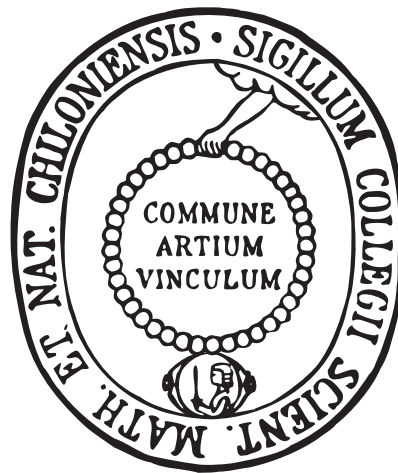


Agent-based models for financial markets

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Zusammenfassung

In dieser Arbeit betrachten wir verschiedene Finanzmarkt-Modelle, deren Dynamik auf dem Kommunikationsverhalten der am Markt handelnden Agenten basiert. Das erste Modell dieser Art hat Kirman in seinen Artikeln [18] und [19] beschrieben. Ein weiteres Modell, das wir betrachten, stammt aus dem Artikel [5] von Alfarano et al. Die Modelle basieren im Wesentlichen auf den folgenden Annahmen: Es gibt eine feste Anzahl Agenten N , die am Markt handeln. Diese Agenten lassen sich in zwei Gruppen aufteilen. Diese Gruppen werden wir als Optimisten und Pessimisten bezeichnen. Die Dynamik des Modells entsteht daraus, dass die Agenten zwischen diesen beiden Gruppen hin und her wechseln. Entscheidend für den Zustand des Systems ist nicht die Gruppenzugehörigkeit jedes einzelnen Agenten, sondern nur das Verhältnis von Optimisten zu Pessimisten. Da die Anzahl der Agenten konstant bleibt, reicht es, die Anzahl der Optimisten zu betrachten. In den beiden oben genannten Modellen wird der Verlauf dieser Anzahl durch einen Geburts- und Todesprozess modelliert. In dieser Arbeit werden wir zunächst das Verhalten dieser Geburts- und Todesprozesse für den Fall betrachten, dass die Anzahl der Agenten N gegen unendlich strebt.

Danach werden wir das Modell um einen weiteren Bestandteil, der weitere auf die Agenten einwirkende Gegebenheiten beschreibt, erweitern; diesen wollen wir als Super-Agenten bezeichnen. Dieser Agent kann verschiedene Meinungen vertreten und ändert seine Meinung im Laufe der Zeit. Die Meinung des Super-Agenten modellieren wir durch einen Markovschen Sprungprozess auf einer endlichen Menge \mathcal{M} . Durch seine Meinung beeinflusst der Super-Agent die anderen am Markt handelnden Agenten. Aus diesem Grund modellieren wir den Prozess der optimistischen Agenten durch einen Geburts- und Todesprozess in dem zufälligen Medium, das von dem Prozess des Super-Agenten erzeugt wird. Zunächst werden wir zeigen, dass dieser Geburts- und Todesprozess im zufälligen Medium durch eine Diffusion im selben zufälligen Medium approximiert wird, wenn die Anzahl der Agenten gegen unendlich strebt. Danach werden wir die stationäre Verteilung des Grenzprozesses betrachten.

Zuletzt werden wir noch den Fall betrachten, dass der Super-Agent seine Meinung sehr schnell ändert. Hierbei werden wir die Grenzprozesse sowohl für den Geburts- und Todesprozess im zufälligen Medium als auch für den Diffusionsprozess im zufälligen Medium bestimmen. Es zeigt sich dabei, dass der resultierende Markt auch bei schnell variierenden äußeren Einflüssen stabil bleibt.

Abstract

In this paper we look at different models of financial markets whose dynamic is based on the herding behaviour of the agents trading on the market. Kirman introduced the first model of this kind in his papers [18] and [19]. We also consider another model that is introduced by Alfarano et al. in the paper [5]. The basic assumptions of these models are that there is a fixed number of agents N , who trade on the market. These agents can be divided into two groups. We call them optimists and pessimists. The dynamic of these models is based on the fact that the agents switch between the two groups. Important for the state of the system is not the membership of each agent in one of the groups but the relation between optimistic and pessimistic agents. Due to the fact that the number of agents is constant, it is sufficient to look at the number of optimistic agents. In the two models mentioned above, the number of optimistic agents is modelled as a birth and death process. At first, we look at the behaviour of these birth and death processes in the case that the number of agents N tends to infinity.

After that, we extend the model with another agent, representing further influences for the behaviour of the optimists and pessimists whom we call super agent. This agent can have different opinions about the market and he changes his opinion from time to time. We model the opinion of the super agent by a Markov jump process on a finite set \mathcal{M} . The super agent does not trade on the market but he influences the behaviour of all other agents. Due to this fact, we model the process of optimistic agents as a birth and death process in the random environment that is created by the super agent.

First, we show that these birth and death processes in random environment converge to a switching diffusion process if the number of agents tends to infinity. After that, we look at the stationary distribution of this limiting switching diffusion.

Finally, we look at the fast switching behaviour of the birth and death processes in random environment and of the switching diffusion. Fast switching means that the super agent switches his opinion very fast. It is shown that the market remains stable under very fast switching.

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Chapter 1

Introduction

Starting with the articles [18] and [19] of Kirman in the years 1991 and 1993, herding models for financial markets are investigated in a variety of papers, see [1], [2], [3], [5], [13], [20]. In the eighties of the last century entomologists analysed in a couple of experiments the behaviour of ants that are faced with two identical food sources. They observed that the ants exploited one food source more intensively than the other. Furthermore, from time to time they switched their attention to the other source. This group behaviour can be explained by interaction between the individual ants. In view of the fact that similar behaviour has been observed on financial markets, Kirman used the observations of the ants for a model of behaviour of agents on a market. The concept of these agent-based models, also called ant models, is that we have $N \in \mathbb{N}$ agents who trade on the market. The agents can be divided into two groups e.g. optimists and pessimists or chartists and fundamentalists. The behaviour of an agent is governed by this classification. Also we assume that all agents communicate with each other. And it is possible that any agent can switch his opinion. This takes place by communication with other agents or without influence from outside.

One main problem of the model of Kirman is the so called N -dependence. This means that the model gives a good approximation for markets with a small number of agents N , but it cannot be used for large markets. The problem here is that the process of optimistic agents, also called herding process, can be approximated by a Gaussian process if the number of agents is large. As already known, this is not realistic for a financial market. To overcome this problem Alfarano et al. introduced a modified model in [5]. In their model they overcome the problem of N -dependence by speeding up the communication between the agents.

In [5] the thermodynamic limit of these two models is considered. The thermodynamic limit describes the case that the number of agents converges to infinity. Starting from this, our first result is a rigorous mathematical proof of this asymptotic behaviour in Chapter 3. In a next step, we introduce a new agent, called super agent, that does not trade on the market but influences all other agents. With this agent we define a generalised herding model. As one of our main results we also prove the behaviour of the thermodynamic limit in this generalised model. Furthermore we look at the stationary distribution of this

generalised model and consider the case of fast switching. This means that the super agent switches his opinion very fast.

In Chapter 2 we start with some basic facts on continuous-time Markov processes. There we give the basic results for the representation of continuous-time Markov processes and introduce some methods to consider the asymptotic behaviour of continuous-time Markov processes. After that, we give the mathematical definition of the Kirman model and the modified Kirman model in Chapter 3. We also look at the asymptotic behaviour of these models and show some characteristics of the limiting process from the modified Kirman model. In Chapter 4, we formulate the generalised agent model and look at its limiting behaviour if the number of agents tends to infinity. The stationary distribution of the limiting process from the generalised agent model is the main aspect in Chapter 5. Finally, we take a look at the convergence of the generalised agent model in the case of fast switching in Chapter 6. In Appendix A, the reader will find some basic information about stochastic processes and their convergence.

Chapter 2

Basic facts on continuous-time Markov processes and their convergence

In this chapter we introduce the basic facts about continuous-time Markov processes. In Section 2.1 we give a short definition of the Markov properties and show different representations of a Markov process with a finite or countable state space in Section 2.2. Finally we show some concepts for the convergence of Markov processes in Section 2.3. We use these concepts in Chapter 3 to look for good approximations of markets with a large number of agents.

2.1 Basic facts on continuous-time Markov processes

2.1.1 Generalities

We consider a continuous-time Markov process $(X_t)_{t \in [0, \infty)}$ with discrete state space S , finite or countably infinite, having right-continuous paths. The Markov property states that

$$P(X_{t+s} = j | X_{t_1} = x_1, \dots, X_{t_n} = x_n, X_t = i) = P(X_s = j | X_0 = i) = P_{ij}(s)$$

for all $t, s \geq 0, i, j \in S$ and $t_1, \dots, t_n \in [0, t), x_1, \dots, x_n \in S$, i.e. the future development, given the past and present, only depends on the present state i and is time-homogeneous. $(P_{ij}(t))_{i, j \in S}$ are stochastic matrices and are called the transition matrices.

The infinitesimal characteristics are given by

$$q_{ij} = \lim_{t \downarrow 0} \frac{P_{ij}(t)}{t}, j \neq i, \quad q_{ii} = \lim_{t \downarrow 0} \frac{P_{ii}(t) - 1}{t},$$

and we assume $0 < q_i = -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ for all states. The q_{ij} are the transition rates and the matrix $Q = (q_{ij})_{i, j \in S}$ is the generator. Q uniquely determines the distribution of

the process and we have the matrix differential equation

$$P'(t) = P(t)Q, \text{ resulting in } P(t) = e^{Qt}.$$

A stationary distribution $\pi = (\pi_i)_{i \in S}$ fulfils the equation $\pi^T Q = 0$. To the generator Q corresponds the infinitesimal generator A which, for any bounded mapping $f : S \rightarrow S$, defines a mapping $Af : S \rightarrow S$ by

$$Af(i) = \lim_{t \downarrow 0} \frac{\sum_{j \in S} P_{ij}(t)(f(j) - f(i))}{t} = \sum_{j \neq i} q_{ij}(f(j) - f(i)).$$

It determines Q uniquely and so the distribution of the process.

The transition rates fulfil

$$\begin{aligned} P(X_t = j | X_0 = i) &= q_{ij}t + o(t), i \neq j \\ P(X_t = i | X_0 = i) &= 1 - q_i t + o(t). \end{aligned}$$

2.2 Representation of continuous-time Markov processes

In this section we introduce some methods to represent continuous-time Markov processes. Later we use these representations to obtain different methods for deriving convergence of continuous-time Markov processes.

2.2.1 Probabilistic representation

Starting from Q we can realise the process in the following way. Let

$$p_{ij} = \frac{q_{ij}}{q_i}, i \neq j, p_{ii} = 0.$$

This is the transition matrix of a discrete-time Markov process, the so-called embedded process. When the process is in state i , it stays there for a random holding time, exponentially distributed with parameter q_i , and then jumps to j according to p_{ij} . An equivalent description is the following. Assume $\sup_i q_i \leq q < \infty$ and define

$$r_{ij} = \frac{q_{ij}}{q}, i \neq j, r_{ii} = 1 - \sum_{j \neq i} r_{ij}.$$

When the process is in state i , it stays there for a random holding time, exponentially distributed with parameter q , and then makes a transition to j according to r_{ij} , so in this second realisation the process may stay put in i with probability r_{ii} . This representation immediately shows how such a process may be simulated by simulation of exponentially distributed random variables and discrete transitions from i to j . It also shows the different roles of q_i and $\frac{q_{ij}}{q_i}$, the first determining the rate at which transitions occur and the second the probabilities according to which the new state is selected.

2.2.2 Representation using a random time transformation

We now assume that S is a subset of \mathbb{Z} ; setting transition rates outside of S equal to zero, S may be assumed to be equal to \mathbb{Z} . The basic building block is the Poisson process which remains in any state $i = 0, 1, 2, \dots$ with exponential holding time with parameter 1 and then jumps to $i + 1$. A continuous-time Markov process $(X_t)_{t \in [0, \infty)}$ fulfils the following stochastic integral equation with $\gamma_l(i) = q_{i, i+l}$

$$X_t = X_0 + \sum_l l Y^l \left(\int_0^t \gamma_l(X_s) ds \right)$$

where the $(Y_t^l)_{t \in [0, \infty)}$ are independent Poisson processes and we write Y_t^l as $Y^l(t)$; see [12], Chapter 6 Section 4. Here the clock for the Poisson process Y^l runs with the random speed $\int_0^t \gamma_l(X_s) ds$. We shall see in Subsection 2.3.3 how this representation immediately sheds light on the limiting behaviour for such processes.

2.2.3 Semimartingale representation

In this subsection, we leave the Markov processes and give a short introduction to the theory of semimartingales with a focus on the definitions and theorems that we need in Subsection 3.3.2 to show the convergence of the birth and death processes. For more information about semimartingales we refer to the book of Jacod and Shiryaev [15]. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $(\mathfrak{F}_t)_{t \geq 0}$ be a right continuous filtration.

Definition 2.1 (cadlag process). For all $\omega \in \Omega$ the function $\alpha^\omega : [0, \infty) \rightarrow E$ with $\alpha_t^\omega = X_t(\omega)$ is called the path of a stochastic process X relative to ω . The stochastic process X is called cadlag (continue à droite, limitée à gauche) process if all paths are right continuous and with existing left limits.

The jump process ΔX of a cadlag process X is defined by

$$\Delta X_t = X_t - X_{t-} = X_t - \lim_{s \uparrow t} X_s.$$

Definition 2.2 (semimartingale).

1. A semimartingale is a process X of the form

$$X = X_0 + M + A, \tag{2.1}$$

where X_0 is finite-valued and \mathfrak{F}_0 -measurable, where M is a cadlag martingale with $M_0 = 0$, and where A is an adapted cadlag process of finite-variation with $A_0 = 0$.

2. A special semimartingale is a semimartingale X which admits a decomposition $X = X_0 + M + A$ as above, with a process A that is predictable. This decomposition is unique and is called the canonical decomposition.

It is easy to show that all semimartingales are cadlag and adapted (see [15], Page 43). The next proposition shows us one condition for a semimartingale to be a special semimartingale. Further conditions can be found in [15].

Proposition 2.3. *If a semimartingale X satisfies $|\Delta X| \leq a$ for some $a > 0$, it is a special semimartingale.*

Proof. See [15], Lemma 4.24, Chapter I.

Characteristics of Semimartingales

We now introduce the characteristic of a semimartingale. This characteristic describes the local behaviour of a semimartingale and determines the semimartingale completely. The concept of the characteristic of a semimartingale is a generalisation of the Levy-Khintchine triple for Levy processes. But before we define the characteristic, we need some additional concepts.

Proposition 2.4. *Any local martingale M admits a unique decomposition*

$$M = M_0 + M^c + M^d$$

where $M_0^c = M_0^d = 0$, M^c is a continuous local martingale, and M^d is a purely discontinuous local martingale. A local martingale is called purely discontinuous if its quadratic variation process is a pure jump process.

Proof. See [15], Theorem 4.18, Chapter I.

Definition 2.5 (continuous martingale part). M^c , from Proposition 2.4, is called the continuous part of M .

Now we use this definition to define the continuous martingale part of a semimartingale X .

Proposition 2.6. *Let X be a semimartingale. There is a unique continuous local martingale X^c with $X_0^c = 0$, such that any decomposition $X = X_0 + M + A$ of type (2.1) meets $M^c = X^c$. X^c is called the continuous martingale part of X .*

Proof. See [15], Proposition 4.27, Chapter I.

Proposition 2.7. *Let X be an adapted cadlag \mathbb{R}^d – valued process. Then*

$$\mu^X(\omega, dt, dx) = \sum_{s>0} 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt, dx)$$

defines an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. We call this the associated measure of the jumps of X .

Proof. See [15], Proposition 1.16, Chapter II.

Definition 2.8.

1. The optional σ -field is the σ -field \mathcal{O} on $\Omega \times \mathbb{R}_+$ that is generated by all cadlag adapted processes.
2. The predictable σ -field is the σ -field \mathcal{P} on $\Omega \times \mathbb{R}_+$ that is generated by all left continuous adapted processes.
3. A process X on $\Omega \times \mathbb{R}_+$ that is \mathcal{O} -measurable is called optional. A process W on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ is called optional, if W is $\mathcal{O} \otimes \mathcal{B}^d$ -measurable.
4. Let μ be a random measure and W an optional process on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ then we define the integral process $W * \mu$ for all $\omega \in \Omega$ by

$$W * \mu_t(\omega) = \begin{cases} \int_{[0,t] \times \mathbb{R}^d} W(\omega, s, x) \mu(\omega; ds, dx) & \text{if } \int_{[0,t] \times \mathbb{R}^d} |W(\omega, s, x)| \mu(\omega; ds, dx) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

5. A random measure μ is optional if $W * \mu$ is optional for all optional functions W .

Proposition 2.9. *Let W be a nonnegative optional function and*

$$\mu^X(\omega, dt, dx) = \sum_{s>0} 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt, dx)$$

be the jump measure of an \mathbb{R}^d -valued adapted cadlag process X as in Proposition 2.7. Then

$$W * \mu_t^X = \sum_{0 < s \leq t} W(s, \Delta X_s) 1_{\{\Delta X \neq 0\}}(s).$$

Proof. See [15], Proposition 1.14 and 1.16, Chapter II.

Theorem 2.10. *Let μ be an optional σ -finite random measure. There exists a random measure, called the compensator of μ and denoted by μ^p , which is unique up to a P -null set, and which is characterised as being a predictable random measure satisfying either one of the two following equivalent properties:*

1. $E(W * \mu_\infty^p) = E(W * \mu_\infty)$ for every nonnegative $\mathcal{P} \otimes \mathcal{B}^d$ -measurable function W on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$.
2. For every $\mathcal{P} \otimes \mathcal{B}^d$ -measurable function W on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ such that $|W| * \mu$ is a locally integrable increasing process, $|W| * \mu^p$ is a locally integrable increasing process, and $W * \mu - W * \mu^p$ is a local martingale.

Proof. See [15], Theorem 1.8, Chapter II.

Definition 2.11. A bounded function $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called truncation function, if $h(x) = x$ in a neighbourhood of 0.

For a truncation function h and a semimartingale X , we define

$$\begin{aligned}\tilde{X}(h)_t &= \sum_{s \leq t} [\Delta X_s - h(\Delta X_s)] \\ X(h) &= X - \tilde{X}(h).\end{aligned}$$

As $\Delta X(h) = h(\Delta X)$ and h is bounded we know from Proposition 2.3 that $X(h)$ is a special semimartingale with canonical decomposition

$$X(h) = X_0 + M(h) + B(h), \quad (2.2)$$

where $M(h)$ is a martingale and $B(h)$ is a predictable adapted process of finite-variation.

Now we have all notions to define the characteristics of a semimartingale.

Definition 2.12 (characteristics of a semimartingale). Let h be a fixed truncation function and $X : \Omega \rightarrow \mathbb{R}^d$ be a semimartingale. We call characteristics of X associated with h the triplet (B, C, ν) , consisting of:

1. $B = (B_i)_{i \leq d}$ is a predictable adapted process of finite-variation, namely the process $B = B(h)$ appearing in (2.2).
2. $C = (C^{ij})_{i,j \leq d}$ is a continuous adapted process of finite-variation, namely

$$C^{ij} = \langle X^{i,c}, X^{j,c} \rangle,$$

where X^c is the continuous martingale part of X .

3. ν is a predictable random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, namely the compensator of the random measure μ^X associated to the jumps of X .

2.2.4 Examples of semimartingale characteristics for continuous-time Markov processes

In this subsection we look at the semimartingale characteristics for two different kinds of continuous-time Markov processes. As a first example we look at a diffusion process on a subset E of \mathbb{R} with drift parameter B and diffusion parameter σ^2 . Under certain conditions on B and σ^2 , see [12], there exists such a diffusion process and its infinitesimal generator is given by

$$Af(x) = B(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$$

for all $f \in C^2(E)$.

As a second example we look at a Markov jump process X on a finite set $S_d \subseteq \mathbb{R}^d$. In Subsection 2.1.1 we have seen that the process X is completely defined by its infinitesimal generator A . This infinitesimal generator is of the form

$$Af(i) = \sum_{j \neq i} q_{ij}(f(j) - f(i))$$

for all $f : S_d \rightarrow S_d$. For the calculation of the semimartingale characteristics of these two processes we look at a general result for the relation between the semimartingale characteristics and the infinitesimal generator of a Markov process from [10].

Let $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a truncation function with $h(x) = x1_{\{\|x\| \leq r\}}$ for $r > 0$.

Theorem 2.13. *Let X be an Ito process with semimartingale characteristic $B(h)_t = \int_0^t b(X_s)ds$, $C_t = \int_0^t c(X_s)ds$ and $\nu(dx, dt)(\omega) = K(X_t(\omega), dy)dt$, where $b = (b^i)_{i \leq d} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measurable function, $c = (c^{ij})_{i, j \leq d}$ is a measurable function with values in the set of symmetric positive semi-definite matrices and $K(x, dy)$ is a positive kernel from \mathbb{R}^d into \mathbb{R}^d with $K(x, \{0\}) = 0$ and $\int (\min\{\|y\|^2, 1\})K(x, dy) < \infty$ for all $x \in \mathbb{R}^d$.*

Then, $C^2(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ and the restriction of the infinitesimal generator A of X to the set $C^2(\mathbb{R}^d)$ has the form

$$\begin{aligned} Af(x) &= \sum_{i \leq d} b^i(x) \frac{d}{dx^i} f(x) + \frac{1}{2} c^{ij} \sum_{i, j \leq d} \frac{d^2}{dx^i dx^j} f(x) \\ &+ \int \left[f(x+y) - f(x) - 1_{[0, r]}(\|y\|) \sum_{i \leq d} y^i \frac{d}{dx^i} f(x) \right] K(x, dy), \end{aligned} \quad (2.3)$$

here x^i and y^i denote the i -th component of the vectors x and y .

Proof. See [10], Theorem 7.16.

With this theorem we are now able to calculate the semimartingale characteristics of our two continuous-time Markov processes.

Theorem 2.14. *Let X be a continuous-time Markov process on a finite set $S_d \subseteq \mathbb{R}^d$ with infinitesimal generator $A : B(S_d) \rightarrow B(S_d)$ given by*

$$Af(x) = \sum_{\tilde{x} \neq x} q_{x\tilde{x}} (f(\tilde{x}) - f(x))$$

for all bounded $f : S_d \rightarrow S_d$. Let $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined by $h(x) = x1_{\|x\| < r}$ with $r < \min_{x, \tilde{x} \in S_d} \|x - \tilde{x}\|$.

Then the semimartingale characteristic of X is given by $B(h) = 0$, $C = 0$ and $\nu(dy, dt)(\omega) = K(X_t(\omega), dy)dt$ with

$$K(x, y) = q_{x, x+y}.$$

Proof. Let \bar{X} be a process on S_d with semimartingale characteristic $B(h) = C = 0$ and $\nu(dy, dt)(\omega) = K(X_t(\omega), dy)dt$ with $K(x, y) = q_{x(x+y)}$. Then it follows from Theorem 2.13 that the infinitesimal generator \bar{A} of \bar{X} is given by

$$\begin{aligned} \bar{A}f(x) &= \int [f(x+y) - f(x)] K(x, dy) \\ &= \sum_{\tilde{x} \neq x} q_{x\tilde{x}} (f(\tilde{x}) - f(x)) = Af(x). \end{aligned}$$

Due to the fact that the infinitesimal generator uniquely characterises the processes \bar{X} and X , it follows that

$$X \stackrel{d}{=} \bar{X}.$$

From this it follows that the processes have the same semimartingale characteristic. \square

Theorem 2.15. *Let X be a diffusion process on $E \subseteq \mathbb{R}$ with drift parameter B and diffusion parameter σ^2 .*

Then the semimartingale characteristic of X is given by the triple $(B(h), C, \nu)$ with $B(h)_t = \int_0^t B(X_s) ds$, $C_t = \int_0^t \sigma^2(X_s) ds$ and $\nu = 0$.

Proof. The form of the semimartingale characteristics follows directly from the infinitesimal generator of X and Theorem 2.13. \square

2.3 Basic facts on the asymptotic behaviour

Let us assume that for $N = 1, 2, \dots$ we have continuous-time Markov processes $(Z_t^N)_{t \geq 0}$ with state space S_N and transition rates q_{ij}^N , depending on N . In typical examples N is the size of a population or of a network. For large N , the exact behaviour is no longer tractable so one has to rely on approximations. In mathematical terms, the limiting behaviour of suitably standardised versions $(X_t^N)_{t \geq 0}$ of $(Z_t^N)_{t \geq 0}$ as $N \rightarrow \infty$ has to be investigated. There are various mathematical methods to achieve this.

2.3.1 Convergence via infinitesimal operators

Suppose that $S_N \subseteq S$ for all N . The convergence of the process $(X_t^N)_{t \geq 0}$ can be reduced to a study of the convergence of the infinitesimal operators A^N . This is a general fact and does not depend on the assumption of discrete state spaces which are discussed here.

Assume that we find an operator A , defined on a suitable set \mathcal{D} of bounded functions $f : S \rightarrow \mathbb{R}$, where Af again is a function $Af : S \rightarrow \mathbb{R}$, such that the following condition holds

$$\lim_{N \rightarrow \infty} \sup_{y \in S_N} |A^N f(y) - Af(y)| \rightarrow 0.$$

If A generates a well-behaved Markov process $(X_t)_{t \geq 0}$, well-behaved here in the mathematical sense of a Feller process, see [12], then

$$(X_t^N)_{t \geq 0} \text{ converges to } (X_t)_{t \geq 0}.$$

In exact mathematical terms, this convergence takes place in the sense of weak convergence in the function space $D_S[0, \infty)$, which includes the convergence of finite-dimensional distributions and e.g. also convergence of stationary distributions and distributions of first hitting times. We refer to [12], 1.6.1, 4.2.11.

In many examples, A will be a differential operator on a certain set \mathcal{D} of twice differentiable functions and the limit process will be a diffusion. We use this method in the next subsection to show the convergence of birth and death processes to a diffusion process.

2.3.2 Convergence of birth and death processes via infinitesimal operators

For all $N \in \mathbb{N}$ let $(Z_t^N)_{t \geq 0}$ be a homogeneous Markov process on $\{0, \dots, N\}$ with transition probabilities

$$P(Z_{t+\delta}^N = j | Z_t^N = i) = q_N(i, j)\delta + o(\delta)$$

with $q_N(i, i+1), q_N(i, i-1) \geq 0$ and $q_N(i, j) = 0$ for all $j \notin \{i, i+1, i-1\}$. Then $(Z_t^N)_{t \geq 0}$ is a birth and death process with birth rate $\lambda_N(i) = q_N(i, i+1)$ and death rate $\mu_N(i) = q_N(i, i-1)$ for all $i \in \{0, \dots, N\}$.

In the following we use the method mentioned in 2.3.1 to look at the limit of the processes $(Z_t^N)_{t \geq 0}$ if N tends to infinity. To show this, we look at the following transformation

$$X_t^N = \frac{2Z_t^N}{N} - 1.$$

Then $X^N = (X_t^N)_{t \geq 0}$ is a homogeneous Markov process on

$$E_N := \left\{ -1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1 \right\}$$

with birth rates

$$\lambda_N(x) = \tilde{q}_N \left(x, x + \frac{2}{N} \right) = q_N \left(\frac{N(x+1)}{2}, \frac{N(x+1)}{2} + 1 \right)$$

and death rates

$$\mu_N(x) = \tilde{q}_N \left(x, x - \frac{2}{N} \right) = q_N \left(\frac{N(x+1)}{2}, \frac{N(x+1)}{2} - 1 \right)$$

for $x \in E_N$. Here \tilde{q}_N are the transition rates of the process X . We need various results from the theory of Markov processes and refer to [12] and [16] as excellent sources.

General definitions

Let L be a Banach space.

Definition 2.16 (semigroup). A family $(T(t))_{t \geq 0}$ of bounded linear operators $T(t) : L \rightarrow L$ is called a semigroup if

- $T(0) = Id$ and
- $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$.

A semigroup $(T(t))_{t \geq 0}$ is called strongly continuous if

$$\lim_{t \rightarrow 0} T(t)f = f$$

for all $f \in L$.

A semigroup $(T(t))_{t \geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ holds for all $t \geq 0$.

Definition 2.17 (infinitesimal generator). Let $(T(t))_{t \geq 0}$ be a semigroup. The infinitesimal generator A of this semigroup is defined by

$$Af = \lim_{t \rightarrow 0} \frac{T(t)f - f}{t}$$

for all $f \in \mathfrak{D}(A)$. $\mathfrak{D}(A)$ is the subspace of L where this limit exists.

Definition 2.18. Let $X = (X_t)_{t \geq 0}$ be a Markov process. Then the semigroup $(T(t))_{t \geq 0}$ with

$$T(t)f(x) = E(f(X_t) | X_0 = x)$$

is called the semigroup of X .

Semigroups and infinitesimal generators of the processes $(X_t^N)_{t \geq 0}$

In this paragraph we calculate the semigroup and the infinitesimal generators of the birth and death processes X^N .

The semigroups

The semigroups of the birth and death processes $(X_t^N)_{t \geq 0}$ are given by $(T_N(t))_t : B(E_N) \rightarrow B(E_N)$ with

$$(T_N(t)f)(x) = E(f(X_t^N) | X_0^N = x).$$

for all $f \in B(E_N)$ and $x \in E_N$. Here $B(E_N)$ denotes the set of bounded functions from E_N to \mathbb{R} .

We have

- $(T_N(0)f)(x) = E(f(X_0^N) | X_0^N = x) = f(x)$ and
- $T_N(s+t) = T_N(s)T_N(t)$.

The last equation is called the Chapman-Kolmogorov equation and follows directly from the time homogeneity and the Markov properties of the processes X^N , see e.g. [17].

$(T_N(t))_{t \geq 0}$ is a contraction semigroup because for all $f \in B(E_N)$, $x \in E_N$

$$|(T_N(t)f)(x)| = |E(f(X_t^N) | X_0^N = x)| \leq \sup_{y \in E_N} |f(y)| = \|f\|.$$

Let $C_N = \max_{x \in E_N \setminus \{-1\}} |f(x) - f(x - \frac{2}{N})|$. Then

$$\begin{aligned}
 |(T_N(t)f)(x) - f(x)| &= |E[f(X_t^N) - f(x) | X_0^N = x]| \\
 &\leq \left| (\lambda_N(x)t + o(t)) \left[f\left(x + \frac{2}{N}\right) - f(x) \right] \right| \\
 &\quad + \left| (\mu_N(x)t + o(t)) \left[f\left(x - \frac{2}{N}\right) - f(x) \right] \right| + o(t) \\
 &\leq \left(\max_{y \in E_N} \lambda_N(y)t + o(t) \right) C_N \\
 &\quad + \left(\max_{y \in E_N} \mu_N(y)t + o(t) \right) C_N \\
 &= C_N \cdot t \left(\max_{y \in E_N} \lambda_N(y) + \max_{y \in E_N} \mu_N(y) + \frac{o(t)}{t} \right).
 \end{aligned}$$

Furthermore

$$\sup_{x \in E_N} |(T_N(t)f)(x) - f(x)| \leq C_N \cdot t \left(\max_{y \in E_N} \lambda_N(y) + \max_{y \in E_N} \mu_N(y) + \frac{o(t)}{t} \right) \xrightarrow{t \rightarrow 0} 0,$$

where μ_N and λ_N denote the birth and death rates of the birth and death processes X^N . So the semigroup $(T_N(t))_{t \geq 0}$ is strongly continuous.

The infinitesimal generators of the semigroups $(T_N(t))_{t \geq 0}$

In the next step we take a look at the infinitesimal generators of our birth and death processes. As mentioned in 2.3.1 the infinitesimal generators are relevant to show the convergence of the birth and death processes. Our first observation is

$$\begin{aligned}
 E[f(X_t^N) - f(x) | X_0^N = x] &= (\lambda_N(x)t + o(t)) \left[f\left(x + \frac{2}{N}\right) - f(x) \right] \\
 &\quad + (\mu_N(x)t + o(t)) \left[f\left(x - \frac{2}{N}\right) - f(x) \right] + o(t).
 \end{aligned}$$

Hence

$$\begin{aligned}
 A_N f(x) &= \lim_{t \downarrow 0} \left(\lambda_N(x) + \frac{o(t)}{t} \right) \left[f\left(x + \frac{2}{N}\right) - f(x) \right] \\
 &\quad + \lim_{t \downarrow 0} \left(\mu_N(x) + \frac{o(t)}{t} \right) \left[f\left(x - \frac{2}{N}\right) - f(x) \right] \\
 &\quad + \lim_{t \downarrow 0} \frac{o(t)}{t} \\
 &= \lambda_N(x) \left[f\left(x + \frac{2}{N}\right) - f(x) \right] + \mu_N(x) \left[f\left(x - \frac{2}{N}\right) - f(x) \right].
 \end{aligned}$$

Therefore the infinitesimal generator is given by

$$A_N f(x) = \lambda_N(x) \left[f\left(x + \frac{2}{N}\right) - f(x) \right] + \mu_N(x) \left[f\left(x - \frac{2}{N}\right) - f(x) \right]$$

for all $x \in E_N$.

Convergence of the processes X^N

In this paragraph we show that the birth and death processes converge to a diffusion process if the number of agents tends to infinity. For this we first show in Theorem 2.20 that the infinitesimal generators A_N of the birth and death processes converge to the infinitesimal generator A of a diffusion process. Then we show in Theorems 2.23-2.25 that there exists a diffusion process X with this infinitesimal generator A and that the birth and death processes X^N weakly converge in $D_{[-1,1]}[0, \infty)$ to this diffusion process X ; see [12] for a detailed discussion of this convergence. To show this convergence of the birth and death processes we need one further condition. Let $E := [-1, 1]$.

Assumption 2.19. There exist two functions $B : [-1, 1] \rightarrow \mathbb{R}$ and $\sigma^2 : [-1, 1] \rightarrow [0, \infty)$ with

$$\begin{aligned} \lim_{N \rightarrow \infty} \sup_{x \in E_N} \left| \frac{2}{N} (\lambda_N(x) - \mu_N(x)) - B(x) \right| &= 0, \\ \lim_{N \rightarrow \infty} \sup_{x \in E_N} \left| \frac{4}{N^2} (\lambda_N(x) + \mu_N(x)) - \sigma^2(x) \right| &= 0 \end{aligned}$$

and $\sigma^2 \in C^2(E)$, $\sigma^{2''}$ is bounded, $B : E \rightarrow \mathbb{R}$ is Lipschitz continuous and

$$\sigma^2(r_i) = 0 \leq (-1)^i B(r_i)$$

for $i \in \{0, 1\}$, where $r_0 = -1$ and $r_1 = 1$.

For all $x \in [-1, 1]$ let

$$Af(x) = B(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

For $k \in \mathbb{N} \cup \{\infty\}$ let $C^k(E)$ be the set of all functions $f \in C^k((-1, 1))$ with finite limits of the derivatives in the points -1 and 1 , denoted by $f^{(l)}(1)$, $f^{(l)}(-1)$ for all $l \leq k$.

Theorem 2.20. For all $f \in C^2(E)$

$$\lim_{N \rightarrow \infty} \sup_{y \in E_N} |A_N f(y) - Af(y)| = 0.$$

Proof. Let $f \in C^2(E)$. Let $x \in E_N \setminus \{-1, 1\}$. Then we get by Taylor approximation

$$\begin{aligned} A_N f(x) &= \lambda_N(x) \left[f(x) + \frac{2}{N} f'(x) + \frac{2}{N^2} f''(x) + o\left(\frac{1}{N^2}\right) - f(x) \right] \\ &+ \mu_N(x) \left[f(x) - \frac{2}{N} f'(x) + \frac{2}{N^2} f''(x) + o\left(\frac{1}{N^2}\right) - f(x) \right] \\ &= \frac{2}{N} (\lambda_N(x) - \mu_N(x)) f'(x) + \frac{2}{N^2} (\lambda_N(x) + \mu_N(x)) f''(x) \\ &+ (\lambda_N(x) + \mu_N(x)) o\left(\frac{1}{N^2}\right). \end{aligned}$$

Due to the fact that $f \in C^2$, the error term $o(\frac{1}{N^2})$ is uniform in x . Because of finite limits of all derivatives in the points $-1, 1$ and the notations above, it is easy to check that this approximation holds also for $x \in \{-1, 1\}$.

Hence for all $x \in E_N$

$$\begin{aligned} |A_N f(x) - A f(x)| &\leq \left| \frac{2}{N} (\lambda_N(x) - \mu_N(x)) - B(x) \right| |f'(x)| \\ &+ \left| \frac{2}{N^2} (\lambda_N(x) + \mu_N(x)) - \frac{1}{2} \sigma^2(x) \right| |f''(x)| \\ &+ \left| (\lambda_N(x) + \mu_N(x)) o\left(\frac{1}{N^2}\right) \right|. \end{aligned}$$

This shows

$$\begin{aligned} \sup_{x \in E_N} |A_N f(x) - A f(x)| &\leq \sup_{x \in E_N} \left| \frac{2}{N} (\lambda_N(x) - \mu_N(x)) - B(x) \right| \sup_{x \in E} |f'(x)| \\ &+ \sup_{x \in E_N} \left| \frac{2}{N^2} (\lambda_N(x) + \mu_N(x)) - \frac{1}{2} \sigma^2(x) \right| \sup_{x \in E} |f''(x)| \\ &+ \sup_{x \in E_N} \left| (\lambda_N(x) + \mu_N(x)) o\left(\frac{1}{N^2}\right) \right|. \end{aligned}$$

Since $f \in C^2[-1, 1]$, $\sup_{x \in E} |f'(x)|$ and $\sup_{x \in E} |f''(x)|$ are finite. By using this and the Assumption 2.19 it follows that the first two terms tend to zero for N tending to infinity. The last term tends to zero due to the fact that $(\lambda_N(x) + \mu_N(x))$ is of order N^2 by Assumption 2.19, and that the error term is uniform in x . \square

In the following, we show that there exists a process X with infinitesimal generator A , so that the birth and death processes X_N weakly converge to X in $D_E[0, \infty)$. But first we need some further results and definitions from [12].

Definition 2.21 (Feller semigroup). A semigroup $(T(t))_{t \geq 0}$ on $C(E)$ is called positive, if $T(t)$ is a positive operator for all $t \geq 0$. A positive contraction semigroup $(T(t))_{t \geq 0}$ on $C(E)$ is called a Feller semigroup, if

1. $T(t)C(E) \subseteq C(E)$ for all $t \geq 0$.
2. $T(t)f(x) \xrightarrow{t \rightarrow 0} f(x)$ for all $f \in C(E), x \in E$.

Definition 2.22 (core). A subset $D \subseteq \mathfrak{D}(A)$ is called core of an infinitesimal generator $A : L \rightarrow L$, if the closure of the restriction of A to D is equal to A i.e.,

$$\overline{\{(f, Af) : f \in D\}} = \{(f, Af) : f \in \mathfrak{D}(A)\}.$$

Next, we will explain that the infinitesimal generator A generates a Feller semigroup $(T(t))_{t \geq 0}$ on $C(E)$.

Theorem 2.23. *The infinitesimal generator $A = B(x)\frac{d}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}$ generates a Feller semigroup $(T(t))_{t \geq 0}$ on $C([r_0, r_1])$ with core $C^\infty([r_0, r_1])$ if*

- (i) $\sigma^2 \in C^2(E), \sigma^2 \geq 0$ and $\sigma^{2''}$ is bounded,
- (ii) $B : E \rightarrow \mathbb{R}$ is Lipschitz continuous and
- (iii) $\sigma^2(r_i) = 0 \leq (-1)^i B(r_i)$ for $i \in \{0, 1\}$.

Proof. See [12], Theorem 2.1, Chapter 8.

The next result shows that the semigroups of the birth and death processes $(T_N(t))_{t \geq 0}$ converge to the semigroup T of the diffusion process.

Define $\eta_N : E_N \rightarrow E$ with $\eta_N(x) = x$ and $\pi_N : B(E) \rightarrow B(E_N)$ with $\pi_N(f) = f \circ \eta_N$.

Theorem 2.24. *Let $(T_N(t))_{t \geq 0}, N \in \mathbb{N}$ and $(T(t))_{t \geq 0}$ be strongly continuous contraction semigroups on $B(E_N)$ and $C(E)$ with infinitesimal generators $A_N, N \in \mathbb{N}$, and A . Let D be a core for A . Then the following are equivalent:*

1. For each $f \in C(E)$

$$T_N(t)\pi_N f \rightarrow T(t)f$$

for all $t \geq 0$, uniformly on bounded intervals.

2. For each $f \in C(E)$ and all $t \geq 0$

$$T_N(t)\pi_N f \rightarrow T(t)f.$$

3. For each $f \in D$, there exists $f_N \in \mathfrak{D}(A_N)$ for each $N \geq 1$ such that $f_N \rightarrow f$ and $A_N f_N \rightarrow Af$.

Proof. See [12], Theorem 6.1, Chapter 1.

Theorem 2.25. *Suppose that $(T(t))_{t \geq 0}$ is a Feller semigroup on $C(E)$ and that for each $t \geq 0$ and $f \in C(E)$*

$$T_N(t)\pi_N f \rightarrow T(t)f.$$

If $X_N(0)$ has limiting distribution ν , then there is a Markov process X corresponding to $(T(t))_{t \geq 0}$ with initial distribution ν and sample paths in $D_E[0, \infty)$, and

$$X_N \rightarrow X \text{ weakly in } D_E[0, \infty).$$

Proof. See [12], Theorem 2.11, Chapter 4.

Conclusion.

By Theorem 2.20 and from Assumption 2.19 it follows that the infinitesimal generators A_N of our birth and death processes converge for $f \in C^\infty(E)$ to the operator

$$Af(x) = B(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

By Theorem 2.23 the operator A generates a Feller semigroup. Furthermore it follows from Theorem 2.23 that $C^\infty(E)$ is a core for this process. So we obtain from Theorems 2.24 and 2.25, that there exists a process X on $E = [-1, 1]$ with infinitesimal generator A so that

$$X^N = \frac{2Z^N}{N} - 1 \rightarrow X \text{ weakly in } D_E[0, \infty).$$

2.3.3 Convergence via the Representation 2.2.2

For a concise discussion we let N denote the size of a population and $S_N = \{0, 1, \dots, N\}$. Z_t^N describes the random number of a certain species within the population. The standardised version is given by the proportion of this species

$$X_t^N = \frac{2Z_t^N}{N} - 1$$

taking values in $\{-1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1\}$. Now assume that the transition rates for $(Z_t^N)_{t \geq 0}$ fulfil

$$\gamma_l^N(i) = q_N(i, i+l) = N\beta_l \left(\frac{2i}{N} - 1 \right)$$

for some bounded function β on $[-1, 1]$. From Subsection 2.2.1 we know that the time between two transitions of the process is exponentially distributed with parameter $q_i^N = \sum_{j \neq i} q_N(i, j)$. Due to the fact that all $q_N(i, j)$ are of order N , q_i^N is also of order N for all $i \in S_N$. Then the mean time between two transitions is of order $\frac{1}{N}$. This implies that the mean number of transitions in one unit of time is of the order N , equal in order to the size of the population. So we expect a law of large numbers to hold for X_t^N , hence a non-random limiting process $(X_t)_{t \geq 0}$, and similarly a central limit theorem. This may be seen with the following arguments, and we refer to [12], 11.2 for a rigorous and complete discussion.

A law of large numbers

Let $\tilde{Y}_t = Y_t - t$ be a centered Poisson process. The law of large numbers for this process states that for all t

$$\sup_{s \leq t} \left| \frac{1}{N} \tilde{Y}_{Ns} \right| \rightarrow 0 \text{ almost surely for } N \rightarrow \infty.$$

Using the representation from 2.2.2 for $(Z_t^N)_{t \geq 0}$ we have, with $f(x) = \sum_l l \beta_l(x)$,

$$X_t^N = \frac{2Z_t^N}{N} - 1 = X_0^N + \sum_l l \frac{1}{N} \tilde{Y}^l \left(N \int_0^t \beta_l(X_s^N) ds \right) + \int_0^t f(X_s^N) ds.$$

So our random clock has a speed of order N . Assuming $X_0^N \rightarrow x_0$ and Lipschitz continuity of f , the law of large numbers for the centered Poisson process implies for all $t \geq 0$

$$\sup_{s \leq t} |X_s^N - X_s| \rightarrow 0 \text{ almost surely for } N \rightarrow \infty,$$

where $(X_t)_{t \geq 0}$ is the non-random solution of the differential equation

$$\frac{d}{dt} X_t = f(X_t), X_0 = x_0.$$

A central limit theorem

The basic central limit theorem states that, for a sum of i.i.d. random variables X_i with finite mean μ and finite positive variance, the normalised sum $\frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - \mu)$ converges to a normal distribution. Writing

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - \mu) = \sqrt{N} \left(\frac{1}{N} \sum_{i=1}^N X_i - \mu \right)$$

we see that the term $\frac{1}{N} \sum_{i=1}^N X_i - \mu$, which tends to zero according to the law of large numbers, has to be enlarged by a factor \sqrt{N} to obtain a non-degenerate asymptotic normal distribution. This is a wide-spread phenomenon, so in our setting it is readily conjectured that the stochastic process $\left(\sqrt{N}(X_t^N - X_t) \right)_{t \geq 0}$ converges to a non-degenerate process which is Gaussian. To see this we use the central limit theorem for the centered Poisson process. This states that for the process $(W_t^N)_{t \geq 0}$ with $W_t^N = \frac{1}{\sqrt{N}}(Y_{Nt} - Nt)$

$$(W_t^N)_{t \geq 0} \text{ converges to a standard Wiener process } (W_t)_{t \geq 0}.$$

Set

$$\begin{aligned} V_t^N &= \sqrt{N}(X_t^N - X_t) = V_0^N + \sum_l l W^{l,N} \left(\int_0^t \beta_l(X_s^N) ds \right) + \int_0^t \sqrt{N}(f(X_s^N) - f(X_s)) ds \\ &= V_0^N + \sum_l l W^{l,N} \left(\int_0^t \beta_l(X_s^N) ds \right) + \int_0^t \sqrt{N} \left(f\left(X_s + \frac{1}{\sqrt{N}} V_s^N\right) - f(X_s) \right) ds. \end{aligned}$$

Letting N tend to infinity the limiting equation looks formally, with $V_0^N \rightarrow v_0$,

$$V_t = v_0 + U_t + \int_0^t f'(X_s)V_s ds$$

where $U_t = \sum_l lW^l(\int_0^t \beta_l(X_s)ds)$ is a Gaussian process. It can be proved rigorously that in fact

$$(\sqrt{N}(X_t^N - X_t))_{t \geq 0} \text{ converges to a process } (V_t)_{t \geq 0}$$

with $(V_t)_{t \geq 0}$ a solution of the limiting equation. Since $(U_t)_{t \geq 0}$ is a Gaussian process, $(V_t)_{t \geq 0}$ also is a Gaussian process. Gaussian processes are uniquely determined by their mean and covariance function. From the limiting equation one can obtain the following expressions: Let $g(t, s)$ be a solution of

$$\frac{d}{dt}g(t, s) = f'(X_t)g(t, s), \quad g(s, s) = 1.$$

Then the mean of V_t is given by $g(t, 0)v_0$ and the covariance is given by

$$Cov(V_t, V_r) = \int_0^{\min\{t, r\}} g(t, s)g(r, s) \sum_l l^2 \beta_l(X_s) ds. \quad (2.4)$$

Here it is important to recall that the process X is non-random, hence the covariation is also non-random.

So we obtain that the process

$$(X_t^N)_{t \geq 0} \text{ is approximated by } \left(X_t + \frac{1}{\sqrt{N}} V_t \right)_{t \geq 0},$$

hence

$$(Z_t^N)_{t \geq 0} \text{ is approximated by } \left(\frac{N}{2} X_t + \frac{\sqrt{N}}{2} V_t + \frac{N}{2} \right)_{t \geq 0}.$$

The distribution of Z_t^N is approximated by a normal distribution with mean

$$\frac{N}{2}(x_0 + 1) + \frac{\sqrt{N}}{2}v_0$$

and variance

$$\frac{N}{4}Var(V_t).$$

2.3.4 Convergence of semimartingals

In this paragraph we show the main theorem for the convergence of step Markov processes to a diffusion process. We look at a family of pure jump Markov processes $(X^N)_{N \in \mathbb{N}}$, so that the infinitesimal generators have the form

$$A^N f(x) = \int [f(x+y) - f(x)] K^N(x, dy). \quad (2.5)$$

The initial distributions are denoted by η_N .

Definition 2.26 (martingale problem). Let X be a cadlag adapted process on a measurable space (Ω, \mathfrak{A}) with a filtration $(\mathfrak{F}_t)_{t \geq 0}$. Let h be a truncation function and (B, C, ν) a triple as in Definition 2.12. Let \mathfrak{H} be a initial σ -field, and P_H an initial probability measure on (Ω, \mathfrak{H}) .

Then a solution to the martingale problem associated with (\mathfrak{H}, X) and $(P_H; B, C, \nu)$ is a probability measure P on (Ω, \mathfrak{F}) such that:

1. The restriction $P|_{\mathfrak{H}}$ of P to \mathfrak{H} equals P_H ;
2. X is an semimartingale on the basis $(\Omega, \mathfrak{A}, (\mathfrak{F}_t)_t, P)$, with characteristics (B, C, ν) relative to the truncation function h .

We denote by $s(\mathfrak{H}, X | P_H; B, C, \nu)$ the set of all solutions P .

With this definition we are able to show the main condition for this convergence. This condition ensures that a unique limiting process exists.

Condition 2.27 (uniqueness-measurability hypothesis).

1. For each $x \in [-1, 1]$ the martingale problem $s(\sigma(X_0), X | \delta_x; B, C, \nu)$ has a unique solution P_x ,
2. $x \mapsto P_x(A)$ is Borel for all $A \in \mathfrak{A}$.

Now we need some definitions before we can formulate the main theorem.

Definition 2.28. Define

1. $b_N(x) = \int y K^N(x, dy)$,
2. $c_N(x) = \int y^2 K^N(x, dy)$,

with K^N from (2.5).

Theorem 2.29. *Assume that 2.27 holds, with $\nu = 0$ (so the limiting process will be a continuous diffusion process), and that b and c are continuous functions on \mathbb{R} . Assume also that,*

1. $b_N \rightarrow b$ and $c_N \rightarrow c$ locally uniformly;
2. $\sup_{x: |x| < \alpha} \int y^2 1_{\{|y| > \varepsilon\}} K_N(x, dy) \rightarrow 0$ as $n \rightarrow \infty$, for all $\varepsilon > 0$,
3. $\eta_N \rightarrow \eta$ weakly on \mathbb{R} .

Then the laws $\mathfrak{L}(X_N)$ weakly converge in $D_{\mathbb{R}}[0, \infty)$ to $P = \int \eta(dx) P_x$, the law of the diffusion process with coefficients b and c and initial distribution η .

Proof. See [15], Theorem 4.21, Chapter IX.

We will use Theorem 2.29 in Subsection 3.3.2 to show the convergence of a birth and death process to a diffusion process.

Chapter 3

Agent-based models and their asymptotic behaviour

In this chapter we look at the definition and the asymptotic properties of ant models. Ant models consist of two components. The first component is a dynamic model for the number of agents in the two groups. This component is called herding process. Here we look at models with a fixed number of agents so it is enough to give a model for the agents of the first group, e.g. the optimistic agents. The second component is a pricing function that translates the herding process into a price for the asset traded on the market. Mathematically the first part is the more challenging part of such models, therefore further work is concentrated on this one. We give a short overview of the second part in the next section, but refer for details to the articles [1], [2], [3], [5]. After that we treat the model of Kirman introduced in [19]. Due to the fact that this model shows N-dependence, so that the asymptotic properties for a large number of agents are not corresponding to the stylized facts. We then look at the model of Alfarano et al. from [5] that overcomes this problem. In Section 3.2 we give a mathematical model for the process of optimistic agents which is described as a birth and death process. In Section 3.3 we use the methods introduced in Section 2.3 to take a mathematical look at the asymptotic behaviour of the processes if the number of agents on the market tends to infinity. Finally we calculate some properties of the limiting process of the second model in Section 3.4.

3.1 Pricing functions in agent-based models

Before we start modelling the herding process and analysing its properties, we short introduce the pricing functions in agent-based models for financial markets. The pricing function links a given herding process to a price on the market. So we assume for this section that the herding process is given by a stochastic process $(X_t)_{t \geq 0}$ on $[-1, 1]$.

Here we introduce the pricing model mentioned in [5]. We assume that there are two different types of agents. The first type is called fundamentalists (F). They buy or sell assets if the current price p is below or above a fundamental price p^F . For a simpler calculation it is assumed in [5] that the fundamental price is constant over time. But for a

more realistic price p we may assume that the logarithm of the fundamental price follows a Brownian motion W that is independent of the herding process. The second type of agents is called noise traders (C). They are driven by a herding instinct. These agents can be either optimists (O) or pessimists (P), depending on their expectation of the future prices.

In the model in [5] Alfarano et al. assume that the number of fundamentalists N_F and the number of noise traders N_C is constant over time. Further for any time point $t \in [0, \infty)$ the number of optimistic agents N_O is given by

$$N_{O_t} = N_C \left(\frac{X_t + 1}{2} \right).$$

Due to the fact that the number of noise traders N_C is constant, the number of pessimistic agents N_P at time t is $N_{P_t} = N_C - N_{O_t}$.

With the assumption that the trading volume of the fundamentalists is the same as the volume of the noise traders, and under market clearing, Alfarano et al. showed in [5] that the price p is approximated given by

$$p_t = p^F e^{X_t}.$$

3.2 Mathematical definition of herding processes

In agent-based models we have a population of N agents interacting on a financial market. These agents are of two different types, e.g. optimists and pessimists. The number of agents of one type, e.g. optimists, is described by a continuous-time Markov chain $(Z_t^N)_{t \geq 0}$ with state space $\{0, 1, \dots, N\}$. Agents may switch from one type to the other and it is the usual assumption that $(Z_t^N)_{t \geq 0}$ follows a birth and death process where

$$q_N(i, i + l) = 0 \text{ for } |l| > 1.$$

Here, a birth means the conversion of a pessimist to an optimist, a death the opposite conversion. The birth rate, and death rate respectively, are

$$\lambda_N(i) = q_N(i, i + 1) \text{ for } i = 0, 1, \dots, N - 1, \quad \mu_N(i) = q_N(i, i - 1) \text{ for } i = 1, \dots, N,$$

with $\lambda_N(N) = \mu_N(0) = 0$. For a birth and death process model the probability of multiple switches in a time interval of length t is of the order $o(t)$ as t tends to zero.

In the two models mentioned in [5], this swing of opinion can happen due to two factors. The first is that the agent changes his opinion without influence of his peers. The second is a change due to group pressure. For this it is important to assume that the agents are connected in a network and communicate with each other. For simplicity we work with a fully connected network, that means that any two agents are connected. We refer to [6] for a detailed discussion of possible underlying network structures.

The two models from [5], let us call them model 1 and model 2, are characterised by the following birth and death rates. Here model 1 is the original Kirman model introduced in [19], with the problem of N -dependence. Model 2 is a slightly modified Kirman model introduced by Alfarano et al. in [5]. Later we will see that this model is able to overcome the problem of N -dependence.

Model 1 looks at the birth and death rates

$$\lambda_N(i) = (N - i)\left(a + b\frac{i}{N}\right), \quad \mu_N(i) = i\left(a + b\frac{N - i}{N}\right),$$

Model 2 looks at the birth and death rates

$$\lambda_N(i) = (N - i)(a + bi), \quad \mu_N(i) = i(a + b(N - i))$$

where a describes the overall tendency to switch, b the tendency due to group pressure. In model 1 $\lambda_N(i)$ is the rate for the switching of a pessimistic agent gets optimistic if we have i optimistic agents at the moment. This means that we have $N - i$ pessimistic agents who can switch, each of them can switch with rate a without influence by the other agents. Also he chooses randomly another agent to meet. With probability $\frac{i}{N}$ this is an optimistic agent and then the agent switches with rate b due to this meeting. Together we get $\lambda_N(i) = (N - i)\left(a + b\frac{i}{N}\right)$. In the same way we can interpret $\mu_N(i)$. The change in Model 2 is that the agents not only choose one agent to meet. They are always influenced by all other agents of the other group. It looks like only a small change but later we will see that this small change is very important for the asymptotic behaviour of the models.

3.3 Asymptotic behaviour of the herding processes

In this section we look at the asymptotic behaviour of the two models. We are interested in the behaviour of the models for a large number of agents. To show the convergence we use the methods mentioned in Chapter 2.

3.3.1 Analysis of model 1

We slightly generalise the model to $\lambda_N(i) = (N - i)\left(a_1 + b_1\frac{i}{N}\right)$, $\mu_N(i) = i\left(a_2 + b_2\frac{N - i}{N}\right)$ and have with $y = \frac{i}{N}$

$$\begin{aligned} q_N(i) &= \lambda_N(i) + \mu_N(i) = N \left[(1 - y)(a_1 + b_1 y) + y(a_2 + b_2(1 - y)) \right] \\ p_N(i, i + 1) &= \frac{\lambda_N(i)}{\lambda_N(i) + \mu_N(i)} = \frac{1}{1 + [(1 - y)(a_1 + b_1 y)]^{-1} y(a_2 + b_2(1 - y))}, \\ p_N(i, i - 1) &= 1 - p_N(i, i + 1). \end{aligned}$$

So we see that the expected number of switches which may be loosely interpreted as being proportional to the expected number of encounters of agents in one unit of time is of the order N . The probabilities $p_N(i, i + 1)$ and $p_N(i, i - 1)$ which describe into which direction switching occurs only depend on the fraction of agents of one type in the population.

Convergence via method in 2.3.3

The limiting behaviour of the normalised process $X_t^N = \frac{2Z_t^N}{N} - 1$ follows readily from 2.3.3. Using the notation from this section we have

$$\gamma_l^N(i) = q_N(i, i+l) = N\beta_l \left(\frac{2i}{N} - 1 \right), \quad l = -1, 1,$$

with $\beta_1(x) = \frac{1}{4}(1-x)(2a_1 + b_1(1+x))$, $\beta_{-1}(x) = \frac{1}{4}(1+x)(2a_2 + b_2(1-x))$ defined for $x \in [-1, 1]$. From this it is clear that 2.3.3 applies. We have

$$f(x) = \beta_1(x) - \beta_{-1}(x) = \frac{1}{4}((1-x)(2a_1 + b_1(1+x)) - (1+x)(2a_2 + b_2(1-x))).$$

The limiting process is the non-random solution of the differential equation, which is an inhomogeneous Bernoulli-differential equation

$$\frac{d}{dt}X_t = f(X_t), \quad X_0 = x_0.$$

As we have seen in 2.3.3, the standardised differences $\sqrt{N}(X_t^N - X_t)$ tend to a Gaussian process V_t . Assume $V_0 = 0$, the mean value function is zero. We have $f'(x) = \frac{1}{2}((b_2 - b_1)x - (a_1 + a_2))$, and the covariance function may be computed, at least numerically, from the linear differential equation

$$\frac{d}{dt}g(t, s) = f'(X_t)g(t, s), \quad g(s, s) = 1,$$

and the resulting expression in 2.3.3. Let us now make the following standard assumption $b_1 = b_2 = b$. Then

$$f(x) = \frac{1}{2}(a_1 + a_2) \left(\frac{a_1 - a_2}{a_1 + a_2} - x \right),$$

and it follows

$$X_t = \frac{a_1 - a_2}{a_1 + a_2} - \left(x_0 - \frac{a_1 - a_2}{a_1 + a_2} \right) e^{-\frac{1}{2}(a_1 + a_2)t}.$$

This shows that X_t converges to $\frac{a_1 - a_2}{a_1 + a_2}$ as $t \rightarrow \infty$.

We have $f'(x) = -\frac{1}{2}(a_1 + a_2)$ and a solution of the differential equation above is given by

$$g(t, s) = e^{-\frac{1}{2}(a_1 + a_2)(t-s)}.$$

From (2.4) the covariance of the Gaussian process is given by

$$\begin{aligned}
 & Cov(V_t, V_r) \\
 &= \int_0^{\min\{t,r\}} g(t,s)g(r,s) \sum_l l^2 \beta_l(X_s) ds \\
 &= \int_0^{\min\{t,r\}} e^{-\frac{1}{2}(a_1+a_2)(t+r-2s)} \left(\frac{1}{2} [(a_1+a_2) - (a_1-a_2)X_s + b(1-X_s^2)] \right) ds \\
 &= \frac{1}{2} e^{-\frac{1}{2}(a_1+a_2)(t+r)} \int_0^{\min\{t,r\}} e^{(a_1+a_2)s} (a_1+a_2) - (a_1-a_2)X_s + b(1-X_s^2) ds \\
 &= \frac{1}{2} e^{-\frac{1}{2}(a_1+a_2)(t+r)} \left[\int_0^{\min\{t,r\}} \left(x_0 - \frac{a_1-a_2}{a_1+a_2} \right)^2 ds \right. \\
 &\quad - \int_0^{\min\{t,r\}} e^{\frac{1}{2}(a_1+a_2)s} \left[\left((a_1-a_2) + 2\frac{a_1-a_2}{a_1+a_2} \right) \left(x_0 - \frac{a_1-a_2}{a_1+a_2} \right) \right] ds \\
 &\quad \left. + \int_0^{\min\{t,r\}} e^{(a_1+a_2)s} \left[(a_1+a_2) - \frac{(a_1-a_2)^2}{a_1+a_2} + b \left(1 - \left(\frac{a_1-a_2}{a_1+a_2} \right)^2 \right) \right] ds \right] \\
 &= \frac{1}{2} e^{-\frac{1}{2}(a_1+a_2)(t+r)} \left[\left(x_0 - \frac{a_1-a_2}{a_1+a_2} \right)^2 \min\{t,r\} \right. \\
 &\quad - 2 \left(\frac{a_1-a_2}{a_1+a_2} + 2\frac{a_1-a_2}{(a_1+a_2)^2} \right) \left(x_0 - \frac{a_1-a_2}{a_1+a_2} \right) \left[e^{\frac{1}{2}(a_1+a_2)\min\{t,r\}} - 1 \right] \\
 &\quad \left. + \left(1 - \left(\frac{a_1-a_2}{a_1+a_2} \right)^2 + \frac{b}{a_1+a_2} \left(1 - \left(\frac{a_1-a_2}{a_1+a_2} \right)^2 \right) \right) \left[e^{(a_1+a_2)\min\{t,r\}} - 1 \right] \right]
 \end{aligned}$$

Convergence via infinitesimal operators

For using the method from 2.3.2 we have to verify the conditions from Assumption 2.19, namely

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \sup_{x \in E_N} \left| \frac{2}{N} (\lambda_N(x) - \mu_N(x)) - B(x) \right| &= 0, \\
 \lim_{N \rightarrow \infty} \sup_{x \in E_N} \left| \frac{4}{N^2} (\lambda_N(x) + \mu_N(x)) - \sigma^2(x) \right| &= 0,
 \end{aligned}$$

and $\sigma^2 \in C^2(E)$, $\sigma^{2''}$ is bounded, $B : E \rightarrow \mathbb{R}$ is Lipschitz continuous and $\sigma^2(r_i) = 0 \leq (-1)^i B(r_i)$ for $i \in \{0, 1\}$, where $r_0 = -1$ and $r_1 = 1$. The standardised processes $X_t^N = \frac{2Z_t^N}{N} - 1$ of the slightly generalised model with $b := b_1 = b_2$ are birth and

death processes on the sets E_N with birth and death rates

$$\begin{aligned}\lambda_N(x) &= \frac{N}{4}(1-x)(2a_1 + b(1+x)), \\ \mu_N(x) &= \frac{N}{4}(1+x)(2a_2 + b(1-x)).\end{aligned}$$

Easy calculations show

$$\begin{aligned}\lambda_N(x) - \mu_N(x) &= \frac{N}{2}((a_1 - a_2) - (a_1 + a_2)x) \text{ and} \\ \lambda_N(x) + \mu_N(x) &= \frac{N}{2}((a_1 + a_2) - (a_1 - a_2)x + b(1 - x^2)).\end{aligned}$$

Hence Assumption 2.19 is fulfilled with

$$B(x) = (a_1 - a_2) - (a_1 + a_2)x \text{ and } \sigma^2(x) = 0.$$

Using the methods of 2.3.2 we know that the birth and death processes X^N converge to a limiting process X with infinitesimal generator

$$Af(x)(x) = ((a_1 - a_2) - (a_1 + a_2)x) f'(x).$$

From this it follows that the limiting process is the non-random process

$$X_t = \frac{a_1 - a_2}{a_1 + a_2} - \left(X_0 - \frac{a_1 - a_2}{a_1 + a_2} \right) e^{-(a_1 + a_2)t}.$$

Conclusion

The two methods have shown that in case of model 1 the herding process converges to a nonrandom process, and that it can be approximated by a Gaussian process if the number of agents is large. From these facts, the model 1 does not seem to be appropriate for modelling markets with a large number of agents as it does not cover stylised facts as heavy tailed returns and volatility clustering. To overcome this problem of the so-called N -dependence Alfarano et al. introduced the model 2 in [5].

3.3.2 Analysis of model 2

The standardised processes $X_t^N = \frac{2Z_t^N}{N} - 1$ of the slightly generalised model 2 are birth and death processes on the sets E_N with birth and death rates

$$\begin{aligned}\lambda_N(x) &= \frac{N^2}{4}(1-x) \left(\frac{2a_1}{N} + b(1+x) \right), \\ \mu_N(x) &= \frac{N^2}{4}(1+x) \left(\frac{2a_2}{N} + b(1-x) \right).\end{aligned}$$

Furthermore we have

$$q_N(x) = \lambda_N(x) + \mu_N(x) = \frac{N^2}{2} \left(\frac{(a_1 + a_2) - (a_1 - a_2)x}{N} + \frac{1}{2}2b(1 - x^2) \right).$$

There are two changes with regard to model 1. The expected number of switches in one time period has increased to the order of N^2 , and the overall tendency of switching has decreased to the order of $\frac{1}{N}$. This, of course, is a dramatic change, as e.g. for $N = 100$, the number of switches in one unit of time is no longer of the order 100 but of the order 10.000. Due to the first change it is clear that a law of large numbers as stated in 3.3.1 for model 1 can no longer hold. To analyse the asymptotic behaviour of the system we use the method of infinitesimal operators as described in 2.3.2 and the method of semimartingale characteristics from 2.3.4.

Convergence via infinitesimal operators

Due to the fact that

$$\begin{aligned} \lambda_N(x) - \mu_N(x) &= \frac{N}{2} ((a_1 - a_2) - (a_1 + a_2)x) \text{ and} \\ \lambda_N(x) + \mu_N(x) &= \frac{N}{2} ((a_1 + a_2) - (a_1 - a_2)x) + \frac{N^2}{4} (2b(1 - x^2)) \end{aligned}$$

Assumption 2.19 is fulfilled with

$$\begin{aligned} B(x) &= (a_1 - a_2) - (a_1 + a_2)x \text{ and} \\ \sigma^2(x) &= 2b(1 - x^2). \end{aligned}$$

Therefore the limiting process X is a diffusion process with infinitesimal generator

$$Af(x) = ((a_1 - a_2) - (a_1 + a_2)x) f'(x) + b(1 - x^2) f''(x).$$

As explained in 3.4.1 this is the infinitesimal generator of a diffusion process with drift $(a_1 - a_2) - (a_1 + a_2)x$ and diffusion parameter $2b(1 - x)^2$.

We can also interpret these birth and death processes as semimartingales as defined in Subsection 2.2.3. If we do so, we can use the convergence method for semimartingales as shown in Subsection 2.3.4.

Convergence using the semimartingale characteristics

In this paragraph, we use Theorem 2.29 to show that the birth and death processes converge to a diffusion process. The infinitesimal generator of the birth and death process for a market with N agents is given by

$$A_N f(x) = \lambda_N(x) \left[f\left(x + \frac{2}{N}\right) - f(x) \right] + \mu_N(x) \left[f\left(x - \frac{2}{N}\right) - f(x) \right]. \quad (3.1)$$

So the transition measures K^N are given by

$$\begin{aligned} K^N(x, \frac{2}{N}) &= \lambda_N(x) = \frac{N^2}{4}(1-x) \left(\frac{2a_1}{N} + b(1+x) \right), \\ K^N(x, -\frac{2}{N}) &= \mu_N(x) = \frac{N^2}{4}(1+x) \left(\frac{2a_2}{N} + b(1-x) \right), \\ K^N(x, y) &= 0 \text{ for all } y \notin \left\{ \frac{2}{N}, -\frac{2}{N} \right\}. \end{aligned}$$

So we get b_N, c_N defined in 2.28 by

$$b_N(x) = \frac{2}{N}\lambda_N(x) - \frac{2}{N}\mu_N(x) = (a_1 - a_2) - (a_1 + a_2)x \quad (3.2)$$

and

$$c_N(x) = \frac{4}{N^2}\lambda_N(x) + \frac{4}{N^2}\mu_N(x) = \frac{2}{N}(a_1 + a_2 - (a_1 - a_2)x) + 2b(1 - x^2). \quad (3.3)$$

From this it follows directly that b_N converges uniformly to

$$B(x) = (a_1 - a_2) - (a_1 + a_2)x$$

and c_N converges uniformly to

$$c(x) = 2b(1 - x^2).$$

In the paragraph before we have shown that there exists a diffusion with drift parameter b and diffusion parameter c . From this we know that Condition 2.27 holds. Furthermore

$$\sup_{x:|x|<\alpha} \int K_N(x, dy) y^2 1_{\{|y|>\varepsilon\}} = 0 \text{ for all } N \in \mathbb{N} \text{ with } N > \frac{2}{\varepsilon}.$$

So it follows from Theorem 2.29 that the birth and death processes converge weakly in $D_{[-1,1]}[0, \infty)$ to a diffusion process with drift coefficient $B(x) = (a_1 - a_2) - (a_1 + a_2)x$ and diffusion coefficient $\sigma^2(x) = 2b(1 - x^2)$.

3.3.3 Conclusion

In this section we used the methods from Chapter 2 to show the convergence of the birth and death processes for our two models. At first glance the two models are looking rather similar. But if we analyse these models, we see that for model 2 the number of switches in one time period increases from the order of N in model 1 to the order of N^2 . In the agent models we can interpret this as an increase in the communication speed because each agent has no longer only one meeting in one time period. In model 2 each agent communicates with N other agents. As we have seen in Subsection 3.3.2 this increase in communication speed helps us to overcome the problem of N -dependence. Due to the fact that model 2 does not have the problem of N -dependence it is possible to use it for

markets with a large number of agents. In Subsection 3.3.2 we have seen that the birth and death processes of model 2 converge to a limiting diffusion process if the number of agents tends to infinity. This helps us because it is usually easier to analyse a diffusion process than a birth and death process with a large number of agents. Hence we analyse the limiting diffusion process in the next section as a good approximation for model 2 if the number of agents is large.

3.4 Characteristics of the limiting process from model 2

From the form of the infinitesimal generator

$$Af(x) = ((a_1 - a_2) - (a_1 + a_2)x)f'(x) + b(1 - x^2)f''(x),$$

we know that the limiting process X is a diffusion process on $[-1, 1]$. In this subsection we calculate the diffusion and drift parameter of the limiting process and derive the boundary behaviour of this process. Further we calculate the stationary distribution of this process. More characteristics of this process are mentioned in [5]

3.4.1 Diffusion and drift parameter

Definition 3.1. The drift parameter B is given by

$$B(x) = \lim_{t \downarrow 0} \frac{E[X_t - X_0 | X_0 = x]}{t}$$

and the diffusion parameter σ^2 is given by

$$\sigma^2(x) = \lim_{t \downarrow 0} \frac{E[(X_t - X_0)^2 | X_0 = x]}{t}.$$

With this definition we can calculate the parameters of the limiting diffusion process.

Proposition 3.2. *The parameters of the limiting diffusion process X are given by*

$$B(x) = (a_1 - a_2) - (a_1 + a_2)x \text{ and } \sigma^2(x) = 2b(1 - x)^2 \text{ for all } x \in [-1, 1].$$

Proof. We calculate the drift parameter from the infinitesimal generator A with the function $f_1(x) = x$. With this function we get

$$B(x) = Af_1(x) = ((a_1 - a_2) - (a_1 + a_2)x) \cdot 1 + b(1 - x^2) \cdot 0 = (a_1 - a_2) - (a_1 + a_2)x.$$

For the calculation of the diffusion parameter we use the functions f_1 and $f_2(x) = x^2$. Then we get

$$\begin{aligned} \sigma^2(x) &= Af_2(x) - 2xAf_1(x) \\ &= ((a_1 - a_2) - (a_1 + a_2)x)(2x) + b(1 - x^2)2 - 2x((a_1 - a_2) - (a_1 + a_2)x) \\ &= 2b(1 - x^2). \end{aligned}$$

□

3.4.2 Boundary behaviour of the limiting process

We now provide a discussion of the boundary behaviour of the limiting diffusion and refer to [16] and [17] for a detailed discussion of the basic results which are used here. For ease of exposition we only look at the case $a := a_1 = a_2$. The general case can be treated in the same way.

The boundary behaviour of a diffusion process is characterised by the following two functions.

Definition 3.3 (scale and speed measure). The scale function S is defined by

$$S(x) = \int_0^x s(y)dy \text{ with } s(y) = \exp \left\{ - \int_0^y \frac{2B(z)}{\sigma^2(z)} dz \right\}.$$

From the scale function we obtain the scale measure by

$$S[c, d] = S(d) - S(c).$$

The speed measure M is defined by

$$M[c, d] = \int_c^d m(x)dx$$

with $m(x) = \frac{1}{\sigma^2(x)s(x)}$.

For the characterisation of the boundaries we define the following functions:

1. $u(x) = \int_0^x M[0, y]dS(y)$,
2. $v(x) = \int_0^x S[0, y]dM(y)$.

With these functions a boundary point $r \in \{-1, 1\}$ is characterised as

1. regular, if $u(r) < \infty$ and $v(r) < \infty$,
2. entrance, if $u(r) = \infty$ and $v(r) < \infty$,
3. exit, if $u(r) < \infty$ and $v(r) = \infty$,
4. natural, if $u(r) = \infty$ and $v(r) = \infty$.

For the process this means the following: A process can both enter and leave a regular boundary. The behaviour of the process in the boundary must be specified additionally, this can be done by assigning a speed $M[\{r\}]$ to a boundary r . If this speed is zero, the boundary is a reflection boundary. If $0 < M[\{r\}] < \infty$, the boundary is a sticky boundary, this means that the process spends a positive time in the boundary point before leaving it. If the speed is infinity, the boundary is an absorption boundary. Due to the fact that our birth and death processes are reflected at the boundaries we set $M[\{r\}] = 0$ for $r \in \{-1, 1\}$. The process cannot reach an entrance boundary from the interior but it is

possible to begin the process in an entrance boundary. An exit boundary can be reached from the interior, but it is not possible to leave it. A process can neither reach nor be started from a natural boundary. For more detailed information we refer to book [17].

We now characterise the behaviour of our limiting diffusion process X at the boundaries $r_0 = -1$ and $r_1 = 1$. We will see how this behaviour is influenced by the parameters a, b of our model. Let $\gamma = \frac{a}{b} - 1$ and $\delta = -\frac{a}{b}$. Then

1. $u(x) = \frac{1}{b} \int_0^x \int_0^y (1 - z^2)^\gamma dz (1 - y^2)^\delta dy$,
2. $v(x) = \frac{1}{b} \int_0^x \int_0^y (1 - z^2)^\delta dz (1 - y^2)^\gamma dy$.

Based on the symmetry of the functions u, v at the boundaries, we only look at the behaviour at the boundary $r_1 = 1$. We have several cases for different behaviour of the parameters a and b .

The case $a < b$

In this case, $-1 < \gamma, \delta < 0$. It follows

$$\begin{aligned}
 u(1) &= \frac{1}{b} \int_0^1 \int_0^y (1 - z)^\gamma (1 + z)^\gamma dz (1 - y)^\delta (1 + y)^\delta dy \\
 &\leq \frac{1}{b} \int_0^1 \int_0^y (1 - z)^\gamma dz (1 - y)^\delta dy \\
 &= \frac{1}{b(\gamma + 1)} \int_0^1 (1 - y)^\delta - (1 - y)^{\gamma+1+\delta} dy \\
 &= \frac{1}{b(\gamma + 1)} \int_0^1 (1 - y)^\delta - 1 dy \\
 &= \frac{1}{b(\gamma + 1)} \left(\frac{1}{b + 1} - 1 \right) < \infty.
 \end{aligned}$$

In the same way we show $v(1) < \infty$. Hence our boundaries are regular.

The case $a = b$

Then $\gamma = 0$ and $\delta = -1$. From this

$$\begin{aligned}
 u(1) &= \frac{1}{b} \int_0^1 \int_0^y (1 - z^2)^\gamma dz (1 - y^2)^\delta dy \\
 &= \frac{1}{b} \int_0^1 y(1 - y^2)^{-1} dy \\
 &\geq c \int_{\frac{1}{2}}^1 (1 - y)^{-1} dy = \infty.
 \end{aligned}$$

Furthermore

$$\begin{aligned}
 v(1) &= \frac{1}{b} \int_0^1 \int_0^y (1 - z^2)^{-1} dz dy \\
 &= \frac{1}{b} \int_0^1 \operatorname{arctanh}(y) dy \\
 &= \frac{1}{b} \ln(2) < \infty.
 \end{aligned}$$

Therefore the boundaries are entrance boundaries.

The case $a > b$

We have $\gamma > 0$ and $\delta < -1$, hence

$$\begin{aligned}
 u(1) &= \frac{1}{b} \int_0^1 \int_0^y (1 - z^2)^\gamma dz (1 - y^2)^\delta dy \\
 &\geq \frac{1}{b} \int_0^1 y (1 - y^2)^\gamma (1 - y^2)^\delta dy \\
 &= \frac{1}{b} \int_0^1 y (1 - y^2)^{-1} dy = \infty.
 \end{aligned}$$

For v we obtain

$$\begin{aligned}
 v(1) &= \frac{1}{b} \int_0^1 \int_0^y (1 - z)^\delta (1 + z)^\delta dz (1 - y)^\gamma (1 + y)^\gamma dy \\
 &\leq \frac{2^\gamma}{b(\delta + 1)} \int_0^1 (1 - y)^\gamma - (1 - y)^{\delta+1+\gamma} dy \\
 &= \frac{2^\gamma}{b(\delta + 1)} \left(\frac{1}{\gamma + 1} - 1 \right) < \infty.
 \end{aligned}$$

Hence our boundaries are entrance boundaries.

Finally there is one special case, the case $a = 0$. In this case switching from one to the other group only takes place by communication with members of the other group. Due to this we expect that the boundaries are exit boundaries in this case. This intuition is right and we can calculate this in the same way as in the case $a = b$ by interchanging γ, δ and u, v .

Conclusion

We have seen in the last subsection that the behaviour on the boundary points only depends on the ratio of the two parameters a and b of our model. We see that the diffusion reaches the boundary points in the case that a is smaller than b . This means in our model

that the rate of autonomous switching is smaller than the rate of switching due to the herding behaviour. That means that we reach extreme situations of the market where all agents are in the same group only in the case when the herding behaviour dominates the autonomous switching. Due to the fact that the boundary points are regular in this case, they are reflecting boundaries, that means that we leave the boundary immediately after reaching it. For the market this shows that the market reaches extreme situations in the case of herding dominance but stays in these states only for a very short time.

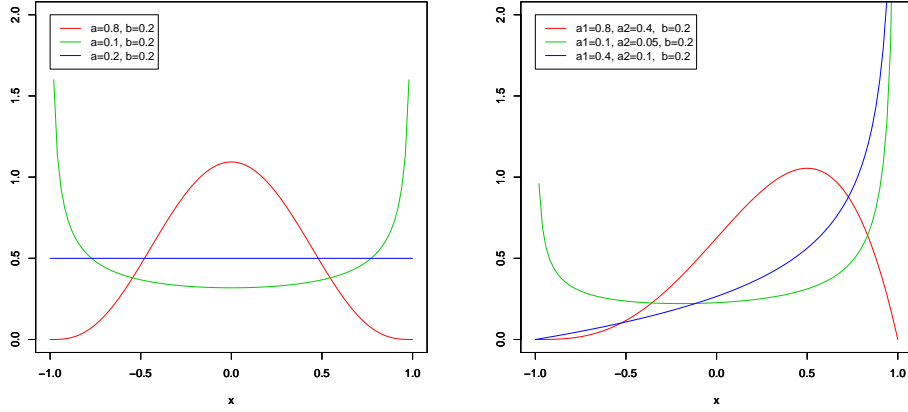


Figure 3.1: Stationary distribution for the limiting distribution of the herding model for different sets of parameters.

3.4.3 Stationary Distribution

In this subsection we look at the stationary distribution of our limiting diffusion process. It is well known, for example see [17], chapter 15, page 220, that the density of the stationary distribution in this case is the solution π of the following differential equation

$$0 = \frac{1}{2} \frac{d^2}{dx^2} [2b(1-x^2)\pi(x)] - \frac{d}{dx} [(a_1 - a_2) - (a_1 + a_2)x]\pi(x) \quad (3.4)$$

with

- $\pi \geq 0$ and
- $\int_{-1}^1 \pi(x) dx = 1$.

As shown in [17] a solution of this differential equation, using the scale and speed measure mentioned above, is given by

$$\pi(x) = m(x)[C_1 S(x) + C_2].$$

The constants C_1, C_2 are determined to guarantee the conditions

$$\pi \geq 0 \text{ and } \int_{-1}^1 \pi(x) dx = 1.$$

As shown in [4] and easy to calculate from (3.4), the stationary distribution for our diffusion process is given by the density

$$\pi(x) = K(1+x)^{\frac{a_1}{b}-1}(1-x)^{\frac{a_2}{b}-1} \quad (3.5)$$

with $K = \frac{\Gamma(\frac{a_1+a_2}{b})}{2^{\frac{a_1+a_2}{b}-1}\Gamma(\frac{a_1}{b})\Gamma(\frac{a_2}{b})}$.

In Figure 3.1 the stationary distribution is shown for three different sets of parameters with $a := a_1 = a_2$ and for three sets of parameters with $a_1 \neq a_2$. In the case of $a_1 = a_2$ we see that we have three different forms of the stationary distribution for different sets of parameters. We have a unimodal distribution in the case of $a > b$, and a bimodal distribution in the case $a < b$. In the case $a = b$ we get the uniform distribution.

Chapter 4

Generalised model as a birth and death process in a random environment

One of the problems of the models shown in Chapter 3 is that the coefficients a and b are constant over time. It would be more realistic if these coefficients would change over time. In this section we look at a hierarchical network model for the herding mechanism. In our models in Chapter 3 we started with an underlying fully connected network, see Figure 4.1, where all agents are connected. In our extension of the model 2 we include a further cluster of agents that influences all other agents, see Figure 4.2. The term agent is used here in a very broad sense for any mechanism which influences the behaviour of the trading agents, like information sources or macroeconomic data.

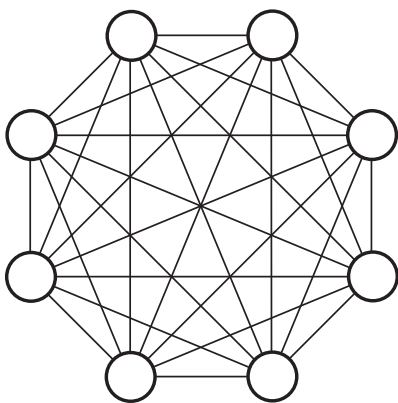


Figure 4.1: A fully connected network

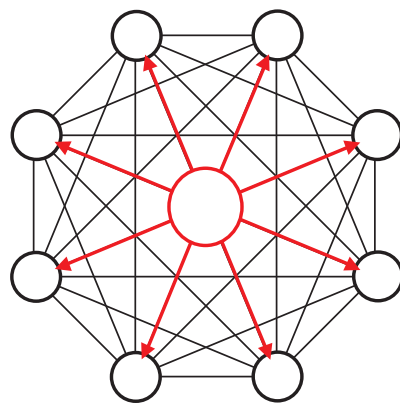


Figure 4.2: The network of our new model

In Section 4.1, we describe the new model mathematically and show in Section 4.2 that these so called birth and death processes in random environment converge to a switching diffusion process if the number of agents grows to infinity. To derive this, we will first take a look at the semigroups and the infinitesimal generators of these processes in Subsection 4.2.1. After that we see in Subsection 4.2.2 that the infinitesimal generators of the birth and death processes in random environment converge to an infinitesimal generator of a switching diffusion process. An important step is to calculate that the infinitesimal generator of a switching diffusion process generates a Feller semigroup. We do this in Subsection 4.2.3. Finally we use all these considerations to obtain in Subsection 4.2.4 that the birth and death processes in random environment converge weakly to a switching diffusion process.

4.1 Mathematical description of the model with a super agent

In this section we give a mathematical definition of the enlarged herding mechanism. Starting with model 2 from Section 3.2 we add a new agent to the network. We call him the super agent. This agent has a finite set of opinions about the future market behaviour. We model the opinion and the switching of the opinions of the super agent by a continuous-time Markov jump process on a finite set $\mathcal{M} = \{1, \dots, M\}$. The state of this process at time t represents the opinion of the super agent at this time. A jump of the process stands for a switching of opinion. The herding mechanism of the other agents is modelled, in the same way as in model 2 of Section 3.2, by a birth and death process. This process stands for the number of optimistic agents. The difference to the cited models is that the birth and death rates are now not only functions of the number of optimistic agents but also depend on the opinion of the super agent. This is stated formally in the following way.

4.1.1 Mathematical model

Let $(Y_t)_{t \geq 0}$, describing the super agent, be a homogeneous Markov process on \mathcal{M} with transition probabilities

$$\begin{aligned} P(Y_{t+\delta} = j | Y_t = i) &= q(i, j)\delta + o(\delta), \quad j \neq i, \\ P(Y_{t+\delta} = i | Y_t = i) &= 1 + q(i, i)\delta + o(\delta), \end{aligned}$$

with $0 > q(i, i) = -\sum_{j \neq i} q(i, j)$.

For all $N \in \mathbb{N}$ the herding of the other agents $(Z_t^N)_{t \geq 0}$ is a process on $\{0, \dots, N\}$ with transition probabilities

$$P(Z_{t+\delta}^N = j | Z_t^N = i, Y_t = y) = q_N(i, j, y)\delta + o(\delta)$$

with

$$\begin{aligned} q_N(i, i+1, y) &= (N-i)(a_1(y) + b(y)i), \\ q_N(i, i-1, y) &= i(a_2(y) + b(y)(N-i)), \\ q_N(i, j, y) &= 0 \quad j \notin \{i, i+1, i-1\}. \end{aligned}$$

Then $(Z_t^N, Y_t)_{t \geq 0}$ is a birth and death process in random environment Y .

To show the convergence to a switching diffusion process we transform the process Z^N . We will look at the transformed process $X_t^N = \frac{2Z_t^N}{N} - 1$ on

$$E_N := \left\{ -1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1 \right\}.$$

Then $(X_t^N, Y_t)_{t \geq 0}$ is a birth and death process in random environment with birth and death rates

$$\begin{aligned} \lambda_N(x, y) &= \frac{N^2}{4}(1-x) \left(\frac{2a_1(y)}{N} + b(y)(1+x) \right), \\ \mu_N(x, y) &= \frac{N^2}{4}(1+x) \left(\frac{2a_2(y)}{N} + b(y)(1-x) \right). \end{aligned}$$

4.2 Diffusion approximation of the birth and death processes in a random environment

In this section we show that the herding process of our new model converges to a switching diffusion process. Therefore we use a method similar to the method in 2.3.2.

4.2.1 Semigroup and infinitesimal generator of (X^N, Y)

In this subsection we calculate the semigroup and the infinitesimal generator of the birth and death processes.

Before we calculate the semigroup of the process (X^N, Y) we show that the processes X^N and Y have no simultaneous jumps.

Proposition 4.1. *It holds for $x \neq x_0$ and $y \neq y_0$*

$$P((X_t^N, Y_t) = (x, y) | (X_0, Y_0) = (x_0, y_0)) = O(t^2).$$

Proof. To prove this proposition we show how the paths of X^N and Y are defined. There is no feedback from the process X^N to the process Y . Therefore we can define Y as a one-dimensional Markov jump process on \mathcal{M} . Let P^Y be the associated probability measure from Y on $D_{\mathcal{M}}[0, \infty)$. Let $0 = \sigma_0 < \sigma_1 < \sigma_2 < \dots$ denote the jump times of the process

Y . For each path of Y the process X^N is defined as follows. In zero we start a birth and death process \tilde{X}^0 with starting point x_0 , birth rate $\lambda_N(\cdot, y_0)$ and death rate $\mu_N(\cdot, y_0)$. In the first jump point σ_1 of Y we start a new birth and death process \tilde{X}^1 with $\tilde{X}_0^1 = \tilde{X}_{\sigma_1}^0$ and birth and death rates $\lambda_N(\cdot, Y_{\sigma_1}), \mu_N(\cdot, Y_{\sigma_1})$. In the same way we define \tilde{X}^i , starting in σ_i for all $i \in \mathbb{N}$. At least we define

$$X_t^N = \sum_{i \in \mathbb{N}} \tilde{X}_{t-\sigma_i}^i 1_{[\sigma_i, \sigma_{i+1})}(t).$$

Starting with this characterisation we get for the first jump time τ_1 of X and for $\sigma_1 \leq t < \sigma_2$

$$P(\tau_1 \leq t | (Y_s)_{s \geq 0}) = P(\tau_1 \leq \sigma_1 | (Y_s)_{s \geq 0}) + P(\sigma_1 < \tau_1 \leq t | (Y_s)_{s \geq 0}).$$

Now $\tau_1 \leq \sigma_1$ if and only if the first jump of \tilde{X}^0 is smaller than σ_1 . This jump time is exponentially distributed with parameter $\lambda_N(x_0, y_0) + \mu_N(x_0, y_0)$. Hence on $\{\sigma_1 \leq t < \sigma_2\}$

$$\begin{aligned} P(\tau_1 \leq \sigma_1 | (Y_s)_{s \geq 0}) &= 1 - e^{-(\lambda_N(x_0, y_0) + \mu_N(x_0, y_0))\sigma_1} \\ &\leq 1 - e^{-(\lambda_N(x_0, y_0) + \mu_N(x_0, y_0))t} \\ &\leq 1 - 1 + (\lambda_N(x_0, y_0) + \mu_N(x_0, y_0))t = O(t). \end{aligned}$$

Also $\sigma_1 < \tau_1 \leq t$ if and only if the first jump of \tilde{X}^0 is greater than σ_1 and the first jump $\tilde{\tau}_1$ of \tilde{X}^1 is smaller than $t - \sigma_1$. Therefore on $\{\sigma_1 \leq t < \sigma_2\}$

$$\begin{aligned} P(\sigma_1 < \tau_1 \leq t | (Y_s)_{s \geq 0}) &\leq P(\tilde{\tau}_1 \leq t - \sigma_1 | (Y_s)_{s \geq 0}) \\ &= 1 - e^{-(\lambda_N(\tilde{X}_0^1, Y_{\sigma_1}) + \mu_N(\tilde{X}_0^1, Y_{\sigma_1}))(t - \sigma_1)} \\ &= O(t). \end{aligned}$$

This implies on $\{\sigma_1 \leq t < \sigma_2\}$

$$P(\tau_1 \leq t | (Y_s)_{s \geq 0}) = O(t).$$

Hence

$$\begin{aligned} P((X_t^N, Y_t) = (x, y) | (X_0, Y_0)) &\leq P(\tau_1 \leq t, \sigma_1 \leq t) \\ &= P(\tau_1 \leq t, \sigma_1 \leq t, \sigma_2 \leq t) + P(\tau_1 \leq t, \sigma_1 \leq t, \sigma_2 > t) \\ &\leq P(\sigma_1 \leq t, \sigma_2 \leq t) + \int_{\{\sigma_1 \leq t, \sigma_2 > t\}} P(\tau_1 \leq t | (Y_s)_{s \geq 0}) dP^Y \\ &= O(t^2) + O(t)P^Y(\{\sigma_1 \leq t, \sigma_2 > t\}) = O(t^2). \end{aligned}$$

□

Now we can start to calculate the semigroups of the processes (X^N, Y) . For the semigroup $T^N, T^N(t) : B(E_N \times \mathcal{M}) \rightarrow B(E_N \times \mathcal{M})$ generated by (X^N, Y) we show that for any

$f \in B(E_N \times \mathcal{M})$

$$\begin{aligned}
 T^N(t)f(x, y) - f(x, y) &= E[f(X_t, Y_t) - f(x, y) | X_0 = x, Z_0 = y] \\
 &= (\lambda_N(x, y)t + o(t)) \left[f\left(x + \frac{2}{N}, y\right) - f(x, y) \right] \\
 &+ (\mu_N(x, y)t + o(t)) \left[f\left(x - \frac{2}{N}, y\right) - f(x, y) \right] \\
 &+ \sum_{\bar{y} \neq y} (q(y, \bar{y})t + o(t)) [f(x, \bar{y}) - f(x, y)] \\
 &+ o(t).
 \end{aligned}$$

Here the last error term has the form

$$\sum_{\bar{x}, \bar{y} \notin I} o(t) [f(\bar{x}, \bar{y}) - f(x, y)]$$

with $I = \{(\tilde{x}, \tilde{y}) \in E_N \times \mathcal{M} : \tilde{x} = x \text{ or } |\tilde{x} - x| = \frac{2}{N}, \tilde{y} = y\}$. These are the jumps of the birth and death process which are larger than one step and the simultaneous jumps of both processes. From the definition of a birth and death process it follows that jumps which are larger than one step appear only with probability $o(t)$. Proposition 4.1 shows that simultaneous jumps of both processes also appear only with probability $o(t)$.

Hence the infinitesimal generator of (X^N, Y) is

$$\begin{aligned}
 A_N f(x, y) &= \lim_{t \rightarrow 0} \frac{T_N(t)f(x, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \left(\lambda_N(x, y) + \frac{o(t)}{t} \right) \left[f\left(x + \frac{2}{N}, y\right) - f(x, y) \right] \\
 &+ \lim_{t \rightarrow 0} \left(\mu_N(x, y) + \frac{o(t)}{t} \right) \left[f\left(x - \frac{2}{N}, y\right) - f(x, y) \right] \\
 &+ \lim_{t \rightarrow 0} \sum_{\bar{y} \neq y} \left(q(y, \bar{y}) + \frac{o(t)}{t} \right) [f(x, \bar{y}) - f(x, y)] \\
 &+ \lim_{t \rightarrow 0} \frac{o(t)}{t} \\
 &= \lambda_N(x, y) \left[f\left(x + \frac{2}{N}, y\right) - f(x, y) \right] \\
 &+ \mu_N(x, y) \left[f\left(x - \frac{2}{N}, y\right) - f(x, y) \right] \\
 &+ \sum_{\bar{y} \neq y} q(y, \bar{y}) [f(x, \bar{y}) - f(x, y)].
 \end{aligned}$$

Using

$$Qf(x, y) = \sum_{\bar{y} \neq y} q(y, \bar{y}) [f(x, \bar{y}) - f(x, y)] = \sum_{\bar{y} \in \mathcal{M}} q(y, \bar{y}) f(x, \bar{y}) \quad (4.1)$$

we get

$$\begin{aligned} A_N f(x, y) &= \lambda_N(x, y) \left[f\left(x + \frac{2}{N}, y\right) - f(x, y) \right] \\ &+ \mu_N(x, y) \left[f\left(x - \frac{2}{N}, y\right) - f(x, y) \right] + Qf(x, y). \end{aligned}$$

4.2.2 Convergence of the infinitesimal generator

In this subsection, we show that the infinitesimal generators from Subsection 4.2.1 converge to the infinitesimal generator of a switching diffusion process, if the number of agents tends to infinity. This is done by a Taylor approximation.

Let $E = [-1, 1]$. Let

$$\begin{aligned} B(x, y) &= (a_1(y) - a_2(y)) - (a_1(y) + a_2(y))x \\ \sigma^2(x, y) &= 2b(y)(1 - x^2). \end{aligned}$$

Theorem 4.2. *Let $f \in C_0(\Omega) = C_0([-1, 1] \times \mathcal{M})$ and $f(\cdot, y) \in C^2([-1, 1])$ (defined in Section 2.3.2) for all $y \in \mathcal{M}$. Let*

$$Af(x, y) = B(x, y) \frac{df(x, y)}{dx} + \frac{1}{2} \sigma^2(x, y) \frac{d^2 f(x, y)}{dx^2} + Qf(x, y).$$

Then

$$A_N f \rightarrow Af \text{ uniformly.}$$

Proof. For all $(x, y) \in (E_N \setminus \{-1, 1\}) \times \mathcal{M}$, we obtain

$$\begin{aligned} A_N f(x, y) &= \lambda_N(x, y) \left[f(x, y) + \frac{2}{N} \frac{df(x, y)}{dx} + \frac{2}{N^2} \frac{d^2 f(x, y)}{dx^2} + o\left(\frac{1}{N^2}\right) - f(x, y) \right] \\ &+ \mu_N(x, y) \left[f(x, y) - \frac{2}{N} \frac{df(x, y)}{dx} + \frac{2}{N^2} \frac{d^2 f(x, y)}{dx^2} + o\left(\frac{1}{N^2}\right) - f(x, y) \right] \\ &+ Qf(x, y) \\ &= \frac{2}{N} (\lambda_N(x, y) - \mu_N(x, y)) \frac{df(x, y)}{dx} + \frac{2}{N^2} (\lambda_N(x, y) + \mu_N(x, y)) \frac{d^2 f(x, y)}{dx^2} \\ &+ (\lambda_N(x, y) + \mu_N(x, y)) o\left(\frac{1}{N^2}\right) + Qf(x, y). \end{aligned}$$

Due to the fact that $f(\cdot, y) \in C^2$ and $|\mathcal{M}| < \infty$, the error term $o(\frac{1}{N^2})$ is uniform in (x, y) . Because of finite limits of all derivatives in the points $-1, 1$ and with the notations from Section 2.3.2, it is easy to check that this approximation holds also for $(x, y) \in \{-1, 1\} \times \mathcal{M}$. Furthermore

$$\lambda_N(x, y) - \mu_N(x, y) = \frac{N}{2} B(x, y)$$

and

$$\lambda_N(x, y) + \mu_N(x, y) = \frac{N^2}{4}\sigma^2(x, y) + o\left(\frac{1}{N^2}\right).$$

From this it follows

$$\begin{aligned} A_N f(x, y) &= -\frac{2}{N} \frac{N}{2} B(x, y) \frac{df(x, y)}{dx} + \frac{2}{N^2} \left(\frac{N^2}{4} \sigma^2(x, y) \right) \frac{d^2 f(x, y)}{dx^2} \\ &+ o(1) + Qf(x, y) \\ &= -\frac{2}{N} \frac{N}{2} B(x, y) \frac{df(x, y)}{dx} + \frac{2}{N^2} \left(\frac{N^2}{4} \sigma^2(x, y) \right) \frac{d^2 f(x, y)}{dx^2} \\ &+ o(1) + Qf(x, y) \\ &= Af(x, y) + o(1). \end{aligned}$$

Obviously, this convergence is uniform in (x, y) . □

4.2.3 The semigroup of the limiting infinitesimal generator

In this subsection we show that the infinitesimal generator A with

$$Af(x, y) = B(x, y) \frac{d}{dx} f(x, y) + \frac{1}{2} \sigma^2(x, y) \frac{d^2}{dx^2} f(x, y) + Qf(x, y) \quad (4.2)$$

generates a Feller semigroup on $C_0(\Omega)$.

Theorem 4.3. *The infinitesimal generator A defined by (4.2) generates a Feller semigroup $(T_t)_{t \geq 0}$ on $C_0(\Omega)$.*

Before we prove this result, we recapitulate some known results. The first is the important theorem of Hille and Yosida that characterises the infinitesimal generators of Feller semigroups.

Theorem 4.4 (Hille and Yosida). *Let A be a linear operator on $C_0(\Omega)$ with domain \mathcal{D} . Then A is closable and its closure \bar{A} is the infinitesimal generator of a Feller semigroup on $C_0(\Omega)$ iff these conditions hold:*

1. \mathcal{D} is dense in $C_0(\Omega)$;
2. the range of $\lambda_0 - A$ is dense in $C_0(\Omega)$ for some $\lambda_0 > 0$;
3. if $f^+ \leq f(x)$ for some $f \in \mathcal{D}$ and $x \in \Omega$, then $Af(x) \leq 0$.

Proof. See [16], Theorem 19.11.

The third condition is known as the positive maximum principle. For the second result we need two further definitions.

Definition 4.5 (dissipative). A linear operator $A : \mathcal{D}(A) \rightarrow X, \mathcal{D}(A) \subseteq X$ is called dissipative if

$$\|\lambda u - Au\| \geq \lambda \|u\|$$

holds for all $\lambda > 0$ and $u \in \mathcal{D}(A)$.

Proposition 4.6. A linear operator A on C_0 satisfying the positive maximum principle is dissipative.

Proof. See [16], p.377.

Definition 4.7 (A -bounded). Let $(A, \mathcal{D}(A))$ and $(Q, \mathcal{D}(Q))$ be two linear operators on X such that $\mathcal{D}(A) \subseteq \mathcal{D}(Q)$ and for some $\alpha \in [0, 1)$ and $\beta \geq 0$

$$\|Qu\| \leq \alpha \|Au\| + \beta \|u\|$$

holds for all $u \in \mathcal{D}(A)$. Then the operator Q is called A -bounded.

With these definitions we can cite a useful result from perturbation theory.

Theorem 4.8. Let $(A, \mathcal{D}(A))$ be a linear operator on a Banach space $(X, \|\cdot\|_X)$ such that $\mathcal{D}(A)$ is dense in X . Suppose that A is dissipative and the range of $\lambda - A$ is dense in X for some $\lambda > 0$. If $(Q, \mathcal{D}(Q))$ is an A -bounded dissipative operator on X , then $(A + Q, \mathcal{D}(A))$ is closable and its closure generates a strongly continuous contraction semigroup.

Proof. See [14], Theorem 4.4.3.

Finally we need a result that shows us the connection between positive semigroups T and the positivity of the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ of the infinitesimal generator A generated by T .

Theorem 4.9. A strongly continuous semigroup $(T_t)_{t \geq 0}$ on a Banach lattice X is positive if and only if the resolvent $R(\lambda, A)$ of its infinitesimal generator A is positive for all sufficient large λ .

Proof. See [11], Theorem 1.3, Chapter VI.

Now we can prove Theorem 4.3.

Proof of Theorem 4.3. First we write $A = \tilde{A} + Q$ with

$$\tilde{A}f(x, y) = B(x, y) \frac{df(x, y)}{dx} + \frac{1}{2} \sigma^2(x, y) \frac{d^2 f(x, y)}{dx^2} \text{ and } Qf(x, y) \text{ as stated above.}$$

Our first step is to show that the operator \tilde{A} generates a Feller semigroup. We do this with Theorem 4.4 of Hille and Yosida.

Define for all $y \in \mathcal{M}$

$$A^y : C^\infty[-1, 1] \rightarrow C^\infty[-1, 1] \text{ with } A^y g(x) = B(x, y)g'(x) + \frac{1}{2} \sigma^2(x, y)g''(x).$$

First we show that the three conditions from Theorem 4.4 hold for the infinitesimal generator \tilde{A} with

$$\mathcal{D} = \{f \in C_0([-1, 1] \times \mathcal{M}) : f(\cdot, y) \in C^\infty[-1, 1] \text{ for all } y \in \mathcal{M}\}.$$

The first condition clearly holds due to the fact that $C^\infty[-1, 1]$ is dense in $C_0[-1, 1]$. For the third condition let $f^+ \leq f(x, y)$ for some $f \in \mathcal{D}$. We know from Subsection 3.3.2 that A^y is the infinitesimal generator of a Feller semigroup. Then it follows from Theorem 4.4 that $A^y f(\cdot, y)(x) \leq 0$. Hence

$$\tilde{A}f(x, y) = A^y f(\cdot, y)(x) \leq 0.$$

For the second condition we show that $(\lambda - \tilde{A})\mathcal{D}$ is dense in $C_0([-1, 1] \times \mathcal{M})$. Let $f \in C_0([-1, 1] \times \mathcal{M})$ and $\varepsilon > 0$. Define $f^y(x, \tilde{y}) = f(x, \tilde{y})\mathbb{1}_y(\tilde{y})$. Due to the fact that A^y is the infinitesimal generator of a Feller semigroup and that $C^\infty[-1, 1]$ is a core for A^y we know from Theorem 4.4 that there exists $\lambda^y > 0$ such that $(\lambda^y - A^y)C^\infty[-1, 1]$ is dense in $C_0([-1, 1])$. By [16] Theorem 19.4 we additionally know that $(\lambda - A^y)C^\infty[-1, 1]$ is dense in $C_0([-1, 1])$ for all $\lambda > 0$. Let $\lambda > 0$, then there exists $f_\varepsilon^y \in C^\infty[-1, 1]$, with

$$\|(\lambda - A^y)f_\varepsilon^y - f(\cdot, y)\|_{C_0([-1, 1])} \leq \frac{\varepsilon}{M}.$$

Furthermore with $\tilde{f}_\varepsilon^y \in C_0([-1, 1] \times \mathcal{M})$ defined by $\tilde{f}_\varepsilon^y(x, \tilde{y}) := f_\varepsilon^y(x)\mathbb{1}_y(\tilde{y})$ it follows

$$\|(\lambda - \tilde{A})\tilde{f}_\varepsilon^y - f^y\| \leq \frac{\varepsilon}{M}.$$

Hence

$$\begin{aligned} \|(\lambda - \tilde{A}) \sum_{y \in \mathcal{M}} \tilde{f}_\varepsilon^y - f\| &= \left\| \sum_{y \in \mathcal{M}} (\lambda - \tilde{A})\tilde{f}_\varepsilon^y - \sum_{y \in \mathcal{M}} f^y \right\| \\ &\leq \sum_{y \in \mathcal{M}} \|(\lambda - \tilde{A})\tilde{f}_\varepsilon^y - f^y\| \\ &\leq \varepsilon. \end{aligned}$$

Furthermore $\sum_{y \in \mathcal{M}} \tilde{f}_\varepsilon^y \in \mathcal{D}$ and $(\lambda - \tilde{A})\mathcal{D}$ is dense in $C_0([-1, 1] \times \mathcal{M})$. Now Theorem 4.4 shows that \tilde{A} generates a Feller semigroup.

Let us take a look at the operator Q . We see that Q is a bounded operator. Furthermore, with equation (4.1), it is easy to see that Q fulfils the positive maximum principle and hence, with Proposition 4.6, Q is dissipative. So with Theorem 4.8, we get that $A = \tilde{A} + Q$ is the infinitesimal generator of a strongly continuous contraction semigroup on $C_0([-1, 1] \times \mathcal{M})$. Now we have to show that A generates a positive semigroup. From Theorem 4.9 we know that this follows if $R(\lambda, A)$ is positive for all $\lambda > 0$. Let $g \in C_0([-1, 1] \times \mathcal{M})$ be a positive function and put $f = R(\lambda, A)g$, so that $g = (\lambda - A)f$. It is easy to check that the infinitesimal generator A fulfils the positive maximum principle for $\mathcal{D} = \mathcal{D}(A)$,

because \tilde{A} and Q fulfil it. If $\inf_{x \in [-1,1] \times \mathcal{M}} f(x) < 0$, we choose some $x_0 \in [-1, 1] \times \mathcal{M}$ with $f(x_0) \leq \min\{f, 0\}$. With the positive maximum principle we get $Af(x_0) \geq 0$, and so

$$\inf_{x \in [-1,1] \times \mathcal{M}} g(x) = \inf_{x \in [-1,1] \times \mathcal{M}} (\lambda - A)f(x) \leq (\lambda - A)f(x_0) \leq \lambda f(x_0) = \lambda \inf_{x \in [-1,1] \times \mathcal{M}} f(x) < 0.$$

But g is a positive function and so $\inf_{x \in [-1,1] \times \mathcal{M}} f(x) \geq 0$. Hence f is a positive function and $R(\lambda, A)$ is positive. \square

4.2.4 Convergence to a switching diffusion

In this subsection we show that the semigroups T^N converge to the semigroup T and that there exists a switching diffusion process X corresponding to A , such that X^N weakly converge to X in $D_{[-1,1] \times \mathcal{M}}[0, \infty)$. For this we use the same methods as in Subsection 2.3.2. To use Theorem 2.24 from Chapter 2, we need that the set

$$\mathcal{D} = \{f \in C_0(E \times \mathcal{M}) : f(\cdot, y) \in C^\infty(E) \text{ for all } y \in \mathcal{M}\}$$

is a core for our infinitesimal generator A . Here $C^\infty(E)$ is the set of all functions $f \in C^\infty((-1, 1))$ with finite limits of the derivatives in the points $\{-1, 1\}$.

Proposition 4.10. \mathcal{D} is a core for A .

Proof. Let $f \in \mathcal{D}(A)$. For all $y \in \mathcal{M}$ define $f^y(x) = f(x, y)$. From Theorem 2.23 we know that for all $y \in \mathcal{M}$ we find a sequence $(f_n^y)_{n \in \mathbb{N}} \in (C^\infty(E))^{\mathbb{N}}$ with $f_n^y \rightarrow f^y$ uniformly and

$$\tilde{A}^y f_n^y \rightarrow \tilde{A}^y f^y \text{ uniformly,}$$

with $\tilde{A}^y g(x) = B(x, y)g'(x) + \frac{1}{2}\sigma^2(x, y)g''(x)$.

Define $f_n \in \mathcal{D}$ by $f_n(x, y) = f_n^y(x)$. Then $f_n \rightarrow f$ and

$$\begin{aligned} \sup_{(x,y) \in E \times \mathcal{M}} |Af_n(x, y) - Af(x, y)| &\leq \sup_{y \in \mathcal{M}} \sup_{x \in E} |\tilde{A}^y f_n^y(x) - \tilde{A}^y f^y(x)| \\ &\quad + \sup_{(x,y) \in E \times \mathcal{M}} |Q(f_n(x, y) - f(x, y))| \\ &\rightarrow 0. \end{aligned}$$

\square

Now we are able to show the convergence of the birth and death process in random environment $(X^N, Y)_{n \in \mathbb{N}}$ to the switching diffusion process (X, Y) .

Theorem 4.11. Let $(X^N, Y)_{N \in \mathbb{N}}$ be a sequence of birth and death processes in random environment on $(E_N \times \mathcal{M})_{N \in \mathbb{N}}$ defined by the infinitesimal generators

$$\begin{aligned} A_N f(x, y) &= \lambda_N(x, y) \left[f\left(x + \frac{2}{N}, y\right) - f(x, y) \right] \\ &\quad + \mu_N(x, y) \left[f\left(x - \frac{2}{N}, y\right) - f(x, y) \right] + Qf(x, y). \end{aligned}$$

Then there exists a switching diffusion process (X, Y) on $[-1, 1] \times \mathcal{M}$ such that

$$(X^N, Y) \rightarrow (X, Y) \text{ weakly in } D_{[-1,1] \times \mathcal{M}}[0, \infty).$$

Proof. Define the infinitesimal generator A on $\mathcal{D}(A) \subset C_0([-1, 1] \times \mathcal{M})$ by

$$Af(x, y) = B(x, y) \frac{d}{dx} f(x, y) + \frac{1}{2} \sigma^2(x, y) \frac{d^2}{dx^2} f(x, y) + Qf(x, y)$$

then Theorem 4.2 implies that

$$A_N f \rightarrow Af \text{ uniformly}$$

for all $f \in \mathcal{D} = \{f \in C_0([-1, 1] \times \mathcal{M}) : f(\cdot, y) \in C^\infty([-1, 1]) \text{ for all } y \in \mathcal{M}\}$. Then it follows from Theorem 2.24 with $E = [-1, 1] \times \mathcal{M}$,

$$E_N = \left\{-1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1\right\} \times \mathcal{M}, L = C_0(E), L_N = C_0(E_N)$$

and $\eta_N = id, \pi_N f = f \circ \eta_N$ that

$$T_t^N \pi_N f \rightarrow T_t f \text{ for all } f \in C_0([-1, 1] \times \mathcal{M}).$$

Finally we can use Theorem 2.25 to obtain a switching diffusion process (X, Y) with semigroup T and infinitesimal generator A such that

$$(X^N, Y) \rightarrow (X, Y) \text{ weakly in } D_{[-1,1] \times \mathcal{M}}[0, \infty).$$

□

4.3 Conclusion

In this chapter we introduced a new agent to the herding model from Chapter 3. Through this agent we overcome the problem of constant switching rates a and b for the agents. The new model is more flexible and allows the switching rates of the agents to differ in time. In this chapter we also showed that this generalised model is not N -dependent. We showed as well that we get a limiting process if the number of agents tends to infinity. In the next chapter we look at this limiting process.

Chapter 5

Stationary distribution of the switching diffusions

In Chapter 4 we saw that the birth and death processes in a random environment of our herding model with a super agent converge to a switching diffusion process. In this chapter we look at the stationary distribution of this switching diffusion process. For this we introduce in Section 5.1 the system of differential equations that determines the stationary distribution. In Section 5.2 we find a numerical approximation for the stationary distribution. Furthermore we investigate the case of fast switching in Section 5.3 when the super agent switches his opinion very rapidly.

5.1 Stationary Distribution

We now look at the stationary distribution of the switching diffusion (X, Y) defined in Chapter 4. In the model with a super agent the limiting process is a switching diffusion process with infinitesimal generator given by equation (4.2). For ease of exposition we only look at the case $a := a_1 = a_2$. The general case can be treated in the same way. The density of the stationary distribution $\pi : (-1, 1) \times \mathcal{M} \rightarrow \mathbb{R}$ is a solution of the following coupled system of differential equation

$$0 = \frac{1}{2} \frac{d^2}{dx^2} (2b(y)(1-x^2)\pi(x, y)) - \frac{d}{dx} (-2a(y)x\pi(x, y)) - \sum_{l=1}^n q(l, y)\pi(x, l), \quad y \in \mathcal{M}, \quad (5.1)$$

with

- $\pi \geq 0$ and
- $1 = \sum_{y=1}^M \int_{(-1,1)} \pi(x, y) dx$.

It is much more difficult to find a solution in this case than in the case of the model without a super agent. Instead of an ordinary differential equation we have to solve a coupled system of M differential equations which is usually not possible in closed form. Due to this fact we find approximations for the stationary distribution by simulation.

5.2 Approximate Solutions for the Stationary Distribution of the Switching Diffusion

To make the problem easier to handle, we assume that the super agent only has two states $\mathcal{M} = \{1, 2\}$. In this case we have to find the solution of the system of the two differential equations

$$\begin{aligned} 0 &= \frac{d^2}{dx^2} (b(1)(1-x^2)\pi_1(x)) + \frac{d}{dx} (2a(1)x\pi_1(x)) - q(1,2)\pi_1(x) + q(2,1)\pi_2(x) \\ 0 &= \frac{d^2}{dx^2} (b(2)(1-x^2)\pi_2(x)) + \frac{d}{dx} (2a(2)x\pi_2(x)) - q(2,1)\pi_2(x) + q(1,2)\pi_1(x) \end{aligned}$$

with

- $\pi \geq 0$ and
- $1 = \int_{(-1,1)} \pi_1(x) + \pi_2(x) dx$.

For easier notation we write the function $\pi : (-1, 1) \times \{1, 2\} \rightarrow \mathbb{R}$ as two functions $\pi_1, \pi_2 : (-1, 1) \rightarrow \mathbb{R}$, with $\pi_i(x) = \pi(x, i)$.

The model depends on six parameters $a(1), a(2), b(1), b(2), q(1, 2), q(2, 1)$. For the numerical examples in this section we look at two different sets of parameters. At first we take

$$a(1) = 0.8, a(2) = 0.1, b(1) = b(2) = 0.2, q(1, 2) = q(2, 1) = 0.1,$$

and the second set of parameters is

$$a(1) = 0.2, a(2) = 0.1, b(1) = 0.1, b(2) = 0.7, q(1, 2) = q(2, 1) = 0.1.$$

Below we use different methods to get an approximation for the stationary distribution for these two sets of parameters.

5.2.1 Approximation by simulation of the process

In this subsection we follow the idea of article [21]. There Mao et al. show that we get a good approximation by using an Euler-Maruyama scheme with variable step sizes. A short introduction of this method is given in the next paragraph. For more information on the simulation method and the convergence to the stationary distribution see [21].

Euler-Maruyama method for simulating switching diffusions

Let $\Delta = (\Delta_k)_{k \geq 1}$ be a nonrandom sequence of positive numbers such that

$$\lim_{k \rightarrow \infty} \Delta_k = 0 \text{ and } \sum_{k=1}^{\infty} \Delta_k = \infty.$$

In general $\Delta_k \neq \Delta_l$ for $k \neq l$, and we call Δ the variable step sizes. Set $t_0 = 0$ and define $t_k = \sum_{l=1}^k \Delta_l$ for $k \geq 1$, then $(t_k)_{k \geq 1}$ defines a nonrandom partition of $(0, \infty)$. We want to approximate our process (X, Y) on these time points. First we simulate the approximated switching process $\tilde{Y} = (\tilde{Y}_{t_k})_{k \geq 0}$.

Let $\tilde{Y}_0 = y \in \mathcal{M} = \{1, \dots, M\}$ and simulate a random number ξ_1 which is uniformly distributed in $[0, 1]$. Then compute the one-step transition probability matrix

$$P_{0,1} = (p_{0,1}(l, j))_{l, j \in \mathcal{M}} = e^{\Delta_1 Q}.$$

Define

$$\tilde{Y}_{k_1} = \begin{cases} z_1 & \text{if } z_1 \in \mathcal{M} \setminus \{M\} \text{ such that } \sum_{l=1}^{z_1-1} p_{0,1}(y, l) \leq \xi_1 < \sum_{l=1}^{z_1} p_{0,1}(y, l), \\ M & \text{if } \sum_{l=1}^{M-1} p_{0,1}(y, l) \leq \xi_1. \end{cases}$$

The next step is to compute the one-step transition matrix

$$P_{1,2} = (p_{1,2}(l, j))_{l, j \in \mathcal{M}} = e^{\Delta_2 Q}$$

and simulate a new random ξ_2 which is again uniformly distributed in $[0, 1]$, and then define

$$\tilde{Y}_{k_2} = \begin{cases} z_2 & \text{if } z_2 \in \mathcal{M} \setminus \{M\} \text{ such that } \sum_{l=1}^{z_2-1} p_{1,2}(\tilde{Y}_{k_1}, l) \leq \xi_2 < \sum_{l=1}^{z_2} p_{1,2}(\tilde{Y}_{k_1}, l), \\ M & \text{if } \sum_{l=1}^{M-1} p_{1,2}(\tilde{Y}_{k_1}, l) \leq \xi_2. \end{cases}$$

Repeating this procedure, a trajectory of $\tilde{Y}_{t_k}, k = 0, 1, 2, \dots$ can be generated.

In the next step we have to simulate an approximation $\tilde{X} = (\tilde{X}_{t_k})_{k \geq 0}$ of the diffusion part X of the switching diffusion. For this set $\tilde{X}_0 = x \in [0, 1]$ and compute

$$\begin{aligned} \tilde{X}_{t_{k+1}} &= \tilde{X}_{t_k} + B(\tilde{X}_{t_k}, \tilde{Y}_{t_k}) \Delta_{k+1} + \sqrt{\sigma^2(\tilde{X}_{t_k}, \tilde{Y}_{t_k})} N_{k+1} \\ &= \tilde{X}_{t_k} - 2a(\tilde{Y}_{t_k}) \tilde{X}_{t_k} \Delta_{k+1} + \sqrt{2b(\tilde{Y}_{t_k})(1 - \tilde{X}_{t_k}^2)} N_{k+1}, \end{aligned}$$

where N_{k+1} is a normal distributed random variable with mean 0 and variance Δ_{k+1} . Together we get an approximation (\tilde{X}, \tilde{Y}) for the switching diffusion (X, Y) .

Approximation of the stationary distribution

We use this method to approximate the density of the stationary distribution for our two sets of parameters. This gives us a first approximation for the density of the stationary distribution of the switching diffusion process. The results of this approximation are shown in Figure 5.1 for the first set of parameters and in Figure 5.2 for the second set.

If we compare the two parts π_1 and π_2 of the stationary distribution, as stated above, from parameter set 1 with the corresponding stationary distributions of the systems without switching in Figure 3.1, we see that there are only small differences between these stationary distributions. The explanation for this is that the switching parameters $q(1, 2) = q(2, 1) = 0.1$ are small. Due to this the process Y switches rarely between the two states. And in every state the agent process has sufficient time to reach the stationary distribution for this set of switching parameter a, b .

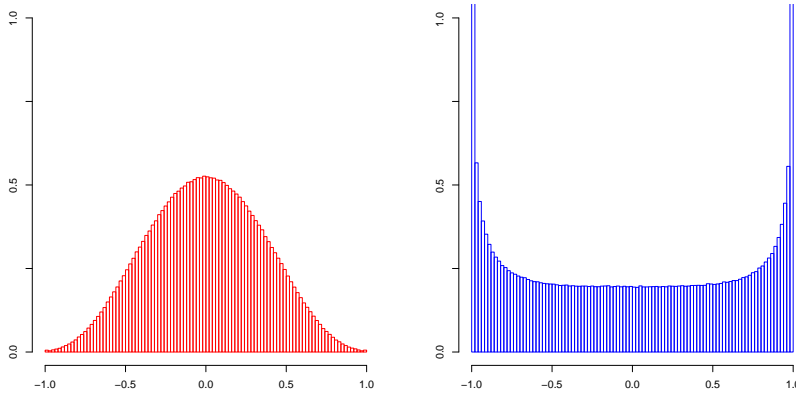


Figure 5.1: Simulated approximation of the two components π_1 (red) and π_2 (blue) of the stationary distribution for our first set of parameters.

5.2.2 Power series approach

In this subsection we approximate the stationary distribution of the switching diffusion by a power series approach. Doing this we get more information about the stationary distribution and a good approximation. For this we assume that the two functions π_1 and π_2 are of the form

$$\begin{aligned}\pi_1(x) &= \sum_{k=0}^{\infty} s_k x^k \\ \pi_2(x) &= \sum_{k=0}^{\infty} t_k x^k.\end{aligned}$$

A simple calculation shows that if $f : (-1, 1) \times \mathcal{M} \rightarrow \mathbb{R}$ is a solution of the system of differential equations (5.1), then $\hat{f} : (-1, 1) \times \mathcal{M} \rightarrow \mathbb{R}$ with

$$\hat{f}(x, z) = f(-x, z)$$

is also a solution of (5.1). From the uniqueness of the stationary distribution it follows that $f = \hat{f}$, so the density of the stationary distribution is symmetric in zero. From this we obtain that π_1 and π_2 are of the form

$$\begin{aligned}\pi_1(x) &= \sum_{k=0}^{\infty} s_k x^{2k} \\ \pi_2(x) &= \sum_{k=0}^{\infty} t_k x^{2k}.\end{aligned}$$

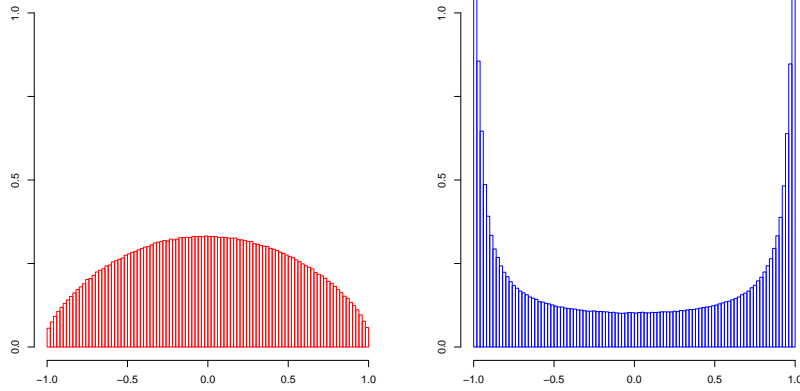


Figure 5.2: Simulated approximation of the two components π_1 (red) and π_2 (blue) of the stationary distribution for our second set of parameters.

The functions π_1 and π_2 are assumed to solve the system (5.1). From this, we get for π_1

$$\begin{aligned}
 0 &= b(1) \frac{d^2}{dx^2} \left[(1-x^2) \sum_{k=0}^{\infty} s_k x^{2k} \right] + 2a(1) \frac{d}{dx} \left[\sum_{k=0}^{\infty} s_k x^{2k+1} \right] \\
 &\quad - q(1,2) \sum_{k=0}^{\infty} s_k x^{2k} + q(2,1) \sum_{k=0}^{\infty} t_k x^{2k} \\
 &= b(1) \sum_{k=0}^{\infty} (2k+2)(2k+1) s_{k+1} x^{2k} - b(1) \sum_{k=0}^{\infty} (2k+2)(2k+1) s_k x^{2k} \\
 &\quad + 2a_1 \sum_{k=0}^{\infty} (2k+1) s_k x^{2k} - q(1,2) \sum_{k=0}^{\infty} s_k x^{2k} + q(2,1) \sum_{k=0}^{\infty} t_k x^{2k}.
 \end{aligned}$$

By the same calculations we get an equivalent equation for π_2 . Equating the coefficients, we get the following recursive equations for the coefficients of π_1 and π_2

$$\begin{aligned}
 s_{k+1} &= \left(1 - \frac{2a(1)(2k+1) + q_{12}}{b(1)(2k+2)(2k+1)} \right) s_k - \frac{q(2,1)}{b(1)(2k+2)(2k+1)} t_k \\
 t_{k+1} &= \left(1 - \frac{2a(2)(2k+1) + q_{21}}{b(2)(2k+2)(2k+1)} \right) t_k - \frac{q(1,2)}{b(2)(2k+2)(2k+1)} s_k.
 \end{aligned} \tag{5.2}$$

The remaining problem is to find $s_0 = \pi_1(0)$ and $t_0 = \pi_2(0)$ in such a way that π_1 and π_2 fulfil the additional conditions in (5.1). Due to the fact that it is not possible to calculate s_0, t_0 analytically, we approximate them numerically. Nevertheless, if we get approximations for s_0 and t_0 , then we can use equations (5.2) to compute an approximation for the stationary distribution. The solution of this approximation is shown in Figure 5.3 for the two sets of parameters.

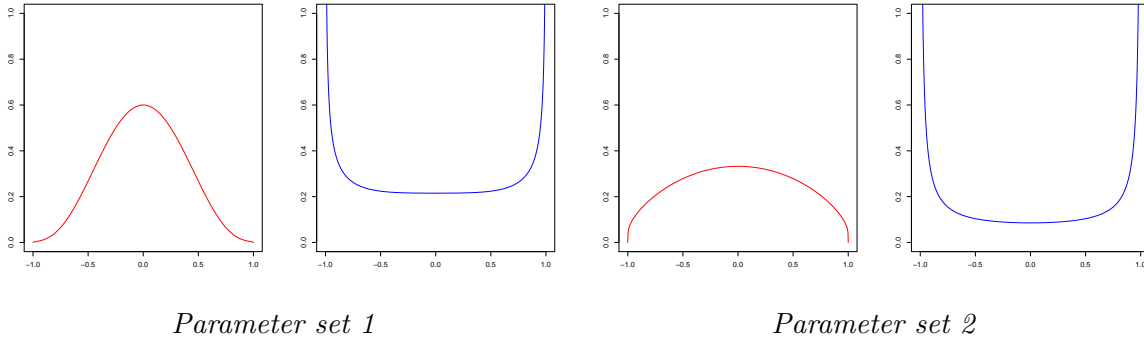


Figure 5.3: Simulated power series approximation of the two components π_1 (red) and π_2 (blue) of the stationary distribution for our two sets of parameters.

We see directly from the equations in (5.2) that the main problem is to find the quotient $\frac{s_0}{t_0}$ so that $\pi_1, \pi_2 \geq 0$ and $\int_{(-1,1)} \pi_1(x) + \pi_2(x) dx < \infty$. Then the integral condition in (5.1) is fulfilled by dividing s_0 and t_0 by $\int_{(-1,1)} \pi_1(x) + \pi_2(x) dx$.

For our two sets of parameters we see that if $q(1, 2)$ and $q(2, 1)$ are small, the two parts π_1 and π_2 from the stationary distribution π of the switching diffusion are nearly of the same form as the stationary distribution of the diffusions with parameter $a(i)$ and $b(i)$, $i \in \{1, 2\}$. We can see this if we compare Figure 3.1 with Figure 5.1 or 5.3. We see this also in Figure 5.4 and 5.5, where the stationary distributions of the models of Chapter 3 are given in black. This is clear because if $q(1, 2)$ and $q(2, 1)$ are small, then this means that the super agent stays for a long time in each state. And so the agents have a long time at the same set of parameters. And the market has time to almost reach the stationary state for this parameter before the super agent switches again.

But what will happen if the super agent switches his opinion faster? We give an answer on this question in the next section.

5.3 Stationary Distribution at Fast Switching

In this section we look at what happens if the super agent switches his opinion very fast. The switching of the super agent is given by the transition rates $q(1, 2), q(2, 1)$. We can use them to obtain the intensity matrix

$$Q = \begin{pmatrix} -q(1, 2) & q(1, 2) \\ q(2, 1) & -q(2, 1) \end{pmatrix}.$$

Now we speed up this switching by multiplying Q with $\frac{1}{\varepsilon}$ for $\varepsilon \in (0, 1)$. Mathematically we look at the set of processes $(X^\varepsilon, Y^\varepsilon)$, depending on $\varepsilon > 0$ with infinitesimal generators

$$Af(x, z) = B(x, z) \frac{df(x, z)}{dx} + \frac{1}{2} \sigma^2(x, z) \frac{d^2 f(x, z)}{dx^2} + \frac{1}{\varepsilon} Qf(x, z). \quad (5.3)$$

Now we are interested in what happens with the stationary distribution of the switching diffusion if the parameter ε tends to zero. First we look at our two sets of parameters from Section 5.2. For this we use the power series method from Subsection 5.2.2. The results of these approximations are shown in Figure 5.4 for the first set of parameters and in Figure 5.5 for the second set.

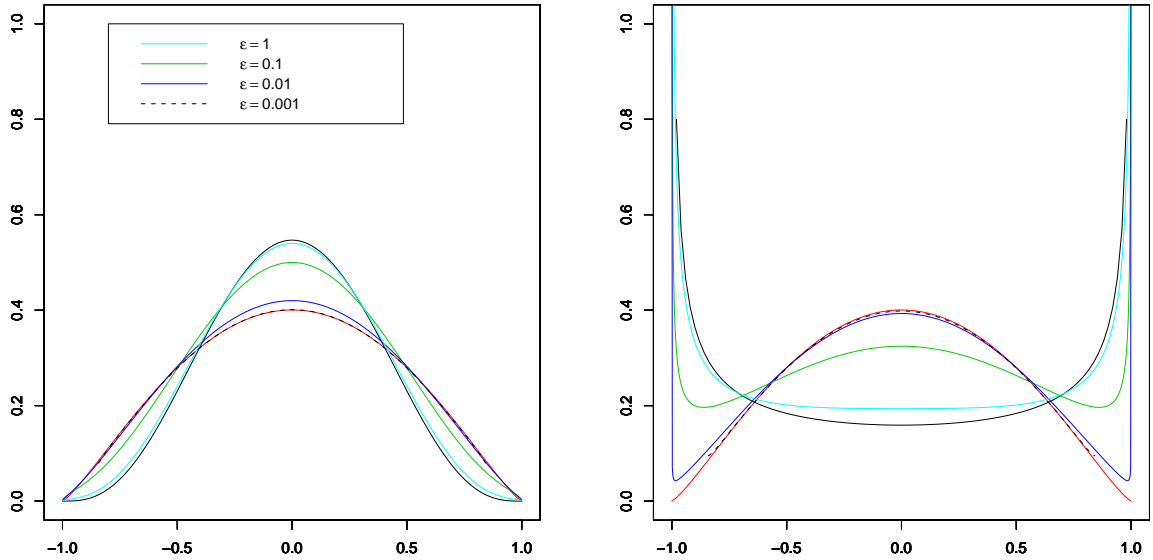


Figure 5.4: Results of the approximation for π_1^ε (left) and π_2^ε (right) using the first sets of parameters for different values of ε .

5.3.1 Convergence of the stationary distribution for $\varepsilon \rightarrow 0$.

In Figure 5.4 and 5.5 we see that for $\varepsilon \rightarrow 0$, the two parts of the stationary distribution tend to the same function. There the limiting function is plotted in red. Now we look again at the system of differential equations (5.1) to see mathematically why this happens, and what the limiting function looks like. The following argument is a heuristic one that shows what the limit of the stationary distribution looks like. In the next chapter we prove that these heuristics are right. There we show that the diffusion processes converge at fast switching. Due to this we show also the convergence of the stationary distributions.

The stationary distribution of the process $(X^\varepsilon, Y^\varepsilon)$ is given as a solution of the following

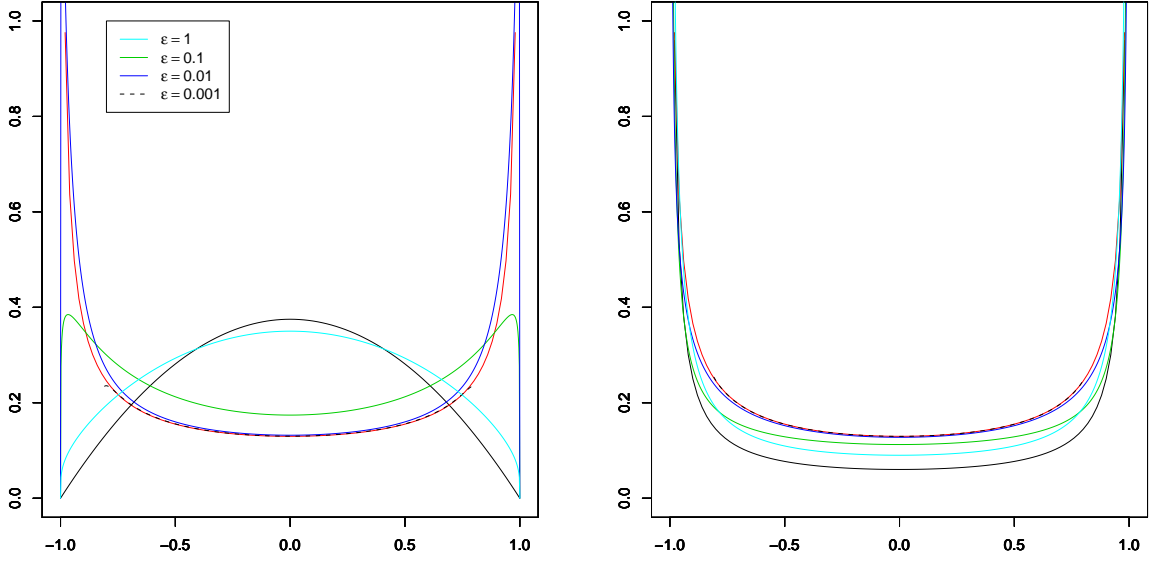


Figure 5.5: Results of the approximation for π_1^ε (left) and π_2^ε (right) using the second sets of parameters for different values of ε .

system of differential equations

$$\begin{aligned}
 0 &= \frac{d^2}{dx^2} (b(1)(1-x^2)\pi_1^\varepsilon(x)) + \frac{d}{dx} (2a(1)x\pi_1^\varepsilon(x)) + \frac{1}{\varepsilon} (q(2,1)\pi_2^\varepsilon(x) - q(1,2)\pi_1^\varepsilon(x)) \\
 0 &= \frac{d^2}{dx^2} (b(2)(1-x^2)\pi_2^\varepsilon(x)) + \frac{d}{dx} (2a(2)x\pi_2^\varepsilon(x)) - \frac{1}{\varepsilon} \underbrace{(q(2,1)\pi_2^\varepsilon(x) - q(1,2)\pi_1^\varepsilon(x))}_{d^\varepsilon(x)}
 \end{aligned}$$

with the conditions given in (5.1). If ε tends to zero, then $d^\varepsilon(x)$ has to converge to zero as well. Otherwise, the last term in the equations grows to infinity in the equation for π_1^ε , and falls to minus infinity in the equation for π_2^ε . That means that $q(1,2)\pi_1^\varepsilon(x) - q(2,1)\pi_2^\varepsilon(x)$ converges to zero for $\varepsilon \rightarrow 0$. Assuming that all the limits exist we may define

1. $\pi(x) = \lim_{\varepsilon \rightarrow 0} q(1,2)\pi_1^\varepsilon(x) = \lim_{\varepsilon \rightarrow 0} q(2,1)\pi_2^\varepsilon(x)$ and
2. $d(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d^\varepsilon(x)$.

Then π fulfils the following two equations

$$0 = \frac{b(1)}{q(1,2)} \frac{d^2}{dx^2} ((1-x^2)\pi(x)) + \frac{2a(1)}{q(1,2)} \frac{d}{dx} (x\pi(x)) + d(x) \quad (5.4)$$

$$0 = \frac{b(2)}{q(2,1)} \frac{d^2}{dx^2} ((1-x^2)\pi(x)) + \frac{2a(2)}{q(2,1)} \frac{d}{dx} (x\pi(x)) - d(x). \quad (5.5)$$

Summing equation (5.4) and (5.5), we get

$$0 = \left(\frac{b(1)}{q(1,2)} + \frac{b(2)}{q(2,1)} \right) \frac{d^2}{dx^2} ((1-x^2)\pi(x)) + \left(\frac{2a(1)}{q(1,2)} + \frac{2a(2)}{q(2,1)} \right) \frac{d}{dx} (x\pi(x)).$$

If we compare this with the differential equation for the stationary distribution of the model without switching (3.4), we see that it has the same form as the stationary distribution $\tilde{\pi}$ of the model without super agent with parameters $\tilde{a} = \left(\frac{a(1)}{q(1,2)} + \frac{a(2)}{q(2,1)} \right)$ and $\tilde{b} = \left(\frac{b(1)}{q(1,2)} + \frac{b(2)}{q(2,1)} \right)$.

This means that our two components $\pi_1^\varepsilon, \pi_2^\varepsilon$ converge, for $\varepsilon \rightarrow 0$, to a stationary distribution of a model without super agent. The two parameters of this model are a combination of the four parameters of the model with a super agent. We obtain

- $\lim_{\varepsilon \rightarrow 0} \pi_1^\varepsilon(x) = \frac{q(2,1)}{q(1,2)+q(2,1)} \tilde{\pi}(x)$ and
- $\lim_{\varepsilon \rightarrow 0} \pi_2^\varepsilon(x) = \frac{q(1,2)}{q(1,2)+q(2,1)} \tilde{\pi}(x)$.

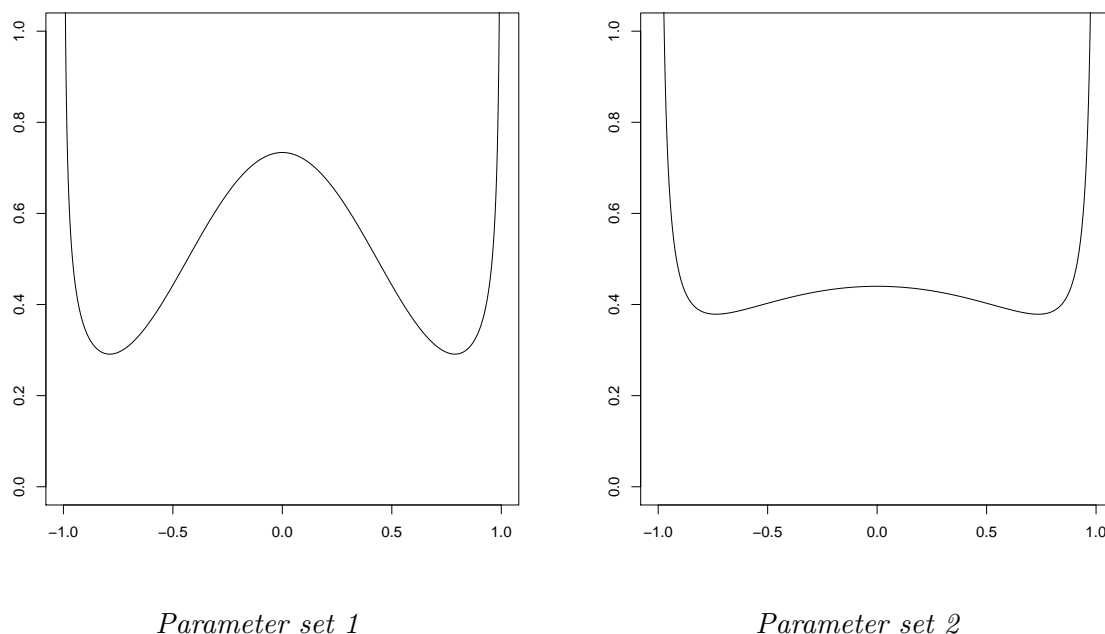


Figure 5.6: Approximation of the distribution $\hat{\pi}$ for the two parameter sets.

5.4 Conclusion

In this chapter we looked at the stationary distribution of the switching diffusion process (X, Y) , but for our financial market only the behaviour of the optimistic agents is of

importance. So we are interested in the stationary behaviour of the first component X of our switching diffusion (X, Y) . Due to this we set $\hat{\pi} : [-1, 1] \rightarrow \mathbb{R}_{\geq 0}$ with

$$\hat{\pi}(x) = \pi_1(x) + \pi_2(x) = \pi(x, 1) + \pi(x, 2).$$

The approximations of $\hat{\pi}$ for the two sets of parameters are shown in Figure 5.6. We see there that the new model with a super agent leads to a more general stationary behaviour of the herding process than the model without a super agent.

Chapter 6

Convergence of fast switching processes

In this chapter we look at our model in the case of fast switching. This means that the super agent switches his opinion very fast. In Section 6.2 we look at the limiting switching diffusion process of our model. Then, in Section 6.3, we look at the fast switching birth and death processes with a fixed number of agents.

6.1 Fast switching of the super agent

As defined in Section 4.1, the process of the super agent $(Y_t)_{t \geq 0}$ is a homogeneous Markov jump process on the finite set $\mathcal{M} = \{1, \dots, M\}$ with transition probabilities

$$\begin{aligned} P(Y_{t+\delta} = j | Y_t = i) &= q(i, j)\delta + o(\delta), \quad j \neq i, \\ P(Y_{t+\delta} = i | Y_t = i) &= 1 + q(i, i)\delta + o(\delta), \end{aligned}$$

with $0 > q(i, i) = -\sum_{j \neq i} q(i, j)$. We now assume that the process Y has an unique stationary distribution $\pi : \mathcal{M} \rightarrow [0, 1]$, this exists e.g., if $q(i, j) > 0$ for all $i \neq j$.

Starting from this, for all $\varepsilon > 0$, the fast switching super agent process $(Y_t^\varepsilon)_{t \geq 0}$ is a homogeneous Markov jump process on the finite set $\mathcal{M} = \{1, \dots, M\}$ with transition probabilities

$$\begin{aligned} P(Y_{t+\delta}^\varepsilon = j | Y_t^\varepsilon = i) &= \frac{1}{\varepsilon} q(i, j)\delta + o(\delta), \quad j \neq i, \\ P(Y_{t+\delta}^\varepsilon = i | Y_t^\varepsilon = i) &= 1 + \frac{1}{\varepsilon} q(i, i)\delta + o(\delta). \end{aligned}$$

If we look at the probabilistic representation from Subsection 2.2.1, the processes are characterised by the transition functions $p_{ij} = \frac{q(i, j)}{-q(i, i)}$ of the embedded processes, and by the holding times. They are exponentially distributed with parameter $-q(i, i)$.

For the fast switching super agent process Y^ε we see that

$$p_{ij}^\varepsilon = \frac{\frac{1}{\varepsilon} q(i, j)}{-\frac{1}{\varepsilon} q(i, i)} = \frac{q(i, j)}{-q(i, i)} = p_{ij},$$

meaning the probability that the process Y^ε switches from i to j is the same as for the process Y . If $\sigma(i), \sigma^\varepsilon(i)$ are the holding times of the processes Y, Y^ε in the point i , it follows

$$P(\sigma^\varepsilon(i) \leq t) = 1 - e^{-\frac{1}{\varepsilon}q(i,i)t} = 1 - e^{q(i,i)\frac{t}{\varepsilon}} = P\left(\sigma(i) \leq \frac{t}{\varepsilon}\right).$$

This shows that the process Y^ε is a time transformation of the process Y , in the sense of

$$(Y_t^\varepsilon)_{t \geq 0} \stackrel{d}{=} (Y_{\frac{t}{\varepsilon}})_{t \geq 0}.$$

Due to this fact we can define $Y_t^\varepsilon := Y_{\frac{t}{\varepsilon}}$. From this it follows that π is also the unique stationary distribution of Y^ε .

6.2 Convergence of fast switching diffusion processes

In this section we give an accurate mathematical verification of the empirical result from Subsection 5.3.1. For this we show that the agent component X^ε of the fast switching diffusion $(X^\varepsilon, Y^\varepsilon)$ defined by the generator in Equation (5.3) converges to an ordinary diffusion process. This convergence is shown for a general class of fast switching diffusion processes in [22]. Our purpose now is to adapt the notations and prove the necessary conditions for this convergence.

6.2.1 Fast switching diffusion process

To define the fast switching diffusion process $(X^\varepsilon, Y^\varepsilon)$ we start with our fast switching super agent process Y^ε defined in Section 6.1. Using

$$B(x, y) = -[(a_1(y) + a_2(y))x + a_2(y) - a_1(y)],$$

$$\sigma^2(x, y) = 2b(y)(1 - x^2) \text{ and } Q_\varepsilon = \frac{1}{\varepsilon}Q,$$

we obtain a fast switching diffusion process $(X^\varepsilon, Y^\varepsilon)$ as in 4.2.3, 4.2.4 with infinitesimal generator given by Equation (5.3).

6.2.2 Notations and conditions from [22]

First we need some notations from [22] to define the necessary conditions for the convergence of the fast switching diffusion process.

Let σ be the first jump time of the super agent process Y . Then we define $c : [-1, 1] \times \mathcal{M} \rightarrow \mathbb{R}$ by

$$P(\sigma > t | \mathcal{F}_t) = e^{\int_0^t c(X_s, Y_0) ds},$$

where \mathcal{F}_t is the σ -algebra generated by (X_s, Y_0) for $s \leq t$. In [22] this notation is needed as there may be influences from the process X to Y . In our case, c is constant on $[-1, 1] \times \{y\}$ for all elements $y \in \mathcal{M}$. Hence we can interpret c as a function on \mathcal{M} that is given by

$$P(\sigma > t | Y_0 = y) = e^{tc(y)} \text{ for all } y \in \mathcal{M}.$$

From the probabilistic representation in Subsection 2.2.1 and the definition of Y in Section 6.1, it follows that

$$c(y) = -q(y, y) = \sum_{\tilde{y} \neq y} q(y, \tilde{y}). \quad (6.1)$$

Furthermore in [22], for all $x \in [-1, 1]$ the operator $\Pi_x : B(\mathcal{M}) \rightarrow B(\mathcal{M})$ are defined by

$$\Pi_x g(y) = -c(x, y)g(y) + c(x, y) \int g(z) \hat{Q}(x, y, dz),$$

where $\hat{Q}(x, y, dy) = P(Y_\sigma \in dy | X_\sigma = X_{\sigma-} = x, Y_{\sigma-} = y)$. Here an integral term is needed because the number of elements in \mathcal{M} may be infinite. In our model Π_x depends not on x and is given by

$$\Pi_x g(y) = -c(y)g(y) + c(y) \sum_{\tilde{y} \neq y} g(\tilde{y}) \frac{q(y, \tilde{y})}{-q(y, y)}.$$

From Equation (6.1) we get

$$\Pi_x g(y) = \sum_{\tilde{y} \neq y} q(y, \tilde{y}) [g(\tilde{y}) - q(y)] \text{ for all } x \in [-1, 1].$$

With these notations we can define the necessary conditions for the convergence of the fast switching diffusion process.

Definition 6.1. In [22] the following conditions are used:

1. $c(x, y) \geq 0$.
2. Π_x is continuous in x .
3. The functions $B(x, y)$ and $\sigma^2(x, y)$ are jointly measurable functions of their variables, they are continuous in x uniformly with respect to y and

$$\sup_{x \in [-1, 1]} \frac{|B(x, y)| + |\sigma^2(x, y)|}{1 + |x|} < \infty.$$

4. There exists a unique homogeneous Markov process to the infinitesimal generator

$$Af(x, y) = B(x, y) \frac{df(x, y)}{dx} + \frac{1}{2} \sigma^2(x, y) \frac{d^2 f(x, y)}{dx^2} + Qf(x, y).$$

5. There exists a constant k such that for all $x \in [-1, 1]$ and $y \in \mathcal{M}$

$$|B(x, y)| + |\sigma^2(x, y)| \leq k(1 + |x|).$$

6. There exists an increasing upwards convex function Ψ on $[0, \infty)$ such that $\Psi(0) = 0$, $\Psi(lx) \leq l\Psi(x)$ for $l > 1$, and for all $x_1, x_2 \in [-1, 1]$

$$\|\Pi_{x_1} - \Pi_{x_2}\| \leq \Psi(|x_1 - x_2|^2).$$

7. The family of Markov processes in \mathcal{M} with generating operators Π_x is uniformly ergodic; if π_x is an ergodic distribution for the process with generating operator Π_x and $P^x(t, y, dz)$ is the transition probability for this process, then there exists a constant $c > 0$ such that for all $T \in [0, \infty)$ holds

$$\left| \frac{1}{T} \int_0^T \int f(z) P^x(t, y, dz) dt \int f(z) \pi_x(dz) \right| \leq \frac{c}{T} \|f\|.$$

6.2.3 Convergence of the fast switching diffusion

In this subsection we introduce the convergence theorem from [22] and show the convergence of the agent components X^ε from the fast switching diffusion processes $(X^\varepsilon, Y^\varepsilon)$ to an ordinary diffusion process.

Theorem 6.2. *Assume conditions 1 – 7 from Definition 6.1 hold. Then the processes X^ε , where $(X^\varepsilon, Y^\varepsilon)$ are as defined in Subsection 6.2.1 with initial conditions $X_0^\varepsilon = x_0$ and $Y_0^\varepsilon = y_0$, weakly converge in $D_{[-1,1]}[0, \infty)$ to a diffusion process \hat{X} with drift coefficient $\hat{B}(x) = \int B(x, y) \pi^x(dy)$ and diffusion coefficient $\hat{\sigma}^2(x) = \int \sigma^2(x, y) \pi^x(dy)$ and initial condition $\hat{X}_0 = x_0$.*

Proof. See [22], Theorem 8, Chapter II.

In our case Theorem 6.2 says that the agent components X^ε of the fast switching diffusion processes $(X^\varepsilon, Y^\varepsilon)$ converge to a diffusion process \hat{X} on $[-1, 1]$ with drift and diffusion coefficient

$$\hat{B}(x) = \sum_{y \in \mathcal{M}} B(x, y) \pi(y) \quad \text{and} \quad \hat{\sigma}^2(x) = \sum_{y \in \mathcal{M}} \sigma^2(x, y) \pi(y).$$

Now, our task is to prove the seven conditions from Definition 6.1.

Condition 1 follows directly from the definition in Section 6.1. Due to the fact that Π_x is constant over all $x \in [-1, 1]$ conditions 2 and 6 hold. Conditions 3 and 5 hold because $B(x, y)$ and $\sigma^2(x, y)$ are by definition continuous in x and bounded. We have shown in Chapter 4 that the infinitesimal generator

$$Af(x, y) = B(x, y) \frac{df(x, y)}{dx} + \frac{1}{2} \sigma^2(x, y) \frac{d^2f(x, y)}{dx^2} + Qf(x, y)$$

generates a Feller semigroup, this proves condition 4. Condition 7 follows again from the definition in Section 6.1 and the fact that process Y is not effected by process X .

6.2.4 Conclusion

By proving all conditions we get by Theorem 6.2 that the agent components from the fast switching diffusion processes weakly converge in $D_{[-1,1]}[0, \infty)$ to an ordinary diffusion process. Due to the fact that this also implies that the stationary distributions converge to the stationary distribution of the limiting diffusion process, this proves the empirical

results from Subsection 5.3.

After showing the convergence at fast switching for the limiting models in this section, we will take a look at the pre-limiting model in the next section.

6.3 Convergence of fast switching birth and death processes

In Section 6.2 we saw that the switching diffusions converge to a limiting diffusion of a model without switching if the super agent switches his opinion very fast. In this section we look at the behaviour in the pre-limiting case. We look at the model from Section 4.1 with a finite number $N \in \mathbb{N}$ of agents. We show that this model converges to a model without switching, introduced in Section 3.2, in the limit of fast switching. To show this convergence we use a result for convergence of semimartingales introduced in [15]. But first we take a look at the model under fast switching.

6.3.1 Fast switching birth and death process

To define the fast switching birth and death process we take, for all $\varepsilon > 0$, the fast switching super agent process Y^ε from Section 6.1 and define the corresponding fast switching agent process X^ε , as in Section 4.1, on $E_N := \{-1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1\}$ by

$$P(X_{t+\delta}^\varepsilon = \bar{x} | X_t^\varepsilon = x, Y_t^\varepsilon = y) = q(x, \bar{x}, y)\delta + o(\delta) \quad (6.2)$$

with

$$\begin{aligned} q\left(x, x + \frac{2}{N}, y\right) &= \lambda(x, y), \\ q\left(x, x - \frac{2}{N}, y\right) &= \mu(x, y), \\ q(x, \bar{x}, y) &= 0 \quad \bar{x} \notin \left\{x, x + \frac{2}{N}, x - \frac{2}{N}\right\}. \end{aligned}$$

Then $(X^\varepsilon, Y^\varepsilon)$ is the fast switching birth and death process. In our model from Section 4.1 the birth and death rates are given by $\lambda(x, y) = \frac{N^2}{4}(1-x) \left(\frac{2a_1(y)}{N} + b(y)(1+x)\right)$ and $\mu(x, y) = \frac{N^2}{4}(1+x) \left(\frac{2a_2(y)}{N} + b(y)(1-x)\right)$.

In the following we will show that the processes X^ε weakly converge in $D_{\mathbb{R}}[0, \infty)$ for $\varepsilon \rightarrow 0$ to a birth and death process X on E_N with birth and death rates

$$\begin{aligned} \lambda(x) &= \sum_{y \in \mathcal{M}} [\lambda(x, y)\pi(y)], \\ \mu(x) &= \sum_{y \in \mathcal{M}} [\mu(x, y)\pi(y)]. \end{aligned}$$

The problem is that the processes X^ε are no longer Markov processes, so we cannot use the convergence methods for Markov processes. But the processes X^ε are semimartingales, and due to this we are able to use the methods for semimartingales from [15]. To use these methods we first calculate the semimartingale characteristics, introduced in Definition 2.12, of the processes X^ε and of the limiting process X .

6.3.2 Semimartingale characteristics of the fast switching agent process

First we fix a continuous truncation function h . To simplify the calculations we take the function

$$h(x) = \begin{cases} x & \text{for } x \leq \frac{1}{N}, \\ \frac{2}{N} - x & \text{for } \frac{1}{N} < x \leq \frac{2}{N}, \\ 0 & \text{for } x > \frac{2}{N}. \end{cases}$$

First we calculate the semimartingale characteristics of X^ε with respect to the truncation function h . As in Definition 2.11 we calculate

$$\begin{aligned} \tilde{X}^\varepsilon(h)_t &= \sum_{s \leq t} [\Delta X_s^\varepsilon - h(\Delta X_s^\varepsilon)] = \sum_{s \leq t} \Delta X_s^\varepsilon, \\ X^\varepsilon(h) &= X^\varepsilon - \tilde{X}^\varepsilon(h) = X^\varepsilon - \sum_{s \leq t} \Delta X_s^\varepsilon. \end{aligned}$$

Then is $X^\varepsilon(h)_t = X_0^\varepsilon$ for all $t \geq 0$. From this and due to the fact that X^ε is a pure jump process it follows that $B^\varepsilon(h)_t = 0$ and $C_t^\varepsilon = 0$ for all $t \geq 0$.

Before we calculate the compensator ν^ε of the random measure μ^{X^ε} we use Theorem 2.14 to calculate the compensator $\hat{\nu}^\varepsilon$ of the random measure $\mu^{(X^\varepsilon, Y^\varepsilon)}$. From Theorem 2.14 we get that $\hat{\nu}^\varepsilon(du, dt)(\omega) = \hat{K}^\varepsilon((X_t^\varepsilon, Y_t^\varepsilon)(\omega), du)dt$ with

$$\hat{K}^\varepsilon((x, y), u) = \lambda(x, y)1_{\{u=(\frac{2}{N}, 0)\}} + \mu(x, y)1_{\{u=(-\frac{2}{N}, 0)\}} + \sum_{\bar{y} \neq y} \frac{1}{\varepsilon} q(y, \bar{y})1_{\{u=(0, \bar{y}-y)\}}.$$

Define $\nu^\varepsilon(dx, dt) = K_t^\varepsilon(dx)dt$ with

$$\begin{aligned} K_t^\varepsilon\left(\frac{2}{N}\right) &= \lambda(X_t^\varepsilon, Y_t^\varepsilon) = \sum_{y \in \mathcal{M}} [\lambda(X_t^\varepsilon, y)1_{\{Y_t^\varepsilon=y\}}], \\ K_t^\varepsilon\left(-\frac{2}{N}\right) &= \mu(X_t^\varepsilon, Y_t^\varepsilon) = \sum_{y \in \mathcal{M}} [\mu(X_t^\varepsilon, y)1_{\{Y_t^\varepsilon=y\}}], \\ K_t^\varepsilon(x) &= 0 \text{ for } x \notin \left\{-\frac{2}{N}, \frac{2}{N}\right\}. \end{aligned}$$

From the definition of the compensator in Theorem 2.10, it is sufficient to show that

$$E(W * \mu_\infty^{X^\varepsilon}) = E(W * \nu_\infty^\varepsilon)$$

for all nonnegative predictable processes $W : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$.

Let $W : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative predictable process. Then define $\hat{W} : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\hat{W}(\omega, s, (x, y)) = \begin{cases} W(\omega, s, x) & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

From the definition of \hat{W} and due to the fact that there are no simultaneous jumps of the processes X^ε and Y^ε , it follows that

$$E(\hat{W} * \mu_\infty^{(X^\varepsilon, Y^\varepsilon)}) = E(W * \mu_\infty^{X^\varepsilon}) \text{ and } E(\hat{W} * \hat{\mu}_\infty^\varepsilon) = E(W * \mu_\infty^\varepsilon).$$

From the fact that $\hat{\nu}^\varepsilon$ is the compensator of $\mu^{(X^\varepsilon, Y^\varepsilon)}$ and that \hat{W} is a nonnegative predictable process, it follows that

$$E(\hat{W} * \mu_\infty^{(X^\varepsilon, Y^\varepsilon)}) = E(\hat{W} * \hat{\mu}_\infty^\varepsilon).$$

Hence

$$E(W * \mu_\infty^{X^\varepsilon}) = E(W * \nu_\infty^\varepsilon).$$

For the limiting birth and death process X we get the characteristics in the same way. They are given by

$$\begin{aligned} B(h) &= 0, \\ C &= 0, \\ \nu(dx, dt) &= K_t(dx)dt \text{ with} \\ K_t\left(\frac{2}{N}\right) &= \lambda(X_t) = \sum_{y \in \mathcal{M}} [\lambda(X_t, y)\pi(y)], \\ K_t\left(-\frac{2}{N}\right) &= \mu(X_t) = \sum_{y \in \mathcal{M}} [\mu(X_t, y)\pi(y)], \\ K_t(x) &= 0 \text{ for } x \notin \left\{-\frac{2}{N}, \frac{2}{N}\right\}. \end{aligned}$$

We have calculated the semimartingale characteristics for the agent process under fast switching and for the limiting birth and death process. In the next subsection we look at the necessary methods and definitions that we need for the convergence of the agent process under fast switching to a birth and death process.

6.3.3 Convergence of semimartingales and a continuous-time ergodic theorem

Before we look at the main theorem for the convergence of semimartingales from [15] we start with some necessary definitions.

Definition 6.3. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ a deterministic function and ν be a predictable random measure on $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Then we define $g * \nu : \Omega \times \mathbb{R}_{\geq 0}$ by

$$(g * \nu_t)(\omega) = \int_{[0,t]} \int_{\mathbb{R}} g(x) \nu(\omega)(ds, dx),$$

see Definition 2.8 for the corresponding definition for stochastic processes instead of deterministic functions. In the case that $\nu(dt, dx) = K_t(dx)dt$ we get

$$(g * \nu_t)(\omega) = \int_{[0,t]} \int_{\mathbb{R}} g(x) K_s(dx) ds.$$

Definition 6.4 (strong majorization hypothesis). Let $\mathcal{V}(B(h))$ be the variation process of $B(h)$. There is a continuous and deterministic function $F : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ which strongly majorizes the functions $\mathcal{V}(B(h))(\omega)$ and $C(\omega) + \min\{x^2, 1\} * \nu(\omega)$ for all $\omega \in \Omega$. Strongly majorizes means that the difference is an increasing function.

The second definition gives us a condition on the big jumps.

Definition 6.5 (condition on the big jumps). For all $t \geq 0$ we have:

$$\limsup_{n \uparrow \infty} \nu([0, t] \times \{x : |x| > n\})(\omega) = 0.$$

Definition 6.6. For a semimartingale triplet $(B(h), C, \nu)$ define

$$\tilde{C}(h) = C + h^2 * \nu - \sum_{s \leq \cdot} \Delta B_s(h)^2.$$

We set $\mathcal{C}_2(\mathbb{R})$ equal to the set of all continuous bounded functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which are zero around zero. From this we define $\mathcal{C}_1(\mathbb{R})$ as a subset of $\mathcal{C}_2(\mathbb{R})$ with some special properties. These properties are not important for our results and can be found in [15]. With these definitions we are able to formulate the theorem for semimartingale convergence.

Theorem 6.7. Let $X^n : \Omega^n \rightarrow D_{\mathbb{R}}[0, \infty)$ be a sequence of semimartingales with characteristics $(B^n(h), C^n, \nu^n)$ and initial distributions η^n . Ω is the canonical space $D_{\mathbb{R}}[0, \infty)$ with the canonical process X and it is equipped with the triplet $(B(h), C, \nu)$. Let η be a probability measure on \mathbb{R} . Let $\phi^n : \Omega^n \rightarrow \Omega$ be measurable functions. Let D be a dense subset in $\mathbb{R}_{\geq 0}$, and assume:

1. The strong majoration hypothesis 6.4.
2. The condition 6.5 on big jumps.
3. Uniqueness: the martingale problem $s(\sigma(X_0), X | \eta; B, C, \nu)$ has a unique solution P .
4. Continuity condition: For all $t \in D$, $g \in \mathcal{C}_1(\mathbb{R})$, the functions

$$B_t(h), \tilde{C}_t(h), g * \nu_t : \Omega = D_{\mathbb{R}}[0, \infty) \rightarrow \mathbb{R}$$

are Skorokhod-continuous.

5. $\eta^n \rightarrow \eta$ in distribution.

6. (a) $P^n(\sup_{t \leq R} |B^n(h)_t - B_t(h) \circ \phi^n| > \xi) \rightarrow 0$ for $n \rightarrow \infty$ and for all $\xi, R > 0$.

(b) $P^n(|\tilde{C}_t^n(h) - \tilde{C}_t(h) \circ \phi^n| > \xi) \rightarrow 0$ for $n \rightarrow \infty$ and for all $\xi, t > 0$.

(c) $P^n(|g * \nu_t^n - (g * \nu_t) \circ \phi^n| > \xi) \rightarrow 0$ for $n \rightarrow \infty$ and for all $\xi, t > 0$ and all $g \in \mathcal{C}_1(\mathbb{R})$.

Then the law of $\mathcal{L}(X^n)$ weakly converges to P in $D_{\mathbb{R}}[0, \infty)$.

Proof. See [15], Theorem 3.21, Chapter IX.

For our special case of a fast switching birth and death process Theorem 6.7 can be reduced to the following theorem.

Theorem 6.8. Let $(X^\varepsilon)_{\varepsilon > 0}, X$ be as defined in Subsection 6.3.1 with $X_0^\varepsilon \stackrel{d}{=} X_0$ for all $\varepsilon > 0$. And let $\phi : D_{[-1,1] \times \mathcal{M}}[0, \infty) \rightarrow D_{[-1,1]}[0, \infty)$ with $\phi(\omega^1 \times \omega^2) = \omega^1$. Assume that the following conditions hold:

1. The strong majoration hypothesis 6.4.

2. Continuity condition: For all $t \in D, g \in \mathcal{C}_1(\mathbb{R})$, the function

$$g * \nu_t : \Omega = D_{[-1,1]}[0, \infty) \rightarrow \mathbb{R}$$

is Skorokhod-continuous.

3. $P^\varepsilon(|g * \nu_t^\varepsilon - (g * \nu_t) \circ \phi| > \xi) \rightarrow 0$ for $n \rightarrow \infty$ and for all $\xi, t > 0$ and all $g \in \mathcal{C}_1(\mathbb{R})$.

Then X^ε converges weakly to X in $D_{[-1,1]}[0, \infty)$.

Proof. We show that under these three conditions the six conditions from Theorem 6.7 are fulfilled. The second condition of Theorem 6.7 holds because the limiting process X has no big jumps. Due to the fact that $B(h) = C = 0$, see 6.3.2 and that

$$\tilde{C}(h) = C + h^2 * \nu - \sum_{s \leq \cdot} \Delta B_s(h)^2 = 0$$

the fourth and the sixth condition of Theorem 6.7 reduce to the second and third condition of Theorem 6.8. By definition of the birth and death process X from Subsection 6.3.1 the process X is the unique process corresponding to triple $(B(h), C, \nu)$. Hence its distribution P on $D_{\mathbb{R}}[0, \infty)$ is the unique solution of the martingale problem $s(\sigma(X_0), X|\eta; B, C, \nu)$. This proves condition 3. At last condition five of Theorem 6.7 follows from the fact that $X_0^\varepsilon \stackrel{d}{=} X_0$ for all $\varepsilon > 0$. \square

We will use this theorem in the next subsection to show the convergence of the fast switching agent processes. For proving the sixth condition of Theorem 6.7 we need the following law of large numbers.

Theorem 6.9 (law of large numbers for stationary Markov processes). *Let $Y : \Omega \rightarrow S$ be a stationary Markov process on a finite set S with unique stationary distribution π . Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t 1_{\{Y_s=y\}} ds = \pi(y) \text{ a.s.}$$

Proof. See [7], Chapter II, Section 4c.

Now we made all necessary preparations to prove the convergence of the fast switching agent processes X^ε .

6.3.4 Convergence of the fast switching agent processes

For all $\varepsilon > 0$ set $\Omega^\varepsilon = D_{[-1,1]}[0, \infty) \times D_{\mathcal{M}}[0, \infty)$ with the probability measure P^ε given by the process $(X_t^\varepsilon, Y_t^\varepsilon)_{t \geq 0} = (X_t^\varepsilon, Y_{\frac{t}{\varepsilon}})_{t \geq 0}$, defined in Subsection 6.3.1. Then $X^\varepsilon : \Omega^\varepsilon \rightarrow D_{[-1,1]}[0, \infty)$ is given by

$$X_t^\varepsilon(\omega^1, \omega^2) = \omega_t^1 \text{ for all } (\omega^1, \omega^2) \in \Omega^\varepsilon.$$

Let $\mathcal{L}(X^\varepsilon)$ be the distribution of X^ε on $D_{\mathbb{R}}[0, \infty)$. Set $\Omega = D_{\mathbb{R}}[0, \infty)$ with the probability measure P introduced by the limiting birth and death process defined in Subsection 6.3.1. Then $X(\omega)_t = \omega_t$ for all $\omega \in \Omega$. Set $\phi : \Omega^\varepsilon \rightarrow \Omega$ with $\phi((\omega^1, \omega^2)) = \omega^1$.

At first we verify the third condition of Theorem 6.8.

Proposition 6.10. *It holds that*

$$P^\varepsilon(|g * \nu_t^\varepsilon - (g * \nu_t) \circ \phi^\varepsilon| > \xi) \rightarrow 0 \text{ if } \varepsilon \rightarrow \infty$$

for all $\xi, t > 0$ and all $g \in \mathcal{C}_1(\mathbb{R})$.

Proof. Let $t > 0$ and $g \in \mathcal{C}_1(\mathbb{R})$. Set $\alpha := g(\frac{2}{N})$ and $\beta := g(-\frac{2}{N})$.

Let \bar{X} denote the canonical process on $D_{[-1,1]}[0, \infty)$ and let \bar{P} be the probability measure on $D_{[-1,1]}[0, \infty)$ induced by a birth and death process on E_N with birth and death rates

$$\bar{\lambda}(x) = \sum_{y \in \mathcal{M}} \lambda(x, y) \text{ and } \bar{\mu}(x) = \sum_{y \in \mathcal{M}} \mu(x, y).$$

Let $0 = \tau_0^\varepsilon < \tau_1^\varepsilon < \tau_2^\varepsilon < \dots$ denote the jump times of X^ε and $\bar{\tau}_1$ the first jump time of \bar{X} . Further n_t^ε and \bar{n}_t are the numbers of jumps of X^ε and \bar{X} up to time t . Due to the fact that the birth and death rates of \bar{X} are bigger than those of X^ε , it is clear that

$$P^\varepsilon(\tau_1^\varepsilon \leq s) \leq \bar{P}(\bar{\tau}_1 \leq s) \text{ for all } s > 0 \text{ and } P^\varepsilon(n_t^\varepsilon \leq n) \geq \bar{P}(\bar{n}_t \leq n) \text{ for all } n \in \mathbb{N}.$$

Let $\xi, \delta > 0$. Let $t_0 > 0$ with $\bar{P}(\bar{\tau}_1 \leq t_0) \leq \frac{\delta}{4}$ and $\bar{n} \in \mathbb{N}$ with $\bar{P}(\bar{n}_t > \bar{n}) \leq \frac{\delta}{4}$.

Due to Theorem 6.9

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s 1_{\{Y_r=y\}} dr = \pi(y) \text{ a.s. for all } y \in \mathcal{M}.$$

Hence there exists an $\varepsilon_0 > 0$ with

$$P^\varepsilon \left(\sup_{T \geq t_0} \max_{y \in \mathcal{M}} \left| \frac{\varepsilon}{T} \int_0^{\frac{T}{\varepsilon}} 1_{\{Y_s=y\}} ds - \pi(y) \right| > \tilde{\xi} \right) \leq \frac{\delta}{2}$$

for all $\varepsilon < \varepsilon_0$ and $\tilde{\xi} = \frac{1}{2MC(\bar{n}+1)t}$, where $\bar{C} = \max_{(x,y) \in E_N \times \mathcal{M}} (\alpha\lambda(x,y) + \beta\mu(x,y))$.
Let $\varepsilon < \varepsilon_0$ and

$$A_\varepsilon := \left\{ (\omega^1, \omega^2) \in \tilde{\Omega} : \sup_{T \geq t_0} \max_{y \in \mathcal{M}} \left| \frac{\varepsilon}{T} \int_0^{\frac{T}{\varepsilon}} 1_{\{Y_s(\omega^2)=y\}} ds - \pi(y) \right| > \tilde{\xi} \right\}.$$

For a simpler notation define $\tilde{\tau}_i^\varepsilon = \min\{\tau_i^\varepsilon, t\}$. Then $\tilde{\tau}_i^\varepsilon(\omega) = \tau_i^\varepsilon(\omega)$ for all $i \leq n^\varepsilon(\omega)$ and $\tilde{\tau}_{n^\varepsilon(\omega)+1}^\varepsilon(\omega) = t$.

Let $\omega = (\omega^1, \omega^2) \in A_\varepsilon^c \cap \{\tau_1^\varepsilon > t_0\} \cap \{n_t^\varepsilon \leq \bar{n}\}$, then follows with Definition 6.3

$$\begin{aligned} & |(g * \nu_t^\varepsilon)(\omega) - (g * \nu_t) \circ \phi(\omega)| \\ &= \left| \int_0^t g \left(\frac{2}{N} \right) \sum_{y \in \mathcal{M}} [\lambda(X_s^\varepsilon(\omega^1), y) 1_{\{Y_s^\varepsilon(\omega^2)=y\}}] + g \left(-\frac{2}{N} \right) \sum_{y \in \mathcal{M}} [\mu(X_s^\varepsilon(\omega^1), y) 1_{\{Y_s^\varepsilon(\omega^2)=y\}}] ds \right. \\ & \quad \left. - \int_0^t g \left(\frac{2}{N} \right) \sum_{y \in \mathcal{M}} [\lambda(X_s(\omega^1), y) \pi(y)] + g \left(-\frac{2}{N} \right) \sum_{y \in \mathcal{M}} [\mu(X_s(\omega^1), y) \pi(y)] ds \right| \\ &= \left| \int_0^t \sum_{y \in \mathcal{M}} [(\alpha\lambda(\omega_s^1, y) + \beta\mu(\omega_s^1, y)) 1_{\{Y_s^\varepsilon(\omega^2)=y\}}] ds \right. \\ & \quad \left. - \int_0^t \sum_{y \in \mathcal{M}} [(\alpha\lambda(\omega_s^1, y) + \beta\mu(\omega_s^1, y)) \pi(y)] ds \right| \\ &= \left| \int_0^t \sum_{y \in \mathcal{M}} (\alpha\lambda(\omega_s^1, y) + \beta\mu(\omega_s^1, y)) [1_{\{Y_s^\varepsilon(\omega^2)=y\}} - \pi(y)] ds \right| \\ &= \left| \sum_{y \in \mathcal{M}} \sum_{i=1}^{n_t^\varepsilon(\omega^1)+1} \int_{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}^{\tilde{\tau}_i^\varepsilon(\omega^1)} (\alpha\lambda(\omega_s^1, y) + \beta\mu(\omega_s^1, y)) [1_{\{Y_s^\varepsilon(\omega^2)=y\}} - \pi(y)] ds \right| \\ &= \left| \sum_{y \in \mathcal{M}} \sum_{i=1}^{n_t^\varepsilon(\omega^1)+1} \underbrace{(\alpha\lambda(\omega_{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}^1, y) + \beta\mu(\omega_{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}^1, y))}_{C(\omega_{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}^1, y)} \int_{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}^{\tilde{\tau}_i^\varepsilon(\omega^1)} [1_{\{Y_s^\varepsilon(\omega^2)=y\}} - \pi(y)] ds \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \sum_{y \in \mathcal{M}} \sum_{i=1}^{n_i^\varepsilon(\omega^1)+1} C(\omega_{\tilde{\tau}_{i-1}^\varepsilon}^1, y) \left(\int_0^{\tilde{\tau}_i^\varepsilon(\omega^1)} [1_{\{Y_s^\varepsilon(\omega^2)=y\}} - \pi(y)] ds \right. \right. \\
 &\quad \left. \left. - \int_0^{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)} [1_{\{Y_s^\varepsilon(\omega^2)=y\}} - \pi(y)] ds \right) \right| \\
 &= \left| \sum_{y \in \mathcal{M}} \sum_{i=1}^{n_i^\varepsilon(\omega^1)+1} C(\omega_{\tilde{\tau}_{i-1}^\varepsilon}^1, y) \left(\tilde{\tau}_i^\varepsilon(\omega^1) \frac{\varepsilon}{\tilde{\tau}_i^\varepsilon(\omega^1)} \int_0^{\frac{\tilde{\tau}_i^\varepsilon(\omega^1)}{\varepsilon}} [1_{\{Y_u(\omega^2)=y\}} - \pi(y)] du \right. \right. \\
 &\quad \left. \left. - \tilde{\tau}_{i-1}^\varepsilon(\omega^1) \frac{\varepsilon}{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)} \int_0^{\frac{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}{\varepsilon}} [1_{\{Y_u(\omega^2)=y\}} - \pi(y)] du \right) \right|.
 \end{aligned}$$

Due to the definition of \bar{C}

$$\begin{aligned}
 &|(g * \nu_t^\varepsilon)(\omega) - (g * \nu_t) \circ \phi(\omega)| \\
 &\leq \sum_{y \in \mathcal{M}} \sum_{i=1}^{n_i^\varepsilon(\omega^1)+1} \bar{C} \left(\tilde{\tau}_i^\varepsilon(\omega^1) \left| \frac{\varepsilon}{\tilde{\tau}_i^\varepsilon(\omega^1)} \int_0^{\frac{\tilde{\tau}_i^\varepsilon(\omega^1)}{\varepsilon}} [1_{\{Y_u(\omega^2)=y\}}] du - \pi(y) \right| \right. \\
 &\quad \left. + \tilde{\tau}_{i-1}^\varepsilon(\omega^1) \left| \frac{\varepsilon}{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)} \int_0^{\frac{\tilde{\tau}_{i-1}^\varepsilon(\omega^1)}{\varepsilon}} [1_{\{Y_u(\omega^2)=y\}}] du - \pi(y) \right| \right) \\
 &\leq \sum_{y \in \mathcal{M}} \sum_{i=1}^{n_i^\varepsilon(\omega^1)+1} \bar{C} 2t\tilde{\xi} \\
 &\leq \sum_{y \in \mathcal{M}} \sum_{i=1}^{\bar{n}+1} \bar{C} 2t\tilde{\xi} \\
 &= 2M\bar{C}(\bar{n}+1)t\tilde{\xi} = \xi.
 \end{aligned}$$

Hence

$$\begin{aligned}
 P^\varepsilon(|g * \nu_t^\varepsilon - (g * \nu_t) \circ \phi| > \xi) &= 1 - P^\varepsilon(|g * \nu_t^\varepsilon - (g * \nu_t) \circ \phi| \leq \xi) \\
 &\leq 1 - P^\varepsilon(A_\varepsilon^c \cap \{\tau_1^\varepsilon > t_0\} \cap \{n_t^\varepsilon \leq \bar{n}\}) \\
 &= P^\varepsilon(A_\varepsilon \cup \{\tau_1^\varepsilon \leq t_0\} \cup \{n_t^\varepsilon > \bar{n}\}) \\
 &\leq P^\varepsilon(A_\varepsilon) + P^\varepsilon(\tau_1^\varepsilon \leq t_0) + P^\varepsilon(n_t^\varepsilon > \bar{n}) \\
 &\leq \frac{\delta}{2} + \bar{P}(\bar{\tau}_1 \leq t_0) + \bar{P}(\bar{n}_t > \bar{n}) \\
 &\leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta.
 \end{aligned}$$

□

For the proof of the second condition of Theorem 6.8 we need a further proposition.

Proposition 6.11. *Let*

$$J(\omega) = \{t \in \mathbb{R}_{\geq 0} : \omega(t) \neq \omega_{t-} := \lim_{t_n \uparrow t} \omega_{t_n}\}$$

for all $\omega \in D_{\mathbb{R}}[0, \infty)$ and $f_s : D_{\mathbb{R}}[0, \infty) \rightarrow \mathbb{R}$ with $f_s(\omega) = \omega_s$. Then f_s is Skorokhod-continuous in all $\omega \in D_{\mathbb{R}}[0, \infty)$ with $s \notin J(\omega)$.

Proof. See [15], Proposition 2.1, Chapter VI.

Now we can prove the second condition.

Proposition 6.12. *Let $t \in \mathbb{R}_{\geq 0}$ and $g \in \mathcal{C}_1(\mathbb{R})$. Then the function*

$$g * \nu_t : D_{[-1,1]}[0, \infty) \rightarrow \mathbb{R}$$

is Skorokhod-continuous.

Proof. Let $\omega \in D_{[-1,1]}[0, \infty)$ and $(\omega^n)_{n \in \mathbb{N}} \in (D_{[-1,1]}[0, \infty))^{\mathbb{N}}$ with ω^n converging to ω with respect to the Skorokhod metric. Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} (g * \nu_t)(\omega^n) \\ &= \lim_{n \rightarrow \infty} \int_{[0,t]} g\left(-\frac{2}{N}\right) \left(\sum_{y \in \mathcal{M}} \mu(\omega_s^n, y) \pi(y) \right) + g\left(\frac{2}{N}\right) \left(\sum_{y \in \mathcal{M}} \lambda(\omega_s^n, y) \pi(y) \right) ds \\ &= g\left(-\frac{2}{N}\right) \lim_{n \rightarrow \infty} \int_{[0,t] \cap J(\omega)^c} \left(\sum_{y \in \mathcal{M}} \frac{N^2}{4} (1 + \omega_s^n) \left(\frac{2a_2(y)}{N} + b(y)(1 - \omega_s^n) \right) \pi(y) \right) ds \\ &+ g\left(\frac{2}{N}\right) \lim_{n \rightarrow \infty} \int_{[0,t] \cap J(\omega)^c} \left(\sum_{y \in \mathcal{M}} \frac{N^2}{4} (1 - \omega_s^n) \left(\frac{2a_1(y)}{N} + b(y)(1 + \omega_s^n) \right) \pi(y) \right) ds. \\ &= g\left(-\frac{2}{N}\right) \int_{[0,t] \cap J(\omega)^c} \left(\sum_{y \in \mathcal{M}} \frac{N^2}{4} (1 + \lim_{n \rightarrow \infty} \omega_s^n) \left(\frac{2a_2(y)}{N} + b(y)(1 - \lim_{n \rightarrow \infty} \omega_s^n) \right) \pi(y) \right) ds \\ &+ g\left(\frac{2}{N}\right) \int_{[0,t] \cap J(\omega)^c} \left(\sum_{y \in \mathcal{M}} \frac{N^2}{4} (1 - \lim_{n \rightarrow \infty} \omega_s^n) \left(\frac{2a_1(y)}{N} + b(y)(1 + \lim_{n \rightarrow \infty} \omega_s^n) \right) \pi(y) \right) ds. \\ &= (g * \nu_t)(\omega). \end{aligned}$$

Here the permutation of the integration and the limit is possible due to the dominated convergence theorem, and $\lim_{n \rightarrow \infty} \omega_s^n = \omega_s$ on $J(\omega)^c$ by Proposition 6.11. \square

Theorem 6.13. X^ε weakly converges to X in $D_{[-1,1]}[0, \infty)$.

Proof. To prove the convergence of X^ε to X we have to verify the three conditions from Theorem 6.8.

For the first condition we have to find a deterministic increasing function F that strictly majorizes the function $\min\{x^2, 1\} * \nu(\omega)$ for all $\omega \in \Omega$.

With $C = \max_{(x,y) \in E_N \times \mathcal{M}} (\lambda(x, y) + \mu(x, y))$ we get

$$\begin{aligned} \min\{x^2, 1\} * \nu_t(\omega) &= \frac{4}{N^2} \int_0^t \sum_{y \in \mathcal{M}} [\lambda(X_s(\omega), y) + \mu(X_s(\omega), y)] \pi(y) ds \\ &\leq \frac{4}{N^2} \int_0^t \sum_{y \in \mathcal{M}} C \pi(y) ds \\ &= \frac{4}{N^2} Ct \end{aligned}$$

This function can clearly be strictly majorized.

The second condition follows from Proposition 6.12 and the third condition is proved in Proposition 6.10. \square

6.3.5 Interpretation of the fast switching convergence in the agent model

We saw in this section that the agent component of our model with a super agent tends to a birth and death process, if the super agent switches his opinion very fast. The behaviour of the trading agents in our starting model is characterised by the functions $a, b : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$. Here $a(y)$ describes the overall tendency to switch and $b(y)$ the tendency due to group pressure if the opinion of the super agent is $y \in \mathcal{M}$. The behaviour of the super agent is given by the transition matrix $Q = (q_{ij})_{i,j \in \mathcal{M}}$. For the limiting case we need the further assumption that there exists a unique limiting distribution $\pi : \mathcal{M} \rightarrow [0, 1]$ for the process of the super agent. This process is a pure jump process on \mathcal{M} with transition matrix Q . Due to the fact that \mathcal{M} is a finite set, this assumption is fulfilled under weak conditions on Q .

Then the limiting process is a birth and death process on $[-1, 1]$ with birth and death rates

$$\begin{aligned} \lambda(x) &= \sum_{y \in \mathcal{M}} \left[\frac{N^2}{4} (1-x) \left(\frac{2a(y)}{N} + b(y)(1+x) \right) \right] \pi(y), \\ \mu(x) &= \sum_{y \in \mathcal{M}} \left[\frac{N^2}{4} (1+x) \left(\frac{2a(y)}{N} + b(y)(1-x) \right) \right] \pi(y). \end{aligned}$$

Let us look at a model without a super agent, as defined in Section 3.2, with parameters

$$\tilde{a} = \sum_{y \in \mathcal{M}} a(y) \pi(y) \quad \text{and} \quad \tilde{b} = \sum_{y \in \mathcal{M}} b(y) \pi(y).$$

Then the agent process for this model is a birth and death process with birth and death rates

$$\begin{aligned}\tilde{\lambda}(x) &= \frac{N^2}{4}(1-x) \left(\frac{2 \left(\sum_{y \in \mathcal{M}} a(y) \pi(y) \right)}{N} + \left(\sum_{y \in \mathcal{M}} b(y) \pi(y) \right) (1+x) \right), \\ \tilde{\mu}(x) &= \frac{N^2}{4}(1+x) \left(\frac{2 \left(\sum_{y \in \mathcal{M}} a(y) \pi(y) \right)}{N} + \left(\sum_{y \in \mathcal{M}} b(y) \pi(y) \right) (1-x) \right).\end{aligned}$$

Obviously we have $\tilde{\lambda} = \lambda$ and $\tilde{\mu} = \mu$. Hence the limiting model in the case of a fast switching model with a super agent may be viewed as a model without super agent. The parameters are weighted sums of the parameters in the model with a super agent, and the weights are constituted by the stationary distribution of the super agent process.

6.3.6 Conclusion

An intuitive guess may be that very fast switching in the environment of the market participants might lead to instability of the market. The mathematical analysis shows that this is not the case. In fact, the market becomes stable and we may even compute for given Q the stationary distribution.

Appendix A

Mathematical basics for stochastic processes and their convergence

Let (Ω, \mathcal{A}, P) be a probability space and (E, \mathfrak{C}) a measurable space. Further let $X : \Omega \times [0, \infty) \rightarrow E$ be a stochastic process.

A.1 Basic definitions

In this section we give a short overview of the basic mathematical definitions for stochastic processes. For more detailed information we refer to [9] and [16].

We start with the flow of information on the probability space (Ω, \mathcal{A}, P) that is given by a filtration.

Definition A.1 (filtration). A family of σ -fields $(\mathcal{F}_t)_{t \geq 0}$ is called filtration, if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$. A filtration $(\mathcal{F}_t)_{t \geq 0}$ is right continuous if $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ for all $t \in [0, \infty)$, and called complete, if $\{A \in \mathcal{A} : P(A) = 0\} \subseteq \mathcal{F}_0$.

For the rest of the appendix let $(\mathcal{F}_t)_{t \geq 0}$ be a right continuous and complete filtration.

Definition A.2 (adapted). The stochastic process $X : \Omega \times [0, \infty) \rightarrow E$ is called adapted to $(\mathcal{F}_t)_{t \geq 0}$, if $X_t : \Omega \rightarrow E$ is \mathcal{F}_t -measurable, i.e. $X_t^{-1}(C) \in \mathcal{F}_t$ for all $C \in \mathfrak{C}$.

Next we define the path of a stochastic process.

Definition A.3 (cadlag, cag, continuous). For all $\omega \in \Omega$ the function $\alpha^\omega : [0, \infty) \rightarrow E$ with $\alpha_t^\omega = X_t(\omega)$ is called the path of X relative to ω . The stochastic process X is called continuous if all paths $(\alpha^\omega)_{\omega \in \Omega}$ are continuous and cadlag (continue à droite, limitée à gauche) if all paths are right continuous and with existing left limits. A process is called cag (continue à gauche) if all paths are left continuous.

From now on we assume that X is a cadlag process.

Definition A.4 (conditional expectation). Let $Y : \Omega \rightarrow E$ be an integrable random variable, i.e. $E|Y| < \infty$, then the conditional expectation of Y given \mathcal{F}_t is denoted by $E(Y|\mathcal{F}_t)$ and is a \mathcal{F}_t -measurable random variable such that

$$\int_A E(Y|\mathcal{F}_t)dP = \int_A YdP, \text{ for all } A \in \mathcal{F}_t.$$

The main properties of the conditional expectation are given in the next proposition.

Proposition A.5. *Let $Y, Y_1, Y_2 : \Omega \rightarrow E$ be integrable random variables and let $s, t \in [0, \infty]$ with $s < t$. Then holds*

1. $E(E(Y|\mathcal{F}_t)) = E(Y)$.
2. $E(aY_1 + bY_2|\mathcal{F}_t) = aE(Y_1|\mathcal{F}_t) + bE(Y_2|\mathcal{F}_t)$ for all $a, b \in \mathbb{R}$.
3. If $Y_1 \leq Y_2$ then $E(Y_1|\mathcal{F}_t) \leq E(Y_2|\mathcal{F}_t)$.
4. $E(hY|\mathcal{F}_t) = hE(Y|\mathcal{F}_t)$ for all \mathcal{F}_t -measurable bounded random variables h .
5. $E(E(Y|\mathcal{F}_t)|\mathcal{F}_s) = E(Y|\mathcal{F}_s)$.

Proof. See [16], Theorem 6.1.

Now we are able to define martingales.

Definition A.6 (martingale). The process X is a martingale, if it is $(\mathcal{F}_t)_{t \geq 0}$ adapted, integrable and for all $t \in [0, \infty)$ holds

$$E(X_t|\mathcal{F}_s) = X_s \text{ for all } s < t.$$

A.2 Weak convergence in $D_E[0, \infty)$

In this section we introduce the space $D_E[0, \infty)$ for a metric space (E, r) and show the main definitions for the weak convergence in $D_E[0, \infty)$. References for more details are [8],[12] and [15].

Definition A.7 ($D_E[0, \infty)$). We denote by $D_E[0, \infty)$ the space of all cadlag functions $\alpha : [0, \infty) \rightarrow E$.

On this space we define the Skorokhod metric.

Definition A.8 (Skorokhod metric). Let Λ be the collection of all strictly increasing surjective functions $\lambda : [0, \infty) \rightarrow [0, \infty)$ which are Lipschitz continuous with

$$\gamma(\lambda) = \sup_{s>t \geq 0} \left| \ln \left(\frac{\lambda(s) - \lambda(t)}{s - t} \right) \right| < \infty.$$

For $\alpha, \beta \in D_E[0, \infty)$, define the Skorokhod metric

$$d(\alpha, \beta) = \inf_{\lambda \in \Lambda} \max \left\{ \gamma(\lambda), \int_0^\infty e^{-u} d(\alpha, y, \beta, u) du \right\},$$

where

$$d(\alpha, \beta, \lambda, u) = \sup_{t \geq 0} \tilde{r}(\alpha_{\min\{t, u\}}, \beta_{\min\{\lambda(t), u\}}).$$

Here \tilde{r} is the metric $\tilde{r}(x, y) = \min\{r(x, y), 1\}$ on E .

Then it follows that $(D_E[0, \infty), d)$ is a metric space and the following theorem holds.

Theorem A.9. *If E is separable, then $D_E[0, \infty)$ is separable. If (E, r) is complete, then $(D_E[0, \infty), d)$ is complete.*

Proof. See [12], Theorem 5.6, Chapter 3.

The Skorokhod topology on $D_E[0, \infty)$ is the topology generated by the Skorokhod metric. To get a probability space we need a σ -field on $D_E[0, \infty)$. For this we take the Borel σ -field \mathcal{D}_E , that is generated by all open sets relative to the Skorokhod topology. Another characterisation of the σ -field \mathcal{D}_E is given in the next proposition.

Proposition A.10. *For all $t \in [0, \infty)$ define $\pi_t : D_E[0, \infty) \rightarrow E$ by $\pi_t(\alpha) = \alpha_t$. Then, if E is separable,*

$$\mathcal{D}_E = \sigma(\{\pi_t : t \in [0, \infty)\}),$$

the σ -field generated by $\{\pi_t : t \in [0, \infty)\}$.

A.2.1 Convergence in $D_E[0, \infty)$

Let (E, r) be a separable and complete metric space. For a stochastic process $X : \Omega \rightarrow D_E[0, \infty)$ the law $\mathcal{L}(X)$ of X is the probability measure on the measurable space $(D_E[0, \infty), \mathcal{D}_E)$ generated by X , i.e.

$$(\mathcal{L}(X))(A) = P(X^{-1}(A)) = P(\{\omega \in \Omega : X(\omega) \in A\}) \text{ for all } A \in \mathcal{D}_E.$$

We consider a sequence $(X^n)_{n \in \mathbb{N}}$ of E defined on probability spaces $(\Omega^n, \mathcal{A}^n, P^n)$.

Definition A.11 (weak convergence in $D_E[0, \infty)$). We say that $(X^n)_{n \in \mathbb{N}}$ weakly converge to X in $D_E[0, \infty)$ or equivalent the laws $\mathcal{L}(X^n)$ weakly converge to $\mathcal{L}(X)$ in $D_E[0, \infty)$, if the laws $\mathcal{L}(X^n)$ converge to $\mathcal{L}(X)$ in distribution, i.e. for all bounded Skorokhod continuous functions $g : D_E[0, \infty) \rightarrow \mathbb{R}$

$$\int g d\mathcal{L}(X^n) \rightarrow \int g d\mathcal{L}(X).$$

Definition A.12 (convergence in finite-dimensional distributions). Let D be a subset of $[0, \infty)$. Then $(X^n)_{n \in \mathbb{N}}$ converges to X in finite-dimensional distributions along D , if

$$(X^n_{t_1}, \dots, X^n_{t_k})_{n \in \mathbb{N}} \text{ converges to } (X_{t_1}, \dots, X_{t_k}) \text{ in distribution for all } t_i \in D, k \in \mathbb{N}.$$

In the next propositions we show connectedness between the weak convergence in $D_E[0, \infty)$ and the convergence in finite-dimensional distributions.

Proposition A.13. *Let $J(X) = \{t \in [0, \infty) : P(\Delta X_t \neq 0) > 0\}$ and let $(X^n)_{n \in \mathbb{N}}$ weakly converges to X in $D_E[0, \infty)$. Then $(X^n)_{n \in \mathbb{N}}$ converges to X in finite-dimensional distributions along $D = [0, \infty) \setminus J(X)$.*

Proof. See [15], Proposition 3.15, Chapter VI.

Definition A.14 (tightness). The sequence $(X^n)_{n \in \mathbb{N}}$ is tight if for every $\varepsilon > 0$ there is a compact set K of E such that $P^n(X^n \notin K) \leq \varepsilon$ for all $n \in \mathbb{N}$.

Proposition A.15. *If $(X^n)_{n \in \mathbb{N}}$ is tight and there exists a dense subset D of $[0, \infty)$ such that $(X^n)_{n \in \mathbb{N}}$ converges to X in finite-dimensional distributions along D , then $(X^n)_{n \in \mathbb{N}}$ weakly converges to X in $D_E[0, \infty)$.*

Proof. See [12], Theorem 7.8, Chapter 3.

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Erklärung

Ich versichere an Eides Statt, die vorliegende Arbeit selbständig angefertigt zu haben. Die einzigen Hilfsmittel waren die angegebene Literatur sowie der Rat meines akademischen Lehrers. Die Arbeit ist unter Einhaltung der Regeln guter wissenschaftlicher Praxis entstanden und ist bisher weder veröffentlicht, noch zur Veröffentlichung eingereicht worden, noch hat sie ganz oder zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Des Weiteren versichere ich, noch keinen Promotionsversuch unternommen zu haben.

Kiel, den 13.07.2010