

Hedging in Affine Stochastic Volatility Models

Dissertation

zur Erlangung des Doktorgrades

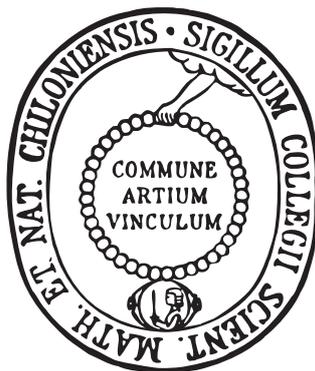
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Zusammenfassung

Ein Investor, der ein Derivat emittiert, geht durch dieses Geschäft üblicherweise ein Risiko ein. Er kann versuchen dieses Risiko zu verringern, indem er in dem zugrundeliegenden Wertpapier gemäß einer – in einem gewissen Sinne – angemessenen Strategie handelt. In einem unvollständigen Finanzmarkt gibt es normalerweise mehrere sinnvolle Möglichkeiten zur Bestimmung einer solchen Absicherungsstrategie. Wir erörtern zwei auf Optimierungskriterien basierende Ansätze: asymptotisches nutzenbasiertes Hedgen unter Verwendung der exponentiellen Nutzenfunktion und varianz-optimales Hedgen. Dies geschieht jeweils unter der Annahme, dass das zugrundeliegende Wertpapier durch eine Teilklasse der affinen Prozesse modelliert werden kann. Diese Klassen enthalten einige gängige affine stochastische Volatilitätsmodelle wie z.B. das Barndorff-Nielsen & Shephard-Modell und das Heston-Modell. Darüber hinaus umfassen sie Modelle, die sowohl Sprünge in dem zugrundeliegenden Wertpapier als auch in dessen Volatilität erlauben. Wir erhalten für beide Absicherungsmethoden semi-explizite Ausdrücke für die relevanten Größen (d.h. lineare Näherungen für die optimale Absicherungsstrategie und den zugehörigen Nutzenindifferenzpreis für das asymptotische nutzenbasierte Hedgen, die optimale Strategie, das optimale Anfangskapital und die damit verbundene Varianz des Hedgefehlers für das varianz-optimale Hedgen). Wir verwenden hauptsächlich allgemeine Strukturaussagen, die Laplace-Transformation und den Semimartingalkalkül zur Bestimmung dieser Ergebnisse. Für das asymptotische nutzenbasierte Hedgen benötigen wir zusätzlich eine Charakterisierung der Lösung des Erwartungsnutzenproblems ohne zusätzliches Derivat und des zugehörigen Dichteprozesses des Martingalmaßes mit minimaler Entropie. Wir bestimmen diese Charakterisierung in dem etwas allgemeineren Rahmen von mehrdimensionalen affinen Prozessen. Wir illustrieren die Ergebnisse der beiden Absicherungsansätze durch einige numerische Beispiele, in denen die entsprechenden Lösungen in einigen konkreten stochastischen Volatilitätsmodellen berechnet werden.

Abstract

Typically an investor incurs risk by issuing a contingent claim. She can try to reduce this risk by trading in the underlying asset according to a strategy which is in some sense appropriate. In an incomplete financial market there usually are several meaningful choices for the determination of such a hedging strategy. We discuss two of them based on optimization criteria: asymptotic exponential utility-based hedging and variance-optimal hedging. We investigate these approaches under the assumption that the underlying can be described by subclasses of affine processes in each case. These classes contain some popular affine stochastic volatility models, e.g. the Barndorff-Nielsen & Shephard model and the Heston model. Furthermore they allow for models which exhibit jumps in the return process as well as in the volatility component. For both hedging approaches we obtain semi-explicit expressions for the objects of interest (i.e. first order approximations of the optimal hedging strategy and the associated utility indifference price for asymptotic utility-based hedging as well as the optimal hedging strategy, optimal initial endowment and the corresponding variance of the hedging error for quadratic hedging). We mainly use general structural results, a Laplace transform technique and semimartingale calculus in order to derive these findings. For the exponential utility-based hedging approach we additionally require a characterization of the density process of the minimal entropy martingale measure and the associated solution to the exponential utility maximization problem without additional claims. We provide these in the more general framework of multivariate affine processes. Finally, some numerical examples illustrate the results of the two hedging approaches by computing the relevant quantities in concrete stochastic volatility models.

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Chapter 1

Introduction

Derivatives are one of the key issues in Mathematical Finance. Besides several standard products (e.g. European and American claims) the financial universe also contains a number of exotic instruments (e.g. path-dependent claims, timer options). This variety of derivatives requires several mathematical techniques in order to answer the questions which typically arise for an issuer of an option. In principle these problems can be classified as follows:

1. *Pricing* of the contingent claim: How much money does the issuer have to demand from the buyer for selling the option?
2. Determination of an appropriate *hedging strategy*: How can the issuer reduce the risk arising from the sold claim by dynamic trading?
3. Quantification of the *remaining risk*: How can the risk which can not be eliminated by hedging be measured and how big is it?

Apparently one needs a mathematical model for the financial market in order to tackle these problems. Whether the answers to these questions are connected to each other then depends on this model and the methods used to solve the problems. The aim of this dissertation is to develop formulas for prices, hedging strategies and remaining risk which can be evaluated semi-explicitly in models which reflect the statistical behaviour of financial markets adequately. We discuss the classical case of a European-type contingent claim H with a fixed finite maturity T written on a single underlying asset (e.g. a stock). Exotic options are beyond the scope of this thesis. However, a treatment of such derivatives probably requires an understanding of the methods used for standard products. Additionally we suppose that the issuer is only allowed to trade in the underlying S and in a riskless asset (e.g. a money account). We ignore almost all frictions (e.g. transaction costs, taxes) which usually arise in financial markets. We only deviate from an idealized market hypothesis by modelling the asset price process as a semimartingale such that the resulting financial market is *incomplete*. More specifically, we assume that the underlying can be described by an *affine stochastic volatility model*. This kind of models are able to capture some of the so-called *stylized facts* observed in financial time series. Furthermore they typically lead to an *incomplete market*.

Here – in contrast to a complete market – it is in general not possible to choose the price of the claim and the hedging strategy in such a way that the option can be replicated. Therefore *perfect replication* is no longer the link between the three problems above. Hence we investigate two approaches – *variance-optimal hedging* and *asymptotic exponential utility-based pricing and hedging* – in the body of this dissertation which partially give answers to the above questions.

Before we start our analysis we try to embed the approaches presented in this thesis into the theory of Mathematical Finance by giving a brief historic review (cf. [96, 51] for more details). The problem of pricing European call and put options was first tackled by Bachelier [4, 170]. He assumed that the underlying asset is a martingale under the physical probability measure and computed the expected value of the payoff of the option under this measure. He obtained an integral representation for this expectation by proposing a Brownian motion as model for the underlying. Several decades later the spirit of his ideas re-entered econometrics and Mathematical Finance. Osborne [147] and Samuelson [165] advocated the usage of a geometric Brownian motion instead of a Brownian motion in order to model stock prices. The Webner-Fechner law was employed by Osborne in order to attain that logarithmic returns are normally distributed. Additionally, he affirmed the hypothesis of this kind of return distribution by investigating the empirical distribution of US stock data. Samuelson calculated among other things the pricing formula for European call options in a more general framework containing a geometric Brownian motion. For this type of model he obtained a representation analogous to the expression of Bachelier by using a martingale argument as well. Merton & Samuelson [135] proposed utility-based pricing (in particular they suggested a kind of marginal utility-based price) in order to enhance this result. They got a similar pricing formula as in [165] but with the main difference that the expected value had to be computed under a measure which does in general not coincide with the physical probability measure. Black & Scholes [14] and Merton [132] combined the concepts of no arbitrage and replication in order to evaluate European call and put options. Beyond the seminal Black-Scholes formula they found a consistent answer to the three main questions noted above if the underlying is modelled as geometric Brownian motion. In this case one can build a dynamic portfolio which replicates the payoff of the option at the maturity of the claim. Therefore the remaining risk is reduced to zero if one follows the hedging strategy suggested by the weights of the underlying and the riskless asset in the dynamic portfolio.

However the brilliant idea of perfect replication relies heavily on the *martingale representation property* (cf. e.g. [75, 76]) holding for the (geometric) Brownian motion. It is shown in [25] that only a Poisson process shares this property within the class of Lévy processes. Unfortunately, a model for the return process built from a single Poisson process seems quite unrealistic. Furthermore, if one just modifies the Black-Scholes framework by introducing stochastic volatility in terms of a single jump of the volatility parameter to one of two predefined values at a fixed date then the resulting model is already incomplete if the volatility process is non-traded, cf. e.g. [75, Section 6.3]. One should note that beyond

the mentioned Brownian motion and Poisson process there are semimartingales exhibiting jumps of random size for which the martingale representation property is also satisfied (e.g. Azéma martingales [155, Chapter IV, Sections 5 & 8]). However this kind of semimartingales are academic examples rather than actually used in Mathematical Finance (but cf. e.g. [39] for an asset price model which is built on such semimartingales). Therefore, apart from pathological examples, completeness of a financial market is essentially linked to the assumption that the returns of the asset price are normally distributed. That a normal distribution provides a good fit for the percentage changes of commodity prices was doubted by several authors already in the first half of the 20th century (e.g. [137, Chapter III], [136]). Cootner [32] noticed that the logarithmic returns of weekly observed stock data are often significantly *leptokurtic*. Therefore he modified the classical random walk hypothesis by introducing (flexible) barriers for the stock price. Another way to accommodate the *fat tails* was proposed by Mandelbrot [129]. He generalized the Gaussian assumption by suggesting a *stable law* for the distribution of logarithmic returns. In the comprehensive survey of Fama [47] this model was examined for time series of the members of the DOW JONES stock index. In view of the leptokurtic daily returns Fama concluded that a non-Gaussian law seems appropriate amongst the stable distributions. Since this leads to a model with infinite variance he advocated also the usage of other long-tailed probability laws. Additionally he found some empirical evidence for another hypothesis of Mandelbrot. This says that the independence assumption does not hold for returns following large fluctuations of returns since the absolute values of such subsequent changes tend also to be high. This stylized fact is nowadays known as *volatility clustering* or *persistence of volatility*.

The previous studies are some of the first contributions to the area of time series analysis of asset prices. In the following decades econometricians laid a focus on the enhancement of models for the dynamics of asset prices with respect to stylized facts. Besides the already noted properties (fat tails and volatility clustering) returns reveal even more characteristic features (e.g. [150, 27] and the references therein). E.g. on the equity markets big slumps typically occur more often than sharp rises. Therefore the empirical distribution of returns is *asymmetric*. Moreover a sharp decline of share prices usually raises the volatility. This phenomenon is called *leverage effect*.

There are several categories of models which are able to reproduce some of these stylized facts. Important classes are GARCH-type models, c.f. e.g. [43, 16, 145, 44], in discrete-time and stochastic volatility models, c.f. e.g. [90, 178, 188, 82, 83, 7, 18, 20, 102], in continuous-time. Although the distinction between discrete and continuous time seems natural one could raise the question of the continuous time limit of a discrete time model. As shown by [144, 114, 128] the answer to this question depends in the case of a GARCH model on the limiting procedure. One could obtain a diffusion-type stochastic volatility model [144] or a stochastic volatility model [114, 128] without a diffusion component in the volatility process but with jumps in the volatility as well as in the return process. The

latter called COGARCH model was introduced by [118] and statistical tests [77] indicate that it is capable to provide a good fit to real financial data with respect to some stylized facts.

Besides empirical features of financial data the choice of a category of models depends also on the tractability of this class for the application in focus (e.g. forecasting volatility, pricing and hedging of derivatives, etc.). The questions arising in Mathematical Finance are often answered in a continuous-time setting. This is probably due to the tractability of such a framework for pricing and hedging issues. In particular this applies to affine processes which find a use in this dissertation in order to determine results. The choice of this class of continuous-time processes is governed by several insights. First, exponential moments of the relevant processes are used for some methods, which are often applied to solve the problems from above (e.g. option pricing). For affine processes the computation of exponential moments and the associated characteristic function is possible via the solution of generalized Riccati equations by the results of [40, 48]. Therefore in principle some of the problems occurring in financial mathematics (cf. e.g. [40, Section 13] and the references therein) can be reduced to the determination of a solution of ordinary differential equations. This finding provides the basis for the applicability of affine processes for our purposes. Moreover the affine structure can be exploited in statistical methods (e.g. the generalized method of moments (c.f. e.g. [97, 190, 105])) in order to estimate the parameters of the process. Finally, affine processes encompass stochastic volatility models which allow for some important stylized facts (e.g. heavy tails, volatility clustering, asymmetric returns, leverage effect).

In the main part of this thesis we use for the illustration of our results a class of models proposed by [18]. There the technique of subordination (c.f. e.g. [26, 68]) is applied to *time-change* Lévy processes. This procedure leads to models exhibiting a kind of *stochastic activity* which is related to stochastic volatility. This activity can be motivated by the idea that trading is not uniform over time in a financial market. Times with a high trading frequency are followed by calm periods and vice versa. This interpretation is connected to the belief that in *business time* trading probably takes place more uniformly than in *real-time* and that the time change relates both time scales. More precisely, let $z = \log(S)$ be the logarithm of the asset price process S . Then the basic model of [18] is given by

$$z(t) = z(0) + \ell(V(t)), \quad (1.1)$$

$$dV(t) = v(t-)dt. \quad (1.2)$$

Here, ℓ denotes a Lévy process, v the activity process and V the time-change. [18] discusses different choices for the activity process, e.g. a non-Gaussian Ornstein-Uhlenbeck process

$$v(t) = -\lambda v(t-)dt + dr(t). \quad (1.3)$$

Here, λ denotes a constant and r an increasing Lévy process independent from ℓ . Note that the leverage effect, which is excluded by the independence assumption in the basic model,

can be incorporated by adding an additional term to the return process z (cf. [18, Sections 6 & 7]).

If the dynamics of the model (1.1-1.3) are expressed via semimartingale characteristics and ℓ is chosen as Brownian motion, we can recognize (cf. [102, Section 4]) the dynamics of the stochastic volatility model proposed by Barndorff-Nielsen & Shephard [7]. Hence the above model is a generalization of [7]. Analogously, the Heston model [82] is generalized by using a square-root process as activity process in [18].

The results of [105, Sections 3 & 4] show that the model (1.1-1.3) can be fitted to real financial data with respect to skewness, kurtosis and the autocorrelation functions of the returns and squared returns. Furthermore it allows for jumps in the return process as well as in the activity process. There is some evidence (cf. [45]) indicating that this is a desirable feature of a stochastic volatility model. The need for jumps in the volatility process can be motivated by the observation that a sharp decline of the asset price is typically followed by very active trading which can be modelled by a rapidly increasing volatility.

In Figure 1.1 we compare the daily logarithmic returns of the German equity index DAX with return series generated from simulations of a Brownian motion and a NIG-IG-OU process in (1.1-1.3). This means that ℓ is chosen as *Normal Inverse Gaussian* process (henceforth NIG process, cf. e.g. [172, Section 5.3.8]) and that the activity process v has a stationary *Inverse Gaussian* law (henceforth IG law, cf. e.g. [7, Section 2.3] or [172, Section 5.5.2] for details about the IG-OU process). The parameters used for the simulations are taken from [105]. Figure 1.1 illustrates that in contrast to normal distributed returns the series based on the simulation of a NIG-IG-OU process exhibits a similar inhomogeneous behaviour as the DAX returns.

Up to now the focus has mainly been laid on two insights. First even if we ignore other frictions like transaction costs and discrete-time trading, real financial time series suggest the use of mathematical models leading to incomplete markets. Second, statistical methods exist which allow us to fit some of these models to the observed data. At this stage the original problem comes into play:

What are answers to the three questions raised above in incomplete financial markets?

A rigorous treatment of all relevant approaches proposed to solve these problems is far beyond the scope of this thesis. Nevertheless we give some brief comments about a few interesting concepts and certain general developments in this area. At first we note that researchers have been discussing the above questions at least since the sixties of the last century. E.g. in view of the results of Mandelbrot, Samuelson [165] allowed stable processes in his framework. Merton [134, 133] modified the framework of Black & Scholes by

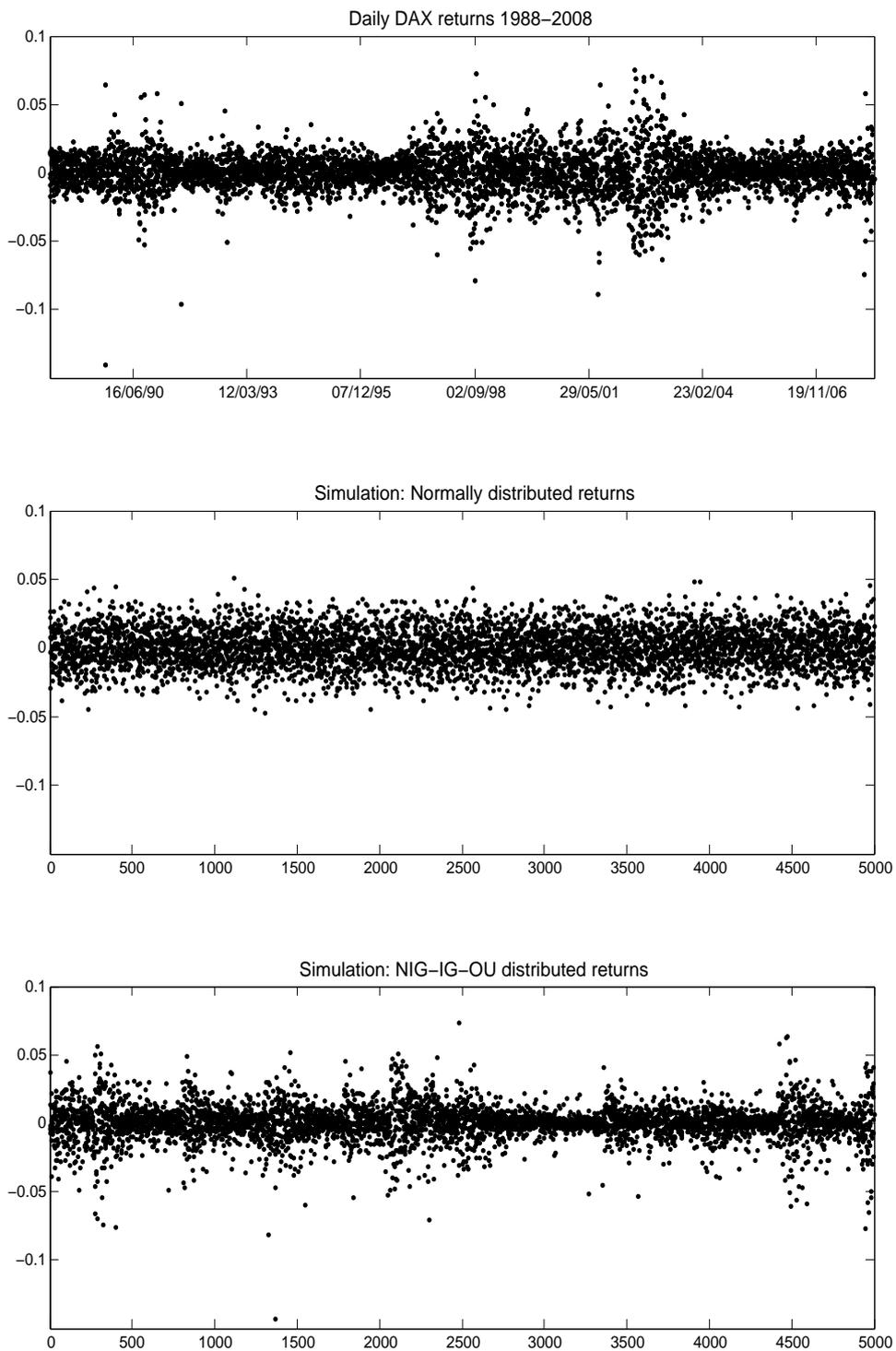


Figure 1.1: Comparison of historic and simulated returns

adding a compound Poisson process in the return process. By supposing that the risk arising from the additional jump component can be diversified completely, he chose a certain equiv-

alent martingale measure in modern terminology. Using a conditioning argument under this measure the value of the option can then be expressed by a weighted sum of Black-Scholes prices (cf. [28, Section 10.1] for further details and for a discussion of the diversification assumption). Furthermore he investigated the misspecification error which occurs if one uses the Black & Scholes model when in reality the underlying follows the process proposed by him. Here, the error was measured in terms of differences between the option prices in the two models.

Beyond these first steps in this area there exist a number of approaches which can be roughly separated as follows. If hedging and risk do not matter then an answer to the pricing issue is given by the evaluation of a contingent claim under an adequately chosen martingale measure (cf. e.g. [103, Sections 2 & 3] for a brief overview). This measure can be determined e.g. via *calibration* (cf. [29, 12, 30] and the references therein) to prices of liquidly traded derivatives observed on the market. We do not go into details about this technique since the focus of this dissertation lies on approaches which are hedging-based. The reason for considering such approaches is due to the fact that a rational investor cannot normally ignore the consequences of her actions, since she has a certain attitude towards risk. This preference can be described e.g. by risk measures or utility functions which are two examples of the approaches discussed below. Here one should note that some of these methods can also be utilized if the investor is only interested in a fair price of a contingent claim (cf. e.g. [84, Section 5] for an overview). The investor can use the calculated price based on her individual preferences, in order to decide whether an observed market price of this derivative seems reasonable to her.

We start our overview of hedging-based approaches by recapitulating that we consider the case of a European claim H with maturity T and that hedging takes place by trading in the underlying asset S . Furthermore we ignore interest rates for the sake of clarity, i.e. we assume that the short rate of the riskless asset is zero. A lot of the methods which are presented here have in common that one builds up a portfolio

$$V(t) := w + \theta \cdot S(t), \quad t \in [0, T], \quad (1.4)$$

endowed with some initial capital w and adjusted according to a strategy θ which belongs to a space of admissible strategies Θ . The value of this portfolio is compared with the payoff of the claim at the maturity T . Besides some technical details (e.g. in general the choice of the space of admissible strategies depends on the method used to solve the problem), which we ignore in this introduction, the key difference between the different approaches is how this comparison takes place. Here the *hedging error* $H - V(T)$ often plays a crucial role.

A main class of the investigated approaches is driven by certain optimization criteria (cf. e.g. [52, 171] for an overview). E.g. it is intuitively appealing to reduce the probability that the *shortfall* $(H - V(T))^+$ is strictly positive to zero. This idea leads to the concept of *superhedging* initially investigated by [42, 119, 53]. Roughly speaking, here one looks for the

minimal amount of money w and a corresponding trading strategy θ such that the portfolio (1.4) delivers at least the payoff of the option H at the maturity, i.e. $V(T) \geq H$. Using an optional decomposition theorem for supermartingales one can show that under some mild conditions such a strategy exists and that the initial endowment is given as the upper bound of the open interval of prices which are consistent with no arbitrage (cf. [119, Theorem 3.2] or [53, Section 1]). However, superhedging leads to undesirable results from the perspective of a practitioner. E.g. [55, Remark 8.1] considered a modification of the Black-Scholes model in which the volatility jumps at a certain date. In contrast to the example of [75, Section 6.3] discussed above here the volatility can take a value from a set which is not bounded from above. In this case the superhedging strategy for a European call option is to hold a share of the underlying stock until the jump of the volatility occurs. Since this strategy requires the current asset price $S(0)$ as initial endowment there is no reason to buy the option instead of the share. This quite simple example shows that one can expect that selling options for the price determined via superhedging would rule out the usage of many derivatives in a lot of models for incomplete markets. This guess is affirmed e.g. by [61, Sections 2 & 3] or [41]. Here it is shown for a class of continuous stochastic volatility models with unbounded volatility [61] (resp. a class of geometric Lévy processes [41]) that the interval of option prices consistent with no arbitrage coincides with the trivial no arbitrage bounds for a European call with strike K , i.e. it has the form $((S(0) - K)^+, S(0))$.

The situation changes if the issuer of the claim accepts to keep a certain amount of risk in her portfolio. In order to follow such an approach she has to make up her mind about two issues:

1. How is risk defined?
2. Which rule describes her attitude towards risk?

The above defined shortfall is one natural answer to the first question. It can be combined with several choices for the second one. The resulting approaches discussed in the literature include the minimization of e.g. the probability of a shortfall, cf. [54], the expectation of an increasing convex function of a shortfall, cf. [55, 33, 152, 154], a *convex* or a *coherent* risk measure (cf. [3, 56, 64] for more details and e.g. [57] for an overview on these kinds of risk measures) associated to a shortfall, cf. [143, 171, 162, 163]. In these methods the measure quantifying the risk is minimized for a given fixed initial endowment w over the set of admissible strategies Θ . Alternatively, one can fix the amount of tolerated risk and afterwards one looks for the minimal initial endowment consistent with this risk specification. Beyond results ensuring the existence of an optimal solution, one of the key insights in this area is that the risk minimization problem can be decomposed into two subproblems. In a first step, one has to solve a static optimization problem which is given in terms of a randomized test. The term *static* refers to the fact that one does not seek for a dynamic hedging strategy here. In order to characterize the solution of this problem, duality approaches and the Neyman-Pearson test theory are applied. The optimizer of this problem is then used to modify the

claim H in a second step. Then one obtains a solution of the original problem via superhedging of the modified claim (e.g. [54, Section 4], [55, Theorem 3.2], [154, Theorem 5.1], [143, Proposition 1.3 & Theorem 1.5], [162, Theorems 3.1 & 4.3 & 4.9], [163, Proposition 2.3 & Theorem 3.3] for precise statements of the main results in each case). Beyond these structural results it turns out to be very hard to compute a solution for this kind of problems allowing for numerical evaluation even in some classical stochastic volatility models. In fact there are very few situations in which one can solve the problem at least partially. Notable exceptions include the following contributions: [167] obtain the minimal expected shortfall and the corresponding optimal strategy in a trinomial model, [92] combine a *Hamilton-Jacobi-Bellman* approach with a finite difference method in order to compute the expected squared shortfall of a claim written on a non-traded asset (the risk arising from this kind of derivatives is called *basis risk* in the literature (e.g. [36] and the references therein)), [151] proposes the usage of a Monte Carlo method for the solution of the risk minimization problem without a claim. The need of the prior knowledge of the initial endowment w can be seen as a further drawback of the minimization of the shortfall risk. Moreover one should add an appropriate surcharge to the initial endowment as compensation for the remaining risk in order to obtain an option price at the end of the shortfall risk minimization procedure.

In the context of risk measures one can avoid the inconvenience of specifying the initial endowment *a priori* by following an approach called *risk indifference pricing*, cf. [189, 117, 146]. Here the principle of utility indifference pricing treated below is transferred to convex risk measures. In order to explain the key idea, let ϱ be a convex risk measure. We assume that an investor who does not hold a claim minimizes her risk by solving the problem

$$\inf_{\theta \in \Theta} \varrho(w + \theta \cdot S(T)). \quad (1.5)$$

Now we apply the indifference concept by demanding that this investor will only sell a claim H if she earns in return at least so much money p such that she can achieve her original risk level:

$$\inf_{\theta \in \Theta} \varrho(w + \theta \cdot S(T)) = \inf_{\theta \in \Theta} \varrho(w + p + \theta \cdot S(T) - H). \quad (1.6)$$

In contrast to the preceding approaches, the risk measure is not applied to the shortfall here. Instead it is used to quantify the risk of portfolio values at the maturity. p can be viewed as an indifference price for the claim. Like for exponential utility indifference pricing this price is independent of the initial endowment due to translation invariance of ϱ . Here one should note that the utility indifference criterion can be interpreted as a convex risk measure (cf. [8, Remark 3.1] for exponential utility and [149, Remark 7.1(iii)] for the general case of utility functions defined on the whole real line). Moreover, one can obtain a similar characterization for p (cf. e.g. [146, Theorem 6.3]) as for the indifference price for exponential utility (cf. e.g. [161, Theorem 2.1]). Additionally, the negative of the *certainty equivalent* (c.f. e.g. [11] for relation between utility indifference and certainty equivalence) is a convex risk measure for an exponential utility function (c.f. e.g. [57, Section 5]).

These observations emphasize the deep relation between exponential utility and convex risk measures and they lead us to utility-based approaches. In comparison to the risk-based methods discussed above the investor now is interested in the expected utility (cf. e.g. [181, 182] for a historic account about utility theory) arising from portfolios with or without an additional claim. The general theory provides us with results quite similar to the risk driven approaches. E.g. if the expected utility of a portfolio including an option is maximized for a given initial endowment, one can show the existence of an optimal solution and characterize the optimal utility via duality theory (cf. e.g. [37, Theorem 2.1] for exponential utility, e.g. [148, Theorem 1.1] and [149, Theorems 1.1 & 4.1] for the general case of a utility function defined on \mathbb{R} , e.g. [34, Theorem 3.1] and [89, Theorems 1 & 2] for utility functions which do not allow negative wealth). These results extend the findings for expected utility maximization without an additional claim (cf. e.g. [169] for an overview, e.g. [120] for utility functions defined on \mathbb{R}_+ and e.g. [168] for utility functions defined on \mathbb{R}). If one combines both methods – expected utility maximization with and without an additional claim – one can obtain the utility indifference approach or the related certainty equivalence rule. E.g. if we replace the risk measure and minimizations in (1.6) by the expectation of a utility function u and maximizations, we get the utility indifference criterion (cf. e.g. [62, 8, 81] for an overview and more details) proposed by [86]:

$$\sup_{\theta \in \Theta} E(u(w + p + \theta \cdot S(T) - H)) = \sup_{\theta \in \Theta} E(u(w + \theta \cdot S(T))).$$

Similarly as for the risk-based approaches, it is typically difficult to derive semi-explicit solutions for this problem. Again basis risk constitutes an exception (cf. [36] and Section 4.1 for further references). Therefore in the case of more general European-type claims asymptotic techniques have been used to characterize an approximate solution to the problem. For example, several authors discuss approximations for a small number of claims: E.g. in the case of an exponential utility function [130, 10, 112] show that one can obtain an approximation of the utility indifference problem by solving a utility maximization problem without a claim and a quadratic hedging problem (see below) under the minimal entropy martingale measure. For utility functions defined on \mathbb{R}_+ e.g. [121, 122, 109] derive similar results for the asymptotics of utility-based prices and hedging strategies. Since we follow the former approach for affine processes in the main part of this dissertation, we give more details about the resulting approximations there. We only note here (c.f. e.g. [8, Remark 3.4]) that the zeroth order approximation of the exponential utility indifference price per unit of the claim coincides with the marginal utility-based *fair price* introduced by [35] for utility functions defined on the positive real line. This fair price is an example of utility-based pricing *rules* (cf. [149, Section 6] or [104] for more details).

Another way to answer the above question "What is risk?" is to consider the deviation of the portfolio V from the claim at T :

$$|H - V(T)|.$$

This is a symmetric criterion for risk and therefore gains and losses expressed in terms of the hedging error will be punished alike by any rule used based on it. The most popular rule for measuring the attitude towards this risk is the expected squared hedging error. Here one minimizes the expectation of the squared hedging error over all initial endowments w and all admissible strategies θ :

$$\inf_{\theta \in \Theta, w \in \mathbb{R}} E((H - V(T))^2). \quad (1.7)$$

Approaches in this spirit are called *variance-optimal hedging*, *mean-variance hedging* or *quadratic hedging* in the literature (cf. e.g. [176, 153, 22, 177] for an overview). Similarly as in the above discussed case of shortfall risk minimization, a practitioner would probably add a further compensation depending on the variance of the hedging error to the initial endowment in order to obtain a price for the claim. The popularity of variance-optimal hedging can be explained by the fact that structure results (e.g. decomposition properties of the claim) allow to find semi-explicit expressions for the quantities characterizing the solution (optimal initial endowment, optimal hedging strategy, variance of the associated hedging error). This approach is discussed for affine processes at several stages in the body of this dissertation. As already noted, a quadratic hedging problem under the minimal entropy martingale measure appears as auxiliary problem in the asymptotic utility-based hedging approach. Independently, in a own chapter we treat variance-optimal hedging for a given physical probability measure P which need not to be a martingale measure. Therefore we give in this introduction only a brief outline about quadratic hedging.

The theory of quadratic hedging has been evolving step by step. The story began with the famous article of [59]. There it is shown (cf. [59, Theorem 1]) that in the case that S is a square-integrable martingale, a solution to (1.7) exists and can be characterized by means of the so called *Galtchouk-Kunita-Watanabe decomposition* (henceforth GKW decomposition, cf. [123, 67]) of the claim H with respect to S . Intuitively speaking, one obtains the optimal strategy from a projection of the claim H onto the space of *attainable* claims and the remaining hedging error is orthogonal to this space. The above article is also the starting point of another quadratic hedging approach called (*local*) *risk minimization*. Here, in contrast to (1.7), the strategies need not to be self-financing. Instead one demands that the portfolio value coincides at T with the claim H and one minimizes the conditional expectation of the squared remaining cost process in the martingale case. Like for the variance-optimal hedging criterion (1.7) the GKW decomposition of H appears in the solution of this problem (cf. [59, Theorem 2]).

If we drop the martingale assumption the situation gets more complicated in both approaches. Since we concentrate on the mean-variance hedging criterion (1.7), we refer for local risk minimization to the literature [176, 153] already mentioned above. The mean-variance hedging problem (1.7) was tackled by requiring additional structure properties of the asset price (e.g. deterministic mean-variance tradeoff, continuous price processes). In the case of a *deterministic mean-variance tradeoff* [174] showed that (1.7) admits an opti-

mal strategy which is given in feedback form. More precisely, an optimal strategy can be represented as a sum of a *pure hedge* coefficient and a feedback component depending on the evolution of the portfolio value V . Moreover the pure hedge coefficient can be characterized by the so-called *Föllmer-Schweizer decomposition* (henceforth FS decomposition, cf. [58, 173]) of the claim H .

If the asset price S is continuous, one can show (cf. e.g. [157, 71]) that the optimal strategy is again given by a feedback equation. Here, the pure hedge coefficient can be calculated by computing the GKW decomposition of H under the *variance-optimal martingale measure* (cf. [175]). Like the minimal entropy martingale measure for exponential utility, this measure is the counterpart for a quadratic criterion related to (1.7). Finally, in the general semimartingale case, [22] obtained a feedback representation of an optimal strategy by introducing a further equivalent measure denoted as *opportunity neutral measure*. The pure hedge coefficient is then given as a kind of FS decomposition of the mean value process of H under this measure. We follow the idea of [22] in Chapter 5. There we give more details about this approach and we briefly summarize some computational approaches using the general structure results in order to solve (1.7) in concrete models. Let us also mention that there are also several other techniques used in order to characterize a solution of the mean-variance hedging problem, cf. e.g. [71] for a change of numéraire approach and [125] for dynamic programming, [15] for a BSDE characterization. However it is beyond the scope of this introduction to describe these approaches in detail.

Instead we give a brief overview of further approaches in order to answer the questions raised at the beginning of this introduction. Beyond methods which are based on optimization criteria, there are a number of other ideas. E.g. for a practitioner it is probably important to be able to compute the risk associated with a chosen strategy, even if this strategy is not optimal. We call strategies which are not necessarily derived from an optimization criterion *suboptimal*. In the case of discrete-time trading the computation of the expected squared hedging error of such strategies is treated for geometric Lévy processes by [2]. Another stream of research focuses on the development of asymptotic results. Besides the already noted asymptotic utility-based pricing and hedging, here a key idea is to isolate the factors causing incompleteness. Afterwards one has to find an expansion of the objects of interest with respect to these factors. E.g. if in the Black-Scholes model trading is restricted to discrete dates the decisive quantity is the time step between the trading dates. One can calculate asymptotic results with respect to this factor for e.g. the distribution of the hedging error or the variance of the hedging error (cf. e.g. [186, 69, 78, 184]). Another example for asymptotic approaches is a fast mean reverting volatility in diffusion models. Here the key idea is that a main component of the stochastic volatility process is rapidly pulled back to its mean reversion level. Such a behaviour of the volatility process allows for an asymptotic expansion with respect to the mean reversion speed (e.g. [98, 60]).

We could continue the list of hedging approaches by discussing further related topics or methods: e.g. good deal bounds, (semi-) static hedging, etc. Instead we only note that our collection is in no sense complete. As noted above, we investigate two of these approaches in the main part of the thesis. We discuss mean-variance hedging and asymptotic exponential utility-based hedging in the framework of affine processes. These approaches lead to semi-explicit solutions for the objects of interest in some affine stochastic volatility models.

More precisely, we continue by providing an overview of the contents of the main part of the thesis. The next chapter introduces the general affine framework which is used to describe the dynamics of the financial market. In particular this setup includes affine stochastic volatility models. The affine structure underlies all subsequent chapters and it is the essence of almost all of our results. Furthermore in this chapter we summarize some frequently used rules of semimartingale calculus. These techniques are our main tools in order to derive our findings. The subsequent chapters are in some sense self contained since each of them begins with a detailed introduction.

The third chapter is devoted to exponential utility maximization for affine processes. Here the hedging issue is not to the fore. Instead, we are interested in a solution to the utility maximization problem and the computation of the density process of the related minimal entropy martingale measure. For affine processes, we obtain a characterization of these objects in terms of a solution of a system of differential algebraic equations. The key idea to obtain this result is to combine a duality approach with an appropriate ansatz for the density process. Since portfolio optimization is usually treated in a multivariate setup we work here – in contrast to the subsequent chapters – in a framework which allows the return process as well as the volatility process to be multidimensional. We demonstrate the applicability of the general findings for affine processes by calculating optimal trading strategies and density processes for some concrete stochastic volatility models (e.g. a generalized Heston model).

The general results of the third chapter are not only of independent interest but also a key ingredient for the asymptotic utility-based hedging approach discussed in the fourth chapter. Here we apply general structural results in order to derive an approximate solution to the utility indifference problem for exponential utility. Using the integral representation technique of [88, 21] we obtain semi-explicit expressions for the quantities appearing in the first-order expansions of the hedging strategy and the indifference price with respect to the number of claims. In particular, we show that in the superposition model proposed by Barndorff-Nielsen and Shephard [7] the integrands of this representation can be computed explicitly by solving linear differential equations. Finally, we illustrate our results by a numerical example.

In the last chapter, we consider quadratic hedging for affine processes without the restriction that the underlying asset has to be a martingale. It is structured similarly as the

preceding chapter. More precisely, again based on general results, we calculate integral representations of the variance-optimal hedging strategy, the optimal initial endowment and the associated variance of the hedging error. Furthermore, in the class of models suggested by Carr et. al. [18], we present numerical examples showing that the described approach leads to formulas allowing for computational evaluation. The appendix contains further auxiliary results from the areas of affine processes, semimartingale calculus and differential equations.

Let us fix some notation. For $n \in \mathbb{N}$, we write $\mathcal{I}^n := \{1, \dots, n\}$, $\mathcal{I}_0^n := \mathcal{I}^n \cup \{0\}$ and $\mathbb{R}_{++}^n := \{x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n : x_i > 0 \ \forall i \in \mathcal{I}^n\}$ and denote by $\tilde{e}_i^n \in \mathbb{R}^n$ the i -th unit vector for $i \in \mathcal{I}^n$ and by 1_d the n -dimensional vector $(1, \dots, 1)^\top$. Moreover, we write $C_1 \odot C_2$ for the *Hadamard product* (cf. e.g. [87, Chapter 5]) of two matrices $C^1, C^2 \in \mathbb{C}^{n_1 \times n_2}$, $n_1, n_2 \in \mathbb{N}$ where the elements of $C := C_1 \odot C_2 \in \mathbb{C}^{n_1 \times n_2}$ are given by $C_{i,j} = C_{i,j}^1 C_{i,j}^2$. Moreover, for $x \in \mathbb{R}^d$, we write $\exp(x) := (\exp(x_1), \dots, \exp(x_d))^\top$ for the ordinary exponential of x . Analogously, the ordinary logarithm is defined componentwise for $x \in \mathbb{R}_{++}^d$. The identity process $I(t) = t$ is denoted by I . Let X be a semimartingale. Then the space of X -integrable predictable processes is denoted by $L(X)$. The stochastic integral of $\theta \in L(X)$ w.r.t. to X is written as $\theta \cdot X$. $\mathcal{E}(X)$ denotes the *stochastic exponential* of X . Additionally, if $X, X(-) \neq 0$ hold, we write $\mathcal{L}(X) := \frac{1}{X^-} \cdot X$ for the *stochastic logarithm* of X (cf. [95, Section II.8.3] for further details). We also refer to [95] for unexplained notation and terminology regarding semimartingales and stochastic processes. Terminology concerning convex analysis is used as in [159].

Chapter 2

Affine Framework and Further Foundations

In this chapter we lay the foundations for our further considerations by introducing some basic concepts, definitions and calculation rules.

We begin with a description of our setup for the investigated financial market model. We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \in [0;T]}, P)$ with finite time horizon $T \in (0, \infty)$. We assume that the filtration satisfies the usual conditions of right-continuity and completeness (cf. [95, Section I.1.3]) and that $\mathcal{F}(0)$ is trivial. The discounted price process $S = (S_1, \dots, S_n)$ of our securities is modelled by an \mathbb{R}^n -valued strictly positive semimartingale.

In the main chapters we will use two different approaches in order to relate such an S with an *affine process*.

On one side, we model S as the stochastic exponential of an \mathbb{R}^n -valued semimartingale z :

$$S_i = S_i^0 \mathcal{E}(z_i), \quad i \in \mathcal{I}^n. \quad (2.1)$$

Here, $S_i^0, i \in \mathcal{I}^n$ denote strictly positive constants. Moreover we demand that $\Delta z_i > -1, i \in \mathcal{I}^n$. Hence $S_i, i \in \mathcal{I}^n$ are strictly positive by [95, Section I.4.62].

Alternatively, we assume that S is given as the ordinary exponential of an \mathbb{R}^n -valued semimartingale z :

$$S_i = \exp(z_i), \quad i \in \mathcal{I}^n. \quad (2.2)$$

In both cases we call the process z a *return process*. Furthermore we suppose that in the financial market another source of randomness exists which describes a kind of *stochastic activity*. It is an \mathbb{R}_+^m -valued semimartingale v called *stochastic volatility*. We assume that the activity process v and the return process z form an *affine process*. We make this notion precise in the following section.

2.1 Semimartingale characteristics and affine processes

In order to describe the relation between the return process z and the activity process v , we use semimartingale characteristics. Additionally, these characteristics are a central tool for our calculations. Therefore we briefly summarize some basic definitions and discuss the connection to affine Markov processes.

Definition 2.1 Let X be an \mathbb{R}^d -valued semimartingale with characteristics (B^X, C^X, ν^X) relative to some continuous truncation function $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ (cf. [95, Section II.2] for more details). By [95, Section II.2.9] there exist some predictable process $A^X \in \mathcal{A}_{\text{loc}}^+$, some predictable \mathbb{R}^d -valued process b^X , some predictable $\mathbb{R}^{d \times d}$ -valued process c^X whose values are non-negative, symmetric matrices, and some transition kernel F^X from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that, for $t \in [0, T]$,

$$\begin{aligned} B^X(t) &= b^X \cdot A^X(t), \\ C^X(t) &= c^X \cdot A^X(t), \\ \nu^X([0, t] \times G) &= F^X(G) \cdot A^X(t) \text{ for all } G \in \mathcal{B}^d. \end{aligned}$$

If one can choose $A^X = I$ then we call the associated triple (b^X, c^X, F^X) *differential characteristics* of X .

The class of stochastic volatility models described below allows the choice $A^X = I$.

We choose the following *continuous* truncation function $h = (h_1, \dots, h_d)^\top : \mathbb{R}^d \mapsto \mathbb{R}^d$:

$$h_k(x) := \begin{cases} x_k & \text{if } X_k \text{ is a special semimartingale,} \\ \frac{x_k}{|x_k|} (1 \wedge |x_k|) & \text{if } X_k \text{ is not a special semimartingale and } x_k \neq 0, \\ 0 & \text{if } X_k \text{ is not a special semimartingale and } x_k = 0, \end{cases} \quad (2.3)$$

for an \mathbb{R}^d -valued semimartingale X and $1 \leq k \leq d$.

This choice is motivated by the fact that nearly all semimartingales appearing in this thesis are special and hence some expressions can be considerably simplified.

The following definition of (time-inhomogeneous) *affine processes* is central for the whole dissertation, since it allows to impose that the return process z and the activity process v are related through a certain affine structure. A detailed exposition of the subject of affine processes can be found in [40] (time-homogeneous affine processes) or [48, 106] (time-inhomogeneous affine processes). Note that, by a slight abuse of notation, we call $(\beta(t), \gamma(t), \varphi(t))_{t \in [0, T]}$ Lévy-Khintchine triplet if $(\beta(t), \gamma(t), \varphi(t))$ is a Lévy-Khintchine triplet for every $t \in [0, T]$. Moreover, for simplicity and for convenience of the reader we sometimes drop the argument t .

Definition 2.2 Let $m, n \in \mathbb{N}_0$, $d = m + n$ and $X = (v, z)$ an $\mathbb{R}_+^m \times \mathbb{R}^n$ -valued semimartingale. If the differential characteristics (b^X, c^X, F^X) of X are affine functions of $v(t-)$ in

the sense that

$$\begin{aligned} b^X(t) &= \beta^0(t) + \sum_{i=1}^m v_i(t-) \beta^i(t), \\ c^X(t) &= \gamma^0(t) + \sum_{i=1}^m v_i(t-) \gamma^i(t), \\ F^X(t, G) &= \varphi^0(t, G) + \sum_{i=1}^m v_i(t-) \varphi^i(t, G) \quad \text{for all } G \in \mathcal{B}^d, \end{aligned}$$

where $(\beta^i(t), \gamma^i(t), \varphi^i(t))_{t \in [0, T]}$, $i \in \mathcal{I}_0^m$ are *strongly admissible* Lévy-Khintchine triplets such that the Lévy measures φ^i , $i \in \mathcal{I}_0^m$ satisfy,

$$\sup_{t \in [0, T]} \int_{\{x_k > 1\}} x_k \varphi^i(t, dx) < \infty \quad \text{if } k \in \mathcal{I}_0^m, \quad (2.4)$$

then X is called a *time-inhomogeneous affine process*. Strong admissibility means that the Lévy-Khintchine triplets $(\beta^i(t), \gamma^i(t), \varphi^i(t))$, $i \in \mathcal{I}_0^m$ fulfill, for every $t \in [0, T]$,

$$\begin{aligned} \beta_k^i(t) - \int h_k(x) \varphi^i(t, dx) &\geq 0 \quad \text{if } k \in \mathcal{I}_0^m \text{ and } k \neq i, \\ \int h_k(x) \varphi^i(t, dx) &< \infty \quad \text{if } k \in \mathcal{I}_0^m \text{ and } k \neq i, \\ \gamma_{k,k}^i(t) &= 0 \quad \text{if } k \in \mathcal{I}_0^m \text{ and } k \neq i, \\ \varphi^i(t, (\mathbb{R}_+^m \times \mathbb{R}^n)^C) &= 0, \end{aligned}$$

and that they satisfy the following continuity conditions:

- β^i, γ^i are continuous on $[0, T]$,
- the measures $h_k(x) \varphi^i(\cdot, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^n) \setminus \{0\}$ are weakly continuous on $[0, T]$ for every $k \in \mathcal{I}_0^m$ with $k \neq i$,
- the measures $|h_k(x)|^2 \varphi^i(\cdot, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^n) \setminus \{0\}$ are weakly continuous on $[0, T]$ for $k = i$ or for every $k \in \mathcal{I}_0^d \setminus \mathcal{I}_0^m$.

If a financial market $X = (v, z)$ is modelled as an affine process, we call X an *affine stochastic volatility model*. If the Lévy-Khintchine triplets do not depend on the time, we speak about *time-homogeneous affine processes*. The notion *affine process* is justified by the results of [40, 48, 106]. E.g. it is shown in [106, Theorem 2.6] using the results of [40, 48], that the affine martingale problem, which can be related to given strongly admissible Lévy-Khintchine triplets satisfying (2.4), admits a solution. This theorem also allows to calculate the conditional characteristic function of X . In order to recall it we introduce some additional terminology. We start with the notion of a *Lévy exponent*:

Definition 2.3 Suppose that $(\beta^i(t), \gamma^i(t), \varphi^i(t))_{t \in [0, T]}$, $i \in \mathcal{J}_0^m$ are Lévy-Khintchine triplets. Then

$$\psi_i(t, u) = u^\top \beta^i(t) + \frac{1}{2} u^\top \gamma^i(t) u + \int \left(e^{u^\top x} - 1 - u^\top h(x) \right) \varphi^i(t, dx), \quad (t, u) \in \bar{D}_i,$$

denote the associated *Lévy exponents*, where

$$\bar{D}_i := \left\{ (t, u) \in [0, T] \times \mathbb{C}^d : \int_{\{|x| \geq 1_d\}} e^{\operatorname{Re}(u)^\top x} \varphi^i(t, dx) < \infty \right\}.$$

If these triplets are strongly admissible in the sense of Definition 2.2 and X is an affine process, we call

$$\psi^X(t, u) := \psi_0(t, u) + \sum_{i=1}^m v(t-) \psi_i(t, u), \quad (t, u) \in \bar{D} := \bigcap_{i \in \mathcal{J}_0^m} \bar{D}_i$$

Lévy exponent of X . Moreover, we write $\psi_i^X := \psi_i$, $i \in \mathcal{J}_0^m$ in this case.

If X is time-homogeneous, the argument t in ψ^X , ψ_i^X , $i \in \mathcal{J}_0^m$ will be omitted. Furthermore in this case we define the sets D_i , $i \in \mathcal{J}_0^m$ by

$$D_i := \left\{ u \in \mathbb{C}^d : \int_{\{|x| \geq 1_d\}} e^{\operatorname{Re}(u)^\top x} \varphi^i(dx) < \infty \right\}$$

and the set $D := \bigcap_{i \in \mathcal{J}_0^m} D_i$. Note that the sets \bar{D}_i (resp. D_i), $i \in \mathcal{J}_0^m$, are closely related to the existence of certain exponential moments of the Lévy measures φ^i , $i \in \mathcal{J}_0^m$.

Example 2.4 Barndorff-Nielsen & Shephard ([7], henceforth BNS) proposed a stochastic volatility model whose dynamics for the asset price $S = \exp(z)$ and the stochastic activity process v are given by:

$$dz(t) = \mu v(t-) dt + \sqrt{v(t-)} dW(t), \quad (2.5)$$

$$dv(t) = -\lambda v(t-) dt + dr(t). \quad (2.6)$$

Here, μ , λ denote constants, W a Wiener process and r a subordinator (i.e. an increasing Lévy process). In this model the Lévy-Khintchine triplets $(\beta^i, \gamma^i, \varphi^i)$, $i = 0, 1$, are given by

$$\beta^0 = \begin{pmatrix} b^r \\ 0 \end{pmatrix}, \quad \gamma^0 = 0, \quad \varphi^0(G) = \int 1_G(x, 0) F^r(dx) \quad \forall G \in \mathcal{B}^2,$$

$$(\beta^1, \gamma^1, \varphi^1) = \left(\begin{pmatrix} -\lambda \\ \mu \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right),$$

cf. [102, Section 4.3]. The corresponding Lévy exponents are

$$\psi_0^X(u_1, u_2) = \psi^r(u_1), \quad u \in D_0,$$

$$\psi_1^X(u_1, u_2) = -\lambda u_1 + \mu u_2 + \frac{1}{2} u_2^2, \quad u \in D_1.$$

Here, D_0 and D_1 are given by

$$D_0 = \left\{ u \in \mathbb{C}^2 : \int_{\{|x| \geq 1_2\}} e^{\operatorname{Re}(u)^\top x} \varphi^0(dx) < \infty \right\}.$$

and $D_1 = \mathbb{C}^2$.

Next, we characterize solutions to a certain class of affine martingale problems.

Definition 2.5 Suppose that P_0 is a distribution on \mathbb{R}^d and that $(\beta^i(t), \gamma^i(t), \varphi^i(t))_{t \in [0, T]}$, $i \in \mathcal{I}_0^m$, are given Lévy-Khintchine triplets on \mathbb{R}^d . Then $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \in [0, T]}, P, X)$ is called *solution to the affine martingale problem*, which is related to P_0 and the Lévy-Khintchine triplets $(\beta^i(t), \gamma^i(t), \varphi^i(t))_{t \in [0, T]}$, $i \in \mathcal{I}_0^m$, if X is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \in [0, T]}, P)$ such that $P^{X(0)} = P_0$ and the differential characteristics (b^X, c^X, F^X) of X are given by

$$\begin{aligned} b^X(t) &= \beta^0(t) + \sum_{i=1}^m X_i(t-) \beta^i(t), \\ c^X(t) &= \gamma^0(t) + \sum_{i=1}^m X_i(t-) \gamma^i(t), \\ F^X(t, G) &= \varphi^0(t, G) + \sum_{i=1}^m X_i(t-) \varphi^i(t, G) \text{ for all } G \in \mathcal{B}^d. \end{aligned}$$

We can now recall [106, Theorem 2.6] and [48, Theorem 2.13].

Theorem 2.6 Suppose that $(\beta^i(t), \gamma^i(t), \varphi^i(t))$, $i \in \mathcal{I}_0^m$, $t \in [0, T]$ are strongly admissible Lévy-Khintchine triplets in the sense of Definition 2.2 and ψ_i , $i \in \mathcal{I}_0^m$ the associated Lévy exponents. Assume that the Lévy measures φ^i , $i \in \mathcal{I}_0^m$ additionally satisfy (2.4). Then the affine martingale problem related to $(\beta^i(t), \gamma^i(t), \varphi^i(t))$, $i \in \mathcal{I}_0^m$, $t \in [0, T]$ and any initial distribution P_0 on $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$ has a solution P such that X is $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$ -valued. For $0 \leq s \leq t \leq T$ the conditional characteristic function of X is given by:

$$E \left(e^{u^\top X(t)} | \mathcal{F}(s) \right) = \exp \left(\Psi_0(s, t, u) + \sum_{j=1}^m \Psi_j(s, t, u) X_j(s) + \sum_{j=m+1}^d u_j X_j(s) \right),$$

for $u \in \mathbb{C}_-^m \times i\mathbb{R}^{d-m}$, where $\Psi_j(\cdot, t, u) : [0, t] \rightarrow \mathbb{C}_-$, $j \in \mathcal{I}_0^m$ solve the generalized Riccati equations

$$\frac{\partial}{\partial s} \Psi_j(s, t, u) = -\psi_j^X(s, \Psi_1(s, t, u), \dots, \Psi_m(s, t, u), u_{m+1}, \dots, u_d), \quad \Psi_j(t, t, u) = u_j$$

and $\Psi_0(\cdot, t, u) : [0, t] \rightarrow \mathbb{C}$ is given by

$$\Psi_0(s, t, u) = \int_s^t \psi_0^X(\hat{s}, \Psi_1(\hat{s}, t, u), \dots, \Psi_m(\hat{s}, t, u), u_{m+1}, \dots, u_d) d\hat{s}.$$

Moreover, if $(\Omega', \mathcal{F}', (\mathcal{F}'(t))_{t \in [0, T]}, P', X')$ is another solution to the affine martingale problem, the distributions of X and X' coincide, i.e. $P^{X'} = P^X$.

In the proofs of the main results we require two results for time-inhomogeneous affine processes which are provided by [106]. For the convenience of the reader we recall these theorems in the present context of a trivial σ -algebra $\mathcal{F}(0)$. We begin with a result which allows to decide whether the stochastic exponential of a component of an affine process is a martingale.

Theorem 2.7 *Let $X = (v, z)$ be a time-inhomogeneous affine process in the sense of Definition 2.2. Suppose that for some $i \in \mathcal{I}^d$ the following holds:*

1. $\varphi^j(t, \{x \in \mathbb{R}^d : x_i < -1\}) = 0$ for every $j \in \mathcal{I}_0^m$ and for all $t \in [0, T]$,
2. $\int_{\{x_i > 1\}} x_i \varphi^j(t, dx) < \infty$ for every $j \in \mathcal{I}_0^m$ and for all $t \in [0, T]$,
3. $b_i^j(t) + \int (x_i - h_i(x)) \varphi^j(t, dx) = 0$ for every $j \in \mathcal{I}_0^m$ and for all $t \in [0, T]$,
4. the measure $h_k(x) x_i \varphi^j(t, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^n) \setminus \{0\}$ is weakly continuous in $t \in [0, T]$ for $j \in \mathcal{I}^m$ and $k \in \mathcal{I}^d$,
5. $\sup_{t \in [0, T]} \int_{\{x_k > 1\}} x_k (1 + x_i) \varphi^j(t, dx) < \infty$ for $j, k \in \mathcal{I}^m$.

Then $\mathcal{E}(X_i)$ is a martingale.

PROOF. [106, Theorem 3.1] □

Additionally, we require a result which guarantees the existence of certain exponential moments of affine processes.

Theorem 2.8 *Assume that $X = (v, z)$ is a time-inhomogeneous affine process in the sense of Definition 2.2. Let $p^v \in \mathbb{R}^m$ and $p^z \in \mathbb{R}^n$. Suppose that there exist mappings $\Psi^0 \in C^1([0, T], \mathbb{R})$ and $\Psi^v = (\Psi_1^v, \dots, \Psi_m^v)^\top \in C^1([0, T], \mathbb{R}^m)$ which satisfy*

1. $\int_{\{|x| > 1_d\}} e^{\Psi^v(t)^\top x^v + (p^z)^\top x^z} \varphi^i(t, d(x^v, x^z)) < \infty$ for all $i \in \mathcal{I}_0^m$ and $t \in [0, T]$,
2. $\Psi^v(T) = p^v$ and $\frac{d}{dt} \Psi_i^v(t) = -\psi_i^X(t, \Psi^v(t), p^z)$ for all $i \in \mathcal{I}^m$,
3. $\sup_{t \in [0, T]} \int_{\{x_j^v > 1\}} x_j^v e^{\Psi^v(t)^\top x^v + (p^z)^\top x^z} \varphi^i(t, d(x^v, x^z)) < \infty$ for all $i, j \in \mathcal{I}^m$,
4. $\Psi^0(t) = \int_t^T \psi_0^X(s, \Psi^v(s), p^z) ds$.

Then we have

$$E \left(\exp \left(v(T)^\top p^v + z(T)^\top p^z \right) \mid \mathcal{F}(t) \right) = e^{\Psi^0(t) + v(t)^\top \Psi^v(t) + z(t)^\top p^z} \quad \text{for all } t \in [0, T].$$

PROOF. [106, Theorem 5.1] □

We summarize some additional auxiliary results concerning affine processes in the Appendix A.

2.2 Semimartingale calculus

The results derived in this thesis heavily rely on some results from the calculus of semimartingale characteristics. Hence, we summarize some of this rules. Moreover, we introduce some notations to increase the readability.

Let $\tilde{m}, \tilde{n} \in \mathbb{N}$ and Y_1 (resp. Y_2) be $\mathbb{R}^{\tilde{m}}$ -valued (resp. $\mathbb{R}^{\tilde{n}}$ -valued) semimartingales. Additionally we impose that $X := (Y_1, Y_2)$ admits differential characteristics (b^X, c^X, F^X) . We use the shorthand notation

$$(b^X, c^X, F^X) =: (b^{(Y_1, Y_2)}, c^{(Y_1, Y_1)}, F^{(Y_1, Y_2)}) =: \left(\begin{pmatrix} b^{Y_1} \\ b^{Y_2} \end{pmatrix}, \begin{pmatrix} c^{Y_1} & c^{Y_1, Y_2} \\ c^{Y_2, Y_1} & c^{Y_2} \end{pmatrix}, F^{(Y_1, Y_2)} \right).$$

For convenience we typically suppress the dependence of the drift term b on the truncation function h . If X is additionally an affine process in the sense of Definition 2.2, we also apply this kind of shorthand notation to the Lévy-Khintchine triplets $(\beta^i(t), \gamma^i(t), \varphi^i(t))_{t \in [0, T]}$, $i \in \mathcal{I}_0^m$

$$\begin{aligned} (\beta^i, \gamma^i, \varphi^i) &=: (\beta^{i; (Y_1, Y_2)}, \gamma^{i; (Y_1, Y_2)}, \varphi^{i; (Y_1, Y_2)}) \\ &=: \left(\begin{pmatrix} \beta^{i; Y_1} \\ \beta^{i; Y_2} \end{pmatrix}, \begin{pmatrix} \gamma^{i; Y_1} & \gamma^{i; Y_1, Y_2} \\ \gamma^{i; Y_2, Y_1} & \gamma^{i; Y_2} \end{pmatrix}, \varphi^{i; (Y_1, Y_2)} \right). \end{aligned}$$

Definition 2.9 Let X be an \mathbb{R}^d -valued semimartingale which admits differential characteristics (b^X, c^X, F^X) . Then the $\mathbb{R}^{d \times d}$ -valued predictable process

$$\tilde{c}^X(t) := c^X(t) + \int xx^\top F^X(t, dx)$$

is called *modified second characteristic* of X if the integral exists.

This modified second characteristic appears in the context of predictable covariation processes.

Lemma 2.10 Let X be an \mathbb{R}^d -valued semimartingale which admits differential characteristics (b^X, c^X, F^X) and modified second characteristics \tilde{c} . If the corresponding integral exists, we have

$$\langle X_i, X_j \rangle(t) = \int_0^t \tilde{c}_{i,j}^X(s) ds$$

for the predictable covariation process of the components of $X = (X_1, \dots, X_d)$.

PROOF. [22, Proposition 1.2] □

In the main part of this thesis, semimartingales are subject to certain transformations (i.e. stochastic integration, C^2 -mappings, change of measure). We need some calculation rules in order to compute differential characteristics of a semimartingale after such transformations. We start with a stochastic integration rule:

Proposition 2.11 (Integration) *Let X be an \mathbb{R}^d -valued semimartingale with differential characteristics (b^X, c^X, F^X) and H an $\mathbb{R}^{n \times d}$ -valued predictable process which is integrable with respect to X . The differential characteristics (b^Y, c^Y, F^Y) of the \mathbb{R}^n -valued integral process*

$$Y := H \cdot X := (H^j \cdot X)_{j=1, \dots, n}$$

are of the form

$$\begin{aligned} b^Y(t) &= H(t)b^X(t) + \int (\tilde{h}(H(t)x) - H(t)h(x))F^X(t, dx), \\ c^Y(t) &= H(t)c^X(t)H(t)^\top, \\ F^Y(t, G) &= \int 1_G(H(t)x)F^X(t, dx) \quad \forall G \in \mathcal{B}^n \text{ with } 0 \notin G. \end{aligned}$$

Here, \tilde{h} denotes the truncation function on \mathbb{R}^n .

PROOF. [113, Lemma 3] □

We continue with a rule for the computation of differential characteristics with respect to C^2 -mappings:

Proposition 2.12 (C^2 -functions) *Let X be an \mathbb{R}^d -valued semimartingale with differential characteristics (b^X, c^X, F^X) . Suppose that $f : U \rightarrow \mathbb{R}^n$ is twice continuously differentiable on some open subset $U \subseteq \mathbb{R}^d$ such that X, X_- are U -valued. Then the differential characteristics (b^Y, c^Y, F^Y) of the \mathbb{R}^n -valued semimartingale $Y := f(X)$ are given by*

$$\begin{aligned} b_i^Y(t) &= J_{f^i}(X(t-))b^X(t) + \frac{1}{2}1_d^\top(H_{f^i}(X(t-)) \odot c^X(t))1_d \\ &\quad + \int \left(\tilde{h}^i(f(X(t-) + x) - f(X(t-))) - J_{f^i}(X(t-))h(x) \right) F^X(t, dx), \\ c^Y(t) &= J_f(X(t-))c^X(t)(J_f(X(t-)))^\top, \\ F^Y(t, G) &= \int 1_G(f(X(t-) + x) - f(X(t-))) F^X(t, dx) \quad \forall G \in \mathcal{B}^n \text{ with } 0 \notin G, \end{aligned}$$

for $i \in \mathcal{I}^n$. Here, J_f denotes the Jacobian matrix of f , H_{f^i} the Hessian matrix of f^i and \tilde{h} the truncation function on \mathbb{R}^n .

PROOF. [70, Corollary A.6] □

Finally, we recall a rule describing the calculation of the differential characteristics under an equivalent probability measure. It is proved in [101, Lemma 5.1].

Lemma 2.13 (Change of measure) *Let X be an \mathbb{R}^d -valued semimartingale and $P^* \stackrel{\text{loc}}{\sim} P$ a probability measure with density process Z . If $Y := (X, \mathcal{L}(Z))$ admits differential characteristics (b^Y, c^Y, F^Y) , the P^* -characteristics (b^*, c^*, F^*) of Y are given by*

$$\begin{aligned} b_i^{Y^*}(t) &= b_i^Y(t) + c_{i, d+1}^Y(t) + \int x_i x_{d+1} F^Y(t, dx) \text{ for } i \in \mathcal{I}^{d+1}, \\ c^{Y^*}(t) &= c^Y(t), \\ F^{Y^*}(t, G) &= \int 1_G(x)(1 + x_{d+1})F^Y(t, dx). \end{aligned}$$

Chapter 3

Exponential Utility Maximization and the Minimal Entropy Martingale Measure in Affine Stochastic Volatility Models

3.1 Introduction

Pricing and hedging of financial instruments is one of the key questions in financial mathematics. In an incomplete market the question arises as to what constitutes an economically reasonable criterion in order to determine the relevant quantities (i.e. option prices and hedging strategies).

A possible approach is *utility-indifference pricing* (cf. e.g. [86]). Unfortunately, it turns out to be very hard to determine semi-explicit formulas allowing for (efficient) numerical evaluation. A possible way out is to consider *asymptotic utility-based hedging strategies* (cf. e.g. [130, 10, 121, 122, 112]). Roughly speaking, the indifference pricing problem is replaced here by a first-order approximation with respect to the number of claims. We investigate this approach in Chapter 4. Since the *minimal entropy martingale measure* (henceforth *MEMM*) Q^* and an optimal trading strategy θ^* for an *exponential utility maximization problem* appear in the solution of this approximation, it seems desirable to have concrete formulas for these quantities in realistic market models. Even if such explicit expressions are unavailable one needs at least criteria under which the MEMM and the solution of the corresponding exponential utility maximization problem exist. In a general semimartingale framework the existence of the MEMM is proved if the asset S is locally bounded and an equivalent martingale measure with finite entropy exists (cf. [63, Theorem 2.1 and Remark 2.1]). Since then the density of Q^* is related to θ^* via a duality result (cf. [99, Theorem 2.1]), one can use this duality in order to prove optimality of candidates for Q^* and θ^* . If one follows this approach at least the following three questions have to be answered:

1. What is an appropriate ansatz in order to come up with candidates for an optimal trading strategy and the density process of the MEMM?
2. Under which (technical) conditions does an equivalent martingale measure with finite entropy exist?
3. How can one verify that the candidates are indeed optimal?

We tackle the above questions in multivariate *affine stochastic volatility models*. On the one hand, this choice is motivated by the fact that these models allow us to capture some so-called stylized facts observed in financial time series. On the other hand, they are analytically tractable which can be observed e.g. from the fact that Q^* and θ^* are (partially) known in some stochastic volatility models.

E.g. Foldes calculates a solution of the utility maximization problem in terms of semimartingale characteristics in the case of a multidimensional geometric Lévy process (cf. [50, Theorem 5.1]). In a geometric Lévy framework the determination of a solution of the utility maximization problem and the corresponding MEMM was discussed in slightly different generality by several other authors (e.g. [138, 24, 100, 139, 66, 46]). A popular stochastic volatility model is the model of Heston [82]. This model is usually described by a system of stochastic differential equations for the return process z and the squared volatility v . In the case of two assets $S = (S_1, S_2)^\top = (\exp(z_1), \exp(z_2))^\top$ this leads to the following equations for $z = (z_1, z_2)^\top$ and v :

$$dv(t) = (\kappa + \lambda v(t))dt + \sqrt{v(t)}dW_1(t), \quad (3.1)$$

$$dz_i(t) = \delta_i v(t)dt + \sqrt{v(t)}dW_{i+1}(t), \quad i = 1, 2. \quad (3.2)$$

Here, $\kappa \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$, $\delta = (\delta_1, \delta_2)^\top \in \mathbb{R}^2$. $W = (W_1, W_2, W_3)^\top$ denotes an \mathbb{R}^3 -valued Wiener process with covariance matrix σ . Since a process following the above dynamics is an example of a time-homogeneous affine process in the sense of Chapter 2 one can represent the differential characteristics of this model like in [102] with two Lévy-Khintchine triplets $(\beta^i, \gamma^i, \varphi^i)$, $i = 0, 1$ given by

$$(\beta^0, \gamma^0, \varphi^0) = \left(\begin{pmatrix} \kappa \\ 0 \end{pmatrix}, 0, 0 \right), \quad (3.3)$$

$$(\beta^1, \gamma^1, \varphi^1) = \left(\begin{pmatrix} \lambda \\ \delta \end{pmatrix}, \sigma, 0 \right). \quad (3.4)$$

Due to the fact that many stochastic volatility models have such a representation (cf. [102]), we can express our general framework in terms of semimartingale characteristics of an affine process. This relation was recently exploited by Kallsen & Pauwels [111, 110] in order to determine variance-optimal hedging strategies in affine stochastic volatility models. In the model of Heston [82] with strictly positive volatility, Hobson [85] determines the density process of the MEMM and shows that Q^* exists for every fixed time horizon $T < \infty$ (cf.

[85, Example 5.3]). Rheinländer [156] calculates the density of the MEMM in the Stein & Stein model [180] with correlation. Since both authors use duality arguments in order to prove the optimality of Q^* , they implicitly calculate the corresponding optimal strategy θ^* as well. This is also true for the article of Benth & Meyer-Brandis [13]. They determine the density process of Q^* in the Barndorff-Nielsen & Shephard ([7], cf. also Example 2.4) model. They show the existence of Q^* for every $T < T^* < \infty$. Here T^* is determined by the parameters of the model. Finally Rheinländer & Steiger [158] prove that the density process of the MEMM can be obtained from the solution of an Integro-PDE in a generalized BNS-model with jumps.

[7] argue that a weighted sum of independent OU-processes exhibits a greater flexibility than a single OU process in order to reflect the behaviour of volatility, because long term as well as short term effects can be described by such a *superposition*. We include this feature in our general setting by allowing for multidimensional volatility processes. We use time-homogeneous affine processes as introduced in Chapter 2 as the general framework for our processes (cf. [40]). This class of processes is rich enough to include both multidimensional assets and multidimensional volatility processes.

Therefore the contribution of this chapter can be summarized as follows. We prove in a class of multidimensional stochastic volatility models that the MEMM exists for *any* time horizon $T > 0$ and characterize the corresponding optimal trading strategy θ^* . The proof is independent from the concrete model and unifies therefore the arguments used to show optimality in the literature. The integrability assumptions needed to establish the above result are weaker than typically assumed in this context. E.g. we do not suppose that S is locally bounded. Furthermore, we provide a criterion which says that a stochastic volatility model belongs to the above class of models if it fulfills a weak no arbitrage condition and a structure condition which can be easily checked in concrete models. By allowing for multivariate volatility processes, our general framework covers both superposition models and non-trivial models in which the assets are affected by different volatilities (cf. a generalization of the Heston model in Section 3.5.1.2). Furthermore we calculate semi-explicit formulas for an optimal trading strategy in some popular stochastic volatility models (e.g. [18, 19]).

This chapter is structured as follows. In Section 3.2 we introduce our framework of affine processes and discuss some of our assumptions. Furthermore we give an overview of exponential utility maximization, the MEMM and the corresponding duality relationship. Finally Lemma 3.6 provides a sufficient criterion to check optimality of candidates for Q^* and θ^* . Subsequently we state the main theorems of this chapter in Section 3.3. In Theorem 3.16 we show that a candidate for θ^* exists, which fulfills certain properties. Then we show that this candidate is optimal, characterize the density process of Q^* and determine the dynamics of the model under Q^* in Theorem 3.17. Section 3.4 contains the proofs of the

above theorems. Finally, in Section 3.5 we apply the results of Section 3.3 in order to solve the exponential utility maximization problem in several specific stochastic volatility models.

3.2 Setup and preliminary results

In this chapter we investigate exponential utility maximization in a multivariate affine framework. However, we do not use the most general affine setup allowed by Chapter 2. We focus on *time-homogeneous* affine processes. Therefore we briefly specialize Definition 2.2 to the time-homogeneous case:

Definition 3.1 Let $m, n \in \mathbb{N}_0$, $d = m + n$ and $X = (v, z)$ be an $\mathbb{R}_+^m \times \mathbb{R}^n$ -valued semimartingale. If the differential characteristics (b^X, c^X, F^X) of X are affine functions of $v(t-)$ in the sense that

$$\begin{aligned} b^X &= \beta^0 + \sum_{i=1}^m v_i(t-) \beta^i, \\ c^X &= \gamma^0 + \sum_{i=1}^m v_i(t-) \gamma^i, \\ F^X(G) &= \varphi^0(G) + \sum_{i=1}^m v_i(t-) \varphi^i(G) \quad \forall G \in \mathcal{B}^d, \end{aligned}$$

where $(\beta^i, \gamma^i, \varphi^i)$, $i \in \mathcal{I}_0^m$ are admissible Lévy-Khintchine triplets such that the Lévy measures φ^i , $i \in \mathcal{I}^m$ satisfy

$$\int_{\{x_k > 1\}} x_k \varphi^i(dx) < \infty \quad \text{if } k \in \mathcal{I}^m$$

then X is called a *time-homogeneous affine process*. *Admissibility* means that the Lévy-Khintchine triplets $(\beta^i, \gamma^i, \varphi^i)$, $i \in \mathcal{I}_0^m$, satisfy the following conditions:

$$\begin{aligned} \beta_k^i - \int h_k(x) \varphi^i(dx) &\geq 0 && \text{if } i \in \mathcal{I}_0^m, \quad k \in \mathcal{I}^m, \quad k \neq i, \\ \varphi^i((\mathbb{R}_+^m \times \mathbb{R}^n)^c) &= 0 && \text{if } i \in \mathcal{I}_0^m, \\ \int h_k(x) \varphi^i(dx) &< \infty && \text{if } i \in \mathcal{I}_0^m, \quad k \in \mathcal{I}^m, \quad k \neq i, \\ \gamma_{k,k}^i &= 0 && \text{if } i \in \mathcal{I}_0^m, \quad k \in \mathcal{I}^m, \quad k \neq i. \end{aligned}$$

We noted in Chapter 2, that there are at least two ways to relate an affine process $X = (v, z)$ to a strictly positive asset price process S . We use both alternatives in this chapter. On one side, we describe S as the stochastic exponential of z as in (2.1) in order to establish our main results. On the other side, for the solution of the exponential utility maximization problem in concrete models we model S as the ordinary exponential of z as in (2.2).

We start with the case that S is given by (2.1). We now equip the affine framework with an assumption which ensures that $\Delta z_i > -1$ for all $i \in \mathcal{I}^n$. Hence S remains strictly positive. Moreover we add a condition on the exponential moments of the Lévy measures φ^i , $i \in \mathcal{I}_0^m$. This condition will turn out to be appropriate for the exponential utility maximization problem investigated below.

Assumption 3.2 Let $X = (v, z)$ be an affine stochastic volatility model in the sense of Definition 3.1. We assume that

$$\varphi^j(\{x \in \mathbb{R}^d : x_i \leq -1\}) = 0 \text{ for every } j \in \mathcal{J}_0^m \text{ and every } i \in \mathcal{J}^d \setminus \mathcal{J}^m. \quad (3.5)$$

Moreover, we suppose that there exists an $\varepsilon > 0$ such that

$$\int_{\{|x| \geq 1_d\}} \exp\left(\varepsilon \left(\sum_{j=1}^m x_j + \sum_{j=m+1}^d x_j^2\right)\right) \varphi^i(dx) < \infty, \quad i \in \mathcal{J}_0^m. \quad (3.6)$$

In exponential utility maximization it is often assumed that S is locally bounded. From Lemma A.2 we know that this boundedness condition means that the jumps of the return process z are bounded from above. Therefore one can deduce that Assumption 3.2 is automatically satisfied if S is locally bounded and there exist an $\varepsilon > 0$ such that

$$\int_{\{|x| \geq 1_d\}} e^{\varepsilon \sum_{j=1}^m x_j} \varphi^i(dx) < \infty, \quad i \in \mathcal{J}_0^m.$$

Assumption 3.2 implies the existence of the Lévy exponent of v on an open set including the imaginary axes (cf. [40, Lemma 5.3]). We collect some additional properties of the Lévy exponent of X :

Lemma 3.3 Suppose that Assumption 3.2 holds. Then some open and convex set $\mathcal{V} \supset \mathbb{R}_-^m$ exists such that

$$\int_{\{|x| \geq 1_d\}} \exp\left(\sum_{j=1}^m u_j^v x_j + \sum_{j=1}^n u_j^z x_{j+m}\right) \varphi^i(dx) < \infty, \quad i \in \mathcal{J}_0^m \quad \forall u = (u^v, u^z) \in \mathcal{U} \quad (3.7)$$

with $\mathcal{U} := \mathcal{V} \times \mathbb{R}^n$. The Lévy exponents ψ_i^X , $i \in \mathcal{J}_0^m$ are analytic on $\tilde{\mathcal{U}} := \{u \in \mathbb{C}^d : \operatorname{Re}(u) \in \mathcal{U}\}$ and convex on \mathcal{U} . Moreover, they are proper functions on \mathbb{R}^d . In particular, the Lévy exponents ψ_i^X , $i \in \mathcal{J}_0^m$ are locally Lipschitz-continuous on \mathcal{U} and for $u \in \mathcal{U}$ their first and second derivatives with respect to u^z are given by

$$\begin{aligned} \partial_{u^z} \psi_i^X(u^v, u^z) &= \beta^{i;z} + \gamma^{i;z} u^z + (u^v)^\top \gamma^{i;v,z} + \\ &\quad \int x^z \left(e^{(u^v)^\top x^v + (u^z)^\top x^z} - 1 \right) \varphi^i(d(x^v, x^z)), \end{aligned} \quad (3.8)$$

$$\partial_{u^z u^z} \psi_i^X(u^v, u^z) = \gamma^{i;z} + \int \left(x^z (x^z)^\top e^{(u^v)^\top x^v + (u^z)^\top x^z} \right) \varphi^i(d(x^v, x^z)). \quad (3.9)$$

PROOF. The first assertions are obvious or follow immediately from [40, Lemma 5.3] since Assumption 3.2 holds. The claim concerning convexity of Lévy exponents and the derivatives ∂_{u^z} , $\partial_{u^z u^z}$ is proved in the univariate case in [166, Lemma 26.4] and can easily be generalized to the multivariate case. Since the Lévy exponents ψ_i^X , $i \in \mathcal{J}_0^m$ are convex and finite on \mathcal{U} the local Lipschitz-property follows from [159, Theorem 10.4]). Keeping in mind Definition 2.3, we finally obtain that these exponents are proper on \mathbb{R}^d . \square

3.2.1 Exponential utility

Via duality results utility maximization is closely related to the determination of certain *equivalent σ -martingale measures* (henceforth *EMMs*). A probability measure Q is called an *equivalent σ -martingale measure* if $Q \sim P$ and S is a σ -martingale under Q . We denote the space of such Q by $\mathcal{M}^e(P)$. For details about σ -martingales and σ -martingale measures we refer to [101]. By [101, Proposition 3.1 and Corollary 3.1] and [94, Proposition 2.18], S is a Q - σ -martingale if and only if S is a local Q -martingale, because S is non-negative and $\mathcal{F}(0)$ is trivial. Therefore $\mathcal{M}^e(P)$ would be not affected by choosing local martingale measures instead of σ -martingale measures. The *relative entropy* of a probability measure Q relative to P is given by

$$H(Q, P) := \begin{cases} E \left(\frac{dQ}{dP} \log \left(\frac{dQ}{dP} \right) \right) & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.10)$$

The *minimal entropy martingale measure* (henceforth *MEMM*) Q^* is the minimizer of

$$\inf_{Q \in \mathcal{M}^e(P)} H(Q, P).$$

The space of all $Q \in \mathcal{M}^e(P)$ such that $H(Q, P) < \infty$ is denoted by $\mathcal{M}(P)$. If $\mathcal{M}(P)$ is non-empty and S is locally bounded then the MEMM exists and is unique (cf. [63, Theorem 2.1]).

The MEMM Q^* is related to exponential utility maximization (cf. [37, Theorem 2.1]). In utility maximization, we consider an investor trying to maximize the expected utility from terminal wealth:

$$U(w) := \max_{\theta} E(u(V_{\theta}^w(T))), \quad (3.11)$$

where $V_{\theta}^w(t) := w + \theta \cdot S(t)$ denotes the wealth process corresponding to θ . w is the initial endowment and θ an admissible strategy (cf. Definition 3.4 below). As noted above we consider the case of an *exponential utility* function

$$u^p(x) = -\exp(-px), \quad p > 0,$$

defined on the whole real line. In this case the optimizer of (3.11) θ_p^* is independent of w and inversely proportional to p :

$$\theta_p^* = \frac{\theta_1^*}{p}.$$

Furthermore the optimal utility $U(w)$ only depends on w via a constant factor:

$$U(w) = U(0)e^{-pw}.$$

Additionally, $-\theta_1^*$ is the optimizer of the following optimization problem:

$$-\min_{\theta} E(e^{\theta \cdot S(T)}). \quad (3.12)$$

Therefore for exponential utility it suffices to solve the Problem (3.12) instead of (3.11). Hence we call (3.12) *standardized exponential utility maximization problem*, though the exponential function $\exp(x)$, $x \in \mathbb{R}$ is not a utility function. In the case of the standardized exponential utility maximization problem we use for simplicity the abbreviation $V_\theta := V_\theta^0$. Moreover the optimizer of (3.12) is denoted by θ^* .

As for the space of admissible strategies we follow Kallsen & Rheinländer (cf. [112, Definition 2.1]).

Definition 3.4 The space of admissible strategies Θ consists of all predictable processes $\theta \in L(S)$ such that $\theta \cdot S$ is a Q -martingale for all $Q \in \mathcal{M}(P)$. More precisely, we consider the quotient space of Θ (which is also denoted by $\bar{\Theta}$) modulo the following equivalence relation: if $\theta, \tilde{\theta} \in \Theta$, then

$$\theta \sim \tilde{\theta} \quad \text{if} \quad \theta \cdot S = \tilde{\theta} \cdot S \quad \text{up to indistinguishability.} \quad (3.13)$$

If $\mathcal{M}(P) = \emptyset$ then Θ consists of *all* predictable and integrable processes. We mention that we do not *a priori* assume that $\mathcal{M}(P)$ is non-empty. Indeed we show in Theorem 3.17 that an EMM with finite entropy exists if our affine framework satisfies Assumption 3.2 and a system of differential equations admits a smooth solution.

In [112, Remark 2.2] for locally bounded price processes the Definition 3.4 is economically motivated by the results of [183], since an optimal strategy $\theta^* \in \Theta$ can be approximated in some sense by simple processes. A second justification is given by the fact that for exponential utility the optimum of (3.11) is invariant with respect to some spaces of admissible strategies (cf. e.g. [37, Theorems 2.2 and 2.3]).

The next two results are the foundation for our calculations of Q^* and an optimal trading strategy θ^* , because they allow us to come up with an appropriate ansatz. The following duality relation between Q^* and θ^* is proved in Kabanov & Stricker (cf. [99, Theorem 2.1]). Similar results can be found e.g. in [72, Proposition 3.2] or [8, Proposition 2.2].

Theorem 3.5 *Assume S is locally bounded and there exists a measure $Q \in \mathcal{M}(P)$. Then $Q \in \mathcal{M}(P)$ is the MEMM if and only if there exists $q \in \mathbb{R}$ and an admissible strategy $\theta \in \Theta$ such that the density of Q has the form*

$$\frac{dQ}{dP} = e^{q + \theta \cdot S(T)}.$$

In this case θ is an optimal trading strategy for (3.12).

If one knows candidates for the MEMM and an optimal strategy, one can proceed as in Rheinländer & Steiger [158] using the preceding theorem for verification of optimality. Hence one has a condition for optimality which can be checked at a single point in time T . However, one has to assume that S is locally bounded which is not satisfied in our general framework. Therefore we use Theorem 3.5 as a guideline for the following result. Here, we

state an optimality condition involving the whole density process. An analogous result was recently established in [107] for power utility.

Lemma 3.6 *Let θ be an admissible strategy. If a process $L^* : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ with $L^*(T) = 1$ a.s. exists such that*

1. \tilde{Z}^* is a martingale,
2. \tilde{Z}^*S is a σ -martingale,

with $\tilde{Z}^* := L^* \exp(V_\theta)$, then $\mathcal{M}^e(P)$ is non-empty. If additionally

$$E(\tilde{Z}^*(T) \log(\tilde{Z}^*(T))) < \infty \quad (3.14)$$

holds, then $\mathcal{M}(P)$ is non-empty, L^* is unique up to P -indistinguishability, θ is optimal and $Z^*(T) := \frac{\tilde{Z}^*(T)}{\tilde{Z}^*(0)}$ is the density of the MEMM.

PROOF. Since \tilde{Z}^* is a positive martingale and \tilde{Z}^*S is a σ -martingale, $Z^*(T)$ is the density of an equivalent σ -martingale measure. Therefore $\mathcal{M}^e(P)$ is non-empty. We now suppose that additionally (3.14) holds. Then it follows immediately that $\mathcal{M}(P)$ is non-empty. Let $\tilde{\theta}$ be an arbitrary admissible strategy. Using the convexity of the exponential function, the optimality of θ is implied by the inequality

$$E(e^{V_{\tilde{\theta}}(T)}) \geq E(e^{V_\theta(T)}) + E(e^{V_\theta(T)}(V_{\tilde{\theta}}(T) - V_\theta(T))) = E(e^{V_\theta(T)})$$

and the finite entropy of $Z(T) = \exp(V_\theta(T))$. $Z(T)$ is the density of the MEMM, since the assertion of [63, Theorem 2.3], showing that a candidate is actually the MEMM, also holds in our slightly different setting. In particular, Q^* is unique, since the argument in the proof of [63, Theorem 2.1] only requires the strict convexity of the relative entropy. Hence the uniqueness of L^* can be deduced from this uniqueness, the uniqueness of the density process Z (cf. [95, Section III.3.4]) and the observation that wealth processes belonging to optimal strategies coincide (cf. e.g. the proof of [22, Lemma 2.11] for a similar argument). \square

Remark 3.7 L^* in Lemma 3.6 is called *opportunity process*. Since the process $\exp(V_\theta)$ is typically not a density process itself, L^* is the link between this process and a density process. Recently L^* was used in this fashion in [22, 107] to which we also refer for an economical interpretation. Moreover, in order to solve a quadratic hedging problem, we determine the opportunity process associated to such a problem in Chapter 5.

In the next section we solve the utility maximization Problem (3.12) by choosing an appropriate ansatz for the opportunity process L^* and for an optimal strategy θ^* .

3.3 General results

In this section we calculate the MEMM and an optimal trading strategy for exponential utility maximization in the affine framework of Definition 3.1. The key idea is to determine candidates for the opportunity process L^* (cf. Lemma 3.6) and for optimal investment strategies in appropriate classes of processes. After that, we verify the optimality of the corresponding candidates for the MEMM and an optimal trading strategy by checking the conditions of Lemma 3.6. In the whole section it is assumed that Assumption 3.2 is satisfied.

3.3.1 An appropriate ansatz for the opportunity process

We begin with the specification of a class of processes for the candidate of the opportunity process:

$$L(t) := e^{\eta^0(t) + \eta^v(t)^\top v(t)}, \quad t \in [0, T]. \quad (3.15)$$

Here, $\eta^0 : [0, T] \rightarrow \mathbb{R}_-$ and $\eta^v : [0, T] \rightarrow \mathbb{R}_-^m$ are continuously differentiable deterministic functions with $\eta^0(T) = 0$ and $\eta^v(T) = (0, \dots, 0)^\top$.

We call L *candidate opportunity process*. This choice is motivated by two considerations. Firstly this approach was successful in the similar situation of utility maximization in stochastic volatility models with power utility (cf. [107]). Secondly using the exponential function for the opportunity process leads to desirable features (e.g. positivity). In particular, one can easily show that $(v, \mathcal{L}(L))$ is a time-inhomogeneous affine process in the sense of Definition 2.2. This sounds reasonable in the present setting and allows us to apply some results concerning affine processes. However the drawback of the approach will be that in our affine framework not every price process has an opportunity process of the above form. Therefore in general we will have to demand an additional *structure condition* (cf. Assumption 3.9) in the affine model of Definition 3.1, which ensures that the opportunity process is actually of the form (3.15).

In order to describe the ansatz for an optimal investment strategy $\theta^* =: (\theta_1^*, \dots, \theta_n^*)^\top$ we introduce the notion of *optimal wealth*. An optimal wealth ξ^* invested in S is defined by $\xi^* := (\xi_1^*, \dots, \xi_n^*)^\top := (\theta_1^* S_1(-), \dots, \theta_n^* S_n(-))^\top$. We make the ansatz that this wealth is a deterministic continuous function ξ on $[0, T]$. The corresponding investment strategy ϑ is then given by

$$\vartheta_i(t) := \frac{\xi_i(t)}{S_i(t-)} \text{ for } t \in [0, T] \text{ and } i \in \mathcal{I}^n. \quad (3.16)$$

This ansatz for an optimal wealth is again motivated by several observations. E.g. an optimal wealth is constant in geometric Lévy models (cf. e.g. [100, Theorem 3.3]) or a deterministic càglàd function in the PII-case (cf. e.g. [65, Theorem 4.2]). Furthermore a similar approach is successful for power utility and stochastic volatility (cf. [107]). Additionally, we will see in the proofs of the main results below that this ansatz leads to a characterization of an optimal wealth ξ^* which is independent of the state of the price process S . Usually this reduces the number of constraints in the associated optimization problem significantly.

3.3.2 Main theorems

After choosing suitable candidate classes for an optimal trading strategy and the opportunity process we are able to assemble optimality conditions by demanding the martingale (resp. σ -martingale) conditions for

$$Z := \frac{L}{L(0)} \exp(V_\vartheta) \quad (3.17)$$

(resp. ZS) in Lemma 3.6. In order to determine these conditions, we calculate the joint differential characteristics of S and K , the stochastic logarithm of the candidate for the opportunity process, i.e. $K := \mathcal{L}(L)$:

Proposition 3.8 *Let L as in (3.15) and assume that Assumption 3.2 holds. Then $K = \mathcal{L}(L)$ is well-defined and the differential characteristics $(b^{(S,K)}, c^{(S,K)}, F^{(S,K)})$ of $(S, K) = (S, \mathcal{L}(L))$ are given by*

$$\begin{aligned} b_i^S(t) &= S_i(t-)b_i^z, \quad i \in \mathcal{I}^n, \\ b^K(t) &= \partial\eta^0(t) + \nabla\eta^v(t)^\top v(t-) + \psi^X(\eta^v(t), 0), \\ c_{i,j}^S(t) &= S_i(t-)S_j(t-)c_{i,j}^z, \quad i, j \in \mathcal{I}^n, \\ c^{S_j, K}(t) &= c^{K, S_j}(t) = S_j(t-) \left(\sum_{i=1}^m \eta_i^v(t) c_{j,i}^{z,v} \right), \quad j \in \mathcal{I}^n, \\ c^K(t) &= \eta^v(t)^\top c^v \eta^v(t), \\ F^{(S,K)}(t, G) &= \int 1_G(S(t-) \odot x^z, e^{\eta^v(t)^\top x^v} - 1) F^{(v,z)}(d(x^v, x^z)) \\ &\quad \forall G \in \mathcal{B}^{n+1} \text{ with } 0 \notin G. \end{aligned}$$

PROOF. K is well-defined because $L, L(-) \neq 0$. We obtain the differential characteristics of (S, K) from an application of Propositions 2.12 and 2.11. More precisely, we begin with the differential characteristics of (v, z, I) . Proposition 2.11 yields those of $(v, z, S, \eta^0, \eta^v)$, Proposition 2.12 those of (v, z, S, L) and again Proposition 2.11 those of (S, K) . \square

Another application of Proposition 2.12 and 2.11 yields the differential characteristics of (Z, ZS) . The drift conditions for Z and ZS arising from the martingale conditions (cf. [101, Lemma 3.1]) lead together with the terminal conditions for η^0 and η^v to the following equations for η^0, η^v and ϑ (resp. ξ):

$$0 = b^K + \vartheta^\top b^S + \vartheta^\top c^{S,K} + \frac{1}{2} \vartheta^\top c^S \vartheta + \quad (3.18)$$

$$\int \left((e^{\vartheta^\top x^S} - 1)(1 + x^K) - \vartheta^\top x^S \right) F^{(S,K)}(d(x^S, x^K)),$$

$$0 = b^S + c^S \vartheta + c^{S,K} + \quad (3.19)$$

$$\int \left(x^S (e^{\vartheta^\top x^S} (1 + x^K) - 1) \right) F^{(S,K)}(d(x^S, x^K)),$$

$$\eta^0(T) = 0, \quad (3.20)$$

$$\eta^v(T) = (0, \dots, 0)^\top. \quad (3.21)$$

Here, condition (3.18) corresponds to the drift condition for Z . If this condition holds then the drift conditions for ZS can be written as equation (3.19). Using Proposition 3.8 we see that the Equations (3.18-3.19) can also be expressed in the following way:

$$0 = \partial\eta^0(t) + \nabla\eta^v(t)^\top v(t-) + \psi^X(\eta^v(t), \xi(t)), \quad (3.22)$$

$$0 = \partial_{u^z}\psi^X(\eta^v(t), \xi(t)). \quad (3.23)$$

We show in the proof of Theorem 3.16 that the system (3.22-3.23) is closely related to a system of *differential algebraic equations* (henceforth *DAEs*, cf. e.g. [124] for more details). Like for ordinary differential equations it is not clear *a priori* whether a solution exists. In some stochastic volatility models the above equations can be solved explicitly (cf. Section 3.5). In general a *structure condition* ensures that equation (3.19), which can be interpreted as the constraint of a DAE, is fulfilled. We prove in Theorem 3.17 that this solution actually leads to the density process of the MEMM Q^* and an optimal investment strategy θ^* . We start by postulating the structure condition in terms of Lévy exponents of the affine process X and the set \mathcal{V} defined in Lemma 3.3.

Assumption 3.9 *Let Assumption 3.2 hold and define for $u^v \in \mathcal{V}$ and $i \in \mathcal{J}_0^m$ the optimality sets*

$$\mathcal{D}_i(u^v) := \{u^z \in \mathbb{R}^n : \psi_i^X(u^v, u^z) \leq \psi_i^X(u^v, \hat{u}^z) \ \forall \hat{u}^z \in \mathbb{R}^n\}$$

of the optimization problems

$$\min_{u^z \in \mathbb{R}^n} \psi_i^X(u^v, u^z). \quad (3.24)$$

We assume that

$$\mathcal{D}(u^v) := \bigcap_{i \in \mathcal{J}_0^m} \mathcal{D}_i(u^v) \neq \emptyset \quad \forall u^v \in \mathcal{V}. \quad (3.25)$$

We require (3.25) in order to ensure that the intersection of certain optimality sets is non-empty. The intuition behind this assumption can be described as follows. Roughly speaking, in some sense our affine model can be characterized in terms of the Lévy exponents ψ_i^X , $i \in \mathcal{J}_0^m$. Each of these exponents restricts the set of optimal strategies. These restrictions will be related in Lemma 3.14 for a subclass of our affine framework to the absence of certain arbitrage opportunities. However Assumption 3.9 is not necessary in order to guarantee the existence of the MEMM for affine models. There exist (at least) degenerate or artificial cases, for which the MEMM exists, L^* (resp. θ^*) has the form (3.15) (resp. (3.16)), and Assumption 3.9 is violated. Apart from that, we will see from Theorems 3.16 and 3.17 that for exponential utility maximization Assumption 3.9 is a sufficient criterion in order to prove the existence of the MEMM and an optimal investment strategy.

Because the non-emptiness of the set $\mathcal{D}(u^v)$ for every $u^v \in \mathcal{V}$ represents a central structure condition to our affine framework we investigate properties of this set in the following

Lemma 3.10 *Let Assumption 3.2 hold. Then the optimization problems (3.24) are well-posed, i.e. for every $u^v \in \mathcal{V}$ and $i \in \mathcal{J}_0^m$ there exist a $u^z \in \mathbb{R}^n$ such that $\psi_i^X(u^v, u^z) < \infty$. In particular, $\mathcal{D}(u^v)$ is given by*

$$\mathcal{D}(u^v) = \{\tilde{\xi} \in \mathbb{R}^n : 0 = \partial_{u^z}\psi_i^X(u^v, \tilde{\xi}) \ \forall i \in \mathcal{J}_0^m\}, \quad u^v \in \mathcal{V}. \quad (3.26)$$

Moreover, if Assumption 3.9 is satisfied, then for every $u^v \in \mathcal{V}$ there exists a $\tilde{\xi}_{u^v} \in \mathbb{R}^n$ such that

$$\mathcal{D}(u^v) = \tilde{\xi}_{u^v} + \mathcal{X}.$$

Here, the set \mathcal{X} is defined by

$$\mathcal{X} := \left\{ \tilde{\xi} \in \mathbb{R}^n : \gamma^{i; z} \tilde{\xi} = 0 \wedge \int (\tilde{\xi}^\top x^z)^2 \varphi^i(d(x^v, x^z)) = 0 \quad \forall i \in \mathcal{I}_0^m \right\}.$$

\mathcal{X} is closed and independent of u^v .

PROOF. Section 3.4.1. □

In an affine framework allowing for multivariate volatilities Assumption 3.9 is usually hard to check. However, if the return process z has no jumps, it is sufficient to verify this condition for certain elements of \mathcal{V} :

Corollary 3.11 *Let Assumption 3.2 hold. Suppose that $\varphi^i((\mathbb{R}_+^m \times \mathbb{R}^n \setminus \{0\})) = 0$, $i \in \mathcal{I}_0^m$. If $\mathcal{D}(0) \neq \emptyset$ and $\mathcal{D}(-\tilde{e}_i^m) \neq \emptyset$ for all $i \in \mathcal{I}^m$ then Assumption 3.9 is satisfied.*

PROOF. The assertion can be easily deduced from (3.26) since every $u^v \in \mathcal{V}$ has a representation $u^v = -\sum_{i=1}^m u_i^v \tilde{e}_i^m$. □

Moreover, there is a class of stochastic volatility models which fulfill Assumption 3.9. In order to describe this class we require two assumptions.

Assumption 3.12 *Assume that Assumption 3.2 holds. We suppose that for a $r \in \mathcal{I}^{m+1}$ there exist a partition $\{\mathcal{I}_0^0, \mathcal{I}_1^0, \dots, \mathcal{I}_r^0\}$ of \mathcal{I}_0^m and a partition $\{\mathcal{I}_1, \dots, \mathcal{I}_r\}$ of \mathcal{I}^n such that*

1. *the Lévy exponents ψ_i^X , $i \in \mathcal{I}_0^0$ only depend on u^v ,*
2. *for every $j \in \mathcal{I}^r$ the Lévy exponents ψ_i^X , $i \in \mathcal{I}_j^0$ only depend on u^v and u_k^z , $k \in \mathcal{I}_j$,*
3. *for every $i, j \in \mathcal{I}_k^0$, $k \in \mathcal{I}^r$ there exists a $\mu \in \mathbb{R}_{++}$ such that $\partial_{u^z} \psi_i^X(u^v, u^z) = \mu \partial_{u^z} \psi_j^X(u^v, u^z) \forall u \in \mathcal{V} \times \mathbb{R}^n$.*

We call this assumption *partition condition*. It enforces an additional structure within the affine framework.

Additionally, a linear combination of the returns z should not lead to a riskless profit in an arbitrage-free setting. In order to exclude such arbitrage opportunities, we first define for every $\lambda \in \mathbb{R}^n$ a process p_λ by

$$p_\lambda := \lambda^\top z. \tag{3.27}$$

Again (v, p_λ) is an affine process in the sense of Definition 3.1 for every $\lambda \in \mathbb{R}^n$. The differential characteristics $(b^{(v, p_\lambda)}, c^{(v, p_\lambda)}, F^{(v, p_\lambda)})$ are given by (cf. Definition 3.1 for b^v and c^v):

$$b^{p_\lambda} = \lambda^\top b^z, \quad (3.28)$$

$$c^{v, p_\lambda} = c^{v, z} \lambda,$$

$$c^{p_\lambda, v} = \lambda^\top c^{z, v},$$

$$c^{p_\lambda} = \lambda^\top c^z \lambda, \quad (3.29)$$

$$F^{(v, p_\lambda)}(G) = \int 1_G(x^v, \lambda^\top x^z) F^{(v, z)}(d(x^v, x^z)) \quad \forall G \in \mathcal{B}^{m+1} \text{ with } 0 \notin G. \quad (3.30)$$

Our *no arbitrage assumption* can be formulated as follows:

Assumption 3.13 *Suppose that Assumptions 3.12 and 3.2 hold. Let p_λ be as in (3.27) and the partitions of \mathcal{I}_0^m and \mathcal{I}^n as in Assumption 3.12. Define for $i \in \mathcal{I}^r$*

$$\Lambda_i := \left\{ \lambda \in \mathbb{R}^n \quad : \quad \lambda_j = 0 \text{ for all } j \notin \mathcal{I}_i \text{ and} \right. \\ \left. \gamma^{j; p_\lambda} = 0 \text{ and } \varphi^{j; (v, p_\lambda)}(\mathbb{R}_+^m \times (0, \infty)) = 0 \text{ for all } j \in \mathcal{I}_0^i \right\}.$$

We assume that for every $i \in \mathcal{I}^r$ and any $\lambda \in \Lambda_i$ one of the following two conditions hold:

1. $\varphi^{j; (v, p_\lambda)}(\mathbb{R}_+^m \times (-\infty, 0)) = 0$ and $\beta^{j; p_\lambda} = 0$ for all $j \in \mathcal{I}_0^i$,
2. $\varphi^{j; (v, p_\lambda)}(\mathbb{R}_+^m \times (-\infty, 0)) > 0$ and $\beta^{j; p_\lambda} - \int x^{p_\lambda} \varphi^{j; (v, p_\lambda)}(d(x^v, x^{p_\lambda})) > 0$ for all $j \in \mathcal{I}_0^i$.

If there exists a λ violating this assumption we call it an *immediate arbitrage opportunity*. This notion is motivated by the results of [115]. There it is proved that for geometric Lévy processes λ or $-\lambda$ lead to an *unbounded increasing profit* if λ is in some sense admissible. We transfer this concept to our situation in Lemma 3.21. We can now formulate a sufficient criterion for Assumption 3.9:

Lemma 3.14 *If the integrability Assumption 3.2, the partition Condition 3.12 and the no arbitrage Assumption 3.13 hold, then the structure Condition 3.9 is satisfied.*

PROOF. Section 3.4.1. □

Remark 3.15 1. If one demands that ψ^0 belongs to \mathcal{I}_0^0 then the partition Assumption 3.12 is satisfied in stochastic volatility models with univariate volatility or in superposition models (cf. Section 3.5.2.1), where only the sum of volatilities affects the asset S and therefore the Lévy exponents of the summands also pertain to \mathcal{I}_0^0 .

2. Lemma 3.14 is only a sufficient condition in order to meet Assumption 3.9. We describe in Section 3.5.1.2 a situation where the structure condition holds and no partition exists fulfilling the above properties.

3. Roughly speaking, the above lemma consists of two building blocks: a weak no arbitrage condition and a restrictive structure condition on the form of the affine model.

We can now state our first main theorem. It shows that (3.18-3.21) admits a solution such that the associated candidates for the opportunity process and an optimal strategy belong to the classes of processes chosen in Section 3.3.1.

Theorem 3.16 *Let Assumptions 3.2 and 3.9 hold. Then the differential algebraic equation*

$$\begin{aligned} 0 &= \partial\eta^0(t) + \psi_0^X(\eta^v(t), \xi(t)), \\ 0 &= \partial\eta_i^v(t) + \psi_i^X(\eta^v(t), \xi(t)), \quad i \in \mathcal{I}^m, \\ 0 &= \partial_{u^z}\psi_i^X(\eta^v(t), \xi(t)), \quad i \in \mathcal{I}_0^m \end{aligned}$$

has a solution with respect to the terminal conditions $\eta^0(T) = 0$ and $\eta^v(T) = 0$ such that $\eta^0 \in C^1([0, T], \mathbb{R}_-)$, $\eta^v \in C^1([0, T], \mathbb{R}_-^m)$ and $\xi \in C([0, T], \mathbb{R}^n)$. These mappings are also a solution to (3.18 -3.21).

PROOF. Section 3.4.2. □

Our following main theorem shows that a solution of the System (3.18-3.21) leads to the MEMM Q^* and an optimal investment strategy θ^* :

Theorem 3.17 *Let Assumption 3.2 hold and suppose that the System (3.18-3.21) has a solution consisting of $\eta^0 \in C^1([0, T], \mathbb{R}_-)$, $\eta^v \in C^1([0, T], \mathbb{R}_-^m)$ and $\xi \in C([0, T], \mathbb{R}^n)$. Then the following statements hold:*

1. *An optimal investment strategy θ^* for the standardized exponential utility maximization Problem (3.12) is given by*

$$\vartheta_i = \frac{\xi_i}{S_i(-)}, \quad i \in \mathcal{I}^n.$$

2. *$L(t) = \exp(\eta^0(t) + \eta^v(t)^\top v(t))$, $t \in [0, T]$ is the opportunity process.*
 3. *The MEMM Q^* exists with density*

$$\frac{dQ^*}{dP} = \frac{e^{\vartheta \bullet S(T)}}{L(0)}$$

and density process Z

$$Z(t) = \frac{L(t)}{L(0)} e^{\vartheta \bullet S(t)}, \quad t \in [0, T].$$

4. *S is not only a Q^* σ -martingale, but even a Q^* martingale.*

5. (v, z) is a time-inhomogeneous affine process under Q^* with differential characteristics $(b^{(v,z)^*}, c^{(v,z)^*}, F^{(v,z)^*})$ given by

$$b^{v^*}(t) = b^v + (c^v, c^{v,z})\nu(t) + \int x^v (e^{\eta^v(t)^\top x^v + \xi(t)^\top x^z} - 1) F^{(v,z)}(d(x^v, x^z)), \quad (3.31)$$

$$b^{z^*}(t) = 0, \quad (3.32)$$

$$c^{(v,z)^*}(t) = c^{(v,z)}, \quad (3.33)$$

$$F^{(v,z)^*}(t, G) = \int \left(1_G(x) e^{\nu(t)^\top x} \right) F^{(v,z)}(dx) \quad \forall G \in \mathcal{B}^d \text{ with } 0 \notin G, \quad (3.34)$$

where $\nu(t) := (\eta^v(t)^\top, \xi(t)^\top)^\top$.

6. The Lévy exponent ψ^{X^*} of X under Q^* is defined for all $(t, u) \in [0, T] \times \mathbb{C}^d$ satisfying $u + \nu(t) \in D$ and is given by

$$\psi^{X^*}(t, u) = \psi^X(u + \nu(t)) - \psi^X(\nu(t)).$$

PROOF. Section 3.4.3. □

Hence for affine processes the MEMM exists, if the Lévy measures admit certain exponential moments and a differential algebraic equation has a smooth solution. The later condition can be verified by means of Theorem 3.16. Alternatively, in some concrete models one can directly determine a solution of these DAEs.

Our market model for the asset price process S is formulated in terms of the stochastic exponential of the process z . Alternatively, one can model S as the ordinary exponential of z . In order to transfer our results to this case we recall [106, Lemma 2.7]:

Proposition 3.18 *Let $X = (v, z)$ be an affine process in the sense of Definition 3.1. Then $\bar{X} := (v, \bar{z}) := (v, \mathcal{L}(e^z))$ is also an affine process in this sense whose differential characteristics $(b^{(v,\bar{z})}, c^{(v,\bar{z})}, F^{(v,\bar{z})})$ are given by*

$$\begin{aligned} b^v &= b^v, \\ b_i^{\bar{z}} &= b_i^z + \frac{1}{2} c_{i,i}^z + \int (h(e^{x_i^z} - 1) - h(x_i^z)) F^{(v,z)}(dx), \quad i \in \mathcal{I}^n, \\ c^{(v,\bar{z})} &= c^{(v,z)}, \\ F^{(v,\bar{z})}(G) &= \int 1_G(x^v, e^{x^z} - 1_n) F^{(v,z)}(d(x^v, x^z)) \quad \forall G \in \mathcal{B}^d \text{ with } 0 \notin G. \end{aligned}$$

Additionally, we have

$$\exp(z_i) = \exp(z_i(0)) \mathcal{E}(\bar{z}_i), \quad i \in \mathcal{I}^n. \quad (3.35)$$

This proposition allows us to prove the following corollary, which parallels Theorem 3.17.

Corollary 3.19 *Let $X = (v, z)$ be an affine process in the sense of Definition 3.1. Suppose that there exists an ε such that*

$$\int_{\{|x| \geq 1_d\}} \exp\left(\varepsilon \left((x^v)^\top 1_m + e^{2(x^z)^\top 1_n}\right)\right) \varphi^i(d(x^v, x^z)) < \infty, \quad i \in \mathcal{I}_0^m. \quad (3.36)$$

Let

$$S_i = \exp(z_i), \quad i \in \mathcal{I}^n \quad (3.37)$$

and define the candidate opportunity process L as in (3.15). Then the differential characteristics $(b^{(S,K)}, c^{(S,K)}, F^{(S,K)})$ of $(S, K) = (S, \mathcal{L}(L))$ are given by

$$\begin{aligned} b_j^S(t) &= S_j(t-) \psi^X(\tilde{e}_{j+m}^d), \quad j \in \mathcal{I}^n, \\ b^K(t) &= \partial \eta^0(t) + \nabla \eta^v(t)^\top v(t-) + \psi^X(\eta^v(t), 0), \\ c_{i,j}^S(t) &= S_i(t-) S_j(t-) c_{i,j}^z, \quad i, j \in \mathcal{I}^n, \\ c^{S_j, K}(t) &= c^{K, S_j}(t) = S_j(t-) \left(\sum_{i=1}^m \eta_i^v(t) c_{j,i}^{z,v} \right), \quad j \in \mathcal{I}^n, \\ c^K(t) &= \eta^v(t)^\top c^v \eta^v(t), \\ F^{(S,K)}(t, G) &= \int 1_G(S(t-) \odot (e^{x^z} - 1_n), e^{\eta^v(t)^\top x^v} - 1) F^{(v,z)}(d(x^v, x^z)) \\ &\quad \forall G \in \mathcal{B}^{n+1} \text{ with } 0 \notin G. \end{aligned}$$

Assume additionally that $\eta^0 \in C^1([0, T], \mathbb{R}_-)$, $\eta^v \in C^1([0, T], \mathbb{R}_-^m)$ and $\xi \in C([0, T], \mathbb{R}^n)$ are a solution of the System (3.18-3.21). Then the following statements hold:

1. The assertions (1), (2), (3) and (4) of Theorem 3.17 remain valid.
2. Under the MEMM Q^* the process $X = (v, z)$ is time-inhomogeneous affine in the sense of Definition 2.2 with differential characteristics $(b^{X^*}, c^{X^*}, F^{X^*})$ given by

$$\begin{aligned} b^{v^*}(t) &= b^v + (c^v, c^{v,z}) \nu(t) + \\ &\quad \int x^v (e^{\eta^v(t)^\top x^v + \sum_{i=1}^n \xi_i(t) (\exp(x_i^z) - 1)} - 1) F^{(v,z)}(d(x^v, x^z)), \\ b^{z^*}(t) &= b^z + (c^{v,z}, c^z) \nu(t) + \\ &\quad \int x^z (e^{\eta^v(t)^\top x^v + \sum_{i=1}^n \xi_i(t) (\exp(x_i^z) - 1)} - 1) F^{(v,z)}(d(x^v, x^z)), \\ c^{(v,z)^*}(t) &= c^{(v,z)}, \\ F^{(v,z)^*}(t, G) &= \int 1_G(x^v, x^z) e^{\eta^v(t)^\top x^v + \sum_{i=1}^n \xi_i(t) (\exp(x_i^z) - 1)} F^{(v,z)}(d(x^v, x^z)) \\ &\quad \forall G \in \mathcal{B}^d \text{ with } 0 \notin G, \end{aligned}$$

where $\nu(t) := (\eta^v(t)^\top, \xi(t)^\top)^\top$.

PROOF. $\bar{X} := (v, \mathcal{L}(e^z))$ is an affine process by Proposition 3.18. Using (3.35) we can replace (3.37) by a model written in terms of the stochastic exponential of $\mathcal{L}(e^z)$. Furthermore, (3.36) implies that (v, \bar{z}) satisfies (3.6). Moreover, using Proposition 3.18 we can express the characteristics of (v, \bar{z}) in terms of the characteristics of (v, z) . Hence, the assertions follow directly from Theorem 3.17. \square

Remark 3.20 1. One may also ask, how Assumption 3.9 and hence Theorem 3.16 can be translated if S is given by (3.37). It is crucial to note that in this case, the corresponding structure condition (3.25) can not directly be expressed via the Lévy exponents of X . Instead, like in the proof of Corollary 3.19, one has to consider the *alternative representation* of S in terms of the stochastic exponential of \bar{z} given by Proposition 3.18. Hence, one can express the structure condition in terms of the Lévy exponents of \bar{X} from Proposition 3.18.

2. Using this process \bar{X} one can also transfer Assumption 3.13 and therefore Lemma 3.14.

3.4 Proof of the main results

This section contains the proofs of the main theorems. We refer to Section 2.2 for some frequently used abbreviations concerning the semimartingale characteristics of an affine process.

3.4.1 Introductory results

We begin with a characterization of the sets $\mathcal{D}(u^v)$, $u^v \in \mathcal{V}$ given in Lemma 3.10.

PROOF OF LEMMA 3.10. From Lemma 3.3 we know that the Lévy exponents ψ_i^X , $i \in \mathcal{I}_0^m$ are proper on \mathbb{R}^d . Therefore the optimization problems (3.24) are well-posed. Additionally, we obtain from Lemma 3.3 that the minimization problems in Assumption 3.9 are convex due to the convexity of the Lévy exponents ψ_i^X , $i \in \mathcal{I}_0^m$. Hence Equality (3.26) follows from this observation and the smoothness of ψ_i^X , $i \in \mathcal{I}_0^m$ with respect to u^z shown in Lemma 3.3. In the following we consider an arbitrary $u^v \in \mathcal{V}$. Choose an arbitrary $\tilde{\xi}_{u^v} \in \bigcap_{i \in \mathcal{I}_0^m} \mathcal{D}_i(u^v)$ and an arbitrary $\tilde{\xi} \in \mathcal{X}$. We first show that $\hat{\xi} := \tilde{\xi}_{u^v} + \tilde{\xi} \in \bigcap_{i \in \mathcal{I}_0^m} \mathcal{D}_i(u^v)$. (3.8) yields

$$\begin{aligned} \partial_{u^z} \psi_i^X(u^v, \hat{\xi}) &= \beta^{i;z} + \gamma^{i;z} \tilde{\xi}_{u^v} + (u^v)^\top \gamma^{i;v,z} + \\ &\quad \int x^z \left(e^{(u^v)^\top x^v + \tilde{\xi}_{u^v}^\top x^z} - 1 \right) \varphi^i(d(x^v, x^z)) = 0 \end{aligned}$$

for $i \in \mathcal{I}_0^m$. Hence we obtain $\tilde{\xi}_{u^v} + \mathcal{X} \subseteq \mathcal{D}(u^v)$. In order to show the \supseteq -inclusion let additionally $\tilde{\xi}_1, \tilde{\xi}_2 \in \bigcap_{i \in \mathcal{I}_0^m} \mathcal{D}_i(u^v)$. Define $\hat{\xi} := \tilde{\xi}_2 - \tilde{\xi}_1$. Then for every $i \in \mathcal{I}_0^m$ the

following equation holds:

$$\begin{aligned} 0 &= \hat{\xi}^\top (\partial_{u^z} \psi_i^X(u^v, \tilde{\xi}_2) - \partial_{u^z} \psi_i^X(u^v, \tilde{\xi}_1)) \\ &= \hat{\xi}^\top \gamma^{i;z} \hat{\xi} + \int \left(\hat{\xi}^\top x^z e^{(u^v)^\top x^v} (e^{\hat{\xi}_1^\top x^z} - 1) e^{\tilde{\xi}_1^\top x^z} \right) \varphi^i(d(x^v, x^z)). \end{aligned}$$

Since $\gamma^{i;z}$, $i \in \mathcal{I}_0^m$ are positive semidefinite and the function $\hat{g} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\hat{g}(x) := x(e^x - 1)$ is \mathbb{R}_+ -valued, the above equation is only fulfilled for every $i \in \mathcal{I}_0^m$ if $\hat{\xi} \in \mathcal{X}$. Hence

$$\tilde{\xi}_2 = \hat{\xi} + \tilde{\xi}_1 = \tilde{\xi}_{u^v} + \hat{\xi} + \tilde{\xi}_1 - \tilde{\xi}_{u^v} = \tilde{\xi}_{u^v} + \bar{\xi}$$

for a $\bar{\xi}$ in \mathcal{X} . This proves $\mathcal{D}(u^v) \subseteq \tilde{\xi}_{u^v} + \mathcal{X}$. Since ψ_i^X , $i \in \mathcal{I}_0^m$ are analytic by Lemma 3.3, the mappings $(u^v, \tilde{\xi}) \mapsto \partial_{u^z} \psi_i^X(u^v, \tilde{\xi})$, $i \in \mathcal{I}_0^m$ are continuous. Therefore \mathcal{X} is closed. \square

Now we prove the sufficient criterion for Assumption 3.9.

PROOF OF LEMMA 3.14. Obviously, we get $\mathcal{D}_i(u^v) = \mathbb{R}^n$, $i \in \mathcal{I}_0^0$, $u^v \in \mathcal{V}$ from the first condition on the partitions in Assumption 3.12. The third condition of Assumption 3.12 implies that for every $u^v \in \mathcal{V}$ and for every k , $1 \leq k \leq r$, the optimality sets $\mathcal{D}_i(u^v)$, $i \in \mathcal{I}_k^0$, coincide. Finally the second condition of this assumption yields that the Intersection (3.25) is non-empty if every optimality set $\mathcal{D}_i(u^v)$, $i \in \mathcal{I}_0^m$ is non-empty for every $u^v \in \mathcal{V}$. Let $i \in \mathcal{I}_0^m \setminus \mathcal{I}_0^0$ be arbitrary and let j be the index such that $i \in \mathcal{I}_j^0$. Then \mathcal{I}_j^0 and \mathcal{I}_j denote the corresponding sets in the partitions. We show that $\mathcal{D}_i(u^v)$ is non-empty for every $u^v \in \mathcal{V}$. Since the corresponding Lévy exponent ψ_i^X is convex, it suffices to prove that for every fixed $u^v \in \mathcal{V}$ every direction of recession of $\psi_i^X(u^v, \cdot)$ is a direction in which $\psi_i^X(u^v, \cdot)$ is constant (cf. [159, Theorem 27.1(b)]). We define for every fixed $u^v \in \mathcal{V}$ and for every direction $w \in \mathbb{R}^n$ a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $g(\lambda) := \psi_i^X(u^v, u^z + \lambda w)$. g is convex, continuous and hence closed on \mathbb{R}_+ , since ψ_i^X is convex and finite on \mathcal{U} by Lemma 3.3. Therefore it suffices to choose $u^z = 0$ in order to determine the directions of recession of g (cf. [159, Theorem 8.6]). Furthermore we can assume that $w_k = 0$ for all $k \notin \mathcal{I}_j$ since ψ_i^X does not depend on w_k for all $k \notin \mathcal{I}_j$. From (3.8) and (3.9) we can deduce g' and g'' :

$$\begin{aligned} g'(\lambda) &= w^\top \beta^{i;z} + \lambda w^\top \gamma^{i;z} w + (u^v)^\top \gamma^{i;v,z} w + \\ &\quad \int w^\top x^z \left(e^{(u^v)^\top x^v + \lambda w^\top x^z} - 1 \right) \varphi^i(d(x^v, x^z)), \\ g''(\lambda) &= w^\top \gamma^{i;z} w + \int \left((w^\top x^z)^2 e^{(u^v)^\top x^v + \lambda w^\top x^z} \right) \varphi^i(d(x^v, x^z)). \end{aligned}$$

Since g is also convex we have $g'' \geq 0$. If $\lim_{\lambda \rightarrow \infty} g''(\lambda) > 0$ then $\lim_{\lambda \rightarrow \infty} g'(\lambda) = \infty$ and w is not a direction of recession. Therefore we have to investigate the situation $\lim_{\lambda \rightarrow \infty} g''(\lambda) = 0$. This is only possible if $w^\top \gamma^{i;z} w = 0$, which implies $\gamma^{i;v,z} w = 0$ since $\gamma^{i;(v,z)}$ and $\gamma^{i;z}$ are positive-semidefinite. According to (3.27) we define the affine process $p_w := w^\top z$. Now g'' tends to zero in only two situations:

1. *Case 1:* $w^\top \gamma^{i;z} w = 0$ and $\int (w^\top x^z)^2 \varphi^i(d(x^v, x^z)) = 0$.

Since the third condition on the partitions holds and $w_k = 0$ for all $k \notin \mathcal{J}_j$ the no arbitrage Assumption 3.13 forces $\beta^{i;p_w} = 0$. Together with the above observation concerning $\gamma^{i;v,z} w$ we obtain that $g'(\lambda) = 0$. Therefore w is a direction of recession in which g is constant.

2. *Case 2:* $w^\top \gamma^{i;z} w = 0$ and $\lim_{\lambda \rightarrow \infty} \int \left((w^\top x^z)^2 e^{(u^v)^\top x^v + \lambda w^\top x^z} \right) \varphi^i(d(x^v, x^z)) = 0$.

Because the situation $\varphi^{i;(v,p_w)}(\mathbb{R}_+^m \times (-\infty; \infty)) = 0$ is covered by Case 1, we may assume $\varphi^{i;(v,p_w)}(\mathbb{R}_+^m \times (-\infty; \infty)) \neq 0$. Then obviously the second case only happens if $\varphi^{i;(v,p_w)}(\mathbb{R}_+^m \times (-\infty; 0)) \neq 0$ and $\varphi^{i;(v,p_w)}(\mathbb{R}_+^m \times (0; \infty)) = 0$. Hence we obtain by monotone convergence

$$\lim_{\lambda \rightarrow \infty} g'(\lambda) = w^\top \beta^{i;z} - \int (w^\top x^z) \varphi^i(d(x^v, x^z)).$$

Thus, similarly as in Case 1, the no arbitrage Assumption 3.13 yields $\lim_{\lambda \rightarrow \infty} g'(\lambda) > 0$. Therefore w is no direction of recession.

This completes the proof. □

Finally, we show that a violation of the no arbitrage Assumption 3.13 indeed leads to an arbitrage opportunity in the models of Assumption 3.12:

Lemma 3.21 *Suppose that Assumptions 3.12 and 3.2 hold. Let p_λ be as in (3.27), the partitions of \mathcal{J}_0^m and \mathcal{J}^n as in Assumption 3.12 and Λ_i , $i \in \mathcal{J}^r$ as in Assumption 3.13. If there exist an $i \in \mathcal{J}^r$, a $j \in \mathcal{J}_i^0$ and a $\lambda \in \Lambda_i$ such that one of the following two conditions hold*

1.

$$\varphi^{j;(v,p_\lambda)}(\mathbb{R}_+^m \times (-\infty, 0)) = 0 \text{ and } \beta^{j;p_\lambda} < 0, \quad (3.38)$$

2.

$$\varphi^{j;(v,p_\lambda)}(\mathbb{R}_+^m \times (-\infty, 0)) > 0 \text{ and } \beta^{j;p_\lambda} - \int x^{p_\lambda} \varphi^{j;(v,p_\lambda)}(d(x^v, x^{p_\lambda})) \leq 0. \quad (3.39)$$

then for $\tilde{v} := (\tilde{v}_1, \dots, \tilde{v}_n)^\top := (-\lambda_1/S_1(-), \dots, -\lambda_n/S_n(-))^\top$ the gains process $G := \tilde{v} \cdot S$ is non-decreasing. Moreover, there exist initial values $X^0 := (v^0, z^0)^\top \in \mathbb{R}_+^m \times \mathbb{R}^n$ of X such that $P_{X^0}(\tilde{v} \cdot S(T) > 0) > 0$.

PROOF. First, we note that $\tilde{v} \in L(S)$. Hence G is well-defined. Moreover we observe that a λ , which satisfies one of the conditions (3.38) or (3.39) for a $j \in \mathcal{J}_i^0$, fulfills this condition for every $j \in \mathcal{J}_i^0$ since the third condition of the partition Assumption 3.12 holds. The differential characteristics of G are given by the characteristics of $p_{-\lambda}$ which we obtain from (3.28-3.30). The definition of Λ_i and the second condition of the partition Assumption

3.12 yield that $\beta^{j:p-\lambda} = 0$ and $\varphi^{j:p-\lambda} = 0$ for every $j \notin \mathcal{J}_i^0$. We obtain from [95, Theorem II.2.34] that

$$G = x * (\mu^G - \nu^G) + b^{p-\lambda} \cdot I. \quad (3.40)$$

We now distinguish between two cases:

1. *Case 1:* $0 \in \mathcal{J}_i^0$.

Then for every initial value of X the following arguments hold. If (3.38) is satisfied then $b^{p-\lambda} > 0$ and we deduce from (3.40) that G is P -a.s. increasing. On the other hand, if (3.39) holds, then we obtain from (3.40) that G is P -a.s. non-decreasing and $P(G(T) > 0) > 0$.

2. *Case 2:* $0 \notin \mathcal{J}_i^0$.

In this case we have to combine the arguments of the first case with a stochastic continuity result. We obtain from [131, Lemma B.1] that for any initial value $X^0 := (v^0, z^0)^\top \in \mathbb{R}_+^m \times \mathbb{R}^n$ of X satisfying $v_j^0 > 0$ for some $j \in \mathcal{J}_i^0$, that there exist a t_{X^0} in $(0, T]$ and an $\varepsilon_{X^0} > 0$ such that $P_{X^0}(v_j(t-) > \varepsilon_{X^0}) > 1 - \delta$ for all $t \in [0, t_{X^0}]$ and any $\delta \in (0, 1]$. Hence the arguments of the first case hold for these initial values, i.e. with respect to P_{X^0} .

□

As noted in Section 3.3.2 a λ satisfying (3.38) or (3.39) is called *immediate arbitrage opportunity* by [115]. The proof of the above lemma reveals that the associated gains process G is non-decreasing. The profit occurring with a positive probability can be scaled up arbitrarily by multiplying $-\lambda$ with some positive constant. One may ask whether a result like Lemma 3.21 can also be shown in the general affine framework without demanding Assumption 3.12. This situation is more involved since the triplets affecting G can "point in different directions". E.g. assume that $(\beta^{0;z}, \gamma^{0;z}, \varphi^{0;z}) = (1, 0, 0)$ and $(\beta^{1;z}, \gamma^{1;z}, \varphi^{1;z}) = (-1, 1, 0)$ and that the activity component v is univariate. Here, the question whether for $v^0 = 0$ the direction 1 leads to an increasing gains process, depends on the behaviour of the volatility. If there exists a $t_0 > 0$ such that $P(v^0(t) = 0) = 1$ for all $t \in [0, t_0]$ then one can construct such a gains process.

3.4.2 Existence of the candidates

PROOF OF THEOREM 3.16. The proof of Theorem 3.16 consists of three steps. In a first step, we show that the System (3.18-3.21) corresponds to a differential algebraic equation for which a local solution exists. After that we show that this solution does not explode on $[0, T]$. Finally, we use a continuous selection principle in order to prove the existence of a solution satisfying the claimed smoothness properties.

Step 1: In order to prove that the equations (3.18-3.19) admit a solution, we analyze the equivalent reformulation

$$0 = \partial \hat{\eta}^0(t) + \nabla \hat{\eta}^v(t)^\top v(t-) + \psi^X(\hat{\eta}^v(t), \hat{\xi}(t)), \quad (3.41)$$

$$0 = \partial_{u^z} \psi^X(\hat{\eta}^v(t), \hat{\xi}(t)), \quad (3.42)$$

with respect to the terminal conditions

$$\hat{\eta}^0(T) = 0, \quad (3.43)$$

$$\hat{\eta}^v(T) = 0. \quad (3.44)$$

Note that here the ansatz for an optimal strategy was replaced by the ansatz for an optimal wealth. Due to this ansatz, Equations (3.41-3.42) are independent from the state of the asset price process. In order to show that they admit a solution, it is sufficient that all coefficients of the activity process v and the term, which is not multiplied with v , vanish. Therefore we obtain the system

$$0 = \partial \hat{\eta}^0(t) + \psi_0^X(\hat{\eta}^v(t), \hat{\xi}(t)), \quad (3.45)$$

$$0 = \partial \hat{\eta}_i^v(t) + \psi_i^X(\hat{\eta}^v(t), \hat{\xi}(t)), \quad i \in \mathcal{I}^m, \quad (3.46)$$

$$0 = \partial_{u^z} \psi_i^X(\hat{\eta}^v(t), \hat{\xi}(t)), \quad i \in \mathcal{I}_0^m. \quad (3.47)$$

The System (3.45-3.47) is a differential algebraic equation. Since the Lévy exponents of X are analytic on $\tilde{\mathcal{U}}$ by Lemma 3.3, they are continuously differentiable with respect to u^z . The other assumptions of Theorem B.3 also follow directly from the structure Assumption 3.9 and Lemma 3.3. Hence we can apply this theorem in order to show that a local solution of the System (3.45-3.47) with respect to the terminal conditions (3.43-3.44) exists.

Step 2: We have to show that the solution $\hat{\eta}^0, \hat{\eta}^v$ does not explode on $[0, T]$. The main idea therefore is to prove that the function $\hat{\eta}^v$ is \mathbb{R}_-^m -valued and that the ordinary differential equation determining $\hat{\eta}^v$ is dominated by a linear differential equation. An application of a comparison result will then yield that $\hat{\eta}^v$ exists on $[0, T]$. If $\hat{\eta}^v$ is \mathbb{R}_-^m -valued, the existence of $\hat{\eta}^0$ on $[0, T]$ trivially follows from the observation that the ordinary integral (3.45) is well-defined. In order to follow this idea we consider another reformulation of equations (3.18-3.21). If we solve (3.19) for b^S , plug this into (3.18) and replace the other terms with the expression from Proposition 3.8, we get

$$\begin{aligned} \partial \hat{\eta}^0(t) + \nabla \hat{\eta}^v(t)^\top v(t-) &= -\psi^X(\hat{\eta}^v(t), 0) + \frac{1}{2} \hat{\xi}(t)^\top c^z \hat{\xi}(t) - \\ &\quad \int e^{\hat{\eta}^v(t)^\top x^v} (e^{\hat{\xi}(t)^\top x^z} (1 - \hat{\xi}(t)^\top x^z) - 1) F^{(v,z)}(d(x^v, x^z)). \end{aligned} \quad (3.48)$$

We transform this terminal value problem into an initial value problem:

$$\begin{aligned} \bar{\eta}^0(t) &:= \hat{\eta}^0(T-t), \quad t \in [0, T], \quad \bar{\eta}^0(0) = 0, \\ \bar{\eta}^v(t) &:= \hat{\eta}^v(T-t), \quad t \in [0, T], \quad \bar{\eta}^v(0) = 0. \end{aligned}$$

Define

$$g^i(\bar{\eta}^v(t), \hat{\xi}(T-t)) := -\frac{1}{2}\hat{\xi}(T-t)^\top \gamma^{i;z} \hat{\xi}(T-t) + \int e^{\bar{\eta}^v(t)^\top x^v} \left(e^{\hat{\xi}(T-t)^\top x^z} (1 - \hat{\xi}(T-t)^\top x^z) - 1 \right) \varphi^i(d(x^v, x^z))$$

for $i \in \mathcal{J}_0^m$ and $t \in [0, T]$. Using this definition we obtain from Equation (3.48) similarly as in Step 1 the following $m+1$ equations:

$$\partial \bar{\eta}^0(t) = \psi_0^X(\bar{\eta}^v(t), 0) - g^0(\bar{\eta}^v(t), \hat{\xi}(T-t)), \quad (3.49)$$

$$\partial \bar{\eta}_i^v(t) = \psi_i^X(\bar{\eta}^v(t), 0) - g^i(\bar{\eta}^v(t), \hat{\xi}(T-t)), \quad i \in \mathcal{J}^m. \quad (3.50)$$

In the sequel we inspect Equations (3.49-3.50) for a wealth $\hat{\xi} = S(-) \odot \hat{v}$ fulfilling the Constraint (3.19). We note that

$$-\infty \leq g^i(\bar{\eta}^v(t), \hat{\xi}(T-t)) \leq 0, \quad i \in \mathcal{J}_0^m, t \in [0, T]$$

because $\gamma^{i;z}$, $i \in \mathcal{J}_0^m$ are positive semidefinite and the function $\hat{g} : \mathbb{R} \rightarrow \mathbb{R}$ with $\hat{g}(x) := e^x(1-x) - 1$ is non-positive.

This allows us to prove the following

Lemma 3.22 *The mappings $\bar{\eta}^0$ and $\bar{\eta}^v$ are $\bar{\mathbb{R}}_-$ -valued (resp. $\bar{\mathbb{R}}_-^m$ -valued) on $[0, T]$.*

PROOF. First, we solve the auxiliary problem,

$$\partial \chi_i(t) = \psi_i^X(\chi(t), 0), \quad \chi_i(0) \in \mathbb{R}_-, \quad i \in \mathcal{J}^m.$$

This differential equation admits a unique solution $\chi = (\chi_1, \dots, \chi_m)^\top$ for every $\chi(0) \in \bar{\mathbb{R}}_-^m$ by [40, Theorem 2.7]. For $\chi(0) = 0$ the solution is given by $\tilde{\chi} := \chi = 0$. This implies $\tilde{\chi}(0) = \bar{\eta}^v(0)$ and

$$0 = \partial \tilde{\chi}_i(t) - \tilde{\chi}_i(t) \geq g^i(\bar{\eta}^v(t), \hat{\xi}(T-t)) = \partial \bar{\eta}_i^v(t) - \psi_i^X(\bar{\eta}^v(t), 0), \quad i \in \mathcal{J}^m.$$

Moreover, one can easily show that the Lévy exponents ψ_i^X , $i \in \mathcal{J}^m$ of an admissible affine model in the sense of Definition 3.1 are quasimonotone increasing (cf. Definition B.4) with respect to $u^v \in \mathcal{V}$. This was also recently observed by [116, Lemma 4.6.]. Hence an application of the comparison Theorem B.5 yields that $\bar{\eta}^v$ is $\bar{\mathbb{R}}_-^m$ -valued. Therefore the assertion concerning $\bar{\eta}^0$ can be easily deduced from another comparison argument. \square

We can now apply Corollary B.6 in order to show that $\bar{\eta}^v$ is bounded from below on $[0, T]$. As noted above, this also yields that $\bar{\eta}^0$ is bounded from below on $[0, T]$.

Step 3: Up to now we focused on the properties of the mappings $\hat{\eta}^0$ and $\hat{\eta}^v$. It remains to show that a solution of (3.45-3.47) exists such that $\hat{\xi}$ is continuous on $[0, T]$. In order to prove this, we define a function $\bar{g} : \bar{\mathbb{R}}_-^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ by $\bar{g}(u^v, \tilde{\xi}) = \sum_{i=0}^n \psi_i^X(u^v, \tilde{\xi})$,

and conclude from Lemma 3.3 that \bar{g} is convex on $\mathbb{R}_-^m \times \mathbb{R}^n$. We consider the parametric optimization problem

$$\min_{\tilde{\xi} \in \mathbb{R}^n} \bar{g}(u^v, \tilde{\xi}).$$

We refer to [5] for unexplained terminology concerning parametric optimization problems and related topics. Since Assumption 3.9 holds the optimality sets

$$\{\tilde{\xi} \in \mathbb{R}^n : \bar{g}(u^v, \tilde{\xi}) \leq \bar{g}(u^v, \bar{\xi}) \text{ for all } \bar{\xi} \in \mathbb{R}^n\}, \quad u^v \in \mathbb{R}_-^m$$

coincide with $\mathcal{D}(u^v)$ for every $u^v \in \mathbb{R}_-^m$. As the parameter u^v is replaced in our framework by an \mathbb{R}_-^m -valued and continuous function $\hat{\eta}^v$ it suffices to show that a function $f \in C(\mathbb{R}_-^m, \mathbb{R}^n)$ exists such that $f(u^v) \in \mathcal{D}(u^v)$ for all $u^v \in \mathbb{R}_-^m$. A criterion for the existence of such a function is given by [5, Theorem 2.3.1]. We now check the conditions (A1-A4) of this theorem. As pointed out in [5, p.34], Assumption (A1) can be relaxed to the non-compact set \mathbb{R}_-^m . Because \mathbb{R}^n is a Banach space with respect to the Euclidean norm, (A2) is obviously fulfilled. (A4) holds since the optimality sets $\mathcal{D}(u^v)$, $u^v \in \mathbb{R}_-^m$ of a convex function are convex and – by assumption – non-empty. Finally it remains to be shown that the optimal set mapping $\mathcal{D}(\cdot)$ is lower semicontinuous in the sense of Berge (cf. [5, Section 2.2]). Therefore we verify the conditions of [5, Lemma 4.3.1]. By Lemma 3.3 \bar{g} is analytic with respect to $\tilde{\xi}$. Hence we only have to prove that the dimension of the vertex set of the recession cone (cf. [5, (3.2.6)] for a definition) of $\mathcal{D}(u^v)$ is independent of u^v . This follows, because we know from Lemma 3.10 that the optimality sets $\mathcal{D}(u^v)$ are related to each other via a translation. Therefore there exists a function $f \in C(\mathbb{R}_-^m, \mathbb{R}^n)$ such that $f(u^v) \in \text{cl}(\mathcal{D}(u^v))$ for all $u^v \in \mathbb{R}_-^m$. Since $\mathcal{D}(u^v)$ is closed for every $u^v \in \mathbb{R}_-^m$ (cf. Lemma 3.10) this completes the proof. \square

3.4.3 Verification of optimality

In this section we prove Theorem 3.17. We begin with some auxiliary results. In order to show admissibility of our candidate strategy we use the following lemma, which ensures the existence of an arbitrary small exponential moment of a square bracket process related to a strategy.

Lemma 3.23 *Let (v, z) be an affine process such that Assumption 3.2 holds and let $g : [0, T] \rightarrow \mathbb{R}^n$ be a predictable and bounded deterministic function on $[0, T]$. Define a process $\varrho = (\varrho_1, \dots, \varrho_n)^\top$ by $\varrho_i := g_i/S_i(-)$, $i \in \mathcal{I}^n$. Then ϱ is a locally bounded predictable process and there exists an $\alpha > 0$ such that*

$$E(e^{\alpha \varrho_i^2 \bullet [S_i, S_i](T)}) < \infty \quad \text{for all } i \in \mathcal{I}^n.$$

PROOF. Obviously it suffices to consider the case $n = 1$. Like in the proof of [95, Theorem II.8.3] one can construct a localizing sequence in order to show that ϱ is locally bounded.

Hence we obtain the first assertion and $\varrho^2 \cdot [S, S]$ makes sense. Let $\hat{g} := \sup_{t \in [0, T]} |g(t)|$ and define the process $\hat{\varrho}$ by $\hat{\varrho} := \hat{g}/S(-)$. Since $[S, S]$ is increasing by [95, I.4.47] it follows that $\varrho^2 \cdot [S, S] \leq \hat{\varrho}^2 \cdot [S, S]$. Therefore it suffices to show $E(\exp(\alpha \hat{\varrho}^2 \cdot [S, S](T))) < \infty$ for an $\alpha > 0$. By the Definition [95, I.4.45] of $[S, S]$ and Proposition 2.12 and 2.11 we easily obtain the differential characteristics $(b^{(v, \hat{z})}, c^{(v, \hat{z})}, F^{(v, \hat{z})})$ (cf. Definition 3.1 for b^v and c^v) of the process $\hat{X} := (v, \hat{z}) := (v, \hat{\varrho}^2 \cdot [S, S])$:

$$\begin{aligned} b^{\hat{z}} &= \hat{g}^2 \left(c^z + \int (x^z)^2 F^{(v, z)}(d(x^v, x^z)) \right), \\ c^{\hat{z}} &= 0, \\ c^{v, \hat{z}} &= 0, \\ c^{\hat{z}, v} &= 0, \\ F^{(v, \hat{z})}(G) &= \int 1_G(x^v, \hat{g}^2(x^z)^2) F^{(v, z)}(d(x^v, x^z)) \quad \forall G \in \mathcal{B}^{m+1} \text{ with } 0 \notin G. \end{aligned}$$

Consequently, \hat{X} is an affine process in the sense of Definition 3.1 with Lévy exponents $\psi_i^{\hat{X}}$, $i \in \mathcal{I}_0^m$. Hence in order to prove the claim, we have to show that there exists an arbitrary small exponential moment $\alpha > 0$ of \hat{z} . Therefore we check the conditions of Theorem 2.8. First, we prove that for some $\alpha > 0$ there exists a function $\Psi^v = (\Psi_1^v, \dots, \Psi_m^v) \in C^1([0, T], \mathbb{R}^m)$ which is the unique solution of the differential equation

$$\partial \Psi_i^v(t) = \psi_i^{\hat{X}}(\Psi^v(t), \alpha), \quad i \in \mathcal{I}_0^m \quad (3.51)$$

with respect to the initial condition $\Psi^v(0) = 0$. Similarly as in Lemma 3.3, we deduce from Assumption 3.2 that there exists an $\hat{\varepsilon} > 0$ such that the Lévy exponents $\psi_i^{\hat{X}}$, $i \in \mathcal{I}_0^m$ are well defined on $\hat{U} := \mathcal{V} \times (-\infty, \hat{\varepsilon})$. In particular, the Lévy exponents are locally Lipschitz-continuous on \hat{U} . Obviously, for $\alpha = 0$ a solution to (3.51) which exists on $[0, T]$ is given by $\Psi^v = 0$. Combining these observations with a result arising in sensitivity analysis of ordinary differential equations (cf. [187, Paragraph 13, Theorem XIII]) we obtain that there exists some small $\alpha > 0$ with $\alpha < \hat{\varepsilon}$ such that (3.51) has a unique \mathcal{V} -valued solution existing on $[0, T]$. Hence, we immediately deduce from the Definition of \hat{U} that Conditions 1 and 2 of Theorem 2.8 hold. One easily obtains Condition 3 from the observations that there exists an $\tilde{\varepsilon} > 0$ such that the mapping $(\Psi_1^v, \dots, \Psi_i^v + \tilde{\varepsilon}, \dots, \Psi_m^v, \alpha)^\top$ is \hat{U} -valued for every $i \in \mathcal{I}^m$ and that there exists an $\tilde{x} > 0$ such that $x < e^{\tilde{\varepsilon}x}$ for every $x \geq \tilde{x}$. Finally, the continuity of $\psi_0^{\hat{X}}$ immediately yields Condition 4. \square

In the following lemmas we show that Z defined by (3.17) is the density process Z of an equivalent change of measure leading to a σ -martingale measure with finite entropy. First, we show that Z is a density process of an equivalent change of measure.

Lemma 3.24 *Let (v, z) be an affine process such that Assumption 3.2 holds and suppose that $\eta^0 \in C^1([0, T], \mathbb{R}_-)$, $\eta^v \in C^1([0, T], \mathbb{R}_-^m)$ and $\xi \in C([0, T], \mathbb{R}^n)$ are a solution of the System (3.18-3.21). Then Z defined by (3.17) is a positive martingale with $E(Z(T)) = 1$.*

PROOF. Because $\mathcal{F}(0)$ is trivial and $\eta^0(0)$ and $\eta^v(0)$ are finite, $L(0) > 0$ exists and $Z(0) = 1$. Observe that $Z, Z(-) \neq 0$. Hence $N := \mathcal{L}(Z)$ is well-defined. The joint differential characteristics $(b^{(v,z,N)}, c^{(v,z,N)}, F^{(v,z,N)})$ of (v, z, N) are given by ($b^{v,z}$ remains unchanged as in Definition 3.1)

$$\begin{aligned} b^N(t) &= \int (h(x_N) - x_N) F^{(v,z,N)}(d(x^v, x^z, x^N)), \\ c^{(v,z,N)}(t) &= \begin{pmatrix} c^v & c^{v,z} & (c^v, c^{v,z})\nu(t) \\ c^{z,v} & c^z & (c^{z,v}, c^z)\nu(t) \\ \nu(t)^\top (c^v, c^{v,z})^\top & \nu(t)^\top (c^{z,v}, c^z)^\top & \nu(t)^\top c^{(v,z)}\nu(t) \end{pmatrix}, \\ F^{(v,z,N)}(t, G) &= \varphi^{0;(v,z,N)}(t, G) + \sum_{i=1}^m v_i(t-) \varphi^{i;(v,z,N)}(t, G) \text{ for all } G \in \mathcal{B}^{d+1}, \end{aligned}$$

where, for all $t \in [0, T]$,

$$\varphi^{i;(v,z,N)}(t, G) = \int 1_G \left(x, e^{\nu(t)^\top x} - 1 \right) \varphi^{i;(v,z)}(dx) \text{ for all } G \in \mathcal{B}^{d+1} \text{ with } 0 \notin G$$

and $\nu(t) := (\eta^v(t)^\top, \xi(t)^\top)^\top$. In order to calculate these characteristics we start with the differential characteristics of (v, z, S, L) determined in the proof of Proposition 3.8. Proposition 2.11 yields those of $(v, z, S, L, \vartheta \cdot S)$, Proposition 2.12 those of (v, z, Z) and finally Proposition 2.11 those of (v, z, N) . Here, b^N is zero, because η^0, η^v and ξ are a solution of the System (3.18-3.21) and N is P -special by Lemma A.4 (2) and [95, Proposition II.2.29]. In a next step, we show that (v, z, N) is a time-inhomogeneous affine process by verifying the conditions of Definition 2.2. Firstly, we note that Lemma A.4 (1) implies that $\varphi^{i;(v,z,N)}$, $i \in \mathcal{I}_0^m$ are Lévy measures for every $t \in [0, T]$. The continuity of ν yields in connection with Lemma A.4 (4) that the continuity properties are fulfilled. The remaining prerequisites are obviously satisfied because (v, z) is still an affine process. Hence, (v, z, N) is a time-inhomogeneous affine process and we can apply Theorem 2.7 in order to show that Z is a martingale. Therefore we check the assumptions of this theorem. The first condition obviously follows from the form of $F^{(v,z,N)}$. Lemma A.4 (2), (3) and (4,5) yield the second, fourth and fifth condition. Therefore Z is a martingale. This proves the claim. \square

In a second step, we prove that the density process Z leads to a martingale measure.

Lemma 3.25 *Suppose that the assumptions of Lemma 3.24 hold and let the density process Z and the time-inhomogeneous affine process (v, z, N) as in this lemma. The measure \bar{Q}^* defined via $\frac{d\bar{Q}^*}{dP} := Z(T)$ is a σ -martingale measure. In particular, S is a martingale under \bar{Q}^* . Moreover (v, z) is a time-inhomogeneous affine process with differential characteristics $(b^{(v,z)^*}, c^{(v,z)^*}, F^{(v,z)^*})$ given by (3.31-3.34).*

PROOF. We obtain the joint differential characteristics $(b^{(v,z,N)^*}, c^{(v,z,N)^*}, F^{(v,z,N)^*})$ of the process (v, z, N) under \bar{Q}^* by an application of Lemma 2.13 to the differential characteris-

tics calculated in Lemma 3.24. They can be written as

$$b^{v*}(t) = b^v + (c^v, c^{v,z})\nu(t) \quad (3.52)$$

$$+ \int x^v (e^{\eta^v(t)^\top x^v + \xi(t)^\top x^z} - 1) F^{(v,z)}(d(x^v, x^z)),$$

$$b^{z*}(t) = 0, \quad (3.53)$$

$$b^{N*}(t) = \nu(t)^\top c^{(v,z)}\nu(t) + \int (e^{\nu(t)^\top x} - 1)^2 F^{(v,z)}(dx), \quad (3.54)$$

$$c^{(v,z,N)*}(t) = \begin{pmatrix} c^v & c^{v,z} & (c^v, c^{v,z})\nu(t) \\ c^{z,v} & c^z & (c^{z,v}, c^z)\nu(t) \\ \nu(t)^\top (c^v, c^{v,z})^\top & \nu(t)^\top (c^{z,v}, c^z)^\top & \nu(t)^\top c^{(v,z)}\nu(t) \end{pmatrix}, \quad (3.55)$$

$$F^{(v,z,N)*}(t, G) = \varphi^{0;(v,z,N)*}(t, G) + \sum_{i=1}^m v_i(t-) \varphi^{i;(v,z,N)*}(t, G) \quad \forall G \in \mathcal{B}^{d+1}, \quad (3.56)$$

where, for all $t \in [0, T]$,

$$\varphi^{i;(v,z,N)*}(t, G) = \int 1_G \left((x, e^{\nu(t)^\top x} - 1) e^{\nu(t)^\top x} \right) \varphi^{i;(v,z)}(dx),$$

for all $G \in \mathcal{B}^{d+1}$ with $0 \notin G$ and $\nu(t) := (\eta^v(t)^\top, \xi(t)^\top)^\top$. Because Assumption 3.2 holds it follows that $\varphi^{i;(v,z,N)*}, i \in \mathcal{I}_0^m$ are Lévy measures for every $t \in [0, T]$ by Lemma A.4 (6). Moreover, $b^{z*} = 0$ since η^v and ξ satisfies (3.18). The above characteristics immediately yield (3.31-3.34). We prove that (v, z, N) is a time-inhomogeneous affine process under \bar{Q}^* by showing that the associated Lévy-Khintchine triplets $(\beta^{i;(v,z,N)*}, \gamma^{i;(v,z,N)*}, \varphi^{i;(v,z,N)*}), i \in \mathcal{I}_0^m$ are strongly admissible in the sense of Definition 2.2 and that (2.4) holds. From Lemma A.4 (4,5) it easily follows that $\beta^{i;(v,z,N)*}, i \in \mathcal{I}_0^m$ are continuous on $[0, T]$. Similarly as for $\varphi^{i;(v,z,N)*}, i \in \mathcal{I}_0^m$ the weak continuity properties of $\varphi^{i;(v,z,N)*}, i \in \mathcal{I}_0^m$ can be deduced by using dominated convergence. Finally, condition (2.4) can be shown by means of Lemma A.4 (8). Hence (v, z, N) is a time-inhomogeneous affine process under \bar{Q}^* . In order to prove that S is a martingale under \bar{Q}^* we verify the conditions (1-5) of Theorem 2.7. First, note that (3.5) is preserved under \bar{Q}^* . Hence the first condition is satisfied. Since $b^{z*} = 0$, the third condition holds. The remaining conditions follow from Lemma A.4 (7,10,11). This completes the proof. \square

The next lemma shows that the martingale measure \bar{Q}^* has finite entropy .

Lemma 3.26 *Suppose that the assumptions of Lemma 3.25 hold and let \bar{Q}^* be the measure defined there. Then \bar{Q}^* has finite entropy.*

PROOF. As in Lemma 3.25 Z denotes the density process of \bar{Q}^* relative to P . Additionally, $\eta^0 \in C^1([0, T], \mathbb{R}_-)$, $\eta^v \in C^1([0, T], \mathbb{R}_-^m)$ and $\xi \in C([0, T], \mathbb{R}^n)$ appearing in the definition of Z are a solution of the System (3.18-3.21). In the sequel we prove that there exists an $\alpha > 0$ such that $E_{\bar{Q}^*}(\exp(\alpha \log(Z(T)))) < \infty$. From this observation it easily follows that $E_{\bar{Q}^*}(\log(Z(T))) < \infty$ since the argumentation below also holds for an $\alpha < 0$. This

yields the assertion. We follow this approach by defining $\tilde{X} := (v, \bar{N}) := (v, \log(Z))$. In order to determine the differential characteristics $(b^{(v, \bar{N})^*}, c^{(v, \bar{N})^*}, F^{(v, \bar{N})^*})$ of \tilde{X} under \bar{Q}^* , we start from the differential characteristics (3.52-3.56) of (v, N) under \bar{Q}^* . An application of Proposition 2.11 yields those of (v, Z) . Finally, Proposition 2.12 delivers those of (v, \bar{N}) . These characteristics are given by

$$\begin{aligned} b^{v^*}(t) &= b^v + (c^v, c^{v,z})\nu(t) \\ &\quad + \int x^v (e^{\eta^v(t)^\top x^v + \xi(t)^\top x^z} - 1) F^{(v,z)}(d(x^v, x^z)), \\ b^{\bar{N}^*}(t) &= \frac{1}{2} \nu(t)^\top c^{(v,z)} \nu(t) + \int \left(e^{\nu(t)^\top x} (\nu(t)^\top x - 1) + 1 \right) F^{(v,z)}(dx), \\ c^{(v, \bar{N})^*}(t) &= \begin{pmatrix} c^v & (c^v, c^{v,z})\nu(t) \\ \nu(t)^\top (c^v, c^{v,z})^\top & \nu(t)^\top c^{(v,z)} \nu(t) \end{pmatrix}, \\ F^{(v, \bar{N})^*}(t, G) &= \varphi^{0; (v, \bar{N})^*}(t, G) + \sum_{i=1}^m v_i(t-) \varphi^{i; (v, \bar{N})^*}(t, G) \quad \forall G \in \mathcal{B}^{m+1}, \end{aligned}$$

where

$$\varphi^{i; (v, \bar{N})^*}(t, G) = \int \left(1_G(x^v, \nu(t)^\top x) e^{\nu(t)^\top x} \right) \varphi^{i; (v,z)}(d(x^v, x^z)) \quad \forall G \in \mathcal{B}^{m+1} \text{ with } 0 \notin G$$

and $\nu(t) := (\eta^v(t)^\top, \xi(t)^\top)^\top$. Note that $\varphi^{i; (v, \bar{N})^*}(t)$, $i \in \mathcal{I}_0^m$ are Lévy measures for every $t \in [0, T]$ by Lemma A.4(8). Hence, for all $t \in [0, T]$ we can associate Lévy-Khintchine triplets $(\beta^{i; (v, \bar{N})^*}(t), \gamma^{i; (v, \bar{N})^*}(t), \varphi^{i; (v, \bar{N})^*}(t))_{t \in [0, T]}$, $i \in \mathcal{I}_0^m$ to the differential characteristics of (v, \bar{N}) . The strong admissibility of these triplets can be deduced very similarly as in Lemma 3.25 or Proposition A.1 from the strong admissibility of the triplets of (v, N) . Additionally, Lemma A.4(9b) implies that (2.4) is satisfied. Hence we conclude that (v, \bar{N}) is a time-inhomogeneous affine process under \bar{Q}^* . We proceed similarly as in the proof of Lemma 3.23 and check the conditions of Theorem 2.8 in order to show that an arbitrary small exponential moment α of \bar{N} exists. Therefore let $\psi_i^{\tilde{X}^*}$, $i \in \mathcal{I}_0^m$, be the time-dependent Lévy exponents which correspond to the Lévy-Khintchine triplets $(\beta^{i; (v, \bar{N})^*}(t), \gamma^{i; (v, \bar{N})^*}(t), \varphi^{i; (v, \bar{N})^*}(t))$, $i \in \mathcal{I}_0^m$, $t \in [0, T]$. From Lemma A.4(12) one can deduce that these Lévy exponents are well-defined on $\tilde{U} := [0, T] \times \mathcal{V} \times (-1, \infty)$ (cf. Lemma 3.3 for the definition of \mathcal{V}). In particular, from the continuity properties of the time-inhomogeneous process \tilde{X} it follows that $\psi_i^{\tilde{X}^*}$, $i \in \mathcal{I}_0^m$ are continuous on \tilde{U} . Moreover, they are convex on $\mathcal{V} \times (-1, \infty)$ for every fixed $t \in [0, T]$, since Lévy exponents are convex functions (cf. Lemma 3.3). Hence $\psi_i^{\tilde{X}^*}(t, \cdot)$, $i \in \mathcal{I}_0^{m+1}$, are locally Lipschitz-continuous functions on $\mathcal{V} \times (-1, \infty)$ for every fixed $t \in [0, T]$. Keeping these properties in mind, we investigate for $\alpha \geq 0$ the differential equation

$$\partial \Psi_i^v(t) = -\psi_i^{\tilde{X}^*}(t, \Psi^v(t), \alpha), \quad i \in \mathcal{I}^m \quad (3.57)$$

with respect to the terminal condition $\Psi^v(T) = 0$. The local Lipschitz-continuity of the Lévy exponents yields that the System (3.57) has the unique solution $\Psi^v = 0$, $t \in [0, T]$

for $\alpha = 0$. Moreover, like in Lemma 3.23 we obtain therefore that for every small $\tilde{\varepsilon} > 0$ there exist $\alpha > 0$ such that the system (3.57) has a unique solution satisfying $|\Psi_i(t)| < \tilde{\varepsilon}$ for every $i \in \mathcal{I}^m$ and all $t \in [0, T]$. Hence we obtain for some $\alpha > 0$ unique \mathcal{V} -valued solutions $\Psi^v = (\Psi_1^v, \dots, \Psi_m^v) \in C^1([0, T], \mathbb{R}^m)$ of (3.57). By Lemma A.4(12), this immediately yields that Conditions 1 and 2 of Theorem 2.8 hold. Since the constructed solution is bounded by an arbitrarily small $\tilde{\varepsilon}$, Condition 3 can be shown by a similar argument as used to prove Lemma A.4(9b). Finally, the Condition 4 is satisfied, because the Lévy exponent $\psi_0^{\tilde{X}^*}$ is continuous on \tilde{U} . This completes the proof. \square

PROOF OF THEOREM 3.17. We show that the candidate for the opportunity process L and the candidate for an optimal investment strategy ϑ satisfy the conditions of Lemma 3.6. First, we prove that ϑ is admissible. Observe that ϑ is a locally bounded predictable process by Lemma 3.23. Therefore $\vartheta \in L(S)$. Fix an arbitrary $Q \in \mathcal{M}(P)$. Since $\vartheta_i, i \in \mathcal{I}^n$ are locally bounded and $S_i, i \in \mathcal{I}^n$ are Q - σ -martingales and hence local martingales as noted in Section 3.2.1, [95, I.4.34(b)] yields that $\vartheta_i \cdot S_i, i \in \mathcal{I}^n$ are local martingales. Similarly as in the proof of [156, Proposition 3.2] we obtain

$$\alpha E_Q(\vartheta_i \cdot [S_i, S_i](T)) \leq H(Q, P) + \frac{1}{e} (1 + E(e^{\alpha \vartheta_i \cdot [S_i, S_i](T)})) < \infty$$

for every $i \in \mathcal{I}^n$ and an $\alpha > 0$ from [37, Lemma 3.5] and Lemma 3.23. Hence $\vartheta_i \cdot S_i, i \in \mathcal{I}^n$ are square-integrable Q -martingales by [95, I.4.54] and [95, Proposition I.4.50(c)]. This shows that ϑ is admissible. From the construction of L we see that L is \mathbb{R}_{++} -valued and $L(T) = 1$ a.s. By Lemmas 3.24, 3.25 and 3.26 the process Z meets the conditions of Lemma 3.6. Therefore the assertions (1-3) of Theorem 3.17 hold and \bar{Q}^* is the MEMM Q^* . (4,5) are shown in Lemma 3.25. Finally, (6) can be deduced from the definition of the Lévy exponent and the characteristics of (v, z) under Q^* . This completes the proof. \square

3.5 Solution in concrete affine stochastic volatility models

In this section, we specialize the general results from Section (3.3.2) in order to solve the utility maximization Problem (3.12) in some popular stochastic volatility models. Hereby according to common practice we formulate the dynamics of these stochastic volatility models by means of stochastic differential equations for the return process z and the activity process v . Afterwards, we translate the models into semimartingale characteristic notation. All investigated models are affine and the asset price S is given as the ordinary exponential of z , i.e.

$$S_i = \exp(z_i), \quad i \in \mathcal{I}^n. \quad (3.58)$$

Hence we can apply Corollary 3.19 in order to determine an optimal investment strategy, the minimal entropy martingale measure and the dynamics under this measure. Additionally, we

know from Remark 3.20 that the other results of Section 3.3.2, which are formulated in terms of the stochastic exponential, can be transferred to the model (3.58). Therefore the affine process $\bar{X} := (v, \bar{z}) := (v, \mathcal{L}(e^z))$ is used. We call \bar{X} *alternative affine process* for the representation of S . Since the examples of this section should demonstrate the principles of utility maximization in (multivariate) volatility models we restrict ourselves typically to a univariate resp. bivariate asset price process S , i.e. $n \leq 2$, and an activity process $v = (v_1, \dots, v_m)^\top$ with $m \leq 3$. As one will see all results can be easily extended to higher dimensions. In [102] a more detailed description of stochastic volatility models is given in the univariate case.

3.5.1 Multidimensional Heston

In the sequel we consider two possible generalizations of the model by Heston [82] to a financial market with two assets S_1, S_2 and a univariate resp. bivariate volatility process v . These two types of models could be captured by a joint Heston-like framework. Nevertheless, we describe both models on their own for notational convenience. As noted in the introduction of this chapter, the MEMM was determined in the univariate Heston model by [85].

3.5.1.1 Univariate volatility

We first consider the case of an one dimensional volatility process. This assumption seems quite plausible for the portfolio optimization problem in a stock market where the trading environment is influenced by the same factors. The dynamics of such a Heston-type model are given by the stochastic differential equations (3.1-3.2). Like in [102, Section 4.2] one easily obtains that $X = (v, z)$ is an affine process in the sense of Definition 3.1 with triplets $(\beta^i, \gamma^i, \varphi^i)$, $i = 0, 1$ given by (3.3-3.4). We see that the Heston-type model (3.1-3.2) is almost the most general model allowed in the affine framework with one volatility process and without jumps. For convenience we assume that the covariance matrix $\gamma^{1;z}$ is positive definite. This ensures that the weak no arbitrage Assumption 3.13 holds for \bar{X} . Because \bar{X} also satisfies the other prerequisites of Lemma 3.14, we directly obtain from Theorem 3.16 and Corollary 3.19 that the MEMM Q^* exists. In order to calculate an explicit solution to the utility maximization Problem (3.12) we use the Lévy exponent of X :

$$\psi^X(u) = \psi_0^X(u) + v(t)\psi_1^X(u), \quad u \in \mathbb{C}^3$$

with $\psi_0^X(u) = u^\top \beta^0$ and $\psi_1^X(u) = u^\top \beta^1 + \frac{1}{2}u^\top \gamma^1 u$. The exponential Ansatz (3.15) for the opportunity process L^* has the following form in the case of a univariate volatility:

$$L(t) = e^{\eta^0(t) + \eta^v(t)v(t)}, \quad t \in [0, T],$$

with continuously differentiable deterministic functions η^0, η^v fulfilling the terminal conditions $\eta^0(T) = 0$ and $\eta^v(T) = 0$. The resulting optimality conditions (3.18-3.19) read as

follows:

$$0 = b^K + \vartheta^\top b^S + \vartheta^\top c^{S,K} + \frac{1}{2} \vartheta^\top c^S \vartheta, \quad (3.59)$$

$$0 = b^S + c^S \vartheta + c^{S,K}. \quad (3.60)$$

We obtain from Corollary 3.19 that the appearing components of the joint characteristics of (S, K) can be written as:

$$\begin{aligned} b^{S_i}(t) &= S_i(t) \psi^X(\tilde{e}_{i+1}^3) = S_i(t) v(t) \psi_1^X(\tilde{e}_{i+1}^3), \quad i \in \mathcal{I}^2, \\ b^K(t) &= \partial_1 \eta^0(t) + \partial_1 \eta^v(t) v(t) + \psi^X(\eta^v(t) \tilde{e}_1^3), \\ c^{S_i, S_j}(t) &= S_i(t) S_j(t) v(t) \gamma_{i,j}^{1;z}, \quad i, j \in \mathcal{I}^2, \\ c^{S_i, K}(t) &= S_i(t) v(t) \gamma_i^{1;z, v} \eta^v(t), \quad i \in \mathcal{I}^2. \end{aligned}$$

Let $\hat{\beta} = (\psi_1^X(\tilde{e}_2^3), \psi_1^X(\tilde{e}_3^3))^\top$. A simple reformulation of Equation (3.60) shows that a candidate for an optimal wealth ξ^* invested in S (and hence for an optimal trading strategy θ^*) is given by

$$\xi(t) = -(\gamma^{1;z})^{-1} \left(\hat{\beta} + \gamma^{1;z, v} \eta^v(t) \right)$$

Therefore we can substitute ϑ into (3.59) in order to determine a candidate for the stochastic logarithm of the opportunity process:

$$0 = b^K - \frac{1}{2} (b^S + c^{S,K})^\top (c^S)^{-1} (b^S + c^{S,K}). \quad (3.61)$$

Corollary 3.19 leads us to the following result:

Theorem 3.27 *Let $\text{rank}(\gamma^{1;z}) = 2$ and define the constants*

$$\begin{aligned} a_1 &= \frac{1}{2} (\gamma^{1;v,z} (\gamma^{1;z})^{-1} \gamma^{1;z,v} - \gamma^{1;v}), \\ a_2 &= \gamma^{1;v,z} (\gamma^{1;z})^{-1} \hat{\beta} - \lambda, \\ a_3 &= \frac{1}{2} (\hat{\beta})^\top (\gamma^{1;z})^{-1} \hat{\beta}, \\ a_4 &= \kappa. \end{aligned}$$

Then the functions $x_a : [0, T] \rightarrow \mathbb{R}$ and $y_a : [0, T] \rightarrow \mathbb{R}$ given by (B.4) resp. (B.5) are well-defined for $a := (a_1, \dots, a_4)^\top$. Furthermore a solution of the standardized exponential utility maximization Problem (3.12) in the Heston-type model (3.3-3.4) is given by

$$\theta^*(t) = \left(\frac{\xi_1^*(t)}{S_1(t)}, \frac{\xi_2^*(t)}{S_2(t)} \right)^\top,$$

where for $t \in [0, T]$,

$$\xi^*(t) = -(\gamma^{1;z})^{-1} \left(\hat{\beta} + \eta^v(t) \gamma^{1;z, v} \right).$$

The associated optimal utility is

$$-E(e^{\theta^* \bullet S(T)}) = -L^*(0) = -e^{\eta^0(0) + \eta^v(0)v(0)},$$

where, for $t \in [0, T]$,

$$\begin{aligned}\eta^v(T-t) &= x_a(t), \\ \eta^0(T-t) &= y_a(t).\end{aligned}$$

PROOF. (3.61) yields the following ordinary (differential) equations:

$$\begin{aligned}0 &= \partial_1 \eta^0(t) + a_4 \eta^v(t), \quad \eta^0(T) = 0 \\ 0 &= \partial_1 \eta^v(t) - a_1 \eta^v(t)^2 - a_2 \eta^v(t) - a_3, \quad \eta^v(T) = 0,\end{aligned}$$

Since $a_1 \leq 0$ by [1, Theorem 9.1.6], we get the solution of these Riccati equations from Lemma B.7. Since this solution also fulfills the desired smoothness requirements we know from Corollary 3.19 that ξ , η^0 and η^v are optimal. \square

Remark 3.28 1. If $a_3 = 0$ then S_1, S_2 are local martingales under the original probability measure P , this measure coincides with Q^* and it is optimal not to invest into the assets at all.

2. If $a_1 = 0$ then the assets S_1, S_2 are fully correlated with the volatility v .

3.5.1.2 A generalized Heston model

As pointed out in the introduction, it is desirable to allow for more than one volatility process. Furthermore, a model should allow correlation between the different assets and between the assets and the volatilities. Additionally, we demand that the utility maximization problem is solvable in the chosen class of opportunity processes. Below we discuss a non-trivial generalization of the model of Heston fulfilling these requirements in the case of two assets $S = (S_1, S_2)^\top = (\exp(z_1), \exp(z_2))^\top$ and two volatilities $v = (v_1, v_2)^\top$. We also describe in Remark 3.29 how this approach can be further generalized. First, we express the dynamics of this generalized Heston model in terms of a system of stochastic differential equations:

$$dv_i(t) = (\kappa_i + \lambda^i v_i(t))dt + \sqrt{v_i(t)}dW_i^i(t), \quad i \in \mathcal{I}^2, \quad (3.62)$$

$$dz_i(t) = \sum_{j=1}^2 \left(\delta_i^j v_j(t)dt + \sqrt{v_j(t)}dW_{i+2}^j(t) \right), \quad i \in \mathcal{I}^2, \quad (3.63)$$

with $\kappa = (\kappa_1, \kappa_2)^\top \in \mathbb{R}_+^2$, $\delta^1, \delta^2 \in \mathbb{R}^2$, $\lambda^i \in \mathbb{R}, i \in \mathcal{I}^2$. $W^i, i \in \mathcal{I}^2$ denote two 4-dimensional Wiener processes with covariance matrices γ^i such that $\gamma_{2,2}^1 = \gamma_{1,1}^2 = 0$. Additionally we suppose that W^1 and W^2 are independent. Like in Section 3.5.1.1 one can associate an affine process $X = (v, z)$ to this model. We note that X has the desired features concerning correlation, but the alternative affine process \bar{X} does not necessarily satisfy Assumption 3.9 which ensures that a solution to the exponential utility maximization

problem exists. In order to establish a reasonable condition under which this assumption is satisfied we initially consider the triplets $(\beta^i, \gamma^i, \varphi^i), i \in \mathcal{I}_0^2$, of the affine process (v, z) :

$$\begin{aligned} (\beta^0, \gamma^0, \varphi^0) &= \left(\begin{pmatrix} \kappa \\ 0 \end{pmatrix}, 0, 0 \right), \\ (\beta^i, \gamma^i, \varphi^i) &= \left(\begin{pmatrix} \lambda^i \tilde{e}_i^2 \\ \delta^i \end{pmatrix}, \gamma^i, 0 \right), i \in \mathcal{I}^2. \end{aligned}$$

The corresponding Lévy exponent of the process (v, z) has the following form:

$$\psi^X(u) = \psi_0^X(u) + \sum_{i=1}^2 v_i(t) \psi_i^X(u), \quad u \in \mathbb{C}^4,$$

with $\psi_0^X(u) = u^\top \beta^0$ and $\psi_i^X(u) = u^\top \beta^i + \frac{1}{2} u^\top \gamma^i u$ for $i \in \mathcal{I}^2$. Let $\hat{\beta}^i := (\psi_i^X(\tilde{e}_3^4), \psi_i^X(\tilde{e}_4^4))^\top$ and $A^i := (\hat{\beta}^i, \gamma^{i;z}, \gamma^{i;z, v_i})$ for $i \in \mathcal{I}^2$. By demanding that

$$\text{rank}(A^i) = 1, \quad i \in \mathcal{I}^2, \quad (3.64)$$

and

$$\text{rank}((\gamma^{1;z}, \gamma^{2;z})) = 2, \quad (3.65)$$

we will see below that Assumption 3.9 holds for \bar{X} and hence (3.18-3.21) admits a solution for X .

We want to discuss these conditions:

Remark 3.29 1. Condition (3.64) can be seen as a kind of structure condition on the generalized Heston model. If this condition is satisfied, Condition (3.65) excludes artificial special cases, since it essentially rules out the case that the model is built up by a single asset.

2. If conditions (3.64) and (3.65) hold then w.l.o.g. one can assume that there exist constants $\mu^i, i \in \mathcal{I}^2$ such that

$$\gamma^{1;z} = \gamma^{1;z_1} \begin{pmatrix} 1 & \mu^1 \\ \mu^1 & (\mu^1)^2 \end{pmatrix}, \quad \gamma^{1;z, v_1} = \gamma^{1;z_1, v_1} \begin{pmatrix} 1 \\ \mu^1 \end{pmatrix}, \quad \hat{\beta}^1 = \psi_1^X(\tilde{e}_3^4) \begin{pmatrix} 1 \\ \mu^1 \end{pmatrix}, \quad (3.66)$$

$$\gamma^{2;z} = \gamma^{2;z_2} \begin{pmatrix} (\mu^2)^2 & \mu^2 \\ \mu^2 & 1 \end{pmatrix}, \quad \gamma^{2;z, v_2} = \gamma^{2;z_2, v_2} \begin{pmatrix} \mu^2 \\ 1 \end{pmatrix}, \quad \hat{\beta}^2 = \psi_2^X(\tilde{e}_4^4) \begin{pmatrix} \mu^2 \\ 1 \end{pmatrix}. \quad (3.67)$$

If $\mu^i = 0, i \in \mathcal{I}^2$, then \bar{X} fulfills the conditions of Lemma 3.14 and the corresponding utility maximization problem has a solution. In this case we have two uncorrelated assets, which we denote by \tilde{S}_1 and \tilde{S}_2 . Each of them is influenced by its own volatility process. One can regard these assets as building blocks of a more complex dynamic allowing for correlation between the created assets. More precisely, if one chooses

non-trivial linear combinations of the stochastic logarithms $\mathcal{L}(\tilde{S}_1)$ and $\mathcal{L}(\tilde{S}_2)$ then conditions (3.66) and (3.67) are satisfied with $\mu^i \neq 0, i \in \mathcal{I}^2$:

$$S = \begin{pmatrix} S_1^0 \mathcal{E}(\mathcal{L}(\tilde{S}_1) + \mu^2 \mathcal{L}(\tilde{S}_2)) \\ S_2^0 \mathcal{E}(\mathcal{L}(\tilde{S}_2) + \mu^1 \mathcal{L}(\tilde{S}_1)) \end{pmatrix}, \quad \mu^1, \mu^2 \neq 0.$$

Here, S_1^0 and S_2^0 denote strictly positive constants.

3. Obviously, this description of the origin of the above model immediately leads to generalizations to more than two volatilities (resp. assets). In general, a condition ensuring optimality could be written in the case of n assets and m volatilities as follows:

$$\sum_{i=1}^m \text{rank}(A^i) = n \text{ and } \text{rank}((\gamma^{1;z}, \dots, \gamma^{m;z})) = n$$

Here A^i and $\hat{\beta}^i, i \in I^m$ are defined analogously as in the 2-dimensional case.

4. One can show that trading strategies $\theta^i = (\theta_1^i, \theta_2^i)^\top \in \Theta, i \in \mathcal{I}^2$, exist such that the characteristics of $\theta^i \cdot S$ are independent of v_i if the Conditions (3.66) and (3.67) are fulfilled for $\mu^i \neq 0, i \in \mathcal{I}^2$. E.g. for $\theta^1 = (\mu^1/S_1, -1/S_2)^\top$ the characteristics of $\theta^1 \cdot S$ are independent of v_1 . Hence the generalized Heston model allows non-trivial *volatility neutral* portfolios with respect to one of the volatility components.

5. As one can see from conditions (3.66) and (3.67) demanding (3.64) and (3.65) reduces the number of model parameters.

Like in Section 3.5.1.1, the exponential Ansatz (3.15) for the opportunity process L^* and the ansatz for an optimal strategy (3.16) lead to the optimality conditions (3.59) and (3.60) with respect to the terminal conditions $\eta^0(T) = 0$ and $\eta^v(T) = 0$. The corresponding characteristics of (S, K) can be deduced from Corollary 3.19. Like in the proof of Theorem 3.16, Equation (3.60) is satisfied if all coefficients of the volatility process vanish. Hence for every $t \in [0, T]$ Equation (3.60) corresponds to the following equations:

$$0 = \psi_1^X(\tilde{e}_{3+i}^4) + \gamma^{1; z_{i+1}} \xi_{i+1}(t) + \gamma^{1; z_1, z_2} \xi_{2-i}(t) + \gamma^{1; z_{i+1}, v_1} \eta_1^v(t), \quad i = 0, 1, \quad (3.68)$$

$$0 = \psi_2^X(\tilde{e}_{3+i}^4) + \gamma^{2; z_{i+1}} \xi_{i+1}(t) + \gamma^{2; z_1, z_2} \xi_{2-i}(t) + \gamma^{2; z_{i+1}, v_2} \eta_2^v(t), \quad i = 0, 1. \quad (3.69)$$

Each of the Pairs (3.68) and (3.69) can be reduced to one equation, since condition (3.64) holds. In order to express the remaining equations, we use the Representations (3.66) and (3.67) and define for $t \in [0, T]$,

$$\hat{A} = \begin{pmatrix} \gamma^{1; z_1} & \mu^1 \gamma^{1; z_1} \\ \mu^2 \gamma^{2; z_2} & \gamma^{2; z_2} \end{pmatrix}, \quad \hat{r}(t) = \begin{pmatrix} \hat{r}_1 \\ \hat{r}_2 \end{pmatrix} = \begin{pmatrix} \psi_1^X(\tilde{e}_3^4) + \gamma^{1; z_1, v_1} \eta_1^v(t) \\ \psi_2^X(\tilde{e}_4^4) + \gamma^{2; z_2, v_2} \eta_2^v(t) \end{pmatrix}.$$

Since \hat{A} is non-singular by Assumption (3.65), the linear system $r = \hat{A}\xi$ has a solution for every $r \in \mathbb{R}^2$, and, in particular, the structure Assumption 3.9 holds for \bar{X} . Therefore we obtain the following candidate for an optimal wealth

$$\xi(t) = -\hat{A}^{-1} \hat{r}(t), \quad t \in [0, T], \quad (3.70)$$

as a function of t . We plug this candidate into equation (3.18) and obtain from Corollary 3.19 the following result:

Theorem 3.30 *Let the Assumptions (3.64) and (3.65) hold. Define the vectors $a^i \in \mathbb{R}^4$, $i \in \mathcal{I}^2$ by*

$$\begin{aligned} a_1^i &:= \frac{1}{2} \left(\frac{(\gamma^{i; z_i, v_i})^2}{\gamma^{i; z_i}} - \gamma^{i; v_i} \right), \\ a_2^i &:= \frac{\gamma^{i; z_i, v_i} \psi_i^X(\tilde{e}_{2+i}^4)}{\gamma^{i; z_i}} - \lambda^i, \\ a_3^i &:= \frac{1}{2} \frac{\psi_i^X(\tilde{e}_{2+i}^4)^2}{\gamma^{i; z_i}}, \\ a_4^i &:= \kappa_i. \end{aligned}$$

Then for a^i , $i \in \mathcal{I}^2$ the functions $x_{a^i} : [0, T] \rightarrow \mathbb{R}$ and $y_{a^i} : [0, T] \rightarrow \mathbb{R}$ given by (B.4) resp. (B.5) are well-defined. Moreover in the generalized Heston model (3.62-3.63) a solution of the standardized exponential utility maximization Problem (3.12) is given by

$$\theta^*(t) = \left(\frac{\xi_1^*(t)}{S_1(t)}, \frac{\xi_2^*(t)}{S_2(t)} \right)^\top, \quad t \in [0, T],$$

and the associated maximal utility is

$$-E(e^{\theta^* \bullet S(T)}) = -L^*(0) = -e^{\eta^0(0) + \eta_1^v(0)v_1(0) + \eta_2^v(0)v_2(0)}.$$

Here, for $t \in [0, T]$ an optimal wealth ξ^* is given by equation (3.70) and

$$\begin{aligned} \eta_i^v(T-t) &= x_{a^i}(t), \quad i \in \mathcal{I}^2, \\ \eta^0(T-t) &= y_{a^1}(t) + y_{a^2}(t). \end{aligned}$$

PROOF. Keeping Equation (3.70) in mind, one can show by straightforward calculations that condition (3.59) leads to three ordinary (differential) equations:

$$\begin{aligned} 0 &= \partial_1 \eta^0(t) + \kappa^\top \eta^v(t), \quad \eta^0(T) = 0 \\ 0 &= \partial_1 \eta_i^v(t) - a_1^i \eta_1^v(t)^2 - a_2^i \eta_i^v(t) - a_3^i, \quad \eta_i^v(T) = 0, \quad i \in \mathcal{I}^2. \end{aligned}$$

Like in the Heston model, Lemma B.7 yields the solution to these equations. The remaining claims then follow from Corollary 3.19. \square

3.5.2 Barndorff-Nielsen & Shephard

We now discuss two versions of the Barndorff-Nielsen & Shephard model (2.5-2.6).

3.5.2.1 Superposition

In this section we start with the BNS-superposition model proposed by [7]. We consider the case of one asset with a BNS-superposition dynamic. For simplicity we assume that the squared volatility influencing the asset is the weighted sum of two independent OU-processes. Such a model can be written as

$$dz(t) = \delta v_3(t-)dt + \sqrt{v_3(t-)}dW(t), \quad (3.71)$$

$$dv_i(t) = -\lambda_i v_i(t)dt + dR_i(t), \quad i \in \mathcal{I}^2, \quad (3.72)$$

$$v_3(t) = w_1 v_1(t) + w_2 v_2(t). \quad (3.73)$$

Here $w_1, w_2 \in \mathbb{R}_+$, $\delta \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$, W denotes a standard Wiener process, R a 2-dimensional subordinator, i.e. a 2-dimensional Lévy process with triplet $(b^R, 0, F^R)$ whose components R_1 and R_2 are independent subordinators with Lévy exponents ψ^{R_i} , $i \in \mathcal{I}^2$. Since we want to apply the results from Section 3.3.2, we demand that Condition (3.36) holds by requiring that arbitrarily small exponential moments of F^{R_i} , $i \in \mathcal{I}^2$ exist:

$$\exists \varepsilon > 0 : \int_{\{|x| \geq 1\}} e^{\varepsilon x} F^{R_i}(dx) < \infty, \quad i \in \mathcal{I}^2. \quad (3.74)$$

Similarly as in [102] one can show that the above SDEs correspond to an affine process $X = (v, z)$ on $\mathbb{R}_+^3 \times \mathbb{R}$ with initial value $(v_1(0), v_2(0), w_1 v_1(0) + w_2 v_2(0), z(0))$ and triplets $(\beta^i, \gamma^i, \varphi^i)$, $i \in \mathcal{I}_0^3$ given by

$$\begin{aligned} \beta^0 &= \begin{pmatrix} b^R \\ w_1 b_1^R + w_2 b_2^R \\ 0 \end{pmatrix}, \\ \gamma^0 &= 0, \\ \varphi^0(G) &= \int 1_G(x_1, x_2, w_1 x_1 + w_2 x_2, 0) F^R(dx) \quad \forall G \in \mathcal{B}^4 \\ (\beta^1, \gamma^1, \varphi^1) &= \left((-\lambda_1, 0, \lambda_2 w_1, 0)^\top, 0, 0 \right), \\ (\beta^2, \gamma^2, \varphi^2) &= \left((0, -\lambda_2, \lambda_1 w_2, 0)^\top, 0, 0 \right), \\ (\beta^3, \gamma^3, \varphi^3) &= \left(\delta \tilde{e}_4^4 - (\lambda_1 + \lambda_2) \tilde{e}_3^4, (0, 0, 0, \tilde{e}_4^4), 0 \right). \end{aligned}$$

The corresponding Lévy exponent is

$$\psi^X(u) = \psi_0^X(u) + \sum_{i=1}^3 v_i(t-) \psi_i^X(u), \quad u \in D,$$

with

$$\begin{aligned}\psi_0^X(u) &= u^\top \beta^0 + \int (e^{u^\top x} - 1 - u^\top x) \varphi^0(dx), \\ \psi_1^X(u) &= -u_1 \lambda_1 + u_3 w_1 \lambda_2, \\ \psi_2^X(u) &= -u_2 \lambda_2 + u_3 w_2 \lambda_1, \\ \psi_3^X(u) &= -u_3 (\lambda_1 + \lambda_2) + u_4 \delta + \frac{1}{2} u_4^2,\end{aligned}$$

and

$$D = \left\{ u \in \mathbb{C}^4 : \int_{\{|x| \geq 1_4\}} e^{\operatorname{Re}(u)^\top x} \varphi^0(dx) < \infty \right\}.$$

Like in the previous paragraphs the optimality conditions for this model can be deduced from the conditions (3.18 -3.19) in the general framework of Section 3.3:

$$0 = b^K + \vartheta^\top b^S + \frac{1}{2} \vartheta^\top c^S \vartheta, \quad (3.75)$$

$$0 = b^S + c^S \vartheta. \quad (3.76)$$

Again, the corresponding characteristics of (S, K) are determined in Corollary 3.19. One can easily see that a solution of (3.76) in terms of wealth ξ is given by

$$\xi = -\psi_1^X(\tilde{e}_4^4) = -\left(\delta + \frac{1}{2}\right).$$

If we plug this in (3.75) and apply comparison of coefficients, then we obtain the following system of linear ordinary differential equations:

$$0 = \partial_1 \eta^0(t) + \psi_0^X \left(\sum_{i=1}^3 \eta_i^v(t) \tilde{e}_i^4 \right), \quad (3.77)$$

$$0 = \partial_1 \eta_1^v(t) - \lambda_1 \eta_1^v(t) + w_1 \lambda_2 \eta_3^v(t), \quad (3.78)$$

$$0 = \partial_1 \eta_2^v(t) - \lambda_2 \eta_2^v(t) + w_2 \lambda_1 \eta_3^v(t), \quad (3.79)$$

$$0 = \partial_1 \eta_3^v(t) - (\lambda_1 + \lambda_2) \eta_3^v(t) - \frac{\psi_1^X(\tilde{e}_4^4)^2}{2}. \quad (3.80)$$

Solving these equations and applying Corollary 3.19 gives us the following result for the BNS-superposition model:

Theorem 3.31 *Assume that Condition (3.74) holds. Then in the BNS-superposition model (3.71-3.73) an optimal strategy for the utility maximization Problem (3.12) is given by*

$$\theta^*(t) = \frac{\xi^*}{S(t)} = \frac{-(\delta + \frac{1}{2})}{S(t)}, \quad t \in [0, T],$$

and the associated optimal utility is

$$-E(e^{\theta^* \bullet S(T)}) = -L^*(0) = -e^{\eta^0(0) + \eta^v(0)^\top v(0)}.$$

where, for $t \in [0, T]$,

$$\begin{aligned}\eta^0(T-t) &= \int_0^t \left(\psi^{R_1} \left(\frac{w_1(\xi^*)^2}{2\lambda_1} (e^{-\lambda_1 s} - 1) \right) + \psi^{R_2} \left(\frac{w_2(\xi^*)^2}{2\lambda_2} (e^{-\lambda_2 s} - 1) \right) \right) ds, \\ \eta_1^v(T-t) &= \frac{w_1(\xi^*)^2}{2(\lambda_1 + \lambda_2)} \left(e^{-\lambda_1 t} \left(1 + \frac{\lambda_2}{\lambda_1} \right) - e^{-(\lambda_1 + \lambda_2)t} - \frac{\lambda_2}{\lambda_1} \right), \\ \eta_2^v(T-t) &= \frac{w_2(\xi^*)^2}{2(\lambda_1 + \lambda_2)} \left(e^{-\lambda_2 t} \left(1 + \frac{\lambda_1}{\lambda_2} \right) - e^{-(\lambda_1 + \lambda_2)t} - \frac{\lambda_1}{\lambda_2} \right), \\ \eta_3^v(T-t) &= \frac{(\xi^*)^2}{2(\lambda_1 + \lambda_2)} (e^{-(\lambda_1 + \lambda_2)t} - 1).\end{aligned}$$

PROOF. One can easily see that η^v and η^0 are unique solutions of the system (3.77-3.80). Like in the proof of Theorem 3.27 the claim concerning optimality follows from Corollary 3.19. \square

Remark 3.32 The univariate BNS-model (2.5-2.6) included in the BNS-superposition model was examined by Benth & Meyer-Brandis [13]. In order to establish the existence of the MEMM they need that a *fixed* exponential moment exists for each time horizon (cf. [13, Proposition 3.2]). This assumption typically limits the time horizon T for which the MEMM exists. In our framework the existence of the MEMM for any arbitrary time horizon $T < \infty$ follows from the weaker condition (3.74).

For some subordinators the integral determining η^0 in Theorem 3.31 can be calculated explicitly:

Corollary 3.33 Suppose that the Lévy exponents of R_1 and R_2 in Theorem 3.31 are given by

$$\psi^{R_i}(u) = \frac{\lambda_i p_i u}{q_i + u}, \quad i \in \mathcal{J}^2, \quad (3.81)$$

where $p_i, q_i, \lambda_i, i \in \mathcal{J}^2$ are strictly positive constants. Then the integrability Condition (3.74) is fulfilled and η^0 in Theorem 3.31 is given by:

$$\eta^0(T-t) = g_1(t) + g_2(t)$$

for $t \in [0, T]$, where the functions $g_i : [0, T] \rightarrow \mathbb{R}_-$, $i \in \mathcal{J}^2$, are defined by

$$g_i(t) = \frac{p_i}{q_i + r_i} \left(-q_i \log \left(1 - \frac{r_i}{q_i} (e^{-\lambda_i t} - 1) \right) + r_i \lambda_i t \right),$$

with $r_i = (w_i(\xi^*)^2)/(2\lambda_i)$.

PROOF. Since one can easily see that Condition (3.74) holds for $\psi^{R_i}, i \in \mathcal{J}^2$, the first claim is obvious. By differentiation of $g_1 + g_2$ one obtains that (3.77) holds. \square

If the Lévy exponents are chosen according to (3.81), the resulting activity processes v_2, v_3 are called Γ -OU processes. This notation is justified because these activity processes follow Gamma laws if the initial values $v_2(0), v_3(0)$ are also chosen according a Gamma distribution (cf.[172, Sections 5.2 and 5.3.3] for details). Since R_1 and R_2 are independent, one could also combine different choices of OU-processes leading to an explicit expression for η^0 . The IG-OU case is investigated in the next paragraph as an example.

3.5.2.2 Bivariate volatility

Another way to include different volatilities in the BNS-model (2.5-2.6) is to consider two assets which are influenced by two volatilities. This case is similar to the generalization of the Heston model discussed in the last section. But we will see below (cf. Remark 3.34) that the optimality conditions are easier to meet since the covariance between volatility and asset return is zero in the BNS-framework. Formally, such a model can again be described by stochastic differential equations for the return process $z = (z_1, z_2)^\top$ and the activity process $v = (v_1, v_2)^\top$:

$$dv_i(t) = -\lambda_i v_i(t-)dt + dR_i(t), \quad i \in \mathcal{I}^2, \quad (3.82)$$

$$dz(t) = \sum_{i=1}^2 \left(\delta^i v_i(t-)dt + \sqrt{v_i(t-)} dW^i(t) \right), \quad (3.83)$$

with $\delta^i \in \mathbb{R}^2, \lambda_i \in \mathbb{R}, i \in \mathcal{I}^2$. $W^i, i \in \mathcal{I}^2$, denote two 2-dimensional Wiener processes with covariance matrices $\gamma^{i;z}$, which are independent from each other, and $R = (R_1, R_2)^\top$ a 2-dimensional subordinator with Lévy-Khintchine triplet $(b^R, 0, F^R)$ and Lévy exponent ψ^R . By these assumptions v is an \mathbb{R}_+^2 -valued Ornstein-Uhlenbeck (OU) process. Furthermore we require that there exists an arbitrary small $\varepsilon > 0$ such that

$$\int_{\{|x| \geq 1\}} e^{\varepsilon(x_1+x_2)} F^R(dx) < \infty, \quad (3.84)$$

in order to ensure that Condition (3.36) holds. Similarly as in [102] for the univariate BNS-model one can show that the model (3.82-3.83) defines an affine process $X = (v, z)$ in the sense of Definition 3.1 with triplets $(\beta^i, \gamma^i, \varphi^i), i \in \mathcal{I}_0^2$ given by

$$\beta^0 = \begin{pmatrix} b^R \\ 0 \\ 0 \end{pmatrix}, \quad \gamma^0 = 0, \quad \varphi^0(G) = \int 1_G(x, 0, 0) F^R(dx) \quad \forall G \in \mathcal{B}^4,$$

$$(\beta^i, \gamma^i, \varphi^i) = \left(\begin{pmatrix} -\lambda_i \tilde{e}_i^2 \\ \delta_i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{i;z} \end{pmatrix}, 0 \right), \quad i \in \mathcal{I}^2.$$

The Ansatz (3.15) for the opportunity process and the Ansatz (3.16) for an optimal strategy lead by comparison of coefficients to the following two systems of linear equations for a wealth ξ invested in S :

$$0 = \hat{\beta}^i + \gamma^{i;z} \xi, \quad i \in \mathcal{I}^2 \quad (3.85)$$

with $\hat{\beta}^i := (\delta_1^i + \frac{1}{2}\gamma^{i;z_1}, \delta_2^i + \frac{1}{2}\gamma^{i;z_2})^\top, i \in \mathcal{I}^2$.

Remark 3.34 A wealth ξ has to fulfill both systems simultaneously. Obviously, one can choose $\hat{\beta}^i, \gamma^{i;z}, i \in \mathcal{I}^2$, such that \bar{X} satisfies the conditions of Lemma 3.14 and hence such a ξ exists. But it is also obvious that the *partition condition* of this lemma is not necessary in order to get a solution to (3.85). E.g. if $\text{rank}(\gamma^{i;z}) = 1, i \in \mathcal{I}^2$, then the situation is similar to the generalized Heston model of Section 3.5.1.2, in which the partition condition is also not demanded. Even in the case $\sum_{i=1}^2 \text{rank}(\gamma^{i;z}) > 2$ one can find solutions of (3.85) without meeting the partition condition. But then one has to include condition (3.85) as constraint in the parameter estimation.

We assume now that one of the situations described in the above remark is fulfilled and that ξ is a solution to (3.85). Once more Corollary 3.19 provides us with the following result:

Theorem 3.35 *Assume that ξ^* is a solution to (3.85) and that Condition (3.84) is fulfilled. Then a solution to the utility maximization Problem (3.12) in the BNS-model (3.82-3.83) is given by*

$$\theta^*(t) = \left(\frac{\xi_1^*}{S_1(t)}, \frac{\xi_2^*}{S_2(t)} \right)^\top$$

and the associated optimal utility is

$$-E(e^{\theta^* \cdot S(T)}) = -L^*(0) = -e^{\eta^0(0) + \eta_1^v(0)v_1(0) + \eta_2^v(0)v_2(0)},$$

where, for $t \in [0, T]$ and $i \in \mathcal{I}^2$,

$$\eta_i^v(T-t) = \begin{cases} \frac{d_i}{2\lambda_i} (e^{-\lambda_i t} - 1) & \text{if } \lambda_i \neq 0, \\ -\frac{d_i}{2} t & \text{if } \lambda_i = 0, \end{cases} \quad (3.86)$$

$$\eta^0(T-t) = \int_0^t \psi^R(\eta_1^v(T-s), \eta_2^v(T-s)) ds. \quad (3.87)$$

Here, $d_i := (\hat{\beta}^i)^\top (\gamma^{i;z})^{-1} \hat{\beta}^i$ and $(\gamma^{i;z})^{-1}$ denotes the generalized inverse of $\gamma^{i;z}$ (cf. [1, Chapter III]).

PROOF. Like in the previous paragraph (cf. Theorem 3.31), one can derive the following system of ordinary differential equations from the equation which corresponds to (3.75) in the BNS-model (3.82-3.83):

$$0 = \partial_1 \eta^0(t) + \psi^R \left(\sum_{i=1}^2 \eta_i^v(t) \tilde{e}_i^2 \right), \quad \eta^0(T) = 0, \quad (3.88)$$

$$0 = \partial_1 \eta_2^v(t) - \lambda_i \eta_i^v(t) - \frac{1}{2} d_i, \quad \eta_i^0(T) = 0, \quad i \in \mathcal{I}^2.$$

One easily verifies that (3.86) and (3.87) are the solutions to these equations. Again the results concerning optimality follow from Corollary 3.19. \square

Finally we determine η^0 in a situation similar to Section 3.5.2.1:

Corollary 3.36 Assume that the Lévy process R in the BNS-model (3.82-3.83) consists of two independent subordinators R_1, R_2 with Lévy exponents

$$\psi^{R_i}(u) = \frac{\lambda_i p_i u}{\sqrt{q_i^2 + 2u}}, \quad i \in \mathcal{I}^2, \quad (3.89)$$

for strictly positive constants $p_i, q_i, \lambda_i, i \in \mathcal{I}^2$. Then (3.84) holds and η^0 in Theorem 3.35 is given by

$$\eta^0(T - t) = g_1(t) + g_2(t)$$

for $t \in [0, T]$, where for $i \in \mathcal{I}^2$ the functions $g_i : [0, T] \rightarrow \mathbb{R}_-$ are defined as

$$g_i(t) = -\frac{p_i}{\sqrt{d_i}} \left(2r_i \left(\tanh^{-1} \left(\frac{q_i}{\sqrt{d_i}} \right) - \tanh^{-1} \left(\frac{d_i - 2r_i e^{-\lambda_i t}}{\sqrt{d_i}} \right) \right) + \sqrt{d_i} \left(\sqrt{d_i - 2r_i e^{-\lambda_i t}} - q_i \right) \right)$$

with $d_i := q_i^2 + 2r_i$ and

$$r_i := \frac{(\hat{\beta}^i)^\top (\gamma^{i;z})^{-1} \hat{\beta}^i}{2\lambda_i}.$$

PROOF. From the form of $\psi^{R_i}, i \in \mathcal{I}^2$ one can see that condition (3.84) holds. By checking that $g_1 + g_2$ fulfills Equation (3.88) the remainder of the claim follows. \square

Similarly as the Γ -OU case treated in Corollary 3.33, the Lévy exponents (3.89) lead to IG-OU processes with stationary Inverse Gaussian distribution (cf. e.g. [172, Sections 5.3.4 and 5.5.2]).

3.5.3 Further tractable models

In this section we discuss the solutions of the Problem (3.12) in models proposed by Carr et al. [18] and by Carr & Wu [19] which can be seen as generalizations of the Heston model and the BNS model, respectively. In both cases the models allow for jumps in the asset price. Therefore in general the determination of an optimal strategy θ^* requires a numerical method in order to find a root of a deterministic function. Beyond this fact the solutions below are very similar compared to the solutions presented in the preceding paragraphs. In this section we consider for simplicity the case of one asset, i.e. $m = 1$, and one volatility, i.e. $n = 1$. The extension to more general settings is straightforward, though.

3.5.3.1 Carr & Wu (2003)

Carr & Wu [19] replace the Brownian motion driving the return process z in the Heston model by an α -stable Lévy process R (cf. e.g. [164, Example 3.1.3]) with *index of stability* $\alpha \in (1, 2)$ and Lévy-Khintchine triplet $(b^R, 0, F^R)$. Additionally, they require that R is

totally skewed to the left (cf. [164, Section 1.2]) and b^R satisfies the martingale property (cf. e.g. [101, Lemma 3.1])

$$b^R = 0. \quad (3.90)$$

Such a kind of model can be written as

$$dz(t) = v(t)^{1/\alpha} dR(t), \quad (3.91)$$

$$dv(t) = (\kappa + \lambda v(t-))dt + \sigma \sqrt{v(t-)} dW(t). \quad (3.92)$$

Here, W denotes a standard Wiener process and $\kappa \geq 0$, λ , σ real-valued constants. We exclude degenerate models by requiring that $\sigma \neq 0$. As the Lévy process R is totally skewed to the left, its Lévy measure is concentrated on the negative half of the real line, i.e. $F^R(\mathbb{R}_+) = 0$. Hence Condition (3.36) is satisfied. It is shown in [102] that $X = (v, z)$ is an affine process with triplets $(\beta^i, \gamma^i, \varphi^i)$, $i = 0, 1$, given by

$$\begin{aligned} (\beta^0, \gamma^0, \varphi^0) &= \left(\begin{pmatrix} \kappa \\ 0 \end{pmatrix}, 0, 0 \right), \\ \beta^1 &= \begin{pmatrix} \lambda \\ b^R \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi^1(G) = \int 1_G(0, x) F^R(dx) \quad \forall G \in \mathcal{B}^2. \end{aligned}$$

In this model the optimality conditions (3.18-3.19) read as

$$0 = b^K + \vartheta b^S + \int (e^{\vartheta x^S} - 1 - \vartheta x^S) F^{(S,K)}(d(x^S, x^K)), \quad (3.93)$$

$$0 = b^S + \int x^S (e^{\vartheta x^S} - 1) F^{(S,K)}(d(x^S, x^K)). \quad (3.94)$$

Here, by Corollary 3.19 the corresponding characteristics of (S, K) are given by

$$\begin{aligned} b^S(t) &= v(t)S(t-) \left(b^R + \int (e^x - 1 - x) F^R(dx) \right), \\ b^K(t) &= \partial \eta^0(t) + \kappa \eta^v(t) + v(t)(\partial \eta^v(t) + \lambda \eta^v(t) + \frac{1}{2} \sigma^2 \eta^v(t)^2), \\ c^S(t) &= 0, \\ F^{(S,K)}(t, G) &= v(t) \int 1_G(S(t-)(e^x - 1), 0) F^R(dx) \quad \forall G \in \mathcal{B}^2 \text{ with } 0 \notin G. \end{aligned}$$

Rewriting condition (3.94) in terms of wealth gives rise to

$$0 = \int ((e^x - 1) \exp((e^x - 1)\xi) - x) F^R(dx). \quad (3.95)$$

One can deduce from the martingale property (3.90) and the characteristics (b^S, c^S, F^S) that \bar{X} satisfies the no arbitrage Assumption 3.13. Therefore, by Lemma 3.14, a solution to Equation (3.95) exists. This solution can be easily determined with a numerical procedure

and is constant on $[0, T]$. Furthermore, one can show that $\xi < 0$ since (3.90) holds and degenerate models are excluded. The remaining Equation (3.93) corresponds to

$$0 = \partial\eta^0(t) + \eta^v(t)\kappa, \quad (3.96)$$

$$0 = \partial\eta^v(t) + \eta^v(t)\lambda + \frac{1}{2}\sigma^2(\eta^v(t))^2 + \int (\exp((e^x - 1)\xi) - 1 - \xi x) F^R(dx). \quad (3.97)$$

Solving these equations (3.96-3.97) with respect to the terminal conditions (3.20-3.21) leads to the following result, when combined with Corollary 3.19:

Theorem 3.37 *Assume that R is totally skewed to the left and the stochastic volatility is non-trivial, i.e. $\sigma \neq 0$, in the model of Carr & Wu (3.91-3.92). Then a solution to the utility maximization Problem (3.12) is given by*

$$\theta^*(t) = \frac{\xi^*}{S(t-)}, \quad t \in [0, T],$$

where ξ^* is the unique root of $f : \mathbb{R} \rightarrow \mathbb{R}$ with

$$f(\nu) = \int ((e^x - 1) \exp((e^x - 1)\nu) - x) F^R(dx).$$

The corresponding maximal utility is given by

$$-E(e^{\theta^* \bullet S(T)}) = -L^*(0) = -e^{\eta^0(0) + \eta^v(0)v(0)},$$

where, for $t \in [0, T]$,

$$\eta^v(T-t) = \frac{\sqrt{d} \tanh\left(\frac{\sqrt{d}}{2}t + q\right) + \lambda}{-\sigma^2}, \quad (3.98)$$

$$\eta^0(T-t) = \frac{\kappa}{-\sigma^2} \left(\lambda t + 2 \log \left(\frac{\cosh\left(\frac{\sqrt{d}}{2}t + q\right)}{\cosh(q)} \right) \right), \quad (3.99)$$

with $q := \tanh^{-1}\left(-\lambda/\sqrt{d}\right)$ and $d := \lambda^2 - 2\sigma^2 \int (\exp(\xi(e^x - 1)) - 1 - \xi x) F^R(dx)$.

PROOF. Similarly as in the proof of Theorem 3.27, one can show that the Riccati equations (3.96-3.97) have a unique solution, which are given by (3.98-3.99) since $d < 0$ (cf. [74, Section 0.9.5, formula 34]). Again the remaining claims follow from Corollary 3.19. \square

3.5.3.2 Carr et al.

Finally, we consider one of the models put forward in Carr et al. [18]. They suggest to use time-changed Lévy processes in order to describe asset returns. As for the time-change,

different choices are possible. If one selects an integrated OU-process then one obtains the following model for the return process z and the volatility process v :

$$z(t) = z(0) + R_2(\tilde{V}(t)), \quad (3.100)$$

$$d\tilde{V}(t) = v(t-)dt, \quad (3.101)$$

$$dv(t) = -\lambda v(t-)dt + R_1(t). \quad (3.102)$$

Here, λ denotes a real-valued constant, R_1, R_2 Lévy processes with Lévy exponents ψ^{R_i} , $i \in \mathcal{I}^2$ and Lévy-Khintchine triplets $(b^{R_1}, c^{R_1}, F^{R_1})$ and $(b^{R_2}(h), c^{R_2}, F^{R_2})$ such that R_1 is a subordinator and

$$\int_{\{|x| \geq 1\}} e^{\varepsilon x} F^{R_1}(dx) < \infty \quad \text{and} \quad \int_{\{|x| \geq 1\}} \exp(\varepsilon e^{2x}) F^{R_2}(dx) < \infty \quad (3.103)$$

for an arbitrary small $\varepsilon > 0$. These properties ensure that Condition (3.36) is satisfied and they are fulfilled if e.g. R_1 is a Γ -OU-process (cf. Corollary 3.33) or an IG-OU-process (cf. Corollary 3.36) and the jumps of R_2 are bounded from above. Like in [102] one can show that $X = (v, z)$ is an affine process with triplets $(\beta^i, \gamma^i, \varphi^i)$, $i = 0, 1$ given by

$$\beta^0 = \begin{pmatrix} b^{R_1} \\ 0 \end{pmatrix}, \quad \gamma^0 = 0, \quad \varphi^0(G) = \int 1_G(x, 0) F^{R_1}(dx) \quad \forall G \in \mathcal{B}^2,$$

$$\beta^1 = \begin{pmatrix} \lambda \\ b^{R_2}(h) \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 \\ 0 & c^{R_2} \end{pmatrix}, \quad \varphi^1(G) = \int 1_G(0, x) F^{R_2}(dx) \quad \forall G \in \mathcal{B}^2.$$

We additionally demand that the associated alternative affine process \bar{X} satisfies the no arbitrage Assumption 3.13. This is satisfied if e.g. $c^{R_i} > 0$. Similarly as in the preceding paragraph we obtain a characterization of an optimal trading strategy and the associated expected utility:

Theorem 3.38 *Suppose that Condition (3.103) holds in the model of Carr et al. (3.100-3.102) and that \bar{X} satisfies the no arbitrage Assumption 3.13. Then a solution to the utility maximization Problem (3.12) is given by*

$$\theta^*(t) = \frac{\xi^*}{S(t-)}, \quad t \in [0, T],$$

where ξ^* is a root of the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ which is defined by

$$f(\xi) = \psi^{R_2}(1) + c^{R_2}\xi + \int ((e^x - 1) \exp((e^x - 1)\xi) - 1) F^{R_2}(dx).$$

The root is unique if R_2 is not constant. The utility corresponding to θ^* is

$$-E(e^{\theta^* \bullet S(T)}) = -L^*(0) = -e^{\eta^0(0) + \eta^v(0)v(0)},$$

where for $t \in [0, T]$,

$$\eta^v(T-t) = \begin{cases} \frac{d}{\lambda} (1 - e^{-\lambda t}) & \text{if } \lambda \neq 0, \\ -dt & \text{if } \lambda = 0, \end{cases}$$

$$\eta^0(T-t) = \int_0^t \psi^{R_1}(\eta^v(T-s)) ds, \quad (3.104)$$

with $d = \xi^* \psi^{R_2}(1) + \frac{1}{2} c^{R_2} (\xi^*)^2 + \int (\exp(\xi^*(e^x - 1)) - 1 - \xi^*(e^x - 1)) F^{R_2}(dx)$.

PROOF. Similarly as in Section 3.5.3.1 the optimality Condition (3.19) can be reduced to

$$0 = f(\xi). \quad (3.105)$$

Since the no arbitrage Assumption (3.13) holds for \bar{X} , we can use Lemma 3.14 in order to prove that Equation (3.105) has a solution denoted by ξ^* . The uniqueness of the root follows from strict monotonicity of f . The remaining optimality Conditions (3.18, 3.20-3.21) give rise to the following equations

$$\begin{aligned} 0 &= \partial\eta^v(t) - \lambda\eta^v(t) + d, & \eta^v(T) &= 0, \\ 0 &= \partial\eta^0(t) + \psi^{R_1}(\eta^v(t)), & \eta^0(T) &= 0. \end{aligned}$$

The remaining claims follow as in the proofs of the previous paragraphs. □

Remark 3.39 1. As in Section 3.5.2 one can determine the integral in equation (3.104) explicitly in the case of an IG-OU-process or a Γ -OU-process.

2. If the OU-process from equation (3.102) is replaced by a square-root process (3.1), one obtains a generalization of the Heston model (cf. [18]). Again one could apply the methodology presented here in order to solve the utility maximization problem. But if the desirable correlation between the return process and the volatility process is allowed, the analogue of the optimality condition (3.105) is time-dependent. Therefore the calculation of an optimal wealth ξ^* requires to determine a root of a deterministic function for every $t \in [0, T]$.

Chapter 4

Asymptotic Exponential Utility-Based Pricing and Hedging in Affine Stochastic Volatility Models

4.1 Introduction

The issuer of a contingent claim faces the risk arising from selling options. Therefore she typically demands a *compensation* for this risk. Furthermore, the issuer will try to reduce this risk by trading in the underlying according to an appropriate *trading strategy*. We suppose that the financial market is incomplete. Hence there are several reasonable criteria to determine these quantities. We assume that the issuer expresses her preferences in terms of exponential utility-indifference pricing (cf. [86, 161, 8, 81] and the references cited therein). Here for q units of an option with discounted payoff H and an exponential utility function $u(x) = 1 - \exp(-px)$ for a $p > 0$, the investor looks for an amount of money $\widehat{\pi}_q^p$ called *indifference price per unit* of H such that the maximal utility of portfolios containing the claim

$$\sup_{\tilde{\theta}} E \left(u(q\widehat{\pi}_q^p + \tilde{\theta} \cdot S(T) - qH) \right), \quad (4.1)$$

is greater than or equal to the maximal utility of portfolios without the option, i.e.

$$\sup_{\tilde{\theta}} E \left(u(\tilde{\theta} \cdot S(T)) \right). \quad (4.2)$$

Here, S denotes the discounted stock price process, T the time horizon and $\tilde{\theta} \cdot S$ the stochastic integral of a strategy $\tilde{\theta}$ with respect to S . In the literature, several sources of incompleteness are discussed. If H is a claim on a non-traded asset correlated with S , one speaks of *basis risk*. Depending on the model for the assets, the form of H and the method to tackle the problem, it is possible to obtain more or less explicit (approximative) solutions in this situation, cf. [185, 80, 141, 140, 36, 73, 126].

Instead, we consider the case that H is a European-type claim on S and that incompleteness arises from stochastic volatility. Using the HJB approach it was shown in [179] for a class of diffusion models that the utility-indifference price satisfies a quasilinear PDE. Martingale duality methods were applied by [9] in order to relate the utility-indifference price to the solution of a semi-linear PDE in a framework of interacting Itô and point processes. In [93] for a very special diffusion model with a bounded stochastic volatility, indifference prices of Barrier options were computed by using a finite-difference method. Besides these examples it seems very hard to find more or less explicit solutions for the utility-indifference price and the corresponding trading strategy in any kind of nontrivial stochastic volatility model. Hence several authors, cf. e.g. [130, 10, 91, 112], investigate the *asymptotics* of (4.1) if q tends to zero. Their main results reveal that one can obtain approximations for $\widehat{\pi}_q^p$ and an optimal trading strategy $\widehat{\theta}$ in (4.1) for a small number of options q as follows:

1. Determine the *optimal investment strategy* θ_0 maximizing (4.2) and the corresponding *minimal entropy martingale measure* Q^* .
2. Solve the *quadratic hedging problem* for the claim H under Q^* .

Both problems are well-known and each of them has been extensively discussed in the literature. For the characterization and determination of the minimal entropy martingale measure and the relation to an optimal investment strategy θ_0 in a general semimartingale framework we refer to [63, 37, 99] and the references therein. Likewise confer [59, 176, 153, 22] for an overview about quadratic hedging in a general semimartingale setting.

In this chapter we will focus on the *computation* of the solution to the above problems in affine stochastic volatility models. We determine semi-explicit formulas, which allow to calculate the first-order approximations to (4.1) for European options. Besides the model of Heston [82] the model proposed by Barndorff-Nielsen & Shephard (cf. Example 2.4) is another popular representative for models of this kind (cf. also [102] for an overview of further similar models). For the BNS model, the minimal entropy martingale measure can be determined by duality methods (cf. [13]). By this duality an optimal strategy θ_0 is also known. Similar results have been obtained in certain other stochastic volatility models (e.g. [85] for the Heston model [82], [156] for the Stein & Stein model [180] and [158] for general BNS models). The results in the body of this chapter rest on Chapter 3. Here, for our affine framework of Chapter 2, specialized to the time-homogeneous case, the MEMM and an optimal investment strategy are characterized in terms of a solution to a differential algebraic equation by martingale duality methods.

There are several approaches to compute the solution of a quadratic hedging problem. Besides PDE and simulation methods [79, 31] a Laplace transform technique is often used to calculate the solution for European options. In [88, 21] this approach was applied to geometric Lévy models and in [111, 110] to affine stochastic volatility models. The latter embedded

their findings in a time-homogeneous affine framework. We generalize their results to time-inhomogeneous affine processes in the sense of [48] since we show (cf. Theorem 3.17 and Corollary 3.19) that the dynamics of an affine stochastic volatility model typically turn out to be time-inhomogeneous under the minimal entropy martingale measure.

We combine these findings with the results of [112] and Chapter 3 in order to determine semi-explicit expressions for the asymptotics of (4.1) in affine stochastic volatility models. For the class of superposition models proposed by Barndorff-Nielsen & Shephard [7] we show that the assumptions needed to establish the results in the general framework can be easily checked. Moreover, in a numerical example we compare the results for different subordinators in the BNS superposition model.

This chapter is structured as follows. In Section 4.2 we present the affine framework and give an overview of general results concerning (asymptotic) exponential utility-indifference pricing and hedging. In Section 4.3 we determine the asymptotic results in the affine framework. In Section 4.4 we present the procedure of the previous section by calculating the asymptotics in the BNS superposition model. Finally we visualize these results by a numerical example in Section 4.5.

4.2 Framework and preliminary results

4.2.1 Asymptotic utility-based pricing and hedging

We give an overview on some general results about (asymptotic) utility-based pricing and hedging in this section. For more details we refer to [112, 37] and the references therein. We assume that the univariate discounted price process S is given by a strictly positive and locally bounded semimartingale. In this section we do not suppose that S is related to an affine process as described in Section 2.1. First, we recall some notions already given in Section 3.2.1. Again, a probability measure $Q \sim P$ is called *equivalent σ -martingale measure* if S is a σ -martingale under Q . We denote by $\mathcal{M}_e(P)$ the space of all equivalent σ -martingale measures and by $\mathcal{M}(P)$ the space of all measures which belong to $\mathcal{M}_e(P)$ and have finite *entropy* (3.10). In this section, we assume that $\mathcal{M}(P)$ is non-empty. By [63, Remark 2.1] this implies that there exists a unique $Q^* \in \mathcal{M}(P)$ minimizing the entropy. Q^* is called *minimal entropy martingale measure (MEMM)*. Via duality, it is closely related (cf. [99, Theorem 2.1]) to the solution of the utility maximization problem

$$\sup_{\tilde{\theta} \in \Theta} E \left(u(\tilde{\theta} \cdot S(T)) \right) \quad (4.3)$$

for an exponential utility function u given by

$$u(x) = 1 - e^{-px}, \quad \text{with } p > 0 \text{ and } x \in \mathbb{R}.$$

Again, the set of admissible strategies Θ is given by Definition 3.4. Note that the uniqueness statements which are given below for optimal strategies have to be understood modulo the

equivalence relation (3.13). The *optimal investment strategy* θ_0 in (4.3) exists and is unique by [99, Theorem 2.1]. Given $q \in \mathbb{R}$ units of a bounded claim H one can formulate the indifference pricing principle for exponential utility as follows: Find a $\widehat{\pi}_q^p$ called *indifference price per unit* of H such that (4.3) coincides with

$$\sup_{\tilde{\theta} \in \Theta} E(u(q\widehat{\pi}_q^p + \tilde{\theta} \cdot S(T) - qH)). \quad (4.4)$$

$\widehat{\pi}_q^p$ and the corresponding optimizer $\widehat{\theta}_q^p \in \Theta$ of (4.4) exist and are unique by [112, Section 2] and [99, Theorem 2.1]. Since one easily verifies that $\widehat{\pi}_q^p = \widehat{\pi}_{pq}^1$ and $\widehat{\theta}_q^p = p\widehat{\theta}_{pq}^1$, it suffices to consider the case $p = 1$. The corresponding solutions are denoted by $\widehat{\pi}_q := \widehat{\pi}_q^1$ and $\widehat{\theta}_q := \widehat{\theta}_q^1$.

Define the *utility-based hedging strategy* per option

$$\theta^H := \frac{\widehat{\theta}_q - \widehat{\theta}_0}{q}, \quad q \neq 0.$$

θ^H is unique (cf. [112, Section 2]) and represents the adjustment in the optimal trading strategy per unit of H due to the presence of q units of H in the portfolio.

The approximations for θ^H and $\widehat{\pi}_q$ obtained by [130, 10, 112] are related to the optimization problem arising in *variance-optimal hedging*. In order to describe this concept we assume that S is a locally square-integrable martingale under a probability measure Q . In variance-optimal hedging one minimizes the expected squared hedging error of a given claim $H \in L^2(\mathcal{F}(T), Q)$, i.e.

$$\varepsilon^2(\bar{\pi}, \bar{\theta}) := E_Q(\bar{\pi} + \bar{\theta} \cdot S(T) - H), \quad (4.5)$$

over all initial endowments $\bar{\pi}$ and all strategies $\bar{\theta} \in L^2(S)$. We call a claim *attainable* if there exists a constant h and an admissible strategy $\bar{\theta} \in L^2(S)$ such that $H = h + \bar{\theta} \cdot S$. The optimal endowment resp. the optimal strategy in (4.5) are denoted by π resp. θ .

As in [111], we make use of an integral representation of the payoff in order to solve the quadratic hedging problem (4.5). More specifically, we assume that H is a European-style claim such that its discounted payoff can be written as $H = f(\log(S(T)))$. Moreover, we suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be represented as

$$f = \int e^{su} \Pi(du), \quad (4.6)$$

where Π is a complex measure on a strip

$$\mathcal{S}_f := \{u \in \mathbb{C} : R' \leq \operatorname{Re}(u) \leq R\} \quad (4.7)$$

with $R', R \in \mathbb{R}$. We require that Π is symmetric, which means that $\Pi(\overline{G}) = \overline{\Pi(G)}$ for $G \in \mathcal{B}(\mathbb{C})$ and $\overline{G} := \{\bar{u} \in \mathbb{C} : u \in G\}$. For many standard options one can choose $R' = R$ and the measure Π has a Lebesgue-density (cf. [88] for some examples). E.g. for a European put with strike K the measure Π is given by

$$\Pi(du) = \frac{1}{2\pi i} \frac{K^{1-u}}{u(1-u)} du$$

on $\mathcal{S}_f := R + i\mathbb{R}$ with $R < 0$. This leads to the integral representation

$$(K - e^s)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{su} \frac{K^{1-u}}{u(1-u)} du. \quad (4.8)$$

As it is well known the results of [59] show that the solution to (4.5) can be expressed as follows:

$$V(t) := E_Q(H | \mathcal{F}(t)), \quad t \in [0, T], \quad (4.9)$$

$$\pi := V(0),$$

$$\theta(t) := \frac{d\langle V, S \rangle^Q(t)}{d\langle S, S \rangle^Q(t)}, \quad t \in [0, T],$$

$$\varepsilon^2 := \varepsilon^2(\pi, \theta) = E_Q(\langle V, V \rangle^Q(T) - \theta^2 \cdot \langle S, S \rangle^Q(T)). \quad (4.10)$$

The main result of [112] tells us that the utility-based hedging strategy θ^H and the indifference price π_q can be approximated by the solution to (4.5) under the minimal entropy martingale measure Q^* :

Theorem 4.1 *Assume that $\mathcal{M}(P) \neq \emptyset$. Let π , θ and ε^2 be the solution of the quadratic hedging problem (4.5) under Q^* . Suppose that*

1. $\theta \cdot S$ has bounded jumps,
2. at least one of the following conditions hold:

(a) S is continuous,

(b) $\frac{1}{S(-)} \cdot S$ has bounded jumps and

$$\|q\theta^H S(-)\|_{L^\infty(\Omega \times [0, T])} = \mathcal{O}(q)$$

for $q \rightarrow 0$.

Then we have

$$\|\theta - \theta^H\|_{L^2(S)} = \left\| \theta - \frac{\hat{\theta}_q - \theta_0}{q} \right\|_{L^2(S)} \xrightarrow{q \rightarrow 0} 0,$$

$$\hat{\pi}_q = \pi + \frac{q}{2} \varepsilon^2 + o(q),$$

where

$$\|\theta\|_{L^2(S)} := \sqrt{E_{Q^*}(\theta^2 \cdot [S, S](T))} = \sqrt{E_{Q^*}((\theta \cdot S(T))^2)}.$$

In particular, we obtain

$$\sup_{t \in [0, T]} |\theta \cdot S(t) - \theta^H \cdot S(t)| \rightarrow 0 \text{ in } L^2(Q^*).$$

PROOF. Section 4 in [112]. □

The preceding theorem allows us to define the *approximate utility-indifference price per unit of H*

$$\pi_q(H) := \pi + \frac{q}{2}\varepsilon^2$$

and the *approximate utility-based trading strategy*

$$\theta_q(H) := \theta_0 + q\theta,$$

for q units of H . Furthermore we call θ the *approximate utility-based hedging strategy per unit of H* and π the *marginal utility-based price*.

In [8, Section 3] some properties of utility-indifference prices are summarized. Obviously the approximate utility-indifference price will violate even trivial arbitrage bounds if $|q|$ is not small enough and $\varepsilon^2 \neq 0$. But the approximation π_q shares a number of other properties with the exact indifference price $\widehat{\pi}_q$:

Remark 4.2 Assume that the prerequisites of Theorem 4.1 hold and let H, H_1, H_2 be bounded claims. Then one easily obtains that the following statements are satisfied:

1. $\pi_q(H)$ does not depend on the initial capital.
2. $q \mapsto \pi_q(H)$ is non-decreasing in q .
3. $\pi_1(qH) = q\pi_q(H)$. *(volume-scaling)*
4. $\pi_q(H + c) = \pi_q(H) + c$ for $c \in \mathbb{R}$. *(translation invariance)*
5. $\pi_q(\lambda H_1 + (1 - \lambda)H_2) \leq \lambda\pi_q(H_1) + (1 - \lambda)\pi_q(H_2)$ for $\lambda \in [0, 1]$. *(convexity)*
6. If H is attainable, then $\pi_q(H) = h$. *(elementary compatibility with no arbitrage)*
7. There exists a constant $\hat{q} > 0$ such that for $|q| \leq \hat{q}$
 - (a) $\pi_q(H_1) \leq \pi_q(H_2)$ if $H_1 \leq H_2$, *(monotonicity)*
 - (b) $\inf_{Q \in \mathcal{M}_e(P)} E_Q(H) \leq \pi_q(H) \leq \sup_{Q \in \mathcal{M}_e(P)} E_Q(H)$.
(compatibility with no arbitrage)
8. $\pi_q(H) \geq E_{Q^*}(H)$ for $q \geq 0$. *(risk-aversion under Q^*)*

For an interpretation of these properties we refer to [8]. We only note that for small q [8, Remark 3.1] remains valid for the approximate utility-indifference price, which means that π_q can be interpreted as a convex risk measure for a finite number of bounded claims.

A second remark concerns a certain *decomposition property* of the above approximation.

Remark 4.3 For exponential utility, [142, 8] investigate indifference prices in a binomial model. The authors develop iterative algorithms for the calculation of the indifference prices. In [142, Section 2] it is noted that the computation scheme decomposes the risk of the claim into hedgeable risk and unhedgeable risk. The hedgeable risk is valued according to a linear pricing rule, whereas the unhedgeable risk is priced non-linearly. The approximation $q\pi_q$ for the indifference price of the whole derivative position also has this property. Here the risk is separated through the Galtchouk-Kunita-Watanabe decomposition of H under Q^* and only the hedging error enters the approximate utility-indifference price quadratically.

4.2.2 Affine framework

Up to now we have assumed that the discounted price process S is a strictly positive and locally bounded semimartingale. Similarly as in the preceding chapter we now add an additional assumption by demanding that S can be written as

$$S = \exp(z),$$

where the return process z is a component of a time-homogeneous affine process. In the following two sections we consider the case of a univariate volatility process v and a single return process z . More precisely, we assume that (v, z) satisfies Definition 3.1 for $m = n = 1$.

In order to show that the MEMM Q^* and the optimal investment strategy θ_0 for affine processes exist we need some additional requirements:

Assumption 4.4 *Let X be an affine process according to Definition 3.1 and denote by $\psi_i^X, i = 0, 1$, the associated Lévy exponents. We impose that the following conditions hold:*

1. $\psi_0^X(0, u_2) = 0 \quad \forall u_2 \in i\mathbb{R}$. *This structure condition is a special case of Assumption 3.9. In particular it ensures that the optimal investment strategy θ_0 exists. It means that the differential characteristics of z depend in a linear fashion on $v(t-)$. It is equivalent to the following conditions*

$$\varphi^0((\mathbb{R}_+ \times \{0\})^C) = 0 \quad \wedge \quad \beta_2^0 = 0 \quad \wedge \quad \gamma_{2,2}^0 = 0.$$

2. *There exists an $\tilde{\varepsilon} > 0$ such that*

$$\int_{\{|x| \geq 1_2\}} e^{\tilde{\varepsilon}x_1} \varphi^i(dx) < \infty, \quad i = 0, 1.$$

The existence of an arbitrary small exponential moment of v is needed in Section 3.3 to show that the MEMM Q^ exists.*

3. *There exists a constant M such that*

$$\varphi^1((\mathbb{R}_+ \times (-\infty, M])^C) = 0.$$

Together with the first assumption this condition implies that S is locally bounded (cf. Lemma A.2 and the subsequent Remark A.3).

4. $\psi_1^X(0, 2) - 2\psi_1^X(0, 1) \neq 0$. By demanding this condition we essentially exclude constant stock price processes.

5.

$$\begin{aligned} \beta_2^1(h) - \int h_2(x)\varphi^1(dx) &< 0 \quad \text{if } \gamma_{1,1}^1 = 0 \text{ and } \varphi^1(\mathbb{R}_+^2) = 0, \\ \beta_2^1 - \int x_2\varphi^1(dx) &> 0 \quad \text{if } \gamma_{1,1}^1 = 0 \text{ and } \varphi^1(\mathbb{R}_+ \times \mathbb{R}_-) = 0. \end{aligned}$$

In view of Assumption 3.13 and Remark 3.20 these conditions can be interpreted as no arbitrage conditions.

We have to require further conditions in order to determine a solution to (4.5). However the formulation of these prerequisites demands the knowledge of the MEMM Q^* . Hence we impose these additional conditions in Section 4.3.

4.3 Main results

By Theorem 4.1 one has to solve a quadratic hedging problem under the minimal entropy measure Q^* in order to obtain approximations for utility-indifference prices and strategies. Therefore we first determine the MEMM Q^* and the related optimal investment strategy θ_0 in the affine framework of Section 4.2.2. For this purpose we recall a slightly generalization of Corollary 3.19 in the case of a single return process z and a single activity process v :

Theorem 4.5 *Suppose that Assumption 4.4 is satisfied. Then the following holds.*

1. *There exist mappings $\alpha_1 \in C^1([0, T], \mathbb{R}_-)$ and $\chi \in C([0, T], \mathbb{R})$ which are the solution to the differential algebraic equation*

$$\alpha_1'(t) = -\psi_1^X(\alpha_1(t), 0) - \chi(t)\psi_1^X(0, 1) - \chi(t)\alpha_1(t)\gamma_{1,2}^1 - \frac{1}{2}\chi^2(t)\gamma_{2,2}^1 \quad (4.11)$$

$$- \int ((\exp(\chi(t)(e^{x_2} - 1)) - 1) e^{\alpha_1(t)x_1} - \chi(t)(e^{x_2} - 1)) \varphi^1(dx),$$

$$0 = \psi_1^X(0, 1) + \gamma_{2,2}^1\chi(t) + \alpha_1(t)\gamma_{1,2}^1 \quad (4.12)$$

$$+ \int (e^{x_2} - 1) (\exp((e^{x_2} - 1)\chi(t) + \alpha_1(t)x_1) - 1) \varphi^1(dx),$$

$$\alpha_1(T) = 0. \quad (4.13)$$

2. *For $p = 1$ the optimal trading strategy θ_0 for (4.3) is given by*

$$\theta_0(t) = -\frac{\chi(t)}{S(t-)}, \quad t \in [0, T]. \quad (4.14)$$

3. *The MEMM Q^* exists with density*

$$\frac{dQ^*}{dP} = \frac{\exp(-\theta_0 \cdot S(T))}{L(0)}$$

and density process

$$Z(t) = \frac{L(t)}{L(0)} \exp(-\theta_0 \bullet S(t)), \quad t \in [0, T].$$

Here, the process L is given by

$$L(t) := \exp(\alpha_0(t) + \alpha_1(t)v(t)), \quad t \in [0, T]$$

for $\alpha_0(t) := \int_t^T \psi_0^X(\alpha_1(s), 0) ds$.

4. X is a time-inhomogeneous affine process under Q^* with differential characteristics $(b^{X^*}, c^{X^*}, F^{X^*})$ given by

$$\begin{aligned} b_1^{X^*}(t) &= b_1^X + c_{1,1}^X \alpha_1(t) + c_{1,2}^X \chi(t) + \int x_1 (e^{\alpha_1(t)x_1 + \chi(t)(\exp(x_2)-1)} - 1) F^X(dx), \\ b_2^{X^*}(t) &= b_2^X + c_{1,2}^X \alpha_1(t) + c_{2,2}^X \chi(t) \\ &\quad + \int h(x_2) (e^{\alpha_1(t)x_1 + \chi(t)(\exp(x_2)-1)} - 1) F^X(dx), \\ c^{X^*}(t) &= c^X, \\ F^{X^*}(t, G) &= \int 1_G(x_1, x_2) e^{\alpha_1(t)x_1 + \chi(t)(\exp(x_2)-1)} F^X(dx), \quad \forall G \in \mathcal{B}^2 \text{ with } 0 \notin G. \end{aligned}$$

PROOF. In order to prove that there exists a solution to the DAE (4.11-4.13), we use the following approach. As noted in Remark 3.20, S has an alternative representation in terms of the stochastic exponential of a process \bar{z} , which can be obtained from Proposition 3.18. Moreover $\bar{X} = (v, \bar{z})$ is an affine process by this proposition. Since Assumption 4.4(1-3,5) holds, one can apply Lemma 3.14 in order to show that \bar{X} satisfies the structure condition (3.25). Hence Theorem 3.16 yields that the differential algebraic equation associated to \bar{X} admits a solution. Using the differential characteristics of X , this DAE can be written as (4.11-4.13). The remaining claims follows from Corollary 3.19. \square

For $t \in [0, T]$ the Lévy-Khintchine triplets associated to the affine process X under Q^* are denoted by $(\beta^{0^*}(t), \gamma^{0^*}(t), \varphi^{0^*}(t))$ and $(\beta^{1^*}(t), \gamma^{1^*}(t), \varphi^{1^*}(t))$. Additionally, we write ψ^{X^*} , $\psi_0^{X^*}$ and $\psi_1^{X^*}$ for the corresponding Lévy exponents under Q^* . Since the Lévy exponent ψ^X is known explicitly for many stochastic volatility models it seems desirable to represent ψ^{X^*} in terms of ψ^X . To this end, one can show that the following series expansion holds:

Proposition 4.6 *Let X be as in Theorem 4.5 with Lévy exponents $\psi_0^{X^*}$ resp. $\psi_1^{X^*}$ under Q^* defined on \bar{D}_0 resp. \bar{D}_1 . Then $[0, T] \times D_i \subset \bar{D}_i, i = 0, 1$, and $\psi_0^{X^*}$ and $\psi_1^{X^*}$ can be written as*

$$\psi_0^{X^*}(t, u_1, u_2) = \psi_0^X(u_1 + \alpha_1(t), 0) - \psi_0^X(\alpha_1(t), 0), \quad (4.15)$$

$$\begin{aligned} \psi_1^{X^*}(t, u_1, u_2) &= \psi_1^X(u_1 + \alpha_1(t), u_2) - \psi_1^X(\alpha_1(t), 0) \\ &\quad + \sum_{j=1}^{\infty} \frac{\chi(t)^j}{j!} \left(\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \widehat{\psi}_1^i(t, u_1, u_2) \right), \end{aligned} \quad (4.16)$$

for $(t, u_1, u_2) \in [0, T] \times D_0$ resp. $(t, u_1, u_2) \in [0, T] \times D_1$, where

$$\widehat{\psi}_1^i(t, u_1, u_2) := \psi_1^X(\alpha_1(t) + u_1, i + u_2) - \psi_1^X(\alpha_1(t), i).$$

PROOF. First, we deduce from Assumption 4.4(3) that there exists a constant C_1 such that $|e^{x^2} - 1| < C_1$ for all $x \in \text{supp}(\varphi^1)$. Since $|\chi|$ is bounded by a constant C_2 and α_1 is non-positive, we conclude that there exists a constant $C_3 > 0$ such that, for $t \in [0, T]$ and $i = 0, 1$,

$$\varphi^{i*}(t, G) \leq C_3 \varphi^i(G) \quad \forall G \in \mathcal{B}^2.$$

This proves the first assertion. (4.15) follows immediately from Theorem 4.5(4). Define $f_n(t, x_2) := 1 + \sum_{j=1}^n \frac{\chi(t)^j}{j!} (e^{x_2} - 1)^j$, $n \geq 0$. Since $|f_n(t, x_2)| \leq \exp(C_1 C_2)$ for all $x \in \text{supp}(\varphi^1)$, dominated convergence yields

$$\lim_{n \rightarrow \infty} \int e^{(u_1 + \alpha_1(t))x_1 + u_2 x_2} f_n(t, x_2) \varphi^1(dx) = \int e^{(u_1 + \alpha_1(t))x_1 + u_2 x_2 + \chi(t)(e^{x_2} - 1)} \varphi^1(dx) \quad (4.17)$$

for every $(t, u_1, u_2) \in [0, T] \times D_1$. We obtain (4.16) from a binomial expansion of $(e^{x_2} - 1)^n$ in the left hand side of (4.17) by using $\sum_{i=0}^j (-1)^i \binom{j}{i} = 0$, $j \geq 1$, and $\sum_{i=1}^j i (-1)^i \binom{j}{i} = 0$, $j \geq 2$. \square

As in [111], the calculation of the variance-optimal hedging strategy θ , the optimal initial value π and the corresponding hedging error ε^2 requires some additional assumptions:

Assumption 4.7 *Suppose that X satisfies Assumption 4.4 and H admits an integral representation in the sense of (4.6). Let Ψ_0 and Ψ_1 be the mappings associated to the time-inhomogeneous affine process X under Q^* according to Theorem 2.6. Additionally, we demand that the following conditions are satisfied:*

1. For some $\tilde{\varepsilon} > 0$ the mappings $(u_1, u_2) \mapsto \Psi_0(s, t, u_1, u_2), \Psi_1(s, t, u_1, u_2)$ have an analytic extension on

$$\mathcal{S} := \{u \in \mathbb{C}^2 : (\text{Re}(u_1), \text{Re}(u_2)) \in \mathcal{V}_{\tilde{\varepsilon}}(0)\}$$

for all $s \in [0, t]$ and all $t \in [0, T]$, where

$$M_0 := \sup\{2\Psi_1(t, T, 0, r) : r \in [R' \wedge 0, R \vee 0], t \in [0, T]\}$$

and

$$\mathcal{V}_{\tilde{\varepsilon}}(a) := (-\infty, (M_0 \vee 0) + \varepsilon) \times ((2R' \wedge 0) - \tilde{\varepsilon}, (2R \vee a) + \tilde{\varepsilon})$$

for $a \in \mathbb{R}_+$. These extensions are again denoted by Ψ_0 resp. Ψ_1 .

2. $\mathcal{V}_{\tilde{\varepsilon}}(2) \subset D_0 \cap D_1$,
3. The mappings $s \mapsto \Psi_0(s, t, u_1, u_2)$ and $s \mapsto \Psi_1(s, t, u_1, u_2)$ are continuous on $[0, t]$ for any $(u_1, u_2) \in \mathcal{S}$ and any $t \in [0, T]$,

Now we can solve the quadratic hedging problem under Q^* .

Theorem 4.8 *Suppose that Assumption 4.7 is satisfied. Then the solution (4.9-4.10) of the quadratic hedging problem (4.5) under the minimal entropy martingale measure Q^* is given by*

$$\pi = V(0), \quad (4.18)$$

$$V(t) = \int V_\zeta(t) \Pi(d\zeta), \quad (4.19)$$

$$\theta(t) = \int \theta_\zeta(t) \Pi(d\zeta), \quad (4.20)$$

$$\varepsilon^2 = \int \int \int_0^T J(t, \zeta_1, \zeta_2) dt \Pi(d\zeta_1) \Pi(d\zeta_2), \quad (4.21)$$

where the processes V_ζ, θ_ζ have the form

$$\theta_\zeta := \frac{V_\zeta(t-)}{S(t)} \frac{\kappa(t, \zeta)}{\psi_1^{X^*}(t, 0, 2)},$$

$$V_\zeta(t) := S(t)^\zeta \exp(\Psi_0(t, T, 0, \zeta) + \Psi_1(t, T, 0, \zeta)v(t)), \quad \zeta \in \mathcal{S}_f$$

and the integrand $J : [0, T] \times \mathcal{S}_f^2 \rightarrow \mathbb{C}$ is given by

$$\begin{aligned} J(t, \zeta_1, \zeta_2) &:= S(0)^{\tilde{\zeta}} e^{\xi_0} \exp(\Psi_0(0, t, \xi_1, \tilde{\zeta}) + \Psi_1(0, t, \xi_1, \tilde{\zeta})v(0)) \times \\ &\quad \times \left(\iota_0 + \iota_1 \left(D_3 \Psi_0(0, t, \xi_1, \tilde{\zeta}) + D_3 \Psi_1(0, t, \xi_1, \tilde{\zeta})v(0) \right) \right). \end{aligned} \quad (4.22)$$

The variable $\tilde{\zeta} \in \mathbb{C}$ and the mappings $\kappa : [0, T] \times \mathcal{S}_f \rightarrow \mathbb{C}$ and $\iota_0, \iota_1, \xi_0, \xi_1 : [0, T] \times \mathcal{S}_f^2 \rightarrow \mathbb{C}$ are defined as

$$\begin{aligned} \tilde{\zeta} &:= \zeta_1 + \zeta_2, \\ \kappa(t, \zeta) &:= \psi_1^{X^*}(t, \Psi_1(t, T, 0, \zeta), \zeta + 1) - \psi_1^{X^*}(t, \Psi_1(t, T, 0, \zeta), \zeta), \\ \xi_i &= \xi_i(t, \zeta_1, \zeta_2) := \Psi_i(t, T, 0, \zeta_1) + \Psi_i(t, T, 0, \zeta_2), \quad i = 0, 1, \\ \iota_i &= \iota_i(t, \zeta_1, \zeta_2) := \psi_i^{X^*}(t, \xi_1(t, \zeta_1, \zeta_2), \tilde{\zeta}) \\ &\quad - \psi_i^{X^*}(t, \Psi_1(t, 0, \zeta_1), \zeta_1) - \psi_i^{X^*}(t, \Psi_1(t, 0, \zeta_2), \zeta_2), \quad i = 0, 1. \end{aligned}$$

The integral (4.21) is to be understood in the sense of the Cauchy principal value (cf. [111, Remark 4.3]).

PROOF. We apply Theorem A.6. Therefore we have to check the conditions of Assumption A.5. Conditions (1), (2) and (4) follow immediately from Assumption 4.7. Condition (3) is obvious in view of Proposition 4.6. (5) holds since Q^* is the MEMM. Moreover (6) holds under Q^* since Assumption 4.4(4) is satisfied. In particular, we obtain $\psi_0^{X^*}(t, 0, 2) = 0, t \in [0, T]$ and thus J is given by (4.22). \square

By using the next corollary the approximate utility-indifference price π_q and the approximate utility-based trading strategy θ_q can be calculated.

Corollary 4.9 *If, in addition to the assumptions of the previous theorem, H is bounded and Assumptions 1 and 2 of Theorem 4.1 hold then the quantities θ_0 , π , θ and ε^2 appearing in this theorem are given by (4.14, 4.18, 4.20, 4.21).*

It is noted in [112] that the Assumption 2 of Theorem 4.1 is very hard to check unless S is continuous. Even for affine models a method to verify this condition does not seem to be available to the best of our knowledge. Therefore we consider a model in which S has no jumps in the next section.

4.4 Solution for the BNS superposition model

In this section we determine the quantities of Theorem 4.1 in the superposition model proposed by Barndorff-Nielsen & Shephard [7]. This model generalizes (2.5-2.6) by representing the squared volatility as a *weighted sum* of independent OU-processes. A further formulation of this model is discussed in Section 3.5.2.1. There the weighted sum was included as additional component in the affine framework. This was due to the desire to demonstrate that the BNS superposition model satisfies the structure Assumption 3.9. In this section we follow a different approach. We only include the different volatility processes, which appear in the sum, in the affine model. More specifically, we treat a superposition model with two independent OU-processes and we suppose that the dynamics of the return process z and the activity process v can be described as follows:

$$dz(t) = \mu v(t-)dt + \sqrt{v(t-)}dW(t), \quad (4.23)$$

$$v(t) = w_1 v_1(t) + w_2 v_2(t),$$

$$dv_i(t) = -\lambda_i v_i(t)dt + dr_i(t), \quad i = 1, 2. \quad (4.24)$$

Here $w_i \geq 0$, $\lambda_i > 0$, and μ are constants, W denotes a Wiener process and r a 2-dimensional Lévy process whose components r_1 and r_2 are independent subordinators with triplets $(b^{r_1}, 0, F^{r_1})$ and $(b^{r_2}, 0, F^{r_2})$ and Lévy exponents ψ^{r_1} and ψ^{r_2} . Again S is given by $S = \exp(z)$. An advantage of the concrete model (4.23-4.24) is that the prerequisites needed to compute the approximate utility-indifference prices and the approximate utility-based strategies can be easily checked under the probability measure P .

Assumption 4.10 *We suppose that the claim H has an integral representation (4.6) and that there exists an $\tilde{\varepsilon} > 0$ such that*

$$\int_{\{|x| \geq 1\}} e^{(M_i + \tilde{\varepsilon})x} F^{r_i}(dx) < \infty, \quad i = 1, 2, \quad (4.25)$$

for

$$M_i := \frac{2w_i(1 - e^{-\lambda_i T})}{\lambda_i} \max\{\psi^W(2R' \wedge 0), \psi^W(2R \vee 2)\}.$$

Here, ψ^W denotes the Lévy exponent of a standard Brownian motion, i.e.

$$\psi^W(u) = \frac{1}{2}u(u - 1)$$

and R and R' are the constants from the integral representation (4.6) of H . In order to exclude the degenerate case without stochastic volatility we assume that $w_1 + w_2 > 0$.

The model (4.23-4.24) described by differential equations corresponds to an affine process $X = (v_1, v_2, z)$ with differential characteristics (b^X, c^X, F^X) in the sense of Definition 3.1 by [102, Section 4.3]. The corresponding triplets $(\beta^j, \gamma^j, \varphi^j)$, $j = 0, \dots, 2$ are given by

$$\begin{aligned}\beta^0 &= (b^{r_1}, b^{r_2}, 0)^\top, \quad \gamma^0 = 0, \quad \varphi^0(G) = \int 1_G(x_1, x_2, 0) F^r(dx) \quad \forall G \in \mathcal{B}^3, \\ (\beta^1, \gamma^1, \varphi^1) &= \left((-\lambda_1, 0, \mu w_1)^\top, (0, 0, w_1 \tilde{e}_3^3), 0 \right), \\ (\beta^2, \gamma^2, \varphi^2) &= \left((0, -\lambda_2, \mu w_2)^\top, (0, 0, w_2 \tilde{e}_3^3), 0 \right).\end{aligned}$$

The Lévy exponent ψ^X is given by

$$\psi^X(u) = \psi_0^X(u) + v_1(t-) \psi_1^X(u) + v_2(t-) \psi_2^X(u), \quad u \in D = \hat{D}_1 \cap \hat{D}_2,$$

with

$$\begin{aligned}\psi_0^X(u) &= \psi^{r_1}(u_1) + \psi^{r_2}(u_2), \\ \psi_i^X(u) &= -u_i \lambda_i + w_i u_3 \left(\mu + \frac{1}{2} u_3 \right),\end{aligned}$$

and

$$\hat{D}_i := \left\{ u \in \mathbb{C}^3 : \int_{\{x \geq 1\}} e^{\operatorname{Re}(u_i)x} F^{r_i}(dx) < \infty \right\}.$$

Following the procedure of the previous section, we first determine the dynamics of the model under the MEMM Q^* and the optimal investment strategy θ_0 .

Theorem 4.11 *Suppose that Assumption 4.10 holds. Then a solution of the utility maximization problem (4.3) is given by*

$$\theta_0(t) = \frac{\mu + \frac{1}{2}}{S(t)}, \quad t \in [0, T] \quad (4.26)$$

for $p = 1$ in the superposition model (4.23-4.24). Moreover, the minimal entropy martingale measure Q^* exists and X is a time-inhomogeneous affine process under Q^* with differential characteristics $(b^{X^*}, c^{X^*}, F^{X^*})$ given by:

$$\begin{aligned}b_i^{X^*}(t) &= -v_i(t-) \lambda_i + b^{r_i} + \int x(e^{\alpha_i(t)x} - 1) F^{r_i}(dx), \quad i = 1, 2, \\ b_3^{X^*}(t) &= -\frac{1}{2} (w_1 v_1(t-) + w_2 v_2(t-)), \\ c^{X^*}(t) &= c^X, \\ F^{X^*}(t, G) &= \int 1_G(x, 0, 0) e^{\alpha_1(t)x} F^{r_1}(dx) + \int 1_G(0, x, 0) e^{\alpha_2(t)x} F^{r_2}(dx), \\ &\quad \forall G \in \mathcal{B}^3 \text{ with } 0 \notin G.\end{aligned}$$

Here the mappings $\alpha_1, \alpha_2 \in C^\infty([0, T], \mathbb{R}_-)$ are given by

$$\alpha_i(t) = \frac{w_i(\mu + \frac{1}{2})^2}{2\lambda_i} (e^{-\lambda_i(T-t)} - 1), \quad i = 1, 2.$$

PROOF. Theorem 3.31 and Corollary 3.19. \square

Remark 4.12 Note that one only has to demand (4.25) with $M_1 = M_2 = 0$ in order to establish the above theorem. For the BNS model (2.5-2.6) the MEMM Q^* and the dynamics of the model under this measure were also calculated in [13] subject to stronger assumptions.

The Lévy exponent ψ^{X^*} under Q^* is given by

$$\psi^{X^*}(t, u) = \psi_0^{X^*}(t, u) + v_1(t-) \psi_1^{X^*}(t, u) + v_2(t-) \psi_2^{X^*}(t, u), \quad (t, u) \in [0, T] \times D,$$

with

$$\begin{aligned} \psi_0^{X^*}(t, u) &= \psi^{r_1}(\alpha_1(t) + u_1) + \psi^{r_2}(\alpha_2(t) + u_2) - \psi^{r_1}(\alpha_1(t)) + \psi^{r_2}(\alpha_2(t)), \\ \psi_i^{X^*}(t, u) &= -u_i \lambda_i + w_i \psi^W(u_3), \quad i = 1, 2. \end{aligned}$$

In the next step, we solve the quadratic hedging problem under Q^* in model (4.23-4.24).

Theorem 4.13 *Suppose that Assumption 4.10 is satisfied. Then the solution (4.9-4.10) of the quadratic hedging problem (4.5) under Q^* can be expressed by (4.18 - 4.21) with*

$$\begin{aligned} \theta_\zeta &:= \zeta \frac{V_\zeta(t-)}{S(t)}, \\ V_\zeta(t) &:= S(t)^\zeta \exp(\Psi_0(t, T, 0, 0, \zeta) + \Psi_1(t, T, 0, \zeta)v_1(t) + \Psi_2(t, T, 0, \zeta)v_2(t)), \\ J(t, \zeta_1, \zeta_2) &:= \exp(\Psi_0(0, t, \xi_1, \xi_2, \tilde{\zeta}) + \Psi_1(0, t, \xi_1, \tilde{\zeta})v_1(0) + \Psi_2(0, t, \xi_2, \tilde{\zeta})v_2(0)) \times \\ &\quad \times S(0)^{\tilde{\zeta}} e^{\xi_0}(\eta_1 + \eta_2), \end{aligned}$$

where

$$\begin{aligned} \Psi_0(s, t, u_1, u_2, u_3) &:= \int_s^t \left(\psi^{r_1}(p_1(\tilde{s}, t, u_1, u_3)) - \psi^{r_1}(p_1(\tilde{s}, t, 0, 0)) \right. \\ &\quad \left. + \psi^{r_2}(p_2(\tilde{s}, t, u_2, u_3)) - \psi^{r_2}(p_2(\tilde{s}, t, 0, 0)) \right) d\tilde{s}, \quad (4.27) \\ \Psi_i(s, t, u_1, u_2) &:= u_1 e^{-\lambda_i(t-s)} + w_i \psi^W(u_2) \left(\frac{1 - e^{-\lambda_i(t-s)}}{\lambda_i} \right), \quad i = 1, 2. \end{aligned}$$

The remaining variables and mappings are defined as

$$\begin{aligned}
p_i(s, t, u_1, u_2) &:= \alpha_i(s) + \Psi_i(s, t, u_1, u_2) = \nu_i(u_1, u_2, t)e^{-\lambda_i(t-s)} + \hat{\nu}_i(u_2), \quad i = 1, 2, \\
\nu_i(t, u_1, u_2) &:= u_1 + \left(\frac{w_i}{\lambda_i}\right) \left(\frac{1}{2} \left(\mu + \frac{1}{2}\right)^2 e^{-\lambda_i(T-t)} - \psi^W(u_2)\right), \quad i = 1, 2, \\
\hat{\nu}_i(u) &:= \left(\frac{w_i}{\lambda_i}\right) \left(\psi^W(u) - \frac{1}{2} \left(\mu + \frac{1}{2}\right)^2\right), \quad i = 1, 2, \\
\xi_i &= \xi_i(t, \zeta_1, \zeta_2) := \Psi_i(t, T, 0, \zeta_1) + \Psi_i(t, T, 0, \zeta_2), \quad 0 \leq i \leq 2, \\
\eta_i &= \eta_i(t, \zeta_1, \zeta_2) := \psi^{r_i}(\alpha_i(t)) - \psi^{r_i}(p_i(t, T, 0, \zeta_1)) \\
&\quad - \psi^{r_i}(p_i(t, T, 0, \zeta_2)) + \psi^{r_i}(\xi_i + \alpha_i(t)), \quad i = 1, 2, \\
\tilde{\zeta} &:= \zeta_1 + \zeta_2.
\end{aligned}$$

PROOF. If one of the weights w_1 and w_2 is zero we recover the BNS model (2.5-2.6) with a single subordinator. In this case we obtain the assertions of the above theorem from Theorem 4.8 and a generalization of [110, Theorem 3.2 and Theorem 3.3] to a time-inhomogeneous BNS model. The arguments of the corresponding proofs hold also in the presence of time dependence. If both weights are unequal to zero the situation is more involved. We have to adapt Theorem A.6 to the superposition model (4.23-4.24). Since this model also has a quite simple structure under Q^* (e.g. S has no jumps, the Riccati equations are linear, z and v are locally independent (cf. Remark 5.8)) all necessary conclusions can be drawn similarly in the related proofs. Additionally, we obtain that [110, Theorem 3.2 and Theorem 3.3] can be also generalized to this case. This proves the claim. \square

From Theorem 4.11 and Theorem 4.13 we obtain the main approximation result for the superposition model (4.23-4.24).

Corollary 4.14 *If the conditions of Theorem 4.13 hold and H is bounded then the quantities θ_0 , π , θ and ε^2 appearing in Theorem 4.1 are given by (4.26, 4.18, 4.20, 4.21). The respective integrands can be written as in Theorem 4.13.*

For some subordinators the integrals appearing in (4.27) can be expressed in closed form. We obtain the following formulas by adopting the results from [110, Proposition 3.6 and Proposition 3.7].

Proposition 4.15 *Suppose that v_i , $i = 1, 2$ are inverse Gaussian (IG) OU-processes (cf. [172]) with Lévy exponents*

$$\psi^{r_i}(u) = \frac{\lambda_i a_i u}{\sqrt{b_i^2 - 2u}}$$

for some $a_i, b_i > 0$ such that

$$\max\{\psi^W(2R' \wedge 0), \psi^W(2R \vee 2)\} < \frac{b_i^2}{4w_i(1 - e^{-\lambda_i T})}.$$

Then (4.25) is satisfied and the integrals in (4.27) can be written as

$$C(s, t, u_1, u_2) := \int_s^t \psi^{r_i}(p_i(\tilde{s}, t, u_1, u_2)) d\tilde{s}$$

with

$$C(s, t, u_1, u_2) = \begin{cases} C_1(t-s, t, u_1, u_2) & \text{if } b_i^2 \neq 2\hat{\nu}_i(u_2) \text{ and } \nu_i(t, u_1, u_2) \neq 0, \\ C_2(t-s, t, u_1, u_2) & \text{if } b_i^2 = 2\hat{\nu}_i(u_2), \\ C_3(t-s, u_2) & \text{if } b_i^2 \neq 2\hat{\nu}_i(u_2) \text{ and } \nu_i(t, u_1, u_2) = 0, \end{cases}$$

and

$$\begin{aligned} C_1(s, t, u_1, u_2) &:= \frac{a_i \hat{\nu}_i(u_2)}{B(u_2)} (\log \Upsilon(s, t, u_1, u_2) + \lambda_i s) \\ &\quad + a_i (A(s, t, u_1, u_2) - A(0, t, u_1, u_2)), \\ C_2(s, t, u_1, u_2) &:= \frac{2a_i}{\alpha(0, t, u_1, u_2)} \left(\nu_i(t, u_1, u_2) \left(1 - e^{-\frac{1}{2}\lambda_i s} \right) + \hat{\nu}_i(u_2) \left(e^{\frac{1}{2}\lambda_i s} - 1 \right) \right), \\ C_3(s, u) &:= \frac{\lambda_i a_i \hat{\nu}_i(u) s}{B(u)}, \end{aligned}$$

where

$$\begin{aligned} A(s, t, u_1, u_2) &:= \sqrt{b_i^2 - 2p_i(t-s, t, u_1, u_2)}, \\ B(u) &:= \sqrt{b_i^2 - 2\hat{\nu}_i(u)}, \\ \Upsilon(s, t, u_1, u_2) &:= \frac{B(u_2)(B(u_2) - \alpha(0, t, u_1, u_2)) - \nu_i(t, u_1, u_2)}{\nu_i^2(t, u_1, u_2)} \times \\ &\quad \times (B(u_2)(\alpha(s, t, u_1, u_2) + B(u_2)) - e^{-\lambda_i s} \nu_i(t, u_1, u_2)). \end{aligned}$$

The mapping $s \mapsto \log \Upsilon(s, t, u_1, u_2)$ denotes the distinguished logarithm in the sense of [166, Lemma 7.6], i.e. the branch is chosen such that the mapping $\log \Upsilon$ is continuous with respect to s .

PROOF. This result can be immediately deduced from [110, Proposition 3.7]. \square

Similarly, we obtain the corresponding result for Gamma-OU processes:

Proposition 4.16 *Assume that v_i , $i = 1, 2$ are Gamma-OU processes (cf. [172]) with Lévy exponents*

$$\psi^{r_i}(u) = \frac{\lambda_i a_i u}{b_i - u}$$

for some $a_i, b_i > 0$ such that

$$\max\{\psi^W(2R' \wedge 0), \psi^W(2R \vee 2)\} < \frac{b_i}{2w_i(1 - e^{-\lambda_i T})}$$

Then (4.25) is satisfied and the integrals in (4.27) are given by

$$C = C(s, t, u_1, u_2) := \int_s^t \psi^{r_i}(p_i(\tilde{s}, t, u_1, u_2)) d\tilde{s},$$

with

$$C = \begin{cases} \frac{a_i}{b_i - \hat{\nu}_i(u_2)} \left(b_i \log \left(\frac{b_i - p_i(s, t, u_1, u_2)}{b_i - p_i(t, t, u_1, u_2)} \right) + \lambda_i \hat{\nu}_i(u_2)(t - s) \right) & \text{if } b_i \neq \hat{\nu}_i(u_2), \\ -a_i \left(\frac{b_i}{\nu_i(t, u_1, u_2)} (e^{\lambda_i(t-s)} - 1) + \lambda_i(t - s) \right) & \text{if } b_i = \hat{\nu}_i(u_2). \end{cases}$$

Here, \log again denotes the distinguished logarithm of $s \mapsto (b_i - p_i(s, t, u_1, u_2)) / (b_i - p_i(t, t, u_1, u_2))$.

PROOF. This again follows from [110, Proposition 3.6]. \square

Following [6] we call a superposition of independent OU processes, which leads to a process with stationary IG law (resp. Gamma law) if the initial values are chosen appropriately, IG-supOU (resp. Gamma-supOU) process. E.g. the superposition of two independent Gamma-OU processes with $b_1 = b_2$ yields a Gamma-supOU process.

4.5 Numerical illustration

We illustrate the results of the last section with a numerical example. For this purpose, we investigate a European put option with strike $K = 100$ and a maturity of three months, i.e. $T = 0.25$ years. Furthermore we choose $p = 1$ as parameter for the risk aversion. We use the parameters estimated in [105] from a German equity index (DAX) time series. Note that the following parameters are quoted as parameters per annum. [105] considers the BNS superposition model (4.23), where v is the sum of two independent subordinators. We compare in this model the results for four different choices of the subordinator: IG-supOU process, Gamma-supOU process, a single IG-OU and a single Gamma-OU. [105] obtains $\hat{\mu} = 0.904$ as an estimator for the drift parameter μ . The estimates for the remaining parameters are summarized in Table 4.5. This table also contains the initial values for the activity processes. They are set equal to the expectations of the stationary laws of the respective activity processes. Note that this choice leads to the same initial value 0.0485 for the resulting activity process v in each model. For comparison, we consider a Black-Scholes model where the variance parameter σ^2 is also set to 0.0485. If R is chosen appropriately, e.g. $R = -1$ in the integral representation of the put option (4.8), then the subordinators corresponding to the above parameters in particular satisfy the conditions of Proposition 4.15 (resp. Proposition 4.16).

First, we compare the approximate utility-indifference prices π_q for several values of q in the BNS superposition model with two IG-OU processes. Figure 4.1 shows these prices as a function of the initial asset value $S(0)$. We see that the marginal utility-based price π is not

subordinator	$\hat{\lambda}_1$	$\hat{\lambda}_2$	\hat{a}_1	\hat{a}_2	\hat{b}_1	\hat{b}_2	$v_1(0)$	$v_2(0)$
IG-supOU process	32.5	1.38	0.0751	0.0933	3.47	3.47	0.0216	0.0269
Γ -supOU process	32.5	1.38	0.261	0.324	12.1	12.1	0.0216	0.0269
IG-OU process	2.54	-	0.203	-	4.18	-	0.0485	-
Γ -OU process	2.54	-	0.847	-	17.5	-	0.0485	-

Table 4.1: Estimated parameters for some subordinators in the BNS superposition model

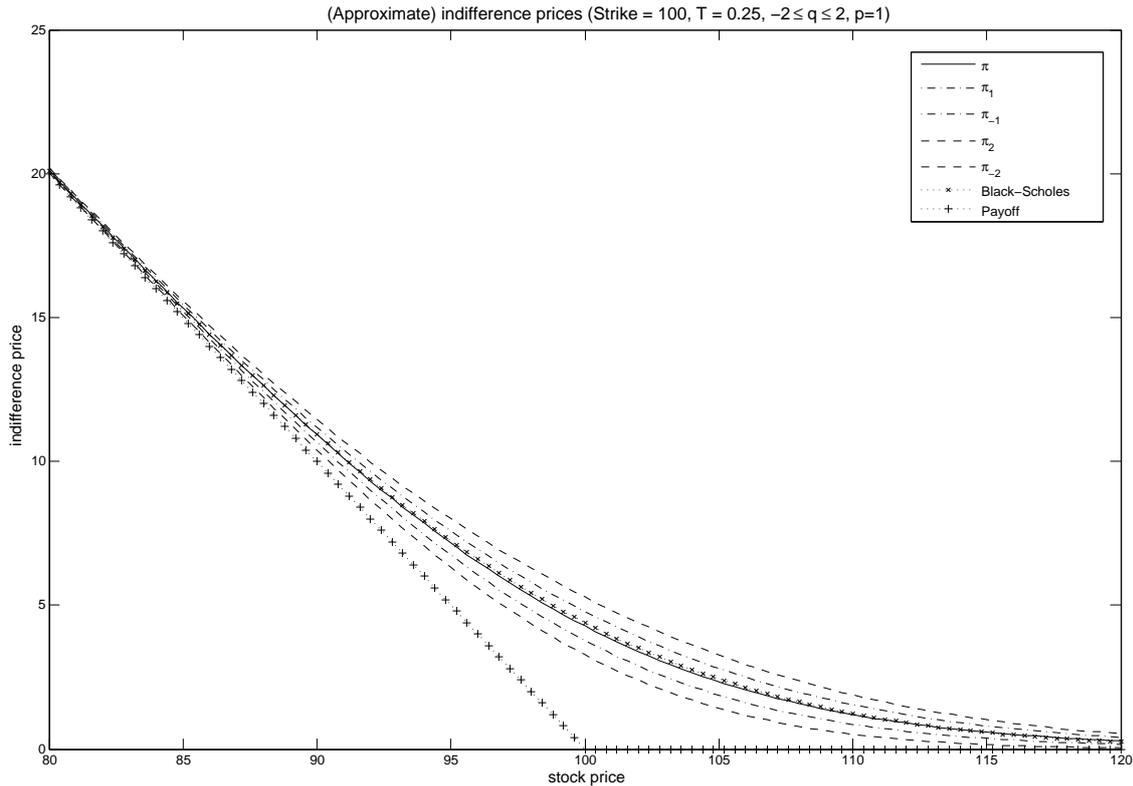


Figure 4.1: Approximate utility-indifference prices

too far away from the Black-Scholes price. As a side remark this observation also holds for the other subordinators. Since π is also the optimal initial endowment in (4.5) this parallels the results of the numerical studies made in [88, 110] and in Section 5.5. There the solution of the quadratic hedging problem (4.5) is computed in several stochastic volatility models. It is observed in the case study of [110] and Section 5.5 that the variance-optimal strategies θ and the optimal initial endowments $V(0)$ are quite similar for the examined models. On the other hand, the expected squared hedging error ε^2 varies significantly for different models. Both observations remain true for our setting. Since the approximate utility-indifference price is affected by the expected squared hedging error ε^2 for $q \neq 0$ we compare in Figure 4.2 these hedging errors for the different subordinators. We note that while the influence of the model selection to the hedging error ε^2 is visible, the impact to approximate utility-

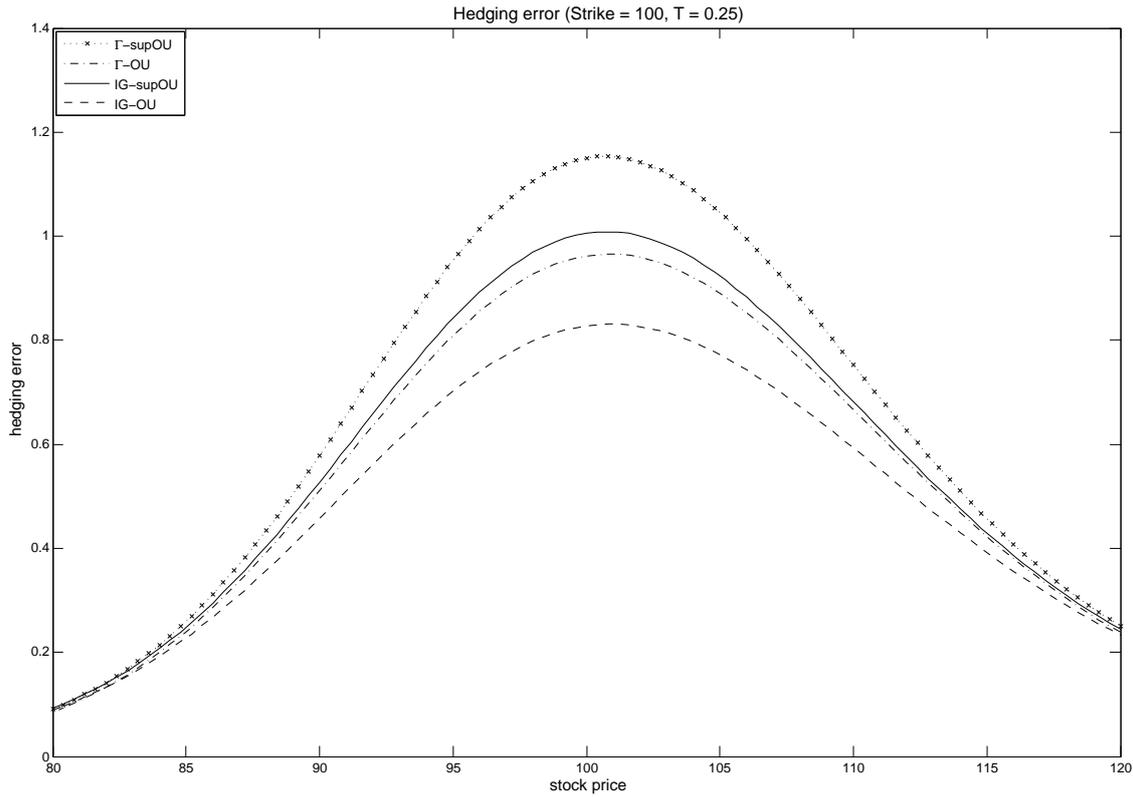


Figure 4.2: Minimal expected squared hedging error

indifference prices is rather limited—at least for the chosen parameters. Finally in Figure 4.3, the relevant strategies are depicted for the IG-supOU case. We see that the difference between the approximate utility-based hedging strategy and the optimal strategy in the Black-Scholes model is small. Moreover the approximate utility-based trading strategy is assembled from the optimal investment strategy and the approximate utility-based hedging strategy.

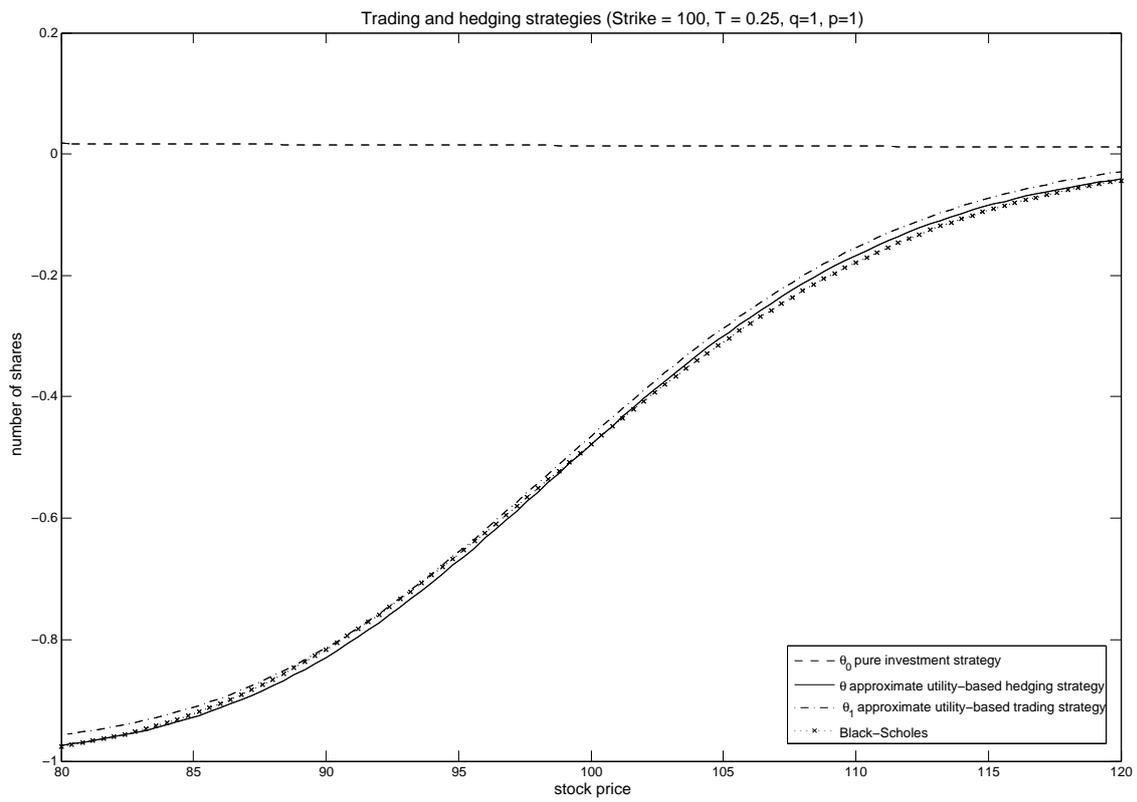


Figure 4.3: Utility-based strategies

Chapter 5

Quadratic Hedging in Affine Stochastic Volatility Models

5.1 Introduction

A classical problem in Mathematical Finance is how to hedge the risk from selling a contingent claim. Since perfect hedging strategies do not exist in incomplete markets, we focus on variance-optimal hedging strategies in this chapter. This concept has been studied intensively in the literature (cf. [176, 153] for an overview). The idea is to choose an initial endowment w^* and a self-financing strategy θ^* such that the expected squared hedging error

$$\varepsilon^2(w, \theta) := E \left(\left(w + \int_0^T \theta(t) dS(t) - H \right)^2 \right)$$

is minimized over all such endowments w and strategies θ . Here, S denotes the discounted price process of a stock, T the time horizon, and H the discounted payoff of the option that is to be hedged. The minimal hedging error $\varepsilon^2(w^*, \theta^*)$ quantifies the residual risk that cannot be avoided. It may enter the premium that the issuer charges the buyer.

We consider the above problem for European-style options in stochastic volatility models of affine structure. This class generalizes Lévy processes and includes e.g. the Heston [82] model. Again the BNS model (cf. Example 2.4) is another example for this kind of models. A generalization of Carr et al. [18] allows for jumps in the stock price as well.

This chapter rests on several pillars. One main ingredient is a general characterization of the variance-optimal hedge in [22]. Moreover, results of [40, 48] on affine processes are used on the way to concrete solutions. These cannot be expressed in closed form. But using the integral transform approach of [88] or similarly [21], we can derive semiexplicit representations which allow for straightforward numerical implementation. The problem has been attacked in [23] for the Heston model and in [111, 110, 108] for the case that the discounted stock is a martingale. A different approach is taken in [79] and [31], which rely

on partial differential equation and partly simulation methods.

The structure of this chapter is as follows. We start in Section 5.2 by introducing the mean-variance hedging problem and by recalling our affine framework. Section 5.3 contains the solution in integral form. Subsequently, it is applied to the time-changed Lévy process model of [18] in Section 5.4. Section 5.5 provides numerical illustration. Finally, Section 5.6 contains the proofs of our results.

For unexplained notation we refer in this chapter also to [22]. Moreover we note that, mathematical formalism is treated liberally here. E.g. we do not state and verify technical conditions concerning interchanging the order of integration in Fubini's theorem, uniform integrability of local martingales, admissibility of trading strategies, existence of analytic extensions of complex functions etc. Arguments and conditions of this type have been worked out in related setups by [88, 111, 110, 71].

5.2 General setup

In this chapter we assume that the discounted stock price S of a univariate asset is of the form (2.2):

$$S = e^z.$$

Like in the preceding chapters, the return process z is a component of an affine process $X = (v, z)$ in the sense of Definition 2.2. We consider the case of a single return process and a single activity process in this chapter. Keeping Definition 2.3 in mind, we express the solution to the quadratic hedging problem in terms of

$$\begin{aligned} \kappa_i(t, x, y_1, y_2, \hat{y}_1, \hat{y}_2) &:= \psi_i^X(t, x + y_1 + y_2, \hat{y}_1 + \hat{y}_2) - \psi_i^X(t, x + y_1, \hat{y}_1) \\ &\quad - \psi_i^X(t, x + y_2, \hat{y}_2) + \psi_i^X(t, x, 0), \end{aligned}$$

for $i = 0, 1$ and $t \in [0, T]$ and $x, y_1, y_2, \hat{y}_1, \hat{y}_2 \in \mathbb{C}$ such that the Lévy exponents exist.

In order to solve the hedging problem for S , we need some additional assumptions: Some of them correspond to the conditions required in Assumption 4.4 in the case of variance-optimal hedging under the minimal entropy martingale measure.

1. $\psi_0^X(t, 0, u_2) = 0$ for any $(t, u_2) \in [0, T] \times i\mathbb{R}$. This condition means that the local dynamics of z depend in a linear (rather than an affine) fashion on $v(t-)$. This restriction is imposed to obtain semiexplicit solutions in the case where S is not a martingale. It is not needed if S is a martingale, cf. [111].
2. S is a locally square-integrable semimartingale (cf. [22, Definition A.1]), which is needed for quadratic hedging to make sense. This condition holds if $[0, T] \times [0, \varepsilon) \times (-\varepsilon, 2 + \varepsilon) \subset \bar{D}_0 \cap \bar{D}_1$ for some $\varepsilon > 0$.

3. $\psi_1^X(t, 0, 2) \neq 2\psi_1^X(t, 0, 1)$ for any $t \in [0, T]$. This condition essentially rules out constant stock price processes.
4. There exists some equivalent local martingale measure $Q \sim P$ for S such that we have $E\left(\frac{dQ}{dP}\right)^2 < \infty$. This can be interpreted as a kind of no-arbitrage condition needed in the framework of quadratic hedging.
5. We assume $\varphi^1(t, \mathbb{R} \times \{x\}) = 0$ for any $t \in [0, T], x \in \mathbb{R}$. This condition means roughly speaking that the jumps of z have a diffuse law. It holds immediately if the involved Lévy measures have Lebesgue densities.

We now turn to the European-style option that is to be hedged. As in Chapter 4 we use an integral representation technique in order to tackle the mean-variance hedging problem. The payoff of the option is denoted as

$$H = f(\log(S(T))) = f(z(T))$$

with some function $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{sp} \hat{f}(p) dp,$$

where $R \in \mathbb{R}$ and $\hat{f} : R + i\mathbb{R} \rightarrow \mathbb{C}$ denotes another function. For the European call with strike K we have $f(s) = (e^s - K)^+$ and

$$\hat{f}(p) = \frac{K^{1-p}}{p(p-1)} \quad (5.1)$$

for arbitrary $R > 1$, cf. [88]. For the put we have $f(s) = (K - e^s)^+$ and the same \hat{f} as in (5.1) but with $R < 0$. \hat{f} is generally obtained as bilateral Laplace transform of f . In comparison with Chapter 4, the integral representation presented here is less general. However they fit to several standard options (cf. [88] for more details).

As noted in the introduction of this chapter, the goal is to minimize the expected squared hedging error

$$\varepsilon^2(w, \theta) := E\left(\left(w + \int_0^T \theta(t) dS(t) - H\right)^2\right) \quad (5.2)$$

over all initial endowments $w \in \mathbb{R}$ and all admissible trading strategies θ . For the proper notion of admissibility in the context of quadratic hedging we refer to [22, Definition 2.2]. It means essentially that the gains of θ can be approximated in an L^2 -sense by the gains of a sequence of simple, i.e. bounded and piecewise constant strategies.

5.3 Main results

In this section we solve the hedging problem (5.2) in the affine setup of the previous section. The approach relies on the general structural results of [22]. A two-step procedure is applied in order to solve the problem. First, the auxiliary problem of optimal investment under quadratic utility is solved. Its solution enters the hedging problem in the second step.

5.3.1 Pure investment problem

The pure investment problem in [22] involves an *opportunity process* L , an *adjustment process* a , and an *opportunity-neutral measure* P^* . The process L satisfies

$$L(t) = \inf_{\theta} E \left(\left(1 - \int_t^T \theta(s) dS(s) \right)^2 \middle| \mathcal{F}(t) \right), \quad (5.3)$$

i.e. it stands for the maximal quadratic utility between t and T of an investor with negative quadratic utility function. The adjustment process is related to the optimizer $\theta^{(t)}$ in (5.3). Specifically, we have

$$\theta^{(t)}(s) = \left(1 - \int_t^{s-} \theta^{(t)}(r) dS(r) \right) a(s)$$

for $s \in [t, T]$. In particular, the strategy $\theta^{(0)}(t) = a(t) \mathcal{E}(-\int_0^t a(s) dS(s))(t-)$ minimizes

$$E \left(\left(1 - \int_0^T \theta(t) dS(t) \right)^2 \right).$$

Hence it is directly related to dynamic portfolio optimization in a Markowitz sense. The opportunity-neutral measure is harder to motivate. It helps to simplify the structure of the hedging problem if S fails to be a martingale. The solution to the pure investment problem in our present setup reads as follows.

Theorem 5.1 *Set*

$$\begin{aligned} \bar{a}(t, x) &:= \frac{\psi_1^X(t, x, 1) - \psi_1^X(t, x, 0)}{\kappa_1(t, x, 0, 0, 1, 1)}, \\ \bar{\alpha}(t, x) &:= \bar{a}(t, x) (\psi_1^X(t, x, 1) - \psi_1^X(t, x, 0)). \end{aligned}$$

Let $\alpha_1 : [0, T] \rightarrow \mathbb{R}$ be the solution to the terminal value problem

$$\alpha_1'(t) = -\psi_1^X(t, \alpha_1(t), 0) + \bar{\alpha}(t, \alpha_1(t)), \quad \alpha_1(T) = 0 \quad (5.4)$$

and set

$$\alpha_0(t) := \int_t^T \psi_0^X(s, \alpha_1(s), 0) ds. \quad (5.5)$$

Then the opportunity process L and the adjustment process a in the sense of [22] satisfy

$$L(t) = \exp(\alpha_0(t) + \alpha_1(t)v(t)), \quad (5.6)$$

$$a(t) = \frac{\bar{a}(t, \alpha_1(t))}{S(t-)}. \quad (5.7)$$

PROOF. Section 5.6. □

The opportunity process L allows for a multiplicative decomposition $L = L(0)ZA$ with a martingale Z and a predictable process of finite variation A satisfying $Z(0) = A(0) = 1$. The martingale $Z^{P^*} := Z$ can be used as density process of some probability measure $P^* \sim P$. This so-called *opportunity-neutral measure* plays a crucial role for determining the variance-optimal hedge. According to [22], we have $A = \exp(\int_0^t b^{\mathcal{L}(L)}(s)ds)$, where $b^{\mathcal{L}(L)}$ denotes the drift part in the differential characteristics of the stochastic logarithm $\mathcal{L}(L) := \int_0^\cdot \frac{1}{L(t-)} dL(t)$.

5.3.2 Variance-optimal hedging

We turn now to the hedging problem introduced at the end of Section 2. According to [22], a first step consists in determining the mean value process V of the option defined as

$$V(t) := E_{P^*} \left(H \mathcal{E} \left(N^* - (N^*)^t \right) (T) \mid \mathcal{F}(t) \right), \quad (5.8)$$

where N^* denotes the P^* -martingale part of the special semimartingale $-\int_0^\cdot a(t)dS(t)$ and $(N^*)^t$ the process N^* stopped at t . In particular, the optimal endowment w^* in (5.2) is given by $V(0)$, cf. [22, Theorem 4.10].

Theorem 5.2 (Mean value process) For fixed $p \in R + i\mathbb{R}$ let $\Phi_1(\cdot, p) : [0, T] \rightarrow \mathbb{C}$ denote the solution to the terminal value problem $\Phi_1(T, p) = 0$,

$$\begin{aligned} \partial_1 \Phi_1(t, p) &= -\psi_1^X(t, \Phi_1(t, p) + \alpha_1(t), p) + \psi_1^X(t, \alpha_1(t), 0) \\ &\quad + \bar{a}(t, \alpha_1(t)) \kappa_1(t, \alpha_1(t), \Phi_1(t, p), 0, p, 1). \end{aligned} \quad (5.9)$$

Moreover, set

$$\Phi_0(t, p) := \int_t^T \left(\psi_0^X(s, \Phi_1(s, p) + \alpha_1(s), 0) - \psi_0^X(s, \alpha_1(s), 0) \right) ds \quad (5.10)$$

and

$$V_p(t) := S(t)^p \exp(\Phi_0(t, p) + \Phi_1(t, p)v(t)) \quad (5.11)$$

for $t \in [0, T], p \in R + i\mathbb{R}$. Then the mean value process V from (5.8) satisfies

$$V(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} V_p(t) \hat{f}(p) dp. \quad (5.12)$$

PROOF. Section 5.6. □

In some concrete models as in Section 5.4, the ordinary differential equation (5.9) and the integral (5.10) can be calculated in closed form. In this case the mean value process (5.12) is obtained by a single numerical integration.

The next step on our way is to determine the optimal hedging strategy θ^* in (5.2). According to [22, Theorem 4.10] it is specified by its feedback form

$$\theta^*(t) = \xi(t) - \left(V(0) + \int_0^{t-} \theta^*(s) dS(s) - V(t-) \right) a(t),$$

where ξ denotes the so-called *pure hedge coefficient*, which solves

$$\langle S, V \rangle^{P^*}(t) = \int_0^t \xi(s) d\langle S, S \rangle^{P^*}(s). \quad (5.13)$$

It is yet to be determined, whereas V and a are already computed in (5.12) and (5.7).

Theorem 5.3 (Hedging strategy) *The pure hedge coefficient is of the form*

$$\xi(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \xi_p(t) \hat{f}(p) dp,$$

where

$$\xi_p(t) := \frac{V_p(t-) \kappa_1(t, \alpha_1(t), \Phi_1(t, p), 0, p, 1)}{S(t-) \kappa_1(t, \alpha_1(t), 0, 0, 1, 1)}.$$

PROOF. Section 5.6. □

Finally, we can use the results of [22] in order to determine the minimal hedging error in (5.2). According to [22, Theorem 4.12] it satisfies

$$\varepsilon^2(w^*, \theta^*) = E \left(\int_0^T L(t) d \left\langle V - \int_0^\cdot \xi(s) dS(s), V - \int_0^\cdot \xi(s) dS(s) \right\rangle^{P^*} (t) \right).$$

Its integral representation in the present setup reads as follows.

Theorem 5.4 (Hedging error) *For fixed $p, q \in \mathbb{C}, t \in [0, T]$ let $\Upsilon_1(\cdot, q, p, t) \rightarrow \mathbb{C}$ denote the solution to the terminal value problem*

$$\partial_1 \Upsilon_1(s, q, p, t) = -\psi_1^X(s, \Upsilon_1(s, q, p, t), p), \quad \Upsilon_1(t, q, p, t) = q.$$

Moreover, set

$$\Upsilon_0(s, q, p, t) := \int_s^t \psi_0^X(r, \Upsilon_1(r, q, p, t), p) dr$$

for $0 \leq s \leq t \leq T$. For $t \in [0, T]$, $p_1, p_2 \in R + i\mathbb{R}$ define

$$\begin{aligned} \varrho(t, p_1, p_2) &:= S(0)^p \exp(\hat{\Phi}_0 + \Upsilon_0(0, \hat{\Phi}_1, p, t) + \Upsilon_1(0, \hat{\Phi}_1, p, t)v(0)) \times \\ &\quad \times \left(\varrho_0 + \varrho_1 \left(\partial_2 \Upsilon_0(0, \hat{\Phi}_1, p, t) + \partial_2 \Upsilon_1(0, \hat{\Phi}_1, p, t)v(0) \right) \right) \end{aligned} \quad (5.14)$$

with

$$\begin{aligned} p &:= p_1 + p_2, \\ \hat{\Phi}_i &:= \hat{\Phi}_i(t, p_1, p_2) := \alpha_i(t) + \Phi_i(t, p_1) + \Phi_i(t, p_2), \quad i = 0, 1, \\ \varrho_0 &:= \varrho_0(t, p_1, p_2) := \kappa_0(t, \alpha_1(t), \Phi_1(t, p_1), \Phi_1(t, p_2), p_1, p_2), \\ \varrho_1 &:= \varrho_1(t, p_1, p_2) := \kappa_1(t, \alpha_1(t), \Phi_1(t, p_1), \Phi_1(t, p_2), p_1, p_2) \\ &\quad - \frac{\kappa_1(t, \alpha_1(t), \Phi_1(t, p_1), 0, p_1, 1)\kappa_1(t, \alpha_1(t), \Phi_1(t, p_2), 0, p_2, 1)}{\kappa_1(t, \alpha_1(t), 0, 0, 1, 1)}. \end{aligned}$$

Then the minimal expected squared hedging error satisfies

$$\varepsilon^2(w^*, \theta^*) = -\frac{1}{4\pi^2} \int_0^T \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \varrho(t, p_1, p_2) \hat{f}(p_1) \hat{f}(p_2) dp_1 dp_2 dt.$$

PROOF. Section 5.6. □

Remark 5.5 As special cases of Theorems 5.2–5.4 we recover results from the literature. If X is time-homogeneous and $\psi_j(u, 0) = 0$ for $j = 0, 1, u \in i\mathbb{R}$, the stock price follows an exponential Lévy process. The results of Theorems 5.2–5.4 correspond to Theorems 3.1, 3.2 in [88]. If, instead, we suppose that X is time-homogeneous and $\psi_1(0, 1) = 0$, then S is a martingale and Theorems 5.2–5.4 correspond to [111, Theorems 4.1 & 4.2].

5.4 Time-changed Lévy processes

We apply the general results from Section 5.3 to a class of stochastic volatility models considered in [18]. In this framework the return process z is modelled by a time-changed Lévy process. Choosing an integrated Ornstein-Uhlenbeck (OU) process as time change leads to the following model for z and the activity process v :

$$z(t) = z(0) + \ell(\hat{v}(t)), \quad (5.15)$$

$$d\hat{v}(t) = v(t-)dt, \quad (5.16)$$

$$dv(t) = -\lambda v(t-)dt + dr(t) \quad (5.17)$$

with $\lambda > 0$ and independent Lévy processes r, ℓ such that r is increasing. Their Lévy-Khintchine triplets and Lévy exponents are denoted by $(b^r, c^r, F^r), \psi^r$ and $(b^\ell, c^\ell, F^\ell), \psi^\ell$, respectively. If ℓ is chosen as Brownian motion with drift, one obtains the dynamics of the

BNS model (2.5-2.6). It is shown in [102] that $X = (v, z)$ is an affine process with triplets $(\beta^i, \gamma^i, \varphi^i), i = 0, 1$ of the form

$$\begin{aligned}\beta^0 &= \begin{pmatrix} b^r \\ 0 \end{pmatrix}, \quad \gamma^0 = 0, \quad \varphi^0(G) = \int 1_G(x, 0) F^r(dx) \quad \forall G \in \mathcal{B}^2, \\ \beta^1 &= \begin{pmatrix} -\lambda \\ b^\ell \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 \\ 0 & c^\ell \end{pmatrix}, \quad \varphi^1(G) = \int 1_G(0, x) F^\ell(dx) \quad \forall G \in \mathcal{B}^2.\end{aligned}$$

The corresponding Lévy exponents are

$$\begin{aligned}\psi_0^X(u_1, u_2) &= \psi^r(u_1), \\ \psi_1^X(u_1, u_2) &= -\lambda u_1 + \psi^\ell(u_2).\end{aligned}$$

Applying the results from the previous section we obtain the solution to the quadratic hedging problem.

Proposition 5.6 *For the present setup (5.15–5.17) the functions in Theorems 5.1–5.4 read as follows.*

$$\Upsilon_0(s, q, p, t) = \int_s^t \psi^r(\Upsilon_1(\tilde{s}, q, p, t)) d\tilde{s}, \quad (5.18)$$

$$\begin{aligned}\Upsilon_1(s, q, p, t) &= qe^{\lambda(s-t)} - \frac{\psi^\ell(p)}{\lambda} (e^{\lambda(s-t)} - 1), \\ \partial_2 \Upsilon_1(s, q, p, t) &= e^{\lambda(s-t)},\end{aligned}$$

$$\Phi_0(t, p) = \int_t^T \left(\psi^r(\Phi_1(s, p) + \alpha_1(s)) - \psi^r(\alpha_1(s)) \right) ds, \quad (5.19)$$

$$\Phi_1(t, p) = \frac{\bar{a}\bar{\kappa}_1(p, 1) - \psi^\ell(p)}{\lambda} (e^{\lambda(t-T)} - 1),$$

$$\alpha_0(t) = \int_t^T \psi^r(\alpha_1(s)) ds, \quad (5.20)$$

$$\alpha_1(t) = \frac{\bar{\alpha}}{\lambda} (e^{\lambda(t-T)} - 1),$$

$$\varrho_0 = \bar{\kappa}_0(\alpha_1(t), \Phi_1(t, p_1), \Phi_1(t, p_2)),$$

$$\varrho_1 = \bar{\kappa}_1(p_1, p_2) - \frac{\bar{\kappa}_1(p_1, 1)\bar{\kappa}_1(p_2, 1)}{\bar{\kappa}_1(1, 1)},$$

$$\bar{\alpha} = \bar{a}\psi^\ell(1),$$

$$\bar{a} = \frac{\psi^\ell(1)}{\bar{\kappa}_1(1, 1)},$$

$$\kappa_1(t, x, y_1, y_2, \hat{y}_1, \hat{y}_2) = \bar{\kappa}_1(\hat{y}_1, \hat{y}_2),$$

$$\bar{\kappa}_1(\hat{y}_1, \hat{y}_2) := \psi^\ell(\hat{y}_1 + \hat{y}_2) - \psi^\ell(\hat{y}_1) - \psi^\ell(\hat{y}_2),$$

$$\kappa_0(t, x, y_1, y_2, \hat{y}_1, \hat{y}_2) = \bar{\kappa}_0(x, y_1, y_2),$$

$$\bar{\kappa}_0(x, y_1, y_2) := \psi^r(x + y_1 + y_2) - \psi^r(x + y_1) - \psi^r(x + y_2) + \psi^r(x).$$

PROOF. One easily verifies that the candidates for $\alpha_1, \Phi_1, \Upsilon_1$ satisfy the corresponding terminal value problems. Moreover, the candidates for $\alpha_0, \Phi_0, \Upsilon_0$ have the proper derivative. \square

In some cases the integrals in (5.18–5.20) and the derivative $\partial_2 \Upsilon_0$ in (5.14) can be expressed in closed form.

Example 5.7 (Carr et al. [18] model with Γ -OU subordinator) Suppose that the Lévy exponent of r equals $\psi^r(u) = \frac{\lambda \zeta u}{\eta - u}$ with constants $\zeta, \eta > 0$. In this case the Ornstein-Uhlenbeck type process v has a stationary gamma law, cf. [28, Example 15.1]. Note that the integrals in (5.18–5.20) are all of the form

$$G(q) := \int_{t_1}^{t_2} \psi^r(g(t, q)) dt$$

with $g(t, q) = w(\exp(\lambda(t - \tilde{t})) - 1) + q \exp(\lambda(t - \tilde{t}))$ for some constants $w \in \mathbb{C}, t_1, t_2, \tilde{t} \in [0, T]$ with $t_1 \leq t_2$. One easily verifies that this integral has the closed-form representation

$$G(q) = \begin{cases} \frac{-\zeta}{\eta+w} \left(\lambda(t_2 - t_1)w - \eta \log \left(\frac{g(t_1, q) - \eta}{g(t_2, q) - \eta} \right) \right) & \text{if } \eta + w \neq 0, \\ \zeta \left(\lambda(t_1 - t_2) + \frac{\eta}{\eta - q} \left(e^{-\lambda(t_1 - \tilde{t})} - e^{-\lambda(t_2 - \tilde{t})} \right) \right) & \text{if } \eta + w = 0. \end{cases}$$

Here, \log denotes the *distinguished logarithm* in the sense of [166], i.e. the branch is chosen such that the resulting function is continuous in t . The derivative of G is needed for $\partial_2 \Upsilon_0$ in (5.14) and it satisfies

$$G'(q) = \begin{cases} \frac{\zeta \eta}{\eta+w} \left(\frac{e^{\lambda(t_1 - \tilde{t})}}{g(t_1, q) - \eta} - \frac{e^{\lambda(t_2 - \tilde{t})}}{g(t_2, q) - \eta} \right) & \text{if } \eta + w \neq 0, \\ \frac{\zeta \eta}{(\eta - q)^2} \left(e^{-\lambda(t_1 - \tilde{t})} - e^{-\lambda(t_2 - \tilde{t})} \right) & \text{if } \eta + w = 0. \end{cases}$$

Remark 5.8 From Proposition 5.6 one can deduce that some expressions can be simplified in the BNS model (2.5-2.6), namely

$$\begin{aligned} \xi_p(t) &= \frac{pV_p(t-)}{S(t)}, \\ \varrho_1 &= 0, \end{aligned}$$

cf. also [31, Section 4]. This structure for ξ_p is obtained for any model in our affine setup such that the asset price process S is continuous and the *local independence condition*

$$\psi_j^X(t, u_1, u_2) = \psi_j^X(t, u_1, 0) + \psi_j^X(t, 0, u_2), \quad j = 0, 1$$

is satisfied (e.g. the Heston [82] model with independent Brownian motions). By (5.11, 5.12), $V_p(t)$ and hence $V(t)$ are deterministic functions of asset price and volatility. If we differentiate this function with respect to the asset price, we obtain $\xi_p(t)$ resp. $\xi(t)$. Therefore the pure hedge ξ can be viewed as a *delta hedge* if V is interpreted as price

process of the claim. Local independence implies that volatility risk cannot be hedged by trading in the stock.

Barndorff-Nielsen and Shephard [7] consider also superpositions of Lévy-driven Ornstein-Uhlenbeck processes. This extension can be treated along the same lines as in the present section if v is replaced by a multivariate process. For ease of notation we do not consider this generalization here. Moreover, we treat such a generalization to a multivariate volatility process in Section 4.4 under the prerequisite that the price process S is a martingale under the minimal entropy martingale measure.

5.5 Numerical illustration

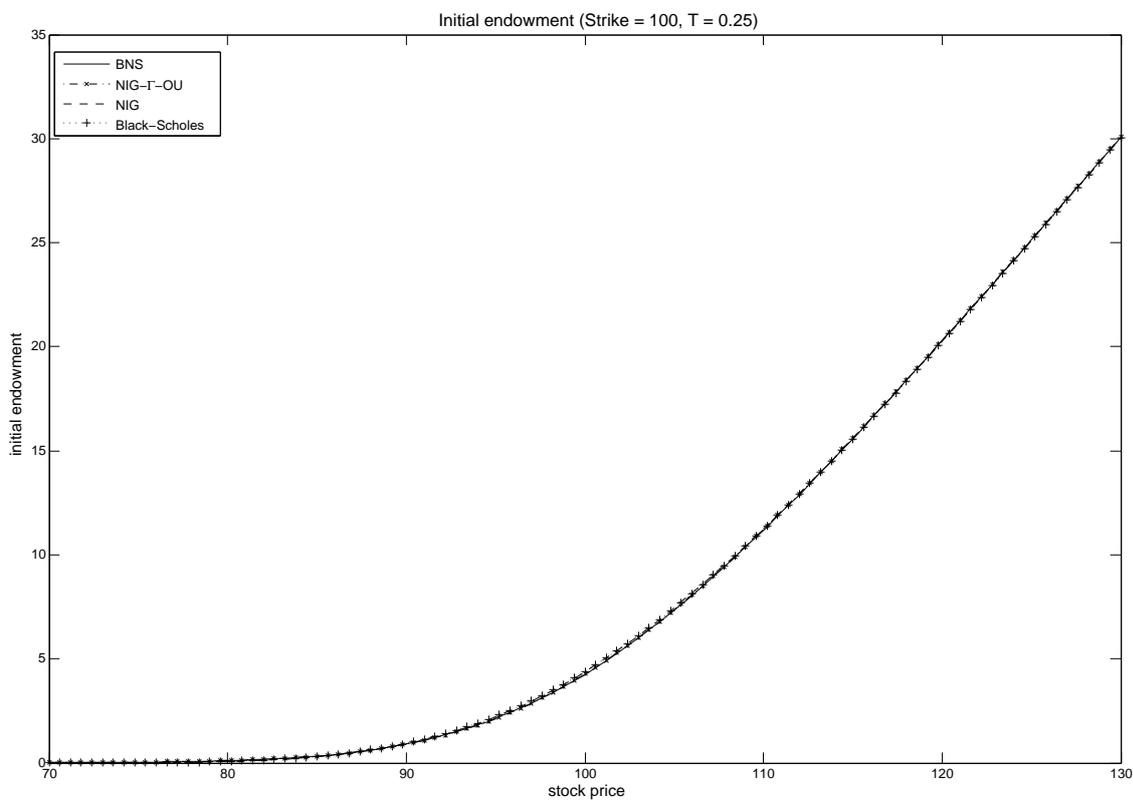


Figure 5.1: Optimal initial endowment

We demonstrate the approach numerically in some concrete stochastic volatility models. We use the parameters estimated in [105] for German stock index data. We compare four different models.

1. Firstly, we consider the model in (5.15–5.17) with a Γ -OU subordinator (cf. Example 5.7) and a normal inverse Gaussian (NIG) Lévy process ℓ . Its Lévy exponent ψ^ℓ is given by

$$\psi^\ell(u) = u\mu + \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right)$$

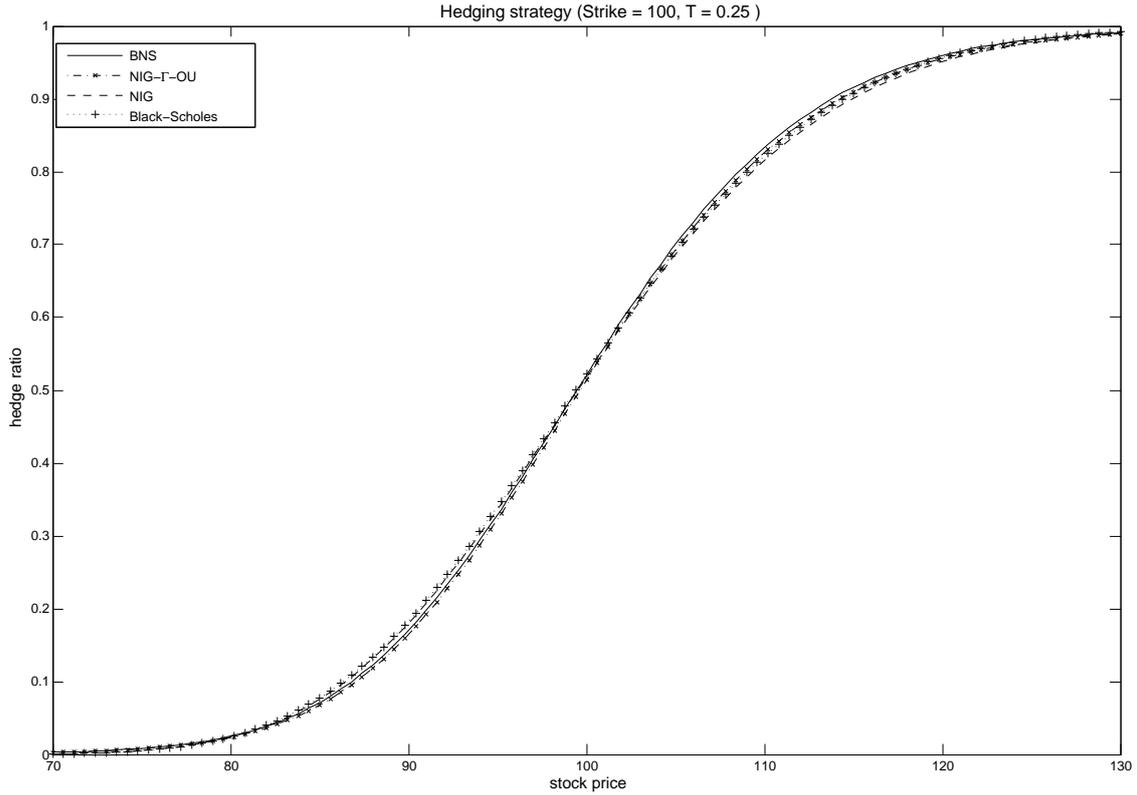


Figure 5.2: Variance-optimal hedging strategy at $t = 0$

with constants $\alpha, \delta > 0$, $\mu, \beta \in \mathbb{R}$ such that $|\beta| < \alpha$. The estimated parameters in [105] are $\hat{\alpha} = 90.1$, $\hat{\beta} = -16.0$, $\hat{\delta} = 85.9$, $\hat{\mu} = 15.5$, $\hat{\lambda} = 2.54$, $\hat{\zeta} = 0.847$, and $\hat{\eta} = 17.5$.

2. In the BNS model (2.5-2.6) with a Γ -OU process as in Example 5.7, [105] obtains the estimate $\hat{\mu} = 0.904$ for the drift parameter of the return process. The estimation procedure in [105] for the volatility process delivers the same values for λ , ζ and η as in the above NIG- Γ -OU case. In both stochastic volatility models we set $v(0) := \hat{\zeta}/\hat{\eta} = 0.0484$, which coincides with the expectation of the stationary law of the activity process.
3. For a comparison with [88] we consider an exponential Lévy model. Specifically, the return process is chosen as $z = z(0) + \ell$ with a NIG Lévy process ℓ . The corresponding formulas follow from Section 5.3 or from [88, Theorems 3.1, 3.2]. The estimated parameters are $\hat{\alpha} = 53.0$, $\hat{\beta} = -5.09$, $\hat{\delta} = 2.53$, $\hat{\mu} = 0.288$.
4. Finally we consider a Black-Scholes model with estimated variance parameter $\hat{\sigma}^2 = 0.0484$.

Note that time is measured in years in the above parameterization.

We compute the solution to the mean-variance hedging problem by evaluating the formulas of the previous sections numerically for a European call with strike $K = 100$ and

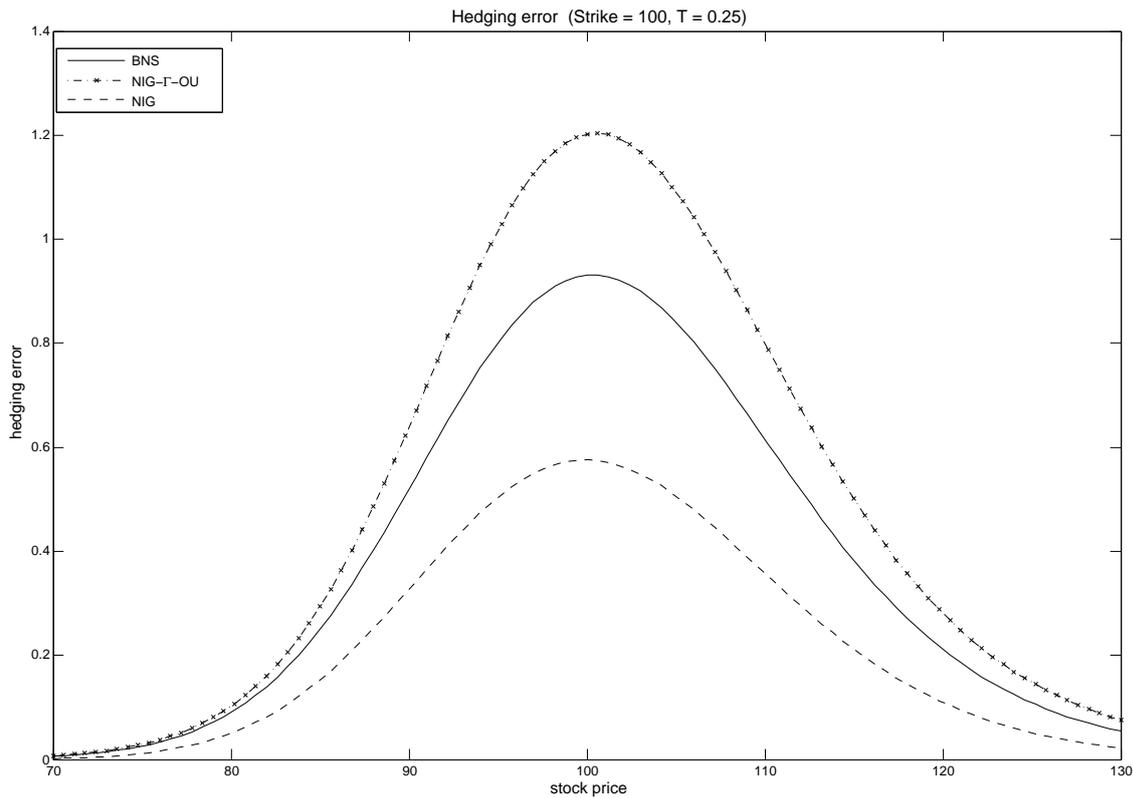


Figure 5.3: Minimal expected squared hedging error

a maturity of three months, i.e. $T = 0.25$ years. Figure 5.1 shows the optimal initial endowment w^* as a function of the initial asset price $S(0)$. The results are remarkably similar for the four models. From Figure 5.2 we see that the same is true for the variance-optimal hedging strategy θ^* at $t = 0$. This is in line with similar observations for exponential Lévy processes [88] or in the martingale case [110]. In Figure 5.3 the resulting hedging errors are shown as a function of $S(0)$. Here the models differ substantially from each other. From an intuitive perspective, incompleteness may result from both jumps in the stock and stochastic volatility. Therefore it does not come as a surprise that a model allowing for both yields the highest expected squared hedging error. Note that the hedging error vanishes in the Black-Scholes case because variance-optimal hedging is perfect.

5.6 Proofs

This section contains the proofs of the results given in Section 5.3. Note that an additional star at the differential characteristics of a semimartingale indicates that this characteristic refers to measure P^* .

PROOF OF THEOREM 5.1. We show that L given by (5.6) meets the conditions of Theorem 3.25 in [22] (up to uniform integrability and admissibility). $L(T) = 1$ and $L > 0$ are

obvious. Note that $\bar{\alpha}(t, \alpha_1(t)) \geq 0$ in (5.4). Moreover, the modified terminal value problem

$$\tilde{\alpha}'(t) = -\psi_1^X(t, \tilde{\alpha}(t), 0), \quad \tilde{\alpha}(T) = 0$$

is solved by $\tilde{\alpha} = 0$. A comparison argument yields that α_1 and hence also α_0 are nonpositive. It remains to prove that

$$b^L(t) = L(t-) \frac{\bar{b}(t)^2}{\bar{c}(t)} \tag{5.21}$$

where

$$\begin{aligned} \bar{b}(t) &:= b^S(t) + \frac{c^{S,L}(t)}{L(t-)} + \int x_1 \frac{x_2}{L(t-)} F^{(S,L)}(t, dx), \\ \bar{c}(t) &:= c^S(t) + \int x_1^2 \left(1 + \frac{x_2}{L(t-)}\right) F^{(S,L)}(t, dx). \end{aligned}$$

The differential characteristics of (v, z) are given. They lead immediately to the characteristics of the process (v, z, I) where $I(t) := t$ denotes the identity process. Application of Proposition 2.12 yields the differential characteristics $(b^{(S,L)}, c^{(S,L)}, F^{(S,L)})$ of (S, L) , namely

$$\begin{aligned} b^S(t) &= S(t-)v(t-)\psi_1^X(t, 0, 1), \\ b^L(t) &= L(t-)(\alpha'_0(t) + \alpha'_1(t)v(t-) + \psi^X(t, \alpha_1(t), 0)), \\ c^{(S,L)}(t) &= \begin{pmatrix} S(t-)^2 c^z(t) & S(t-)L(t-)\alpha_1(t)c^{v,z}(t) \\ S(t-)L(t-)\alpha_1(t)c^{v,z}(t) & L(t-)^2\alpha_1(t)^2 c^v(t) \end{pmatrix}, \\ F^{S,L}(t, G) &= \int 1_G(S(t-)(e^{x_2} - 1), L(t-)(e^{\alpha_1(t)x_1} - 1)) F^{(v,z)}(t, dx) \\ &\quad \forall G \in \mathcal{B}^2 \text{ with } 0 \notin G. \end{aligned}$$

We obtain

$$\begin{aligned} \bar{b}(t) &= S(t-)v(t-) (\psi_1^X(t, \alpha_1(t), 1) - \psi_1^X(t, \alpha_1(t), 0)), \\ \bar{c}(t) &= S(t-)^2 v(t-) \kappa_1(t, \alpha_1(t), 0, 0, 1, 1). \end{aligned}$$

By (5.4, 5.5) we have that (5.21) holds. According to [22, Theorem 3.25] the adjustment process is given by \bar{b}/\bar{c} , which coincides with (5.7). \square

PROOF OF THEOREM 5.2. Since

$$H = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{z(T)p} \hat{f}(p) dp,$$

Fubini theorem yields

$$V(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} V_p(t) \hat{f}(p) dp$$

with

$$V_p(t) := E_{P^*} \left(e^{pz(T)} \mathcal{E}(N^* - (N^*)^t)(T) \middle| \mathcal{F}(t) \right).$$

From the proof of Theorem 5.1 we know the differential characteristics of (S, L) . This immediately yields an expression for the drift coefficient $b^{\mathcal{L}(L)}(t) = \frac{1}{L(t-)} b^L(t)$ (cf. Proposition 2.11). Recall that $Z^{P^*}(t) = \exp(-\int_0^t b^{\mathcal{L}(L)}(s) ds) \frac{L(t)}{L(0)}$. By definition we have

$$N^*(t) = - \int_0^t a(s) dS(s) + \int_0^t a(s) b^{S^*}(s) ds,$$

where b^{S^*} denotes the drift coefficient in the differential characteristics of S relative to probability measure P^* . Fix $t \in [0, T]$. Since $\int_0^t a(s) b^{S^*}(s) ds$ is continuous and of finite variation, we have

$$\mathcal{E}(N^* - (N^*)^t)(T) = \mathcal{E}(U_1 + U_2)(T) = \mathcal{E}(U_1)(T) \mathcal{E}(U_2)(T) = \mathcal{E}(U_1)(T) \exp(U_2(T))$$

where $U_1(s) := -\int_{t \wedge s}^s a(r) dS(r)$ and $U_2(s) := \int_{t \wedge s}^s a(r) b^{S^*}(r) dr$. The conditions on ψ_0^X and the Lévy measure φ_1 imply that U_1 has almost surely no jumps of size -1 , which implies that its stochastic exponential $\mathcal{E}(U_1)$ does not vanish. By [95, I.4.64], $\mathcal{E}(U_1)$ changes its sign whenever $\Delta U_1(s) < -1$, i.e. we have

$$\mathcal{E}(U_1)(s) = \exp \left(\log |\mathcal{E}(U_1)(s)| + i\pi \sum_{r \leq s} 1_{\{\Delta U_1(r) < -1\}} \right).$$

We determine the differential characteristics of the semimartingale

$$Y := (v, z, \log |\mathcal{E}(U_1)|, \sum_{s \leq \cdot} 1_{\{\Delta U_1(s) < -1\}}, U_2)$$

relative to P^* by successive application of Propositions 2.11 and 2.12 and Lemmas 2.13 and A.7. Indeed, we start with the differential characteristics of $(X, I) = (v, z, I)$ relative to P . Proposition 2.11 yields those of $(v, z, I, \int_0^\cdot b^{\mathcal{L}(L)}(s) ds)$, Lemma 2.12 those of (v, z, I, Z^{P^*}) , again Proposition 2.11 those of $(v, z, I, \mathcal{L}(Z^{P^*}))$. Lemma 2.13 now leads to the P^* -differential characteristics of $(v, z, I, \mathcal{L}(Z^{P^*}))$. Proposition 2.12 yields those of (v, z, S, I) , Proposition 2.11 those of (v, z, U_1, U_2) and finally Proposition A.7 the P^* -differential characteristics of $Y = (Y_1, Y_2, Y_3, Y_4, Y_5)$. Very lengthy but straightforward

calculations yield that they are given by

$$\begin{aligned}
b^{Y_1^*}(s) &= b^v(s) + c^v(s)\alpha_1(s) + \int x_1(e^{\alpha_1(s)x_1} - 1)F^X(s, dx), \\
b^{Y_2^*}(s) &= b^z(s) + c^{v,z}(s)\alpha_1(s) + \int x_2(e^{\alpha_1(s)x_1} - 1)F^X(s, dx), \\
b^{Y_3^*}(s) &= -v(s-)\bar{\alpha}(s, \alpha_1(s)) - \frac{1}{2}\bar{a}^2c^z(s) \\
&\quad + \int (\log |1 - \bar{a}(e^{x_2} - 1)| + \bar{a}(e^{x_2} - 1))e^{\alpha_1(s)x_1}F^X(s, dx), \\
b^{Y_4^*}(s) &= \int 1_{(1,\infty)}(\bar{a}(e^{x_2} - 1))e^{\alpha_1(s)x_1}F^X(s, dx), \\
b^{Y_5^*}(s) &= v(s-)\bar{\alpha}(s, \alpha_1(s)), \\
c^{Y_1^*}(s) &= c^v(s), \\
c^{Y_1, Y_2^*}(s) &= c^{v,z}(s), \\
c^{Y_2^*}(s) &= c^z(s), \\
c^{Y_1, Y_3^*}(s) &= -\bar{a}c^{v,z}(s), \\
c^{Y_2, Y_3^*}(s) &= -\bar{a}c^z(s), \\
c^{Y_3^*}(s) &= \bar{a}^2c^z(s), \\
c^{Y_i, Y_j^*}(s) &= 0 \quad \text{for } i = 1, \dots, 5 \text{ and } j = 4, 5, \\
F^{Y^*}(s, G) &= \int 1_G(x_1, x_2, \log |1 - \bar{a}(e^{x_2} - 1)|, 1_{(1,\infty)}(\bar{a}(e^{x_2} - 1)), 0) e^{\alpha_1(s)x_1}F^X(s, dx)
\end{aligned}$$

for any $G \in \mathcal{B}^5$ and $s \in [t, T]$, where

$$\bar{a} := \bar{a}(s, \alpha_1(s)).$$

In particular, Y is a time-inhomogeneous affine process whose characteristic function is obtained from Theorem 2.6. Moreover, we have that

$$e^{z(T)p} \mathcal{E}(N^* - (N^*)^t)(T) = \exp(pY_2(T) + Y_3(T) + i\pi Y_4(T) + Y_5(T)).$$

Hence

$$V_p(t) = E_{P^*}(\exp(pY_2(T) + Y_3(T) + i\pi Y_4(T) + Y_5(T)) | \mathcal{F}(t)),$$

which is obtained from the generalized characteristic or moment generating function of Y (cf. Theorem 2.6). Note that the extension to $u \notin i\mathbb{R}^d$ requires sufficient regularity, cf. [48, Lemma 6.5], Theorem 2.8, [49, Theorem 3.3], [38, (10.8.2)] in this respect.

Another lengthy but straightforward calculation shows that

$$\begin{aligned}
\psi_1^Y(s, q, p, 1, i\pi, 1) &= \psi_1^X(s, q + \alpha_1(s), p) - \psi_1^X(s, \alpha_1(s), 0) \\
&\quad - \bar{a}(s, \alpha_1(s))\kappa_1(s, \alpha_1(s), q, 0, p, 1), \\
\psi_0^Y(s, q, p, 1, i\pi, 1) &= \psi_0^X(s, q + \alpha_1(s), p) + \psi_0^X(s, \alpha_1(s), 0)
\end{aligned}$$

This yields that $V_p(t)$ is of the form stated in the assertion. \square

PROOF OF THEOREM 5.3. In view of (5.13) we need to compute $\langle S, V \rangle^{P^*}$ and $\langle S, S \rangle^{P^*}$. Linearity of the predictable covariation yields

$$\langle S, V \rangle^{P^*}(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \langle S, V_p \rangle^{P^*}(t) \hat{f}(p) dp. \quad (5.22)$$

In an intermediate step in the proof of Theorem 5.2 we have determined the differential characteristics of (v, z, I) relative to P^* . Proposition 2.12 yields those of (S, V_p) . This leads to the modified second P^* -characteristics

$$\tilde{c}^{(S, V_{p^*})} = \begin{pmatrix} \tilde{c}^{S^*} & \tilde{c}^{S, V_{p^*}} \\ \tilde{c}^{S, V_{p^*}} & \tilde{c}^{V_{p^*}} \end{pmatrix}$$

of (S, V_p) , which satisfy

$$\tilde{c}^{S^*}(t) = S(t-)^2 \left(c^z(t) + \int (e^{x^2} - 1)^2 e^{\alpha_1(t)x_1} F^{(v,z)}(t, dx) \right), \quad (5.23)$$

$$\begin{aligned} \tilde{c}^{S, V_{p^*}}(t) &= V_p(t-) S(t-) \left(p c^z(t) + \Phi_1(t, p) c^{v,z}(t) \right. \\ &\quad \left. + \int (e^{x^2} - 1) (e^{\Phi_1(t,p)x_1 + px_2} - 1) e^{\alpha_1(t)x_1} F^{(v,z)}(t, dx) \right). \end{aligned} \quad (5.24)$$

By

$$\begin{aligned} \langle S, S \rangle^{P^*}(t) &= \int_0^t \tilde{c}^{S^*}(s) ds, \\ \langle S, V_p \rangle^{P^*}(t) &= \int_0^t \tilde{c}^{S, V_{p^*}}(s) ds \end{aligned}$$

and using (5.13, 5.22), the assertion follows. \square

PROOF OF THEOREM 5.4. Like in the proof of Theorem 5.2 we obtain for $p, q \in \mathbb{C}$ the moment generating function of the affine process $X = (v, z)$ from Theorem 2.6. It is of the form

$$E(e^{qv(t)+pz(t)}) = \exp(\Upsilon_0(0, q, p, t) + \Upsilon_1(0, q, p, t)v(0) + pz(0)) \quad (5.25)$$

with Υ_0, Υ_1 as in the assertion. Differentiation relative to q yields

$$\begin{aligned} E(e^{qv(t)+pz(t)}v(t)) &= \exp(\Upsilon_0(0, q, p, t) + \Upsilon_1(0, q, p, t)v(0) + pz(0)) \times \\ &\quad \times (\partial_2 \Upsilon_0(0, q, p, t) + \partial_2 \Upsilon_1(0, q, p, t)v(0)). \end{aligned} \quad (5.26)$$

An alternative representation of the expected squared hedging error in [22, Theorem 4.12] is

$$\varepsilon^2(w^*, \theta^*) = E \left(\int_0^T \left(\tilde{c}^{V^*}(t) - \frac{(\tilde{c}^{S, V^*}(t))^2}{\tilde{c}^{S^*}(t)} \right) L(t) dt \right),$$

where we use the notation of the proof of Theorem 5.3 with V instead of V_p . Bilinearity of the predictable covariation yields

$$\tilde{c}^{V^*}(t) = -\frac{1}{4\pi^2} \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \tilde{c}^{V_{p_1}, V_{p_2^*}}(t) \hat{f}(p_1) \hat{f}(p_2) dp_1 dp_2.$$

The modified second characteristics $\tilde{c}^{V_{p_1}, V_{p_2^*}}$ can be calculated similarly as in the proof of Theorem 5.3. In particular, we obtain

$$\begin{aligned} \tilde{c}^{V_{p_1}, V_{p_2^*}}(t) &= V_{p_1}(t-) V_{p_2}(t-) \left((p_1 \Phi_1(t, p_2) + p_2 \Phi_1(t, p_1)) c^{v, z}(t) \right. \\ &\quad + p_1 p_2 c^z(t) + \Phi_1(t, p_1) \Phi_1(t, p_2) c^v(t) \\ &\quad \left. + \int (e^{\Phi_1(t, p_1) x_1 + p_1 x_2} - 1) (e^{\Phi_1(t, p_2) x_1 + p_2 x_2} - 1) e^{\alpha_1(t) x_1} F^{(v, z)}(t, dx) \right). \end{aligned} \quad (5.27)$$

It remains to determine

$$E \left(\left(\tilde{c}^{V_{p_1}, V_{p_2^*}}(t) - \frac{\tilde{c}^{S, V_{p_1^*}}(t) \tilde{c}^{S, V_{p_2^*}}(t)}{\tilde{c}^{S^*}(t)} \right) L(t) \right).$$

From (5.23, 5.24, 5.27) and $v(t) = v(t-)$ almost surely we obtain after some calculations

$$\left(\tilde{c}^{V_{p_1}, V_{p_2^*}}(t) - \frac{\tilde{c}^{S, V_{p_1^*}}(t) \tilde{c}^{S, V_{p_2^*}}(t)}{\tilde{c}^{S^*}(t)} \right) L(t) = e^{\hat{\Phi}_0 + \hat{\Phi}_1 v(t) + p z(t)} (\varrho_0 + \varrho_1 v(t)) \quad \text{a.s.}$$

with $\hat{\Phi}_0, \hat{\Phi}_1, p, \varrho_0, \varrho_1$ as defined in the assertion. By (5.25, 5.26) we have

$$E \left(\left(\tilde{c}^{V_{p_1}, V_{p_2^*}}(t) - \frac{\tilde{c}^{S, V_{p_1^*}}(t) \tilde{c}^{S, V_{p_2^*}}(t)}{\tilde{c}^{S^*}(t)} \right) L(t) \right) = \varrho(t, p_1, p_2).$$

This yields the assertion. □

Appendix A

Affine Processes and Semimartingale Calculus

In the body of this dissertation we need some additional notions and certain auxiliary results. We summarize these findings and definitions in this part of the appendix. We begin with some results concerning affine processes, variance-optimal hedging and semimartingale calculus.

A.1 Properties of time-inhomogeneous affine processes

In Chapter 3 the stock price is partially modelled as the stochastic exponential of some components of an affine process. It is sometimes more convenient to describe the asset price in terms of the ordinary exponential. In order to translate the corresponding dynamics, one can use the following result.

Proposition A.1 *Let $X = (v, z)$ be a time-inhomogeneous affine process in the sense of Definition 2.2 which satisfies $\varphi^j(t, \{x \in \mathbb{R}^d : x_i \leq -1\}) = 0$ for every $j \in \mathcal{I}_0^m$, $i \in \mathcal{I}^d \setminus \mathcal{I}^m$ and for all $t \in [0, T]$. Then $\bar{X} := (v, \bar{z}) := (v, \log(\mathcal{E}(z_i)), \dots, \log(\mathcal{E}(z_n)))$ is a time-inhomogeneous affine process. The differential characteristics $(b^{\bar{X}}, c^{\bar{X}}, F^{\bar{X}})$ of \bar{X} are given for all $t \in [0, T]$ by*

$$\begin{aligned} b^v(t) &= b^v(t), \\ b_i^{\bar{z}}(t) &= b_i^z(t) - \frac{1}{2}c_{i,i}^z(t) + \int (h_i(\log(1_m + x^z)) - h_i(x^z)) F^{(v,z)}(t, d(x^v, x^z)), \\ c^{(v,\bar{z})}(t) &= c^{v,z}(t), \\ F^{(v,\bar{z})}(t, G) &= \varphi^0(t, G) + \sum_{i=1}^m v_i(t-) \varphi^i(t, G) \quad \forall G \in \mathcal{B}^d, \end{aligned}$$

where

$$\varphi^{i;(v,\bar{z})}(t, G) = \int 1_G(x^v, \log(x^z + 1_m)) \varphi^{i;(v,z)}(t, d(x^v, x^z)) \quad \forall G \in \mathcal{B}^d \text{ with } 0 \notin G. \quad (\text{A.1})$$

PROOF. This proposition is in some sense the converse of [106, Lemma 2.7]. As there we obtain the differential characteristics from an application of Propositions 2.11 and 2.12. As in [106, Lemma 2.7] the strong admissibility of the associated Lévy-Khintchine triplets $(\beta^{i:(v,\bar{z})}, \gamma^{i:(v,\bar{z})}, \varphi^{i:(v,\bar{z})})$, $i \in \mathcal{I}_0^m$ can be deduced from the strong admissibility of Lévy-Khintchine triplets $(\beta^{i:(v,z)}, \gamma^{i:(v,z)}, \varphi^{i:(v,z)})$, $i \in \mathcal{I}_0^m$. Observe that obviously (2.4) also holds for \bar{X} . Hence \bar{X} is an affine process. \square

In the affine framework locally bounded price processes correspond to bounded support of the associated components in the Lévy measures:

Lemma A.2 *Let $X = (v, z)$ be an $\mathbb{R}_+^m \times \mathbb{R}^n$ -valued time-homogeneous affine process in the sense of Definition 3.1 satisfying additionally $\varphi^i(\{x \in \mathbb{R}^d : x_j \leq -1\}) = 0$ for every $i \in \mathcal{I}_0^m$ and for every $j \in \mathcal{I}^d \setminus \mathcal{I}^m$. Then $S_i = \mathcal{E}(z_i)$, $i \in \mathcal{I}^n$ are locally bounded for any initial value $X^0 = (v^0, z^0)^\top \in \mathbb{R}_+^m \times \mathbb{R}^n$ of X if and only if numbers $M_j^i \in \mathbb{R}_+$ exist such that*

$$\varphi^i((\mathbb{R}_+^m \times \mathbb{R} \times \dots \times (-1; M_j^i] \times \dots \times \mathbb{R})^C) = 0 \quad \forall i \in \mathcal{I}_0^m, \forall j \in \mathcal{I}^n. \quad (\text{A.2})$$

PROOF. \Rightarrow : Define $\bar{z}_i := \log(S_i)$, $i \in \mathcal{I}^n$. Note that $\bar{X} := (v, \bar{z})$ is an affine process by Proposition A.1. The initial value of this process is given by $(v^0, 0)^\top =: \bar{X}^0$. Assume that, for an arbitrary $i \in \mathcal{I}_0^m$ and an arbitrary $j \in \mathcal{I}^n$, (A.2) is not satisfied. Since S_j is locally bounded, \bar{z}_j is locally bounded from above and there exist a sequence of constants $(\bar{M}_k)_{k \in \mathbb{N}}$ such that $\bar{M}_k < \infty$ for $k \in \mathbb{N}$ and a localizing sequence of stopping times $(\hat{T}_k)_{k \in \mathbb{N}}$ such that $(\bar{z}_j)^{\hat{T}_k} < \bar{M}_k$. $\tilde{T}_k := \inf\{t \in [0, T] : \bar{z}_j(t) \leq -k\}$ defines an increasing sequence of stopping times $(\tilde{T}_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} \tilde{T}_k \uparrow T$ P -a.s. Then $(T_k)_{k \in \mathbb{N}}$ with $T_k := \tilde{T}_k \wedge \hat{T}_k$ is a localizing sequence of stopping times such that $(\bar{z}_j)^{T_k}$ is bounded from above by \bar{M}_k . Define a function $W : \mathbb{R}_+ \times \mathbb{R}_+^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ by $W(t, x_1, \dots, x_d) := 1_{(\bar{M}_k + k, \infty)}(x_{m+j})$. Since the logarithm is strictly increasing and (A.2) is not satisfied for $i \in \mathcal{I}_0^m$ and $j \in \mathcal{I}^n$, we obtain from (A.1)

$$0 < W * (\varphi^{i:\bar{X}} \otimes dt)(T_k). \quad (\text{A.3})$$

We now distinguish between two cases:

1. $i = 0$: In this case Inequality (A.3) immediately yields

$$0 < E_{\bar{X}^0} \left(W * \nu^{\bar{X}}(T_k) \right) = E_{\bar{X}^0} \left(\sum_{t \leq T_k} 1_{\{\Delta \bar{z}_j(t) \geq \bar{M}_k + k\}} \right).$$

Hence there is a positive probability of a jump higher than $\bar{M}_k + k$ in $[0, T_k]$.

2. $i > 0$: Similarly as in the proof of [131, Lemma 4.1] one can use a stochastic continuity result [131, Lemma B.1] in order to show that for any initial value \bar{X}^0 of \bar{X} satisfying $v_i^0 > 0$ there exist a $t_{\bar{X}^0}$ in $(0, T]$ and an $\varepsilon_{\bar{X}^0} > 0$ such that $P_{\bar{X}^0}(v_i(t-) >$

$\varepsilon_{\bar{X}^0} > 1 - \delta$ for all $t \in [0, t_{\bar{X}^0}]$ and any $\delta \in (0, 1]$. Again, using (A.3) we obtain that

$$0 < E_{\bar{X}^0} \left(W * \nu^{\bar{X}}(T_k \wedge t_{\bar{X}^0}) \right) = E_{\bar{X}^0} \left(\sum_{t \leq T_k \wedge t_{\bar{X}^0}} 1_{\{\Delta \bar{z}_j(t) \geq \bar{M}_k + k\}} \right).$$

Similarly as in the first case there is a positive probability of a jump higher than $\bar{M}_k + k$ in $[0, T_k \wedge t_{\bar{X}^0}]$

Hence both cases lead to a contradiction to the local boundedness of \bar{z}_j . Therefore for the process (v, \bar{z}) and \bar{X}^0 there exist numbers $\bar{M}_j^i \in \mathbb{R}_+$ such that

$$\varphi^i((\mathbb{R}_+^m \times \mathbb{R} \times \dots \times (-\infty; \bar{M}_j^i] \times \dots \times \mathbb{R})^C) = 0 \quad \forall i \in \mathcal{I}_0^m, \forall j \in \mathcal{I}^n.$$

From (A.1) we conclude that (A.2) holds.

\Leftarrow : Let $j \in \mathcal{I}^n$ be arbitrary and define $M_j := \max_{i \in \mathcal{I}_0^m} \{M_j^i\}$. A localizing sequence of stopping times $(T_k)_{k \in \mathbb{N}}$ is defined by $T_k := \inf\{t \in [0, T] : S(t) \geq k\}$. [95, I.4.62] yields

$$(S_j(t))^{T_k} = (S_j(t-))^{T_k} (1 + \Delta(z_j(t))^{T_k}) \leq k(1 + M_j) < \infty$$

and hence the assertion. \square

Remark A.3 An inspection of the proof of the preceding lemma reveals that the result remains valid if (v, z) is a time-homogeneous affine process in the sense of Definition 3.1 and S is modelled as the ordinary exponential of z , i.e. $S_i = \exp(z_i)$, $i \in \mathcal{I}^n$.

The affine processes in Chapter 3 of this thesis undergo certain transformations (e.g. change of measure, C^2 -mappings). In the following lemma we show that the Lévy measures related to these affine processes remain Lévy measures under the applied transformations. Moreover these measures satisfy some weak continuity properties.

Lemma A.4 *Let $m, n \in \mathbb{N}_0$ and $d = m + n$. Suppose that $\eta \in C([0, T], \mathbb{R}^m)$, $\xi \in C([0, T], \mathbb{R}^n)$ and let φ be a Lévy measure on $\mathbb{R}_+^m \times \mathbb{R}^n$ such that*

$$\varphi(\{x \in \mathbb{R}^d : x_i < -1\}) = 0 \text{ for every } i \in \mathcal{I}^d \setminus \mathcal{I}^m. \quad (\text{A.4})$$

Additionally, we assume that there exists some open and convex set $\bar{\mathcal{V}} \supset \mathbb{R}_+^m$ such that

$$\int_{\{|x| \geq 1_d\}} e^{u^\top x} \varphi(dx) < \infty \quad \forall u \in \bar{\mathcal{V}} \times \mathbb{R}^n. \quad (\text{A.5})$$

Define $\nu(t) := (\eta^v(t)^\top, \xi(t)^\top)^\top$, $t \in [0, T]$. Then the following assertions hold:

1. *for every $t \in [0, T]$ a Lévy measure is given by*

$$\hat{\varphi}(t, G) := \int 1_G(x, e^{\nu(t)^\top x} - 1) \varphi(dx) \text{ for all } G \in \mathcal{B}^{d+1} \text{ with } 0 \notin G,$$

2. $\int_{\{x_{d+1}>1\}} x_{d+1} \hat{\varphi}(t, dx) < \infty$ for all $t \in [0, T]$,
3. $\sup_{t \in [0, T]} \left(\int_{\{x_i>1\}} x_i (1 + x_{d+1}) \hat{\varphi}(t, dx) \right) < \infty$ for every $i \in \mathcal{I}^m$,
4. the measure $x_{d+1}^2 \hat{\varphi}(t, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^{n+1}) \setminus \{0\}$ is weakly continuous in $t \in [0, T]$,
5. the measure $x_i x_{d+1} \hat{\varphi}(t, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^{n+1}) \setminus \{0\}$ is weakly continuous in $t \in [0, T]$ for $i \in \mathcal{I}^d$,
6. for every $t \in [0, T]$ a Lévy measure $\bar{\varphi}$ is given by

$$\bar{\varphi}(t, G) := \int 1_G(x) (1 + x_{d+1}) \hat{\varphi}(t, dx) \quad \forall G \in \mathcal{B}^{d+1} \text{ with } 0 \notin G,$$

7. $\int_{\{x_i>1\}} x_i \bar{\varphi}(t, dx) < \infty$ for every $i \in \mathcal{I}^d \setminus \mathcal{I}^m$ and all $t \in [0, T]$,
8. for every $t \in [0, T]$ a Lévy measure $\tilde{\varphi}$ is given by

$$\tilde{\varphi}(t, G) := \int 1_G(x_1, \dots, x_m, \log(1 + x_{d+1})) (1 + x_{d+1}) \hat{\varphi}(t, dx) \\ \forall G \in \mathcal{B}^{m+1} \text{ with } 0 \notin G,$$

9. (a) $\sup_{t \in [0, T]} \left(\int_{\{x_i \geq 1\}} x_i \bar{\varphi}(t, dx) \right) < \infty$ for every $i \in \mathcal{I}^m$,
- (b) $\sup_{t \in [0, T]} \left(\int_{\{x_i \geq 1\}} x_i \tilde{\varphi}(t, dx) \right) < \infty$ for every $i \in \mathcal{I}^m$,
10. $\sup_{t \in [0, T]} \left(\int_{\{x_j \geq 1\}} x_j (1 + x_i) \bar{\varphi}(t, dx) \right) < \infty$ for every $j \in \mathcal{I}^m, i \in \mathcal{I}^d \setminus \mathcal{I}^m$,
11. the measure $x_j x_i \bar{\varphi}(t, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^{n+1}) \setminus \{0\}$ is weakly continuous in $t \in [0, T]$ for every $j \in \mathcal{I}^d, i \in \mathcal{I}^d \setminus \mathcal{I}^m$,
12. $\int_{\{|x| \geq 1\}} e^{u^\top x} \tilde{\varphi}(t, dx) < \infty \quad \forall u \in \mathcal{V} \times (-1, \infty)$.

PROOF. Define $\mathcal{R}^n := \mathbb{R}_+^m \times \mathbb{R}^n$, the set D^ε by $D^\varepsilon := \{x \in \mathcal{R}^n : |x| \leq \varepsilon\}$, $\varepsilon > 0$ and the functions $g : [0, T] \times \mathcal{R}^n \rightarrow \mathbb{R}_+$ by $g(t, x) := e^{\nu(t)^\top x}$. Note that there exist constants $M_1, M_2 \in [0, \infty)$ such that

$$\int_{\mathcal{R}^n} g(t, x) \varphi(dx) \leq \int_{\mathcal{R}^n} M_1 e^{M_2 \sum_{j=m+1}^d x_j} \varphi(dx), \quad (\text{A.6})$$

for all $t \in [0, T]$, because η^ν is \mathbb{R}_+^m -valued, ξ is continuous on $[0, T]$ and (A.4) holds. Moreover, since η^ν and ξ are continuous on $[0, T]$, one can deduce from a Taylor expansion of g that for all $x \in D^\varepsilon$ and all $t \in [0, T]$,

$$|g(t, x) - 1| \leq M_3 |x| \text{ for some } M_3 \in [0, \infty). \quad (\text{A.7})$$

(1): Since φ is a Lévy measure and (A.5) holds, we obtain from (A.6-A.7) that

$$\begin{aligned} \int x_{d+1}^2 \hat{\varphi}(t, dx) &\leq \int_{\mathcal{R}^n \setminus D^\varepsilon} \left(1 + M_1^2 e^{2M_2 \sum_{j=m+1}^d x_j} + 2M_1 e^{M_2 \sum_{j=m+1}^d x_j}\right) \varphi(dx) \\ &\quad + \int_{D^\varepsilon} M_3 |x|^2 \varphi(dx) < \infty. \end{aligned} \quad (\text{A.8})$$

Since we have $\hat{\varphi}(t, \{0\}) = 0, t \in [0, T]$ by the definition of $\hat{\varphi}$, we obtain (1).

(2): Define the sets $\hat{\mathcal{R}}(t), t \in [0, T]$, by $\hat{\mathcal{R}}(t) := \{x \in \mathcal{R}^n : g(t, x) > 2\}$. Since η^v is \mathbb{R}_-^m -valued and ξ is continuous on $[0, T]$, one can show that for every $t \in [0, T]$ there exists an $\tilde{\varepsilon} > 0$ such that $\hat{\mathcal{R}}(t) \cap D^\varepsilon = \emptyset$. This observation and (A.6) yield

$$\begin{aligned} \int_{\{x_{d+1} > 1\}} x_{d+1} \hat{\varphi}(t, dx) &= \int_{\hat{\mathcal{R}}(t) \setminus D^\varepsilon} (g(t, x) - 1) \varphi(dx) \\ &\leq \int_{\hat{\mathcal{R}}(t) \setminus D^\varepsilon} \left(M_1 e^{M_2 \sum_{j=m+1}^d x_j} + 1\right) \varphi(dx) < \infty. \end{aligned}$$

Here, the last inequality follows, because φ is a Lévy measure and (A.5) holds.

(3): Since (A.5) holds, one easily deduces from (A.6) that

$$\begin{aligned} \sup_{t \in [0, T]} \left(\int_{\{x_i > 1\}} x_i (1 + x_{d+1}) \hat{\varphi}(t, dx) \right) &= \sup_{t \in [0, T]} \left(\int_{\{x_i > 1\}} x_i g(t, x) \varphi(dx) \right) \\ &\leq \int_{\{x_i > 1\}} M_1 e^{M_2 \sum_{j=m+1}^d x_j} x_i \varphi(dx) < \infty, \end{aligned}$$

for every $i \in \mathcal{I}^m$.

(4) and (5): Let $f_2 \in C(\mathbb{R}^{d+1}, \mathbb{R})$ and observe that if a function $\hat{f}_2 \in C(\mathbb{R}^{m+n}, \mathbb{R}_+)$ exists such that the inequality

$$\int |f_2(x)| \hat{\varphi}(t, dx) \leq \int |\hat{f}_2(x)| \varphi(dx) < \infty$$

holds, then $f_2(x) \hat{\varphi}(t, dx)$ is weakly continuous by dominated convergence, since $g(\cdot, x)$ is a continuous function on $[0, T]$. Hence, (A.8) immediately yields (4). Similarly, since (A.5) holds, one easily shows using (A.6-A.7) that

$$\int |x_{d+1}| |x_i| \hat{\varphi}(t, dx) \leq M_3 \int_{D^\varepsilon} |x|^2 \varphi(dx) + \int_{\mathcal{R}^n \setminus D^\varepsilon} (g(t, x) + 1) |x_i| \varphi(dx) < \infty,$$

for $i \in \mathcal{I}^d$. Hence (5) follows.

(6): Similarly as (1), we obtain (6) from

$$\begin{aligned} &\int_{\mathcal{R}_\varepsilon^{n+1}} \left((1 \wedge x_i^2) g(t, x) \right) \hat{\varphi}(t, dx) + \int_{\mathcal{R}^{n+1} \setminus \mathcal{R}_\varepsilon^{n+1}} \left((1 \wedge x_i^2) g(t, x) \right) \hat{\varphi}(t, dx) \\ &\leq \int_{\mathcal{R}_\varepsilon^{n+1}} \left((1 \wedge x_i^2) g(t, x) \right) \hat{\varphi}(t, dx) + \int_{\mathcal{R}^n \setminus D^\varepsilon} M_1 e^{M_2 \sum_{j=m+1}^d x_j} \varphi(dx) < \infty, \end{aligned}$$

for $i \in \mathcal{I}^{d+1}$ and $t \in [0, T]$, because $g(t, x)$ is bounded on $[0, T] \times D^\varepsilon$ and $\hat{\varphi}$ is a Lévy measure resp. (A.6) and (A.5) hold.

(7): Since (A.5) is satisfied and (A.6) holds, we obtain

$$\int_{\{x_i > 1\}} x_i g(t, x) \varphi(dx) < \int_{\{x_i > 1\}} x_i M_1 e^{M_2 \sum_{j=m+1}^d x_j} \varphi(dx) < \infty,$$

for every $i \in \mathcal{I}^d \setminus \mathcal{I}^m$ and all $t \in [0, T]$.

(8): Since ν is continuous on $[0, T]$, there exists a constant $M_4 \in [0, \infty)$ such that

$$\int_{\mathcal{R}_\varepsilon^{n+1}} (1 \wedge x_{d+1}^2) \tilde{\varphi}(t, dx) \leq \int_{D^\varepsilon} ((1 \wedge M_4 |x|^2) g(t, x)) \varphi(dx).$$

Hence the assertion follows similarly as (6).

(9): Note that (9a) and (9b) are equivalent. Since (A.5) is satisfied, we deduce by using (A.6) that

$$\sup_{t \in [0, T]} \left(\int_{\{x_i \geq 1\}} x_i \bar{\varphi}(t, dx) \right) \leq \int_{\{x_i \geq 1\}} \left(x_i M_1 e^{M_2 \sum_{j=m+1}^d x_j} \right) \varphi(dx) < \infty,$$

for $i \in \mathcal{I}^m$. Hence (9a) holds.

(10): Note that there exist an $\bar{x} \geq 1$ and an $\bar{\varepsilon} > 0$ such that $\bar{\varepsilon} \bar{e}_j^m \in \bar{\mathcal{V}}$ and $x \leq e^{\bar{\varepsilon} x}$ for $x \geq \bar{x}$ and every $j \in \mathcal{I}^m$. This yields

$$\begin{aligned} \sup_{t \in [0, T]} \left(\int_{\{x_j \geq 1\}} (x_j(1 + x_i)) \bar{\varphi}(t, dx) \right) &\leq \int_{\{x_j \geq \bar{x}\}} \left(M_1 e^{1+x_i+\bar{\varepsilon}x_j+M_2 \sum_{j=m+1}^d x_j} \right) \varphi(dx) \\ &\quad + \int_{\{1 \leq x_j < \bar{x}\}} \left(M_1 \bar{x} e^{1+x_i+M_2 \sum_{j=m+1}^d x_j} \right) \varphi(dx) \\ &< \infty, \end{aligned}$$

for every $j \in \mathcal{I}^m$ and every $i \in \mathcal{I}^d \setminus \mathcal{I}^m$ since (A.5) and (A.6) hold.

(11): We proceed as in the proof of (5). First, we note that there exists a constant $M_5 \in [0, \infty)$ such that $\int_{D^\varepsilon} |x_j| |x_i| g(t, x) \varphi(dx) \leq \int_{D^\varepsilon} M_5 |x|^2 g(t, x) \varphi(dx)$ for every $j \in \mathcal{I}^d$ and every $i \in \mathcal{I}^d \setminus \mathcal{I}^m$. We obtain from this observation and (A.6) that

$$\int |x_j| |x_i| \bar{\varphi}(t, dx) \leq \int_{D^\varepsilon} M_5 |x|^2 g(t, x) \varphi(dx) + \int_{\mathcal{R}^n \setminus D^\varepsilon} |x_j| |x_i| M_1 e^{M_2 \sum_{j=m+1}^d x_j} \varphi(dx),$$

for every $j \in \mathcal{I}^d$ and every $i \in \mathcal{I}^d \setminus \mathcal{I}^m$. Here, the last inequality follows by a similar estimate as in the proof of (10) and since φ is a Lévy measure.

(12): Let $u \in \bar{\mathcal{V}} \times (-1, \infty)$. A similar argument as used to prove (A.6) yields

$$\begin{aligned} \int_{\{|x| \geq 1_d\}} e^{u^\top x} \tilde{\varphi}(t, dx) &\leq \int_{\{|x| \geq 1_d\}} e^{\sum_{i=1}^m u_i x_i + (u_{m+1} + 1)x^\top \nu(t)} \varphi(dx) \\ &\leq \int_{\{|x| \geq 1_d\}} e^{\sum_{i=1}^m u_i x_i + (u_{m+1} + 1)(\log(M_1) + M_2 \sum_{i=m+1}^d x_i)} \varphi(dx) < \infty, \end{aligned}$$

because (A.5) holds. \square

A.2 Variance-optimal hedging

In [111] the quadratic hedging problem (4.5) is solved for time-homogeneous affine processes. These results can easily be extended to time-inhomogeneous affine processes by a slight modification of the respective prerequisites:

Assumption A.5 *Let $X = (v, z)$ be an $\mathbb{R}_+ \times \mathbb{R}$ -valued time-inhomogeneous affine process with Lévy exponent ψ^X and let Ψ_0 and Ψ_1 be the mappings appearing in the calculation of the conditional characteristic function of X in Theorem 2.6. We suppose that the following conditions hold*

1. *The claim H admits an integral representation (4.6).*
2. *For some $\tilde{\varepsilon} > 0$ the mappings $(u_1, u_2) \mapsto \Psi_0(s, t, u_1, u_2), \Psi_1(s, t, u_1, u_2)$ have an analytic extension on*

$$\mathcal{S} := \{u \in \mathbb{C}^2 : (\operatorname{Re}(u_1), \operatorname{Re}(u_2)) \in \mathcal{V}_{\tilde{\varepsilon}}(0)\}$$

for all $s \in [0, t]$ and all $t \in [0, T]$, where

$$M_0 := \sup\{2\Psi_1(t, T, 0, r) : r \in [R' \wedge 0, R \vee 0], t \in [0, T]\}$$

and

$$\mathcal{V}_{\tilde{\varepsilon}}(a) := (-\infty, (M_0 \vee 0) + \varepsilon) \times ((2R' \wedge 0) - \tilde{\varepsilon}, (2R \vee a) + \tilde{\varepsilon})$$

for $a \in \mathbb{R}_+$. These extensions are again denoted by Ψ_0 resp. Ψ_1 . Here R and R' denote the constants related to the strip \mathcal{S}_f in (4.7).

3. $[0, T] \times \mathcal{V}_{\tilde{\varepsilon}}(2) \subset \bar{D}_0 \cap \bar{D}_1$.
4. *The mappings $s \mapsto \Psi_0(s, t, u_1, u_2)$ and $s \mapsto \Psi_1(s, t, u_1, u_2)$ are continuous on $[0, t]$ for any $(u_1, u_2) \in \mathcal{S}$ and any $t \in [0, T]$,*
5. $\psi_0^X(t, 0, 1) = \psi_1^X(t, 0, 1) = 0$ for all $t \in [0, T]$,
6. $\psi_0^X(t, 0, 2) + \psi_1^X(t, 0, 2) \neq 0$ for all $t \in [0, T]$.

For time-inhomogeneous affine processes [111, Theorem 4.1 and Theorem 4.2] can then be generalized as follows.

Theorem A.6 *If Assumption A.5 is satisfied, the variance-optimal initial capital π and the variance-optimal hedging strategy θ are given by*

$$\begin{aligned}\pi &= \int V_\zeta(0)\Pi(d\zeta), \\ \theta &= \int \frac{V_\zeta(t-)}{S(t-)} \frac{\kappa_0(t, \zeta) + \kappa_1(t, \zeta)v(t-)}{\delta_0(t) + \delta_1(t)v(t-)} \Pi(d\zeta),\end{aligned}$$

where the process $V_\zeta(t)$ satisfies

$$V_\zeta(t) = S(t)^\zeta \exp(\Psi_0(t, T, 0, \zeta) + \Psi_1(t, T, 0, \zeta)v(t)), \quad \zeta \in S_f,$$

and the functions $\delta_0, \delta_1 : [0, T] \rightarrow \mathbb{R}_+$ and $\kappa_0, \kappa_1 : [0, T] \times S_f$ are defined by

$$\begin{aligned}\delta_j(t) &:= \psi_j^X(t, 0, 2), \quad j = 0, 1, \\ \kappa_j(t) &:= \psi_j^X(t, \Psi_1(t, T, 0, \zeta), \zeta + 1) - \psi_j^X(t, \Psi_1(t, T, 0, \zeta), \zeta).\end{aligned}$$

If additionally

$$(\delta_0(t) = 0 \text{ for all } t \in [0, T]) \text{ or } (\delta_0(t) \neq 0 \text{ for all } t \in [0, T])$$

and

$$(\delta_1(t) = 0 \text{ for all } t \in [0, T]) \text{ or } (\delta_1(t) \neq 0 \text{ for all } t \in [0, T])$$

hold, then the minimal hedging error ε^2 is given by

$$\varepsilon^2 = \begin{cases} \int \int \int_0^T J_1(t, \zeta_1, \zeta_2) dt \Pi(d\zeta_1) \Pi(d\zeta_2), & \text{if } \delta_0(t) \neq 0, \delta_1(t) \neq 0, t \in [0, T], \\ \int \int \int_0^T J_2(t, \zeta_1, \zeta_2) dt \Pi(d\zeta_1) \Pi(d\zeta_2), & \text{if } \delta_0(t) = 0, t \in [0, T], \\ \int \int \int_0^T J_3(t, \zeta_1, \zeta_2) dt \Pi(d\zeta_1) \Pi(d\zeta_2), & \text{if } \delta_1(t) = 0, t \in [0, T]. \end{cases} \quad (\text{A.9})$$

The integrals over S_f in (A.9) have to be understood in the sense of the Cauchy principal value (cf. [111, Remark 4.3]). The integrands $J_k : [0, T] \times S_f^2 \rightarrow \mathbb{C}, k = 1, 2, 3$ in these

expressions are defined as

$$\begin{aligned}
J_1(t, \zeta_1, \zeta_2) &= S(0)^{\tilde{\zeta}} e^{\xi_0} \left(\exp(\Psi_0(0, t, \xi_1, \tilde{\zeta}) + \Psi_1(0, t, \xi_1, \tilde{\zeta})v(0)) \left(\frac{\eta_2}{\delta_1} (D_3\Psi_0(0, t, \xi_1, \tilde{\zeta}) \right. \right. \\
&\quad \left. \left. + D_3\Psi_1(0, t, \xi_1, \tilde{\zeta})v(0)) + \frac{\eta_1\delta_1 - \eta_2\delta_0}{\delta_1^2} \right) + \frac{\eta_0\delta_1^2 - \eta_1\delta_0\delta_1 + \eta_2\delta_0^2}{\delta_1^3} e^{-\frac{\xi_0}{\delta_1}\xi_1} \times \right. \\
&\quad \left. \times \int_0^1 \left(\frac{\delta_1}{\delta_0} + \xi_1 s \right) e^{\frac{\xi_0}{\delta_1}\xi_1 s + \Psi_0\left(0, t, \frac{\delta_1}{\delta_0} \log(s) + \xi_1 s, \tilde{\zeta}\right) + \Psi_1\left(0, t, \frac{\delta_1}{\delta_0} \log(s) + \xi_1 s, \tilde{\zeta}\right)v(0)} ds \right), \\
J_2(t, \zeta_1, \zeta_2) &= \frac{S(0)^{\tilde{\zeta}} e^{\xi_0}}{\delta_1} \exp(\Psi_0(0, t, \xi_1, \tilde{\zeta}) + \Psi_1(0, t, \xi_1, \tilde{\zeta})v(0)) \times \\
&\quad \times (\eta_1 + \eta_2(D_3\Psi_0(0, t, \xi_1, \tilde{\zeta}) + D_3\Psi_1(0, t, \xi_1, \tilde{\zeta})v(0))), \\
J_3(t, \zeta_1, \zeta_2) &= \frac{S(0)^{\tilde{\zeta}} \eta_0}{\delta_1} \exp(\xi_0 + \Psi_0(0, t, 0, \tilde{\zeta})).
\end{aligned}$$

The remaining variables are given by:

$$\begin{aligned}
\tilde{\zeta} &:= \zeta_1 + \zeta_2, \\
\delta_i &:= \delta_i(t), \quad i = 0, 1, \\
\eta_0 &= \eta_0(t, \zeta_1, \zeta_2) := \delta_0\iota_0 - \kappa_0(t, \zeta_1)\kappa_0(t, \zeta_2), \\
\eta_1 &= \eta_1(t, \zeta_1, \zeta_2) := \delta_0\iota_1 + \delta_1\iota_0 - \kappa_1(t, \zeta_1)\kappa_0(t, \zeta_2) - \kappa_1(t, \zeta_2)\kappa_0(t, \zeta_1), \\
\eta_2 &= \eta_2(t, \zeta_1, \zeta_2) := \delta_1\iota_1 - \kappa_1(t, \zeta_1)\kappa_1(t, \zeta_2), \\
\iota_i &= \iota_i(t, \zeta_1, \zeta_2) := \psi_i^X(t, \xi_1(t, \zeta_1, \zeta_2), \tilde{\zeta}) \\
&\quad - \psi_i^X(t, \Psi_1(t, T, 0, \zeta_1), \zeta_1) - \psi_i^X(t, \Psi_1(t, T, 0, \zeta_2), \zeta_2), \quad i = 0, 1, \\
\xi_i &= \xi_i(t, \zeta_1, \zeta_2) := \Psi_i(t, T, 0, \zeta_1) + \Psi_i(t, T, 0, \zeta_2), \quad i = 0, 1,
\end{aligned}$$

where the mappings $\eta_0, \eta_1, \eta_2, \iota_0, \iota_1, \xi_0, \xi_1$ are defined on $[0, T] \times \mathcal{S}_f^2$.

PROOF. The assertions of this theorem follow by similar arguments as in the proofs of [111, Theorem 4.1 and Theorem 4.2]. The proofs of these theorems consist of several steps. Only one of these needs a mentionable modification. Namely, in [111, Lemma 5.2], Itô's formula formulated in Proposition 2.12 is used to calculate the differential characteristics of $g(I, v, z)$. g has the form

$$g(s, x_1, x_2) := \begin{pmatrix} \exp(\Psi_0(s, t, 0, \zeta) + \Psi_1(s, t, 0, \zeta)x_1 + \zeta x_2) \\ \exp(x_2) \end{pmatrix}$$

for time-inhomogeneous affine processes. Since Ψ_0 and Ψ_1 are in general not C^2 with respect to s , we can not directly apply Proposition 2.12. Therefore we first use an integration rule for characteristics (cf. Proposition 2.11) in order to obtain the characteristics of

(v, z, Ψ_0, Ψ_1) . In a second step we obtain the differential characteristics of $g(I, v, z)$ from Proposition 2.12. The remaining steps in the proof of [111, Lemma 5.2] can then be carried through. \square

A.3 Semimartingale calculus

The following result is needed in a proof of Chapter 5 (cf. Theorem 5.2).

Lemma A.7 *Let $X = (X_1, \dots, X_d)$ be an \mathbb{R}^d -valued semimartingale such that X_d does not have jumps of size -1 . If (b^X, c^X, F^X) denote the differential characteristics of X , then the differential characteristics (b^Y, c^Y, F^Y) of the \mathbb{R}^{d+1} -valued semimartingale*

$$Y = (Y_1, \dots, Y_d, Y_{d+1}) := \left(X_1, \dots, X_{d-1}, \log |\mathcal{E}(X_d)|, \sum_{s \leq \cdot} 1_{\{\Delta X_d < -1\}} \right)$$

are given by

$$\begin{aligned} b_i^Y(t) &= b_i^X(t), \quad i = 1, \dots, d-1, \\ b_d^Y(t) &= b_d^X(t) - \frac{1}{2} c_{d,d}^X(t) + \int (h_d(\log |1+x_d|) - h_d(x)) F^X(t, dx), \\ b_{d+1}^Y(t) &= \int 1_{(-\infty, -1)}(x_d) F^X(t, dx), \\ c_{i,j}^Y(t) &= c_{i,j}^X(t), \quad i, j = 1, \dots, d, \\ c_{d+1,i}^Y(t) &= c_{i,d+1}^Y(t) = 0, \quad i = 1, \dots, d+1, \\ F^Y(t, G) &= \int 1_G(x_1, \dots, x_{d-1}, \log |1+x_d|, 1_{(-\infty, -1)}(x_d)) F^X(t, dx) \quad \forall G \in \mathcal{B}^{d+1}. \end{aligned}$$

PROOF. Denote by (B^X, C^X, ν^X) the (integral) characteristics of X in the sense of [95, Definition II.2.6]. The canonical representation of Y_d and Y_{d+1} in the sense of [95, Theorem II.2.34] equals

$$\begin{aligned} Y_d &= X_d^C + h_d(\log |1+x_d|) * (\mu^X - \nu^X) + B^X - \frac{1}{2} \langle X_d^C, X_d^C \rangle + \\ &\quad (\log |1+x_d| - h_d(\log |1+x_d|)) * \mu^X + (h_d(\log |1+x_d|) - h_d(x)) * \nu^X, \\ Y_{d+1} &= 1_{(-\infty, -1)}(x_d) * (\mu^X - \nu^X) + 1_{(-\infty, -1)}(x_d) * \nu^X. \end{aligned}$$

This yields the differential characteristics of Y above. \square

Appendix B

Differential Equations

In Chapter 3 we need some auxiliary results for differential algebraic equations and ordinary differential equations. Therefore, we prove an existence and uniqueness result for a class of differential algebraic equations (henceforth DAEs) in this appendix. The general theory of DAEs is discussed in some textbooks (e.g. [17, 124]) and we refer to these books for unexplained notation and terminology.

We consider an autonomous semi-explicit DAE:

$$\nabla x = f(x, y), \quad (\text{B.1})$$

$$0 = h(x, y), \quad (\text{B.2})$$

where $f : X \times Y \rightarrow \mathbb{R}^n$ and $h : X \times Y \rightarrow \mathbb{R}^k$, $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. In order to prove existence and uniqueness of a solution to (B.1-B.2), often regularity or smoothness of the functions h and f is assumed in the literature. If the System (B.1-B.2) appears in connection with exponential utility maximization we can deduce existence and uniqueness from a smoothness condition on f and the *convex structure* of the problem. These prerequisites can be captured by the following assumption

Assumption B.1 *X and Y are open convex sets and $f = (f^1, \dots, f^n)$ is a convex function, i.e. f^i , $i \in \mathcal{I}^n$ are convex functions. Moreover, on $X \times Y$ the function f is finite and continuously differentiable with respect to y . $h : X \times Y \rightarrow \mathbb{R}^{nm}$ is given by the column vector of these partial derivatives, i.e. $h := (f_y^1, \dots, f_y^n)^\top$.*

Furthermore we need a condition which ensures that a solution satisfying the constraint (B.2) exists.

Assumption B.2 *The equation (B.2) has a finite solution for every $x \in X$.*

If these two assumptions are fulfilled we can prove the following existence and uniqueness result:

Theorem B.3 *If the Assumptions B.1 and B.2 are satisfied the DAE (B.1-B.2) with initial value $x(0) = \xi \in X$ has a unique solution. The solution can be extended to the left and right up to the boundary of X .*

PROOF. We consider the convex minimization problems

$$g^i(x) := \min_{y \in Y} f^i(x, y), \quad i \in \mathcal{I}^n. \quad (\text{B.3})$$

By convexity and finiteness, f is a continuous function on $X \times Y$ (cf. [159, Theorem 10.1]). Since f is continuously differentiable with respect to y and Assumption B.2 is fulfilled, for every $x \in X$ there exists a $y^* \in Y$ which is simultaneously a stationary point of $f^i(x, \cdot)$ for $i \in \mathcal{I}^n$. By [127, Kapitel 6.4, Theorem 2], y^* is a (global) minimizer of the optimization problem (B.3). On X the objective functions g^i , $i \in \mathcal{I}^n$ are convex by [160, Theorem 1] and finite since the optima of (B.3) are attained. Since the constraint in the DAE arises from the optimality condition in the convex optimization problem (B.3), we can replace the original problem (B.1-B.2) by the equivalent differential equation:

$$\nabla x(t) = g(x(t)), \quad g = (g^1, \dots, g^n).$$

g is locally Lipschitz-continuous on X , because the components of g are convex and finite on X and [159, Theorem 10.4] holds. Hence the assertion follows by a standard result for ordinary differential equations (cf. [187, Chapter III, Paragraph 10, Section VI]). \square

It is sometimes convenient to know a condition implying that there is no blow-up of the solution x in the preceding theorem. In order to prove such a kind of result, we recall from [187, Chapter III, Paragraph 10, Section XII] the following definition and theorem.

Definition B.4 A function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be *quasimonotone increasing* if f_i , $i \in \mathcal{I}^n$ are increasing in x_j for $i \neq j$, i.e. if for $i \in \mathcal{I}^n$

$$x_j \leq \tilde{x}_j \text{ for all } j \in \mathcal{I}^n \text{ and } x_i = \tilde{x}_i \text{ for } x, \tilde{x} \in D \text{ implies } f_i(x) \leq f_i(\tilde{x}).$$

Theorem B.5 Assume that $f : D \rightarrow \mathbb{R}^n$ is quasimonotone increasing, locally Lipschitz-continuous and that the functions v, w are differentiable in $[0, T]$. Then $v(0) \leq w(0)$ and $\nabla v(t) - f(v(t)) \leq \nabla w(t) - f(w(t))$ for $t \in [0, T]$ implies $v \leq w$ on $[0, T]$.

Theorem B.5 allows us to prove the following result:

Corollary B.6 Let the assumptions of Theorem B.3 be satisfied. If the solution x is \mathbb{R}_-^n -valued in its lifetime $[0, T)$, then $T = \infty$.

PROOF. Let g be as in the proof of Theorem B.3. Since the components of g are convex and finite on X , their subdifferentials are non-empty on X by [159, Theorem 23.4]. Hence there exist affine global underestimators for g^i , $i \in \mathcal{I}^n$. Therefore one can choose quasimonotone increasing affine functions L^i , $i \in \mathcal{I}^n$ such that

$$g^i(x) \geq L^i(x) \quad \forall x \in X \cap \mathbb{R}_-^n, \quad i \in \mathcal{I}^n.$$

The linear differential equation

$$\nabla \hat{x} = L(\hat{x}), \quad \hat{x}(0) = \xi, \quad L = (L^1, \dots, L^n)$$

has a solution existing on $[0, \infty)$ by [187, Chapter IV, Paragraph 14, Section VI]. Since $\nabla \hat{x} - L(\hat{x}) = 0 = \nabla x - g(x) \leq \nabla x - L(x)$ on $[0, T)$, the comparison Theorem B.5 implies that

$$x_i(t) \geq \hat{x}_i(t), \quad i = 1, \dots, n, \quad t \in [0, \infty).$$

This proves the claim. \square

Finally, in the body of the thesis we require the explicit solution of a certain Riccati equation.

Lemma B.7 *Let $a := (a_1, \dots, a_4)^\top \in \mathbb{R}^4$ be such that $a_2, a_4 \in \mathbb{R}$, $a_1 \in \mathbb{R}_-$, $a_3 \in \mathbb{R}_+$. Define for*

$$q := \tanh^{-1} \left(\frac{a_2}{\sqrt{d}} \right) \quad \text{and} \quad d := a_2^2 - 4a_1a_3$$

the function

$$g : [0, T] \rightarrow \mathbb{R} \quad \text{by} \quad g(t) = \frac{\sqrt{d}}{2}t + q.$$

Then for $t \in [0, T]$

$$x_a(t) := \begin{cases} 0 & \text{if } a_3 = 0, \\ -a_3t & \text{if } a_3 > 0 \wedge a_1 = 0 \wedge a_2 = 0, \\ \frac{a_3}{a_2} (e^{-a_2t} - 1) & \text{if } a_3 > 0 \wedge a_1 = 0 \wedge a_2 \neq 0, \\ \frac{\sqrt{d} \tanh(g(t)) - a_2}{2a_1} & \text{if } a_3 > 0 \wedge a_1 < 0, \end{cases} \quad (\text{B.4})$$

$$y_a(t) := \begin{cases} 0 & \text{if } a_3 = 0, \\ -\frac{1}{2}a_4a_3t^2 & \text{if } a_3 > 0 \wedge a_1 = 0 \wedge a_2 = 0, \\ -\frac{a_4a_3}{a_2^2} (e^{-a_2t} - 1 + a_2t) & \text{if } a_3 > 0 \wedge a_1 = 0 \wedge a_2 \neq 0, \\ \frac{a_4}{2a_1} \left(-a_2t + 2 \ln \left(\frac{\cosh(g(t))}{\cosh(q)} \right) \right) & \text{if } a_3 > 0 \wedge a_1 < 0 \end{cases} \quad (\text{B.5})$$

solve the following system of Riccati equations:

$$\begin{aligned} \partial x_a(t) &= -a_1x_a(t)^2 - a_2x_a(t) - a_3, & x_a(0) &= 0, \\ \partial y_a(t) &= a_4x_a(t), & y_a(0) &= 0. \end{aligned}$$

PROOF. The above result is a reformulation of [23, Lemma 6.1]. \square

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Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation eigenständig angefertigt habe. Davon ausgenommen ist der fachliche Rat durch Prof. Dr. Jan Kallsen, der diese Arbeit betreut hat. Darüber hinaus basiert der Inhalt von Kapitel 5 auf der Preprint-Fassung eines gemeinsam mit meinem Betreuer in der Fachzeitschrift *Review of Derivatives Research* veröffentlichten Artikels:

- J. Kallsen und R. Vierthauer. Quadratic hedging in affine stochastic volatility models. *Review of Derivatives Research*, 12(1):3-27, 2009.

Bei der Erstellung der Dissertation habe ich die Regeln guter wissenschaftlicher Praxis der *Deutschen Forschungsgemeinschaft* eingehalten. Außerdem versichere ich, dass diese Arbeit weder ganz noch teilweise einem anderen Prüfungsverfahren zu Grunde lag. Des Weiteren erkläre ich, dass ich noch keinen Promotionsversuch unternommen habe.

Richard Vierthauer

Kiel, den 20.05.2010

Curriculum Vitae

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1991 - 1995	Realschule Kemnath (Abschluß: Mittlere Reife)
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1997 - 1998	Mathematik, Fachhochschule Regensburg, Abschluß: Vordiplom (fachgebundene Hochschulreife)
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