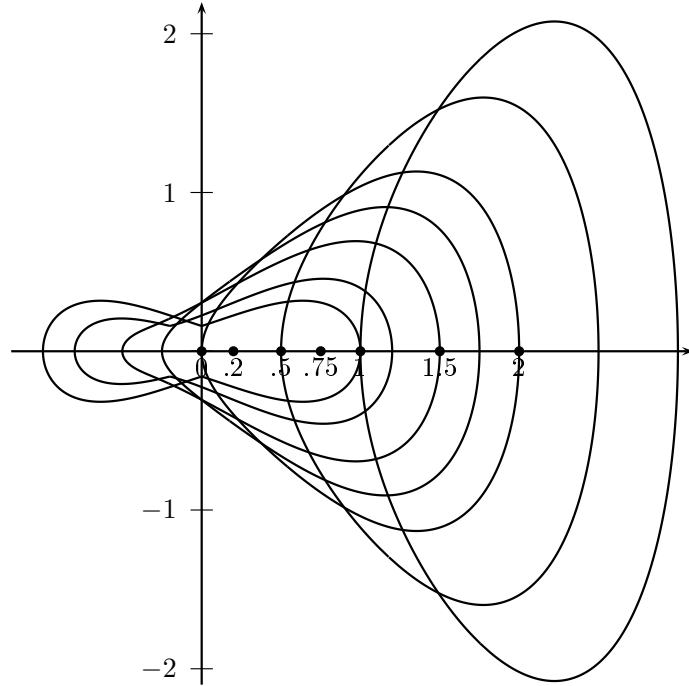


Figure 3.9: Sub-Riemannian spheres $S(p, r) := \{q : d_{CC}(p, q) = r\}$ with radius $r = 1$ and different center points p .

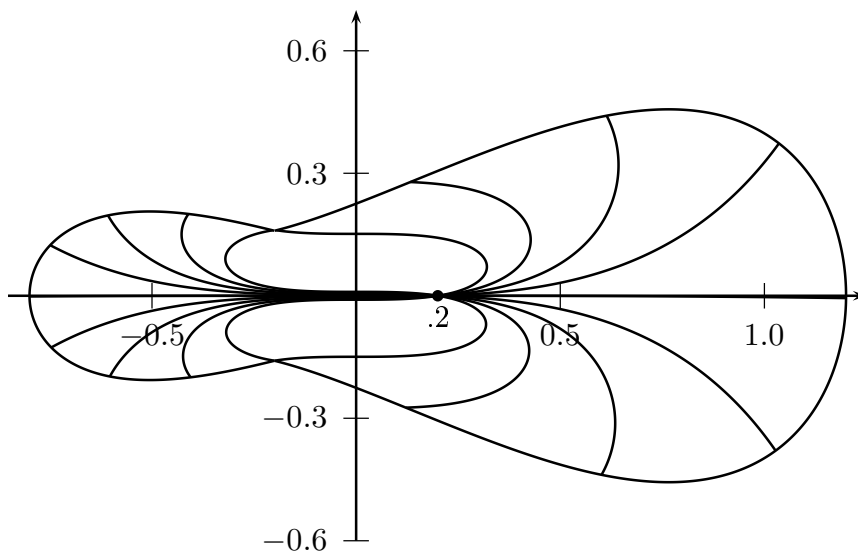


Figure 3.10: The sub-Riemannian sphere $S(0.2, 1)$ with some minimizers.

3.3 The heat kernel associated to the Grušin operator

We now turn our attention to the heat equation, i.e. the Cauchy problem

$$\begin{cases} (\partial_t - G)v(t, x, u) = 0 \\ v(0, x, u) = f(x, u) \end{cases} \quad (3.50)$$

with some function $f \in \mathcal{S}(\mathbb{R}^{n+1})$. Taking the Fourier-transform in the u -variable we see that this is equivalent to

$$\begin{cases} (\partial_t - H_\lambda)w(t, x, \lambda) = 0 \\ w(0, x, \lambda) = g(x, \lambda) \end{cases}, \quad (3.51)$$

where $g(x, \lambda) := f(x, \hat{\lambda})$ denotes the partial Fourier-transform of f in the u -variable and $H_\lambda = \Delta_x - |\lambda|^2|x|^2$ denote the Hermite operator with parameter $\lambda \in \mathbb{R}$, which is the partial Fourier-transform of G . Thanks to the Plancherel formula, any solution w of (3.51) defines a solution v to (3.50) by taking the inverse Fourier-transform, and vice versa.

But the latter equation is well known and we give a brief discussion of this equation. The solution of this equation is given by $e^{tH_\lambda}g$, where e^{tH_λ} is a bounded operator defined by

the functional calculus, since H_λ is a self-adjoint operator. By the Schwartz kernel theorem e^{tH_λ} is given by an integral kernel, i.e.

$$e^{tH_\lambda}g(x) = \int_{\mathbb{R}^n} k_t^\lambda(x, \xi)g(\xi)d\xi. \quad (3.52)$$

And, since H_λ is hypoelliptic, we have $k_t^\lambda \in C^\infty(\mathbb{R}^{2n})$.

The Hermite functions

$$h_\alpha(x) := \frac{2^{n/4}}{\sqrt{\alpha!}} \left(\frac{-1}{2\sqrt{\pi}} \right)^{|\alpha|} e^{\pi|x|^2} \left(\frac{\partial}{\partial x} \right)^\alpha e^{-2\pi|x|^2}, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n, \quad (3.53)$$

which give an orthonormal basis of $L^2(\mathbb{R}^n)$, are eigenfunctions of $H_{2\pi}$:

$$H_{2\pi}h_\alpha = -2\pi(2|\alpha| + n)h_\alpha. \quad (3.54)$$

Since

$$\left(\Delta - \mu^2|x|^2 \right) f(r \cdot) = r^2 \left(\left(\Delta - \left(\frac{\mu}{r^2} \right)^2 |\cdot|^2 \right) f \right) (r \cdot), \quad (3.55)$$

we can rescale the Hermite functions

$$h_\alpha^\lambda(x) := \left(\frac{|\lambda|}{2\pi} \right)^{\frac{n}{4}} h_\alpha \left(\left(\frac{|\lambda|}{2\pi} \right)^{\frac{1}{2}} x \right) \quad (3.56)$$

to get an orthonormal basis $(h_\alpha^\lambda)_{\alpha \in \mathbb{N}_0^n}$ of eigenfunctions of H_λ :

$$H_\lambda h_\alpha^\lambda = -|\lambda|(2|\alpha| + n)h_\alpha^\lambda. \quad (3.57)$$

Using Mehler's formula, we can calculate the kernel k_t^λ :

$$\begin{aligned}
 k_t^\lambda(x, \xi) &= \sum_{|\alpha| \geq 0} e^{tH_\lambda} h_\alpha^\lambda(x) h_\alpha^\lambda(\xi) \\
 &= \sum_{|\alpha| \geq 0} e^{-|\lambda|(2|\alpha|+n)t} \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} h_\alpha \left(\sqrt{\frac{|\lambda|}{2\pi}}x\right) h_\alpha \left(\sqrt{\frac{|\lambda|}{2\pi}}\xi\right) \\
 &= \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} e^{-n|\lambda|t} \sum_{|\alpha| \geq 0} e^{-2|\lambda||\alpha|t} h_\alpha \left(\sqrt{\frac{|\lambda|}{2\pi}}x\right) h_\alpha \left(\sqrt{\frac{|\lambda|}{2\pi}}\xi\right) \\
 &= \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} e^{-n|\lambda|t} \left(\frac{2}{1-e^{-4|\lambda|t}}\right)^{\frac{n}{2}} \\
 &\quad \cdot \exp\left(\frac{-\pi(1+e^{-4|\lambda|t})\frac{|x|^2+|\xi|^2}{2\pi}|\lambda| + 4\pi\frac{x \cdot \xi}{2\pi}|\lambda|e^{-2|\lambda|t}}{1-e^{-4|\lambda|t}}\right) \\
 &= (2\pi)^{-n/2} \left(\frac{|\lambda|}{\sinh(2|\lambda|t)}\right)^{\frac{n}{2}} \\
 &\quad \cdot \exp\left(-\frac{|x|^2+|\xi|^2}{2}|\lambda| \coth(2|\lambda|t) + \frac{|\lambda|}{\sinh(2|\lambda|t)}x \cdot \xi\right) \\
 &= (4\pi t)^{-n/2} \left(\frac{2\lambda t}{\sinh(2\lambda t)}\right)^{\frac{n}{2}} \\
 &\quad \cdot \exp\left(-\frac{1}{4t}\left(2\lambda t \coth(2\lambda t)(|x|^2+|\xi|^2) - \frac{4\lambda t}{\sinh(2\lambda t)}x \cdot \xi\right)\right).
 \end{aligned}$$

Now, by taking the Fourier inverse

$$\begin{aligned}
 K_t(x, \xi, u) &= (2\pi)^{-1} \int_{-\infty}^{\infty} k_t^\lambda(x, \xi) e^{i\lambda u} d\lambda \tag{3.58} \\
 &= (2\pi)^{-1} (4\pi t)^{-n/2} \int_{-\infty}^{\infty} \left(\frac{2\lambda t}{\sinh(2\lambda t)}\right)^{\frac{n}{2}} \\
 &\quad \cdot \exp\left(-\frac{1}{4t}\left(2\lambda t \coth(2\lambda t)(|x|^2+|\xi|^2) - \frac{4\lambda t}{\sinh(2\lambda t)}x \cdot \xi\right)\right) e^{i\lambda u} d\lambda \\
 &= (4\pi t)^{-n/2-1} \int_{-\infty}^{\infty} \left(\frac{\lambda}{\sinh \lambda}\right)^{\frac{n}{2}} \\
 &\quad \exp\left(-\frac{1}{4t}\left(\lambda \coth \lambda(|x|^2+|\xi|^2) - \frac{2\lambda}{\sinh \lambda}x \cdot \xi - 2i\lambda u\right)\right) d\lambda,
 \end{aligned}$$

one gets the heat kernel for the Grušin operator. Now the solution of (3.50) is given by

$$v(x, u) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} K_t(x, \xi, u - \lambda) f(\xi, \lambda) d\xi d\lambda. \quad (3.59)$$

3.4 Heat kernel estimates

We may now use the results of section 3.2 to give some estimates for the heat kernel. We set

$$\begin{aligned} h(x, \xi, u) &:= \int_{-\infty}^{\infty} \left(\frac{\lambda}{\sinh \lambda} \right)^{\frac{n}{2}} \exp \left(- \left(\lambda \coth \lambda (|x|^2 + |\xi|^2) - \frac{2\lambda}{\sinh \lambda} x \cdot \xi - 2i\lambda u \right) \right) d\lambda \\ &= \int_{-\infty}^{\infty} \left(\frac{\lambda}{\sinh \lambda} \right)^{\frac{n}{2}} \exp \left(- \left(\lambda \coth \lambda - \frac{\lambda}{\sinh \lambda} a \right) R^2 + 2i\lambda u \right) d\lambda \end{aligned} \quad (3.60)$$

with

$$R := R(x, \xi) := \sqrt{|x|^2 + |\xi|^2} \text{ and } a := a(x, \xi) := \frac{2x \cdot \xi}{R^2} \in [0, 1].$$

Then

$$K_t(x, \xi, u) = (4\pi t)^{-n/2-1} h \left(\frac{x}{2\sqrt{t}}, \frac{\xi}{2\sqrt{t}}, \frac{u}{4t} \right). \quad (3.61)$$

It is also convenient to set

$$V(\lambda) := \left(\frac{\lambda}{\sinh \lambda} \right)^{\frac{n}{2}} \quad (3.62)$$

$$\begin{aligned} \psi(\lambda) &:= \psi(\lambda, a) \\ &:= \lambda \coth \lambda - \frac{\lambda}{\sinh \lambda} a, \quad \lambda \in \mathbb{R}. \end{aligned} \quad (3.63)$$

One may see that $\psi(ib) = b \cot b - \frac{b}{\sin b} a$, so that if b is a solution of (3.49), then the exponent

$$\left(\lambda \coth \lambda - \frac{\lambda}{\sinh \lambda} a \right) R^2 - 2i\lambda u$$

at $\lambda = ib$ gives exactly the square of the Carnot-Carathéodory distance $d_{CC}((x, 0), (\xi, u))^2$. So we expect Gaussian-type estimates of the form

$$|h(x, \xi, u)| \leq F(x, \xi, u) e^{-d_{CC}((x,0),(\xi,u))^2},$$

where $F(x, \xi, u) > 0$ is a function depending on x, ξ, u . It will turn out that F has polynomial growth, more precisely

$$F(x, \xi, u) \lesssim (1 + d_{CC}((x, 0), (\xi, u))^2)^\alpha,$$

where $\alpha = \frac{n}{2} - 1$, if $n > 2$ and $\alpha = 0$ if $n \leq 2$. Compared to the euclidean case, where $\alpha = 0$, one has some additional growth, if the dimension is greater than 2.

The strategy is as follows: Move the line of integration in (3.60) from the real axis to the line $\mathbb{R} + ib$, where b is near to the solution b_0 of (3.49).

Lemma 3.4.1. *The function*

$$V(\lambda) := \left(\frac{\lambda}{\sinh \lambda} \right)^{\frac{n}{2}},$$

where the square root is the principal branch in $\mathbb{C} \setminus (-\infty, 0]$, is holomorphic in $\{z \in \mathbb{C} : |\Im z| < \pi\}$. And for $\nu + ib \in \mathbb{C}$, $-\pi < b < \pi$, one has

$$|V(\nu + ib)| \leq \left(\frac{b}{\sin b} \right)^{\frac{n}{2}}, \tag{3.64}$$

$$|V(\nu + ib)| \leq \left(1 + \frac{b^2}{\nu^2} \right)^{\frac{n}{4}} \left(\frac{\nu}{\sinh \nu} \right)^{\frac{n}{2}}. \tag{3.65}$$

Proof. To show that V is holomorphic, we have to show that

$$\frac{\lambda}{\sinh \lambda} \notin (-\infty, 0],$$

for all $\lambda \in \mathbb{C}$ with $-\pi < \Im \lambda < \pi$. Let $\nu + ib \in \mathbb{C}$, $|b| < \pi$. Then

$$\begin{aligned} \frac{\nu + ib}{\sinh(\nu + ib)} &= \frac{(\nu + ib) \sinh(\nu - ib)}{|\sinh(\nu + ib)|^2} \\ &= \frac{\nu \sinh \nu \cos b + b \cosh \nu \sin b + i(b \sinh \nu \cos b - \nu \cosh \nu \sin b)}{|\sinh(\nu + ib)|^2}. \end{aligned} \tag{3.66}$$

So

$$\begin{aligned} \frac{\nu + ib}{\sinh(\nu + ib)} \in \mathbb{R} &\Leftrightarrow b \sinh \nu \cos b = \nu \cosh \nu \sin b \\ &\Leftrightarrow b \cot b = \nu \coth \nu \vee \nu = 0. \end{aligned}$$

But $b \cot b \leq 1$ and $\nu \coth \nu > 1$ for $\nu \neq 0$; and $\nu = 0$ means

$$\frac{\nu + ib}{\sinh(\nu + ib)} = \frac{b}{\sin b} > 0.$$

We can conclude

$$\frac{\nu + ib}{\sinh(\nu + ib)} \notin (-\infty, 0], \quad \nu \in \mathbb{R}, \quad -\pi < b < \pi. \quad (3.67)$$

The second inequality is easy, since

$$\begin{aligned} \left| \frac{\nu + ib}{\sinh(\nu + ib)} \right|^2 &= \frac{\nu^2 + b^2}{\sinh^2 \nu + \sin^2 b} \\ &\leq \frac{\nu^2 + b^2}{\sinh^2 \nu} \\ &= \left(1 + \frac{b^2}{\nu^2}\right) \left(\frac{\nu}{\sinh \nu}\right)^2. \end{aligned}$$

To verify the first inequality we show that

$$f(\nu) := \left| \frac{\nu + ib}{\sinh(\nu + ib)} \right|^2 \leq \frac{b^2}{\sin^2 b}, \quad \text{for all } \nu \in \mathbb{R}.$$

This is obviously true for $\nu = 0$, and since $f(-\nu) = f(\nu)$, we may restrict our analysis to the case $\nu \geq 0$:

$$\begin{aligned} f'(\nu) &= \frac{2\nu(\sinh^2 \nu + \sin^2 b) - (\nu^2 + b^2) \sinh(2\nu)}{(\sinh^2 \nu + \sin^2 b)^2} \\ &= \frac{2\nu \sinh \nu (\sinh \nu - \nu \cosh \nu) + 2\nu \sin^2 b - b^2 \sinh(2\nu)}{(\sinh^2 \nu + \sin^2 b)^2} \\ &\leq 0, \end{aligned}$$

since $\sinh \nu \leq \nu \cosh \nu$ and $2\nu \sin^2 b \leq \sinh(2\nu) \sin^2 b \leq b^2 \sinh(2\nu)$. This means that f stays below $\frac{b^2}{\sin^2 b}$ for all $\nu \in \mathbb{R}$. \square

Lemma 3.4.2. *The function*

$$\psi(\lambda) = \lambda \coth \lambda - a \frac{\lambda}{\sinh \lambda} \quad (3.68a)$$

$$= (1 - a)\lambda \coth \lambda + a\lambda \tanh \frac{\lambda}{2} \quad (3.68b)$$

$$= (1 + a)\lambda \coth \lambda - a\lambda \coth \frac{\lambda}{2} \quad (3.68c)$$

is holomorphic in $\mathbb{C} \setminus \{\pi ki : k \in \mathbb{Z}^*\}$ for $a \neq \pm 1$.

If $a = 1$, then $\psi(\lambda) = \lambda \tanh \frac{\lambda}{2}$ is holomorphic in $\mathbb{C} \setminus \{(2k + 1)\pi i : k \in \mathbb{Z}\}$;

if $a = -1$, then $\psi(\lambda) = \lambda \coth \frac{\lambda}{2}$ is holomorphic in $\mathbb{C} \setminus \{2k\pi i : k \in \mathbb{Z}^*\}$.

In particular ψ is holomorphic in the strip $\{z \in \mathbb{C} : |\Im z| < \pi\}$ for any $-1 \leq a \leq 1$.

Proof. The holomorphic properties of ψ follow easily from those of the trigonometric function \coth , \tanh , and the equalities (3.68), which are easy to verify:

$$\begin{aligned}\psi(\lambda) &= \lambda \coth \lambda - a \frac{\lambda}{\sinh \lambda} \\ &= (1-a)\lambda \coth \lambda + a\lambda \frac{\cosh \lambda - 1}{\sinh \lambda} \\ &= (1-a)\lambda \coth \lambda + a\lambda \tanh \frac{\lambda}{2},\end{aligned}$$

and

$$\begin{aligned}\psi(\lambda) &= (1+a)\lambda \coth \lambda - a\lambda \frac{\cosh \lambda + 1}{\sinh \lambda} \\ &= (1+a)\lambda \coth \lambda - a\lambda \coth \frac{\lambda}{2}.\end{aligned}$$

□

Lemma 3.4.3. For $\nu + ib \in \mathbb{C}$, $|b| < \pi$, $\psi(ib)$ is real and

$$\Re(\psi(\nu + ib)) \geq \psi(ib), \quad (3.69)$$

where

$$\psi(ib) = b \cot b - a \frac{b}{\sin b} \quad (3.70a)$$

$$= (1-a)b \cot b + a \left(-b \tan \frac{b}{2} \right) \quad (3.70b)$$

$$= (1+a)b \cot b - ab \cot \frac{b}{2}. \quad (3.70c)$$

Proof. (3.70) is clear. Let $\nu + ib \in \mathbb{C}$, $|b| \leq \pi$, then

$$\begin{aligned}\Re((\nu + ib) \coth(\nu + ib)) &= \frac{1}{2} \frac{\nu \sinh(2\nu) + b \sin(2b)}{\sinh^2 \nu + \sin^2 b} \\ &= \frac{\nu \coth \nu \sinh^2 \nu + b \cot b \sin^2 b}{\sinh^2 \nu + \sin^2 b} \\ &= b \cot b + \sinh^2 \nu \frac{\nu \coth \nu - b \cot b}{\sinh^2 \nu + \sin^2 b} \\ &\geq b \cot b,\end{aligned}$$

and for $|b| \leq \frac{\pi}{2}$:

$$\begin{aligned} \Re((\nu + ib) \tanh(\nu + ib)) &= \frac{1}{2} \frac{\nu \sinh(2\nu) - b \sin(2b)}{\sinh^2 \nu + \cos^2 b} \\ &= \frac{\nu \coth \nu \sinh^2 \nu - b \tan b \cos^2 b}{\sinh^2 \nu + \cos^2 b} \\ &= -b \tan b + \sinh^2 \nu \frac{\nu \coth \nu + b \tan b}{\sinh^2 \nu + \cos^2 b} \\ &\geq -b \tan b. \end{aligned}$$

This gives for $\lambda = \nu + ib \in \mathbb{C}$, $|b| < \pi$:

$$\begin{aligned} \Re(\psi(\lambda)) &= \begin{cases} \Re\left((1-a)\lambda \coth \lambda + a\lambda \tanh \frac{\lambda}{2}\right) & , \text{ if } a \geq 0 \\ \Re\left((1+a)\lambda \coth \lambda - a\lambda \coth \frac{\lambda}{2}\right) & , \text{ if } a < 0 \end{cases} \\ &\geq \begin{cases} (1-a)b \cot b + a(-b \tan \frac{b}{2}) & , \text{ if } a \geq 0 \\ (1+a)b \cot b - ab \cot \frac{b}{2} & , \text{ if } a < 0 \end{cases} \\ &= \psi(ib). \end{aligned}$$

□

With these preliminary lemmata we can start estimating h by moving the path of integration to $\lambda \mapsto \lambda + ib$:

Lemma 3.4.4. *There is a constant $C > 0$, such that for all $x, \xi \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $b \in \mathbb{R}$, $|b| < \pi$ the following estimate holds:*

$$|h(x, \xi, u)| \leq C \left(\frac{b}{\sin b} \right)^\alpha \exp \left(-2bu - \left(b \cot b - \frac{b}{\sin b} \frac{2x \cdot \xi}{|x|^2 + |\xi|^2} \right) (|x|^2 + |\xi|^2) \right), \quad (3.71)$$

with $\alpha = \max(0, \frac{n}{2} - 1)$.

Proof. We have

$$\begin{aligned}
 |h(x, \xi, u)| &= \left| \int_{-\infty}^{\infty} V(\lambda) e^{-\psi(\lambda)R^2 + 2i\lambda u} d\lambda \right| \\
 &= \left| \int_{-\infty}^{\infty} V(\lambda + ib) e^{-\psi(\lambda+ib)R^2 + 2i(\lambda+ib)u} d\lambda \right| \\
 &\leq \int_{-\infty}^{\infty} |V(\lambda + ib)| e^{-\Re\psi(\lambda+ib)R^2 - 2bu} d\lambda \\
 &= e^{-\psi(ib)R^2 - 2bu} \int_{-\infty}^{\infty} |V(\lambda + ib)| d\lambda.
 \end{aligned}$$

Now we split the remaining integral with $r := \pi - |b|$:

$$\begin{aligned}
 \int_{-\infty}^{\infty} |V(\lambda + ib)| d\lambda &= 2 \int_0^{\infty} |V(\lambda + ib)| d\lambda \\
 &\lesssim \int_0^r \left| \frac{b}{\sin b} \right|^{\frac{n}{2}} d\lambda + \int_r^{\pi} \left(1 + \frac{b^2}{\lambda^2} \right)^{\frac{n}{4}} d\lambda + \int_{\pi}^{\infty} \left| \frac{\lambda}{\sinh \lambda} \right|^{\frac{n}{2}} d\lambda \\
 &\lesssim r \left(\frac{b}{\sin b} \right)^{\frac{n}{2}} + \int_r^{\pi} \frac{1}{\lambda^{n/2}} d\lambda + 1 \\
 &\lesssim \left(\frac{b}{\sin b} \right)^{\frac{n}{2}-1} + \left(\frac{1}{r} \right)^{\frac{n}{2}-1} + 1 \\
 &\lesssim \left(\frac{b}{\sin b} \right)^{\alpha},
 \end{aligned}$$

since

$$\frac{b}{\sin b} \sim \frac{1}{\pi - |b|} = \frac{1}{r}. \quad (3.72)$$

□

Observe that we can assume $u \geq 0$, since $h(x, \xi, -u) = h(x, \xi, u)$.

Setting $b = b_0$, where b_0 parametrizes the corresponding (shortest) normal extremal, would not give good results in every case. If b tends to π , the estimate would blow up. To avoid this problem, just set $b = b_0 - \epsilon$ with an appropriate $\epsilon > 0$. (Since we assumed $u \geq 0$, we have $b_0 \geq 0$.)

Proposition 3.4.5. For $\zeta = (x, 0)$, $\eta = (\xi, u) \in \mathbb{R}^{n+1}$ we have

$$|h(x, \xi, u)| \lesssim \min \left\{ 1 + \frac{d_{CC}(\zeta, \eta)}{|x + \xi|}, 1 + d_{CC}(\zeta, \eta)^2 \right\}^\alpha e^{-d_{CC}(\zeta, \eta)^2}, \quad (3.73)$$

with $\alpha = \max(\frac{n}{2} - 1, 0)$.

Proof. First assume that $x \neq -\xi$. Then the shortest normal extremal and therefore the Carnot-Carathéodory distance will be parametrized by $b_0 \in (\pi, \pi)$. We claim that

$$\frac{b_0^2}{\sin^2 b_0} \leq 3 \left(1 + \frac{d_{CC}(\zeta, \eta)^2}{|x + \xi|^2} \right).$$

To see this, first assume that $a < 0$, i.e. $\xi \cdot x < 0$. Then

$$\begin{aligned} d_{CC}(\zeta, \eta)^2 &= \frac{b_0^2}{\sin^2 b_0} (|x|^2 + |\xi|^2 - 2\xi \cdot x \cos b_0) \\ &\geq \frac{b_0^2}{\sin^2 b_0} (|x|^2 + |\xi|^2 + 2\xi \cdot x) \\ &= \frac{b_0^2}{\sin^2 b_0} |x + \xi|^2. \end{aligned}$$

If $a \geq 0$, i.e. $\xi \cdot x \geq 0$, then

$$\begin{aligned} d_{CC}(\zeta, \eta)^2 &\geq \frac{1}{3} \frac{b_0^2}{\sin^2 b_0} (1 - a \cos b_0) (|x|^2 + 2\xi \cdot x + |\xi|^2) \\ &\geq \frac{1}{3} \left(\frac{b_0^2}{\sin^2 b_0} - a \frac{b_0^2 \cos b_0}{\sin^2 b_0} \right) |x + \xi|^2. \end{aligned}$$

If $b_0 > \frac{\pi}{2}$, then $a \frac{b_0^2}{\sin^2 b_0} \cos b_0 < 0$ and if $b_0 \geq \frac{\pi}{2}$, then $a \frac{b_0^2}{\sin^2 b_0} \cos b_0 \leq \frac{b_0^2}{\sin^2 b_0} \leq \frac{\pi^2}{4}$. So

$$\begin{aligned} d_{CC}(\zeta, \eta)^2 &\geq \frac{1}{3} \left(\frac{b_0^2}{\sin^2 b_0} - \frac{\pi^2}{4} \right) |x + \xi|^2 \\ &\geq \frac{1}{3} \left(\frac{b_0^2}{\sin^2 b_0} - 3 \right) |x + \xi|^2. \end{aligned}$$

This proves the claim. Now use Lemma 3.4.4 with $b = b_0$ to get

$$|h(x, \xi, u)| \lesssim \left(1 + \frac{d_{CC}(\zeta, \eta)}{|x + \xi|} \right)^\alpha e^{-d_{CC}(\zeta, \eta)^2}. \quad (3.74)$$

But this remains true, if $x = -\xi$ (and $x = \xi = 0$), since the right side becomes $+\infty$ in this case. To prove the remaining estimate

$$|h(x, \xi, u)| \lesssim (1 + d_{CC}(\zeta, \eta)^2)^\alpha e^{-d_{CC}(\zeta, \eta)^2}, \quad (3.75)$$

we, again, use lemma 3.4.4. Let $b_0 \in [0, \pi]$ be the parameter for the Carnot-Carathéodory distance. As mentioned above assume that $u \geq 0$. If $b_0 \leq \frac{\pi}{2}$, then $\frac{b_0}{\sin b_0} \leq \frac{\pi}{2}$, and one can immediately use Lemma 3.4.4 with $b = b_0$. So assume that $b_0 > \frac{\pi}{2}$. Now set $b = b_0 - \epsilon$ with $\epsilon < \frac{\pi}{2}$, which will be specified later.

If $b_0 \neq \pi$, we have

$$\begin{aligned} b \cot b &= b_0 \cot b_0 + b (\cot b - \cot b_0) - \epsilon \cot b_0 \\ &\geq b_0 \cot b_0 - \epsilon \cot b_0, \end{aligned}$$

since \cot is monotone decreasing on $(0, \pi)$; and

$$\begin{aligned} -b \tan \frac{b}{2} &= -b_0 \tan \frac{b_0}{2} + b \left(\tan \frac{b_0}{2} - \tan \frac{b}{2} \right) + \epsilon \tan \frac{b_0}{2} \\ &\geq -b_0 \tan \frac{b_0}{2} + \epsilon \tan \frac{b_0}{2}, \end{aligned}$$

since \tan is monotone increasing on $(0, \pi)$. Therefore

$$\begin{aligned} \psi(ib) &= \begin{cases} (1-a)b \cot b + a(-b \tan \frac{b}{2}) & , a \geq 0 \\ (1+a)b \cot b - ab \cot \frac{b}{2} & , a < 0 \end{cases} \\ &\geq \psi(ib_0) - \frac{\epsilon}{b_0} \psi(ib_0). \end{aligned}$$

The use of lemma 3.4.4 with this b yields:

$$\begin{aligned} |h(x, \xi, u)| &\lesssim \left(\frac{b}{\sin b} \right)^\alpha e^{-\psi(ib_0)R^2 - 2b_0u + \frac{\epsilon}{b_0}(\psi(ib_0) + 2b_0u)} \\ &\lesssim \left(\frac{1}{\epsilon} \right)^\alpha e^{-(1 - \frac{\epsilon}{b_0})d_{CC}(\zeta, \eta)^2}, \end{aligned} \quad (3.76)$$

first for $b_0 \neq \pi$. But, if $b_0 = \pi$, then we have $x = \xi = 0$ or $x = -\xi$. We claim that in this case estimate (3.76) remains true: If $x = \xi = 0$ then

$$\begin{aligned} |h(x, \xi, u)| &\lesssim \left(\frac{b}{\sin b} \right)^\alpha e^{-2b_0u + 2b_0u \frac{\epsilon}{b_0}} \\ &\lesssim \left(\frac{1}{\epsilon} \right)^\alpha e^{-(1 - \frac{\epsilon}{b_0})d_{CC}(\zeta, \eta)^2}, \end{aligned}$$

since $2b_0u = 2\pi u = d_{CC}(\zeta, \eta)^2$. And, if $x = -\xi$, then $a = -1$ and $\psi(ib) = b \cot \frac{b}{2} \geq 0$, so that

$$\begin{aligned} |h(x, \xi, u)| &\lesssim \left(\frac{b}{\sin b}\right)^\alpha e^{-\psi(ib)R^2 - 2b_0u + 2b_0u\frac{\epsilon}{b_0}} \\ &\lesssim \left(\frac{1}{\epsilon}\right)^\alpha e^{-2b_0u + 2b_0u\frac{\epsilon}{b_0}} \\ &= \left(\frac{1}{\epsilon}\right)^\alpha e^{-(1-\frac{\epsilon}{b_0})d_{CC}(\zeta, \eta)^2}, \end{aligned}$$

since again $2b_0u = 2\pi u = d_{CC}(\zeta, \eta)^2$.

Setting $\epsilon = \frac{1}{1+d_{CC}(\zeta, \eta)^2}$, which optimizes this inequality (up to a constant), gives us the desired result:

$$|h(x, \xi, u)| \lesssim (1 + d_{CC}(\zeta, \eta)^2)^\alpha e^{-d_{CC}(\zeta, \eta)^2}. \quad (3.77)$$

□

This gives the following main result:

Theorem 3.4.6. For $\zeta = (x, 0)$, $\eta = (\xi, u) \in \mathbb{R}^{n+1}$ we have

$$|K_t(x, \xi, u)| \lesssim t^{-\frac{n}{2}-1} \min \left\{ 1 + \frac{d_{CC}(\zeta, \eta)}{|x + \xi|}, 1 + \frac{d_{CC}(\zeta, \eta)^2}{4t} \right\}^\alpha e^{-\frac{1}{4t}d_{CC}(\zeta, \eta)^2}, \quad (3.78)$$

with $\alpha = \max(\frac{n}{2} - 1, 0)$.

Proof. The proof is almost trivial, if one uses (3.61) and the homogeneity of d_{CC} . □

Curriculum Vitae

Persönliche Daten

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch meinen akademischen Lehrer, selbst angefertigt habe. Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Des Weiteren versichere ich, noch keinen Promotionsversuch unternommen zu haben.

Kiel, den 18. März 2008

Nomenclature

\mathbb{N}	set of all positive integers
\mathbb{N}_0	$\mathbb{N} \cup \{0\}$
\mathbb{Z}	set of all integers
\mathbb{Z}^*	$\mathbb{Z} \setminus \{0\}$
\mathbb{R}	set of all real numbers
$\mathbb{R}_{>0}$	set of all positive real numbers
\mathbb{C}	set of all complex numbers
$\delta_{i,j}$	Kronecker-delta, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$
\Re	real part of a complex number
\Im	imaginary part of a complex number
e_i	i -th standard unit vector in \mathbb{R}^n
I	identity matrix. The dimension depends on the context.
A^t	transpose of a matrix A
span	\mathbb{R} -linear span
kern	kernel of a linear map (or matrix)
im	image of a linear map (or matrix)
$ES(A, \lambda)$	eigenspace of a matrix A corresponding to the eigenvalue λ
TM, T_pM	tangent bundle (space at $p \in M$) of a manifold M
T^*M, T_p^*M	cotangent bundle (space at $p \in M$) of a manifold M
$T^\#M, T_p^\#M$	cotangent bundle (space at $p \in M$) with the zero section removed

- $|\cdot|$ euclidian norm
- $d_{CC}(\cdot, \cdot)$ Carnot-Carathéodory distance, page 17
- $|\cdot|_{CC}$ Carnot-Carathéodory norm, page 49
- \mathfrak{h}_ν Heisenberg algebra of dimension $2\nu + 1$, page 49
- \mathbb{H}_ν Heisenberg group of dimension $2\nu + 1$, page 49
- $\mathfrak{h}_{\nu,m}$ Heisenberg type algebra of dimension $2\nu + m$, page 49
- $\mathbb{H}_{\nu,m}$ Heisenberg type group of dimension $2\nu + m$, page 50
- \mathfrak{f} free nilpotent Lie algebra of step 2, page 50
- F free nilpotent Lie group of step 2, page 50
- Δ Laplacian on \mathbb{R}^n . $\Delta = \sum_{j=1}^n \partial_{x_j}^2$.
- \mathcal{L} sub-Laplacian, page 84
- G Grušin operator, page 95
- \lesssim $X \lesssim Y$ denotes the estimate $X \leq CY$ for some absolute constant $C > 0$.

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