



Hausdorff measure of escaping and Julia sets of exponential maps

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Zusammenfassung

Im Jahr 1987 bewies Curt McMullen, dass die Hausdorffdimension der Juliamenge $\mathcal{J}(E_\lambda)$ einer beliebigen Exponentialfunktion $E_\lambda(z) = \lambda e^z$ gleich 2 ist, jedoch andererseits Juliamengen hyperbolischer Exponentialfunktionen Lebesguesche Nullmengen sind. Dies wirft die Frage nach einer geeigneten 'Eichfunktion' für Juliamengen von hyperbolischen Exponentialfunktionen auf. Eine Eichfunktion ist hierbei eine monoton wachsende, rechtsseitig stetige Funktion $h : [0, \varepsilon) \rightarrow \mathbb{R}_{\geq 0}$. Es kann gezeigt werden, dass jeder Eichfunktion durch

$$\mathcal{H}^h(A) := \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} h(\text{diam } A_i) \mid \bigcup_{i=1}^{\infty} A_i \supset A, \text{ diam } A_i < \delta \text{ für alle } i \right\}$$

ein metrisches äußeres Maß auf \mathbb{C} zugeordnet werden kann, das Hausdorffmaß bezüglich h (die Eichfunktionen $h^s(t) = t^s$ liefern das s -dimensionale äußere Hausdorffmaß). McMullens Resultate besagen also, dass im hyperbolischen Fall

$$\mathcal{H}^{h^2}(\mathcal{J}(E_\lambda)) = 0$$

und

$$\mathcal{H}^{h^s}(\mathcal{J}(E_\lambda)) = \infty \text{ für } s < 2.$$

Diese Arbeit gibt eine Antwort auf die Frage, welches Wachstum eine Eichfunktion, bezüglich der das Hausdorffmaß der Juliamenge einer Exponentialfunktion E_λ mit $0 < \lambda < 1/e$ positiv und endlich ist, besitzen muss:

Satz. Sei $\lambda_0 \in (0, 1/e)$ und Φ_{λ_0} die inverse Funktion der Linearisierungsabbildung von E_{λ_0} nahe des reellen abstoßenden Fixpunkts β_{λ_0} . Dann existiert eine Konstante $K_{\lambda_0} > 0$ mit der folgenden Eigenschaft: Sei $h(t) = t^2 g(t)$ eine Eichfunktion.

(a) Falls

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_{\lambda_0},$$

so gilt $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$ für alle $\lambda \in \mathbb{C} \setminus \{0\}$. Das Maß \mathcal{H}^h ist sogar nicht σ -endlich auf $\mathcal{J}(E_\lambda)$.

(b) Falls

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_{\lambda_0},$$

so gilt $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$ falls $\lambda \in (0, 1/e)$.

Ferner zeigen wir, dass der obige Satz richtig bleibt, wenn wir anstatt der Juliamenge einer Exponentialfunktion E_λ mit $0 < \lambda < 1/e$ die entkommene Menge $I(E_\lambda)$ einer *beliebigen* oder die Juliamenge einer *hyperbolischen* Exponentialfunktion betrachten. Allerdings muss dann a priori die Konstante K_{λ_0} in (b) durch eine kleinere Konstante K ersetzt werden, welche nicht nur von λ_0 , sondern auch von λ abhängt.

Abstract

In 1987, Curt McMullen proved that the Hausdorff dimension of the Julia set $\mathcal{J}(E_\lambda)$ of an arbitrary exponential map $E_\lambda(z) = \lambda e^z$ is 2, whereas Julia sets of hyperbolic exponentials have zero Lebesgue measure. This gives rise to the question about a suitable 'gauge function' for Julia sets of hyperbolic exponential maps. A gauge function is a monotonically increasing function $h : [0, \varepsilon) \rightarrow \mathbb{R}_{\geq 0}$ which is continuous from the right. It can be shown that every gauge function defines a metric outer measure on \mathbb{C} by

$$\mathcal{H}^h(A) := \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} h(\text{diam } A_i) \mid \bigcup_{i=1}^{\infty} A_i \supset A, \text{ diam } A_i < \delta \text{ for every } i \right\},$$

the Hausdorff measure with respect to h (the gauge functions $h^s(t) = t^s$ lead to s -dimensional Hausdorff measure). So McMullen's results show that in the hyperbolic case,

$$\mathcal{H}^{h^2}(\mathcal{J}(E_\lambda)) = 0$$

and

$$\mathcal{H}^{h^s}(\mathcal{J}(E_\lambda)) = \infty \text{ for } s < 2.$$

This thesis gives an answer to the question what order of growth a gauge function with respect to which the Hausdorff measure of the Julia set of an exponential map E_λ (where $0 < \lambda < 1/e$) is positive and finite must have:

Theorem. Let $\lambda_0 \in (0, 1/e)$ and Φ_{λ_0} the inverse function of the linearizing map of E_{λ_0} near the real repelling fixed point β_{λ_0} . Then there exists a constant $K_{\lambda_0} > 0$ with the following property: Let $h(t) = t^2 g(t)$ be a gauge function.

(a) If

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_{\lambda_0},$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. The measure \mathcal{H}^h is not even σ -finite on $\mathcal{J}(E_\lambda)$.

(b) If

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_{\lambda_0},$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$ if $\lambda \in (0, 1/e)$.

We prove further that the above theorem remains true if we consider the escaping set $I(E_\lambda)$ of an *arbitrary* or the Julia set of a *hyperbolic* exponential map instead of the Julia set of E_λ with $0 < \lambda < 1/e$. However, the constant K_{λ_0} in (b) a priori has to be replaced by a smaller constant K which not only depends on λ_0 , but also on λ .

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1 Introduction

The theory of iteration of analytic functions began at the end of the 19th century with the work of Schröder and Koenigs, who investigated how the iterates of an analytic function behave near a fixed point. After Montel's work on normal families in 1912 and 1916 ([22],[23]), the global theory started with independent works by Pierre Fatou and Gaston Julia. While Julia [16] restricted his work to the case of rational maps, Fatou [12] was the first who studied the iteration of transcendental entire functions. For a long time after that, the subject lay dormant. In the 1960s, it was mainly Baker who proved many results in transcendental entire dynamics, but these were not really noticed by the mathematical community at first. This situation changed dramatically in the 1980s with the ability to produce stunning computer pictures of Julia sets, Sullivan's results (who introduced quasiconformal mappings into the theory of holomorphic dynamics in the proof of his non-wandering-domains theorem) and the work of Douady and Hubbard on polynomial dynamics (especially on the Mandelbrot set). By now, a lot of interesting results have been proven, but there are still many open questions to solve. We now give a short introduction into the field of holomorphic dynamics.

For every entire function f , we can partition the complex plane into the Fatou set $\mathcal{F}(f)$, which consists of all $z \in \mathbb{C}$ such that the family of iterates (f^n) is normal in a neighborhood of z , and its complement, the Julia set $\mathcal{J}(f)$. The Julia set usually is topologically a quite complicated set, and many (at the first glance) astonishing results concerning this set hold. One of them will be mentioned later. Another set which has been worked on frequently is the set $I(f)$ of escaping points, i.e. the points which tend to infinity under iteration of f . Eremenko [8] showed (using Wiman-Valiron-theory) that for any entire function f , $I(f)$ is non-empty. It is then straightforward to show that $\partial I(f) = \mathcal{J}(f)$.

In this thesis, we will mostly be concerned with the exponential family, which consists of all functions $E_\lambda(z) := \lambda e^z$, where $\lambda \in \mathbb{C} \setminus \{0\}$. Among all one-parameter families in transcendental dynamics, the exponential family is the simplest and most important one. There are various results in complex dynamics saying that the dynamics of a function is more or less determined by its singular (i.e. critical and asymptotic) values. As each member of the exponential family has only one singular value (namely, the asymptotic value 0), this family is kind of an analogue to the quadratic family in polynomial dynamics. In fact, if a transcendental entire function has only one singular value, it is conjugated to some exponential function. The fact that every exponential function has only one singular value has important consequences. Sullivan's non-wandering-domains theorem can be carried over to the class of entire functions with finitely many singular values, and a general result of Eremenko and Lyubich [10] implies that $I(E_\lambda)$ is always contained in $\mathcal{J}(E_\lambda)$ (in fact, this result is true for any function for which the set of its singular values is bounded). They prove this result by introducing logarithmic coordinates, a method that is still used frequently.

The work on the Julia and escaping sets of exponential functions started with Devaney and Krych [5] who proved that for $0 < \lambda < 1/e$, the escaping set consists

of curves (so-called 'Devaney hairs' or 'rays'), and $\mathcal{J}(E_\lambda)$ consists of these curves together with their endpoints. This result was later generalized by Schleicher and Zimmer [28], who showed that the escaping set of every exponential function consists of curves. Although this suggests that the structure of Julia sets of exponential maps is fairly easy to understand, it is far from being simple. We mention one theorem here that perfectly describes this fact: Karpińska showed that for $0 < \lambda < 1/e$, the Hausdorff dimension of the set of endpoints of the hairs is 2 [17], and shortly after that, she proved that the Hausdorff dimension of the (uncountably many) hairs which form $\mathcal{J}(E_\lambda)$ with their endpoints removed is 1 [18] (the first result follows from the second and a theorem by McMullen stated below). This means that the set of endpoints of the hairs is a larger set than the union of all hairs without their endpoints from the point of view of Hausdorff dimension. This result was generalized later by Schleicher and Zimmer to the hairs which form $I(E_\lambda)$ for arbitrary parameters [28] and by Schleicher to the cosine family [27].

Julia sets are an (if not the most important) example of 'fractals', and a major question in the theory is to determine (or at least to estimate) their Hausdorff dimension. In the case of the exponential family, this problem was solved by McMullen in 1987. He proved [20] that $\mathcal{J}(E_\lambda)$ has Hausdorff dimension 2 for every λ . This result was our starting point, and some of the techniques McMullen used to prove it will be applied in this thesis as well. He also proved that if E_λ is hyperbolic, i.e. there exists an attracting periodic cycle, then $\mathcal{J}(E_\lambda)$ has zero Lebesgue-measure. The latter result was obtained independently by Eremenko and Lyubich in [9].

McMullen's results immediately lead to the question what an 'optimal' Hausdorff gauge function for a hyperbolic exponential might be. Recall that Hausdorff measure of a set $A \subset \mathbb{C}$ with respect to a gauge function $h : [0, \varepsilon) \rightarrow \mathbb{R}_{\geq 0}$ (i.e. a function which is continuous to the right and increasing) is defined as

$$\mathcal{H}^h(A) := \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} h(\text{diam } A_i) \mid \bigcup_{i=1}^{\infty} A_i \supset A, \text{diam } A_i < \delta \text{ for all } i \right\},$$

which generalizes the usual α -dimensional Hausdorff measure (see section 2.4). In [20], McMullen remarks (without proof) that for every λ , the Julia set of E_λ has infinite Hausdorff measure with respect to the gauge function $h(t) = t^2 \log^k(1/t)$, where \log^k denotes the k -th iterate of the logarithm. So it seems natural to consider a function that grows slower than any iterate of the logarithm, and it turns out that the inverse function of the linearizing map for a real hyperbolic parameter (with respect to the repelling fixed point) is the correct choice: Recall that for $\lambda_0 \in (0, 1/e)$, E_{λ_0} has a unique real repelling fixed point β_{λ_0} , and the well-known linearization theorem of Koenigs says that there exists a function S_{λ_0} , defined in a neighborhood U of 0 such that

$$E_{\lambda_0}(S_{\lambda_0}(z)) = S_{\lambda_0}(\beta_{\lambda_0} z) \text{ if } z, \beta_{\lambda_0} z \in U.$$

This functional equation admits an analytic continuation of S_{λ_0} to the whole complex plane (this result is due to Poincaré). It is not difficult to show that $S_{\lambda_0}|_{\mathbb{R}}$ grows faster than any iterate of the (standard) exponential function (see section 5.2), so the inverse function Φ_{λ_0} grows slower than any iterate of the logarithm. The main result of this thesis is the following:

Theorem 8.2.1 *For every $\lambda_0 \in (0, 1/e)$, there exists a constant $K_{\lambda_0} > 0$ with the following property: Let $h(t) = t^2g(t)$ be a gauge function.*

(a) *If*

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_{\lambda_0},$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. The measure \mathcal{H}^h is not even σ -finite on $\mathcal{J}(E_\lambda)$.

(b) *If*

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_{\lambda_0},$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$ if $\lambda \in (0, 1/e)$.

This result shows that if $h(t) = t^2\Phi_{\lambda_0}(1/t)^K$ for some $K > 0$, then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$ if $K > K_{\lambda_0}$, and $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$ if $K < K_{\lambda_0}$. The case $K = K_{\lambda_0}$ remains open. If we combine this result with recent results by Rempe [25] and Astala and Clop [2], then we get the following two theorems, which fully describe the situation for escaping sets and Julia sets of hyperbolic exponentials (in this case, we need another result by Urbański and Zdunik [30]):

Theorem 8.2.3 *For every $\lambda_0 \in (0, 1/e)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, there exist constants $K_1, K_2 > 0$ (where K_1 depends only on λ_0) with the following property: Let $h(t) = t^2g(t)$ be a gauge function.*

(a) *If*

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_1,$$

then $\mathcal{H}^h(I(E_\lambda)) = \infty$. The measure \mathcal{H}^h is not even σ -finite on $I(E_\lambda)$.

(b) *If*

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_2,$$

then $\mathcal{H}^h(I(E_\lambda)) = 0$.

Theorem 8.2.4 *For every $\lambda_0 \in (0, 1/e)$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that E_λ is hyperbolic, there exist constants $K_1, K_2 > 0$ (where K_1 depends only on λ_0) with the following property:*

Let $h(t) = t^2g(t)$ be a gauge function.

(a) *If*

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_1,$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$. The measure \mathcal{H}^h is not even σ -finite on $\mathcal{J}(E_\lambda)$.

(b) If

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_2,$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$.

The thesis is organized as follows: In section 2, we review some basic results from complex dynamics as well as from function and measure theory. We introduce Hausdorff measure with respect to gauge functions and prove a simple but powerful lemma due to Frostman, which will help us to find a lower bound for Hausdorff measure of Julia sets.

Section 3 is concerned with distortion of holomorphic maps. We show that if a map has small distortion, then the image of a square is again 'almost' a square - a result which seems quite clear, but we could not find a proof of the result needed in the literature, so we include it here. We also give elementary upper bounds for distortion of maps which can be continued univalently to a domain that compactly contains their domain of definition (these arise from the Koebe theorems) and prove that if f is expanding, the distortion of any inverse branch φ of $(f^n)^{-1}$ does not depend on n (this result was also proved by McMullen in [20], but we prove it differently, using the Koebe theorems).

Section 4 describes how we can use the lemma by Frostman to obtain a sufficient condition for a gauge function h and a set A such that A has infinite Hausdorff measure with respect to h . The ideas in this section (especially how the set A is constructed) are mainly due to McMullen.

In section 5, we prove some elementary results concerning the linearizing map for real hyperbolic parameters, in particular that this function grows faster than any iterate of the exponential map.

In section 6, we develop a sufficient condition for a gauge function h such that $\mathcal{J}(E_\lambda)$ has infinite (not even σ -finite) Hausdorff measure with respect to h with the methods introduced in section 4. The ideas of the proof are again due to McMullen, but we have to estimate things more carefully.

Section 7 gives a sufficient condition for a gauge function h such that $\mathcal{J}(E_\lambda)$ has zero Hausdorff measure with respect to h . In contrast to section 6, this h a priori depends on λ and λ is restricted to values between 0 and $1/e$. The main idea here is that Julia sets of such exponentials are 'almost porous', so the concept of porosity will be explained in this section as well.

In section 8, we get rid of the condition that the function h we developed in section 7 depends on λ by showing that if $\beta_{\lambda_1}^{\gamma_1} = \beta_{\lambda_2}^{\gamma_2}$, the two resulting gauge functions from section 7 have the same growth, i.e. their quotient can be estimated from above and below by positive constants. After that, we formulate and prove the main Theorem 8.2.1 and describe how Theorem 8.2.3 and Theorem 8.2.4 follow from this.

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2 Preliminaries

2.1 Notations

Let \mathbb{C} be the complex plane. If $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function, we write

$$f^n := \underbrace{f \circ \dots \circ f}_{n \text{ times}}$$

for the n -th iterate and $f^{(n)}$ for the n -th derivative of f . For $z \in \mathbb{C}$, let $\Re z$ and $\Im z$ denote the real and imaginary parts of z . If $z_0 \in \mathbb{C}$ and $r > 0$, we write $D(z_0, r)$ for the disc in \mathbb{C} with center z_0 and radius r with respect to the euclidean metric. By $\mathbb{D} := D(0, 1)$ we denote the open unit disc in \mathbb{C} . For $\theta \in \mathbb{R}$, let $Q(z_0, r, \theta)$ be the disc with center z_0 and radius $r/2$ with respect to the l_∞ -metric, rotated by the angle θ . In other words, $Q(z_0, r, \theta)$ is a rotated square with center z_0 and side length r . Thus we have

$$Q(0, r, 0) = \left\{ z \in \mathbb{C} \mid \max \{ \Re z, \Im z \} < \frac{r}{2} \right\}$$

and

$$Q(z_0, r, \theta) = z_0 + e^{i\theta} Q(0, r, 0).$$

If the angle θ is not important, we will suppress it and just write $Q(z_0, r)$ in order to increase readability. If $z, w \in \mathbb{C}$, let $|z - w|$ be the euclidean distance between z and w and $[z, w]$ be the set of points which lie on the euclidean geodesic connecting z and w . For $A, B \subset \mathbb{C}$, we define the distance between A and B as

$$\text{dist}(A, B) := \inf \{ |a - b| \mid a \in A, b \in B \}.$$

If A consists of only one point z , we will write $\text{dist}(z, B)$ instead of $\text{dist}(\{z\}, B)$. For any set $\emptyset \neq A \subset \mathbb{C}$, let $\text{diam}(A)$ be its diameter, i.e. $\text{diam}(A) := \sup_{z, w \in A} |z - w|$, and let ∂A and \bar{A} denote the boundary and closure of A . If $A \subset \mathbb{C}$ is (Lebesgue-) measurable, we denote its Lebesgue measure by $|A|$. If $A, B \subset \mathbb{C}$ are (Lebesgue-) measurable and $0 < |B| < \infty$, we write $\text{dens}(A, B)$ for the density of A in B , i.e.

$$\text{dens}(A, B) := \frac{|A \cap B|}{|B|}.$$

We end each proof with the symbol \square , each remark with a \spadesuit and every definition with a \diamond . Other notations are introduced when they are used for the first time. These include: The class S (Definition 2.3.2), Hausdorff measure (Definition 2.4.1), the distortion of a function (Definition 3.1.1), the exponential family (Definition 5.1.1) and the linearizing map (Theorem 2.2.5).

2.2 Some results from complex dynamics

We start with a short overview of the results in complex dynamics that we will use throughout this thesis. Almost all of the results in this chapter can be found in standard monographs on complex dynamics (e.g. [21], [4]). Recall that for a transcendental

entire function f , we can divide the complex plane into two disjoint sets, the **Fatou set** $\mathcal{F}(f)$ and the **Julia set** $\mathcal{J}(f)$. The Fatou set consists of those points which have a neighborhood on which the family of iterates $\{f^n\}$ is normal, or equivalently equicontinuous (by the Arzela-Ascoli theorem). The Julia set is the complement of the Fatou set and it can be shown that it is always a perfect set. Both sets are completely invariant, and we have $\mathcal{F}(f^n) = \mathcal{F}(f)$ for every n . Another easy, but useful property is that if g is conformally conjugated to f via T , i.e. $g = T \circ f \circ T^{-1}$, then we have $\mathcal{F}(g) = T(\mathcal{F}(f))$. We will also need the **escaping set** $I(f)$, which consists of all points whose orbit tends to ∞ under iteration.

Let $\text{sing}(f^{-1})$ denote the set of all points which have a neighborhood on which at least one branch of f^{-1} cannot be defined. A simple fact concerning the singularities of the n -th iterate of a function is

Lemma 2.2.1 *For every transcendental entire function f and every n ,*

$$\text{sing}((f^n)^{-1}) \subset \bigcup_{j=0}^{n-1} f^j(\text{sing}(f^{-1})).$$

The **postcritical set** $P(f)$ is defined as

$$P(f) := \overline{\bigcup_{n=1}^{\infty} \text{sing}((f^n)^{-1})}.$$

We say that an entire function f is of **finite type** if $\text{sing}(f^{-1})$ is finite. We have the following two results:

Theorem 2.2.2 ([10]) *For every f of finite type, $I(f) \subset \mathcal{J}(f)$.*

Theorem 2.2.3 ([14]) *If f is of finite type, then every connected component U of the Fatou set is preperiodic, i.e. there exists a component V of $\mathcal{F}(f)$ and $m \in \mathbb{N}_0, n \in \mathbb{N}$ such that $f^m(U) \subset V$ and $f^n(V) \subset V$.*

The latter result is an analogue of Sullivan's non-wandering-domains theorem for transcendental entire maps with finitely many singular values. The periodic Fatou components can be classified as follows:

Theorem 2.2.4 *Let f be an entire function and U be a connected component of $\mathcal{F}(f)$. If U is periodic with minimal period n , then exactly one of the following four statements holds:*

- (a) U is the immediate basin of an **attracting or superattracting periodic point**, i.e. there exists $z_0 \in U$ with $f^n(z_0) = z_0$, $|(f^n)'(z_0)| < 1$ and $f^{nk} \rightarrow z_0$ locally uniformly in U as $k \rightarrow \infty$.
- (b) U is a **parabolic basin**, i.e. there exists $z_0 \in \partial U$ such that $(f^n)'(z_0) = 1$ and $f^{nk} \rightarrow z_0$ locally uniformly in U as $k \rightarrow \infty$.

- (c) U is a Baker domain, i.e. $f^{nk} \rightarrow \infty$ locally uniformly (with respect to the spherical metric) in U as $k \rightarrow \infty$.
- (d) U is a Siegel disc, i.e. $f^n|_U$ is conjugated to an irrational rotation of the unit disc.

If f is of finite type, then type (c) does not occur and every non-periodic component U is some preimage of a periodic component of type (a), (b) or (d).

As in the case of a Siegel disc, we can also conjugate f to a linear map near an attracting or repelling periodic point (a periodic point z_0 of period n is called **attracting** resp. **repelling** if $0 < |(f^n)'(z_0)| < 1$ resp. $|(f^n)'(z_0)| > 1$): We prove this theorem, which is due to Kœnigs, only for fixed points. The general case follows immediately. We include a proof here (which is different from the one in most textbooks) because we will need to know later how the conjugating function is constructed.

Theorem 2.2.5 *Let f be a transcendental entire function and let z_0 be an attracting or repelling fixed point of f . Set $\lambda := f'(z_0)$. Then there exists a neighborhood U of 0 , a neighborhood V of z_0 and a uniquely determined biholomorphic function $\Phi : V \rightarrow U$ which satisfies*

$$\Phi(z_0) = 0, \Phi'(z_0) = 1 \text{ and } \Phi(f(z)) = \lambda\Phi(z) \text{ for all } z \text{ such that } z, f(z) \in V. \quad (2.2.1)$$

Φ is called Kœnigs' function, and the inverse function $S := \Phi^{-1}$ is called the linearizing map (or linearizer) of f near z_0 and satisfies $S(0) = z_0, S'(0) = 1$ and the functional equation

$$f(S(z)) = S(\lambda z) \text{ for all } z \text{ such that } z, \lambda z \in U.$$

Proof. First we prove uniqueness by showing that S is uniquely determined. We may assume without loss of generality that $z_0 = 0$. Let S_1 and S_2 be two functions that satisfy $S_i(0) = 0, S_i'(0) = 1$ and $f(S_i(z)) = S_i(\lambda z)$ in a neighborhood of 0 for $i = 1, 2$. Then $h := S_2^{-1} \circ S_1$ satisfies $h(0) = 0$ and $h'(0) = 1$, as well as

$$\lambda h(z) = h(\lambda z) \quad (2.2.2)$$

for z in a neighborhood of 0 . If $z + \sum_{k=2}^{\infty} a_k z^k$ denotes the Taylor expansion of h around 0 , it follows from (2.2.2) that

$$\lambda z + \sum_{k=2}^{\infty} a_k \lambda^k z^k = \lambda z + \sum_{k=2}^{\infty} a_k \lambda z^k.$$

But this is only possible if $a_k = 0$ for $k \geq 2$, which implies that $h(z) = z$ and hence $S_1(z) = S_2(z)$ in a neighborhood of 0 .

Now we prove existence: First, we can assume without loss of generality that z_0 is attracting (if z_0 is repelling, it is attracting for the branch of f^{-1} that fixes z_0). By

conjugating with $z \mapsto \frac{1}{z-z_0}$, we get a function g that fixes ∞ and (with $\nu := 1/\lambda$) has the form

$$g(z) = \nu z + R(z)$$

in a neighborhood W of ∞ . It is easy to see that there exists $C > 0$ such that $|R(z)| \leq C$ for all $z \in W$, which implies that we may assume without loss of generality that W is chosen such that $g(W) \subset W$. Now we define

$$\Phi_n(z) := \frac{g^n(z)}{\nu^n}$$

for all $z \in U$. We now show that Φ_n converges locally uniformly on U to a function Φ . We have

$$\begin{aligned} |\Phi_n(z) - \Phi_{n+1}(z)| &= \left| \frac{1}{\nu^n} g^n(z) - \frac{1}{\nu^{n+1}} g(g^n(z)) \right| \\ &= \left| \frac{1}{\nu^n} g^n(z) - \frac{g^n(z)}{\nu^n} - \frac{R(g^n(z))}{\nu^{n+1}} \right| \\ &= \left| \frac{1}{\nu^{n+1}} R(g^n(z)) \right| \\ &\leq \frac{C}{|\nu|^{n+1}}. \end{aligned}$$

It follows that for given $\varepsilon > 0$, we have for all $m < n$ large enough that

$$|\Phi_m(z) - \Phi_n(z)| \leq \sum_{k=m+1}^n \left(\frac{1}{|\nu|} \right)^k C < \varepsilon,$$

which implies that (Φ_n) converges locally uniformly to a function Φ which satisfies

$$\Phi(g(z)) = \lim_{n \rightarrow \infty} \frac{g^{n+1}(z)}{\nu^n} = \nu \lim_{n \rightarrow \infty} \frac{g^{n+1}(z)}{\nu^{n+1}} = \nu \Phi(z).$$

After reversing the conjugation, it is immediate that Φ satisfies (2.2.1). The fact that $\Phi'(0) = 1$ follows from Weierstrass' theorem. Further, the corresponding statements about S follow directly from the properties of Φ . \square

It was proven independently by Fatou and Julia that the Julia set of a rational map is the closure of repelling periodic points. For entire maps, this is also true and was first shown by Baker [3]. To formulate a consequence this result, we need the following definition:

Definition 2.2.6 (exceptional points) Let f be a transcendental entire function. A point $z_0 \in \mathbb{C}$ is called **exceptional** if its backward orbit

$$O^-(z_0) := \{w \in \mathbb{C} \mid \exists n \in \mathbb{N}_0 : f^n(w) = z_0\}$$

is finite. \diamond

Remark: Note that there can be at most one exceptional point by Montel's Theorem (if there were two, the Julia set would consist of only finitely many points, which contradicts its perfectness). ♠

We are now able to formulate the beforementioned consequence of the density of repelling periodic points in the Julia set:

Theorem 2.2.7 (blow-up property of Julia sets) *Let f be a transcendental entire function and $U \subset \mathbb{C}$ be a domain with $U \cap \mathcal{J}(f) \neq \emptyset$. Let $K \subset \mathbb{C}$ be a compact set which does not contain any exceptional points. Then there exists $n \in \mathbb{N}$ such that*

$$f^n(U) \supset K.$$

Last but not least, we introduce the following special class of maps: We call f **hyperbolic** if $P(f)$ is bounded and $\text{dist}(\mathcal{J}(f), P(f)) > 0$. This is equivalent to saying that every singularity of f^{-1} is absorbed by a cycle of attracting periodic points, since it can be shown that the boundary of a Siegel disc is always contained in the postcritical set.

2.3 Tools from function theory

In this subsection, we provide some definitions, notations and theorems from function theory. Although these are quite elementary, it turns out that they are the main tools to prove our results. We will introduce class S , which is the object of interest in univalent function theory, then state the well-known Koebe distortion and growth theorems and finally prove an easy lemma which we will need in the succeeding sections. But first, we state an easy result which follows directly from Cauchy's integral formula.

Lemma 2.3.1 *If a holomorphic function f maps $D(z_0, r)$ into a disc of radius s , then*

$$|f'(z_0)| \leq \frac{s}{r}.$$

Now we define univalent functions and the important class S .

Definition 2.3.2 (univalent, Class S) We say that a function f is **univalent** if it is holomorphic and injective in its domain. The class S consists of all univalent functions $f : \mathbb{D} \rightarrow \mathbb{C}$, normalized such that $f(0) = 0$ and $f'(0) = 1$:

$$S := \{f : \mathbb{D} \rightarrow \mathbb{C} \mid f \text{ univalent, } f(0) = 0, f'(0) = 1\}.$$

◇

Because every simply connected domain $U \neq \mathbb{C}$ can be mapped biholomorphically onto \mathbb{D} by the Riemann mapping theorem, class S is the main object of research in

the theory of univalent functions (see for example [24]). In our work, we only need some basic facts and well-known theorems about class S . The following two theorems can be found in the book mentioned above, as well as in many other monographs about univalent functions.

Theorem 2.3.3 (Koebe growth theorem) *For every $f \in S$ and $z \in \mathbb{D}$, we have*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3} \quad (2.3.1)$$

Theorem 2.3.4 (Koebe distortion theorem) *For every $f \in S$ and $z \in \mathbb{D}$, we have*

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2} \quad (2.3.2)$$

An immediate consequence of the growth and distortion theorems is the following corollary.

Corollary 2.3.5 *Let $z_0 \in \mathbb{C}$, $r > 0$, $f : D(z_0, r) \rightarrow \mathbb{C}$ be a univalent function and $z \in D(z_0, r)$. Then the following two double inequalities are true:*

$$r^2 |f'(z_0)| \frac{r - |z - z_0|}{(r + |z - z_0|)^3} \leq |f'(z)| \leq r^2 |f'(z_0)| \frac{r + |z - z_0|}{(r - |z - z_0|)^3} \quad (2.3.3)$$

$$r^2 |f'(z_0)| \frac{|z - z_0|}{(r + |z - z_0|)^2} \leq |f(z) - f(z_0)| \leq r^2 |f'(z_0)| \frac{|z - z_0|}{(r - |z - z_0|)^2} \quad (2.3.4)$$

Proof. The first double inequality follows from (2.3.1) and the second from (2.3.2) by considering the function

$$g(z) := \frac{f(z_0 + rz) - f(z_0)}{rf'(z_0)}.$$

Then g belongs to S and using the Koebe theorems for g leads to the desired inequalities for f . \square

The Koebe distortion theorem has another immediate corollary.

Theorem 2.3.6 (S compact) *S is a compact set with respect to the topology of locally uniform convergence.*

Proof. For $f \in S$, we have

$$|f(z)| = |f(z) - f(0)| \leq \frac{|z|}{(1 - |z|)^2}$$

by the Koebe distortion theorem. It follows that S is locally bounded and hence normal by Montel's theorem. So every sequence (f_n) in S has a locally uniformly convergent subsequence (f_{n_k}) . By Weierstrass' theorem, the limit function g is holomorphic and $f'_{n_k} \rightarrow g'$ locally uniformly. It follows that $g(0) = 0$ and $g'(0) = 1$.

By Hurwitz' theorem, g is either injective or constant. The latter cannot be (because $g'(0) = 1$), and so g is univalent and the theorem is proved. \square

We now prove a consequence of the above theorem which - in contrast to the Koebe theorems - shows that if z is near 0 and $f \in S$, then $|f(z) - z|$ is $\mathcal{O}(|z|)$, the constant being independent of f (from the Koebe distortion theorem it only follows that $|f(z)|$ is 'near' $|z|$ if $|z|$ is small). We do not give sharp estimates like they appear in the Koebe theorems because they are not needed in the future. This makes the proof a lot easier.

Theorem 2.3.7 *For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following statement is true: For every $f \in S$ and every $z \in \mathbb{D}$ with $|z| < \delta$, we have*

$$\left| \frac{f(z)}{z} - 1 \right| < \varepsilon.$$

Proof. This is an application of the compactness property of S . Suppose that the statement is wrong. Then there exists a sequence (z_n) converging to zero and a sequence (f_n) in S such that

$$\left| \frac{f_n(z_n)}{z_n} - 1 \right| \geq \varepsilon \text{ for all } n. \quad (2.3.5)$$

Because S is compact, (f_n) has a locally uniformly convergent subsequence. Without loss of generality we may assume that (f_n) itself converges to a function $f \in S$. Because $f'(0) = 1$ and $f(0) = 0$, there exists $\delta > 0$ such that $|f(z)/z - 1| < \varepsilon/2$ for all z with $|z| < \delta$. In other words, because $z_n \rightarrow 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{f(z_n)}{z_n} - 1 \right| < \frac{\varepsilon}{2} \text{ for every } n \geq n_0. \quad (2.3.6)$$

Furthermore, there exists $n_1 \in \mathbb{N}$ with

$$\left| \frac{f_n(z_n) - f(z_n)}{z_n} \right| < \frac{\varepsilon}{2} \text{ for every } n \geq n_1, \quad (2.3.7)$$

because $|f_n(z) - f(z)| = \mathcal{O}(|z|^2)$ as $z \rightarrow 0$. It follows by (2.3.6) and (2.3.7) that

$$\left| \frac{f_n(z_n)}{z_n} - 1 \right| \leq \left| \frac{f_n(z_n) - f(z_n)}{z_n} \right| + \left| \frac{f(z_n)}{z_n} - 1 \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } n \geq \max\{n_0, n_1\},$$

a contradiction to (2.3.5). \square

Finally, we will at some point use the famous Koebe 1/4-theorem:

Theorem 2.3.8 (Koebe 1/4-theorem) *Let $z_0 \in \mathbb{C}$ and $r > 0$. If $f : D(z_0, r) \rightarrow \mathbb{C}$ is univalent, then*

$$f(D(z_0, r)) \supset D\left(f(z_0), \frac{1}{4}|f'(z_0)|r\right).$$

2.4 Tools from measure theory

In this subsection, we provide the measure-theoretical definitions and foundations that we will need. We introduce the concept of Hausdorff measures and then prove two results that will help us to estimate Hausdorff measure of Julia sets from below.

Definition 2.4.1 (gauge function, δ -covering, \mathcal{H}^h) A **gauge function** is a non-decreasing function $h : [0, \infty) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is continuous from the right. For any $A \subset \mathbb{C}$ and $\delta > 0$, we call a countable family $\{A_i\}$ of subsets of \mathbb{C} which satisfies

$$\bigcup_{i=1}^{\infty} A_i \supset A \text{ and } \text{diam } A_i < \delta \text{ for all } i$$

a **(countable) δ -covering** of the set A . We define

$$h(A) := \begin{cases} h(\text{diam } A) & , A \neq \emptyset \\ 0 & , A = \emptyset \end{cases}$$

We then set

$$\mathcal{H}_\delta^h(A) := \inf \left\{ \sum_{i=1}^{\infty} h(A_i) \mid \{A_i\} \text{ is a } \delta\text{-covering of } A \right\}.$$

Because \mathcal{H}_δ^h increases as δ decreases, the limit $\lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(A)$ exists in $\mathbb{R} \cup \{\infty\}$ and we define

$$\mathcal{H}^h(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(A).$$

◇

It is quite remarkable that this fairly general definition always leads to a metric outer measure on \mathbb{C} [26]:

Theorem 2.4.2 *For any gauge function h , the set function \mathcal{H}^h defines a metric outer measure on \mathbb{C} , i.e. \mathcal{H}^h satisfies*

- (a) $0 \leq \mathcal{H}^h(A) \leq \infty$ for all $A \subset \mathbb{C}$
- (b) $\mathcal{H}^h(\emptyset) = 0$
- (c) If $A \subset B \subset \mathbb{C}$ then $\mathcal{H}^h(A) \leq \mathcal{H}^h(B)$
- (d) If (A_n) is a sequence of subsets of \mathbb{C} then

$$\mathcal{H}^h \left(\bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mathcal{H}^h(A_i)$$

and \mathcal{H}^h is a measure on the Borel- σ -field generated by the open subsets of \mathbb{C} . Further, considering only countable coverings consisting of open (or closed) subsets of \mathbb{C} (instead of arbitrary sets as in the definition) leads to an equivalent outer measure.

We call the outer measure \mathcal{H}^h the **Hausdorff measure** with respect to the gauge function h .

Remarks: (i) Even if the condition that h is continuous from the right was dropped, \mathcal{H}^h would still be a metric outer measure on \mathbb{C} . But the second property of \mathcal{H}^h fails to hold: If one only considers countable coverings consisting of just *open* subsets (instead of coverings by arbitrary sets), the two resulting measures would not be equivalent any more. A simple example is given by the function

$$h(t) := \begin{cases} 1 & , t > 0 \\ 0 & , t = 0 \end{cases}$$

If A is any subset of \mathbb{C} consisting of just one point, then clearly $\mathcal{H}^h(A) = 0$, but considering only coverings consisting of open sets leads to measure 1, which shows that the two resulting outer measures are not equivalent.

(ii) In the succeeding sections, the gauge function will often be defined only on a small interval $[0, \varepsilon)$. But this does not matter, because it follows immediately from the definition that \mathcal{H}^h is completely determined by the values of h on $[0, \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small. Therefore, any increasing non-negative function which is continuous from the right and defined on an interval $[0, \varepsilon)$ will be called a gauge function.

(iii) For $s \geq 0$, the well-known s -dimensional Hausdorff measure is a special case of the above definition - one gets it by considering the gauge function $h^s(t) := t^s$. The Hausdorff dimension of a set A , $\text{HD}(A)$, is then defined by

$$\text{HD}(A) := \sup \{s \geq 0 \mid \mathcal{H}^{h^s}(A) = \infty\}.$$



We now define a partial order on the family of all gauge functions.

Definition 2.4.3 For two gauge functions h_1 and h_2 , we write

$$h_1 \prec h_2 \Leftrightarrow \frac{h_2(t)}{h_1(t)} \rightarrow 0 \text{ as } t \rightarrow 0$$

and we say that h_1 corresponds to a smaller generalized dimension than h_2 . ◇

Remarks: (i) The partial order \prec is *not* a total order as there exist gauge functions h_1 and h_2 such that

$$\liminf_{t \rightarrow 0} \frac{h_2(t)}{h_1(t)} = 0 \text{ and } \limsup_{t \rightarrow 0} \frac{h_2(t)}{h_1(t)} = \infty. \quad (2.4.1)$$

An example is given by the following construction:
We define for a small $\varepsilon > 0$

$$h_1(t) := \begin{cases} \frac{1}{2^{(4^k)}} & , t \in \left[\frac{1}{4(k+1)-\varepsilon}, \frac{1}{4k+\varepsilon} \right] \\ \text{linear} & , \text{otherwise} \end{cases}$$

and

$$h_2(t) := \begin{cases} \frac{1}{2^{(4^k/2)}} & , t \in \left[\frac{1}{4k+2-\varepsilon}, \frac{1}{4k-2+\varepsilon} \right] \\ \text{linear} & , \text{otherwise} \end{cases}$$

Here 'linear' means that on $(1/(4(k+1)+\varepsilon), 1/(4k-\varepsilon))$, h_1 is a linear map such that h_1 becomes continuous (and analogously for h_2).

If

$$t \in \left(\frac{1}{4(k+1)-\varepsilon}, \frac{1}{4k+2+\varepsilon} \right),$$

then it follows that

$$\frac{h_1(t)}{h_2(t)} = \frac{2^{(4^{k+1}/2)}}{2^{(4^k)}} = 2^{(4^{k+1})/2-4^k} = 2^{(4^k)} \rightarrow \infty \text{ as } k \rightarrow \infty.$$

On the other hand, if

$$t \in \left(\frac{1}{4k+2-\varepsilon}, \frac{1}{4k+\varepsilon} \right),$$

then

$$\frac{h_1(t)}{h_2(t)} = \frac{2^{(4^k/2)}}{2^{(4^k)}} = 2^{-(4^k/2)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Because $k \rightarrow \infty$ corresponds to $t \rightarrow 0$, (2.4.1) holds for h_1 and h_2 .

(ii) The last remark has an unpleasant consequence: Eggleston [6],[7] showed that the lack of a total ordering among gauge functions is reflected by the lack of a total ordering among sets: One cannot say in general that a set A is smaller or larger than a set B from the point of view of Hausdorff measures.

(iii) Suppose that h_1 and h_2 are gauge functions such that there exist constants c, C with

$$0 < c \leq \liminf_{t \rightarrow 0} \frac{h_2(t)}{h_1(t)} \leq \limsup_{t \rightarrow 0} \frac{h_2(t)}{h_1(t)} \leq C < \infty.$$

Then the Hausdorff measures \mathcal{H}^{h_1} and \mathcal{H}^{h_2} satisfy

$$c\mathcal{H}^{h_1}(A) \leq \mathcal{H}^{h_2}(A) \leq C\mathcal{H}^{h_1}(A)$$

for every $A \subset \mathbb{C}$. This follows directly from the definition. ♠

Next, we introduce the well-known concept of σ -finiteness and prove a theorem which we will need later on.

Definition 2.4.4 (σ -finite sets) Let h be a gauge function and $A \subset \mathbb{C}$. We say that A has σ -finite \mathcal{H}^h -measure if A is the union of countably many sets A_i with finite \mathcal{H}^h -measure. \diamond

Theorem 2.4.5 Let h_1 and h_2 be gauge functions with $h_1 \prec h_2$ and $A \subset \mathbb{C}$. If A has σ -finite \mathcal{H}^{h_1} -measure, then A has zero \mathcal{H}^{h_2} -measure.

Proof. First suppose that A has finite \mathcal{H}^{h_1} -measure M . Let $\varepsilon, \delta > 0$ be arbitrary. Because

$$\frac{h_2(t)}{h_1(t)} \rightarrow 0 \text{ as } t \rightarrow 0$$

and h_2 is continuous from the right, it follows that $h_2(0) = 0$ and we can choose t_0 with $0 < t_0 < \delta$ and

$$h_2(t) \leq \frac{\varepsilon}{M+1} h_1(t) \text{ for } 0 \leq t \leq t_0.$$

Now choose a countable covering $\{A_i\}$ of A such that $\text{diam}(A_i) < t_0$ for all i and $\sum h_1(\text{diam } A_i) < M+1$. Thus

$$h_2(\text{diam } A_i) \leq \frac{\varepsilon}{M+1} h_1(\text{diam } A_i) \text{ for all } i$$

and summing over i leads to

$$\sum_{i=1}^{\infty} h_2(\text{diam } A_i) \leq \frac{\varepsilon}{M+1} \sum_{i=1}^{\infty} h_1(\text{diam } A_i) < \varepsilon.$$

Since $t_0 < \delta$, it follows that $\mathcal{H}_\delta^{h_2}(A) < \varepsilon$. Because $\varepsilon, \delta > 0$ are arbitrary, this implies that $\mathcal{H}^{h_2}(A) = 0$.

Now, if A has σ -finite \mathcal{H}^{h_1} -measure, $A = \bigcup_{i=1}^{\infty} A_i$ for some countable family $\{A_i\}$ of sets of finite \mathcal{H}^{h_1} -measure. Because \mathcal{H}^{h_2} is an outer measure, it follows that

$$\mathcal{H}^{h_2}(A) \leq \sum_{i=1}^{\infty} \mathcal{H}^{h_2}(A_i) = 0$$

and $\mathcal{H}^{h_2}(A) = 0$ as required. \square

Corollary 2.4.6 Let h_0, h_1, h_2 be gauge functions with $h_0 \prec h_1 \prec h_2$. If a set $A \subset \mathbb{C}$ has σ -finite nonzero \mathcal{H}^{h_1} -measure, then A has zero \mathcal{H}^{h_2} -measure and non- σ -finite \mathcal{H}^{h_0} -measure.

Proof. The first part is just a repetition of the statement of the above theorem. If A had σ -finite \mathcal{H}^{h_0} -measure, it would follow from the theorem that A would have zero \mathcal{H}^{h_1} -measure, a contradiction. \square

2.5 A lemma by Frostman

From the definition of Hausdorff measure it is quite obvious that in order to estimate Hausdorff measure from above, it is 'enough' to find suitable coverings of a given

set with sets of arbitrarily small diameter. This will in fact be the strategy in the final part of this thesis - it is a quite elementary method, but it turns out that this method works well in the case of Julia sets of exponential maps. On the other hand, if one wants to estimate Hausdorff measure from below, one covering is not enough. All possible coverings must be taken into account. To solve this problem, we will use a simple but powerful lemma due to Frostman ([13]) which will allow us to estimate Hausdorff measure from below.

Theorem 2.5.1 *Let h be a gauge function and μ be a mass distribution (i.e. a finite measure) on \mathbb{C} . Let $A \subset \mathbb{C}$ be μ -measurable and $0 < c < \infty$. If*

$$\limsup_{r \rightarrow 0} \frac{\mu(D(z, r))}{h(r)} < c \text{ for all } z \in A, \quad (2.5.1)$$

then $\mathcal{H}^h(A) \geq \mu(A)/c$.

Proof. For $\delta > 0$, define

$$A_\delta := \{z \in A \mid \mu(D(z, r)) < ch(r) \text{ for all } 0 < r \leq \delta\}.$$

Clearly, $A_\delta \subset A_{\delta'}$ whenever $\delta > \delta'$, and (because of (2.5.1)) we have $\bigcup_{\delta > 0} A_\delta = A$. Let $\delta > 0$ and $\{A_i\}$ be a δ -covering of A . It follows that $\{A_i\}$ is also a δ -covering of A_δ . For every A_i with $A_i \cap A_\delta \neq \emptyset$ and $z \in A_i \cap A_\delta$ we have

$$D := D(z, \text{diam}(A_i)) \supset A_i.$$

Hence $\mu(A_i) \leq \mu(D) < ch(\text{diam}(A_i))$ by the definition of A_δ and it follows that

$$\mu(A_\delta) \leq \sum_{i: A_i \cap A_\delta \neq \emptyset} \mu(A_i) < c \sum_{i: A_i \cap A_\delta \neq \emptyset} h(\text{diam}(A_i)).$$

Because $\{A_i\}$ was an arbitrary δ -covering, we have

$$\mu(A_\delta) \leq c\mathcal{H}_\delta^h(A_\delta) \leq c\mathcal{H}^h(A_\delta) \leq c\mathcal{H}^h(A).$$

Because $\bigcup_{\delta > 0} A_\delta = A$ and A is μ -measurable, we have $\lim_{\delta \rightarrow 0} \mu(A_\delta) = \mu(A)$ and therefore $\mu(A) \leq c\mathcal{H}^h(A)$. \square

The construction of a suitable mass distribution which can be used to estimate Hausdorff measure of Julia sets from below will be given in section 4. This construction will be the same as the one McMullen used for his proof that the Julia set of each exponential map has Hausdorff dimension 2 ([20]).

3 Distortion

In this section, we introduce the concept of distortion of holomorphic maps. After giving the definition and some basic results, we prove two helpful lemmas which we will need quite a couple of times. One of them is a consequence of the Koebe distortion and growth theorems and will be used to substitute McMullen's 'Distortion Lemma' in the proof for the estimation of Hausdorff measure from below, the other one states that 'small' squares are mapped to 'close-to-square' sets under maps with bounded distortion. In the third subsection, we give an application to expanding dynamics.

3.1 Definition and basic facts

We begin by defining maps with bounded distortion and the distortion of a function itself.

Definition 3.1.1 (Distortion) Let $A \subset \mathbb{C}$ be a bounded and Lebesgue-measurable set. Let $f : A \rightarrow \mathbb{C}$ be a map. We say that f has **bounded distortion** if f is a bilipschitz mapping, i.e.

$$0 < c_f := \inf_{\substack{z, w \in A \\ z \neq w}} \frac{|f(z) - f(w)|}{|z - w|} \leq \sup_{\substack{z, w \in A \\ z \neq w}} \frac{|f(z) - f(w)|}{|z - w|} =: C_f < \infty.$$

If f has bounded distortion, we call

$$D(f) := \frac{C_f}{c_f}$$

the **distortion** of f on A . ◇

Remarks: (i) The above setting is not the most general. One does not need that A is bounded or measurable - A could be any subset of \mathbb{C} . But since we will mainly be interested in how measures and densities of sets behave under maps with bounded distortion, it makes sense to be sure that the sets we consider are Lebesgue-measurable and have finite measure.

(ii) Often, the distortion is defined differently: It is only defined for holomorphic maps f on bounded open sets, and such a map f is said to have bounded distortion if

$$0 < \inf_{z \in U} |f'(z)| \leq \sup_{z \in U} |f'(z)| < \infty.$$

The distortion of f is then defined as

$$L(f) := \frac{\sup_{z \in U} |f'(z)|}{\inf_{z \in U} |f'(z)|}.$$

But it turns out that this concept of distortion is too weak for our purposes - we will need that maps with small distortion nearly preserve relative lengths and areas, which the above concept does not give us. Of course, it is always true that $L(f) \leq D(f)$, because if

$$\frac{|f(z) - f(w)|}{|z - w|} \leq C_f \text{ (resp. } \geq c_f) \text{ for all } z, w \in U,$$

we have

$$|f'(z)| \leq C_f \text{ (resp. } \geq c_f) \text{ for all } z \in U.$$

It is easily seen (using the mean value theorem) that if both U and $f(U)$ are convex sets, then $D(f) = L(f)$, but even if U is convex, there may exist a function $f : U \rightarrow \mathbb{C}$ for which $L(f)$ is finite, but f does not have bounded distortion. We now give an example:

Let U be the interior of the ellipse in $\mathbb{C} = \mathbb{R}^2$ defined by

$$\frac{(x-1)^2}{\left(\frac{1}{2}\right)^2} + y^2 = 1$$

and $f : U \rightarrow \mathbb{C}, z \mapsto z^4$. Because $1+i, 1-i \in \partial U$ and $(1+i)^4 = (1-i)^4$, f does not have bounded distortion. But, since $\sup_{z \in U} |z| = \sqrt{2}$ and $\inf_{z \in U} |z| = \frac{1}{2}$, it follows that

$$L(f) = \frac{\sqrt{2}^3}{\left(\frac{1}{2}\right)^3} = 16\sqrt{2} < \infty.$$

However, it should be mentioned that there exist results in the other direction: For example, John [15] proved that if f is univalent on a disc $D(z_0, r)$ with $0 < c = \inf |f'| \leq \sup |f'| = C < \infty$, then $D(f|_{D(z_0, rc/C)}) \leq L(f)$. ♠

Now, we show that if A is a bounded measurable set and $f : A \rightarrow \mathbb{C}$ has bounded distortion d , then there exists an extension of f to \bar{A} , also with distortion d (and vice versa).

Lemma 3.1.2 *Let $A \subset \mathbb{C}$ be bounded and measurable. Then*

- (a) *If $f : A \rightarrow \mathbb{C}$ has bounded distortion, then f extends to a map $\bar{f} : \bar{A} \rightarrow \mathbb{C}$ such that $D(f) = D(\bar{f})$.*
- (b) *If $f : \bar{A} \rightarrow \mathbb{C}$ is a map with bounded distortion, then $D(f|_A) = D(f)$.*

Proof. We first prove part (a). Fix $z_0 \in \partial A$ and let (z_n) be a sequence in A which converges to z_0 . Because f is a bilipschitz mapping, the sequence $(f(z_n))$ is a Cauchy-sequence which converges to some $w_0 \in \mathbb{C}$. We want to define

$$\bar{f}(z_0) := w_0. \tag{3.1.1}$$

To be able to do this, we must show that taking a different sequence (z'_n) with $z'_n \rightarrow z_0$ leads to the same w_0 . Suppose on the contrary that this is wrong, $f(z'_n) \rightarrow w'_0 \neq w_0$. Then we have $|z_n - z'_n| \rightarrow 0$ as $n \rightarrow \infty$, but $|f(z_n) - f(z'_n)|$ is bounded below by a

positive constant, a contradiction to the fact that f has bounded distortion. Therefore the extended function \bar{f} is well-defined by (3.1.1). To complete the proof of (a), we show that \bar{f} has the same distortion as f does. To see this, note that the function

$$g : (\bar{A} \times \bar{A}) \setminus \Delta\bar{A} \rightarrow \mathbb{C}, (z, w) \mapsto \frac{|f(z) - f(w)|}{|z - w|}$$

is continuous (where $\Delta\bar{A}$ denotes the diagonal). Let $z, w \in \bar{A}$ with $z \neq w$. If we choose arbitrary sequences $(z_n), (w_n)$ in A such that $z_n \rightarrow z, w_n \rightarrow w$ and $z_n \neq w_n$ for all $n \in \mathbb{N}$, it follows by continuity of g that $g(z_n, w_n) \rightarrow g(z, w)$ as $n \rightarrow \infty$, and thus we obtain

$$c_f \leq g(z, w) = \frac{|f(z) - f(w)|}{|z - w|} \leq C_f$$

and part (a) follows.

Part (b) is proved in the same way. Because $g(z_n, w_n)$ converges to $g(z, w)$ if (z_n, w_n) is a sequence in $(A \times A) \setminus \Delta A$ converging to $(z, w) \in (\bar{A} \times \bar{A}) \setminus \Delta\bar{A}$, the supremum and infimum of g do not change if we restrict g to $(A \times A) \setminus \Delta A$. \square

Remark: The above lemma shows in particular that if U is an open bounded set and $f : U \rightarrow \mathbb{C}$ is a holomorphic map with bounded distortion (this will be our setting in the following), it makes no difference (in terms of distortion) whether we look at the map f or at the extension \bar{f} on \bar{U} (which always exists). \spadesuit

From now on, we will only consider holomorphic maps defined on bounded domains U . In the next lemma, we show that if a univalent function f can be extended univalently to a domain V which compactly contains U , f must have bounded distortion.

Lemma 3.1.3 *Let $U, V \subset \mathbb{C}$ be bounded domains such that $\bar{U} \subset V$. Let $f : V \rightarrow \mathbb{C}$ be a univalent function. Then $f|_U$ has bounded distortion.*

Proof. Suppose that $f|_U$ does not have bounded distortion. Then, without loss of generality, there exist sequences $(z_n), (w_n)$ in U such that

$$\frac{|f(z_n) - f(w_n)|}{|z_n - w_n|} \rightarrow 0 \text{ as } n \rightarrow \infty \tag{3.1.2}$$

(the limit could of course also be ∞ , but then we do the proof for f^{-1} instead, and it is clear that f has bounded distortion if and only if f^{-1} has). Because \bar{U} is compact, we may assume that (z_n) and (w_n) converge to some points z_0 and w_0 in \bar{U} . Because $|f(z_n) - f(w_n)| \rightarrow 0$ and f is injective, we have $z_0 = w_0$. Now choose $\varepsilon > 0$ such that

$D(f(z_0), \varepsilon) \subset f(V)$. If $k_0 \in \mathbb{N}$ is so large that $z'_k, w'_k \in D(f(z_0), \varepsilon)$ for $k \geq k_0$, we have

$$\begin{aligned}
\frac{|f(z_k) - f(w_k)|}{|z_k - w_k|} &= \frac{|z'_k - w'_k|}{|f^{-1}(z'_k) - f^{-1}(w'_k)|} \\
&= \frac{1}{\frac{|f^{-1}(z'_k) - f^{-1}(w'_k)|}{|z'_k - w'_k|}} \\
&\geq \frac{1}{\max_{z' \in [z'_k, w'_k]} |(f^{-1})'(z')|} \\
&= \min_{z' \in [z'_k, w'_k]} \frac{1}{|(f^{-1})'(z')|} \\
&= \min_{z \in f^{-1}([z'_k, w'_k])} |f'(z)| \text{ for } k \geq k_0. \tag{3.1.3}
\end{aligned}$$

Because of (3.1.2) and (3.1.3), we have $\min_{z \in f^{-1}([z'_k, w'_k])} |f'(z)| \rightarrow 0$ as $k \rightarrow \infty$, and together with $\bigcap_{k=1}^{\infty} f^{-1}([z'_k, w'_k]) = \{z_0\}$, it follows that $f'(z_0) = 0$. But this is impossible since f is univalent on V . \square

Remark: In the statement of the lemma, 'f holomorphic and injective' cannot be weakened to 'f continuous and injective'. An example is given by the following: For $r > 0$, let

$$f_r : D(0, r) \rightarrow \mathbb{C}, z \mapsto \begin{cases} |z|^2 e^{i \arg z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

f_r is a continuous and injective for all r , but we have

$$\frac{|f_r(z) - f_r(0)|}{|z - 0|} = |z| \rightarrow 0 \text{ as } z \rightarrow 0,$$

which means that f_r does not have bounded distortion. \spadesuit

In the proof of the preceding lemma, we used the fact that a univalent function f has bounded distortion if and only if f^{-1} has bounded distortion. In the next lemma, which gives some simple properties of functions with bounded distortion, we show that distortions of f and f^{-1} are in fact equal.

Lemma 3.1.4 *Let $U \subset \mathbb{C}$ be a domain and $f : U \rightarrow \mathbb{C}$ be a univalent function with bounded distortion. Then the following statements hold:*

- (a) $D(f) = D(f^{-1})$
- (b) *If $V \supset f(U)$ is a domain and $g : V \rightarrow \mathbb{C}$ is univalent with bounded distortion, then we have $D(g \circ f) \leq D(g)D(f)$*
- (c) *If $A \subset U$ is Lebesgue-measurable, then $f(A)$ is Lebesgue-measurable and*

$$\text{dens}(A, U) \leq D(f)^2 \text{dens}(f(A), f(U))$$

(d) For every $z \in U$,

$$c_f \leq |f'(z)| \leq c_f D(f)$$

Proof. We start with the proof of (a). From the definition of C_f and c_f , it is obvious that

$$\frac{1}{C_f} = \inf_{\substack{z, w \in U \\ z \neq w}} \frac{|z - w|}{|f(z) - f(w)|} = \inf_{\substack{u, v \in f(U) \\ u \neq v}} \frac{|f^{-1}(u) - f^{-1}(v)|}{|u - v|}$$

and

$$\frac{1}{c_f} = \sup_{\substack{z, w \in U \\ z \neq w}} \frac{|z - w|}{|f(z) - f(w)|} = \sup_{\substack{u, v \in f(U) \\ u \neq v}} \frac{|f^{-1}(u) - f^{-1}(v)|}{|u - v|}.$$

It follows that

$$D(f^{-1}) = \frac{1/c_f}{1/C_f} = \frac{C_f}{c_f} = D(f).$$

For the proof of (b), let $z, w \in U$ be given. It is immediate that

$$\frac{|(g \circ f)(z) - (g \circ f)(w)|}{|z - w|} = \frac{|(g \circ f)(z) - (g \circ f)(w)|}{|f(z) - f(w)|} \frac{|f(z) - f(w)|}{|z - w|} \begin{cases} \leq C_g C_f \\ \geq c_g c_f \end{cases}$$

and hence

$$D(g \circ f) \leq \frac{C_g C_f}{c_g c_f} = D(g) D(f).$$

Now we prove (c). First note that because f is bilipschitz, $f(A)$ is Lebesgue-measurable (see e.g. [1]). We have by the transformation formula and by the fact that $L(f) \leq D(f)$ (see part (b) of the remark after Definition 3.1.1):

$$\begin{aligned} \text{dens}(f(A), f(U)) &= \frac{|f(A)|}{|f(U)|} \\ &= \frac{\int_{f(A)} 1 \, dz \wedge d\bar{z}}{\int_{f(U)} 1 \, dz \wedge d\bar{z}} \\ &= \frac{\int_A |f'|^2 \, dz \wedge d\bar{z}}{\int_U |f'|^2 \, dz \wedge d\bar{z}} \\ &\leq \frac{|A|}{|U|} \left(\frac{\sup_{z \in A} |f'(z)|}{\inf_{z \in U} |f'(z)|} \right)^2 \\ &\leq \text{dens}(A, U) L(f)^2 \\ &\leq \text{dens}(A, U) D(f)^2 \end{aligned}$$

and (c) follows.

Finally, it is clear that $|f'(z)| \geq c_f$ and $|f'(z)| \leq C_f = c_f D(f)$ for $z \in U$, which shows (d). \square

3.2 Two distortion lemmas

We now prove the two useful lemmas announced at the beginning of this section. The first one is an application of the Koebe growth and distortion theorems (more precisely of Corollary 2.3.5) and gives us an elementary upper bound for the distortion of a univalent function defined on a disc if we can extend the function univalently to a larger disc (see also Lemma 3.1.3).

Lemma 3.2.1 *Let $z_0 \in \mathbb{C}$, $r > 0$, $K > 3$. Let $f : D(z_0, Kr) \rightarrow \mathbb{C}$ be a univalent function. Then*

$$D(f|_{D(z_0, r)}) < \left(\frac{K+1}{K-3}\right)^6.$$

Proof. Let $z, w \in D(z_0, r)$. Because of (2.3.3), we have

$$\frac{Kr - |w - z_0|}{(Kr + |w - z_0|)^3} (Kr)^2 |f'(z_0)| \leq |f'(w)| \leq \frac{Kr + |w - z_0|}{(Kr - |w - z_0|)^3} (Kr)^2 |f'(z_0)|. \quad (3.2.1)$$

Further, since f is univalent on $D(w, (K-1)r)$, it follows from (2.3.4) that

$$\begin{aligned} \frac{((K-1)r)^2}{((K-1)r + |z - w|)^2} |f'(w)| &\leq \frac{|f(z) - f(w)|}{|z - w|} \\ &\leq \frac{((K-1)r)^2}{((K-1)r - |z - w|)^2} |f'(w)| \end{aligned} \quad (3.2.2)$$

Combining (3.2.1) and (3.2.2) and using the facts that $|w - z_0| < r$ and $|z - w| < 2r$, we get

$$\begin{aligned} \frac{((K-1)r)^2}{((K-1)r + 2r)^2} \frac{Kr - r}{(Kr + r)^3} (Kr)^2 |f'(z_0)| &\leq \frac{|f(z) - f(w)|}{|z - w|} \\ &\leq \frac{((K-1)r)^2}{((K-1)r - 2r)^2} \frac{Kr + r}{(Kr - r)^3} (Kr)^2 |f'(z_0)| \end{aligned}$$

It follows for the distortion of f on $D(z_0, r)$:

$$\begin{aligned} D(f|_{D(z_0, r)}) &\leq \frac{Kr + r}{Kr - r} \frac{(Kr + r)^3 ((K-1)r + 2r)^2}{(Kr - r)^3 ((K-1)r - 2r)^2} \\ &= \frac{(Kr + r)^4 ((K+1)r)^2}{(Kr - r)^4 ((K-3)r)^2} \\ &= \frac{(K+1)^4 (K+1)^2}{(K-1)^4 (K-3)^2} \\ &< \left(\frac{K+1}{K-3}\right)^6. \end{aligned}$$

□

In the next lemma, we prove that if a given square is small enough and f is univalent has bounded distortion, then the image of this square 'looks almost like a square', i.e.

it is contained in some square Q_1 and contains some square Q_2 , and the side lengths of Q_1 and Q_2 are 'almost the same'.

Lemma 3.2.2 *For every $1/\sqrt{2} > \varepsilon > 0$, there exists a constant $K > 1$ such that the following statement is true: Let some $z_0 \in \mathbb{C}$, $r > 0$, $K' \geq K$, an arbitrary univalent function $\tilde{f} : D(z_0, K'r/\sqrt{2}) \rightarrow \mathbb{C}$ and some square $Q = Q(z_0, r, \theta)$ be given. Let $f := \tilde{f}|_Q$ and $d := D(f)$ be the distortion of f (which is finite because of Lemma 3.1.2 and Lemma 3.1.3). Then we have*

$$Q\left(f(z_0), |f'(z_0)| r \frac{1}{d} (1 - \sqrt{2}\varepsilon), \theta + \arg f'(z_0)\right) \subset f(Q)$$

and

$$f(Q) \subset Q\left(f(z_0), |f'(z_0)| rd (1 + \sqrt{2}\varepsilon), \theta + \arg f'(z_0)\right).$$

Proof. Since the argument function is continuous in 1, we can find a $\delta' > 0$ such that

$$|\arg(z)| < \varepsilon \text{ if } |z - 1| < \delta'. \quad (3.2.3)$$

It now follows from Theorem 2.3.7 that there exists $\delta > 0$ with

$$\left| \frac{g(z)}{z} - 1 \right| < \delta' \text{ whenever } |z| < \delta \text{ and } g \in S. \quad (3.2.4)$$

Now we choose $K > 1$ so large that $1/(\sqrt{2}K) < \delta$. Let $K' \geq K$, $r > 0$, a univalent function $\tilde{f} : D(z_0, K'r/\sqrt{2}) \rightarrow \mathbb{C}$ and a square $Q = Q(z_0, r, \theta)$ be given. Without loss of generality, we may assume that $z_0 = 0 = \tilde{f}(z_0)$. Let $f := \tilde{f}|_Q$ and $R := K'r$. The transformation

$$M(z) := \frac{1}{R} e^{-i\theta} z$$

maps $D(0, R)$ biholomorphically onto \mathbb{D} and Q biholomorphically onto $Q(0, 1/K', 0)$. We define the function

$$\tilde{g} : \mathbb{D} \rightarrow \mathbb{C}, z \mapsto \frac{(f \circ M^{-1})(z)}{f'(0)Re^{i\theta}} = \frac{f(Re^{i\theta}z)}{f'(0)Re^{i\theta}}.$$

An easy calculation gives $\tilde{g}'(0) = 1$ and hence $\tilde{g} \in S$. Set

$$g := \tilde{g}|_{\overline{Q(0, 1/K', 0)}}$$

in order to increase readability. Fix $z, w \in \overline{Q(0, 1/K', 0)}$ and define $u := M^{-1}(z)$ and $v := M^{-1}(w)$. Then we have

$$\begin{aligned} \frac{|g(z) - g(w)|}{|z - w|} &= \frac{|f(Re^{i\theta}z) - f(Re^{i\theta}w)|}{|z - w| |f'(0)Re^{i\theta}|} \\ &= \frac{|f(M^{-1}(z)) - f(M^{-1}(w))|}{|M^{-1}(z) - M^{-1}(w)| |f'(0)|} \\ &= \frac{1}{|f'(0)|} \frac{|f(u) - f(v)|}{|u - v|} \end{aligned}$$

which implies

$$d = D(f) = D(g) \quad (3.2.5)$$

and

$$c_g = \frac{1}{|f'(0)|} c_f. \quad (3.2.6)$$

Now let

$$\gamma : \left[-\frac{1}{2K'}, \frac{1}{2K'} \right] \rightarrow \mathbb{C}, t \mapsto \frac{1}{2K'} + it$$

be a parametrization of the 'right edge' of $Q(0, 1/K', 0)$. Clearly,

$$\Re g(\gamma(t)) = \cos \arg g(\gamma(t)) |g(\gamma(t))| = \cos \arg g(\gamma(t)) |g(\gamma(t)) - g(0)|,$$

and it follows from (3.2.5) that

$$\Re g(\gamma(t)) \leq \cos \arg g(\gamma(t)) |\gamma(t)| c_g d \quad (3.2.7)$$

and

$$\Re g(\gamma(t)) \geq \cos \arg g(\gamma(t)) |\gamma(t)| c_g. \quad (3.2.8)$$

Because \cos is Lipschitz with Lipschitz-constant 1 and

$$|\gamma(t)| \leq \frac{1}{\sqrt{2K'}} \leq \frac{1}{\sqrt{2K}} < \delta,$$

we deduce from (3.2.3) and (3.2.4) that

$$\begin{aligned} |\cos \arg g(\gamma(t)) - \cos \arg \gamma(t)| &\leq |\arg g(\gamma(t)) - \arg \gamma(t)| \\ &= \left| \arg \frac{g(\gamma(t))}{\gamma(t)} \right| \\ &= \left| \arg \frac{g(\gamma(t))}{\gamma(t)} - \arg 1 \right| < \varepsilon, \end{aligned}$$

hence

$$\cos \arg \gamma(t) - \varepsilon < \cos \arg g(\gamma(t)) < \cos \arg \gamma(t) + \varepsilon. \quad (3.2.9)$$

Using (3.2.9) and $|\gamma(t)| < 1/(\sqrt{2K'})$, it follows from (3.2.7) that

$$\begin{aligned} \Re g(\gamma(t)) &\leq (\cos \arg \gamma(t) + \varepsilon) |\gamma(t)| c_g d \\ &= (\Re \gamma(t) + \varepsilon |\gamma(t)|) c_g d \\ &= \left(\frac{1}{2K'} + \varepsilon |\gamma(t)| \right) c_g d \\ &\leq c_g d \frac{1}{2K'} (1 + \sqrt{2}\varepsilon) \end{aligned}$$

and analogous it follows from (3.2.8) that

$$\Re g(\gamma(t)) \geq c_g \frac{1}{2K'} (1 - \sqrt{2}\varepsilon).$$

Summarizing, we have

$$c_g \frac{1}{2K'} (1 - \sqrt{2\varepsilon}) \leq \Re g(\gamma(t)) \leq c_g d \frac{1}{2K'} (1 + \sqrt{2\varepsilon}) \quad (3.2.10)$$

Now let

$$\tilde{\gamma}(t) := Re^{i\theta} \gamma(t)$$

be a parametrization of the edge of Q corresponding to the 'right edge' of $Q(0, 1/K', 0)$ under M . Because

$$f(z) = f'(0) Re^{i\theta} g \left(\frac{1}{R} e^{-i\theta} z \right),$$

we have

$$\begin{aligned} \Re \left(e^{-i\theta} e^{-i \arg f'(0)} f(\tilde{\gamma}(t)) \right) &= \Re \left(e^{-i\theta} e^{-i \arg f'(0)} f'(0) g(\gamma(t)) Re^{i\theta} \right) \\ &= \Re (R |f'(0)| g(\gamma(t))) \\ &= R |f'(0)| \Re g(\gamma(t)). \end{aligned}$$

Combining this with (3.2.10), (3.2.6) and using $R/K' = r$, we get

$$c_f \frac{r}{2} (1 - \sqrt{2\varepsilon}) \leq \Re \left(e^{-i\theta} e^{-i \arg f'(0)} f(\tilde{\gamma}(t)) \right) \leq c_f d \frac{r}{2} (1 + \sqrt{2\varepsilon}).$$

We now use Lemma 3.1.4 (d) - which gives us that $c_f \leq |f'(0)|$ and $c_f \geq |f'(0)|/d$ - to conclude that

$$|f'(0)| \frac{1}{d} \frac{r}{2} (1 - \sqrt{2\varepsilon}) \leq \Re \left(e^{-i\theta} e^{-i \arg f'(0)} f(\tilde{\gamma}(t)) \right) \leq |f'(0)| d \frac{r}{2} (1 + \sqrt{2\varepsilon}).$$

For the other edges of Q , we get similar results by absolutely analogous estimations, so that we end up with the fact that

$$\begin{aligned} Q \left(0, |f'(0)| \frac{1}{d} r (1 - \sqrt{2\varepsilon}), 0 \right) &\subset e^{-i\theta} e^{-i \arg f'(0)} f(Q) \\ &\subset Q \left(0, |f'(0)| dr (1 + \sqrt{2\varepsilon}), 0 \right) \end{aligned}$$

and the claim of the lemma follows by applying the rotation $z \mapsto e^{i(\theta + \arg f'(0))} z$ to this formula. \square

3.3 An application to expanding dynamics

An important application of distortion is the fact that if an entire function f is expanding, then the inverse of f^n has bounded distortion independent of $n \in \mathbb{N}$, i.e. all inverse branches of any iterate have uniformly bounded distortion on a fixed set A . This fact is made precise in this section and is an application of Lemma 3.2.1.

Lemma 3.3.1 *Let Q_1, \dots, Q_n be a finite sequence of squares in \mathbb{C} with the same side length r , $Q_i = Q(a_i, r)$. Let $K > 3$ and $f_i : Q_i \rightarrow \mathbb{C}$ be univalent maps such that $Q_{i+1} \subset f_i(Q_i)$ for every $1 \leq i \leq n-1$. Suppose that both f_n and $f_1^{-1} \circ \dots \circ f_{n-1}^{-1}|_{Q_n}$ can be extended univalently to $D(a_n, Kr/\sqrt{2})$. Let*

$$F := (f_n \circ \dots \circ f_1)^{-1} : f_n(Q_n) \rightarrow Q_1.$$

Then we have

$$D(F) \leq \left(\frac{K+1}{K-3} \right)^{12}$$

independent of n .

Proof. The proof is a very simple application of Lemma 3.2.1 and the basic properties of distortion. We write

$$F = (f_1^{-1} \circ \dots \circ f_{n-1}^{-1}) \circ f_n^{-1}.$$

Because of Lemma 3.1.4 (b), we have that

$$D(F) \leq D(f_1^{-1} \circ \dots \circ f_{n-1}^{-1}|_{Q_n}) D(f_n^{-1}).$$

It follows directly from Lemma 3.1.4 (a) and Lemma 3.2.1 that

$$D(f_n^{-1}) = D(f_n) \leq \left(\frac{K+1}{K-3} \right)^6$$

and

$$D(f_1^{-1} \circ \dots \circ f_{n-1}^{-1}|_{Q_n}) \leq \left(\frac{K+1}{K-1} \right)^6,$$

and the claim follows. □

4 Helpful tools for the estimate from below

In this section, we explain how to construct a suitable mass distribution μ on the (nonempty) intersection A of nested compact sets such that it can be shown (using Theorem 2.5.1) that the Hausdorff measure of A with respect to certain gauge functions is ∞ (see Lemma 4.2.1). In our later applications of this lemma, the set A will be a subset of the Julia set of a given exponential map. This section is structured as follows: We first define the mass distribution we will use and introduce some notations we will need, and after that, we formulate and prove the 'infinite-measure-lemma'. As already mentioned, the ideas in this sections are not new - McMullen used them for his proof that the Hausdorff dimension of the Julia set of each exponential map is 2.

4.1 A mass distribution on the intersection of nested sets

We start by defining the nesting conditions and thereby introducing Δ_k and d_k as notations for special densities and diameters, which we will use frequently in section 6.

Definition 4.1.1 (nesting conditions) For $k \in \mathbb{N}$, let

$$\mathcal{A}_k = \{F_k^1, \dots, F_k^{n_k}\}$$

be a finite collection of compact, disjoint and connected subsets of \mathbb{C} with positive Lebesgue-measure. We define

$$A_k := \bigcup_{i=1}^{n_k} F_k^i$$

and say that the family $\{\mathcal{A}_k\}$ satisfies the **nesting conditions** if it has the following three properties:

- (a) For every $k \in \mathbb{N}$ and $F \in \mathcal{A}_{k+1}$, there exists some $F' \in \mathcal{A}_k$ such that $F \subset F'$.
- (b) There exists a monotone decreasing sequence (d_k) converging to 0 such that

$$\max_{1 \leq i \leq n_k} \text{diam}(F_k^i) \leq d_k \text{ for all } k \in \mathbb{N}$$

- (c) There exists a sequence (Δ_k) of positive reals such that

$$\text{dens}(A_{k+1}, F) \geq \Delta_k \text{ for all } k \in \mathbb{N}, F \in \mathcal{A}_k.$$

We then define the nonempty and compact set A by

$$A := \bigcap_{k=1}^{\infty} A_k.$$

◇

Remarks: (i) Because of condition (c), we also have that every set $F' \in \mathcal{A}_k$ contains at least one set $F \in \mathcal{A}_{k+1}$, which is kind of a converse statement to condition (a).

(ii) In condition (b), the sequence of the maximal diameters is itself decreasing. This follows immediately from condition (a). \spadesuit

In the remaining part of this subsection, let $\{\mathcal{A}_k\}$ be a family that satisfies the nesting conditions, and let A be defined as above.

Now we want to construct a mass distribution μ on A . We do this in the most natural way: First we define a set function τ_k on each \mathcal{A}_k , obtain a metric outer measure η by a well-known limit process, and define μ as the restriction of η to all η -measurable sets (which include all Borel sets because η is a metric measure).

The sequence (τ_k) of set functions is defined inductively as follows: For $F \in \mathcal{A}_1$, we define

$$\tau_1(F) := \frac{|F|}{|A_1|}.$$

For $F \in \mathcal{A}_{k+1}$, there exists a unique $G \in \mathcal{A}_k$ such that $F \subset G$. Denoting by F_{k+1}^i the elements of \mathcal{A}_{k+1} that are contained in G , we define

$$\tau_{k+1}(F) := \frac{|F|}{\sum_i |F_{k+1}^i|} \tau_k(G).$$

From this definition, it is immediate that $\tau_{k+1}(A_{k+1} \cap F) = \tau_k(F)$, that

$$\tau_k(F) = \sum_{\substack{j=1 \\ F_{k+1}^j \subset F}}^{n_{k+1}} \tau_{k+1}(F_{k+1}^j) \quad (4.1.1)$$

and that $\tau_k(F) > 0$ for all $F \in \mathcal{A}_k$. Now we set

$$\mathcal{X} := \bigcup_{k \in \mathbb{N}} \mathcal{A}_k \cup \bigcup_{k \in \mathbb{N}} \{M \subset \mathbb{C} \mid M \cap A_k = \emptyset\} \quad (4.1.2)$$

and define τ by

$$\tau : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}, B \mapsto \begin{cases} \tau_k(B) & , B \in \mathcal{A}_k \\ 0 & , B \notin \bigcup_{k \in \mathbb{N}} \mathcal{A}_k \end{cases} \quad (4.1.3)$$

Now we extend τ to an outer measure η on \mathbb{C} . The construction is very similar to the construction of Hausdorff measure. For $\delta > 0$ and $B \subset \mathbb{C}$, let

$$\eta_\delta(B) := \inf \left\{ \sum_{i=1}^{\infty} \tau(B_i) \mid B \subset \bigcup_{i \in \mathbb{N}} B_i, B_i \in \mathcal{X}, \text{diam}(B_i) < \delta \right\}.$$

$\eta_\delta(B)$ increases as δ decreases and we set

$$\eta(B) := \lim_{\delta \rightarrow 0} \eta_\delta(B).$$

It follows by standard arguments that η is a metric outer measure on \mathbb{C} . Let μ be the restriction of η to all η -measurable sets, which - as mentioned above - include the Borel sets since η is a metric outer measure. We now show that μ coincides with τ on $\bigcup_{k \in \mathbb{N}} \mathcal{A}_k$ and $\text{supp}(\mu) = A$. To show this, we first prove a lemma about η .

Lemma 4.1.2 *For every $0 < \varepsilon < \delta$ and every $B \subset \mathbb{C}$, $\eta_\delta(B) = \eta_\varepsilon(B)$.*

Proof. It is clear from the definition that $\eta_\delta(B) \leq \eta_\varepsilon(B)$. For the opposite inequality, let (B_n^δ) be a sequence of sets in \mathcal{X} such that

$$B \subset \bigcup_{n \in \mathbb{N}} B_n^\delta \text{ and } \text{diam}(B_n^\delta) < \delta \text{ for all } n.$$

Set

$$M := \bigcup_{\tau(B_n^\delta) = 0} B_n^\delta.$$

M can be partitioned into countably many sets C_l^ε with $\text{diam}(C_l^\varepsilon) < \varepsilon$ and we have

$$\sum_{l \in \mathbb{N}} \tau(C_l^\varepsilon) = 0.$$

Every set B_m^δ with $\tau(B_m^\delta) > 0$ belongs to some \mathcal{A}_k . For such a set B_m^δ , there exists some $N(m) > k$ such that $\text{diam}(F) < \varepsilon$ for every $F \in \mathcal{A}_{N(m)}$. By a simple induction, it follows easily from (4.1.1) that

$$\sum_{\substack{j=1 \\ F_{N(m)}^j \subset B_m^\delta}}^{n_{N(m)}} \tau(F_{N(m)}^j) = \tau(B_m^\delta). \quad (4.1.4)$$

Now we have

$$\tau(B_m^\delta \setminus A_{N(m)}) = 0$$

and we can again partition this set into countably many sets $C_{m,l}^\varepsilon$ such that

$$\text{diam}(C_{m,l}^\varepsilon) < \varepsilon \text{ for all } l.$$

Summarizing, we have that

$$B \subset \left(\bigcup_{l \in \mathbb{N}} C_l^\varepsilon \cup \bigcup_{\substack{m \in \mathbb{N} \\ \tau(B_m^\delta) > 0}} \left(\bigcup_{\substack{j=1 \\ F_{N(m)}^j \subset B_m^\delta}}^{N(m)} F_{N(m)}^j \cup \bigcup_{l \in \mathbb{N}} C_{m,l}^\varepsilon \right) \right)$$

and each set on the right side has diameter less than ε . From (4.1.4) and because $\tau(C_l^\varepsilon) = \tau(C_{m,l}^\varepsilon) = 0$ for all m, l , it follows that

$$\eta_\varepsilon(B) \leq \sum_{n=1}^{\infty} \tau(B_n^\delta).$$

Since (B_n^δ) is an arbitrary covering of B , we conclude that $\eta_\varepsilon(B) \leq \eta_\delta(B)$, which is what we wanted to show. \square

Now we can prove that μ coincides with τ on $\bigcup_{k \in \mathbb{N}} \mathcal{A}_k$ and that the support of μ is equal to A :

Theorem 4.1.3 *For every $B \in \bigcup_{k \in \mathbb{N}} \mathcal{A}_k$, we have $\mu(B) = \tau(B)$.*

Proof. Let $B \in \bigcup_{k \in \mathbb{N}} \mathcal{A}_k$ be given. Because $\{B\}$ is a valid covering of B , it follows from Lemma 4.1.2 that

$$\mu(B) = \eta(B) = \eta_{\text{diam}(B)+1}(B) \leq \tau(B).$$

If $\eta_{\text{diam}(B)+1}(B)$ was strictly smaller than $\tau(B)$, there would (in view of the definition of η) exist a covering $\{B_i\}$ of B such that

$$\sum_{i=1}^{\infty} \tau(B_i) < \tau(B). \quad (4.1.5)$$

Let such a covering be given. Now consider only the B_i with $\tau(B_i) > 0$, i.e. the B_i which belong to some \mathcal{A}_k . Call this collection of sets \mathcal{C} and define

$$\mathcal{C}_n := \{C \in \mathcal{C} \mid C \in \mathcal{A}_k \text{ for some } k \leq n\}.$$

We will show that there exists an index $n_0 \in \mathbb{N}$ such that

$$\mathcal{B}_{n_0} := \mathcal{C}_{n_0} \cup \bigcup_{\tau(B_i)=0} B_i$$

is a covering of B .

Suppose on the contrary that this is wrong. Let $k \in \mathbb{N}$ with $B \in \mathcal{A}_k$. Because \mathcal{B}_{n_0} is not a covering of B for any n_0 , the following statement holds:

For every $n \in \mathbb{N}_{\geq k}$, there exists some $F \in \mathcal{A}_n$ with $F \subset B$, but $F \notin \mathcal{C}$ and $G \notin \mathcal{C}$ whenever $F \subset G \in \mathcal{A}_m$ for some m between k and n .

It follows that we get a sequence $(F_n)_{n \geq k}$ of such sets with $F_{n+1} \subset F_n$ for every n . Because all these sets are compact, the intersection $\bigcap_{n \geq k} F_n$ is non-empty - in fact, it consists of a single point $a \in A$ because the diameters of the F_n tend to 0. But then $\{B_i\}$ cannot be a covering of B , since any of the B_i that contains a must satisfy $\tau(B_i) > 0$ (every admissible set in a covering that contains a point in A must be in some \mathcal{A}_k , see (4.1.2) and (4.1.3)). This is a contradiction.

Hence it follows by (4.1.1) that

$$\tau(B) = \sum_{\substack{F \in \mathcal{A}_{n_0} \\ F \subset B}} \tau(F) = \sum_{C \in \mathcal{C}_{n_0}} \sum_{\substack{F \in \mathcal{A}_{n_0} \\ F \subset C}} \tau(F) = \sum_{C \in \mathcal{C}_{n_0}} \tau(C) \leq \sum_{C \in \mathcal{C}} \tau(C) = \sum_{i=1}^{\infty} \tau(B_i),$$

a contradiction to (4.1.5) which finishes the proof. \square

Corollary 4.1.4 *The support of μ is A .*

Proof. We first show that the support of μ is contained in A , i.e. $\mathbb{C} \setminus A$ is a set of zero μ -measure. We have

$$\mu(\mathbb{C} \setminus A) = \mu\left(\mathbb{C} \setminus \bigcap_{k \in \mathbb{N}} A_k\right) = \mu\left(\bigcup_{k \in \mathbb{N}} (\mathbb{C} \setminus A_k)\right) = \mu(\mathbb{C} \setminus A_1) + \mu(A_1 \setminus A_2) + \dots$$

Because $\{A_k \setminus A_{k+1}\} \in \mathcal{X}$, $\{A_k \setminus A_{k+1}\}$ is a valid covering of $A_k \setminus A_{k+1}$ for all $k \in \mathbb{N}$. Since $\tau(A_k \setminus A_{k+1}) = 0$, we have by Lemma 4.1.2 that

$$\mu(A_k \setminus A_{k+1}) = \eta_{\text{diam}(A_k)+1}(A_k \setminus A_{k+1}) = 0$$

for all $k \in \mathbb{N}$.

Further, we have that

$$\mu(\mathbb{C} \setminus A_1) = \lim_{R \rightarrow \infty} \mu(D(0, R) \setminus A_1).$$

The same argument as above shows that $\mu(D(0, R) \setminus A_1) = 0$ for all $R > 0$. Hence

$$\mu(\mathbb{C} \setminus A) = 0$$

as required.

Now we show that the support of μ is in fact equal to A . Suppose that this is not the case. Then there exists an open set U such that $U \cap A \neq \emptyset$ and $\mu(U) = 0$. Because the maximal diameter of sets in \mathcal{A}_k tends to 0 as $k \rightarrow \infty$ and $U \cap A \neq \emptyset$, there exists some $k \in \mathbb{N}$ and $F \in \mathcal{A}_k$ such that $F \subset U$. By Theorem 4.1.3, we must have $\mu(F) = \tau(F)$, but this cannot be:

Because $F \in \mathcal{A}_k$, we have $\tau(F) > 0$. On the other hand, it follows from $F \subset U$ that $\mu(F) = 0$. This is a contradiction. \square

4.2 A sufficient condition for infinite measure

We now prove the main lemma which will help us estimating Hausdorff measure from below. The statement and proof of this Lemma are quite similar to McMullen's Lemma which he uses for estimating the Hausdorff dimension of Julia sets of exponential maps ([20], Proposition 2.2).

Lemma 4.2.1 *Let $\{\mathcal{A}_k\}$ be a collection of families of sets which satisfies the nesting conditions (with properly chosen sequences (d_k) and (Δ_k)). Let A be defined as above. Let $\varepsilon > 0$ and $g : (0, \varepsilon) \rightarrow \mathbb{R}_{\geq 0}$ be a decreasing continuous function such that $t^2 g(t)$ is increasing. Further, suppose that*

$$\lim_{t \rightarrow 0} t^2 g(t) = 0$$

and

$$\lim_{k \rightarrow \infty} g(d_k) \prod_{j=1}^k \Delta_j = \infty. \quad (4.2.1)$$

Define

$$h : [0, \varepsilon) \rightarrow \mathbb{R}, t \mapsto \begin{cases} t^2 g(t) & , t > 0 \\ 0 & , t = 0 \end{cases}$$

Then h is a gauge function and we have $\mathcal{H}^h(A) = \infty$.

Proof. By the remark (ii) after Theorem 2.4.2 and the conditions on g , it is clear that h is a gauge function. For the proof that A has infinite \mathcal{H}^h -measure, we use Theorem 2.5.1. Let τ be the set function and μ be the mass distribution connected to the family $\{\mathcal{A}_k\}$ which were both defined in the previous subsection. Let $z \in A, r > 0$ and $D := D(z, r)$. Choose $k \in \mathbb{N}$ such that

$$d_k > r \geq d_{k+1}. \quad (4.2.2)$$

Let \tilde{D} be the union of all sets in \mathcal{A}_{k+1} which intersect D , i.e.

$$\tilde{D} := \bigcup_{\substack{F \in \mathcal{A}_{k+1} \\ F \cap D \neq \emptyset}} F.$$

It follows from (4.2.2) that

$$\text{diam}(\tilde{D}) \leq \text{diam}(D) + 2d_{k+1} \leq 2r + 2r = 4r. \quad (4.2.3)$$

Because $(D \setminus \tilde{D}) \cap \mathcal{A}_{k+1} = \emptyset$ and $\text{supp}(\mu) = A$ by Theorem 4.1.4, we have $\mu(D \setminus \tilde{D}) = 0$ and hence

$$\mu(D) \leq \mu(\tilde{D}) + \mu(D \setminus \tilde{D}) = \mu(\tilde{D}). \quad (4.2.4)$$

We now show that

$$\mu(\tilde{D}) \leq \frac{|\tilde{D}|}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k}. \quad (4.2.5)$$

To establish this inequality, we prove by induction that for every $k \in \mathbb{N}_0$ and every $F \in \mathcal{A}_{k+1}$, we have

$$\mu(F) \leq \frac{|F|}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k}. \quad (4.2.6)$$

For $k = 0$ and $F \in \mathcal{A}_1$, we have

$$\mu(F) = \tau(F) = \frac{|F|}{|A_1|}$$

by Theorem 4.1.3 and the definition of τ . Now let some $k \in \mathbb{N}_0$ be given and let (4.2.6) hold for every $F \in \mathcal{A}_{k+1}$. Choose some $F' \in \mathcal{A}_{k+2}$. Let $F \in \mathcal{A}_{k+1}$ such that $F' \subset F$. Again by Theorem 4.1.3 and the induction hypothesis, we have

$$\mu(F') = \frac{|F'|}{\sum_{\substack{j=1 \\ F_{k+2}^j \subset F'}}^{n_{k+2}} |F_{k+2}^j|} \mu(F) \leq |F'| \frac{|F|}{\sum_{\substack{j=1 \\ F_{k+2}^j \subset F'}}^{n_{k+2}} |F_{k+2}^j|} \frac{1}{\Delta_1 \cdots \Delta_k} \frac{1}{|A_1|}.$$

Since

$$\frac{|F|}{\sum_{\substack{j=1 \\ F_{k+2}^j \subset F}}^{n_{k+2}} |F_{k+2}^j|} = \frac{1}{\text{dens}(A_{k+2}, F)} \leq \frac{1}{\Delta_{k+1}},$$

it follows that

$$\mu(F') \leq |F'| \frac{1}{\Delta_1 \cdots \Delta_{k+1}} \frac{1}{|A_1|},$$

which is what we wanted to show.

Now (4.2.5) follows easily because

$$\mu(\tilde{D}) = \sum_{\substack{F \in \mathcal{A}_{k+1} \\ F \cap D \neq \emptyset}} \mu(F) \leq \frac{1}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k} \sum_{\substack{F \in \mathcal{A}_{k+1} \\ F \cap D \neq \emptyset}} |F| = \frac{|\tilde{D}|}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k}.$$

By (4.2.4), (4.2.5) and (4.2.3), we have

$$\mu(D) \leq \mu(\tilde{D}) \leq \frac{|\tilde{D}|}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k} \leq \frac{|D(z, 4r)|}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k}.$$

Further, we estimate

$$\begin{aligned} \frac{|D(z, 4r)|}{|A_1|} \frac{1}{\Delta_1 \cdots \Delta_k} &= \frac{\pi}{|A_1|} (4r)^2 \frac{1}{\Delta_1 \cdots \Delta_k} \\ &= \frac{16\pi}{|A_1|} r^2 g(r) \frac{1}{g(r) \Delta_1 \cdots \Delta_k} \\ &= C \frac{1}{g(r) \Delta_1 \cdots \Delta_k} h(r) \\ &\leq C \frac{1}{g(d_k) \Delta_1 \cdots \Delta_k} h(r). \end{aligned}$$

Because

$$C \frac{1}{g(d_k) \Delta_1 \cdots \Delta_k} \rightarrow 0 \text{ as } k \rightarrow \infty$$

by our hypothesis and $r \rightarrow 0$ if and only if $k \rightarrow \infty$, it follows that A has infinite \mathcal{H}^h -measure by Theorem 2.5.1. \square

5 The exponential family and the linearizing map

In this section, we first define the exponential family and prove some simple facts we will need in the following. In the second part, we state some facts about the linearizing map for exponential maps with a real attracting fixed point. This linearizer will be our main object of interest in the future - we will later use it to construct the gauge functions we will be concerned with.

5.1 Definition of the exponential family

Although the exponential family was already defined in the introduction, we give the definition here once again and state McMullen's results as well as some simple properties of exponential maps.

Definition 5.1.1 (exponential family) For $\lambda \in \mathbb{C} \setminus \{0\}$, let

$$E_\lambda(z) := \lambda \exp(z)$$

be the **exponential map with parameter** λ . The collection of all exponential maps, i.e. $\{E_\lambda\}_{\lambda \in \mathbb{C} \setminus \{0\}}$, is known as the **exponential family**. \diamond

It is immediate that every member of the exponential family has only 0 as a singular value (0 is also the only exceptional value). Later, we will often use the following easy fact, in which another notation is introduced:

Lemma 5.1.2 *Let $0 < \lambda < 1/e$. Then E_λ has exactly one real attracting fixed point α_λ and one real repelling fixed point β_λ .*

Proof. This follows directly by the intermediate value theorem: We have $E_\lambda(0) > 0$, $E_\lambda(1) = \lambda e < 1$ and $\lim_{x \rightarrow \infty} E_\lambda(x)/x = \infty$. Because E'_λ is a strictly increasing function, E_λ has exactly one fixed point $\alpha_\lambda \in (0, 1)$ and one fixed point $\beta_\lambda > 1$. Since $E'_\lambda = E_\lambda$, α_λ is attracting and β_λ is repelling. \square

The next easy lemma describes the escaping set $I(E_\lambda)$ of an arbitrary exponential map.

Lemma 5.1.3 *For every $\lambda \in \mathbb{C} \setminus \{0\}$, we have*

$$z \in I(E_\lambda) \Leftrightarrow \Re E_\lambda^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Proof. This follows immediately by $|E_\lambda(z)| = \lambda e^{\Re z}$. \square

As we are concerned with Hausdorff measure of Julia sets, we want to state the two results by McMullen [20] which were the starting point for this thesis (the second result also follows by a general argument of Eremenko and Lyubich [9]):

Theorem 5.1.4 (Hausdorff dimension of Julia set is 2) *Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then the Hausdorff dimension of $\mathcal{J}(E_\lambda)$ is 2.*

Theorem 5.1.5 (Julia sets for hyperbolic exponentials have zero measure) *Let $\lambda \in \mathbb{C} \setminus \{0\}$ such that E_λ is a hyperbolic map, i.e. 0 is absorbed by an attracting periodic cycle. Then $|\mathcal{J}(E_\lambda)| = 0$, i.e. the two-dimensional Hausdorff measure of $\mathcal{J}(E_\lambda)$ is 0.*

Remark: From the first theorem and the definition of Hausdorff dimension, it follows that $\mathcal{H}^{h^s}(\mathcal{J}(E_\lambda)) = \infty$ for every $s < 2$, and by the second theorem, we have $\mathcal{H}^{h^2}(\mathcal{J}(E_\lambda)) = 0$ if E_λ is hyperbolic. This immediately leads to the question if one can give a gauge function h for which - at least in the hyperbolic case - one has $h^s \prec h \prec h^2$ for any $s < 2$ and $\mathcal{H}^h(\mathcal{J}(E_\lambda))$ is positive and finite (or σ -finite). A first step in this direction was done by McMullen [20]. He remarks (without proof) that for every λ and every $k \in \mathbb{N}$, the Julia set of E_λ has infinite Hausdorff measure with respect to the gauge function

$$h(t) = t^2 \log^k \left(\frac{1}{t} \right).$$

This result suggests to consider gauge functions $h(t) = t^2 g(t)$, where g is a function which satisfies

$$g(t) = o\left(\log^k \left(\frac{1}{t} \right)\right) \text{ as } t \rightarrow 0.$$

One function with this property is the inverse of the linearizing map considered in the next subsection. ♠

5.2 The linearizing map for real hyperbolic parameters

Before we prove the main lemma (the linearizing function grows faster than any fixed iterate of the (standard) exponential function E_1), we extend Theorem 2.2.5, fit it to our setting and thereby introduce some notations which we will use in the remaining sections of this thesis.

Theorem 5.2.1 (Existence of globally defined linearizing map) *Let an arbitrary $\lambda \in (0, 1/e)$ be given. Let $\beta_\lambda > 1$ be the unique real repelling fixed point of E_λ . Let $\Phi_\lambda : V \rightarrow U$ be the Kœnigs function (see Theorem 2.2.5) with respect to β_λ . Let $S_\lambda := (\Phi_\lambda)^{-1}$ be the linearizing map. Then $S_\lambda(0) = \beta_\lambda$, $S'_\lambda(0) = 1$ and S_λ satisfies the functional equation*

$$E_\lambda(S_\lambda(z)) = S_\lambda(\beta_\lambda(z)) \text{ for all } z \text{ such that } z, \beta_\lambda z \in U. \quad (5.2.1)$$

Further, S_λ can be continued analytically to an entire function (which we also denote by S_λ).

Proof. Let S_λ be the linearizing map (defined in a neighborhood of 0) as in Theorem 2.2.5. Every statement of the theorem follows directly by Theorem 2.2.5, except the

claim that S_λ can be continued analytically to the whole complex plane. But this is also very easy to see: For $z \in \mathbb{C}$, we choose $n \in \mathbb{N}$ such that $z/\beta_\lambda^n \in U$ and we define

$$S_\lambda(z) := E_\lambda^n \left(S_\lambda \left(\frac{z}{\beta_\lambda^n} \right) \right). \quad (5.2.2)$$

This function is well-defined because of the functional equation (5.2.1) and satisfies exactly this equation by the permanence of functional relations. \square

Now we can formulate the announced main lemma of this section about the growth of the linearizing map.

Lemma 5.2.2 (Properties of S_λ) *Let $0 < \lambda < 1/e$ and S_λ be the linearizing map. Then the following three statements hold:*

- (a) *For all $z \in \mathbb{C}$, we have $S'_\lambda(z) \neq 0$.*
- (b) *$S_\lambda(\mathbb{R}) \subset \mathbb{R}$ and $S_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function.*
- (c) *For all $n \in \mathbb{N}$, we have*

$$\frac{S_\lambda(x)}{\exp^n(x)} \rightarrow \infty \text{ as } x \rightarrow \infty.$$

Proof. For the proof of (a), suppose that there exists $z_0 \in \mathbb{C}$ with $S'_\lambda(z_0) = 0$. By differentiating (5.2.1), we have

$$S'_\lambda(\beta_\lambda z) \beta_\lambda = E'_\lambda(S_\lambda(z)) S'_\lambda(z) \text{ for all } z \in \mathbb{C},$$

and since $E'_\lambda(z) \neq 0$ for all z , it follows that $S'_\lambda(z_0/\beta_\lambda) = 0$. By induction, we have

$$S'_\lambda \left(\frac{z_0}{\beta_\lambda^n} \right) = 0 \text{ for all } n \in \mathbb{N}$$

and the identity theorem yields that S'_λ is identically 0, which is a contradiction to $S'_\lambda(0) = 1$.

We prove (b) and (c) by showing first that every coefficient of the Taylor expansion of S_λ in 0 is real and positive, i.e. $S_\lambda^{(n)}(0) > 0$ for all $n \in \mathbb{N}_0$. In order to show this, we prove by induction that for every $n \in \mathbb{N}_{\geq 2}$ and every $x \in \mathbb{R}$, we have

$$S_\lambda^{(n)}(x) \frac{\beta_\lambda^n}{E_\lambda \left(S_\lambda \left(\frac{x}{\beta_\lambda} \right) \right)} - S_\lambda^{(n)} \left(\frac{x}{\beta_\lambda} \right) = P_n \left(S'_\lambda \left(\frac{x}{\beta_\lambda} \right), \dots, S_\lambda^{(n-1)} \left(\frac{x}{\beta_\lambda} \right) \right), \quad (5.2.3)$$

where P_n is a polynomial with positive coefficients.

We define

$$y := \frac{x}{\beta_\lambda}$$

and first prove formula (5.2.3) for $n = 2$. Again by differentiating the functional equation (5.2.1), we get

$$S'_\lambda(x) = \frac{1}{\beta_\lambda} E_\lambda(S_\lambda(y)) S'_\lambda(y) \text{ for all } x \in \mathbb{R}, \quad (5.2.4)$$

and hence by the product rule (and using $E'_\lambda = E_\lambda$)

$$\begin{aligned} S''_\lambda(x) &= \frac{1}{\beta_\lambda} \left(E'_\lambda(S_\lambda(y)) S'_\lambda(y)^2 \frac{1}{\beta_\lambda} + E_\lambda(S_\lambda(y)) S''_\lambda(y) \frac{1}{\beta_\lambda} \right) \\ &= \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^2} (S'_\lambda(y)^2 + S''_\lambda(y)). \end{aligned}$$

It follows that

$$S''_\lambda(x) \frac{\beta_\lambda^2}{E_\lambda(S_\lambda(y))} - S''_\lambda(y) = S'_\lambda(y)^2 =: P_2(S'_\lambda(y))$$

for all x , and P_2 is a polynomial with positive coefficients.

Now suppose that (5.2.3) is true for some $n \in \mathbb{N}_{\geq 2}$ with some polynomial P_n . By differentiation, we get

$$\begin{aligned} S_\lambda^{(n+1)}(x) &= \frac{d}{dx} S_\lambda^{(n)}(x) \\ &= \frac{d}{dx} \left(\left(P_n(S'_\lambda(y), \dots, S_\lambda^{(n-1)}(y)) + S_\lambda^{(n)}(y) \right) \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^n} \right) \\ &= \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^n} \frac{d}{dx} \left(P_n(S'_\lambda(y), \dots, S_\lambda^{(n-1)}(y)) + S_\lambda^{(n)}(y) \right) \\ &\quad + \left(P_n(S'_\lambda(y), \dots, S_\lambda^{(n-1)}(y)) + S_\lambda^{(n)}(y) \right) \frac{d}{dx} \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^n}. \end{aligned}$$

It is clear that we can find a polynomial $\overline{P_{n+1}}$ with positive coefficients such that

$$\frac{d}{dx} P_n(S'_\lambda(y), \dots, S_\lambda^{(n-1)}(y)) = \frac{1}{\beta_\lambda} \overline{P_{n+1}}(S'_\lambda(y), \dots, S_\lambda^{(n)}(y)).$$

Hence we obtain

$$\begin{aligned} S_\lambda^{(n+1)}(x) &= \frac{1}{\beta_\lambda} \left(\overline{P_{n+1}}(S'_\lambda(y), \dots, S_\lambda^{(n)}(y)) + S_\lambda^{(n+1)}(y) \right) \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^n} \\ &\quad + \frac{1}{\beta_\lambda} \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^n} S'_\lambda(y) \left(P_n(S'_\lambda(y), \dots, S_\lambda^{(n-1)}(y)) + S_\lambda^{(n)}(y) \right) \\ &= \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^{n+1}} \left(\overline{P_{n+1}}(S'_\lambda(y), \dots, S_\lambda^{(n)}(y)) \right. \\ &\quad \left. + \left(P_n(S'_\lambda(y), \dots, S_\lambda^{(n-1)}(y)) + S_\lambda^{(n)}(y) \right) S'_\lambda(y) \right. \\ &\quad \left. + S_\lambda^{(n+1)}(y) \right). \end{aligned}$$

If we set

$$P_{n+1}(x_1, \dots, x_n) := \overline{P_{n+1}}(x_1, \dots, x_n) + (P_n(x_1, \dots, x_{n-1}) + x_n)x_1,$$

then P_{n+1} is a polynomial with positive coefficients and it follows by the above calculations that

$$S_\lambda^{(n+1)}(x) = \frac{E_\lambda(S_\lambda(y))}{\beta_\lambda^{n+1}} \left(P_{n+1}(S'_\lambda(y), \dots, S_\lambda^{(n)}(y)) + S_\lambda^{(n+1)}(y) \right).$$

Thus we have

$$S_\lambda^{(n+1)}(x) \frac{\beta_\lambda^{n+1}}{E_\lambda(S_\lambda(y))} - S_\lambda^{(n+1)}(y) = P_{n+1}(S_\lambda(y), \dots, S_\lambda^{(n)}(y))$$

as desired. From (5.2.3), it now follows easily by induction that

$$S_\lambda^{(n)}(0) > 0 \text{ for all } n \in \mathbb{N}_0. \quad (5.2.5)$$

Note that this is true for $n = 0$ and $n = 1$ by definition of S_λ , and for $n \geq 2$ we have by (5.2.3) that

$$S_\lambda^{(n)}(0) = \frac{P_n(S'_\lambda(0), \dots, S_\lambda^{(n-1)}(0))}{\beta_\lambda^{n-1} - 1}.$$

We have $S_\lambda^{(k)} > 0$ for all $k \leq n - 1$ by the induction hypothesis. Because $\beta_\lambda > 1$ and P_n has only positive coefficients, (5.2.5) follows.

It is clear that (b) now follows immediately from (a), $S'_\lambda(0) = 1$ and the fact that all coefficients in the Taylor series are real. So it remains to prove (c). It is very easy to see that

$$\lim_{x \rightarrow \infty} \frac{E_\lambda^n(x)}{\exp^{n-1}(x)} = \infty. \quad (5.2.6)$$

This is true because E_λ 'behaves' like the exponential function if x is large. For a short proof, note that we have

$$\frac{E_\lambda(x)}{x} = \lambda \frac{e^x}{x} \rightarrow \infty \text{ as } x \rightarrow \infty,$$

which shows (5.2.6) for $n = 1$. If we suppose that (5.2.6) is true for some $n \in \mathbb{N}$, we have

$$\frac{E_\lambda^{n+1}(x)}{\exp^n(x)} = \lambda \frac{\exp(E_\lambda^n(x))}{\exp(\exp^{n-1}(x))} = \lambda \exp(E_\lambda^n(x) - \exp^{n-1}(x)) \rightarrow \infty \text{ as } x \rightarrow \infty$$

by the induction hypothesis.

Now we can prove the statement of (c). Fix $n \in \mathbb{N}$. By (5.2.5), we can find $x_0 > 0$ such that

$$S_\lambda\left(\frac{x}{\beta_\lambda^{n+1}}\right) > \left(\frac{x}{\beta_\lambda^{n+1}}\right)^2 > x \text{ for all } x \geq x_0. \quad (5.2.7)$$

It follows for every $x \geq x_0$ by (5.2.1) and (5.2.7) that

$$\frac{S_\lambda(x)}{\exp^n(x)} = \frac{E_\lambda^{n+1}\left(S_\lambda\left(\frac{x}{\beta_\lambda^{n+1}}\right)\right)}{\exp^n(x)} > \frac{E_\lambda^{n+1}(x)}{\exp^n(x)}.$$

By (5.2.6), the right side tends to ∞ as $x \rightarrow \infty$, and (c) is proven. \square

Because of part (a) and (b) of the preceding lemma, the function Φ_λ can be defined on $S_\lambda(\mathbb{R}) \subset \mathbb{R}$ and is smooth there. It follows from (c) that Φ_λ grows slower than any iterate of the logarithm, i.e.

$$\lim_{x \rightarrow \infty} \frac{\Phi_\lambda(x)}{\log^n(x)} = 0 \text{ for all } n \in \mathbb{N}, \quad (5.2.8)$$

and (again by the permanence of functional relations) Φ_λ satisfies the functional equation

$$\Phi_\lambda(E_\lambda(x)) = \beta_\lambda \Phi_\lambda(x). \quad (5.2.9)$$

With the help of Φ_λ , we construct the gauge functions we will consider in the next sections. The fact that these functions really are gauge functions is straightforward and is now proven as a final result of this section.

Lemma 5.2.3 *Let $0 < \lambda < 1/e$ and Φ_λ defined as above. Let $\gamma > 0$. Then the function $g(t) := \Phi_\lambda(1/t)^\gamma$ satisfies the monotony and continuity conditions from Lemma 4.2.1, in particular*

$$h_\lambda^\gamma(t) := \begin{cases} t^2 \Phi_\lambda\left(\frac{1}{t}\right)^\gamma & , t > 0 \\ 0 & , t = 0 \end{cases}$$

is a gauge function.

Proof. Because Φ_λ is increasing and $\gamma > 0$, g is a decreasing function. The fact that $t^2 g(t) \rightarrow 0$ as t tends to 0 follows easily by (5.2.8) and $t^2 \log(1/t) \xrightarrow{t \rightarrow 0} 0$, and the fact that $t^2 g(t)$ is increasing on a small interval $[0, \varepsilon]$ is trivial because the function

$$x \mapsto \frac{\Phi_\lambda(x)^\gamma}{x^2}$$

is decreasing for x large enough. \square

6 The estimate from below

We now combine the results of the previous sections to show that the Julia set of E_λ has infinite (not even σ -finite) Hausdorff measure with respect to the gauge function

$$h_{\lambda_0}^\gamma(t) := t^2 \Phi_{\lambda_0}(1/t)^\gamma,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda_0 \in (0, 1/e)$ are arbitrary and γ is chosen such that $\beta_{\lambda_0}^\gamma > 2$. In section 7, we will prove that this result is optimal and that the condition on γ is really needed.

6.1 Idea of the proof and preparations

We introduce some notations and prove a few lemmas which we will use in the proof of the main theorem of this section (and in the proof of the main theorem of section 7 as well).

For the remaining part of this section, let $\lambda \in \mathbb{C}$ and $\lambda_0 \in (0, 1/e)$ be fixed. For $\delta > 0$, we define

$$I_\delta := \left[-\frac{\pi}{2} + \delta - \arg \lambda, \frac{\pi}{2} - \delta - \arg \lambda \right]$$

and

$$J_\delta := \{z \in \mathbb{C} \mid \Im z \in I_\delta + 2\pi i k \text{ for some } k \in \mathbb{Z}\}.$$

If $z = x + iy \in J_\delta$, we have $\arg E_\lambda(z) = \arg \lambda + y$, hence

$$\Re E_\lambda(z) = |\lambda| e^{\Re z} \cos \arg E_\lambda(z) \geq |\lambda| \cos \left(\frac{\pi}{2} - \delta \right) e^{\Re z} =: C e^{\Re z}. \quad (6.1.1)$$

Thus we obtain: As long as the iterates $E_\lambda^n(z)$ stay in J_δ and $\Re z$ is large enough, the real part of $E_\lambda^n(z)$ grows exponentially fast. Hence we have $z \in I(E_\lambda)$ if $E_\lambda^n(z) \in J_\delta$ for all $n \in \mathbb{N}$ and z belongs to a suitable right half plane. Because $I(E_\lambda) \subset \mathcal{J}(E_\lambda)$ by Theorem 2.2.2, the set of all such points z belongs to the Julia set of E_λ .

McMullen's proof that the Julia set of E_λ has Hausdorff dimension 2 (and our proof as well) uses the following idea: We start with a small square $Q \subset J_\delta$ which lies far to the right and set $\mathcal{A}_0 := \{Q\}$. Then we construct a suitable family $\{\mathcal{A}_n\}$ which satisfies the nesting conditions, and which is chosen such that $E_\lambda^n(F)$ is a square contained in J_δ for every $F \in \mathcal{A}_n$. It turns out that for our choice of the \mathcal{A}_k , the Δ_k are near $1/2$ and the d_k tend to 0 exponentially fast, which (combined with Lemma 4.2.1) proves our theorem.

In this subsection, we prove some Lemmas which will help us estimating the densities and diameters of the sets in \mathcal{A}_k . We start with a lemma which is quite obvious, but nevertheless it is an important lemma for our main theorem. We introduce the following notation:

Definition 6.1.1 (r-box, r-packing) For $r > 0$, a subset $Q \subset \mathbb{C}$ is an **r-box** if $Q = \overline{Q}(z_0, r, 0)$ for some $z_0 \in \mathbb{C}$, i.e. a square of side length r with sides parallel to the coordinate axes. For a set $B \subset \mathbb{C}$ and $r > 0$, an **r-packing** of B is a union of disjoint r-boxes which are contained in B . \diamond

Lemma 6.1.2 *For every $\delta, \delta' > 0$, there exists $r_0 > 0$ with the following property: If $r < r_0$ is given, we can find $c(r) > 0$ such that for every $z_0 \in \mathbb{C}$, $c > c(r)$ and $\theta \in \mathbb{R}$, the set $Q := Q(z_0, cr, \theta) \cap J_\delta$ has an r -packing pack_Q which satisfies*

$$\text{dens}(\text{pack}_Q, Q) > \frac{1}{2} - \frac{\delta}{\pi} - \delta'.$$

Proof. We first show that for every $r > 0$, we can find a c such that the density of J_δ in an arbitrary $Q(z_0, rc, \theta)$ is at least $\frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2}$. So fix $r > 0$ and let $c > 0$. If we choose some $Q = Q(z_0, rc, \theta)$, we may assume without loss of generality that $z_0 = 0$. Of course we can also restrict ourselves to the case where $0 \leq \theta < \pi/2$, because $Q(0, rc, 0) = Q(0, rc, \pi/2)$. Here, we give the proof only for $0 \leq \theta \leq \pi/4$, the proof for $\theta > \pi/4$ will work analogous. We divide Q into three parts which we denote by I, II and III (compare the figure below). Let

$$z_l = x_l + iy_l = \left(-\frac{rc}{2} + i\frac{rc}{2} \right) e^{i\theta}$$

be the leftmost,

$$z_r = x_r + iy_r = \left(\frac{rc}{2} - i\frac{rc}{2} \right) e^{i\theta}$$

be the rightmost and

$$z_t = x_t + iy_t = \left(\frac{rc}{2} + i\frac{rc}{2} \right) e^{i\theta}$$

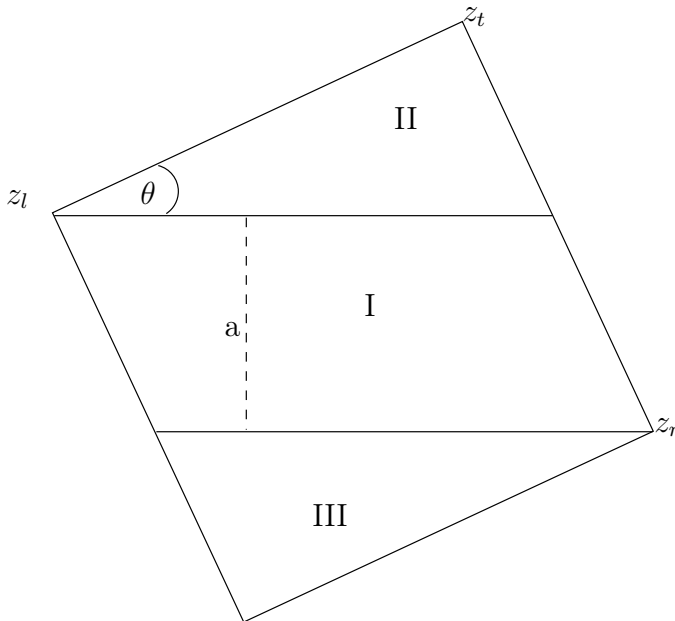
be the topmost point in Q . A simple calculation gives

$$z_l = \frac{rc}{2} \left((-\cos \theta - \sin \theta) + i(\cos \theta - \sin \theta) \right),$$

$$z_r = \frac{rc}{2} \left((\cos \theta + \sin \theta) + i(\sin \theta - \cos \theta) \right)$$

and

$$z_t = \frac{rc}{2} \left((\cos \theta - \sin \theta) + i(\sin \theta + \cos \theta) \right).$$



There are now two cases which will be treated separately: In the first case, θ is 'small', so that the areas of II and III are 'insignificant'. In the second case, neither the area of II and III nor the area of I dominates.

It follows by the above calculations that the height a of I is given by

$$a = \frac{rc}{2}(-\sin \theta + \cos \theta - \sin \theta + \cos \theta) = rc(\cos \theta - \sin \theta).$$

The linear measure of each horizontal line in I is $rc/\cos \theta$, and it follows therefore that every strip T in J_δ which intersects I with its full height satisfies

$$|T \cap I| = (\pi - 2\delta) \frac{rc}{\cos \theta}. \quad (6.1.2)$$

Let n_1 denote the number of strips in J_δ that intersect I with their full height. Then we have by simple considerations that

$$n_1 \geq \lfloor \frac{rc(\cos \theta - \sin \theta)}{2\pi} - 2 \rfloor, \quad (6.1.3)$$

so by (6.1.2) and (6.1.3), we obtain

$$|J_\delta \cap I| \geq \lfloor \frac{rc(\cos \theta - \sin \theta)}{2\pi} - 2 \rfloor (\pi - 2\delta) \frac{rc}{\cos \theta}. \quad (6.1.4)$$

We can now do the first case. There exists $\varepsilon_1 > 0$ such that if $\theta < \varepsilon_1$, then

$$\left(\frac{1}{2} - \frac{\delta}{\pi}\right) \frac{\cos \theta - \sin \theta}{\cos \theta} > \frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{4}.$$

So assume that $\theta < \varepsilon_1$. Let c be so large that

$$\frac{rc(\cos \theta - \sin \theta)}{2\pi} - 3 > 0.$$

Then we have by (6.1.4) and the fact that $1/\cos \theta$ is bounded on $[0, \varepsilon_1]$ that

$$\begin{aligned} \text{dens}(J_\delta, Q) &\geq \frac{1}{r^2 c^2} \left(\frac{rc(\cos \theta - \sin \theta)}{2\pi} - 3 \right) (\pi - 2\delta) \frac{rc}{\cos \theta} \\ &\geq \frac{1}{r^2 c^2} \left(\frac{r^2 c^2 \cos \theta - \sin \theta}{2\pi \cos \theta} - \frac{3rc}{\cos \theta} \right) (\pi - 2\delta) \\ &\geq \left(\frac{1}{2} - \frac{\delta}{\pi} \right) \frac{\cos \theta - \sin \theta}{\cos \theta} - \frac{3(\pi - 2\delta)}{rc \cos \theta} \\ &\geq \frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{4} + \mathcal{O}\left(\frac{1}{rc}\right) \\ &> \frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2} \end{aligned}$$

if rc is large enough, which finishes the first case.

Now let $\theta \geq \varepsilon_1$, $z = x_t + iy \in II$ and $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ be the points on ∂Q

with $\Im z_1 = \Im z_2 = y$.

If we set $b := y_t - y$, then we have by simple trigonometry that

$$x_t - x_1 = b \tan\left(\frac{\pi}{2} - \theta\right) = \frac{b}{\tan \theta}$$

and

$$x_2 - x_t = b \tan \theta.$$

It follows that

$$x_1 = \frac{rc}{2}(\cos \theta - \sin \theta) + b \tan \theta$$

and

$$x_2 = \frac{rc}{2}(\cos \theta - \sin \theta) - \frac{b}{\tan \theta}.$$

The maximal value for b (i.e. the height of II) is $rc \sin \theta$, so we have (as above) that the number n_2 of strips in J_δ that intersect II with their full height satisfies

$$n_2 \geq \lfloor \frac{rc \sin \theta}{2\pi} - 2 \rfloor. \quad (6.1.5)$$

If T is such a strip and we have $\text{dist}(\partial T, z_t) = y_0$, then it follows that

$$\begin{aligned} |T \cap II| &= \int_{y_0}^{y_0 + \pi - 2\delta} \int_{(\cos \theta - \sin \theta)rc/2 - y/\tan \theta}^{(\cos \theta - \sin \theta)rc/2 + y \tan \theta} 1 \, dx dy = \int_{y_0}^{y_0 + \pi - 2\delta} y \tan \theta + \frac{y}{\tan \theta} dy \\ &= \left((\pi - 2\delta)y_0 + \frac{1}{2}(\pi - 2\delta)^2 \right) \left(\tan \theta + \frac{1}{\tan \theta} \right). \end{aligned} \quad (6.1.6)$$

If we combine (6.1.5) and (6.1.6), we get

$$|J_\delta \cap II| \geq \sum_{k=1}^{\lfloor \frac{rc \sin \theta}{2\pi} - 2 \rfloor} \left((\pi - 2\delta)(2k\pi + \delta) + \frac{1}{2}(\pi - 2\delta)^2 \right) \left(\tan \theta + \frac{1}{\tan \theta} \right). \quad (6.1.7)$$

For $|J_\delta \cap III|$, we obviously get exactly the same estimate as for $|J_\delta \cap II|$. Now we can deal with the second case. Let c be so large that

$$\frac{rc(\cos \theta - \sin \theta)}{2\pi} - 3 > 0$$

and

$$\frac{rc \sin \theta}{2\pi} - 3 > 0 \text{ for every } \theta \in [\varepsilon_1, \pi/4].$$

By (6.1.4), (6.1.7) and the fact that $\tan \theta + \frac{1}{\tan \theta}$ is bounded on $[\varepsilon_1, \pi/4]$, we obtain

$$\begin{aligned}
\text{dens}(J_\delta, Q) &\geq \frac{1}{r^2 c^2} \left(\left(\frac{rc(\cos \theta - \sin \theta)}{2\pi} - 3 \right) (\pi - 2\delta) \frac{rc}{\cos \theta} \right. \\
&\quad \left. + 2 \sum_{k=1}^{\lfloor \frac{rc \sin \theta}{2\pi} - 2 \rfloor} \left((\pi - 2\delta)(2k\pi + \delta) + \frac{1}{2}(\pi - 2\delta)^2 \right) \left(\tan \theta + \frac{1}{\tan \theta} \right) \right) \\
&\geq \frac{1}{r^2 c^2} \left(\left(\frac{rc(\cos \theta - \sin \theta)}{2\pi} - 3 \right) (\pi - 2\delta) \frac{rc}{\cos \theta} \right. \\
&\quad \left. + \left((\pi - 2\delta)^2 \left(\frac{rc \sin \theta}{2\pi} - 3 \right) \right. \right. \\
&\quad \left. \left. + 2(\pi - 2\delta) \left(\left(\frac{rc \sin \theta}{2\pi} - 3 \right) \delta + \pi \left(\frac{rc \sin \theta}{2\pi} - 3 \right) \left(\frac{rc \sin \theta}{2\pi} - 2 \right) \right) \right) \right. \\
&\quad \left. \left(\tan \theta + \frac{1}{\tan \theta} \right) \right) \\
&= \frac{1}{r^2 c^2} \left(\frac{r^2 c^2 (\cos \theta - \sin \theta)}{\cos \theta} \frac{\pi - 2\delta}{2\pi} \right. \\
&\quad \left. + 2\pi(\pi - 2\delta) \frac{r^2 c^2 \sin(\theta)^2}{(2\pi)^2} \left(\tan \theta + \frac{1}{\tan \theta} \right) \right) + \mathcal{O}\left(\frac{1}{rc}\right) \\
&= \frac{\cos \theta - \sin \theta}{\cos \theta} \left(\frac{1}{2} - \frac{\delta}{\pi} \right) + \sin(\theta)^2 \left(\tan \theta + \frac{1}{\tan \theta} \right) \left(\frac{1}{2} - \frac{\delta}{\pi} \right) + \mathcal{O}\left(\frac{1}{rc}\right) \\
&= \left(\frac{1}{2} - \frac{\delta}{\pi} \right) (1 - \tan \theta + \sin(\theta)^2 \tan \theta + \sin \theta \cos \theta) + \mathcal{O}\left(\frac{1}{rc}\right) \\
&= \left(\frac{1}{2} - \frac{\delta}{\pi} \right) (1 - \tan \theta \cos(\theta)^2 + \sin \theta \cos \theta) + \mathcal{O}\left(\frac{1}{rc}\right) \\
&= \frac{1}{2} - \frac{\delta}{\pi} + \mathcal{O}\left(\frac{1}{rc}\right) \\
&> \frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2}
\end{aligned}$$

if rc is large enough, and the second case is done.

Now let $r > 0$ be given and choose some $c > 0$ which satisfies

$$\text{dens}(J_\delta, Q(z_0, rc, \theta)) > \frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2} \text{ for every } z_0 \in \mathbb{C}, \theta \in \mathbb{R},$$

and let $Q = Q(z_0, rc, \theta)$ be arbitrary. In order to find our r -packing of Q , we fix some small $\varepsilon > 0$ and cover \mathbb{C} with an $(r + \varepsilon)$ -grid of $(r + \varepsilon)$ -boxes. We define an r -packing pack_Q of $Q \cap J_\delta$ as follows: An r -box $Q(z, r, 0)$ is contained in pack_Q if and only if $Q(z, r + \varepsilon, 0)$ is a member of the $(r + \varepsilon)$ -grid which is completely contained in $J_\delta \cap Q$. We want to estimate the amount of $(r + \varepsilon)$ -boxes which are contained in pack_Q . First, note that at least $\lfloor \frac{(rc)^2}{(r + \varepsilon)^2} \rfloor$ squares from the grid intersect Q , because otherwise the sum of the areas of the grid squares would be less than $(rc)^2 = |Q|$ which is a

contradiction. Because we have

$$|J_\delta \cap Q| > \left(\frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2}\right) |Q|,$$

it follows that the number of grid squares that intersect $J_\delta \cap Q$ is at least

$$\lfloor \left(\frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2}\right) \frac{(rc)^2}{(r+\varepsilon)^2} \rfloor.$$

The number of grid squares that intersect Q and the boundary of a particular strip T in J_δ is bounded by $\mathcal{O}\left(\frac{rc}{r+\varepsilon}\right) = \mathcal{O}(c)$, and there are $\mathcal{O}(rc)$ strips in J_δ intersecting Q . Hence it follows that the amount of grid squares that intersect $\partial J_\delta \cap Q$ is bounded by $\mathcal{O}(rc^2)$. So we have

$$\begin{aligned} \text{dens}(\text{pack}_Q, Q) &\geq \frac{(r+\varepsilon)^2}{r^2 c^2} \left(\left(\frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2}\right) \frac{r^2 c^2}{(r+\varepsilon)^2} - 1 - \mathcal{O}(rc^2) \right) \\ &= \frac{1}{2} - \frac{\delta}{\pi} - \frac{\delta'}{2} + \mathcal{O}\left(\frac{1}{c^2}\right) + \mathcal{O}(r) \\ &> \frac{1}{2} - \frac{\delta}{\pi} - \delta' \end{aligned}$$

if r is small enough (and c is large enough). This finishes the proof. \square

The next result, which is mainly an application of the preceding lemma, will help us to estimate the densities of our nested sets.

Lemma 6.1.3 *For every $\tilde{\varepsilon} > 0$, there exist $\delta > 0$ and $r_0 > 0$ with the following property: If $r \leq r_0$, we can find $x > 0$ such that if*

$$Q := Q(z_0, r, \theta) \subset (J_\delta \cap \{z \mid \Re z > x\}),$$

there exists an r -packing $\text{pack}_{E_\lambda(Q)}$ of $E_\lambda(Q) \cap J_\delta$ with

$$\text{dens}(\text{pack}_{E_\lambda(Q)}, E_\lambda(Q)) > \frac{1}{2} - \tilde{\varepsilon}.$$

Proof. Let $\varepsilon > 0$. Because E_λ is univalent on $D(z, \pi)$ for all $z \in \mathbb{C}$, there exists $r_1 > 0$ such that for every $r' < r_1$ and every $Q = Q(z, r', \theta)$, we can find some

$$Q\left(E_\lambda(z), |E'_\lambda(z)| r' \frac{1}{D(E_\lambda|_Q)} (1 - \sqrt{2\varepsilon})\right)$$

which is contained in $E_\lambda(Q)$ and some

$$Q\left(E_\lambda(z), |E'_\lambda(z)| r' D(E_\lambda|_Q) (1 + \sqrt{2\varepsilon})\right)$$

that contains $E_\lambda(Q)$ (this follows from Lemma 3.2.2). By Lemma 3.2.1, we can choose r_2 so small that $D(E_\lambda|_Q) < 1 + \varepsilon$ for every $Q = Q(z, r', \theta)$, $r' < r_2$. Hence we have

$$Q\left(E_\lambda(z), |E'_\lambda(z)| r' \frac{1}{1+\varepsilon} (1 - \sqrt{2\varepsilon})\right) \subset E_\lambda(Q) \subset Q\left(E_\lambda(z), |E'_\lambda(z)| r' (1+\varepsilon) (1 + \sqrt{2\varepsilon})\right)$$

for all $r' < \min\{r_1, r_2\}$ and all $Q = Q(z, r', \theta)$. Now we choose $\delta, \delta' > 0$ such that

$$\frac{1}{2} - \frac{\delta}{\pi} - \delta' > \frac{1}{2} - \varepsilon.$$

By Lemma 6.1.2, there exists $r_3 > 0$ such that for every $r' < r_3$, we can find $c(r') > 0$ such that for every $c \geq c(r')$ and every $Q = Q(z, cr', \theta)$, there is an r' -packing pack_Q of $Q \cap J_\delta$ such that

$$\text{dens}(\text{pack}_Q, Q) > \frac{1}{2} - \varepsilon. \quad (6.1.8)$$

If we choose $r < \min\{r_1, r_2, r_3\}$ and $c(r) > 0$ with the above property, we can find $x > 0$ such that

$$|E_\lambda(z)| > c(r) \frac{1 + \varepsilon}{1 - \sqrt{2}\varepsilon}$$

for every $z \in \mathbb{C}$ with $\Re z > x$. Now fix some $Q = Q(z_0, r, \theta) \subset \{z \mid \Re z > x\}$. It follows that we have

$$\begin{aligned} Q_1 &:= Q\left(E_\lambda(z_0), |E_\lambda(z)| r \frac{1 - \sqrt{2}\varepsilon}{1 + \varepsilon}\right) \subset E_\lambda(Q) \\ &\subset Q\left(E_\lambda(z_0), |E_\lambda(z)| r(1 + \varepsilon)(1 + \sqrt{2}\varepsilon)\right) =: Q_2. \end{aligned}$$

Let an r -packing pack_{Q_1} of $Q_1 \cap J_\delta$ which satisfies (6.1.8) (with Q_1 instead of Q) be given. Then pack_{Q_1} is also an r -packing of $E_\lambda(Q) \cap J_\delta$. Thus

$$\begin{aligned} \text{dens}(\text{pack}_{Q_1}, E_\lambda(Q)) &= \frac{|\text{pack}_{Q_1} \cap E_\lambda(Q)|}{|E_\lambda(Q)|} \\ &\geq \frac{|\text{pack}_{Q_1} \cap Q_1|}{|Q_2|} \\ &= \frac{|\text{pack}_{Q_1} \cap Q_1|}{|Q_1|} \frac{|Q_1|}{|Q_2|} \\ &> \left(\frac{1}{2} - \varepsilon\right) \frac{(1 - \sqrt{2}\varepsilon)^2}{(1 + \sqrt{2}\varepsilon)^2(1 + \varepsilon)^4}. \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, we can choose ε so small that the last term is smaller than $\frac{1}{2} - \tilde{\varepsilon}$, which finishes the proof. \square

Whereas the preceding lemmas will allow us to estimate densities of the nested sets, the following lemma will enable us to estimate their diameters.

Lemma 6.1.4 *For every $\delta > 0$, there exists $x_0 > 0$ with*

$$\Re E_\lambda^n(z) > E_{\lambda_0}^n(2\beta_{\lambda_0})$$

for all $z \in J_\delta$ and all $n \in \mathbb{N}$ such that $\Re z > x_0$ and $E_\lambda^k(z) \in J_\delta$ for every $0 \leq k \leq n$.

Proof. We choose x_0 so large that

$$x_0 > 2\beta_{\lambda_0} \quad (6.1.9)$$

and that

$$\frac{e^{\Re z}}{e^{2\beta_{\lambda_0}}} \underbrace{\frac{|\lambda| \cos\left(\frac{\pi}{2} - \delta\right)}{\lambda_0}}_{=:C} > \frac{\Re z}{2\beta_{\lambda_0}} \quad (6.1.10)$$

for all $z \in \mathbb{C}$ with $\Re z > x_0$.

Now choose $z \in J_\delta \cap \{z \mid \Re z > x_0\}$ and let $n \in \mathbb{N}_0$ such that $E_\lambda^k(z) \in J_\delta$ for all $0 \leq k \leq n$. We prove

$$\frac{e^{\Re E_\lambda^k(z)}}{e^{E_{\lambda_0}^k(2\beta_{\lambda_0})}} C > \frac{\Re E_\lambda^k(z)}{E_{\lambda_0}^k(2\beta_{\lambda_0})} \quad \text{for all } 0 \leq k \leq n. \quad (6.1.11)$$

From (6.1.11), the lemma follows easily because we have

$$\frac{\Re E_\lambda^{k+1}(z)}{E_{\lambda_0}^{k+1}(2\beta_{\lambda_0})} > \frac{e^{\Re E_\lambda^k(z)}}{e^{E_{\lambda_0}^k(2\beta_{\lambda_0})}} C \quad \text{for all } 0 \leq k \leq n$$

by (6.1.1), and $\Re z > 2\beta_{\lambda_0}$ by (6.1.9).

We prove (6.1.11) by induction. For $k = 0$, the statement is true because of (6.1.10).

Suppose that the inequality holds for some $0 \leq k \leq n - 1$. Thus we have

$$\frac{\Re E_\lambda^{k+1}(z)}{E_{\lambda_0}^{k+1}(z)} > \frac{e^{\Re E_\lambda^k(z)}}{e^{E_{\lambda_0}^k(2\beta_{\lambda_0})}} C > \frac{\Re E_\lambda^k(z)}{E_{\lambda_0}^k(2\beta_{\lambda_0})} > 1. \quad (6.1.12)$$

Further, it is clear that

$$E_{\lambda_0}^{k+1}(2\beta_{\lambda_0}) > E_{\lambda_0}^k(2\beta_{\lambda_0}) > 1 \quad (6.1.13)$$

because $E_{\lambda_0}(x) > x$ whenever $x > \beta_{\lambda_0}$. If we set

$$a := \Re E_\lambda^{k+1}(z), b := E_{\lambda_0}^{k+1}(2\beta_{\lambda_0}), c := \Re E_\lambda^k(z) \text{ and } d := E_{\lambda_0}^k(2\beta_{\lambda_0}),$$

it suffices to show (in view of (6.1.12) and (6.1.13)) that for $a, b, c, d > 0$, the three conditions

- $\frac{e^c}{e^d} C > \frac{c}{d}$
- $\frac{a}{b} > \frac{c}{d} > 1$ and
- $b > d > 1$

imply that

$$\frac{e^a}{e^b} C > \frac{a}{b}.$$

So let $a, b, c, d > 0$ which satisfy the three conditions above be given. Because of the first condition, it is enough to show that

$$e^{a-b} \frac{b}{a} > e^{c-d} \frac{d}{c}. \quad (6.1.14)$$

Define

$$f : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}, (x, y) \mapsto \int_x^y 1 - \frac{1}{t} dt.$$

Then (6.1.14) is equivalent to $f(b, a) > f(d, c)$. For every $x, y, z \in \mathbb{R}_{>0}$, we have

$$\begin{aligned} f(xz, yz) &= \int_{xz}^{yz} 1 - \frac{1}{t} dt = \int_x^y z \left(1 - \frac{1}{zt}\right) dt \\ &= \int_x^y z - \frac{1}{t} dt \\ &= (z-1)(y-x) + f(x, y). \end{aligned}$$

It follows that

$$f(1, a/b) = (1/b - 1)(a - b) + f(b, a)$$

and

$$f(1, c/d) = (1/d - 1)(c - d) + f(d, c),$$

hence

$$f(b, a) - f(d, c) = f\left(1, \frac{a}{b}\right) - f\left(1, \frac{c}{d}\right) + \left(1 - \frac{1}{b}\right)(a - b) - \left(1 - \frac{1}{d}\right)(c - d).$$

Because of the second condition, we have

$$f\left(1, \frac{a}{b}\right) = \int_1^{a/b} 1 - \frac{1}{t} dt > \int_1^{c/d} 1 - \frac{1}{t} dt = f\left(1, \frac{c}{d}\right).$$

So it suffices to show that

$$\left(1 - \frac{1}{b}\right)(a - b) - \left(1 - \frac{1}{d}\right)(c - d) \geq 0,$$

which is equivalent to

$$(a - b) - (c - d) \geq \frac{a}{b} - \frac{c}{d}.$$

Because of the second and the third condition, we have

$$(a - b) - (c - d) > 0. \tag{6.1.15}$$

Thus we obtain by $b > d > 0$ and (6.1.15) that

$$\begin{aligned} \frac{a}{b} - \frac{c}{d} \leq (a - b) - (c - d) &\Leftrightarrow \frac{a - b}{b} - \frac{c - d}{d} \leq (a - b) - (c - d) \\ &\Leftrightarrow \frac{a - b}{d} - \frac{c - d}{d} \leq (a - b) - (c - d) \\ &\Leftrightarrow 1 \leq d, \end{aligned}$$

which is true by the third condition, and the proof is finished. \square

If we want to estimate the distortion of inverse branches of the exponential map with the help of Lemma 3.3.1, we have to be sure that these inverse branches are well defined on a large region. This is not very difficult to see and is the statement of the following lemma.

Lemma 6.1.5 *Let $\delta > 0$ and J_δ be defined as above. Let $K > 0$. Then there exists $x_0 > 0$ such that for all $z \in \mathbb{C}$, $n \in \mathbb{N}_0$ with $\Re z > x_0$ and $E_\lambda^k(z) \in J_\delta$ for all $0 \leq k \leq n$, we have*

$$\Re E_\lambda^n(z) - \Re E_\lambda^n(0) \geq K \text{ for all } 0 \leq k \leq n.$$

Proof. Choose $x_0 > K$ so large that $Ce^{x_0} - |\lambda| > x_0$ and

$$Ce^{\Re E_\lambda(z)} > \Re E_\lambda(z) \text{ for all } z \in J_\delta \text{ with } \Re z > x_0, \quad (6.1.16)$$

where C is defined as in (6.1.1). Because (6.1.16) implies that

$$\Re E_\lambda^{k+1}(z) \geq Ce^{\Re E_\lambda^k(z)} \geq \Re E_\lambda^k(z),$$

it suffices to show that

$$\Re E_\lambda^n(z) - \Re E_\lambda^n(0) \geq x_0 \quad (6.1.17)$$

for all $n \in \mathbb{N}_0$ such that $E_\lambda^k(z) \in J_\delta$ for all $0 \leq k \leq n$. We show (6.1.17) by induction on n . For $n = 0$, this is trivial. Now suppose that (6.1.17) holds for some $n \in \mathbb{N}_0$. Because of

$$\Re E_\lambda^{n+1}(0) \leq |\lambda| e^{\Re E_\lambda^n(0)},$$

we have

$$\Re E_\lambda^{n+1}(z) - \Re E_\lambda^{n+1}(0) \geq Ce^{\Re E_\lambda^n(z)} - |\lambda| e^{\Re E_\lambda^n(0)} \geq Ce^{x_0} - |\lambda| > x_0,$$

which finishes the proof. \square

6.2 A gauge function that leads to infinite measure

After these preparations, we are now able to prove a theorem which gives us a gauge function h such that $\mathcal{J}(E_\lambda)$ has infinite \mathcal{H}^h -measure.

Theorem 6.2.1 *Let $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda_0 \in (0, 1/e)$. Let β_{λ_0} be the unique repelling fixed point of E_{λ_0} and S_{λ_0} be the corresponding linearizing map. Let $\Phi_{\lambda_0} := (S_{\lambda_0}|_{\mathbb{R}})^{-1}$. Let $\gamma > 0$ such that $\beta_{\lambda_0}^\gamma > 2$ and define*

$$h_{\lambda_0}^\gamma(t) := t^2 \Phi_{\lambda_0} \left(\frac{1}{t} \right)^\gamma.$$

Then we have

$$\mathcal{H}^{h_{\lambda_0}^\gamma}(\mathcal{J}(E_\lambda)) = \infty,$$

in fact $\mathcal{H}^{h_{\lambda_0}^\gamma}$ is not even a σ -finite measure on $\mathcal{J}(E_\lambda)$.

Proof. The fact that $h_{\lambda_0}^\gamma$ is a gauge function was proved in Lemma 5.2.3, hence $\mathcal{H}^{h_{\lambda_0}^\gamma}$ is a well defined Hausdorff measure. We want to use Lemma 4.2.1 to prove the theorem. First of all, let $\varepsilon > 0$ such that

$$\beta_{\lambda_0}^\gamma \frac{1/2 - \varepsilon}{(1 + \varepsilon)^2} > 1. \quad (6.2.1)$$

Further, let $K > 0$ be so large that

$$\left(\frac{K+1}{K-3}\right)^{12} < 1 + \varepsilon. \quad (6.2.2)$$

By Lemma 6.1.3, there exist $\delta > 0$, $x_0 > 0$ and $r > 0$ such that the following three conditions are satisfied:

- (a) $Kr/\sqrt{2} < \pi$
- (b) If Q is an r -box with $Q \subset J_\delta \cap \{\Re z > x_0\}$, then there exists an r -packing $\text{pack}_{E_\lambda(Q)}$ of $E_\lambda(Q) \cap J_\delta$ such that

$$\text{dens}(\text{pack}_{E_\lambda(Q)}, E_\lambda(Q)) > \frac{1}{2} - \varepsilon.$$

- (c) $r\sqrt{2}\lambda_0/|\lambda| < 1$

For every Q as in (b), let $\mathcal{B}(E_\lambda(Q))$ denote the collection of the r -boxes that form $\text{pack}_{E_\lambda(Q)}$. By Lemma 6.1.4, we can find $x_1 \geq x_0$ such that

$$\Re E_\lambda^n(z) > E_{\lambda_0}^n(2\beta_{\lambda_0}) \quad (6.2.3)$$

for all $z \in J_\delta$ with $\Re z > x_1$ and all $n \in \mathbb{N}$ such that $E_\lambda^k(z) \in J_\delta$ for $0 \leq k \leq n$. Then, by Lemma 6.1.5, we can choose $x_2 \geq x_1$ such that for all $z \in J_\delta$, $n \in \mathbb{N}_0$ with $\Re z > x_2$ and $E_\lambda^k(z) \in J_\delta$ for $0 \leq k \leq n$, we have

$$\Re E_\lambda^n(z) - \Re E_\lambda^k(0) \geq \frac{Kr}{\sqrt{2}} \text{ whenever } 0 \leq k \leq n. \quad (6.2.4)$$

We define $T := J_\delta \cap \{z \mid \Re z > x_2\}$ and choose an r -box $Q \subset T$. Now we define a family of nested sets as follows: Set

$$\mathcal{A}_0 := \{Q\}$$

and

$$\mathcal{A}_k := \{G \mid G \subset F \text{ for some } F \in \mathcal{A}_{k-1} \text{ and } E_\lambda^k(G) \in \mathcal{B}(E_\lambda^k(F))\}$$

for $k \in \mathbb{N}$, i.e. \mathcal{A}_1 consists of all preimages (under E_λ) of the r -boxes that are contained in $\mathcal{B}(E_\lambda(Q))$, \mathcal{A}_2 consists of those subsets of elements $F \in \mathcal{A}_1$ which are mapped onto

an r -box in $\mathcal{B}(E_\lambda^2(F))$ under E_λ^2 and so on. Then $\{\mathcal{A}_k\}$ is a family of nested sets. As in section 4, we define $A_k := \bigcup_{F \in \mathcal{A}_k} F$ and $A := \bigcap_{k \in \mathbb{N}} A_k$. Then $z \in A$ implies that

$$E_\lambda^n(z) \in J_\delta \text{ for all } n \in \mathbb{N}_0.$$

It follows from (6.2.3) that $\Re E_\lambda^n(z) \rightarrow \infty$ as $n \rightarrow \infty$, and therefore $z \in \mathcal{J}(E_\lambda)$ by Theorem 2.2.2.

We now want to estimate the densities and diameters of the sets in \mathcal{A}_k . If $k \in \mathbb{N}$ and $F \in \mathcal{A}_{k-1}$, then $E_\lambda^{k-1}(F)$ is an r -box $Q = \overline{Q(z_Q, r, 0)}$. First, we estimate the distortion of $(E_\lambda^k)^{-1}|_{E_\lambda(Q)} : E_\lambda(Q) \rightarrow F$. Because of condition (a), E_λ is univalent on $D(z_Q, Kr/\sqrt{2})$. Further, the branch φ of $(E_\lambda^{k-1})^{-1}|_Q$ with $\varphi(z_Q) \in F$ can be continued univalently to $D(z_Q, Kr/\sqrt{2})$ because of (6.2.4) and Lemma 2.2.1, since 0 is the only singular value of E_λ . It follows by Theorem 3.3.1 and (6.2.2) that

$$D((E_\lambda^k)^{-1}|_{E_\lambda^k(F)}) < 1 + \varepsilon. \quad (6.2.5)$$

Thus we obtain by condition (b) and Lemma 3.1.4 (c) that

$$\begin{aligned} \frac{1}{2} - \varepsilon &< \text{dens} \left(\bigcup_{Q \in \mathcal{B}(E_\lambda^k(F))} Q, E_\lambda^k(F) \right) \\ &\leq D((E_\lambda^k)^{-1}|_{E_\lambda^k(F)})^2 \text{dens} \left((E_\lambda^k)^{-1} \left(\bigcup_{Q \in \mathcal{B}(E_\lambda^k(F))} Q \right), F \right) \\ &\leq (1 + \varepsilon)^2 \text{dens} \left(\bigcup_{Q \in \mathcal{B}(E_\lambda^k(F))} (E_\lambda^k)^{-1}(Q), F \right). \end{aligned}$$

Since

$$A_k \cap F = \bigcup_{Q \in \mathcal{B}(E_\lambda^k(F))} (E_\lambda^k)^{-1}(Q) \cap F,$$

we have

$$\text{dens}(A_k, F) = \text{dens} \left(\bigcup_{Q \in \mathcal{B}(E_\lambda^k(F))} (E_\lambda^k)^{-1}(Q), F \right) \geq \frac{1/2 - \varepsilon}{(1 + \varepsilon)^2} =: \Delta_k. \quad (6.2.6)$$

Now we turn to the diameters of the sets in \mathcal{A}_k . Let $F \in \mathcal{A}_k$ and $u, v \in F$. Because $E_\lambda^k(F)$ is an r -box and hence convex, it follows by the mean value theorem that

$$|u - v| \leq |E_\lambda^k(u) - E_\lambda^k(v)| \max_{z \in [E_\lambda^k(u), E_\lambda^k(v)]} |((E_\lambda^k)^{-1})'(z)|.$$

If $E_\lambda^k(w)$ is the value where the above maximum is taken, we have by (6.2.3) and

condition (c) that

$$\begin{aligned}
|u - v| &\leq |E_\lambda^k(u) - E_\lambda^k(v)| |((E_\lambda^k)^{-1})'(E_\lambda^k(w))| \\
&\leq r\sqrt{2} \frac{1}{|(E_\lambda^k)'(w)|} \\
&< r\sqrt{2} \frac{1}{|E_\lambda^k(w)|} \\
&= \frac{r\sqrt{2}}{|\lambda|} \frac{1}{e^{\Re E_\lambda^{k-1}(w)}} \\
&< \frac{r\sqrt{2}}{|\lambda|} \frac{1}{e^{E_{\lambda_0}^{k-1}(2\beta_{\lambda_0})}} \\
&= \frac{r\sqrt{2}\lambda_0}{|\lambda|} \frac{1}{E_{\lambda_0}^k(2\beta_{\lambda_0})} \\
&< \frac{1}{E_{\lambda_0}^k(2\beta_{\lambda_0})} =: d_k.
\end{aligned} \tag{6.2.7}$$

So $\{\mathcal{A}_k\}$ satisfies the nesting conditions. To finish the proof, we now only have to check that $t \mapsto \Phi_{\lambda_0}(1/t)^\gamma$ satisfies the condition (4.2.1) in Lemma 4.2.1. But this is now straightforward and follows mainly from the functional equation (5.2.9), combined with (6.2.6) and (6.2.7). In fact,

$$\begin{aligned}
\Phi_{\lambda_0}\left(\frac{1}{d_k}\right)^\gamma \prod_{j=1}^k \Delta_j &= \Phi_{\lambda_0}\left(\frac{1}{d_k}\right)^\gamma \left(\frac{1/2 - \varepsilon}{(1 + \varepsilon)^2}\right)^k \\
&= \Phi_{\lambda_0}(E_{\lambda_0}^k(2\beta_{\lambda_0}))^\gamma \left(\frac{1/2 - \varepsilon}{(1 + \varepsilon)^2}\right)^k \\
&= \left(\beta_{\lambda_0}^k \Phi_{\lambda_0}(2\beta_{\lambda_0})\right)^\gamma \left(\frac{1/2 - \varepsilon}{(1 + \varepsilon)^2}\right)^k \\
&= \left(\beta_{\lambda_0}^\gamma \left(\frac{1/2 - \varepsilon}{(1 + \varepsilon)^2}\right)\right)^k \Phi_{\lambda_0}(2\beta_{\lambda_0})^\gamma
\end{aligned}$$

which tends to ∞ as $k \rightarrow \infty$ by (6.2.1). Thus Lemma 4.2.1 shows that $\mathcal{J}(E_\lambda)$ has infinite $\mathcal{H}^{h_{\lambda_0}^\gamma}$ -measure. The claim now follows from Theorem 2.4.5: Suppose that $\mathcal{J}(E_\lambda)$ has σ -finite $\mathcal{H}^{h_{\lambda_0}^\gamma}$ -measure. Then we can choose some $\gamma' < \gamma$ such that $\beta_{\lambda_0}^{\gamma'} > 2$. Because $h_{\lambda_0}^\gamma \prec h_{\lambda_0}^{\gamma'}$, it follows from Theorem 2.4.5 that $\mathcal{H}^{h_{\lambda_0}^{\gamma'}}(\mathcal{J}(E_\lambda)) = 0$, a contradiction. \square

Remark: The proof of the theorem shows that in fact, $\mathcal{H}^{h_{\lambda_0}^\gamma}$ is a non- σ -finite measure on $I(E_\lambda)$. \spadesuit

7 The estimate from above

In this section, we show that the condition on γ stated in Theorem 6.2.1 cannot be dropped: For $\lambda \in (0, 1/e)$, we can find γ such that $\mathcal{H}^{h_\lambda^\gamma}(\mathcal{J}(E_\lambda)) = 0$ (here $h_\lambda^\gamma(t) = t^2\Phi_\lambda(1/t)^\gamma$ as before). As already mentioned, it suffices to construct a suitable family of coverings of $\mathcal{J}(E_\lambda)$ explicitly. The idea of the proof that $\mathcal{J}(E_\lambda)$ has zero Hausdorff measure with respect to h_λ^γ with properly chosen γ is based on the concept of porosity, which will be explained in the first part of this chapter. The second part is concerned with a theorem of Urbański and Zdunik which - together with another lemma - gives us that zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure of a suitable subset of the escaping set already implies zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure of the whole Julia set. In the third part, we will prove a sufficient condition for a gauge function h such that $\mathcal{J}(E_\lambda)$ has zero \mathcal{H}^h -measure.

7.1 Porosity

We start with the definition of porosity and then mention two theorems which have led us to the idea that this concept might be useful.

Definition 7.1.1 (Porosity) Let $A \subset \mathbb{C}$. Then A is **porous** if there exists $c \in (0, 1)$ and $\delta > 0$ such that for all $z \in A$ and all $r \in (0, \delta]$, we can find $z' \in \mathbb{C}$ with

$$D(z', cr) \subset D(z, r) \setminus A. \quad (7.1.1)$$

◇

The following result was the starting point for us to consider the concept of porosity.

Theorem 7.1.2 *If f is a rational map which is hyperbolic (i.e. $P(f) \cap \mathcal{J}(f) = \emptyset$), then $\mathcal{J}(f)$ is porous.*

This was shown by Sullivan in [29]. The proof strongly uses the fact that $\mathcal{J}(f)$ is compact, which of course is no longer true in our situation. In fact, Julia sets of exponential maps are not porous, as shown by

Theorem 7.1.3 *If $A \subset \mathbb{C}$ is porous, then $\text{HD}(A) < 2$.*

So it seems that this concept is not good enough for our purposes, but a closer look at the proof of this theorem gave us an idea how to construct a suitable family of coverings of $\mathcal{J}(E_\lambda)$ and estimate its Hausdorff measure. So we want to present the proof here.

Proof. Let c, δ be as in Definition 7.1.1. Then we can find $N \in \mathbb{N}$ (which only depends on c) with the following property: If Q is a square of side length $r \leq \delta$ and if we divide Q into N^2 squares Q_j of side length r/N , $j = 1, \dots, N^2$, then there exists j_0 such

that $Q_{j_0} \cap A = \emptyset$. Let

$$s_0 := \frac{\log N^2 - 1}{\log N}.$$

Then clearly $s_0 < 2$. We show that $\text{HD}(A \cap Q) \leq s_0$ for every square Q of side length δ . Let Q be such a square and divide it into N^2 squares of side length δ/N . By the above, $N^2 - 1$ of these squares suffice to cover $A \cap Q$. Then we divide each of the $N^2 - 1$ 'subsquares' into N^2 squares of side length δ/N^2 . The above argument applies to each of the subsquares, and inductively, for every $k \in \mathbb{N}$, we can find $q_k := (N^2 - 1)^k$ squares $Q_j^k, j = 1, \dots, q_k$, of side length $r_k := \delta/N^k$ such that

$$A \cap Q \subset \bigcup_{j=1}^{q_k} Q_j^k.$$

Recall that $h^s(t) = t^s$. Hence it follows for $s > s_0$ that

$$\begin{aligned} \mathcal{H}^{h^s}(A \cap Q) &\leq \lim_{k \rightarrow \infty} \mathcal{H}^{h^s} \left(\bigcup_{j=1}^{q_k} Q_j^k \right) \\ &\leq \lim_{k \rightarrow \infty} \sum_{j=1}^{q_k} \mathcal{H}^{h^s}(Q_j^k) \\ &\leq \lim_{k \rightarrow \infty} q_k (\sqrt{2} r_k)^s \\ &= \sqrt{2}^s \lim_{k \rightarrow \infty} (N^2 - 1)^k \left(\frac{\delta}{N^k} \right)^s \\ &= \sqrt{2}^s \delta^s \lim_{k \rightarrow \infty} \left(\frac{N^2 - 1}{N^s} \right)^k \\ &= 0 \end{aligned}$$

by the way that s and s_0 were defined. It follows that $A \cap Q$ has Hausdorff dimension at most s_0 . Because the Hausdorff dimension of the union of a countable number of sets equals the supremum of the Hausdorff dimensions of these sets (see for example [11]) and we can clearly cover A with a countably many squares of side length δ , the claim follows. \square

7.2 Measure of the set of non-escaping points

In this subsection, we state a theorem of Urbański and Zdunik about the Hausdorff dimension of the set of non-escaping points in the hyperbolic case and use this theorem to prove that zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure of a suitable subset of $I(E_\lambda)$ implies that in fact $\mathcal{J}(E_\lambda)$ has zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure.

The theorem by Urbański and Zdunik is the following [30]:

Theorem 7.2.1 *Let $\lambda \in \mathbb{C} \setminus \{0\}$ such that E_λ is hyperbolic. Then*

$$\text{HD}(\mathcal{J}(E_\lambda) \setminus I(E_\lambda)) < 2.$$

In particular, this theorem implies that $\mathcal{J}(E_\lambda) \setminus I(E_\lambda)$ has zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure for every $\gamma > 0$, which means that we can restrict ourselves to $I(E_\lambda)$ if we want to prove that $\mathcal{J}(E_\lambda)$ has zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure.

We now show that it is in fact sufficient to consider a suitable subset of $I(E_\lambda)$. For this, we need the following definition:

Definition 7.2.2 For $\lambda \in \mathbb{C} \setminus \{0\}$, we define

$$I_R(E_\lambda) := \{z \in I(E_\lambda) \mid \Re E_\lambda^n(z) \geq R \text{ for all } n \in \mathbb{N}_0\}.$$

◇

Remark: It is clear (because $I(E_\lambda)$ is completely invariant and $z \in I(E_\lambda)$ if and only if $\Re E_\lambda^n(z) \rightarrow \infty$ by Lemma 5.1.3) that

$$I(E_\lambda) = \bigcup_{n=0}^{\infty} E_\lambda^{-n}(I_R(E_\lambda)).$$

♠

The following two simple observations will be useful in the proof of the final lemma of this subsection.

Lemma 7.2.3 For every $\lambda_0 \in (0, 1/e)$, $K > 0$ and $\gamma > 0$, there exists $K' > 0$ such that

$$h_{\lambda_0}^\gamma(Kt) \leq K' h_{\lambda_0}^\gamma(t).$$

Proof. If $K \leq 1$, we can clearly choose $K' = 1$. If $K > 1$, we define $K' := K^2$ and obtain

$$h_{\lambda_0}^\gamma(Kt) = K^2 t^2 \Phi_{\lambda_0} \left(\frac{1}{Kt} \right)^\gamma \leq K^2 t^2 \Phi_{\lambda_0} \left(\frac{1}{t} \right)^\gamma = K' h_{\lambda_0}^\gamma(t).$$

□

Corollary 7.2.4 Let $\lambda_0 \in (0, 1/e)$, $A \subset \mathbb{C}$, $\gamma > 0$ and $f : A \rightarrow f(A)$ be a bilipschitz mapping, i.e. there exist constants $c, C > 0$ such that

$$c \leq \frac{|f(z) - f(w)|}{|z - w|} \leq C.$$

Then

$$\mathcal{H}^{h_{\lambda_0}^\gamma}(A) = 0 \Leftrightarrow \mathcal{H}^{h_{\lambda_0}^\gamma}(f(A)) = 0.$$

Proof. Clearly, it suffices to prove one direction. So suppose that $\mathcal{H}^{h_{\lambda_0}^\gamma}(A) = 0$. Let $\varepsilon > 0$ and choose $\delta > 0$ as well as a δ -covering $\{A_i\}$ of A with

$$\sum h_{\lambda_0}^\gamma(\text{diam } A_i) < \varepsilon.$$

Then $\{f(A_i)\}$ is a $C\delta$ -covering of $f(A)$ and we obtain from Lemma 7.2.3 that

$$\mathcal{H}^{h_{\lambda_0}^\gamma}(f(A)) \leq \sum h_{\lambda_0}^\gamma(\text{diam } f(A_i)) \leq \sum h_{\lambda_0}^\gamma(C \text{diam } A_i) \leq C' \sum h_{\lambda_0}^\gamma(\text{diam } A_i) < C'\varepsilon,$$

which proves the statement of the corollary. \square

Now we can prove that for $\lambda_0 \in (0, 1/e)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, $\mathcal{H}^{h_{\lambda_0}^\gamma}(I_R(E_\lambda)) = 0$ already implies that the whole escaping set $I(E_\lambda)$ has zero $\mathcal{H}^{h_{\lambda_0}^\gamma}$ -measure.

Lemma 7.2.5 *Let $\lambda_0 \in (0, 1/e)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $\gamma, R > 0$. Then $\mathcal{H}^{h_{\lambda_0}^\gamma}(I_R(E_\lambda)) = 0$ implies that $\mathcal{H}^{h_{\lambda_0}^\gamma}(I(E_\lambda)) = 0$.*

Proof. We show that for any set $A \subset \mathbb{C}$, $\mathcal{H}^{h_{\lambda_0}^\gamma}(E_\lambda^{-1}(A)) = 0$ implies that $\mathcal{H}^{h_{\lambda_0}^\gamma}(A) = 0$. The claim of the lemma then follows easily from the remark after Definition 7.2.2. So let $A \subset \mathbb{C}$ and suppose that $\mathcal{H}^{h_{\lambda_0}^\gamma}(E_\lambda^{-1}(A)) = 0$. We cover $E_\lambda^{-1}(A)$ by countably many small discs $U_n = D(z_n, \delta)$ with $\delta \ll \pi$. Then $\mathcal{H}^{h_{\lambda_0}^\gamma}(U_n \cap E_\lambda^{-1}(A)) = 0$ for all $n \in \mathbb{N}$. Clearly, E_λ is bilipschitz on $U_n \cap E_\lambda^{-1}(A)$. It follows from Corollary 7.2.4 that

$$\mathcal{H}^{h_{\lambda_0}^\gamma}(f(U_n) \cap A) = 0 \text{ for all } n \in \mathbb{N}.$$

Thus $\mathcal{H}^{h_{\lambda_0}^\gamma}(A) = 0$, which is what we wanted to show. \square

7.3 A gauge function that leads to zero measure

Although Julia sets of exponential maps cannot be porous (as explained in the first subsection), we use a similar strategy to prove that if $0 < \lambda < 1/e$ and $\gamma > 0$ such that $\beta_\lambda^\gamma < 2$, then $\mathcal{J}(E_\lambda)$ has zero Hausdorff measure with respect to suitable functions h_λ^γ : In our application, a disc satisfying (7.1.1) (with $A = \mathcal{J}(E_\lambda)$ and some c) will not exist for all small values of r , but only for certain sequences of r -values tending to 0 (Koskela and Rohde developed a similar condition in [19], but their condition still leads to Hausdorff dimension less than two). In other words, $\mathcal{J}(E_\lambda)$ satisfies a condition like porosity only on certain scales. But this is sufficient for our purposes, and we obtain

Theorem 7.3.1 *Let $\lambda \in (0, 1/e)$, β_λ be the unique real repelling fixed point of E_λ and let S_λ be the corresponding linearizing map. Let $\Phi_\lambda := (S_\lambda|_{\mathbb{R}})^{-1}$ and $\gamma > 0$ such that $\beta_\lambda^\gamma < 2$. Then we have*

$$\mathcal{H}^{h_\lambda^\gamma}(\mathcal{J}(E_\lambda)) = 0,$$

where $h_\lambda^\gamma(t) := t^2 \Phi_\lambda(1/t)$.

Before we start proving this theorem, we have to introduce a notation. In the last section, we were mostly interested in strips that may contain points in the Julia set (the strips that form J_δ). In this section, we are interested in strips which are

completely contained in the Fatou set: For $\delta > 0$, we define

$$F_\delta := \left\{ z \in \mathbb{C} \mid \Im z \in \left[\frac{(4k+1)\pi}{2} + \delta, \frac{(4k+3)\pi}{2} - \delta \right] \pmod{2\pi} \right\}.$$

It is easy to see that for $0 < \lambda < 1/e$, we have $F_\delta \subset \mathcal{F}(E_\lambda)$. Further it is immediate that Lemma 6.1.3 remains valid if we replace J_δ by F_δ .

Now we are ready to prove Theorem 7.3.1:

Proof of Theorem 7.3.1. By Theorem 7.2.1 and Lemma 7.2.5, it suffices to show that

$$\mathcal{H}^{h_\lambda}(I_R(E_\lambda)) = 0$$

for some $R > 0$. Let $\varepsilon' > 0$ such that $(1/2 + \varepsilon')\beta_\lambda^\gamma < 1$. Let $\varepsilon > 0$ and $M \in \mathbb{N}$ such that

$$(1 - \varepsilon) \frac{(1 - 2\varepsilon)^4}{(1 + 2\varepsilon)^{18}} \left(\frac{1 - \sqrt{2\varepsilon}}{1 + \sqrt{2\varepsilon}} \right)^4 \left(\frac{M}{M+1} \right)^2 \left(\frac{1}{2} - \varepsilon \right) \geq \frac{1}{2} - \varepsilon'$$

and

$$(1 - \varepsilon) \left(\frac{1 - \sqrt{2\varepsilon}}{1 + \sqrt{2\varepsilon}} \right)^4 \frac{(1 - 2\varepsilon)^4}{(1 + 2\varepsilon)^{16}} \left(\frac{1}{2} - 2\varepsilon \right) \geq \frac{1}{2} - \varepsilon'.$$

Now let $K > 3$ with

$$\left(\frac{K+1}{K-3} \right)^6 < 1 + 2\varepsilon \tag{7.3.1}$$

and such that K satisfies the property of Lemma 3.2.2: If $r' > 0, z \in \mathbb{C}, \theta \in \mathbb{R}$ and $f : D(z, Kr'/\sqrt{2}) \rightarrow \mathbb{C}$ is univalent, then $f(Q(z, r', \theta))$ is contained in some $Q(f(z), |f'(z)|r'(1 + \sqrt{2\varepsilon})d)$ and contains some $Q(f(z), |f'(z)|r'(1 - \sqrt{2\varepsilon})/d)$, where $d = D(f|_{Q(z, r', \theta)})$.

Let $r_0 > 0$ with

$$KM \left(\frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^2 \frac{r_0}{\sqrt{2}} < \pi. \tag{7.3.2}$$

Further, we can assume that r_0 is so small that (by Lemma 6.1.2) we can find a constant $A > \frac{(\pi+1)\sqrt{\pi}}{r_0\sqrt{\varepsilon}}$ such that if $Q = Q(z, Ar_0, \theta)$, then there exists an r_0 -packing pack_Q of $Q \cap F_\delta$ such that

$$\text{dens}(\text{pack}_Q, Q) > \frac{1}{2} - \varepsilon. \tag{7.3.3}$$

Because of Lemma 6.1.3, we can also assume that r_0 is so small that we can find $R_1 > 0$ and $\delta > 0$ with the following property: If

$$Q := Q(z_0, r_0, \theta) \subset \{\Re z > R_1\},$$

then $E_\lambda(Q) \cap F_\delta$ has an r_0 -packing $\text{pack}_{E_\lambda(Q)}$ which satisfies

$$\text{dens}(\text{pack}_{E_\lambda(Q)}, E_\lambda(Q)) > \frac{1}{2} - \varepsilon. \tag{7.3.4}$$

Further it is clear that there exists a universal constant C' with the following property: If we partition \mathbb{C} into a grid of squares with side lengths r_- and take any $Q(z, r_+, \theta)$

with $r_+ > r_-$, then at most $C'r_+/r_-$ squares of the r_- -grid intersect the boundary of $Q(z, r_+, \theta)$. Now let

$$B := 90(A + 1)r_0$$

and choose $C > 0$ such that

$$C > Mr_0\sqrt{2} + B$$

and $N := (C - Mr_0\sqrt{2})r_0 \frac{1-\sqrt{2}\varepsilon}{1+2\varepsilon}$ satisfies

$$1 - \frac{C'}{N} > 1 - \varepsilon.$$

In particular, we have $C > 4KMr_0$. Let

$$L := C - Mr_0\sqrt{2} > B$$

and

$$\kappa := \frac{1}{LB}.$$

Finally, let $R \geq R_1 + (A + 1)r_0\sqrt{2}$ such that

$$\lambda e^{R-(A+1)r_0\sqrt{2}} - \alpha_\lambda \geq \text{dist}(\{\Re z \geq R\}, P(E_\lambda)) \geq C + \sqrt{2}Mr_0. \quad (7.3.5)$$

After these preparations, choose some $z'_0 \in \mathbb{C}$ such that

$$Q_0 := Q(z'_0, r_0, 0) \cap I_R(E_\lambda) \neq \emptyset,$$

and some $x = x(Q_0) \in \mathbb{R}$ with the following five properties:

- $x \geq L(E_\lambda^2)'(\max\{\Re z \mid z \in Q_0\})$
- $-\frac{\log \kappa}{\log x} \leq \sqrt{\kappa} - \kappa$
- $E_\lambda^n(2x) \geq (E_\lambda^n)'(x)$ for all $n \in \mathbb{N}$
- $1/E_\lambda(x) < r_0$
- $(E_\lambda^{n+1})'(x) + \sqrt{2}(E_\lambda^n)'(x) < \sqrt{2}(E_\lambda^{n+1})'(x)$ for all $n \in \mathbb{N}$

(of course, x depends on Q_0 only by the first property). From the first condition, it follows immediately that

$$(E_\lambda^m)'(x) \geq L |(E_\lambda^{m+2})'(z)| \text{ for all } m \in \mathbb{N}_0 \text{ and } z \in Q_0. \quad (7.3.6)$$

For $n \in \mathbb{N}$, we define inductively a sequence of side lengths r_n : Let

$$r_1 := \frac{1}{E_\lambda(x)}$$

and for $n \in \mathbb{N}$, let r_{n+1} be such that

$$\frac{r_n}{r_{n+1}} \in \mathbb{N} \text{ and } r_n(E_\lambda^{n+1})'(x) \in \left(\frac{r_n}{r_{n+1}} - 1, \frac{r_n}{r_{n+1}} \right].$$

We prove inductively that

$$\frac{1}{\sqrt{2}(E_\lambda^n)'(x)} < r_n \text{ for all } n \in \mathbb{N}.$$

The case $n = 1$ is clear by the definition of r_1 . The step $n \rightarrow n + 1$ is also easy: By the fifth property of x , we have

$$\begin{aligned} r_{n+1} &> \frac{r_n}{r_n(E_\lambda^{n+1})'(x) + 1} \\ &= \frac{1}{(E_\lambda^{n+1})'(x) + 1/r_n} \\ &\geq \frac{1}{(E_\lambda^{n+1})'(x) + \sqrt{2}(E_\lambda^n)'(x)} \\ &> \frac{1}{\sqrt{2}(E_\lambda^{n+1})'(x)} \end{aligned}$$

Thus we obtain

$$\frac{1}{\sqrt{2}(E_\lambda^n)'(x)} < r_n \leq \frac{1}{(E_\lambda^n)'(x)} \text{ for all } n \in \mathbb{N}. \quad (7.3.7)$$

Now choose some $n \in \mathbb{N}$ and any $Q = Q(z_0, r_n, 0) \subset Q_0$ such that $Q \cap I_R(E_\lambda) \neq \emptyset$. The idea of the proof is as follows: Because $r_n/r_{n+1} \in \mathbb{N}$, we can partition Q into exactly r_n^2/r_{n+1}^2 squares of side length r_{n+1} , and we show that at most $(1/2 + \varepsilon)r_n^2/r_{n+1}^2$ of these squares are sufficient to cover $I_R(E_\lambda) \cap Q$.

Because of (7.3.5), we have $\text{dist}(Q, P(E_\lambda)) \geq C$. By Theorem 2.2.7, there exists a minimal $k \in \mathbb{N}$ such that $E_\lambda^{k-1}(Q)$ is contained in some $Q(E_\lambda^{k-1}(z_0), Mr_0)$, but $E_\lambda^k(Q)$ is not. We now show that $E_\lambda^{k-1} : Q \rightarrow \mathbb{C}$ has a univalent continuation to $D(z_0, Kr_n/\sqrt{2})$. Let φ be the branch of $(E_\lambda^{k-1})^{-1}$ which is defined on a neighborhood of $E_\lambda^{k-1}(z_0)$ and satisfies $\varphi(E_\lambda^{k-1}(z_0)) = z_0$. By the monodromy theorem and since

$$\text{dist}(E_\lambda^{k-1}(z_0), P(E_\lambda)) \geq \text{dist}(\{\Re z \geq R\}, P(E_\lambda)) - \sqrt{2}Mr_0 \geq C > 4KMr_0,$$

we conclude by (7.3.5) that φ has an analytic continuation to $D(E_\lambda^{k-1}(z_0), 4KMr_0)$. Because $E_\lambda^{k-1}(Q)$ is contained in a disc of radius $Mr_0/\sqrt{2}$, it follows by Lemma 2.3.1 that

$$|(E_\lambda^{k-1})'(z_0)| \leq \frac{Mr_0/\sqrt{2}}{r_n/2} = M\sqrt{2}\frac{r_0}{r_n},$$

hence

$$|\varphi'(E_\lambda^{k-1}(z_0))| \geq \frac{1}{Mr_0} \frac{r_n}{\sqrt{2}}.$$

By the Koebe 1/4-Theorem, we obtain

$$\varphi(D(E_\lambda^{k-1}(z_0), 4KMr_0)) \supset D(z_0, |\varphi'(E_\lambda^{k-1}(z_0))| KMr_0) \supset D(z_0, Kr_n/\sqrt{2}),$$

and therefore E_λ^{k-1} can be continued univalently to $D(z_0, Kr_n/\sqrt{2})$. By Lemma 3.2.1 and (7.3.1), the distortion of E_λ^{k-1} satisfies

$$D(E_\lambda^{k-1}|_Q) \leq \left(\frac{K+1}{K-3}\right)^6 < 1 + 2\varepsilon. \quad (7.3.8)$$

It follows now by Lemma 3.2.2 that $E_\lambda^{k-1}(Q)$ contains some

$$\begin{aligned} Q_1 &:= Q(E_\lambda^{k-1}(z_0), |(E_\lambda^{k-1})'(z_0)| r_n(1 - \sqrt{2}\varepsilon)/(1 + 2\varepsilon)) \\ &\supset Q(E_\lambda^{k-1}(z_0), |(E_\lambda^{k-1})'(z_0)| r_n(1 - 2\varepsilon)^2) \end{aligned}$$

and is contained in some

$$\begin{aligned} Q_2 &:= Q(E_\lambda^{k-1}(z_0), |(E_\lambda^{k-1})'(z_0)| r_n(1 + \sqrt{2}\varepsilon)(1 + 2\varepsilon)) \\ &\subset Q(E_\lambda^{k-1}(z_0), |(E_\lambda^{k-1})'(z_0)| r_n(1 + 2\varepsilon)^2). \end{aligned}$$

Because the side length l_1 of Q_1 is at most Mr_0 , the side length l_2 of Q_2 satisfies

$$l_2 < \left(\frac{1+2\varepsilon}{1-2\varepsilon}\right)^2 Mr_0.$$

Since

$$\left(\frac{1+2\varepsilon}{1-2\varepsilon}\right)^2 KMr_0 \frac{1}{\sqrt{2}} < \pi$$

by (7.3.2), E_λ is univalent on

$$D\left(E_\lambda^{k-1}(z_0), \left(\frac{1+2\varepsilon}{1-2\varepsilon}\right)^2 KMr_0/\sqrt{2}\right) \supset D\left(E_\lambda^{k-1}(z_0), \frac{l_2 K}{\sqrt{2}}\right).$$

It follows by the same arguments as before that

$$D(E_\lambda|_{Q_2}) < 1 + 2\varepsilon, \quad (7.3.9)$$

and hence $E_\lambda(Q_1)$ contains a square of side length $|E'_\lambda(E_\lambda^{k-1}(z_0))| l_1(1 - \sqrt{2}\varepsilon)/(1 + 2\varepsilon)$ and $E_\lambda(Q_2)$ is contained in a square of side length $|E'_\lambda(E_\lambda^{k-1}(z_0))| l_2(1 - \sqrt{2}\varepsilon)(1 + 2\varepsilon)$. Thus, summarizing the previous estimates, $E_\lambda(Q_1)$ contains some

$$Q'_1 := Q\left(E_\lambda^k(z_0), |(E_\lambda^k)'(z_0)| r_n \frac{(1 - \sqrt{2}\varepsilon)(1 - 2\varepsilon)^2}{1 + 2\varepsilon}\right)$$

and $E_\lambda(Q_2)$ is contained in some

$$Q'_2 := Q\left(E_\lambda^k(z_0), |(E_\lambda^k)'(z_0)| r_n(1 + \sqrt{2}\varepsilon)(1 + 2\varepsilon)^3\right).$$

Since $E_\lambda^k(Q)$ is not contained in a square of side length Mr_0 , it follows that the side length l'_2 of Q'_2 satisfies

$$l'_2 > Mr_0.$$

If ε is chosen small enough, we can assume that the side length l'_1 of Q'_1 is also at least Mr_0 . Now there are two cases:

Case 1: $l'_1 \leq Ar_0$.

Then (if A is chosen large enough) it follows that $l'_2 \leq (A+1)r_0$.

We partition Q'_1 into squares of side length r_0 . Choose $m \in \mathbb{N}$ such that $mr_0 \leq r'_1$, but $(m+1)r_0 > r'_1$. Then $m \geq M$ and at least $m^2r_0^2$ of these squares are completely contained in Q'_1 . Let $P = Q(p, r_0)$ be one of these squares. Then, because $\Re z \geq R - (A+1)r_0\sqrt{2} \geq R_1$ for all $z \in P$ and by the way R_1 and r_0 are defined, there exists an r_0 -packing $\text{pack}_{E_\lambda(P)}$ of $E_\lambda(P) \cap F_\delta$ such that

$$\text{dens}(\text{pack}_{E_\lambda(P)}, E_\lambda(P)) > \frac{1}{2} - \varepsilon.$$

Since E_λ is clearly injective on $D(p, KMr_0/\sqrt{2})$ (because $KMr_0/\sqrt{2} < \pi$), the distortion satisfies

$$D(E_\lambda|_P) < 1 + 2\varepsilon.$$

If φ_P denotes the branch of E_λ^{-1} such that $\varphi_P(\text{pack}_{E_\lambda(P)}) \subset P$, we thus obtain by Lemma 3.1.4 (d) that

$$\text{dens}(\varphi_P(\text{pack}_{E_\lambda(P)}), P) \geq \frac{1}{(1+2\varepsilon)^2} \left(\frac{1}{2} - \varepsilon \right).$$

If we take the union over all P , it follows that

$$\text{dens}\left(\bigcup_P E_\lambda^{-1}(\text{pack}_{E_\lambda(P)}), Q'_1\right) \geq \frac{1}{(1+2\varepsilon)^2} \left(\frac{1}{2} - \varepsilon \right) \left(\frac{m}{m+1} \right)^2$$

and therefore

$$\begin{aligned} \text{dens}\left(\bigcup_P \varphi_P(\text{pack}_{E_\lambda(P)}), E_\lambda^k(Q)\right) &\geq \frac{1}{(1+2\varepsilon)^2} \left(\frac{1}{2} - \varepsilon \right) \left(\frac{m}{m+1} \right)^2 \frac{|Q'_1|}{|Q'_2|} \\ &= \left(\frac{m}{m+1} \right)^2 \left(\frac{1}{2} - \varepsilon \right) \frac{(1-2\varepsilon)^4(1-\sqrt{2\varepsilon})^2}{(1+2\varepsilon)^{10}(1+\sqrt{2\varepsilon})^2}. \end{aligned}$$

Taking preimages, it follows from (7.3.8), (7.3.9) and $m \geq M$ that

$$\begin{aligned} &\text{dens}\left((E_\lambda^k)^{-1}\left(\bigcup_P \varphi_P(\text{pack}_{E_\lambda(P)})\right) \cap Q, Q\right) \\ &\geq \frac{1}{D(E_\lambda^k|_Q)^2} \left(\frac{m}{m+1} \right)^2 \left(\frac{1}{2} - \varepsilon \right) \frac{(1-2\varepsilon)^4(1-\sqrt{2\varepsilon})^2}{(1+2\varepsilon)^{10}(1+\sqrt{2\varepsilon})^2} \\ &\geq \frac{1}{D(E_\lambda^{k-1}|_Q)^2} \frac{1}{D(E_\lambda|_{Q_2})^2} \left(\frac{m}{m+1} \right)^2 \left(\frac{1}{2} - \varepsilon \right) \frac{(1-2\varepsilon)^4(1-\sqrt{2\varepsilon})^2}{(1+2\varepsilon)^{10}(1+\sqrt{2\varepsilon})^2} \\ &\geq \left(\frac{m}{m+1} \right)^2 \left(\frac{1}{2} - \varepsilon \right) \underbrace{\frac{(1-2\varepsilon)^4(1-\sqrt{2\varepsilon})^2}{(1+2\varepsilon)^{14}(1+\sqrt{2\varepsilon})^2}}_{=:h(\varepsilon)} \\ &\geq \left(\frac{1}{2} - \varepsilon \right) \left(\frac{M}{M+1} \right)^2 h(\varepsilon) \end{aligned} \tag{7.3.10}$$

Now we show that these preimages (which all belong to the Fatou set of E_λ) are 'large' compared to a square of side length r_{n+1} . Let P be a square in Q'_1 , and let $z \in P$ be arbitrary. Then we have $\Re z \geq R - (A+1)\sqrt{2}r_0$, and it follows that

$$|E_\lambda(z)| \geq \lambda e^{R-(A+1)\sqrt{2}r_0},$$

which implies that

$$\text{dist}(E_\lambda(z), P(E_\lambda)) \geq |E_\lambda(z)| - \alpha_\lambda > C$$

by the definition of R . Hence the branch ψ of $(E_\lambda^{k+1})^{-1}$ with $\psi(E_\lambda(z)) \in Q$ can be continued univalently to $D(E_\lambda(z), Kr_0/\sqrt{2})$. We denote (as in section 6) the collection of boxes that form $\text{pack}_{E_\lambda(P)}$ by $\mathcal{B}(E_\lambda(P))$. So if $P' = Q(z_{P'}, r_0, 0) \in \mathcal{B}(E_\lambda(P))$ and $\psi_{P'}$ denotes the branch of $(E_\lambda^{k+1})^{-1}$ with $\psi_{P'}(P') \subset Q$, then $D(\psi_{P'}|_{P'}) < 1 + 2\varepsilon$ and $\psi(P')$ is contained in a square $Q'_{P'}$ of side length

$$\rho'_{P'} := |\psi'_{P'}(z_{P'})| r_0(1 + 2\varepsilon)(1 + \sqrt{2}\varepsilon)$$

and contains a square $Q_{P'}$ of side length

$$\rho_{P'} := |\psi'_{P'}(z_{P'})| r_0 \frac{1}{1 + 2\varepsilon} (1 - \sqrt{2}\varepsilon).$$

Let

$$w_{P'} := \psi_{P'}(z_{P'}) \in Q.$$

We now show that

$$(E_\lambda^{n+1})'(x) \geq L |(E_\lambda^{k+1})'(w_{P'})|, \quad (7.3.11)$$

which means that r_{n+1} is much smaller than the size of the preimages of $\text{pack}_{E_\lambda(P)}$ in Q . If $k \leq n + 2$, then (7.3.11) is clear by (7.3.6), so we may assume that $k - n \geq 3$. Since $C > 90(A+1)r_0$, we trivially obtain

$$\text{dist}(E_\lambda^k(Q), P(E_\lambda)) \geq \text{dist}(Q'_2, P(E_\lambda)) > 6(A+1)r_0.$$

Hence the branch φ_0 of $(E_\lambda^k)^{-1}$ with $\varphi_0(E_\lambda^k(z_0)) = z_0$ can be continued analytically to $D(E_\lambda^k(z_0), 6(A+1)r_0)$. Because

$$|(E_\lambda^k)'(z_0)| \leq \sqrt{2}(A+1)\frac{r_0}{r_n} \quad (7.3.12)$$

by the same arguments as above, it follows again by the Koebe 1/4-theorem that

$$\varphi_0(D(E_\lambda^k(z_0), 6(A+1)r_0)) \supset D\left(z_0, \frac{3}{2}\frac{r_n}{\sqrt{2}}\right).$$

Thus E_λ^k is univalent on $D(z_0, \frac{3r_n}{2\sqrt{2}})$. Corollary 2.3.5 implies that

$$|(E_\lambda^k)'(w_{P'})| \leq \frac{3r_n/2\sqrt{2} + r_n/\sqrt{2}}{(3r_n/2\sqrt{2} - r_n/\sqrt{2})^3} (3r_n/2\sqrt{2})^2 |(E_\lambda^k)'(z_0)|. \quad (7.3.13)$$

Combining (7.3.12) and (7.3.13), we obtain together with (7.3.7) that

$$|E_\lambda^k(w_{P'})| \leq \frac{5/2(3/2)^2 \sqrt{2}(A+1)r_0}{(1/2)^3 r_n} = \frac{B}{\sqrt{2}r_n} \leq B(E_\lambda^n)'(x). \quad (7.3.14)$$

For the proof of (7.3.11), it remains to show that

$$\kappa E_\lambda^{n+1}(x) \geq |E_\lambda^{k+1}(w_{P'})|. \quad (7.3.15)$$

In order to prove (7.3.15), first note that

$$2^j(\kappa^{1/2^{j+1}} - \kappa^{1/2^j}) \geq \sqrt{\kappa} - \kappa \text{ for all } j \in \mathbb{N}_0, \quad (7.3.16)$$

which we prove by induction on j .

The case $j = 0$ is clear. The step $j \rightarrow j + 1$ follows easily by

$$\begin{aligned} 2^j(\kappa^{1/2^{j+1}} - \kappa^{1/2^j}) &= 2^j \underbrace{(\kappa^{1/2^{j+2}} + \kappa^{1/2^{j+1}})}_{\leq 2} (\kappa^{1/2^{j+2}} - \kappa^{1/2^{j+1}}) \\ &\leq 2^{j+1}(\kappa^{1/2^{j+2}} - \kappa^{1/2^{j+1}}). \end{aligned}$$

We now use (7.3.16) to prove that

$$x^{\kappa^{1/2^j} - 1} \leq \kappa^{1/2^{j-1}} \text{ for all } j \in \mathbb{N}_0, \quad (7.3.17)$$

which is equivalent to

$$x \geq \frac{x^{(\kappa^{1/2^j})}}{\kappa^{1/2^{j-1}}} \text{ for all } j \in \mathbb{N}_0.$$

The right side tends to x as $j \rightarrow \infty$. In fact, it is an increasing sequence: We have that

$$\begin{aligned} \frac{x^{(\kappa^{1/2^{j+1}})}}{\kappa^{1/2^j}} &\geq \frac{x^{(\kappa^{1/2^j})}}{\kappa^{1/2^{j-1}}} \Leftrightarrow x^{\kappa^{1/2^{j+1}} - \kappa^{1/2^j}} \geq \kappa^{1/2^j - 1/2^{j-1}} \\ &\Leftrightarrow (\kappa^{1/2^{j+1}} - \kappa^{1/2^j}) \log x \geq -\frac{1}{2^j} \log \kappa \\ &\Leftrightarrow 2^j(\kappa^{1/2^{j+1}} - \kappa^{1/2^j}) \geq -\frac{\log \kappa}{\log x}. \end{aligned}$$

Because

$$-\frac{\log \kappa}{\log x} \leq \sqrt{\kappa} - \kappa,$$

(7.3.17) follows from (7.3.16). From (7.3.17), it is clear that

$$E_\lambda^i(x)^{\kappa^{1/2^j} - 1} \leq \kappa^{1/2^{j-1}} \text{ for all } i, j \in \mathbb{N}_0. \quad (7.3.18)$$

Now suppose that

$$\kappa E_\lambda^{n+1}(x) < |E_\lambda^{k+1}(w_{P'})|. \quad (7.3.19)$$

We show that this implies that

$$\kappa^{1/2^j} E_\lambda^{n+1-j}(x) < \left| E_\lambda^{k+1-j}(w_{P'}) \right| \text{ for all } 1 \leq j \leq n.$$

This can be done inductively as well: For $j = 1$, suppose that $|E_\lambda^k(w_{P'})| \leq \kappa^{1/2} E_\lambda^n(x)$. Then we have

$$\begin{aligned} |E_\lambda^{k+1}(w_{P'})| &= \lambda e^{\Re E_\lambda^k(w_{P'})} \\ &\leq \lambda e^{\kappa^{1/2} E_\lambda^n(x)} \\ &\leq (\lambda e^{E_\lambda^n(x)})^{\kappa^{1/2}} \\ &= E_\lambda^{n+1}(x)^{\kappa^{1/2}} \\ &\leq \kappa E_\lambda^{n+1}(x), \end{aligned}$$

where the second inequality is true because $\lambda < 1$, and the last inequality follows from (7.3.18). The step $j \rightarrow j + 1$ follows in exactly the same way. So it follows from (7.3.19) that

$$\kappa^{1/2+\dots+1/2^n} \prod_{j=1}^n E_\lambda^j(x) < \prod_{j=k-n+1}^k |E_\lambda^j(w_{P'})|,$$

which is equivalent to

$$\kappa^{1-1/2^n} (E_\lambda^n)'(x) < \prod_{j=k-n+1}^k |E_\lambda^j(w_{P'})|. \quad (7.3.20)$$

We have

$$\begin{aligned} \prod_{j=k-n+1}^k |E_\lambda^j(w_{P'})| &= \frac{1}{\prod_{j=1}^{k-n} |E_\lambda^j(w_{P'})|} |(E_\lambda^k)'(w_{P'})| \\ &\leq \frac{B}{\prod_{j=1}^{k-n} |E_\lambda^j(w_{P'})|} (E_\lambda^n)'(x) \end{aligned} \quad (7.3.21)$$

by (7.3.14). Further, because $E_\lambda^j(Q)$ is contained in a square of side length Mr_0 for $j \leq k - 1$ and $I_R(E_\lambda) \cap Q \neq \emptyset$, we have

$$|E_\lambda^j(w_{P'})| > R - Mr_0\sqrt{2} > C - Mr_0\sqrt{2} = L \text{ for } j \leq k - n. \quad (7.3.22)$$

Combining (7.3.20), (7.3.21) and (7.3.22), we obtain

$$\kappa^{1-1/2^n} (E_\lambda^n)'(x) < \frac{B}{L^{k-n}} (E_\lambda^n)'(x),$$

and hence

$$\frac{L^{k-n} \kappa^{1-1/2^n}}{B} < 1.$$

We now can show easily that this is impossible: It suffices to prove that

$$\frac{L^{k-n} \kappa}{B} = \frac{L^{k-n-1}}{B^2} \geq 1.$$

But this is clear since $k - n - 1 \geq 2$ and $L > B$ by our assumption, and (7.3.11) is proven. So it follows that

$$\rho_{P'} \geq L \frac{1}{(E_\lambda^{n+1})'(x)} r_0 \frac{1}{1 + 2\varepsilon} (1 - \sqrt{2\varepsilon}) = N \frac{1}{(E_\lambda^{n+1})'(x)} \geq N r_{n+1}.$$

Now we show that $Q_{P'}$ contains at least $(1 - \varepsilon) \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2$ squares of the r_{n+1} -grid: At least $\left(\frac{\rho_{P'}}{r_{n+1}}\right)^2$ squares in the r_{n+1} -grid intersect $Q_{P'}$, whereas only $C' \frac{\rho_{P'}}{r_{n+1}}$ squares of the grid intersect $\partial Q_{P'}$, which implies that

$$\left(\frac{\rho_{P'}}{r_{n+1}}\right)^2 - C' \frac{\rho_{P'}}{r_{n+1}} = \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2 \left(1 - C' \frac{r_{n+1}}{\rho_{P'}}\right) \geq \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2 \left(1 - C' \frac{1}{N}\right) > \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2 (1 - \varepsilon)$$

squares are contained in $Q_{P'}$. Because all the previous estimates were independent of P' and P , we can do the same for every P and every $P' \in \mathcal{B}(E_\lambda(P))$ and obtain that at least

$$(1 - \varepsilon) \sum_P \sum_{P' \in \mathcal{B}(E_\lambda(P))} \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2$$

squares of the r_{n+1} -grid are contained in the Fatou set of E_λ . It follows now from (7.3.10) that

$$\begin{aligned} & (1 - \varepsilon) \sum_P \sum_{P' \in \mathcal{B}(E_\lambda(P))} \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2 \\ &= \frac{1}{r_{n+1}^2} (1 - \varepsilon) \sum_P \sum_{P' \in \mathcal{B}(E_\lambda(P))} (\rho_{P'})^2 \\ &= \frac{1}{r_{n+1}^2} (1 - \varepsilon) \frac{1}{(1 + 2\varepsilon)^4} \left(\frac{1 - \sqrt{2\varepsilon}}{1 + \sqrt{2\varepsilon}}\right)^2 \sum_P \sum_{P' \in \mathcal{B}(E_\lambda(P))} (\rho'_{P'})^2 \\ &\geq \frac{1}{r_{n+1}^2} (1 - \varepsilon) \frac{1}{(1 + 2\varepsilon)^4} \left(\frac{1 - \sqrt{2\varepsilon}}{1 + \sqrt{2\varepsilon}}\right)^2 \sum_P \sum_{P' \in \mathcal{B}(E_\lambda(P))} |\psi_{P'}(P')|^2 \\ &\geq \frac{1}{r_{n+1}^2} (1 - \varepsilon) \frac{1}{(1 + 2\varepsilon)^4} \left(\frac{1 - \sqrt{2\varepsilon}}{1 + \sqrt{2\varepsilon}}\right)^2 \left(\frac{1}{2} - \varepsilon\right) \left(\frac{M}{M + 1}\right)^2 h(\varepsilon) r_n^2 \\ &= \left(\frac{r_n}{r_{n+1}}\right)^2 (1 - \varepsilon) \frac{(1 - 2\varepsilon)^4}{(1 + 2\varepsilon)^{18}} \left(\frac{1 - \sqrt{2\varepsilon}}{1 + \sqrt{2\varepsilon}}\right)^4 \left(\frac{M}{M + 1}\right)^2 \left(\frac{1}{2} - \varepsilon\right) \\ &\geq \left(\frac{r_n}{r_{n+1}}\right)^2 \left(\frac{1}{2} - \varepsilon'\right), \end{aligned}$$

so at least

$$\left(\frac{r_n}{r_{n+1}}\right)^2 \left(\frac{1}{2} - \varepsilon'\right)$$

squares from the r_{n+1} -grid which belong to $\mathcal{F}(E_\lambda)$ are contained in Q .

Case 2: $l'_1 > Ar_0$.

In this case, we can find an r_0 -packing $\text{pack}_{Q'_1}$ of $Q'_1 \cap F_\delta$ such that

$$\text{dens}(\text{pack}_{Q'_1}, Q'_1) > \frac{1}{2} - \varepsilon.$$

Because we have to be sure that $(E_\lambda^k)^{-1}$ is univalent on a domain which is large enough, we consider only the squares of $\text{pack}_{Q'_1}$ which are contained in $Q'_1 \setminus Y$, where $Y := \{z \in \mathbb{C} \mid \text{dist}(z, P(E_\lambda)) < \pi\} \subset D(0, \pi + 1)$. By the definition of A , we have

$$\text{dens}(Y, Q'_1) < \varepsilon,$$

hence if $\text{pack}_{Q'_1 \setminus Y}$ denotes only the elements of $\text{pack}_{Q'_1}$ which are contained in $Q'_1 \setminus Y$, we easily obtain

$$\text{dens}(\text{pack}_{Q'_1 \setminus Y}, Q'_1) \geq \text{dens}(\text{pack}_{Q'_1}, Q'_1) - \text{dens}(Y, Q'_1) > \frac{1}{2} - 2\varepsilon.$$

Because $D(E_\lambda^k|_Q) < (1 + 2\varepsilon)^2$, it follows that

$$\begin{aligned} \text{dens}((E_\lambda^k)^{-1}(\text{pack}_{Q'_1 \setminus Y}) \cap Q, Q) &\geq \frac{1}{(1 + 2\varepsilon)^4} \text{dens}(\text{pack}_{Q'_1 \setminus Y}, E_\lambda^k(Q)) \\ &= \frac{1}{(1 + 2\varepsilon)^4} \text{dens}(\text{pack}_{Q'_1 \setminus Y}, Q'_1) \text{dens}(Q'_1, E_\lambda^k(Q)) \\ &\geq \frac{1}{(1 + 2\varepsilon)^4} \left(\frac{1}{2} - 2\varepsilon\right) \text{dens}(Q'_1, Q'_2) \\ &= \frac{(1 - 2\varepsilon)^4 (1 - \sqrt{2}\varepsilon)^2}{(1 + 2\varepsilon)^{12} (1 + \sqrt{2}\varepsilon)^2} \left(\frac{1}{2} - 2\varepsilon\right). \end{aligned} \quad (7.3.23)$$

Again, denote by $\mathcal{B}(Q'_1 \setminus Y)$ the collection of r_0 -boxes that form $\text{pack}_{Q'_1 \setminus Y}$ and let $P' = Q(z_{P'}, r_0, 0) \in \mathcal{B}(Q'_1 \setminus Y)$. By the definition of Y , it follows that the branch $\psi_{P'}$ of $(E_\lambda^k)^{-1}$ that maps P' into Q can be continued univalently to $D(z_R, KM r_0 / \sqrt{2})$. Hence we have as before that $\psi(P')$ contains a square $Q_{P'}$ of side length

$$\rho_{P'} := |\psi'(z_{P'})| r_0 \frac{1 - \sqrt{2}\varepsilon}{1 + 2\varepsilon}$$

and is contained in a square of side length

$$\rho'_{P'} := |\psi'(z_{P'})| r_0 (1 + 2\varepsilon) (1 + \sqrt{2}\varepsilon).$$

Let

$$w_{P'} := \psi_{P'}(z_{P'}) \in Q.$$

We now show that

$$(E_\lambda^{n+1})'(x) \geq L |(E_\lambda^k)'(w_{P'})|. \quad (7.3.24)$$

First note that because E_λ^{k-1} is injective on $D(z_0, Kr_n/\sqrt{2})$, we conclude as in case 1 (using Corollary 2.3.5 again) that

$$|(E_\lambda^{k-1})'(w_{P'})| \leq B(E_\lambda^n)'(x),$$

and by the same arguments as before, (7.3.24) follows. Again we see (analogously to case 1) that $Q_{P'}$ contains at least $(1 - \varepsilon)\left(\frac{\rho_{P'}}{r_{n+1}}\right)^2$ squares in the r_{n+1} -grid. Again, because all of our estimates were independent of P' , we obtain that at least

$$(1 - \varepsilon) \sum_{P' \in \mathcal{B}(Q'_1 \setminus Y)} \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2$$

squares in the r_{n+1} -grid are contained in $\mathcal{F}(E_\lambda)$, and thus we have (using (7.3.23)) that

$$\begin{aligned} (1 - \varepsilon) \sum_{P' \in \mathcal{B}(Q'_1 \setminus Y)} \left(\frac{\rho_{P'}}{r_{n+1}}\right)^2 &= \frac{1}{r_{n+1}^2} (1 - \varepsilon) \sum_{P' \in \mathcal{B}(Q'_1 \setminus Y)} (\rho_{P'})^2 \\ &= \frac{1}{r_{n+1}^2} (1 - \varepsilon) \sum_{P' \in \mathcal{B}(Q'_1 \setminus Y)} (\rho'_{P'})^2 \left(\frac{1 - \sqrt{2}\varepsilon}{1 + \sqrt{2}\varepsilon}\right)^2 \frac{1}{(1 + 2\varepsilon)^4} \\ &\geq \frac{1}{r_{n+1}^2} (1 - \varepsilon) \left(\frac{1 - \sqrt{2}\varepsilon}{1 + \sqrt{2}\varepsilon}\right)^2 \frac{1}{(1 + 2\varepsilon)^4} \sum_{P' \in \mathcal{B}(Q'_1 \setminus Y)} |\psi_{P'}(P')| \\ &\geq \frac{1}{r_{n+1}^2} (1 - \varepsilon) \left(\frac{1 - \sqrt{2}\varepsilon}{1 + \sqrt{2}\varepsilon}\right)^4 \frac{(1 - 2\varepsilon)^4}{(1 + 2\varepsilon)^{16}} \left(\frac{1}{2} - 2\varepsilon\right) r_n^2 \\ &\geq \left(\frac{r_n}{r_{n+1}}\right)^2 \left(\frac{1}{2} - \varepsilon'\right). \end{aligned}$$

So in each of the two cases,

$$\left(\frac{1}{2} + \varepsilon'\right) \left(\frac{r_n}{r_{n+1}}\right)^2$$

squares from the r_{n+1} -grid suffice to cover $I_R(E_\lambda) \cap Q$. Because n was chosen to be an arbitrary natural number, this result does not depend on n . Hence we obtain the following: Starting with a square Q of side length r_1 (where r_1 depends on Q , of course), we can cover $I_R(E_\lambda) \cap Q$ by

$$\left(\frac{1}{2} + \varepsilon'\right)^n \left(\frac{r_1}{r_{n+1}}\right)^2$$

squares in the r_{n+1} -grid. It follows (with $h_\lambda^\gamma(t) = t^2\Phi_\lambda(1/t)^\gamma$) that

$$\begin{aligned}
\mathcal{H}^{h_\lambda^\gamma}(I_R(E_\lambda) \cap Q) &\leq \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \varepsilon'\right)^n \left(\frac{r_1}{r_{n+1}}\right)^2 h_\lambda^\gamma(\sqrt{2}r_{n+1}) \\
&= \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \varepsilon'\right)^n r_1^2 2\Phi_\lambda\left(\frac{1}{\sqrt{2}r_{n+1}}\right)^\gamma \\
&< 2r_1^2 \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \varepsilon'\right)^n \Phi_\lambda((E_\lambda^{n+1})'(x))^\gamma \\
&\leq 2r_1^2 \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \varepsilon'\right)^n \Phi_\lambda(E_\lambda^{n+1}(2x))^\gamma \\
&= 2r_1^2 \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \varepsilon'\right)^n (\beta_\lambda^{n+1})^\gamma \Phi_\lambda(2x)^\gamma \\
&= 2r_1^2 \beta_\lambda^\gamma \Phi_\lambda(2x)^\gamma \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \varepsilon'\right)^n (\beta_\lambda^\gamma)^n \\
&= 2r_1^2 \beta_\lambda^\gamma \Phi_\lambda(2x)^\gamma \lim_{n \rightarrow \infty} \left(\left(\frac{1}{2} + \varepsilon'\right)\beta_\lambda^\gamma\right)^n \\
&= 0,
\end{aligned}$$

because $\beta_\lambda^\gamma(1/2 + \varepsilon') < 1$. We now only need to show that $I_R(E_\lambda)$ can be covered by countably many of such squares Q (compare the proof of Theorem 7.1.3). But this is not difficult to see: For some $Q_0 = Q(z_0, r_0, 0)$, choose $x = x(Q_0)$ as before, which defines r_1 , and every square of side length r_1 that is completely contained in Q_0 has zero $\mathcal{H}^{h_\lambda^\gamma}$ -measure. Because Q_0 can be covered by finitely many squares of side length r_1 and $I_R(E_\lambda)$ can clearly be covered by countably many squares of side length r_0 , the theorem is proved. \square

Remark: Because $I(E_\lambda) \subset \mathcal{J}(E_\lambda)$, the previous theorem is clearly also true when we replace $\mathcal{J}(E_\lambda)$ by $I(E_\lambda)$. \spadesuit

8 The main theorem

8.1 Equivalence of the gauge functions

In the preceding two sections, we estimated $\mathcal{H}^{h_{\lambda_0}^\gamma}(\mathcal{J}(E_\lambda))$ for certain γ which depended on β_{λ_0} . One major difference between section 6 and section 7 is that λ and λ_0 , which were arbitrary elements of $\mathbb{C} \setminus \{0\}$ and $(0, 1/e)$ in section 6, had to be equal (and between 0 and $1/e$) in section 7. In this section, we will prove that if $\lambda \in (0, 1/e)$, then $\mathcal{J}(E_\lambda)$ still has zero measure with respect to the gauge function $h_{\lambda_0}^\gamma$, where $\lambda_0 \in (0, 1/e)$ is arbitrary and γ is chosen such that $\beta_{\lambda_0}^\gamma < 2$.

We prove this by showing that two gauge functions $h_{\lambda_1}^{\gamma_1}$ and $h_{\lambda_2}^{\gamma_2}$ are equivalent (in the sense that there exist constants $c, C > 0$ with $ch_{\lambda_1}^{\gamma_1}(t) \leq h_{\lambda_2}^{\gamma_2}(t) \leq Ch_{\lambda_1}^{\gamma_1}(t)$ if t is small) whenever $\beta_{\lambda_1}^{\gamma_1} = \beta_{\lambda_2}^{\gamma_2}$.

To prove this result, it is more convenient to consider a different parametrization of the exponential family. Let $\lambda \in (0, 1/e)$ be fixed. If $\zeta \in (1, \infty)$ is chosen such that $\zeta/e^\zeta = \lambda$ (in fact, we have $\zeta = \beta_\lambda$), we conjugate E_λ by $z \mapsto z + \zeta$ and get a corresponding function

$$\tilde{E}_\zeta(z) = \zeta(e^z - 1).$$

Then 0 is a repelling fixed point of \tilde{E}_ζ with multiplier ζ . Further, it is immediate that

$$\tilde{S}_\zeta(z) = S_\lambda(z) - \zeta,$$

where \tilde{S}_ζ is the linearizing map for \tilde{E}_ζ near the repelling fixed point 0. If $\tilde{\Phi}_\zeta$ denotes the inverse function of $\tilde{S}_\zeta|_{\mathbb{R}}$, it follows that $\tilde{\Phi}_\zeta(x) = \Phi_\lambda(x + \zeta)$, and therefore (using the fact that Φ_λ grows slower than any iterate of the logarithm), we have

$$\frac{\tilde{\Phi}_\zeta(x)}{\Phi_\lambda(x)} = \frac{\Phi_\lambda(x + \zeta)}{\Phi_\lambda(x)} \leq \frac{\zeta + \Phi_\lambda(x)}{\Phi_\lambda(x)} \searrow 1$$

as $x \rightarrow \infty$. It is also trivial that

$$\frac{\tilde{\Phi}_\zeta(x)}{\Phi_\lambda(x)} = \frac{\Phi_\lambda(x + \zeta)}{\Phi_\lambda(x)} \geq 1$$

because Φ_λ is increasing. So if we choose $\gamma > 0$ and define

$$\tilde{h}_\zeta^\gamma(t) := t^2 \tilde{\Phi}_\zeta\left(\frac{1}{t}\right)^\gamma,$$

we deduce by remark (iii) after Definition 2.4.3 that \tilde{h}_ζ^γ and h_λ^γ lead to the same Hausdorff measure. Thus, if we want to prove that there exist constants $c, C > 0$ such that $ch_{\lambda_1}^{\gamma_1}(t) \leq h_{\lambda_2}^{\gamma_2}(t) \leq Ch_{\lambda_1}^{\gamma_1}(t)$ for t small, it is enough to prove that the corresponding functions for ζ_1 and ζ_2 satisfy $c\tilde{h}_{\zeta_1}^{\gamma_1}(t) \leq \tilde{h}_{\zeta_2}^{\gamma_2}(t) \leq C\tilde{h}_{\zeta_1}^{\gamma_1}(t)$, which we want to prove now.

Theorem 8.1.1 *Let $1 < \zeta_1, \zeta_2 < \infty$ and $\gamma_1, \gamma_2 > 0$ such that $\zeta_1^{\gamma_1} = \zeta_2^{\gamma_2}$. Let $\tilde{\Phi}_{\zeta_1}, \tilde{\Phi}_{\zeta_2}$ and $\tilde{h}_{\zeta_1}^{\gamma_1}, \tilde{h}_{\zeta_2}^{\gamma_2}$ be defined as above. Then there exist constants $c, C > 0$ such that*

$$c < \frac{\tilde{h}_{\zeta_1}^{\gamma_1}(t)}{\tilde{h}_{\zeta_2}^{\gamma_2}(t)} < C$$

for every $t > 0$ small enough.

Proof. First, for an arbitrary $\zeta \in (1, \infty)$, we define

$$L_\zeta := (\tilde{E}_\zeta|_{\mathbb{R}})^{-1}.$$

Then L_ζ is defined on $(-\zeta, \infty)$ and a simple computation shows that

$$L_\zeta(x) = \log \left(1 + \frac{x}{\zeta} \right).$$

Choose $x_1(\zeta)$ so large that

$$L_\zeta(x) \leq \log x \text{ whenever } x \geq x_1(\zeta) \tag{8.1.1}$$

and

$$x_0(\zeta) := L_\zeta(x_1(\zeta)) \gg 1.$$

The equation (8.1.1) can be satisfied because we have

$$L_\zeta(x) \leq \log x \Leftrightarrow \frac{x}{x-1} \leq \zeta,$$

which of course holds for x large enough. The functional equation for $\tilde{\Phi}_\zeta$ (compare (5.2.9)) gives

$$\tilde{\Phi}_\zeta(L_\zeta^n(x)) = \frac{1}{\zeta^n} \tilde{\Phi}_\zeta(x) \text{ for all } n \in \mathbb{N}.$$

Now choose some $x \geq x_1(\zeta)$. Let $n_\zeta(x)$ be defined as the natural number such that

$$L_\zeta^{n_\zeta(x)}(x) \in [x_0(\zeta), x_1(\zeta)].$$

Then we have

$$\begin{aligned} \tilde{\Phi}_\zeta(x) &= \lim_{n \rightarrow \infty} \zeta^n L_\zeta^n(x) \\ &= \lim_{n \rightarrow \infty} \zeta^n L_\zeta^{n-n_\zeta(x)}(L_\zeta^{n_\zeta(x)}(x)) \\ &= \zeta^{n_\zeta(x)} \lim_{n \rightarrow \infty} \zeta^{n-n_\zeta(x)} L_\zeta^{n-n_\zeta(x)}(L_\zeta^{n_\zeta(x)}(x)) \\ &= \zeta^{n_\zeta(x)} \tilde{\Phi}_\zeta(\underbrace{L_\zeta^{n_\zeta(x)}(x)}_{\in [x_0(\zeta), x_1(\zeta)]}). \end{aligned} \tag{8.1.2}$$

By induction, it follows easily from (8.1.1) and the monotonicity of the logarithm that

$$L_\zeta^n(x) \leq \log^n(x) \text{ if } n \leq n_\zeta(x). \quad (8.1.3)$$

We now prove (also by induction) an inequality in the other direction: We claim that

$$L_\zeta^n(x) \geq \log^n(x) - \log(\zeta) \text{ if } n \leq n_\zeta(x). \quad (8.1.4)$$

First of all, observe that

$$\begin{aligned} L_\zeta(x) &= \log\left(1 + \frac{x}{\zeta}\right) \\ &= \log\left(\frac{x}{\zeta}\left(1 + \frac{\zeta}{x}\right)\right) \\ &= \log\left(\frac{x}{\zeta}\right) + \log\left(1 + \frac{\zeta}{x}\right) \\ &> \log x - \log \zeta. \end{aligned}$$

So the inequality is true for $n = 1$. Now suppose that it holds for some $n < n_\zeta(x)$. Then it follows by (8.1.3) that $\log^{n+1}(x) \geq x_0(\zeta) > 1$, in particular $\log^{n+1}(x)$ is well defined. We compute

$$\begin{aligned} L_\zeta^{n+1}(x) &= L_\zeta(L_\zeta^n(x)) \\ &> L_\zeta(\log^n(x) - \log(\zeta)) \\ &= \log\left(1 + \frac{\log^n(x) - \log(\zeta)}{\zeta}\right) \\ &= \log\left(\frac{\log^n(x)}{\zeta}\left(1 + \frac{\zeta}{\log^n(x)} - \frac{\log(\zeta)}{\log^n(x)}\right)\right) \\ &= \log^{n+1}(x) - \log(\zeta) + \log\left(1 + \frac{\zeta}{\log^n(x)} - \frac{\log(\zeta)}{\log^n(x)}\right) \\ &> \log^{n+1}(x) - \log(\zeta), \end{aligned}$$

which finishes the proof of (8.1.4). We have thus proved that for every $\zeta \in (1, \infty)$, we have

$$\log^n(x) - \log \zeta \leq L_\zeta^n(x) \leq \log^n(x) \quad (8.1.5)$$

for all $x \geq x_1(\zeta)$ and $n \leq n_\zeta(x)$.

Let such $x_1(\zeta_1)$ and $x_1(\zeta_2)$ be chosen for ζ_1 and ζ_2 and define

$$x_1 := \max\{x_1(\zeta_1), x_1(\zeta_2)\}.$$

We now show that the difference between $n_{\zeta_1}(x)$ and $n_{\zeta_2}(x)$ is uniformly bounded in x . Let $x \geq x_1$. Without loss of generality we may assume that $n_{\zeta_1}(x) \geq n_{\zeta_2}(x)$. Using (8.1.5), we obtain

$$x_0(\zeta_1) \leq \log^{n_{\zeta_1}(x)}(x) \leq L_{\zeta_1}^{n_{\zeta_1}(x)}(x) + \log \zeta_1 \leq x_1 + \log \zeta_1$$

as well as the corresponding statement for ζ_2 . If we define $x_0 := \min \{x_0(\zeta_1), x_0(\zeta_2)\}$, it follows that

$$\log^{n_{\zeta_j}(x)}(x) \in [x_0, x_1 + \log \max_i \zeta_i] \text{ for all } x \geq x_1, j = 1, 2.$$

Now suppose that (x_k) is a sequence of real numbers tending to ∞ such that

$$|(n_{\zeta_1} - n_{\zeta_2})(x_k)| \rightarrow \infty.$$

Because $\log^{n_{\zeta_1}(x_k)}(x_k)$ is bounded below by x_0 and $\log^{n_{\zeta_2}(x_k)}(x_k)$ is bounded above by $M = x_1 + (1 + \varepsilon) \log \max_i \zeta_i$, we obtain

$$x_0 \leq \log^{n_{\zeta_1}(x_k)}(x_k) = \log^{(n_{\zeta_1} - n_{\zeta_2})(x_k)}(\log^{n_{\zeta_2}(x_k)}(x_k)) \leq \log^{(n_{\zeta_1} - n_{\zeta_2})(x_k)}(M),$$

which is clearly impossible. It follows that there exists some constant K such that

$$|n_{\zeta_1}(x) - n_{\zeta_2}(x)| \leq K \text{ for all } x \geq x_1 \quad (8.1.6)$$

and thus this difference is uniformly bounded in x . Let $n(x) := n_{\zeta_1}(x)$. We now use (8.1.2) to deduce that for $i = 1, 2$, we have

$$\tilde{\Phi}_{\zeta_i}(x) \begin{cases} \leq \zeta_i^{n_{\zeta_i}(x)} \tilde{\Phi}_{\zeta_i}(x_1) =: \zeta_i^{n_{\zeta_i}(x)} B_i \\ \geq \zeta_i^{n_{\zeta_i}(x)} \tilde{\Phi}_{\zeta_i}(x_0) =: \zeta_i^{n_{\zeta_i}(x)} A_i \end{cases}$$

for all $x \geq x_1$. By passing to $A := \min_i A_i$ and $B := \max_i B_i$ and using that we have $n_{\zeta_2}(x) \in [n(x) - K, n(x) + K]$ by (8.1.6), it follows that

$$\zeta_2^{n(x)} A \zeta_2^{-K} \leq \tilde{\Phi}_{\zeta_2}(x) \leq \zeta_2^{n(x)} B \zeta_2^K$$

and that

$$\zeta_1^{n(x)} A \leq \tilde{\Phi}_{\zeta_1}(x) \leq \zeta_2^{n(x)} B$$

for all $x \geq x_1$. Thus there exist constants $K_1, K_2 > 0$ such that

$$K_1 \leq \frac{\tilde{\Phi}_{\zeta_j}(x)}{\zeta_j^{n(x)}} \leq K_2 \text{ for all } x \geq x_1, j = 1, 2.$$

Now we compare $\tilde{\Phi}_{\zeta_1}$ and $\tilde{\Phi}_{\zeta_2}$. We have

$$\begin{aligned} \tilde{\Phi}_{\zeta_1}(x) &\leq K_2 \zeta_1^{n(x)} \\ &= K_2 e^{\log(\zeta_1)n(x)} \\ &= K_2 e^{\log(\zeta_2) \frac{\log(\zeta_1)}{\log(\zeta_2)} n(x)} \\ &= K_2 (\zeta_2^{n(x)})^{\frac{\log(\zeta_1)}{\log(\zeta_2)}} \\ &\leq K_2 \frac{1}{K_1^{\frac{\log(\zeta_1)}{\log(\zeta_2)}}} (\tilde{\Phi}_{\zeta_2}(x))^{\frac{\log(\zeta_1)}{\log(\zeta_2)}} \\ &=: C' (\tilde{\Phi}_{\zeta_2}(x))^{\frac{\log(\zeta_1)}{\log(\zeta_2)}} \end{aligned}$$

and analogously

$$\tilde{\Phi}_{\zeta_1}(x) \geq K_1 \frac{1}{K_2^{\frac{\log(\zeta_1)}{\log(\zeta_2)}}} (\tilde{\Phi}_{\zeta_2}(x))^{\frac{\log(\zeta_1)}{\log(\zeta_2)}} =: c (\tilde{\Phi}_{\zeta_2}(x))^{\frac{\log(\zeta_1)}{\log(\zeta_2)}}.$$

Let $a := \zeta_1^{\gamma_1} = \zeta_2^{\gamma_2}$, i.e.

$$\gamma_i = \frac{\log a}{\log \zeta_i} \text{ for } i = 1, 2.$$

It follows that

$$\begin{aligned} \tilde{\Phi}_{\zeta_1}(x)^{\gamma_1} &= \tilde{\Phi}_{\zeta_1}(x)^{\frac{\log a}{\log \zeta_1}} \\ &\leq C' \frac{\log a}{\log \zeta_1} \tilde{\Phi}_{\zeta_2}(x)^{\frac{\log a}{\log \zeta_1} \frac{\log \zeta_1}{\log \zeta_2}} \\ &=: C \tilde{\Phi}_{\zeta_2}(x)^{\frac{\log a}{\log \zeta_2}} \\ &= C \tilde{\Phi}_{\zeta_2}(x)^{\gamma_2} \end{aligned}$$

Similarly, there exists $c > 0$ such that

$$\tilde{\Phi}_{\zeta_1}(x)^{\gamma_1} \geq c \tilde{\Phi}_{\zeta_2}(x)^{\gamma_2}$$

for all $x \geq x_1$. Thus, by the definition of $\tilde{h}_{\zeta_i}^{\gamma_i}$, we have

$$c \tilde{h}_{\zeta_2}^{\gamma_2}(t) \leq \tilde{h}_{\zeta_1}^{\gamma_1}(t) \leq C \tilde{h}_{\zeta_2}^{\gamma_2}(t)$$

for all $t > 0$ such that $1/t \geq x_1$, which proves the claim of the theorem. \square

At the beginning of this subsection, we convinced ourselves that this theorem has the following immediate corollary:

Corollary 8.1.2 *Let $0 < \lambda_1, \lambda_2 < 1/e$ and $\gamma_1, \gamma_2 > 0$ such that $\beta_{\lambda_1}^{\gamma_1} = \beta_{\lambda_2}^{\gamma_2}$. Let $\Phi_{\lambda_1}, \Phi_{\lambda_2}$ and $h_{\lambda_1}^{\gamma_1}, h_{\lambda_2}^{\gamma_2}$ be defined as before. Then there exist constants $c, C > 0$ such that*

$$c < \frac{h_{\lambda_1}^{\gamma_1}(t)}{h_{\lambda_2}^{\gamma_2}(t)} < C$$

if $t > 0$ is small enough.

8.2 The main theorem and its consequences

Now we combine the preceding results and formulate the main theorem of this thesis. First, we easily obtain the following result from the theorems we have proven so far:

Theorem 8.2.1 *For every $\lambda_0 \in (0, 1/e)$, there exists a constant $K_{\lambda_0} > 0$ with the following property: Let $h(t) = t^2 g(t)$ be a gauge function.*

(a) If

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_{\lambda_0},$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. The measure \mathcal{H}^h is not even σ -finite on $\mathcal{J}(E_\lambda)$.

(b) If

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_{\lambda_0},$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$ for every $\lambda \in (0, 1/e)$.

Proof. We define

$$K_{\lambda_0} := \frac{\log 2}{\log \beta_{\lambda_0}}.$$

Let $\lambda \in \mathbb{C} \setminus \{0\}$. If

$$\frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} \geq \gamma > K_{\lambda_0} \text{ for } t \text{ small enough,} \quad (8.2.1)$$

then it follows that

$$g(t) \geq \Phi_{\lambda_0}(1/t)^\gamma$$

if t is small enough, which clearly implies that

$$\mathcal{H}^h(\mathcal{J}(E_\lambda)) \geq \mathcal{H}^{h_{\lambda_0}}(\mathcal{J}(E_\lambda)).$$

Further, (8.2.1) shows that $\beta_{\lambda_0}^\gamma > 2$. Thus $\mathcal{H}^{h_{\lambda_0}}(\mathcal{J}(E_\lambda)) = \infty$ by Theorem 6.2.1, and the first statement is proven.

Now we prove the second statement. Let $\lambda_0 \in (0, 1/e)$ and suppose that

$$\frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} \leq \gamma_0 < K_{\lambda_0} \text{ for } t \text{ small enough.} \quad (8.2.2)$$

From (8.2.2), it follows that $\beta_{\lambda_0}^{\gamma_0} < 2$. Now choose $\lambda \in (0, 1/e)$ and $\gamma > 0$ such that $\beta_\lambda^\gamma = \beta_{\lambda_0}^{\gamma_0}$. It follows from Corollary 8.1.2 that there exists a constant $C > 0$ such that

$$\Phi_{\lambda_0}\left(\frac{1}{t}\right)^{\gamma_0} \leq C \Phi_\lambda\left(\frac{1}{t}\right)^\gamma$$

if t is small enough. Hence

$$g(t) \leq \Phi_{\lambda_0}\left(\frac{1}{t}\right)^{\gamma_0} \leq C \Phi_\lambda\left(\frac{1}{t}\right)^\gamma,$$

which implies that

$$\mathcal{H}^h(\mathcal{J}(E_\lambda)) \leq C \mathcal{H}^{h_\lambda}(\mathcal{J}(E_\lambda)) = 0$$

by Theorem 7.3.1. Thus the second statement is proven. \square

Remark: By the remarks after Theorem 6.2.1 and Theorem 7.3.1, the same statement remains true if we replace $\mathcal{J}(E_\lambda)$ by $I(E_\lambda)$. \spadesuit

We now show how results of Astala and Clop [2] and Rempe [25], combined with Theorem 7.2.1, can be used to give similar estimates for Julia sets of hyperbolic

and escaping sets of arbitrary exponential maps, but at the expense of the optimal constant K_{λ_0} .

Rempe [25] proved that for every $\lambda \in (0, 1/e)$, $\lambda' \in \mathbb{C} \setminus \{0\}$, there exists $R' > 0$ and a K -quasiconformal map $\phi : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$E_\lambda \circ \phi = \phi \circ E_{\lambda'} \text{ on } I_{R'}(E_{\lambda'}).$$

In fact, he proved a much more general theorem, but here we only need this result. Because ϕ is a conjugacy on a subset of $I(E_{\lambda'})$, it follows easily that

$$\phi(I_{R'}(E_{\lambda'})) \subset I(E_\lambda). \quad (8.2.3)$$

In fact, for $z \in I_{R'}(E_{\lambda'})$, we have $E_\lambda^n(\phi(z)) = \phi(E_{\lambda'}^n(z))$, and since $E_{\lambda'}^n(z) \rightarrow \infty$ as n tends to infinity, it is clear that then also $\phi(E_{\lambda'}^n(z)) \rightarrow \infty$ as $n \rightarrow \infty$, since ϕ is a homeomorphism. But this means that

$$E_\lambda^n(\phi(z)) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and thus $\phi(z) \in I(E_\lambda)$.

Recently, Astala and Clop [2] proved the following result:

Theorem 8.2.2 *Let $\gamma > 0$ and $h^\gamma(t) = t^2 g(1/t)^\gamma$ be a function such that g satisfies the following three assumptions:*

- g is monotonically increasing and smooth
- $\lim_{t \rightarrow \infty} g(t)/t^\alpha = \begin{cases} \infty & \alpha \leq 0 \\ 0 & \alpha > 0 \end{cases}$
- For each $\alpha > 0$, there exists $C_\alpha > 0$ and $t_\alpha > 0$ such that

$$\frac{1}{C_\alpha} g(t) \leq g(t^\alpha) \leq C_\alpha g(t)$$

whenever $t \geq t_\alpha$.

Let $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal mapping and let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^{h^\gamma}(E) = 0$. Then

$$\mathcal{H}^{h^{\gamma'}}(\varphi(E)) = 0 \text{ for every } \gamma' < \frac{\gamma}{K}.$$

It is easy to see that Φ_λ satisfies the three assumptions from the theorem. By Theorem 7.3.1, we have $\mathcal{H}^{h^\lambda}(\mathcal{J}(E_\lambda)) = 0$, in particular $\mathcal{H}^{h^\lambda}(\mathcal{J}(E_\lambda) \cap \overline{D(0, n)}) = 0$ for every $n \in \mathbb{N}$. Let $\gamma' < \gamma/K$. Because $\mathcal{J}(E_\lambda) \cap \overline{D(0, n)}$ is a compact set and ϕ^{-1} is also a K -quasiconformal mapping, it follows by Theorem 8.2.2 that

$$\mathcal{H}^{h^{\gamma'}}(\phi^{-1}(\mathcal{J}(E_\lambda) \cap \overline{D(0, n)})) = 0$$

for every $n \in \mathbb{N}$. Since $I(E_\lambda) \subset \mathcal{J}(E_\lambda)$, it is also true that

$$\mathcal{H}^{h_\lambda'}(\phi^{-1}(I(E_\lambda) \cap \overline{D(0, n)})) = 0$$

for each $n \in \mathbb{N}$. Because

$$\phi^{-1}(I(E_\lambda)) = \bigcup_{n \in \mathbb{N}} \phi^{-1}(I(E_\lambda) \cap \overline{D(0, n)}),$$

we obtain

$$\mathcal{H}^{h_\lambda'}(\phi^{-1}(I(E_\lambda))) = 0.$$

Now, (8.2.3) implies that

$$\mathcal{H}^{h_\lambda'}(I_{R'}(E_{\lambda'})) = 0,$$

and Lemma 7.2.5 yields

$$\mathcal{H}^{h_\lambda'}(I(E_{\lambda'})) = 0.$$

Thus we obtain the following result:

Theorem 8.2.3 *For every $\lambda_0 \in (0, 1/e)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, there exist constants $K_1, K_2 > 0$ (where K_1 depends only on λ_0) with the following property: Let $h(t) = t^2 g(t)$ be a gauge function.*

(a) *If*

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_1,$$

then $\mathcal{H}^h(I(E_\lambda)) = \infty$. The measure \mathcal{H}^h is not even σ -finite on $I(E_\lambda)$.

(b) *If*

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_2,$$

then $\mathcal{H}^h(I(E_\lambda)) = 0$.

The question whether K_2 actually depends on λ remains open, but I conjecture that K_2 is also independent of λ , because it does not depend on λ for $\lambda \in (0, 1/e)$.

In the hyperbolic case, we get the same result for $\mathcal{J}(E_\lambda)$ by Theorem 7.2.1, because in this situation, the set $\mathcal{J}(E_\lambda) \setminus I(E_\lambda)$ has zero $\mathcal{H}^{h_{\lambda_0}}$ -measure for every $\lambda_0 \in (0, 1/e)$ and every $\gamma > 0$. We state this result for completeness:

Theorem 8.2.4 *For every $\lambda_0 \in (0, 1/e)$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that E_λ is hyperbolic, there exist constants $K_1, K_2 > 0$ (where K_1 depends only on λ_0) with the following property:*

Let $h(t) = t^2 g(t)$ be a gauge function.

(a) *If*

$$\liminf_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_1,$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = \infty$. The measure \mathcal{H}^h is not even σ -finite on $\mathcal{J}(E_\lambda)$.

(b) If

$$\limsup_{t \rightarrow 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K_2,$$

then $\mathcal{H}^h(\mathcal{J}(E_\lambda)) = 0$.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung guter wissenschaftlicher Praxis selbstständig angefertigt und keine anderen als die angegebenen Hilfsmittel verwendet habe. Diese Arbeit hat weder ganz noch in Teilen an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Desweiteren versichere ich, noch keinen Promotionsversuch unternommen zu haben.

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