

# **Asset Price and Wealth Dynamics with Heterogeneous Expectations**

**A Deterministic Nonlinear Structural Model Approach**

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with Heterogeneous Expectations**  
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# Introduction

Since its beginnings more than one century ago, the investigation of the behavior of financial markets has become one of the primary and most thriving fields in economic research, which has led to a huge variety of work being published in economic literature and in related fields. This field of research has two main aims. Firstly, it seeks to systematically observe diverse and, particularly, universal characteristics in different types of financial markets, such as commodity markets, precious metal markets, stock markets, foreign exchange markets or derivative markets. Secondly, it attempts to find substantiated theoretical explanations for a multitude of related phenomena through the development of models and generally accepted theories. Notable examples of these, though by no means an exhaustive list, are the following seminal and outstanding publications:

L. BACHELIER, who, in his thesis entitled ‘Théorie de la Spéculation’, conducted both theoretical work on stochastic processes, most notably the first formulation of a theory of random walks, and empirical analysis of actual market data (see Bachelier, 1900);

M. G. KENDALL, who was the first to find empirical evidence that stock prices follow a random walk (see Kendall, 1953) and M. F. M. OSBORNE, who rediscovered the Brownian motion for stock markets (see Osborne, 1959);

P. A. SAMUELSON, who replaced the assumptions proposed by the random walk hypothesis with the more general ‘martingale property’ and particularly linked this to the framework of economic pricing equilibria (see Samuelson, 1965);

E. F. FAMA and B. MANDELBROT, who systematically investigated the behavior of daily stock and cotton prices (see Fama, 1965; Mandelbrot, 1963*b*, 1967), disproved the assertion of normal distributed daily returns as proposed by the random walk model (see Fama, 1963; Mandelbrot, 1963*b*) and proposed the Lévy stable distributions as an alternative model (see Mandelbrot, 1963*a,b*);

E. F. FAMA, who reviewed in his classic survey for the first time the theory and empirical work on the ‘Efficient Market Hypothesis’, which mainly states that prices reflect all available information on the underlying asset at any point of time, and that they respond instantaneously and in an unbiased fashion to any new asset information entering the market (see Fama, 1970);

F. BLACK, M. SCHOLES and R. MERTON, who derived a theoretically based pricing rule for options and corporate liabilities, which enabled the regular trade of such instruments on

financial markets and which can therefore be regarded as the birth of modern finance (see Black and Scholes, 1973; Merton, 1973*b*, and also Black, 1976a);

J. M. HARRISON, D. M. KREPS and S. R. PLISKA, who formulated and proved the ‘Fundamental Theorem of Asset Pricing’, in which the economic concept of arbitrage-free markets is equivalently linked to the mathematical property of the existence of an ‘equivalent martingale measure’ or ‘risk-neutral measure’ (see Harrison and Kreps, 1979; Harrison and Pliska, 1981, 1983). Their general theory of arbitrage-free pricing of derivative instruments using the martingale formalism also opened up the asset pricing theory and the investigation of financial markets to a research field of stochastics;

R. F. ENGLE and C. W. J. GRANGER, who won the Nobel prize in Economics in 2003 for developing methods to analyze economic time series with time-dependent volatilities (ARCH) (see Engle, 1982, and for a further generalization see Bollerslev, 1986) and non-stationary variables, nowadays known as the concept of ‘cointegration’ (see Granger, 1981, 1986; Engle and Granger, 1987).

The above have all led to a generally better and fundamentally new understanding of the behavior of financial markets and have encouraged other researchers to follow them in these newly discovered research directions. Moreover, some of the ideas and techniques originally used for investigating financial markets have gained influence in other areas of economics as well as in different, adjoined sciences.

It might seem remarkable that, while most of this work investigating financial markets is motivated by essentially *economic* problems, many of these publications also come from non-economists such as stochastics, mathematicians and physicists, as well as from scientists in different applied research fields. As a physicist without a formal education in economics, it is of great interest to me that the very notion of physicists contributing to financial research is actually as old as the two separate fields themselves (see Voit, 2003, for a more detailed historical review). Indeed, besides the work of I. NEWTON and C. F. GAUSS relating to financial problems, it is generally accepted that the French mathematician L. BACHELIER’s thesis of 1900 (see above) is one of the first and most important pieces of work on the behavior of financial markets. In his thesis, Bachelier developed and formulated the ‘Random Walk Process’ for the first time, which is nowadays regarded as one of the best-known concepts in both statistical physics and quantitative finance. Hence it now serves as a prominent example of a shared concept which is well established and widely used in both research fields, albeit applied to different problems. Moreover, in the last 10 years it seems that using models, concepts and techniques from physics (and of course from mathematics) to describe phenomena in financial markets and other socio-economic systems has become even more popular for a growing group of physicists. Proof of this popularity can be seen in the marked increase in finance-related publications contributed by physicists in recent years, as well as by the coining of the term ‘Econophysics’ (for a textbook treatment of Econophysics-related topics, see Mantegna and Stanley, 2000; Bouchaud and Potters, 2003; Voit, 2003, as well as Levy, Levy and Solomon, 1994; Mantegna and Stanley, 1994; Levy, Levy and Solomon, 1995; Mantegna and Stanley, 1995; Levy and Levy, 1996; Stanley, Amaral, Buldyrev, Havlin, Leschhorn, Maass, Salinger

and Stanley, 1996; Mantegna and Stanley, 1997; Stauffer, de Oliveira and Bernardes, 1999; Cont and Bouchaud, 2000; Cont, 2001; Bouchaud, Matacz and Potters, 2001; Lux and Sornette, 2002; Plerou, Gopikrishnan, Rosenow, Amaral and Stanley, 2000 for a truly modest selection of related paper publications strongly restricted by my own knowledge).

It is vital to stress that there is a huge variety of publications from mathematicians, which have also led to a better understanding of the behavior of financial markets and to a new strand of research activities within the applied mathematics community, nowadays called ‘Mathematical Finance’. Nevertheless, as I am no expert in this strand of literature, the only example cited in this thesis are those of Harrison and Kreps (1979); Harrison and Pliska (1981, 1983) (see above). Whilst this brief introduction prevents me from going into more comprehensive detail, it should nevertheless have become clear that the investigation of financial markets and their empirical data is largely based on concepts stemming from mathematics, and on techniques used and formulated by both economists and non-economists. These methods would be employed, for example, to identify typical events or characteristic processes in the market, or to find explanations for financial time series phenomena and to find modeling approaches for certain observed behavior in financial markets.

However, there is also considerable and growing criticism of the increased and almost exclusive use of mathematical (or ‘quantitative’) concepts and techniques to investigate and model financial markets. Indeed, there is broader criticism of the overreaching mathematical character of economic theories applying to a wider spectrum of economic research fields, such as game theory or the principal elements of macroeconomics. In this sense it could be seen that economists are currently faced with rather general criticism. Nevertheless, it is the investigation and modeling of financial markets which is presently being criticized most fiercely. There are two main reasons for that. On the one hand, the quantitative concepts in the financial industry, developed and implemented with highly sophisticated methods, have exacerbated the currently active financial crisis, which will likely impact dramatically on the worldwide economy. Some critics would even say these concepts caused the crisis, along with other factors. On the other hand, it is becoming obvious that the current financial crisis, with its catastrophic and disastrous implications for the individual, cannot be explained by the traditional economic theories which are still mainly based on mathematical concepts, such as ‘market efficiency’ and ‘market rationality’. Therefore, critics argue that using a purely quantitative approach to explain the behavior of financial markets, or the economy as a whole, neglects the fact that almost none of the observed fundamental economic events can ever be fully explained if the psychologically motivated component of those events is disregarded. Instead, an economic theory should also include factors such as confidence, reliance, suspiciousness, fairness, deception and fraud, because such factors seem to play a greater role in economically based decision making processes than a purely quantitative approach pretends. Some critics even argue that the increasing and decreasing fluctuations in the economy are mainly based on the interplay between an exceeding perception and an absence of confidence. Thus confidence is the main driver for economic activities (see R. J. SHILLER in Akerlof and Shiller (2009) for a very prominent and up-to-date example of this statement).

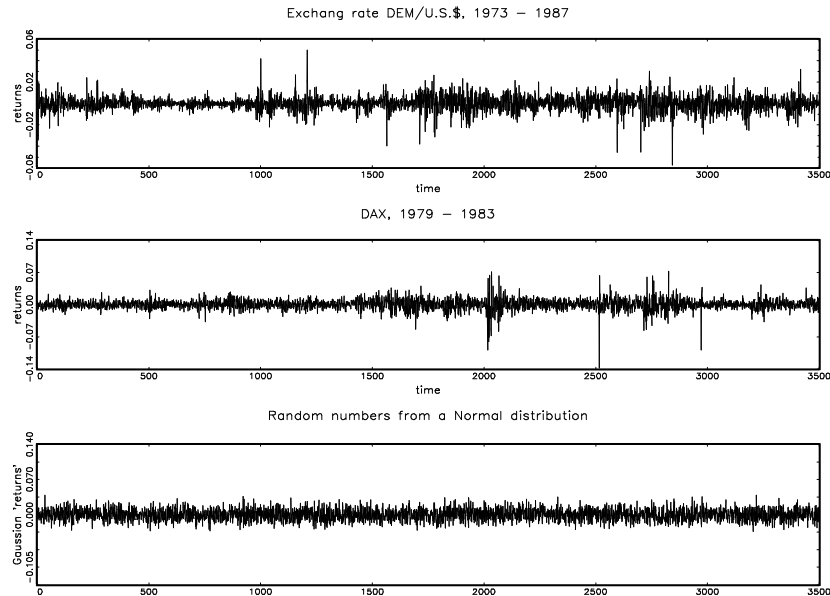
Although this thesis will also mainly build on mathematical concepts and techniques and can thus be criticized for the same reason to some extent, it aims, at least, to improve the traditional economic theories by investigating whether nonlinear economic laws can explain a significant part of observed (irregular) fluctuations in economic and financial markets. In particular, different types of market participants will be considered, whose heterogeneity seems to be one of the driving economic mechanisms of the price dynamics in financial markets.

### Motivation for the Thesis

Despite the general belief in the fundamental economic principle of balancing demand and supply in different kinds of economic markets, very few economists or market participants would question the need for and the usefulness of a basic understanding of the behavior and the dynamics of modern day financial markets. In this context, one of the main interests is focused on the fluctuations of prices (or relative price changes) observed in financial markets, which seem to be more volatile than would be expected by the standard paradigms of asset price dynamics based on the geometric Brownian motion price process and on the long established assumption that economic agents have homogeneous beliefs. For this reason, different approaches now coexist emphasizing different viewpoints to explain the observed fluctuations in financial markets.

Classic examples of these are the concept of ‘Rational Expectations’ (RE) and the ‘Efficient market Hypothesis’ (EMH). The RE concept states that asset prices are completely determined by their fundamentals, meaning that asset price fluctuations can only be traced back to exogenous, randomly driven shocks to fundamentals. As a consequence, prices converge to a steady-state like growth path if no shocks enter the market. Furthermore, ‘irrational’ traders using simple speculatively driven forecasting rules to determine asset prices and ignoring information about the underlying asset’s fundamental value should eventually be driven out of the market by rational traders as already argued by M. FRIEDMAN (see Friedman, 1953). Closely related to this concept of rational expectations is the EMH. It states that current prices already contain all the available information on the underlying asset and therefore cannot be used to predict future asset prices. Examples of stochastic processes consistent with this theory are the random walk processes as well as the ARCH and GARCH processes, which have been extensively investigated in the empirical literature. However, observations in real markets give reason to doubt that these paradigms can fully explain the behavior of asset prices. For instance, if we look at the stock market ‘Millennium Boom’ starting in the late 1990s and continuing into the early years of the next decade, the value of all major stock indices rises dramatically (more than 100 percent!) and decreases accordingly in the following. Unfortunately, neither the RE concept nor the EMH could deliver a satisfactory explanation for the Millennium Boom. Furthermore, detailed investigation of statistical properties of prices of stocks, commodities and market indices on different kinds of markets during the last 40 years has discovered a set of properties for asset *returns*, termed ‘stylized facts’, which are common across many instruments, markets and time periods (see Pagan, 1996; Cont, 2001, and references given therein for details). Interestingly, these stylized facts are partly inconsistent with the geometric Brownian motion price process favored by the classic viewpoints mentioned above. Examples of these inconsistencies are the absence of linear autocorrelations of asset returns, fat-

tailed asset return distributions with clustered volatility and slow decay rates of the autocorrelation of absolute and squared returns (see Figure 1 for an illustration of some of these features).



**Figure 1:** Time series of returns generated from empirical and ‘synthetic’ data. Both daily returns of the Deutsche Mark vs. U.S. Dollar exchange rate (upper panel) and the German stock price index DAX (middle panel) form a higher concentration of large returns (fat tails) and switches between tranquil and volatile sequences (cluster volatility) compared with Gaussian distributed returns (lower panel). The plot can originally be found in Lux and Heitger (2002).

Therefore, a different, more modern view on asset price fluctuations also considers the *interaction* of market participants as well as herding effects influencing the trading process. Associated to this idea, different trader types pursuing different trading strategies and thus forming different expectations of future price movements do coexist in the market regularly, which might lead to nonlinear price dynamics even in the absence of any external shocks. Proponents of this viewpoint hold these mechanisms mainly responsible for the high volatility observed in financial markets, a view which has become popular in the literature under the notion of ‘Animal Spirits’ or ‘Market Psychology’ and ‘Heterogeneous Expectation Hypothesis’. Indeed, in contrast to Friedman’s assertion mentioned above, De Long, Shleifer, Summers and Waldmann (1990) show that traders who do not use or who misperceive the fundamentals can survive in financial markets. More precisely, they show that a constant fraction of ‘noise traders’ (i.e., irrational traders) may on average earn a higher expected return than ‘smart traders’ (i.e., rational traders). Indeed, the noise traders may survive in the market with positive probability. Interestingly, one of the most prominent quotes supporting this result stems from J. M. KEYNES, a long time before the RE concept and the EMH was even formulated and became a popular approach in finance. He stated (see Keynes, 1997,

p. 157):

### **Keynes's Beauty Contest Analogy of the Stock Market**

Investment based on genuine long-term expectation is so difficult [...] as to be scarcely practicable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and, given equal intelligence, he may make more disastrous mistakes.

(Keynes, 1936)

Obviously, this statement is in stark contrast to the EMH and instead emphasizes the role of non-fundamental analysis in financial markets. Keynes argued that stock prices are less determined by an objective view of fundamental values than by 'what average opinion expects average opinion to be'.

In recent years, the debate about the competing kinds of volatility concepts has led to prospering research activities, resulting in a broad strand of literature that has developed in different directions. Notable publications from the 1980s investigating the influence of non-fundamental trading activities are Shiller, Fischer and Friedman (1984); Frankel and Froot (1986, 1990). Important empirically motivated contributions to this discussion which are based on survey data are Frankel and Froot (1987); Froot and Frankel (1989); Allen and Taylor (1990); Taylor and Allen (1992). Furthermore, since the beginning of the 1990s the number of related publications has soared, with a large part of this work using the structural agents model approach, in which the market dynamics is characterized by the interaction of individual heterogeneous agents. These kind of artificial financial market models can be classified into two groups, namely the more theoretical and the more computational orientated models. As it happens, most of them are computational orientated, due to the tremendous increase in computer power and the increasing affordability of appropriate computing technology in recent years. Prominent examples of computer simulated market models are established by Arthur, Holland, LeBaron, Palmer and Tayler (1997); Chen and Yeh (1997, 2002); Cont and Bouchchaud (2000); Day and Huang (1990); De Grauwe and Grimaldi (2006*b*); Farmer and Joshi (2002); Gaunersdorfer and Hommes (2007); LeBaron, Arthur and Palmer (1999); LeBaron (2000); Levy et al. (1994, 1995); Levy and Levy (1996); Levy, Persky and Solomon (1996); Lux (1995, 1997, 1998); Lux and Marchesi (1999, 2000); Kim and Markowitz (1989); Kirman (1991, 1993); Palmer, Arthur, Holland, LeBaron and Tayler (1994); Westerhoff (2003); Wilpert (2004). Although the study of agent-based models by computer simulations is a promising and prosperous way of analyzing complex adaptive systems of many groups of heterogeneous agents, it is potentially flawed because it does not always seem obvious how certain generated simulation results can be traced back to the economic 'ingredients' implemented in the computer model. Therefore, slightly different approaches try mainly to build on a more theoretical framework, combining an analytical and numerical treatment, to analyze both simple and more complex versions of their artificial market models (see Brock and Hommes, 1997*b,a*, 1998, 1999; Brock, Hommes and Wagener, 2005; Chiarella and He, 2001, 2002*a,b,c*; Chiarella, Dieci and Gardini, 2006; Diks and von der Weide, 2002; Gaunersdorfer, 2000; Gaunersdorfer, Hommes and Wagener, 2003, 2008).



The research proposed in this thesis is largely motivated by the recent work of Brock and Hommes (1997*b,a*, 1998, 1999), Levy et al. (1994, 1995); Levy and Levy (1996); Levy et al. (1996) and in particular by the work of Chiarella, Dieci and Gardini (2001); Chiarella and He (2001, 2002*a,b,c*); Chiarella et al. (2006), which studies the dynamics of financial asset prices resulting from the interaction of heterogeneous agents that have different expectations about the future evolution of prices. The aim of this thesis is twofold. Firstly, I will analyze the asset price dynamics of an extended version of the market model with heterogeneous agents under a Walrasian market clearing scenario proposed by Chiarella and He (2001). Allowing agents for different risk-aversion coefficients and assuming trading strategies that slightly differ from the original publication, I shall investigate the impact of these newly introduced model variants on the overall dynamic behavior of the original model framework. Secondly, I will use these results to propose a market model under a market maker scenario which brings the model framework of the Chiarella and He model to a more consistent model structure. For this improved model configuration, I will select stylized representations of different types of market participants, namely fundamentalists and chartists, and investigate how the heterogeneous expectations of investors drive economic mechanisms of the price and how they drive investors' wealth dynamics. I will show to which extent those heterogeneous expectations in the agent-based market model can explain observed fluctuations in real financial markets and lead to the emergence of complicated dynamics of growing asset price paths. Overall, by proposing and analyzing a novel variant of a structural heterogeneous agents model, the thesis aims to provide further useful insight and intuition into the understanding of the behavior of financial markets.

### Structure of the Thesis

The thesis is composed of four chapters divided into two parts. It also includes four excursions which are relegated to the appendices of the thesis. As might be expected from a typical 'Dissertationsschrift', it includes three main components all related to the topic:

- the preparation of relevant concepts
- a literature review
- my own contributions providing new results.

The first part, consisting of Chapter 1 and Chapter 2 gives the reader a comprehensive introduction to the basic concepts used in the remainder of the thesis and reviews the relevant literature. Chapter 1 is devoted to traditional and widely accepted approaches for modeling financial markets and also considers these from a historical perspective. This chapter mainly focuses on the concept of rational expectations equilibria (i.e., RE, see above) as well as on the related concept of the efficiency of a market (i.e., EMH, see above), which can be regarded as two of the fundamental paradigms of traditional finance theory. The appendix of Chapter 1 closes with a first excursus, which gives a description of the standard expected utility maximization scheme. The standard expected utility maximization scheme will be one of the main building blocks of the artificial behavioral market models outlined in the second part of the thesis. Chapter 2 gives an overview of two prominent and representative examples of behavioral models of financial markets

proposed by Brock and Hommes (1997*b,a*, 1998, 1999) and Levy et al. (1994, 1995); Levy and Levy (1996); Levy et al. (1996) respectively. Both model frameworks clarify the interplay of both traditional ideas (i.e., RE and EMH) with more current concepts of modeling financial markets, each of them highlighting slightly different facets of modern concepts in their approaches. The study of the framework of the Brock and Hommes (BH) model as well as of the Levy, Levy and Solomon (LLS) model also helps to give early insights into the specific class of behavioral models on which my thesis is concentrated. It also serves as a motivation and as a starting point for my own model framework, which will be proposed in the next part of the thesis. Additionally, I will propose and analyze my own model variant according to the Brock and Hommes model, which brings the original model framework to a more consistent model structure. Apart from my own contribution mentioned above, the first part of the thesis includes the ‘result-invariant’ part of the thesis. Finally, three further excursions on selected topics in nonlinear dynamics are provided at the end of the appendix of the second chapter, which are mainly intended to familiarize readers with some theoretical background and useful techniques in nonlinear dynamics that will also be applied in the following part of the thesis.

The second part, consisting of Chapter 3 and Chapter 4, largely builds on the asset pricing model with heterogeneous agents established by Chiarella and He (2001), which is mainly based on the ideas of the BH model mixed with some assumptions made in the LLS model framework. In these models, heterogeneous economic agents are faced with a standard asset allocation problem in each trading period, where heterogeneity among agents is expressed in terms of different perceptions of the expectations of future price movements. Besides a comprehensive model description of the original Chiarella and He (CH) model, Chapter 3 focuses on a model extension in which agents are allowed to have different attitudes towards risk according to their investment decisions integrated in their CRRA utility function. In the course of an extensive sensitivity analysis with respect to the risk parameter using both analytical and numerical tools, the level of functionality of the original model framework is inspected by means of testing the plausibility of the model structure and reproducing important and already published results for the enriched model. Chapter 4 contains the core chapter of this work and proposes a simple asset pricing model with heterogeneous investors under a market maker scenario. The model framework is developed by including concepts of the three artificial market models mentioned above (i.e., the BH model, the LLS model and the CH model), but also incorporates new ideas partly motivated by the conclusions drawn from the analysis of the enriched CH model outlined in Chapter 3. One of the new ideas includes the explicit consideration of a risky-asset supply side within the model framework. This extension in the model structure allows for modeling the risk premium demanded by the market participants for taking market risk which appears to be endogenously driven by the market over time. After a detailed model description of this proposed market maker model, a comprehensive analytical as well as numerical treatment of both homogeneous and heterogeneous model designs is performed, giving new insights into the understanding of asset price and wealth dynamics in agent-based models with heterogeneous expectations.

The thesis closes with a concluding summary of the main results of this thesis and also indicates some possible routes for future research.

## **Part One**

### Basic Concepts and Literature Review

## CHAPTER 1

### Prologue: Three Paradigms of Economic Modeling

This chapter is subdivided into five sections. After some preliminary remarks on the different use of the idea of expectations in economics and natural science, three crucial paradigms of economic modeling are proposed in the subsequent sections of the chapter. The ‘Efficient Market Hypothesis’ has been the basic building block of financial theory for nearly thirty years since its first formal development in the mid 1960s, while the concept of rational expectations first formulated approximately in the same decade has become an indispensable tool in almost all fields of economics. The third paradigm of economic modeling is the ‘Representative Agent Hypothesis’. Its importance in financial theory is emphasized by its being one of the cornerstone assumptions in the prominent Capital Asset Pricing Model (CAPM). This model is therefore briefly summarized in this context. Although for many economists these paradigms seem nowadays to be too restrictive to build a modern economic and finance theory, they can still serve as an important benchmark in the sense that more realistic and very often more complex market scenarios should still contain the homogeneous and rational expectation schemes as a special subcase. This is because, in spite of their limited ability to explain realistic market scenarios, they are generally more tractable analytically and therefore might help the researcher to better understand even the more complicated dynamics of the more comprehensive underlying system. Thus the three hypotheses are still helpful as benchmarks precisely *because* of their limitations. The chapter concludes with a summary.

#### 1.1 Expectations in Economics and Physics

Before the three paradigms of financial modeling are outlined in the following sections it might be worth making some basic remarks on the importance of the idea of expectation formation in economic modeling since, at first glance, it very clearly displays one of the key differences between economics and natural science. For this reason let us think for a moment about the equation of motion in physics, which is used to describe the behavior of the dynamic process in mathematical terms. In almost all cases it will never depend on expectations of future developments of the relevant dynamic variables. However, in economics the dynamic process might indeed depend on expectations as, for instance, in financial theory the decisions of agents today depend on their expectations or ‘beliefs’ about the future. For instance, investment decisions are very often determined solely by expectations of high future profits. The Dutch ‘tulip mania’ can serve as a prominent and early economic example of this phenomenon. In 1636 the price of Dutch tulip bulbs exploded by a factor of more than 20 within a short space of time, until it fell back to its

former ‘realistic’ price level at the end of the year, when it had become clear that the dramatic increase in tulip prices had only been caused by the investors’ dreams and beliefs of invariably increasing returns on their investments in tulip bulbs. But even nowadays, over-evaluation of certain asset prices and overreaction of certain investors can be detected at almost any time in financial markets. Hence it can be concluded that a theory of expectation formation is an essential component of any economic model or theory, which is totally untrue for models in classical physics, such as the KEPLER’s Law or the MAXWELL’s Laws of Electrodynamics.

But even though this statement has to be treated with much greater care when regarding modern physics, crucial differences in the understanding of the notion of expectation still remain. In modern physics the idea of expectation is indeed well established, but it is *exclusively* understood as the mathematical or, in other words, statistical expectation value that agrees with our intuitive notion of ‘average measured value’. Of course, such an expectation value really does play a crucial rule in modern physics; the whole theory of Quantum Mechanics, for example, which represents the birth of modern physics, is based on this concept. However, such a narrow view of expectations in economics coincides with the concept of ‘rational’ expectations, which is just a specific case of the term ‘expectations’ used in economics. In other words, the notion of expectation in economics can generally have a broader meaning, since the expectation formation mainly accounts for modeling agent’s behavior or his/her decision making process, which inevitably need not be rational. I will refer to this interesting fact in the following section in greater detail.

## 1.2 Rational Expectation Hypothesis (REH)

The concept of rational expectations was first introduced to economists in 1961 by J. F. MUTH<sup>1</sup>, whereas the basic idea of the rational expectation hypothesis had already been established seven years earlier in a paper written by Grunberg and Modigliani (1954).<sup>2</sup> Although there is an overall consensus on the enormous importance and capacity of rational expectations in both microeconomic and macroeconomic theory, there are today still verbally and formally slightly different versions of the Rational Expectation Hypothesis, some of which are quoted in the following discussion. As a first definition we can recapitulate Muth’s version of rational expectations as it is literally stated in Redman (1992):

### Definition 1.1 (Muth’s Version of Rational Expectations)

Expectations can be represented by the subjective probability distribution and tend to be distributed about the prediction of the theory, that is, about the objective probability distribution of outcomes. In other words, the weighted arithmetic mean of the expectations is equal to the prediction of the relevant economic model. The expected values of the variable to be forecast and the actual variables of the variable *have a common mean value*.

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<sup>1</sup>See Muth (1961).

<sup>2</sup>For this statement and the following discussion see Redman (1992) and Sheffrin (1983). There is an exhaustive bibliography of the literature on rational expectations in the latter.

Muth therefore claims that, under rational expectations, agents do not make any systematic forecasting errors, since agents' subjective expectations are, on average, equal to the true model variables.<sup>3</sup> Note that in this definition, Muth restricts his consideration to the expectation (i.e., first moment) of the probability distribution of a random variable. Obviously, higher moments are negligible in this version of the REH.

While Muth initially applied this concept to microeconomic problems and specifically to the cobweb model, in macroeconomics the REH was adopted for the first time about ten years later (Walters, 1971; Lucas, 1972*a,b*; Sargent and Wallace, 1975; Lucas, 1977). Subsequent work on macroeconomic problems during the 1970s was mainly associated with the name of R. E. LUCAS, who is regarded as the main force in popularizing the concept of rational expectations in macroeconomics.<sup>4</sup> In the course of this development, a second, slightly different definition of rational expectations was proposed, which is attributed to Lucas and Prescott (1971).<sup>5</sup> According to Redman (1992), they give the following definition:

**Definition 1.2 (Narrow Version of Rational Expectations)**

Subjective expectations held by economic agents are the same as the conditional mathematical expectations of the true probability model of the economy. In other words, agents' subjective probability distributions coincide with the objective probability distribution of events. The expected values of the variable to be forecast and the actual values *have the same probability distribution*.

Thus in contrast to Muth's definition, the subjective distribution (and not its mean) is exactly equal to the true conditional distribution based on all available information at time  $t$ .

For the sake of completeness, I will offer a third and more generalized REH version from Redman (1992):

**Definition 1.3 (Weak Version of Rational Expectations)**

Rational agents form expectations by acquiring and using information to the point where expected marginal cost and the expected marginal benefit of gathering and using this information are equal.

One major drawback immediately becomes obvious with this definition. As forecasts need not to be on average correct, the weak version of the REH is not free from systematic error. For further reading on the different definitions of the REH and their implications on economic modeling we can refer once again to Redman (1992). With this reference, I would like to close the review of alternative versions of the REH and now stress the main logic of the concept of rational expectation.

In the previous statement, Muth realized that economic models are extremely sensitive to the expectation formation of agents who act in these models. Hence Muth's basic idea was to link the beliefs of individual economic agents to the actual behavior of the underlying economic model.

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<sup>3</sup>See also Muth (1961, p. 316) for the original statement.

<sup>4</sup>In the 1970s traditional macroeconomic models lost their predictive power because of the breakdown of the Phillips curve and the arising phenomenon of stagflation.

<sup>5</sup>See also Lucas (1978).

He proposed that agents should exploit their knowledge of the economic model to form their expectations with respect to the relevant variables of the model. He called such an expectation formation ‘rational’, as it allows for the economic system to behave consistently according to the agents’ individual beliefs. Let us state this idea now in a more formal way. In statistical notation, Muth’s REH can be expressed in terms of conditional, mathematical expectation values in the following manner:

**Definition 1.4 (Statistical Notation of Rational Expectation)**

Let  $\mathcal{I}_t$  be the information set about the underlying economic model at time  $t$ . Then the individual, subjective expectation about a stochastic, economic variable  $X_{t+1}$  formed at time  $t$  denoted by  $X_{t+1}^e$  equals, on average, the true value of this variable conditioned on the set of total information available at time  $t$ . Mathematically, this can be expressed by

$$X_{t+1}^e = E_t[X_{t+1}], \quad (1.1)$$

where  $E_t[X_{t+1}] \equiv E[X_{t+1} | \mathcal{I}_t]$  is the expectation value of a stochastic variable  $X_{t+1}$  given the information set  $\mathcal{I}_t$  at time  $t$  into.<sup>6</sup>

There are two important properties related to conditional expectations. On the one hand, the conditional forecast error, which is defined as

$$\epsilon_{t+1} = X_{t+1} - E_t[X_{t+1}],$$

can be shown to be zero on average, as it is

$$E_t[\epsilon_{t+1}] = E_t[X_{t+1}] - E_t[X_{t+1}] = 0.$$

On the other hand, the forecast error is uncorrelated with all the information given at time  $t$  (and earlier), which is known as the ‘orthogonality property of conditional expectations’, and can be mathematically formalized as

$$E[\epsilon_{t+1} | \mathcal{I}_t] = 0. \quad (1.2)$$

Thus, from these properties of conditional mathematical expectations, it follows that, under rational expectations (according to Definition 1.4), agents’ expectations are an unbiased forecast of the economic stochastic variable,<sup>7</sup> and that the information set  $\mathcal{I}_t$  cannot be used to improve the forecast. Finally, it is worth mentioning that the REH does not imply perfect foresight. Of course, if there were no uncertainty, rational expectations would coincide with perfect foresight, which would lead to unrealistic economic outcomes. Instead, in a given economic environment, rational expectations generate model outcomes which equal these expectations only on average. Thus the REH implies that deviations from agents’ expectations can solely be caused by non-anticipated (or unpredictable) uncertainty of the economic model. In this sense the REH is closely connected to the EMH as we will see in the next section.<sup>8</sup>

<sup>6</sup>For a formal mathematical definition of conditional probabilities, see Schmitz (1996) or Feller (1971).

<sup>7</sup>This property is also incorporated in Definition 1.1.

<sup>8</sup>For a more detailed discussion about the mathematical version of the REH according to Definition 1.4, see Sheffrin (1983) and Cuthbertson (1996).

Two further remarks are worth mentioning here. First, the REH is not only valid in economic models with homogeneous agents, i.e., where all agents use exactly the same expectation formation rule. In fact, the agents' expectations simply have to be distributed about the true value, so that no systematic forecasting errors exist and the average of all individual expectations equals the expected true value given by the model.

Second, in contrast to the perfect, deductive rationality outlined above, there is a rapidly growing new strand of literature on 'bounded rationality'<sup>9</sup>, where agents are assumed to be provided with limited knowledge and limited cognitive capacity and thus only behave in a manner that is at most 'nearly optimal'. Indeed, according to Arthur (1994) there are two main problems that come along with the assumption of perfect or deductive rationality. On the one hand, it is conceivable that the complexity of the underlying problem, e.g., given in terms of the form of an economic system, can easily exceed the agents' ability to cope with to a fully high degree. On the other hand, in interactive situations of sufficient complexity, the agents cannot fully believe that all other agents form their beliefs under perfect rationality. As a consequence, they have to guess the other agents' behavior. This brings the agents themselves away from totally deductive reasoning and might force them to form subjective beliefs. In this sense, people act only 'partly rationally' and are in fact emotional (or irrational) in the remaining part of their actions. To summarize, bounded rationality deals with the fact that agents often do not have a perfectly logical approach to making their decisions but also form (partly) subjective beliefs, e.g., from a pool of (slightly) different models or hypotheses. This also implies that agents are allowed to use learning models for their expectation formation, which enables them to occasionally replace their poorly performing beliefs. Furthermore, the concept of bounded rationality is concerned with the ways the actual decision-making process influences decisions and is a central theme in behavioral economics.

### 1.2.1 Rational Expectations Equilibrium

As we have seen above, the concept of rational expectations takes the aspect of aggregate information into consideration (see Eq. (1.1)). Interestingly, even though the REH has primarily been applied to microeconomic problems, the idea of aggregating information had been ignored in microeconomic theory for a long time, until its potential capability was recognized for macroeconomic problems in the 1970s.

Conventional microeconomic theory does not take the informational issue into account. For instance, in the standard model of a pure exchange economy, where agents are characterized by their preferences and their initial endowments, prices are determined simply by eliminating the excess demands caused by the agents' buy and sell orders. This situation does not change even when uncertainty is introduced. Thus in the conventional microeconomic model, the economic agents ignore the information of the other agents' beliefs, which is potentially revealed through the market price. Instead, the agents are assumed to be myopic, i.e., they simply determine their demands under uncertainty and without using the aggregate information about the probability assessment of all economic agents for a certain event which is embodied in the price system.<sup>10</sup>

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<sup>9</sup>H. SIMON coined the notion of 'bounded rationality' in 1957 (see Simon, 1957). See also Simon (1955) for an earlier publication and Simon (1982) for an exhaustive paper collection.

<sup>10</sup>For more detailed remarks of this statement and the following discussion see also Sheffrin (1983).



This perception changed in the mid-1970s, particularly in connection with a simple model proposed by S. J. GROSSMAN<sup>11</sup>, in which the agents are assumed to use the information revealed by the market price. The model consists of  $I$  traders which can choose between two assets, one risk-free asset and one risky asset. The price of the risky asset at time  $t + 1$  is uncertain, i.e., a random variable, and denoted by  $p_{t+1}$ . At time  $t$  every agent receives information  $\mathcal{I}_{i,t}$  about the true price  $p_{t+1}$  realized in  $t + 1$ . The information is normally distributed around the actual price  $p_{t+1}$  and is different for every agent, which represents the agents' uncertainty. In the naive equilibrium in which the agents do not exploit the fact that the market price does reveal *additional* information from their counterparts, all agents determine their demand  $N_i$  of the risky asset by maximizing their utility function for the time period  $t + 1$  with respect to their information  $\mathcal{I}_{i,t}$  received one period before. Hence, the market clearing price  $p_t$  is determined by the naive equilibrium condition given by

$$\sum_{i=1}^I N_i [p_t | \mathcal{I}_{i,t}] = N,$$

where  $N$  denotes the fixed supply of the risky asset.

In a next step, Grossman (1976) assumed that the agents do use the market price for acquiring other agents' information. Obviously, the equilibrium condition now changes, as the demand  $N_i$  of each agent now depends also on the market clearing price  $p_t$  which, in turn, reveals all the available information. In such a situation, the market clearing price can be determined by solving

$$\sum_{i=1}^I N_i [p_t(\mathcal{I}_{1,t}, \dots, \mathcal{I}_{I,t}) | \mathcal{I}_{i,t}; p_t(\mathcal{I}_{1,t}, \dots, \mathcal{I}_{I,t})] = N. \quad (1.3)$$

Under the assumption of a CARA utility function<sup>12</sup> for all agents and jointly normal distributed and uncorrelated agents' prediction errors, Grossman (1976) showed that such a price function  $p_t$  exists. Furthermore, it has the property of being a 'sufficient statistic' for all the information. In other words, the solution of Eq. (1.3) contains all the information of the economy to predict  $p_{t+1}$ , which implies that all agents' individual information  $\mathcal{I}_{i,t}$  can be neglected as it does not reveal any more information than  $p_t$  itself.<sup>13</sup> Grossman named this equilibrium a 'rational expectations equilibrium', whose solution is a price that is a function of all the underlying information available in the economy.

### 1.3 Efficient Market Hypothesis (EMH)

As in the case of the REH, there is an entire collection of alternative definitions of the Efficient Market Hypothesis, which I do not want to discuss extensively in this outline. Instead, I would again like to stress the basic idea of the EMH, that is perhaps sometimes hidden behind all the different captious or at least technical ways of expressing market efficiency.<sup>14</sup> The discussion will

<sup>11</sup>See Grossman (1976, 1978).

<sup>12</sup>The term CARA denotes 'constant relative risk-aversion'. See Appendix A 1.3 for details.

<sup>13</sup>In Grossman (1978), some of the assumptions could be weakened without losing the main results obtained in Grossman (1976).

<sup>14</sup>For a prominent discussion of a proper definition of market efficiency, see the debate between LeRoy and Fama (see LeRoy, 1976; Fama, 1976a).

mainly follow the review article of LeRoy (1989) and the references given therein.<sup>15</sup>

Nowadays, every modern definition of the EMH mainly consists of two basic statements, namely that

- (i) prices reflect all available information about the underlying asset
- (ii) prices respond instantaneously and in an unbiased fashion to any new information about the asset.

This was formulated for the first time by E. F. FAMA<sup>16</sup> in his classic survey from 1970:

**Definition 1.5 (Fama’s Definition of the EMH)**

A market in which prices always ‘fully reflect’ the available information is called *efficient*.

Obviously, this definition is quite compact and leaves room for interpretation as hinted by the quotation marks around the expression ‘fully reflect’. For instance, without specifying the underlying price and information process, it is hardly possible to verify this assertion empirically.

For a long time, market efficiency was closely linked to the random walk model<sup>17</sup> and serves as a theoretical background for the statistical behavior of asset prices, whose origin can be traced back to an eminent PhD thesis composed by L. BACHELIER<sup>18</sup> at the very beginning of the 20th century. The empirical analysis began in the 1930s with the work of H. WORKING and A. COWLES.<sup>19</sup> While Working claims that commodity and stock prices resemble cumulations of purely random changes, Cowles focused attention on the use of fundamental analysis, finding that there was little evidence that market analysts and financial services outperformed the market. Nevertheless, it is generally held that the first appearance of the ‘Random Walk Hypothesis’ is associated with a publication by M. G. KENDALL<sup>20</sup>, who seriously examined the assertion that stock prices follow a random walk. In his statistical study, he found that weekly changes in a wide variety of financial prices are practically independent from former price changes and thus cannot be predicted from either past changes in the series or from changes in other price series. According to these findings,

<sup>15</sup>For a textbook-like treatment, see Fama (1976*b*); Sheffrin (1983); Cuthbertson (1996); Campbell, Lo and MacKinlay (1997) or Lo and MacKinlay (2002).

<sup>16</sup>See Fama (1970). For a more recent publication on the EMH by the same author see also Fama (1991).

<sup>17</sup>A stochastic variable  $X_t$  is said to follow a random walk (without drift), if  $X_{t+1} = X_t + \epsilon_{t+1}$ , where  $\epsilon_{t+1}$  follows an ‘independent white noise process’, i.e., it has the following properties

$$\epsilon_{t+1} \text{ is i.i.d. , } \quad E[\epsilon_{t+1}] = 0 \quad \text{and} \quad Cov[\epsilon_m, \epsilon_s] = \begin{cases} \sigma^2 & \text{for } m = s \\ 0 & \text{for } m \neq s \end{cases} . \quad (1.4)$$

Note that it is also possible to consider processes as defined by Eq. (1.4), that lack the i.i.d. property, i.e., processes that only have a zero mean and a finite variance. Such processes are simply called ‘white noise processes’ and might have nonlinear dependence in higher moments. For the random walk (without drift) the mean and variance is given by  $E[X_t] = 0$  and  $Var[X_t] = \sigma^2 t$ . For further details see Campbell et al. (1997) or Franke, Härdle and Hafner (2000).

<sup>18</sup>See Bachelier (1900).

<sup>19</sup>See Working (1934); Cowles (1933, 1944) and Cowles and Jones (1937).

<sup>20</sup>See Kendall (1953). However, note that about 60 years after its first mention by Bachelier, a *theoretical* exposition of the Gaussian hypothesis was made by Osborne (1959), who rediscovered the Brownian Motion for stock markets.

he supported Working's results.<sup>21</sup> More technically, Kendall's conclusion can be summarized by stating that the price itself follow a random walk, which is expressed by

$$p_{t+1} = p_t + \epsilon_{t+1} , \quad (1.5)$$

where  $\epsilon_t$  denotes the price increments, which are assumed to be i.i.d. stochastic variables with mean zero and finite variance. Obviously, Eq. (1.5) implies

$$E_t[p_{t+1}] \equiv E[p_{t+1} | \mathcal{I}_t] = p_t , \quad (1.6)$$

which means that the best forecast for tomorrow's price is the price observed today.<sup>22</sup> In other words, price changes are unpredictable and thus the forecast errors should be zero on average (and uncorrelated with any information  $\mathcal{I}_t$ ) as price changes are only caused by the arrival of 'news' or unanticipated events. This argument shows the close link between the concept of the REH and the EMH, which has already been noted in the previous section.<sup>23</sup>

The modern literature of market efficiency in economics began in 1965 with a theoretical piece of work by P. A. SAMUELSON, who generalized the assumptions of the random walk model and particularly linked it to the framework of economic equilibria.<sup>24</sup> He claimed to have weakened the assumptions connected with the random walk model and replaced it with a martingale process<sup>25</sup>, which still has the property of Eq. (1.6), but with the weaker assumption that for the price increments only  $E[\epsilon_t] = 0$  holds. Thus price changes still remain unpredictable as proposed by the random walk model (and the empirical findings) but martingale models also allow for diverse dependencies in higher moments of the increments and do not restrict their distributions to being identical.

Initially, Samuelson (1965) showed that the martingale property holds for futures prices<sup>26</sup>. Assuming rational expectations by postulating that the futures price  $f(t, T)$  at time  $t$  equals the conditional expected spot price of the underlying asset at the maturity date  $T$ , i.e.,

$$f(t, T) = E_t[p_T] ,$$

he concluded by using the law of iterated expectations<sup>27</sup> that

$$E_t[f(t+1, T) - f(t, T)] = 0 \quad \text{for } t = 1, \dots, T-1 ,$$

<sup>21</sup>Analogous results was founded in an econometric study by Granger and Morgenstern (1963) using spectral analysis.

<sup>22</sup>Quite often the Random Walk Hypothesis can be found to be formulated in the logarithm of prices instead of prices themselves, i.e.,  $\ln p_{t+1} = \ln p_t + \epsilon_{t+1}$ .

<sup>23</sup>For this reason the REH is sometimes even identified with the EMH or, is at least called the 'rational expectations element' of the EMH (see Cuthbertson, 1996).

<sup>24</sup>See Samuelson (1965, 1973). Similar results were also published in 1966 by B. Mandelbrot (see Mandelbrot, 1966).

<sup>25</sup>A stochastic process  $X_t$  is said to be a martingale with respect to a sequence of information set  $\mathcal{I}_t$  if  $X_t$  has the property  $E[X_{t+1} | \mathcal{I}_t] = X_t$ . Note that stochastic processes following a random walk have the martingale property. Thus martingale processes are less restrictive than random walk processes. Indeed, as in random walk processes, the error terms  $\epsilon_t$  are independent variables, which rules out any dependence between different error terms. In martingale processes the  $\epsilon_t$  only have to be uncorrelated, i.e., dependence in higher moments is allowed.

<sup>26</sup>For a definition of 'futures price' and connected terms as 'spot price', 'underlying asset' and 'maturity date' as mentioned in the following, see Hull (2000).

<sup>27</sup>See a note in Chapter 2, Appendix A 2.2 for details.

which states that there is no way of making an expected profit by extrapolating past changes in the futures prices or, in mathematical language, future prices follow a fair game<sup>28</sup>.

In his subsequent publication (Samuelson, 1973), he carried over his result to stock prices and linked the concept of describing the EMH by martingale processes to the notion of determining of fundamental pricing equilibria. Maintaining that the current price  $p_t$  equals the sum of the conditional expected price of tomorrow and the conditional expected dividend yield discounted by a discounting rate  $\rho$ , i.e.,

$$p_t = \frac{1}{1 + \rho} E_t[p_{t+1} + d_{t+1}], \quad (1.7)$$

it can be easily shown (this simple transformation is relegated to Appendix A 1.1) that the excess returns have the fair game property, namely

$$E_t[\rho_{t+1}] - \rho = 0, \quad (1.8)$$

where  $\rho_{t+1} \equiv (p_{t+1} + d_{t+1} - p_t)/p_t$  denotes the return at time  $t + 1$ . Now, Samuelson concludes that, in this case, prices are equal to the expected discounted sum of their future dividends, thus

$$p_t = \sum_{i=1}^{\infty} \frac{1}{(1 + \rho)^i} E_t[d_{t+i}]. \quad (1.9)$$

Note that also the reverse assertion is true. If the economy follows the (fundamental) expected present-value model, which is defined by Eq. (1.9), then the excess returns have the fair-game property. To summarize, Samuelson showed that the framework of the EMH expressed by the theory of martingale is consistent with an economy in which agents use fundamental pricing rules, which leads to vanishing excess returns caused by the competing behavior among the agents.<sup>29</sup>

Two further remarks are necessary. First, it should be mentioned that until now, the question has not been asked, whether the sum in Eq. (1.9) is in fact absolutely convergent. Indeed, one further assumption has to be made, generally known in economics as the ‘transversality condition’. Although this discussion might seem incomplete, I want to leave this item out for now and instead refer to the description of the Brock-Hommes (BH) model outlined in the next chapter (see Section 2.1, Eq. (2.7) and Eq. (2.8)), in which we will recover an expression like in Eq. (1.9) when dealing with the ‘fundamental equilibrium solution’ of the BH model. Second, no variable defined in the previous comments has yet had the martingale property. In particular,  $p_t$  is *not* a martingale. Instead, it can be deduced that the discounted value of a mutual fund that reinvests received dividends follows a martingale process (see Appendix A 1.2 for details). However, in applied research it is quite common to assume that prices (or their logs) follow a martingale process. This can be justified by the fact that when dealing, with e.g., daily data, the dividends and discounting rates can be neglected, as their values are relatively small on this time scale.

<sup>28</sup>A stochastic process  $Y_t$  is said to be a fair game if it is  $E[Y_{t+1} | \mathcal{I}_t] = 0$ . Obviously,  $X_t$  is a martingale if and only if  $X_{t+1} - X_t$  is a fair game.

<sup>29</sup>In Samuelson (1965, 1973), the martingale property of rates of returns on stock (or futures) prices is derived under the assumption of risk-neutrality. LeRoy (1973) showed that the martingale property does not carry over to the risk-averse case.

### 1.3.1 Fair Price and Martingale Property in Mathematical Finance Literature

In this subsection, the results of the previous discussion are restated in a slightly different way. The expected present value given in Eq. (1.7) as well as Eq. (1.9) can also be seen as the fair or arbitrage-free price<sup>30</sup> of the investment at the current time  $t$ . Thus Samuelson's result also links the 'no-arbitrage principle', which is the crucial tool in determining the fair (or fundamental) price of an investment opportunity, to the concept of martingales.<sup>31</sup> Interestingly, this perception is more pronounced in modern mathematical finance literature, which started in 1979 with a publication from J. M. HARRISON and D. M. KREPS<sup>32</sup> and was extended two years later by J. M. HARRISON and S. R. PLISKA<sup>33</sup>. Here, the concept of the equivalent martingale measure was introduced for the first time, thus allowing for fair price determination via expectation value calculations, which constituted mathematical finance as a new branch of stochastics. In the past, the fair price determination of assets and derivatives was mainly based on solving differential equation problems (see the determination of the Black-Scholes formula by F. BLACK and M. SCHOLESS<sup>34</sup>) and therefore was *not* a research field for stochastics. However, Harrison and Pliska (1981) then proposed a theory which reduced the problem of contingent claim valuation to an equivalent problem in martingale theory for markets operating in continuous time, while Harrison and Kreps (1979) enabled this work by developing this theory, at first in the case of a finite market, in which agents are only allowed to trade at discrete points in time and in which the underlying probability space is finite.

In this strand of literature, martingale processes are introduced to extend results initially proved for one-period models to the  $n$ -period case. More precisely, in an  $n$ -period model with  $B_i$  being the (unique) discounting process and  $P_i$  being the price of the investment at period  $i$ ,  $(B_i P_i)_{i=0, \dots, n}$  is shown to be a martingale with respect to a certain probability measure  $\mathbb{Q}$ , i.e., it is

$$E_{\mathbb{Q}}[B_{i+1} P_{i+1} \mid \mathcal{I}_i] = B_i P_i, \quad i = 0, \dots, n-1,$$

whereby  $\mathbb{Q}$  is said to be an 'equivalent martingale measure' or an 'equivalent risk-neutral probability measure', which is equivalent to the (initial) probability measure  $\mathbb{P}$ , which describes the model in the 'real world'.<sup>35</sup>

Before I quote the 'fundamental theorem of asset pricing', which is one of the cornerstone results in mathematical finance literature, I briefly want to give some further comments on the notion of the 'equivalent martingale measure', since it might be less common in economic literature. The basic idea of stochastic modeling in finance is the transition from the 'real world probability measure'  $\mathbb{P}$ ,

<sup>30</sup>A transaction is said to be an 'arbitrage' if it is possible to make a risk-free profit by entering simultaneously into transactions in two or more markets, e.g., to buying at the lower price on one market and simultaneously selling at the higher price on the other market. In this case, the 'no-arbitrage principle' states that the price determination of investments has to guarantee that no arbitrage opportunities are present on different markets. See Hull (2000) for more detailed comments on this topic.

<sup>31</sup>To be precise, it is important to note that an arbitrage-free market is not necessarily equivalent to an efficient one. However, the existence of an efficient market implies that there are also no arbitrage opportunities in the market.

<sup>32</sup>See Harrison and Kreps (1979).

<sup>33</sup>See Harrison and Pliska (1981). See also Harrison and Pliska (1983) for a following publication on this topic.

<sup>34</sup>See Black and Scholes (1973). See also Merton (1973b) and Black (1976b) for related publications.

<sup>35</sup>See for example Irle (1998) or Reitz, Schwarz and Martin (2004) for a definition and further details.

which describes the phenomenon in terms of a probabilistic model, to the ‘risk-neutral probability measure’  $\mathbb{Q}$ , which is stochastically equivalent to  $\mathbb{P}$  and in which the prices are assigned to their ‘natural’ values in the risk-neutral market in the sense that all discounted assets in the market have the same expectation value. This property of assets is formalized by postulating that discounted assets follow a martingale process. Let us consider Eq. (1.7). Since the current fair price of an asset is assumed to be the discounted expected sum of tomorrow’s price and dividend payment, an investor will be indifferent to investing in this asset or in another risk-free asset, realizing a certain profit of  $(1 + \rho)$  in the next period, which is also expressed by Eq. (1.8). In financial mathematics, this assumption exactly mirrors the changeover to the equivalent martingale measure  $\mathbb{Q}$ , which is the underlying probability measure with which the expectation value in Eq. (1.7) is to be determined. Of course, this assumption simplifies reality, since  $\mathbb{Q}$  does not take certain things to account, for example different market perceptions of the market participants, or individuals’ risk aversion covered by a risk premium in real markets as it would do the real-world measure  $\mathbb{P}$ .

After this short excursus I continue as earlier proposed. The ‘fundamental theorem of asset pricing’ guarantees the existence of such a measure  $\mathbb{Q}$  and states that an  $n$ -period model is arbitrage-free if and only if such an equivalent martingale measure  $\mathbb{Q}$  exists.<sup>36</sup> Furthermore, under slightly more restrictive assumptions, the fundamental price of the investment is *uniquely* determined as the expected value of future discounted yields in the ‘risk-neutral world’, which can also be expressed by

$$p(C) = E_{\mathbb{Q}} \left[ \sum_{i=1}^n B_i C_i \mid \mathcal{I}_0 \right],$$

where again  $B_i$  denotes the discounting process,  $E_{\mathbb{Q}}[\cdot]$  denotes the expectation operator with respect to the equivalent martingale measure  $\mathbb{Q}$ , and  $C_i$  stands for ‘claim’, i.e., the payouts of the investment at time  $i$ . For a more technical treatment and further details we can refer to Irle (1998) and Korn and Korn (2001).<sup>37</sup>

### 1.3.2 Classification of the Information Set

In the previous discussion we have seen that the price process is linked to the process of information transmission by the martingale property

$$E[p_{t+1} \mid \mathcal{I}_t] = p_t. \quad (1.10)$$

However, nothing has yet been said about the structure of the information set  $\mathcal{I}_t$  although, obviously, for empirical tests of the EMH, the information set would have to be specified. For this reason, Fama (1970) proposed three subgroups of information sets, with which he classified the following three different forms of efficiency:

- *Weak-form efficiency*, i.e., the information set  $\mathcal{I}_t$  includes only the history of prices or returns.

<sup>36</sup>In general the equivalent martingale measure need not be unique.

<sup>37</sup>Although the link between the determination of an arbitrage-free price and the martingale property of properly discounted payouts has been stressed in the preceding discussion, the concept of market efficiency can also be found in mathematical finance literature, where it is modeled by a Markov process that mainly states that future evolution of a stochastic process given its history solely depends on its current value.

- *Semistrong-form efficiency*, i.e., the information set  $\mathcal{I}_t$  includes all publicly available information, i.e., all information known to all market participants.
- *Strong-form efficiency*, i.e., the information set  $\mathcal{I}_t$  includes all information known to any market participant, i.e., it includes also any private information.

It follows directly from the law of iterated expectations (see Appendix A 2.2) that strong-form efficiency implies semi-strong form efficiency, which in turn implies weak-form efficiency. Obviously, this statement seems trivial, as of course, *all* information (i.e., including private information) contains all public information, which in turn contains the historical prices and returns.

As we know from Eq. (1.8) that the expected excess return conditioned on a given information set is always zero, it can be concluded that, in a weak-form efficient market, trading strategies based on past price movements (i.e., ‘chartist strategies’) are not capable of outperforming the market. In such a market, agents who use fundamental analysis and/or public or private information should dominate the agents using only chartist strategies, since their information set is a superset of the weak-form information set of the chartists, which would allow them to make better predictions by more accurately deriving the expectation values via Eq. (1.10).

In the same fashion, it can be argued that for the semi-strong efficiency case only private information can help to gain excess returns, or, in other words, neither fundamental nor chartist based strategies should be able to outperform the market. Finally, in the strong-form efficiency case it is even impossible to profit from private information. To see this, note that all other agents at least know all publicly available information and, additionally, conclude the same value of the fundamental price, which means that the market participants already interpret any simple appearance of an insider as revealed information. Consequently, the insider’s information is instantaneously reflected in the price process. To sum up, under the strong-form efficiency it is not possible to make any excess return profit.

### 1.3.3 The Grossman-Stiglitz-Paradoxon

Based on the martingale framework, the EMH implies that the movement of financial prices are an immediate and unbiased reflection of incoming news and unanticipated information about future earning prospects, which can be alternatively expressed by the fact that future price changes evolve in a totally random fashion and are thus unpredictable.

This conclusion seems to answer a frequently asked question also formulated in Fama’s outstanding piece of work, *The Behavior of Stock-Market Prices* (Fama, 1965), namely:

‘To what extent can the past history of a common stock price be used to make meaningful predictions concerning the future price of the stock?’

However, a puzzle remains with this implication, which was stated for the first time in economic literature by S. J. GROSSMAN and J. E. STIGLITZ<sup>38</sup> in the following paradox:

#### Theoretical Argument 1.6 (The Grossman-Stiglitz-Paradoxon)

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<sup>38</sup>See Grossman and Stiglitz (1980).

According to the EMH, at any time all information is incorporated into the current price, which leads to a fully revealing equilibrium, thus there is no incentive for anyone to collect information if gathering the information is assumed to be costly. Thus, in turn, no information can be incorporated into the prices if there is no one willing to obtain it.

Grossman and Stiglitz overcame this theoretical argument against the EMH in the same publication by pursuing the following idea. While in a market in which prices fully reveal information no individual will acquire it by positive costs, he/she will do so in a market, in which the prices only partially reveals relevant information. Therefore, Grossman and Stiglitz proposed a model with rational expectations, which consists of two types of traders, namely informed and uninformed ones. While the informed traders receive private information about the return of a risky asset in the form of a noisy signal, the uninformed traders do not. Thus the rational, uninformed traders have to use the equilibrium pricing rule for the risky asset, in which information from the informed traders' trades is conveyed, to gain any information about the payout of the risky asset. Since the supply of risky assets is assumed to be an additional source of uncertainty in the model, the uninformed traders cannot obtain all the private information of their counterparts from the market price because they observe it only as a noisy signal. Grossman and Stiglitz showed that in such a model scenario, a rational expectations equilibrium exists in which the equilibrium price only partially reveals the privately informed traders' information already paid for by these traders. In other words, in the equilibrium, the uninformed traders know less than the informed traders and thus are outperformed by them. The costs incurred by the informed traders for acquiring private information are equalized by an increase in their utility, which is caused by their superior information level. As a consequence, the Grossman-Stiglitz-Paradoxon does not appear in this model framework.<sup>39</sup>

#### 1.4 Representative Agent Hypothesis (RAH)

In contrast to the REH and EMH, which are two cornerstone concepts of economic modeling and financial theory, the Representative Agent Hypothesis can be more properly classified as a useful and popular assumption in equilibrium models. Its proposition is summarized in the following definition:

**Definition 1.7 (Extreme Version of the RAH)**

An economic model is said to be provided with a *representative agent* framework if all agents in the model are assumed to be homogeneous with respect to their preferences, their expectation formation and their investment strategies.

Obviously, weaker forms of the RAH can be obtained by lowering the assumptions for example by allowing for different preferences, investment strategies, etc.

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<sup>39</sup>See also Lyons (2001) for a detailed description of the model.



For decades and in almost all economic fields, homogeneity among agents has been the crucial assumption in economic equilibrium models.<sup>40</sup> A prominent example in financial literature is the ‘Capital Asset Pricing Model’ (CAPM), proposed in the 1960s by several academics, namely W. F. SHARPE, J. LINTNER, J. MOSSIN and R. C. MERTON.<sup>41</sup> The CAPM is a classic equilibrium model which seeks to determine risky-asset prices (e.g., stocks) and is based on the following assumptions:<sup>42</sup>

- (i) *Homogeneity of information among agents*, i.e., all agents have free and immediate access to all available information in the market.
- (ii) *Homogeneity of agents’ expectations*, i.e., all traders have homogeneous expectations about returns, variance and covariances (correlation) between the various returns, and base their decisions on that information.
- (iii) *Risk aversion of agents*, i.e., all agents are risk-averse and maximize their expected utility at the end of a one-period horizon investment.
- (iv) *Agents are price takers*, i.e., agents’ investment decisions do not lead to changes in the equilibrium prices.
- (v) *Existence of  $K$  assets and one risk-free asset in the market*, i.e., there are  $K$  different assets available in the market, whose individual risk can be measured by the variance of its return  $\sigma_i^2$ . Furthermore, agents can borrow and lend as much as they want at the risk-free rate  $r$ .
- (vi) *No transactions costs*, i.e., transaction costs and taxes are assumed to be zero.
- (vii) *Infinite divisibility of assets*, i.e., arbitrary fractions of assets can be purchased.
- (viii) *Short selling is allowed*.

Thus, unlike in Definition 1.7, the CAPM allows investors to have different preferences (as long as they remain risk-averse), but the assumption of agents’ homogeneous expectations is crucial in deriving the equilibrium of the CAPM, which, the other way round, will collapse if this assumption is given up. The other crucial assumption in the model is that agents use the so-called  $(\mu - \sigma)$ -criterion for their optimal investment decision, which means that the agents only use the information about the expectation value  $\mu_i$  and variance  $\sigma_i^2$  of the risky asset for their utility maximization problem, also called ‘mean-variance maximization’ for short.<sup>43</sup>

Solving the CAPM leads us to a remarkable result, widely known as the ‘(two-fund) separation theorem’, which I shall briefly discuss. Each agent’s investment decision can be subdivided into two different parts. Since the perception of the objective market variables, which are the expectation

<sup>40</sup>Since the RAH cannot be attributed to certain authors as it is possible in the case of the REH and the EMH, instead, I give two examples found in Levy and Levy (1996), namely Ramsey (1928) and Cass (1965).

<sup>41</sup>See Sharpe (1964); Lintner (1965); Mossin (1966) and Merton (1973a).

<sup>42</sup>The model description mainly follows Cuthbertson (1996) and Franke et al. (2000).

<sup>43</sup>It can be shown that sufficient conditions for the portfolio selection based on the mean-variance maximization being *exact* are, on the one hand, that the investor has a quadratic function, or, on the other hand, that the returns of the risky asset are normally distributed. Otherwise, the mean-variance approach can be viewed as a second-order approximation.

about asset returns and its variances and covariances, are assumed to be homogeneous among all agents, all individuals therefore will hold the same proportions  $\pi_i^*$  of the risky assets. Thus, in the equilibrium, each agent is willing to hold the ‘market portfolio’  $X_{m,t}$ , which is given by

$$X_{m,t} = \sum_{i=1}^K \pi_i^* X_{i,t},$$

where  $X_{i,t}$  is the price of asset  $i$  at time  $t$ . Then, in terms of returns, the return of the market portfolio can be determined as<sup>44</sup>

$$R_{m,t} = \ln X_{m,t} - \ln X_{m,t-1}.$$

This completes the first step of the agent’s investment decision problem. In the second step, the individual preferences of each single agent enter the agents’ decision problem. According to their chosen risk aversion, the agents determine their wealth proportion, they are willing to invest in the market portfolio, and they invest the remainder of their wealth in the risk-free asset, providing them with a risk-free rate  $r$ . Note that this agent-varying partitioning of their wealth does not affect their *relative demand* for the risky assets, i.e., the proportions  $\pi_i^*$ . Thus the equilibrium expected returns of the  $K$  assets are independent of the individuals’ preferences, i.e., their magnitude of risk aversion, and are actually only determined by the market variables such as variances and covariances. It can be shown that the equilibrium expected return  $\mu_i$  of asset  $i$  is given by

$$\mu_i \equiv E[R_{i,t}] = r + \beta_i(E[R_{m,t}] - r), \quad i = 1, \dots, K, \quad (1.11)$$

where  $R_{i,t} = \ln(X_{i,t}/X_{i,t-1})$ .  $\beta_i$  denotes the ‘beta-factor’ (or simply ‘beta’) of the risky asset  $i$  given by

$$\beta_i = \frac{\text{Cov}[R_{i,t}, R_{m,t}]}{\text{Var}[R_{m,t}]},$$

which measures the relative impact of asset  $i$  on the systematic risk of the portfolio of all assets as a proportion of the total variance of the portfolio. Indeed, the CAPM implies that the covariance of the return of asset  $i$  with the market return rather than its own variance  $\sigma_i$  determines the asset’s expected return. Thus the covariance between asset  $i$  and the market seems to be the more relevant measure of its risk. Rewriting Eq. (1.11) by

$$(E[R_{i,t}] - r) \sim (E[R_{m,t}] - r)$$

with  $\beta_i$  being the proportion factor, the CAPM implies that the excess return of asset  $i$  is proportional to the excess return of the market portfolio and, in addition, that only the covariance of returns between asset  $i$  and the market portfolio (and the variance of the market portfolio) influences the excess return on asset  $i$ .

To sum up, Eq. (1.11) states that the expected return of asset  $i$  consists of two components. More exactly, it is the sum of the risk-free asset’s return and a risk premium, which is determined for each asset by its specific beta-factor. If there is a positive correlation between the asset  $i$  and the

<sup>44</sup> $R_{m,t}$  is also called ‘log-return’. The log-return  $\hat{r}_t$  during the period  $t$  is defined as  $\hat{r}_t \equiv \ln(1+r_t) = \ln p_t - \ln p_{t-1}$  where  $r_t \equiv (p_t - p_{t-1})/p_{t-1}$ . The log-return is defined for the case of continuous compounding and therefore is also called the ‘continuously compounded return’ or ‘continuous return’.

market portfolio, the risk premium will also be positive, and vice versa. Furthermore, the stronger the correlation between asset  $i$  and the market portfolio, the more significant its contribution to the market portfolio risk. For a formal derivation of the CAPM model see for instance Cuthbertson (1996).

It is no wonder that a lot of effort has put into finding a proper approach for testing the empirical implications of the CAPM. Although even a brief discussion on this issue goes beyond the scope of this thesis, it is widely known that different versions of the CAPM that are even more general in their construction<sup>45</sup> are rejected by certain empirical tests. Seen from a theoretical point of view, this is mainly caused by the overly restrictive assumptions made in the CAPM. For an introduction to these test procedures, see again Cuthbertson (1996).

Although the previous model was just an example of a financial market model under the assumption of homogeneity among agents, albeit a rather seminal one, there is some general dissatisfaction with the assumption of representative investors, for example in Kirman (1992) and, again referring to financial market models, in Levy and Levy (1996). While Kirman questions generally the applicability of the representative agents approach in macroeconomic models and stresses the importance of introducing heterogeneity and interaction between individuals in such models (an approach which is well known from game theory), Levy and Levy focus on the fact that assuming homogeneity among investors in financial models may lead to unrealistic and inefficient prices and market behavior, which can be overcome by allowing for heterogeneity in agents' expectations. Note that their conclusion will be discussed in a more detailed fashion in Chapter 2, Section 2.3, when I outline the Levy-Levy-Solomon (LLS) Model.

## 1.5 Summary

This chapter serves as the introductory chapter of this thesis, in which some basic and essential economic ideas and concepts of modeling financial markets are summarized. It purely concentrates on traditional and generally accepted approaches to financial markets and also considers these from a historical perspective. The content of this chapter can be resumed as follows:

- For nearly the last forty years, two fundamental paradigms of the traditional theory of financial markets are the concept of 'Rational Expectations Equilibria' (REE) and the 'Efficient Market Hypothesis' (EMH). While the REE assumes that all agents act perfectly rationally in the sense that they have access to all relevant information on their economic environment *and* always perfectly use it to make their economic trading decisions, the EMH states that at any time the price of an asset incorporates all relevant information and thus price changes simply occur when new information enters the market.
- Per definition, the 'Rational Expectation Hypothesis' (REH) states that agents' subjective expectations of future variables coincide with the (objective) mathematically computable expectation value of these variables by using the underlying economic theory.
- Per definition, the 'Efficient Market Hypothesis' (EMH), closely connected to the REH, is based on the assumption that the current price of an asset incorporates all relevant and

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<sup>45</sup>For instance, in the first step, the betas can assumed to be time-varying.

hitherto available information or, in other words, that future price changes are only caused by unanticipated events and are therefore unpredictable.

- A third assumption often used in modeling financial markets is the representative agent hypothesis (RAH), which is one of the crucial building blocks in the eminent CAPM model, which still prevails to some extent. The ‘Representative Agent Hypothesis’ (RAH) assumes the homogeneity of agents with respect to their preferences and their expectation formation.

How these traditional and classical concepts of financial economics are still included in newer approaches of financial theory, and in which directions these concepts are generalized in more modern theories of financial markets, can be seen to some extent in Chapter 2 where two different model frameworks of prominent behavioral financial market models will be studied.

## Appendix to Chapter 1

### A 1.1 Derivation of Eq. (1.8)

Rewriting Eq. (1.7) in a slightly different way leads directly to the wanted expression as it is

$$\begin{aligned}\frac{E_t[p_{t+1} + d_{t+1}]}{p_t} &= 1 + \varrho \\ \frac{E_t[p_{t+1} + d_{t+1}] - p_t}{p_t} &= \varrho \\ \frac{E_t[p_{t+1} + d_{t+1}] - p_t}{p_t} - \varrho &= 0,\end{aligned}$$

which can also be written with the definition of the return  $\rho_{t+1}$  as

$$E_t[\rho_{t+1}] - \varrho = 0.$$

### A 1.2 Further Notes on the Martingale Property of Prices

Let  $q_t$  be the number of assets held by an agent at time  $t$ . Then reinvesting the returns at time  $t + 1$  means that

$$p_{t+1}q_{t+1} = (p_{t+1} + d_{t+1})q_t.$$

Furthermore, define

$$x_t \equiv \delta^t p_t q_t \quad \text{with} \quad \delta \equiv \frac{1}{1 + \varrho},$$

which is the value of the stock at time  $t$  discounted on the time period 0. Then it can be concluded that  $x_t$  follows a martingale process as can be shown from the following derivation:

$$\begin{aligned}E_t[x_{t+1}] &= E_t[\delta^{t+1} p_{t+1} q_{t+1}] \\ &= E_t[\delta^{t+1} (p_{t+1} + d_{t+1}) q_t] \\ &= \delta^{t+1} q_t E_t[p_{t+1} + d_{t+1}] \\ &= \delta^{t+1} q_t \cdot (1 + \varrho) p_t \\ &= \delta^{t+1} q_t \delta^{-1} p_t \\ &= \delta^t q_t p_t = x_t.\end{aligned}$$

Thus, it is

$$E_t[x_{t+1}] \equiv E[x_{t+1} | \mathcal{I}_t] = x_t,$$

which states that  $x_t$  has the martingale property.

### A 1.3 Excursion I: Basic Concepts of Utility Theory under Uncertainty

The concept of expected utility was first introduced by D. BERNOULLI<sup>46</sup> in 1738, when he proposed a solution for the St. Petersburg Paradoxon. Nowadays, it is one of the cornerstone concepts of the ‘decision theory’. About 200 years after this idea first appeared in the literature, the concept of expected utility was formalized in an axiomatic way by J. VON NEUMANN and O. MORGENSTERN<sup>47</sup>, when they began to develop a framework for the analysis of strategical situations. This became known in economics in the following years under the notion of ‘game theory’.<sup>48</sup> I will here explain the basic idea behind this concept. In many fields, individuals are faced with the problem of making decisions in situations in where future outcome of the decision cannot be assessed with certainty. For instance, let us assume that each of these possible outcomes of a decision can be measured by a value, or more precisely, that the future outcome of the decision is a random variable. In such a situation of uncertainty, one appropriate strategy would be to derive the expectation values of the future outcome for all possible decisions and choose from among all decisions that alternative whose expectation value is the highest (if a higher value means also ‘better’). Restricting our considerations to investment decisions in the remainder of this discussion, as is common in financial literature, we can use a scenario where the investor chooses the alternative which gives the highest expected value of investor’s wealth given his/her chosen investment horizon. However, one crucial drawback to this simple decision-making rule becomes immediately obvious. Since the degree of uncertainty of the investor’s investment decision is ignored when using such a simple strategy, the investor might choose an alternative where the expected outcome is only moderately higher but, in turn, bears a drastically higher risk relative to the other alternatives. The expected utility framework helps to overcome this drawback and also to implement the inherent risk of the different investment alternatives in the decision-making process.

In this framework, it is assumed that people are endowed with certain preferences (for instance, they prefer more rather than less), which enable them to find a solution for their decision-making problem. To make this approach mathematically tractable, the investor’s preferences are mapped on an analytic function  $U(W)$ . Furthermore, the following general assumptions are made:

- Each individual can be characterized by a utility function  $U(W)$ .
- In any uncertain situation, the individuals maximize the expected value of their utility function, i.e., they derive

$$\max \{E[U(W)]\} . \quad (\text{A 1.1})$$

- All the information about individual’s preferences is represented by the analytic form of  $U(W)$ .

By deriving the utility of wealth for each possible investment decision the investor is able to rank the different investment alternatives according to his personal preferences. Therefore the utility function is simply a tool to rank random wealth levels and thus to compare risky investments.

In the first step, the huge number of possible forms that utility functions might take can be restricted by the following two reasonable assumptions:

- (i) More wealth is better than less, that is, if  $W_2 > W_1$  then  $U(W_2) > U(W_1)$ , which implies

$$U'(W) > 0 .$$

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<sup>46</sup>See Bernoulli (1954).

<sup>47</sup>See von Neumann and Morgenstern (1944, 1947).

<sup>48</sup>For the following discussion see Copeland and Weston (1988); Gollier (2001) and Levy, Levy and Solomon (2000). For an original report about the collaboration between O. Morgenstern and J. von Neumann on that topic see Morgenstern (1976).

- (ii) The higher the current level of wealth the less a further increase in wealth is worth or, in other words, marginal utility decreases as wealth rises, i.e.,

$$U''(W) < 0 .$$

These assumptions imply that  $U(W)$  is a concave function and has a positive but decreasing slope. The concavity of  $U(W)$  implies that investors are ‘risk-averse’, which means that among investments with identical expected yields, they choose the investment with the least risk.<sup>49</sup>

The characteristic property of  $U''(W)$  being negative for a risk-averse individual can also be concluded by a more analytically based argument. Following Eq. (A 1.1), the expected utility of an investor after an investment with a probability density function  $p(W)$  is given by

$$E[U(W)] = \int dw p(w)U(w) . \quad (\text{A } 1.2)$$

Let us assume that  $p(W)$  is approximately symmetric and relatively well concentrated around  $W$  and let  $\sigma_W^2 \equiv \text{Var}[W]$ , that can be interpreted as a risk measure. Approximating the utility of wealth by a Taylor expansion up to the second order Eq. (A 1.2) can be written as

$$\begin{aligned} E[U(W)] &= \int dw p(w) \left[ U(W) + U'(W)(w - W) + \frac{U''(W)}{2}(w - W)^2 + \mathcal{O}(w^3) \right] \\ &= U(W) + U'(W) \int dw p(w)(w - W) + \frac{U''(W)}{2} \int dw p(w)(w - W)^2 + \mathcal{O}(w^3) . \end{aligned}$$

Equating the first integral to zero by using the standard symmetry argument and the second integral to  $\sigma_W^2$ , we end up with

$$E[U(W)] = U(W) + \frac{U''(W)}{2} \sigma_W^2 + \mathcal{O}(W^3) . \quad (\text{A } 1.3)$$

Eq. (A 1.3) implies that if  $U''(W)$  is positive, the utility would rise by increasing the measure of risk, which would contradict the assumption that the investor has a risk-averse behavior. Therefore,  $U''(W)$  has to be negative, which is in harmony with Assumption (ii).

As the assumption of risk aversion still includes many irrelevant utility functions, K. J. ARROW and J. W. PRATT<sup>50</sup> suggest defining two measures of risk aversion to restrict the analytic forms of meaningful utility functions again, namely:

- an absolute measure  $R_A(W)$  which is defined as

$$R_A(W) = -\frac{U''(W)}{U'(W)}$$

- a relative measure  $R_R(W)$  which is given by

$$R_R(W) = -W \frac{U''(W)}{U'(W)} .$$

For each measure, three different subcases are to be distinguished as there is decreasing, constant and increasing absolute (relative) risk-aversion:

- A decreasing absolute (relative) risk-aversion is defined as

$$\frac{\partial R_{\star}(W)}{\partial W} < 0 , \quad \text{and denoted by DARA (DRRA)} .$$

<sup>49</sup>The concavity of  $U(W)$  is equivalent to the relation  $U(E[W]) > E[U(W)]$ , which implies that this relation can also be used as a definition for risk aversion.

<sup>50</sup>See Arrow (1965) and Pratt (1964).

- A constant absolute (relative) risk-aversion is defined as

$$\frac{\partial R_{\star}(W)}{\partial W} = 0, \quad \text{and denoted by CARA (CRRA).}$$

- An increasing absolute (relative) risk-aversion is defined as

$$\frac{\partial R_{\star}(W)}{\partial W} > 0, \quad \text{and denoted by IARA (IRRA).}$$

In all three subcases the star  $\star$  has to be substituted by A (R). Note that absolute risk-aversion measures risk aversion for a given level of wealth, while relative risk-aversion is obtained by multiplying the measure of absolute risk-aversion by the level of wealth. From empirical and experimental studies it can be hypothesized that basically three different types of risk aversion can be detected in reality,<sup>51</sup> namely IRRA, CRRA and DARA, on which I will here offer some remarks:

- An increasing relative risk-aversion (IRRA) implies that an investor will lower his/her proportion of wealth invested in risky prospects as he/she becomes wealthier.
- A constant relative risk-aversion (CRRA) is characterized by the fact that an investor maintains a constant proportion of wealth invested in risky prospects as his/her wealth changes.
- A decreasing absolute risk-aversion (DARA) implies that an individual is willing to invest more (absolute) money in risky prospects as he is getting wealthier.

Thus,

- the DARA assumption is a generalization of the CRRA case, i.e.,

$$\text{CRRA} \subset \text{DARA} .^{52}$$

Prominent candidates for utility functions are the quadratic function<sup>53</sup>

$$U(W) = \alpha W - \beta W^2 ,$$

the negative exponential utility function

$$U(W) = -\exp(-\nu W) , \tag{A 1.4}$$

the logarithmic function

$$U(W) = \ln(W)$$

and, as a generalization of the latter one, the power function<sup>54</sup>

$$U(W) = \frac{W^{1-\nu}}{1-\nu} , \tag{A 1.5}$$

where  $\alpha, \beta, \nu > 0$  are constant and denote the risk preferences. It is easy to see that the negative exponential utility function (A 1.4) is of the CARA type whereas the power function and logarithmic function have the CRRA properties. A summary of the Arrow-Pratt classification of utility of wealth functions is given in Table A 1.1 (taken from Levy et al., 2000). Figure A 1.1 exhibits these two prominent examples of

<sup>51</sup>See Levy et al. (2000) and Chapter 3, Section 3.1 for details.

<sup>52</sup>This implication can easily be shown analytically. Since  $\partial R_R / \partial W = W \partial R_A / \partial W - U''(W) / U'(W)$ , it is obvious that  $\partial R_R / \partial W = 0$  implies  $\partial R_A / \partial W = U''(W) / (W U'(W)) < 0$ .

<sup>53</sup>A quadratic utility function only makes sense for an positive W-range, that is, for  $W \leq \alpha / (2\beta)$ .

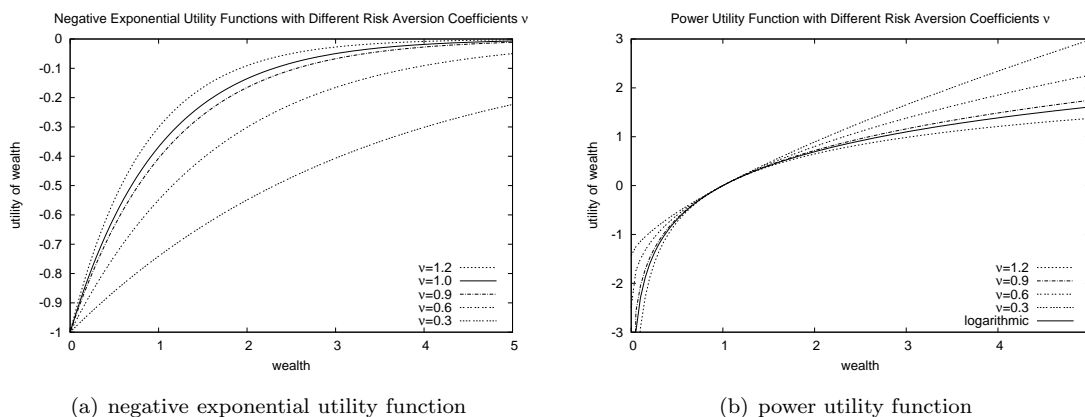
<sup>54</sup>Using the rule of de l'Hospital it can be easily shown that the power utility function coincides with the logarithmic function by setting the risk-aversion parameter  $\alpha$  equal to 1. Let  $g(x) = W^{1-x}$  and  $h(x) = 1 - x$ , then it is  $g'(x) = -\ln(W) \cdot W^{1-x}$  and  $h'(x) = -1$  and, thus,  $g'(x) / h'(x) = \ln(W) \cdot W^{1-x} \rightarrow \ln(W)$  as  $x \rightarrow 1$ .

| Utility Functions    | Absolute Risk Measure | Relative Risk Measure |
|----------------------|-----------------------|-----------------------|
| quadratic            | IARA                  | IRRA                  |
| negative exponential | CARA                  | IRRA                  |
| logarithmic          | DARA                  | CRRA                  |
| power                | DARA                  | CRRA                  |

**Table A 1.1:** Risk-aversion properties for different analytic forms of utility functions.

utility functions. In the left panel the negative exponential CARA utility function is shown for different risk-aversion coefficients  $\nu$ , while in the right panel the power CRRA utility function for different degrees of risk aversion is plotted.

It is worth noting that the concept of expected utility provides a cardinal rank order for risky alternatives. Thus cardinal utility functions map different degrees of utility on a scale according to the individual's preferences. Therefore, in contrast to an ordinal ranking, it can be determined to which extent various utility values differ from each other. From the mathematical point of view, this implies that the rank order is invariant against linear transformations, i.e., transformations of the form  $U^T = a + bU$  where  $b > 0$ . For instance,  $U(W) = (W^{1-\nu} - 1)/(1 - \nu)$  is very often used instead of the standard power utility function given in Eq. (A 1.5).<sup>55</sup>



**Figure A 1.1:** Negative exponential and power utility function for different risk-aversion coefficients. The left panel exhibits a negative exponential utility function which is a CARA type and has the form  $U(W) = -\exp(-\nu W)$ . The right panel shows a CRRA utility type and has the form  $U(W) = (W^{1-\nu} - 1)/(1 - \nu)$ . In both cases a decrease in the risk-aversion coefficient reduces the curvature of the utility function.

As already hinted above, there has been much debate about which analytic form of utility functions describes the investors' preferences best, although no final conclusion has yet been reached. Therefore, choosing an appropriate utility function in economic modeling was for a long time mainly a question of the function's analytical manageability. For this reason, the negative exponential utility function is still often used, since under the assumption of Gaussian distributed wealth the corresponding integral from deriving  $E[U(W)]$  can be solved analytically. Furthermore, it can be shown that this approach is equivalent to the standard mean-variance maximization approach, which takes solely the first and second moments of the

<sup>55</sup>See Chapter 3, Section 3.3.



wealth distribution into account (see Appendix A 2.1 for details). For the same reason, the quadratic form of utility functions is another particularly popular assumption. With this approach,  $E[U]$  also depends simply on the mean and variance of the wealth distribution hence higher moments can be disregarded.

Nowadays, the implementation of microscopic simulation methods in modeling financial markets allows one to enlarge the number of feasible forms of utility functions in general. As they are supported by empirical and experimental surveys, the DARA and CRRA class of utility function are of particular interest. For this reason, a remarkable result is stated in Levy et al. (2000):

**Theorem 1.8** *If CRRA and DARA prevail, the only possible utility function is the power function, i.e.,*

$$U(W) = \frac{W^{1-\nu}}{1-\nu} .$$

PROOF: A proof can be found in Levy et al. (2000).

The theorem clearly states that the power function is the *only* common function, that combines the two properties, i.e., CRRA and DARA.

## CHAPTER 2

# Behavioral Models of Financial Markets

This chapter reviews two prominent examples of behavioral financial market models from the 1990s established by Brock and Hommes and Levy, Levy and Solomon. Both model frameworks are outlined here to a certain extent as they can be understood as a starting point for my own investigations summarized in the next part of this thesis. The Brock-Hommes (BH) model comes from the economic branch of literature. It is a nonlinear evolutionary adaptive model of a financial market with different trader types, which is nested in a widely applicable analytical framework. Due to its crucial relevance for the remainder of this thesis, some modified versions of the original model proposed by other authors and by me are also considered in a subsequent section of this chapter. The Levy-Levy-Solomon (LLS) model outlined afterwards can be classified as a simple microscopic multi-agent model that is mainly computationally orientated. In contrast to the first model, it is published and frequently cited and discussed in both economics as well as physics and computer science journals. The chapter concludes with a summary of the results of the analysis of my own model modifications of the BH model and a final comparison of the two model frameworks presented in the chapter.

### 2.1 The Brock-Hommes (BH) Model

In this section I will present a structural heterogeneous agents model established by W. A. BROCK and C. H. HOMMES that is based on earlier contributions to nonlinear, complex asset pricing modeling of the first author (see Brock, 1993, 1997). In numerous publications, Brock and Hommes propose to model an economic or financial market as an adaptive evolutionary system where boundedly rational agents use different trading strategies to compete against each other (see Brock and Hommes, 1997*b,a*, 1998, 1999). As in many other models with heterogeneous interacting agents, the market participants are classified according to the main ‘prototypes’ of trading strategies, namely fundamental and chartist strategies.<sup>1</sup> While fundamental based strategies assume that asset prices follow, at least in the long run, their fundamental equilibrium price, chartist strategies are basically based on extrapolating future asset prices from past price patterns. The agents choose their trading strategies according to a ‘fitness measure’, such as average accumulated wealth or accumulated past profits, and update their strategies occasionally when other trading strategies seem to promise a higher fitness. In this way, an evolutionary competition between different trader

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<sup>1</sup>See the introduction of Chapter 4 for further publications using the chartist-fundamentalist framework.

types is established in the market and traders can be termed ‘bounded rational’, in the sense that most of them pursue the most profitable strategy.<sup>2</sup> To complete the model framework, Brock and Hommes connect the dynamics of traders’ choice of strategies with the market price dynamics by allowing for endogenous interaction between the equilibrium price dynamics and the evolution of strategies. This means that changes in traders’ beliefs about which strategy is more favorable for them leads to new equilibrium prices, which in turn may lead the traders to update their choice of strategy.

Brock and Hommes coined the notion of ‘Adaptive Belief System’ (ABS) for this modeling concept. In recent years, a remarkable branch of publications has followed, dealing with extensions of the original ABS (see Brock et al., 2005; Brock and Hommes, 2001; Chiarella and He, 2002c; Gaunersdorfer, 2000; Gaunersdorfer et al., 2003; Gaunersdorfer and Hommes, 2007; Gaunersdorfer et al., 2008; Hommes, 2002) or using the ABS framework as a proper starting point for their own slightly different approaches or modifications (see Chiarella and He, 2001; Chiarella, Dieci and Gardini, 2002; Chiarella and He, 2002a,b, 2003; De Grauwe and Grimaldi, 2006b,a; Diks and von der Weide, 2002; Hommes, Huang and Wang, 2005; Westerhoff, 2003). My own presentation will mainly concentrate on the original ABS model and is mostly a condensation of both a review article written by Hommes (2001) and the original publications on the ABS literature mentioned therein. At the end of this section I will also give a brief summary of some publications from Gaunersdorfer (2000) and Gaunersdorfer and Hommes (2007) as well as my own suggestion of a model variant. All of these are closely related to the original BH model.

### 2.1.1 Model Description

The BH model is based on a simple discrete present value asset pricing model that is extended to allow for heterogeneous trading strategies. Traders can choose between two different investment opportunities:<sup>3</sup>

- a risky asset with an uncertain dividend payment  $d_t$  and a current price  $p_t$
- a risk-free asset that is assumed to be perfectly elastically supplied at gross return  $R \equiv 1 + r$ , where  $r$  denotes the interest rate.

In every period, the traders derive their optimal demand function from a standard mean-variance utility maximization approach.<sup>4</sup> Denoting  $W_{i,t}$  the wealth and  $N_{i,t}$  the number of shares of trader  $i$  at time  $t$  respectively, the wealth dynamics of each single trader evolves according to

$$\begin{aligned} W_{i,t+1} &= \underbrace{N_{i,t}(p_{t+1} + d_{t+1})}_{\text{wealth invested in risky asset}} + \underbrace{(1+r)(W_{i,t} - p_t N_{i,t})}_{\text{wealth invested in risk-free asset}} \\ &= RW_{i,t} + (p_{t+1} + d_{t+1} - Rp_t)N_{i,t}. \end{aligned} \quad (2.1)$$

<sup>2</sup>See also Chapter 1, Section 1.2 for details.

<sup>3</sup>For the following discussion, see also the Subsection 1.2.1 in Chapter 1 and Grossman (1976) as well as Grossman and Stiglitz (1980) and Bray (1982).

<sup>4</sup>See Appendix A 2.1 for details.

As each trader is assumed to be a myopic mean-variance optimizer maximizing expected risk adjusted wealth, their optimal demand for risky assets is determined by

$$N_{i,t} = \arg \max_{N_{i,t}} \left\{ E_{i,t} [W_{i,t+1}(N_{i,t})] - \frac{a}{2} \text{Var}_{i,t} [W_{i,t+1}(N_{i,t})] \right\}, \quad (2.2)$$

where  $E_{i,t}[\cdot]$  and  $\text{Var}_{i,t}[\cdot]$  denote the conditional expectation and conditional variance of trader  $i$  at time  $t$  based on a publicly available information set  $\mathcal{I}_t$ .<sup>5</sup> Furthermore,  $a$  denotes the risk-aversion parameter which is the (constant) Arrow-Pratt measure of absolute risk-aversion.<sup>6</sup> Using Eq. (2.1) and Eq. (2.2), the optimal demand of trader  $i$  can be derived as

$$N_{i,t} = \frac{E_{i,t}[p_{t+1} + d_{t+1}] - Rp_t}{a \text{Var}_{i,t}[p_{t+1} + d_{t+1}]}. \quad (2.3)$$

Note that, for each trader, the optimal demand for risky assets is independent of their initial wealth.<sup>7</sup>

In the BH model, the number of interacting traders is assumed to be infinitely large. The traders are classified into  $H$  groups according to the different trading strategies used in the model market. Furthermore, in each time period every trader can be uniquely allocated to one of the  $H$  trading strategies. The fraction of group  $h = 1, \dots, H$  at time  $t$  is denoted by  $n_{h,t}$ <sup>8</sup> and the conditional variance  $\text{Var}_{i,t}[\cdot] = \sigma^2$  is assumed to be constant for all trader types.<sup>9</sup>

Let  $z^s$  be the supply of risky shares per investor (assumed to be independent of  $t$ ), meaning that the equilibrium of supply and demand implies

$$\sum_{h=1}^H n_{h,t} \cdot N_{h,t} = \sum_{h=1}^H n_{h,t} \frac{E_{h,t}[p_{t+1} + d_{t+1}] - Rp_t}{a\sigma^2} = z^s, \quad (2.4)$$

where now  $N_{h,t}$  denotes the number of shares held by one trader belonging to trading strategy group  $h$  and  $E_{h,t}[\cdot]$  denotes the ‘beliefs’ or forecasts of trader type  $h$  with respect to conditional expectations. Making the further simplifying assumption of zero supply of risky shares, i.e.,  $z^s = 0$ , the equilibrium price equation (2.4) can be simplified to<sup>10</sup>

$$\sum_{h=1}^H n_{h,t} E_{h,t}[p_{t+1} + d_{t+1}] = Rp_t. \quad (2.5)$$

Hence the price of the risky asset is the sum of tomorrow’s expected asset price and tomorrow’s expected dividend yield discounted by the discounting rate  $1/R$  and averaged over all trader types  $h$ .

In contrast to many other computationally orientated heterogeneous multi-agent models, the perfect rational EMH benchmark is nested in the ABS framework. Considering the homogeneous

<sup>5</sup>Remember, that  $E_t[\cdot]$  is a short notation for  $E[\cdot | \mathcal{I}_t]$ , where  $\mathcal{I}_t$  is the publicly available information set consisting of past prices and past dividends, i.e.,  $\mathcal{I}_t = \{p_{t-1}, p_{t-2}, \dots, d_{t-1}, d_{t-2}, \dots\}$ .

<sup>6</sup>See Chapter 1, Appendix A 1.3 for details.

<sup>7</sup>See Chapter 1, Appendix A 1.3 and Appendix A 2.1 for details.

<sup>8</sup>This model structure is the same as assuming  $H$  representative investors (see Section 1.4 in Chapter 1) whose impact on the market varies with the current number of investors in the different groups.

<sup>9</sup>In Subsection 2.2.1 we briefly reproduce the results given by Gaunersdorfer (2000), who investigates the case with time varying beliefs about variances. She shows that the results are quite similar to those for constant variances.

<sup>10</sup>A motivation for this special choice of  $z^s$  can be found in Brock (1997). See also Hommes (2001, p. 153), Footnote 8 for a short discussion on this topic.

case, i.e.,  $H = 1$ , in which all traders are identical and form fully rational expectations about future asset prices and dividend yields, the market equilibrium Eq. (2.5) is determined by

$$E_t[p_{t+1} + d_{t+1}] = (1 + r)p_t, \quad (2.6)$$

where  $R = 1 + r$  is used. It is well known that in general Eq. (2.6) has infinitely many solutions. However, assuming that the ‘transversality condition’

$$\lim_{k \rightarrow \infty} \frac{E_t[p_{t+k}]}{(1 + r)^k} = 0 \quad (2.7)$$

holds, for positive interest rates (i.e.,  $r > 0$ ), only one unique fundamental solution exists, which is given by

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[d_{t+k}]}{(1 + r)^k}, \quad (2.8)$$

provided the limit series on the right-hand side is absolutely convergent. The price  $p_t^*$  is also called the ‘fundamental price’ and is completely determined by economic fundamentals, i.e., the discounted sum of expected future dividends.<sup>11</sup> In the special case of an i.i.d. dividend process  $d_t$  with constant mean  $E[d_t] = \bar{d}$ , the fundamental price  $p_t^*$  becomes constant over time, which leads to a simple form of the fundamental solution given by<sup>12</sup>

$$p^* = \frac{\bar{d}}{r}.$$

As already mentioned above, Eq. (2.6) has infinitely many solutions and only the transversality condition ensures that the fundamental solution  $p_t^*$  is unique. Conversely, if this condition does not hold, further solutions, called ‘rational bubble solutions’, exist.<sup>13</sup> Interestingly, although some of these bubble solutions can even lead to continuously growing prices, traders that follow such explosive price paths still have rational expectations. Nevertheless, as perfect rational traders know that in real markets such speculative, explosive bubbles cannot exist forever, they will never jump on these rational bubble solutions and thus the finite fundamental price  $p_t^*$  is uniquely determined. In other words, in a perfectly rational environment, rational traders with homogeneous expectations believe in the parity of current asset price and its fundamental value, and changes in risky-asset values can only be caused by randomly driven ‘news’ about the fundamentals, i.e., the unexpected changes in the dividends.

If we now turn back to the heterogeneous case with evolutionary changes in the traders’ beliefs about future price movements, the situation is quite different. Here, temporary deviations from the fundamental price, which is denoted by

$$x_t = p_t - p_t^*$$

<sup>11</sup>The fundamental price  $p_t^*$  in Eq. (2.8) has exactly the same form as the fundamental price  $p_t$  in Eq. (1.9) in Chapter 1, Section 1.3. For further details on the technical part of the determination of Eq. (2.8), see also Appendix A 2.2.

<sup>12</sup>Brock and Hommes also investigate more complicated versions of the dividend process  $d_t$  (see Brock and Hommes, 1997b; Hommes, 2002).

<sup>13</sup>For this reason, the transversality condition Eq. (2.7) is also called the ‘no bubbles’ condition. It can be shown that  $p_t^* + b_t$  with  $p_t^*$  being a specific solution (namely the ‘fundamental solution’) of Eq. (2.6) and  $b_t$  being arbitrary, but fulfilling  $E_t[b_{t+1}] = (1 + r)b_t$ , is also a solution of Eq. (2.6). In this case,  $b_t$  is also called a ‘rational bubble’ term (see Cuthbertson, 1996, pp. 157–158).

in the following, can occur if the fraction of traders believing in an upwards or downwards price trend is large enough. Contagion effects caused by the updating mechanism of traders' beliefs even enhance the current price trends, at least until fundamental forces bring the asset price back to near its benchmark fundamental.

Before the traders' evolutionary updating process is characterized in detail, the ABS is restricted by the following simplifying assumptions. For the sake of analytical tractability, Brock and Hommes assume that

- $Var_{h,t}[p_{t+1} + d_{t+1}] \equiv \sigma^2$  for all  $h, t$  (as already stated above), i.e., the beliefs about the conditional variance are constant and the same for all trader types,
- $E_{h,t}[d_{t+1}] \equiv E_t[d_{t+1}]$  ( $= \bar{d}$  for i.i.d. dividends) for all  $h, t$ , i.e., the expected future dividends are the same for all trader types, which implies that each trader is able to derive the underlying fundamental value  $p_t^*$  of the risky asset from Eq. (2.8) and
- all beliefs  $E_{h,t}[p_{t+1}]$  are of the form

$$E_{h,t}[p_{t+1}] = E_t[p_{t+1}^*] + f_h(x_{t-1}, \dots, x_{t-L}), \quad L \in \mathbb{N} \quad (2.9)$$

for all  $h, t$ , i.e., traders' beliefs are restricted to time stationary functions  $f_h$  of past deviations of the risky asset from a commonly shared view of the fundamental. The function  $f_h$  is called a 'forecasting rule' or 'trading rule' and only depends on the trading type  $h$ .

Using the fact that obviously  $p_t = x_t + p_t^*$  and  $p_t^*$  satisfy Eq. (2.5) and introducing the shortening notation  $f_{h,t} \equiv f_h(x_{t-1}, \dots, x_{t-L})$ , the market equilibrium equation Eq. (2.5) can be rewritten subject to the deviation  $x_t$  as

$$(1 + r)x_t = \sum_{h=1}^H n_{h,t} E_{h,t}[x_{t+1}] \equiv \sum_{h=1}^H n_{h,t} f_{h,t}.$$

Note again that for  $f_h \equiv 0$  the homogeneous rational expectations model is nested in the more general BH model.

To complete the model description, the evolutionary part of the BH model has to be specified in the next step. The fraction of traders following a certain trading strategy  $h$  is time-dependent and evolves over time (in the case of infinitely many traders) according to the discrete choice probabilities or 'Gibb's probabilities'<sup>14</sup>

$$n_{h,t} = \frac{\exp\{\beta \Phi_{h,t-1}\}}{\sum_{h=1}^H \exp\{\beta \Phi_{h,t-1}\}}, \quad (2.10)$$

where  $\beta$  denotes the 'intensity of choice' and measures the traders' sensitivity of strategy selection. If  $\beta = 0$ , the fraction  $n_{h,t}$  is constant over time and fixed to  $1/H$ . Otherwise, if  $\beta \rightarrow \infty$ , the entire mass of traders uses the strategy which has the highest fitness.  $\Phi_{h,t}$  is called the 'evolutionary fitness function' or 'performance measure' and is assumed to be publicly available to all traders.

<sup>14</sup>See Manski and MacFadden (1981) and Anderson, de Palma and Thisse (1992) for an extensive discussion of discrete choice models and their applications in economics. See also Brock (1997) for the use of discrete choice models in ABS and Hommes (2001, p. 155) for a very brief statement on this topic.

Obviously, the structure of the  $n_{h,t}$ -probabilities guarantees that the higher the current fitness of the certain trading strategy, the more agents will be willing to switch to it. For an appropriate fitness criterion, Brock and Hommes propose to measure the accumulated realized profits. Hence they compose the concrete form of the fitness function as

$$\Phi_{h,t} = \phi_{h,t} - C_h + \gamma\Phi_{h,t-1} \quad (2.11)$$

with

$$\phi_{h,t} = \{p_t + d_t - Rp_{t-1}\} \left( \frac{E_{h,t-1}[p_t + d_t] - Rp_{t-1}}{a\sigma^2} \right), \quad (2.12)$$

where  $\gamma$  is a memory parameter which measures the weight of past strategy selections in the fitness function<sup>15</sup> and  $C_h$  accounts for costs to obtain the forecasting strategy  $h$ . Notice that the term in the curly brackets reveals the traders' realized excess return in period  $t$ , whereas the second term simply mirrors the optimal demand for the risky asset (see Eq. (2.4)). After some simple transformations (see Appendix A 2.3 for details), the realized profits  $\phi_{h,t}$  can also be expressed depending on the deviation  $x_t$  from the fundamental as

$$\phi_{h,t} = \{x_t - Rx_{t-1} + \delta_t\} \left( \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2} \right), \quad (2.13)$$

where  $\delta_t \equiv p_t^* + d_t - E_{t-1}[p_t^* + d_t]$  is a martingale difference sequence with respect to  $\mathcal{I}_{t-1}$ <sup>16</sup>. Considering the curly brackets in Eq. (2.13), the excess return in the ABS framework consists of a conventional 'EMH term'  $\delta_t$  and an additional 'speculative term'  $(x_t - Rx_{t-1})$ , which takes the traders' deviation from perfectly rational behavior into account and thus allows for excess volatility in periods when asset prices increase by more or less than the risk-free rate of return. Hence, in other words, the speculative term represents the specific contribution of the ABS theory expounded here.

To summarize the model description of the BH model outlined in this subsection in terms of its main equations that determine the evolution of the observed price and the fraction of different trader types, we can collect the Adaptive Rational Equilibrium Equations of the ABS, which are given by

$$Rx_t = \sum_{h=1}^H n_{h,t} f_{h,t}, \quad (2.14)$$

$$n_{h,t} = \frac{\exp\{\beta\Phi_{h,t-1}\}}{\sum_{h=1}^H \exp\{\beta\Phi_{h,t-1}\}} \quad (2.15)$$

with

$$\Phi_{h,t} = \phi_{h,t} - C_h + \gamma\Phi_{h,t-1}, \quad (2.16)$$

$$\phi_{h,t} = \{x_t - Rx_{t-1} + \delta_t\} \left( \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2} \right). \quad (2.17)$$

<sup>15</sup>For the extreme case  $\gamma=0$ , the traders have no memory and simply take the last realized net profits into account, whereas if  $\gamma=1$  a uniform weighting of previous realized profits is assumed for the performance measure, i.e.,  $\Phi_{h,t} = \phi_{h,t} + \Phi_{h,t-1} = \phi_{ht} + \phi_{h,t-1} + \Phi_{h,t-2} = \dots$ , which equals the accumulated wealth over time. In the intermediate case ( $0 < \gamma < 1$ ), the weight given to past realized profits decays exponentially with time.

<sup>16</sup>A martingale difference equation is also called a 'fair game' and has the property  $E[\delta_{t+1} | \mathcal{I}_t] = 0$  for all  $t$  as it is already stated in Chapter 1, Section 1.3. Furthermore, note that for the case of i.i.d. dividends, i.e.,  $d_t = \bar{d} + \epsilon_t$  the expression for  $\delta_t$  simplifies to  $\delta_t = \epsilon_t$ .

## 2.1.2 Selected Main Results

Having outlined the ABS framework in the last subsection, we are still left with the specification of the traders' forecasting rules  $f_{h,t}$ , which enable us to investigate the BH model for different market scenarios. In their first publications (see Brock and Hommes, 1997*b*, 1998, 1999), Brock and Hommes restricted their considerations to the simplest linear trading rules with only one lag, i.e.,

$$f_{h,t} = g_h x_{t-1} + b_h, \quad (2.18)$$

where  $g_h$  is a trend parameter and  $b_h$  denotes a bias parameter.<sup>17</sup> For  $g_h = b_h = 0$ , Eq. (2.18) reduces to a forecast rule of fundamentalists, i.e.,

$$f_{h,t} \equiv 0$$

which leads to the beliefs

$$E_{h,t}[p_{t+1}] = E_{h,t}[p_{t+1}^*].$$

about the expected future asset price. Trend chasers (contrarians) will select for a positive (negative) trend parameter, i.e.,  $g_h > 0$  ( $g_h < 0$ ), which leads to the trading rule

$$f_{h,t} = g_h x_{t-1}, \quad g_h \neq 0.$$

Furthermore, the perfect-foresight forecasting rule can also be interpreted in this framework in a simple way. Assuming

$$f_{h,t} = x_{t+1},$$

this forecasting rule models rational traders who are assumed to have perfect knowledge of all relevant information of the equilibrium equation at any point in time, which, of course, might seem to be quite an unrealistic assumption.<sup>18</sup>

To report all the relevant results on all the different versions of the BH model would go far beyond the scope of this review. As stated in Chiarella and He (2001),

[t]he resulting dynamic system [i.e., the ABS] is capable of generating the entire 'zoo' of complex behavior from local stability to high-order cycles and chaos as various key parameters of the model change. [...]

Instead, let us consider some illustrative examples. We will restrict our consideration to the purely deterministic, nonlinear asset pricing model in the remainder of this subsection, with  $\delta_t = 0$  for all  $t$  and constant dividend  $\bar{d}$  per time period. In Brock and Hommes (1998), for instance, the authors investigate 2-trader-type examples, as well as examples with three and four different trader types. Just to get an idea of their analysis let us consider a scenario with a fundamentalist trading group vs. a group of trend chasers. Assuming the no memory case, i.e.,  $\gamma=0$ , they show for this case that, whenever the average per period costs for trend chasers are less than the average per period

<sup>17</sup>The following argument can serve as a motivation for the choice of such simple trading rules. Since it is quite unlikely that more complicated forecasting rules will be used by enough traders to affect market equilibrium prices, most simple trading rules should be more relevant for realistic market scenarios.

<sup>18</sup>See Hommes (2001, p. 155) for a brief discussion of this topic.



costs for fundamentalists, an increase in the intensity of choice  $\beta$  creates a bifurcation route to chaotic asset price fluctuations, which Brock and Hommes call a ‘rational route to randomness’ in their work. More precisely, starting with a unique, globally stable fundamental steady state for low values of  $\beta$ , this steady state becomes unstable and turns into a pitchfork bifurcation<sup>19</sup> while increasing the bifurcation parameter  $\beta$ . Via a further Hopf-bifurcation, quasi-periodic and even chaotic asset price movements eventually arise for large values of the intensity of choice. For a more detailed summary of the most relevant results on the ABS with simple linear trading rules, we can refer to an excellent review article from Hommes (2001) and the original publications cited therein.

Even so, in the remainder of this section I want to more carefully elaborate on one simple example of an ABS with two different trading types before three modified ABS are presented in the next section. This is done for the following reasons. On the one hand, this first example will give us an opportunity to get some primary insights into the theoretically based analysis of these specific types of nonlinear behavioral models of financial markets. On the other hand, the analytical treatment also serves as a useful preparation for my own investigations conducted in the next parts of this thesis since all the main tools for nonlinear dynamic analysis will already be utilized in the following discussion.<sup>20</sup>

#### *A Simple Example with Two Different Trader Types*

In this subsection I will investigate a 2-traders-type example with fundamentalists and trend chasers.<sup>21</sup> In contrast to the previous discussion, we now allow for  $\gamma$  being between 0 and 1. The forecasting rules of the two different trading types are given,

- for the fundamentalist group, by

$$f_{1t} \equiv 0 \tag{2.19}$$

- and, for the trend-following group, by

$$f_{2t} = gx_{t-1}, \quad g > 0. \tag{2.20}$$

It will be convenient to work with the difference in the fraction of the two different trader types, which is defined as

$$m_t \equiv n_{1t} - n_{2t} = \tanh \left( \frac{\beta}{2} [\Phi_{1,t-1} - \Phi_{2,t-1}] \right),$$

where the last transformation is obtained by using Eq. (2.10) and the identity

$$\tanh \left\{ \frac{\beta}{2} (a - c) \right\} = \frac{\exp(\beta a) - \exp(\beta c)}{\exp(\beta a) + \exp(\beta c)}.$$

For the same reason we define the difference in fitness of the two trader types as

$$u_t \equiv \Phi_{1,t-1} - \Phi_{2,t-1}.$$

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<sup>19</sup>For a brief overview of different types of bifurcations in a nonlinear system of difference equations in discrete time, see Appendix A 2.11 and the references given therein.

<sup>20</sup>See also Appendices A 2.10, A 2.11 and A 2.12 for an introduction to the dynamics of nonlinear systems.

<sup>21</sup>See also Brock and Hommes (1999) for the following discussion.

Positive, average costs of fundamentalists are introduced in the fitness function  $\Phi_{1,t}$  (see Eq. (2.11)), here accounting for training costs, to understand how the market behaves according to the fundamental pricing rule. The training costs are denoted by  $C \equiv C_1$ . Now, plugging Eq. (2.19) and Eq. (2.20) in Eq. (2.14)

$$Rx_t = \sum_{h=1}^2 n_{h,t} f_{h,t} = n_{2,t} g x_{t-1}$$

and using

$$n_{2,t} = \frac{n_{2,t} + n_{2,t}}{2} = \frac{n_{2,t} + (1 - n_{1,t})}{2} = \frac{1 - m_t}{2}$$

as well as

$$\begin{aligned} \Phi_{1,t-1} - \Phi_{2,t-1} &= (\phi_{1,t-1} - \phi_{2,t-1}) - C + \gamma(\Phi_{1,t-2} - \Phi_{2,t-2}) \\ &\stackrel{(2.17)}{=} \{x_{t-1} - Rx_{t-2} + \delta_{t-1}\} \left(-\frac{g x_{t-3}}{a\sigma^2}\right) - C + \gamma u_{t-1} \end{aligned}$$

leads to the following ABS equilibrium dynamics in the case of fundamentalists vs. trend chasers:

$$Rx_t = \frac{1 - m_t}{2} g x_{t-1}, \quad (2.21)$$

$$m_t = \tanh\left(\frac{\beta}{2} \left[\gamma u_{t-1} - \frac{g x_{t-3}}{a\sigma^2} (x_{t-1} - Rx_{t-2} + \delta_{t-1}) - C\right]\right), \quad (2.22)$$

$$u_t = \gamma u_{t-1} - \frac{g x_{t-3}}{a\sigma^2} (x_{t-1} - Rx_{t-2} + \delta_{t-1}) - C, \quad (2.23)$$

where  $\delta_{t-1}$  is the stochastic component in the realized excess return. Assuming i.i.d. dividends, i.e.,  $d_{t-1} = \bar{d} + \epsilon_{t-1}$  with  $\epsilon_{t-1} \sim \mathcal{N}(0, \sigma_\epsilon^2)$ , the stochastic component  $\delta_{t-1}$  simply coincides with  $\epsilon_{t-1}$ . Obviously, the resulting dynamics can be described as a system of nonlinear autonomous, ordinary difference equations, which is of third order in  $x_t$  and of first order in  $u_t$ , or equivalently, as a four dimensional system of first-order ordinary difference equations.<sup>22</sup>

Brock and Hommes (1999) prove the following theorem on the existence and stability results for the steady states of the system Eq. (2.21)–Eq. (2.23):

**Theorem 2.1 (Result taken from Brock and Hommes (1999))**

*Denote*

$$m^{eq} = \tanh\left(-\frac{\beta C}{2(1-\gamma)}\right)$$

*and*

$$m^* = 1 - \frac{2R}{g}.$$

*Let  $x^*$  be the positive solution (if existing) of*

$$\tanh\left(\frac{\beta}{2(1-\gamma)} \left[\frac{(R-1)g}{a\sigma^2} (x^*)^2 - C\right]\right) = m^*.$$

*(i) For  $0 < g < R$ ,  $E_1 = (0, m^{eq})$  is the unique, globally stable steady state.*

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<sup>22</sup>One of the most crucial results in the theory of autonomous, ordinary differential and difference equation systems states that dynamic systems of any order can always be transformed in a system of first-order equations. See for instance Medio and Lines (2001, Chapter 1) for further details.

(ii) For  $g > 2R$ , there are three steady states

$$E_1 = (0, m^{eq}), \quad E_2 = (x^*, m^*) \quad \text{and} \quad E_3 = (-x^*, m^*)$$

with  $E_1 = (0, m^{eq})$  being an unstable saddle point.

(iii) For  $R < g < 2R$ , two possibilities have to be distinguished:

(a) if  $m^* < m^{eq}$ , then  $E_1$  is the unique, locally stable steady state.

(b) if  $m^* > m^{eq}$ , then there are three steady states  $E_1, E_2$  and  $E_3$ ; the steady state  $E_1 = (0, m^{eq})$  is an unstable saddle point.

PROOF: See Appendix A 2.4.

Theorem 2.1 states that if the trend chasers only extrapolate weakly (i.e.,  $0 < g < R$ ), the fundamental steady state  $E_1$  is globally stable. For no trading costs ( $C = 0$ ) the difference in the fraction of the trader type populations is zero for any  $\beta$ . If  $C > 0$ , the fundamentalists vanish from the market when the switching intensity of choice  $\beta$  goes to  $+\infty$  or when the memory parameter  $\gamma$  goes to  $+1$  (i.e., for infinite memory). Since it is useless to pay for a trading strategy when staying in a stable steady state, this result is economically reasonable.<sup>23</sup> For strong extrapolation values ( $g > 2R$ ), two further steady states  $E_2$  and  $E_3$  arise regardless of positive or zero information costs. Finally, if  $R < g < 2R$ , a bifurcation scenario occurs. If the intensity of choice  $\beta$  or the memory parameter is high enough, the stable steady state  $E_1$  becomes unstable and two further (stable) steady states arise. Thus both an increase in memory and an increase in the intensity of choice up to a certain critical value lead to a pitchfork bifurcation and leave the steady state  $E_1$  as an unstable saddle point.

Since an analytical treatment of the local stability of the non-fundamental steady states is no longer an easy task, Brock and Hommes also study the system by numerical simulations. As in the case of no memory ( $\gamma = 0$ ) and increasing the intensity of choice  $\beta$  (see the previous discussion in Subsection 2.1.2) they show that a Hopf-bifurcation again arises when the memory strength  $\gamma$  now serves as a bifurcation parameter and reaches a certain bifurcation value  $\gamma^*$ . After this, secondary bifurcation periodic or quasi-periodic fluctuations occur. Finally, increasing  $\gamma$  even up to values very close to 1, the orbits converge to the fundamental steady state. Interestingly, they find that the attractors for the system Eq. (2.21)–Eq. (2.23) in the  $(x_t, m_t)$ -plane are very similar to the attractors generated in Brock and Hommes (1998) by investigating  $\beta$  as a bifurcation parameter. They conjecture that in both cases the system gets close to having a homoclinic orbit<sup>24</sup>.

## 2.2 Selected Modified Versions of the BH Model

In this section I will consider three modifications of the original BH model. First I will present an extension of the BH model in which the traders are allowed to form time dependent beliefs

<sup>23</sup>In this scenario the fundamentalists can never dominate over trend chasers since we always find  $m^{eq} \in ] -1, 0]$ , which is a result of both the shape of the  $\tanh(x)$  function and the (economic reasonably) positively chosen parameters therein.

<sup>24</sup>In the simplest case, homoclinic orbits are orbits that connect a fixed point to itself. More formally put, the orbit  $p$  is said to be *homoclinic to an invariant set  $S$  of a flow  $\phi_t$  (a map  $G$ )* if  $\phi_t(p) \rightarrow S$  as  $t \rightarrow +\infty$  and as  $t \rightarrow -\infty$  (resp.,  $G^n(p) \rightarrow S$  as  $n \rightarrow +\infty$  and as  $n \rightarrow -\infty$ ). See Medio and Gallo (1992, pp. 38–39) for details.

about the first *and* second moments of the risky asset's price movements. In this more complex model framework, the bifurcation routes to chaos will again be analyzed analytically, but, numerical aspects of investigation will also become more important. Additionally, I will also investigate my own slightly different version of this model framework, where more realistic assumptions are implemented. Afterwards, I will briefly discuss a further modification of the BH framework by Gaunersdorfer and Hommes allowing to generate quite realistic properties in both prices and returns, which are well known from empirical time series such as the volatility clustering phenomenon and the fat-tail property of the returns distribution.

### 2.2.1 Gaunersdorfer's Extension of the BH Model

As in the example before, the model framework proposed by A. GAUNERSDORFER<sup>25</sup> contains the same trader types, namely fundamentalists and trend chasers. But compared to the BH model outlined above, two main extensions are implemented which overcome the following two limitations of the original model. On the one hand, the fitness measure assumed by Brock and Hommes (see Eq. (2.12) and Eq. (2.13)) do not take into account the risk with which the traders are faced in every trading period when choosing their optimal demand for the risky asset. This might be slightly unrealistic as the BH framework is originally based on a myopic mean-variance approach, which means that in their optimal portfolio decisions traders do take this risk into account.<sup>26</sup> Thus Gaunersdorfer proposes to use the 'realized risk adjusted profits' as an adequate candidate for the performance measure instead of using the 'accumulated realized profits' proposed in the original publication. On the other hand, besides the updating of traders' beliefs according to a risk adjusted performance measure, the traders should also update their expectations of conditional variances of returns over time.<sup>27</sup>

More precisely, in Gaunersdorfer (2000) the realized risk adjusted profits are formulated as

$$\phi_{riskadj,t} = R_t z^{opt} - \frac{a}{2} (z^{opt})^2 Var_{h,t-1}[R_t], \quad (2.24)$$

where  $R_t$  denotes the realized excess returns in period  $t$  and  $z^{opt}$  denotes the optimal demand for the risky asset.<sup>28</sup> Since subtracting a constant term from the performance measure does not change the discrete choice fractions Eq. (2.10) the expression Eq. (2.24) can be transformed into the following fitness function

$$\phi_{h,t} = - \frac{1}{2a Var_{t-1}[x_t - Rx_{t-1} + \delta_t]} (x_t - f_{h,t-1})^2, \quad (2.25)$$

which uses (risk adjusted) squared prediction errors.<sup>29</sup> Note that for analytical tractability reasons, homogeneous expectations about conditional variances of returns are assumed in Eq. (2.25), i.e.,

$$Var_{h,t}[\cdot] = Var_t[\cdot].$$

Assuming furthermore that

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<sup>25</sup>See Gaunersdorfer (2000).

<sup>26</sup>See also the comments given in Brock and Hommes (1999); Hommes (2001) on that topic.

<sup>27</sup>For my own model framework proposed in Chapter 4 a similar approach is used.

<sup>28</sup>The first term of Eq. (2.24) has an identical meaning as the right-hand side of Eq. (2.12). See also Appendix A 2.3 for the notation.

<sup>29</sup>We relegate the respective derivations to the Appendix A 2.5.

- agents have homogeneous, *time varying* expectations of conditional variances of returns, i.e.,

$$\text{Var}_{t-1}[x_t - Rx_{t-1}] = \sigma_{t-1}^2 ,$$

- the variance of the martingale difference sequence (see Eq. (2.13)) is constant over time, i.e.,

$$\text{Var}_{t-1}[\delta_t] = \sigma_\delta^2 ,$$

and the covariance of  $x_t - Rx_{t-1}$  and  $\delta_t$  is always zero, i.e.,

$$\text{Cov}[x_t - Rx_{t-1}, \delta_t] \equiv 0$$

the performance measure Eq. (2.25) of trader  $h$  at time  $t$  can be written as

$$\phi_{h,t} = -\frac{1}{2a(\sigma_{t-1}^2 + \sigma_\delta^2)}(x_t - f_{h,t-1})^2 .$$

Finally, the traders' time varying beliefs about variances are assumed to be weighted averages according to

$$\sigma_t^2 = \theta_\sigma \sigma_{t-1}^2 + (1 - \theta_\sigma)(x_{t-1} - Rx_{t-2} - \mu_{t-1})^2 , \quad (2.26)$$

$$\mu_t = \theta_\mu \mu_{t-1} + (1 - \theta_\mu)(x_{t-1} - Rx_{t-2}) , \quad (2.27)$$

where  $0 < \theta_\sigma, \theta_\mu < 1$ .  $\mu_t$  defines the exponential moving averages of returns. As in the last example, fundamentalists are faced with positive costs  $C \equiv C_1 > 0$  while for the trend chasers  $C_2 = 0$  is assumed. Introducing an additional 'stabilizing force'  $+\alpha x_t^2$  to the performance measure  $\phi_{1,t}$  of the fundamentalists and choosing the memory parameter  $\gamma$  equal to zero, the Adaptive Rational Equilibrium Equations of the ABS are given by

$$Rx_t = \sum_{h=1}^2 n_{h,t} f_{h,t} , \quad (2.28)$$

$$n_{h,t} = \frac{\exp\{\beta \Phi_{h,t-1}\}}{\sum_{h=1}^2 \exp\{\beta \Phi_{h,t-1}\}} , \quad h = 1, 2 , \quad (2.29)$$

$$\sigma_t^2 = \theta_\sigma \sigma_{t-1}^2 + (1 - \theta_\sigma)(x_{t-1} - Rx_{t-2} - \mu_{t-1})^2 , \quad (2.30)$$

$$\mu_t = \theta_\mu \mu_{t-1} + (1 - \theta_\mu)(x_{t-1} - Rx_{t-2}) \quad (2.31)$$

with

$$f_{1,t} = 0 ,$$

$$f_{2,t} = gx_{t-1} , \quad g > 0 ,$$

$$\Phi_{h,t} = \phi_{h,t} , \quad h = 1, 2 ,$$

$$\phi_{1,t} = -\frac{1}{2a(\sigma_{t-1}^2 + \sigma_\delta^2)} x_t^2 - C + \alpha x_t^2 ,$$

$$\phi_{2,t} = -\frac{1}{2a(\sigma_{t-1}^2 + \sigma_\delta^2)} (x_t - gx_{t-2})^2 .$$

After setting  $m_t \equiv n_{1,t} - n_{2,t} = 1 - 2n_{2,t}$  and introducing new variables  $\check{\sigma}_t \equiv \sigma_{t-1}$  and  $\check{\mu}_t \equiv \mu_{t-1}$ , which reduces the dimension of the system by one,<sup>30</sup> the dynamic system Eq. (2.28)–Eq. (2.31)

<sup>30</sup>See also Gaunersdorfer (2000, pp. 807–808) for details.

yields

$$\begin{aligned} x_t &= \frac{1 - m_t}{2R} g x_{t-1}, \\ m_t &= \tanh \left( \frac{\beta}{2} \left[ \frac{g}{2a(\check{\sigma}_{t-1}^2 + \sigma_\delta^2)} x_{t-3} (g x_{t-3} - 2x_{t-1}) + \alpha x_{t-1}^2 - C \right] \right), \\ \check{\sigma}_t^2 &= \theta_\sigma \check{\sigma}_{t-1}^2 + (1 - \theta_\sigma) (x_{t-2} - R x_{t-3} - \check{\mu}_{t-1})^2, \\ \check{\mu}_t &= \theta_\mu \check{\mu}_{t-1} + (1 - \theta_\mu) (x_{t-2} - R x_{t-3}) \end{aligned}$$

which has quite a similar structure to system Eq. (2.21)–Eq. (2.23). In the last step, this system can be reformulated as a five dimensional autonomous ordinary first-order difference equation by setting  $y_t \equiv x_{t-1}$  and  $z_t \equiv y_{t-1}$  which leads to<sup>31</sup>

$$x_t = \frac{g}{2R} x_{t-1} \left\{ 1 - \tanh \left( \frac{\beta}{2} \left[ \frac{g}{2a(\check{\sigma}_{t-1}^2 + \sigma_\delta^2)} z_{t-1} (g z_{t-1} - 2x_{t-1}) + \alpha x_{t-1}^2 - C \right] \right) \right\}, \quad (2.32)$$

$$y_t = x_{t-1}, \quad (2.33)$$

$$z_t = y_{t-1}, \quad (2.34)$$

$$\check{\sigma}_t^2 = \theta_\sigma \check{\sigma}_{t-1}^2 + (1 - \theta_\sigma) (y_{t-1} - R z_{t-1} - \check{\mu}_{t-1})^2, \quad (2.35)$$

$$\check{\mu}_t = \theta_\mu \check{\mu}_{t-1} + (1 - \theta_\mu) (y_{t-1} - R z_{t-1}). \quad (2.36)$$

Thus, in comparison to the three dimensional system of the previous subsection, Gaunersdorfer included two further dimensions in the model by introducing two further variables, namely the average return  $\mu_t$  (or  $\check{\mu}_t$ ) and the estimate of the conditional variance  $\sigma_t^2$  (or  $\check{\sigma}_t^2$ ).

A rigorous two-parameter bifurcation analysis with respect to the intensity of choice  $\beta$  and the strength of stabilizing force  $\alpha$  reveals a very complicated dynamics along the bifurcation route to chaos.<sup>32</sup> A restriction on the case with a constant beliefs on conditional variance,  $\theta_\sigma = \theta_\mu = 1$ , leads to identical results concerning the steady states analysis like in the original BH model with a memory lag of one period (see Brock and Hommes (1998) and explanations in the previous discussion on this case). Again, one *fundamental steady state* and two *non-fundamental steady states* exist, whose existence and stability depend on both the extrapolation parameter  $g$  and on the intensity of choice parameter  $\beta$ , which we will mainly concentrate on in the following discussion. While for small values of  $\beta$  the fundamental steady state is unique and stable, an increase in  $\beta$  leads to a pitchfork bifurcation and two non-fundamental steady states. These two non-fundamental steady states are either both unstable (referring to a subcritical pitchfork bifurcation) or both stable (referring to a supercritical pitchfork bifurcation) depending on the value of the additional parameter  $\alpha$ . Furthermore, due to the additional parameter  $\alpha$  the situation becomes even more extensive for the secondary bifurcation than in the original BH model proposed by Brock and Hommes (1998). For relatively small  $\alpha$ , the non-fundamental steady states again undergo a Hopf bifurcation, while for larger realizations of  $\alpha$ , a flip bifurcation is obtained. Thus the specification of the stabilizing force  $\alpha$  serves as a further bifurcation parameter. Remarkably, the primary bifurcation of the fundamental steady state and the secondary bifurcations of the non-fundamental steady states even hold in the case of time varying beliefs, i.e., for  $0 < \theta_\sigma, \theta_\mu < 1$ .<sup>33</sup>

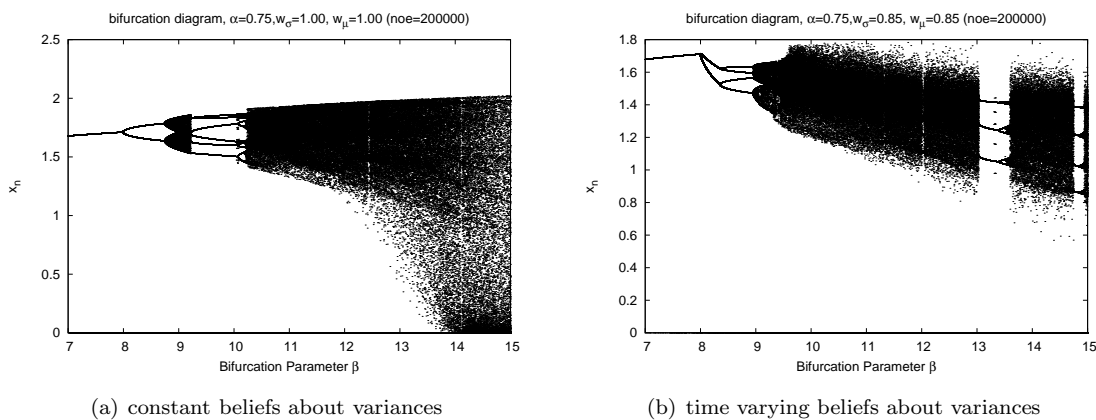
<sup>31</sup>Again we refer to Medio and Lines (2001, Chapter 1) for details.

<sup>32</sup>See Gaunersdorfer (2000) for this statement and the following discussion.

<sup>33</sup>For a detailed discussion on the existence and the stability conditions of the steady states in both cases we can refer to Gaunersdorfer (2000).

Of course, high-order bifurcations might be different for both cases as numerical simulations show.

Figure 2.1 refers to these results. It exhibits a bifurcation diagram with respect to the intensity of choice parameter  $\beta$  for a large value of  $\alpha$ . For example, while for the constant beliefs case

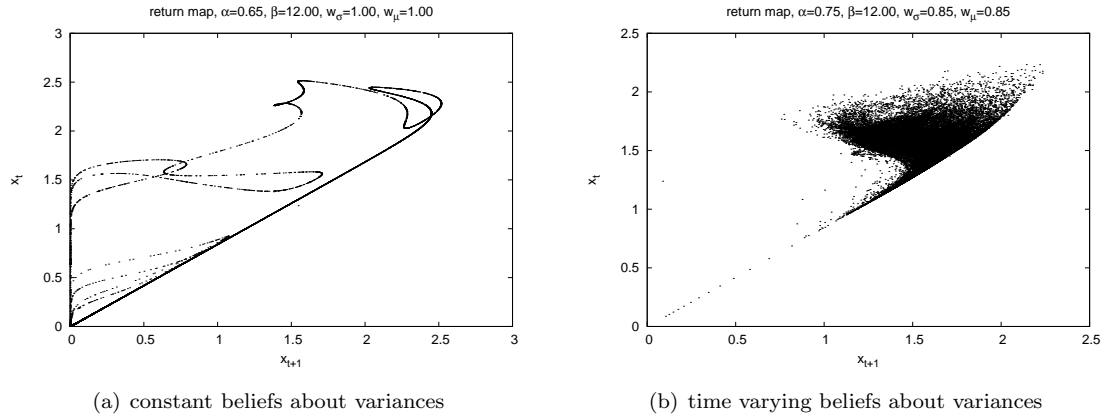


**Figure 2.1:** Bifurcation diagrams with respect to the intensity of choice parameter  $\beta$ . The left panel exhibits the case of constant beliefs on conditional variances of returns ( $\theta_\sigma = \theta_\mu = 1$ ), whereas the right panel displays the time varying case ( $\theta_\sigma = \theta_\mu = 0.85$ ). In both cases  $\alpha$  is chosen to be large ( $\alpha = 0.75$ ) and the secondary flip bifurcation of one of the non-fundamental steady states can be observed. As in all following plots on this model, the plot is generated by a GAUSS program which I wrote based on the model description given in the original publication. For each plot 200,000 experiments with different values of  $\beta$  (between 7 and 15) are conducted where in each experiment 5000 iterations are performed. The initial conditions for the dynamic variables are chosen randomly for each experiment. Both plots reproduce the bifurcation diagrams found by Gaunersdorfer (2000) upon a visual inspection. The parameters are chosen identically to the original article by Gaunersdorfer, i.e.,  $g=1.2$ ,  $C=1.0$ ,  $R=1.01$ ,  $a=10.0$ ,  $\sigma^2=0.1$  (left panel) and  $\sigma_\beta^2=0.1$  (right panel).

the flip bifurcation is destabilized by a Hopf bifurcation until a 6-cycle is created, in the case of time varying beliefs on conditional variances of returns a further period doubling occurs before it is destabilized by a Hopf bifurcation. For even larger values of  $\beta$  (but not too large), strange attractors occur. Again, both cases, i.e., time varying and constant beliefs about variances, have to be distinguished. As shown in Figure 2.2, the phase portraits in the  $(x_{t+1}, x_t)$ -plane and the time series of strange attractors differ in both scenarios.

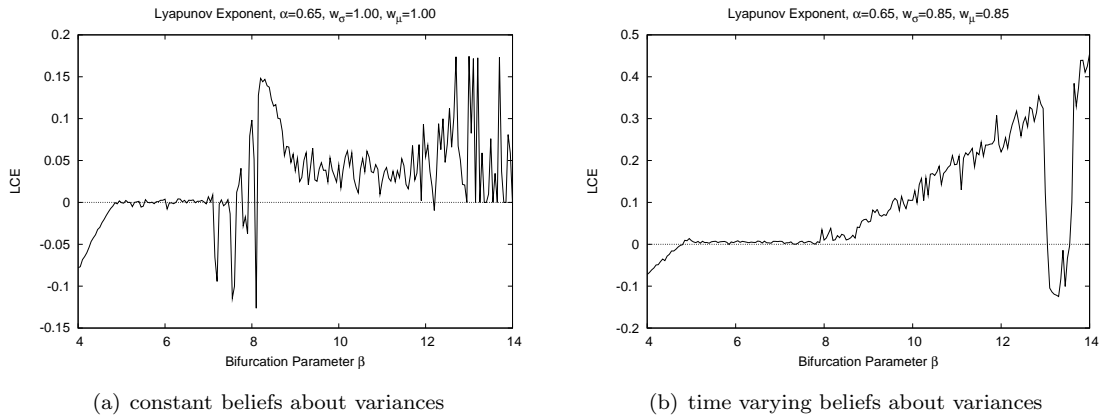
The bifurcation diagrams in Figure 2.1 seem to indicate that chaos arises in our dynamic system. To confirm this conjecture the largest Lyapunov characteristic exponents (LCE) are computed for the system (see Figure 2.3). As outlined in Appendix A 2.12 the LCE measures the asymptotic exponential rate of divergence (or convergence) of orbits. Thus a positive LCE proves a sensitive dependence on initial conditions of the system because, in this case, trajectories which were initially close to each other now separate exponentially as time evolves. In both model variations, with constant and time varying beliefs on conditional variances of returns, positive LCE arise for sufficiently large values of the intensity of choice  $\beta$ . Hence the system exhibits chaotic behavior in such parameter ranges.

Finally, Figure 2.4 shows the time series of the dynamic variables, namely the price deviations from the fundamental  $x_t$  and the difference in the population fraction of the two different trading types  $m_t$ . Furthermore, the return process is exhibited, where the return is defined as  $(x_t - Rx_{t-1})$ ,



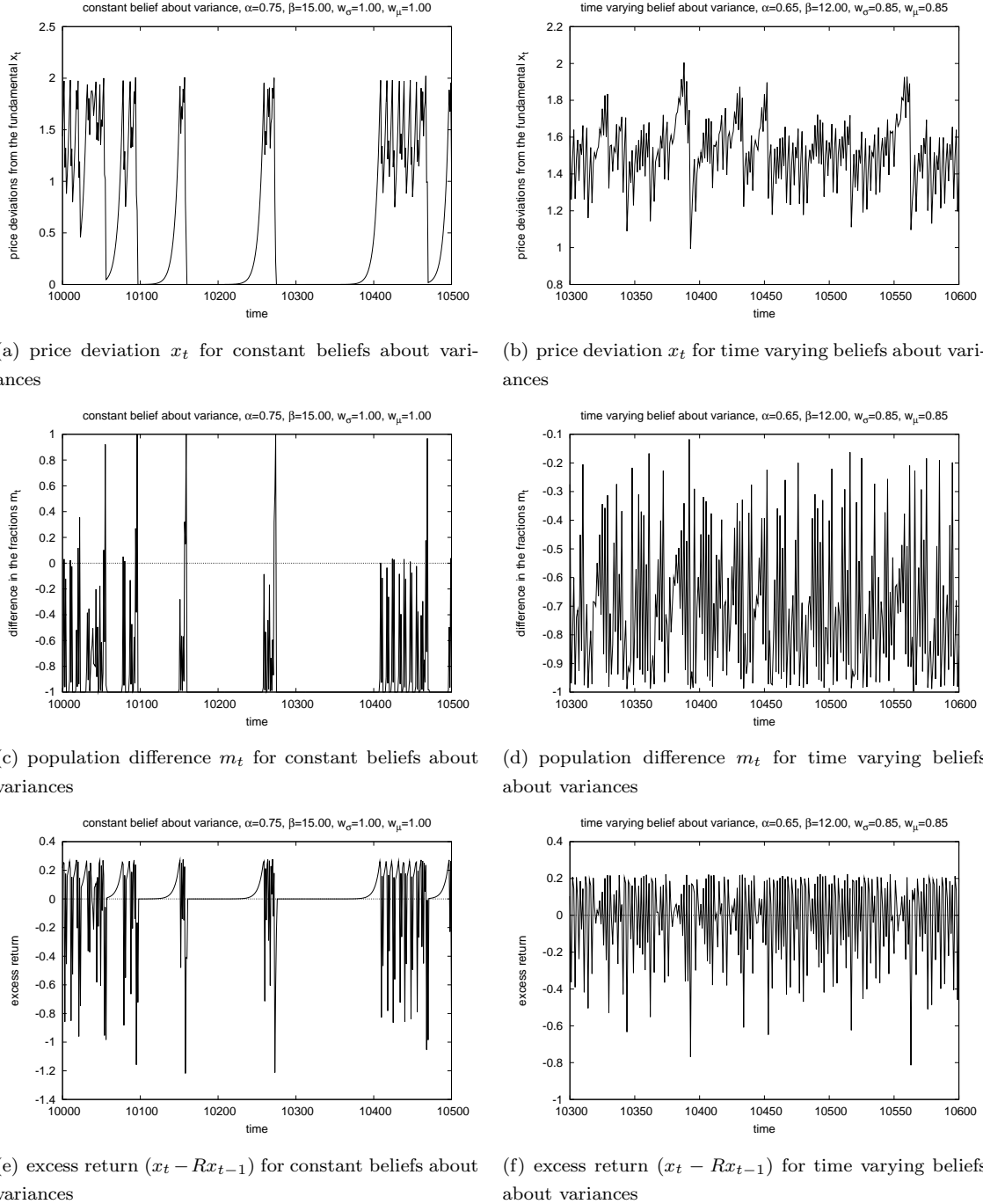
**Figure 2.2:** Projection of strange attractors on the  $(x_{t+1}, x_t)$ -plane. As shown in Gaunersdorfer (2000) the shape of the attractors may differ for the case with constant and time varying beliefs about variances. The return maps are generated by 50,000 iterations and appear visually identical to the findings by Gaunersdorfer. Parameters are  $\beta=12.0$ ,  $\alpha=0.65$ ,  $\theta_\sigma = \theta_\mu = 1$  (left panel) and  $\theta_\sigma = \theta_\mu = 0.85$  (right panel). The remaining parameter set is chosen as in Figure 2.1.

i.e., as excess returns. Let us first consider the case of constant beliefs on conditional variances of returns. Obviously, the price evolution is characterized by irregular switching between periods where prices are close to the fundamental steady state (i.e.,  $x_t=0$ ), followed by episodes of upwards trends and interim stronger deviations from it, which are again replaced after a while by a return back to the fundamental. This price pattern evolves rather persistently over time. The model based explanation of such a phenomenon is quite straightforward. Assume the dynamic system is close to the fundamental steady state. Then both predictors provide good forecasts for future



**Figure 2.3:** Largest Lyapunov characteristic exponents. After the secondary bifurcation of the non-fundamental steady states (quasi-)periodic dynamics arises ( $\lambda=0$ ). For large parameter values of  $\beta$  chaotic behavior is exhibited interspersed with stable cycles for higher  $\beta$  values ( $\lambda < 0$ ). All parameters are chosen as in Figure 2.2. In each experiment (i.e., for all  $\beta$  values), the system is run for 500 iterations before the Lyapunov calculation is started. The iteration length for the determination of LCE is chosen as 500. The initial condition of the dynamic system is drawn randomly and then fixed for all experiments. Again as concerns the visual appearance, the plots reproduce the results from Gaunersdorfer (2000).



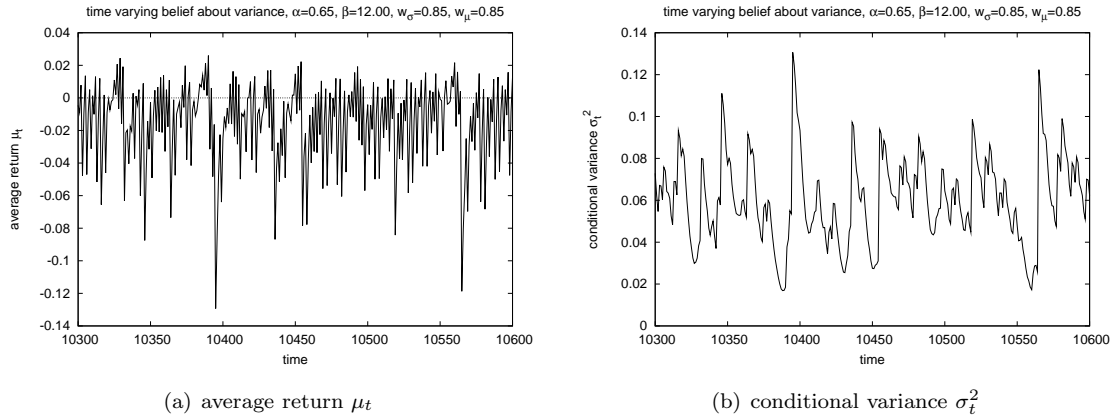


**Figure 2.4:** Time series of dynamic variables in the constant and time varying case. On the left-hand side the time series of price deviations from the fundamental  $x_t$ , difference in the fractions  $m_t$  and returns are shown for the case of constant beliefs on conditional variances of returns with  $\alpha=0.75$ ,  $\beta=15.0$ ,  $\theta_\sigma = \theta_\mu=1$ , while the right-hand side exhibits the same dynamic variables for time varying beliefs with  $\alpha=0.65$ ,  $\beta=12.0$ ,  $\theta_\sigma = \theta_\mu=0.85$ . In all plots the dynamic variables are randomly initialized. The remaining parameters are again identical to the previous plots. On a visual level the plots reproduce the findings in the original publication.

price developments. Since the fundamental trading strategy implies costs, most traders will choose the alternative strategy which will move the current price away from its fundamental value. Hence

an upward moving price trend is created. But at a certain point the stabilizing force  $\alpha$  gives the fundamentalist strategy enough weight to push the price back to the fundamental steady state. This mechanism gives an economic motivation for the observed irregular price movements. Note also that a deviation from the fundamental steady state comes along with an increasing fluctuation of returns.

In the case of time varying beliefs on conditional variances of returns, the situation is somehow more regular (see Figure 2.4, right panels). Figure 2.5 also refers to this case and exhibits the time series of average returns and conditional variances. The price evolution no longer exhibits such



**Figure 2.5:** Time series of average return  $\mu_t$  and conditional variance  $\sigma_t^2$ . The data are taken from the same simulation run as shown on the right-hand side of Figure 2.4.

clearly apparent changes between periods where prices equal the fundamental value and converse periods where rapidly upwards trend arises. Instead, the observed price is permanently above the fundamental price. However, episodes with stronger deviations from the average price level can still be detected. In addition, periods with low and high levels of volatilities can be observed.

One final remark should be made on the price evolution in this model structure. It might be striking that the price deviations from the fundamental  $x_t$  never become negative (see Figure 2.4 (a) and (b)). This is due to the fact that the evolution of  $x_t$  mainly depends on a tangens hyperbolicus term which can only take values between the range of -1 and +1. More precisely, the equation of motion for  $x_t$  is given by Eq. (2.32), from which it can immediately be seen that  $x_t$  can never become negative whenever  $x_t$  is initialized with non negative values.

### 2.2.2 A More Realistic Model Setup of Gaunersdorfer's Version of the BH Model

In this subsection I want to introduce a slightly different version of the previous model framework, where traders use a modified estimate of the conditional variance. In Gaunersdorfer (2000) it is assumed that all investors are *homogeneous* in their time varying expectations of conditional variances of returns using exponential moving averages of past realized excess returns according to Eq. (2.26) and Eq. (2.27). In other words, both investor types use the history of realized deviations from the fundamental price in a uniform manner to extrapolate the future conditional variances of returns. In this subsection a slightly different version of the fitness function for the two trading

strategy groups  $h$  is considered which in my view is more in line with their *different* beliefs about the deviations from the fundamental price.<sup>34</sup> As before, the memory parameter  $\gamma$  is restricted to being zero, which leads to

$$\Phi_{h,t} = \phi_{h,t}, \quad h = 1, 2.$$

Then, the fitness function is given by

$$\phi_{h,t} = -\frac{1}{2a\text{Var}_{h,t-1}[x_t - Rx_{t-1} + \delta_t]} (x_t - f_{h,t-1})^2, \quad (2.37)$$

where now

$$\begin{aligned} \text{Var}_{h,t-1}[x_t - Rx_{t-1}] &= \sigma_{h,t-1}^2, \\ \text{Var}_{h,t-1}[\delta_t] &= \sigma_\delta^2, \\ \text{Cov}[x_t - Rx_{t-1}, \delta_t] &\equiv 0 \end{aligned}$$

with

$$\sigma_{h,t}^2 = \theta_{h,\sigma} \sigma_{h,t-1}^2 + (1 - \theta_{h,\sigma})(x_{t-1} - Rx_{t-2} - \mu_{h,t-1})^2 \quad (2.38)$$

and  $0 < \theta_{h,\sigma} < 1$  for  $h = 1, 2$  is assumed. Therefore the traders' time varying beliefs about variances are again formulated as exponential moving averages but now not longer necessarily have to be identical for both trading strategies. Instead, the expected return  $\mu_{h,t-1} \equiv E_{h,t-2}[x_{t-1} - Rx_{t-2}]$  is allowed to be different for both strategy types and specified as follows. Note that

$$E_{h,t}[x_{t+1}] = E_{h,t}[p_{t+1} - p_{t+1}^*] = E_t[p_{t+1}^*] + f_{h,t} - E_t[p_{t+1}^*] = f_{h,t}$$

with  $f_{1,t} = 0$  and  $f_{2,t} = gx_{t-1}$  for  $h = 1, 2$ , which states that all traders expect the deviations  $x_{t+1}$  from the fundamental price for the next time period to be equal to their own forecasting rule  $f_{h,t}$  at time  $t$ . Hence, in this modified version of Gaunersdorfer's model, both trader types are assumed to use their corresponding forecasting rules to derive the expectation value for the relevant expression  $(x_{t-1} - Rx_{t-2})$  in Eq. (2.38), which can be expressed in terms of the expected return  $\mu_{h,t-1}$  as

$$\begin{aligned} \mu_{1,t-1} &= 0 \\ \mu_{2,t-1} &= gx_{t-3} - Rgx_{t-4}. \end{aligned}$$

Making these assumptions, the Adaptive Rational Equilibrium Equations of the ABS are given by

$$Rx_t = \sum_{h=1}^2 n_{h,t} f_{h,t}, \quad (2.39)$$

$$n_{h,t} = \frac{\exp\{\beta\Phi_{h,t-1}\}}{\sum_{h=1}^2 \exp\{\beta\Phi_{h,t-1}\}} \quad (2.40)$$

$$\sigma_{h,t}^2 = \theta_{h,\sigma} \sigma_{h,t-1}^2 + (1 - \theta_{h,\sigma})(x_{t-1} - Rx_{t-2} - \mu_{h,t-1})^2, \quad h = 1, 2, \quad (2.41)$$

with

$$f_{1,t} = 0,$$

<sup>34</sup>Chiarella and He (2002c) also introduce heterogeneity in beliefs about variances.

$$\begin{aligned}
f_{2,t} &= gx_{t-1}, \\
\mu_{1,t} &= 0, \\
\mu_{2,t} &= g(x_{t-2} - Rx_{t-3}), \quad g > 0, \\
\Phi_{h,t} &= \phi_{h,t}, \\
\phi_{1,t} &= -\frac{1}{2a(\sigma_{1,t-1}^2 + \sigma_\delta^2)} x_t^2 - C + \alpha x_t^2, \\
\phi_{2,t} &= -\frac{1}{2a(\sigma_{2,t-1}^2 + \sigma_\delta^2)} (x_t - gx_{t-2})^2.
\end{aligned}$$

Introducing again appropriate variables  $\check{\sigma}_{1,t} \equiv \sigma_{1,t-1}$ ,  $\check{\sigma}_{2,t} \equiv \sigma_{2,t-1}$ ,  $y_t \equiv x_{t-1}$ ,  $z_t \equiv y_{t-1}$ ,  $a_t \equiv z_{t-1}$  and  $b_t \equiv a_{t-1}$  to formulate the dynamic system as an autonomous ordinary first-order difference equation, Eq. (2.39)–Eq. (2.41) can be rewritten as

$$x_t = \frac{g}{2R} x_{t-1} \left\{ 1 - \tanh \left( \frac{\beta}{2} \left[ \frac{(x_{t-1} - gz_{t-1})^2}{2a(\check{\sigma}_{2,t-1}^2 + \sigma_\delta^2)} - \frac{x_{t-1}^2}{2a(\check{\sigma}_{1,t-1}^2 + \sigma_\delta^2)} + \alpha x_{t-1}^2 - C \right] \right) \right\}, \quad (2.42)$$

$$y_t = x_{t-1}, \quad (2.43)$$

$$z_t = y_{t-1}, \quad (2.44)$$

$$a_t = z_{t-1}, \quad (2.45)$$

$$b_t = a_{t-1}, \quad (2.46)$$

$$\check{\sigma}_{1,t}^2 = \theta_{1,\sigma} \check{\sigma}_{1,t-1}^2 + (1 - \theta_{1,\sigma})(y_{t-1} - Rz_{t-1})^2, \quad (2.47)$$

$$\check{\sigma}_{2,t}^2 = \theta_{2,\sigma} \check{\sigma}_{2,t-1}^2 + (1 - \theta_{2,\sigma})[(y_{t-1} - Rz_{t-1}) - g(a_{t-1} - Rb_{t-1})]^2, \quad (2.48)$$

which is a seven dimensional system. Hence, in contrast to the five dimensional system investigated in Gaunersdorfer (2000), the dimension of the dynamic system has increased by two dimensions. Furthermore, in comparison to the original BH model, four further dimensions are included in the model by introducing two variables which describe the different estimates of the conditional variance  $\sigma_{h,t}^2$  of both trader types, and two further time lags of the deviation  $x_t$  from the fundamental price which account for the investors former beliefs about this deviation  $x_t$ .

A rigorous two-parameter bifurcation analysis with respect to the parameters  $\beta$  and  $\alpha$  reveals similar results as proved in Gaunersdorfer (2000). Again it is helpful to first investigate the reduced five dimensional system with constant beliefs about variances, i.e.,  $\theta_{1,\sigma} = \theta_{2,\sigma} = 1$  and  $\check{\sigma}_{1,t}^2 = \check{\sigma}_{2,t}^2 \equiv \sigma^2$ , first.<sup>35</sup> Due to this more simple case, the following theorem about the steady states and their stability properties can be obtained:

**Theorem 2.2** *Let  $R, a, g, \alpha, \beta, \sigma^2 > 0$  as well as  $C > 0$  and  $\theta_{1,\sigma} = \theta_{2,\sigma} = 1$ . Furthermore, assume that  $\check{\sigma}_{1,t}^2 = \check{\sigma}_{2,t}^2 \equiv \sigma^2$  and  $\sigma_\delta^2 = 0$ .*

- (i) *If  $0 < g < R$ , the system Eq. (2.42)–Eq. (2.46) has a unique, globally stable (fundamental) steady state  $E_1 = (0, 0, 0, 0, 0)$ .*

<sup>35</sup>More precisely, in the case of constant beliefs about variances  $Var_{i,t}[\cdot] \equiv \sigma^2$  is assumed in Eq. (2.37) and consequently  $\sigma_\delta^2$  is assumed to be zero in this case. Instead, in examples where the variance is time dependent,  $\sigma^2$  has to be replaced by  $\sigma_\delta^2$  in the following.

- (ii) If  $R < g < 2R$ , the system Eq. (2.42)–Eq. (2.46) has at least one (fundamental) steady state  $E_1$  which undergoes a pitchfork bifurcation at

$$\beta^* \equiv \frac{1}{C} \ln \left( \frac{R}{g-R} \right).$$

The fundamental steady state  $E_1$  is stable for  $\beta < \beta^*$  and unstable for  $\beta > \beta^*$ . Furthermore, in this case the following subcases have to be considered:

- (a) if  $\alpha < \alpha^* \equiv -g(g-2)/(2a\sigma^2)$ , then the system Eq. (2.42)–Eq. (2.46) has two further unstable (non-fundamental) steady states  $E_{2,3}$  for  $\beta < \beta^*$ , i.e., the pitchfork bifurcation is subcritical.
- (b) if  $\alpha > \alpha^*$ , then the system Eq. (2.42)–Eq. (2.46) has two further stable (non-fundamental) steady states  $E_{2,3}$  for  $\beta^* < \beta < \min(\beta_H, \beta_F)$ , i.e., the pitchfork bifurcation is supercritical.<sup>36</sup>
- (iii) If  $g \geq 2R$ , the system Eq. (2.42)–Eq. (2.46) has three steady states. The fundamental steady state  $E_1$  is unstable for all  $\beta > 0$ . The two non-fundamental steady states are stable for  $\beta < \min(\beta_H, \beta_F)$  (see Footnote 36).

PROOF: See Appendix A 2.6.

Theorem 2.2 describes the existence and stability of steady states as well as the local behavior of the system near the steady states with respect to the intensity of choice parameter  $\beta$ . Consequently, the results coincide with the results found by Gaunersdorfer (2000) since the system Eq. (2.42)–Eq. (2.48) equals the System (10) in Gaunersdorfer (2000) for the case of constant beliefs about variances. Hence, when the trend-following group only extrapolates weakly ( $0 < g < R$ ), then the fundamental steady state  $E_1$  is globally stable. Furthermore, defining the difference in the population of the two different trading strategies in the same way as in the previous discussion, namely

$$m_t \equiv n_{1,t} - n_{2,t} = \tanh \left( \frac{\beta}{2} \left[ \frac{(x_{t-1} - gx_{t-3})^2}{2a(\tilde{\sigma}_{2,t-1}^2 + \sigma_\delta^2)} - \frac{x_{t-1}^2}{2a(\tilde{\sigma}_{1,t-1}^2 + \sigma_\delta^2)} + \alpha x_{t-1}^2 - C \right] \right),$$

it can be shown that at the fundamental steady state  $E_1$  all traders are evenly distributed into both trading groups if there are no information costs ( $C = 0$ ) for the fundamentalist group. Indeed, it is

$$m^{eq} \equiv m_{t|E_1} = \tanh \left( \frac{-\beta C}{2} \right)$$

which is zero for  $C = 0$  for all choices of  $\beta$ . Otherwise, if the costs for the fundamentalists are positive ( $C > 0$ ), more traders switch to the group of trend chasers, which makes sense economically, as there is no reason to choose a costly strategy if the dynamic system has settled down to a fundamental equilibrium in which it is  $x_t = p_t - p_t^* = 0$  for all  $t$ . When the trend-following group extrapolate very strongly ( $g > 2R$ ), the fundamental steady state becomes unstable and the dynamic system converges to a non-fundamental equilibrium. The results of the intermediate

<sup>36</sup>See Theorem 2.3 for a definition of  $\beta_H$  and  $\beta_F$ .

case, i.e., when the trend chasers extrapolate strongly ( $R < g < 2R$ ), are also reasonable. If the stabilizing force measured by the parameter  $\alpha$  is small and the switching intensity  $\beta$  is large enough, but not too large, the prices of the artificial market eventually explode since the fundamentalist strategy is costly and therefore, near the fundamental steady state, the traders switch to the other trading strategy. This makes the prices increase and this will not stop if the stabilizing force  $\alpha$  is chosen too small.

The secondary bifurcations for the non-fundamental steady states  $E_{2,3}$  can also be determined analytically. The next theorem shows that  $E_2$  and  $E_3$  become unstable either by a Hopf or a flip bifurcation when the intensity of choice parameter  $\beta$  is even increased beyond the bifurcation value  $\beta^*$  defined in the previous Theorem 2.2:

**Theorem 2.3** *Let  $g > R$  and  $\beta > \beta^*$ . Under the remaining assumptions of Theorem 2.2*

(i) *if  $\alpha^* < \alpha < \alpha^{**} \equiv g^2/(2a\sigma^2)$ , the non-fundamental steady states  $E_{2,3}$  undergo a Hopf bifurcation at  $\beta_H(\alpha)$  which is implicitly defined by Eq. (A 2.25) given in the Appendix A 2.7.*

(ii) *if  $\alpha > \alpha^{**}$ , the non-fundamental steady states  $E_{2,3}$  undergo a flip bifurcation at*

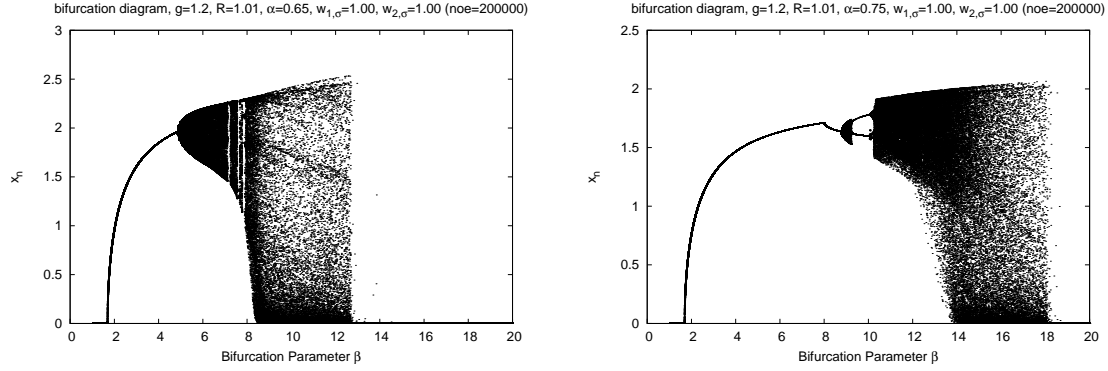
$$\beta_F \equiv \frac{1}{C} \left[ \frac{g}{g-R} + \ln \left( \frac{g-R}{R} \right) \right].$$

PROOF: See Appendix A 2.7.

Theorem 2.3 states that for the secondary bifurcation of the non-fundamental steady states, too, the stabilizing force plays a crucial role for the local behavior of the dynamic system near to its steady states. Depending on the strength of the stabilizing force measured by  $\alpha$ , both  $E_2$  and  $E_3$  undergo a Hopf or a flip bifurcation at  $\beta_H(\alpha)$  and  $\beta_F$  respectively. Figure 2.6 illustrates these results for four different parameter sets. The upper panels show bifurcation diagrams with respect to the intensity of choice parameter  $\beta$  when the trend-following group extrapolates strongly ( $R < g < 2R$ ). In both plots the positive branch of the primary (supercritical) pitchfork bifurcation can be clearly detected. Depending on the choice of the strength of the forcing term, i.e.,  $\alpha$ , the non-fundamental steady state is destabilized either by a Hopf bifurcation (upper left panel) or a flip bifurcation (upper right panel). Note that in the case of the flip bifurcation scenario ( $\alpha$  large), further Hopf bifurcations and period doubling emerge when  $\beta$  is increased even further (see also lower right panel). When the trend chasers extrapolate very strongly ( $g > 2R$ ),  $\beta^*$  becomes negative and  $E_1$  is no longer stable. Hence for small  $\beta$  the system converges immediately to the non-fundamental steady states until they are destabilized by a Hopf bifurcation (lower left panel) or a flip bifurcation (lower right panel).

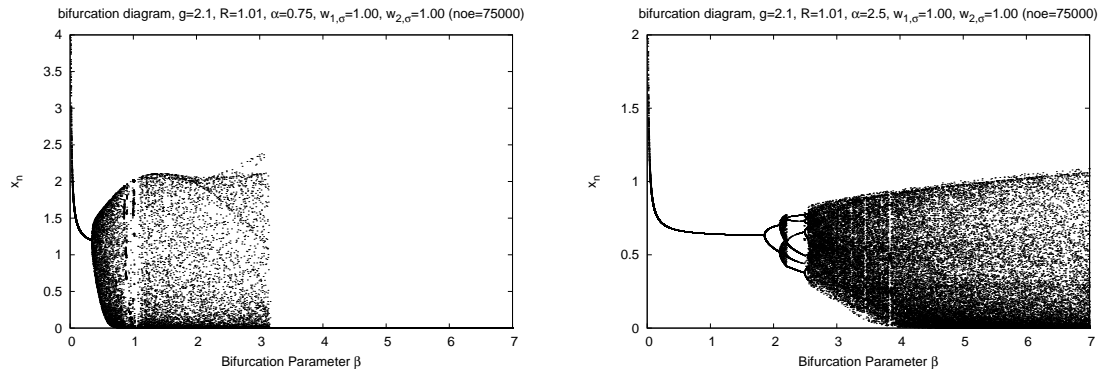
Let us now consider the general case, when the traders are allowed to form heterogeneous time dependent beliefs on the conditional variances of returns. In this case the complete seven dimensional system Eq. (2.42)–Eq. (2.48) has to be investigated. The following theorem can be proved:

**Theorem 2.4** *Let  $R, a, g, \alpha, \beta, \sigma_\sigma^2 > 0$  as well as  $C > 0$ . Assume that  $0 < \theta_{1,\sigma}, \theta_{2,\sigma} < 1$ , i.e., both trader groups form different time-varying beliefs on variances of returns. Then the primary*



(a) supercritical pitchfork and Hopf bifurcation, ( $R < g < 2R$ ,  $\alpha^* < \alpha < \alpha^{**}$ )

(b) supercritical pitchfork and flip bifurcation, ( $R < g < 2R$ ,  $\alpha^* < \alpha^{**} < \alpha$ )



(c) stable non-fundamental steady state and Hopf bifurcation, ( $g > 2R$ ,  $\alpha^* < \alpha < \alpha^{**}$ )

(d) stable non-fundamental steady state and flip bifurcation, ( $g > 2R$ ,  $\alpha^{**} < \alpha$ )

**Figure 2.6:** Bifurcation diagrams with respect to the intensity of choice parameter  $\beta$  for constant beliefs on conditional variances of returns. The upper panels exhibit the case when the trend-following group extrapolate strongly ( $R < g < 2R$ ). Both the upper branch of the pitchfork bifurcation (i.e., the primary bifurcation) and the Hopf bifurcation (i.e., the secondary bifurcation of the non-fundamental steady states for  $\alpha$  being small, upper left panel) or flip bifurcation (i.e., the secondary bifurcation of the non-fundamental steady states for  $\alpha$  being large, upper right panel) can clearly be detected. The lower panels display the case when the trend-chasing group extrapolates very strongly ( $g > 2R$ ). For small  $\alpha$  the non-fundamental steady state undergoes a Hopf bifurcation while for large  $\alpha$  a flip bifurcation occurs. For each plot 200000 (75000) experiments with different  $\beta$  values between 0 and 20 (0 and 7) are conducted, where in each experiment 5000 iterations are performed. The initial conditions for the dynamic variables are chosen randomly for each experiment. The remaining parameters are  $R = 1.01$ ,  $a = 10$ ,  $\sigma^2 = 0.1$ ,  $C = 1$  and  $\theta_{1,\sigma} = \theta_{2,\sigma} = 1$ .

*bifurcation of the fundamental steady state  $E_1$  as well as the secondary bifurcations of the non-fundamental steady states  $E_{2,3}$  coincide with the bifurcations found in Theorems 2.2 and 2.3 for the case of constant beliefs about variances.*

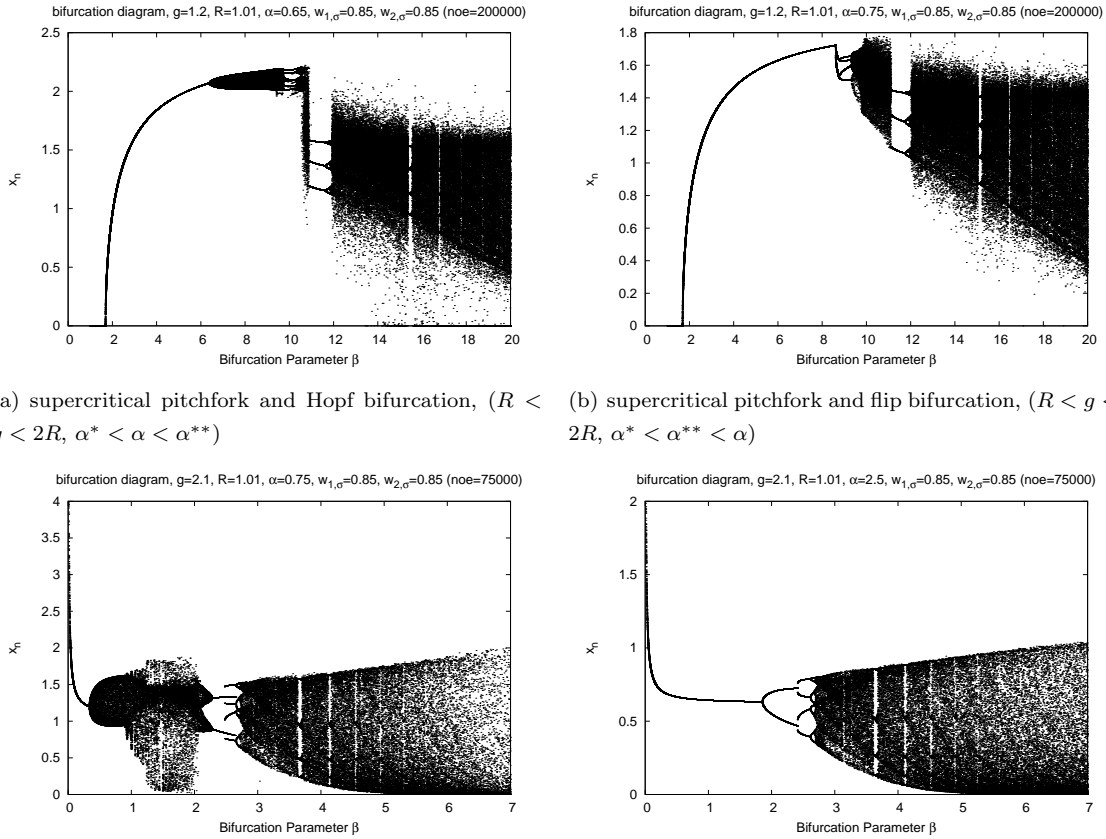
PROOF: See Appendix A 2.8.

Thus, Theorem 2.4 states that for the origin of the bifurcation routes of the fundamental and non-fundamental steady states, the dynamic system Eq. (2.42)–Eq. (2.48) has local behavior similar to that of the reduced system Eq. (2.42)–Eq. (2.46). In the time varying case, the

fundamental steady state  $E_1$  is given by  $E_1 = (0, 0, 0, 0, 0, 0)$  while the non-fundamental equilibria are determined by

$$E_{2,3} = (\pm x^*, \pm x^*, \pm x^*, \pm x^*, \pm x^*, (1-R)^2(x^*)^2, (1-R)^2(1-g^2)(x^*)^2)$$

where  $x^*$  is given by Eq. (A 2.18) when  $\sigma^2$  is replaced by  $\sigma_\delta^2$ .<sup>37</sup> Hence, the steady states also have a similar structure in both the constant beliefs case and the time-varying beliefs case. Nevertheless, Figure 2.7 shows that higher-order bifurcations might be different from the reduced system Eq. (2.42)–Eq. (2.46) as already observed by Gaunersdorfer (2000) (see Subsection 2.2.1, Figure 2.1). For instance, in the lower right panel the flip bifurcation undergoes further period doublings



(a) supercritical pitchfork and Hopf bifurcation, ( $R < g < 2R$ ,  $\alpha^* < \alpha < \alpha^{**}$ )

(b) supercritical pitchfork and flip bifurcation, ( $R < g < 2R$ ,  $\alpha^* < \alpha^{**} < \alpha$ )

(c) stable non-fundamental steady state and Hopf bifurcation, ( $g > 2R$ ,  $\alpha^* < \alpha < \alpha^{**}$ )

(d) stable non-fundamental steady state and flip bifurcation, ( $g > 2R$ ,  $\alpha^{**} < \alpha$ )

**Figure 2.7:** Bifurcation diagrams with respect to the intensity of choice parameter  $\beta$  for time varying beliefs about variances. The primary bifurcation of the fundamental steady state and the secondary bifurcations of the non-fundamental steady states are identical to those in the case of constant beliefs about variances for all chosen parameter sets (see Figure 2.6). Nevertheless, higher-order bifurcations might be different in the time-varying case. The experiments are conducted in exactly the same way as described in the previous Figure 2.6. The weight factors  $\theta_{1,\sigma}$  and  $\theta_{2,\sigma}$  are chosen as  $\theta_{1,\sigma} = \theta_{2,\sigma} = 0.85$ . The remaining parameters are the same as in the constant beliefs case.

instead of being destabilized by a Hopf bifurcation as can be observed in the previous Figure 2.6.

<sup>37</sup>The non-fundamental steady states  $E_{2,3}$  slightly deviate from the non-fundamental steady states found in Gaunersdorfer (2000).



Interestingly, for the time-varying case the bifurcation diagram also exhibits in each case at least one quite expanded region (around  $\beta=11.0$ ) where the system becomes stable again and undergoes a 3-cycle. Note that this property is also shared by the findings reported in Gaunersdorfer (2000).

If we compare our results of an extended BH model with a modified estimate of the conditional variance to those of the extended BH model proposed by Gaunersdorfer (2000) (which has also been outlined in the previous subsection), the following summary can be made. According to the bifurcation route to strange attractors, the local and global behavior of both systems are surprisingly similar and both model outcomes only differ from each other in higher-order bifurcations. For example, comparing the right panel of Figure 2.1 in Subsection 2.2.1 with the upper right panel of Figure 2.7, those deviations in higher-order bifurcations can clearly be detected. Obviously, after entering a chaotic region, stable cycles occur for smaller values of  $\beta$  in our model variant than in the model extension of Gaunersdorfer (2000). Figure 2.8 shows the time series of price deviations  $x_t$  and the corresponding return map for different choices of the bifurcation parameter  $\beta$ . Increasing  $\beta$  leads to increasingly complex behavior in the dynamic system. For  $\beta$  being high enough limit cycles and strange attractors occur.

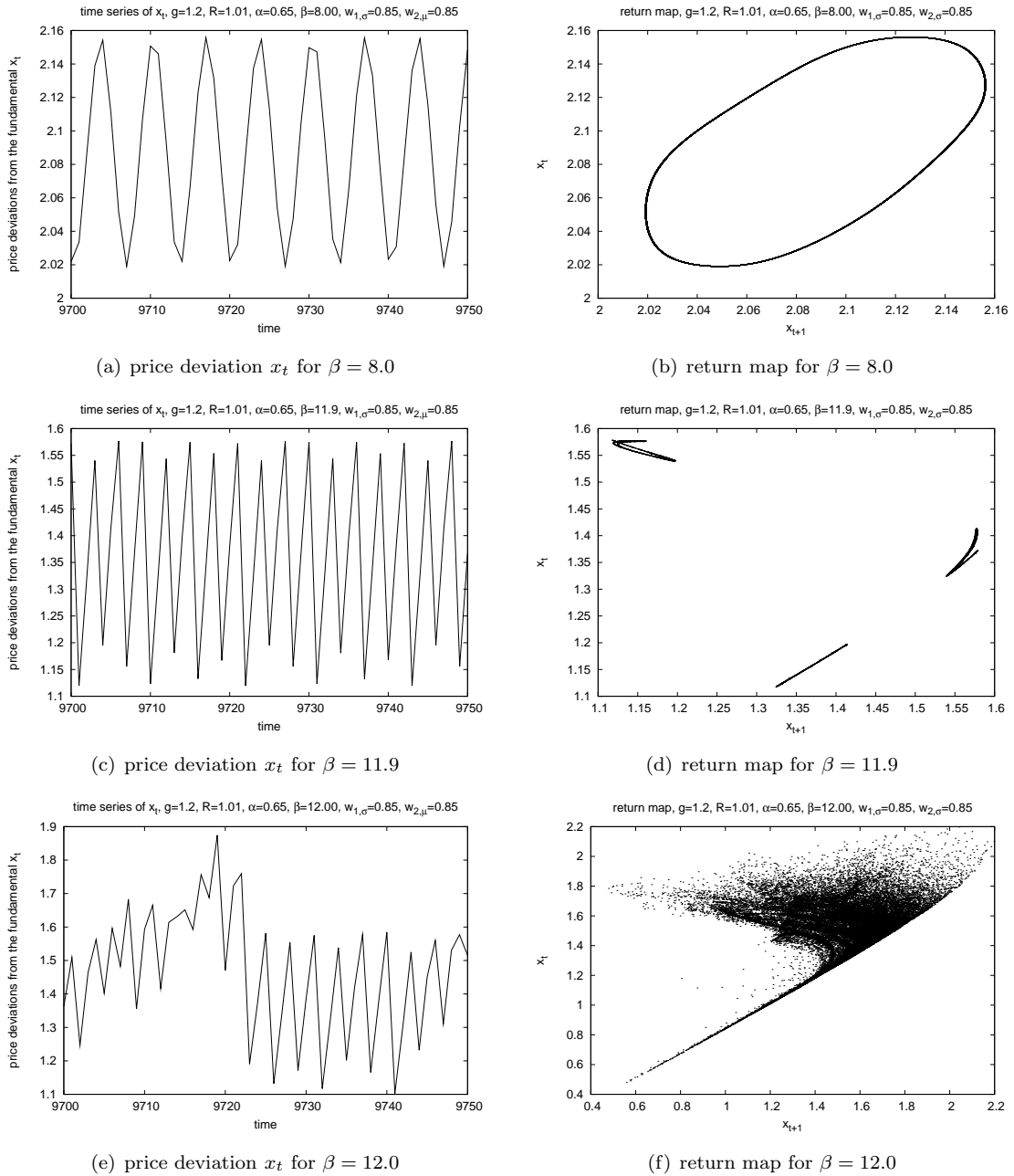
Finally, Figure 2.9 exhibits the largest Lyapunov characteristic exponents for two different parameter sets of our model variant and corresponds to the bifurcation diagrams shown in the upper panels of Figure 2.7. Note that a comparison of the LCEs of both model designs for an identical parameter set (e.g., see Figure 2.3, right panel and Figure 2.9, left panel) again highlights the existence of a different occurrence of stable regions for relatively large values of  $\beta$ . While in Figure 2.3 (b) the system becomes stable again in a region where values of  $\beta$  are slightly larger than 13.0 (i.e., negative LCEs arise), in Figure 2.9 (a) the system already becomes stable for  $\beta$  being slightly smaller than 12.0.

### 2.2.3 Gaunersdorfer and Hommes' Extension of the BH Model

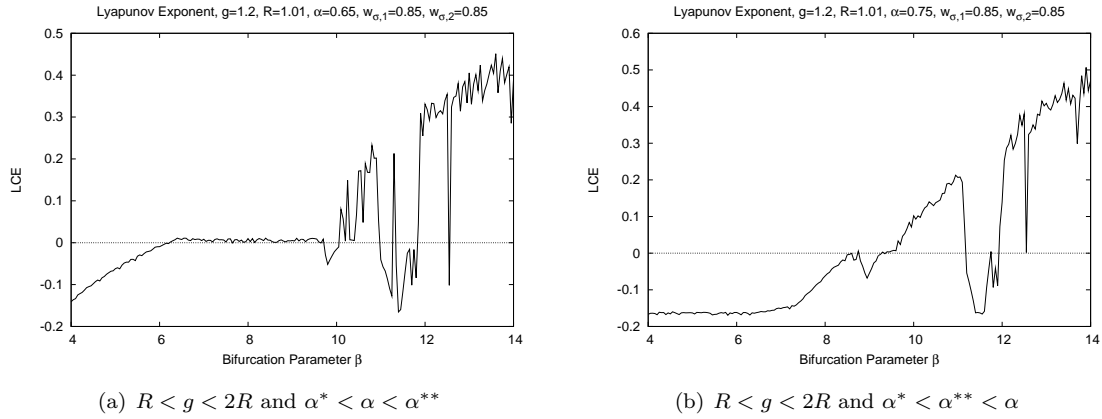
In this subsection I want to very briefly present a further modification of the BH model proposed by A. GAUNERSDORFER and C. H. HOMMES<sup>38</sup>, that mainly aims at generating more realistic time series for both prices and returns in the sense that characteristic properties of real financial markets are reproduced by this artificial market.<sup>39</sup> The main reason for obtaining such artificial time series with more realistic properties is to add some dynamic noise to the nonlinear system. Although we want to concentrate on this noisy evolutionary ABS in this section it might be important to mention that the deterministic nonlinear system exhibits periodic as well as chaotic fluctuations

<sup>38</sup>See Gaunersdorfer and Hommes (2007).

<sup>39</sup>As already mentioned briefly in the introduction of this thesis, almost all empirical data from financial markets exhibit the 'stylized facts' which denotes certain universal statistical properties of financial time series: while prices seem to follow a random walk or a martingale process ('unit root property') the unconditional distribution of returns are characterized by a non-Gaussian behavior with a higher concentration of the probability mass in the center and the tail of the distribution ('leptokurtosis'). Furthermore, the magnitude of volatility is clustered exhibiting periods of both tranquil and more turbulent episodes ('clustered volatility') which leads to a slow decay rate of the positive autocorrelation for absolute and squared returns used as a measure for volatility. We can refer to Pagan (1996) and Cont (2001) for further reading on this topic. For a discussion about the time-varying nature of the volatility of financial assets originally recognized in the 1960s, see Fama (1965) and Mandelbrot (1963a,b, 1967).



**Figure 2.8:** Time series of price deviations from the fundamental  $x_t$  and corresponding return maps. The left-hand side exhibits the time series of price deviations from the fundamental  $x_t$  for different choices of the switching intensity  $\beta$ . On the right-hand side the corresponding projection of the strange attractor on the  $(x_{t+1}, x_t)$ -plane is shown. The upper panels correspond to a parameter set for which the dynamic system stays in a region of the first Hopf bifurcation. The middle and lower panel on the right-hand side shows different shapes of attractors for two higher values of  $\beta$  near a region where the LCEs are around zero (see also Figure 2.9 for details). The return maps are generated by 50,000 iterations from which the first 250 are deleted afterwards. In all plots the dynamic variables are randomly initialized. Parameters are  $R=1.01, a=10, \sigma^2=0.1, C=1, \alpha=0.65$  and  $\theta_{1,\sigma} = \theta_{2,\sigma}=0.85$ .



**Figure 2.9:** Largest Lyapunov characteristic exponents for time-varying beliefs about variances. Both plots correspond to the bifurcation diagrams shown in the upper panels of Figure 2.7, i.e., when the trend chasers extrapolate strongly ( $g = 1.2$ ,  $R = 1.01$ ). As expected, for small parameter values of  $\beta$  the dynamic system is stable ( $LCE < 0$ ) because of the primary bifurcation, which is a supercritical pitchfork bifurcation, and the secondary bifurcation, which is a Hopf bifurcation (for  $\alpha$  being small, left panel) or a flip bifurcation (for  $\alpha$  being large, right panel). For larger values of  $\beta$ , chaotic behavior is exhibited interspersed with stable cycles for higher values of  $\beta$ . The experiments are conducted in exactly the same way as described in Figure 2.3. The weight factors  $\theta_{1,\sigma}$  and  $\theta_{2,\sigma}$  are chosen as  $\theta_{1,\sigma} = \theta_{2,\sigma} = 0.85$ . The remaining parameters are chosen as in the previous three plots (see Figures 2.6, 2.7 and 2.8).

of asset prices and returns with coexisting attractors<sup>40</sup> between which the system starts to switch when noise is introduced into the model structure (see also the discussion at the end of this section for more details).

The model modifications compared to the BH model are as follows. Besides using the evolutionary fitness measure based on risk adjusted realized profits (as in the previous discussion) two further modifications are proposed in comparison to the original BH model. On the one hand the model allows for one further time lag in the trend following forecasting rule of chartists.<sup>41</sup> Again two trading strategies are assumed, namely

- a fundamentalist trading strategy forecasting via partial adjustment

$$E_{1,t}[p_{t+1}] = f_{1,t} = vp_{t-1} + (1-v)p^* = p^* + v(p_{t-1} - p^*), \quad 0 \leq v \leq 1,$$

- and the trading strategy applied by the trend extrapolators

$$E_{2,t}[p_{t+1}] = f_{2,t} = p_{t-1} + g(p_{t-1} - p_{t-2}), \quad g \geq 0.$$

In the following consideration the strategy of fundamentalists is restricted to the simple case  $v = 1$ . Thus fundamentalists believe that tomorrow's price will be equal to the last observable price  $p_{t-1}$ ,<sup>42</sup>

<sup>40</sup>For a detailed mathematical analysis of the bifurcation routes to strange attractors and coexisting attractors see Gaunersdorfer et al. (2008).

<sup>41</sup>This assumption leads to a four dimensional system (in contrast to a three dimensional) in the simplest case of zero memory, i.e.,  $\gamma=0$ .

<sup>42</sup>In Gaunersdorfer and Hommes (2007) this trader type is also denoted as an EMH believer since his/her naive forecast is consistent with a random walk of prices.

while the other trader type is a simple trend extrapolator, extrapolating the latest observed price change. Assuming an i.i.d. dividend process with  $E_t[d_{t+1}] = \bar{d}$  and expressing the dynamics in form of prices rather than in price deviations leads to the market equilibrium according to Eq. (2.5), namely

$$Rp_t = n_{1,t}p_{t-1} + n_{2,t}(p_{t-1} + g(p_{t-1} - p_{t-2})) + \bar{d} + \epsilon_t, \quad (2.49)$$

where  $n_{1,t}$ ,  $n_{2,t}$  denote the fractions of fundamentalists and trend extrapolators respectively and  $\epsilon_t$  is an i.i.d. random variable representing model approximation errors<sup>43</sup> which turns this model modification into a noisy evolutionary ABS.

On the other hand a second updating scheme for the fractions of different trader types is introduced where trend extrapolators now also condition their rules upon deviations from the fundamental. This is modeled as follows. In the first step the fraction of traders following the forecasting rule  $h = 1, 2$  is again determined by the risk adjusted realized profits of the past trading periods and given, as before, by

$$\tilde{n}_{h,t} = \frac{\exp\{\beta(\phi_{h,t-1} + \gamma\Phi_{h,t-2})\}}{\sum_{h=1}^2 \exp\{\beta(\phi_{h,t-1} + \gamma\Phi_{h,t-2})\}}$$

with<sup>44</sup>

$$\phi_{h,t} = -\frac{1}{2a\sigma^2}(p_t - f_{h,t-1} + \delta_t)^2. \quad (2.50)$$

To complete the updating scheme in a second step the fraction of traders is assumed to evolve according to

$$\begin{aligned} n_{2,t} &= \tilde{n}_{2,t} \exp\left\{-\frac{1}{\alpha}(p_{t-1} - p^*)^2\right\}, & \alpha > 0, \\ n_{1,t} &= 1 - n_{2,t}, \end{aligned} \quad (2.51)$$

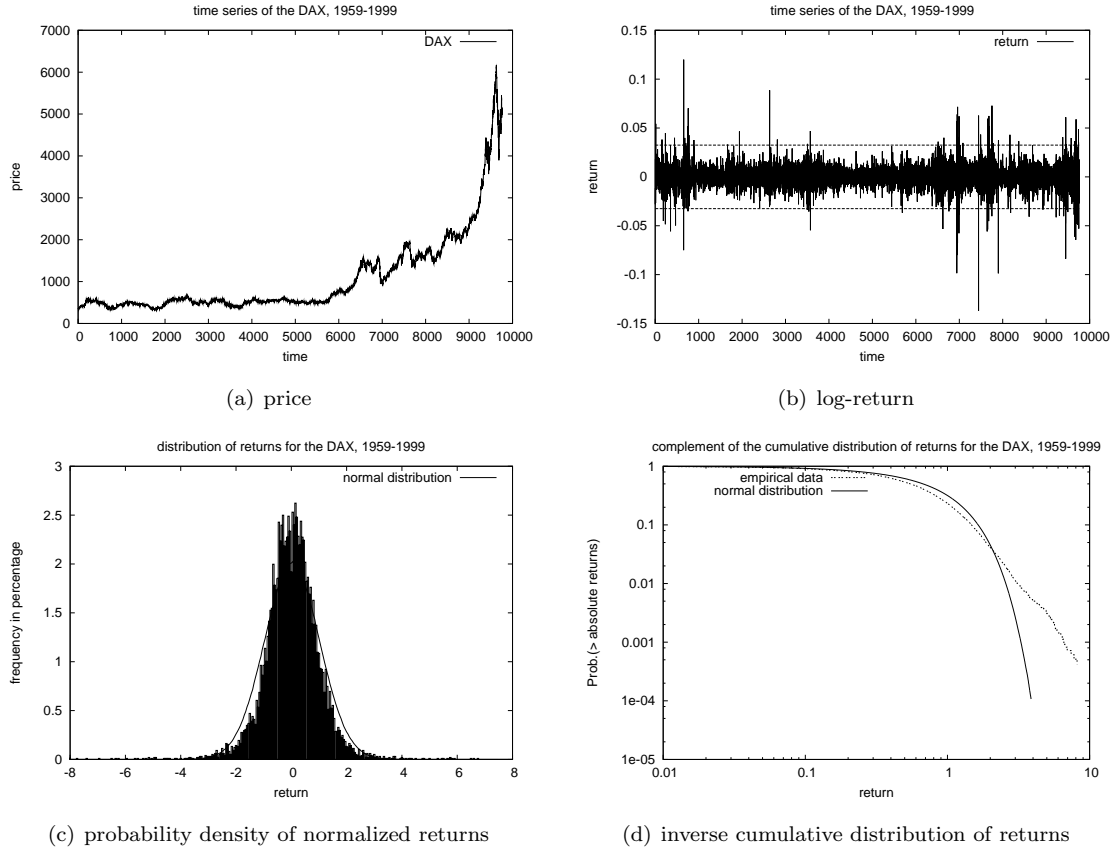
where the additional exponential term in Eq. (2.51) assures that the population of trend chasers decreases more than proportionally, the further prices deviate from their fundamental value.<sup>45</sup> Hence, the exponential term can be seen as a correction term, that becomes more important the further prices move away from the fundamental.

Figure 2.10 and Figure 2.11 show the time series of prices and returns (upper panels) as well as properties of the return distribution (lower panels) of a real financial time series and from a typical simulation run of the noisy ABS model. In Figure 2.10 the empirical data from Deutsche Aktien Index (DAX) serves as a typical example for a time series of a financial market. Since the modified BH model (as well as the original BH model) is a stationary model in prices by its construction the price series is not strongly increasing as would be expected from empirical data (compare the upper left panels of both figures). Nevertheless it can be shown, that prices are highly persistent and are close to having a unit root (see Gaunersdorfer and Hommes, 2007). As in real markets the time series of returns (compare the upper right panels of both figures) of the artificial market is

<sup>43</sup>The error term  $\epsilon_t$  can also be interpreted as exogenous random shocks from outside.

<sup>44</sup>See also Appendix A 2.9 for details.

<sup>45</sup>Hommes (2001) states in his review article, that the conditioning on deviations from the fundamental may be interpreted as a stabilizing transversality condition in a heterogeneous world. Temporarily, speculative bubbles can arise that push the current price away from its fundamental but at some point, a correction towards the fundamental price will occur.



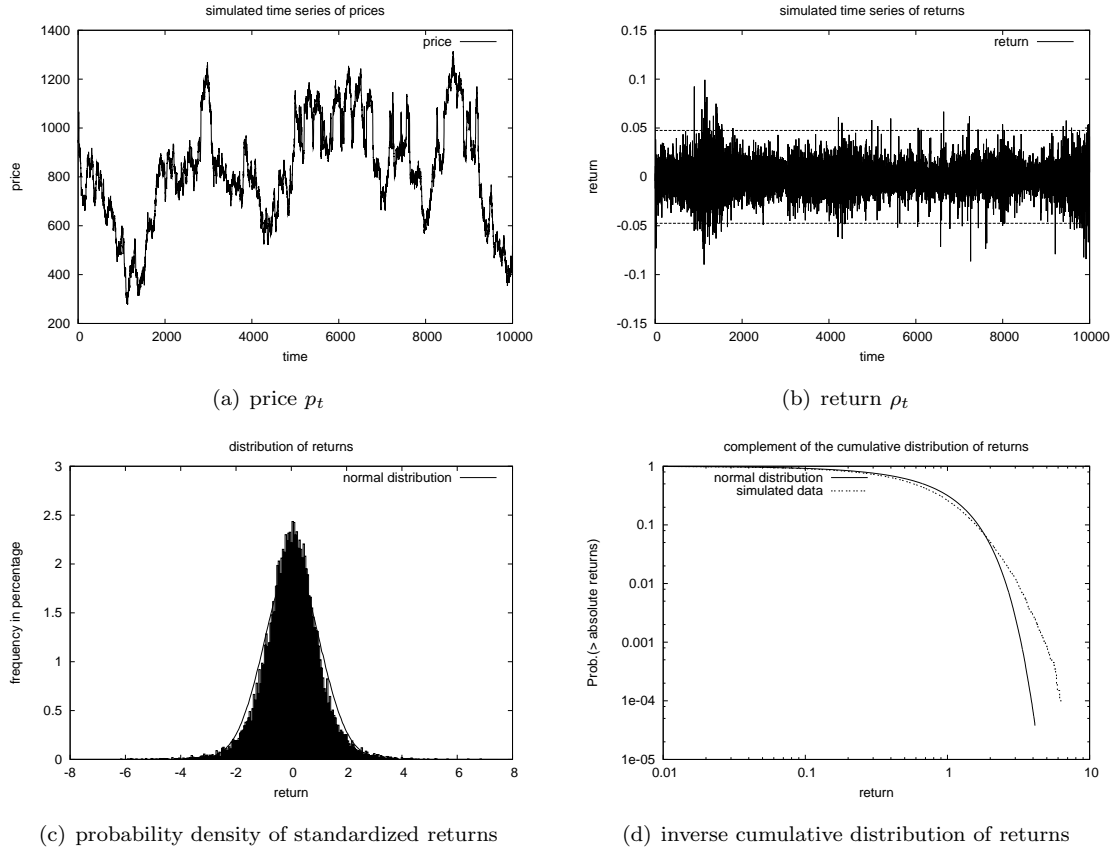
**Figure 2.10:** Time series of price and return properties for the Deutsche Aktien Index (DAX). The sample consists of 9762 daily observations spanning the period from January 1959 to January 1999. The upper left panel shows the price evolution of the index. The upper right panel shows the time series of the daily log-returns for the DAX. The dashed lines refer to the  $\pm 3\sigma$  region. The lower left panel exhibits a histogram of the sample probability density of normalized returns compared to a standard normal distribution. For a better visibility of the leptokurtic shape of the empirical distribution the inverse cumulative distribution is shown for the empirical as well as for the normal distribution in the lower right panel.

characterized by time-varying fluctuations with phases of low volatility and counterparts of high volatility and, thus, clearly exhibits the phenomenon of ‘clustered volatility’. Note, that in contrast to the empirical time series the returns of the simulated data are determined as net returns, i.e., relative price changes of the form

$$\rho_t = \frac{p_t - p_{t-1}}{p_{t-1}},$$

which simply follows the original publication and does not play a crucial role.<sup>46</sup> Another characteristic which is typically observed in empirical time series of financial markets is the ‘fat-tails property’ which is also shared by the artificial market. The lower panels of Figures 2.10 and 2.11 serve as an illustration for this feature. Using normalized returns, the sample probability density function of returns  $\rho_t$  are compared with the density function of a standard normal distribution.

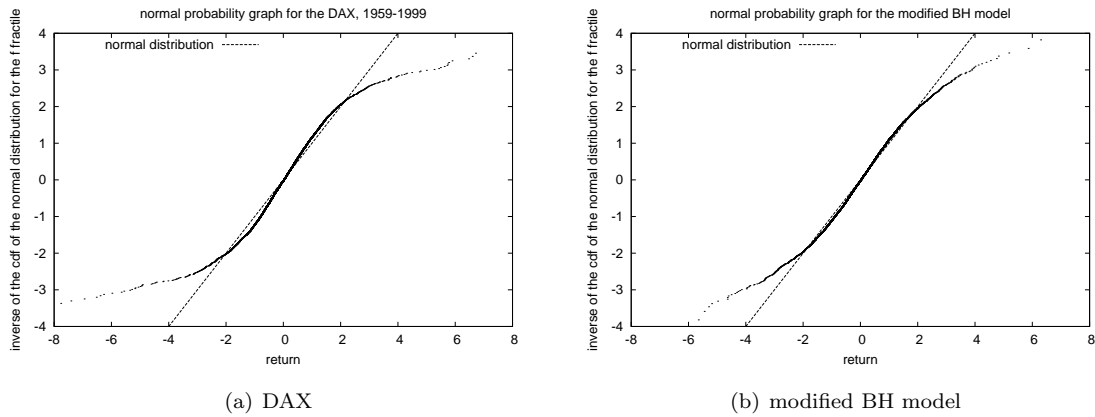
<sup>46</sup>Return values defined as relative price changes do not deviate much from using log-returns on a daily level since from a Taylor expansion it directly follows that the log-return  $\hat{r}_t \equiv \ln p_t - \ln p_{t-1} \approx r_t$  with  $r_t \equiv (p_t - p_{t-1})/p_{t-1}$  is a proper approximation for  $r_t$  close to zero.



**Figure 2.11:** Simulation results for prices and returns in a modified BH model. The plots exhibit time series of prices and returns (upper panels) and standardized return distribution properties (lower panels) in a modified BH model. Initial prices are drawn from a normal distribution  $\mathcal{N}(950, 100)$  while the initial population fractions is set to 0.5. The parameter set is chosen as  $r=0.001$ ,  $p^*=1000$ ,  $\bar{d}=1.0$ ,  $a=1.0$ ,  $\sigma=1.0$ ,  $v=1.0$ ,  $g=1.9$ ,  $\beta=2.0$ ,  $\gamma=0.99$ ,  $\alpha=1800$ ,  $\delta_t \equiv 0.0$  and  $\epsilon_t \sim \mathcal{N}(0, 121)$ . The time series of prices and returns are generated by a GAUSS program which I wrote following the model description in Gaunersdorfer and Hommes (2007). The parameter are chosen according to the parameter values given in the original publication.

Obviously, in comparison to a normal distribution, which is the chosen distribution for the noise term in the equilibrium equation for the price dynamics, the return distribution of the modified BH model has a higher probability for large events and, consequently, contains also more probability mass around the symmetry axis of the distribution (see lower left panel in Figure 2.11). This leptokurtic character of the entire distribution can also be assisted by investigating the outer part of the empirical distribution more carefully. In the lower right panels of Figures 2.10 and 2.11 the inverse cumulative distribution  $P_{>}(|\rho_t|) \equiv 1 - F(|\rho_t|)$  of absolute returns is plotted against the return  $\rho_t$ .<sup>47</sup> It can be seen that the probability for large returns decay significantly more slowly than under the assumption of a Gaussian distribution with the same standard deviation. This phenomenon is coined by the term ‘fat-tail phenomenon’ which is one crucial regularity of

<sup>47</sup>Absolute returns are used to exhibit the presence of (positive and negative) extreme events in the distribution, which is appropriate since both distributions show a symmetric behavior around zero as well as in the outer part of the distribution.



**Figure 2.12:** Normal probability graphs for the DAX and a modified BH model. The departures from normality can easily be detected for both the empirical distribution of the DAX (left panel) and the return distribution of the simulated artificial data (right panel). The dashed line refers to Gaussian distributed returns. The data for the modified BH model are taken from the same simulation run as shown in Figure 2.11.

the behavior of financial data.<sup>48</sup> This clearly departure from the Gaussian behavior can also be illustrated in a slightly different way, namely by plotting the theoretical fractiles (or quantiles) of a Gaussian distribution against the empirical fractiles of the return data which is done in a ‘normal probability graph’ (or ‘qq-plot’). This procedure is used for investigating financial data, for instance, in the seminal survey of stock market prices in Fama (1965). The idea of this method is as follows. Let  $u$  be a Gaussian random variable with mean  $\mu$  and standard deviation  $\sigma$  and let  $z$  be the standardized variable  $z = (u - \mu)/\sigma$ . Then, the graph of  $z$  against  $u$  is just a straight line, since  $z$  results from  $u$  by a simple linear transformation. If  $u_i$ ,  $i = 1, \dots, N$  are  $N$  sample values of the random variable  $u$  ranked from smallest to largest, then each particular  $u_i$  is an estimate of the  $f_i$  fractile of the distribution of  $u$ , where the values of  $f_i$  can be estimated by<sup>49</sup>

$$f_i = \frac{3i - 1}{3N + 1}. \quad (2.52)$$

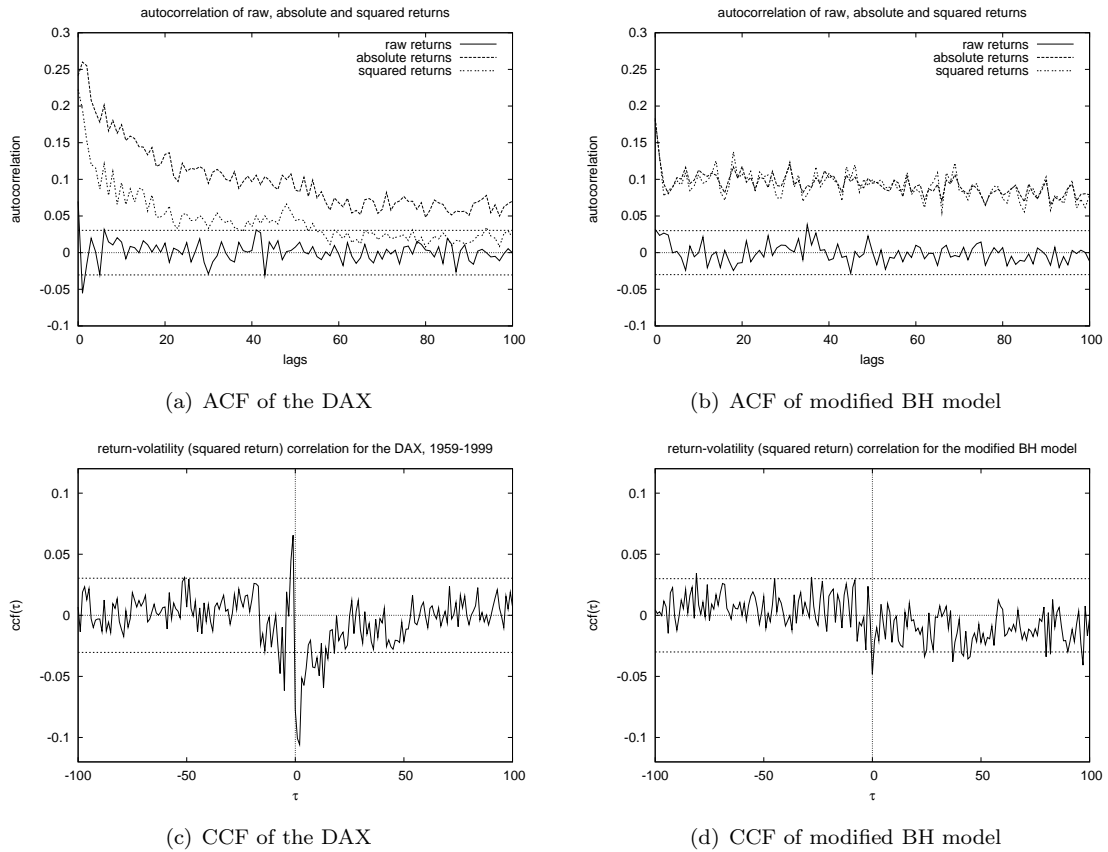
Moreover, for the standardized normal distribution the exact value of  $z$  for the  $f_i$  fractile can easily be derived by using standard computer software or can be found in any standard table. Then, for the case of a Gaussian random variable the plot of the sample values of  $u$  against the theoretical determined values of  $z$  should be a straight line. Conducting the same procedure for the return data of the DAX and the simulation data from the modified BH model, i.e., ranking the  $N$  sample values from smallest to largest, deriving the  $f_i$  fractile,  $i = 1, \dots, N$ , via Eq. (2.52) and comparing the  $N$  sample values with the inverse of the cdf of the normal distribution evaluated for each  $f_i$ , shows an extreme departure from linearity for the outer part of the return distribution in both cases (see Figure 2.12). To conclude, the heaviness of the tails and the leptokurtic behavior of the empirical and simulated distributions confirm that the Gaussian hypothesis for the distribution of both the returns of the DAX and the returns of the artificial market should be discarded. In addition, these findings support the ability of the modified BH model to reproduce at least some

<sup>48</sup>Again we refer to Pagan (1996) and Cont (2001) for further details.

<sup>49</sup>There are many conventions for estimating the  $f_i$  fractile. Other popular conventions are  $f_i = i/(N + 1)$ ,  $f_i = (i - 0.5)/N$  and  $f_i = (i - 0.375)/(N + 0.25)$ . See also Fama (1965) for details.

crucial properties of returns of financial data. The descriptive time series statistics of returns for the DAX and the modified BH model are summarized in the first two columns of Table 2.1. The first and second moments of both time series are of the same order of magnitude, as well as the third moment indicates a symmetric distribution in both cases. The fourth moment, i.e., the kurtosis, is also increased in comparison to the Gaussian distribution, but less significant for the simulated time series than for the empirical data.

Finally, Figure 2.13 is devoted to the autocorrelation function of returns, absolute returns and squared returns as well as to the return-volatility correlation. The upper left panel exhibits the



**Figure 2.13:** Return-return and return-volatility correlation for the DAX and a modified BH model. In the upper panels the autocorrelation function of raw, absolute and squared returns are shown. The upper left panel shows the typical decay behavior of the autocorrelation function of absolute and squared returns as qualitatively observed in almost all time series of empirical financial data. The modified BH model shares the slow decay rate of absolute and squared returns (upper right panel). The lower panels show the return-volatility correlation of both time series, where the squared return is used as a volatility measure in both plots. While a short ranged negative correlation is detected for the empirical data, the modified BH model hardly shares this feature. The dashed lines refer to the  $\pm 3\sigma$  confidence bands. The data are taken from the same simulation run as shown in Figures 2.11 and 2.12.

decay rate of the autocorrelation functions for the DAX which shows the typical decay behavior of empirical time series of financial markets. The autocorrelation function of the raw returns permanently moves around zero within the  $\pm 3\sigma$  confidence bands. For the absolute and squared returns the autocorrelation pattern is crucially different. Both autocorrelation functions show a signifi-

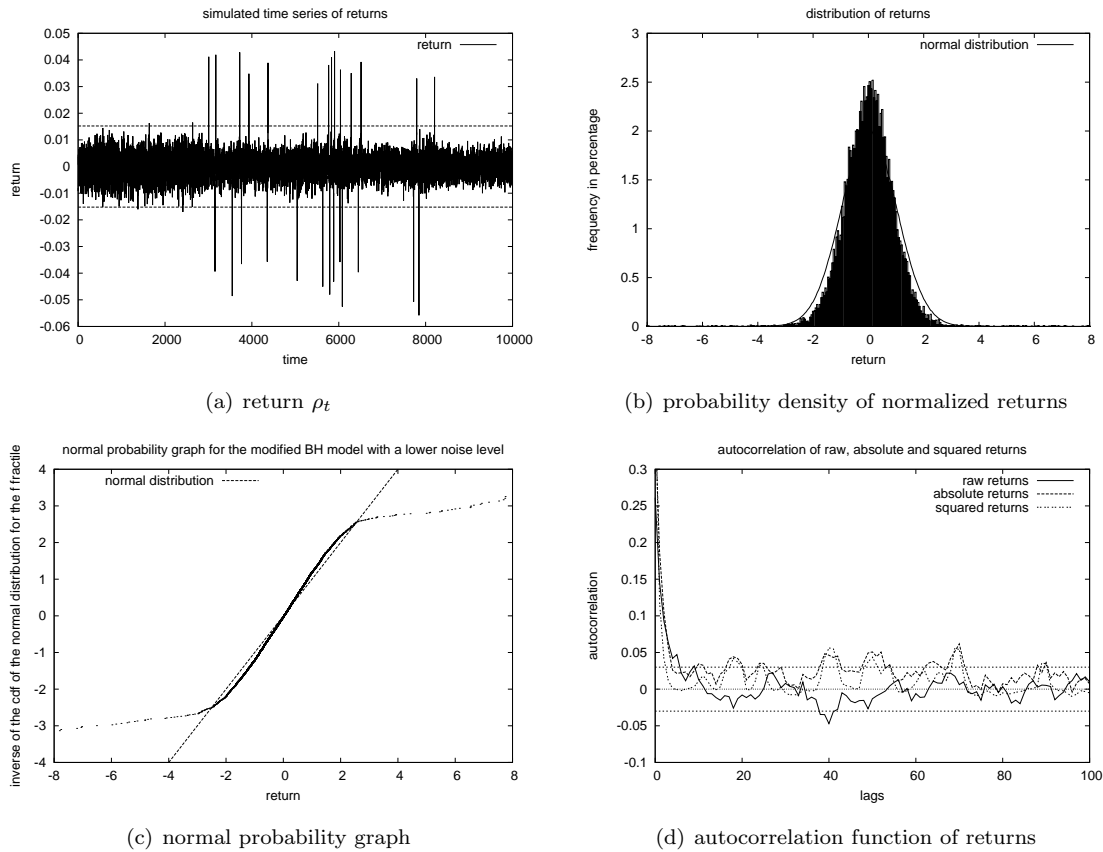


cantly positive autocorrelation which is persistent over a quite long time horizon. The modified BH model is able to reproduce these empirical findings as it exhibits no significant autocorrelations of returns but slowly decaying autocorrelations of absolute and squared returns (upper right panel). Furthermore, we have also found some empirical evidence for correlations between future volatilities and past price changes. The ‘leverage effect’<sup>50</sup>, due originally to Black (1976*a*), states that a price drop increases the risk of a firm going bankrupt, which induces its stock to become more volatile. However, there is also a counterargument, which states, that an increase in volatility makes the stock less attractive which, pushing its price down. We can refer to Bouchaud et al. (2001), Campbell et al. (1997, Chapter 12) and Bouchaud and Potters (2003, Chapter 8) and the references given therein for further details on this debate. However, the lower left panel of Figure 2.13 shows that negative shocks to returns drive up volatility within a short time period for the DAX index. The modified BH model does not share this feature. Instead, current price drops seem to be negatively correlated to the current level of volatility, which can be confirmed by performing further simulation runs with identical parameter sets and averaging over many of those experiments (not shown here).

I would like to make one final remark, not mentioned in the original publications, but which might give further insights into the basic question of how to model financial markets generating realistic time series properties. Running simulations of the modified BH model with different levels of noise, it turns out that due to the statistical properties of the return series, the model outcome seems to be very sensitive with respect to the chosen noise amplitude (see Lux (2003), i.e., private communication with T. Lux). More precisely, the system loses its main features (particularly volatility clustering, fat-tail phenomenon, typical autocorrelation pattern of raw, absolute and squared returns as described above) when the variance of the approximation error  $\epsilon_t$  of the system falls below a certain threshold. In such a situation the time series of returns become less volatile and move permanently within the  $\pm 3\sigma$  region except for very rare and very ‘synthetic’ spikes simply caused by numerically based negative effects during the simulation run. Furthermore, the autocorrelation of returns, absolute returns and squared returns becomes insignificant (see Figure 2.14). The descriptive statistics of the simulated time series in the right column of Table 2.1 confirms these findings. The standard deviation (and the mean) is reduced by a factor 10 in this case. Moreover, even though the skewness and the kurtosis seem to fit better to empirical data, a further look at the normal probability graph (see Figure 2.14 (c)) indicates that the relatively high kurtosis is only caused by the very rare outliers mentioned above. Indeed the normal probability graph shows a very sharp cut off slightly above and below the  $\pm 2.5\sigma$  region. For a possible explanatory approach for this sensitive behavior of the system for the chosen noise level we might have to return to the dynamic behavior of the model in the absence of noise. As already mentioned above, in the deterministic case ( $\epsilon_t = \delta_t = 0$ ) the nonlinear dynamic system Eq. (2.49) has two coexisting attractors namely a locally stable fundamental steady state and an attracting quasi-periodic cycle.<sup>51</sup> Turning on the noise at a sufficiently high level makes the system start to switch between these two attractors which generates dynamics that allows for quite realistic time series

<sup>50</sup>The ‘leverage effect’ is also a phenomenon attributed to the category of ‘stylized facts’ of financial time series (see Bouchaud et al., 2001; Cont, 2001).

<sup>51</sup>For details see Gaunersdorfer and Hommes (2007).



**Figure 2.14:** Return properties in a modified BH model with a lower noise level. However, the parameter set is chosen as in the previous simulation run, where the noise level is reduced to  $\epsilon_t \sim \mathcal{N}(0, 16)$ . The dashed lines (upper left panel and lower right panel) refer to the  $\pm 3\sigma$  region. Obviously, the time series of return and the distribution of return lose their main characteristics which they shared with the empirical financial data (see Figures 2.11 and 2.12). Instead, with the exception of very few and very ‘synthetic’ outliers, the return pattern of both the time series of returns and its distribution is very similar to a white noise process, i.e., Gaussian distributed returns. Therefore, note that in contrast to Figure 2.11 (b) the normal probability graph only indicates a deviation from the Gaussian distribution for *very* rare and *very* extreme events (lower left panel). The lower right panel shows, that the autocorrelation functions of raw, absolute and squared returns decay rapidly to zero which is caused by an undersized level of noise.

properties. At present the dependence between the amplitude of noise and the occurrence of such switching regimes among different attractors of the system is an interesting but open question and might be investigated more carefully in the future.

### 2.3 The Levy-Levy-Solomon (LLS) Model

Unlike the asset pricing framework of Brock and Hommes, the model formulated by M. LEVY, H. LEVY and S. SOLOMON<sup>52</sup> is based on the more realistic assumption that agents choose their optimal investment decisions from an underlying CRRA utility function, which gives the agents’

<sup>52</sup>See Levy et al. (1994, 1995); Levy and Levy (1996); Levy et al. (1996).

| DAX vs. Modified BH Model |           |                   |  |
|---------------------------|-----------|-------------------|--|
| return statistics         | DAX       | modified BH model | modified BH model with lower noise level |
| mean                      | 0.000270  | -0.000185         | 0.000001                                 |
| median                    | 0.000373  | -0.000002         | 0.000058                                 |
| maximum                   | 0.120038  | 0.099353          | 0.043286                                 |
| minimum                   | -0.137061 | -0.089737         | -0.055846                                |
| std. dev.                 | 0.010824  | 0.015827          | 0.005071                                 |
| skewness                  | -0.321050 | -0.072112         | -0.263523                                |
| kurtosis                  | 10.347282 | 1.902256          | (14.744482)                              |

**Table 2.1:** Descriptive statistics for the DAX and a modified BH model with two different levels of noise. The modified BH model shares some crucial features of real financial data as uni-modal and symmetric behavior around zero and an increased kurtosis. In the case of reduced input noise the standard deviation of the output signal, i.e., the returns, is also reduced by the same magnitude. Note that the relatively high kurtosis is simply a result of the occurrence of a few price explosions during the simulation run caused by the limitations of the used computer system why its value is placed in parentheses.

current wealth a role in their portfolio optimization considerations.<sup>53</sup> As a result, asset price and wealth dynamics become growing processes in time, which makes the analytical tractability of the model more challenging. Moreover, in the specific case of the LLS model, there is no analytical solution for most of the model variants and therefore the greater part of the model investigations are performed by numerical simulations.<sup>54</sup> Levy, Levy and Solomon show that, under the assumption of homogeneous expectations, highly unrealistic price variations with periodic booms and crashes are obtained, whereas the heterogeneity of agents can generate more realistic price movements. In the following, only the basic aspects of this model are outlined, which can be found in two textbooks from the ‘econophysics’ strand (see Levy et al. (2000) and Voit (2003)), as well as in the original literature.

### 2.3.1 Model Description

The LLS model is a microscopic model of a stock market with a variable but finite number of agents, who pursue either all the same trading strategy (i.e., homogeneous model version) or different trading strategies (i.e., heterogeneous model version). The microscopic ‘element’ of the system is the individual investor. Each investor has the same attitude to risk and is characterized by a standard CRRA utility-of-wealth function  $U(W) = \ln(W)$  from which their investment decision are derived.

Like in the BH model the stock market consists of two investment options:

<sup>53</sup>See Chapter 1, Appendix A 1.3 for some details on CRRA utility functions and their impacts on portfolio optimization. See also the Introduction 3.1 in Chapter 3 for a more general discussion on the different types of utility functions and their empirical validation.

<sup>54</sup>The following part of the thesis deals with a quite similar asset pricing model which shares most of the main ideas of the LLS model but which, in turn, is analytically tractable. Within this discussion the LLS model framework will be revisited at the end of Chapter 3 in Section 3.7.

- one risky asset (stock) with an uncertain return

$$\rho_t = \frac{p_t - p_{t-1} + d_t}{p_{t-1}},$$

which is composed of the variations of the stock price  $p_t$  and a fixed positive dividend payment  $d_t$

- one riskless asset (bond) that returns a fixed interest with rate  $r$  in each period.

The time period between two time steps is taken as one year.<sup>55</sup> In each trading period the investors now have to split their wealth between these two investment opportunities where, of course, the objective of each investor is to maximize their expected value of his/her utility. Cash, credit and short selling of stocks are not allowed. For the expectations formation it is assumed that each investor memorizes the last  $k$  returns of the risky assets and weights each of them with probability  $1/k$  to reappear in the next trading period. From these recent observations, they eventually determine their order volumes for the risky and for the riskless asset. Again, as in the former model, each trader is assumed to be a price taker, so that the stock price in each trading round is determined by equating supply and demand according to a Walrasian scenario.

In order to describe the architecture and the emerging dynamics of the model more precisely, the whole trading and market clearing procedure within two time steps will now be discussed. Note that Figure 2.15 summarizes the main aspects of the model's time structure. Let us start at time  $t$  directly after a trade (note that the next trade takes place exactly at time  $t + 1$ ) and let  $W_{i,t}$  and  $N_{i,t}$  denote the wealth and number of shares of the  $i$ th investor at period  $t$  respectively. Hence between  $t$  and  $t + 1$ , *after* receiving dividends and interest payments and *before* the trading action, the wealth of investor  $i$  has changed to

$$W_{i,t}^d = W_{i,t} + N_{i,t}d_t + (W_{i,t} - N_{i,t}p_t)r,$$

where the superscript  $d$  labels the included dividend return (and interest payments) in investors wealth  $W_{i,t}$ . All traders now have to make their investment decisions for the next trading period  $t + 1$  based on their preferences, i.e., they must derive their optimal investment proportion  $\pi_{i,t+1}$  which they want to invest in the risky asset via

$$\max_{\pi_{i,t+1}} E_{i,t+1}[U(W_{i,t+2})], \quad (2.53)$$

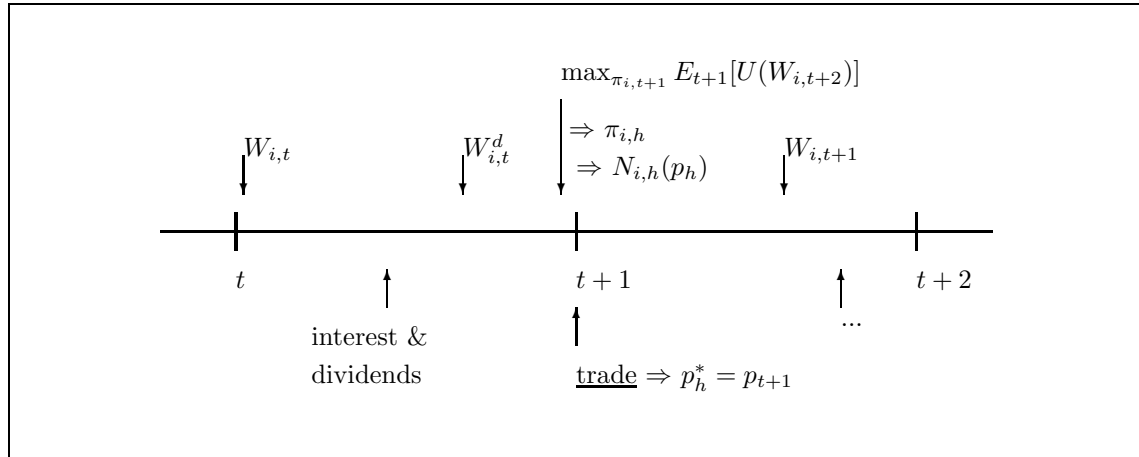
where  $\pi_{i,t+1}$  is restricted to being between some lower and upper boundary.<sup>56</sup> Since at this moment (i.e., *before*  $t + 1$ ) the stock price  $p_{t+1}$  is not yet revealed, the investors have to determine their optimal order volumes in a slightly sophisticated way. They derive their optimal demand function for different hypothetical prices  $p_h$ , that would lead to a total, hypothetical wealth  $W_{i,h}$  that is defined as

$$\begin{aligned} W_{i,h} &= W_{i,t}^d + N_{i,t}(p_h - p_t) \\ &= W_{i,t} + N_{i,t}d_t + (W_{i,t} - N_{i,t}p_t)r + N_{i,t}(p_h - p_t). \end{aligned}$$

<sup>55</sup>In later versions of the model the time period (together with relevant parameters) is modified to quarterly (see Levy et al., 2000) or daily (see Levy et al., 1996) time steps.

<sup>56</sup>In most of their simulations it is assumed that  $0.01 < \pi_{i,t+1} < 0.99$  for all  $t$ .

Note that  $W_{i,h}$  is the wealth the investor will hold *directly after* the trade at  $t + 1$ . Thus only the gain/loss of the  $N_{i,t}$  stocks held before the trade are to be included in his/her consideration. As already mentioned before, it is assumed that each investor has a memory span of  $k$  time steps



**Figure 2.15:** Model time line of the LLS model. Below the time line all market actions are indicated, while their impact on trader  $i$ , i.e., the change in his/her variables is illustrated above the time line.

and that he/she expects for the next time step a price taken from the past  $k$  prices with equal probability  $1/k$ . Using Eq. (2.53) and the explicit form of the investors' utility function, the optimal (hypothetical) investment proportion at time  $t + 1$ , which, in general, depends on  $p_h$  through  $W_h$ , is given by

$$\pi_{i,h} = \arg \max_{\pi_{i,h}} \left\{ \frac{1}{k} \sum_{j=t}^{t-k+1} \ln[(1 - \pi_{i,h})W_{i,h}(1 + r) + \pi_{i,h}W_{i,h}(1 + \rho_j)] \right\},$$

which can be solved at least numerically. Therefore the number of shares investor  $i$  wants to hold at the hypothetical price  $p_h$  is given by

$$N_{i,h}(p_h) = \frac{\pi_{i,h}W_{i,h}}{p_h}$$

from which the aggregate demand function can be determined via summation that gives

$$N_h(p_h) = \sum_{i=1}^I N_{i,h}(p_h),$$

where  $I$  denotes the number of investors in the market. As it is assumed that the market has a fixed number of available shares, denoted by  $N$ , at time  $t + 1$  the market clearing equilibrium price  $p_h^* = p_{t+1}$  is simply determined by the intersection point of the aggregate demand function and the supply function, which is a vertical line.<sup>57</sup> Thus, as in real markets, the price  $p_{t+1}$  and the wealth of investors  $W_{t+1} \equiv (\dots, W_{i,t+1}, \dots)$  is determined simultaneously. In the last step the traders have to update their history of past price movements. This completes one single trading procedure and a new trading period starts with the payout of dividends and interest payments.

<sup>57</sup>The riskless asset (bond) is assumed to be in infinite supply.

So far, the model is purely deterministic. Now, randomness will be introduced by adding a noise term to the investors' optimal investment proportion by

$$\tilde{\pi}_{i,t} = \pi_{i,t} + \epsilon_i,$$

where  $\epsilon_i$  is drawn separately for each investor from a Gaussian distribution with zero mean and standard deviation  $\sigma$ . This random factor is assumed to take into account all individual (e.g., psychological) factors as well as those that cannot be mapped into the rational utility maximization part of the model. Expressed in economic language, the random term captures the deviation from utility maximization of each single investor and makes the individual's and aggregated demand function become a stochastic variable.

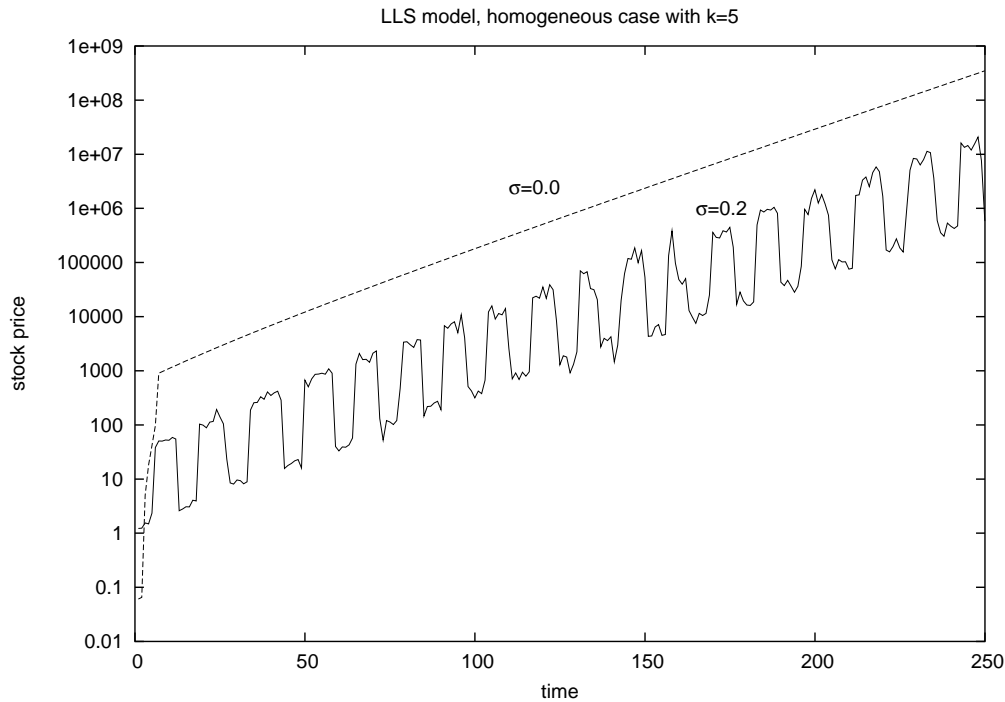
In most papers on the LLS model, the authors compare two different scenarios of the model, one with homogeneous traders and one with a heterogeneous population. This will be discussed in some detail in the following subsection.

### 2.3.2 Main Results

Early academic work on the LLS model is mainly concentrated on the unrealistic model outcomes under homogeneous expectations in a representative investors framework (see Levy et al., 1994, 1995; Levy and Levy, 1996). Accordingly, Levy, Levy and Solomon conclude that different forms of heterogeneity have to be introduced into the model structure to allow the model to be used for more realistic market scenarios.

In the homogeneous model the only individual component of each investor is the random factor  $\epsilon_i$  added to each investors' optimal diversification proportion. Assuming realistic parameter values (annual interest rates are taken as 4%, initial dividend yield is fixed at 5% per year and increases by 5% annually<sup>58</sup>) Levy et al. (1995) show that in such a model structure unrealistic booms and crashes of the market price are generated. A typical time series of the stock price for the homogeneous model is shown in Figure 2.16. The explanation of these market phenomena is rather evident. In the no-noise case (i.e.,  $\sigma = 0$ ), all investors behave in exactly the same way, which leads to identical demand functions. As a result, in every trading period, the outstanding shares are divided equally between the traders and all individuals will eventually hold the same number of shares, leading to a market price which simply grows exponentially at the constant growth rate of the dividend. In other words, there is no real trade and thus this market scenario is, of course, somewhat pathological. Now let us turn to the homogeneous model with noise (i.e.,  $\sigma > 0$ ). Again, after a transient period the stock price rises at the same exponential growth rate of the dividends as in the no-noise case. However, the price does not increase in a smooth manner as in the latter case, but now fluctuates around the trend. These price fluctuations are caused by fluctuations in the optimal investment proportions, which in turn are caused by the noise term. Because of the rise in the market price, investors become bullish about the stock and will invest in the risky asset as much as possible. As the history of returns becomes relatively homogeneous in such a situation, a small change in return caused by the noise term can induce a dramatic change in the investment proportion, which

<sup>58</sup>For the sake of completeness, in this model design the market consists of 100 investors with 10,000 outstanding shares, the initial share price is set to 4 units of cash and each investor is initially endowed with 1000 units of cash (see Levy et al., 1995).



**Figure 2.16:** Time series of stock price in the LLS model (homogeneous case). With only one investor group, the market price simply grows exponentially at the constant growth rate of the dividend in the case of no noise i.e.,  $\sigma = 0.0$  (dashed line). With investor-specific noise  $\sigma = 0.2$ , periodic booms and crashes arises (solid line). As the plot is generated from the original LLS C-program provided by the authors, the visual appearance coincides with the plot originally found in Levy et al. (1995). In this plot 100 investors with memory span 15 are chosen.

results in a market crash. Note, that this ‘discontinuity property’ (see Levy et al., 1995) is one of the main features of the LLS model.<sup>59</sup> After a market crash, investors again become more and more homogeneous, and again start to buy the stock and sell the bond if a small increase in the return is detected. The stock price increases sharply and a new boom is created. This pattern recurs periodically, with the periodicity depending on the memory span  $k$  of the investor.

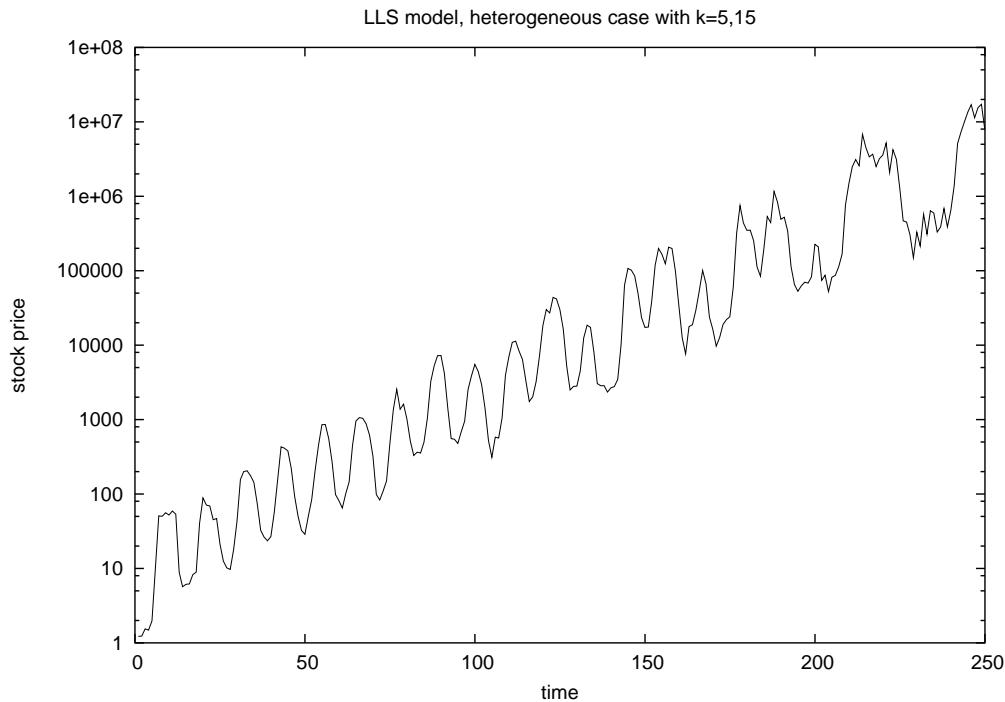
Heterogeneity can be introduced in different forms into the model (see Levy et al., 1995). Firstly, agents can be grouped into investor types with different memory spans. Secondly, their risk preferences can be diversified by allowing them to use different risk-aversion coefficients in their utility function.<sup>60</sup> Thirdly, Levy, Levy and Solomon use a mixture of both types of heterogeneity and choose the investors’ memory span ( $1 \leq k \leq 20$ ) and risk preferences ( $0 < \nu \leq 4$ ) randomly. In the last step, the latter scenario is coupled with investors following a long-run investment strategy. This means that some agents keep their investment proportion in the risky asset fixed at a certain rate (for instance, 20%).<sup>61</sup> To summarize the results from all these different market scenarios

<sup>59</sup>As we will see soon, such discontinuous changes in investment preferences can be also detected in the heterogeneous framework.

<sup>60</sup>In this case, a more general version of a CRRA utility function is used that has the form  $U(W) = W^{1-\nu}/1-\nu$ , where  $\nu$  is the risk-aversion coefficient. For  $\nu \rightarrow 1$  this version converges to the logarithmic utility function that has been assumed so far. See also Chapter 1, Appendix A 1.3 for the details.

<sup>61</sup>This strategy is related to the EMH (see Chapter 1, Section 1.3). As future prices cannot be forecasted by using recent price observation, a buy-and-hold strategy should be at least as promising as any artificial trading strategy

briefly, in all scenarios the resulting time series of the stock price are generally more irregular than in the homogeneous case. The periodic occurrence of booms and crashes is relaxed, but price movements still remain rather predictable (see Figure 2.17 for a typical example). Not surprisingly, the time series of the stock price become more irregular the more elements of heterogeneity enters the market. To sum up, in their early publications the authors focus on the necessity of heterogeneity as the essential constituent of a more or less realistic model of financial markets. In other words, in the microscopic, artificial world of the LLS model, the assumption of homogeneous investors, although it is common and useful in many theoretical arguments and economic models,<sup>62</sup> leads overall to unrealistic time series of stock prices.



**Figure 2.17:** Time series of stock price in the LLS model (heterogeneous case). With two different investor groups (memory span  $k = 5, 15$ ), the market price becomes more irregular but still shows rather predictable price patterns. Again, the plot is made by the original LLS C-program and looks like the results in Levy et al. (1995). In this plot 100 investors and  $\sigma = 0.2$  are chosen.

In the authors' more recent publications, the base line version of the model is exploited again (see Levy et al., 1996) and a slightly more general version of the model is investigated (see Levy et al., 2000). In Levy et al. (1996) the dynamics of the system with one, two and three investor subgroups is analyzed in a more exhaustive manner. It is assumed that all agents have the same utility function and only differ in their memory spans. Additionally, the time step between the trades is changed into days, and relevant parameters are adjusted to appropriately realistic values. As the homogeneous case has already entirely been examined above, in the two population market Levy et al. (1996) find that the dominance of one subgroup over the other in terms of the fraction of total wealth depends significantly on the ratio of their memory spans. Furthermore, they give

based on past price movements.

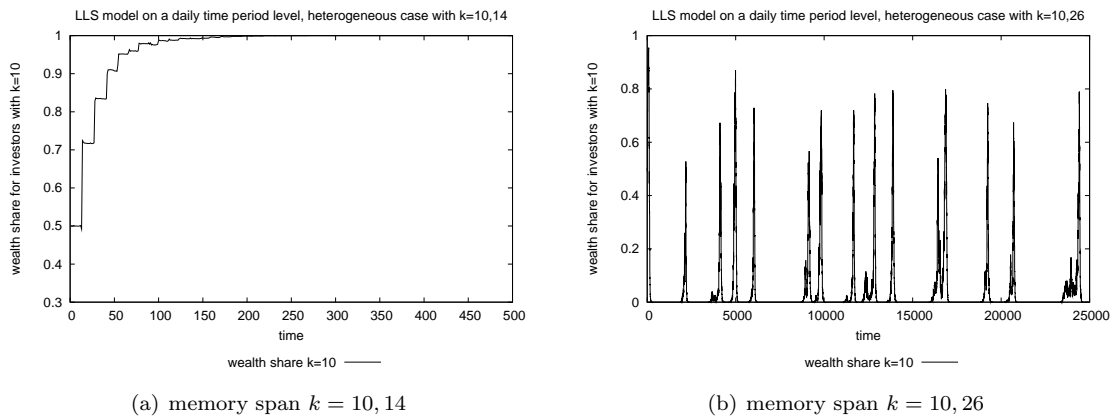
<sup>62</sup>See the discussion about the notion of the 'representative agent' in Chapter 1, Section 1.4.



very intuitive explanations for such phenomena in different scenarios that are all based on the fact that each population remembers recent booms and crashes in a slightly but crucially different way. Restricting their considerations to cycles up to length  $2k_0$ ,<sup>63</sup> the results can be divided into the following cases assuming a starting point where one population  $k_0$  dominates the market<sup>64</sup> and dictates the dynamics (see Levy et al., 1996):

- (i) If  $k_0 < k_1 < 2k_0$ ,  $k_0$  prevails over  $k_1$ .
- (ii) If  $2nk_0 < k_1 < (2n + 1)k_0$ ,  $n \in \mathbb{N}$ ,  $k_1$  performs better than  $k_0$ .
- (iii) If  $(2n + 1)k_0 < k_1 < 2nk_0$ ,  $n > 1$ ,  $k_1$  still dominates, but less strongly than in (ii).
- (iv) If  $k_1 < k_0$ ,  $k_1$  always prevails over  $k_0$ .

Exemplarily, cases with memory spans  $k = 10, 14$ ,  $k=10, 26$  and  $k = 10, 36$  are considered in more detail, with Figure 2.18 showing some results on this. For a larger number of investor populations, e.g., a three population case with  $k = 10, 141$  and 256 or a six population case with  $k = 10, 36, 141, 193, 256$  and 420, the dynamics of the system qualitatively changes. No specific cycle length can be detected any longer in the time series of the stock price, and the model shows a more irregular structure. The system has become complex (see Levy et al., 1996).



**Figure 2.18:** Time series of wealth share with two investor groups in the LLS model. In a market with two equal investor populations and memory spans  $k = 10, 14$ , the group with the lower memory will dominate the market in the long run (left panel). In case of  $k = 10, 26$ , no population can prevail over the other one forever. Nevertheless, memory 26 dominates most of the time (right panel). The plot reproduces the findings in Levy et al. (1996) where all parameters are discounted on a daily time period level. Parameters are  $I=100$ ,  $N=10000$ ,  $r=0.001$ ,  $\sigma=0.2$ ,  $W_0=1000$  and  $p_0=4$ . The dividend process is assumed to evolve according to  $d_t = 0.004 \cdot 1.00015^t(t - 1)$ , which corresponds to a realistic annual growth rate.

A recent book written by the authors of the LLS model also dedicates one chapter to their financial market model, and this has already become well established in the relevant physics and

<sup>63</sup>Remember that in the heterogeneous case the evolution of the system is still determined by almost periodic dynamics determined by the chosen memory spans.

<sup>64</sup>In this context by definition the dominating population is just starting to create a trend. Interestingly, dominance over stock price evolution does not necessarily mean dominance over total wealth as it is shown in a more recent publication by Zschischang and Lux (2001).

economics branches of literature (see Levy et al., 2000). However, I wish only to summarize and therefore only the main differences will be discussed here. Roughly speaking, the model is modified in two directions. On the one hand, the dividend stream is assumed to be a stochastic variable following a multiplicative random walk

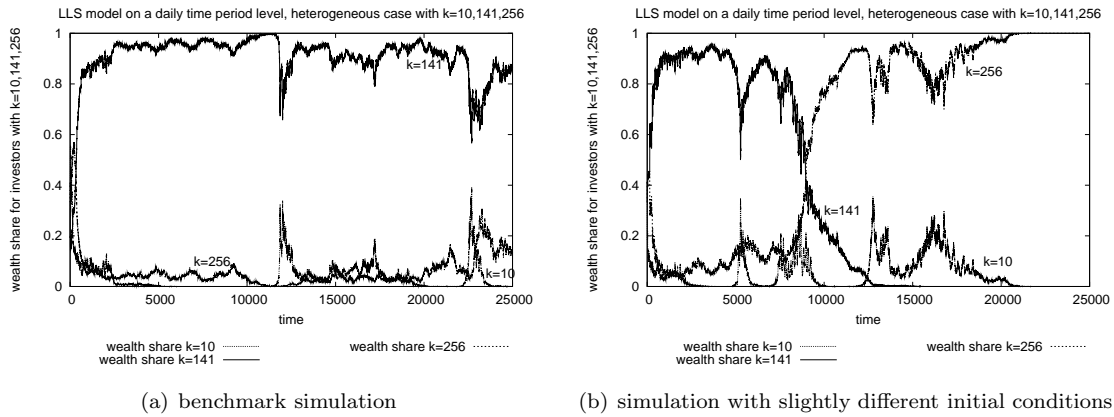
$$\tilde{d}_t = d_{t-1}(1 + \tilde{z})$$

where  $\tilde{z}$  is a stochastic variable with an appropriate probability density function. The tilde denotes the stochastic nature of the variables. On the other hand, in addition to the well known trading strategy using the last  $k$  historical returns for price prediction, which are denoted by ‘Efficient Market Believers’ (EMB), so-called ‘Rational Informed Identical’ (RII) investors are also introduced. They know about the dividend process and derive the ‘fundamental value’ of the stock as the discounted stream of future dividends. Most of the previous results qualitatively remain the same with these modifications. In the case of a whole ensemble of EMB investors (and the RII population) the time series of the stock price becomes quite irregular and seems, at least visually, quite similar to empirical data. No periodicity in the time series can be detected any longer in this scenario. Furthermore, the authors investigate their model version for different market scenarios with respect to typical market anomalies such as heavy trading volume, autocorrelation of returns and excess volatility. The results of these investigations are not stated here but we can refer to Levy et al. (2000) for further reading.

Instead, some criticism or at least some limitation of the LLS model should be mentioned at the end of this section. Some work has been published on this topic also giving new insights into the dynamic structure of the model (e.g., see Hellthaler, 1995; Kohl, 1997; Zschischang and Lux, 2001). Using a model design identical to the one in the original papers, E. ZSCHISCHANG and T. LUX<sup>65</sup> show that the long-run model outcomes are extremely dependent on the initial conditions, at least in the case of a three investor population scenario. Providing the investor populations with slightly different (random) return histories in different simulation runs in order to let them determine their optimal investment proportion at time  $t = 0$ , the authors observe that different population types can dominate their counterparts. Figure 2.19 shows a typical example of this result using the same combination  $k = 10, 141$  and  $256$  of memory spans as in Levy et al. (1996). Furthermore, introducing a power utility function (see Footnote 60) instead of logarithmic utility preferences, Zschischang and Lux detect the degree of risk aversion as the more relevant parameter for investors’ dominance in terms of wealth share. In their own experiments they generate scenarios in which investor populations that have been dominated by other groups in former runs prevail over the others when their degree of risk aversion is decreased. Note that this result is not surprising from an economics perspective. Indeed, it is well known in economics that agents who are willing to take a higher risk level will gain a higher profit (at least on average). Another similar finding is certainly worthy of note. Introducing a small group of traders that follows a buy-and-hold strategy leads to the result that these agents increase their wealth proportion and eventually dominate the whole population in the long run in all experiment outcomes.<sup>66</sup> Although the model cannot be interpreted as an efficient market because of its generally regular price patterns and periodic price

<sup>65</sup>See Zschischang and Lux (2001).

<sup>66</sup>A simple explanation for this phenomenon can be found in Zschischang and Lux (2001).



**Figure 2.19:** Time series of wealth share with three investor groups in the LLS model. In the case of a three investor population with  $k = 10, 141$  and  $256$ , either the group with memory span  $141$  (left panel) or the group with  $256$  (right panel) may dominate the market. Note that this plot is based on the GAUSS program of the LLS model also used in Zschischang and Lux (2001).

movements, this result nevertheless agrees with the EMH, confirming its conjecture that trading strategies based on trend detection from past price movements should not be more profitable than more simple and more fundamental trading rules (see Chapter 1, Section 1.3). In other words, the LLS model also leaves the survival of trend-based trading strategies as observed in real markets (see LeRoy, 1989) as a puzzle.

There are two further publications dealing with the LLS model which I want to mention here very briefly. T. HELLTHALER<sup>67</sup> investigated the effect of increasing the number of investors in the original model. In his own experiments he observed a strong sensitivity of the model to the system size (i.e., number of agents). He shows that an increase in the number of agents (from 96 to 4 million) leads to periodic price movements in all model variants, while at the same time, the main characteristics of the model (e.g., irregular price patterns in the heterogeneous cases) vanishes.<sup>68</sup>

R. KOHL<sup>69</sup> simulates the LLS model for more than one stock and analyzes the model behavior for small and large investor groups. In the case of small investor groups (100 investors) he shows that, like in the original model with only one stock, irregular price developments can be observed. He also demonstrates that an increase in the number of investors leads to periodic price patterns (at least for almost all stocks) as already observed by Hellthaler for the one-stock scenarios.

The LLS model has another limitation of a different nature. Like many other multi-agent models, the LLS framework relies exclusively on numerical simulations since the model framework is more or less analytically intractable due to the fact that the resulting dynamics describes a

<sup>67</sup>See Hellthaler (1995).

<sup>68</sup>The LLS model shares this phenomenon with several other microscopic models of financial markets e.g., the Kirman model (see Kirman, 1991, 1993), the SantaFe Artificial Stock Market Model (see Palmer et al., 1994; Arthur et al., 1997; LeBaron et al., 1999; Wilpert, 2004), the Cont-Bouchaud percolation model (see Cont and Bouchaud, 2000), the Kim-Markowitz model (see Kim and Markowitz, 1989) and the Lux-Marchesi model (see Lux and Marchesi, 1999, 2000). See also Egenter, Lux and Stauffer (1999) for some results on this topic for the last two models.

<sup>69</sup>See Kohl (1997).

growing process in time. However, this purely numerical treatment is somewhat unsatisfactory because very often it cannot be precisely clarified which model effects cause certain simulation outcomes. One possible response to shortcoming in these kinds of financial market models is to reformulate the original model structure in such a way that a stationary version of the model framework is obtained. Such a financial market model which also shares some crucial ideas of the LLS model will be outlined in the next chapter following Chiarella and He (2001). Hence this model overcomes the drawback of solely relying on numerical simulation results, yet we will also see that it still does not solve some of the limitations mentioned above for the LLS model framework.

Although I have summarized some critical statements and academic publications in the past few paragraphs, it would also be useful to consider two citations emphasizing the value of the LLS model before I close the section. The authors themselves state in Levy et al. (1995) that

[...] microscopic simulations [as performed in the LLS model] hold great potential for new insights into the complex behavior of the stock market;

which can definitely be seen as a first resume for this section. Furthermore, this statement is to some extent confirmed by two different authors working in the same field (see Lux and Ausloos, 2002) when discussing different model approaches proposed in physics and economic literature in their review article, i.e., more precise models without any key economic concepts proposed by physicists versus rather ‘complicated’ models favored by economists. Here they write that

[...], in order to take into account the peculiarities of economic systems (namely, in order to account for the nonmechanical nature of individual decisions even within a multi-agent framework), it seems indispensable to attempt a synthesis of both approaches. The models by Levy, Levy, and Solomon [...] may constitute some modest steps in this direction. [...]

In this sense the LLS model perhaps connects the physicists’ view (and maybe even the physicists’ way of thinking) for the first time with concepts hitherto used mainly by economists and thus makes a valuable contribution to this branch of research.

## 2.4 Summary

This chapter presents two prominent economic models of financial markets as well as some related model modifications of the original models, which reveal basic concepts of modern financial modeling. Furthermore, both models are closely connected to the model frameworks that will be outlined in the following chapters of this thesis. Both the model framework by Brock and Hommes (1998) and the model framework by Levy et al. (1994, 1995) are unified by the idea of removing some of the limitations which come with the traditional view of theoretical finance. This revision of the concepts of financial modeling is mainly based on a paradigmatic shift from a rational representative agent model to a boundedly rational, heterogeneous agents framework. All financial market models presented in this chapter use an agent-based modeling approach, where heterogeneous agents employ competing trading strategies and where prices are, at least partly, driven by the ‘psychology’ of the market, i.e., prices are not anymore fully determined by their underlying

economic fundamentals as proposed by the traditional theory (see Chapter 1). Instead, they may also be influenced by the market perception and the individual behavior of the market participants. But also a change in the research methodology occurs. As a result of the increased complexity of the modeling framework caused by the implementation of agents' heterogeneity, which very often leads to its analytical intractability, a shift from a purely analytical treatment of the models to a more computationally orientated approach becomes necessary.

To some extent, the model structure of both models is comparable. The heterogeneous agents in the model frameworks proposed by Brock and Hommes (1998) and Levy et al. (1994, 1995) are faced with a typical asset allocation problem in each discrete time step and therefore use a straightforward myopic utility maximization scheme to determine their optimal demand for the risky asset. This implies that all agents are assumed to be endowed with a certain amount of wealth which they want to maximize via their utility function and which, in turn, determines their portfolio decisions. Here, heterogeneity is introduced by positioning different trading groups with different beliefs or expectations about future price movements. Thus in these model frameworks there is an attempt to model the agents' individual behavior and decision-making process in a rather explicit and concrete way, in order to subsequently investigate whether the arising dynamics of the risky asset's equilibrium price (which is here determined by a Walrasian scenario) can reproduce some of the important observed stylized facts often seen in real financial data. However, there are also some crucial differences in the main building blocks of the two model frameworks presented in this chapter. While the Levy-Levy-Solomon (LLS) model does not allow the agents to switch between the different trading strategies even if one trading strategy is performing significantly better than the other one(s), in the Brock-Hommes (BH) model the interaction of agents is implemented by a switching mechanism between different trading strategies of agents. Depending on the current state of the market, agents are allowed to switch between fundamental and chartist based strategies which, in turn, again influence the price determination in the following time period.

Although this chapter mainly reviews some prominent examples of behavioral models of financial markets, it also contains my own contribution to the discussion in Gaunersdorfer (2000); Gaunersdorfer and Hommes (2007) relating to interesting model extensions of the original BH model. While Gaunersdorfer (2000) studies the case of time varying *homogeneous beliefs* about conditional variances, I propose to introduce *heterogeneity in beliefs* about variances, which makes the resulting nonlinear dynamic system more complex, but also draws a more realistic picture of the behavior of the two different strategy groups. Since the agents do believe in different price deviations from the fundamental price, they should therefore use also their own expectations for the determination of their time varying beliefs on conditional variances of returns. The results of our modified version of Gaunersdorfer's extension of the BH model might be summarized as follows:

- The implementation of heterogeneous time varying beliefs about variances of traders increases the dimension of the nonlinear system by two dimensions and thus leads to a seven dimensional system with two further variables which describes the different estimates of the conditional variance of both trader types. Otherwise, when restricting our model modification to the case of constant beliefs about variances it equals the constant variance case of the

model extension proposed by Gaunersdorfer (2000).

- Apart from the difference in the dimensions of the two considered model versions, the local and global picture of asset price dynamics in our approach is surprisingly similar to Gaunersdorfer (2000). In both versions a bifurcation route to strange attractors occurs when the intensity of choice to switch strategies increases.
- Concerning the local behavior of the system near the steady states, the primary bifurcation (with respect to the intensity of choice parameter) of the fundamental steady state as well as the secondary bifurcations of the non-fundamental steady states coincides with the bifurcations found in the case of constant beliefs about variances and is thus the same for our model modification as it is in Gaunersdorfer (2000).
- Higher-order bifurcation routes to chaos, however, might be different from the findings presented in Gaunersdorfer (2000).

In summary, introducing heterogeneous time varying beliefs into the model framework of Gaunersdorfer (2000) generalizes her results to a more realistic but also more complex market scenario where the main characteristics of the resulting price dynamic system still become valid. Nevertheless, there might several further ways of investigating our modified model version in even more general and more realistic model variants. First, the model can be studied for the case of a non-zero memory parameter  $\gamma$ , as is already the case for the original BH model (see also the simple example in Subsection 2.1.2). Furthermore, investigating the case of a positive supply of risky shares of the risky asset as well as introducing an asynchronous updating of beliefs mechanism might be an interesting task.<sup>70</sup> Since all these proposals to generalize the current model version also require an extensively complex analysis of the arising nonlinear dynamic system, we can leave these problems for future research.

This chapter closes the first part of the thesis, which has presented some mathematical background (mainly relegated to the appendices) as well as the basic economic concepts of modeling financial markets and some prominent model frameworks. At the beginning of the next part of this thesis I will introduce the constitutive asset pricing framework formulated by Chiarella and He (2001), which will serve as a point of departure for my own contribution to the branch of research presented in this thesis.

## Appendix to Chapter 2

### A 2.1 The Standard Mean-Variance Maximization Approach

The standard myopic<sup>71</sup> mean-variance maximization approach assumes that the expected utility of investor's wealth solely depends on the expected value of wealth  $\mu_W \equiv E[W]$  and the variance of wealth  $\sigma_W^2 \equiv Var[W]$  and has the form

$$E[U] = \mu_W - \frac{1}{2}a\sigma_W^2,$$

<sup>70</sup>See also Hommes et al. (2005) for a discussion on this topic for the original BH model.

<sup>71</sup>A single-period decision rule is said to be 'myopic' when it is independent of the future (i.e., beyond the current period) in a sequential decision problem. In other words subjects behave as if each trading period is the last decision period.

which can be motivated by the following, simple derivation.<sup>72</sup> It shows that the mean-variance maximization approach is equivalent to the maximization of a negative exponential utility function, which is a utility function of CARA type, under the assumption of normally distributed wealth  $W$ .

To be precise, assume that the individual maximizes his/her expected utility of wealth at time  $t + 1$  conditional on the information set  $\mathcal{I}_t$  available one period before. For the sake of simplicity denote the uncertain wealth  $W_{t+1}$  of the individual with  $W'$ . Furthermore assume that  $W'$  is normally distributed conditional on  $\mathcal{I}_t$  and let  $\sigma_{t,W}^2$  be the variance of  $W'$  conditional on  $\mathcal{I}_t$ . Then the expected utility of wealth is given by

$$\begin{aligned}
E_t[U(W')] &= E_t[-\exp\{-aW'\}] \\
&= -\frac{1}{\sqrt{2\pi\sigma_{t,W}^2}} \int_{-\infty}^{\infty} dW' \exp\left\{-\frac{1}{2\sigma_{t,W}^2}(W' - E_t[W'])^2 - aW'\right\} \\
&= -\frac{1}{\sqrt{2\pi\sigma_{t,W}^2}} \int_{-\infty}^{\infty} dW' \exp\left\{-\frac{1}{2\sigma_{t,W}^2}[W' - (E_t[W'] - a\sigma_{t,W}^2)]^2\right\} \\
&\quad \cdot \exp\left\{-aE_t[W'] + \frac{1}{2}a^2\sigma_{t,W}^2\right\} \\
&= -\exp\left\{-aE_t[W'] + \frac{1}{2}a^2\sigma_{t,W}^2\right\} \\
&= -\exp\left\{-a\left(E_t[W'] - \frac{1}{2}a\sigma_{t,W}^2\right)\right\}. \tag{A 2.1}
\end{aligned}$$

Since in general it is

$$\frac{\partial}{\partial x} \left\{ -\exp(-af(x)) \right\} = -\exp\{-af(x)\} \cdot (-a) \frac{\partial}{\partial x} f(x)$$

the maximization of Eq. (A 2.1) is equivalent to maximizing simply its argument, i.e.,

$$\begin{aligned}
\max\{E[-\exp(-aW')]\} &= \max\left\{-\exp\left\{-a\left(E_t[W'] - \frac{1}{2}a\sigma_{t,W}^2\right)\right\}\right\} \\
&= \max\left\{E_t[W_{t+1}] - \frac{1}{2}a\text{Var}_t[W_{t+1}]\right\},
\end{aligned}$$

which becomes obvious by using the first-order condition.

Furthermore, it can be shown in this framework that an investor who is faced with two investment alternatives namely a riskless asset giving the fixed return  $r$  and one risky asset giving the return  $\rho_t = (p_t + d_t - p_{t-1})/p_{t-1}$  determines his/her demand of the risky assets independently of his/her wealth level, which is a crucial result in asset allocation theory. To see this, note that the wealth  $W_{t+1}$  of the investor at time  $t + 1$  is given by

$$W_{t+1} = (1 - \pi_t)W_t(1 + r) + \pi_t W_t(1 + \rho_{t+1}), \tag{A 2.2}$$

where  $\pi_t$  denotes the investment proportion invested in the risky asset at time  $t$ . At each time period  $t$  the investor has to maximize his expected utility by deriving

$$\max_{\pi_t} \left\{ E_t[W_{t+1}(\pi_t)] - \frac{1}{2}a\text{Var}_t[W_{t+1}(\pi_t)] \right\}$$

which leads to the following determination problem

$$\frac{\partial}{\partial \pi_t} \left\{ W_t[(1 + r) + \pi_t(E_t[\rho_{t+1}] - r)] - \frac{1}{2}aW_t^2\pi_t^2\text{Var}_t[\rho_{t+1}] \right\} = 0, \tag{A 2.3}$$

where

$$\begin{aligned}
E_t[W_{t+1}] &= W_t[(1 + r) + \pi_t(E_t[\rho_{t+1}] - r)] \\
\text{Var}_t[W_{t+1}] &= W_t^2\pi_t^2\text{Var}_t[\rho_{t+1}]
\end{aligned}$$

<sup>72</sup>See also Grossman (1976).

is used. The solution of Eq. (A 2.3) is given by

$$\pi_t = \frac{1}{W_t} \frac{E_t[\rho_{t+1}] - r}{a \text{Var}_t[\rho_{t+1}]},$$

which states that the wealth proportion invested in the risky asset has an inversely proportional dependence on the current level of investor's wealth, which is consistent with the property of an originally chosen IRRA utility function. The investor's demand of the risky asset at time  $t$  (i.e., the *number* of risky assets which are denoted by  $N_t$ ) can simply be calculated using the relation

$$N_t p_t = \pi_t W_t$$

and is given by

$$N_t = \frac{1}{W_t} \frac{E_t[\rho_{t+1}] - r}{a \text{Var}_t[\rho_{t+1}]} \cdot \frac{W_t}{p_t} = \frac{E_t[\rho_{t+1}] - r}{a \text{Var}_t[\rho_{t+1}]} \cdot \frac{1}{p_t}, \quad (\text{A 2.4})$$

which is obviously independent of investor's wealth. Using the definition  $\rho_{t+1} = (p_{t+1} + d_{t+1} - p_t)/p_t$  leads to another version of Eq. (A 2.4) namely

$$N_t = \frac{1}{p_t} \frac{\frac{1}{p_t} E_t[p_{t+1} + d_{t+1}] - 1 - r}{a \frac{1}{p_t^2} \text{Var}_t[p_{t+1} + d_{t+1}]} = \frac{E_t[p_{t+1} + d_{t+1}] - (1+r)p_t}{a \text{Var}_t[p_{t+1} + d_{t+1}]},$$

which is the typical demand curve for an investor whose investment preferences are described by a CARA utility function using the mean-variance maximization approach.

## A 2.2 Further Notes on Eq. (2.8)

Rewriting Eq. (2.6) as

$$p_t = \frac{1}{1+r} E_t[p_{t+1} + d_{t+1}]$$

and shifting  $p_t$  one period leads to

$$p_{t+1} = \frac{1}{1+r} E_{t+1}[p_{t+2} + d_{t+2}].$$

By iterated substitution we obtain

$$p_t = \sum_{i=1}^k \frac{1}{(1+r)^i} E_t[d_{t+i}] + \frac{1}{(1+r)^k} E_t[p_{t+k}], \quad (\text{A 2.5})$$

where the iterated conditional expectations have been simplified by the following rule (e.g., see Schmitz, 1996, p. 143): let  $\mathcal{I}_1 \subseteq \mathcal{I}_2$ , then the conditional expectation value has the following property

$$E[E[X|\mathcal{I}_2] | \mathcal{I}_1] = E[X | \mathcal{I}_1] \quad P | \mathcal{I}_1 - \text{asymptotically stable}. \quad (\text{A 2.6})$$

Hence Eq. (A 2.5) can be obtained by setting  $X \equiv d_{t+i}$  and  $X \equiv p_{t+i}$  respectively, as well as  $\mathcal{I}_1 \equiv \mathcal{I}_t$  and  $\mathcal{I}_2 \equiv \mathcal{I}_{t+i}$  for  $i = 1, \dots, k$ . Note that this property of conditional expectations is known as the 'law of iterated expectations' in economic literature. By rewriting Eq. (A 2.6) as  $E[X - E[X|\mathcal{I}_2] | \mathcal{I}_1] = 0$  it suggests an intuitive interpretation. In words, it is not possible to use limited information  $\mathcal{I}_1$  to predict the forecast error one would make if one had superior information  $\mathcal{I}_2$  (e.g., see Cuthbertson, 1996; Campbell et al., 1997). Obviously, for Eq. (A 2.5) to converge for  $k \rightarrow \infty$  the remaining term  $E_t[p_{t+k}]/(1+r)^k$  has to vanish in the limit, which is known as a terminal condition or 'transversality condition' and which rules out rational speculative bubbles (see Cuthbertson, 1996, p. 78 for details).



### A 2.3 Derivation of Eq. (2.13)

Denote the realized excess return over period  $t-1$  to period  $t$  by  $R_t$ . Then,  $R_t$  can be obtained depending on the deviation  $x_t$  from the fundamental by using the identity  $p_t = x_t + p_t^*$  and adding a ‘zero-supplement’ which leads to

$$\begin{aligned} R_t &\equiv p_t + d_t - Rp_{t-1} = x_t + p_t^* + d_t - Rx_{t-1} - Rp_{t-1}^* \\ &= x_t - Rx_{t-1} + p_t^* + d_t - E_{t-1}[p_t^* + d_t] + \{E_{t-1}[p_t^* + d_t] - Rp_{t-1}^*\} \\ &= x_t - Rx_{t-1} + \delta_t, \end{aligned} \quad (\text{A 2.7})$$

where  $\delta_t \equiv p_t^* + d_t - E_{t-1}[p_t^* + d_t]$ . Note that the term in the curly brackets is zero since the fundamental price  $p_t^*$  satisfies the market equilibrium equation (2.6). To determine the optimal demand for the risky asset with respect to  $x_t$  by trader type  $h$  the following maximization has to be considered

$$\max_z \left\{ E_{h,t-1}[R_t]z - \frac{a}{2}z^2 \text{Var}[R_t] \right\} = \max_z \left\{ E_{h,t-1}[R_t]z - \frac{a}{2}z^2 \sigma^2 \right\}, \quad (\text{A 2.8})$$

where  $z$  denotes the number of risky assets in a short-hand notation. As this maximization problem equals the maximization problem in Eq. (2.2) up to a constant (i.e.,  $RW_{*,t-1}$ , see Eq. (2.1)), the optimal solution is given by

$$\begin{aligned} z^{opt} &= \frac{E_{h,t-1}[p_t + d_t - Rp_{t-1}]}{a\sigma^2} = \frac{\{E_{h,t-1}[p_t^*] + f_{h,t-1}\} + E_{h,t-1}[d_t] - Rp_{t-1}}{a\sigma^2} \\ &= \frac{f_{h,t-1} - R(p_{t-1} - p_{t-1}^*)}{a\sigma^2} = \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}, \end{aligned}$$

where again Eq. (2.6) is used for the fundamental  $p_t^*$ . For further details see Brock and Hommes (1999).

### A 2.4 Proof of Theorem 2.1

The proof proceeds in three steps.

#### (i) Existence of Steady States

From Eq. (2.21) a necessary condition for the steady state  $(x^*, m^*, u^*)$  to exist is given by

$$Rx^* = \frac{1 - m^*}{2}gx^*,$$

which implies

$$x^* = 0 \quad \vee \quad \left( x^* \neq 0 \wedge R = \frac{1 - m^*}{2}g \quad \left[ \text{which is equivalent to } m^* = 1 - \frac{2R}{g} \right] \right), \quad (\text{A 2.9})$$

whereas, additionally from Eq. (2.23) the steady state  $(x^*, m^*, u^*)$  has to fulfill<sup>73</sup>

$$u^* = \gamma u^* - \frac{gx^*}{a\sigma^2} (x^* - Rx^*) - C,$$

that implies

$$u^* = \frac{1}{1 - \gamma} \left[ \frac{(R - 1)g}{a\sigma^2} (x^*)^2 - C \right]. \quad (\text{A 2.10})$$

Finally, according to Eq. (2.22) the steady state  $(x^*, m^*, u^*)$  also has to satisfy

$$m^* = \tanh \left( \frac{\beta}{2} u^* \right), \quad (\text{A 2.11})$$

<sup>73</sup>In the following, a (fully) deterministic scenario (i.e.,  $\delta_{t-1} = 0$ ) is assumed.

which can be combined by using Eq. (A 2.10) as

$$m^* = \tanh \left( \frac{\beta}{2(1-\gamma)} \left[ \frac{(R-1)g}{a\sigma^2} (x^*)^2 - C \right] \right). \quad (\text{A 2.12})$$

Thus, neglecting the  $u^*$  variable in the further considerations (because it can be simply derived by Eq. (A 2.10)), if  $x^* = 0$  we get a *fundamental steady state*

$$E_1 = (x^*, m^*) = (0, \tanh \left( \frac{-\beta C}{2(1-\gamma)} \right)) \equiv (x^*, m^{eq}),$$

whereas if  $x^* \neq 0$  two *non-fundamental steady states* arise

$$E_2 = (x_+^*, m^*) \quad \text{and} \quad E_3 = (x_-^*, m^*) = (-x_+^*, m^*),$$

where  $m^*$  is given by Eq. (A 2.9) and  $x_{\pm}^*$  is the solution (if it exists) of Eq. (A 2.12) given by

$$x_{\pm}^* = \pm \sqrt{\frac{a\sigma^2}{(R-1)g} \left\{ \frac{2(1-\gamma)}{\beta} [\operatorname{arctanh}(m^*)] + C \right\}}. \quad (\text{A 2.13})$$

### (ii) Uniqueness of Steady States

For  $0 < g < R$  the fundamental steady state is unique as  $R/g > 1$  implies

$$m^* = 1 - \frac{2R}{g} < -1$$

which is in contradiction to the necessary condition  $m^* = \tanh \left( \frac{\beta}{2} u^* \right) \in [-1, 1]$ .

For  $g > 2R$  we get

$$0 < m^* = 1 - \frac{2R}{g} < 1$$

since  $1 > 2R/g > 0$ . Hence, Eq. (A 2.12) has two solutions, one positive solution  $x_+^*$  and one negative solution  $x_-^* \equiv -x_+^*$ . Overall, there are three steady states in this case.

For  $R < g < 2R$  consider  $m^*$  at the boundaries  $g = R$  and  $g = 2R$  first, which satisfies  $m^* = -1$  or  $m^* = 0$  respectively. From this, it is obvious, that

$$m^* \in ]-1, 0[$$

for  $R < g < 2R$ . Now, we show that Eq. (A 2.12) has two solutions if and only if

$$m^* > m^{eq}.$$

Considering the following equivalences

$$\begin{aligned} m^* > m^{eq} &= \tanh \left( -\frac{\beta C}{2(1-\gamma)} \right) \\ \Leftrightarrow \frac{2(1-\gamma)}{\beta} \operatorname{arctanh}(m^*) &> -C \\ \Leftrightarrow \frac{2(1-\gamma)}{\beta} \operatorname{arctanh}(m^*) + C &> 0 \\ \Leftrightarrow \frac{a\sigma^2}{(R-1)g} \cdot \frac{2(1-\gamma)}{\beta} \operatorname{arctanh}(m^*) + C &> 0 \end{aligned}$$

we end with a positive expression equal to the discriminant of Eq. (A 2.13). Since only equivalent transformations are used, it is therefore shown that Eq. (A 2.12) has two solutions if and only if  $m^* > m^{eq}$ .

(iii) Stability of Fundamental Steady State  $E_1$ 

The system Eq. (2.21)–Eq. (2.23) can be equivalently transformed into the following canonical form of ordinary first-order difference equations

$$\begin{aligned} x_t &= \frac{1 - \tanh\left(\frac{\beta}{2} \left[\gamma u_{t-1} - \frac{gz_{t-1}}{a\sigma^2}(x_{t-1} - Ry_{t-1}) - C\right]\right)}{2R} g x_{t-1}, \\ y_t &= x_{t-1}, \\ z_t &= y_{t-1}, \\ u_t &= \gamma u_{t-1} - \frac{gz_{t-1}}{a\sigma^2}(x_{t-1} - Ry_{t-1}) - C, \end{aligned}$$

where an appropriate number of auxiliary variables has been introduced.<sup>74</sup> The Jacobian can be derived as

$$J = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{gz_{t-1}}{a\sigma^2} & \frac{gRz_{t-1}}{a\sigma^2} & -\frac{g}{a\sigma^2}(x_{t-1} - Ry_{t-1}) & \gamma \end{pmatrix}$$

with

$$\begin{aligned} a_1 &= -\frac{1}{2R}(\tanh[\dots])' \cdot \frac{\beta g z_{t-1}}{2a\sigma^2} \cdot g x_{t-1} \\ &\quad + \frac{1 - \tanh\left(\frac{\beta}{2R} \left[\gamma u_{t-1} - \frac{gz_{t-1}}{a\sigma^2}(x_{t-1} - Ry_{t-1}) - C\right]\right)}{2} g, \\ a_2 &= -\frac{g x_{t-1}}{2R}(\tanh[\dots])' \cdot \frac{\beta g R z_{t-1}}{2a\sigma^2}, \\ a_3 &= -\frac{g x_{t-1}}{2R}(\tanh[\dots])' \cdot \left(-\frac{\beta g}{2a\sigma^2}(x_{t-1} - Ry_{t-1})\right), \\ a_4 &= -\frac{g x_{t-1}}{2R}(\tanh[\dots])' \cdot \frac{\beta \gamma}{2}, \end{aligned}$$

where in each case  $(\cdot)'$  denotes the first derivative with respect to the currently appropriate variable. Evaluating  $J$  at the fundamental state  $E_1 = (x^*, m^{eq}) = (0, \tanh\left(\frac{-\beta C}{2(1-\gamma)}\right))$ , leads to

$$\begin{aligned} J|_{(x^*, m^{eq}, u^*)} &= \begin{pmatrix} \frac{1 - \tanh\left(\frac{-\beta C}{2(1-\gamma)}\right)}{2R} g & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \\ &= \begin{pmatrix} \frac{1 - m^{eq}}{2R} g & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \end{aligned}$$

where  $u^* = -C/(1-\gamma)$  and  $(\tanh x)' = 1 - \tanh^2 x$  are used for simplification. Now, the characteristic polynomial can be derived by exploiting Laplace's expansion rule as

$$\det(\lambda \mathbb{I} - J) = (-1)^6 \lambda \cdot \left\{ \left( \lambda - \frac{1 - m^{eq}}{2R} g \right) \cdot \lambda \cdot (\lambda - \gamma) \right\}$$

from which the eigenvalues of the fundamental steady state can be determined as

$$\lambda_1 = \gamma \quad \text{and} \quad \lambda_2 = \frac{1 - m^{eq}}{2R} g \quad \text{as well as} \quad \lambda_3 = \lambda_4 = 0.$$

<sup>74</sup>As before  $\delta_{t-1}$  is assumed to be 0.

As  $\gamma$  is always assumed to be in the interval  $]0, 1[$ , the fundamental steady state is locally asymptotically stable, if and only if  $-1 < \lambda_2 < 1$  that is equivalent to

$$\begin{aligned} & -1 < \frac{1 - m^{eq}}{2R}g < 1 \\ \Leftrightarrow & -\frac{2R}{g} < 1 - m^{eq} < \frac{2R}{g} \\ \Leftrightarrow & 1 + \frac{2R}{g} > m^{eq} > 1 - \frac{2R}{g} = m^* . \end{aligned}$$

Since it is  $m^{eq} \in [-1, 1]$  by definition, it can be deduced that  $\lambda_2 \in ]0, 1[$  if and only if  $m^* < m^{eq}$ . To sum up and conclude, for  $R < g < 2R$   $E_1$  is locally asymptotically stable if and only if  $m^* < m^{eq}$ . Otherwise  $E_1$  is an unstable saddle point since  $|\lambda_j| < 1$  for  $j = 1, 3, 4$  and  $|\lambda_2| > 1$ . For  $g > 2R$  it is  $m^{eq} = \tanh\{-\beta C/(2(1 - \gamma))\} \in ]-1, 0[$ . Hence  $E_1$  is an unstable saddle point since

$$\lambda_2 = (1 - m^{eq})\frac{g}{2R} > 1 .$$

Finally, for  $0 < g < R$  it is  $m^* = 1 - (2R/g) < -1$ . Therefore, it is always  $m^{eq} > m^*$  and thus  $E_1$  is locally asymptotically stable. Furthermore, in this case  $E_1$  is even globally asymptotically stable, since Eq. (2.21) can be written as  $x_t = A_t x_{t-1}$ , where  $A_t = [(1 - m_t)/2]g/R$ . Because the expression in the square brackets is  $\in [0, 1]$ , so  $|A_t| \leq g/R < 1$  and hence  $|x_t| \leq (g/R)^t |x_0| \rightarrow 0$  as  $t \rightarrow \infty$  for every choice of  $x_0$ . This completes the proof.

### A 2.5 Gaunersdorfer's Risk Adjusted Profits Term

Using the same notation for the realized excess return  $R_t = p_t + d_t - Rp_{t-1}$  as in similar considerations in Appendix A 2.3 we obtain for  $\phi_{h,t}$

$$\phi_{h,t} = R_t z^{opt} - \frac{a}{2} (z^{opt})^2 \text{Var}_{h,t-1}[R_t] ,$$

where again,  $z^{opt}$  denotes the optimal demand of risky assets in a short-hand notation. In Appendix A 2.3  $z^{opt}$  has already been derived from Eq. (A 2.8) as

$$z^{opt} = \frac{E_{h,t-1}[R_t]}{a \text{Var}_{h,t-1}[R_t]} ,$$

which can also be expressed in deviation from the fundamental as

$$z^{opt} = \frac{f_{h,t-1} - R x_{t-1}}{a \text{Var}_{h,t-1}[x_t - R x_{t-1} + \delta_t]} ,$$

where the relation  $R_t = x_t - R x_{t-1} + \delta_t$  is used (see Eq. (A 2.7)). Now, the performance measure can be written as

$$\begin{aligned} \phi_{h,t} &= R_t \frac{E_{h,t-1}[R_t]}{a \text{Var}_{h,t-1}[R_t]} - \frac{a}{2} \left( \frac{E_{h,t-1}[R_t]}{a \text{Var}_{h,t-1}[R_t]} \right)^2 \text{Var}_{h,t-1}[R_t] \\ &= R_t \frac{E_{h,t-1}[R_t]}{a \text{Var}_{h,t-1}[R_t]} - \frac{1}{2a} \frac{(E_{h,t-1}[R_t])^2}{\text{Var}_{h,t-1}[R_t]} . \end{aligned}$$

Note that for rational traders, i.e., traders who form their expectations according to  $E_{R,t-1}[R_t] = R_t$ , the performance measure simplifies to

$$\phi_{R,t} = \frac{1}{2a} \frac{R_t^2}{\text{Var}_{R,t-1}[R_t]} .$$

For analytical tractability, homogeneous expectations of conditional variances of returns are assumed in the following derivation, i.e.,

$$\text{Var}_{h,t}[\cdot] = \text{Var}_t[\cdot] .$$

To be able to use squared prediction errors for the performance measure, which might be an economically more reasonable variable, we subtract  $\phi_{R,t}$  from  $\phi_{h,t}$  which leads to

$$\begin{aligned}\phi_{h,t} - \phi_{R,t} &= R_t \frac{E_{h,t-1}[R_t]}{a \text{Var}_{t-1}[R_t]} - \frac{1}{2a} \frac{E_{h,t-1}^2[R_t]}{\text{Var}_{t-1}[R_t]} - \frac{1}{2a} \frac{R_t^2}{\text{Var}_{t-1}[R_t]} \\ &= -\frac{1}{2a \text{Var}_{t-1}[R_t]} (R_t - E_{h,t-1}[R_t])^2.\end{aligned}\quad (\text{A } 2.14)$$

Using the relations

$$\begin{aligned}R_t &\stackrel{!}{=} E_{R,t-1}[R_t] = E_{R,t-1}[x_t - Rx_{t-1} + \delta_t] = x_t - Rx_{t-1}, \\ E_{h,t-1}[R_t] &= f_{h,t-1} - Rx_{t-1},\end{aligned}\quad (\text{A } 2.15)$$

finally, we end up with<sup>75</sup>

$$\begin{aligned}\phi_{h,t} - \phi_{R,t} &= -\frac{1}{2a \text{Var}_{t-1}[x_t - Rx_{t-1} + \delta_t]} \left( \underbrace{(x_t - Rx_{t-1})}_{\text{Eq. (A } 2.15)} - (f_{h,t-1} - Rx_{t-1}) \right)^2 \\ &= -\frac{1}{2a(\sigma_{t-1}^2 + \sigma_\delta^2)} (x_t - f_{h,t-1})^2.\end{aligned}\quad (\text{A } 2.16)$$

where, for the last step, the following assumptions

$$\begin{aligned}\text{Var}_{t-1}[x_t - Rx_{t-1}] &= \sigma_{t-1}^2, \\ \text{Var}_{t-1}[\delta_t] &= \sigma_\delta^2, \\ \text{Cov}[x_t - Rx_{t-1}, \delta_t] &\equiv 0\end{aligned}$$

are made. Note that in the main text the notation  $\phi_{h,t}$  stands for the difference ( $\phi_{h,t} - \phi_{R,t}$ ) calculated in Eq. (A 2.16). Furthermore, the justification for this linear transformation is the following: since  $\phi_{R,t}$  is independent from  $h$  and since the discrete choice probabilities  $n_{h,t}$  (see Eq. (2.10)), which are the fractions of trader type  $h$ , are independent of the level of fitness, subtracting the term  $\phi_{R,t}$  has no real effect on the dynamics.

## A 2.6 Proof of Theorem 2.2

The proof proceeds in three steps.

### (i) Existence and Uniqueness of Steady States

Let  $x^* = x_t = \dots = x_{t-4}$  be the steady state of the system Eq. (2.42)–Eq. (2.46) and denote

$$m^* \equiv \tanh \left( \frac{\beta}{2} \left[ \frac{g(g-2)(x^*)^2}{2a\sigma^2} + \alpha(x^*)^2 - C \right] \right).$$

From Eq. (2.42) a necessary condition for the steady state to exist is given by

$$x^* = \frac{g}{2R} x^* (1 - m^*) \quad (\text{A } 2.17)$$

which implies

$$x^* = 0 \quad \vee \quad \left( x^* \neq 0 \wedge m^* = 1 - \frac{2R}{g} \right).$$

<sup>75</sup>Contrarily, Hommes (2001) also uses Eq. (A 2.7) for the left part in the big parentheses and consequently end up with  $\phi_{h,t} - \phi_{R,t} = -(2a \text{Var}_{t-1}[x_t - Rx_{t-1} + \delta_t])^{-1} (x_t - f_{h,t-1} + \delta_t)^2$ . Obviously, for the ‘deterministic skeleton model’, i.e., when all noise terms are set to zero, both transformations do not make any difference.

Using the definition for  $m^*$  given above for the non-trivial solution of Eq. (A 2.17) leads to a quadratic equation, i.e.,

$$(x^*)^2 = \frac{2a\sigma^2 [\ln\{(g/R) - 1\} + \beta C]}{\beta(g(g-2) + 2a\alpha\sigma^2)}, \quad (\text{A 2.18})$$

if  $g > R$ . Thus, besides a *fundamental steady state*  $E_1 = (0, 0, 0, 0, 0)$  which exists for all  $g > 0$ , additionally, (only) two *non-fundamental steady states*  $E_{2,3} = (\pm x^*, \pm x^*, \pm x^*, \pm x^*, \pm x^*)$  can occur if and only if

$$\frac{\ln\{(g/R) - 1\} + \beta C}{g(g-2) + 2a\alpha\sigma^2} \equiv \frac{N}{D} > 0 \quad \wedge \quad g > R.$$

More precisely, it can easily be shown that

$$\begin{aligned} D > 0 & \quad \text{if and only if} \quad \alpha > \alpha^* \equiv \frac{-g(g-2)}{2a\sigma^2} \quad \text{for all } g > 0, \\ N > 0 & \quad \text{if and only if} \quad \beta > \beta^* \equiv \frac{1}{C} \ln\left(\frac{R}{g-R}\right) \quad \text{for } R < g < 2R, \\ N > 0 & \quad \text{for } g > 2R, \end{aligned}$$

which implies that for  $0 < g < R$  the fundamental steady state  $E_1$  is the unique steady state, while for  $R < g < 2R$  and  $\alpha > \alpha^*$  ( $\alpha < \alpha^*$ )  $E_{2,3}$  exist if and only if  $\beta > \beta^*$  ( $\beta < \beta^*$ ). Finally, for  $g > 2R$  all three steady states  $E_1$  and  $E_{2,3}$  exist for all choices of  $\alpha, \beta > 0$ . Note that in that case  $\alpha^*$  becomes negative.

### (ii) Stability of Fundamental Steady State $E_1$

The Jacobian of the fundamental steady state  $E_1$  can be determined by

$$J = \begin{pmatrix} \frac{g}{2R} (1 + \tanh(\frac{\beta C}{2})) & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

from which the the eigenvalues of  $E_1$  can be derived as

$$\lambda_1 = \frac{g}{2R} \left[ 1 + \tanh\left(\frac{\beta C}{2}\right) \right] \quad \text{and} \quad \lambda_2 = \dots \lambda_5 = 0.$$

Obviously,  $E_1$  is locally asymptotically stable for  $0 < g < R$  since then  $\lambda_1$  lies in the unit circle while for  $g > 2R$  the eigenvalue  $\lambda_1 > 1$  which implies that in this case  $E_1$  is unstable. For the intermediate case, i.e.,  $R < g < 2R$ , the eigenvalue  $\lambda_1 \in (0, 1)$  if and only if  $\beta < \beta^*$ . Furthermore, the global stability property of  $E_1$  for  $0 < g < R$  can be concluded in a totally analogous way as already outlined in the proof of Theorem 2.1 in Appendix A 2.4.

### (iii) Stability of Non-Fundamental Steady States $E_{2,3}$

The determination of the stability properties of the non-fundamental steady states is more comprehensive and therefore supported by the software package MATHEMATICA 4.1, which is able to give numerical as well as analytical solutions for a wide range of standard mathematical problems. Thus the characteristic polynomial of the Jacobian of the non-fundamental steady states  $E_2$  and  $E_3$  can be determined by

$$\Gamma(\lambda) = -\lambda^2 \left\{ \lambda^3 - \left( 1 + \frac{g - 2a\alpha\sigma^2}{g} \Psi \right) \lambda^2 - (1 - g) \Psi \right\} \quad (\text{A 2.19})$$

with

$$\Psi \equiv 2(g - R) \frac{\ln\{(g/R) - 1\} + \beta C}{g(g-2) + 2a\alpha\sigma^2}. \quad (\text{A 2.20})$$

Hence  $E_{2,3}$  have two trivial eigenvalues  $\lambda_1 = \lambda_2 = 0$  and three further ones which are implicitly given by the expression in the curly brackets of Eq. (A 2.19).<sup>76</sup>

For  $R < g < 2R$  two subcases have to be considered. If  $\alpha < \alpha^*$  the non-fundamental steady states exist for  $\beta < \beta^*$ . In this subcase the three non-trivial eigenvalues  $\lambda_3, \lambda_4$  and  $\lambda_5$  are all real. Two eigenvalues (one negative and one positive) lie inside the unit circle while the third (positive) eigenvalue does not. At  $\beta = \beta^*$  two eigenvalues become zero and one eigenvalue is equal to one. Hence the non-fundamental steady states are not stable in this subcase. Contrastingly, if  $\alpha > \alpha^*$  the non-fundamental steady states exist for  $\beta > \beta^*$  and  $\beta$  being not too large. In this subcase all three non-trivial eigenvalues of  $E_{2,3}$  are real (two positive which increase and one negative which decreases) and lie inside the unit circle. Hence the non-fundamental steady states are stable. Increasing  $\beta$  up to a certain value the two positive eigenvalues become complex and  $E_{2,3}$  undergo the secondary bifurcation. Note that in this case the third negative non-trivial eigenvalue might already have crossed the unit circle (see next Theorem 2.3 and its proof for details).

For  $g > 2R$  the non-fundamental steady states are stable since (besides the two trivial eigenvalues) one negative and two complex conjugate eigenvalues lie inside the unit circle for all  $\alpha$  and  $\beta$  being small enough. Increasing  $\beta$  sufficiently, the complex eigenvalues as well as the negative eigenvalue cross the unit circle and, therefore,  $E_{2,3}$  undergoes a Hopf or a flip bifurcation (see next Theorem 2.3 and its proof for details). This completes the proof.

### A 2.7 Proof of Theorem 2.3

As already mentioned in the previous proof, for  $g > R$ ,  $\alpha > \alpha^*$  and  $\beta$  being sufficiently large the three non-trivial eigenvalues of the system Eq. (2.42)–Eq. (2.46) cross the unit circle. More precisely, either the modulus of the two complex conjugate eigenvalues become greater than 1, or the negative eigenvalue becomes equal to -1. Thus the proof has to consider two different bifurcation scenarios.

#### (i) Hopf Bifurcation of the Non-Fundamental Steady States $E_{2,3}$

As stated in the previous proof, the characteristic polynomial of the Jacobian of the non-fundamental steady states  $E_{2,3}$  are given by Eq. (A 2.19), which has two trivial eigenvalues  $\lambda_1 = \lambda_2 = 0$  and three non-trivial eigenvalues  $\lambda_3, \lambda_4$  and  $\lambda_5$ . Since the cubic term in the curly brackets has no linear term, one eigenvalue can be expressed by the other two, i.e.,

$$\lambda_3 = -\frac{\lambda_4\lambda_5}{\lambda_4 + \lambda_5}.$$

A Hopf bifurcation occurs if a pair of complex conjugate eigenvalues cross the unit circle. Thus assume that  $\lambda_{4,5} = \mathcal{A} \pm i\mathcal{B}$  and note that  $|\lambda_i|^2 = \lambda_i\lambda_i^* = \mathcal{A}^2 + \mathcal{B}^2$ ,  $i = 4, 5$ , and  $\lambda_4 = \lambda_5^*$  where the  $*$  denotes the complex conjugate counterpart. Then  $\lambda_3$  is given by

$$\lambda_3 = -\frac{\mathcal{A}^2 + \mathcal{B}^2}{2\mathcal{A}}. \tag{A 2.21}$$

Writing  $\Gamma(\lambda)$  as

$$\begin{aligned} \Gamma(\lambda) &= -\lambda^2 \{(\lambda - \lambda_3)(\lambda - \lambda_4)(\lambda - \lambda_5)\} \\ &= -\lambda^2 \{(\lambda - \lambda_3)[\lambda^2 - (\lambda_4 + \lambda_5)\lambda + \lambda_4\lambda_5]\} \end{aligned} \tag{A 2.22}$$

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<sup>76</sup>The ‘fundamental theorem of algebra’ assures that all non-constant polynomials with integer and real or complex coefficients can be decomposed into linear factors in the complex space, which, in other words, guarantees as many roots as the order of the polynomial is.

and inserting Eq. (A 2.21) into Eq. (A 2.22) leads to

$$\Gamma(\lambda) = -\lambda^2 \left\{ \lambda^3 + \left[ \frac{\mathcal{A}^2 + \mathcal{B}^2}{2\mathcal{A}} - 2\mathcal{A} \right] \lambda^2 + \frac{(\mathcal{A}^2 + \mathcal{B}^2)^2}{2\mathcal{A}} \right\}. \quad (\text{A } 2.23)$$

Comparing the coefficients of Eq. (A 2.23) with the coefficients of Eq. (A 2.19) yields

$$\left[ \frac{\mathcal{A}^2 + \mathcal{B}^2}{2\mathcal{A}} - 2\mathcal{A} \right] = - \left( 1 + \frac{g - 2a\alpha\sigma^2}{g} \Psi \right) \quad \wedge \quad \frac{(\mathcal{A}^2 + \mathcal{B}^2)^2}{2\mathcal{A}} = -(1 - g)\Psi,$$

which simplifies at the Hopf bifurcation point (where  $\mathcal{A}^2 + \mathcal{B}^2 = 1$ ) to

$$\left[ \frac{1}{2\mathcal{A}} - 2\mathcal{A} \right] = - \left( 1 + \frac{g - 2a\alpha\sigma^2}{g} \Psi \right) \quad \wedge \quad \frac{1}{2\mathcal{A}} = -(1 - g)\Psi.$$

Eliminating the real part  $\mathcal{A}$  of the complex conjugate eigenvalues  $\lambda_{4,5}$  from these equations leads to an equation which describes the Hopf bifurcation curve on the  $(\beta, \alpha)$ -plane implicitly by

$$\frac{(g^2 - 2a\alpha\sigma^2)(g - 1)}{g} \Psi^2 + (g - 1)\Psi - 1 = 0 \quad (\text{A } 2.24)$$

which is only valid if

$$\Psi = \frac{-g(g - 1) \pm \sqrt{g^2(g - 1)^2 + 4g(g - 1)(g^2 - 2a\alpha\sigma^2)}}{2(g^2 - 2a\alpha\sigma^2)(g - 1)}$$

is real. It can easily be shown that  $\Psi$  is real for

$$\alpha < \bar{\alpha} \equiv \frac{g(5g - 1)}{8a\sigma^2}.$$

Furthermore Eq. (A 2.24) can be re-expressed in terms of the original parameters of the dynamic system Eq. (2.42)–Eq. (2.46) by using Eq. (A 2.20) which leads to a quite complicated expression, i.e.,

$$\frac{2(g - 1)(g - R)[\beta C + \ln \{(g/R) - 1\}]}{(g - 2)g + 2a\alpha\sigma^2} + \frac{4(g - 1)(g - R)^2(g^2 - 2a\alpha\sigma^2)[\beta C + \ln \{(g/R) - 1\}]^2}{g[(g - 2)g + 2a\alpha\sigma^2]^2} = 1 \quad (\text{A } 2.25)$$

which again describes the Hopf bifurcation curve on the  $(\beta, \alpha)$ -plane in an implicit fashion.

### (ii) Flip Bifurcation of the Non-Fundamental Steady States $E_{2,3}$

In general, a flip bifurcation occurs if the eigenvalue of the dynamic system crosses the unit circle at -1, which is  $\lambda_3$  in our case. Thus from Eq. (A 2.21) the non-fundamental steady states  $E_{2,3}$  undergo a flip bifurcation for

$$\lambda_3 = -\frac{\mathcal{A}^2 + \mathcal{B}^2}{2\mathcal{A}} = -1.$$

Then, Eq. (A 2.23) can be rewritten as

$$\Gamma(\lambda) = -\lambda^2 \left\{ \lambda^3 + [1 - 2\mathcal{A}] \lambda^2 + (\mathcal{A}^2 + \mathcal{B}^2) \right\}. \quad (\text{A } 2.26)$$

Comparing again the coefficients with Eq. (A 2.19) yields

$$[1 - 2\mathcal{A}] = - \left( 1 + \frac{g - 2a\alpha\sigma^2}{g} \Psi \right) \quad \wedge \quad \underbrace{(\mathcal{A}^2 + \mathcal{B}^2)}_{=2\mathcal{A}} = -(1 - g)\Psi.$$

Eliminating again  $\mathcal{A}$  leads to

$$\Psi = \frac{2g}{g(g - 2) - 2a\alpha\sigma^2}$$

which can be solved for  $\beta$  by using Eq. (A 2.20) and yields

$$\beta \equiv \beta_F = \frac{1}{C} \left[ \frac{g}{g - R} - \ln \left( \frac{g - R}{R} \right) \right].$$



For  $\beta = \beta_F$ , Eq. (A 2.25) gives two solutions for  $\alpha$ , namely

$$\begin{aligned}\alpha_1 &= -\frac{g(5g-6)}{2a\sigma^2}, \\ \alpha_2 &= \frac{g^2}{2a\sigma^2} \equiv \alpha^{**}.\end{aligned}$$

It can be shown that  $\alpha_1 < \alpha^*$  if and only if  $g > 1$ . Therefore, the first solution  $\alpha_1$  is an irrelevant solution for our proof since on the one hand  $E_{2,3}$  do only exist for  $g > R(> 1)$  and, on the other hand, are only stable for  $\alpha > \alpha^*$ . Instead, the second solution  $\alpha_2 = \alpha^{**}$  defines the intersection point  $(\beta_F, \alpha^{**})$  where the flip bifurcation curve  $\beta_F$  intersects the Hopf bifurcation curve in the  $(\beta, \alpha)$ -plane.

Taking into account the fact that  $\alpha^{**} < \bar{\alpha}$  if and only if  $g > 1$  (which states that the same as  $E_{2,3}$  exist) the following summarizing conclusion can be drawn. If  $\alpha \in (\alpha^*, \alpha^{**})$ , it is  $\beta_H(\alpha) < \beta_F$  and the non-fundamental steady states undergo a Hopf bifurcation at  $\beta = \beta_H(\alpha)$  which is implicitly given by Eq. (A 2.25). Otherwise, if  $\alpha \in (\alpha^{**}, \bar{\alpha}]$ , it is  $\beta_F < \beta_H(\alpha)$  and then the non-fundamental steady states  $E_{2,3}$  are destabilized by a flip bifurcation at  $\beta = \beta_F$ . Finally, if  $\alpha > \bar{\alpha}$ , then  $\beta_H(\alpha) \neq \mathbb{R}$  and  $E_{2,3}$  undergo a flip bifurcation at  $\beta = \beta_F$ . This completes the proof.

### A 2.8 Proof of Theorem 2.4

Following the lines of the proof of Theorem 2.2 (see Appendix A 2.6) it can easily be shown that the fundamental steady state  $E_1$  of the seven dimensional system Eq. (2.42)–Eq. (2.48) is given by  $E_1 = (0, 0, 0, 0, 0, 0, 0)$ , and that two non-fundamental steady states  $E_2$  and  $E_3$  exist with

$$E_{2,3} = (\pm x^*, \pm x^*, \pm x^*, \pm x^*, \pm x^*, (1-R)^2(x^*)^2, (1-R)^2(1-g)^2(x^*)^2).$$

Fortunately, the structure of the dynamic system has the property that allows the Jacobian matrix  $J$  of the system to be decomposed into two parts, i.e., one  $5 \times 5$ -matrix part which is exactly the same as in the case of constant beliefs about variances, and one additional  $2 \times 2$ -matrix part which has  $\theta_{1,\sigma}$  and  $\theta_{2,\sigma}$  as its eigenvalues. Since  $\theta_{1,\sigma}, \theta_{2,\sigma} \in (0, 1)$  is assumed, these two eigenvalues are stable and therefore the stability properties of the fundamental and non-fundamental steady states are solely determined by the characteristic polynomial of the Jacobian  $J_{5 \times 5}$  of the reduced five dimensional system's steady state, which has already been investigated in the previous two theorems. In other words, the characteristic polynomial of  $J$  is given by

$$\det(\lambda \mathbb{I}_7 - J) = (\lambda - \theta_{1,\sigma})(\lambda - \theta_{2,\sigma})(\lambda \mathbb{I}_5 - J_{5 \times 5}).$$

Hence, in the case of time varying beliefs about variances the system has the same eigenvalues as in the case of constant beliefs about variances and two further eigenvalues  $\theta_{1,\sigma}$  and  $\theta_{2,\sigma}$ , which always lie inside the unit circle. This completes the proof.

### A 2.9 Notes on Gaunersdorfer and Hommes' Risk Adjusted Profits Term

The risk adjusted profit term Eq. (2.50) can be derived in analogy to the considerations made in Appendix A 2.5 as follows. Using Eq. (A 2.14) and assuming a constant conditional variance  $\sigma^2$ , the performance measure  $(\phi_{h,t} - \phi_{R,t})$  can be written in prices as

$$\begin{aligned}\phi_{h,t} - \phi_{R,t} &= -\frac{1}{2a\sigma^2} (p_t + d_t - Rp_{t-1} - E_{h,t-1}[p_t + d_t - Rp_{t-1}])^2 \\ &= -\frac{1}{2a\sigma^2} (p_t + d_t - E_{h,t-1}[p_t] - E_{h,t-1}[d_t])^2 \\ &\stackrel{!}{=} -\frac{1}{2a\sigma^2} (p_t - f_{h,t-1} + d_t - E_{h,t-1}[d_t])^2 \\ &= -\frac{1}{2a\sigma^2} (p_t - f_{h,t-1} + \delta_t)^2,\end{aligned}$$

where  $\delta_t \equiv d_t - \bar{d}$  is again a martingale difference sequence. Note that in contrast to the previous calculations (see also Eq. (2.9)) in this model modification,  $E_{h,t}[p_{t+1}] = f_{h,t}$  and i.i.d. dividends are assumed. In the main text the notation  $\phi_{h,t}$  again stands for the difference ( $\phi_{h,t} - \phi_{R,t}$ ) calculated in this appendix.<sup>77</sup>

### A 2.10 Excursion II: Theoretical Analysis in Nonlinear Dynamics

A general strategy for analyzing a nonlinear dynamic system starts with the determination of the steady states (i.e., fixed points) and their stability properties. In particular, the analysis of steady states concentrates on how the stability behavior is influenced when crucial model parameters are varied. If the existence or stability of a steady state changes by such a variation of the model parameters, a ‘bifurcation’ occurs which very often leads to a region of more complicated dynamics (e.g., cycles) maybe even culminating in chaotic behavior of the system. Hence, a ‘bifurcation’ means a change of the qualitative behavior of the trajectories, and thus of the dynamic system, caused by a parameter passing through a critical value. Steady states and primary bifurcation points in which the steady state becomes unstable can typically be determined analytically even in higher dimensional nonlinear systems. The common procedure of this theoretically based analysis is briefly discussed in this and the subsequent excursion. However, analyzing the stability of cycles and detecting secondary and subsequent bifurcations of high-order cycles in a purely analytical way is, generally, no longer an easy task. Consequently, numerical tools become important, which I will refer to in Excursion IV.

Since only nonlinear difference equation systems will be examined in this thesis, we will restrict our discussion to the theoretical and numerical analysis of discrete-time systems.

#### *Stability of Fixed Points for Discrete Maps*

In general, the stability of fixed points in nonlinear dynamic systems can be investigated analytically. One of the most often used techniques in applications is the method of linear approximation, i.e., the nonlinear system is replaced by its linear version around the examined fixed point, which preserves most of the crucial properties of the system, in particular its stability properties relating to this fixed point. The main idea behind this method provides the Hartman-Grobman Theorem<sup>78</sup>. But before I explain this procedure in further detail, I will provide some precise and formal definitions of different types of stability of equilibrium points in discrete-time dynamic systems.

Consider the difference equation system

$$x_{n+1} = G(x_n), \quad x_n \in \mathbb{R}^m. \quad (\text{A 2.27})$$

Assume that  $x^*$  is an isolated fixed point of the map  $G$ , i.e., it has a surrounding neighborhood containing no other fixed points, so that  $x^* = G(x^*)$ .

#### **Definition 2.5 (Lyapunov stable)**

The fixed point  $x^*$  is *Lyapunov stable* (or simply *stable*) if for every  $\epsilon > 0$ , there exists  $\delta(\epsilon)$  so that for  $\|x_0 - x^*\| < \delta(\epsilon)$  it is<sup>79</sup>

$$\|G^n(x_0) - x^*\| < \epsilon \quad \forall n > 0.$$

<sup>77</sup>See also Appendix A 2.5 for details.

<sup>78</sup>Roughly speaking, the Hartman-Grobman Theorem states that the qualitative properties of the solutions of a nonlinear system  $\dot{x} = f(x)$ ,  $x \in U \subset \mathbb{R}^n$  remain the same as those of the linearized system  $\dot{x} = Ax$  in a neighborhood of a fixed point  $x^*$  where  $A$  is a  $(n \times n)$ -matrix of constant coefficients. For further details and proof of the theorem see Guckenheimer and Holmes (1983).

<sup>79</sup> $\|\cdot\|$  denotes the Euclidean norm, which defines a distance (or a metric) on the Euclidean space.

While Lyapunov stability only guarantees that a trajectory starting in a neighborhood of the fixed point  $x^*$  stays in an  $\epsilon$ -neighborhood (which can be larger than the  $\delta$ -neighborhood) forever, the following stability concept eventually claims a convergence to the fixed point.

**Definition 2.6 (asymptotically stable)**

The fixed point  $x^*$  is *asymptotically stable* if

- (i) it is stable and
- (ii) there exists  $\eta > 0$  such that for  $\|x_0 - x^*\| < \eta$  it is

$$\lim_{n \rightarrow \infty} \|G^n(x_0) - x^*\| = 0.$$

Thus a trajectory starting in a  $\delta$ -neighborhood of  $x^*$  converges towards the fixed point.

**Remark 2.7** Property (ii) can be replaced by the following equivalent property:

- (ii') there exists  $\eta > 0$  and for each  $\epsilon > 0$  there exists an integer  $T(\eta, \epsilon) > 0$  so that for  $\|x_0 - x^*\| < \eta$  it is

$$\|G^n(x_0) - x^*\| < \epsilon \quad \forall n \geq T.$$

Note that the previous two concepts of stability are both local ones, which is usually the relevant view of stability for economic applications. Globally asymptotic stability can be defined in an analogous way to Definition 2.6, with the only difference being that the  $\delta$ -neighborhood can be arbitrarily large.

**Definition 2.8 (globally asymptotically stable)**

The fixed point  $x^*$  is *globally asymptotically stable* if

- (i) it is stable and
- (ii) for every  $x_0$  in the domain of definition of Eq. (A 2.27) it is

$$\lim_{n \rightarrow \infty} \|G^n(x_0) - x^*\| = 0.$$

It is one of the most crucial characteristics of nonlinear dynamic systems that the notion of local and global stability is not the same and has to be carefully distinguished in every stability analysis. While in linear systems local stability always implies global stability<sup>80</sup>, nonlinear systems can have any finite, or countable infinite, number of fixed points and, therefore, locally asymptotic stability does not imply global stability. In the following we will mainly refer to the local stability concept.

The notion of instability of fixed points is quite straightforward and simply means the violation from stability.

**Definition 2.9 (unstable)**

A fixed point of the system Eq. (A 2.27) is called *unstable* if it is not stable.

### *The Method of Linear Approximation*

Since closed form solutions for nonlinear dynamic systems are scarce, it is necessary to find methods to determine the stability of fixed points and which do not explicitly require the knowledge of their solution. The method of linearization has this feature. The Hartman-Grobman Theorem<sup>81</sup> guarantees that the local behavior of a nonlinear dynamic system near to a fixed point can be investigated by analyzing the

<sup>80</sup>Linear systems generally have a unique equilibrium point from which the statement directly follows.

<sup>81</sup>Again we refer to Guckenheimer and Holmes (1983).

behavior of the linear part<sup>82</sup> of the system. The linearization procedure can be conducted by performing a Taylor expansion of the system. The expansion of the coordinate functions  $G_i$  in a Taylor series up to the first-order term leads to

$$\begin{aligned} x_{1,n+1} &= G_1(x^*) + \frac{\partial G_1(x^*)}{\partial x_{1,n}}(x_{1,n} - x_1^*) + \dots + \frac{\partial G_1(x^*)}{\partial x_{m,n}}(x_{m,n} - x_m^*), \\ x_{2,n+1} &= G_2(x^*) + \frac{\partial G_2(x^*)}{\partial x_{1,n}}(x_{1,n} - x_1^*) + \dots + \frac{\partial G_2(x^*)}{\partial x_{m,n}}(x_{m,n} - x_m^*), \\ &\vdots \\ x_{m,n+1} &= G_m(x^*) + \frac{\partial G_m(x^*)}{\partial x_{1,n}}(x_{1,n} - x_1^*) + \dots + \frac{\partial G_m(x^*)}{\partial x_{m,n}}(x_{m,n} - x_m^*), \end{aligned}$$

where  $x_{i,n}$ ,  $x_i^*$  denote the  $i$ th element of the vector  $x_n$  and  $x^*$  respectively, and  $G_i$  is the  $i$ th coordinate function of the map  $G$ . In vector notation the linear Taylor expansion can be written in a compact form as

$$x_{n+1} = G(x^*) + J_{|x_n=x^*}(x_n - x^*) \tag{A 2.28}$$

with  $J_{|x_n=x^*}$  being the Jacobian matrix evaluated at  $x^*$  and

$$J = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \dots & \frac{\partial G_1}{\partial x_m} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \dots & \frac{\partial G_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial x_1} & \frac{\partial G_m}{\partial x_2} & \dots & \frac{\partial G_m}{\partial x_m} \end{pmatrix}.$$

Obviously, if  $x^*$  is a fixed point, it is  $x^* = G(x^*)$ . Hence, Eq. (A 2.28) can be rewritten as

$$x_{n+1} = x^* + J_{|x_n=x^*}(x_n - x^*)$$

or in another, likewise intuitive notation

$$\xi_{n+1} = DG(x^*) \xi_n \tag{A 2.29}$$

where  $\xi_n \equiv x_n - x^*$  and  $DG(x^*) \equiv J_{|x_n=x^*}$ .

So far, we have reduced the problem of determining the stability of a fixed point of a nonlinear dynamic system to a corresponding problem in the linear world with the Jacobian matrix of  $G$  being the linear map, which becomes particularly evident by considering Eq. (A 2.29). The theory of linear dynamic systems is often subject to academic review and can be found in many textbook, hence only the basic principles need be repeated here.<sup>83</sup> Writing Eq. (A 2.29) more generally as

$$x_{n+1} = A x_n, \quad x_n \in \mathbb{R}^m, \quad z \in \mathbb{Z} \tag{A 2.30}$$

where  $A$  is a  $(m \times m)$ -matrix of constants, it is obvious that  $x^* = 0$  is an equilibrium solution (as a trivial solution) and that it is, in general, the unique equilibrium.<sup>84</sup> Furthermore, it can easily be shown that

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<sup>82</sup>It is assumed, that a nonlinear system can be decomposed into a linear and nonlinear part, i.e.,  $x_{n+1} = Ax_n + g(x_n)$  where  $A$  is a  $(m \times m)$ -matrix with constant coefficients and  $g$  is a nonlinear vector valued function of  $x_n$ .

<sup>83</sup>See for instance Medio and Lines (2001, Chapter 2) and Gandolfo (1997) for a comprehensive introduction.

<sup>84</sup>A solution of the system Eq. (A 2.30) is called an 'equilibrium solution' if a system starting at the equilibrium point stays there forever. Note that the equilibrium solution is unique if  $A$  is nonsingular, i.e., the determinant  $\det(A)$  is nonzero and thus there are no zero eigenvalues. In this case,  $A$  is also invertible.

searching for non-trivial solutions of Eq. (A 2.30) directly leads to the eigenvalue problem<sup>85</sup>

$$A u = \lambda u \quad \text{or} \quad (\lambda \mathbb{I}_m - A) u = 0, \quad (\text{A 2.31})$$

which has nontrivial solutions, i.e.,  $u \neq 0$ , if and only if

$$\det(\lambda \mathbb{I} - A) = 0. \quad (\text{A 2.32})$$

Eq. (A 2.32) can typically be written in form of a polynomial in the variable  $\lambda$ , thus

$$\Gamma(\lambda) \equiv \det(\lambda \mathbb{I} - A) = a_0 \lambda^m + a_1 \lambda^{m-1} + \dots + a_{m-1} \lambda + a_m = 0, \quad (\text{A 2.33})$$

where  $\Gamma(\lambda)$  is called the ‘characteristic polynomial’ and Eq. (A 2.32) is called the ‘characteristic equation’. Hence we can conclude that nontrivial solutions of Eq. (A 2.30) exist if  $\lambda$  is an eigenvalue and  $u$  is the associated eigenvector of  $A$ . This is a quite interesting statement. Obviously, solving the (linear) functional equation Eq. (A 2.30) can be identified with the *algebraic* problem of finding the roots of the characteristic polynomial given by Eq. (A 2.33) and, afterwards, solving the system Eq. (A 2.31), which is linear for every fixed  $\lambda$ .

Having determined the nontrivial solutions of the linear system as described above, the solutions fulfill the stability condition, i.e., they are (asymptotically) stable solutions, if they converge to the (unique) equilibrium point  $x^* = 0$  for  $n$  goes to infinity. Looking at the structure of the non-trivial solutions of Eq. (A 2.30) it becomes obvious that the eigenvalues  $\lambda_i$  of the matrix  $A$  play the crucial role for stability conditions of the system as stated, for instance, in Gandolfo (1997, p. 115).

**Theorem 2.10 (basic principle to check for stability in linear discrete systems)**

*For a linear system of difference equations, the solution’s stable or unstable behavior over time depends exclusively on the roots  $\lambda_1, \lambda_2, \dots, \lambda_m$ . To analyze the stability of the system we can examine only the nature of such roots, without any need to compute the elements of eigenvectors  $u_{i,j}$ ,  $1 \leq i, j \leq m$ . For this purpose we have to check whether the roots of the characteristic polynomial are in absolute value less than unity, without finding them explicitly.*

As a matter of course, explicit stability conditions for linear systems of difference (and differential) equations depend in particular on the order of the system and can be found in Gandolfo (1997).

To close this discussion, I will briefly summarize the procedure of analyzing the system with respect to its existing fixed points from the practitioner’s point of view. Given a nonlinear system, the following steps have to be performed to determine the stability properties of the fixed points:

STEP 1: Determine the fixed points of the nonlinear system (existence and uniqueness).

then investigate the stability property for each fixed point, i.e.,

STEP 2: Transform the system into its canonical form of ordinary first-order difference equations.

STEP 3: Derive the Jacobian  $J$  and evaluate  $J$  at the fixed point.

STEP 4: Determine the corresponding characteristic equation and solve for the roots.

STEP 5: If all roots lie inside the complex unit circle, the fixed point is locally asymptotically stable, otherwise it is not.

STEP 6: If it exists continue with the next fixed point and start with STEP 2.

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<sup>85</sup>Without going into detail, a non-trivial solution of Eq. (A 2.30) can generally be obtained by exploiting an Ansatz like  $x_n = \lambda^n u$  (or considering superpositions of corresponding ones) where  $\lambda$  can be identified by a real or complex eigenvalue and  $u$  by the corresponding real or complex eigenvector. Again we can refer to Medio and Lines (2001, Chapter 2).

### A 2.11 Excursion III: Short Review of Bifurcation Theory for Discrete Maps

Consider a system of difference equations

$$x_{n+1} = G(x_n; \alpha), \quad x_n \in \mathbb{R}^n \quad (\text{A 2.34})$$

where  $\alpha \in \mathbb{R}$  is a parameter. Hence Eq. (A 2.34) describes an entire family of systems with  $x_n$  being the independent variable and whose members are identical except for the parameter  $\alpha$ . It is obvious that certain properties of the system Eq. (A 2.34), such as the number of fixed points or their stability, may change if  $\alpha$  varies. Assume that such a property holds for  $\alpha \in ]a, b[$ ,  $a, b \in \mathbb{R}$  but is violated for any larger interval. Then the points  $a$  and  $b$  are called ‘bifurcation points’.

Bifurcations can be classified as ‘local’, ‘global’ and ‘local/global’. Since the remainder of this thesis mainly deals with local bifurcations we can refer to Medio and Lines (2001) for a discussion about the latter ones or, for a more mathematical and technical treatment, to Guckenheimer and Holmes (1983) and Kuznetsov (1995). Local bifurcations occur if the qualitative changes in the orbit structure can be analyzed in a neighborhood of a fixed or a periodic point on a map (or a flow). More precisely, local bifurcations are closely related to the loss of hyperbolicity in fixed points<sup>86</sup>. Consider two systems, one of which one is slightly perturbed in the parameter space, i.e., a parameter value is slightly changed. Generally, near each hyperbolic fixed point of the unperturbed system there will be a hyperbolic fixed point in the perturbed system with similar properties. In particular, the Jacobian matrix of the perturbed system evaluated at the ‘new’ fixed point will have the same structure of eigenvalues (i.e., for maps the same number of eigenvalues inside and outside of the complex unit circle) as the Jacobian of the unperturbed system at the original fixed point. However, at a bifurcation point the hyperbolicity of fixed points is lost and at least one eigenvalue of the Jacobian evaluated at the fixed point crosses the unit circle in the complex plane.

Local bifurcations in a system like Eq. (A 2.34) which depends on one parameter is always connected with one real eigenvalue or a pair of complex conjugate eigenvalues. Interestingly, they can completely be characterized in a reduced one- or two-dimensional space even when the underlying system is originally defined in a higher dimensional state space. This is guaranteed by the ‘centre manifold theorem’ which involves quite technical details beyond the scope of this discussion.<sup>87</sup> Thus, in the following, our brief discussion about possible types of bifurcations can be reduced to a one- or two-dimensional setting. More precisely, at the bifurcation value, three different types of bifurcation can generally occur:

- If an eigenvalue becomes  $+1$ , either a ‘fold bifurcation’ or a ‘transcritical bifurcation’ will be created with a pair of fixed points, one of which is stable and the other unstable; or a ‘pitchfork bifurcation’ occurs in which two additional fixed points are created.
- If an eigenvalue becomes  $-1$ , a ‘period-doubling’ or ‘flip bifurcation’ will be created with two additional fixed points.
- If a pair of complex eigenvalues reach the unit circle, a ‘Hopf bifurcation’ or ‘Neimark-Sacker bifurcation’ occurs, in which an invariant circle with periodic or quasi-periodic dynamics is created.

Let us now consider these types of bifurcation in further detail.

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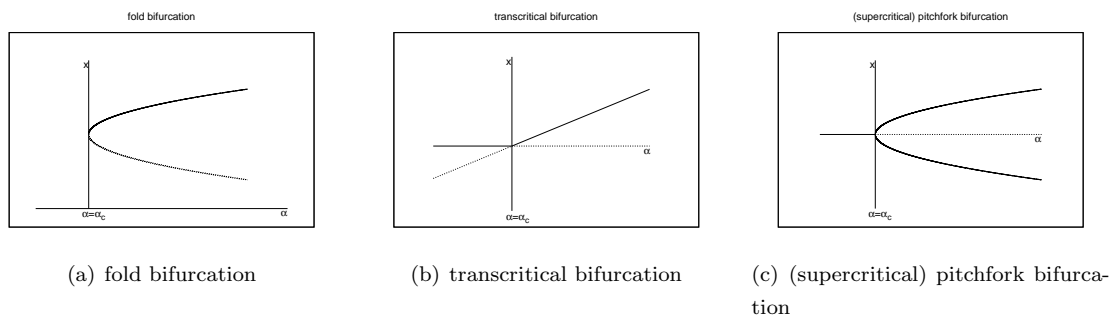
<sup>86</sup>A fixed point of a system of differential equations is ‘hyperbolic’ if the Jacobian matrix evaluated at that point has no zero or purely imaginary eigenvalues (i.e., no eigenvalue has a real part equal to zero).

<sup>87</sup>For an introduction to this concept we can refer to Medio and Lines (2001, Section 5.2). A more elaborated and formal description can be found in Guckenheimer and Holmes (1983).

*Fold, Transcritical and Pitchfork Bifurcation*

If in a system like Eq. (A 2.34) at a critical parameter value  $\alpha_c$  an eigenvalue of the Jacobian evaluated at the fixed point  $x^*$  is equal to  $+1$ , a fold, a transcritical or a pitchfork bifurcation occur. A ‘fold bifurcation’ is characterized by the fact that for  $\alpha > \alpha_c$  two fixed points exist (one stable and one unstable) while for  $\alpha$  decreasing through  $\alpha_c$  these fixed points, at first, collapse (exactly at  $\alpha = \alpha_c$ ) and then disappear. The fold bifurcation phenomenon is said to be a ‘discontinuous’ or ‘catastrophic’ bifurcation since the bifurcation point signifies the endpoint of the stable branch. Thus the fold bifurcation is useful in order to find parameter ranges in which equilibrium points exist. Contrastingly, a bifurcation is called ‘continuous’ if there is a continuous path on the stable branch through a bifurcation point. Such a bifurcation is the ‘transcritical bifurcation’. No fixed points appear or disappear at the critical value  $\alpha_c$ , but only the stability properties change among each other, i.e., the stable branch becomes unstable and the unstable one becomes stable.

A ‘pitchfork bifurcation’ generates both appearance and disappearance of fixed points as well as changes in their stability properties. Precisely, in addition to the fixed points  $(x^* = 0, \alpha)$  two further (non-trivial) branches of fixed points emerge which are only located on one side of the critical value  $\alpha_c$ . If the non-trivial fixed points appear for  $\alpha > \alpha_c$  the bifurcation is called a ‘supercritical’ pitchfork bifurcation, otherwise, if the nontrivial equilibria exist for  $\alpha < \alpha_c$ , the pitchfork bifurcation is called ‘subcritical’. Obviously, supercritical pitchfork bifurcations are continuous bifurcations while subcritical pitchfork bifurcations are discontinuous ones. Figure A 2.1 illustrates the qualitative change of the behavior of the fixed points at the critical value  $\alpha_c$  for the three types of bifurcations in a ‘bifurcation diagram’ (see Appendix A 2.12 for details) which is simply a plot of the fixed points in the  $(\alpha, x)$ -plane.



**Figure A 2.1:** Fold, transcritical and pitchfork bifurcation. The equilibrium value  $x^*$  is plotted against the bifurcation parameter  $\alpha$ . The left panel shows a fold bifurcation. Two fixed points occur for  $\alpha > \alpha_c$ , whereas for a transcritical bifurcation the number of fixed points do not change (middle panel). The right panel reveals that the shape of the bifurcation diagram of a pitchfork bifurcation gives this bifurcation type its name. In all diagrams, stable fixed points are represented as a solid curve (stable branch) whereas unstable fixed points are plotted as a dashed curve (unstable branch).

One further remark should be added at this point. Of course, necessary and sufficient conditions for these bifurcation types to exist can easily be derived for one-parameter families of systems like Eq. (A 2.34) in an analytic form by taking into account mainly the first three derivatives of  $G$  with respect to  $x_n$  and  $\alpha$  (or combinations of them), evaluated at the fixed point and the critical parameter value. The explicit form of these conditions (and also those for bifurcation types mentioned below) can be found in Medio and Linares (2001) or Gandolfo (1997).

*Flip Bifurcation*

A ‘flip bifurcation’ emerges when a real eigenvalue of the Jacobian matrix calculated at the equilibrium point goes through minus one.<sup>88</sup> In this case, the non-zero fixed point loses its stability and a stable period-2 cycle appears. Flip bifurcations and pitchfork bifurcations can easily be confused. Indeed, the flip bifurcation of the map  $G$  corresponds to a pitchfork bifurcation for the map  $G \circ G = G^2$ , i.e.,  $G$  composed with itself. Typically, the period-doubling phenomenon continues when the bifurcation parameter  $\alpha$  is further increased and the period-2 cycle of the map  $G$  loses its stability through a flip bifurcation of the map  $G^3$ . Thus period-4, period-8, period-16, etc. cycles occur step by step, which is often called a ‘route to chaos’ since at some time an infinite number of periodic orbits with periods equal to  $2^k$ ,  $k \in \mathbb{N}^+$  appear where all of them are unstable.

*Neimark-Sacker Bifurcation*

A ‘Neimark-Sacker Bifurcation’ or (‘Hopf bifurcation’ for maps) appears when a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point of a map have modulus one.<sup>89</sup> This might be the case for both maps and flows. For flows, periodic solutions occur near the bifurcation value of the parameter  $\alpha$ . The dynamics of the system near to the bifurcation point is characterized by the eminent ‘Hopf-Theorem’ in which the conditions of stability of the periodic orbits are stated. An analogous theorem for maps also exists. The ‘Neimark-Sacker-Theorem’ proves that, under certain assumptions, invariant circles exist in a neighborhood of the critical parameter value, i.e., at the bifurcation point, bifurcating from the fixed point  $x^*$  and enclosing it. For further details about these extremely demanding theorems once again we refer to the textbooks on nonlinear dynamics and bifurcation theory mentioned earlier in this chapter and the even more specialized references given therein.

**A 2.12 Excursion IV: Numerical Analysis in Nonlinear Dynamics**

Theoretical analysis of nonlinear dynamics is frequently restricted to determining (the stability of) fixed points and the corresponding first bifurcation points in ‘concise’ (or ‘manageable’) regions of the system. However, for higher-dimensional systems and more complicated problems as subsequent bifurcations of higher-order cycles, the analytical tractability of such systems comes rapidly close to its limits. In this case, numerical simulations might give further insights into the behavior of the system. One easy procedure is to plot the state variable  $x_n$  for a certain choice of the parameter  $\alpha$  in a  $(x_{n+1}, x_n)$ -plane.<sup>90</sup> Such a plot is called a ‘phase diagram’ or ‘return map’.

A more systematic, numerical technique for finding stable cycles is very often preferable. A ‘bifurcation diagram’ plots the long-term behavior of the dynamic system as a multivalued function of a parameter. More precisely, such a diagram is obtained by plotting e.g., 200,000 points of an orbit (after a transient of e.g., 5000 periods) for uniformly drawn parameter values from the parameter interval under consideration. Thus the horizontal axis represents the parameter while the vertical axis represents higher iterates  $G^n(x_0)$  of a specific initial point  $x_0$ , so that the diagram shows the limiting behavior of the orbit starting with  $x_0$ .

Although phase diagrams and bifurcation diagrams are simple numerical techniques, they reveal already a lot of information about the global behavior of a dynamic system. For instance, an infinite sequence

<sup>88</sup>For this bifurcation type, there is no analogon for corresponding flows, i.e., for one-parameter families of continuous-time systems. However, the case of an eigenvalue passing through  $+1$  for maps is analogous to the case for flows, in which the eigenvalue goes through zero.

<sup>89</sup>In this case the system of difference equations have to have at least two dimensions (i.e., two independent state variables).

<sup>90</sup>For the sake of simplicity we still assume the simple case of  $x_{n+1} = G(x_n; \alpha)$  where  $x_n, \alpha \in \mathbb{R}$ .



of period doubling leading eventually to chaotic dynamics can easily be discovered in such numerical simulations as is well known for the so-called logistic map. However, further numerical tools should be applied to analyze whether strange attractors arise as might be suggested by the phase and bifurcation diagrams. In this connection, a quantitative tool to diagnose the presence of ‘sensitive dependence on initial conditions’ (SDIC), and thereby of chaoticity, would be particularly useful. However, this can be provided by the numerical computation of the Lyapunov characteristic exponents, which will be outlined in the next subsections. Since this concept of Lyapunov is of particular importance in the analysis of nonlinear dynamics, a more general treatment of the theory described in continuous-time is outlined first. Afterwards, we will restrict our more practical considerations to the largest Lyapunov exponent in the discrete-time case.

### *Lyapunov Characteristic Exponents*

For almost 40 years, the investigation of the spectrum of Lyapunov characteristic exponents (LCEs) has been one of the most useful tools for detecting and quantifying chaos in nonlinear dynamic systems or experimental time series. While at first many papers on LCE came predominantly from physics and mathematics (see Oseledec (1968); Benettin, Froeschle and Scheidecker (1979); Benettin, Galgani, Giorgilli and Strelcyn (1980); Shimada and Nagashima (1979); Wolf, Swift, Swinney and Vastano (1985); Eckmann and Ruelle (1985)), in recent years the research in the calculation of LCEs has also become popular in economics and has led to a significant increase in work published on LCEs (see Bajo-Rubio, Fernández-Rodríguez and Sosvilla-Rivera (1992); Dechert and Gençay (1992); Eckmann, Ollifson Kamphorst, Ruelle and Scheinkamp (1988); Frank, Gençay and Stengos (1988); Frank and Stengos (1988); Bask (1996, 2002)). For instance, since the LCEs of an unknown dynamic system cannot be derived straightforwardly from pure observations, a great effort has been made to improve Lyapunov exponent algorithms and to establish a statistical framework for testing chaotic dynamics via Lyapunov exponents by both physicists and economists (see Wolf et al. (1985); Brown, Bryant and Abarbanel (1991); Abarbanel, Brown and Kennel (1992); Abarbanel, Brown, Sidorowich and Tsmiring (1993); Rosenstein, Collins and De Luca (1993); Gençay (1996); Bask and Gençay (1998)).<sup>91</sup>

The Lyapunov characteristic exponents measure the average exponential rate of divergence or convergence of nearby orbits in the phase space. The idea of this concept can be easily illustrated for a one-dimensional nonlinear system. Consider two points  $x$  and  $\hat{x}$  in the phase space  $P$  which are arbitrarily close to each other and determine their distance,  $d_0$ . Applying a flow  $\phi_t$  to both of these points leads to trajectories  $x(t)$  and  $\hat{x}(t)$  respectively. Then, the ratio between their current distance  $d_t$  and their initial distance  $d_0$  can be expressed in terms of  $e^{\lambda t}$ , where  $\lambda$  indicates the rate of divergence (or convergence). If  $\lambda$  eventually converges to a limit for  $t \rightarrow \infty$ , this limit will be called the ‘Lyapunov characteristic exponent’ (LCE).<sup>92</sup> Obviously, a positive limit of the LCE implies that orbits which are initially close to each other eventually diverge, while a negative LCE means that the nearby orbits stay close to each other when the system evolves over time. Hence, in other words, since nearby orbits can generally be identified with almost identical states, the Lyapunov exponents quantitatively characterize the sensitive dependence on initial conditions of the dynamic system. As a consequence, a system in which at least one positive Lyapunov exponent can be determined is said to be chaotic, where the (positive) exponent reflects the time scale on which the dynamics of the system become unpredictable.

<sup>91</sup>For dynamic systems whose equations of motions are explicitly available there is a straightforward procedure for determining a complete set of LCE (see Benettin et al. (1980); Shimada and Nagashima (1979)).

<sup>92</sup>In the case of a higher-dimensional system multiple LCEs can be determined. See the discussion below for details.

More precisely, the general theory of LCEs can briefly be outlined as follows.<sup>93</sup> Let

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (\text{A 2.35})$$

be a system of ordinary differential equations and let  $\phi_t : P \rightarrow \mathbb{R}^n$  be the corresponding flow, where  $P \subset \mathbb{R}^n$  is the phase space of the system.<sup>94</sup> The solution of the system can be written as

$$x(t) = \phi_t(x_0),$$

where  $x_0$  is the initial condition of Eq. (A 2.35) and  $\phi_t$  is taken as a map which describes the time- $t$ -evolution of all phase points. To quantify the evolution of the rate of change in the distance of initially nearby points  $x, \hat{x} \in P$  (when applying the flow  $\phi_t$  to these points), the first variation of the orbit is considered, which can be formulated analytically by a set of non-autonomous linear differential equations, namely

$$\delta\dot{x} = \frac{\partial f}{\partial x}[\phi_t(x_0)] \delta x. \quad (\text{A 2.36})$$

Note that the matrix

$$\frac{\partial f}{\partial x}(x) = \left( \frac{\partial f_i}{\partial x_j} \right), \quad i, j = 1, \dots, n \quad (\text{A 2.37})$$

is the Jacobian matrix evaluated at  $x$ . Denoting the Jacobian in Eq. (A 2.37) by  $D_x f$  the solution of Eq. (A 2.36) is given by

$$\delta x(t) = D_{x_0} \phi_t(x_0) \delta x_0. \quad (\text{A 2.38})$$

Hence, in other words, the time-dependent matrix  $B(t) \equiv D_{x_0} \phi_t(x_0)$  satisfies the matrix differential equation

$$\frac{d}{dt} B(t) = A(t) B(t)$$

where  $A(t) \equiv D_{x_0} f[\phi_t(x_0)]$ , which is simply a ‘variational equation’ with a matrix  $A(t)$  representing the action of the vector field  $f$  for small displacements along  $\delta x_0$ .<sup>95</sup>

The solution of Eq. (A 2.36) can also be written more compactly by using the notion of the ‘fundamental matrix’ which is defined in the case of Eq. (A 2.36) as

$$U_x^t \equiv D_x \phi_t(x) = \frac{\partial \phi_t}{\partial x}(x),$$

where the following chain rule holds<sup>96</sup>

$$U_{x_0}^{t+s} = U_{\phi_t(x_0)}^s \circ U_{x_0}^t.$$

Then, Eq. (A 2.38) can be written in terms of the fundamental matrix by

$$\delta x(t) = U_{x_0}^t \delta x_0,$$

where  $\delta x_0$  is a initial deviation at  $t = 0$ . Obviously, the asymptotic property of a small deviation is determined by the asymptotic property of the fundamental, time-varying matrix  $U_{x_0}^t$  for  $t \rightarrow \infty$ . Now, it can be shown that the asymptotic behavior of the fundamental matrix for  $t \rightarrow \infty$  can be characterized by the following exponents (see Shimada and Nagashima (1979) and references given therein)

$$\lambda(E^k, x_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|U_{x_0}^t e_1 \wedge U_{x_0}^t e_2 \wedge \dots \wedge U_{x_0}^t e_k\|}{\|e_1 \wedge e_2 \wedge \dots \wedge e_k\|} \quad (\text{A 2.39})$$

<sup>93</sup>For the following discussion see Shimada and Nagashima (1979) and Medio and Gallo (1992).

<sup>94</sup>For technical reasons  $\phi_t$  is also assumed to be measure preserving due to a probability measure  $\mu$  on the phase space  $P$ .

<sup>95</sup>Note that  $D_{x_0} \phi_t(x_0)$  is the tangent (linearized) mapping at  $x_0$  of the nonlinear map  $\phi_t$ , which maps the tangent space  $T_{x_0} P$  onto  $T_{\phi_t(x_0)} P$ , which is generally not equal to  $T_{x_0} P$ .

<sup>96</sup>See Oseledec (1968) for details.

for  $k = 1, 2, \dots, n$ , where  $E^k$  is a  $k$ -dimensional subspace in the tangent space  $T_{x_0}P$  at  $x_0$ ,  $\{e_1, \dots, e_k\}$  is a set of bases of  $E^k$ ,  $\wedge$  is an exterior product (so-called ‘wedge product’) and  $\|\cdot\|$  is the Euclidean norm (or, more generally, a norm with respect to some Riemannian metric) on  $T_{x_0}P$ .<sup>97</sup> More concretely, the expression in the logarithmic function on the right-hand side of equation Eq. (A 2.39) can be interpreted as the volume of a  $k$ -dimensional parallelepiped in the tangent space along the orbit starting in  $x_0$ . Consequently,  $\lambda(E^k, x_0)$  is the expansion rate of this geometric object and is called ‘ $k$ -dimensional Lyapunov exponent’. Furthermore, from this definition it can be concluded that  $\lambda(E^k, x_0)$  only depends on the  $k$ -dimensional subspace  $E^k$  and not on both the choice of a set of bases and the choice of norms. Note that the proof of the existence of the limit in Eq. (A 2.39) for  $\mu$ -almost all points  $x \in P$  is a crucial and demanding task and has been performed by Oseledec (1968) under rather general conditions.

It is worth mentioning that an interesting relation between  $k$ -dimensional Lyapunov exponents and their one-dimensional counterparts exists. Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the no more than  $n$  distinct values of the 1-dimensional exponent  $\lambda(E^1, x)$ . Then it can be shown that for a set of bases  $\{e_1, \dots, e_k\}$  chosen at random in the tangent space  $T_xP$ , the  $k$ -dimensional exponents  $\lambda(E^k, x)$  can be expressed by the sum of the  $k$  largest LCEs of order one, i.e.,

$$\lambda(E^k, x) = \lambda_1 + \dots + \lambda_k .$$

However, this can also be reversed. Each LCE of order  $k$  can be connected to any  $k$ -sum of certain 1-dimensional LCEs if  $k$  linear independent vectors  $w_i \in E^k$  are chosen appropriately. The latter means that in the case of  $n = 3$ , for instance,

- $\lambda(e^1, x)$  may take one of the values in  $\{\lambda_1, \lambda_2, \lambda_3\}$ ,
- $\lambda(e^2, x)$  may take one of the values in  $\{(\lambda_1 + \lambda_2), (\lambda_1 + \lambda_3), (\lambda_2 + \lambda_3)\}$  and
- $\lambda(e^3, x)$  may take the value  $(\lambda_1 + \lambda_2 + \lambda_3)$ .

So far we have considered the most general case in the theory of LCEs in which orbits are not necessarily periodic. To get a better and more intuitively understanding of the concept of LCEs, we will hereafter restrict our considerations to 1-dimensional LCEs, i.e., LCEs of a vector<sup>98</sup> and consider the particular case of a periodic orbit with periodicity  $\tau$  again. Then,  $D_{x_0}\phi_\tau(x_0)$  is a linear map on the tangent space  $T_{x_0}P$  onto itself and can be written as a matrix  $e^{A\tau}$ , where  $A$  is a  $(n \times n)$ -matrix with constant coefficients. Assume that this matrix has  $n$  independent eigenvectors  $u_1, \dots, u_n$  with corresponding eigenvalues  $\zeta_1 \dots, \zeta_n$  which obey the property  $|\zeta_1| \geq \dots \geq |\zeta_n|$  and let  $\lambda_i \equiv \tau^{-1} \ln |\zeta_i|$ . This directly implies that

$$\frac{\|D_{x_0}\phi_{k\tau}(x_0)u_i\|}{\|u_i\|} \simeq e^{\lambda_i k \tau}, \quad i = 1, \dots, n .$$

Then, for a vector  $w$  with nonvanishing component along  $u_1$  it follows asymptotically for large  $t$  that

$$\frac{\|D_{x_0}\phi_t(x_0)w\|}{\|w\|} \simeq e^{\lambda_1 t}$$

in the sense that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|D_{x_0}\phi_t(x_0)w\|}{\|w\|} = \lambda_1(x_0, w) \equiv \lambda_{max} . \tag{A 2.40}$$

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<sup>97</sup>Some further remarks might be useful. Note, that  $E^k \equiv \text{span}\{e_1, \dots, e_k\}$ , i.e., the space which is spanned by all linear combinations of the vectors  $e_1, \dots, e_k$ . Without going into detail beyond the scope of this discussion, the wedge product  $\wedge$  is a mathematical object related to the domain of differential forms and can be used to calculate determinants and volumes of parallelepipeds. For further reading we can refer to standard textbooks on multilinear algebra and tensor calculus.

<sup>98</sup>Note that 1-dimensional LCEs are also known as ‘LCEs of order one’.

$\lambda_{max}$  is called the (largest) ‘Lyapunov characteristic exponent of vector  $w$ ’. Obviously, as long as at least the largest eigenvalue  $\zeta_1$  is positive, one is faced with asymptotic exponential divergence of initially nearby orbits and in this case, the periodic orbit is unstable. Of course, in the one-dimensional case with only one single LCE this coincides with the motivation given at the beginning of this excursion, where the variation of distances between points under the dynamic process in time has been considered. As a last remark, in principle, all other LCEs can be derived in a similar manner. For further reading consult Benettin et al. (1979) and Medio and Gallo (1992).

### *LCE for Discrete Maps*

First, we discuss the concept of LCEs for the simplest case, i.e., a one-dimensional setting in discrete time.<sup>99</sup> Consider the difference equation

$$x_{n+1} = G(x_n), \quad x_n \in \mathbb{R}, \quad n \in \mathbb{N}$$

with  $f : U \rightarrow \mathbb{R}$ ,  $U$  being an open subset of  $\mathbb{R}$  and  $f$  is continuously differentiable. After  $n$  iterations, two initially nearby points  $x_0$  and  $\hat{x}_0$  are separated according to

$$\begin{aligned} \hat{x}_n - x_n &= G^n(\hat{x}_0) - G^n(x_0) \\ &\approx \left[ \frac{d}{dx}(G^n)(x_0) \right] (\hat{x}_0 - x_0), \end{aligned} \quad (\text{A 2.41})$$

where a Taylor expansion around  $\hat{x}_0 = x_0$  is used in the second step and  $G^n$  is the composition of  $G$  with itself  $n$  times. Applying the chain rule of differentiation on  $(d/dx)(G^n)(x_0)$  yields

$$\frac{d}{dx}(G^n)(x_0) = \frac{d}{dx}G(x_{n-1}) \cdot \frac{d}{dx}G(x_{n-2}) \cdot \dots \cdot \frac{d}{dx}G(x_0), \quad (\text{A 2.42})$$

where  $x_i = G^i(x_0)$ . Assuming that all factors of the right-hand side of Eq. (A 2.42) are of comparable size,  $(d/dx)(G^n)$  should grow (or decay) exponentially with  $n$ . Since the same is true for  $|\hat{x}_n - x_n|$ , i.e.,

$$|\hat{x}_n - x_n| \sim e^{n\lambda(x_0)} |\hat{x}_0 - x_0|,$$

the average rate of growth can be defined as

$$\begin{aligned} \lambda(x_0) &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{d}{dx}G(x_{n-1}) \cdot \frac{d}{dx}G(x_{n-2}) \cdot \dots \cdot \frac{d}{dx}G(x_0) \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \left| \frac{d}{dx}G(x_j) \right|, \end{aligned}$$

where  $\lambda(x_0)$  is the (single) Lyapunov characteristic exponent for a one-dimensional map. Note that for periodic orbits of period  $k$ ,  $k > 1$ , the corresponding LCE is given by

$$\lambda(x_0) = \ln \left| \frac{\partial G^k(x^*)}{\partial x_n} \right|,$$

where  $x^*$  is any periodic point and which can be interpreted as the eigenvalue of the matrix  $D(G^k)(x^*)$ . If  $x^*$  is stable, then  $|\partial G^k(x^*)/(\partial x_n)| < 1$  and the LCE  $\lambda(x_0)$  becomes negative.

The generalization to the multi-dimensional case is straightforward. Consider

$$x_{n+1} = G(x_n), \quad x_n \in \mathbb{R}^m, \quad n \in \mathbb{N},$$

where  $G : U \rightarrow \mathbb{R}^m$ ,  $U$  being a subset of  $\mathbb{R}^m$ . Then, the derivative  $(d/dx)(G^n)$  in Eq. (A 2.41) has to be replaced by the Jacobian matrix  $D(G^n)$ , where

$$D(G^n)(x_0) = DG(x_0)DG(x_1) \dots DG(x_{n-1})$$

<sup>99</sup>See Eckmann and Ruelle (1985) and Medio and Lines (2001) for the following discussion.

with  $x_i = G^i(x_0)$  and  $DG(x_i) = (\partial G_k/\partial x_{l,i})$ ,  $k, l = 1, \dots, m$ ,  $i = 0, \dots, n-1$ . Then, analogous to continuous-time case (see Eq. (A 2.40)) the LCEs of the vector  $w$  are defined as

$$\lambda(x_0, w) = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\|D(G^n)(x_0)w\|}{\|w\|} = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\|\prod_{i=0}^{n-1} DG(x_i)w\|}{\|w\|},$$

where the matrix  $D(G^n)(x_0)$  describes how the vector  $w$  will be stretched or contracted (and rotated) by the action of the map  $G$ . Assume, without loss of generality,  $\|w\| = 1$  and adopting the vector norm  $\|\cdot\| = (\cdot, \cdot)^{\frac{1}{2}}$ , where  $(\cdot, \cdot)$  denotes the scalar product operation, the image of  $w$  under the matrix  $D(G^n)(x_0)$  is given by

$$\begin{aligned} \|D(G^n)(x_0)w\| &= \left( D(G^n)(x_0)w, D(G^n)(x_0)w \right)^{\frac{1}{2}} \\ &= \left( w^T \left[ (D(G^n)(x_0))^T D(G^n)(x_0) \right] w \right)^{\frac{1}{2}} \end{aligned}$$

where the square root can be taken since the matrix within the square brackets is positive definite. Hence the average rate of stretching (or contraction) per iteration step is given by  $(w^T [(D(G^n)(x_0))^T D(G^n)(x_0)] w)^{1/(2n)}$  and thus the eigenvalues of the matrix

$$\Psi(x_0) = \lim_{n \rightarrow \infty} \left[ (D(G^n)(x_0))^T D(G^n)(x_0) \right]^{\frac{1}{2n}} \tag{A 2.43}$$

are called the ‘Lyapunov characteristic numbers’ and their logarithms the ‘Lyapunov characteristic exponents’ provided that the limit in Eq. (A 2.43) exists.

### *LCEs and the Ergodic Approach*

So far, the question of how or even whether the LCEs depend on the initial conditions has been left out. Indeed, in all formulas of LCEs the initial condition  $x(0) = x_0$  has always been treated as a further parameter. To answer this question, dynamic systems have to be reconsidered from a slightly different, non-geometrical perspective. The ‘ergodic approach’ is based on the concept of measures and investigates the statistical properties of ensembles of orbits. Since an introduction to this approach would go beyond the scope of this chapter, only a few main results according to LCEs are quoted here. For a more detailed discussion on that topic we can refer again to Medio and Lines (2001, Chapter 9).

Roughly speaking, elementary mathematical concepts from measure theory, such as measure space, measurable maps, measure preserving maps, invariant measures, are applied to the analysis of dynamic systems. In this context, the ‘ergodic measures’, a special class of invariant measures, play a crucial role since they guarantee that the average of a certain property of a system calculated along a certain, randomly chosen history of the system is a typical representative of the averages evaluated over most of the other possible histories.

With respect to the determination of LCEs, two results should be stated now. In the simple case of a one-dimensional map preserving a measure  $\mu$  with  $\mu$  being ergodic, it can be shown that the LCE is given by

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{d}{dx} G[G^i(x_0)] \right| = \int \ln \left| \frac{d}{dx} G[G^i(x_0)] \right| \mu(dx),$$

where the integral is taken over the state space. Obviously, in this case the LCE no longer depend on the initial condition  $x_0$ .

For a multi-dimensional analogous result much more effort has to be made. Again, I will only quote the result established in the eminent theorem by Oseledec (1968), which is known as the ‘multiplicative

ergodic theorem' and states that for a diffeomorphism  $G : M \rightarrow M$  preserving a measure  $\mu$ , and  $M$  being a  $m$ -dimensional manifold, the limit

$$\lambda(x, w) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|D(G^n)(x)w\|,$$

where  $w$  is a vector in the tangent space at  $x$  and  $\|w\| = 1$ , exists for  $\mu$ -almost all points  $x \in M$ . Furthermore,  $\lambda(x, w)$  takes  $s \leq m$  different values and, if  $\mu$  is ergodic, the  $\lambda$ s are  $\mu$ -almost everywhere constant, i.e., they do *not* depend on initial conditions (modulo sets of  $\mu$ -measure zero).

Hence, in both cases given above, we can speak about *the* LCE(s) of the map  $G$  with respect to the  $G$ -invariant measure  $\mu$ .

### *A Practical Procedure to Derive the Largest LCE*

Apart from few specific cases, the LCEs of a nonlinear map (or a flow) cannot be calculated analytically and therefore numerical procedures have to be utilized. There are several numerical methods to compute LCEs more or less accurately and efficiently.<sup>100</sup> Indeed, in principle, Eq. (A 2.43) provides a passable procedure to calculate the whole spectrum of LCEs for maps by iteration. However, in practice, the implementation of this method causes some problems as stated in Brown et al. (1991):

The only problem in implementing this [see Eq. (A 2.43)] comes from the ill-conditioned nature of the matrices  $Df^K$  [in our notation:  $D(G^n)(x_0)$ ], but with care this can be handled. [...]

Furthermore, in applications the determination of LCEs mostly aims to detect chaotic attractors which coincide with the presence of at least one positive LCE in both discrete and continuous-time systems. Therefore, a method to compute the largest LCE of the whole spectrum of them is, for the most part, completely sufficient. Moreover, its determination is comparatively simple since almost any arbitrary vector will have some components in the direction of the eigenvector corresponding to the largest eigenvalue. Hence the quantity

$$\lambda_{max}(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|D(G^n)(x_0)w\|$$

should provide a good approximation of the largest LCE, where  $w$  is chosen at random.

In the following, I want to briefly give a 'recipe' to compute the largest LCE for discrete-time nonlinear systems in  $n$  dimensions as we will use it in the remainder parts of this thesis (see Wagner (2003), i.e., private communication with F. Wagner). Consider the map

$$\begin{aligned} x_{1,n+1} &= G_1(x_n; \alpha), \\ &\vdots \\ x_{m,n+1} &= G_m(x_n; \alpha) \end{aligned} \tag{A 2.44}$$

with  $m \in \mathbb{N}$ ,  $\alpha \in \mathbb{R}$  and  $G : U \rightarrow \mathbb{R}^m$ ,  $U$  being an open subset of  $\mathbb{R}^m$ .

STEP 1: If possible, choose the initial condition of the system so that the orbit starting in  $x_0$  is taken from the attractor under consideration.

STEP 2: Discard the first 500 iteration steps of the system.

STEP 3: Then compute in each subsequent iteration step the Jacobian matrix

$$S_{ik}(x_n) = \frac{\partial G_i}{\partial x_k} \Big|_{x_n}, \quad 1 < i, k < m$$

evaluated at  $x_n$  for the next  $N$  (e.g.,  $N=5000$ ) iterations.

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<sup>100</sup>For some general remarks consult Medio and Gallo (1992) and Medio and Lines (2001). For a more detailed discussion about that topic see also Abarbanel et al. (1993).

STEP 4: In each single iteration step compute

$$\phi_n = S(x_n)\phi_{n-1}, \quad n \in \mathbb{N}^+,$$

where  $\phi_n$  is a  $m$ -dimensional vector with  $\phi_0$  chosen at random.<sup>101</sup>

STEP 5: After the last iteration step  $N$  the largest LCE of the system can approximately be calculated as

$$\lambda_{max} = \frac{1}{2N} \ln |\phi_N|^2 = \frac{1}{2N} \ln (\phi_{1,N}^2 + \dots + \phi_{m,N}^2),$$

where, in general,  $\lambda_{max}$  should not depend on the initial condition  $x_0$  as long as  $x_0$  is chosen from the underlying attractor.

STEP 6: To determine a whole set of largest LCE for *different* values of a model parameter  $\alpha$  (see Eq. (A 2.44)) STEPS 1–5 have to be repeated for all different  $\alpha$ s. In practice, one often starts with a lower bound  $\alpha_1$  for the parameter  $\alpha$  and increases in each experiment  $\alpha$  by a certain step width, e.g., by 0.0001, up to an upper bound  $\alpha_2$ .

To close this excursion, I want to summarize the different types of attractors with respect to the sign patterns of their LCEs. While for fixed points, all LCEs have to be negative, periodic orbits (limit cycles) have one LCE equal to zero. A torus can emerge only in at least 3-dimensional phase space where the number of LCEs equal to zero coincides with the dimension of the torus. Finally in the case of strange or chaotic attractors at least one LCE has to be positive which, is conversely also very often used as one working definition for the phenomenon of chaotic motion. These findings are also summarized in Table A 2.1.

| Type of Attractor            | Sign Pattern of LCEs                                  |
|------------------------------|---|
| fixed points                 | $(-, \dots, -)$                                       |
| limit cycles                 | $(0, -, \dots, -)$                                    |
| $T^n$ torus                  | $(\underbrace{0, -, 0, - \dots, -}_{n\text{-times}})$ |
| strange or chaotic attractor | $(+, \dots, +, 0, - \dots, -)$                        |

**Table A 2.1:** Sign patterns of LCEs to classify different types of attractors.

<sup>101</sup>Note that  $\phi_n = [\prod_{k=1}^n S(x_k)] \phi_0$  for  $n \in \mathbb{N}^+$  where  $n$  is the current iteration step of the simulation run.

## **Part Two**

### Two Nonlinear Stationary Models of Asset Return



## CHAPTER 3

# The Chiarella-He (CH) Model with a Power Utility Function

### 3.1 Introduction

This chapter outlines a simple asset pricing model with heterogeneous agents proposed by C. CHIARELLA and X.-Z. HE<sup>1</sup> and investigates how the dynamics of the model is affected by allowing the agents to have different attitudes towards risk, which is characterized by the risk-aversion coefficient in their utility function. It directly builds on the original publication and borrows some of its structure as well as most of the notation. This analysis is inspired by another work published by the same authors (see Chiarella and He, 2002*c*), in which they pursue the same idea within the framework of the asset pricing model proposed by Brock and Hommes (1998).

The Chiarella-He (CH) model is based on both the asset pricing framework of the BH model established in Brock and Hommes (1998) and the LLS model established in Levy et al. (1994, 1995), which have been both extensively reviewed in the first part of this thesis. Both models consider heterogeneous economic agents who are faced with a standard asset allocation problem in each trading period. The heterogeneity of agents is expressed in terms of them having different expectations of future price movements on the risky asset. Brock and Hommes developed a financial market as an ‘Adaptive Belief System’ (ABS), where agents can permanently choose between different predictors or expectation functions for the future values of endogenous variables based on their past performance as measured by realized profits (see Brock and Hommes, 1997*a*, 1998). In the LLS model, however, agents are initially arranged in different strategy groups and constrained to stay within their strategy group during the whole trading process. Another key difference between these two models concerns the asset price and wealth processes. As a result of different types of utility functions, the time series of the asset price in the BH model is different from the time series in the LLS model. In the BH model the time series of the asset price is stationary and the fundamental price equilibrium can be examined analytically. Furthermore, the resulting nonlinear dynamic system can be characterized in terms of equilibrium equations which determine the current endogenous variables of the system. Contrastingly, both price and wealth processes grow over time in the LLS model, which is, of course, fairly consistent with observations made in real financial markets. Therefore, the dynamic system in this model can only be analyzed by numerical simulations, which is somewhat unsatisfying as it gives few insights into the underlying

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<sup>1</sup>See Chiarella and He (2001).

structure responsible for the more realistic time series (see Chapter 2, Sections 2.1 and 2.3 for details).

The CH model is useful in that it combines the analytical tractability of the BH model with some of the more realistic assumptions of the LLS model. The main idea of the CH model is to formulate the resulting (non-stationary) growth model, which stems from the more realistic underlying CRRA utility function taken from the LLS model, in terms of the return on the risky asset and the wealth proportion among heterogeneous investors, instead of describing it in terms of the price and the investors' total wealth. This change in the state variables reduces the growth model to a stationary model, which is a crucial step for the non-numerical analysis of the dynamics. In the case of heterogeneous agents, grouped as fundamentalists and chartists, the arising dynamics of the stationary model exhibits multiple steady states whose local stability mainly depends on the extrapolation rate of the chartists. Furthermore, the convergence to the steady state follows a 'market selection principle', which states that the return and wealth proportions tend to converge to the steady state, where a relatively higher return can be obtained.

The aim of this chapter is to investigate how the dynamic system of the CH model is influenced by the relative risk attitudes of different trader types. More precisely, my approach is to take the framework of Chiarella and He and to introduce one further parameter to characterize the investors' degree of risk aversion. This is done by replacing the CRRA logarithmic utility function with the more general CRRA power utility function which includes the logarithmic utility function as a special case. I will then seek to determine the robustness of their findings given changes in the relative risk-aversion coefficient.

The link between differing attitudes towards risk and various types of investors has been highlighted in earlier literature. A prominent example is the discussion about the 'smart money' investors and 'noise traders' in De Long et al. (1990), who show that (irrational) noise traders can gain higher returns than risk-averse rational traders if the noise traders are willing to bear a higher amount of risk than their sophisticated counterparts. Further discussion of this topic can be seen in Black (1986); Campbell and Kyle (1993); Miller (1977); Poterba and Summers (1988) and Summers (1986). However, the precise way in which the risk attitude of investors affects the dynamics of asset prices has not received a great deal of attention in economic literature. Nevertheless, in the case of the BH model, such an investigation has been carried out (see Chiarella and He, 2002c) and has produced some interesting results. For instance, assuming that fundamentalists are relatively more risk-averse than their chartist counterparts, the dynamics of the system becomes unstable and leads to complicated dynamics even to the point of chaotic behavior, which cannot be detected if all traders are assumed to have identical risk-aversion coefficients. Hence allowing for different risk preferences among the agents enriches the dynamics of the system and exhibits some significant differences in the resulting dynamic behavior.

A discussion about relevant functional forms commonly used in the expected utility framework can be found in Levy et al. (2000) and the references given therein. It also gives some justification for the chosen type favored in this chapter. In summary, empirical and experimental studies confirm that utility functions of the DARA type<sup>2</sup> dominate within all possible characterizations of

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<sup>2</sup>See Chapter 1, Appendix A 1.3 for a definition.

investor's preferences in terms of absolute risk measure (see Arrow, 1971; Friend and Blume, 1975; Gordon, Paradis and Rorke, 1972; Kroll, Levy and Rapoport, 1988; Latané, 1959; Hakansson, 1971; Markowitz, 1976; Levy, 1994). Regarding the relative risk-aversion measures, most studies support the CRRA type assumption (see Latané, 1959; Hakansson, 1971; Markowitz, 1976; Levy, 1994), although the agreement is not as complete as in the case of the absolute measure.<sup>3</sup> Nevertheless, these findings should be treated with care. It should be noted that there is a basic problem with testing absolute and relative risk aversion by empirical analysis. As outlined in Chapter 1, Appendix A 1.3 different analytic forms of utility functions imply different demand functions, which generally depend on the investor's current wealth. In other words, the *amount* (or *proportion*) of wealth invested in the risky asset<sup>4</sup> might change when the investors' wealth increases. This means that, in order to achieve an accurate testing procedure, the investment behavior of the *same* investor has to be examined at various levels of their wealth (and thus at various points of their lifetime). Of course, these studies are hard to perform and, even if such data collections can be generated, it is almost impossible to isolate the pure wealth effect in the investors' investment behavior from general changes in the investors' preferences caused, for instance, by their growing age or a modified perception of their current environment.<sup>5</sup> For this reason, experimental studies seem to be an appropriate tool to overcome those difficulties. In such laboratory experiments, the participants can be endowed with different levels of wealth within a very short time period by letting them play a sequential series of games, where the wealth changes from one game to another. In spite of this advantage compared to empirical studies, only a few experiments have been conducted on this topic in modern finance (see Gordon et al., 1972; Kroll et al., 1988; Levy et al., 1994). As already stated above, all three studies support the DARA hypothesis, while the CRRA type is only favored by Levy et al. (1994). This might be explained by the fact that only the study by Levy et al. (1994) accounts for financial penalty and financial reward in their experimental setup. For more details about the empirical and experimental studies mentioned in this discussion, we can refer to Levy et al. (2000).

Inspired by these findings, Chiarella and He follow the approach used in the LLS framework. They propose to let the agents derive their optimal demand for the risky asset from a CRRA utility function rather than from a CARA mean-variance approach as originally assumed in the BH model. As a consequence, the optimal demand function for the risky asset is no longer independent of the agents' wealth level (see Chapter 1, Appendix A 1.3 and Chapter 2, Section 2.1) and causes growing price and wealth processes. Chiarella and He's model assumes the same logarithmic utility function for all investors. This assumption simplifies matters, and can also be supported by the findings of H. A. LATANÉ<sup>6</sup>, who theoretically claims that long-run investors (i.e., investors whose investment horizon is very long) should be characterized by a logarithmic utility function.<sup>7</sup> In his publication

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<sup>3</sup>This result seems to be in some respects consistent as the DARA assumption is a generalization of the CRRA case. Again we can refer to Appendix A 1.3 for further explanations.

<sup>4</sup>Let us here assume a standard asset diversification problem, where the investors have to diversify their wealth between one risky asset and one riskless asset as assumed, for instance, in the BH and the LLS model.

<sup>5</sup>Instead, cross-sectional empirical studies are very often employed, e.g., in Friend and Blume (1975).

<sup>6</sup>See Latané (1959).

<sup>7</sup>The idea of logarithmic preferences was introduced 200 years ago by D. BERNOULLI (see Bernoulli, 1954). In his famous publication he proposed that gamblers should evaluate bets on the basis of mathematical expectations of the utility of winnings instead of simply deriving the mathematical expectation value of their winnings. Giving

he concluded that the investment strategy with the highest geometric mean is almost certain to be more valuable than any other investment strategy in the long run. Moreover, he argued that this result implies a logarithmic utility function. In contrast, R. C. MERTON and P. A. SAMUELSON<sup>8</sup> disagree with Latané's conclusion that many-period expected utility maximizers should maximize the expected logarithm of portfolio outcomes. Instead, they give some counterexamples to the assertion and argue that Latané's result is caused by a fallacious use of the Law of Large Numbers (see Merton and Samuelson, 1974). They also claim that, for the power utility function, the investment strategy is independent of the number of periods engaged in the investment, thus contradicting Latané's assertion still further. Hence, this discussion might allow us to reasonably conjecture that the assumption of the logarithmic preference of investors seems too restrictive and must be relaxed. This can be achieved by assuming the more general power utility function, which coincides with the logarithmic function for a relative risk aversion equal to unity. This conjecture is also supported by experimental studies (see Gordon et al., 1972), as well as by empirical surveys (see Friend and Blume, 1975). In both publications it is shown that the risk-aversion coefficient falls approximately in the range of 0.6–1.07 and 1.0–2.0 respectively, which is, nevertheless, also in harmony with the assumption of logarithmic preferences. Given these findings, the following analysis is not only an obvious extension of the CH model which simply enlarges the dynamic system by one additional parameter, but also a means of bringing the current model to a more realistic level.

The chapter is organized as follows. The first section gives a detailed model description of the CH model and summarizes some crucial results from the original publication. In the subsequent section I will implement the risk-aversion measure in the utility function of investors. Three further sections are then dedicated to my own investigations, where I will perform an extensive analytic and numerical study of the extended CH model under homogeneous and heterogeneous expectations for both the deterministic and a noisy model variant. In the last section of the chapter the LLS model is revisited (see Chapter 2, Section 2.3), but from a slightly different perspective. This will involve the dynamic system of the CH model being investigated by numerical simulations, but now combined with *all* the assumptions made in the original LLS model framework. The chapter concludes with a summary of the results of the analysis, while the detailed proofs of the theorems can be found in the appendix.

### 3.2 Model Description of the Chiarella-He Model

In this section I will present the original framework of the CH model by referring to Chiarella and He (2001). In the first step, I will summarize all necessary assumptions for the model structure and develop the equilibrium equation in terms of growing prices and investors' wealth. In the second step, I will introduce new state variables and now describe the dynamics in terms of return and proportions of wealth. This will provide us with the basis to extend the model in the subsequent section.

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very intuitive examples, he also argued for utility measures having a logarithmic form.

<sup>8</sup>See Merton and Samuelson (1974).

### 3.2.1 Development of the Growth Model

The artificial financial market consists of two investment alternatives, namely:

- one risky asset with the uncertain return

$$\rho_t = \frac{p_t + d_t - p_{t-1}}{p_{t-1}} \quad (3.1)$$

at period  $t$ , where  $p_t$  and  $d_t$  denotes the price and the dividend of the risky asset at time  $t$  respectively

- one risk-free asset, which realizes the constant gross return  $R = 1 + r$  at each trading period, where  $r$  denotes the risk-free rate.

The trading periods are assumed to be mapped on an equidistant, discrete time line and investors are allowed to rearrange their portfolios at each trading interval. As already mentioned above, all investors are identical in their preferences towards risk, which are analytically summarized in their utility function  $U(W) = \ln(W)$ . Following the framework of Brock and Hommes and Levy, Levy and Solomon, the wealth of investor  $i$  at time  $t + 1$  can be derived by

$$\begin{aligned} W_{i,t+1} &= (1 - \pi_{i,t})W_{i,t}R + \pi_{i,t}W_{i,t}(1 + \rho_{t+1}) \\ &= W_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)], \end{aligned} \quad (3.2)$$

where  $\pi_{i,t}$  denotes the proportion of wealth of agent  $i$  invested in the risky asset at time  $t$ .<sup>9</sup> The demand equation for the risky asset is assumed to be determined by a Walrasian scenario. Thus we assume that each trader is a price taker and that the market finds the price  $p_t$  that equates the sum of investors' demand to supply. In other words, at each trading period  $t$  all investors treat the price  $p_t$  as fixed when solving their optimization problem to determine the investment proportion  $\pi_{i,t}$ . As the investors' wealth at time  $t + 1$  is an uncertain variable provided that the price of the risky asset is not revealed for the current trading period  $t$ , the investors have to form expectations about their wealth based on the currently available information set  $\mathcal{I}_t$ , which is given by  $\mathcal{I}_t = \{p_{t-1}, p_{t-2}, \dots; d_t, d_{t-1}, \dots\}$ . Let  $E_t$  and  $Var_t$  denote respectively the conditional expectation and variance based on  $\mathcal{I}_t$ , and let  $E_{i,t}$  and  $Var_{i,t}$  denote the belief about the conditional expectation and variance of investor  $i$ . Hence the beliefs of investor  $i$  about the conditional expectation and variance on their wealth at time  $t + 1$  follow directly from Eq. (3.2) as

$$E_{i,t}(W_{i,t+1}) = W_{i,t}[R + \pi_{i,t}(E_{i,t}(\rho_{t+1}) - r)], \quad (3.3)$$

$$Var_{i,t}(W_{i,t+1}) = W_{i,t}^2 \pi_{i,t}^2 Var_{i,t}(\rho_{t+1}). \quad (3.4)$$

According to the standard expected-utility-of-wealth optimization scheme, each investor has to solve the following optimization problem

$$\max_{\pi_{i,t}} E_{i,t}[U(W_{i,t+1})]$$

<sup>9</sup>Note that Eq. (3.2) coincides with Eq. (2.1) of the BH model in Chapter 2 by using the identity  $\pi_{i,t} = (N_{i,t} p_t) / W_{i,t}$  which will be motivated below.

at time  $t$ . Assuming that all investors believe that the asset return  $\rho_t$  is conditionally normally distributed with mean  $E_{i,t}(\rho_{t+1})$  and variance  $Var(\rho_{t+1})$ , the optimal investment proportion  $\pi_{i,t}$  is given by<sup>10</sup>

$$\pi_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r}{Var_{i,t}(\rho_{t+1})}. \quad (3.5)$$

Now, heterogeneous beliefs can be introduced by making the assumption

$$E_{i,t}(\rho_{t+1}) = f_i(\rho_{t-1}, \dots), \quad (3.6)$$

$$Var_{i,t}(\rho_{t+1}) = g_i(\rho_{t-1}, \dots) \quad (3.7)$$

for  $i = 1, \dots, I$ , where  $I$  denotes the number of investors in the market and  $f_i, g_i$  are some deterministic functions which can be different within the set of investors. Eq. (3.5) implies that the optimum proportion of investment  $\pi_{i,t}$  is a function of the past prices and dividends, i.e.,<sup>11</sup>

$$\pi_{i,t} = \text{fct.}(p_{t-1}, p_{t-2}, \dots; d_t, d_{t-1}, \dots).$$

To conclude this subsection, the market clearing equilibrium price has to be determined. The number of shares that investor  $i$  wishes to hold at time  $t$  is given by the optimal investment proportion via the equality

$$N_{i,t} p_t = \pi_{i,t} W_{i,t}, \quad (3.8)$$

where  $N_{i,t}$  denotes the number of shares investor  $i$  wishes to hold at price  $p_t$ . Different to the framework of Brock and Hommes, in this model structure agents' optimal demand for the risky asset *does* depend on their wealth level (as a result of the underlying CRRA utility function), which has a decisive impact on the resulting dynamics of the model as we will see in the equation following the next.<sup>12</sup> The market clearing equilibrium price is obtained by summing the demands of all investors and assuming a fixed supply of risky assets, denoted by  $N$ , which leads to the following constraint

$$\sum_{i=1}^I N_{i,t} = N.$$

Thus the market clearing process leads finally to the conditional equation for the equilibrium price  $p_t$ , which is given by

$$\sum_{i=1}^I \pi_{i,t} W_{i,t} = N p_t. \quad (3.9)$$

Obviously, Eq. (3.9) shows that the equilibrium price  $p_t$  and the wealth of the investors

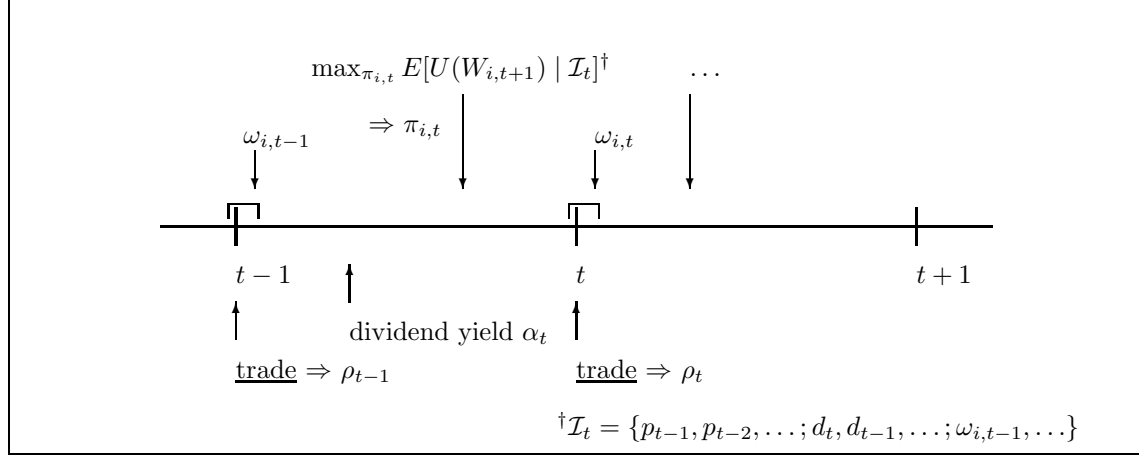
$$W_t \equiv (W_{1,t}, \dots, W_{I,t})$$

<sup>10</sup>A heuristic explanation for the formula can be found in Appendix A 3.1. A more formal derivation of Eq. (3.5) is given in Chiarella and He (2001).

<sup>11</sup>In this model framework the currently unrevealed price  $p_t$  is not included in the conditional expectations of the risky-asset return. Thus the construction is in harmony with the BH model, while in the LLS framework the optimal investment proportion of each investor explicitly depends on the 'hypothetical' price  $p_t$ , which makes the determination of the market clearing price analytically almost unsolvable.

<sup>12</sup>For a comparison with the demand function of the investors in the framework of Brock and Hommes, see also Eq. (A 2.4) in Chapter 2, Appendix A 2.1.

are determined simultaneously as in real markets. Furthermore, it can be seen from Eq. (3.2) and by the fact that dividends are paid in each trading period (which interfere via  $\rho_t$ ) that the model outlined above is a growth model in both the asset price and wealth, as already emphasized in the introduction of this chapter. In Figure 3.1 a model time line summarizes the main aspects of the Walrasian scenario.



**Figure 3.1:** Model time line for the CH model under a Walrasian scenario. Some variables will be first introduced in the following subsection. The horizontal brackets indicate that the state variables are determined at the same point in model time. The small shifts of the state variables within the brackets should simply give the reader a hint as to the order in which the state variables are calculated in the simulation program.

### 3.2.2 Stationary Formulation of the Model

In order to transform the growth model into a stationary model, Chiarella and He reformulate the market clearing process in terms of the return  $\rho_t$  of the risky asset and introduce a new state variable  $\omega_{i,t}$ , which denotes the wealth proportion of investor  $i$  at time  $t$ . Thus, instead of the total wealth  $W_t$  of all investors which is a growing variable over time, a normalized version of a wealth measure is used to describe the dynamics of the model, which always stays between the fixed interval  $[0,1]$ .

The analytic derivation of the dynamic system in terms of the new state variables  $\rho_t$  and  $\omega_{i,t}$  is straightforward, as outlined in Chiarella and He (2001).  $\omega_{i,t}$  denotes the proportion of wealth of investor  $i$  relative to the total wealth of all investors at time  $t$ , that is

$$\omega_{i,t} = \frac{W_{i,t}}{W_t}, \quad \text{where } W_t = \sum_{i=1}^I W_{i,t}.$$

Then, using Eq. (3.2)

$$\omega_{i,t+1} = \frac{W_{i,t+1}}{W_{t+1}} = \frac{W_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]}{W_{t+1}} = \frac{\omega_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]}{W_{t+1}/W_t}.$$

Rewriting the denominator of the last expression as

$$\frac{W_{t+1}}{W_t} = \frac{\sum_{j=1}^I W_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)]}{W_t} = \sum_{j=1}^I \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)] \quad (3.10)$$

leads to the following equation

$$\omega_{i,t+1} = \frac{\omega_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]}{\sum_{j=1}^I \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)]}, \quad (3.11)$$

which describes the evolution of the wealth proportions among the investors in terms of the return of the risky asset.<sup>13</sup> Now we turn to the market clearing equation. Note that from Eq. (3.9)

$$Np_t = \sum_{i=1}^I \pi_{i,t} W_{i,t} = W_t \sum_{i=1}^I \pi_{i,t} \omega_{i,t}. \quad (3.12)$$

Rewriting this equation for the following trading period and considering the ratio of both leads to

$$\frac{W_{t+1}}{W_t} \frac{\sum_{i=1}^I \pi_{i,t+1} \omega_{i,t+1}}{\sum_{i=1}^I \pi_{i,t} \omega_{i,t}} = \frac{p_{t+1}}{p_t}.$$

Now, using Eq. (3.1) and Eq. (3.10) and defining  $\alpha_{t+1} \equiv d_{t+1}/p_t$  leads to

$$\sum_{j=1}^I \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)] \frac{\sum_{i=1}^I \pi_{i,t+1} \omega_{i,t+1}}{\sum_{i=1}^I \pi_{i,t} \omega_{i,t}} = 1 + \rho_{t+1} - \alpha_{t+1}, \quad (3.13)$$

which still depends on the wealth proportion  $\omega_{i,t+1}$ , which is somewhat unsatisfactory. Therefore, note that the numerator can be expressed in terms of  $\omega_{i,t}$  using Eq. (3.11), which gives

$$\sum_{i=1}^I \pi_{i,t+1} \omega_{i,t+1} = \sum_{i=1}^I \pi_{i,t+1} \left\{ \frac{\omega_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]}{\sum_{j=1}^I \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)]} \right\}. \quad (3.14)$$

Inserting Eq. (3.14) in Eq. (3.13) results in

$$\sum_{i=1}^I \pi_{i,t+1} \omega_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)] = (1 + \rho_{t+1} - \alpha_{t+1}) \left( \sum_{i=1}^I \pi_{i,t} \omega_{i,t} \right), \quad (3.15)$$

from which the return of the risky asset  $\rho_{t+1}$  at time  $t + 1$  can be determined. Finally, with Eq. (3.11) and Eq. (3.15) it has been shown that a stationary model in the return and wealth proportions exists. This result is summarized in the following theorem, which is quoted from the original publication of the CH model:

**Theorem 3.1 (First Result taken from Chiarella and He (2001))**

*The wealth proportions evolve according to*

$$\omega_{i,t+1} = \frac{\omega_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]}{\sum_{j=1}^I \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)]}, \quad i = 1, \dots, I, \quad (3.16)$$

*where the return  $\rho_{t+1}$  is given by*

$$\rho_{t+1} = r + \frac{\sum_{i=1}^I \omega_{i,t}[(1+r)(\pi_{i,t} - \pi_{i,t+1}) - \alpha_{t+1}\pi_{i,t}]}{\sum_{i=1}^I \pi_{i,t}\omega_{i,t}(\pi_{i,t+1} - 1)} \quad (3.17)$$

*with*

$$\alpha_{t+1} = \frac{d_{t+1}}{p_t} \quad (3.18)$$

<sup>13</sup>The optimum proportion of investment  $\pi_{i,t}$  is also a function of the past returns, see Eq. (3.5).



and

$$\pi_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r}{\text{Var}_{i,t}(\rho_{t+1})}, \quad i = 1, \dots, I.$$

If it is assumed that the dividend yield  $\alpha_{t+1}$  follows a normal distribution with mean  $\alpha_0$  and variance  $\sigma_0^2$ , the system Eq. (3.16), Eq. (3.17) together forms a stationary model in the return and wealth proportion.

The homogeneous scenario can be immediately concluded from Theorem 3.1 as a special case. Assume that all investors form the same expectations about both mean and variance about the return of the risky asset, i.e.,

$$E_{i,t}(\rho_{t+1}) = E_t(\rho_{t+1}) \quad (3.19)$$

$$\text{Var}_{i,t}(\rho_{t+1}) = \text{Var}_t(\rho_{t+1}), \quad i = 1, \dots, I. \quad (3.20)$$

Then, according to Eq. (3.5), the optimal investment proportion at time  $t$  is the same for all the investors, that is,

$$\pi_t \equiv \pi_{i,t} = \frac{E_t(\rho_{t+1}) - r}{\text{Var}_t(\rho_{t+1})} \quad \text{for } i = 1, \dots, I.$$

Therefore, we get for the resulting dynamics of  $(\omega_{i,t+1}, \rho_{t+1})$  from Eq. (3.16) and Eq. (3.17)

$$\omega_{i,t+1} = \frac{\omega_{i,t}[R + \pi_t(\rho_{t+1} - r)]}{\sum_{j=1}^I \omega_{j,t}[R + \pi_t(\rho_{t+1} - r)]} = \frac{\omega_{i,t}}{\sum_{j=1}^I \omega_{j,t}} = \omega_{i,t} (= \omega_{i,0}), \quad (3.21)$$

and

$$\rho_{t+1} = r + \frac{(1+r)(\pi_t - \pi_{t+1}) - \alpha_{t+1}\pi_t}{\pi_t(\pi_{t+1} - 1)} \quad (3.22)$$

where in both cases  $\sum_{i=1}^I \omega_{i,t} = 1$  is used. The above analysis can be summarized in the following corollary:

**Corollary 3.2 (Second Result taken from Chiarella and He (2001))**

*Under the assumption of homogeneous expectation about mean Eq. (3.19) and variance Eq. (3.20), the wealth proportion of each investor stays at his/her initial level Eq. (3.21), while the return dynamics is governed by Eq. (3.22).*

*The Model with Representative Agents*

As a second application of Theorem 3.1, consider the case where all investors can be grouped by their view of the conditional expectation and variance of the risky asset's return distribution. Suppose there are  $h$  different perceptions of the mean and the variance among the investors and let  $n_j$ ,  $j = 1, \dots, h$  be the fraction of investors belonging to trading group  $j$ . The fractions  $n_j$  are assumed to be fixed during the whole trading process. Thus every single investor can uniquely be associated with one of the  $h$  different trading types and the optimal investment proportion denoted by  $\bar{\pi}_{j,t}$  is identical within each single trading group. It is straightforward to express the system Eq. (3.16) and Eq. (3.17) in terms of the population proportion  $n_j$ . This implies that the individual wealth level  $W_{i,t}$  of investor  $i$  is no longer an appropriate variable to describe the dynamics of the system. Instead, Chiarella and He propose to consider the *average wealth* of each investor within each group denoted by  $\bar{W}_{j,t}$ , where  $j = 1, \dots, h$ . Hence the following wealth measures have to be distinguished in the following analysis:

- $W_{i,t}$  being the individual wealth of investor  $i$ , where  $i = 1, \dots, I$
- $\bar{W}_{j,t}$  being the average wealth of *each* investor within group  $j$ , where  $j = 1, \dots, h$
- $W_t$  being the total wealth in the market, i.e.,  $W_t = \sum_{i=1}^I W_{i,t}$ .

Following the line of derivation in Theorem 3.1, the average wealth proportion of trading group  $j$  can be defined as<sup>14</sup>

$$\bar{\omega}_{j,t} = \frac{\bar{W}_{j,t}}{\bar{W}_t}, \quad \text{where} \quad \bar{W}_t = \sum_{j=1}^h \bar{W}_{j,t}.$$

A totally analogous derivation (which is relegated to Appendix A 3.2) leads to the following theorem:

**Theorem 3.3 (Third Result taken from Chiarella and He (2001))**

*The dynamics of average wealth proportions evolves according to*

$$\bar{\omega}_{j,t+1} = \frac{\bar{\omega}_{j,t}[R + \bar{\pi}_{j,t}(\rho_{t+1} - r)]}{\sum_{\ell=1}^h \bar{\omega}_{\ell,t}[R + \bar{\pi}_{\ell,t}(\rho_{t+1} - r)]}, \quad j = 1, \dots, h \quad (3.23)$$

*with the return  $\rho_{t+1}$  given by*

$$\rho_{t+1} = r + \frac{\sum_{j=1}^h n_j \bar{\omega}_{j,t} [(1+r)(\bar{\pi}_{j,t} - \bar{\pi}_{j,t+1}) - \alpha_{t+1} \bar{\pi}_{j,t}]}{\sum_{j=1}^h n_j \bar{\pi}_{j,t} \bar{\omega}_{j,t} (\bar{\pi}_{j,t+1} - 1)}. \quad (3.24)$$

*Recall that the (average) optimum wealth proportion  $\bar{\pi}_{j,t}$  is calculated according to Eq. (3.5). If it is assumed that the dividend yield  $\alpha_{t+1}$  follows a normal distribution with mean  $\alpha_0$  and variance  $\sigma_0^2$ , the system Eq. (3.23), Eq. (3.24) together forms a stationary model.*

Referring to the Representative Agent Hypothesis outlined in Chapter 1 in the first part of the thesis, Theorem 3.3 can be read in the following way. The system Eq. (3.23) and Eq. (3.24) describe the dynamics of  $h$  different representative investors, whose individual wealth level is given by  $\bar{W}_{j,t}$ . Hence the only difference between the system Eq. (3.23), Eq. (3.24) and the system Eq. (3.16), Eq. (3.17) of Theorem 3.1 are the weighting factors  $n_j$ , which take into account the different market power of each individual representative agent. The close connection of both theorems are also obvious by setting  $h = I$  and  $n_j = 1/I$  for  $j = 1, \dots, h$ . Then, Theorem 3.3 leads to Theorem 3.1, which is not surprising because all investors are here pursuing their own trading strategies, which implies that every trading group consists only of one single investor. Thus each single trading group  $j$  can be 1:1 mapped to a single investor  $i$ . In other words the average wealth  $\bar{W}_{j,t}$  of each investor  $i$  within group  $j$  coincides with his/her individual wealth level denoted by  $W_{i,t}$  and, therefore, it is  $\bar{\omega}_{j,t} = \omega_{i,t}$  with investor  $i$  belonging to trading group  $j$  for all trading groups  $j$ .

The assumption of different trading groups having fixed proportions seems unrealistic. As proposed in the framework of Brock and Hommes (1998), traders should be allowed to adjust their

<sup>14</sup>Obviously, the average wealth proportion again adds up to unity, i.e., it is  $\sum_{i=j}^h \bar{\omega}_{j,t} = 1$ . However, note that  $\bar{W}_t$  has less of an intuitive meaning, but rather serves as a normalization quantity. Nevertheless, it can be denoted as the total average wealth (see Chiarella and He, 2001).

trading strategies occasionally according to some performance measure or fitness function, which would lead to an adaptive model with changing population proportions.<sup>15</sup> Since implementing the population proportions as endogenous state variables causes insuperable difficulties when trying to study the model analytically (because of the highly nonlinear nature of the CH model), it will not be considered here.<sup>16</sup> Instead, following the approach of Chiarella and He, I will keep the proportions of different groups fixed, since our main interest is to investigate whether certain trading strategies will prevail over others, which is much easier to keep track of when switching between different strategies is forbidden for the investors. Therefore, the remainder of this chapter concentrates exclusively on the fixed proportion model as proposed in Chiarella and He (2001). In the next section a generalized form of utility function will be introduced. What then follows is an investigation of this enriched CH model with respect to homogeneous and heterogeneous expectation schemes on the conditional mean and variance of return of the risky asset.

### 3.3 Implementation of the Risk Aversion Coefficient

The implementation of a risk-aversion measure (or Arrow-Pratt measure) in the utility function and the derivation of the resulting demand function of the investors for the risky asset can be carried out easily. Having determined the new demands  $\pi_{i,t}$  (or  $\bar{\pi}_{j,t}$  in the case of trading groups), the dynamic system is of course changed via Eq. (3.16) and Eq. (3.17) (or Eq. (3.23) and Eq. (3.24)).

In the first step, the originally assumed utility function  $U(W) = \ln(W)$  is replaced by the power utility function, which allows for different risk-aversion coefficients, that is<sup>17</sup>

$$U(W) = \frac{W^{1-\nu_i} - 1}{1 - \nu_i}, \quad i = 1, \dots, I \quad (\text{or indexed by } j \text{ with } j = 1, \dots, h). \quad (3.25)$$

Hence investors can be assumed to have different attitudes to risk, characterized by their risk-aversion parameter  $\nu_i$ . In general,  $\nu_i$  can be any finite number.<sup>18</sup> For  $\nu_i < 0$  investors are risk lovers (which might be quite unrealistic in our framework), for  $\nu_i > 0$  investors are risk averters, which coincides with empirical and experimental findings briefly outlined in the introduction of this chapter. Nevertheless, in the remainder of this chapter the risk preference parameter is not restricted to being positive in principle although our analysis will, of course, mostly focus on positive risk-aversion coefficients. As with the logarithmic utility function, the power utility function is also one of the CRRA type. In fact, it coincides with the logarithmic utility function for  $\nu_i = 1$ , which can be shown by applying the rule of L' Hospital.<sup>19</sup> Thus the replacement of the utility function simply introduces one further parameter into the CH model.

In the second step, the influence of the newly introduced risk-aversion coefficient on the optimal

<sup>15</sup>See also Chapter 2, Subsection 2.1 in the first part of the thesis for details.

<sup>16</sup>Some effort has already been made to solve this problem, albeit with slightly unsatisfactory (see Chiarella and He, 2002a,b, 2005).

<sup>17</sup>The (standard) power utility function is often defined as  $U(W) = W^{1-\nu}/(1-\nu)$ , which causes no difference since preferences are invariant under linear transformations of the utility function. See Levy et al. (2000) or Chapter 1, Appendix A 1.3 in the first part of the thesis for details.

<sup>18</sup>The parameter  $\nu_i$  is the (constant) Arrow-Pratt measure of relative risk aversion introduced in Chapter 1, Appendix A 1.3. Indeed, a short calculation shows that  $R_R(W) = -W(U''(W)/U'(W)) = \nu_i$ . A plot of the graph of Eq. (3.25) for different choices of risk-aversion coefficients can also be found in Chapter 1, Appendix A 1.3.

<sup>19</sup>See Chapter 1, Appendix A 1.3 for the derivation.

investment proportion of the investors has to be considered. Keeping the assumption that all investors believe that the asset return  $\rho_t$  is conditionally normally distributed with mean  $E_{i,t}(\rho_{t+1})$  and variance  $Var_{i,t}(\rho_{t+1})$ , the optimal investment proportion  $\pi_{i,t}$  is now determined by<sup>20</sup>

$$\pi_{i,t} = \frac{1}{\nu_i} \frac{E_{i,t}(\rho_{t+1}) - r}{Var_{i,t}(\rho_{t+1})}, \tag{3.26}$$

which can be derived analogous to Eq. (3.5). Eq. (3.26) indicates that the optimal investment proportion is inversely proportional to the risk preference parameter  $\nu_i$ , which implies that more risk-averse investors are willing to invest only a lower fraction of their wealth in the risky asset. Once more, it should be stressed that the optimal demand  $N_{i,t}$  for the risky asset does depend on the individual wealth level of the investor in this scenario (see Eq. (3.8)), which results from the underlying utility function that is still of the CRRA type.

The following sections closely proceed along the line of Chiarella and He (2001) and shows an enlarged analytical and numerical study of the homogeneous model and a heterogeneous model with two investors. Resulting from the model framework, almost all the results of the CH model (see Chiarella and He, 2001) can be recovered by setting the risk-aversion coefficient of investors to  $\nu = 1$ . Nevertheless, introducing the new model parameter  $\nu_i$  gives new insights into the behavior of the underlying model dynamics and reveals some interesting conclusions regarding the CH model's limitations caused by its model design, which might be less clear from the original publication.

### 3.4 The Model under Homogeneous Expectations: An Analytical Treatment

This section deals with the simplest case, which is the price and wealth dynamics under homogeneous preferences. First, investors' expectation formation is concretized. Second, I will investigate the existence of steady states for different ranges of the risk-aversion coefficient  $\nu$ . Third, I will consider the local stability properties of the steady states. As we will see that, even in the homogeneous case, the analytical tractability of the study rapidly attains to its boundaries, I will conduct a numerical analysis based on computer simulations for the heterogeneous case in the following section.

#### 3.4.1 Specification of the Trading Strategies

It is assumed that all investors are homogeneous with respect to their risk preferences and their expectation formation. Therefore, the investors' beliefs about the conditional mean and variance of the return distribution of the risky asset are formed identically among all investors as proposed in Eq. (3.19) and Eq. (3.20). Furthermore, the deterministic functions  $f_i$  and  $g_i$  (see Eq. (3.6) and Eq. (3.7)) have to be determined to enable us to perform an analytic and numerical analysis of the model. Following Chiarella and He (2001), we assume that the common perception of the distribution of the risky asset can be mathematically expressed as

$$E_t(\rho_{t+1}) = r + \delta + d\bar{\rho}_t, \tag{3.27}$$

---

<sup>20</sup>The calculation is relegated to the Appendix A 3.1.

$$Var_t(\rho_{t+1}) = \sigma^2, \tag{3.28}$$

where  $\delta \geq 0$  and  $\sigma^2 > 0$  are constants,  $d \in \mathbb{R}$  and

$$\bar{\rho}_t = \frac{1}{L} \sum_{k=1}^L \rho_{t-k}$$

with the integer  $L \geq 1$ . In addition, it is supposed that all investors possess the same degree of risk aversion, i.e.,  $\nu \equiv \nu_i$  for  $i = 1, \dots, I$  in their utility function given by Eq. (3.25).

Assumption (3.27) indicates that for  $d \neq 0$  the agents' common expectation about the return distribution mainly depends on the moving average over the past  $L$  realizations of the return. In this case the traders are called 'chartists'. For  $d > 0$  the investors are further classified as 'trend followers', who extrapolate the future return from the past, i.e., they believe that future returns follow the return trend in the same direction. In contrast, 'contrarians' extrapolate in the reverse direction and are thus characterized by a negative extrapolation parameter ( $d < 0$ ). For  $d = 0$ , traders are called 'fundamentalists'. In this case, the moving average term vanishes and the first and second term of Eq. (3.27) become decisive. Rewriting Eq. (3.27) as

$$E_t(\rho_{t+1}) - r = \delta, \tag{3.29}$$

it obviously states that the excess conditional mean from the risk-free rate  $r$  is a constant term, denoted by  $\delta$ . As the excess return should compensate for the riskiness of realizing losses by investing in the risky asset,  $\delta$  is assumed to be positive. Of course, a constant excess return is a simplified assumption. Nevertheless, the parameter  $\delta$  can be motivated by a slightly different interpretation. In principle, the current long-run return of the risky asset, e.g., derived from the past returns, can serve as a rough estimation for the future long-run return of the asset. Then,  $\delta$  can be understood as an estimation value for the difference between the average (long-run) future return of the risky asset and the risk-free rate of the riskless asset. Therefore, although we do not totally share the interpretation of the parameter  $\delta$  as a 'risk premium' (see Chiarella and He, 2001, p. 513), we keep its notation the same to avoid awkward and useless confusion. According to Eq. (3.28), the conditional variance is assumed to be constant, which is also a simplification, but one again caused by the need for analytical tractability.<sup>21</sup>

### 3.4.2 Determination of the Steady States

Under the homogeneous expectations Eq. (3.27) and Eq. (3.28), the corresponding dynamics of the stationary model is given by Eq. (3.21) and Eq. (3.22). Chiarella and He has shown that, depending on the extrapolation rate  $d$ , the system can have multiple fixed equilibria. Allowing for one further degree of freedom (namely the risk-aversion coefficient) in my analysis, the following enriched theorem can be derived, which, of course, contains the findings documented in Chiarella and He (2001) as a special case:<sup>22</sup>

<sup>21</sup>However, note that the following chapter will present a model framework in which a time-dependent market variance is used to describe the investors' conditional variance of returns.

<sup>22</sup>Although the CH model is in all respects the intellectual progenitor of this work, at some marginal points we find slightly different results as noted in Chiarella and He (2001), e.g., the determination of the upper boundary  $d_{oU,1}$  for  $d \neq 0$ . According to the upper and lower boundaries  $d_{oU,i}$ ,  $d_{oL,i}$  see also the remark in Footnote 23.

**Theorem 3.4** Let  $\bar{\delta} = \delta/\sigma^2$ ,  $\bar{d} = d/\sigma^2$ . Assume Eq. (3.27) and Eq. (3.28) hold and  $\alpha_t = \alpha_0 > 0$ ,  $\bar{\delta} < 1$ ,  $\nu \neq 0$ ,  $\bar{\delta} \neq \nu$ .

(i) If  $d = 0$ , then the system Eq. (3.22) has a steady state

$$\rho_0^* = r + \frac{\alpha_0}{1 - \bar{\delta}/\nu} \quad (3.30)$$

which satisfies

- (1)  $\rho_0^* > r + \alpha_0$  for  $\nu > \bar{\delta}$  and  $\rho_0^* \rightarrow +\infty$  as  $\nu \rightarrow \bar{\delta}^+$ ;
- (2)  $\rho_0^* < 0$  for  $\bar{\delta} > \nu > \nu_0 \equiv \bar{\delta}/(1 + \alpha_0/r)$  and  $\rho_0^* \rightarrow -\infty$  as  $\nu \rightarrow \bar{\delta}^-$ ;
- (3)  $r > \rho_0^* \geq 0$  for  $\nu_0 \geq \nu > 0$  and  $\rho_0^* \rightarrow r$  as  $\nu \rightarrow 0^+$ ;
- (4)  $r + \alpha_0 > \rho_0^* > r$  for  $\nu < 0$  and  $\rho_0^* \rightarrow r$  as  $\nu \rightarrow 0^-$ , while  $\rho_0^* \rightarrow r + \alpha_0$  as  $\nu \rightarrow -\infty$ .

(ii) If  $d \neq 0$ , then the system Eq. (3.22) has two steady states

$$\rho_{\pm}^* = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{\bar{d}} \pm \sqrt{\left[ r - \frac{\nu - \bar{\delta}}{\bar{d}} \right]^2 - \frac{4\nu\alpha_0}{\bar{d}}} \right] \quad (3.31)$$

if and only if

$$\left[ r - \frac{\nu - \bar{\delta}}{\bar{d}} \right]^2 \geq \frac{4\nu\alpha_0}{\bar{d}}. \quad (3.32)$$

In this case the following subcases have to be considered:

(a) if  $\nu > \bar{\delta} > 0$ , then condition Eq. (3.32) holds for all  $d < 0$  and the system Eq. (3.22) has two steady states  $\rho_{\pm}^*$  which satisfy

- (1)  $\rho_-^* < 0$  and  $\rho_-^*$  is a decreasing function of  $\bar{d}$ ;
- (2)  $r < \rho_+^* < \rho_0^*$  and  $\rho_+^*$  is an increasing function of  $\bar{d}$  satisfying  $\rho_+^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$ .

For  $d > 0$  the system Eq. (3.22) has also two steady states  $\rho_{\pm}^*$  if and only if Eq. (3.32) holds, and whenever they exist, they satisfy

- (1)  $0 < \rho_0^* < \rho_{\pm}^*$  for  $0 < \bar{d} < d_{oL,1} \equiv \frac{(\nu - \bar{\delta})^2}{r(\nu - \bar{\delta}) + 2\alpha_0\nu}$  and  $\rho_-^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$ ;
- (2)  $0 < \rho_{\pm}^* < \rho_0^*$  for  $\bar{d} > d_{oU,1} \equiv \frac{(\nu - \bar{\delta})}{r}$ ;
- (3)  $\rho_-^*$  increases for  $0 < \bar{d} < d_{oL,1}$  and decreases for  $\bar{d} > d_{oU,1}$ ;
- (4)  $\rho_+^*$  decreases for  $0 < \bar{d} < d_{oL,1}$  and increases for  $\bar{d} > d_{oU,1}$ .

(b) if  $\bar{\delta} > \nu > 0$ , then Eq. (3.32) holds for all  $d < 0$  and the system Eq. (3.22) has two steady states  $\rho_{\pm}^*$  which satisfy

- (1)  $\rho_0^* < r < \rho_+^*$  and  $\rho_+^*$  is an increasing function of  $\bar{d}$ ;
- (2)  $0 \leq \rho_-^* \leq \rho_0^*$  for  $\nu \leq \nu_0$  and  $\rho_-^*$  is an increasing function of  $\bar{d}$  satisfying  $\rho_-^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$ ;
- (3)  $\rho_0^* < \rho_-^* < 0$  for  $\nu > \nu_0$  and  $\rho_-^*$  is a decreasing function of  $\bar{d}$  satisfying  $\rho_-^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$ .

For  $d > 0$  the system Eq. (3.22) has two steady states  $\rho_{\pm}^*$  if and only if Eq. (3.32) holds and, whenever they exist, they satisfy, for  $\nu \leq \nu_0$ ,

(1)  $0 \leq \rho_0^* < \rho_+^* < r$  and  $\rho_+^*$  is an increasing function of  $\bar{d}$  satisfying  $\rho_+^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$ ;

(2)  $\rho_-^* < 0$  and  $\rho_-^*$  is an increasing function of  $\bar{d}$ ;

and, for  $\nu > \nu_0$ ,

(3)  $\rho_{\pm}^* < \rho_0^* < 0$  for  $0 < \bar{d} < d_{oL,2} \equiv \frac{(\nu-\bar{\delta})^2}{r(\nu-\bar{\delta})+2\alpha_0\nu}$  and  $\rho_+^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$ ;

(4)  $\rho_0^* < 0 < \rho_{\pm}^* < r$  for  $\bar{d} > d_{oU,2} \equiv -\frac{(\nu-\bar{\delta})}{r}$ ;

(5)  $\rho_-^*$  increases for  $0 < \bar{d} < d_{oL,2}$  and decreases for  $\bar{d} > d_{oU,2}$ ;

(6)  $\rho_+^*$  decreases for  $0 < \bar{d} < d_{oL,2}$  and increases for  $\bar{d} > d_{oU,2}$ .

(c) if  $\nu < 0$  and  $d < 0$ , then  $\rho_{\pm}^*$  are also steady states if and only if Eq. (3.32) holds, and whenever they exist, they satisfy

(1)  $0 < \rho_{\pm}^* < r < \rho_0^*$  for  $\bar{d} < d_{oL,3} \equiv \frac{(\nu-\bar{\delta})}{r}$ ;

(2)  $r < \rho_0^* < \rho_{\pm}^*$  for  $\bar{d} > d_{oU,3} \equiv \frac{(\nu-\bar{\delta})^2}{r(\nu-\bar{\delta})+2\alpha_0\nu}$  and  $\rho_-^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$ ;

(3)  $\rho_-^*$  increases for  $\bar{d} < d_{oL,3}$  and decreases for  $0 > \bar{d} > d_{oU,3}$ ;

(4)  $\rho_+^*$  decreases for  $\bar{d} < d_{oL,3}$  and increases for  $0 > \bar{d} > d_{oU,3}$ .

For  $d > 0$  condition Eq. (3.32) holds invariably and the system Eq. (3.22) has two steady states  $\rho_{\pm}^*$  which satisfy

(1)  $r < \rho_+^* < \rho_0^*$  and  $\rho_+^*$  is a decreasing function of  $\bar{d}$  satisfying  $\rho_+^* \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$ ;

(2)  $\rho_-^* < 0$  and  $\rho_-^*$  is an increasing function of  $\bar{d}$ .

PROOF: See Appendix A 3.3.

Theorem 3.4 characterizes various market scenarios on the existence of steady states when the dividend yield is assumed to be a positive constant  $\alpha_0$ . It also contains many market situations which are quite unrealistic. Nevertheless, according to the dynamic system  $(\omega_{t+1}, \rho_{t+1})$  given by Eq. (3.21) and Eq. (3.22), they do exist analytically.<sup>23</sup> Thus the meaningful economic results have to be selected from the complex appearance of the Theorem 3.4, which I will come on to shortly. Before I highlight and illustrate the theorem's main findings, it is necessary to give several technical remarks:

- Following Chiarella and He (2001) once more, it is more convenient to work with the parameters  $\bar{\delta}$  and  $\bar{d}$ , which measure the extrapolation rate  $d$  and the parameter  $\delta$  per unit of the variance  $\sigma^2$ .
- In the case of fundamentalist investors, the conditional equation of the fixed point is not defined for  $\nu = 0$  (no risk aversion) and  $\nu = \bar{\delta}$ . As these cases are generally rather pathological, they will be mostly excluded from my analysis in the interests of simplicity.

<sup>23</sup>As the wealth proportion remains at its initial level in the homogeneous case, the dynamic system is solely described by the return  $\rho_t$  of the risky asset, the optimal investment proportion  $\pi_t$  and the exogenous model parameters. Particularly, the system does not depend on the price  $p_t$  of the risky asset and the number of risky assets  $N$  that are available in the market as initially proposed by Eq. (3.9). Therefore, considering exclusively the dynamic system Eq. (3.21) and Eq. (3.22), market situations in which the price for the risky asset becomes non-positive and/or  $N$  becomes negative cannot be detected in this framework.

- For the risk-aversion coefficient being equal to  $\nu = 1$ , the power utility function Eq. (3.25) coincides with the logarithmic utility function and has to be replaced by  $U(W) = \ln W$ .
- In the case of chartist investors ( $d \neq 0$ ), Eq. (3.31) can be written as

$$\rho_{\pm}^* = \frac{1}{2} \left[ r + \frac{1 - \check{\delta}}{\check{d}} \pm \sqrt{\left[ r - \frac{1 - \check{\delta}}{\check{d}} \right]^2 - \frac{4\alpha_0}{\check{d}}} \right],$$

where  $\check{\delta} \equiv \bar{\delta}/\nu = \delta/\nu\sigma^2$  and  $\check{d} \equiv \bar{d}/\nu = d/\nu\sigma^2$ . Since it is  $\check{\delta} < 1$  for Subcase (a), i.e., if the risk-aversion coefficient is higher than  $\bar{\delta}$ , note that all results from the corresponding theorem maintained in Chiarella and He (2001, see Proposition 3.1.) can simply be adopted by substituting  $\check{\delta}$  for  $\bar{\delta}$  and  $\check{d}$  for  $\bar{d}$ .

Obviously, the latter remark is of particular interest. However, let us first examine the case where all investors are fundamentalists.

When the investors are assumed not to extrapolate from the past ( $d = 0$ ), a unique steady state  $\rho_0^*$  exists for any choice of the risk-aversion coefficient.<sup>24</sup>  $\rho_0^*$  is determined by Eq. (3.30) and can be called the ‘fundamental steady state’. Depending on the risk-aversion coefficient  $\nu$ , the only fixed equilibrium can be higher as well as lower than the dividend yield or can even be negative (which is one of the unrealistic market scenarios already mentioned above). Figure 3.2 illustrates the outcome of fundamental steady states for different risk preferences when all investors are fundamentalists. As Eq. (3.30) indicates, only risk preferences that exceed the excess conditional mean from the risk-free rate, namely  $\bar{\delta}$  (measured per units of variance), lead to realistic market scenarios in which the steady-state return is higher than the risk free rate and the dividend yield.<sup>25</sup> In those cases a higher  $\bar{\delta}$  causes a higher steady-state return. For risk preferences lower than  $\bar{\delta}$ , only unrealistic situations occur. Moreover, for  $\nu < 0$ , i.e., when the investors are risk lovers, the corresponding optimal investment proportion  $\pi^*$  becomes negative, which leads to a non-positive price process.<sup>26</sup>

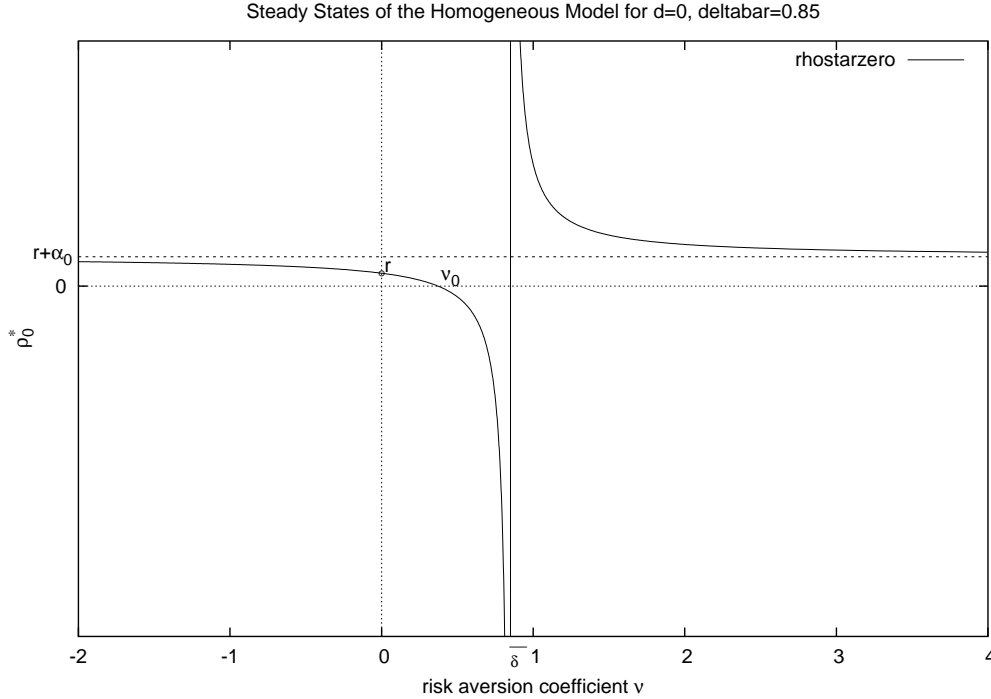
In addition to the analytic form of Eq. (3.30), from which the curve of Figure 3.2 can easily be understood, an economic explanation for the curve’s qualitative development can also be given. If the investors are totally risk-averse, they do not invest in the risky asset, i.e.,  $\pi_t$  is approximately zero, and therefore the fundamental steady state is simply the risk free rate  $r$  plus the guaranteed dividend yield  $\alpha_0$ . If the investors become less risk-averse, they start to invest in the risky asset and, as a consequence, the fundamental return  $\rho_0^*$  increases. Otherwise, if the excess conditional mean from the risk-free rate  $\bar{\delta}$  has a higher value than the investors’ risk aversion measured by the coefficient  $\nu$ , i.e., if  $\bar{\delta} > \nu$ , it follows that there is a mismatch between these two parameters since the expected excess return  $\bar{\delta}$  is higher than it has to be with regard to the risk perception of the investors, which, therefore, has to be compensated by being  $\rho_0^* < r$ . In other words, for  $\bar{\delta} > \nu$  the

<sup>24</sup>It can be shown that  $\rho_0^*$  is locally asymptotically stable for all choices of  $\nu$ . For details see Corollary 3.6 below.

<sup>25</sup>This market situation coincides with those considered in Chiarella and He (2001) as for the logarithmic utility function  $\nu = 1$ , which is always higher than  $\bar{\delta}$  constrained to be smaller than unity.

<sup>26</sup>In the following, the case  $\nu < 0$  will not be considered in detail as it is not in harmony with the assumption of risk-averse investors, which is a crucial assumption in almost all asset pricing models. Nevertheless, it will be quoted in some of the following theorems and corollaries for the sake of completeness.



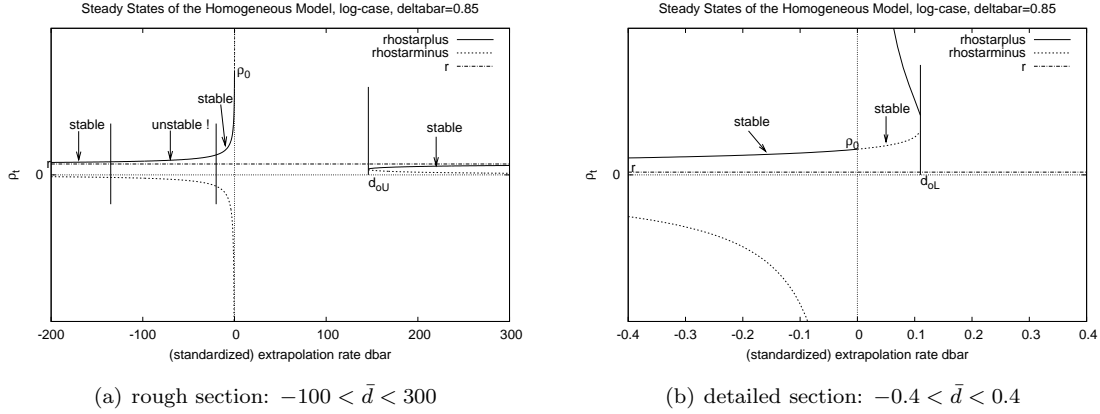


**Figure 3.2:** Steady state distribution of the homogeneous model for  $d = 0$ . When all investors are fundamentalists (no extrapolation,  $d = 0$ ) a unique steady state  $\rho_0^*$  exists for all possible risk preferences  $\nu$  of the investors. Note that the fixed equilibrium is above the risk-free rate  $r$  only for  $\nu > \bar{\delta}$  and  $\nu < 0$ , although the latter case leads to unrealistic market situations.  $\bar{\delta}$  is fixed and set to 0.85. Realistic values for the risk free rate  $r$  (3.7% p.a.) and the dividend yield  $\alpha_0$  (4.7% p.a.) are chosen to generate this plot.

investors’ *expectations* about the development of the risky asset and the investors’ basic *attitude* towards investing in risky prospects is modeled in the opposite direction. This is an unrealistic assumption and thus leads to unrealistic market scenarios (see also the summary section below for a more detailed discussion on this topic).

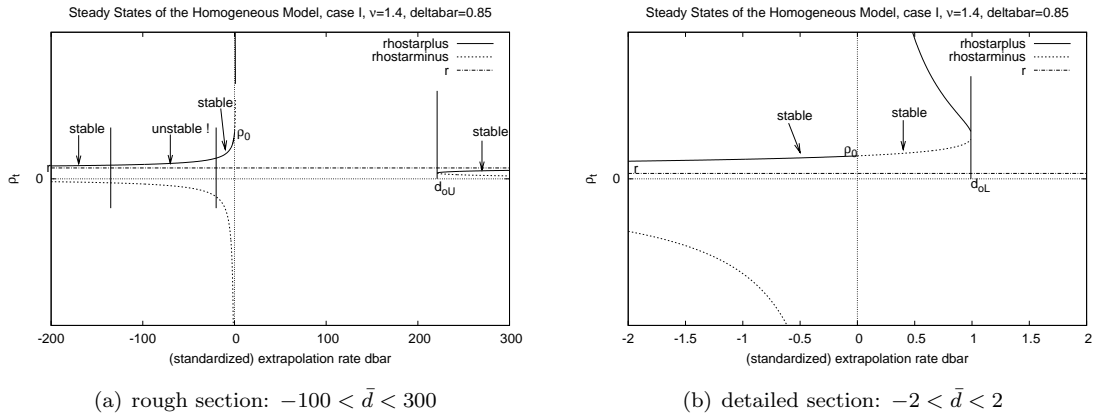
For  $d \neq 0$ , investors are called chartists. When all investors act as contrarians (i.e.,  $\bar{d} < 0$ , while for trend chasers  $\bar{d} > 0$ ) they believe that the difference of excess conditional mean and the parameter  $\delta$  which is  $[E_t(\rho_{t+1}) - r] - \delta$  is negatively (or positively for trend chasers) proportional to the moving average of returns over the last  $L$  time periods. In Figures 3.3, 3.4 and 3.5, the existence of fixed equilibria for the homogeneous model and their stability in terms of the (standardized) extrapolation rate  $\bar{d}$  is depicted for several risk preference levels. In all cases, two ‘non-fundamental steady states’  $\rho_+^*$  and  $\rho_-^*$  exist. We will concentrate on two of the three different cases in the following discussion.

First, if the risk-aversion coefficient  $\nu$  of the homogeneous investors exceeds the parameter  $\bar{\delta}$  (i.e.,  $\nu > \bar{\delta}$ ), the visual appearance of the fixed equilibria and their stability properties coincides with the findings outlined in Chiarella and He (2001). When all investors are contrarians,  $\rho_+^*$  is the only positive steady state below the fundamental steady state  $\rho_0^*$  (for fixed  $\bar{\delta}$ ). When all investors are trend chasers both (non-fundamental) steady states are positive. For small extrapolation rates  $\bar{d} > 0$ , both steady states are above the corresponding fundamental steady state  $\rho_0^*$ , otherwise both



**Figure 3.3:** Steady state distribution of the homogeneous model for  $d \neq 0$  and  $\nu = 1$ . Investors are described by a logarithmic utility function ( $\nu = 1$ ). Two steady states exist. Stable branches of the fixed equilibria are indicated. Transitions of regions of stability and instability are indicated by (solid) vertical lines. The plot replicates the findings in Chiarella and He (2001) and can thus serve as a benchmark for market situations under power utility risk preferences different from  $\nu = 1$ .  $\bar{\delta}$  is fixed and set to 0.85. Realistic values for the risk free rate  $r$  (3.7% p.a.) and the dividend yield  $\alpha_0$  (4.7% p.a.) are chosen to generate these plots.

are below  $\rho_0^*$ . Figures 3.3 and 3.4 indicate these findings. The left panels show a rough section of

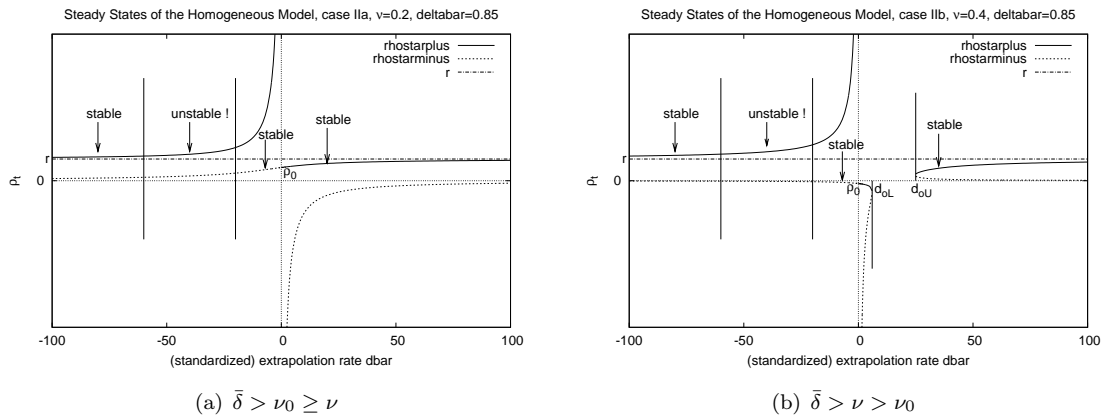


**Figure 3.4:** Steady state distribution of the homogeneous model for  $d \neq 0$  and  $\nu > \bar{\delta}$ . Stable branches of the fixed equilibria are indicated. The visual appearance of the steady states and their stability regions coincide with the visual appearance of steady states and the stability regions in the case of logarithmic preferences (see Figure 3.3). In this plot  $\nu = 1.4$  and  $\bar{\delta} = 0.85$  is chosen. Realistic values for the risk free rate  $r$  (3.7% p.a.) and the dividend yield  $\alpha_0$  (4.7% p.a.) are chosen to generate these plots.

the extrapolation rate  $\bar{d}$ . Particularly, the upper boundary  $d_{oU}$  for (positive) large extrapolation rates is emphasized. The right panels concentrate on the region of small extrapolation rates. They exhibit the lower boundary  $d_{oL}$ .<sup>27</sup>

<sup>27</sup>The condition Eq. (3.32) is in general much more restrictive than the boundaries  $d_{oL,i}, d_{oU,i}, i = 1, 2, 3$  given in Theorem 3.4. Therefore, the actual boundaries denoted by  $d_{oL}$  and  $d_{oU}$  in Figures 3.3–3.5 do not necessarily coincide with the primary boundaries  $d_{oL,i}, d_{oU,i}, i = 1, 2, 3$  defined in Theorem 3.4.

Second, if the risk-aversion coefficient  $\nu$  is smaller than the parameter  $\bar{\delta}$  (i.e.,  $\bar{\delta} > \nu > 0$ ), the visual appearance of the non-fundamental steady states differs from the findings in Chiarella and He (2001). Figure 3.5 relates to this case. Interestingly, positive (non-fundamental) steady states occur even in the case where the corresponding fundamental steady state  $\rho_0^*$  becomes negative (see Figure 3.5, right panel). Again,  $\rho_+^*$  is above the risk-free rate  $r$  for negative extrapolation rates  $\bar{d}$  (as in the latter case above). Two subcases have to be distinguished. For small risk-aversion coefficients (i.e.,  $\nu < \nu_0 \equiv \bar{\delta}/(1 + \alpha_0/r)$ ) both non-fundamental steady states  $\rho_{\pm}^*$  exist for any choice of extrapolation rates  $\bar{d}$ . In addition,  $\rho_+^*$  exceeds the corresponding fundamental steady state  $\rho_0^*$  for all  $\bar{d}$ . For larger risk-aversion coefficients (i.e.,  $\bar{\delta} > \nu > \nu_0$ ) the visual appearance of the steady states slightly resembles the findings for large risk-aversion coefficients  $\nu > \bar{\delta}$  (first case), although notable differences occur for (positive) small extrapolation rates as the corresponding fundamental steady state becomes negative.



**Figure 3.5:** Steady state distribution of the homogeneous model for  $d \neq 0$  and  $\bar{\delta} > \nu > 0$ . Due to the change of sign of the fundamental steady state in this parameter region, two subcases (small  $\nu$ , left panel and relatively large  $\nu$ , right panel) have to be distinguished. The visual appearance of fixed equilibria differs from the findings exhibited in Figures 3.3 and 3.4 in both subcases. Nevertheless, relatively high (non-fundamental) fixed equilibrium return can again be detected when investors extrapolate as contrarians. As in the benchmark model, the stability of the higher fixed equilibrium is only guaranteed for low and very high extrapolation rates in both subcases. Transitions of regions of stability and instability are indicated by (solid) vertical lines. In this plot  $\nu = 0.2$  and  $\bar{\delta} = 0.85$  (left panel) and  $\nu = 0.4$  and  $\bar{\delta} = 0.85$  (right panel) are chosen. Realistic values for the risk free rate  $r$  (3.7% p.a.) and the dividend yield  $\alpha_0$  (4.7% p.a.) are chosen to generate these plots.

The third case, where all investors are risk lovers, is neglected in the discussion as it does not conform with the generally accepted assumption of risk-averse investors.

Before I investigate the local stability of the steady states, one final remark should be made. It should again be stressed that I have not distinguished between realistic and unrealistic market situations in the previous discussion. Obviously, whenever the steady-state return falls below the risk free rate  $r$ , an unrealistic market situation occurs which eventually leads to non-positive prices and/or unrealistic optimal investment proportions.

3.4.3 Local Stability of the Steady States

The local stability of the steady states in the homogeneous model is determined by the corresponding characteristic equation. It depends on the lag length  $L$ , the risk premium  $\delta$ , the extrapolation rate  $d$ , the variance  $\sigma^2$  and, in contrast to the CH model, on the risk-aversion coefficient  $\nu$ . We find a similar result to that in Chiarella and He (2001):

**Lemma 3.5** *Let  $\bar{\delta} = \delta/\sigma^2$ ,  $\bar{d} = d/\sigma^2$ . Assume Eq. (3.27) and Eq. (3.28) hold and  $\alpha_t = \alpha_0 > 0$ ,  $\bar{\delta} < 1$  and  $\nu \neq 0$ . Then the fixed equilibrium return of the homogeneous system Eq. (3.22), say  $x_0$ , is locally asymptotically stable if and only if all the eigenvalues  $\lambda_i$  of the polynomial*

$$\Gamma_L(\lambda) \equiv \lambda^{L+1} - \frac{\bar{d}}{\nu L} [(C - A)\lambda^L + C\lambda^{L-1} + C\lambda^{L-2} + \dots + C\lambda + A] \tag{3.33}$$

satisfy  $|\lambda_i| < 1$  for  $i = 1, \dots, L + 1$ , in which

$$A = \frac{1+r}{\pi_0(\pi_0 - 1)}, \quad C = \frac{\alpha_0}{(\pi_0 - 1)^2}, \quad \pi_0 = \frac{1}{\nu}(\bar{\delta} + \bar{d}x_0).$$

PROOF: See Appendix A 3.4.

For realistic time lags (e.g.,  $L \geq 3$ ), the roots of the characteristic equation Eq. (3.33) cannot be derived analytically. Instead, a numerical approach has to be used to investigate the stability properties of the fixed equilibria in the homogeneous case. On a visual level the stability regions of the non-fundamental steady states have already been highlighted in Figures 3.3–3.5 as they have been obtained by numerical simulations. Otherwise, even in the homogeneous case a rigorous bifurcation analysis with respect to the extrapolation parameter  $\bar{d}$  is already a quite extensive exercise and, moreover, does not give significantly new qualitative insights into the behavior of the dynamic system. Nevertheless, it is evident that the results outlined in the previous subsection can, in principle, be identified with typical forms of bifurcations (see Appendix A 2.11 for an overview). Let us simply consider the case  $\nu > \bar{\delta}$  in detail as an example (see Figures 3.3 and 3.4). In this case a numerical analysis shows that, in the region of negative large values of  $\bar{d}$ , the dynamic system undergoes a supercritical Hopf bifurcation, i.e., a pair of complex conjugate eigenvalues of the corresponding Jacobian evaluated at the non-fundamental steady state  $\rho_+^*$  has modulus one, whereas for negative sufficiently small values of  $\bar{d}$  a subcritical flip bifurcation occurs.<sup>28</sup> For positive parameter values of  $\bar{d}$  the opposite is true. Hence in the region of small extrapolation parameters the system undergoes a supercritical Hopf bifurcation if  $\bar{d}$  becomes sufficiently large, while the  $\rho_+^*$  becomes stable again under a subcritical flip bifurcation for sufficiently large values of  $\bar{d}$ . Of course, similar results can be obtained for the second case, i.e.,  $\bar{\delta} > \nu > 0$  (see Figure 3.5).<sup>29</sup>

However, for the fundamental steady states (i.e.,  $d = 0$ ) and for the very restrictive case  $d \neq 0$ ,  $L = 1$ , the following theoretical result on the local stability of the homogeneous model can be

<sup>28</sup>The numerical results are obtained by using the algebraic manipulation software MATHEMATICA 4.1.

<sup>29</sup>For a meaningful economic interpretation the (non-fundamental) steady states also have to be a real number, which constitutes another constraint to the existence of the steady states. For this reason the bifurcation values obtained by a numerical analysis do not necessarily coincide with the lower and upper boundaries  $d_{oL}$  and  $d_{oU}$  given in Theorem 3.4 in all cases. Therefore, the (solid) vertical lines in Figures 3.4–3.5 only approximately indicate the transition from a stable region to an unstable one.

obtained directly from the previous Lemma 3.5:

**Corollary 3.6** *Under the assumption of Theorem 3.4,*

(i) *if  $d = 0$ , then the unique steady state  $\rho_0^*$  is locally asymptotically stable;*

(ii) *if  $d \neq 0$  and  $L = 1$ , then for  $\nu > 0$  the steady state, say  $x_0$  is locally asymptotically stable if*

$$-\bar{d}A < 0, \quad \bar{d}C < \nu, \quad \bar{d}(2A - C) < \nu \tag{3.34}$$

*and for  $\nu < 0$  the steady state, say  $x_0$  is locally asymptotically stable if*

$$\bar{d}A < 0, \quad -\bar{d}C < -\nu, \quad \bar{d}(C - 2A) < -\nu. \tag{3.35}$$

PROOF: See Appendix A 3.5.

Corollary 3.6 states that the fundamental steady state (i.e., no extrapolation) is locally asymptotically stable for all choices of the lag length  $L$  and the risk-aversion coefficient  $\nu$ . For non-zero extrapolation rates, several (sufficient) conditions on the risk-aversion coefficient  $\nu$  have to be fulfilled for the steady state being locally asymptotically stable. Corollary 3.6 again seems to simply enrich the findings of Chiarella and He by one additional parameter, namely  $\nu$ . However, a closer study of the conditions in Eq. (3.34) and Eq. (3.35) indicates that this is not the case since they are drastically more complex, because the parameters  $A, B, C$  themselves do depend on the risk-aversion coefficient  $\nu$  via the parameter  $\pi_0$ . Thus we can conclude that, in contrast to the findings in Chiarella and He (2001), a theoretical analysis of the stability of steady states becomes analytical intractable even in the very special and restricted cases.

### 3.5 The Model under Heterogeneous Expectations: A Numerical Treatment

In the last section the robustness of the homogeneous stationary CH model with respect to the investors' risk preference was investigated. In this section a straightforward generalization of the homogeneous case, the heterogeneous model with only two different investors in the market, will be considered, this representing also the most simple case. Since a theoretical treatment of the model could only be managed with quite complex and exhausting derivations, even in the homogeneous case, the two-investors model will be investigated by using solely numerical simulations. However, for logarithmic utility preferences, theoretical results of the heterogeneous model are derived in the original publication of the CH model.<sup>30</sup> Consequently, they will serve as benchmark results for my own numerically based investigations.

Analogous to Eq. (3.27) and Eq. (3.28) in the homogeneous model, the trading strategies of the two investors are assumed to be

$$E_{i,t}(\rho_{t+1}) = r + \delta_i + d_i \bar{\rho}_{i,t}, \tag{3.36}$$

$$Var_{i,t}(\rho_{t+1}) = \sigma^2, \tag{3.37}$$

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<sup>30</sup>The following discussion about the heterogeneous CH model under power utility preferences is closely related to the discussion outlined in Chiarella and He (2001).

where  $\delta_i \geq 0$  and  $\sigma^2 > 0$  are constants,  $d_i \in \mathbb{R}$  and

$$\bar{\rho}_{i,t} = \frac{1}{L_i} \sum_{k=1}^{L_i} \rho_{t-k}$$

with the integers  $L_i \geq 1$ ,  $i = 1, 2$ . Thus the investors may differ from each other with respect to their extrapolation rate  $d_i$  for the moving average of the last  $L_i$  returns of the risky asset, the window width  $L_i$  of the moving average procedure and the parameter  $\delta_i$ , which measures the difference between the expected rate of return and the risk-free rate for the fundamentalist strategy. Using the following notations

$$\bar{\delta}_i = \delta_i / \sigma^2, \quad \bar{d}_i = d_i / \sigma^2 \quad i = 1, 2,$$

the optimal investment proportion of investor  $i$  can be expressed according to Eq. (3.26) by

$$\pi_{i,t} = \frac{1}{\nu_i} \frac{E_{i,t}(\rho_{t+1}) - r}{Var_{i,t}(\rho_{t+1})} = \frac{1}{\nu_i} (\bar{\delta}_i + \bar{d}_i \bar{\rho}_{i,t}). \quad (3.38)$$

In the case of two investors it is convenient to introduce a new state variable

$$\omega_t \equiv \omega_{1,t} - \omega_{2,t},$$

which denotes the difference in the wealth proportions of each investor. Note that from the new variable  $\omega_t$  the investors' wealth proportions can still be determined by using the following relations

$$\omega_{1,t} = \frac{1 + \omega_t}{2}, \quad \omega_{2,t} = \frac{1 - \omega_t}{2}.$$

Therefore, the dynamic system of the two heterogeneous investors case can be derived in terms of the state variables  $\omega_{t+1}$  and  $\rho_{t+1}$  by applying Theorem 3.1 as

$$\begin{aligned} \omega_{t+1} &= \omega_{1,t+1} - \omega_{2,t+1} \\ &= \frac{\omega_{1,t}[R + \pi_{1,t}(\rho_{t+1} - r)] - \omega_{2,t}[R + \pi_{2,t}(\rho_{t+1} - r)]}{\sum_{j=1}^2 \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)]} \\ &= \frac{\frac{1+\omega_t}{2}[R + \pi_{1,t}(\rho_{t+1} - r)] - \frac{1-\omega_t}{2}[R + \pi_{2,t}(\rho_{t+1} - r)]}{\frac{1+\omega_t}{2}[R + \pi_{1,t}(\rho_{t+1} - r)] + \frac{1-\omega_t}{2}[R + \pi_{2,t}(\rho_{t+1} - r)]} \end{aligned}$$

and

$$\begin{aligned} \rho_{t+1} &= r + \frac{\sum_{i=1}^2 \omega_{i,t}[(1+r)(\pi_{i,t} - \pi_{i,t+1}) - \alpha_{t+1}\pi_{i,t}]}{\sum_{i=1}^2 \pi_{i,t}\omega_{i,t}(\pi_{i,t+1} - 1)} \\ &= r + \frac{\frac{1+\omega_t}{2}[(1+r)(\pi_{1,t} - \pi_{1,t+1}) - \alpha_{t+1}\pi_{1,t}] + \frac{1-\omega_t}{2}[(1+r)(\pi_{2,t} - \pi_{2,t+1}) - \alpha_{t+1}\pi_{2,t}]}{\frac{1+\omega_t}{2}\pi_{1,t}(\pi_{1,t+1} - 1) + \frac{1-\omega_t}{2}\pi_{2,t}(\pi_{2,t+1} - 1)} \end{aligned}$$

respectively. In sum, the dynamic system  $(\omega_{t+1}, \rho_{t+1})$  is thus given by

$$\omega_{t+1} = \frac{(1 + \omega_t)[R + \pi_{1,t}(\rho_{t+1} - r)] - (1 - \omega_t)[R + \pi_{2,t}(\rho_{t+1} - r)]}{(1 + \omega_t)[R + \pi_{1,t}(\rho_{t+1} - r)] + (1 - \omega_t)[R + \pi_{2,t}(\rho_{t+1} - r)]} \quad (3.39)$$

$$\rho_{t+1} = r + \frac{(1 + \omega_t)[R(\pi_{1,t} - \pi_{1,t+1}) - \alpha_{t+1}\pi_{1,t}] + (1 - \omega_t)[R(\pi_{2,t} - \pi_{2,t+1}) - \alpha_{t+1}\pi_{2,t}]}{(1 + \omega_t)\pi_{1,t}(\pi_{1,t+1} - 1) + (1 - \omega_t)\pi_{2,t}(\pi_{2,t+1} - 1)}, \quad (3.40)$$

where  $R = (1 + r)$  and  $\pi_{i,t}$  is given by Eq. (3.38) for  $i = 1, 2$ . Since the optimal investment proportions  $\pi_{i,t}$  and  $\pi_{i,t-1}$  are functions of the lagged values of returns, the dynamic system Eq. (3.39) and Eq. (3.40) is a nonlinear system of order  $\max(L_1, L_2) + 2$ . This means that the dynamic behavior of the system will become rather complex for realistic time lags (e.g.,  $L_i > 1$ ). However, before I provide a theoretical result for the stability of the heterogeneous model's steady states, the following remark confines the possible (realistic) fixed equilibria  $(\omega^*, \rho^*)$ , at least with respect to the possible outcomes of the state variable  $\omega_t$  in its steady-state level:

**Remark 3.7** *Assume Eq. (3.36) and Eq. (3.37) hold. Let the dynamics of  $(\omega_{t+1}, \rho_{t+1})$  be determined by Eq. (3.39) and Eq. (3.40) with  $\pi_{1,t} \neq \pi_{2,t}$  for all  $t$ . Let  $\alpha_t$  be a positive constant  $\alpha_0$ . Then all steady states of the dynamic system are given by*

$$(\omega^* = \pm 1, \rho^*) \quad \text{if and only if} \quad \rho^* \neq r .$$

*Otherwise, if  $\rho^* = r$  is a fixed point of the system, then Eq. (3.39) is valid for any arbitrary value of  $\omega^*$ . Nevertheless, according to Eq. (3.40), the fixed point  $\omega^*$  has to fulfill the condition*

$$\omega^* = -\frac{\pi_1^* + \pi_2^*}{\pi_1^* - \pi_2^*} , \tag{3.41}$$

where  $\pi_i^*$  is given by

$$\pi_i^* = \frac{1}{\nu_i} (\bar{\delta}_i + \bar{d}_i r) \quad i = 1, 2 .$$

PROOF: See Appendix A 3.6.

The first part of Remark 3.7 states that, in the case of heterogeneous investors (i.e., when the optimal investment proportion  $\pi_{i,t}$  of the two investors are different), the wealth proportion of investors  $\omega_{i,t}$  either converges to 1 or to 0 in any steady-state situation unless the return of the risky asset converges to the degenerated steady state  $\rho^* = r$ . In other words, if the dynamic system  $(\omega_{t+1}, \rho_{t+1})$  settles down to a stable equilibrium, the total wealth of the market is passed over to one of the investors.<sup>31</sup> In terms of the model behavior this result can be interpreted in a slightly different way. The heterogeneous model does not provide stable market situations in which both investors can survive in the long run. Once one investor dominates the market (measured by his/her wealth proportion  $\omega_{i,t}$ ), there is no counterforce implemented in the model which can slow down the ruin of the other investor. The second part of the Remark 3.7 is relevant only because the risk-free rate can, in principle, also become a steady state in the heterogeneous model. Of course, this situation is an unrealistic market scenario since a risk-averse investor should gain a higher return from a risky investment in a long-run steady-state situation than from the risk-free asset. Furthermore, the condition  $\omega^* \in [-1, 1]$  might be violated in such scenarios, which means that the wealth proportions of the two investors can become smaller than 0 or larger than 1. These are unrealistic model outcomes. Nevertheless, the dynamic system Eq. (3.39) and Eq. (3.40) can mathematically converge to such a degenerated steady state and actually *does*, as can be observed when conducting numerical simulations. I will refer to this point in more

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<sup>31</sup>The condition  $\pi_{1,t} \neq \pi_{2,t}$  is equivalent to  $(\delta_1 - \delta_2) + (d_1 \bar{\rho}_{1,t} - d_2 \bar{\rho}_{2,t}) \neq 0$ , which is essentially fulfilled if the investors differ from their extrapolation rates  $d_1$  and  $d_2$  and/or their choice of the parameter value  $\delta_1$  and  $\delta_2$ .

detail when such market scenarios occur in the following discussion, which is outlined in the next subsections.

According to Remark 3.7, the investigation of the local stability of the heterogeneous model's steady states can be restricted to cases with  $\omega^* = \pm 1$  for realistic market situations. Analogous to the homogeneous model (see Lemma 3.5), the condition for the local stability of steady states can be given by the characteristic equation for the linear approximation of the underlying dynamic system as stated in the following Lemma 3.8:<sup>32</sup>

**Lemma 3.8** *Assume Eq. (3.36) and Eq. (3.37) hold. Let  $L = \max(L_1, L_2)$  and  $E(w^*, \rho^*)$  be a fixed equilibrium of the system Eq. (3.39) and Eq. (3.40) with  $\omega^* = \pm 1$ . Then the characteristic polynomial for the linearized system of Eq. (3.39) and Eq. (3.40) at  $E(w^*, \rho^*)$  is given by*

(i)  $\Gamma_{L_1}(\lambda)$  if  $\omega^* = 1$  and

(ii)  $\Gamma_{L_2}(\lambda)$  if  $\omega^* = -1$ ,

where

$$\Gamma_{L_i}(\lambda) \equiv (\lambda - \beta_i)\lambda^{L-L_i} \left\{ \lambda^{L_i+1} - \frac{\bar{d}_i}{\nu_i L_i} [(C_i - A_i)\lambda^{L_i} + C_i\lambda^{L_i-1} + C_i\lambda^{L_i-2} + \dots + C_i\lambda + A_i] \right\} \quad (3.42)$$

and

$$A_i = \frac{1+r}{\pi_{i,0}(\pi_{i,0}-1)}, \quad C_i = \frac{\alpha_0}{(\pi_{i,0}-1)^2}, \quad (3.43)$$

$$\pi_{i,0} = \frac{1}{\nu_i} (\bar{\delta}_i + \bar{d}_i \rho^*), \quad (3.44)$$

$$\beta_1 = \frac{(1+r) + (\rho^* - r)\pi_{2,0}}{(1+r) + (\rho^* - r)\pi_{1,0}}, \quad \beta_2 = \frac{1}{\beta_1} \quad (3.45)$$

for  $i = 1, 2$ .

PROOF: See Appendix A 3.7.

Compared to the original publication, the consideration of the investors' risk-aversion coefficients  $\nu_i$  again does not significantly change the stability condition for the steady states from a visual perspective. The structure of the characteristic equations  $\Gamma_{L_i}(\lambda)$ ,  $i = 1, 2$  is still almost the same as in the case of logarithmic risk preferences assumed in Chiarella and He (2001). Nevertheless, the risk-aversion coefficient  $\nu_i$  *does* influence the stability behavior of the system Eq. (3.39) and Eq. (3.40) since the characteristic equation Eq. (3.42) depends on  $\nu_i$  for any parameter choice. Note that, even in the case where  $d_i = 0$ , i.e., investor  $i$  is assumed to behave as a fundamentalist, the characteristic equation  $\Gamma_{L_i}(\lambda)$  is a function of  $\nu_i$  because  $\beta_i$  depends on  $\nu_i$  via the expression of  $\pi_{i,0}$ .

As already mentioned in the homogeneous case, the polynomial cannot easily be solved analytically for time lags  $L_i$  larger than 1. Therefore, it is generally impossible to obtain explicit

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<sup>32</sup>Remember the general result which states that a system is locally asymptotically stable if and only if its eigenvalues lie inside of the unit circle, equivalent to the conclusion which states that all the roots of the corresponding characteristic equation lie inside of the unit circle.



stability conditions. Instead, numerical tools must be used to investigate the stability region of the dynamic system. Nevertheless, it is interesting to note that, apart from the higher dimension of the resulting characteristic equation, the component of the polynomial in the curly brackets has the same analytical form as the characteristic equation Eq. (3.33) given in Lemma 3.5 in the homogeneous case. In other words, the eigenvalues of the heterogeneous model mirror the eigenvalues of the corresponding homogeneous model (where both investors follow the same trading strategy  $i$ ) and also possesses the eigenvalue  $\beta_i$  and  $(L - L_i)$  eigenvalues equal to zero.

In the following subsections four different market scenarios with two different investors will be investigated: two fundamentalists with different risk-aversion coefficients, two fundamentalists with different choices of the parameter  $\delta_i$ , fundamentalist versus trend trader and a two trend-trader scenario. In all model variants a sensitivity analysis with respect to the risk-aversion coefficients chosen by the investors will be performed, which provides an insight into the qualitative robustness of the findings from the original CH model.

### 3.5.1 Fundamentalists with Different Risk Aversion Coefficients

Fundamentalists believe that the conditional expected return of the risky asset is a composition of the risk-free rate  $r$  and the conditional excess return from the risk-free rate measured by the parameter  $\delta_i$ . In this subsection it is assumed that both traders are fundamentalists with the same view of the conditional excess return from the risk-free rate, i.e.,  $\delta_1 = \delta_2 \equiv \delta$ , but who are characterized by different attitudes towards the inherent risk in the market, specified by different risk-aversion coefficients  $\nu_i$ ,  $i = 1, 2$ . Thus the conditional expectations about mean and variance of the risky asset are identical for both investors and simply given by

$$E_{i,t}(\rho_{t+1}) = r + \delta \tag{3.46}$$

$$Var_{i,t}(\rho_{t+1}) = \sigma^2 \quad i = 1, 2, \tag{3.47}$$

where  $\delta \geq 0$  and  $r, \sigma^2 > 0$  are constants. The heterogeneity of the two investors therefore becomes apparent by their optimal investment proportions  $\pi_{i,t}$ , which are different for both traders

$$\pi_{i,t} = \frac{r + \delta - r}{\nu_i \sigma^2} = \frac{\bar{\delta}}{\nu_i}$$

with  $\nu_i > 0$ .<sup>33</sup> Under these assumptions, it can be shown that two steady states exist, in which, according to the previous Remark 3.7, the total wealth of the market is concentrated on either of the single investors. Furthermore, the lower the risk-aversion coefficient  $\nu_i$  chosen by investor  $i$ , the higher the (fundamental) steady-state return  $\rho_{0,i}^*$ . The findings are summarized in the following theorem:

**Theorem 3.9** *Assume Eq. (3.46) and Eq. (3.47) hold and  $\alpha_t = \alpha_0 > 0$ ,  $\bar{\delta} < 1$  and  $\nu_i > 0$  for  $i = 1, 2$ . Furthermore, neglect the case  $\rho^* = r$ . The system Eq. (3.39) and Eq. (3.40) has two steady states  $E_i(\omega_i^*, \rho_{0,i}^*)$  for  $i = 1, 2$  with*

$$\omega_1^* = 1, \quad \omega_2^* = -1,$$

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<sup>33</sup>In the following discussion I restrict the risk-aversion coefficient to being positive, which is in harmony with the assumption of risk-averse market participants. Otherwise, the optimal demand function of the investor would become negative, which could lead to non-positive prices  $p_t$  for the risky asset according to Eq. (3.9).

and

$$\rho_{0,i}^* = r + \frac{\alpha_0}{1 - \bar{\delta}/\nu_i}. \tag{3.48}$$

If  $\nu_1 = \nu_2$ , then the system reduces to the homogeneous case with a steady state  $E_3(\omega_3^*, \rho_3^*)$ , where  $\rho_3^*$  is given by Eq. (3.30) and  $\omega_3^* = \omega_{1,0} - \omega_{2,0}$  (which is the difference in the initial wealth proportions).

PROOF: See Appendix A 3.8.

Applying Lemma 3.8 to the model of two fundamentalists with different risk-aversion coefficients but homogeneous perceptions of the conditional excess return from the risk-free rate, the following result for the stability behavior of the dynamic system Eq. (3.39) and Eq. (3.40) can be obtained:

**Corollary 3.10** Define  $\epsilon \equiv \alpha_0/(1+r)$  and  $\nu_{o,2} \equiv -(\alpha_0\bar{\delta}\nu_1)/[2(1+r)(\nu_1 - \bar{\delta}) + \alpha_0\bar{\delta}]$ . Furthermore, let  $\nu_1 \notin [\bar{\delta}(1 - \epsilon), \bar{\delta}]$ . Under the assumption of Theorem 3.9, a necessary and sufficient condition for  $E_1$  to be locally asymptotically stable is

$$(i) \quad \nu_1 < \nu_2 \quad (\text{which is equivalent to } \rho_{0,1}^* > \rho_{0,2}^*) \quad \text{for} \quad \nu_i > \bar{\delta} > 0, \quad i = 1, 2$$

and

$$(ii) \quad \left\{ \begin{array}{l} \nu_1 > \nu_2 \quad (\text{which is equivalent to } \rho_{0,1}^* < \rho_{0,2}^*) \\ \wedge \quad \nu_2 > \nu_{o,2} > 0 \end{array} \right\} \quad \text{for} \quad \bar{\delta} > \nu_i > 0, \quad i = 1, 2.$$

Otherwise, if

$$(iii) \quad \nu_{i_1} > \bar{\delta} > \nu_{i_2} > 0, \quad i_j = 1, 2, j = 1, 2, i \neq j,$$

then  $E_1$  is not locally asymptotically stable (and nor is  $E_2$ ).

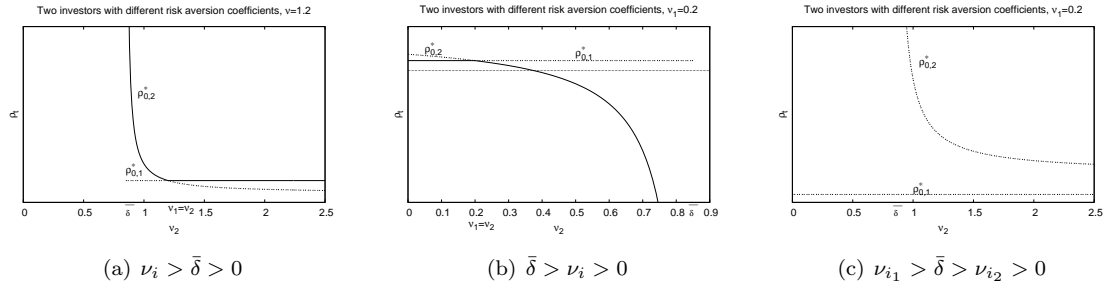
PROOF: See Appendix A 3.9.

Corollary 3.10 distinguishes between three different market scenarios. In Case (i) which is the simplest case, the risk-aversion coefficients of both investors are higher than the parameter  $\bar{\delta}$ , which is measured per unit of variance. In this case Investor 1 prevails over Investor 2 if and only if he/she chooses a lower risk-aversion coefficient than his/her counterpart, which, in turn, means that he/she chooses a higher investment proportion  $\pi_i$  for the risky asset in the long run. Note that the resulting fixed equilibrium  $\rho_{0,1}$  is then also higher than the steady state  $\rho_{0,2}$ . This result can be described as a ‘market selection principle’, which states that the investor with the lower risk-aversion coefficient will be the winner, which coincides with an equilibrium  $E_1$  in which the long-term return of the market is higher than in the other existing steady state  $E_2$ .<sup>34</sup> Otherwise,

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<sup>34</sup>A different term was originally used for this result in Chiarella and He (2001), where the authors named it ‘optimal selection principle’. However, the term ‘market selection principle’, which is well established and is mainly identified with Blume and Easley (1992), might be slightly more appropriate since Blume and Easley (1992) show that fitter investment rules survive in the market at the expense of the less fit rules, where the fitness criterion is expected growth rates of wealth share accumulation. Thus market forces favor the survival of investors whose decisions are more optimal than those of other investors. Because of a certain similarity between this statement and our findings, this term is borrowed for our modified CH model version. For further publications that build on the paper by Blume and Easley (1992), see Evstigneev, Hens and Schenk-Hoppé (2002) and Hens and Schenk-Hoppé (2005).

if the market return  $\rho_{0,2}$  of  $E_2$  becomes higher than  $\rho_{0,1}$  (e.g., by leaving  $\nu_1$  fixed and decreasing the parameter  $\nu_2$ ), the situation changes. Now  $E_2$  becomes locally asymptotically stable. Figure 3.6 illustrates this phenomenon for a fixed  $\nu_1$  (left panel).



**Figure 3.6:** Stability properties of the two-fundamentalist model with different risk-aversion coefficients. The solid curve indicates a locally asymptotically stable equilibrium and the dotted curve indicates an unstable steady state. In the left panel the ‘market selection principle’ holds. The middle and the right panel exhibits unrealistic market scenarios. In all three cases  $\nu_1$  is fixed and set to 1.2 (left panel) and 0.2 (middle and right panel). Furthermore, it is  $\bar{\delta} = 0.85$ .

If both risk-aversion coefficients are lower than  $\bar{\delta}$ , the ‘market selection principle’ no longer holds. The fixed equilibrium  $E_1$  now becomes locally asymptotically stable if and only if the corresponding return  $\rho_{0,1}$  is lower than the return of the second steady state  $E_2$  and vice versa (middle panel). Furthermore, these long-run returns are always below the risk-free rate  $r$  or even become negative, as already mentioned in the discussion of the homogeneous case. Therefore, it also describes an unrealistic market situation. Nevertheless, note that the conditions for  $\nu_1$  and  $\nu_2$  are rather unrestrictive in this case. They simply define a parameter space in which the result can be proved in a mathematically correct way while still giving a realistic range to the parameter values. To illustrate this fact we can consider the following example. Assume that the parameters are

$$r = 0.037 \text{ (i.e., 3.7\% p.a.)}, \quad \alpha_0 = 0.047 \text{ (i.e., 4.7\% p.a.)}, \quad \bar{\delta} = 0.85,$$

which is of a realistic magnitude on an annual basis.<sup>35</sup> It then follows that  $\epsilon = 0.045$ .<sup>36</sup> According to the assumptions in Corollary 3.10,  $\nu_1$  has to be smaller than  $(1 - 0.045) \cdot 0.85 = 0.81$ , which does not dramatically restrict the choice of  $\nu_1$ . It only states that  $\nu_1$  must not be chosen very close to the fixed exogenous parameter  $\bar{\delta}$ . For instance, let  $\nu_1 = 0.2$  so that  $\rho_{0,1}$  is (at least) positive. Then  $\nu_{0,2} = 0.006$ , which means in principle, that the lower boundary for  $\nu_2$  coincides with the overall restriction  $\nu_2 > 0$ .

Finally, if one of the risk-aversion coefficients is above  $\bar{\delta}$  and the other one below  $\bar{\delta}$ , neither  $E_1$  nor  $E_2$  is locally asymptotically stable for any market situation (right panel). Instead, numerical simulation shows that the dynamic system Eq. (3.39) and Eq. (3.40) converges to the risk-free rate  $r$  and to  $\omega^*$  given as proposed in Eq. (3.41). Note that this case is explicitly excluded in Theorem

<sup>35</sup>Note that  $\bar{\delta} = \delta/\sigma^2 = 0.85$  can be split, for instance into  $\delta = 0.034$  and  $\sigma = 0.2$ , which is a quite realistic assumption on an annual basis. Similar splits can always be performed whenever certain parameter values are chosen for the parameter  $\bar{\delta}$  (and, analogously, for the parameter  $\bar{d}$ ) in the following.

<sup>36</sup>Remember that  $\epsilon$  is defined in Corollary 3.10 as  $\alpha_0/(1+r)$ .

3.9, because it would again lead to an unrealistic market situation.<sup>37</sup> Nevertheless, simulation results confirm that this fixed equilibrium seems to be locally asymptotically stable under the assumptions given in Case (iii).

In conclusion, this subsection shows that only situations in which the risk-aversion coefficients  $\nu_i$  are larger than  $\bar{\delta}$  lead to realistic market situations. The ‘market selection principle’ then holds and the steady-state return of the risk-aversion coefficient is higher than the risk-free rate  $r$ .

### 3.5.2 Fundamentalists with Different Choices of the Parameter $\delta$

In this subsection we assume that the two fundamentalists differ only in terms of their perception of the conditional excess return of the risky asset from the risk-free rate, which they measure by the parameter  $\delta_i$ . Hence Eq. (3.36) and Eq. (3.37) can be simplified as

$$E_{i,t}(\rho_{t+1}) = r + \delta_i \tag{3.49}$$

$$Var_{i,t}(\rho_{t+1}) = \sigma^2 \quad i = 1, 2, \tag{3.50}$$

where  $\delta \geq 0$  and  $r, \sigma^2 > 0$  are constants. The risk-aversion coefficient is assumed to be the same for both investors. The optimal investment proportions of the two investors are then given by

$$\pi_{i,t} = \frac{\bar{\delta}_i}{\nu}$$

with  $\nu > 0$ . Similar to the market scenario outlined in the last subsection, multiple steady states exist which are summarized in the following theorem:

**Theorem 3.11** *Assume Eq. (3.49) and Eq. (3.50) hold and  $\alpha_t = \alpha_0 > 0$ ,  $\bar{\delta}_i < 1$  and  $\nu > 0$  for  $i = 1, 2$ . Furthermore, neglect the case  $\rho^* = r$ . Then the system Eq. (3.39) and Eq. (3.40) has two steady states  $E_i(\omega_i^*, \rho_{0,i}^*)$  for  $i = 1, 2$  with*

$$\omega_1^* = 1, \quad \omega_2^* = -1,$$

and

$$\rho_{0,i}^* = r + \frac{\alpha_0}{1 - \bar{\delta}_i/\nu}. \tag{3.51}$$

*In particular, if  $\bar{\delta}_1 = \bar{\delta}_2$ , then the system reduces to the homogeneous case with a steady state  $E_3(\omega_3^*, \rho_3^*)$ , where  $\rho_3^*$  is given in Eq. (3.30) and  $\omega_3^* = \omega_{1,0} - \omega_{2,0}$  (which is the difference in the initial wealth proportions).*

PROOF: The proof can be conducted analogous to the proof of Theorem 3.9.

As in the previous subsection, different stability results for different choices of the risk-aversion coefficient  $\nu$  can be obtained according to Lemma 3.8:

**Corollary 3.12** *Define  $\epsilon \equiv \alpha_0/(1+r)$  and  $\bar{\delta}_{o,2} \equiv 1/\epsilon \cdot (\bar{\delta}_1 - \nu)$ . Furthermore, let  $\bar{\delta}_1 \notin [\nu, \nu(1+\epsilon)]$ . Under the assumption of Theorem 3.11, a necessary and sufficient condition for  $E_1$  to be locally asymptotically stable is*

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<sup>37</sup>Some remarks are also given on that discussion in the proof of Theorem 3.9 in Appendix A 3.8.

$$(i) \quad \bar{\delta}_1 > \bar{\delta}_2 \quad (\text{which is equivalent to } \rho_{0,1}^* > \rho_{0,2}^*) \quad \text{for} \quad \nu > \bar{\delta}_i > 0, \quad i = 1, 2$$

and

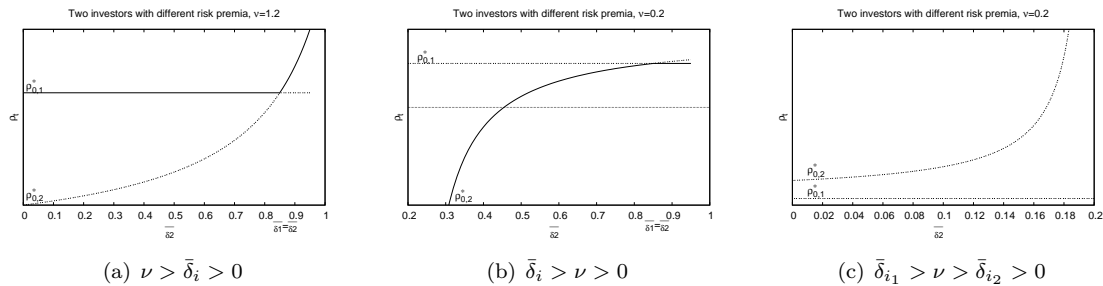
$$(ii) \quad \left\{ \begin{array}{l} \bar{\delta}_1 < \bar{\delta}_2 \quad (\text{which is equivalent to } \rho_{0,1}^* < \rho_{0,2}^*) \\ \wedge \quad \bar{\delta}_2 < \bar{\delta}_{0,2} \end{array} \right\} \quad \text{for} \quad \bar{\delta}_i > \nu > 0, \quad i = 1, 2.$$

Otherwise, if

$$(iii) \quad \bar{\delta}_{i_1} > \nu > \bar{\delta}_{i_2} > 0, \quad i_j = 1, 2, j = 1, 2, i \neq j, \text{ then } E_1 \text{ is not locally asymptotically stable (nor } E_2).$$

PROOF: See Appendix A 3.10.

Corollary 3.12 can be interpreted in a similar way as Corollary 3.10. As in the previous corollary, the constraints on the variables  $\bar{\delta}_1$  and  $\bar{\delta}_2$ , which have to be done in some cases, are not as restrictive.<sup>38</sup> Figure 3.7 illustrates the results for a fixed  $\bar{\delta}_1$  for different values of the risk-aversion coefficient. Whenever the risk-aversion coefficient is higher than the difference between the expected rate of return and the risk-free rate, the ‘market selection principle’ holds. The investor who chooses a higher value for  $\bar{\delta}_i$  will accumulate the total wealth of the market. In this situation the dynamic system converges to that fixed equilibrium in which the steady-state return  $\rho_{0,i}^*$  of the risky asset is higher than its counterpart. This result confirms the findings in Chiarella and He (2001), where the risk-aversion coefficient was restricted to being equal to 1.



**Figure 3.7:** Stability properties of the two-fundamentalist model with different risk premia. The solid curve indicates a locally asymptotically stable equilibrium and the dotted curve indicates an unstable steady state. In the left panel the ‘market selection principle’ holds. The middle and the right panel exhibits unrealistic market scenarios. In all three cases  $\bar{\delta}_1$  is fixed and set to 0.85. Risk-aversion coefficients are chosen to be 1.2 (left panel) and 0.2 (middle and right panel).

Contrastingly, when the risk-aversion coefficient  $\nu$  falls at least below one of the parameters  $\bar{\delta}_i$ , the ‘market selection principle’ is violated. Either the system settles down to steady state which comes along with a relatively low return  $\rho_{0,i}$  ( see Case (ii)) or even neither of the fixed equilibria determined in Theorem 3.11 is locally asymptotically stable (see Case (iii)). Then, as already

<sup>38</sup>For instance, the second condition in Case (ii) basically states that  $\bar{\delta}_2$  must not be chosen too exceedingly far away from the parameter  $\bar{\delta}_1$ . But this can be motivated by the fact that the parameters  $\bar{\delta}_i, i = 1, 2$  are assumed to be determined mainly through an estimation of the risky asset’s past returns. Therefore, the investors’ estimations of this parameter should not deviate dramatically from each other.

observed in the previous subsection, the return  $\rho_t$  of the risky asset converges to the risk-free rate  $r$ , which leads to an degenerated fixed equilibrium of the dynamic system Eq. (3.39) and Eq. (3.40).

So far, it has been shown that the findings on the CH model seem to indicate that the model framework is not robust against the newly introduced parameter, which measures the level of risk aversion in their utility function. If the risk-aversion coefficients of the two fundamentalists (i.e.,  $d_i = 0$ ) is higher than the excess return of the risky asset from the risk-free rate, similar results can be found to those in the original publication. In all other cases the model behavior does not exhibit realistic and desired market outcomes. In other words, the investors' risk aversion has to be sufficiently high in order to generate a realistic level of steady-state returns. If such a situation is guaranteed, the model is robust against changes in the risk-aversion preferences of the investors.

### 3.5.3 Fundamentalist versus Trend Trader

In the following discussion the two investors are allowed to incorporate the history of past observed returns in their trading strategies. More precisely, the trading strategies of the two investors are given by Eq. (3.36) and Eq. (3.37) where  $\bar{d}_i \neq 0$  for at least one  $i = 1, 2$ . While this subsection concentrates on the case with one fundamentalist ( $\bar{d}_1 = 0$ ) and one trend-trader ( $\bar{d}_2 \neq 0$ ), the next subsection deals with the case where both investors are assumed to extrapolate the excess return of the risky asset from the risk-free rate  $r$  by using a moving average over the current past returns. We can see from the previous discussion that analytical results can hardly be obtained for this model variant. Instead, numerical simulation has to be conducted to investigate the behavior of this complex two-investors model. Nevertheless, Remark 3.7 and the findings in the previous subsections have already indicated that (desired) fixed equilibria are characterized by the fact that only one investor survives in the artificial market. Furthermore, it can already be conjectured that, in such steady states, the equilibrium return of the dynamic system is given simply by the corresponding analogous steady state of the homogeneous model, where only the dominating investor acts on the market.<sup>39</sup> Indeed, such results have already been derived in the original publication under the assumption of logarithmic utility preferences for the investors (see Chiarella and He, 2001).

The procedure of the analysis is as follows. Using the theoretical results proposed by Chiarella and He as a benchmark, comprehensive numerical experiments are conducted which involve a systematic analysis of how the local stability of the fixed equilibria is influenced by different levels of the investors' risk aversion. For this purpose, experiments with three different designs are performed for every investor trading strategy combination, which are then enumerated in the following discussion by #1, #2 and #3. As a first experimental setup  $\bar{\delta}_1 = \bar{\delta}_2 = 0.85$  is chosen, which means that both investors are identical in their perception of the constant part of the excess rate of return from the risk-free asset  $r$ . In Experiment #2 and #3 both investors slightly deviate from each other in this respect. While in the second experiment Investor 1 is characterized by a relatively higher choice of the parameter  $\bar{\delta}_1$ , in the third experiment the exact opposite assumption is made. In each experiment  $\bar{d}_1$  is fixed at a representative value, while  $\bar{d}_2$  varies from -4.0 to 0.0

<sup>39</sup>For instance compare Eq. (3.48) and Eq. (3.51) of the heterogeneous model with Eq. (3.30) of the homogeneous model.

(from 0.0 to 4.0) by using steps of with a width of 0.1 in the case of contrarians (trend followers). Each of these runs are performed for 5 different levels of investors' risk aversion (including the benchmark level) for each single experiment. All the results are summarized in Tables 3.1–3.5. For the remainder of this section the parameter set is given by

$$r = 0.037, \quad \alpha_0 = 0.047, \quad \omega_0 = 0.00, \quad L_i = 15 \quad \text{with } i = 1, 2$$

if not otherwise stated.

*Fundamentalist versus Contrarian*

Let us first consider the case of one fundamentalist-trader versus one contrarian-trader. In this case it is assumed that

$$\bar{d}_1 = 0 \quad \text{and} \quad \bar{d}_2 < 0.$$

In accordance with theoretical results of the CH model outlined in Chiarella and He (2001), three fixed equilibria exist<sup>40</sup>, which are given by

$$E_1(\omega_1^*, \rho_1^*), \quad E_{2,\pm}(\omega_2^*, \rho_{2,\pm}^*),$$

where

$$\omega_1^* = 1, \quad \rho_1^* = r + \frac{\alpha_0}{1 - \bar{d}_1/\nu} \tag{3.52}$$

and

$$\omega_2^* = -1, \quad \rho_{2,\pm}^* = \frac{1}{2} \left[ r + \frac{\nu - \bar{d}_2}{\bar{d}_2} \pm \sqrt{\left[ r - \frac{\nu - \bar{d}_2}{\bar{d}_2} \right]^2 - \frac{4\nu\alpha_0}{\bar{d}_2}} \right]. \tag{3.53}$$

Note that, in contrast to Chiarella and He (2001), both the fundamental steady state  $E_1$  as well as the non-fundamental steady states  $E_{2,\pm}$  depend on the risk-aversion coefficient  $\nu$ . Two of them are positive and one of them might be negative.<sup>41</sup> As already mentioned before, they have the same analytical structure as in the homogeneous model, in which only the dominating investor acts in the market. Table 3.1 shows the overall results for this case. Simulation runs in which the ‘market selection principle’ holds are marked by bold type. The results are again in harmony with the earlier findings outlined in the previous subsection and the subsection before that. Whenever the risk-aversion coefficient  $\nu$  exceeds both  $\bar{d}_1$  and  $\bar{d}_2$ , the ‘market selection principle’ holds, i.e.,  $E_1$  ( $E_{2,\pm}$ ) is locally asymptotically stable if and only if  $\rho_1^* \geq \rho_{2,+}^*$  ( $\rho_1^* \leq \rho_{2,+}^*$ ). Then, for Experiment #1 and #2 the fundamental steady state is locally asymptotically stable for any choice of extrapolation rate  $d_2$  of the contrarian within the assumed range of the parameter value. For Experiment #3 the fundamental steady state becomes unstable for relatively low extrapolation rates (measured in absolute values). In this case the risk-aversion coefficient influences the stability property of the system. The lower the risk-aversion coefficient is chosen the smaller the stability region of the (positive) non-fundamental steady state  $E_{2,+}$  will be.

<sup>40</sup>For the sake of precision note that from the theoretical analysis an additional steady state  $E_4(\omega_4^*, \rho_4^*)$  exists under quite restrictive conditions. More precisely, if  $(\nu_2\bar{d}_1/\nu_1 - \bar{d}_2)/\bar{d}_2 = r + \alpha_0/(1 - \bar{d}_1/\nu_1)$  holds,  $E_4$  is given by  $\omega_4^* = \omega_{1,0} - \omega_{2,0}$  (which is the difference in the initial wealth proportions) and  $\rho_4^* = (\nu_2\bar{d}_1/\nu_1 - \bar{d}_2)/\bar{d}_2$ . Since in almost all cases it would lead to a degenerated fixed equilibrium anyway, it will be neglected in the following discussion. For details consult Chiarella and He (2001).

<sup>41</sup>Note that  $\rho_{2,-}^*$  can become positive for certain settings of parameters in this enlarged CH model.

| <b>Fundamentalist vs. Contrarian Model</b> |   |                          |   |   |   |  |
|--|---|--------------------------|---|---|---|--|
|  | Results of the Simulation Experiments             |                          |   |   |   |  |
|  | Experiment #1                                     |                          | Experiment #2                                   |   | Experiment #3                                   |  |
|  | $(\bar{\delta}_1, \bar{\delta}_2) = (0.85, 0.85)$ |                          | $(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.4)$ |   | $(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.6)$ |  |
| risk avers.<br>coeff.                      | $\bar{d}_2 \in$                                   | stable<br>fixed<br>point | $\bar{d}_2 \in$                                 | stable<br>fixed<br>point                          | $\bar{d}_2 \in$                                 | stable<br>fixed<br>point   |
| $\nu = 1.4$                                | $(-4.0, 0)$                                       | <b><math>E_1</math></b>  | $(-4.0, 0)$                                     | <b><math>E_1</math></b>                           | $(-4.0, -1.0]$                                  | <b><math>E_1</math></b><br>$[-0.9, 0)$ <b><math>E_{2,+}</math></b> |
| $\nu = 1$<br>(benchmark)                   | $(-4.0, 0)$                                       | <b><math>E_1</math></b>  | $(-4.0, 0)$                                     | <b><math>E_1</math></b>                           | $(-4.0, -0.8]$                                  | <b><math>E_1</math></b><br>$[-0.7, 0)$ <b><math>E_{2,+}</math></b> |
| $\nu = 0.7$                                | $(-4.0, 0)$                                       | $E_1 < 0^\dagger$        | $(-4.0, 0)$                                     | <b><math>E_1</math></b>                           | $(-4.0, -0.5]$                                  | <b><math>E_1</math></b><br>$[-0.4, 0)$ <b><math>E_{2,+}</math></b> |
| $\nu = 0.4$                                | $(-4.0, 0)$                                       | $E_1 < 0^\dagger$        | $(-4.0, -0.7]$                                  | $E_1 < 0^\dagger$<br>$[-0.6, 0)$ $E_{2,-} < 0^\S$ | $(-4.0, 0)$                                     | $E_1 < 0^\dagger$  |
| $\nu = 0.2$                                | $(-4.0, 0)$                                       | $E_{2,-}^\ddagger$       | $(-4.0, -3.4]$                                  | all unstable<br>$[-3.3, 0)$ $E_{2,-} < 0^\P$      | $(-4.0, 0)$                                     | $E_1^{\dagger\dagger}$   |

$^\dagger$  In this case  $\rho_1^* < \rho_{2,-}^* < 0 < \rho_{2,+}^*$ .  $^\P$  In this case  $\rho_{2,-}^* < 0 < \rho_1^* < \rho_{2,+}^*$ .  
 $^\ddagger$  In this case  $0 < \rho_{2,-}^* < \rho_1^* < \rho_{2,+}^*$ .  $^{\dagger\dagger}$  In this case  $0 < \rho_1^* < \rho_{2,-}^* < \rho_{2,+}^*$ .  
 $^\S$  In this case  $\rho_{2,-}^* < \rho_1^* < 0 < \rho_{2,+}^*$ .

**Table 3.1:** Stability properties of the fundamentalist vs. contrarian model. Three qualitatively different scenarios in terms of the pair  $(\bar{\delta}_1, \bar{\delta}_2)$  are considered. Such cases in which the ‘market selection principle’ holds are indicated by bold type. The steady states which have a negative value are also indicated. Fixed points without any visual highlighting are positive but do not fulfill the ‘market selection principle’.

In all other cases the dynamic system Eq. (3.39) and Eq. (3.40) converges to degenerated fixed equilibria. Sometimes the steady-state return even becomes negative. Otherwise, the steady-state return is lower than the risk-free rate  $r$ . It thus leads to totally unrealistic market situations in any case.

*Fundamentalist versus Trend Follower*

Now we assume that the trend trader extrapolates positively from the current past returns of the risky asset, i.e.,

$$\bar{d}_1 = 0, \quad \text{and} \quad \bar{d}_2 > 0.$$



Then one fundamental steady state  $E_1(\omega_1^*, \rho_1^*)$  and two non-fundamental steady states  $E_2(\omega_2^*, \rho_{2,\pm}^*)$  again exist and are given by Eq. (3.52) and Eq. (3.53) if they do exist.<sup>42</sup> Numerical experiments on stability indicate that the system becomes unstable for relatively high extrapolation rates  $\bar{d}_2$  in almost all cases (see Table 3.2). Furthermore,  $E_{2,+}$  is never locally asymptotically stable as already proposed in Chiarella and He (2001).<sup>43</sup> According to Experiment #3 a lower risk-aversion coefficient seems to destabilize the system in any case if  $\nu > \bar{\delta}_i$ ,  $i = 1, 2$  since the stability region decreases dramatically when lowering the parameter  $\nu$ . Concerning the robustness of the original CH model's findings, this market scenario does not show different results from the previous discussions. Since  $E_{2,+}$  is always unstable, the stability switching follows a '(quasi-)market selection principle', i.e.,  $E_1$  ( $E_{2,-}$ ) is locally asymptotically stable whenever  $\rho_1^* > (<) \rho_{2,-}^*$  for sufficiently high levels of risk aversion. If the risk-aversion coefficient is lower than (at least) one  $\bar{\delta}_i$ , degenerated steady states are reached, as could already be expected. Nevertheless, it is worth mentioning that the '(quasi-)market selection principle' does hold for  $\nu = 0.2$  if  $\bar{\delta}_1 \leq \bar{\delta}_2$ , although in this case the steady-state return does not exceeds the risk-free return  $r$ .

### 3.5.4 Two Trend-Traders

The experiment design for the two trend-trader case is identical to the previous experiments. While the extrapolation rate  $\bar{d}_1 \neq 0$  of Trader 1 is fixed for the whole experiment procedure, the extrapolation rate  $\bar{d}_2$  of Trader 2 varies during the experiments. In general, three different subcases are considered:

$$\bar{d}_i < 0 \text{ for } i = 1, 2, \quad \bar{d}_i > 0 \text{ for } i = 1, 2 \quad \text{and} \quad \bar{d}_1 < 0, \bar{d}_2 > 0.$$

It can be shown that four steady states exist, which are denoted by  $E_i(\omega_i^*, \rho_{i,\pm}^*)$ ,  $i = 1, 2$ .<sup>44</sup> They are given by

$$\omega_1^* = 1, \quad \omega_2^* = -1,$$

$$\rho_{i,\pm} = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}_i}{\bar{d}_i} \pm \sqrt{\left[ r - \frac{\nu - \bar{\delta}_i}{\bar{d}_i} \right]^2 - \frac{4\nu\alpha_0}{\bar{d}_i}} \right].$$

So as not to overburden the reader, I will shorten the discussion as much as possible and refer mainly to the Tables 3.3–3.5 for the detailed results. Therefore, only some selected important facts are stressed in the following subsections.

<sup>42</sup>A further steady state  $E_4(\omega_4^*, \rho_4^*)$  is again neglected in the following discussion for the same reason as in the subsection before. See also Footnote 40 for an explicit expression for  $E_4$ .

<sup>43</sup>More precisely, this is true at least as long as  $\rho_{2,+}^*$  exceeds both  $\rho_1^*$  and  $\rho_{2,-}^*$ .

<sup>44</sup>Chiarella and He (2001) show that, under quite limiting conditions, one further steady state  $E_3(\omega_3^*, \rho_3^*)$  once again exists. More precisely, the steady state  $E_3$  is given by  $\omega_3^* = \omega_{1,0} - \omega_{2,0}$  (which is the difference in the initial wealth proportions) and  $\rho_3^* = \Delta$ , where  $\Delta \equiv (\bar{\delta}_1/\nu_1 - \bar{\delta}_2/\nu_2)/(\bar{d}_2/\nu_2 - \bar{d}_1/\nu_1)$  has to fulfill the condition  $(r - \Delta)(1 - \bar{\delta}_1/\nu_1 - \bar{d}_1\Delta/\nu_1) + \alpha_0 = 0$ . This steady state can again be neglected.

| <b>Fundamentalist vs. Trend-Follower Model</b>    |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| Results of the Simulation Experiments             |   |   |   |   |   |   |
| Experiment #1                                     |   | Experiment #2   |   | Experiment #3                                   |   |   |
| $(\bar{\delta}_1, \bar{\delta}_2) = (0.85, 0.85)$ |   | $(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.4)$   |   | $(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.6)$ |   |   |
| risk avers.<br>coeff.                             | $\bar{d}_2 \in$   | stable<br>fixed<br>point  | $\bar{d}_2 \in$   | stable<br>fixed<br>point                        | $\bar{d}_2 \in$   | stable<br>fixed<br>point  |
| $\nu = 1.4$                                       | (0.0, 0.9] <b><math>E_{2,-}</math></b><br>[1.0, 4.0) all unstable | (0.0, 0.9] <b><math>E_1</math></b><br>[1.0, 2.5] <b><math>E_{2,-}</math></b><br>[2.6, 4.0) all unstable | (0.0, 2.0] <b><math>E_{2,-}</math></b><br>[2.1, 4.0) all unstable   | $\nu = 1$<br>(benchmark)                        | (0.0, 0.1] <b><math>E_{2,-}</math></b><br>[0.2, 4.0) all unstable | (0.0, 0.7] <b><math>E_1</math></b><br>[0.8, 1.5] <b><math>E_{2,-}</math></b><br>[1.6, 4.0) all unstable |
| $\nu = 0.7$                                       | (0.0, 0.1] $E_{2,+} < 0^\dagger$<br>[0.2, 4.0) all unstable       | (0.0, 0.4] <b><math>E_1</math></b><br>{0.5} <b><math>E_{2,-}</math></b><br>[0.6, 4.0) all unstable      | (0.0, 0.07] <b><math>E_{2,-}</math></b><br>[0.08, 4.0) all unstable | $\nu = 0.4$                                     | (0.0, 4.0) $E_{2,+} < 0^\dagger$                                  | (0.0, 0.5] 0.037 <sup>§</sup><br>[0.6, 4.0) all unstable  |
| $\nu = 0.2$                                       | (0.0, 4.0) <b><math>E_1</math></b> ¶                              | (0.0, 1.9] $E_{2,+} < 0^{\dagger\dagger}$<br>[2.0, 4.0) all unstable                                    | (0.0, 4.0) <b><math>E_1</math></b> ¶                                |   |   |   |

<sup>†</sup> In this case  $\rho_{2,-}^* < \rho_{2,+}^* < \rho_1^* < 0$ . ¶ In this case  $\rho_1^* < r$ .  
<sup>‡</sup> In this case  $\rho_{2,-}^* < \rho_1^* < \rho_{2,+}^* < 0$ . †† In this case  $\rho_{2,-}^* < \rho_{2,+}^* < 0 < \rho_1^*$ .  
<sup>§</sup> In this case  $\rho_{2,\pm}^*$  becomes complex.

**Table 3.2:** Stability properties of the fundamentalist vs. trend-follower model. Three qualitatively different scenarios in terms of the pair  $(\bar{\delta}_1, \bar{\delta}_2)$  are considered. Such cases in which the ‘(quasi-)market selection principle’ holds are indicated by bold type. Those steady states which have a negative value are also indicated. Fixed points without any visual highlighting are positive but do not fulfill the ‘(quasi-)market selection principle’.

*Two Contrarians*

For this market scenario we assume that Trader 1 extrapolates moderately. More precisely, it is assumed that

$$\bar{d}_1 = -1, \quad \text{and} \quad \bar{d}_2 \in [-0.4, 0.0).$$

While for less risk-averse investors the dynamic system does not exhibit realistic market behavior, the stability properties for sufficiently high levels of risk aversion are in harmony with the ‘market selection principle’. If the risk-aversion coefficient decreases, unstable regions might be generated at some parameter settings (see Experiment #3 for  $\nu = 0.7$ ). Furthermore, an increase in the

risk-aversion coefficient leads to indifferent results. While for Experiment #3 an increase in  $\nu$  seems to increase the range of parameter values in which the non-fundamental steady state  $E_{2,+}$  is locally asymptotically stable, exactly the opposite is true for Experiment #2. All the results are summarized in Table 3.3.

| Two-Contrarian Model                  |                 |   |  |  |  |   |  |
|---------------------------------------|-----------------|---|--|--|--|---|--|
| Results of the Simulation Experiments |                 |   |  |  |  |   |  |
|                                       |                 | Experiment #1   |  | Experiment #2  |  | Experiment #3   |  |
|                                       |                 | $(\bar{\delta}_1, \bar{\delta}_2) = (0.85, 0.85)$                                       |  | $(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.4)$                              |  | $(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.6)$   |  |
| risk avers.<br>coeff.                 | $\bar{d}_2 \in$ | stable  |  | stable   |  | stable  |  |
|                                       |                 | fixed point   |  | fixed point  |  | fixed point   |  |
| $\nu = 1.4$                           |                 | $(-4.0, -1.0)$ $E_{1,+}$<br>$(-1.0, 0)$ $E_{2,+}$                                       |  | $(-4.0, 0.0)$ $E_{1,+}$  |  | $(-4.0, -2.0)$ $E_{1,+}$<br>$[-1.9, 0.0)$ $E_{2,+}$   |  |
| $\nu = 1$<br>(benchmark)              |                 | $(-4.0, -1.0)$ $E_{1,+}$<br>$(-1.0, 0)$ $E_{2,+}$                                       |  | $(-4.0, -0.2)$ $E_{1,+}$<br>$[-0.1, 0.0)$ $E_{2,+}$                          |  | $(-4.0, -1.9)$ $E_{1,+}$<br>$[-1.8, 0.0)$ $E_{2,+}$   |  |
| $\nu = 0.7$                           |                 | $(-4.0, -1.0)$ $E_{1,-} < 0^\dagger$<br>$(-1.0, 0.0)$ $E_{2,-} < 0^\ddagger$            |  | $(-4.0, -0.3)$ $E_{1,+}$<br>$[-0.2, 0.0)$ $E_{2,+}$                          |  | $(-4.0, -1.8)$ $E_{1,+}$<br>$[-1.7, -1.2]$ all unstable <sup>§</sup><br>$[-1.1, 0)$ $E_{2,+}$ |  |
| $\nu = 0.4$                           |                 | $(-4.0, -1.0)$ $E_{1,-} < 0^\dagger$<br>$(-1.0, 0.0)$ $E_{2,-} < 0^\ddagger$            |  | $(-4.0, -1.2)$ all unstable <sup>§</sup><br>$[-1.1, 0)$ $E_{2,-} < 0^\nabla$ |  | $(-4.0, 0.0)$ $E_{1,-} < 0^\dagger$   |  |
| $\nu = 0.2$                           |                 | $(-4.0, -1.0)$ $E_{2,-}^{\ddagger\ddagger}$<br>$(-1.0, 0.0)$ $E_{1,-}^{\dagger\dagger}$ |  | $(-4.0, -3.3)$ all unstable<br>$[3.2, 0.0)$ $E_{2,-} < 0^\parallel$          |  | $(-4.0, 0.0)$ $E_{1,-}^{\dagger\dagger}$  |  |

<sup>†</sup> In this case  $\rho_{1,-}^* < \rho_{2,-}^* < 0 < \rho_{2,+}^* < \rho_{1,+}^*$ .

<sup>‡</sup> In this case  $\rho_{2,-}^* < \rho_{1,-}^* < 0 < \rho_{1,+}^* < \rho_{2,+}^*$ .

<sup>‡‡</sup> In this case  $0 < \rho_{2,-}^* < \rho_{1,-}^* < \rho_{2,+}^* < \rho_{1,+}^*$ .

<sup>§</sup> Sometimes  $\rho_t$  converges to the risk free rate  $r$  (e.g., for  $d_2 = 1.5, 1.6$ ).

<sup>¶</sup> In this case  $\rho_{2,-}^* < \rho_{1,-}^* < 0 < \rho_{1,+}^* < \rho_{2,+}^*$ .

<sup>††</sup> In this case  $0 < \rho_{1,-}^* < \rho_{2,-}^* < \rho_{2,+}^* < \rho_{1,+}^*$ .

<sup>||</sup> In this case  $\rho_{2,-}^* < 0 < \rho_{1,-}^* < \rho_{2,+}^* < \rho_{1,+}^*$ .

**Table 3.3:** Stability properties of the two-contrarian model. The table can be read in the same way as the previous ones. Note that in Experiment #1 the case  $\bar{d}_2 = -1$  is not considered since it would lead to the homogeneous case.

*Two Trend-Followers*

Now the opposite market situation is considered. Both trend traders act as trend followers, i.e.,

$$\bar{d}_1 = 1 \quad \text{and} \quad \bar{d}_2 \in (0.0, 4.0] .$$

All the results for this case can be found in Table 3.4. The stability behavior is now mainly characterized by unstable situations. As in the homogeneous case,  $E_{i,+}$  are unstable for any reasonably chosen positive extrapolation rates  $\bar{d}_i$ . Meanwhile the steady states  $E_{i,-}$  are locally

asymptotically stable for certain choices of parameters (see Experiment #2 and #3). Thus the ‘(quasi-)market selection principle’ holds in these situations, as already observed in the similar case with one fundamentalist and one trend-follower. Furthermore, an increase in the risk-aversion coefficient seems to increase the stability region of  $E_{2,-}$  (see Experiment #2 and #3).

| Two Trend-Follower Model |   |  |   |
|--------------------------|---|--|---|
|                          | Results of the Simulation Experiments   |  |   |
|                          | Experiment #1<br>$(\bar{\delta}_1, \bar{\delta}_2) = (0.85, 0.85)$                  | Experiment #2<br>$(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.4)$                             | Experiment #3<br>$(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.6)$  |
| risk avers. coeff.       | $\bar{d}_2 \in$   | stable fixed point   | $\bar{d}_2 \in$ stable fixed point  |
| $\nu = 1.4$              | $(0.0, 1.0]$ all unstable <sup>§</sup><br>$[0.9, 0.0)$ all unstable <sup>¶</sup>    | $(0.0, 1.8]$ $E_{1,-}$<br>$[1.9, 2.7]$ $E_{2,-}$<br>$[2.8, 4.0)$ all unstable                | $(0.0, 0.1]$ $E_{1,-}$<br>$[0.2, 2.0]$ $E_{2,-}$<br>$[2.1, 4.0)$ all unstable <sup>  </sup>                   |
| $\nu = 1$<br>(benchmark) | $(0.0, 4.0)$ all unstable <sup>§</sup>  | $(0.0, 1.5]$ $E_{1,-}$<br>$[1.6, 4.0)$ all unstable <sup>  </sup>                            | $(0.0, 0.4]$ $E_{1,-}$<br>$[0.5, 0.7]$ $E_{2,-}$<br>$[0.8, 4.0)$ all unstable <sup>  </sup>                   |
| $\nu = 0.7$              | $(0.0, 0.1]$ all unstable <sup>¶</sup><br>$[0.2, 4.0)$ all unstable <sup>§</sup>    | $(0.0, 0.5]$ all unstable <sup>¶</sup><br>$[0.6, 4.0)$ all unstable <sup>§</sup>             | $(0.0, 0.5]$ all unstable <sup>§</sup><br>$[0.6, 2.3]$ $0.037^{\S}$<br>$[2.4, 4.0)$ all unstable <sup>§</sup> |
| $\nu = 0.4$              | $(0.0, 1.0)$ $E_{1,+} < 0^\dagger$<br>$(1.0, 4.0)$ $E_{2,+} < 0^\ddagger$           | $(0.0, 1.5]$ $0.037^{\S}$<br>$[1.6, 4.0)$ all unstable <sup>§</sup>                          | $(0.0, 0.6]$ all unstable <sup>¶</sup><br>$[0.7, 4.0)$ all unstable <sup>§</sup>                              |
| $\nu = 0.2$              | $(0.0, 1.0)$ $E_{2,+}^{\dagger\dagger}$<br>$(1.0, 4.0)$ $E_{1,+}^{\ddagger\dagger}$ | $(0.0, 2.0]$ all unstable <sup>  </sup><br>$[1.9, 4.0)$ $E_{2,+} < 0^{\backslash\backslash}$ | $(0.0, 4.0)$ $E_{1,+}^{\backslash\backslash}$   |

<sup>†</sup> In this case  $\rho_{2,-}^* < \rho_{1,-}^* < \rho_{1,+}^* < \rho_{2,+}^* < 0$ .

<sup>††</sup> In this case  $\rho_{2,-}^* < \rho_{1,-}^* < 0 < \rho_{2,+}^* < \rho_{1,+}^*$ .

<sup>||</sup> In this case  $0 < \rho_{1,+}^* < \rho_{2,+}^*$ .

<sup>§</sup> In this case  $\rho_{1,\pm}^*$  and  $\rho_{2,\pm}^*$  become complex.

<sup>||</sup> In this case only  $\rho_{2,\pm}^*$  becomes complex.

<sup>‡</sup> In this case  $\rho_{1,-}^* < \rho_{2,-}^* < \rho_{2,+}^* < \rho_{1,+}^* < 0$

<sup>‡‡</sup> In this case  $\rho_{1,-}^* < \rho_{2,-}^* < 0 < \rho_{1,+}^* < \rho_{2,+}^*$ .

<sup>\backslash\backslash</sup> In this case  $0 < \rho_{2,-}^* < \rho_{1,-}^* < \rho_{2,+}^* < 0 < \rho_{1,+}^*$ .

<sup>¶</sup> In this case only  $\rho_{1,\pm}^*$  becomes complex.

**Table 3.4:** Stability properties of the two trend-follower model. The table can be read in the same way as the previous ones. Note that in Experiment #1 the case  $\bar{d}_2 = 1$  is not considered since it would lead to the homogeneous case.

*Contrarian and Trend Follower*

As a last case one contrarian and one trend-follower is assumed

$$\bar{d}_1 = -0.6 \quad \text{and} \quad \bar{d}_2 \in (0.0, 4.0] .$$

As in the case of the homogeneous model  $E_{2,+}$  is always unstable for realistic market situations. As proposed in Chiarella and He (2001), the local stability switches between  $E_{1,+}$  and  $E_{2,-}$  for small extrapolation rates  $\bar{d}_i$  and follows the '(quasi-)market selection principle'. An increase in the risk-aversion coefficient  $\nu$  increases (see Experiments #1–#3) and shifts (see Experiment #2) the range of  $\bar{d}_2$ , in which the second non-fundamental steady state  $E_{2,-}$  becomes stable. In addition, the stability region of  $E_{1,+}$  also slightly increases in this case. Otherwise, whenever the risk-aversion coefficient  $\nu$  is too low (lower than  $\bar{\delta}_i$ ), unrealistic market situations occur immediately (see Table 3.5 for details).

### 3.6 Numerical Simulations with a Noise Term

The previous sections have shown a purely deterministic model. This allowed me to apply standard methods from the tool box of nonlinear dynamics to give a precise description of the main model characteristics, like the existence of fixed points and their local stability properties. In this section I will briefly give some few examples in which I investigate through numerical simulations how the model behavior is influenced by introducing a noise term into the system. Noise will be implemented in two ways into the model framework. First I will follow Chiarella and He and assume a stochastic dividend yield  $\alpha_t$ . Note that, according to Eq. (3.18), the dividend yield is defined as  $\alpha_t = d_t/p_{t-1}$ , which is generally non-negative and depends on the current realization of the dividend and the preceding stock price. Hence the noise dividend yield process is assumed to be normally distributed around its mean  $\alpha_0$ , i.e.,

$$\frac{\alpha_t - \alpha_0}{q} \sim \mathcal{N}(0, 1) ,$$

where  $\mathcal{N}(0, 1)$  denotes the standard normal distribution and  $q > 0$  is a parameter that controls the standard deviation of the dividend yield process.

In the second step an additional noise term will be introduced. Following the lines of Levy, Levy and Solomon in their LLS model (see Chapter 2, Section 2.3), the investors' optimal investment proportion will be perturbed by a random term, i.e.,

$$\tilde{\pi}_{i,t} = \pi_{i,t} + \epsilon_{i,t} , \quad i = 1, 2 ,$$

where  $\epsilon_{i,t}/b \sim \mathcal{N}(0, 1)$  with  $b > 0$  is a constant. Note that  $\epsilon_{i,t}$  is drawn separately for each investor. From an economic point of view, the deviation from the investors' optimal investment proportion can be justified by assuming that, in each trading period, the traders' investment decision is partly determined by an 'irrational' or psychological component or simply by a slightly erroneous perception of the market situation or of their trading strategy.

The numerical simulations are based on two different trading periods. In the first numerical experiment the time period between each trade is one year, as assumed in the previous sections and

| Contrarian vs. Trend-Follower Model |  |  |   |
|-------------------------------------|--|--|---|
|                                     | Results of the Simulation Experiments                                    |  |   |
|                                     | Experiment #1<br>$(\bar{\delta}_1, \bar{\delta}_2) = (0.85, 0.85)$       | Experiment #2<br>$(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.4)$                     | Experiment #3<br>$(\bar{\delta}_1, \bar{\delta}_2) = (0.5, 0.6)$  |
| risk avers.<br>coeff.               | $\bar{d}_2 \in$<br>stable<br>fixed<br>point                              | $\bar{d}_2 \in$<br>stable<br>fixed<br>point  | $\bar{d}_2 \in$<br>stable<br>fixed<br>point   |
| $\nu = 1.4$                         | (0.0, 0.9] $E_{2,-}$<br>[1.0, 4.0) all unstable <sup>§</sup>             | (0.0, 0.3] $E_{1,+}$<br>[0.4, 2.7] $E_{2,-}$<br>[2.8, 4.0) all unstable              | (0.0, 2.0] $E_{2,-}$<br>[2.1, 4.0) all unstable <sup>§</sup>  |
| $\nu = 1$<br>(benchmark)            | (0.0, 0.1] $E_{2,-}$<br>[0.2, 4.0) all unstable <sup>§</sup>             | (0.0, 0.2] $E_{1,+}$<br>[0.3, 1.5] $E_{2,-}$<br>[1.6, 4.0) all unstable <sup>§</sup> | (0.0, 0.7] $E_{2,-}$<br>[0.8, 4.0) all unstable <sup>§</sup>  |
| $\nu = 0.7$                         | (0.0, 0.1] $E_{2,+} < 0^\dagger$<br>[0.2, 4.0) all unstable <sup>§</sup> | (0.0, 0.5] $E_{2,-}$<br>[0.6, 4.0) all unstable <sup>§</sup>                         | (0.0, 4.0) all unstable <sup>§</sup>  |
| $\nu = 0.4$                         | (0.0, 4.0) $E_{2,+} < 0^\dagger$   | (0.0, 4.0) all unstable <sup>§</sup>   | (0.0, 0.5] $E_{1,-} < 0^\ddagger$<br>{0.6} $E_{2,+} < 0^\dagger$<br>[0.7, 4.0) all unstable <sup>  </sup> |
| $\nu = 0.2$                         | (0.0, 4.0) $E_{1,-}^{\dagger\dagger}$                                    | (0.0, 1.9] $E_{2,+} < 0^{\ddagger\dagger}$<br>[2.0, 4.0) all unstable <sup>§</sup>   | (0.0, 4.0) $E_{1,-}^{\dagger\dagger}$   |

<sup>†</sup> In this case  $\rho_{2,-}^* < \rho_{2,+}^* < \rho_{1,-}^* < 0 < \rho_{1,+}^*$ .

<sup>††</sup> In this case  $\rho_{2,-}^* < 0 < \rho_{1,-}^* < \rho_{2,+}^* < \rho_{1,+}^*$ .

<sup>‡</sup> In this case  $0 < \rho_{2,-}^* < \rho_{1,-}^* < \rho_{2,+}^* < 0 < E_{1,+}$ .

<sup>‡‡</sup> In this case  $\rho_{2,-}^* < \rho_{2,+}^* < 0 < \rho_{1,-}^* < \rho_{1,+}^*$ .

<sup>§</sup> In this case  $\rho_{2,\pm}^*$  become complex.

<sup>¶</sup> In almost all cases  $\rho_{2,\pm}^*$  is complex.

<sup>||</sup> In this case  $\rho_{2,\pm}^*$  become complex. Sometimes it converges to the risk free rate  $r$ .

**Table 3.5:** Stability properties for the contrarian vs. trend-follower model. The table can be read in the same way as the previous ones.

in the LLS model (see Levy et al., 1994, 1995; Levy and Levy, 1996). Of course, simulation runs over 5000 trading rounds then seems to be a quite unrealistic idea. However, it brings the behavior of the stochastic nonlinear dynamic system in line with the results of the purely deterministic system investigated before. In the second experiment the time period is one day. It is here assumed that the number of trading days per year is 250. The simulations are conducted over twenty years. Obviously, for this scenario some parameters of the dynamic system have to be rescaled on a daily basis to obtain realistic values for the return and the price time series of the risky asset. Using the standard analytic manipulation rule for sums of i.i.d. normally distributed random variables, the annual parameters are thus rescaled according to

$$r_{\text{daily}} = r/td, \quad \alpha_{0,\text{daily}} = \alpha_0/td, \quad \bar{\delta}_{i,\text{daily}} = \bar{\delta}_i/td, \quad q_{\text{daily}} = q/\sqrt{td}, \quad (3.54)$$

where  $td$  denotes the trading days per year.

In the first experiment a two-contrarian model on an annual basis ( $td = 1$ ) is considered. This experiment investigates how the model is influenced by introducing three different levels of noise into the model framework. As already explained above, noise is implemented in two different ways. In the first and second investigation a dividend yield process with a lower and a higher noise level is assumed. Then, in the third investigation, an additional random term is added to the optimal investment proportion of each investor in the case of the higher noise dividend yield process. In other words, the noise level will be systematically increased in a quantitative as well as in a qualitative way. For the whole experiment the parameter set is assumed to be

$$\begin{aligned} r = 0.037, \quad \alpha_0 = 0.047, \quad L_1 = L_2 = 15, \quad \omega_0 = 0.00, \quad td = 1, \\ \bar{d}_1 = -0.7, \quad \bar{d}_2 = -1.0, \quad \bar{\delta}_1 = \bar{\delta}_2 = 0.4, \quad \nu_1 = 1.2. \end{aligned}$$

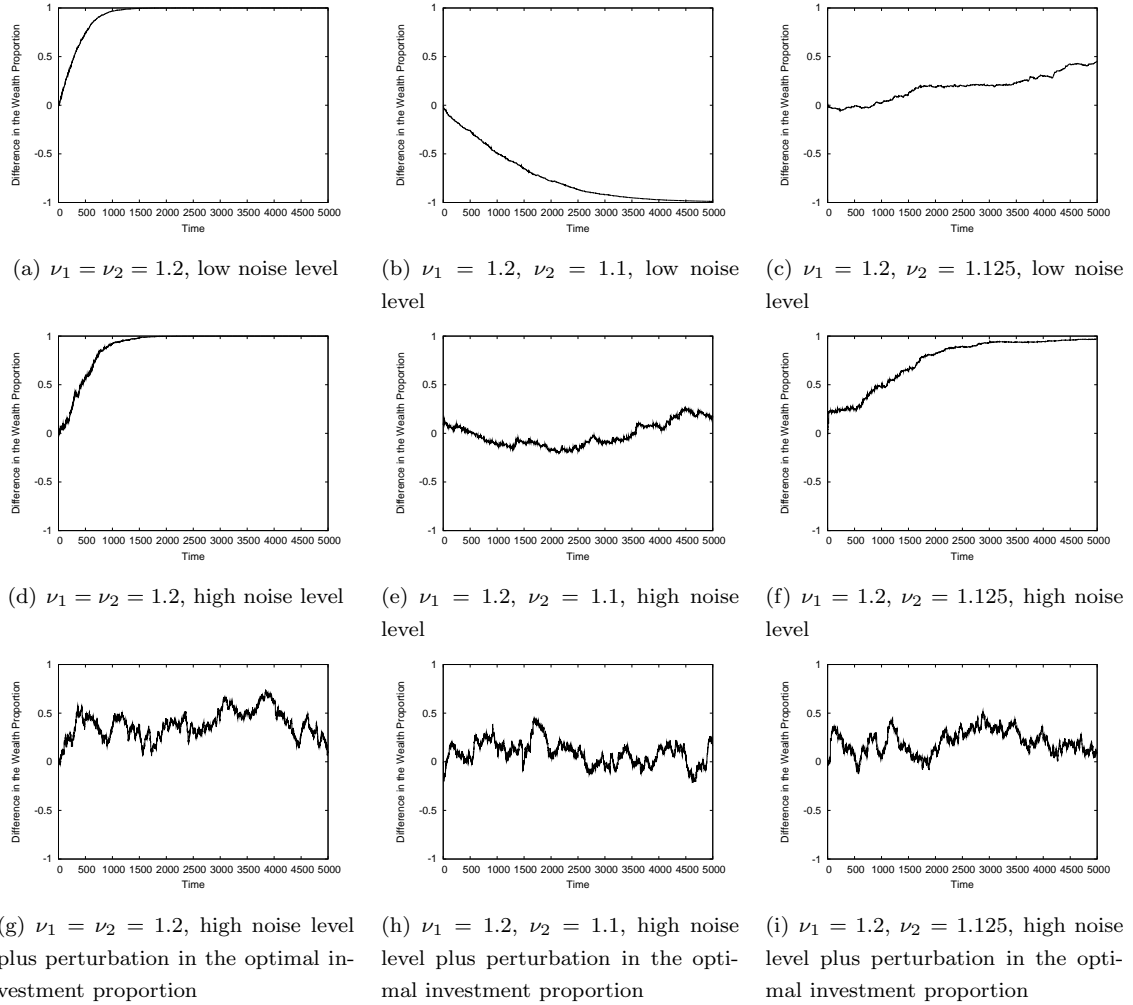
Thus the second investor extrapolates stronger than his/her counterpart. The risk-aversion coefficient of Investor 1 is fixed at  $\nu_1 = 1.2$  while the risk preference of Investor 2 changes during each investigation from  $\nu_2 = 1.2$  to  $\nu_2 = 1.1$  and then to  $\nu_2 = 1.125$ . Figure 3.8 shows the results from the first experiment. The three horizontal rows correspond to the three different investigations, in which the noise level is successively increased. From the left to the right panels the risk-aversion coefficient of Investor 2 changes. All plots show the difference in the wealth proportion of the two investors over time, i.e.,  $\omega_t = \omega_{1,t} - \omega_{2,t}$ .

Let us first consider the case with a low noise level of the dividend yield process in the model. According to Table 3.3 (see Subsection 3.5.4, Experiment #1), in the deterministic model Investor 1 will dominate the market if  $\bar{d}_1 > \bar{d}_2$  and both investors are described by the same risk preference ( $\nu_1 = \nu_2 > \bar{\delta}_i$ ,  $i = 1, 2$ ). It is thus obvious to presume that this result still holds whenever the noise level is modest ( $q = 0.05$ ). The conjecture is confirmed by the first plot of Figure 3.8 (a). Interestingly, if the risk aversion of Investor 2 falls below a certain threshold, the situation totally changes immediately. As a lower risk-aversion coefficient implies a higher investment proportion invested in the risky asset (see Eq. (3.38))<sup>45</sup>, Investor 2 now accumulates more wealth than Investor 1. Hence the wealth proportion of Investor 1 vanishes in the long run (see Figure 3.8 (b)). Of course, parameter settings can be found in which both effects which act in the opposite direction are balanced. On average, both investors then stay at their initial wealth level.<sup>46</sup> Figure 3.8 (c) is related to this case.

When the noise level is increased, the behavior of the model partly sees a qualitative change. Figure 3.8 (d)–(f) shows the results for a standard deviation of 12% in the dividend yield process

<sup>45</sup>This phenomenon holds for the deterministic model in particular. Although these results are here not reported in detail, numerical simulations indicate that this effect can be observed in general. In market situations in which the investors' risk-aversion coefficients  $\nu_i$  (which have to exceed  $\bar{\delta}_i$ ) deviate from each other sufficiently, the investor with the lower  $\nu_i$  always prevails. However, this result might be rather unsurprising because the proposed model framework does not take the second moment of the risky-asset return dynamics into account. Obviously, this leads to the conclusion that taking higher risk (via a higher investment proportion  $\pi_{i,t}$ ) implies higher gains as a rule.

<sup>46</sup>The parameters have to be chosen very specifically to generate such 'balanced' market situations. Hence this situation seems quite artificial and hardly realistic.



**Figure 3.8:** Influence of different risk preferences in a two-contrarian model with noise. All plots show the difference in the wealth proportions of the two investors on an annual basis. Investor 2 extrapolates more strongly than Investor 1. The left panels exhibit the benchmark situation, in which both investors are assumed to have the same risk-aversion coefficient. In the middle and right panels Investor 1 is assumed to be more risk-averse. Furthermore, the noise is increased from the upper panels to the lower panels. The upper panels show a market situation in which the noise level of the dividend yield is low ( $q = 0.05$ ). In the middle panels the noise of the dividend yield process is higher ( $q = 0.12$ ). In the lower panels the same dividend yield process ( $q = 0.12$ ) is combined with a second noise term in the optimal investment proportion ( $b = 0.04$ ). The parameters are  $td = 1$ ,  $\bar{d}_1 = -0.7$ ,  $\bar{d}_2 = -1.0$ ,  $\bar{\delta}_1 = \bar{\delta}_2 = 0.85$ ,  $\omega_0 = 0.00$ ,  $L_1 = L_2 = 15$ .

and the remaining parameters as before. In the case of homogeneous risk preferences, a higher noise level simply leads to a slightly less smooth transition of  $\omega_t$  to its steady-state value of the deterministic model, which is eventually still +1 (see Figure 3.8 (d)). For heterogeneous risk preferences Investor 2 does not dominate the market, either for assuming  $\nu_2 = 1.1$  or for  $\nu_2 = 1.125$  (see Figure 3.8 (e) and (f)). Instead, both investors stay more or less at their initial wealth level for  $\nu_2 = 1.1$  and Investor 1 dominates the market in the long run for  $\nu_2 = 1.125$ . Thus a higher noise level in the model seems to lead to a stabilization of Investor 1's strategy, or, in other words, Investor 2 has to choose a risk-aversion coefficient  $\nu_2$  that is lower than would be necessary in the



case of a lower noise level. To sum up, a higher noise level of the dividend yield process increases the range in which Investor 1 dominates the market and, therefore, his/her counterpart has to become less risk-averse than in the previous scenario to overcome his/her inferiority.

The lower panels concentrate on the case where the dividend yield process has a high standard deviation of 12% and, in addition, the optimal investment proportions of the two investors are slightly perturbed by a random term with a standard deviation of 4%. Figure 3.8 (g)–(i) clearly indicates that the appearance of the deterministic model's time series is superposed by the incorporated second noise term noise. While in the previous scenarios one investor eventually dominates the market in nearly all cases, the investors' wealth proportions now fluctuates around its initial value. None of the investors can take over the total market power. Nevertheless, looking at the simulation data it can be detected that the lower the assumed risk aversion of Investor 2, the better he/she performs. In other words, the behavior of the deterministic model can still be observed in the noisy model. However, this effect is hardly visible in Figure 3.8 (g)–(i) and less distinctive than in the upper and middle panels. One last remark is in order. From the last investigation it once again seems obvious that the investment proportion of the investors is a decisive determinant of the CH model. While implementing noise in the dividend yield process merely leads to a shift in the turning point at which the investors change their superiority, introducing noise in the investors' optimal investment proportion has a direct impact on the qualitative model outcome. Hence the optimal investment proportion  $\pi_i$  is an extremely important variable in the conditional equations of the deterministic dynamic system and crucially influences the overall dynamics. If such a variable undergoes a perturbation, the dynamic system will behave in a totally different way as observed by comparing the upper and middle panels with the lower panels of Figure 3.8.

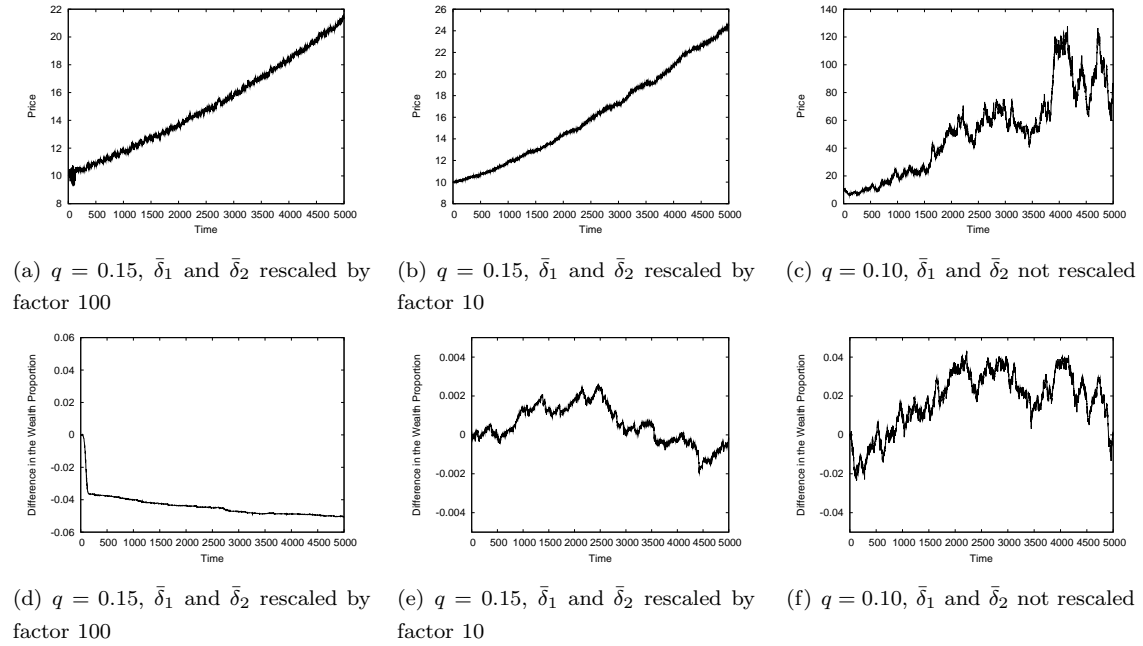
Although the previous discussion only considers an experiment with one specific parameter set, the results outlined above appear to be quite general. Indeed, further numerical simulations with different parameter sets (not reported here) show that the model performs in a similar way for a wide range of parameters and also for different trading strategies. In this sense, the findings represent the typical model behavior.

Before I close this section, a second experiment will briefly be considered. As a time series of 5000 time steps on an annual basis describes a rather unrealistic market situation, I will attempt to scale the model on a daily basis according to Eq. (3.54). The results obtained in this experiment can again be generalized to other market scenarios. The parameter set for the second experiment is assumed to be

$$r = 0.037, \quad \alpha_0 = 0.047, \quad td = 250, \quad \omega_0 = 0.00, \quad L_1 = L_2 = 15, \quad p_0 = 10, \\ \bar{d}_1 = -1.6, \quad \bar{d}_2 = 1.4, \quad \bar{\delta}_1 = 0.95, \quad \bar{\delta}_2 = 0.87, \quad \nu_1 = \nu_2 = 1.2,$$

which describes a contrarian versus trend-follower model with investors who are characterized by the same risk-aversion coefficient. Numerical simulations reveal some limitations in the CH model. When the model parameters are scaled on a daily level ( $td = 250$ ) the dynamic system eventually explodes in almost all cases. No realistic market dynamics can be obtained. Although numerical results on a daily basis are presented for the homogeneous and heterogeneous case in Chiarella and

He (2001), these findings are still in harmony with the original publication, because Chiarella and He did not rescale the parameter  $\bar{\delta}_i$ . However, since  $\delta_i$  is the excess return of the risky asset from the risk-free rate  $r$  (see Eq. (3.29)), this might be rather inconsistent.<sup>47</sup> Instead,  $\delta_i$  (or  $\bar{\delta}_i$ ) should also be rescaled on a daily basis as proposed in Eq. (3.54). As the dynamics of the model in this case does not behave in a reasonable way, or even leads to explosions of the time series of returns, the scaling factor of  $\delta_i$  is chosen differently. Figure 3.9 shows the result for  $\delta_i$  being rescaled by a factor 100, 10 and not being rescaled.



**Figure 3.9:** Contrarian vs. trend-follower model with noise on a daily level. Time series of prices (upper panels) and difference in the wealth proportion of investors (lower panels) are depicted for differently rescaled model parameters  $\bar{\delta}_i$ ,  $i = 1, 2$ . Parameters are  $r = 0.037$ ,  $\alpha_0 = 0.047$ ,  $td = 250$ ,  $\omega_0 = 0.00$ ;  $L_1 = L_2 = 15$ ,  $p_0 = 10$ ,  $\bar{d}_1 = -1.6$ ,  $\bar{d}_2 = 1.4$ ,  $\bar{\delta}_1 = 0.95$ ,  $\bar{\delta}_2 = 0.87$ ,  $\nu_1 = \nu_2 = 1.2$ .

If the model parameter  $\delta_i$ ,  $i = 1, 2$  stays at its annual level (right panels), the time series of prices qualitatively resemble realistic time series from real data from a visual perspective (upper right panel). Over 20 years the price of the risky asset has risen from its initial value 10 up to its highest value of about 120. Furthermore, the growing price trend can easily be detected and is accompanied by local deviations from the overall growing process since higher-than-average increases and declines of the asset price are also visible in the time series. The corresponding lower time series (see Figure 3.9 (f)) shows the difference of investors' wealth proportions  $\omega_i$ . It is quite obvious that, on a daily level of returns, the dominance of one investor over the other on an annual basis is not as distinct as in the preceding first experiment. Hence Investor 1 accumulates at best 4% more wealth than his/her counterpart.

Contrastingly, if the parameter  $\delta_i$  is rescaled by a factor 10 (middle panels) or a factor 100 (left panels) the realistic appearance of the time series of asset prices vanishes and the model

<sup>47</sup>However, that rescaling the parameters  $\delta_i$  leads to rather small and unrealistic optimal investment proportions  $\pi_i$ . Therefore, it can be argued that  $\delta_i$  should stay at its original level as proposed in Chiarella and He (2001).

outcome becomes more insubstantial. The price development is simply characterized by the growing dividend process without any significant deviation from the overall price trend (see Figure 3.9 (a) and (b)). Interestingly, rescaling  $\delta_i$  leads to a shift in the dominance of investors (compare Figure 3.9 (d) and Figure 3.9 (f)), which is caused by the fact that a rescaled  $\delta_i$  directly influences the investors' trading strategies and thus also their optimal investment proportion. It can again be observed that a higher investment proportion invested in the risky asset leads to a dominance in the market in all numerical experiments.

### 3.7 The CH Model under LLS-like Assumptions

Before I close this chapter I will investigate the CH model under the assumption that the investors' optimal investment proportions are bounded within an interval slightly smaller than 0 and 1 so that short selling is forbidden in every trading period. Here, I will mainly concentrate on whether the typical results of the LLS model, such as periodic booms and crashes and certain behavior of the cycle length depending on the investors' memory length (see Chapter 2, Section 2.3 for details), are reproduced by the CH model, or whether more realistic time series of prices and returns can be generated by such a modified CH model. For this reason some further model assumptions of the original CH model must be adjusted to get a model design for the CH model which more closely resembles the LLS model. As a direct consequence, the analysis of the dynamics is based exclusively on numerical simulations, as is the case for the LLS model.

To make the results of the CH model comparable with the results of the original LLS model, the CH model design is modified as follows. In both model frameworks the investors are myopic utility maximizers using a CRRA utility function for their portfolio maximization procedure. Remember that in the case of the CH model the optimal investment proportion  $\pi_{i,t}$  of investor  $i$  at time  $t$  is given by

$$\pi_{i,t} = \frac{1}{\nu_i} \frac{E_{i,t}(\rho_{t+1}) - r}{Var_{i,t}(\rho_{t+1})}, \quad (3.55)$$

where heterogeneous beliefs can be introduced by specifying  $E_{i,t}(\rho_{t+1})$  and  $Var_{i,t}(\rho_{t+1})$  as already outlined in Sections 3.2 and 3.3. Furthermore, we have seen that the wealth proportions of the investors evolve according to Eq. (3.16):

$$\omega_{i,t+1} = \frac{\omega_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]}{\sum_{j=1}^I \omega_{j,t}[R + \pi_{j,t}(\rho_{t+1} - r)]}, \quad i = 1, \dots, I,$$

where the return  $\rho_{t+1}$  is given by Eq. (3.17):

$$\rho_{t+1} = r + \frac{\sum_{i=1}^I \omega_{i,t}[(1+r)(\pi_{i,t} - \pi_{i,t+1}) - \alpha_{t+1}\pi_{i,t}]}{\sum_{i=1}^I \pi_{i,t}\omega_{i,t}(\pi_{i,t+1} - 1)} \quad (3.56)$$

with  $\alpha_{t+1} = d_{t+1}/p_t$  being the dividend yield. As in the LLS model, we assume that our artificial market consists of  $I$  different investors with the belief of each investor being mathematically

expressed by<sup>48</sup>

$$E_{i,t}(\rho_{t+1}) = \bar{\rho}_{i,t}, \quad (3.57)$$

$$Var_{i,t}(\rho_{t+1}) = \sigma^2 + \eta \bar{\sigma}_{i,t}^2, \quad (3.58)$$

where  $\sigma^2 > 0$  and  $\eta \geq 0$  are constants and

$$\bar{\rho}_{i,t} = \frac{1}{L_i} \sum_{k=1}^{L_i} \rho_{t-k} \quad \text{and} \quad \bar{\sigma}_{i,t} = \sqrt{\frac{1}{L_i - 1} \sum_{k=1}^{L_i} (\rho_{t-k} - \bar{\rho}_{i,t})^2} \quad (3.59)$$

with integers  $L_i \geq 1$ ,  $i = 1, \dots, I$ . Note that while the CH model framework uses a slightly different approach to the LLS framework to determine the optimal investment proportion  $\pi_{i,t}$ , the investors' beliefs about the next period's price (or return) are still economically formulated in as similar a fashion as possible. As in the case of the LLS model, all investors only use the last few returns of the risky asset to form their beliefs and weight each of them uniformly to reappear in the next trading period. Note that the parameter  $L_i$  in Eq. (3.59) corresponds to the memory span  $k$  used in the LLS model.

One of the main building blocks of the LLS model is the manipulation of the investors' optimal investment proportion; there will now be an additional random term which is restricted between some lower and upper boundary  $b_l$  and  $b_u$ . This is implemented in the following way. As already considered in the previous section, the investors' optimal investment proportion is first perturbed by a random term via

$$\tilde{\pi}_{i,t} = \pi_{i,t} + \epsilon_{i,t}, \quad i = 1, \dots, I,$$

where  $\epsilon_{i,t}$  is drawn separately for each investor from a Gaussian distribution with zero mean and standard deviation  $\sigma_\epsilon$ . Then, we must check if every single investor's individual investment proportion  $\tilde{\pi}_{i,t}$  fulfills the constraint

$$b_l \leq \tilde{\pi}_{i,t} \leq b_u. \quad (3.60)$$

Where this is not the case, new random terms  $\epsilon_{i,t}$  are drawn for the relevant investors. This procedure may be repeated until the restriction Eq. (3.60) holds for all investors.

Following the line of investigation proposed by the authors of the LLS model, two different model versions are considered in the next two subsections: a homogeneous version where all investors use the same trading strategy, and, in the second subsection, a heterogeneous version where the investors can be subdivided into two groups by their pursuing different trading strategies.

### 3.7.1 Numerical Results for the Homogeneous Model Version

In the homogeneous model version all the investors only deviate from each other by their different random factor  $\epsilon_{i,t}$  added to the optimal diversification proportion  $\pi_t$ , the latter being identical for every investor.<sup>49</sup> To get some comparable numerical results for this scenario, all model parameters

<sup>48</sup>There is also a technical reason for the specific choice of the conditional variance  $Var_{i,t}(\rho_{t+1})$  in Eq. (3.58). Since the random sampling variance  $\bar{\sigma}_{i,t}^2$  might in principle become zero, for which  $\pi_{i,t}$  is not defined, e.g., in the case where the system converges to a locally stable steady state, an additional non-zero fixed variance term  $\sigma^2$  is added.

<sup>49</sup>For convenience, the index  $i$  labelling the single investor is omitted for the homogeneous model.

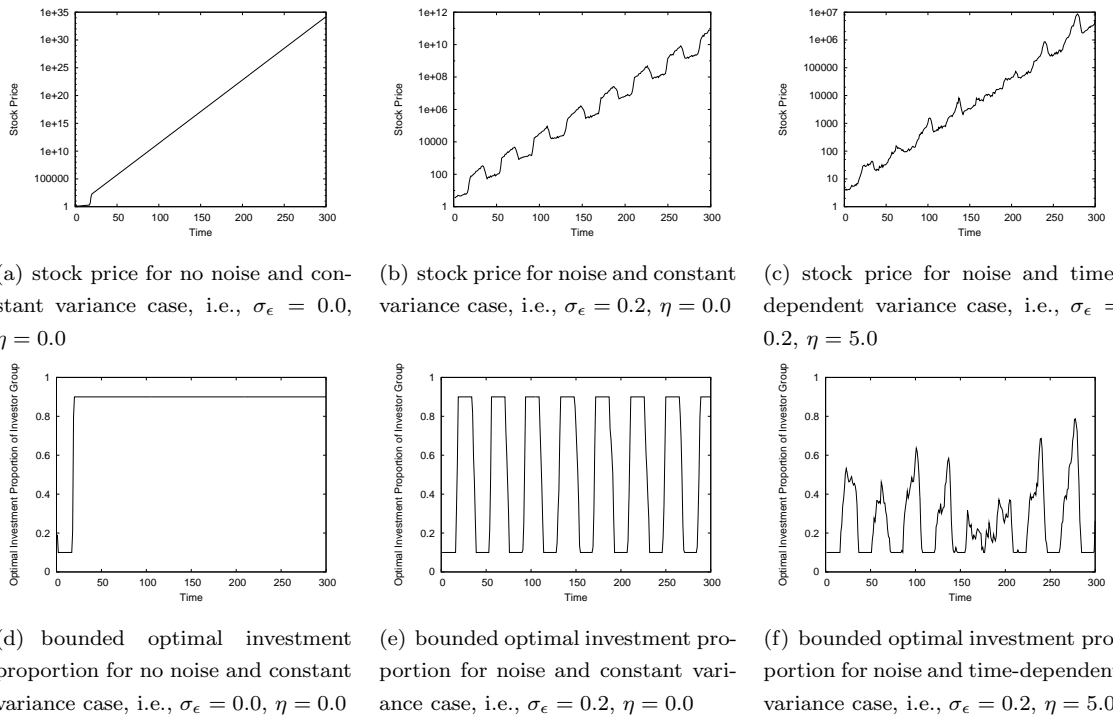
have to be specified in a similar way to in the original publications of the LLS model. Hence we assume that 100 investors are in the market and that all use a memory span of 15 time lags to determine their optimal investment proportion  $\pi_t$ . The lower and upper boundaries for the optimal investment proportions of the investors are chosen as 0.1 and 0.9 respectively. Furthermore, the trading period between two time steps is assumed to be one year. For the first experiment the parameter set of the homogeneous CH model is chosen as

$$r = 0.03, \quad \alpha_0 = 0.03, \quad L_1 = L_2 = 15, \quad I = 100, \quad td = 1,$$

$$b_l = 0.1, \quad b_u = 0.9, \quad \nu_1 = \nu_2 = 1.0, \quad \sigma = 0.4, \quad p_0 = 10,$$

whereas the parameters  $\sigma_\epsilon$  and  $\eta$  are varied during the simulation experiment.

Figure 3.10 shows the results. It presents typical time series of prices together with the cor-



**Figure 3.10:** CH model with bounded optimal investment proportions (homogeneous case). Time series of prices (upper panels) and bounded optimal investment proportions (lower panels) are shown for different choices of  $\sigma_\epsilon$  and  $\eta$  if only one investor group is in the market. In the case of no noise ( $\sigma_\epsilon = 0.0$ ), the stock price simply grows exponentially. In the case of a noisy demand ( $\sigma_\epsilon = 0.2$ ), the time series of prices exhibits periodic behavior (upper middle and upper right panel) caused by the constraints imposed for the investors' optimal investment proportions. The remaining parameters are chosen as  $r = 0.03$ ,  $\alpha_0 = 0.03$ ,  $L_1 = L_2 = 15$ ,  $I = 100$ ,  $td = 1$ ,  $b_l = 0.1$ ,  $b_u = 0.9$ ,  $\nu_1 = \nu_2 = 1.0$ ,  $\sigma = 0.4$ ,  $p_0 = 10$ .

responding time series of the optimal investment proportions  $\pi_t$  of the investor group for the homogeneous model. In the no-noise case (i.e.,  $\sigma_\epsilon = 0.0$ ) the prices of the risky asset simply grow exponentially due to an identical demand function for all investors, which thus leads to a no-trade situation (see Figure 3.10 (a)). In such a situation all investors want to invest as much

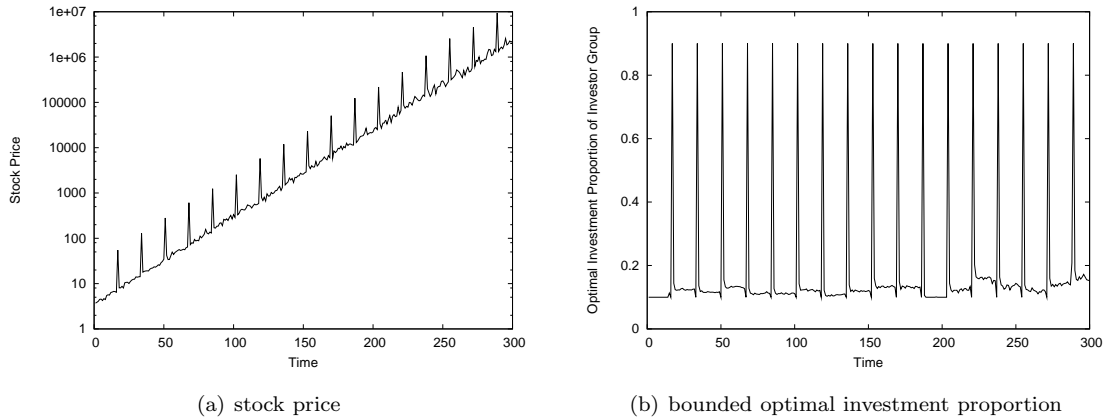
as possible in the risky stock, which is restricted to 90% of their total wealth in our experimental setup. Therefore, the time series of the optimal investment proportion rapidly converges to the upper bound  $b_u = 0.9$  after passing through a short transient period (see Figure 3.10 (d)). Of course, such market behavior is identical to the findings in the original LLS model (see Figure 2.16 in Chapter 2). Nevertheless, note that the constant growth rates of prices are different in both models for this pathological case. While in the LLS model the growth rate is simply determined by the chosen level of dividends, the growth rate of the price level is significantly higher in the CH model. This can be explained by the different model frameworks. In the case of the CH model the dynamic system is formulated in terms of the return of the risky asset, from which the asset price can be determined afterwards via Eq. (3.1). In the homogeneous no-noise case the asset return converges to a steady state according to Eq. (3.56) with  $\pi^* = 0.9$ , which leads to a relatively high fixed return and thus to a asset prices with a higher growth rate than in the LLS model framework.

However, if an individual noise term is introduced into the investors' demand functions, the stock price starts to fluctuate around the exponential price trend. While Figure 3.10 (b) and (e) exhibits a simulation run for the case where all investors use the same constant perception of the conditional variance of the risky-asset return, Figure 3.10 (c) and (f) presents results for a time-dependent case of the variance term (see Eq. (3.58)). Interestingly, the widely known and 'noted' periodic booms and crashes in the stock price for almost all model variants generated by the LLS model also occur for certain parameter settings in our modified CH model version. An explanation for this phenomenon is similar to the LLS case and, therefore, straightforward. If prices rise, the investors will become increasingly bullish until all of them invest as much as possible in the stock. A small number of 'outliers' in the investors' noise term, who are almost homogeneous in this market situation, can cause a drastic change in the investor's individual investment proportion, leading to a market crash from which all investors will recover step by step so that the market can become bullish again. This 'discontinuity property' (see Levy et al. (1995)) can be immediately detected by looking at the time series of the (homogeneous) optimal investment proportions  $\pi_t$ , where dramatic changes can be observed, as already seen in the LLS model.

Nevertheless, further simulation experiments indicate that those LLS-like results are not as robust against changes in the parameters as in the original LLS model. The main reason for this fact might be the analytic form of the optimal investment proportion  $\pi_t$  given in Eq. (3.55). Beside the numerator, in which the (the first moment of the) trading strategy is defined, it is obvious that the time-depend realization of  $\pi_t$  also crucially depends on the denominator, which can be adjusted particularly by the parameters  $\sigma$  and  $\eta$ . It is evident that an increase in the denominator leads to a lower optimal investment proportion (see Figure 3.10 (f)) while choosing the conditional variance of the risky-asset return too small is associated with  $\pi_t$  values near or even at the upper boundary  $b_u$ , which directly leads to exponentially growing stock prices.

Figure 3.11 exhibits the results for  $\sigma = 0$  and  $\eta = 1$  (and  $\sigma_\epsilon = 0.2$  as before), i.e., all investors simply using the purely random sampling variance  $\bar{\sigma}_t$  for the determination of their optimal investment proportion. Again, periodic price patterns can be detected where market crashes directly occur one trading period after one single day of a market boom. Since in this market scenario the model assumptions are adjusted to the LLS case to the highest possible degree, these results can

be seen as the corresponding CH-results to the LLS model assumptions.



**Figure 3.11:** CH model with most possible LLS-like assumptions (homogeneous case). Time series of prices (left panel) and corresponding bounded optimal investment proportions (right panel) are shown for  $\sigma = 0.0$  and  $\eta = 1.0$  if only one investor group is in the market. The stock price grows exponentially with periodic booms, each of them immediately followed by a commensurate market crash in the next trading period. The remaining parameters are chosen as  $r = 0.03$ ,  $\alpha_0 = 0.03$ ,  $L_1 = L_2 = 15$ ,  $I = 100$ ,  $td = 1$ ,  $b_l = 0.1$ ,  $b_u = 0.9$ ,  $\nu_1 = \nu_2 = 1.0$ ,  $\sigma_\epsilon = 0.2$ ,  $p_0 = 10$ .

As already stated above, further simulation runs indicate a certain sensitivity of the model dynamics with respect to the parameter values. (These are not presented in detail here.) For instance, an increase in the dividend yield  $\alpha_0$  as well as an expansion of the lower and upper boundaries  $b_l$  and  $b_u$  leads to a tendency of higher optimal investment proportions and exponentially growing stock prices. Nevertheless, one further accordance with the LLS model should be remarked. As already known from the LLS model framework, the cycle length of the stock price's periodic behavior crucially depends on the investors' memory span.

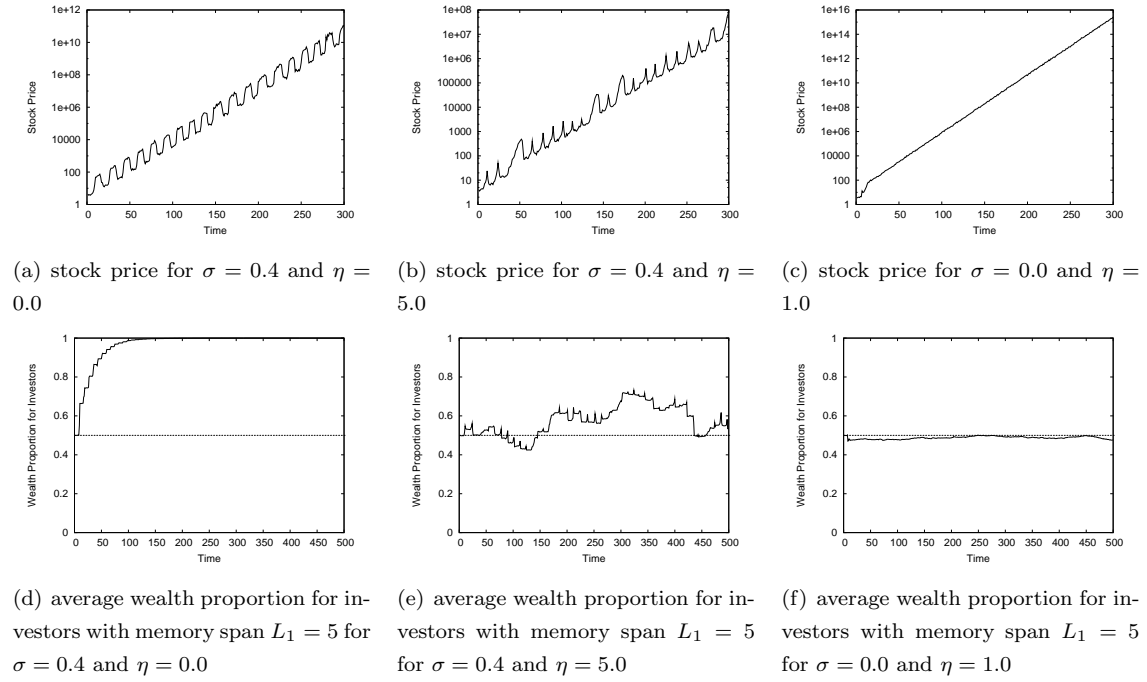
### 3.7.2 Numerical Results for the Heterogeneous Model Version

In the heterogeneous model version we mainly concentrate on a model structure with two investor types in the market each type using different memory spans  $L_i$ ,  $i = 1, 2$  to form their expectations. The investors are assumed to be evenly distributed among the two groups. Now three different experiments will be considered.

Similar to the initial experiment for the homogeneous model, in the first experiment the parameter set is chosen as

$$\begin{aligned}
 r &= 0.03, & \alpha_0 &= 0.03, & L_1 &= 5, & L_2 &= 8, & I &= 100, & td &= 1, \\
 b_l &= 0.1, & b_u &= 0.9, & \nu_1 &= 1.0, & \nu_2 &= 1.0, & p_0 &= 10, & \sigma_\epsilon &= 0.2,
 \end{aligned}$$

whereas the parameters  $\sigma$  and  $\eta$  are varied during the experiment. It describes a market where investors can be grouped into investor types with memory span  $L_1 = 5$  and  $L_2 = 8$ . The results of the simulation runs are summarized in Figure 3.12. The upper left and upper middle panel shows that the market price becomes slightly more irregular in the heterogeneous model. Nevertheless, the



**Figure 3.12:** CH model with bounded optimal investment proportions (heterogeneous case). Time series of prices (upper panels) and time series of average wealth proportion (lower panels) are shown in a two investor-group model with memory spans  $L_i = 5, 8$  for different choices of the parameters  $\sigma$  and  $\eta$ . The remaining parameters are  $r = 0.03$ ,  $\alpha_0 = 0.03$ ,  $L_1 = 5$ ,  $L_2 = 8$ ,  $I = 100$ ,  $td = 1$ ,  $b_l = 0.1$ ,  $b_u = 0.9$ ,  $\nu_1 = 1.0$ ,  $\nu_2 = 1.0$ ,  $p_0 = 10$ ,  $\sigma_\epsilon = 0.2$ .

price patterns are still rather predictable. These findings are shared with the original LLS model (see Figure 2.17 in Chapter 2, Section 2.3). The lower panels exhibit the average wealth proportion of the investor group using the smaller memory span  $L_1 = 5$ . Obviously, if the parameters are chosen in such a way that periodic booms and crashes are generated by our modified CH model, the behavior of the wealth share of investors will behave in the same way as stated in the publications of the original LLS model: the population with the smaller memory span will eventually dominate the market (see Chapter 2, Subsection 2.3.2). Otherwise, if the denominator of the optimal investment proportion is increased (i.e., for  $\eta = 5.0$ ), which makes the market crashes follow immediately in the subsequent trading periods, this phenomenon becomes less strong since the investor group with memory span 5 cannot exploit their dominating trading strategy as strongly. Finally, if the ‘most LLS-like’ assumptions are assumed (i.e.,  $\sigma = 0.0$ ,  $\eta = 1.0$ ), the stock price simply grows exponentially and, therefore, no trading group can prevail over the other.

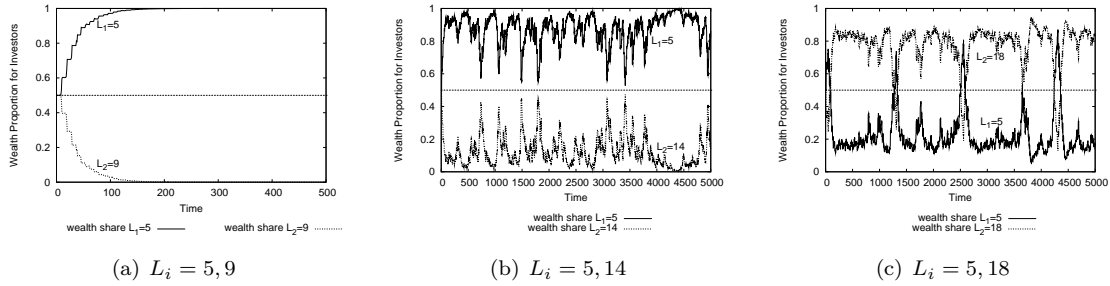
The second experiment investigates the influence of the memory spans for a parameter set which generates LLS-like booms and crashes as shown in Figure 3.12 (a). Hence the parameter set is assumed to be

$$r = 0.03, \quad \alpha_0 = 0.03, \quad I = 100, \quad td = 1, \quad \sigma = 0.4, \quad \eta = 0.0,$$

$$b_l = 0.1, \quad b_u = 0.9, \quad \nu_1 = 1.0, \quad \nu_2 = 1.0, \quad p_0 = 10, \quad \sigma_\epsilon = 0.2.$$



Figure 3.13 shows results for three scenarios, namely  $L_i = 5, 9$ ,  $L_i = 5, 14$  and  $L_i = 5, 18$  with  $i = 1, 2$ . While investor groups with slightly larger memory spans than their counterpart lose all the wealth to the other group with a smaller memory span (see Figure 3.13 (a)), they start to perform better or even dominate the market most of the time when they increase their memory span (see Figure 3.13 (b) and (c)). This result slightly deviates from the findings reported in Levy et al. (1996, p. 103).<sup>50</sup>



**Figure 3.13:** Time series of wealth share with two investor-groups and a constant variance term. The three panels show a CH model with two investor-groups acting in the market and bounded optimal investment proportions. The investor group with a memory span of 5 time lags quite quickly prevails over the investor group using a moderately higher memory span, e.g.,  $L_2 = 9$  (left panel). However, if the second investor group increases their memory span slightly, e.g.,  $L_2 = 14$ , they perform better (middle panel). If the second investor group increases their memory span slightly more, e.g.,  $L_2 = 18$ , they even become able to dominate the market (right panel). The remaining parameters are  $r = 0.03$ ,  $\alpha_0 = 0.03$ ,  $I = 100$ ,  $td = 1$ ,  $\sigma = 0.4$ ,  $\eta = 0.0$ ,  $b_l = 0.1$ ,  $b_u = 0.9$ ,  $\nu_1 = 1.0$ ,  $\nu_2 = 1.0$ ,  $p_0 = 10$ ,  $\sigma_\epsilon = 0.2$ .

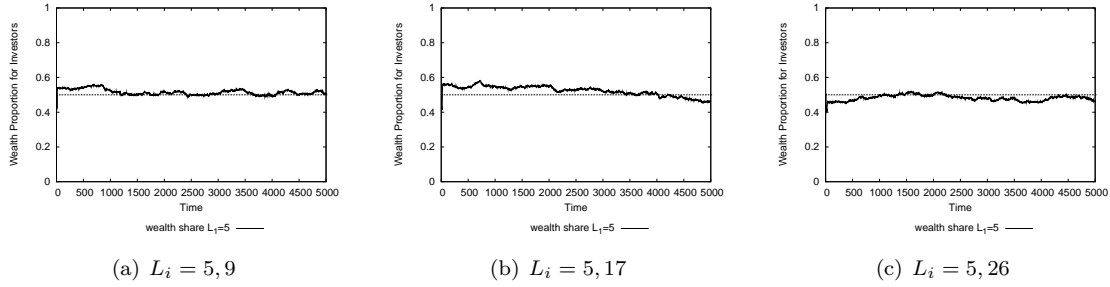
In the final experiment the same investigation is repeated for the ‘most LLS-like’ assumptions, i.e.,  $\sigma = 0.0$  and  $\eta = 1.0$ . Consequently, the parameter set is chosen as follows:

$$\begin{aligned}
 r &= 0.03, & \alpha_0 &= 0.03, & I &= 100, & td &= 1, & \sigma &= 0.0, & \eta &= 1.0, \\
 b_l &= 0.1, & b_u &= 0.9, & \nu_1 &= 1.0, & \nu_2 &= 1.0, & p_0 &= 10, & \sigma_\epsilon &= 0.2.
 \end{aligned}$$

The overall results are straightforward. Since in this scenario the stock price simply grows exponentially (see Figure 3.12 (c)), the influence of differently chosen memory spans is almost insignificant. Nevertheless, the results of the previous experiment can be confirmed at least on a qualitative level. While investors types with slightly lower memory spans (i.e.,  $L_1 = 5$  vs  $L_2 = 9$ ) again dominate the market by holding no more than 60% of the total wealth of the market at its maximum, the second investor group can improve their performance by using an increased memory span (see Figure 3.14).

Furthermore, it can be shown in this scenario that investors who decrease their risk-aversion coefficient  $\nu_i$  prevail over the other investors. This can be simply explained by the fact that a lower risk-aversion coefficient leads to a higher demand function for the risky asset, which, in turn, directly results in a better performance if the stock price grows without any significant booms and crashes. (These results are not shown here in details.)

<sup>50</sup>See also Chapter 2, Subsection 2.3.2 for details.



**Figure 3.14:** Time series of wealth share with two investor-groups and a time-dependent variance. In the case of strict LLS-like assumptions, i.e.,  $\sigma = 0.0$  and  $\eta = 1.0$ , the characteristics of the findings from the second experiment (see Figure 3.13) are minor apparent although their qualitative tendencies can still be detected. The remaining parameters are  $r = 0.03$ ,  $\alpha_0 = 0.03$ ,  $I = 100$ ,  $td = 1$ ,  $\sigma = 0.0$ ,  $\eta = 1.0$ ,  $b_l = 0.1$ ,  $b_u = 0.9$ ,  $\nu_1 = 1.0$ ,  $\nu_2 = 1.0$ ,  $p_0 = 10$ ,  $\sigma_\epsilon = 0.2$ .

### 3.8 Summary

This chapter adopts the framework developed by Chiarella and He (2001) as a starting point and incorporates a measure of investors' risk aversion in a growth model of both the asset price and total wealth of investors with heterogeneous expectations. Following the line of research proposed by Chiarella and He (2001), I reformulate the growth model in terms of the return of the risky asset and the wealth proportion of heterogeneous investors, which makes the model stationary as well as analytically tractable. Replacing the originally assumed logarithmic utility function with a more general power utility function which accounts for differing risk aversion, this chapter provides an explicit study of how the Chiarella-He (CH) model is affected by different types of investors having different risk attitudes. As in almost all other behavioral models of financial markets, two 'prototypes' of investors are considered: fundamentalists and chartists, with the latter investor group being again subdivided in two trader types, namely trend followers and contrarians. An extensive sensitivity analysis with respect to the risk-aversion coefficient in the power utility function of both investor types was performed in both a homogeneous and heterogeneous model version. We might summarize the results of this chapter as follows:

- The dynamics of the system is affected by the investors' risk attitudes (measured by the risk-aversion coefficient of the underlying utility function) and shows a striking dependence on the ratio of the risk-aversion coefficient and the excess conditional mean from the risk-free rate in the homogeneous model as well in the case of two investor-groups. In general, if the risk-aversion coefficients are above the excess conditional mean from the risk-free rate (i.e., ratio  $> 1$ ), all the results found by Chiarella and He can be confirmed in the enriched version of the CH model. Otherwise, if the risk-aversion coefficient is below the excess conditional mean from the risk-free rate (ratio  $< 1$ ), the return and wealth proportion dynamics becomes either unstable or leads to degenerated fixed equilibria in most cases.
- Whenever the ratio  $> 1$ , the fundamentalist who is less risk-averse than his/her fundamental counterpart dominates the market in the long run, i.e., he/she accumulates all the wealth of the market (measured in terms of wealth proportion of investors), which also comes along

with a relatively higher long-term return of the market compared to the long-term return should the more risk-averse fundamentalist have dominated the market. Instead, for ratio  $< 1$  this ‘market selection principle’ does not hold in any model variants for these cases.

- With the analytically manageable size of the homogeneous model version, some slightly inconsistent model assumptions (or at least crucially limiting model assumptions) become obvious. Although these are already incorporated in the original model design, they become particularly evident when studying the influence of risk on the return and wealth proportion dynamics of the CH model. First, the assumption of a constant excess conditional mean from the risk-free rate might be too restrictive because market scenarios can be generated in which the investors’ expectations about the development of the risky asset and about the investors’ basic attitude towards investing in risky prospects are modeled in the opposite way, which leads to the model exhibiting absurd market behavior. Second, the expectations about the development of the return of the risky asset that are formed by the different investor groups will never be fulfilled even if the dynamic system reaches one of its fixed equilibria, which also represents a rather unsatisfying model characteristic.
- In the heterogeneous two-investors model with different combinations of fundamentalist and chartist investors, the ‘market selection principle’ is only valid if the investors’ attitude towards risk (measured by their risk-aversion coefficient in their utility function) is higher than the investors’ expectations about the excess return of the risky asset (measured by the excess conditional mean from the risk-free rate).
- Introducing external noise to the dynamic system leads to partly qualitative changes in the model behavior. For heterogeneous risk preferences of investors, the dominating investor (in terms of accumulated wealth proportion) in the deterministic case becomes less dominating when a sufficiently high noise level is switched on. Thus introducing noise to the model has a stabilizing effect to the inferior investor.
- The attempt to scale the enriched CH model on a daily level (250 trading days per year) leads mostly to explosive price paths and unrealistic market behavior. Such limitation with regard to the calibration of the model parameters is quite common in artificial financial markets and is thus shared by many other behavioral models of this type.<sup>51</sup>
- For a noisy multi-investors version of the CH model with 100 investors acting in the market in which short selling is suppressed by appropriate lower and upper boundaries for the investment proportions, periodic booms and crashes can be generated for specific choices of the parameter set as already observed in a similar model framework proposed by Levy et al. (1994, 1995, 1996). Nevertheless, these LLS-like results are not as robust against changes in the parameters as in the original LLS model.

In summary, the study provides further insights into the dynamic behavior of the CH model. The overall result of our investigation of an enriched version of the original model is that the only feasible parameter space is restricted to market situations in which the investors’ risk-aversion

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<sup>51</sup>For some more general remarks on this discussion, see LeBaron (2000).

coefficient is assumed to be sufficiently high compared to the expected excess return of the risky asset. Furthermore, the explicit incorporation of risk into the CH model reveals some limitations and inconsistencies in the main building blocks of the model.

Finally, one further assumption in the CH model might be doubtful. A normally distributed or even constant dividend yield (originally defined as  $\alpha_t \equiv d_{t+1}/p_t$  in the CH model) implies that price variations (e.g., caused by speculative activities of the investors) are directly compensated by opposite dividend variations. This is not only an unrealistic assumption but also counter-intuitive, since investigating speculative activities which may arise, generated by certain trading strategies of the investors, are of special interest in this kind of behavioral models and, therefore, should not be automatically smoothed away by the model framework.

As a consequence, the procedure that follows is straightforward. We have seen an interesting asset pricing model which still requires some improvements in its own model structure, and now we want to know if we can find some modifications of the model's main building blocks which still leads us to an analytically and numerically manageable dynamic system, but which also makes some more realistic and reasonable assumptions about the economic intuition behind the technical model framework. Such a model framework will be proposed in the last part of this thesis.

## Appendix to Chapter 3

### A 3.1 Heuristic Explanation for Formula (3.26)

Let  $W_t$  be the wealth of an investor at time  $t$  and assume he/she is a (myopic) expected utility of wealth maximizer, which means that he/she solves the maximization problem  $E_t U(W_{t+1})$  in every time period  $t$ .<sup>52</sup> Due to the fact that in most cases the expectation value cannot be derived analytically, a Taylor expansion for  $W_{t+1}$  around  $c \cdot W_t$  up to the second order is considered, which is given by<sup>53</sup>

$$U(W_{t+1}) = U(cW_t) + U'(cW_t)(W_{t+1} - cW_t) + \frac{U''(cW_t)}{2}(W_{t+1} - cW_t)^2 + \mathcal{O}(W_{t+1}^3).$$

Then, the expected utility can (approximately) be derived by

$$E_t[U(W_{t+1})] = U(cW_t) + U'(cW_t)(E_t(W_{t+1}) - cW_t) + \frac{U''(cW_t)}{2}Var_t(W_{t+1}), \quad (\text{A } 3.1)$$

where  $E[(W_{t+1} - cW_t)^2] \approx Var_t(W_{t+1})$  is used apart from the Taylor approximation. As expected utility preferences are invariant under linear transformations, Eq. (A 3.1) can be rewritten as

$$E_t[U(W_{t+1})] \approx U(W_t) + U'(W_t)(E_t(W_{t+1}) - W_t) + \frac{U''(W_t)}{2}Var_t(W_{t+1}).$$

From Eq. (3.3) and Eq. (3.4) it follows that

$$\begin{aligned} E_t[U(W_{t+1})] &\approx U(W_t) + U'(W_t) \{W_t[R + \pi_t(E_t(\rho_{t+1}) - r)] - W_t\} + \\ &\quad + \frac{U''(W_t)}{2}W_t^2\pi_t^2Var_t(\rho_{t+1}) \\ &= \{U(W_t) + U'(W_t)W_t r\} + \{U'(W_t)W_t(E_t(\rho_{t+1}) - r)\}\pi_t + \\ &\quad + \left\{\frac{1}{2}U''(W_t)W_t^2Var_t(\rho_{t+1})\right\}\pi_t^2. \end{aligned}$$

<sup>52</sup>For the sake of convenience, the index  $i$  which labels the single investor is omitted here.

<sup>53</sup>From (Eq. (3.2))  $W_{t+1}$  should be (approximately) distributed around  $c \cdot W_t$ , where  $c$  is assumed to be deterministic.

Hence the maximization problem  $\max_{\pi_t} E_t[U(W_{t+1})]$  can be solved by using the first-order condition, which leads to the (approximately) optimal solution

$$\pi_t = -\frac{U'(W_t)}{W_t U''(W_t)} \frac{E_t(\rho_{t+1}) - r}{\text{Var}_t(\rho_{t+1})}.$$

Applying Eq. (A 3.1) to  $U(W) = \ln(W)$ , the optimal investment proportion for an investor preferring a logarithmic utility function is given by

$$\pi_t = \frac{E_t(\rho_{t+1}) - r}{\text{Var}_t(\rho_{t+1})}$$

as claimed in Eq. (3.5). Assuming a (standard) power utility function instead, that is

$$U(W) = \frac{W^{1-\nu}}{1-\nu},$$

leads to

$$\pi_t = \frac{1}{\nu} \frac{E_t(\rho_{t+1}) - r}{\text{Var}_t(\rho_{t+1})},$$

which is in harmony with Eq. (3.26).

### A 3.2 Proof of Theorem 3.3

The proof of Theorem 3.3 is totally analogous to the derivation of Theorem 3.1 and can be simply obtained by replacing the variables therein by the newly introduced variables provided with a bar. In addition, the sums have to be adapted. Thus,

$$\bar{\omega}_{j,t+1} = \frac{\bar{W}_{j,t+1}}{\bar{W}_{t+1}} = \frac{\bar{W}_{j,t}[R + \bar{\pi}_{j,t}(\rho_{t+1} - r)]}{\bar{W}_{t+1}} = \frac{\bar{\omega}_{j,t}[R + \bar{\pi}_{j,t}(\rho_{t+1} - r)]}{\bar{W}_{t+1}/\bar{W}_t}, \quad (\text{A } 3.2)$$

where again the denominator can be rewritten by

$$\frac{\bar{W}_{t+1}}{\bar{W}_t} = \frac{\sum_{\ell=1}^h \bar{W}_{\ell,t}[R + \bar{\pi}_{\ell,t}(\rho_{t+1} - r)]}{\bar{W}_t} = \sum_{\ell=1}^h \bar{\omega}_{\ell,t}[R + \bar{\pi}_{\ell,t}(\rho_{t+1} - r)]. \quad (\text{A } 3.3)$$

Combining Eq. (A 3.2) and Eq. (A 3.3) directly leads to Eq. (3.23).

Noting that  $n_j \cdot I$  is the number of investors in trading group  $j$ , the market clearing equilibrium can be written as

$$Np_t = \sum_{i=1}^I \pi_{i,t} W_{i,t} = \sum_{j=1}^h (n_j I) \cdot \bar{\pi}_{j,t} \bar{W}_{j,t},$$

which leads to an expression similar to Eq. (3.12), that is

$$\frac{Np_t}{I} = \sum_{j=1}^h n_j \bar{\pi}_{j,t} \bar{W}_{j,t} = \bar{W}_t \sum_{j=1}^h n_j \bar{\pi}_{j,t} \bar{\omega}_{j,t}. \quad (\text{A } 3.4)$$

Replacing  $t$  with  $t + 1$  in Eq. (A 3.4) and dividing the result by Eq. (A 3.4) gives

$$\frac{\bar{W}_{t+1}}{\bar{W}_t} \frac{\sum_{j=1}^h n_j \bar{\pi}_{j,t+1} \bar{\omega}_{j,t+1}}{\sum_{j=1}^h n_j \bar{\pi}_{j,t} \bar{\omega}_{j,t}} = \frac{p_{t+1}}{p_t},$$

which can also be written as

$$\sum_{\ell=1}^h \bar{\omega}_{\ell,t}[R + \bar{\pi}_{\ell,t}(\rho_{t+1} - r)] \frac{\sum_{j=1}^h n_j \bar{\pi}_{j,t+1} \bar{\omega}_{j,t+1}}{\sum_{j=1}^h n_j \bar{\pi}_{j,t} \bar{\omega}_{j,t}} = 1 + \rho_{t+1} - \alpha_{t+1} \quad (\text{A } 3.5)$$

using  $\alpha_{t+1} = d_{t+1}/p_t$ . Note that from Eq. (3.23) the numerator on the left-hand side is given by

$$\sum_{j=1}^h n_j \bar{\pi}_{j,t+1} \bar{\omega}_{j,t+1} = \sum_{j=1}^h n_j \bar{\pi}_{j,t+1} \left\{ \frac{\bar{\omega}_{j,t}[R + \bar{\pi}_{j,t}(\rho_{t+1} - r)]}{\sum_{\ell=1}^h \bar{\omega}_{\ell,t}[R + \bar{\pi}_{\ell,t}(\rho_{t+1} - r)]} \right\}. \quad (\text{A } 3.6)$$

Substitution of Eq. (A 3.6) into Eq. (A 3.5) and simplification of the corresponding expression leads to the equation

$$\sum_{j=1}^h n_j \bar{\pi}_{j,t+1} \bar{\omega}_{j,t} [R + \bar{\pi}_{j,t} (\rho_{t+1} - r)] = (1 + \rho_{t+1} - \alpha_{t+1}) \left( \sum_{j=1}^h n_j \bar{\pi}_{j,t} \bar{\omega}_{j,t} \right),$$

from which Eq. (3.24) can directly be determined.

### A 3.3 Proof of Theorem 3.4

It follows from Eq. (3.22) and Eq. (3.21) that the fixed point  $\rho^*$  satisfies

$$(\rho^* - r)(1 - \pi^*) = \alpha_0,$$

where  $\pi^* = \frac{1}{\nu}(\bar{\delta} + \bar{d}\rho^*)$ . If  $d = 0$ , then  $\rho_0^* = r + \frac{\alpha_0}{1 - \bar{\delta}/\nu}$ . Let  $A = \frac{\alpha_0}{1 - \bar{\delta}/\nu}$ . Then  $A < -r$  if and only if  $\bar{\delta} > \nu > \nu_0 \equiv \bar{\delta}/(1 + \alpha_0/r)$  and  $-r \leq A < 0$  if and only if  $\nu_0 \geq \nu > 0$ . Furthermore it is  $0 < A < \alpha_0$  if and only if  $\nu < 0$  and  $A > \alpha_0$  if and only if  $\nu > \bar{\delta}$ . Let  $f(\nu) = \rho_0^*$ . It can be verified that  $f(\nu) \rightarrow +\infty$  as  $\nu \rightarrow \bar{\delta}^+$ ,  $f(\nu) \rightarrow -\infty$  as  $\nu \rightarrow \bar{\delta}^-$ ,  $f(\nu) \rightarrow r$  as  $\nu \rightarrow 0$ , and  $f(\nu) \rightarrow r + \alpha_0$  as  $\nu \rightarrow -\infty$ .

For  $d \neq 0$ ,  $\rho^*$  fulfills

$$(\rho^* - r) \left[ 1 - \frac{1}{\nu}(\bar{\delta} + \bar{d}\rho^*) \right] - \alpha_0 = 0,$$

which has two solutions given by Eq. (3.31). Define  $D \equiv [r - \frac{\nu - \bar{\delta}}{d}]^2 - \frac{4\nu\alpha_0}{d}$ . Then  $\rho_{\pm}^* \in \mathbb{R}$  if and only if  $D \geq 0$ .

Now, consider the Subcases (a)–(c). Firstly, if  $\nu > \bar{\delta} > 0$ , for  $d < 0$  it is  $D \geq 0$ , so  $\rho_{\pm}^* \in \mathbb{R}$ . Furthermore,

$$\rho_-^* < \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} - \left| r - \frac{\nu - \bar{\delta}}{d} \right| \right] = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} - r + \frac{\nu - \bar{\delta}}{d} \right] = \frac{\nu - \bar{\delta}}{d} < 0,$$

as well as,

$$\rho_+^* > \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} + \left| r - \frac{\nu - \bar{\delta}}{d} \right| \right] = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} + r - \frac{\nu - \bar{\delta}}{d} \right] = r.$$

It can also be verified that  $\rho_+^* < \rho_0^*$  if  $d < 0$ . Assume now  $d > 0$  and Eq. (3.32) holds. Then  $\rho_{\pm}^* \in \mathbb{R}$  and we can verify that  $\rho_0^* < \rho_-^*$  if and only if  $(\nu - \bar{\delta})/\bar{d} > r + 2\alpha_0\nu/(\nu - \bar{\delta})$ , which is,  $\bar{d} < d_{oL,1}$ .  $\rho_+^* < \rho_0^*$  if and only if  $r - (\nu - \bar{\delta})/\bar{d} > 0$ , which is,  $\bar{d} > d_{oU,1}$ . Let  $g(\bar{d}) = \rho_+^*$  and  $h(\bar{d}) = \rho_-^*$ . Then we can check that  $g(\bar{d}) \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$  for  $d < 0$ , while for  $d > 0$ ,  $h(\bar{d}) \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$ .

Secondly, assume  $\bar{\delta} > \nu > 0$  and  $d < 0$ . Then  $D \geq 0$ , so that  $\rho_{\pm}^* \in \mathbb{R}$ . Furthermore,

$$\rho_+^* > \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} + \left| r - \frac{\nu - \bar{\delta}}{d} \right| \right] = \begin{cases} r & \text{if } \bar{d} \leq \frac{\nu - \bar{\delta}}{r} \\ \frac{\nu - \bar{\delta}}{d} > r & \text{if } \bar{d} > \frac{\nu - \bar{\delta}}{r} \end{cases}$$

and  $\rho_0^* < r$ , i.e.,  $\rho_+^* > r > \rho_0^*$ . It can also be verified that  $\rho_-^* \leq \rho_0^*$  if and only if  $\rho_0^* \geq 0$ , which is  $\nu \leq \nu_0$ .  $\rho_-^* > \rho_0^*$  if and only if  $\rho_0^* < 0$ , which is,  $\nu > \nu_0$ . Assume now  $d > 0$  and Eq. (3.32) holds. Then  $\rho_{\pm}^* \in \mathbb{R}$ . For  $\nu \leq \nu_0$ , it is

$$\rho_+^* < \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} + \left| r - \frac{\nu - \bar{\delta}}{d} \right| \right] = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} + r - \frac{\nu - \bar{\delta}}{d} \right] = r. \quad (\text{A 3.7})$$

We can also verify that  $\rho_+^* > \rho_0^*$  and  $\rho_-^* < 0$  when  $d > 0$  and  $\nu \leq \nu_0$ . If for  $d > 0$ ,  $\nu > \nu_0$  Eq. (A 3.7) still holds, but we can check that  $\rho_+^* < \rho_0^*$  if and only if  $r + 2\alpha_0\nu/(\nu - \bar{\delta}) > (\nu - \bar{\delta})/\bar{d}$ , which is  $\bar{d} < d_{oL,2}$ .  $\rho_-^* > 0$  if and only if  $r + (\nu - \bar{\delta})/\bar{d} > 0$ , which is  $\bar{d} > d_{oU,2}$ . Furthermore, it is  $h(\bar{d}) \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$  for  $d < 0$ , while for  $d > 0$ ,  $g(\bar{d}) \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$ .

Finally, assume  $\nu < 0$ . Then  $r < \rho_0^*$ . With  $d < 0$  and if Eq. (3.32) holds, it is

$$\rho_+^* < \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{d} + \left| r - \frac{\nu - \bar{\delta}}{d} \right| \right] = r \quad \text{if and only if} \quad r > \frac{\nu - \bar{\delta}}{d},$$

that is  $\bar{d} < d_{oL,3}$ . It can also be verified that  $\rho_-^* > \rho_0^*$  if and only if  $-r - 2\alpha_0\nu/(\nu - \bar{\delta}) > -(\nu - \bar{\delta})/\bar{d}$ , that is  $\bar{d} > d_{oU,3}$ . Suppose now  $d > 0$ . Then  $D \geq 0$  and  $\rho_{\pm}^* \in \mathbb{R}$ . Also,

$$\rho_+^* > \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{\bar{d}} + \left| r - \frac{\nu - \bar{\delta}}{\bar{d}} \right| \right] = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{\bar{d}} + r - \frac{\nu - \bar{\delta}}{\bar{d}} \right] = r$$

and

$$\rho_-^* < \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{\bar{d}} - \left| r - \frac{\nu - \bar{\delta}}{\bar{d}} \right| \right] = \frac{1}{2} \left[ r + \frac{\nu - \bar{\delta}}{\bar{d}} - r + \frac{\nu - \bar{\delta}}{\bar{d}} \right] = \frac{\nu - \bar{\delta}}{\bar{d}} < 0.$$

After this, it can be verified that  $\rho_+^* < \rho_0^*$  when  $d > 0$ . It is  $h(\bar{d}) \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^-$  for  $d < 0$  and  $g(\bar{d}) \rightarrow \rho_0^*$  as  $\bar{d} \rightarrow 0^+$  for  $d > 0$ .

All the results referring to the behavior of monotony can be evidenced by deriving  $g'(\bar{d})$  and  $h'(\bar{d})$  and checking in which case  $g'(\bar{d}) \geq 0$  and  $h'(\bar{d}) \geq 0$ . This completes the proof.

### A 3.4 Proof of Lemma 3.5

Rewrite Eq. (3.22) in the form

$$\rho_t = F(\pi_t, \pi_{t-1}),$$

where  $\alpha_t = \alpha_0$  and

$$F(u, v) \equiv r + \frac{(1+r)(v-u) - \alpha_0 v}{v(u-1)}.$$

Note  $F(\pi_t, \pi_{t-1}) = \text{fct.}(\rho_{t-1}, \dots, \rho_{t-L}, \rho_{t-L-1})$ . For a fixed time lag  $L$ , introduce a new set of variables:

$$x_{1,t} = \rho_t, \quad x_{2,t} = x_{1,t-1} = \rho_{t-1}, \quad \dots, \quad x_{L+1,t} = x_{L,t-1} = \rho_{t-L},$$

and Eq. (3.22) can be reformulated as an  $L+1$ -dimensional system:

$$\begin{cases} x_{1,t+1} & = F(\pi_{t+1}, \pi_t), \\ x_{2,t+1} & = x_{1,t}, \\ & \vdots \\ x_{L+1,t+1} & = x_{L,t}. \end{cases}$$

As

$$\pi_t = \frac{1}{\nu} \{ \bar{\delta} + \bar{d}\bar{\rho}_t \} = \frac{1}{\nu} \{ \bar{\delta} + (\bar{d}/L)[x_{2,t} + x_{3,t} + \dots + x_{L+1,t}] \}, \quad (\text{A 3.8})$$

and thus

$$\pi_{t+1} = \frac{1}{\nu} \{ \bar{\delta} + (\bar{d}/L)[x_{1,t} + x_{2,t} + \dots + x_{L,t}] \}, \quad (\text{A 3.9})$$

it is

$$\begin{aligned} \frac{\partial \pi_t}{\partial x_{1,t}} &= 0 \quad \text{and} \quad \frac{\partial \pi_t}{\partial x_{i,t}} = \frac{\bar{d}}{\nu L} \quad \text{for } i = 2, \dots, L+1, \\ \frac{\partial \pi_{t+1}}{\partial x_{L+1,t}} &= 0 \quad \text{and} \quad \frac{\partial \pi_{t+1}}{\partial x_{j,t}} = \frac{\bar{d}}{\nu L} \quad \text{for } j = 1, \dots, L. \end{aligned}$$

Let  $x_0$  be the steady state of Eq. (3.22). Then, it can be verified that at the fixed equilibrium

$$\frac{\partial F}{\partial \pi_t} = A \quad \text{and} \quad \frac{\partial F}{\partial \pi_{t+1}} = C - A$$

with  $A$  and  $C$  as defined in Eq. (3.5). Now set

$$a_i \equiv \left( \frac{\partial F}{\partial x_{i,t}} \right)_{|(x_0, \dots, x_0)} \quad \text{for } i = 1, \dots, L+1.$$

and note that

$$\left( \frac{\partial F}{\partial x_{i,t}} \right)_{|(x_0, \dots, x_0)} = \left( \frac{\partial F}{\partial u} \frac{\partial u}{\partial x_{i,t}} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x_{i,t}} \right)_{|(x_0, \dots, x_0)},$$

since  $F$  depends on  $u(x_{i,t})$  and  $v(x_{i,t})$ . Then we can obtain

$$a_1 = (C - A) \frac{\bar{d}}{\nu L}, \quad a_2 = \dots = a_L = C \frac{\bar{d}}{\nu L} \quad \text{and} \\ a_{L+1} = A \frac{\bar{d}}{\nu L}.$$

The Jacobian matrix of the system evaluated at the steady state  $x_0$  is

$$J = \begin{pmatrix} a_1 & a_2 & \dots & a_L & a_{L+1} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

from which the characteristic polynomial can be derived as

$$\det(\lambda \mathbb{I} - J) = \lambda^{L+1} - \{a_1 \lambda^L + a_2 \lambda^{L-1} + \dots + a_L \lambda + a_{L+1}\}$$

by induction. This completes the proof.

### A 3.5 Proof of Corollary 3.6

For  $d = 0$ , the verification is trivial as  $\Gamma_L(\lambda) = \lambda^{L+1}$ . Now, suppose  $d \neq 0$  and  $L = 1$ . Then,

$$\Gamma_1(\lambda) = \lambda^2 - \frac{\bar{d}}{\nu} [(C - A)\lambda + A],$$

and the eigenvalues satisfy

$$\lambda^\pm = \frac{1}{2\nu} \left\{ \bar{d}(C - A) \pm \sqrt{\bar{d}^2(C - A)^2 + 4\nu \bar{d}A} \right\}.$$

Consider  $\nu > 0$  first. Then  $\lambda^\pm \in \mathbb{R}$  if  $\bar{d}A > 0$ . Furthermore,

$$\sqrt{\bar{d}^2(C - A)^2 + 4\nu \bar{d}A} > |\bar{d}(C - A)| = |-\bar{d}(C - A)| \geq \bar{d}(C - A),$$

so that  $\lambda^+ > 0$  and  $\lambda^- < 0$ . We can verify that  $\lambda^+ < 1$  if  $\bar{d}C < \nu$  and  $-\bar{d}A < \nu$ . In addition,  $-\lambda^- < 1$  if  $-\bar{d}(C - 2A) < \nu$  and  $-\bar{d}A < \nu$ . Summarizing all assumptions,  $|\lambda^\pm| < 1$  if  $-\bar{d}A < 0$ ,  $\bar{d}C < \nu$  and  $\bar{d}(2A - C) < \nu$ .

Analogously, for  $\nu < 0$ ,  $\lambda^\pm \in \mathbb{R}$  if  $\bar{d}A < 0$ . But now,  $\lambda^+ < 0$  and  $\lambda^- > 0$  for all  $d \neq 0$ . Verifying  $\lambda^- < 1$  if  $\bar{d}C > \nu$ ,  $\bar{d}A < -\nu$  hold, and  $-\lambda^+ < 1$  if  $-\bar{d}(C - 2A) > \nu$ ,  $\bar{d}A < -\nu$  hold, we end up with  $|\lambda^\pm| < 1$  if  $\bar{d}A < 0$ ,  $-\bar{d}C < -\nu$  and  $\bar{d}(C - 2A) < -\nu$ . This completes the proof.

### A 3.6 Proof of Remark 3.7

The first assertion can be simply proved by investigating the structure of Eq. (3.39) at the fixed point  $(\omega^*, \rho^*)$ . Denote

$$y_1 \equiv [R + \pi_{1,t}(\rho^* - r)], \quad y_2 \equiv [R + \pi_{2,t}(\rho^* - r)]. \quad (\text{A 3.10})$$

Then, Eq. (3.39) can be rewritten as

$$\omega^* = \frac{(1 + \omega^*)y_1 - (1 - \omega^*)y_2}{(1 + \omega^*)y_1 + (1 - \omega^*)y_2} \quad (\text{A 3.11})$$



from which it is obvious that  $\omega^* = \pm 1$  solves the equation. Now assume that  $\omega^* \neq \pm 1$ . Then, Eq. (A 3.11) can be simplified as

$$\begin{aligned}\omega^*[(1 + \omega^*)y_1 + (1 - \omega^*)y_2] &= (1 + \omega^*)y_1 - (1 - \omega^*)y_2 \\ \omega^*(1 + \omega^*)y_1 + \omega^*(1 - \omega^*)y_2 &= (1 + \omega^*)y_1 - (1 - \omega^*)y_2 \\ (1 - \omega^*)(1 + \omega^*)y_1 &= (1 + \omega^*)(1 - \omega^*)y_2 .\end{aligned}$$

As  $\omega^* \neq \pm 1$  is assumed, it follows that  $y_1 = y_2$ , which is only fulfilled if  $\rho^* = r$  or  $\pi_{1,t} = \pi_{2,t}$ .

Now assume that  $\rho^* = r$ . Obviously, then  $y_1 = y_2 = R$  and Eq. (A 3.10) leads to a universal equation. Furthermore, at the steady state with  $\rho^* = r$ , the enumerator of Eq. (3.40) has to vanish, and this is only fulfilled if  $-\alpha_0\{(1 + \omega^*)\pi_1^* + (1 - \omega^*)\pi_2^*\} = 0$ , which directly leads to the condition Eq. (3.41). This completes the proof.

### A 3.7 Proof of Lemma 3.8

As the proof of Lemma 3.5 can be almost assigned to the two heterogenous investor case described in Lemma 3.8, only the main steps are sketched out here.

Reformulate the system Eq. (3.39) and Eq. (3.40) as

$$\begin{aligned}\rho_t &= F(\pi_{1,t}, \pi_{1,t-1}, \pi_{2,t}, \pi_{2,t-1}; \omega_{t-1}) \\ \omega_t &= G(\pi_{1,t}, \pi_{1,t-1}, \pi_{2,t}, \pi_{2,t-1}; \omega_{t-1}) ,\end{aligned}$$

where  $F = F(u, v, m, n, o)$  and  $G = G(u, v, m, n, o)$  are defined in an appropriate way as in Lemma 3.5. Let us restrict to the steady state  $E(\omega^*, \rho^*)$  with  $\omega^* = 1$ . Introduce a new set of variables:

$$x_{1,t} = \rho_t , \quad x_{2,t} = \rho_{t-1} , \quad \dots , \quad x_{L+1,t} = \rho_{t-L} , \quad x_{L+2,t} = \omega_t .$$

where  $L \equiv \max(L_1, L_2)$ . Thus the system Eq. (3.39) and Eq. (3.40) can be rewritten as an  $(L + 2)$ -dimensional system:

$$\left\{ \begin{array}{l} x_{1,t+1} = F(\pi_{1,t+1}, \pi_{1,t}, \pi_{2,t+1}, \pi_{2,t}; \omega_t) , \\ x_{2,t+1} = x_{1,t} , \\ \vdots \\ x_{L+1,t+1} = x_{L,t} , \\ x_{L+2,t+1} = G(\pi_{1,t+1}, \pi_{1,t}, \pi_{2,t+1}, \pi_{2,t}; \omega_t) . \end{array} \right.$$

Set

$$f_i(\pi_{i,t+1}, \pi_{i,t}) \equiv r + \frac{(1+r)(\pi_{i,t} - \pi_{i,t+1}) - \alpha_0 \pi_{i,t}}{\pi_{i,t}(\pi_{i,t+1} - 1)} .$$

Analogously to Lemma 3.5 one can obtain

$$\begin{aligned}\frac{\partial \pi_{1,t}}{\partial x_{i,t}} = 0 & \quad \text{for } i = 1, L_1 + 2, \dots, L + 2 & \quad \text{and} & \quad \frac{\partial \pi_{1,t}}{\partial x_{i,t}} = \frac{\bar{d}}{\nu_1 L_1} & \quad \text{for } i = 2, \dots, L_1 + 1 , \\ \frac{\partial \pi_{1,t+1}}{\partial x_{L+1,t}} = 0 & \quad \text{for } i = L_1 + 1, \dots, L + 2 & \quad \text{and} & \quad \frac{\partial \pi_{1,t+1}}{\partial x_{j,t}} = \frac{\bar{d}}{\nu_1 L_1} & \quad \text{for } j = 1, \dots, L_1 , \\ \frac{\partial \pi_{2,t}}{\partial x_{i,t}} = 0 & \quad \text{for } i = 1, L_2 + 2, \dots, L + 2 & \quad \text{and} & \quad \frac{\partial \pi_{2,t}}{\partial x_{i,t}} = \frac{\bar{d}}{\nu_2 L_2} & \quad \text{for } i = 2, \dots, L_2 + 1 , \\ \frac{\partial \pi_{2,t+1}}{\partial x_{L+1,t}} = 0 & \quad \text{for } i = L_2 + 1, \dots, L + 2 & \quad \text{and} & \quad \frac{\partial \pi_{2,t+1}}{\partial x_{j,t}} = \frac{\bar{d}}{\nu_2 L_2} & \quad \text{for } j = 1, \dots, L_2 .\end{aligned}$$

Furthermore, at the fixed equilibrium  $(\omega^* = 1, \rho^*)$  we can calculate

$$\begin{aligned} \frac{F}{\partial \pi_{1,t}} &= \frac{\partial f_1}{\partial \pi_{1,t}} = A_1 \quad \text{and} \quad \frac{F}{\partial \pi_{1,t+1}} = \frac{\partial f_1}{\partial \pi_{1,t+1}} = C_1 - A_1, \\ \frac{F}{\partial \pi_{2,t}} &= \frac{F}{\partial \pi_{2,t+1}} = 0 \quad \text{and} \quad \frac{F}{\partial \omega_t} = \frac{\alpha_0 \pi_{2,0} (\pi_{1,0} - \pi_{2,0})}{2\pi_{1,0} (\pi_{1,0} - 1)^2}, \\ \frac{G}{\partial \pi_{1,t}} &= \frac{G}{\partial \pi_{1,t+1}} = \frac{G}{\partial \pi_{2,t}} = \frac{G}{\partial \pi_{2,t+1}} = 0 \quad \text{and} \quad \frac{G}{\partial \omega_t} = \beta_1. \end{aligned}$$

with  $\alpha_t = \alpha_0$  and  $A_1, C_1$  and  $\beta_1$  defined in Lemma 3.8. Now define

$$a_i \equiv \left( \frac{\partial F}{\partial x_{i,t}} \right)_{|E(\omega^*=1, \rho^*)} \quad \text{and} \quad b_i \equiv \left( \frac{\partial G}{\partial x_{i,t}} \right)_{|E(\omega^*=1, \rho^*)} \quad \text{for } i = 1, \dots, L+2.$$

Based on the calculation in Lemma 3.5 it can be verified that

$$\begin{aligned} a_1 &= \frac{\bar{d}}{\nu_1 L_1} (C_1 - A_1) \quad \text{and} \quad a_2 = \dots = a_{L_1} = \frac{\bar{d}}{\nu_1 L_1} (A_1 + (C_1 - A_1)) = \frac{\bar{d}}{\nu_1 L_1} C_1, \\ a_{L_1+1} &= \frac{\bar{d}}{\nu_1 L_1} A_1 \quad \text{and} \quad a_{L_1+2} = \dots = a_{L+1} = 0, \quad a_{L+2} = \frac{\alpha_0 \pi_{2,0} (\pi_{1,0} - \pi_{2,0})}{2\pi_{1,0} (\pi_{1,0} - 1)^2}, \\ b_1 &= \dots = b_{L+1} = 0 \quad \text{and} \quad b_{L+2} = \beta_1. \end{aligned}$$

Then, the Jacobian matrix of the system evaluated at the steady state  $(\omega^* = 1, \rho^*)$  is

$$\begin{aligned} J_1 &= \begin{pmatrix} a_1 & a_2 & \dots & a_{L_1} & a_{L_1+1} & a_{L_1+2} & \dots & a_{L+2} \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ b_1 & b_2 & \dots & b_{L_1} & b_{L_1+1} & b_{L_1+2} & \dots & b_{L+2} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & \dots & a_{L_1} & a_{L_1+1} & 0 & \dots & a_{L+2} \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \beta_1 \end{pmatrix} \end{aligned}$$

from which the characteristic polynomial can be derived using Laplace's expansion rule and Lemma 3.5 as

$$\begin{aligned} \det(\lambda \mathbb{I} - J_1) &= (-1)^{2(L+2)} (\lambda - \beta_1) \det((J'_1)_{L+2, L+2}) \\ &= (\lambda - \beta_1) \left\{ \lambda^{L-L_1} [\lambda^{L_1+1} - \{a_1 \lambda^{L_1} + a_2 \lambda^{L_1-1} + \dots + a_{L_1} \lambda + a_{L_1+1}\}] \right\} \\ &= (\lambda - \beta_1) \lambda^{L-L_1} [\lambda^{L_1+1} - a_1 \lambda^{L_1} - a_2 \lambda^{L_1-1} - \dots - a_{L_1} \lambda + a_{L_1+1}], \end{aligned}$$

where  $J'_{i,j}$  is the matrix in which the  $i$ th row and  $j$ th column is deleted. The case where  $\omega^* = -1$  can be derived analogously. This completes the proof.

### A 3.8 Proof of Theorem 3.9

Using Eq. (3.39) and Eq. (3.13) instead of Eq. (3.40), the dynamic system can be written at the fixed point  $(\omega^*, \rho^*)$  by

$$\omega^* = \frac{[1 + \omega^*][R + \pi_1^*(\rho^* - r)] - [1 - \omega^*][R + \pi_2^*(\rho^* - r)]}{[1 + \omega^*][R + \pi_1^*(\rho^* - r)] + [1 - \omega^*][R + \pi_2^*(\rho^* - r)]}, \quad (\text{A 3.12})$$

$$\rho^* = r + \frac{\alpha_0}{1 - \sum_{k=1}^2 \pi_k^* \omega_k^*}, \quad (\text{A 3.13})$$

where  $\omega_1^* = (1 + \omega^*)/2$  and  $\omega_2^* = (1 - \omega^*)/2$ . Note that Eq. (A 3.13) directly follows from Eq. (3.13) if  $\sum_{i=1}^2 \pi_i^* \omega_i^* \neq 0$  and  $\alpha_t = \alpha_0 > 0$  is assumed. Hence the assertion can be concluded directly from Remark 3.7.

It is necessary to make one last remark in order to understand why the case  $\rho^* = r$  is neglected in the theorem. Note that using Eq. (3.13) instead of Eq. (3.40) implies that  $\rho^* = r$  as a fixed point is excluded in Eq. (A 3.13) since  $\alpha_0$  is assumed to be strictly positive. But this reasoning depends crucially on the value of the sum  $\sum_{i=1}^2 \pi_i^* \omega_i^*$ . Note if, in turn,  $\sum_{i=1}^2 \pi_i^* \omega_i^* = 0$  then Eq. (3.13) is not defined at the fixed point. Therefore, this case demands that the original dynamic equation Eq. (3.40) be used (instead of Eq. (3.13)), which has been shown to be equivalent to Eq. (3.17) in Theorem 3.1 for  $I = 2$ . From this equation, it can again easily be seen that it is  $\rho^* = r$  at the fixed point if it is  $\sum_{i=1}^2 \pi_i^* \omega_i^* = 0$  and  $\alpha_0 > 0$ . This completes the proof.

### A 3.9 Proof of Corollary 3.10

Consider the following 3 cases

$$\begin{aligned} (i) \quad & \nu_i > \bar{\delta} > 0, & (ii) \quad & \bar{\delta} > \nu_i > 0, \quad i = 1, 2 \\ (iii) \quad & \nu_{i_1} > \bar{\delta} > \nu_{i_2} > 0, & & i_j = 1, 2, j = 1, 2, i \neq j, \end{aligned}$$

and note that according to Eq. (3.48) for Case (i) and Case (ii)

$$\rho_{0,1}^* > \rho_{0,2}^* \quad \text{if and only if} \quad \nu_2 > \nu_1 \quad (\text{A 3.14})$$

and for Case (iii)

$$\rho_{0,1}^* > \rho_{0,2}^* \quad \text{if and only if} \quad \nu_2 < \nu_1. \quad (\text{A 3.15})$$

Under the assumptions of Theorem 3.9 the characteristic polynomial Eq. (3.42) from Lemma 3.8 is given by

$$\Gamma_{L_i}(\lambda) = (\lambda - \beta_i)\lambda^{L+1}.$$

Hence,  $E_1$  is locally asymptotically stable if and only if  $|\beta_1| < 1$  with  $\beta_1$  given in Eq. (3.45), which is equivalent to

$$\left| (1+r) + (\rho_{0,1}^* - r) \frac{\bar{\delta}}{\nu_2} \right| < \left| (1+r) + (\rho_{0,1}^* - r) \frac{\bar{\delta}}{\nu_1} \right|$$

and finally

$$\left| (1+r) + \frac{\alpha_0 \bar{\delta} \nu_1}{(\nu_1 - \bar{\delta}) \nu_2} \right| < \left| (1+r) + \frac{\alpha_0 \bar{\delta}}{(\nu_1 - \bar{\delta})} \right|. \quad (\text{A 3.16})$$

Note that the right-hand side is strictly positive for all Cases (i)–(iii) if  $\nu_1 < \bar{\delta}(1 - \epsilon)$  or  $\nu_1 > \bar{\delta}$ . Then the result can be obtained simply for (i) by solving for the norm and using Eq. (A 3.14).

Now assume that  $\bar{\delta} > \nu_i > 0$  and  $E_1$  is locally asymptotically stable. The left-hand side is either positive, which leads to the conditions  $\nu_1 > \nu_2$  and  $\nu_2 > \epsilon(\bar{\delta}\nu_1)/(\bar{\delta} - \nu_1)$ , or the left-hand side becomes negative, which implies the conditions  $\nu_2 > \nu_{0,2}$  and  $\nu_2 < \epsilon(\bar{\delta}\nu_1)/(\bar{\delta} - \nu_1)$ . Summarizing these two subcases leads to the desired result.

Finally, consider the Case (iii). For both assumptions  $\nu_2 > \bar{\delta} > \nu_1$  and  $\nu_1 > \bar{\delta} > \nu_2$ , the condition Eq. (A 3.16) leads to a contradiction. Thus  $E_1$  is not locally asymptotically stable. This completes the proof.

### A 3.10 Proof of Corollary 3.12

The proof is similar to the proof of Corollary 3.10. Again, three cases have to be considered:

$$\begin{aligned}
(i) \quad & \nu > \bar{\delta}_i > 0, & (ii) \quad & \bar{\delta}_i > \nu > 0, \quad i = 1, 2 \\
(iii) \quad & \bar{\delta}_{i_1} > \nu > \bar{\delta}_{i_2} > 0, & & i_j = 1, 2, j = 1, 2, i \neq j.
\end{aligned}$$

Assume Case (i) or Case (ii) holds. From Eq. (3.51) it can be derived that

$$\rho_{0,1}^* > \rho_{0,2}^* \quad \text{if and only if} \quad \bar{\delta}_1 > \bar{\delta}_2.$$

Otherwise, if Case (iii) is fulfilled

$$\rho_{0,1}^* > \rho_{0,2}^* \quad \text{if and only if} \quad \bar{\delta}_1 < \bar{\delta}_2.$$

From Eq. (3.45) it can be verified, that  $|\beta_1| < 1$  if and only if

$$\left| (1+r) + \frac{\alpha_0 \bar{\delta}_2}{\nu - \bar{\delta}_1} \right| = \left| (1+r) + \frac{\alpha_0 \bar{\delta}_1}{\nu - \bar{\delta}_1} \right|.$$

If we suppose that the arguments of the left-hand side and right-hand side are positive, all results can be obtained in a similar way to Corollary 3.10. Note that the argument of the right-hand side is positive for Case (ii) if and only if  $\nu < \bar{\delta}_1(1 - \epsilon)$  with  $\epsilon \equiv \alpha_0/(1+r)$ , which is approximately  $\nu(1 + \epsilon) < \bar{\delta}_1$ . In the same way, it can be argued that the left-hand side is positive if and only if  $\bar{\delta}_2 < \bar{\delta}_{0,2}$  with  $\bar{\delta}_{0,2}$  defined as in the corollary. Furthermore, if Case (iii) holds, it can be shown that  $|\beta_1| > 1$  for all choice of  $\bar{\delta}_i > 0$ . This completes the proof.

## CHAPTER 4

# Asset Price and Wealth Dynamics under a Market Maker Scenario

### 4.1 Introduction

In recent years a broad and growing strand of literature addressing price dynamics in financial markets has arisen, which is based on observations that indicate selective departures from the standard assumptions of unbounded rationality in market participants. The work is motivated by the accordance, or common view, that phenomena such as excess volatility, mean-reversion and other universal properties of speculative prices (now widely known as ‘stylized facts’) cannot be fully explained by the traditional theory of finance. While the efficient market hypothesis (see Fama, 1970) assumes that speculative prices always reflect the fundamental value, diverse and detailed observations across many markets, instruments and time periods show many noticeable deviations from fundamental values. A great deal of this research uses a chartist-fundamentalist framework in which two (or even more) trader types coexist in the artificial market and where each trader type pursues a distinctly different trading strategy. Examples of important work related to this topic are Beja and Goldman (1980); Frankel and Froot (1986); Day and Huang (1990); De Long et al. (1990); Kirman (1991); Chiarella (1992); Brock and Hommes (1997*b,a*, 1998); Franke and Sethi (1998); Lux and Marchesi (1999, 2000); Chiarella et al. (2002); Westerhoff (2003). While fundamentalists (or informed traders) are mostly assumed to react to the current price deviating from its fundamental value, chartists (or noise traders) extrapolate expected future price movements from their observations of historical time series data. Interestingly, empirical work also supports this approach, as shown in the classic survey studies by Allen and Taylor (1990); Taylor and Allen (1992) in which traders from the London Foreign Exchange Market admit to using technical analysis and trend-following strategies for short-term speculative trading.

Due to these findings one strand of literature describes financial markets as nonlinear structural agents models that emphasize heterogeneity in allowing agents to have different beliefs, different endowments or different market attitudes, for example towards risk. In these multi-agents systems, financial markets are considered to be complex evolutionary systems between competing boundedly-rational trading strategies.<sup>1</sup> Following this line of research, my approach involves a simple nonlinear market fraction model with heterogeneous traders, fundamentalists and chartists, who all have different trading strategies and expectations about future prices and dividends of a

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<sup>1</sup>See Hommes (2001) for an overview about related literature.

risky asset. My model mainly builds on the work by Brock and Hommes (1997*b,a*, 1998, 1999), who propose to model economic markets as an ‘Adaptive Belief System’ (ABS), but assumes a more realistic framework where the agents’ demand depends on their current wealth level (as a result of the underlying CRRA utility) as proposed by Chiarella and He (2001) and Chiarella et al. (2006). As in these two cited papers, I also assume a growing dividend process and a trend in the fundamental price of the risky asset leading to a growth model in both risky-asset prices and agents’ wealth. However, the chapter aims to contribute to the development and analysis of such models by proposing a slightly different and, arguably, a slightly more realistic model structure which explicitly takes into account the risky-asset supply side. This extension in the model structure allows me to model the risk premium demanded by the market participants for taking market risk, which appears to be endogenously driven by the market over time. The analysis of the resulting dynamics of asset price and agents’ wealth within this quite advanced chartist-fundamentalist framework is very similar to the methods used for the models developed by Chiarella and He (2001) and Chiarella et al. (2006). Indeed, we are also able to characterize the equilibria and the other kinds of asymptotic behavior in terms of the long-run evolution of wealth proportions and risky-asset returns, as was done in these two papers.

The chapter is organized as follows. The next section presents the basic ideas and main building blocks of the proposed model and then the following three sections outline a comprehensive study of different model variants. This includes both homogeneous and heterogeneous expectations of investors with adaptive and non-adaptive expectation formation rules, different time scales for the underlying trading processes and finally a simple stochastic model version. All the model variants are investigated by starting with an analytical treatment and continuing with the performance of extensive numerical simulations. The chapter concludes by summarizing the results of the analysis, while proofs and more technical remarks are relegated to the appendix.

## 4.2 Model Description

This section outlines the model framework of an asset pricing model with heterogeneous agents using CRRA utility for their portfolio optimization problem under a market maker scenario in discrete time<sup>2</sup>, where heterogeneity among investors is established by assuming them to use different and competing trading strategies. The model shares some basic ideas with the CH model (see Chiarella and He, 2001), which was introduced in the previous part of this thesis. In addition, some ideas are adopted from Chiarella et al. (2006). Therefore, it can be seen as a continuative development in this specific way of modeling financial markets. For instance, my presentation will see a nonlinear stationary model being derived in the same way as in the CH model, i.e., by expressing the dynamics of the market model in terms of the return of the risky asset and the wealth proportion among the heterogeneous investor groups instead of describing it in terms of the asset price and the total wealth of investors. However, with respect to the main building blocks of the model framework, my model sometimes differs crucially from the original CH model and from the publication by Chiarella et al. (2006). For this reason, we will see the asset return and wealth dynamics displaying different behavior; this will become evident in the following discussion

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<sup>2</sup>For an example of a continuous-time model with heterogeneous beliefs see Cabrales and Hoshi (1996).

outlined in the remainder of this chapter.

### 4.2.1 Notations and Portfolio Optimization

Our hypothetical financial market is based on a simple discrete asset pricing model and contains two investment choices: a stock (or index of stocks) and a bond. The bond is assumed to be a risk-free asset and to be perfectly elastically supplied and the stock is a risky asset. Let  $r_f$  be the risk-free rate per trading period,  $I$  be the total number of investors in the market, and  $W_{i,0}$  be the initial wealth of agent  $i$ . At time  $t$ , denote by  $p_t$  the price (ex dividend) per share of the risky asset,  $d_t$  the dividend and  $W_{i,t}$  the wealth of agent  $i$  for  $i = 1, 2, \dots, I$ . It is assumed that all agents are myopic and are using a power utility function with different risk-aversion coefficients  $\nu_i$ , which is

$$U_i(W) = \frac{W^{1-\nu_i} - 1}{1 - \nu_i}, \quad \nu_i > 0. \quad (4.1)$$

The return of the risky asset at period  $t$  is defined by

$$\rho_t = \frac{p_t - p_{t-1} + d_t}{p_{t-1}} = r_t + \frac{d_t}{p_{t-1}} \quad (4.2)$$

where

$$r_t = \frac{p_t - p_{t-1}}{p_{t-1}} \quad (4.3)$$

denotes the return of the stock (ex dividend), while the last summand in Eq. (4.2) denotes the dividend yield.

Following the standard portfolio approach, the wealth of agent (or investor)  $i$  at time period  $t + 1$  is given by<sup>3</sup>

$$W_{i,t+1} = W_{i,t} [1 + r_f + \pi_{i,t}(\rho_{t+1} - r_f)], \quad (4.4)$$

where  $\pi_{i,t}$  is the wealth proportion of agent  $i$  invested in the risky asset. Given the information set  $\mathcal{I}_t$  at time  $t$ , let

$$E_{i,t}(\rho_{t+1}) \quad \text{and} \quad \sigma_{i,t}^2 = \text{Var}_{i,t}(\rho_{t+1})$$

be the ‘beliefs’ of investor  $i$  about the conditional expectation of the mean and variance of the return  $\rho_{t+1}$ . For the given power utility function, the optimum investment proportion at time  $t$ ,  $\pi_{i,t}$  is given by<sup>4</sup>

$$\pi_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r_f}{\nu_i \sigma_{i,t}^2}. \quad (4.5)$$

Finally, let  $\beta_{i,t}$  be the growth rate of wealth of agent  $i$ . Then, it follows from Eq. (4.4) that the wealth dynamics of each single investor evolves according to

$$W_{i,t+1} = (1 + \beta_{i,t+1})W_{i,t}, \quad (4.6)$$

where the growth rate of wealth is determined by

$$\beta_{i,t+1} = r_f + \pi_{i,t}(\rho_{t+1} - r_f). \quad (4.7)$$

<sup>3</sup>See also Eq. (3.2) in Chapter 3, Section 3.2 for more details.

<sup>4</sup>See also remarks in the previous chapter on Eq. (3.5) and Eq. (3.26) for further details.

### 4.2.2 Excess Demand

It follows from Eq. (4.5) that the optimal wealth of investor  $i$  invested in the risky asset is given by

$$\zeta_{i,t} = \pi_{i,t} W_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r_f}{\nu_i \sigma_{i,t}^2} W_{i,t}. \quad (4.8)$$

Given the growth rate of wealth at time  $t$ , the individual excess demand on the wealth invested in the risky asset by investor  $i$  can be written as

$$\tilde{\zeta}_{i,t} = \zeta_{i,t} - (1 + r_t) \zeta_{i,t-1}. \quad (4.9)$$

Note that the subtrahend on the right-hand side of Eq. (4.9) considers that the wealth share invested in the risky asset in the previous period grows by the return on the risky asset (ex dividend) at period  $t$  according to Eq. (4.3), as *no retention of the dividends is assumed* here. In other words this term represents the investor's current wealth share invested in the risky asset at time  $t$  resulting from his/her investment decision made in period  $t-1$ , while the minuend represents his/her (nominal) risky-asset demand<sup>5</sup> calculated from the myopic utility maximizing scheme at time  $t$ . Obviously, the difference between minuend and subtrahend defines the individual excess demand of each investor for the period  $t$ . Note also that the investors' (nominal) demand for the risky asset depends on their current wealth level; this dependence is a direct consequence of the assumption of a CRRA utility function given in Eq. (4.1).<sup>6</sup> Because of the underlying CRRA utility function, each investor will invest the same proportion of his/her wealth in the risky asset in each trading period while the individual wealth level  $W_{i,t}$  is principally a growing process over time. Thus the investor's (nominal) demand for the risky asset grows with his/her growth rate of individual wealth, which is  $\beta_{i,t}$ . Using Eq. (4.6) and Eq. (4.8), Eq. (4.9) can be written as follows:

$$\tilde{\zeta}_{i,t} = \pi_{i,t} W_{i,t} - (1 + r_t) \pi_{i,t-1} W_{i,t-1} = W_{i,t-1} [\pi_{i,t}(1 + \beta_{i,t}) - \pi_{i,t-1}(1 + r_t)].$$

The investors' aggregate excess demand in wealth units denoted by  $\tilde{\zeta}_t$  is simply the sum of all individual investors' excess demands and is therefore given by

$$\tilde{\zeta}_t = \sum_{i=1}^I \tilde{\zeta}_{i,t} = \sum_{i=1}^I W_{i,t-1} [\pi_{i,t}(1 + \beta_{i,t}) - \pi_{i,t-1}(1 + r_t)]. \quad (4.10)$$

Let  $H_t$  be the total (nominal) risky-asset supply at time  $t$ , which is determined by

$$H_t = N_t^s p_t, \quad (4.11)$$

where  $N_t^s$  denotes the number of outstanding risky assets at time period  $t$ . It is assumed that  $N_t^s$  is subject to a linearly growing process over time, which means that the number of outstanding risky assets increases by a fixed quantity in each trading period. Although it is quite evident that the assumption of an increasing number of outstanding risky assets is made simply to balance the growing investors' aggregate excess demand for risky assets, a more detailed motivation for this

<sup>5</sup>The investor's asset demand (as well as his/her asset supply) will be described for the most part of the model framework in wealth shares (i.e., measured in wealth units) instead of wealth proportions. This will then be denoted by 'nominal' asset demand (or supply) in the following discussion.

<sup>6</sup>See also Appendix A 1.3 in Chapter 1 for details of different types of utility functions.



assumption will be given in the following subsections. However, this assumption implies that the total (nominal) risky-asset supply  $H_t$  will also be a process growing over time, because  $p_t$  will in general also grow over time (see Eq. (4.11)). Hence the overall *market* excess demand  $\check{\zeta}_t$  in our artificial financial market is determined by

$$\check{\zeta}_t = \tilde{\zeta}_t - [H_t - (1 + r_t)H_{t-1}] , \quad (4.12)$$

where the term in the square brackets can be explained as follows. As already stated,  $H_t$  is the current (nominal) risky-asset supply in the market while  $H_{t-1}$  is the (nominal) risky-asset supply of the preceding trading period. During the trading period  $t$  the (nominal) risky-asset supply  $H_{t-1}$  already existing obviously grows by the return of the risky asset (ex dividends), which implies that the difference of  $H_t$  and  $(1 + r_t)H_{t-1}$  is exactly the (nominal) amount of risky assets (and not the *number* of risky assets) exogenously fed into the market system (by increasing  $N_{t-1}^s$  to  $N_t^s$  during the time period  $t$ ) to meet the assumption of having a (nominal) risky-asset supply of  $H_t$ . Consequently, this (nominal) amount of risky assets has to be subtracted from the investors' aggregated excess demand in order to determine the market excess demand, i.e., the (nominal) risky asset amount desired by the investors but not supplied by the market condition, which is here denoted by  $\check{\zeta}_t$ .<sup>7</sup>

Finally, let  $W_t$  be the total wealth of all investors at time  $t$ , which is

$$W_t = \sum_{i=1}^I W_{i,t} . \quad (4.13)$$

Then it follows from Eq. (4.6) and Eq. (4.13) that the total wealth  $W_t$  can be expressed by the investors' individual wealth levels, namely

$$W_t = \sum_{i=1}^I (1 + \beta_{i,t}) W_{i,t-1} . \quad (4.14)$$

In the remainder of this chapter we will concentrate on the special case with two investor-groups acting in the market. Suppose that all investors can be subdivided into two groups with  $N_1$  investors in the first group and  $N_2$  in the second group. Within each group, agents are heterogeneous in their wealth but homogeneous in their risk-aversion coefficients and beliefs in terms of the conditional mean and variance of the return  $\rho_{t+1}$ . Under these assumptions the investors' aggregate excess demand in wealth is given by

$$\tilde{\zeta}_t = [(1 + \beta_{1,t})\pi_{1,t} - (1 + r_t)\pi_{1,t-1}] W_{1,t-1} + [(1 + \beta_{2,t})\pi_{2,t} - (1 + r_t)\pi_{2,t-1}] W_{2,t-1} , \quad (4.15)$$

<sup>7</sup>Of course, the term in the square brackets of Eq. (4.12) can also be explained in terms of outstanding risky assets. Let us assume that the number of outstanding risky assets grows linearly over time according to  $N_t^s = (1 + n^s)N_{t-1}^s$ . Then, the term in the square brackets can be written as  $[H_t - (1 + r_t)H_{t-1}] = [(1 + n^s)N_{t-1}^s p_t - (1 + r_t)N_{t-1}^s p_{t-1}] = [(1 + n^s)N_{t-1}^s p_t - N_{t-1}^s p_t] = n^s N_{t-1}^s p_t$  by using Eq. (4.11). Hence the term in the square brackets is simply the supplemental number of risky assets which are newly available in the market system at time period  $t$  multiplied by the current price level of risky assets. Thus this expression is in total a nominal value, namely the newly additional risky-asset supply generated by the increased number of risky assets which has to be subtracted from the investors' aggregated excess demand  $\tilde{\zeta}_t$  to obtain the market excess demand in the model as already stated above.

where  $W_{1,t}$  and  $W_{2,t}$  are respectively the total wealth of Groups 1 and 2 at time  $t$ . Using the same notation as proposed by Chiarella and He (2001) (and as already seen in the previous chapter of this thesis) I will now denote  $\bar{W}_{1,t}$ ,  $\bar{W}_{2,t}$  as the average wealth of agents within Groups 1 and 2 respectively and  $\bar{\omega}_{1,t}$ ,  $\bar{\omega}_{2,t}$  as the average wealth proportions of the corresponding group. Then it is

$$W_{1,t} = N_1 \bar{W}_{1,t}, \quad W_{2,t} = N_2 \bar{W}_{2,t} \quad (4.16)$$

and

$$\bar{\omega}_{i,t} = \frac{\bar{W}_{i,t}}{\bar{W}_t} \quad \text{with} \quad \bar{W}_t = \bar{W}_{1,t} + \bar{W}_{2,t}, \quad i = 1, 2. \quad (4.17)$$

Note that

$$W_t = N_1 \bar{W}_{1,t} + N_2 \bar{W}_{2,t}. \quad (4.18)$$

Furthermore let  $n_i$  be the fixed proportion of the number of investors in group  $i$  relative to the total number of investors  $I$ , i.e.,

$$n_1 = \frac{N_1}{I} \quad \text{and} \quad n_2 = \frac{N_2}{I} \quad (4.19)$$

with  $I = N_1 + N_2$ . Finally, by defining  $q_t$  as the market excess demand proportional to the total wealth at time  $t$  as

$$q_t \equiv \frac{\check{\zeta}_t}{\bar{W}_t}, \quad (4.20)$$

it follows from Eq. (4.15)–Eq. (4.20) that the market excess demand proportion  $q_t$  satisfies<sup>8</sup>

$$q_t = \frac{\sum_{i=1}^2 [(1 + \beta_{i,t})\pi_{i,t} - (1 + r_t)\pi_{i,t-1}] n_i \bar{\omega}_{i,t-1} - [H_t - (1 + r_t)H_{t-1}] / (\bar{W}_{t-1} I)}{\sum_{i=1}^2 (1 + \beta_{i,t}) n_i \bar{\omega}_{i,t-1}}, \quad (4.21)$$

while the average wealth proportions of investor group  $i$  at time  $t + 1$  are given by

$$\bar{\omega}_{i,t+1} = \frac{\bar{\omega}_{i,t}(1 + \beta_{i,t+1})}{\bar{\omega}_{1,t}(1 + \beta_{1,t+1}) + \bar{\omega}_{2,t}(1 + \beta_{2,t+1})}, \quad i = 1, 2. \quad (4.22)$$

Similar to the CH model outlined in Chapter 3, it is convenient to introduce new variables

$$\bar{\omega}_t \equiv \bar{\omega}_{1,t} - \bar{\omega}_{2,t} \quad \text{and} \quad \bar{n} \equiv n_1 - n_2 \quad (4.23)$$

in the case of a two investor-groups model. While the state variable  $\bar{\omega}_t$  denotes the difference in the average wealth proportions of Investor Groups 1 and 2, the variable  $\bar{n}$  accounts for the difference in the fixed population proportion of the two investor groups. Of course, the investors groups' average wealth proportions and the corresponding population proportions can still be re-determined from the new variables  $\bar{\omega}_t$  and  $\bar{n}$  using the relations

$$\bar{\omega}_{1,t} = \frac{1 + \bar{\omega}_t}{2}, \quad \bar{\omega}_{2,t} = \frac{1 - \bar{\omega}_t}{2}, \quad n_1 = \frac{1 + \bar{n}}{2}, \quad \text{and} \quad n_2 = \frac{1 - \bar{n}}{2}.$$

Thus the time evolution of the market excess demand proportion  $q_t$  and the average wealth proportions  $\bar{\omega}_{i,t}$  can be written in terms of the newly introduced variables  $\bar{\omega}_t$  and  $\bar{n}_t$  as

$$q_t = \frac{[(1 + \beta_{1,t})\pi_{1,t} - (1 + r_t)\pi_{1,t-1}] (1 + \bar{n}) [1 + \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n}) [1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n}) [1 - \bar{\omega}_{t-1}]} + \frac{[(1 + \beta_{2,t})\pi_{2,t} - (1 + r_t)\pi_{2,t-1}] (1 - \bar{n}) [1 - \bar{\omega}_{t-1}] - 4[H_t - (1 + r_t)H_{t-1}] / (\bar{W}_{t-1} I)}{(1 + \beta_{1,t})(1 + \bar{n}) [1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n}) [1 - \bar{\omega}_{t-1}]} \quad (4.24)$$

<sup>8</sup>The calculations of Eq. (4.21) and Eq. (4.22) are relegated to the Appendix A 4.1.

and

$$\bar{\omega}_{t+1} = \frac{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) - [1 - \bar{\omega}_t](1 + \beta_{2,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{2,t+1})}$$

respectively.

One further remark on this notation should be added at this point. In the two investor-groups model, note that the optimal investment proportions  $\pi_{i,t}$  of group  $i$  (and thus also the growth rate of wealth  $\beta_{i,t}$ ,  $i = 1, 2$ ) should be provided with a bar to emphasize that these are the (average) optimal investment proportions of the investor group  $i$ .<sup>9</sup> Nevertheless, since the optimum demands for wealth proportions to be invested in the risky asset are the same for each investor within each group  $i$ , the bar is omitted to avoid superfluous notation.<sup>10</sup>

### 4.2.3 Fundamental Price and Return

One crucial goal in my model framework is to incorporate the fundamental pricing benchmark as a special case of my more general heterogeneous multi-agents model.<sup>11</sup> For this reason, the model will first be elaborated on the fundamental price path. Let us start with the dividend process. In the most general case, the growth rate of dividends  $g_t$  is assumed to be time-dependent, so that  $d_t$  is specified by

$$d_{t+1} = (1 + g_{t+1})d_t . \quad (4.25)$$

In the remainder of the chapter it will be assumed that agents are homogeneous in their beliefs about the dividend process, but that they are heterogeneous with respect to the price component of the risky asset. To obtain the fundamental price path for the fundamental price  $p_t^*$ , we assume that the dividends evolve in a deterministic way ('deterministic skeleton'), i.e.,  $g_t = g$  with  $g < r_f$ .

To determine the equilibrium condition in our model framework, asset demand has to be equal to asset supply. According to Eq. (4.8) investor  $i$ 's (nominal) demand for risky assets at time  $t$  can be written as

$$\zeta_{i,t} = \pi_{i,t} W_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r_f}{\nu_i \sigma_{i,t}^2} W_{i,t} ,$$

where  $\pi_{i,t}$  is the optimum investment proportion given by Eq. (4.5) and  $W_{i,t}$  is the wealth of investor  $i$  at time  $t$ . Remember that  $E_{i,t}(\rho_{t+1})$  and  $\sigma_{i,t}^2$  denote the beliefs of investor  $i$  about the conditional expectation of the mean and variance of the risky-asset return  $\rho_{t+1}$  respectively. It follows that

$$\begin{aligned} \zeta_{i,t} &= \frac{E_{i,t}(\rho_{t+1}) - r_f}{\nu_i \sigma_{i,t}^2} W_{i,t} \\ &= \frac{E_{i,t}(p_{t+1} + d_{t+1}) - (1 + r_f)p_t}{\nu_i \sigma_{i,t}^2} \frac{W_{i,t}}{p_t} , \end{aligned} \quad (4.26)$$

where  $\rho_{t+1} = (p_{t+1} + d_{t+1} - p_t)/p_t$  is used. Now, denoting  $H_t$  as the positive (nominal) supply of risky assets in our artificial market and equalizing agents' demand on risky assets to the supply of

<sup>9</sup>This notation has already been used in Theorem 3.3 in Chapter 3.

<sup>10</sup>An identical optimum demand  $\pi_{i,t}$  for each investor within each group  $i$  implies that the wealth evolution (or monetary gain) is also identical within each investor group (see Eq. (4.7)). This will be often used as a crucial fact in the following discussion and investigation of this model framework.

<sup>11</sup>See also the BH model framework presented in Chapter 2, Section 2.1 for a similar approach.

risky assets leads to

$$H_t = \sum_{i=1}^I \zeta_{i,t}, \quad (4.27)$$

where the right-hand side denotes the aggregated demand for risky assets at time  $t$ . Combining Eq. (4.27) with Eq. (4.26) yields

$$H_t = \sum_{i=1}^I \frac{E_{i,t}(p_{t+1} + d_{t+1}) - (1 + r_f)p_t}{\nu_i \sigma_{i,t}^2 p_t} W_{i,t}, \quad (4.28)$$

which describes the market equilibrium of our artificial market.

Let us assume for a moment that all investors form correct (and thus homogeneous) expectations as reasonably hypothesized on the fundamental price level  $p_t^*$ . In this case it is

$$H_t = [E_t(p_{t+1} + d_{t+1}) - (1 + r_f)p_t] \frac{1}{p_t} \sum_{i=1}^I \frac{W_{i,t}}{\nu_i \sigma_{i,t}^2}. \quad (4.29)$$

Rewriting Eq. (4.29), it follows that

$$E_t(p_{t+1} + d_{t+1}) = p_t \left\{ 1 + r_f + H_t \left[ \sum_{i=1}^I \frac{W_{i,t}}{\nu_i \sigma_{i,t}^2} \right]^{-1} \right\}. \quad (4.30)$$

Assuming also a homogeneous perception of the risk-aversion coefficient and constant expectations of the variance of the return of the risky asset, Eq. (4.30) reads

$$p_t = \frac{1}{1 + r_f + \frac{H_t}{W_t} \nu \sigma^2} E_t(p_{t+1} + d_{t+1}), \quad (4.31)$$

where  $W_t = \sum_{i=1}^I W_{i,t}$  denotes the total wealth of all investors in the market. Thus Eq. (4.31) describes the evolution of the fundamental price  $p_t^*$  in our model framework, taking into account the simplified assumptions made above. Comparing this result with the fundamental pricing formula often cited by academics<sup>12</sup>, which is

$$p_t = E_t \left[ \frac{p_{t+1} + d_{t+1}}{1 + \varrho_{t+1}} \right], \quad (4.32)$$

with  $\varrho_t$  being the one-period holding period return of the risky asset in the discounting factor, we can conclude that the fundamental pricing benchmark is now embedded in our model outline. Referring to Eq. (4.31), today's price is calculated by adding the conditional expected dividend yield to the conditional expected price of tomorrow and then discounting this by a factor, which is here the sum of the risk-free rate and a 'risk premium' given by

$$RP_t \equiv \frac{H_t}{W_t} \nu \sigma^2. \quad (4.33)$$

Obviously,  $RP_t$  depends on the outstanding (nominal) supply of risky assets, the total wealth in the market as well as the agents' risk-aversion coefficient  $\nu$  defined in their utility function and the agents' (constant) perception of the variance of the risky-asset return, namely  $\sigma^2 \equiv \sigma_{i,t}^2$ . The risk premium is time-dependent in this case.

<sup>12</sup>See Cuthbertson (1996, Chapter 4). See also Eq. (1.7) in Subsection 1.3 of Chapter 1.

Conversely, if we assume a zero supply of outside stocks (as in the BH model in Subsection 2.1.1 of Chapter 2)<sup>13</sup>, i.e.,  $H_t = 0$ , the risk premium  $RP_t$  collapses to zero and Eq. (4.31) simplifies to

$$p_t = \frac{1}{1 + r_f} E_t(p_{t+1} + d_{t+1}) . \quad (4.34)$$

Then the fundamental price is determined by<sup>14</sup>

$$p_t^* = md_t , \quad (4.35)$$

where  $m$  is given by

$$m = \frac{1 + g}{r_f - g} .$$

Hence the fundamental stock price at time  $t$  is given in this simplified case by the cash flow generated by the dividends at time  $t$  multiplied by a factor that depends on the ex-ante rate of return and the growth rate of dividends.<sup>15</sup> From Eq. (4.35) it follows that

$$p_{t+1}^* = (1 + g)p_t^* , \quad (4.36)$$

which states that the fundamental price  $p_t^*$  grows with the (deterministic) growth rate of dividends. The fundamental return (ex and including dividends) as well as the dividend yield can be calculated in a similar way. Hence it is

$$r^* \equiv r_{t+1}^* = \frac{p_{t+1}^* - p_t^*}{p_t^*} = \frac{md_{t+1} - md_t}{md_t} = \frac{(1 + g)d_t - d_t}{d_t} = g ,$$

$$\rho^* \equiv \rho_{t+1}^* = \frac{p_{t+1}^* + d_{t+1} - p_t^*}{p_t^*} = \frac{p_{t+1}^* - p_t^*}{p_t^*} + \frac{(1 + g)d_t}{md_t} = g + (1 + g)\frac{r_f - g}{1 + g} = r_f$$

and

$$\frac{d_{t+1}}{p_t^*} = (1 + g)\frac{r_f - g}{1 + g} = r_f - g .$$

In this case the optimal demand proportion  $\pi^* \equiv \pi_{i,t}^*$  is given by

$$\pi^* = \frac{r_f - r_f}{\nu\sigma^2} = 0 , \quad (4.37)$$

which is a straightforward result for the following reason. If all investors are homogeneous in their expectations and no risky assets are supplied in our artificial market, it follows from Eq. (4.27) and Eq. (4.8) that

$$\pi_{i,t} W_{i,t} = 0$$

for each investor  $i$  and thus, with the evident assumption  $W_{i,t} \neq 0$ , it is

$$E_{i,t}(\rho_{t+1}) = r_f$$

for each investor  $i$ .<sup>16</sup> To sum up the above analysis, the equilibrium price of our market model for the simple case of homogeneous agents and zero asset supply coincides with the fundamental

<sup>13</sup>See also Chiarella et al. (2006).

<sup>14</sup>The derivation of Eq. (4.35) is straightforward and relegated to the Appendix A 4.2.

<sup>15</sup>This model is also known as the static Gordon model. See Gordon (1962, Chapter 4).

<sup>16</sup>Note that Eq. (4.37) as well as Eq. (4.31) contains the assumption that  $\sigma_{i,t}^2 = \text{Var}_{i,t}(\rho_{t+1})$  is equal to a constant value  $\sigma$  for market situations in which the fundamental price path is hit. See also Section 4.3 for further details on the investors' beliefs about the conditional expectation of the variance of the return  $\rho_{t+1}$  in our market model.

price  $p_t^*$  given as a solution of Eq. (4.34) with  $p_t^*$  increasing with the dividend's growth rate  $g$ . As a consequence, the fundamental return (ex dividends)  $r^*$  is equal to this growth rate and the fundamental return  $\rho^*$  is given by the of the risk-free asset return  $r_f$ .

From this straightforward result we have to come back to the more sophisticated case where the risky-asset supply is positiv, i.e.,  $H_t > 0$  (see Eq. (4.31)). Note that we still assume an identical risk-aversion coefficient and a constant expectation of the risky asset's variance. We define

$$r_t^{RP} \equiv r_f + RP_t = r_f + \frac{H_t}{W_t} \nu \sigma^2, \quad (4.38)$$

so Eq. (4.31) leads to

$$p_t = \frac{1}{1 + r_t^{RP}} E_t(p_{t+1} + d_{t+1}). \quad (4.39)$$

By iterated substitution we obtain<sup>17</sup>

$$p_t = E_t \left[ \sum_{j=1}^k \left( \prod_{i=0}^{j-1} \frac{1}{1 + r_{t+i}^{RP}} \right) d_{t+j} \right] + E_t \left[ \left( \prod_{i=1}^k \frac{1}{1 + r_{t+i-1}^{RP}} \right) p_{t+k} \right]. \quad (4.40)$$

Assuming that the transversality condition

$$\lim_{k \rightarrow \infty} E_t \left[ \left( \prod_{i=1}^k \frac{1}{1 + r_{t+i-1}^{RP}} \right) p_{t+k} \right] = 0$$

holds Eq. (4.40) yields

$$p_t = E_t \left[ \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} \frac{1}{1 + r_{t+i}^{RP}} \right) d_{t+j} \right], \quad (4.41)$$

which is generally an analytically non-tractable expression. Eq. (4.41) states that the current stock price depends on the expectations of all time-dependent discount rates  $1/(1 + r_{t+i}^{RP})$  in the future and all future expected dividends. Since it follows that no fundamental price can be calculated in a closed form, we need some additional hypotheses about investors' forecasts of dividends and about the discount rate in order to simplify this problem. One possible simplification is to assume that the agents form the same expectations (or 'beliefs') about the risk premium  $r_t^{RP}$  for all time periods  $t$ , but to revise the estimates of  $r_t^{RP}$  at each point in time.<sup>18</sup> Allowing this assumption, we end up with the fundamental solution for Eq. (4.39), which is then given by

$$p_t^* = \sum_{j=1}^{\infty} \left( \frac{1}{1 + r_t^{RP}} \right)^j E_t(d_{t+j}).$$

Applying the same calculation as outlined for Eq. (4.35) in Appendix A 4.2 to the equation above leads to

$$p_t^* = \frac{1 + g}{r_t^{RP} - g} d_t, \quad (4.42)$$

from which it directly follows that

$$p_{t+1}^* = (1 + g) p_t^* \frac{r_t^{RP} - g}{r_{t+1}^{RP} - g}, \quad (4.43)$$

<sup>17</sup>For further details on this technique, see also the discussion on Eq. (2.8) and Appendix A 2.2 in Chapter 2.

<sup>18</sup>This idea is adopted from a publication relating to this discussion which appeared in a working paper version of Chiarella et al. (2006).

where  $d_{t+1} = (1 + g)d_t$  is used. Obviously, Eq. (4.42) and Eq. (4.43) account for the fact that the risk premium  $r_t^{RP}$  is time-dependent and thus not necessarily the same for different time periods. Therefore, the investors revise their perception of the risk premium in every time step if necessary. Otherwise, if the risk premium is constant over time, i.e., it is  $r_{t+1}^{RP} = r_t^{RP}$  for all  $t$ , as it is in a steady-state situation, the fundamental price  $p_t^*$  evolves over time in the same way as for  $H_t = 0$  (see Eq. (4.36)) and increases with the (deterministic) growth rate of dividends  $g$ .

Although Subsection 4.2.2 contains the assumption that only two investor-groups act in our artificial market, our discussion has not yet taken this fact into account. To remedy this, Eq. (4.30) has to be slightly revised. Following the notation outlined in Subsection 4.2.2, the total wealth of Investor Groups 1 and 2 at time  $t$  is denoted by  $W_{1,t}$  and  $W_{2,t}$  respectively. Thus, Eq. (4.30) has to be rewritten as

$$E_t(p_{t+1} + d_{t+1}) = p_t \left\{ 1 + r_f + H_t \left[ \sum_{i=1}^2 \frac{W_{i,t}}{\nu_i \sigma_{i,t}^2} \right]^{-1} \right\},$$

where it is worth remarking that the variable  $W_{i,t}$  has a different meaning in the sum of both equations. Using the notations Eq. (4.16)–Eq. (4.19) as well as the variables  $\bar{\omega}_t$  and  $\bar{n}_t$  defined in Eq. (4.23) leads to

$$\begin{aligned} E_t(p_{t+1} + d_{t+1}) &= p_t \left\{ 1 + r_f + H_t \left[ \sum_{i=1}^2 \frac{W_{i,t}}{\nu_i \sigma_{i,t}^2} \right]^{-1} \right\} \\ &= p_t \left\{ 1 + r_f + H_t \left[ \sum_{i=1}^2 \frac{N_i \bar{W}_{i,t}}{\nu_i \sigma_{i,t}^2} \right]^{-1} \right\} \\ &= p_t \left\{ 1 + r_f + \frac{H_t}{\bar{W}_t I} \left[ \sum_{i=1}^2 \frac{n_i \bar{\omega}_{i,t}}{\nu_i \sigma_{i,t}^2} \right]^{-1} \right\} \\ &= p_t \left\{ 1 + r_f + \frac{H_t}{\bar{W}_t I} \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu_1 \sigma_{1,t}^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu_2 \sigma_{2,t}^2} \right]^{-1} \right\}. \end{aligned} \quad (4.44)$$

The risk premium is therefore given by

$$RP_t = \frac{H_t}{\bar{W}_t I} \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu_1 \sigma_{1,t}^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu_2 \sigma_{2,t}^2} \right]^{-1},$$

where in contrast to Eq. (4.33) the total wealth  $W_t$  is replaced by the less intuitive variable  $\bar{W}_t$  multiplied by the number of investors in the market denoted by  $I$ .

#### 4.2.4 Risk Premium

According to the market equilibrium equation for the fundamental price path given by Eq. (4.31), the investors' expectation of the risky-asset return is determined by

$$E_t(\rho_{t+1}) = \frac{H_t}{\bar{W}_t} \nu \sigma^2 + r_f, \quad (4.45)$$

which merely describes the fact that investors want to be compensated for taking risk by the (time-dependent) risk premium given by

$$RP_t = \frac{H_t}{\bar{W}_t} \nu \sigma^2. \quad (4.46)$$

In the following discussion I want to constitute some further assumptions on this specific risk-premium term, particularly with regard to the (nominal) risky-asset supply  $H_t$ . It is again assumed that all investors form ‘correct’ expectations of the fundamental price path, i.e., their expectations of the return and variance of the risky asset, in particular, are homogeneous in such a market equilibrium (or fundamental steady state) as already presumed by Eq. (4.45). Furthermore, it is here assumed that all investors use the same risk-aversion coefficient  $\nu$  and a constant  $\sigma^2$  in order to avoid an overly laborious notation.

As already outlined in the previous subsection, the risk premium  $RP_t$  vanishes for  $H_t = 0$ , i.e., for a zero supply of risky assets. Otherwise, if  $H_t$  is assumed to be positive, the dynamics of  $RP_t$  can vary quite considerably depending on the assumption made about the dynamic behavior of  $H_t$ . Therefore, let us briefly consider which assumptions about the dynamics of  $H_t$  might be the most sensible. For instance, assuming  $H_t$  to be a constant would lead to  $RP_t$  decreasing over time (even on the fundamental price path), since the total wealth in the market will increase (at least under ordinary market behavior) because of the chosen CRRA power utility function used by the investors. Moreover, a constant (nominal) risky-asset supply does not perfectly fit to the increasing (nominal) risky-asset demand established in our model framework. Letting  $H_t$  grow linearly over time will also lead to problems, since  $H_t$  is the *nominal* risky-asset supply, meaning that  $H_t$  incorporates the current price level of the risky assets. Thus the assumption of a linear growth of  $H_t$  would in some respects foil the speculative price movements that might emerge in our financial market, induced by the trading activities of the different investor types.<sup>19</sup> To avoid this problem in general, I uncouple quantity and price level within this variable and replace the fraction  $H_t/W_t$  in the risk-premium term with the expression

$$\frac{N_t^s p_t}{W_t}, \quad (4.47)$$

where  $N_t^s > 0$  is the number of outstanding risky assets at time period  $t$ . While  $N_t^s = 0$  again directly leads to a vanishing risk premium, the assumption of a constant  $N_t^s$  for all  $t$  is also not a realistic assumption.<sup>20</sup> Instead, we assume that  $N_t^s$  is linearly growing over time, i.e.,

$$N_{t+1}^s = (1 + n^s)N_t^s. \quad (4.48)$$

As a consequence, the (nominal) risky-asset supply  $H_t$  will increase over time for this model specification where the growth of the (nominal) risky-asset supply is based on the increasing number of risky assets via the  $N_t^s$  process and the evolution of the risky asset’s price level  $p_t$ , which is determined by the investors’ trading activities but which should also be a growing process in general. Hence both (nominal) demand for risky assets as well as (nominal) supply of risky assets are growing processes over time in the model framework that will now be proposed.

Concurrently, we assume that the risk premium  $RP_t$  should be a constant on the fundamental price path, which seems appropriate since the investors should expect a constant return of the risky asset in these ‘stable’ and ‘predictable’ market situations, provided that they form ‘correct’ (or

<sup>19</sup>This effect would be quite similar to the problem with the  $\alpha$ -term used in the CH model (see Eq. (3.13), Theorems 3.1 and 3.4 as well as, in particular, the penultimate paragraph in the summarizing Section 3.8 of Chapter 3).

<sup>20</sup>For details on the inappropriateness of this assumption see the calculations outlined below, first of all Eq. (4.55) and the subsequent explanation.



consistent) expectations of the risky asset's return evolution.<sup>21</sup> The latter assumption implies that the fraction  $H_t/W_t = N_t^s p_t/W_t$  is constant for those market states, which means that the (nominal) risky-asset supply  $H_t$  is growing similarly to the total wealth  $W_t$ .<sup>22</sup> For better readability,

$$RP^* \equiv RP_t|_{\text{fund. price path}} \quad \text{and} \quad z^* \equiv N_t^s p_t/W_t|_{\text{fund. price path}} \quad (4.49)$$

denote the constant risk premium  $RP_t$  and the constant fraction  $H_t/W_t = N_t^s p_t/W_t$  on the fundamental price path. This implies that

$$RP^* = z^* \nu \sigma^2 \quad (4.50)$$

(see Eq. (4.38)) and

$$\rho^* = r_f + RP^* . \quad (4.51)$$

Using this notation, it follows that the (nominal) investors' aggregated risky-asset demand generally given by

$$\sum_{i=1}^I \zeta_{i,t} = \sum_{i=1}^I \pi_{i,t} W_{i,t} = \sum_{i=1}^I \frac{E_{i,t}(\rho_{t+1}) - r_f}{\nu_i \sigma_{i,t}^2} W_{i,t}$$

(see Eq. (4.8)) can be simplified to

$$\sum_{i=1}^I \zeta_{i,t}|_{\text{fund. price path}} = z^* \sum_{i=1}^I W_{i,t} = z^* W_t ,$$

where Eq. (4.45) is used.<sup>23</sup> Obviously, this equation states that the (nominal) aggregated risky-asset demand grows with the growth rate of total wealth on the fundamental price path.<sup>24</sup> This growth rate can be determined according to Eq. (4.7) together with Eq. (4.50) and Eq. (4.51) by

$$\beta^* = r_f + z^*(\rho^* - r_f) = r_f + z^*([r_f + z^* \nu \sigma^2] - r_f) = r_f + (z^*)^2 \nu \sigma^2 . \quad (4.52)$$

In other words, this expression gives us the growth rate for the denominator of the risk-premium term Eq. (4.46) on the fundamental price path. Now let us consider the nominator of this term. Using Eq. (4.47) and Eq. (4.48) as well as Eq. (4.43), the time evolution of the (nominal) risky-asset supply  $H_t$  can be determined by

$$\begin{aligned} H_{t+1} &= N_{t+1}^s p_{t+1}^* \\ &= (1 + n^s) N_t^s \cdot (1 + g) p_t^* \\ &\approx (1 + n^s + g) N_t^s p_t^* = (1 + n^s + g) H_t , \end{aligned} \quad (4.53)$$

where  $n^s \cdot g \ll 1$  is assumed. According to the assumption of a constant risk premium on the fundamental price path, i.e.,  $H_t/W_t$  is constant for all  $t$ , the growth rate of investors' total wealth

<sup>21</sup>Due to our specific choice concerning the investors' belief about the conditional variance of risky-asset return, the variance  $\sigma_{i,t}^2$  is also constant on the fundamental price path (see Eq. (4.38)). See also Section 4.3 for details.

<sup>22</sup>A growing supply of risky assets can be associated with the idea of a company raising capital in order to increase its productivity or market power, which might be a reasonable assumption in my growing economy model design. Instead, the assumed growing dividend process refers then to investors' revenues (or gains) that are siphoned off from the firm's wealth.

<sup>23</sup>It is also  $\pi_{i,t}|_{\text{fund. price path}} = z^*$ .

<sup>24</sup>This result is rather unsurprising because of the underlying CRRA power utility function used by the investors.

$\beta^*$  given by Eq. (4.52), which is the growth rate of the aggregated demand function, should be equal to the growth rate of the risky-asset supply, which is  $n^s + g$ , in accordance with Eq. (4.53). From this it can be concluded that

$$\beta^* = n^s + g, \quad (4.54)$$

which leads to

$$n^s = (r_f - g) + \frac{(RP^*)^2}{\nu\sigma^2}, \quad (4.55)$$

where Eq. (4.50) is used. Obviously, the growth rate of risky-asset supply  $n^s$  has to be positive since  $r_f > g$  and  $RP^* > 0$ , where the latter assumption directly follows from  $H_t > 0$ .<sup>25</sup> Furthermore, if  $n^s$  is fixed, the constant  $RP^*$  on the equilibrium path can be determined by

$$RP^* = \pm \sqrt{[n^s - (r_f - g)] \nu\sigma^2},$$

where only the positive solution is economically reasonable. Furthermore, the relation

$$n^s \geq (r_f - g) \quad (4.56)$$

has to be fulfilled, which makes perfect sense, since Eq. (4.52) states that the growth rate of risky-asset demand  $\beta^*$  exceeds  $r_f$ . Thus the assertion directly follows from Eq. (4.54), as  $r_f > g$  was also assumed previously. From an economic point of view, Eq. (4.56) is also reasonable because the supply of risky assets, or better its growth rate  $n^s$ , has to be the higher the larger the difference between the return of the risk-free asset and the dividends' growth rate. This is simply because a higher growth rate  $n^s$  makes an investment in the risky assets more attractive (since its price decreases relatively) and a larger difference between the return of the risk-free asset and the dividends' growth rate increases the attractiveness of an investment in the bond.

Concluding the procedure to this point, I have established a market equilibrium for my asset pricing model with a non-zero asset supply where the following assumptions have been made on its *fundamental* price path:

- All investors (i.e., both investor groups) form the same rational expectations of the conditional mean and variance of the risky asset and thus derive the same risk premium for investing in the risky prospect.
- The resulting non-zero risk premium is assumed to be constant, which implies that the (nominal) risky-asset supply grows in the same manner as the investors' total wealth in the market. Since the growth rate of total wealth naturally coincides with the growth rate of the aggregated investors' demand for risky assets, the growth rate of (nominal) risky-asset supply also equals the growth rate of (nominal) risky-asset demand, which seems an appropriate and reasonable assumption.
- The supplied number of risky assets increases linearly over time, where the growth rate has to be chosen exogenously, *but* also has to be sufficiently high.

With these three bullet points I want to close my explanations of the risk premium and continue with the implementation of a market maker scenario.

<sup>25</sup>Remember that for  $H_t = 0$  it is  $\beta_t^* = r_f$  and thus  $RP^*$  would collapse to zero.

### 4.2.5 Market Maker Equilibrium

In the last step the market clearing procedure has to be specified. In contrast to the CH model outlined in the previous chapter we assume a market maker scenario in which a ‘market maker’ changes the price of the risky asset according to the observed excess demand. More precisely, the market maker is to take an offsetting long (short) position if the excess demand is negative (positive) to clear the market. At the end of each trading period  $t$ , after the market maker has carried out all transactions, he/she adjusts the price for the next trading period in the direction of the overall observed excess demand. Assuming a constant growth rate of the dividend (i.e., deterministic skeleton), which is

$$d_{t+1} = (1 + g)d_t ,$$

the return of the risky asset at period  $t$  can be rewritten as

$$\rho_{t+1} = \frac{p_{t+1} - p_t + d_{t+1}}{p_t} = r_{t+1} + \frac{(1 + g)d_t}{p_t} = r_{t+1} + (r_t^{RP} - g) \frac{p_t^*}{p_t} . \quad (4.57)$$

Now, we introduce a new state variable  $y_t$  by

$$y_t \equiv \frac{p_t^*}{p_t} ,$$

which is the ‘fundamental/price ratio’ and evolves according to<sup>26</sup>

$$y_{t+1} = \frac{p_{t+1}^*}{(\rho_{t+1} + 1 - d_{t+1}/p_t)p_t} = \frac{(1 + g) \frac{p_t^{RP} - g}{r_{t+1}^{RP} - g} p_t^*}{(1 + r_{t+1})p_t} = \frac{1 + g}{1 + r_{t+1}} \frac{r_t^{RP} - g}{r_{t+1}^{RP} - g} y_t .$$

Note that, according to Eq. (4.7) and Eq. (4.57), the agent  $i$ ’s growth rate of wealth can now be written as

$$\beta_{i,t+1} = r_f + \pi_{i,t}[r_{t+1} - r_f + (r_t^{RP} - g)y_t] .$$

We assume that the price of the risky asset is adjusted by a market maker as follows:

$$p_{t+1} = p_t [(1 + g) + \lambda(q_t + \epsilon_t)] + \alpha(p_t^* - p_t) , \quad (4.58)$$

where  $\lambda \geq 0$  is a constant, measuring the speed of the price adjustment from the market maker, and  $\epsilon_t$  stands for noisy excess demand (proportional to the current total wealth in the market). The last term can be interpreted as a ‘stabilizing term’, which prevents the current price moving too far away from its fundamental value. Indeed, this assumption is equivalent to

$$r_{t+1} = \frac{p_{t+1} - p_t}{p_t} = g + \lambda(q_t + \epsilon_t) + \alpha(y_t - 1) , \quad (4.59)$$

implying that the market maker adjusts the relative return of the risky asset according to the return of dividend (i.e., its growth rate  $g$ ), the overall aggregate excess demand proportion  $q_t$ , the noisy demand proportion  $\epsilon_t$  at time  $t$  and a correction term depending on the deviation of the current price of the risky asset from the underlying fundamental price. Note that, in the case of neither excess nor noisy demand (i.e.,  $q_t = \epsilon_t = 0$ ), the return of the risky asset is a constant and coincides with the fundamental return (ex dividends)  $r_{t+1}^*$  which is equal to the growth rate of dividends  $g$ .

<sup>26</sup>This approach is adopted from a quite similar contribution to this discussion from Chiarella et al. (2006).

#### 4.2.6 Summary of the Stationary Model in its State Variables (Simplified Version)

So far, an artificial financial market model under a market maker scenario has been proposed, where asset prices and investors' wealth levels are growing processes over time. Concentrating in the first instance on the simplified model version in the subsequent two sections, where zero supply of outside stocks (i.e.,  $H_t = 0$ , risk premium  $RP_t = 0$ ) is assumed, it has been shown that the growth model can be reformulated as a stationary model in three state variables, namely the return  $r_t$ , the fundamental/price ratio  $y_t$  and the difference in the wealth proportions  $\bar{\omega}_t$ . This result is resumed in the following theorem:

**Theorem 4.1** *The fundamental/price ratio evolves according to*

$$y_{t+1} = \frac{1+g}{1+r_{t+1}} y_t, \quad (4.60)$$

where the return (ex dividends) of the risky asset is given by

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1) \quad (4.61)$$

with

$$q_t = \frac{[(1 + \beta_{1,t})\pi_{1,t} - (1 + r_t)\pi_{1,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} + \frac{[(1 + \beta_{2,t})\pi_{2,t} - (1 + r_t)\pi_{2,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]}$$

and

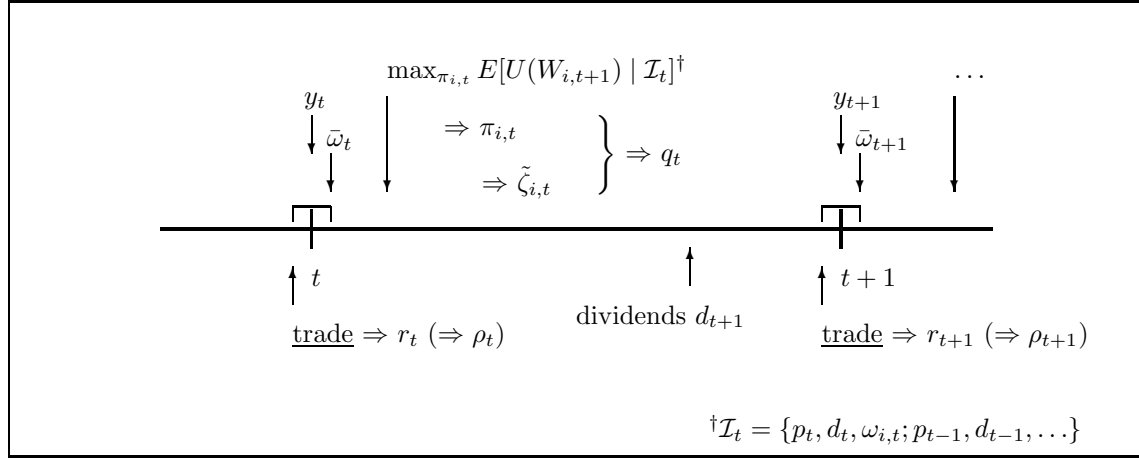
$$\begin{aligned} \beta_{i,t+1} &= r_f + \pi_{i,t}[r_{t+1} - r_f + (r_f - g)y_t], \\ \pi_{i,t} &= \frac{E_{i,t}(r_{t+1}) + (r_f - g)y_t - r_f}{\nu_i \sigma_{i,t}^2}, \quad i = 1, 2. \end{aligned}$$

The difference in the average wealth proportions of the two investor groups evolves according to

$$\bar{\omega}_{t+1} = \frac{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) - [1 - \bar{\omega}_t](1 + \beta_{2,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{2,t+1})}. \quad (4.62)$$

The system Eq. (4.60)–Eq. (4.62) forms a stationary model in three state variables, namely the fundamental/price ratio, the return (ex dividends) of the risky asset and the difference in the average wealth proportions of the two investor groups.

To conclude this section, the model time line depicted in Figure 4.1 summarizes the main aspects of the market maker model outlined above for the  $H_t = 0$  case. Although the time flow of the trading activities seems visually quite similar to that of the Walrasian scenario proposed in the CH model (see Figure 3.1 in Chapter 3), some crucial differences have to be noted. In a Walrasian scenario the equilibrium price and the investors' wealth are determined simultaneously, implying that the information set  $\mathcal{I}_t$  used by the investors for their optimal investment proportion at time  $t$  does not include the current equilibrium price  $p_t$ . In a market maker scenario, however, the information set  $\mathcal{I}_t$  does include the price at the current time period  $t$  since the price adjustment rule applied by the market maker is publicly known to all market participants.



**Figure 4.1:** Model time line for the asset price model under a market maker scenario ( $H_t = 0$  case). Although the diagram seems to be almost identical to the time line of the CH model (see Figure 3.1, Chapter 3) small but important differences can be noticed. The horizontal brackets indicate that the state variables are determined at the same point in model time. The small shifts in the state variables within the brackets should simply give the reader an idea of the order in which the state variables are calculated in the simulation program.

### 4.3 Market Maker Model with Fundamentalists and Chartists

In this section I will consider one homogeneous and two heterogeneous beliefs model with a market maker. As already mentioned above, a zero supply of outside stocks is proposed. It is assumed that the market contains two different groups of investors following different trading strategies, namely ‘fundamentalists’ and ‘chartists’. Both groups of traders know about the dividend growth process but form different beliefs about the price evolution over time:

- Fundamentalists believe that the returns (ex dividends) of the risky asset will eventually move back to the relative price change of the fundamental price, i.e.,  $r^* = g$ . The fundamentalists’ expectations about the conditional mean of the return  $r_t$  are mathematically expressed by

$$E_{f,t}(r_{t+1}) = g - \beta_1(r_t - g), \quad \beta_1 \in [0, 1]. \quad (4.63)$$

Furthermore, it is assumed that fundamentalists have constant beliefs about the variance of the return. Therefore, according to Eq. (4.57), the fundamentalists’ beliefs about the conditional mean and variance of risky-asset return *including* dividends is given by

$$\begin{aligned} E_{f,t}(\rho_{t+1}) &= g - \beta_1(r_t - g) + (r_f - g)y_t \\ \sigma_{f,t}^2 \equiv Var_{f,t}(\rho_{t+1}) &= \sigma^2, \end{aligned}$$

where  $\beta_1$  and  $\sigma^2 > 0$  are constants.

- Chartists extrapolate the future return from the last two realized returns of the risky asset via

$$E_{c,t}(r_{t+1}) = \beta_2 r_t + (1 - \beta_2)r_{t-1}, \quad \beta_2 \in [0, 1], \quad (4.64)$$

which leads to

$$E_{c,t}(\rho_{t+1}) = \beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t$$

for the chartists' expectations about the mean of the return  $\rho_{t+1}$ . The chartists' beliefs about the conditional variances of returns is assumed to be

$$\sigma_{c,t}^2 \equiv \text{Var}_{c,t}(\rho_{t+1}) = \sigma^2 + \beta_3 \bar{\sigma}_{c,t}^2$$

with

$$\bar{\sigma}_{c,t}^2 \equiv [E_{c,t}(\rho_{t+1}) - r_f]^2 = [\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t - r_f]^2, \quad (4.65)$$

where  $\beta_2$  and  $\beta_3 \geq 0$  are constants. Hence the chartists believe that the variance of the returns consists of two components, a constant part (as used for the fundamentalists) and a time-dependent part. The time-dependent part becomes relevant when the current belief about the mean of the next period's return deviates from the fundamental return (including dividends), which is  $r_f$ . Thus it accounts for higher-volatility market periods in which the current return of the risky asset may significantly move away from the fundamental price level. In these market situations the chartists become more careful about their investments and, therefore, are not willing to invest too much of their wealth in the risky prospect. The parameter  $\beta_3$  is a weighting factor with which chartists can adjust the time-dependent component of their conditional variance to their overall perception of the variance of returns.

Note that it can be simply verified that both trading strategies are consistent with the model framework proposed in the previous section since both the fundamentalists and the chartists always form the correct expectations at the fundamental price level  $p_t^*$ . This means that both trading strategies evaluated at the fundamental price  $p_t^*$  give consistent expectations with respect to the classical finance theory of fundamental pricing nested in our model framework.

### 4.3.1 Homogeneous Beliefs – Fundamentalists

In this subsection I will investigate the homogeneous beliefs model with only fundamentalists acting in the market. Hence summing up all relevant equations for the deterministic case (with  $\nu \equiv \nu_f$ ), namely

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.66)$$

$$y_{t+1} = \frac{1+g}{1+r_{t+1}} y_t, \quad (4.67)$$

where

$$q_t = \frac{(1 + \beta_{f,t})\pi_{f,t} - (1 + r_t)\pi_{f,t-1}}{1 + \beta_{f,t}}, \quad (4.68)$$

$$\beta_{f,t+1} = r_f + \pi_{f,t} [r_{t+1} - r_f + (r_f - g)y_t], \quad (4.69)$$

$$\pi_{f,t} = \frac{1}{\nu\sigma^2} [(g - r_f) - \beta_1(r_t - g) + (r_f - g)y_t] \quad (4.70)$$

leads to a four-dimensional nonlinear system. Due to this most simple market scenario, the following theorem relating to the existence of steady state can be obtained:

**Theorem 4.2** *Let  $g, r_f, \lambda, \alpha, \nu, \sigma^2 > 0$  and  $g < r_f$ . Furthermore, assume that  $0 \leq \beta_1 \leq 1$  and Eq. (4.68)–Eq. (4.70) hold. Then the system Eq. (4.66)–Eq. (4.67) has a fundamental steady state  $E_1$  given by*

$$r^* = g \quad \wedge \quad y^* = 1. \quad (4.71)$$

PROOF: See Appendix A 4.3.

Theorem 4.2 states that one economically reasonable steady state exists for the homogeneous fundamental-belief model.<sup>27</sup> The fundamental steady state  $E_1$  corresponds to a market scenario in which the return (ex dividends) of the risky asset equals the growth rate of dividends, i.e.,  $r^* = g$  and so the price of the risky asset increases pursuant to its fundamental value  $p_t^*$  for all time periods, i.e.,  $y^* = 1$ . In such a market situation the optimal investment proportion  $\pi_f^*$  of the investors for investing in the risky asset is determined by

$$\pi_f^* = \frac{E_{f,t}(\rho^*) - r_f}{\nu\sigma^2} = \frac{E_{f,t}(r^*) + (r_f - g)y^* - r_f}{\nu\sigma^2} = \frac{g + (r_f - g) - r_f}{\nu\sigma^2} = 0,$$

which is in total accordance with the request of a consistent model framework.<sup>28</sup> This is summarized in the following remark:

**Remark 4.3** *Note that the total agent's demand at the fundamental steady state is exactly equal to the total supply (which is assumed to be zero), i.e.,  $\pi_f^* = 0$ .*

The next theorem characterizes the local stability properties of the steady state  $E_1$ :

**Theorem 4.4** *Let  $g, r_f, \lambda, \alpha, \nu, \sigma^2 > 0$ ,  $g < r_f$  and  $0 \leq \beta_1 \leq 1$ . Then, the fundamental steady state  $E_1 = (g, g, 1, 1)$  of the system Eq. (4.66)–Eq. (4.67) is locally asymptotically stable if and only if all the eigenvalues  $\xi_i$  of the polynomial*

$$\Gamma_1(\xi) = \frac{\xi}{\nu\sigma^2 GF} \left\{ \nu\sigma^2 GF \xi^3 + [\lambda F(\beta_1 G + \Delta) + \nu\sigma^2 F(\alpha - G)] \xi^2 - \lambda G [\beta_1(G + F) + \Delta] \xi + \beta_1 \lambda G^2 \right\} \quad (4.72)$$

satisfy  $|\xi_i| < 1$  for  $i = 1, \dots, 4$ , in which

$$G = 1 + g, \quad F = 1 + r_f, \quad \text{and} \quad \Delta = r_f - g.$$

PROOF: See Appendix A 4.4.

The local stability of the fundamental steady state of the homogeneous fundamental-belief model is determined by the corresponding characteristic equation, which obviously depends on all the model parameters, i.e., the growth rate of dividends  $g$ , the risk-aversion coefficient  $\nu$ , the variance  $\sigma^2$ , the speed of price adjustment of the market maker  $\lambda$ , the strength of the stabilizing force term  $\alpha$ , the ‘risk adjusted’ discounting factor  $r_f$  (which is the risk-free rate in this simplified model version) and the extrapolation rate of the fundamentalists  $\beta_1$ . Since  $\Gamma_1(\xi)$  is a polynomial of order four, or an ordinary polynomial of order three if we ignore the trivial eigenvalue  $\xi_1 = 0$ , it is almost impossible to calculate its roots analytically. Hence numerical simulations are performed

<sup>27</sup>From a purely mathematical point of view three further steady states with  $y^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  can be derived. However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.2 in Appendix A 4.3 for details). In this respect the fundamental steady state is ‘quasi’ unique. Note further, that although our model framework has some similarities to the model framework proposed by Chiarella et al. (2006) the (‘quasi’) uniqueness of the fundamental steady state is different from the results outlined in the cited literature.

<sup>28</sup>See also Eq. (4.37) and the brief discussion thereafter.

instead. These simulations show that  $E_1$  is locally asymptotically stable for a wide range of parameters if the extrapolation rate  $\beta_1$  is chosen sufficiently small. Contrastingly, a sufficient raising of  $\beta_1$  leads to a flip bifurcation where the bifurcation value increases if the risk-aversion coefficient  $\nu$  increases or if the speed of price adjustment  $\lambda$  decreases. The overall result of the primary bifurcation of the fundamental steady state is summarized in the following theorem:

**Theorem 4.5** *Under the assumptions of Theorem 4.4 the fundamental steady state  $E_1 = (g, g, 1, 1)$  of the system Eq. (4.66)–Eq. (4.67) undergoes a flip bifurcation at*

$$\beta_1^* = \frac{\nu\sigma^2(1+r_f)[2(1+g)-\alpha] - \lambda(r_f-g)(2+r_f+g)}{2\lambda(1+g)(2+r_f+g)}. \quad (4.73)$$

PROOF: See Appendix A 4.5.

Obviously, the results stated in the previous two theorems correspond to economic intuition. If fundamentalists are alone in the market, the market dynamics only temporary moves away from the fundamental price and eventually converges to its fundamental value if the fundamentalists do not extrapolate too strongly, but rather align their beliefs to fundamentals. Looking at Eq. (4.73), a high investors' risk-aversion coefficient as well as a low price adjustment speed of the market maker stabilize the system. Instead, an increase in the risk-free rate makes an investment in the risky asset less attractive and thus the system is destabilized even for a lower extrapolation rate  $\beta_1$  of the fundamentalists. Interestingly, a relatively high strength of the stabilizing force term leads to a more unstable market dynamics.<sup>29</sup> Figure 4.2 (a) exhibits a section from a representative time series of the risky-asset return (ex dividends)  $r_t$  for the extrapolation parameter  $\beta_1$  exceeding the bifurcation value  $\beta_1^*$  of the primary flip bifurcation. The 2-cycle characteristic of the time series of the risky-asset return can clearly be detected visually. Figure 4.2 (b) shows the corresponding bifurcation diagram with respect to the fundamentalists' extrapolation parameter  $\beta_1$ . While the flip bifurcation point lies at about  $\beta_1^* = 0.36$  for the specific parameter set used in this figure, it can also clearly be seen that the dynamic system shows diffuse behavior for  $\beta_1$  being between 0.5 and 0.8 as well as a tendency to explode for  $\beta_1$  being sufficiently close to 1.

The following subsection concentrates on the other case of a homogeneous beliefs model where all investors use chartist trading strategies in the market.

#### 4.3.2 Homogeneous Beliefs – Chartists

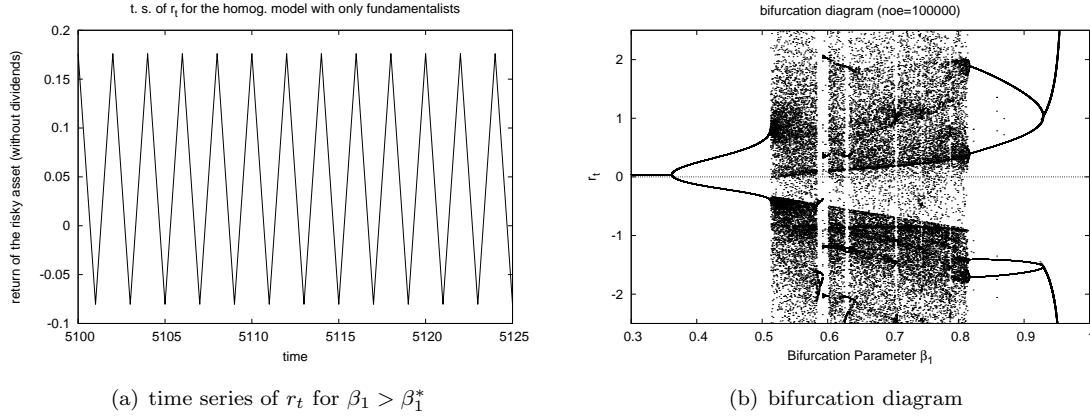
For the homogeneous beliefs model with only chartists in the market, the resulting dynamic system is given by

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.74)$$

$$y_{t+1} = \frac{1+g}{1+r_{t+1}} y_t, \quad (4.75)$$

<sup>29</sup>This result is somehow puzzling, though there is a possible explanation. On the one hand, it is obvious from Eq. (4.58) that the  $\alpha$ -term can stabilize the price evolution within the price-setting rule whenever the current price is too far away from the fundamental price level. On the other hand, for certain market situations, the  $\alpha$ -term can also disturb the convergence of the dynamic system to its fundamental steady state. Therefore, the value of  $\alpha$  in the stabilizing force term has to be chosen moderately.





**Figure 4.2:** Market maker model with only fundamentalists in the market. For  $\beta_1 > \beta_1^*$ , a flip bifurcation occurs, leading to 2-cycles in the time series of risky-asset returns (left panel). The right panel shows a bifurcation diagram with respect to the fundamentalists' extrapolation parameter  $\beta_1$ . For  $\beta_1$  being beyond  $\beta_1^*$ , a flip bifurcation occurs. 100,000 experiments with different values of  $\beta_1$  (between 0.3 and 1.0) are conducted for this plot, with 5,000 iterations being performed in each experiment. The initial conditions for the dynamic variables are chosen randomly for each experiment. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\lambda = 0.05$  and  $\alpha = 0.5$ .

where

$$q_t = \frac{(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1}}{1 + \beta_{c,t}}, \quad (4.76)$$

$$\beta_{c,t+1} = r_f + \pi_{f,t} [r_{t+1} - r_f + (r_f - g)y_t], \quad (4.77)$$

$$\pi_{c,t} = \frac{\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t - r_f}{\nu [\sigma^2 + \beta_3 \bar{\sigma}_{c,t}^2]}, \quad (4.78)$$

$$\bar{\sigma}_{c,t}^2 = [\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t - r_f]^2 \quad (4.79)$$

and  $\nu \equiv \nu_c$ . Obviously, the system Eq. (4.74)–Eq. (4.75) is a five dimensional system (compared to four dimensions in the homogeneous fundamentalists model), which is caused by the fact that the chartists extrapolate the next period's asset return from the last *two* observable return values of the risky asset. Again, it can easily be verified that the fundamental steady state  $E_1 = (g, g, g, 1, 1)$  is the only economically reasonable steady state which can be obtained for this specific model version. Theorem 4.6 summarizes the results of the steady state and its stability property:

**Theorem 4.6** *Let  $0 \leq \beta_2 \leq 1$  and  $\beta_3 \geq 0$ . Assume that Eq. (4.76)–Eq. (4.79) hold. Under the assumptions of Theorem 4.2, the system Eq. (4.74)–Eq. (4.75) has a steady states  $E_1$ , which coincide with the fundamental steady state given in Theorem 4.2 by Eq. (4.71). The fundamental steady state  $E_1$  is locally asymptotically stable if and only if all the eigenvalues  $\xi_i$  of the polynomial*

$$\Gamma(\xi) = \frac{\xi}{\nu \sigma^2 G F} \left\{ \nu \sigma^2 F [\alpha + (\xi - 1)G] \xi^3 - \lambda (F\xi - G) [(\beta_2(\xi - 1)^2 - 1)G + (G - \Delta\xi)\xi] \right\}$$

satisfy  $|\xi_i| < 1$  for  $i = 1, \dots, 5$ , in which

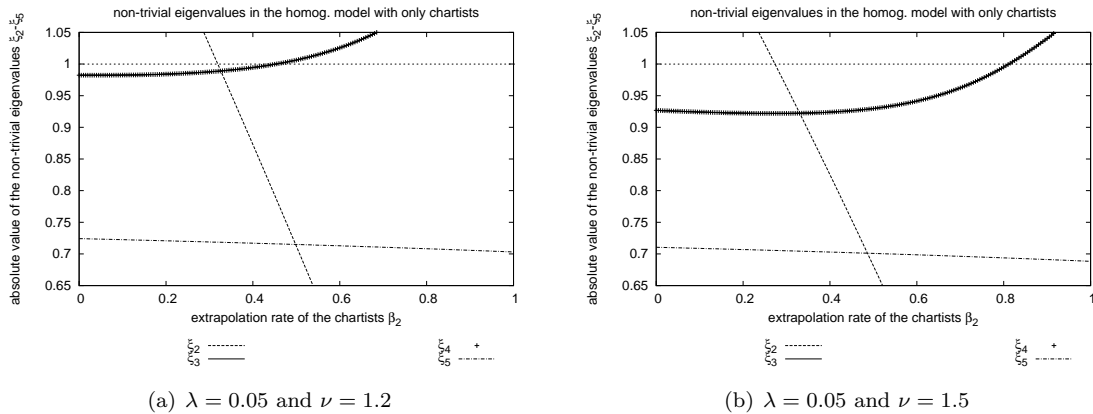
$$G = 1 + g, \quad F = 1 + r_f, \quad \text{and} \quad \Delta = r_f - g.$$

PROOF: See Appendix A 4.6.

The local stability of the fundamental steady state in Theorem 4.6 is characterized by a polynomial of order five having one trivial and four non-trivial roots which cannot be solved analytically.<sup>30</sup> However, numerical simulations suggest that the dynamic system might be stable for specific choices of parameters even in the case where only chartists are in the market. While the parameter set fixed by

$$\begin{aligned} \nu = 1.2, \quad r_f = 0.05, \quad g = 0.04, \quad \sigma = 0.2, \\ \lambda = 0.1, \quad \alpha = 0.5, \quad \beta_3 = 5.0, \end{aligned}$$

makes the fundamental steady state  $E_1$  unstable for all  $\beta_2 \in [0, 1]$ , a decrease in the price adjustment speed of the market maker stabilizes the dynamic system. For example, for  $\lambda = 0.05$  (and remaining parameters as above), the fundamental steady state  $E_1$  becomes stable for  $0.32 < \beta_2 < 0.45$ , where a simultaneous increase in the chartists' risk-aversion coefficient even enhances this effect significantly and leads to a wider stability region (see Figure 4.3). As in the



**Figure 4.3:** Stability region of fundamental steady state with only chartists in the market. Both panels exhibit the absolute values of the four non-trivial eigenvalues  $\xi_2, \dots, \xi_5$  for  $0 \leq \beta_2 \leq 1$ . Note that  $\xi_3$  and  $\xi_4$  are complex conjugate eigenvalues; therefore, for better visualization,  $\xi_3$  is denoted by a line while  $\xi_4$  is denoted by points in both panels. An increase in the chartists' risk-aversion coefficient  $\nu$  clearly enlarges the stability region of the fundamental steady state. In both cases the fundamental steady state  $E_1$  is stable if chartists extrapolate the future risky-asset return from the last two realized returns in a balanced way. Otherwise, if one of the last two returns is heavily weighted relative to its counterpart,  $E_1$  becomes unstable. The remaining parameters are  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\alpha = 0.5$  and  $\beta_3 = 5.0$ .

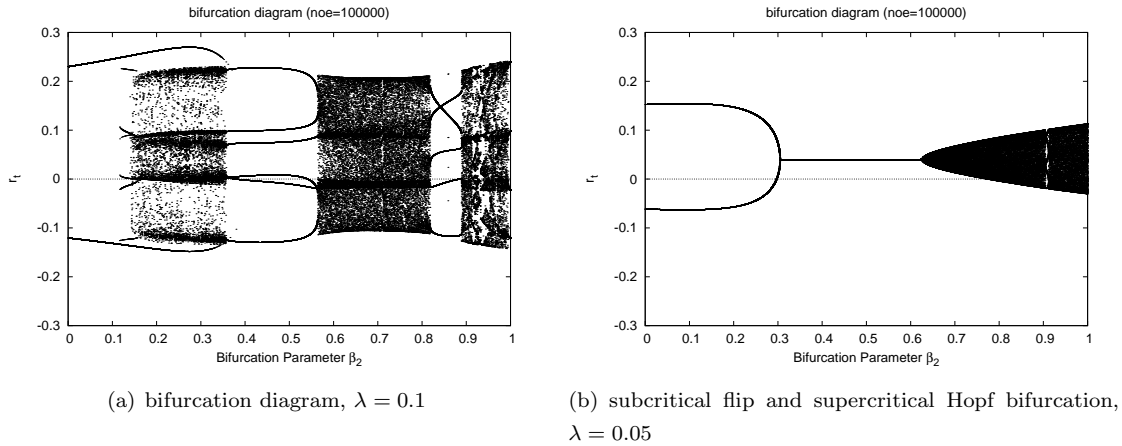
case of the homogeneous beliefs model with only fundamentalists in the market, a low value of  $\alpha$  also contributes to greater stability in the system,<sup>31</sup> whereas different choices of  $\beta_3$  do not seem

<sup>30</sup>From a purely mathematical point of view five further steady states with  $y^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  can be derived. However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.6 in Appendix A 4.6 for details). In this respect the fundamental steady state is 'quasi' unique.

<sup>31</sup>Results for the influence of the  $\alpha$  value for different choices of  $\lambda$  are not shown in detail here. However, this point is taken up in the discussion about the heterogeneous beliefs model outlined in the next subsection.

to greatly influence the local stability property of  $E_1$ . In such stable market situations the fundamental steady state undergoes a (supercritical) Hopf bifurcation at a critical value  $\beta_2^H$ , which can, in principle, be determined numerically. At the lower boundary of the stable region numerical simulations confirm that  $E_1$  undergoes a flip bifurcation for  $\beta_2 < \beta_2^L$ , i.e., the bifurcation is subcritical.<sup>32</sup>

Figure 4.4 shows two bifurcation diagrams for two different parameter sets for the homogeneous beliefs model with only chartists in the market. While the diagram in the left panel corresponds

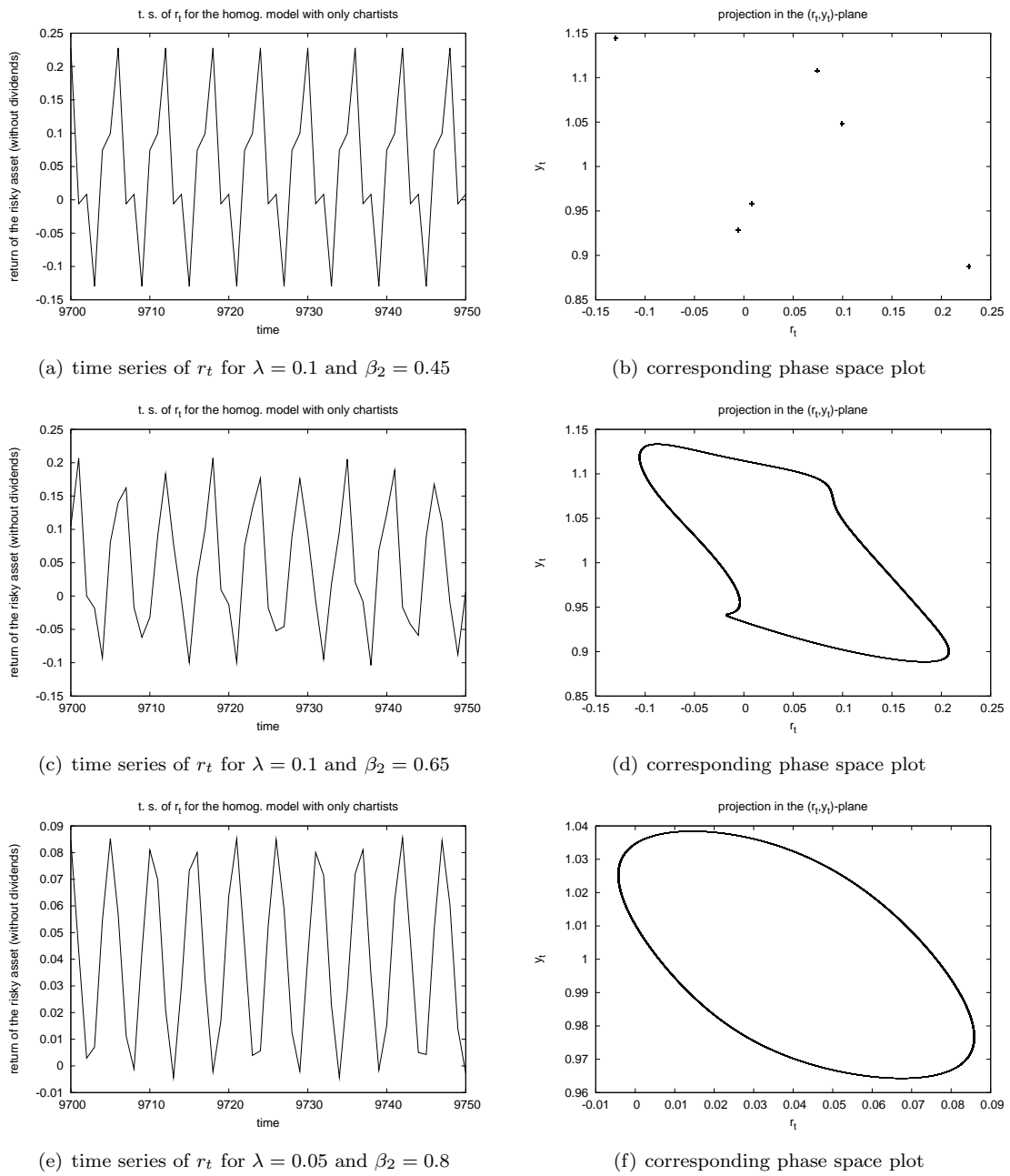


**Figure 4.4:** Bifurcation diagrams with respect to the chartists' extrapolation parameter  $\beta_2$ . The left panel exhibits the case for a parameter set for which the fundamental steady state  $E_1$  is unstable for all choices of  $\beta_2$ . The right panel displays a market situation for which  $E_1$  is stable for  $\beta_2$  being not too close to 0 and 1. For small values of  $\beta_2$  the fundamental steady state undergoes a flip bifurcation, while for  $\beta_2$  being large enough a Hopf bifurcation occurs. The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $\nu = 1.3$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\alpha = 0.5$  and  $\beta_3 = 5.0$ .

to a parameter set very similar to the one fixed above,<sup>33</sup> the right panel concentrates on a market situation with a stable region of the fundamental steady state  $E_1$  by using the same parameters but with  $\lambda$  decreased to 0.05. Let us consider the left panel first. High-order cycles (see also Figure 4.5, upper panels) and more complex dynamic behavior occur alternately. For instance, for  $\beta_2$  being approximately between 0.55 and 0.8, the return of the risky asset fluctuates around the fundamental return with slightly less vigor, which corresponds to limit cycles in the  $(r_t, y_t)$ -plane (see also Figure 4.5, middle panels). However, for  $\beta_2$  between 0.8 and 0.9, 5-cycles with less fluctuation occur. In the right panel, the stable region of the fundamental steady state can clearly be detected. For chartists' extrapolation parameters below the critical value  $\beta_2^L$ , a period doubling is shown, while for relatively high extrapolation parameters a Hopf bifurcation occurs leading to small fluctuations of the risky-asset return about its fundamental value. Finally, Figure 4.5 exhibits the same results from a slightly different perspective. It shows the time series of the risky-asset return  $r_t$  and corresponding phase space plots for three different market scenarios in

<sup>32</sup>Note that  $\beta_2^L$  is given by  $\beta_2^L = [\nu\sigma^2(1+r_f)\{\alpha - 2(1+g)\} + \lambda(2+g+r_f)^2]/[4\lambda(1+g)(2+g+r_f)]$ , which can be determined in a similar way as performed in the proof of Theorem 4.5 in Appendix A 4.5.

<sup>33</sup>I have slightly increased the chartists' risk-aversion coefficient from  $\nu = 1.2$  to  $\nu = 1.3$  simply to achieve a better visualization of the stability properties of  $E_1$ .



**Figure 4.5:** Time series of risky-asset returns and corresponding phase space plots. The left panels exhibit the time series of the risky-asset return (ex dividends)  $r_t$  for different choices of the pair of parameters  $\lambda$  and  $\beta_2$  for the homogeneous beliefs model with only chartists in the market. On the right-hand side, the projection of the corresponding attractor on the  $(r_t, y_t)$ -plane is shown. While the upper panels correspond to a parameter set for which the dynamic system stays in a region with 6-cycles, the middle and lower panels are chosen from parameter sets providing more complex dynamics. In both market scenarios limit cycles occur, leading to fluctuating asset returns. In the lower panels, which correspond to the region of a Hopf bifurcation, the return of the risky asset is less volatile than shown in the middle panels. The phase space representations are generated by 50,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.3$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\alpha = 0.5$  and  $\beta_3 = 5.0$ .

the parameter region constrained above. Again, a complex market dynamics can be observed in both representations leading to the occurrence of periodic behavior and limit cycles.

### 4.3.3 Heterogeneous Beliefs – Fundamentalists and Chartists

In this subsection I will investigate the most general case in which both fundamentalists and chartists participate in the artificial financial market. This will clearly result in a dynamic system even more complex than that seen in the homogeneous cases. Summing up all relevant equations from the model description and the specification of the investors' trading strategies outlined in the beginning of this section and in the previous section the dynamics of the return  $r_t$ , the fundamental/price ratio  $y_t$  and the difference in the average wealth proportion of investors  $\bar{\omega}_t$  evolve over time according to<sup>34</sup>

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.80)$$

$$y_{t+1} = \frac{1+g}{1+r_{t+1}} y_t, \quad (4.81)$$

$$\bar{\omega}_{t+1} = \frac{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) - [1 - \bar{\omega}_t](1 + \beta_{c,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{c,t+1})}, \quad (4.82)$$

where

$$q_t = \frac{[(1 + \beta_{f,t})\pi_{f,t} - (1 + r_t)\pi_{f,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} + \frac{[(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]}$$

and

$$\beta_{i,t+1} = r_f + \pi_{i,t} [r_{t+1} - r_f + (r_f - g)y_t], \quad i \in \{f, c\}, \quad (4.83)$$

$$\pi_{f,t} = \frac{1}{\nu\sigma^2} [(g - r_f) - \beta_1(r_t - g) + (r_f - g)y_t], \quad (4.84)$$

$$\pi_{c,t} = \frac{\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t - r_f}{\nu [\sigma^2 + \beta_3 \bar{\sigma}_{c,t}^2]}, \quad (4.85)$$

$$\bar{\sigma}_{c,t}^2 = [\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t - r_f]^2. \quad (4.86)$$

Obviously, the system Eq. (4.80)–Eq. (4.82) is a seven dimensional system whose dimension can be reduced by one dimension using the transformation  $\check{\omega}_{t+1} \equiv \bar{\omega}_t$ , which leads to

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.87)$$

$$y_{t+1} = \frac{1+g}{1+r_{t+1}} y_t, \quad (4.88)$$

$$\check{\omega}_{t+1} = \frac{[1 + \check{\omega}_t](1 + \beta_{f,t}) - [1 - \check{\omega}_t](1 + \beta_{c,t})}{[1 + \check{\omega}_t](1 + \beta_{f,t}) + [1 - \check{\omega}_t](1 + \beta_{c,t})}, \quad (4.89)$$

where

$$q_t = \frac{[(1 + \beta_{f,t})\pi_{f,t} - (1 + r_t)\pi_{f,t-1}](1 + \bar{n})[1 + \check{\omega}_t]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \check{\omega}_t] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \check{\omega}_t]} + \frac{[(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1}](1 - \bar{n})[1 - \check{\omega}_t]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \check{\omega}_t] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \check{\omega}_t]} \quad (4.90)$$

<sup>34</sup>For the remainder of this chapter we assume that both investor types are characterized by a uniform risk-aversion coefficient  $\nu \equiv \nu_f = \nu_c$ , where possible exceptions from this general assumption will clearly be marked in the text.

and  $\beta_{f,t+1}, \beta_{c,t+1}, \pi_{f,t}, \pi_{c,t}, \bar{\sigma}_{c,t}^2$ , as given above. Due to this six-dimensional nonlinear dynamic system, the following results for the steady states can be obtained:

**Theorem 4.7** *Let  $g, r_f, \lambda, \alpha, \nu, \sigma^2 > 0$ ,  $g < r_f$ ,  $\bar{n} \in [-1, 1]$  as well as  $0 \leq \beta_1, \beta_2 \leq 1$  and  $\beta_3 \geq 0$ . Assume that Eq. (4.83)–Eq. (4.86) and Eq. (4.90) hold. Then the system Eq. (4.87)–Eq. (4.89) has infinitely many steady states, namely fundamental steady states, which are given by*

$$r^* = g \quad \wedge \quad y^* = 1 \quad \wedge \quad \bar{\omega}^* \in [-1, 1].$$

PROOF: See Appendix A 4.7.

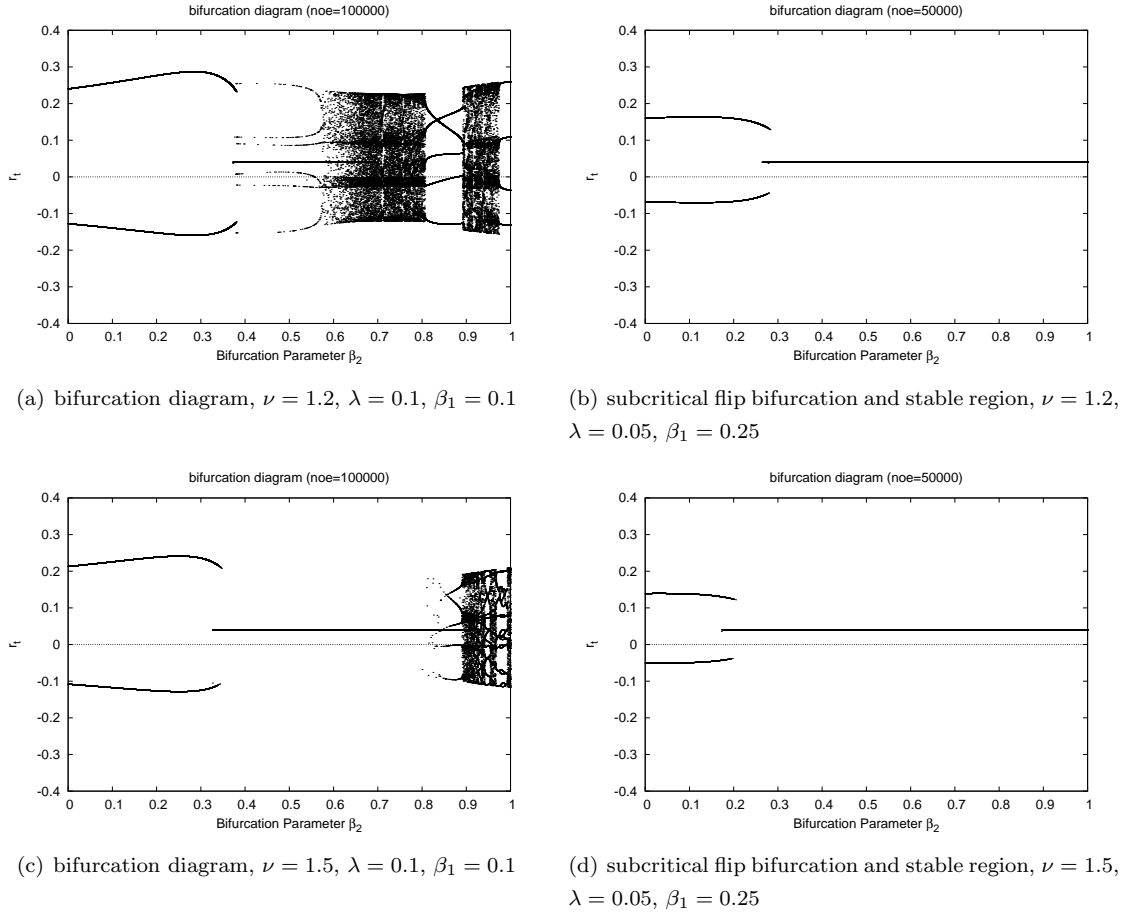
Theorem 4.7 states that the dynamic system of the heterogeneous beliefs model has infinitely many steady states, termed ‘fundamental steady states’.<sup>35</sup> As already observed in the homogeneous case, the fundamental steady states characterize market situations in which the price of the risky asset is at the fundamental ( $p_t = p_t^*$  for all  $t$ ) and the return of the risky asset (ex dividends) equals the growth rate of dividends.<sup>36</sup> The long-term difference in the average wealth proportion of fundamentalists and chartists at a fundamental steady state may be given, in general, by any  $\bar{\omega}_t$  between -1 and 1.<sup>37</sup> However, numerical simulations of the dynamic system confirm that the steady-state distribution of the investors’ average wealth proportion for the fundamental steady states depends on the initial condition.

To concisely sum up the numerical simulation results for this market scenario, the heterogeneous beliefs model with fundamentalists and chartists in the market is characterized by stable situations and 2-cyclic behavior for a wide range of parameter settings whenever  $\beta_1$  is sufficiently small (see Figures 4.6 and 4.8). Otherwise, if the fundamentalists extrapolate too much (i.e.,  $\beta_1$  is close to one), numerical simulations show that the dynamic system explodes in almost all cases. Interestingly, these situations show that an increase in the investor proportion of chartists (i.e., a decrease in  $\bar{n}$ ) might stabilize the system or at least lead to cyclic behavior. Furthermore, high-order cycles or even limit cycles emerge only in very few cases. In contrast to the two homogeneous-beliefs models outlined previously, the wealth distribution in the long run among fundamentalists and chartists is another interesting issue in a heterogeneous model version. Therefore, the main simulation outcomes for the evolution of the two different trading groups’ average wealth share is illustrated in Figure 4.7, which concentrates on two typical market situations. Whenever a fundamental steady state is reached the (stationary) average wealth proportion of investors  $\bar{\omega}^*$  stays almost at its initial condition and both trading groups survive in the market in the long run. However, whenever the system exhibits periodic behavior, chartists’ profits are always higher than the profits made by the fundamentalists. This eventually leads to the chartist group clearly dominating in terms of average wealth share.

<sup>35</sup>A continuum of steady states implies that one eigenvalue of the corresponding characteristic equation is ‘neutral’, i.e.,  $\pm 1$ . I have checked this result by using the software package MATHEMATICA 4.1.

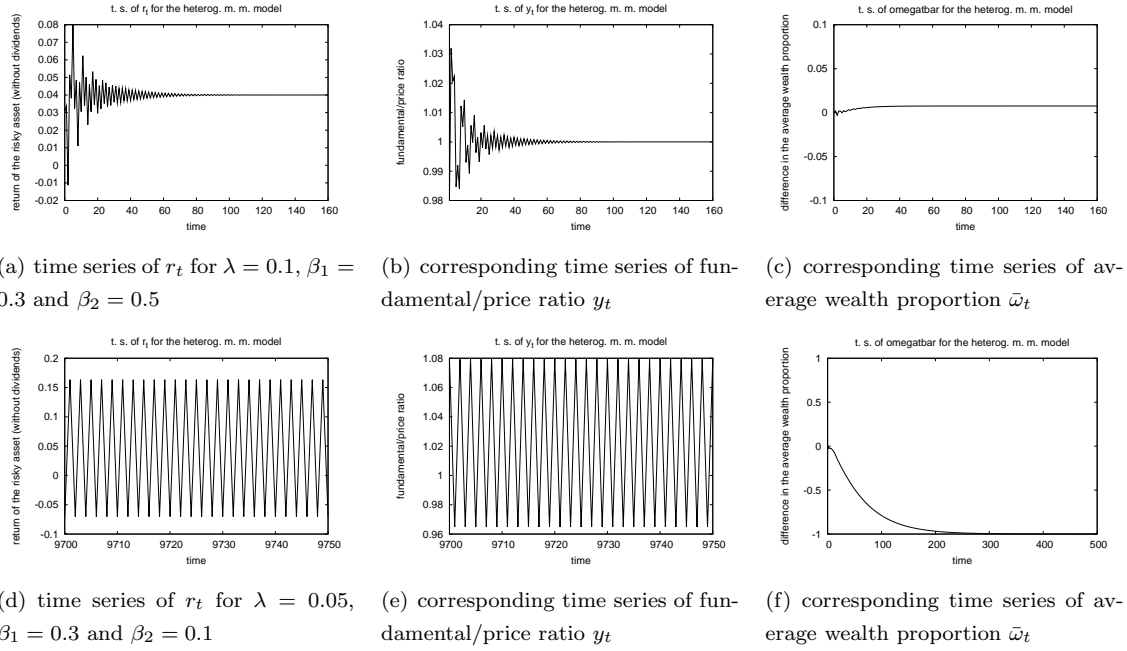
<sup>36</sup>From a purely mathematical point of view, eight further steady states with  $y^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t}{p_t^*} = 0$  can be derived. However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.7 in Appendix A 4.7 for details). In this respect only fundamental steady states exist for the heterogeneous beliefs model.

<sup>37</sup>Note that  $\bar{\omega}_t = \bar{\omega}_t$  at the steady state for all  $t$ .



**Figure 4.6:** Bifurcation diagrams with respect to  $\beta_2$  for the heterogeneous market maker model. For a relatively high adjustment speed ( $\lambda = 0.1$ ) of the market maker, 2-cycles and high-order cycles occur for the chartists' extrapolation parameter  $\beta_2$  being sufficiently close to 0 and 1 respectively (left panels). However, there is also a region where a fundamental steady state is reached. The right panels exhibit a market situation when  $\lambda$  is chosen as 0.05. The system converges to a fundamental steady state whenever  $\beta_2$  is sufficiently large. For small values of  $\beta_2$  the fundamental steady state undergoes a flip bifurcation. An increase in the risk-aversion coefficient stabilizes the system (lower panels). All plots show market situations in which the fundamentalists' group strongly believe in the fundamentals, i.e.,  $\beta_1$  being quite close to 0. The experiments are conducted in exactly the same way as described in Figure 4.2 whereas only 50,000 experiments are performed for the right panels. The remaining parameters are  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

In the following discussion the summarized simulation results are outlined in further details. Figures 4.6 and 4.8 depict representative bifurcation diagrams for the heterogeneous beliefs model. The upper panels of Figure 4.6 exhibit two bifurcation diagrams for similar parameter sets to those used for the homogeneous beliefs case (see Figures 4.2 and 4.4). Again, a smaller value of the market maker's price adjustment speed  $\lambda$  stabilizes the system (see upper right panel). When closely comparing the left panel of Figure 4.4 with the upper left panel of Figure 4.6, it becomes obvious that the fundamentalist group is able to stabilize the market. While high-order cycles and even more complex dynamics arise for any choice of  $\beta_2$  values in a homogeneous chartists' beliefs market (see Figure 4.4 (a)), introducing an investor group of fundamentalists might bring the system to



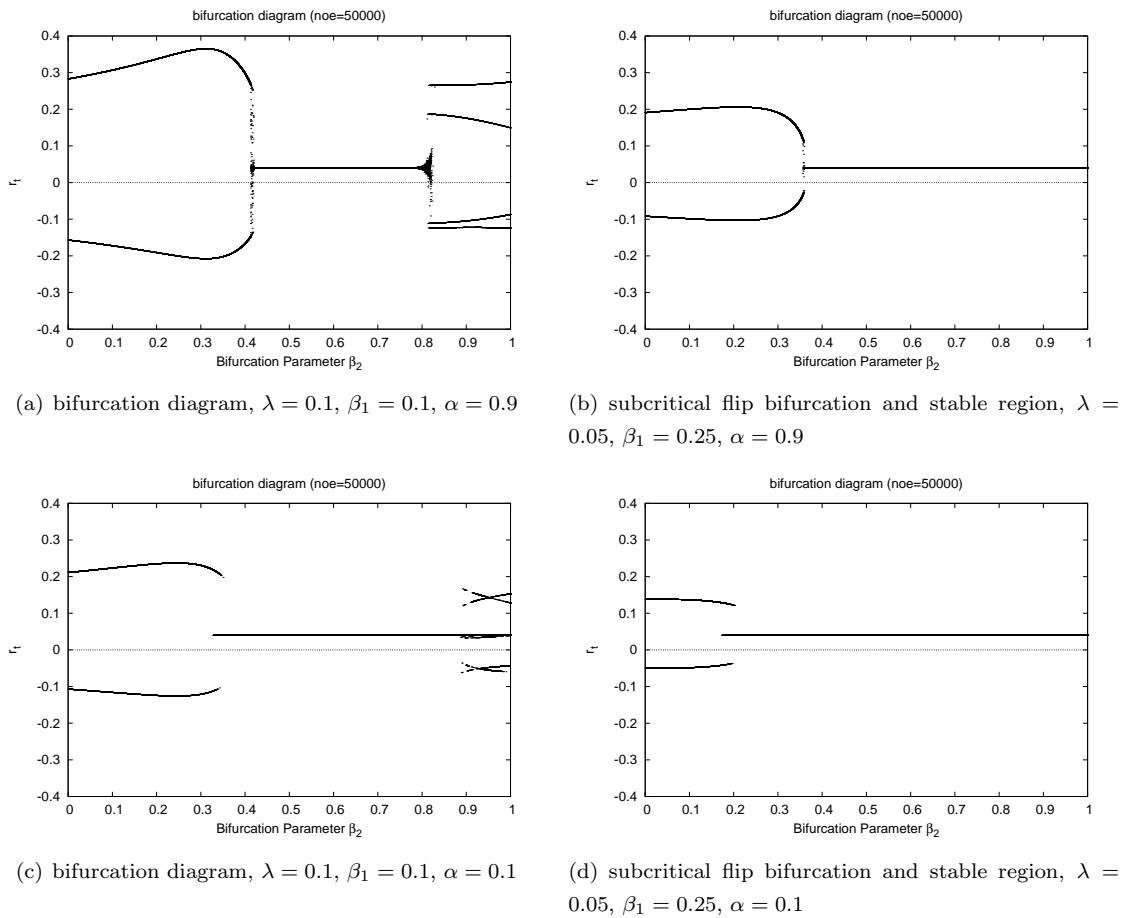
**Figure 4.7:** Time series of state variables for the heterogeneous market maker model. The plot exhibits the time series of the risky-asset return  $r_t$ , the fundamental/price ratio  $y_t$  and the average investors' wealth proportion of  $\bar{\omega}_t$  for two typical market situations for the heterogeneous beliefs model. While the upper panels correspond to a parameter set for which a fundamental steady state is reached, the lower panels show a 2-cycle behavior of the market dynamics. In the latter case the chartists accumulate the entire wealth share in the long run while a stable market situation leaves the initial wealth conditions unaffected. Parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

converge to a fundamental steady state whenever fundamentalists use an extrapolation parameter  $\beta_1$  quite close to 0 (i.e., they strongly believe in the fundamentals) and chartists extrapolate the future risky-asset return from the last two realized returns in a balanced way (i.e.,  $\beta_2$  approximately between 0.38 and 0.58). Indeed, in the upper left panel of Figure 4.6 it can be detected that the system also converges to a fundamental steady state for this specific choice of parameters.<sup>38</sup> Note that the upper panels of Figure 4.7 also give an example of such a market situation. The lower panels of Figure 4.6 clearly show the stabilizing effect of an increase in the investors' risk-aversion coefficient  $\nu$ , as already observed in the homogeneous beliefs models outlined above. Finally, further numerical simulations suggest that, in all non-explosive simulation runs, a decrease in the  $\alpha$  value leads to an increased stability region in the fundamental steady states, as shown in Figure 4.8 (see lower panels) and as already observed for the homogeneous beliefs models. However, the effect of increasing the value of  $\alpha$  for a relatively high value of the adjustment speed  $\lambda$  deviates from doing the same for a small value of the price adjustment speed  $\lambda$  (see upper panels). While the stable region is slightly reduced for the latter case, an increased value of  $\alpha$  even leads to wider

<sup>38</sup>A closer inspection of the bifurcation diagram given in Figure 4.6 (a) confirms that, for a significant number of the 100,000 experiments carried out, the system converges to a fundamental steady state for  $\beta_2$  being in the interval of 0.38 and 0.58. See also Footnote 69 and the discussion on the 'sensitive dependence on initial conditions' outlined thereabouts.



stable region for relatively high choices of  $\lambda$ .<sup>39</sup> Figure 4.9 exhibits the time series of the risky-asset

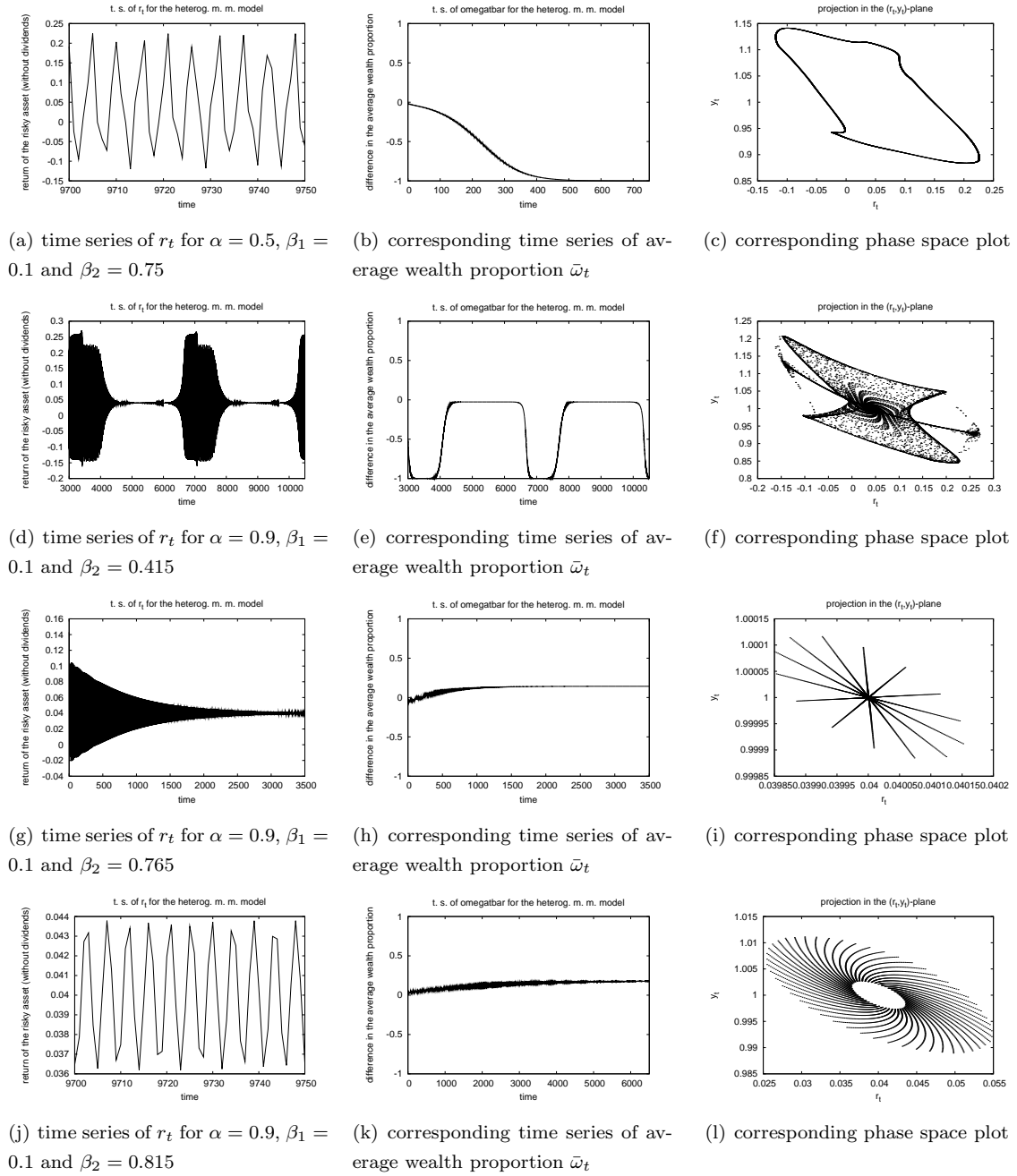


**Figure 4.8:** Bifurcation diagrams with respect to  $\beta_2$  for the heterogeneous market maker model with different choices of  $\alpha$ . While increasing or decreasing  $\alpha$  respectively destabilizes or stabilizes the market for a relatively low adjustment speed ( $\lambda = 0.05$ , right panels) of the market maker, the effect is not as obvious for a relatively high adjustment speed ( $\lambda = 0.1$ , left panels). The experiments are conducted in exactly the same way as described in Figure 4.2 whereas only 50,000 experiments are performed here. The remaining parameters are  $\nu = 1.2, r_f = 0.05, g = 0.04, \sigma = 0.2, \beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

return  $r_t$ , the average investors' wealth proportion of  $\bar{\omega}_t$  and the projection of the corresponding attractor on the  $(r_t, y_t)$ -plane for four different market outcomes. Note that all these four market outcomes can be assigned to their corresponding regions in the bifurcation diagrams shown in Figures 4.6 and 4.8, since the parameter sets are chosen appropriately.<sup>40</sup> Whenever the dynamic system creates a limit cycle, the chartists eventually dominate the market by accumulating the entire wealth share (see Figure 4.9 (a)–(c)). For certain parameter combinations a strange attractor with cyclical movements of returns and wealth shares may occur (see Figure 4.9 (d)–(f)). In phases

<sup>39</sup>See also Footnote 29 for a related discussion.

<sup>40</sup>The results shown in Figure 4.9 are not 'particular' simulation results but describe the typical market behavior of the heterogeneous market maker model for different parameter sets. Nevertheless, the value of  $\beta_2$  has to be calibrated by hand to find all these different shapes in the phase space plots. However only Figure 4.9 (d)–(f) represents a model outcome which is a rather rare event.



**Figure 4.9:** Time series of state variables and corresponding phase space plots for the heterogeneous market maker model. The plot exhibits the time series of the risky-asset return  $r_t$ , the average investors' wealth proportion of  $\bar{\omega}_t$  and the projection of the corresponding attractor on the  $(r_t, y_t)$ -plane for four market situations for the heterogeneous beliefs model. While the first and second row of plots correspond to parameter sets for which the market dynamics can be described by a limit cycle and strange attractor behavior, the third and fourth row of plots correspond to market scenarios in which a fundamental steady state is eventually reached. In the latter two cases the stable market situation leaves the initial wealth conditions more or less unaffected. Contrastingly, unstable situations see chartists accumulate the entire wealth share in the long run or even lead to cyclical movements in wealth share. The phase space representations are generated by 10,000 iterations (50,000 for Panel (f)) from which the first 5,000 (25,000 for Panel (f)) are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $\lambda = 0.1$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

where the return of the risky asset is far from its fundamental value, the chartists prevail over the fundamentalists in terms of wealth share while the fundamentalists are able to recover their initial wealth proportion in more tranquil market situations. Otherwise, if a fundamental steady state is reached, fundamentalists and chartists coexist with respect to their average wealth proportion in the long run, the fundamentalists generally accumulating slightly more wealth than the chartists (see Figure 4.9 (g)–(l)) as already observed in Figure 4.7 (c).<sup>41</sup>

#### 4.4 Adaptive Version of the Market Maker Model

In this section a slightly different model version is considered. I will introduce an adaptive form of expectation formation for the chartists and the beliefs of the fundamentalists are maintained. This modification in the chartists' trading strategy leads to noticeably more complex dynamics in the underlying nonlinear system, with typical features such as the occurrence of high-order cycles and convergence to limit cycles. Furthermore, the dynamics of the adaptive model will turn out to be more stable and robust against parameter changes, leading to a considerably lower number of explosive simulation events. In the next subsection the modified trading strategies of the two investor groups will be specified.

##### 4.4.1 Specification of the Trading Strategies in the Adaptive Model Version

As before, the fundamentalists' conditional mean and variance of the risky asset are mathematically expressed by

$$\begin{aligned} E_{f,t}(\rho_{t+1}) &= g - \beta_1(r_t - g) + (r_f - g)y_t, & \beta_1 &\in [0, 1], \\ \sigma_{f,t}^2 \equiv \text{Var}_{f,t}(\rho_{t+1}) &= \sigma^2, \end{aligned}$$

where  $\sigma^2$  is a constant and the conditional mean corresponds to the expectation about the return ex dividends

$$E_{f,t}(r_{t+1}) = g - \beta_1(r_t - g)$$

as already proposed in Eq. (4.63) in Section 4.3.

The adaptive expectations of the chartists are assumed to be

$$E_{c,t}(r_{t+1}) = r_t + \beta_2(r_t - \phi_t), \quad \beta_2 \in [0, 1], \quad (4.91)$$

$$\sigma_{c,t}^2 \equiv \text{Var}_{c,t}(\rho_{t+1}) = \sigma^2 + \beta_3 v_t, \quad \beta_3 \geq 0, \quad (4.92)$$

where

$$\phi_t = \delta \phi_{t-1} + (1 - \delta)r_t, \quad (4.93)$$

$$v_t = \delta v_{t-1} + \delta(1 - \delta)(r_t - \phi_{t-1})^2 \quad (4.94)$$

or, for the expectations about the risky-asset return including dividends, it is

$$E_{c,t}(\rho_{t+1}) = r_t + \beta_2(r_t - \phi_t) + (r_f - g)y_t, \quad \beta_2 \in [0, 1].$$

---

<sup>41</sup>The phase space plot depicted in Figure 4.7 (l) indicates that the parameter set used for this figure would have required more than 10,000 iterations to visualize the eventual convergence to a fundamental steady state.

The newly introduced variables  $\phi_t$  and  $v_t$  are sample mean and variance and follow some learning processes. The parameter  $\beta_2$  measures the extrapolation rate. High values of  $\beta_2$  correspond to strong extrapolation from the chartists. The parameter  $\beta_3$  weights the influence of the sample variance on the conditional variance estimated by the chartists. Eq. (4.93)–Eq. (4.94) can be considered as a limiting process of a geometric decay process when the memory lag length tends to infinity (see Appendices A 4.8 and A 4.9 for details). Then, the coefficient  $\delta$  measures the geometric decay rate. From Eq. (4.5) and the beliefs  $E_{c,t}(\rho_{t+1})$  and  $Var_{c,t}(\rho_{t+1})$  it follows that the optimal investment proportion  $\pi_{c,t}$  of the chartists can be determined by

$$\pi_{c,t} = \frac{r_t + \beta_2(r_t - \phi_t) + (r_f - g)y_t - r_f}{\nu[\sigma^2 + \beta_3 v_t]},$$

where  $\phi_t$  and  $v_t$  are given by Eq. (4.93) and Eq. (4.94) respectively.

#### 4.4.2 Heterogeneous Beliefs – Fundamentalists and Chartists

To avoid an overly long discussion of homogeneous market situations in this chapter, I will not go into so much detail about the homogeneous, adaptive beliefs model with only chartists in the market, but will instead continue directly with the heterogeneous investors' model. Nevertheless, it should be remarked that, in contrast to the discussion outlined in Subsection 4.3.2, the homogeneous market model with only chartists following the adaptive form of expectation formation specified above is even less stable in the sense that only a simultaneous increase in  $\nu$  and decrease in  $\lambda$  and  $\alpha$  might lead to a locally stable fundamental steady state  $E_1$ . Furthermore, the corresponding dynamic system exhibits quite complex behavior in all unstable situations, leading mostly to limit cycles in the phase space of  $r_t$  and  $y_t$  (see Figure 4.5 (d) and (f) for similar plots).

However, let us now consider the heterogeneous model. The resulting dynamic system with fundamentalists and chartists is given by

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.95)$$

$$y_{t+1} = \frac{1+g}{1+r_{t+1}} y_t, \quad (4.96)$$

$$\check{\omega}_{t+1} = \frac{[1 + \check{\omega}_t](1 + \beta_{f,t}) - [1 - \check{\omega}_t](1 + \beta_{c,t})}{[1 + \check{\omega}_t](1 + \beta_{f,t}) + [1 - \check{\omega}_t](1 + \beta_{c,t})}, \quad (4.97)$$

$$\phi_{t+1} = \delta \phi_t + (1 - \delta)r_{t+1}, \quad (4.98)$$

$$v_{t+1} = \delta v_t + \delta(1 - \delta)(r_{t+1} - \phi_t)^2, \quad (4.99)$$

where

$$q_t = \frac{[(1 + \beta_{f,t})\pi_{f,t} - (1 + r_t)\pi_{f,t-1}](1 + \bar{n})[1 + \check{\omega}_t]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \check{\omega}_t] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \check{\omega}_t]} + \frac{[(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1}](1 - \bar{n})[1 - \check{\omega}_t]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \check{\omega}_t] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \check{\omega}_t]} \quad (4.100)$$

and

$$\beta_{i,t+1} = r_f + \pi_{i,t}[r_{t+1} - r_f + (r_f - g)y_t], \quad i \in \{f, c\}, \quad (4.101)$$

$$\pi_{f,t} = \frac{1}{\nu\sigma^2} [(g - r_f) - \beta_1(r_t - g) + (r_f - g)y_t], \quad (4.102)$$

$$\pi_{c,t} = \frac{r_t + \beta_2(r_t - \phi_t) + (r_f - g)y_t - r_f}{\nu[\sigma^2 + \beta_3 v_t]}. \quad (4.103)$$

Note that  $\check{\omega}_{t+1} \equiv \bar{\omega}_t$  is again used to reduce the dimension of the dynamic system by one dimension. Hence, Eq. (4.95)–Eq. (4.99) exhibits a nine-dimensional nonlinear system with five independent state variables. The following theorem characterizes the possible steady states of the adaptive beliefs system:

**Theorem 4.8** *Let  $g, r_f, \lambda, \alpha, \nu, \sigma^2, \beta_3 > 0$ ,  $g < r_f$ ,  $\bar{n} \in [-1, 1]$  as well as  $0 \leq \beta_1, \beta_2, \delta \leq 1$ . Assume that Eq. (4.100)–Eq. (4.103) hold. Then the system Eq. (4.95)–Eq. (4.99) has infinitely many steady states, namely fundamental steady states, which are given by*

$$\phi^* = g \quad \wedge \quad v^* = 0 \quad \wedge \quad r^* = g \quad \wedge \quad y^* = 1 \quad \wedge \quad \check{\omega}^* \in [-1, 1].$$

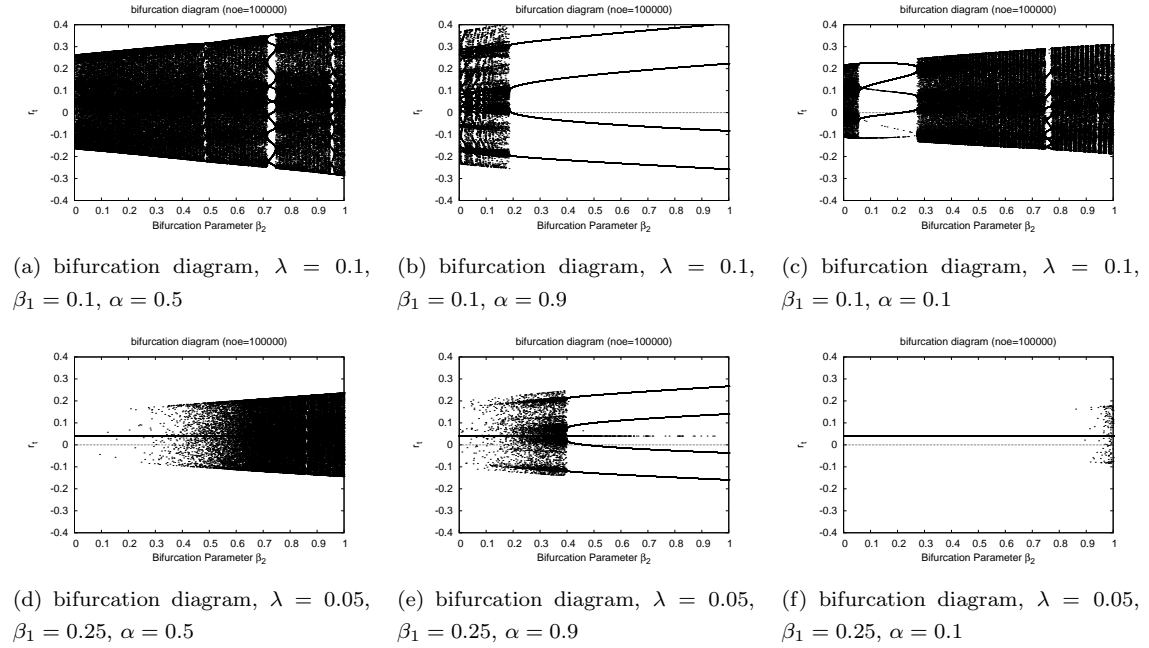
PROOF: See Appendix A 4.10.

The results of Theorem 4.8 are quite similar to the results of Theorem 4.7. Only fundamental steady states exist.<sup>42</sup> Nevertheless, numerical simulations reveal some notable differences between both model versions. As stated in Subsection 4.3.3 for the heterogeneous beliefs model with a chartists' strategy, which only takes into account the last two realized returns of the risky asset for the chartists' portfolio optimization, the dynamic system explodes for almost all market situations in which the extrapolation parameter of the fundamentalists (i.e.,  $\beta_1$ ) is chosen too large. Interestingly, this undesirable market behavior vanishes for almost all corresponding parameter sets for the adaptive market model modification. Hence it can be concluded that the adaptive model is more robust with respect to parameter changes than the heterogeneous market maker model outlined before. An increase in the fundamentalists' extrapolation rate even seems to stabilize the market, i.e., fundamental steady states can be reached for a wider range of  $\beta_2$  values.<sup>43</sup> As in the heterogeneous market maker model outlined in Subsection 4.3.3, numerical experiments suggest that the parameter  $\alpha$ , which measures the impact of the stabilizing force term in the price adjustment mechanism by the market maker, plays a crucial role for the long-term behavior of the market. Both Figures 4.10 and 4.11 concentrate on this feature. They exhibit typical results of the adaptive market model.

Figure 4.10 shows bifurcation diagrams for different parameter sets investigating combinations of  $\lambda$  and  $\alpha$ . Note that the left panels can be compared to Figure 4.6 (a) and (b), since the parameter sets are chosen as identical. Although the resulting dynamics becomes even more complex in the adaptive market maker model framework, the qualitative change from a highly complex dynamics to a more simple one (or even to a stable situation) by decreasing  $\lambda$  from 0.1 to 0.05 is still obvious (see Figure 4.10 (a) and (d)). Moreover, a low value of  $\alpha$  might cause convergence to a fundamental

<sup>42</sup>From a purely mathematical point of view, six further steady states with  $y^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  can be derived. However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.8 in Appendix A 4.10 for details). In this respect only fundamental steady states exist for the adaptive heterogeneous-beliefs model.

<sup>43</sup>These results are determined by evaluating the eigenvalues of the characteristic equation by using the software package MATHEMATICA 4.1 as already explained in the appendices of the proofs for the previous theorems. Since a continuum of steady states are derived for the dynamic system, the corresponding characteristic equation has one 'neutral' eigenvalue (see Footnote 35).



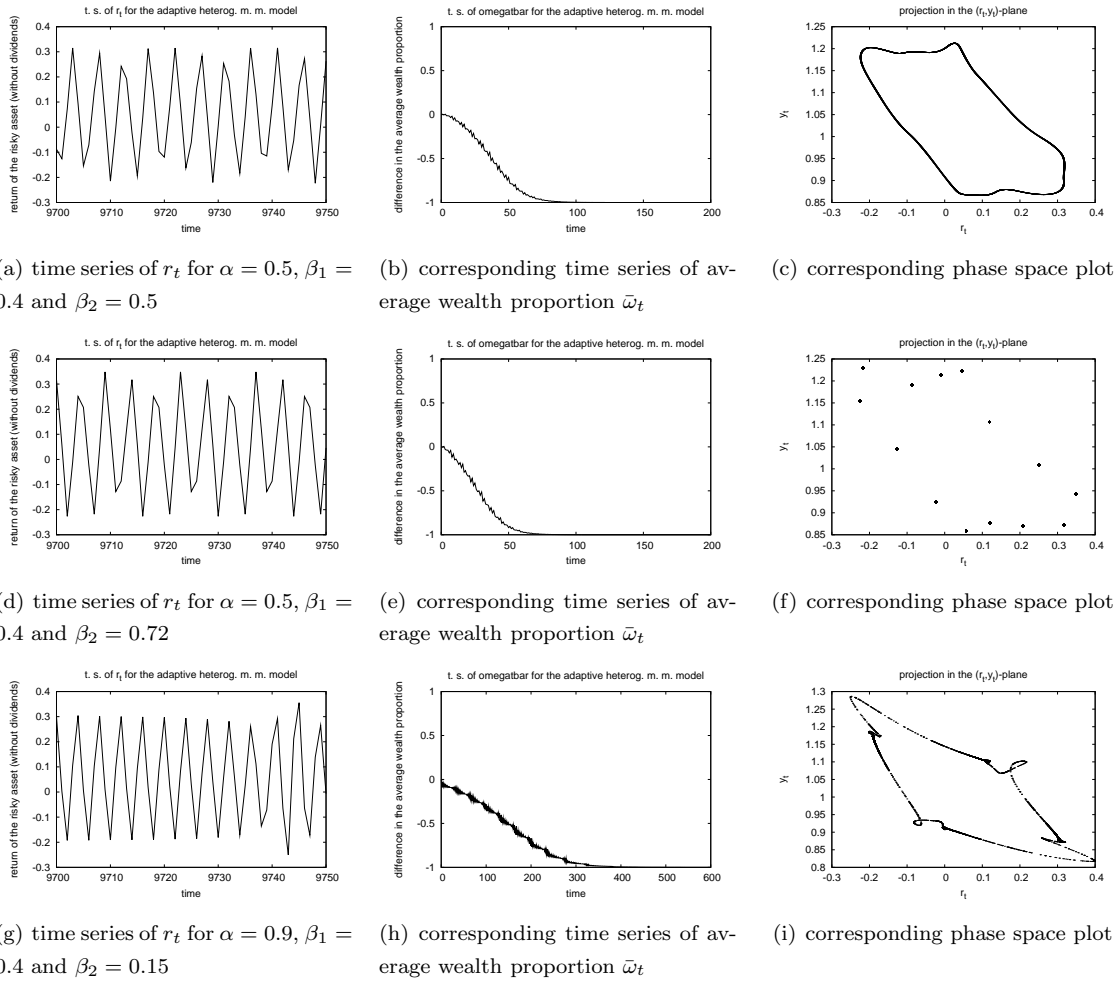
**Figure 4.10:** Bifurcation diagrams with respect to  $\beta_2$  for the adaptive market maker model. For a relatively high adjustment speed ( $\lambda = 0.1$ ) of the market maker, the dynamic system becomes rather complex (upper panels). If  $\lambda$  is decreased to 0.05, a fundamental steady state might be reached for the chartists' extrapolation parameter  $\beta_2$  being sufficiently close to 0 (lower panels). Furthermore, the system becomes less complex for a low value of  $\alpha$ , or even becomes stable (right panels), while a relatively high value of  $\alpha$  causes 4-cycles for almost all choices of  $\beta_2$  (middle panels). To make the plots comparable to the heterogeneous market model outlined in Subsection 4.3.3, the extrapolation rate of the fundamentalists  $\beta_1$  are chosen as 0.1 and 0.25. This describes market situations in which the fundamentalists' group strongly believe in the fundamentals, i.e.,  $\beta_1$  being relatively close to 0. The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

steady state (see Figure 4.10 (f)), while a high value of  $\alpha$  corresponds in almost all cases to a 4-cycle behavior of the dynamic system (see Figure 4.10 (b) and (e)). From a qualitative perspective, similar results have also been found in the heterogeneous market maker model (see Figure 4.8).

Figure 4.11 uses the same type of visualization as Figure 4.9 and corresponds to certain regions of the bifurcation diagrams given in Figure 4.10 (a) and (b). Although the fundamentalists' extrapolation parameter  $\beta_1$  is chosen higher than in the bifurcation diagrams, the dynamic behavior remains qualitatively the same as in Figure 4.10. For almost all parameter sets there are  $n$ -cycles or limit cycles in which chartists eventually dominate the market by accumulating the entire wealth share. Otherwise, if a fundamental steady state is reached, fundamentalists and chartists coexist with respect to their average wealth proportion in the long run.<sup>44</sup>

Again, a mutual increase in the risk-aversion coefficient of fundamentalists and chartists might stabilize the system. Figure 4.12 refers to this aspect and can be compared with Figure 4.10 (a)–(c) since all other parameters are chosen identically. This shows that a fundamental steady state is reached for sufficiently low extrapolation rates of the chartists and values of  $\alpha$  (see Figure

<sup>44</sup>Results are not shown in detail here.

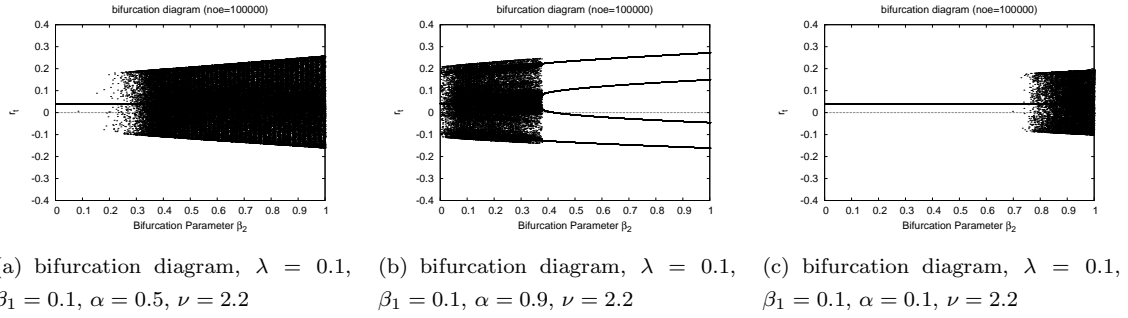


**Figure 4.11:** Time series of state variables and corresponding phase space plots for the adaptive market maker model. The plot exhibits the time series of the risky-asset return  $r_t$ , the average investors' wealth proportion of  $\bar{\omega}_t$  and the projection of the corresponding attractor on the  $(r_t, y_t)$ -plane for three non-stable market situations for the adaptive heterogeneous-beliefs model. As already observed, in the case of high-order cycles or limit cycle behavior, chartists accumulate the entire wealth share in the long run. The phase space representations are again generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $\lambda = 0.1$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

4.12 (a) and (c) whenever both trading groups use a relatively high risk-aversion coefficient to determine their optimal investment decision, which means that they invest in the risky asset only moderately. Furthermore, numerical simulations suggest that a unilateral increase in the risk-aversion coefficient (e.g., for the chartists) might also lead to a stable market situation. Indeed, for quite a large variety of parameter sets, a unilateral increase in  $\nu_c$  stabilizes the market.<sup>45</sup>

Finally, the influence of the starting conditions of the dynamic system (with respect to the investors' population and wealth proportion) on the long-run behavior of the market should be investigated. The results are straightforward and are visualized in Figure 4.13. Starting with an

<sup>45</sup>Results are not shown in details here.



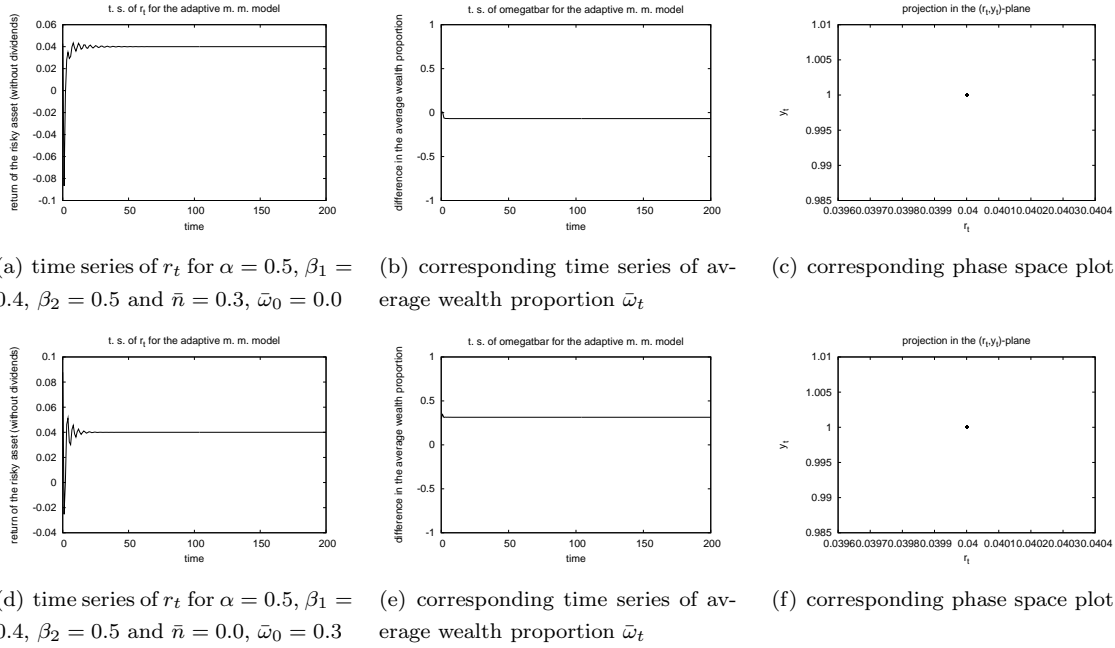
**Figure 4.12:** Bifurcation diagrams with respect to  $\beta_2$  for the adaptive market maker model with an increased risk-aversion coefficient  $\nu$ . Stable regions occur for values of  $\beta_2$  and  $\alpha$  being sufficiently low (left and right panel). The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

unstable market situation (assume Figure 4.11 (d)–(f) as a benchmark), an increase in the population proportion of fundamentalists might bring the dynamic system back to a fundamental steady state (upper panels). Furthermore, an unsymmetrically distributed initial wealth proportion also allows the dynamic system to eventually converge to one of its fundamental steady states, where the fundamentalists only lose a marginal wealth share to the chartist group in the first few trading periods (lower panels). Of course, both results should have been expected. Whenever the market is exposed to disequilibrium dynamics by an appropriate choice of parameters, a decrease in the chartists' market power restabilizes the market. There are two ways of lowering chartists' market power: their endowment (i.e., their wealth) could be reduced, or the fraction of fundamentalists could be increased (or the fraction of chartists decreased), which should also lead to a stabilization of the market model.

## 4.5 Results of the Fully Developed Market Maker Model

The model versions that have been investigated so far assume a zero supply of outside stocks, i.e.,  $H_t = 0$ , which might still be considered as an overly simple assumption for two reasons. On the one hand, it leads to the agents' risk premium vanishing (see Eq. (4.33)), which contradicts the natural fact that the risk premium in real financial markets for investing in risky prospects should be positive and should be determined endogenously by the risky-asset price evolution and the agents' market behavior. On the other hand, in a growth model in which the investors' wealth increases over time, the supply of the risky asset should also grow simply because the investors should be offered the opportunity to invest their increasing wealth in an appropriate number of supplied risky assets. This justifies the assumption of a similarly growing asset supply. For this reason, this final section outlines a fully-developed model version of the market maker model which solves this evident inconsistency in the model framework. Obviously, taking this new model feature into account will lead to a more complex and even higher dimensional dynamic system than those investigated in the previous sections. Therefore, analytic results for the stability of steady states will be hard to obtain even in the most simple cases of a homogeneous beliefs system and so numerical simulations have to be performed for almost all market scenarios.





**Figure 4.13:** Time series of state variables and corresponding phase space plots for the adaptive market maker model with different population proportions and initial wealth proportions. The plot exhibits the time series of the risky-asset return  $r_t$ , the average investors’ wealth proportion of  $\bar{w}_t$  and the projection of the corresponding attractor on the  $(r_t, y_t)$ -plane for the same parameter set as used in Figure 4.11 (d)–(f), but with different starting conditions. Sufficiently Increasing either the fundamentalists’ population or the fundamentalists’ initial wealth proportion leads to stable market situations where fundamentalists and chartists coexist in the long run. The remaining parameters are  $\nu = 1.2, r_f = 0.05, g = 0.04, \sigma = 0.2, \delta = 0.5, \lambda = 0.1, \beta_3 = 5.0$ .

Let us briefly recall the main assumptions made in Subsection 4.2.4, where a market equilibrium for the asset pricing model with a non-zero asset supply was established on its *fundamental* price path. These assumptions are:

- All investors form the same rational expectations of the conditional mean and variance of the risky asset and thus derive the same risk premium for investing in the risky prospect.
- The resulting non-zero risk premium is assumed to be constant, which implies that the (nominal) risky-asset supply grows in the same manner as the investors’ total wealth in the market. Since the growth rate of total wealth naturally coincides with the growth rate of the aggregated investors’ demand on risky assets, the growth rate of (nominal) risky-asset supply also equals the growth rate of (nominal) risky-asset demand, which seems an appropriate and reasonable assumption.
- The supplied number of risky assets increase linearly over time, where the growth rate has to be chosen exogenously, but has to be sufficiently high.

### 4.5.1 Model Dynamics of the Revised Market Maker Model

For the overall model dynamics of the market maker model I apply the assumption of the asset supply process introduced in Eq. (4.48), i.e., that the number of risky assets increases uniformly even beyond the market equilibrium. It is again assumed that only two investor groups act in the market (see Subsections 4.2.2 and 4.2.3 for the notation and details), where both investor groups share the same knowledge of the asset supply process. Furthermore, as for the more simplified model versions investigated in the previous sections, I want to reformulate the resulting dynamic system as a stationary model in terms of asset return, fundamental/price ratio and investors' wealth proportions. However, due to the newly-introduced variable  $H_t$  the model framework has to be enlarged by another state variable which will incorporate this new variable  $H_t$ . For this reason the 'asset supply/total wealth ratio' is defined:

$$z_t \equiv \frac{H_t}{\bar{W}_t I}. \quad (4.104)$$

To determine the time evolution of the state variable  $z_t$ , the following approximation is useful:

$$\begin{aligned} H_{t+1} &= N_{t+1}^s p_{t+1} \\ &= (1 + n^s) N_t^s \cdot (1 + r_{t+1}) p_t \\ &\approx (1 + n^s + r_{t+1}) N_t^s p_t = (1 + n^s + r_{t+1}) H_t, \end{aligned} \quad (4.105)$$

where  $n^s \cdot g \ll 1$  is assumed.<sup>46</sup> Using Eq. (4.105) the time evolution of the state variable  $z_t$  away from the fundamental price path can be determined by

$$\begin{aligned} z_{t+1} = \frac{H_{t+1}}{I \bar{W}_{t+1}} &= \frac{N_{t+1}^s p_{t+1}}{I [(1 + \beta_{1,t+1}) \bar{W}_{1,t} + (1 + \beta_{2,t+1}) \bar{W}_{2,t}]} \\ &\approx \frac{(1 + n^s + r_{t+1})}{(1 + \beta_{1,t+1}) \bar{\omega}_{1,t} + (1 + \beta_{2,t+1}) \bar{\omega}_{2,t}} \frac{N_t^s p_t}{I \bar{W}_t} \\ &= \frac{(1 + n^s + r_{t+1})}{(1 + \beta_{1,t+1}) \bar{\omega}_{1,t} + (1 + \beta_{2,t+1}) \bar{\omega}_{2,t}} z_t, \end{aligned} \quad (4.106)$$

where  $r_{t+1} = (p_{t+1}/p_t) - 1$  and  $\bar{\omega}_{i,t} = \bar{W}_{i,t}/\bar{W}_t$ ,  $i = 1, 2$  is used. Using once more the notation  $\bar{\omega}_t$  from Eq. (4.23) we end up with

$$z_{t+1} = \frac{2(1 + n^s + r_{t+1})}{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{2,t+1})} z_t.$$

Finally, the market excess demand proportion  $q_t$  (see Eq. (4.24)) can be written approximately by using the asset supply/total wealth ratio  $z_t$  as<sup>47</sup>

$$\begin{aligned} q_t \approx & \frac{[(1 + \beta_{1,t}) \pi_{1,t} - (1 + r_t) \pi_{1,t-1}] (1 + \bar{n}) [1 + \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n}) [1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n}) [1 - \bar{\omega}_{t-1}]} \\ & + \frac{[(1 + \beta_{2,t}) \pi_{2,t} - (1 + r_t) \pi_{2,t-1}] (1 - \bar{n}) [1 - \bar{\omega}_{t-1}] - 4n^s z_{t-1}}{(1 + \beta_{1,t})(1 + \bar{n}) [1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n}) [1 - \bar{\omega}_{t-1}]} . \end{aligned} \quad (4.107)$$

The remaining model features are carried over from the model framework outlined in Section 4.2. Thus heterogeneity is introduced by allowing the investors to form different time-varying

<sup>46</sup>An analogous approximation has already been used in Eq. (4.53) in Subsection 4.2.4.

<sup>47</sup>The derivation of Eq. (4.107) is relegated to the Appendix A 4.11.

beliefs about the first and second moment of the risky-asset return, though they are assumed to share the same beliefs about the dividends process according to Eq. (4.25). More precisely, the investors are assumed to update their beliefs about the *price change* over the next period by forming beliefs about the expectation and variance of the excess return (see Eq. (4.63)–Eq. (4.65)). However, due to the fact that both investor groups know the dividend process, they form correct expectations about the fundamental price path which therefore evolves according to Eq. (4.43), which is

$$p_{t+1}^* = (1 + g)p_t^* \frac{r_t^{RP} - g}{r_{t+1}^{RP} - g},$$

where

$$r_t^{RP} = r_f + \frac{H_t}{\bar{W}_t I} \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu_1 \sigma_{1,t}^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu_2 \sigma_{2,t}^2} \right]^{-1} \quad (4.108)$$

and  $p_t^*$  being the fundamental solution of the law of motion on the fundamental price path (see Eq. (4.44)) given by

$$E_t(p_{t+1} + d_{t+1}) = p_t \left\{ 1 + r_f + \frac{H_t}{\bar{W}_t I} \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu_1 \sigma_{1,t}^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu_2 \sigma_{2,t}^2} \right]^{-1} \right\}$$

under certain simplifying assumptions outlined in detail in Subsection 4.2.3.

#### 4.5.2 Summary of the Revised Stationary Model in its State Variables

Based on the artificial financial market model under a market maker scenario proposed in the previous sections of this chapter, a non-zero risky-asset supply has been added to this model framework, which leads to a non-vanishing and time-dependent quantity representing the risk premium required by the investors to hold the risky asset even on the fundamental price path. Again, the resulting growth model can now be reformulated as a stationary model in four state variables: the return  $r_t$ , the fundamental/price ratio  $y_t$ , the asset supply/total wealth ratio  $z_t$  and the difference in the wealth proportions  $\bar{\omega}_t$ . This result is resumed in the following theorem:

**Theorem 4.9** *The fundamental/price ratio evolves according to*

$$y_{t+1} = \frac{1 + g}{1 + r_{t+1}} \frac{r_t^{RP} - g}{r_{t+1}^{RP} - g} y_t, \quad (4.109)$$

where the return (ex dividends) of the risky asset is given by

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1) \quad (4.110)$$

with

$$q_t = \frac{[(1 + \beta_{1,t})\pi_{1,t} - (1 + r_t)\pi_{1,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} + \frac{[(1 + \beta_{2,t})\pi_{2,t} - (1 + r_t)\pi_{2,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}] - 4n^s z_{t-1}}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]}$$

and

$$\beta_{i,t+1} = r_f + \pi_{i,t}[r_{t+1} - r_f + (r_t^{RP} - g)y_t],$$

$$\begin{aligned}\pi_{i,t} &= \frac{E_{i,t}(r_{t+1}) + (r_t^{RP} - g)y_t - r^f}{\nu_i \sigma_{i,t}^2}, & i = 1, 2, \\ r_t^{RP} &= r_f + \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu_1 \sigma_{1,t}^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu_2 \sigma_{2,t}^2} \right]^{-1} z_t.\end{aligned}$$

The asset supply/total wealth ratio evolves according to

$$z_{t+1} = \frac{2(1 + n^s + r_{t+1})}{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{2,t+1})} z_t, \quad (4.111)$$

while the difference in the average wealth proportions of the two investor groups evolves according to

$$\bar{\omega}_{t+1} = \frac{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) - [1 - \bar{\omega}_t](1 + \beta_{2,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{1,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{2,t+1})}. \quad (4.112)$$

The system Eq. (4.109)–Eq. (4.112) forms a stationary model in four state variables: the fundamental/price ratio, the return (ex dividends) of the risky asset, the asset supply/total wealth ratio and the difference in the average wealth proportions of the two investor groups.

In relation to these model equations, the following remarks are necessary in order to clarify some more technical or mathematical issues of the dynamic system. The first remark follows immediately while the other two will appear in the next subsections:

#### Technical Remark 4.10 (On the Specific Choice of the New State Variable)

The newly introduced variable  $r_t^{RP}$  is obviously crucial in the revised model framework. It describes the time evolution of a market-inherent risk-premium term that arises because all investors are assumed to be risk-averse due to their CRRA utility optimization procedure, and because the supply of risky assets is non-vanishing. However, it cannot be used as the fourth state variable for the dynamic system since  $r_t^{RP}$  depends on the variables summarized in the variable  $z_t$  as well as on the difference in the investors' average wealth proportions  $\bar{\omega}_t$  and  $\sigma_{1,t}^2$  and  $\sigma_{2,t}^2$  (see Eq. (4.108)). Thus the time evolution of  $r_t^{RP}$  cannot be determined in a compactly closed form. For this reason the variable  $z_t$  is used as the fourth independent state variable instead. Nevertheless, the definition of  $z_t$  is slightly inconvenient from a different more technical perspective since it involves the variable  $\bar{W}_t$  (and not the total wealth of the economy  $W_t$ ), which is defined as the sum of the average wealth of each agent within group  $i$ . Thus it makes a difference in the homogeneous model for the actual values of  $z_t$  (e.g., derived in the simulation runs) if only *one* investor type is assumed to act in the market (i.e.,  $\bar{n} = \bar{\omega}_t = \pm 1$ ) or if *two* investor groups pursuing *identical* trading strategies are assumed (i.e.,  $\bar{n} = \bar{\omega}_t = 0$ ).<sup>48</sup> This is because the numerical value of  $\bar{W}_t$  (as well as the value of the  $[\dots]^{-1}$ -expression in Eq. (4.108)) will be different in both cases. However, the values of  $r_t^{RP}$  and  $q_t$  will be identical for both cases, which is satisfactory as both are the more interpretable variables from an economic point of view. Obviously, this phenomenon has not yet occurred because the assumption of a zero supply of outside stocks canceled out the  $\bar{W}_t$ -terms in the previous model versions discussed in the two last sections.

<sup>48</sup>Both scenarios can easily be investigated by simulating the corresponding dynamic system. Of course, the numerical results of the main economic (state) variables, such as  $r_t$ ,  $y_t$ ,  $q_t$ ,  $r_t^{RP}$  remain the same for both scenarios. Unfortunately, this is not true for all relevant variables in the dynamic system, for instance in the case of the state variable  $z_t$ .

Specifying the agents' beliefs about the risky-asset return will conclude the model structure and will lead to the resulting dynamic system showing different market and price behaviors. This will be done in the next subsections, where different model variants with homogeneous and heterogeneous beliefs systems are investigated.

### 4.5.3 Homogeneous Beliefs – Fundamentalists

As already defined in Eq. (4.63), we assume that fundamentalists believe the returns of the risky asset will eventually move back to the relative price change of the fundamental price, i.e.,  $r^* = g$ . Mathematically, this is expressed by

$$\begin{aligned} E_{f,t}(r_{t+1}) &= g - \beta_1(r_t - g), \\ \sigma_{f,t}^2 &= \sigma^2, \end{aligned}$$

where  $0 \leq \beta_1 \leq 1$  and  $\sigma^2 > 0$  are constants. Hence Theorem 4.9 can be reduced to following dynamic system:

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.113)$$

$$y_{t+1} = \frac{(1+g)(r_t^{RP} - g)}{(1+r_{t+1})(r_{t+1}^{RP} - g)} y_t, \quad (4.114)$$

$$z_{t+1} = \frac{1+n^s+r_{t+1}}{1+\beta_{f,t+1}} z_t, \quad (4.115)$$

where

$$q_t = \frac{(1+\beta_{f,t})\pi_{f,t} - (1+r_t)\pi_{f,t-1} - n^s z_{t-1}}{1+\beta_{f,t}}, \quad (4.116)$$

$$\beta_{f,t+1} = r_f + \pi_{f,t}[r_{t+1} - r_f + (r_t^{RP} - g)y_t], \quad (4.117)$$

$$\pi_{f,t} = \frac{1}{\nu\sigma^2} [(g - r_f) - \beta_1(r_t - g) + (r_t^{RP} - g)y_t], \quad (4.118)$$

$$r_t^{RP} = r_f + \nu\sigma^2 z_t \quad (4.119)$$

and  $\nu \equiv \nu_f$ . Obviously, the system Eq. (4.113)–Eq. (4.115) is a six-dimensional nonlinear system.<sup>49</sup> Due to this most simple market scenario in the revised model framework, the following theorem can be obtained, which relates to the existence of steady states:

**Theorem 4.11** *Let  $g, r_f, \lambda, \alpha, \nu, \sigma^2, n^s > 0$  and  $n^s \geq (r_f - g)$ . Furthermore, assume that  $0 \leq \beta_1 \leq 1$  and Eq. (4.116)–Eq. (4.119) hold. Then the system Eq. (4.113)–Eq. (4.115) has a steady state  $E_1$  given by*

$$r^* = g \quad \wedge \quad y^* = 1 \quad \wedge \quad z^* = \sqrt{\frac{n^s + g - r_f}{\nu\sigma^2}}. \quad (4.120)$$

PROOF: See Appendix A 4.12.

<sup>49</sup>For a comparison with the results generated by the corresponding model framework with zero supplied risky assets (i.e.,  $H_t = 0$ ), we refer to Subsection 4.3.1.

Theorem 4.11 states that one economically reasonable steady state exists for the homogeneous fundamental-belief model with a non-zero risky-asset supply.<sup>50</sup> The ‘quasi’-unique steady state  $E_1 = (g, g, 1, 1, \sqrt{(n^s + g - r_f)/(\nu\sigma^2)}, \sqrt{(n^s + g - r_f)/(\nu\sigma^2)})$  corresponds to the fundamental steady state where  $z^*$  is exactly the same steady-state variable as has already introduced in Eq. (4.49). Thus it can again be concluded that  $z^*$  is only an economically realistic steady-state variable if the relation  $n^s \geq (r_f - g)$  holds (see Eq. (4.56) and the related discussions). Otherwise, the radical of the square root would become negative, which would be equivalent to a pure imaginary steady-state variable  $z^*$ . In contrast to the  $H_t = 0$  case, where the optimal investment proportion  $\pi_f^*$  of the investors for investing in the risky asset is equal to zero at the fundamental steady state (see Remark 4.3),  $\pi_f^*$  is here determined by

$$\pi_f^* = \frac{E_{f,t}(\rho^*) - r_f}{\nu\sigma^2} = \frac{E_{f,t}(r^*) + (r^{RP^*} - g)y^* - r_f}{\nu\sigma^2} = \frac{g + (r_f + \nu\sigma^2 z^* - g) - r_f}{\nu\sigma^2} = z^*$$

for the revised model framework.<sup>51</sup> This result is summarized in the following remark:

**Remark 4.12** *Note that the total agent’s demand at the fundamental steady state  $E_1$  is exactly equal to the asset supply/total wealth ratio, i.e.,  $\pi_f^* = z^*$ .*

Remark 4.12 implies that the optimal investment proportion  $\pi_f^*$  at the fundamental steady state might even exceed 100%, which means that investors invest more money in the risky asset than they actually possess. Note that this procedure is not strictly forbidden by our model framework because  $\pi_{i,t}$  is assumed to be unbounded in principle. Therefore, a low risk-aversion coefficient of investors or a relatively high growth rate in the number of outstanding shares might generally lead to the investors taking such investment decisions. This interrelation is relatively unsurprising since  $z_t$  can also be taken as the market risk-premium  $RP_t$  divided by  $\nu\sigma^2$  (see Eq. (4.119) and also Eq. (4.50)). Thus an increase in  $z^*$  caused, for instance, by an increase in  $n^s$  is equal to a higher risk premium  $RP^*$ , which will, of course, lead immediately to a higher demand amongst investors for the risky assets if all other market parameters are left unchanged. Of course, the same argument can be used when the growth rate of dividends  $g$  increases or the risk-free rate  $r_f$  decreases, since the risky asset becomes more competitive in both cases. However, whenever the risk-aversion coefficient changes, the behavior of  $z^*$  and  $RP^*$  is slightly different. While  $z^*$  (and therefore also  $\pi_f^*$ ) increases whenever  $\nu$  decreases, which is economically reasonable as the investor becomes less risk-averse, the value of the risk premium  $RP^* = \nu\sigma^2 z^*$  decreases<sup>52</sup> because a lower risk premium is now necessary to convince the investors to invest in the risky prospect. However, in the following discussion we will mainly concentrate on market situations in which the optimal

<sup>50</sup>From a purely mathematical point of view two further steady states with  $y^* = 0$  and  $z^* > 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  and  $\lim_{t \rightarrow \infty} z_t = z^* > 0$ , as well as three further steady states with  $y^* = 0$  and  $z^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  and  $\lim_{t \rightarrow \infty} z_t = z^* = 0$  can be derived. However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.11 in Appendix A 4.12 for details). In this respect the fundamental steady state is ‘quasi’ unique. Note that the ‘quasi’ uniqueness of the steady state coincides with the results found for the  $H_t = 0$  case outlined in Subsection 4.3.1.

<sup>51</sup>See also the proof of Theorem 4.11 in Appendix A 4.12 for further details of the calculation.

<sup>52</sup>Note that  $RP^* = \nu\sigma^2 z^* = \nu\sigma^2 \pi_f^*$  where  $\pi_f^*$  is a function of  $1/\sqrt{\nu}$ .

investment proportion  $\pi^*$  would be between 0 and 1. This can easily be guaranteed by choosing appropriate parameter sets for our artificial market.

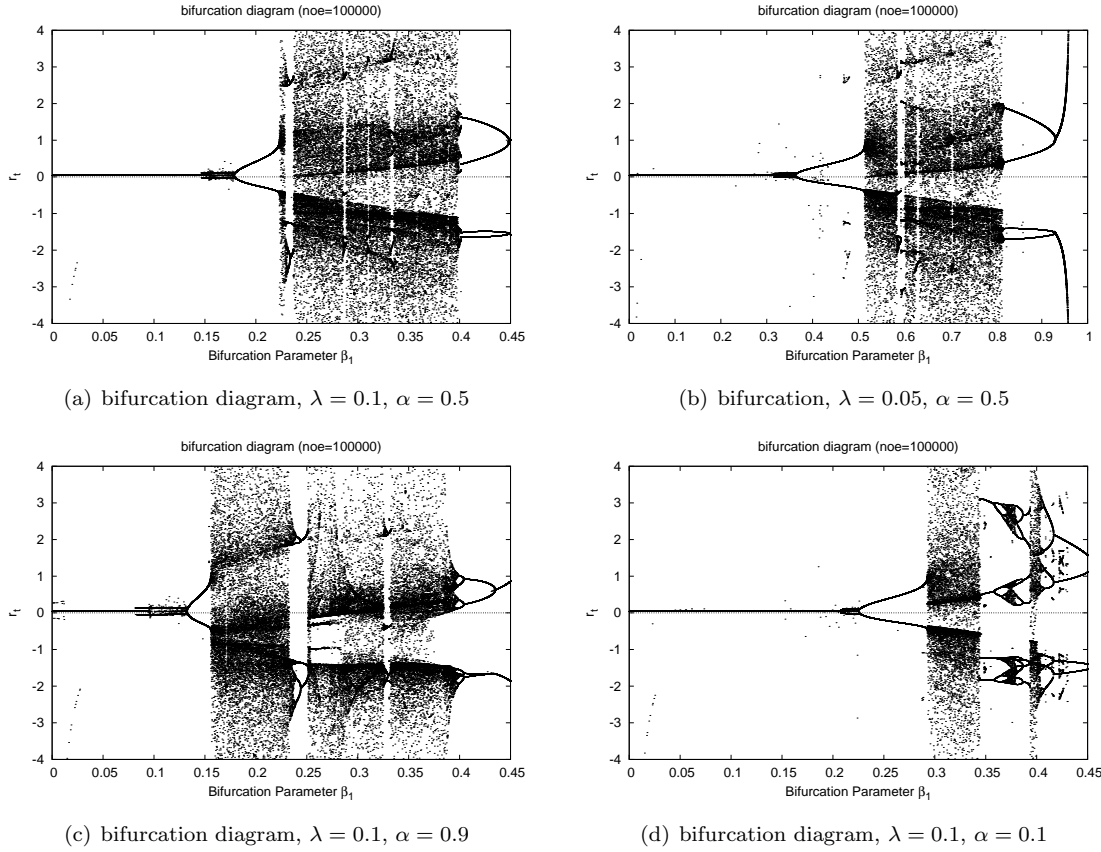
As in the case of the heterogeneous beliefs model variants outlined in the previous sections of this chapter, the local stability of the fundamental steady state  $E_1$  can only be investigated either with the help of a mathematical manipulation software package such as MATHEMATICA 4.1 or by using numerical simulations. The reason for this procedure is simply the high dimension of the underlying dynamic system, which makes an analytical approach almost infeasible. Therefore, all subsequent statements are based on these two methods of investigation, i.e., using MATHEMATICA 4.1 to evaluate the eigenvalues of the associated characteristic equation of  $E_1$  as well as performing extensive numerical simulations.

The overall results are comparable to those found for the corresponding  $H_t = 0$  model framework. As long as fundamentalists do not extrapolate too strongly, i.e., for  $\beta_1$  being sufficiently close to zero, the fundamental steady state  $E_1$  is locally asymptotically stable and thus the dynamic system eventually converges to its unique steady state for wide range of parameter sets. As before, a decrease in the price adjustment speed  $\lambda$  of the market maker stabilizes the dynamic system in two ways. Firstly, the range in which the system converges to its fundamental steady state increases significantly. Secondly, even behind this range of convergence the dynamic system at least does not explode whenever  $\lambda$  is chosen relatively small. Figure 4.14 refers to this fact. It shows four bifurcation diagrams with respect to the fundamentalists' extrapolation parameter  $\beta_1$ . It can clearly be detected that the choice of a lower value of  $\lambda$  leads to wider range of convergence (see upper right panel). Furthermore, numerical simulations show that the dynamic system even explodes for certain parameter sets. This is visualized in Figure 4.14, where all panels are plotted only in the range from 0 to 0.45, apart from the upper right panel, where  $\lambda$  is chosen smaller, in this case 0.05.<sup>53</sup> As for the corresponding  $H_t = 0$  case, a relatively strong stabilizing force term  $\alpha$  leads to more unstable market dynamics (see lower left panel), while a decrease in  $\alpha$  stabilizes the system (see lower right panel).<sup>54</sup> But there is also another feature that both model variants have in common. For all parameter sets that have been investigated, the fundamental steady state  $E_1$  undergoes a flip bifurcation when  $\beta_1$  increases sufficiently. This primary bifurcation can also be easily detected in the bifurcation plots outlined here.<sup>55</sup> Behind the flip bifurcation, limit cycles as well as 2-cycles and high-order cycles occur whenever the dynamic system does not explode, although it does explode for a relatively high number of parameter sets. Lastly, it is worth highlighting that the bifurcation plot shown in the right panel of Figure 4.2 looks almost identical to the bifurcation plot given in the upper right panel of Figure 4.14. However, a striking difference between these plots lies in the fact that the passage from the stable dynamics to the primary flip bifurcation seems to be slightly noisy in all bifurcation plots of Figure 4.14. We will come back to this phenomenon in the following model variants where this feature will be discussed in further

<sup>53</sup>Notice also that the labeling of the  $y$ -axis in these plots corresponds to a significantly larger range than normally used in almost all other bifurcation plots exhibited before in which the terminal points of the 100,000 simulation experiments fall for each of the four bifurcation plots of Figure 4.14.

<sup>54</sup>See also Footnotes 29 and 39 for a related discussion.

<sup>55</sup>The bifurcation point  $\beta_1^*$  of the primary flip bifurcation can in principle be analytically calculated, as already done for the corresponding  $H_t = 0$  case (see Eq. (4.73)). However, the resulting conditional equation is almost unmanageable. Thus an analogous theorem of Theorem 4.5 is not given here.



**Figure 4.14:** Bifurcation diagrams with respect to  $\beta_1$  for the homogeneous market maker model ( $H_t > 0$  case). The fundamental steady state  $E_1$  of the revised homogeneous market maker model is locally asymptotically stable for  $\beta_1$  being sufficiently small, where a decrease in the parameter  $\alpha$  slightly stabilizes the system and an increase has the opposite effect (lower panels). By increasing  $\beta_1$ , the fundamental steady state  $E_1$  undergoes a flip bifurcation in all cases. For values of  $\beta_1$  being larger than 0.45, the dynamic system explodes in almost all simulations in which  $\lambda$  is sufficiently large. The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $\nu = 1.2, r_f = 0.05, g = 0.04, \sigma = 0.2$  and  $n^s = 0.03$ .

detail.

Before a heterogeneous beliefs model is investigated, the next subsection briefly outlines the other case of a homogeneous belief model where all investors pursue a chartist trading strategy.

#### 4.5.4 Homogeneous Beliefs – Chartists

This subsection concentrates on the homogeneous beliefs model with only chartists in the market. Obviously it would appear to make sense to again use the chartist beliefs given by Eq. (4.64)–Eq. (4.65), where

$$E_{c,t}(\rho_{t+1}) = \beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t$$

has to be replaced by

$$E_{c,t}(\rho_{t+1}) = \beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_t^{RP} - g)y_t \tag{4.121}$$



and

$$\bar{\sigma}_{c,t}^2 \equiv [E_{c,t}(\rho_{t+1}) - r_f]^2 = [\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_f - g)y_t - r_f]^2$$

has to be replaced by

$$\bar{\sigma}_{c,t}^2 \equiv [E_{c,t}(\rho_{t+1}) - r_t^{RP*}]^2 = [\beta_2 r_t + (1 - \beta_2)r_{t-1} + (r_t^{RP} - g)y_t - r_t^{RP*}]^2, \quad (4.122)$$

where  $r_t^{RP*}$  denotes the sum of the risk-free rate and the risk premium on the fundamental price path, which is a constant in our model structure. With these assumptions, the resulting dynamic system is given by

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.123)$$

$$y_{t+1} = \frac{(1+g)(r_t^{RP} - g)}{(1+r_{t+1})(r_{t+1}^{RP} - g)} y_t, \quad (4.124)$$

$$z_{t+1} = \frac{1+n^s+r_{t+1}}{1+\beta_{c,t+1}} z_t, \quad (4.125)$$

where

$$\begin{aligned} q_t &= \frac{(1+\beta_{c,t})\pi_{c,t} - (1+r_t)\pi_{c,t-1} - n^s z_{t-1}}{1+\beta_{c,t}}, \\ \beta_{c,t+1} &= r_f + \pi_{c,t}[r_{t+1} - r_f + (r_t^{RP} - g)y_t], \\ \pi_{c,t} &= \frac{[\beta_2 r_t + (1-\beta_2)r_{t-1} + (r_t^{RP} - g)y_t - r_f]}{\nu[\sigma^2 + \beta_3 \bar{\sigma}_{c,t}^2]}, \\ r_t^{RP} &= r_f + \nu[\sigma^2 + \beta_3 \bar{\sigma}_{c,t}^2] z_t \end{aligned} \quad (4.126)$$

with  $\nu \equiv \nu_c$  and  $\bar{\sigma}_{c,t}^2$  as given in Eq. (4.122). However, due to the variable  $r_t^{RP}$ , the resulting dynamic system Eq. (4.123)–Eq. (4.125) would become almost mathematically intractable. This will now be outlined in more detail in the following remark.

#### Technical Remark 4.13 (Infeasibility of a Non-Adaptive Expectation Formation)

In our revised model framework the market participants have to determine, inter alia, the variable  $r_{t+1}^{RP}$  in order to derive the fundamental/price ratio  $y_{t+1}$  of the next trading period (see Eq. (4.124)), which obviously depends on the investors' beliefs about the conditional variances of returns at time  $t+1$ , which is  $\bar{\sigma}_{c,t+1}^2$ , according to Eq. (4.126). Because Eq. (4.122) states that the variable  $\bar{\sigma}_{c,t+1}^2$  itself depends on  $r_{t+1}^{RP}$  and  $y_{t+1}$  as well, we cannot simply derive  $y_{t+1}$  via Eq. (4.124) and iterate the dynamic system as done for the corresponding simplified model variant with  $H_t = 0$  (see Eq. (4.74)–Eq. (4.79) in Subsection 4.3.2). Instead, we would have to solve the newly arising equation for  $y_{t+1}$  first, which is not an easy task and mathematically quite laborious and therefore left out here. To conclude, the revised trading strategies of the chartists given by Eq. (4.121) and Eq. (4.122) cannot be simply applied to the fully-developed market maker model.

Instead, we will use the adaptive expectation formation for the chartists, which has already been introduced in Subsection 4.4.1. Remember that chartists extrapolate the future return of the risky asset from the sample mean and variance of the past realized returns, which follow some learning processes. Precisely, the adaptive expectations of the chartists are assumed to be<sup>56</sup>

$$E_{c,t}(r_{t+1}) = r_t + \beta_2(r_t - \phi_t), \quad \beta_2 \in [0, 1],$$

<sup>56</sup>See Eq. (4.92) and the discussion thereon for further details.

and

$$\sigma_{c,t}^2 \equiv \text{Var}_{c,t}(\rho_{t+1}) = \sigma^2 + \beta_3 v_t, \quad \beta_3 \geq 0,$$

where

$$\begin{aligned} \phi_t &= \delta \phi_{t-1} + (1 - \delta) r_t, \\ v_t &= \delta v_{t-1} + \delta(1 - \delta)(r_t - \phi_{t-1})^2. \end{aligned}$$

Therefore, after collecting all relevant equations for this homogeneous beliefs case, the resulting dynamic system can be summarized as follows (with  $\nu \equiv \nu_c$ ):

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.127)$$

$$y_{t+1} = \frac{(1+g)(r_t^{RP} - g)}{(1+r_{t+1})(r_{t+1}^{RP} - g)} y_t, \quad (4.128)$$

$$z_{t+1} = \frac{1+n^s+r_{t+1}}{1+\beta_{c,t+1}} z_t, \quad (4.129)$$

$$\phi_{t+1} = \delta \phi_t + (1 - \delta) r_{t+1}, \quad (4.130)$$

$$v_{t+1} = \delta v_t + \delta(1 - \delta)(r_{t+1} - \phi_t)^2, \quad (4.131)$$

where

$$q_t = \frac{(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1} - n^s z_{t-1}}{1 + \beta_{c,t}}, \quad (4.132)$$

$$\beta_{c,t+1} = r_f + \pi_{c,t}[r_{t+1} - r_f + (r_t^{RP} - g)y_t], \quad (4.133)$$

$$\pi_{c,t} = \frac{r_t + \beta_2(r_t - \phi_t) + (r_t^{RP} - g)y_t - r_f}{\nu[\sigma^2 + \beta_3 v_t]}, \quad (4.134)$$

$$r_t^{RP} = r_f + \nu[\sigma^2 + \beta_3 v_t] z_t. \quad (4.135)$$

In comparison to the homogeneous beliefs model in the previous subsection, the dimension of the system Eq. (4.127)–Eq. (4.131) has increased by four, i.e., it is a ten dimensional system, caused by the chartists extrapolating the next period's asset return  $r_{t+1}$  from the sample mean and the sample variance of the prior observable return values of the risky asset measured by  $\phi_t$ ,  $\phi_{t-1}$ ,  $v_t$ , and  $v_{t-1}$ . Furthermore, it can easily be verified that a similar steady state exists, as it does in the homogeneous case with only fundamentalists in the market exist:

**Theorem 4.14** *Let  $0 \leq \beta_2, \delta \leq 1$  and  $\beta_3 \geq 0$ . Assume that Eq. (4.132)–Eq. (4.135) hold. Under the assumptions of Theorem 4.11 the system Eq. (4.127)–Eq. (4.131) has a steady state  $E_1$  which is given by*

$$\phi^* = g \quad \wedge \quad v^* = 0 \quad \wedge \quad r^* = g \quad \wedge \quad y^* = 1 \quad \wedge \quad z^* = \sqrt{\frac{n^s + g - r_f}{\nu \sigma^2}}. \quad (4.136)$$

PROOF: See Appendix A 4.13.

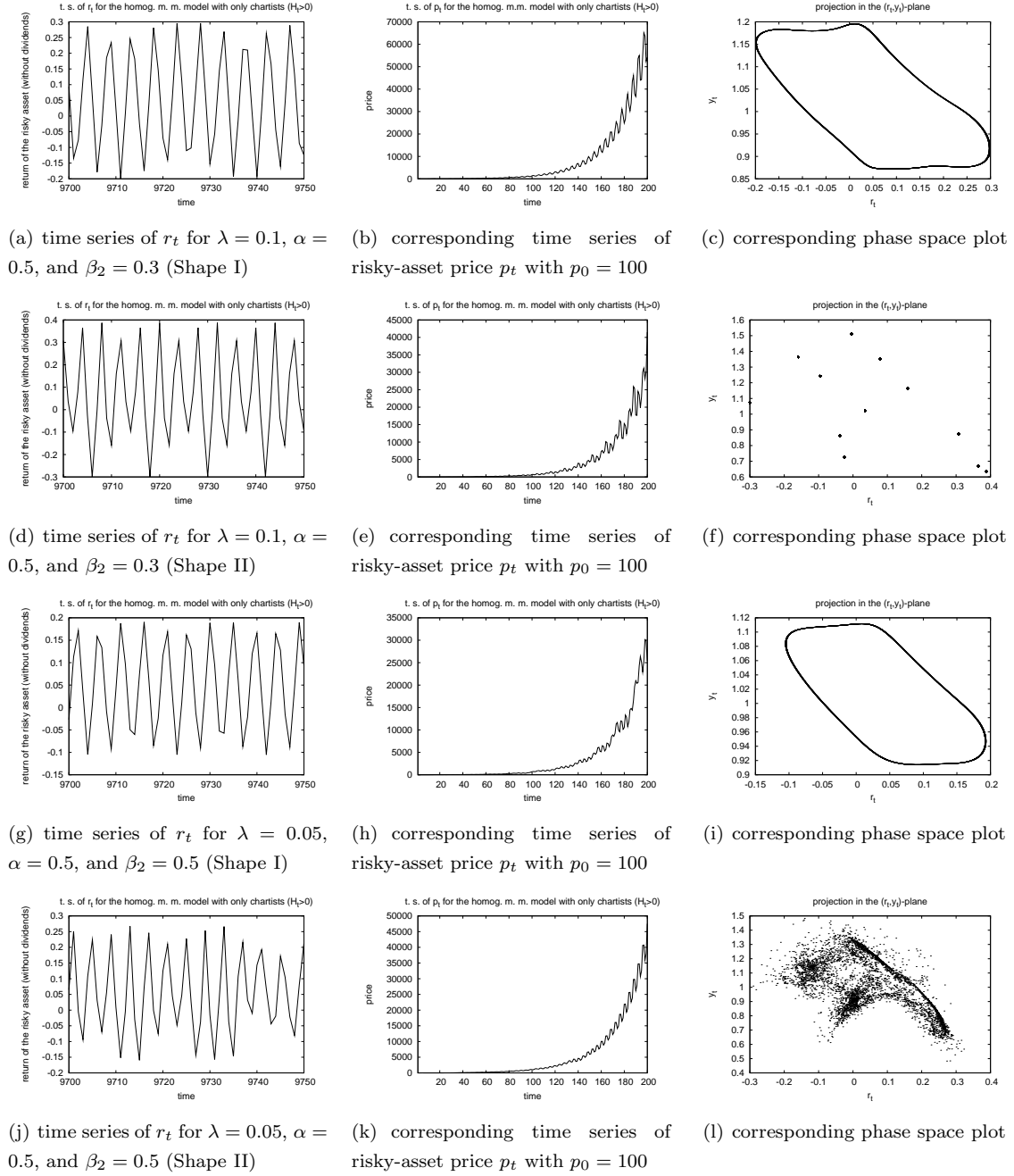
According to Theorem 4.14 the 'quasi'-unique steady state of the homogeneous beliefs model with only chartists in the market is the same as for the homogeneous model with only fundamentalists

in the market.<sup>57</sup> Thus, according to Remark 4.12 the investors' optimal investment proportion  $\pi_c^*$  at the steady state  $E_1$  is again equal to the asset supply/total wealth ratio, i.e.,  $\pi_c^* = z^*$ . However, numerical simulations indicate that the fundamental steady state  $E_1$  is unstable for almost all reasonable parameter sets regardless of the choice of  $\beta_2$ . Note that this result was also shown in the analogous  $H_t = 0$  case briefly outlined at the beginning of Subsection 4.4.2.<sup>58</sup> Furthermore, the dynamic system is subject to explosive behavior. For instance, generating bifurcation diagrams is almost impossible since there is a huge number of experiments for which the dynamic system eventually explodes. For this reason, Figure 4.15 displays time series and phase space plots for different market scenarios instead of bifurcation diagrams. The time series of the risky-asset return  $r_t$  and the risky-asset price  $p_t$  as well as the corresponding phase space plot in the  $(r_t, y_t)$ -plane is exhibited. Plots from two different parameter sets are shown (see Figure 4.15 (a)–(f) and (g)–(l)), which represent typical market outcomes similar to results already observed in the preceding model variants (i.e., high-order cycles, limit cycles and strange attractors). Nevertheless, two aspects are worth mentioning. Firstly, the corresponding time series of the risky-asset price  $p_t$  is depicted since the price evolution of the risky asset shows a notable volatility for certain parameter sets in contrast to *all* other price runs of previous model versions (see Figure 4.15 (e) and (h)).<sup>59</sup> While the exponential increase in the risky-asset price is due to the fact that the market maker model is a growth model in both the asset price and the overall investors' wealth level, the irregular upward and downward jumps of the risky-asset price is caused by the chartists' revised trading strategy. The second observation is maybe even more surprising. Note that the Panels (a)–(c) and (d)–(f) of Figure 4.15 are generated in two experiments using the same parameter set (Shape I vs. Shape II). The same is true for the Panels (g)–(i) and (j)–(l) of Figure 4.15. The only explanation for these occurrences can be that the eventual outcome of the dynamic system depends on the initial conditions of the system, which are chosen randomly in our model design. In other words the randomly generated initial values of the underlying dynamic variables can lead to considerably different results in the long run, which is a well-known characteristic in nonlinear chaotic systems under the notion of 'sensitive dependence on initial conditions' (SDIC). Interestingly, this phenomenon has not been observed in the experiments conducted for the preceding model variants. One reason might be that, for the  $H_t > 0$  case, a further state variable has been introduced which is also initialized randomly and which simply leads to a wider basin of potential start positions for the state variables and thus to a wider variety of possible orbits. Moreover, the fully-developed market maker model might simply be a system of even greater complexity. It should finally be noted that the parameter sets shown in Figure 4.15 are typical examples of simulation results obtained for this homogeneous chartists model.

<sup>57</sup>From a purely mathematical point of view two further steady states with  $y^* = 0$  and  $z^* > 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  and  $\lim_{t \rightarrow \infty} z_t = z^* > 0$ , as well as three further steady states with  $y^* = 0$  and  $z^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{p_t} = 0$  and  $\lim_{t \rightarrow \infty} z_t = z^* = 0$  can be derived. However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.14 in Appendix A 4.13 for details). In this respect the fundamental steady state is 'quasi' unique.

<sup>58</sup>An adaptive expectation formation scheme is used here as in Section 4.4.

<sup>59</sup>The time series of the risky-asset price has always been investigated for all the previous model variants although these results have not been shown in detail in the corresponding subsections.



**Figure 4.15:** Time series of risky-asset returns and corresponding risky-asset prices and phase space plots for the homogeneous market maker model with only chartists in the market ( $H_t > 0$  case). The plot exhibits the time series of the risky-asset return  $r_t$ , the risky-asset price  $p_t$  and the projection of the corresponding attractor on the  $(r_t, y_t)$ -plane for four market situations. While the first and second row of plots correspond to the *same* parameter set for which the market dynamics can be described by both a limit cycle and a high-order cycle, the third and fourth row of plots also correspond to *one* parameter set but show market scenarios in which limit cycles and strange attractor behavior occurs. As before the dynamic variables (i.e.,  $r_t$ ,  $y_t$  and  $z_t$ ) are randomly initialized in all plots. Thus the dynamic system reacts sensitively to the chosen initial conditions of these variables. Furthermore, the time evolution of risky-asset prices shows volatile behavior for certain market scenarios (see Panels (e) and (h)). The phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$  and  $\beta_3 = 5.0$ .

In the following subsection a heterogeneous beliefs model is investigated in which both previous trading strategies are combined.

#### 4.5.5 Heterogeneous Beliefs – Fundamentalists and Chartists

In this subsection the fully-developed market maker model will be analyzed. It is obvious that the resulting dynamic system will become even more complex than in the preceding subsections. Nevertheless, numerical simulations and an analytical treatment with the aid of the software package MATHEMATICA 4.1 will help to manage the investigation of the behavior of the underlying dynamic system. Again the subsection begins with a technical remark:

##### Technical Remark 4.15 (Necessity of a Uniform Choice of $\nu$ and $\sigma^2$ )

Due to the complexity of the heterogeneous beliefs model with a non-vanishing risky-asset supply, we will now assume that both fundamentalists and chartists can be characterized by identical risk-aversion coefficients, i.e., it is assumed that  $\nu \equiv \nu_f = \nu_c$ . Furthermore, the beliefs about the conditional variances of returns are assumed to be given by  $\sigma_{f,t}^2 = \sigma^2$  and  $\sigma_{c,t}^2 = \sigma^2 + \beta_3 v_t$  as already defined above. Of course, these assumptions are known and have already been used in the preceding two subsections. However, the remark is still necessary because the resulting dynamic system reveals that the absence of these assumptions would make it impossible to determine the steady state of the dynamic system, since  $\pi_f^* \neq \pi_c^*$  would directly lead to  $\beta_f^* \neq \beta_c^*$ , making the dynamic system almost mathematically intractable (see Eq. (4.142)).<sup>60</sup>

Taking this remark into account, the dynamic system of the heterogeneous beliefs model is given by

$$r_{t+1} = g + \lambda q_t + \alpha(y_t - 1), \quad (4.137)$$

$$y_{t+1} = \frac{(1+g)(r_t^{RP} - g)}{(1+r_{t+1})(r_{t+1}^{RP} - g)} y_t, \quad (4.138)$$

$$z_{t+1} = \frac{2(1+n^s + r_{t+1})}{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{c,t+1})} z_t, \quad (4.139)$$

$$\phi_{t+1} = \delta \phi_t + (1 - \delta)r_{t+1}, \quad (4.140)$$

$$v_{t+1} = \delta v_t + \delta(1 - \delta)(r_{t+1} - \phi_t)^2, \quad (4.141)$$

$$\bar{\omega}_{t+1} = \frac{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) - [1 - \bar{\omega}_t](1 + \beta_{c,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{c,t+1})}, \quad (4.142)$$

where

$$q_t = \frac{[(1 + \beta_{f,t})\pi_{f,t} - (1 + r_t)\pi_{f,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} + \frac{[(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}] - 4n^s z_{t-1}}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} \quad (4.143)$$

and

$$\beta_{i,t+1} = r_f + \pi_{i,t} [r_{t+1} - r_f + (r_t^{RP} - g)y_t], \quad i \in \{f, c\}, \quad (4.144)$$

<sup>60</sup>Remember that this is not true for the  $H_t = 0$  case. Instead, for  $H_t = 0$  it is  $\pi_f^* = \pi_c^* = 0$  and thus  $\beta_f^* = \beta_c^* = r_f$  at a steady state of the corresponding dynamic system (see also the proof of Theorem 4.2 in Appendix A 4.3 for details).

$$\pi_{f,t} = \frac{1}{\nu\sigma^2} [(g - r_f) - \beta_1(r_t - g) + (r_t^{RP} - g)y_t] , \quad (4.145)$$

$$\pi_{c,t} = \frac{r_t + \beta_2(r_t - \phi_t) + (r_t^{RP} - g)y_t - r_f}{\nu[\sigma^2 + \beta_3v_t]} , \quad (4.146)$$

$$r_t^{RP} = r_f + \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu\sigma^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu(\sigma^2 + \beta_3v_t)} \right]^{-1} z_t . \quad (4.147)$$

Obviously, the system Eq. (4.137)–Eq. (4.142) is a twelve dimensional system for which the following result for the steady states can be obtained:

**Theorem 4.16** *Let  $g, r_f, \lambda, \alpha, \nu, \sigma^2, n^s > 0$ ,  $g < r_f$ ,  $n^s \geq (r_f - g)$ ,  $\bar{n} \in [-1, 1]$  as well as  $0 \leq \beta_1, \beta_2, \delta \leq 1$  and  $\beta_3 \geq 0$ . Assume that Eq. (4.143)–Eq. (4.147) hold. Then the system Eq. (4.137)–Eq. (4.142) has infinitely many steady states, namely fundamental steady states, which are given by*

$$\phi^* = g \quad \wedge \quad v^* = 0 \quad \wedge \quad r^* = g \quad \wedge \quad y^* = 1 \quad \wedge \quad z^* = \frac{1 + \bar{n}\bar{\omega}^*}{2} \sqrt{\frac{n^s + g - r_f}{\nu\sigma^2}} \quad \wedge \quad \bar{\omega}^* \in [-1, 1] .$$

PROOF: See Appendix A 4.14.

Theorem 4.16 states that the dynamic system of the heterogeneous beliefs model has infinitely many steady states.<sup>61</sup> Note that  $z^*$  depends on the difference of the average wealth proportion  $\bar{\omega}^*$  which itself can take a value between -1 and 1. Thus the variety of steady-state values might become rather rich in our fully-developed market maker model. According to Technical Remark 4.15 the following conclusion can be drawn:<sup>62</sup>

**Remark 4.17** *Note that the optimal investment proportion of fundamentalists and chartists is the same at the fundamental steady states, i.e., it is  $\pi^* \equiv \pi_f^* = \pi_c^*$ . Furthermore, the relationship between the total demand of the agent and the asset supply/total wealth ratio is given by  $\pi^* = 2z^*/(1 + \bar{n}\bar{\omega}^*)$ .*<sup>63</sup>

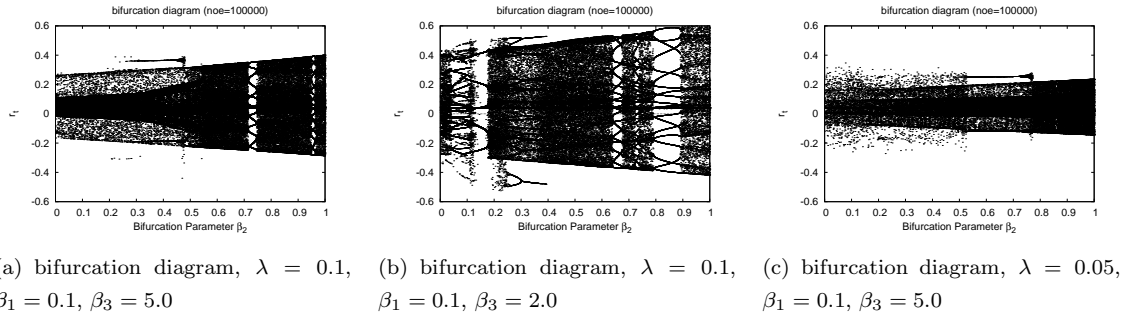
Numerical simulations of the underlying dynamic system indicate that an increase in the fundamentalists' extrapolation parameter  $\beta_1$  leads to the dynamic system exhibiting intensified explosive behavior. For this reason we will concentrate on parameter sets in which  $\beta_1$  is chosen as 0.1, 0.25 and 0.4. This can be also seen as economically reasonable since the fundamentalists should quite strongly believe in the fundamentals to distinguish their trading strategy from that of the chartists', who are just assumed to use an extrapolation scheme for their investment decisions. Obviously, this distinction can be guaranteed by a sufficiently low extrapolation parameter  $\beta_1$  for the fundamentalists.

<sup>61</sup>From a purely mathematical point of view, four further steady states can be derived with  $y^* = 0$  and  $z^* > 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{P_t} = 0$  and  $\lim_{t \rightarrow \infty} z_t = z^* > 0$ , as well as six further steady states with  $y^* = 0$  and  $z^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} \frac{p_t^*}{P_t} = 0$  and  $\lim_{t \rightarrow \infty} z_t = z^* = 0$ . However, a numerical analysis indicates that, for a very wide range of reasonable parameter sets, these non-fundamental steady states do not represent economically reasonable results (see the proof of Theorem 4.16 in Appendix A 4.14 for details). In this respect only fundamental steady states exist for the fully-developed heterogeneous beliefs model.

<sup>62</sup>See Appendix A 4.14 for details.

<sup>63</sup>Remember that the optimal investment proportion  $\pi^*$  at the fundamental steady states might even exceed 100%. See also Remark 4.12 and the discussion around for further details.

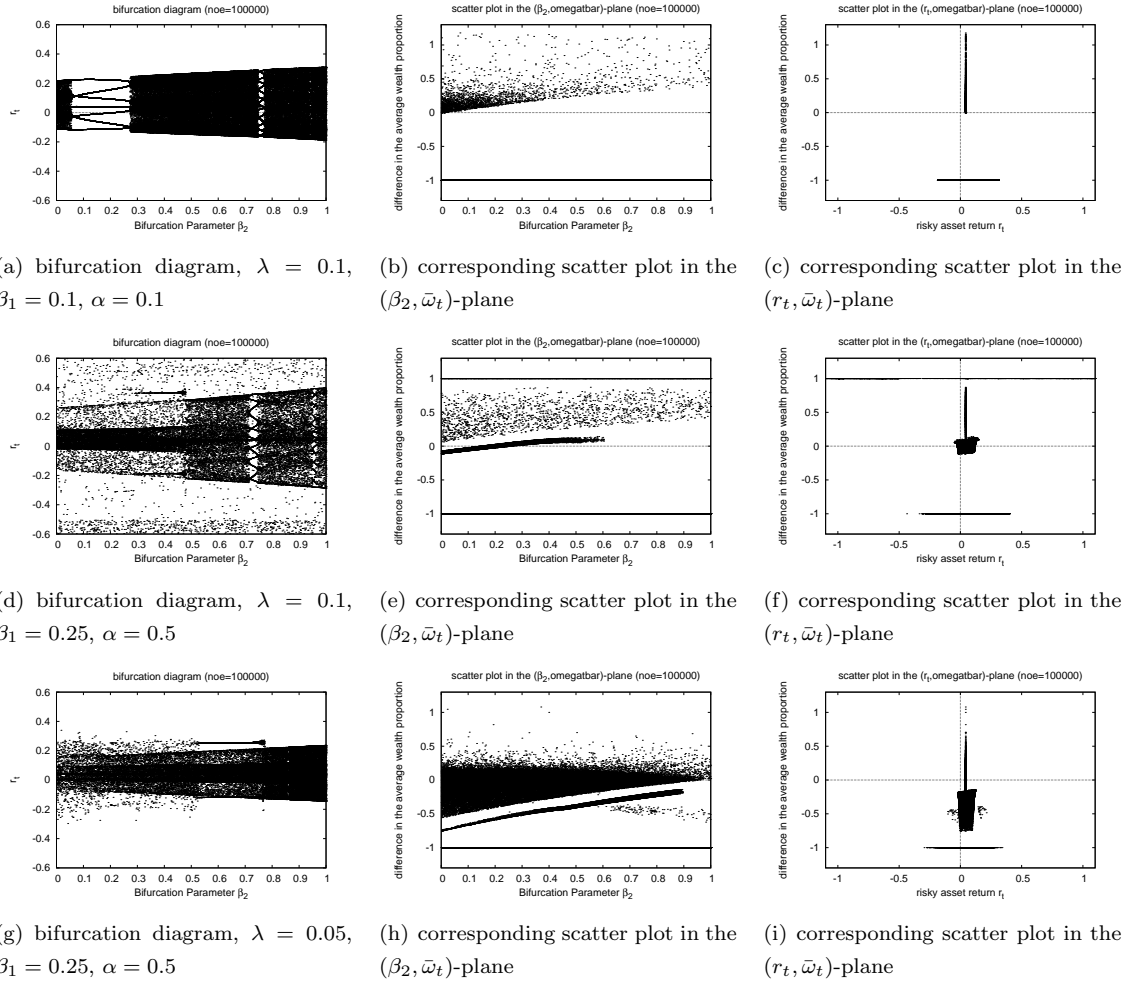
Figure 4.16 and Figure 4.17 (left panels) show six typical bifurcation diagrams for the heterogeneous market maker model with a non-zero risky-asset supply for different parameter sets. Almost all parameter sets have already been used in previous model variants so that the results of different model frameworks can easily be compared with each other (see Figures 4.6 and 4.10). The bifurcation diagrams look slightly more noisy than the corresponding bifurcation diagrams for



**Figure 4.16:** Bifurcation diagrams with respect to  $\beta_2$  for the heterogeneous market maker model ( $H_t > 0$  case). As in the case of the adaptive market maker model with a vanishing risky-asset supply (see Figure 4.10) the resulting dynamic system becomes rather complex for a wide range of parameter sets. Moreover, the bifurcation diagrams look rather noisy, presumably caused by randomly chosen initial conditions for the dynamic variables (see discussion and figures below for further details). The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

the  $H_t = 0$  case. The reason for this noise might be that the equilibrium value  $z^*$  for the newly introduced state variable  $z_t$  depends on the steady-state value of the difference of the average wealth proportion  $\bar{\omega}^*$ , which itself can take any value between -1 and 1. Note that the initial conditions for the dynamic state variables are again chosen randomly for the bifurcation experiments, which might lead to a wide variety of different behavior in the underlying dynamic system. It should also be noted that the presence of SDIC has already been observed for the homogeneous model with only chartists in the market, i.e., the eventual outcome of the dynamic system clearly depends on the initial conditions (see Subsection 4.5.4). For this reason it might be useful to investigate these features in more details first: this will be done in the next paragraphs and figures. While Figure 4.17 provides additional information on the behavior and interrelation of the chartists' extrapolation rate  $\beta_2$ , the risky-asset return  $r_t$  and the difference of the average wealth proportion  $\bar{\omega}_t$ , Figures 4.18 and 4.19 visualize the influence of the specific choice of initial conditions on the resulting behavior of the underlying dynamic system.

As already mentioned above, Figure 4.17 (left panels) shows three typical bifurcation diagrams with the fundamentalists' extrapolation parameter  $\beta_1$  being chosen moderately low. While the bifurcation diagrams are performed in exactly the same way as in all other previous cases, the middle and right panels exhibit corresponding scatter plots in the  $(\beta_2, \bar{\omega}_t)$ -plane and in the  $(r_t, \bar{\omega}_t)$ -plane respectively. They are generated from the same data set as used for the bifurcation diagrams and are conducted as follows. As described in Figure 4.2, 100,000 experiments with randomly chosen values of  $\beta_2$  are performed for each bifurcation plot, with 5,000 iterations being conducted in each experiment. Now, for all 100,000 experiments, the 5000th realization of  $r_t$  and  $\bar{\omega}_t$  is stored



**Figure 4.17:** More bifurcation diagrams with respect to  $\beta_2$  and corresponding scatter plots for the heterogeneous market maker model ( $H_t > 0$  case). While for all possible values of  $\beta_2$  market outcomes can occur in which the chartists are able to prevail over the fundamentalists in the long run (see straight line at -1 in Panels (b), (e) and (h)), the fundamentalists tend to do better and sustain their position in the market whenever the chartists' extrapolation parameter  $\beta_2$  is chosen not too high or a relatively overflowed risky-asset return is realized (Panels (d)–(f)). Obviously, an asset return  $r_t$  around the steady-state value (here:  $r^* = g = 0.04$ ) mostly corresponds to a more balanced wealth distribution among the two investor groups (right panels). While the bifurcation diagrams are conducted in exactly the same way as described in Figure 4.2, a description of how to perform the scatter plots can be found in the continuous text. Note that the middle row of plots, i.e., Panels (d)–(f), only shows about 45,500 experiments; this is because the underlying dynamic system has exploded during the numerical simulations. Note that the tendency of the system to explode becomes quite apparent by the widely scattered realizations of the risky-asset return  $r_t$  (see Panel (d)). The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

and plotted as a scatter plot in both the  $(\beta_2, \bar{\omega}_t)$ -plane and the  $(r_t, \bar{\omega}_t)$ -plane. In other words, for each of the 100,000 experiments a triple of  $(\beta_2, r_t, \bar{\omega}_t)$  values is generated and displayed in all possible 2-dimensional planes, i.e., the  $(\beta_2, r_t)$ -plane, the  $(\beta_2, \bar{\omega}_t)$ -plane and finally the  $(r_t, \bar{\omega}_t)$ -plane. While the first scatter plot is simply the bifurcation diagram (left panels), the other two scatter plots can help to investigate whether a certain realization of  $\beta_2$  implies a certain value



of  $\bar{\omega}_t$  (middle panels) or whether a certain risky-asset return  $r_t$  can always be associated with a certain value of  $\bar{\omega}_t$  (right panels).<sup>64</sup> Indeed, the scatter plots reveal interesting insights into the behavior of the dynamic system.<sup>65</sup> Obviously, in many experiments a risky-asset return close to its fundamental steady-state value corresponds to a market situation in which neither fundamentalists nor chartists accumulate all the wealth. This result coincides with the findings from the previous model variants. However, all scatter plots in the  $(\beta_2, \bar{\omega}_t)$ -plane clearly show that any  $\beta_2$  realization makes it possible for the chartists to prevail over the fundamentalists (see straight line at -1). Otherwise, if the risky-asset return hits exactly its fundamental steady-state value then, to some degree, there is a tendency for fundamentalists to accumulate more wealth than the chartists in the long run (see vertical line in the right panels). These findings clearly indicate that the precise outcome of the revised market maker model depends significantly on the chosen initialization of the dynamic system (i.e., the presence of SDIC). Since this seems to be a crucial issue for the fully-developed market maker model, this fact will be addressed in further detail in the following figures. Finally, it is worth mentioning that in almost all bifurcation diagrams a straight line can be identified at the fundamental steady-state level  $r_t = g = 0.04$ , although this is sometimes hard to detect visually.<sup>66</sup> This fact will become more obvious in the following figures.

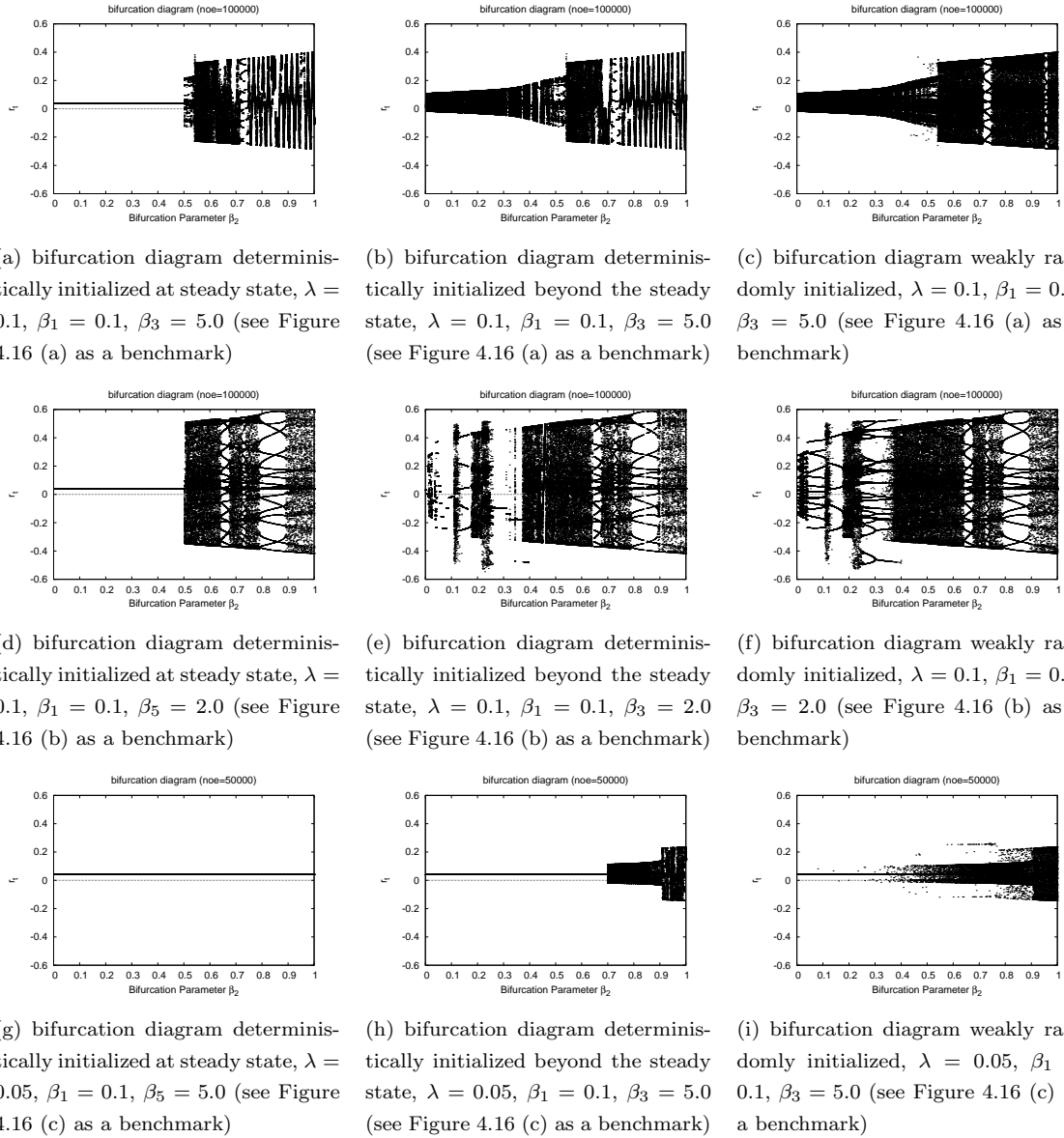
Figure 4.18 shows three different ways of initializing the dynamic system for the three bifurcation diagrams in Figure 4.16. The left panels depict the resulting bifurcation plots for the system starting deterministically at its equilibrium point. The middle panels (i.e., Figure 4.18 (b), (e) and (h)) also correspond to a deterministic model version but with start values initialized beyond the fundamental steady states. Finally, in the right panels the dynamic system *is* initialized randomly, but with a more limited pool of start values than in the ‘regular’ bifurcation experiments shown so far.<sup>67</sup> Let us start with the right panels. It can clearly be detected that a decreased randomization of the initial conditions reduces the noise in the bifurcation diagrams (see especially Figure 4.18 (c) and (i) vs. Figure 4.16 (a) and (c) respectively). In other words, a dynamic system with a fairly strong random initialization leads to a wider range of possible market behavior. This is of course a straightforward result because, from a more mathematical perspective, intensively randomizing the initial conditions produces a richer variety of possible orbits (or solution curves) that can be assumed by the underlying dynamic system. Contrastingly, if the initial conditions for a system

<sup>64</sup>In all experiments the difference in the average wealth proportions of the investor groups is deterministically initialized at  $\bar{\omega}_0 = 0.0$ , which means that the wealth share of investors is equally distributed among the two investor groups at the beginning of each experiment.

<sup>65</sup>The scatter plot results depicted in the middle and lower panels of Figure 4.17 (where  $\beta_1 = 0.25$  is chosen) can be carried over to the parameter sets chosen for Figure 4.16 (a) and (c), where the fundamentalists’ extrapolation parameter is assumed to be  $\beta_1 = 0.1$  (all other parameters are identical). However, the fundamentalists perform slightly less well for the parameter set used in Figure 4.16 (a).

<sup>66</sup>This finding has also been verified by looking at the corresponding data sets, where it can be materially better observed.

<sup>67</sup>More precisely, for the systems exhibited in the left panels the initial conditions are chosen as  $r_{t-1} = r_t = \mu_{t-1} = \mu_t = g$ ,  $y_{t-1} = y_t = 1$ ,  $v_{t-1} = v_t = \bar{\omega}_{t-1} = \bar{\omega}_t = 0.0$  and  $z_{t-1} = z_t = 0.5(1 + \bar{n}\bar{\omega}_{t-1})\sqrt{(n^s + g - r_f)/(\nu\sigma^2)}$ , i.e., at the equilibrium point. For the middle panels the initial conditions remain exactly same besides  $r_{t-1}$  and  $r_t$  which are chosen beyond the equilibrium, namely as  $r_{t-1} = g + 0.02$  and  $r_t = g - 0.01$ . For the right panels the initial condition for  $z_{t-1}$  and  $z_t$  (as well as for  $\bar{\omega}_{t-1}$  and  $\bar{\omega}_t$ ) are chosen as above, while all other variables are now again (randomly) initialized as in all previous bifurcation diagrams.



**Figure 4.18:** Bifurcation diagrams with respect to  $\beta_2$  for the heterogeneous market maker model ( $H_t > 0$  case) with different initial conditions. The parameter set is identical to the parameter set used in Figure 4.16. It can analytically be shown, by checking the eigenvalues of the corresponding characteristic equation, that the fundamental steady state of the upper two dynamic systems cannot be reached outside an initialization at their steady-state values for any realization of  $\beta_2$  while the fundamental steady state of the system shown in the lower panels can even be reached by an initialization outside its steady-state value for approximately  $\beta_2 < 0.7$ . Obviously, all three systems tend to persist at their steady-state values (at least for sufficiently small values of  $\beta_2$ ) whenever they start from this equilibrium point, while a deterministic initialization beyond the steady state reveals the bifurcation point if it exists (see Panel (h)). A random initialization of the system leads to more noisy bifurcation diagrams (right panels). Again, 100,000 experiments are performed whereas for the lower panels only 50,000 experiments are conducted. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ , and  $\bar{n} = 0.0$ . A description of the experiments and the initial conditions can be found in the continuous text.

are chosen deterministically, the possible orbits of the system are rather limited.<sup>68</sup> This effect can

<sup>68</sup>Nevertheless, *different* orbits can be reached even in the case of a deterministic initialization of the dynamic system, simply because the bifurcation parameter  $\beta_2$  is still randomly chosen to produce the bifurcation diagram.

be pretty well observed in the lower panels of Figure 4.18, i.e., Panels (g)–(i). Concentrating on the range of 0.5 and 0.7 for a moment and checking the simulated data, it is evident that the dynamic system converges quite often to a fundamental steady state in the right Panel (i) (and even more frequently than in Figure 4.16 (c)) but not in all cases.<sup>69</sup> Indeed, if the system starts outside the basin of attraction for an equilibrium point of the system, it will not inevitably converge to this equilibrium. However, a sharp and exact division of areas of convergence and divergence can be detected in the middle Panel (h) (where the initial conditions are chosen deterministically slightly beyond the fundamental steady-state values).<sup>70</sup> Moreover, the left Panel (g) shows that the dynamic system persists in its equilibrium point, even if the fundamental steady state is not locally stable for such a parameter set.<sup>71</sup> The diagrams shown in Figure 4.18 (a)–(f) can be interpreted in a similar way. From Figure 4.18 (b) and (e) it becomes apparent that for both parameter sets the fundamental steady state is not locally stable for any choice of  $\beta_2$ .<sup>72</sup> Both upper left panels again indicate a certain persistence of the underlying system to remain at the equilibrium point even if the fundamental steady state is not locally stable. This is at least true whenever  $\beta_2$  is sufficiently small.

The corresponding scatter plots are shown in Figure 4.19 while the top panels correspond to the bifurcation diagrams shown in Figure 4.16. Note that, in comparison to Figure 4.18, the parameter sets are arranged in a transposed order since I want to depict four subfigures, i.e., panels, per parameter set. Let us first concentrate on the findings shown in Figure 4.19 (d)–(l). The results can be summarized as follows. For the systems shown in the upper and lower panels of Figure 4.18 (see Panels (a)–(c) and (g)–(i) respectively), the corresponding scatter plots in Figure 4.19 indicate that fundamentalists and chartists eventually coexist in the market in the long run whenever the system remains close to a fundamental steady state (see Figure 4.19 (d), (g) and (j)) as well as Figure 4.19 (f), (i) and (l)). However, the exact model outcome does clearly depend on the way in which the initial conditions are chosen (i.e., the presence of SDIC). But whenever the system clearly departs from a fundamental steady state, the chartists accumulate more wealth on average (see the right-hand side regions within the corresponding plots). For the systems shown in Figure 4.18 (d)–(f), the results are slightly different. According to the Panels (e), (h) and (k) of Figure 4.19, it becomes evident that the chartists do better in the long run for sufficiently small values of  $\beta_2$  (except for the system shown in Panel (e) where fundamentalists and chartists remain at their initial wealth proportion), while sufficiently large values for the chartists' extrapolation rate  $\beta_2$  can lead to chartists and fundamentalists both prevailing, depending on the specific choice of initial conditions (i.e., the presence of SDIC).

As already mentioned above, the corresponding scatter plots of Figure 4.16 are depicted in Figure 4.19 (a)–(c). Obviously, these scatter plots are more noisy than the scatter plots of the

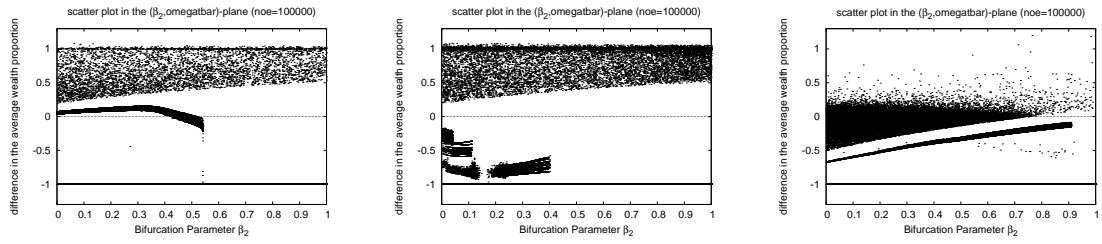
Therefore, even Figure 4.18 (a), (b), (d), (e) and (h) shows noise in the bifurcation diagram.

<sup>69</sup>See the straight line at  $r_t = 0.4$  in Figure 4.18 (i). This feature has already been observed and stated in a previous model variant. See Footnote 38 in Subsection 4.3.3 for further details.

<sup>70</sup>This division line can be determined exactly with the help of the software package MATHEMATICA 4.1.

<sup>71</sup>'Locally stable' here means, that all eigenvalues of the corresponding characteristic equation lie inside the unit circle except for one eigenvalue, which is 'neutral', i.e.,  $\pm 1$ . This eigenvalue corresponds to the state variable  $\bar{\omega}_t$ . Note, that therefore a continuum of fundamental steady states exists.

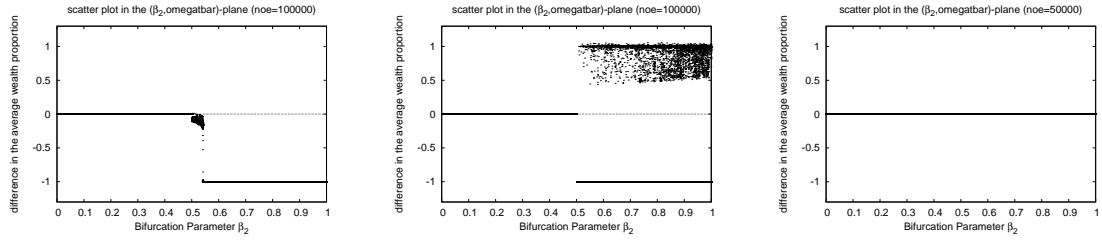
<sup>72</sup>Again this result can be analytically proved with the aid of MATHEMATICA 4.1.



(a) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.1, \beta_1 = 0.1, \beta_3 = 5.0$  (see Figure 4.16 (a))

(b) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.1, \beta_1 = 0.1, \beta_3 = 2.0$  (see Figure 4.16 (b))

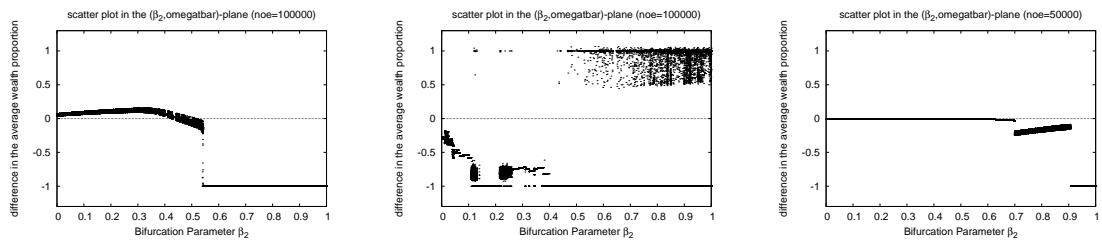
(c) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.05, \beta_1 = 0.1, \beta_3 = 5.0$  (see Figure 4.16 (c))



(d) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.1, \beta_1 = 0.1, \beta_5 = 5.0$  (see Figure 4.18 (a))

(e) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.1, \beta_1 = 0.1, \beta_3 = 2.0$  (see Figure 4.18 (d))

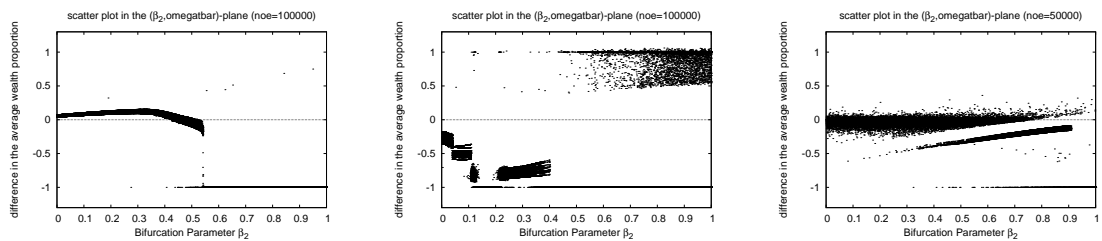
(f) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.05, \beta_1 = 0.1, \beta_3 = 5.0$  (see Figure 4.18 (g))



(g) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized beyond the steady state,  $\lambda = 0.1, \beta_1 = 0.1, \beta_5 = 5.0$  (see Figure 4.18 (b))

(h) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized beyond the steady state,  $\lambda = 0.1, \beta_1 = 0.1, \beta_3 = 2.0$  (see Figure 4.18 (e))

(i) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized beyond the steady state,  $\lambda = 0.05, \beta_1 = 0.1, \beta_3 = 5.0$  (see Figure 4.18 (h))



(j) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane weakly randomly initialized,  $\lambda = 0.1, \beta_1 = 0.1, \beta_3 = 5.0$  (see Figure 4.18 (c))

(k) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane weakly randomly initialized,  $\lambda = 0.1, \beta_1 = 0.1, \beta_3 = 2.0$  (see Figure 4.18 (f))

(l) scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane weakly randomly initialized,  $\lambda = 0.05, \beta_1 = 0.1, \beta_3 = 5.0$  (see Figure 4.18 (i))

**Figure 4.19:** Corresponding scatter plots in the  $(\beta_2, \bar{\omega}_t)$ -plane for Figures 4.16 and 4.18. The same data sets are used for the plots. The remaining parameters are  $\nu = 1.2, r_f = 0.05, g = 0.04, \sigma = 0.2, \delta = 0.5, n^s = 0.03, \alpha = 0.5,$  and  $\bar{n} = 0.0$ .

systems shown below in the same figure where the stronger noise is caused by a less moderate random initialization of the initial condition. Note that the same observation can be made in the related bifurcation diagrams (see Figure 4.16 (a) vs. Figure 4.18 (c) or Figure 4.16 (c) vs. Figure 4.18 (i)). The scatter plots in the top panels of Figure 4.19 indicate that, for some parameter sets and certain initial conditions, the fundamentalists might do even better than the chartists in the more noisy environment (see Panel (a) and left-hand side region of Panel (b) vs. Figure 4.19 (j) and (k)).<sup>73</sup> More precisely, from the corresponding bifurcation plots shown in Figure 4.16 it can be concluded that, whenever the system is initialized further away from a fundamental steady state, it leads eventually to a risky-asset return that is also quite far away from its fundamental value. And this, obviously from a visual perspective, favors the fundamentalists.<sup>74</sup> Contrastingly, a less moderate random initialization of the initial condition also leads to chartists possibly accumulating all the wealth in the market in the long run for sufficiently small values of  $\beta_2$ . For this purpose, compare the straight line at -1 in Figure 4.19 (a)–(c) for all values of  $\beta_2$  with Panes (j)–(l) in the same figure, where the straight line at -1 only occurs for sufficiently large values of  $\beta_2$ .

Summarizing all the findings from Figure 4.16–Figure 4.19, it can be concluded that, for the fully-developed market maker model with a non-vanishing risky-asset supply, the random initialization of the dynamic system leads to a considerably broader range of possible model outcomes than under deterministic initial conditions and, perhaps more interestingly, than in the previous model variants where a zero asset supply was assumed.<sup>75</sup> Obviously, this means that an all-encompassing description of all possible market outcomes for the revised market maker model is even more difficult to outline. Finally, it should also be noted that, from a qualitative perspective, all the findings mentioned above can be carried over to market situations in which the fundamentalists use a slightly stronger extrapolation rate, i.e.,  $\beta_1 = 0.25$  or  $\beta_1 = 0.4$ .<sup>76</sup> Nevertheless, yet another example of striking model behavior is shown in Figure 4.21, which is rather typical for a relatively high fundamentalists' extrapolation rate  $\beta_1$ . Note that a description of the behavior can be found in the caption of the figure.

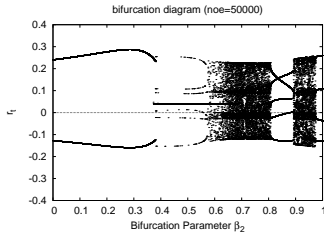
So far the heterogeneous market maker model has been investigated with respect to the phenomenon of 'sensitive dependence on initial conditions' (SDIC), which is a common feature of

<sup>73</sup>It can clearly be visually detected that in some experiments the difference in the average wealth proportions of investor groups  $\bar{\omega}_t$  even exceed +1. These results correspond to market situations in which one investor group loses even more money than they had at the beginning of the trading activities. Of course, such market situations can occur in the artificial market since no restrictions (or boundaries) for the dynamic variables are defined. However, checking the related data set of the numerical simulations shows that the occurrence of such outcomes are very rare within the experiments.

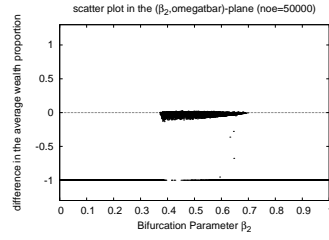
<sup>74</sup>For similar results see Figure 4.21 as well as Figure 4.34 in the following subsection.

<sup>75</sup>Although these results are not shown in details here, there is a similar sensitivity analysis with different system initialization choices for the  $H_t = 0$  model variants. The  $H_t = 0$  model variants do *not* show a notable sensitive dependence on initial conditions, probably because of the less complex model framework (see Figure 4.20 for a quick view). This is true at least when the pool of allowed start values for the dynamic system is not chosen overly large, but rather in an appropriate and similar range as done above.

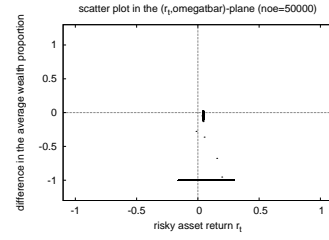
<sup>76</sup>Higher values of  $\beta_1$  beyond 0.4 enhance the tendency of the model to explode. Furthermore, an increase in  $\beta_1$  might lead to market situations for which fundamentalists do significantly better than with lower values of  $\beta_1$  for certain parameter sets. However, these parameter sets are not shown in detail here since fundamentalists with rather high values of  $\beta_1$  generally equal their chartist colleagues with respect to their trading strategy.



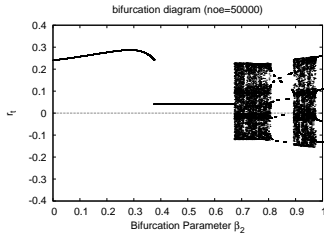
(a) bifurcation diagram (for  $H_t = 0$  case!), benchmark,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$  (see Figure 4.6 (a))



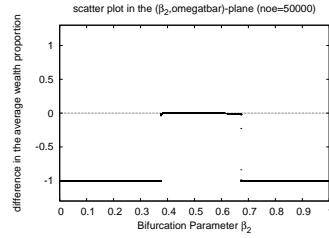
(b) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



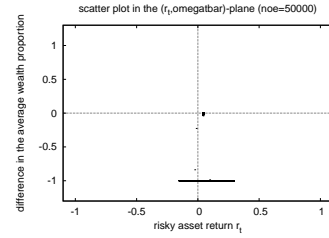
(c) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



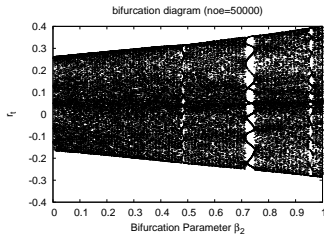
(d) bifurcation diagram (for  $H_t = 0$  case!) deterministically initialized at steady state,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$  (see Figure 4.6 (a) as a benchmark)



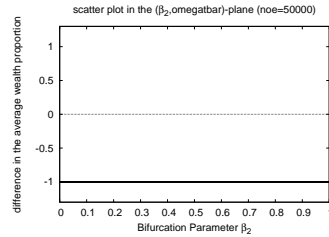
(e) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



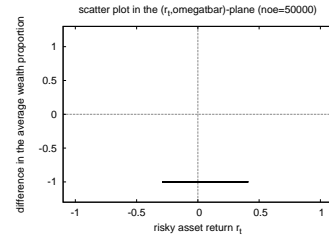
(f) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



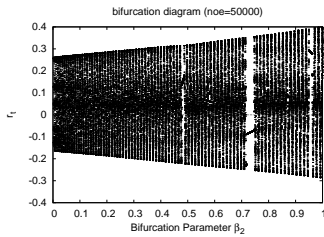
(g) bifurcation diagram (for adaptive  $H_t = 0$  case!), benchmark,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$  (see Figure 4.10 (a))



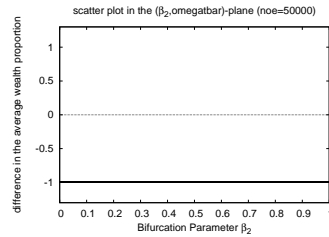
(h) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



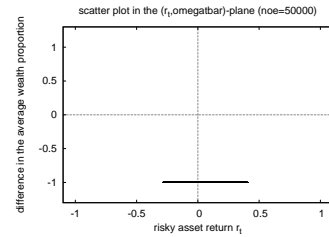
(i) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane, benchmark,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



(j) bifurcation diagram (for adaptive  $H_t = 0$  case!) deterministically initialized at steady state,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$  (see Figure 4.10 (a) as a benchmark)

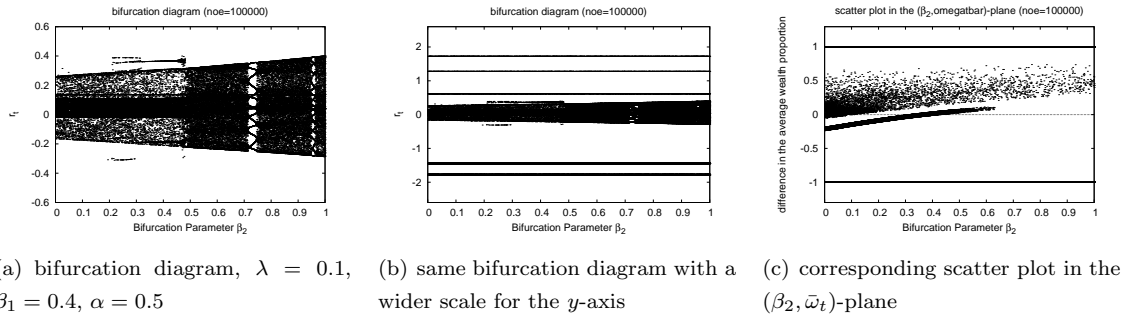


(k) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$



(l) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane deterministically initialized at steady state,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$

**Figure 4.20:** Corresponding scatter plots for Figures 4.6 (a) and 4.10 (a) of the heterogeneous market maker model ( $H_t = 0$  case) for different initial conditions. Obviously an apparent presence of SDIC cannot be detected visually for these dynamic systems. The experiments are conducted in exactly the same way as described for Figure 4.19, where 50,000 experiments are shown in the plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $\alpha = 0.5$ ,  $\beta_3 = 0.5$  and  $\bar{n} = 0.0$ .



**Figure 4.21:** Bifurcation diagrams with respect to  $\beta_2$  for a slightly higher value of  $\beta_1$  for the heterogeneous market maker model ( $H_t > 0$  case). The middle panel exhibits the same bifurcation diagram as the left panel but uses a wider scale for the  $y$ -axis. It can clearly be detected that the dynamic system allows for rather high and low risky-asset returns for this specific parameter set, which is not true for the same parameter set with lower values of  $\beta_1$  (see Figures 4.16 (a) and 4.17 (d) for a comparison). From the right panel it can be seen that, as in Figure 4.17 (e), the fundamentalists might prevail over the chartist group for certain model outcomes. By checking the corresponding data set it again becomes evident that this is the case whenever the risky-asset return assumes these high or low values in the long run. However, if the risky-asset return remains close (but not too close) to its fundamental steady-state value  $g$ , the chartists eventually prevail. The experiments are conducted in exactly the same way as described in Figure 4.17. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

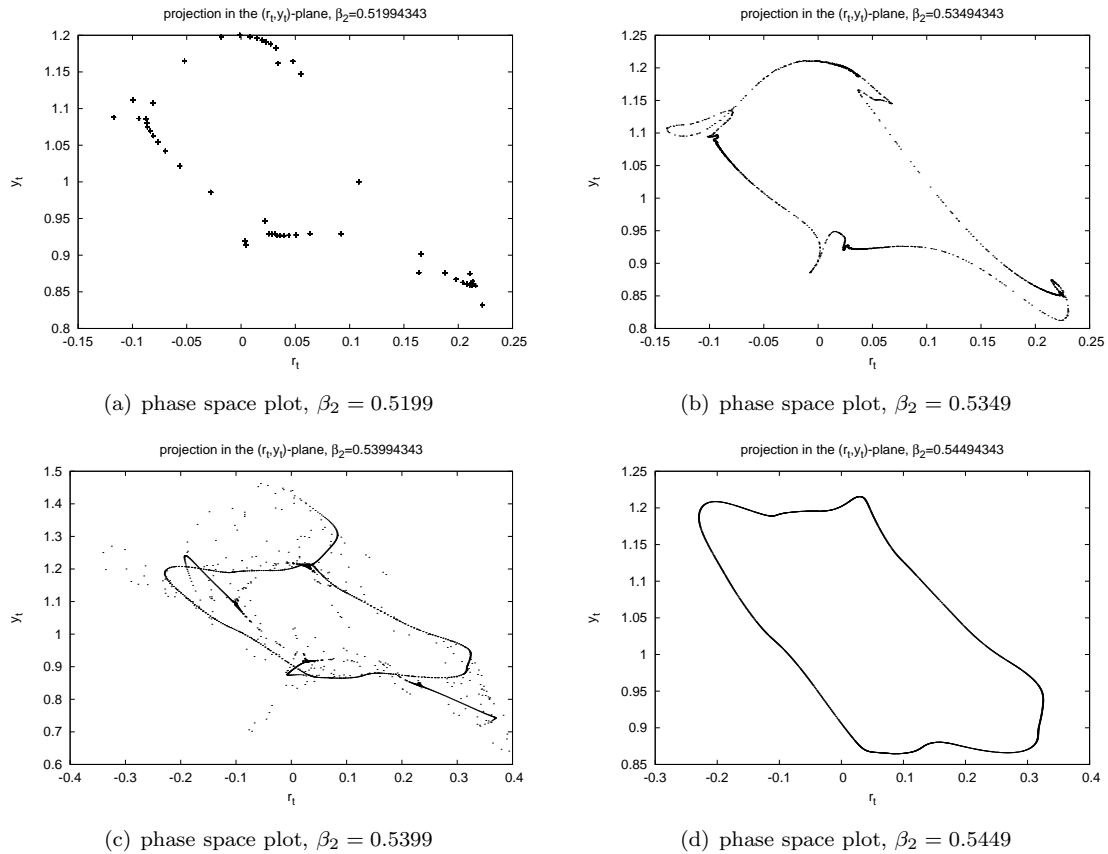
nonlinear dynamic systems.<sup>77</sup> However, Figure 4.22 refers to a slightly different but related feature, namely the ‘sensitive dependence on parameters’ (SDP). More precisely, Figure 4.22 shows four phase space plots in the  $(r_t, y_t)$ -plane for different values of  $\beta_2$  which are, nevertheless, very close to each other. To avoid confusing SDIC and SDP, the dynamic system is deterministically initialized at its steady state. Depending on the specific choice of the parameter  $\beta_2$ , different shapes occur for the phase space plots in the  $(r_t, y_t)$ -plane. Moreover, a further look at the related data sets reveals that for the upper panels fundamentalists and chartists can coexist in the market, while for both parameter sets in the lower panels the chartists are able to rule out the fundamentalists in the long run.<sup>78</sup>

One advantage of the bifurcation and scatter plots shown above is that main aspects of the dynamic system’s behavior can be described in one single plot for more than one realization of a parameter (here the highly relevant parameter  $\beta_2$  is used as a bifurcation parameter). In other words, these single plots offer a lot of information on the behavior of the dynamic system in a rather compact way. However, from an economical point of view, it might be even more interesting to look at certain specific parameter sets and see the *dynamic evolution* of more than one relevant dynamic variable in further detail. Therefore, I will now concentrate on this issue and discuss some typical market outcomes where the focus will be on the evolution of the state variables and some other important model variables of the underlying dynamic system. Furthermore, the influence of initial conditions with respect to the state variables as well as to the model parameters on the long-term behavior of the dynamic system will be investigated.<sup>79</sup>

<sup>77</sup>See Medio and Lines (2001, p. 165) and Medio and Gallo (1992) for a more detailed discussion. See also Chapter 2, Appendix A 2.12.

<sup>78</sup>Results are not shown in detail here.

<sup>79</sup>See Chiarella et al. (2006) for similar results.



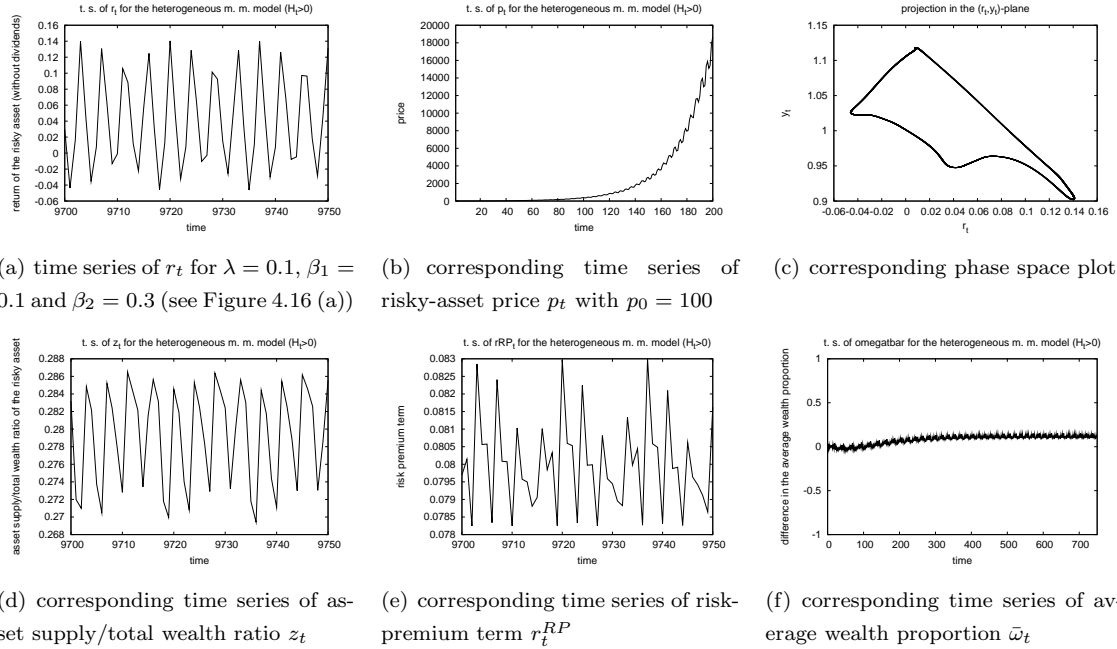
**Figure 4.22:** Sensitive dependence study of parameters with respect to  $\beta_2$  for the heterogeneous market maker model ( $H_t > 0$  case). The plot exhibits four phase space plots in the  $(r_t, y_t)$ -plane for different parameter sets. Although all parameter sets are almost identical (only the values of  $\beta_2$  slightly differs from each other), the resulting dynamics varies significantly in the long run. This feature is also known as ‘sensitive dependence on parameters’ (SDP). The phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\lambda = 0.1$ ,  $\alpha = 0.5$ ,  $\beta_1 = 0.1$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The state variables of the dynamic system are deterministically initialized at their steady-state values where  $\bar{\omega}_t$  is initialized with  $\bar{\omega}_0 = 0.0$ .

The bifurcation plots shown above provide evidence that the fundamentalists generally do not accumulate more wealth than the chartists. In fact, for almost all parameter sets investigated so far, the investor group of chartists is able to survive in the artificial financial market and, moreover, quite often prevails over the fundamentalists in the long run.<sup>80</sup> Figures 4.23–4.26 also illustrate this result quite clearly. Note that the parameter sets used for these figures are chosen in accordance with the bifurcation diagrams depicted in Figures 4.16 (a) and 4.17 (g). Thus the outcome of the time series can be directly related to and compared with the results shown in the bifurcation diagrams and corresponding scatter plots above. Figures 4.23 and 4.24 belong to the same parameter set and only differ in the choice of the chartists’ extrapolation parameter  $\beta_2$ . While for Figure 4.23 a lower value for the extrapolation parameter of the chartists is used (see

<sup>80</sup>There are parameter sets for which the investor group of fundamentalists prevail over the chartist group (see straight line at +1 in some scatter plots in the  $(\beta_2, \bar{\omega}_t)$ -plane in the preceding figures). Nevertheless, a further look at the data sets of the corresponding simulations confirms that these results are quite rare.



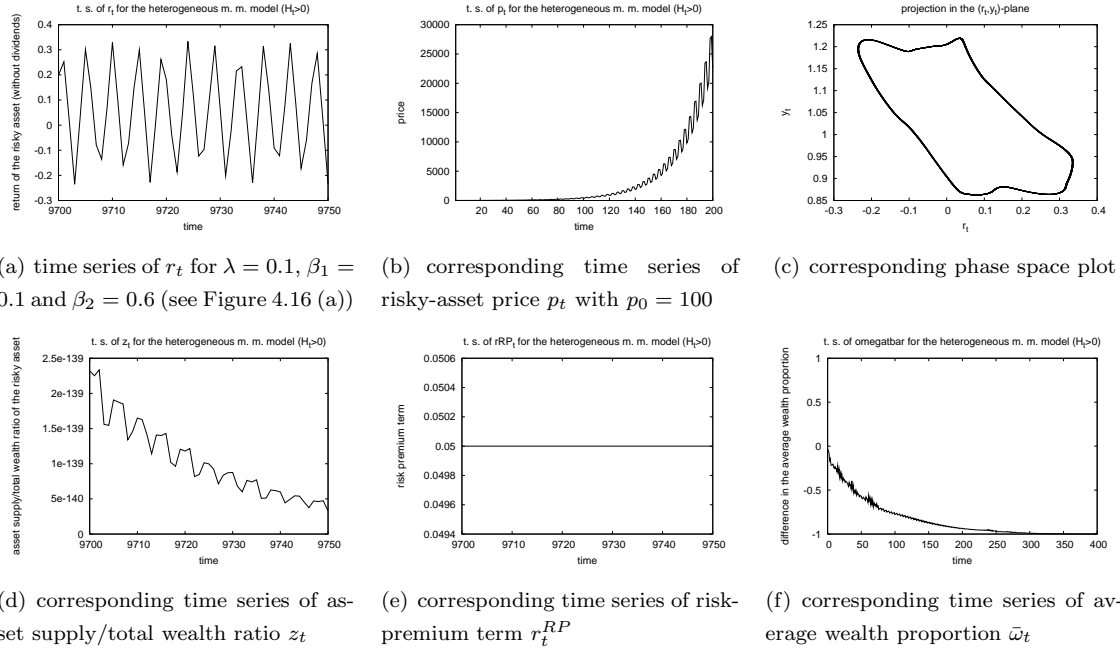
also left-hand side range of Figure 4.16 (a)) the plots in Figure 4.24 refer to a higher value of  $\beta_2$  (see also right-hand side range of Figure 4.16 (a)). More notably, they show two typical model outcomes of the revised heterogeneous market maker model, which can even be considered as quite universal in the sense that these model outcomes occur for a wide range of realistic parameter sets (see also the discussion below). More precisely, whenever the phase space plot in the  $(r_t, y_t)$ -plane



**Figure 4.23:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model ( $H_t > 0$  case) associated to Figure 4.16 (a) with a relatively low value of  $\beta_2$ . For this parameter set both investor groups coexist in the market. The phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

of the dynamic system for a given parameter set has a shape as shown in Figure 4.23 (c), the variables  $r_t$ ,  $z_t$  and  $r_t^{RP}$  (as well as  $y_t$ , which are not shown here in detail) oscillate around their fundamental equilibrium value and neither fundamentalists nor chartists accumulate more wealth than their counterpart in the long run. Otherwise, if the shape of the phase space plot looks like a limit cycle as depicted in Figure 4.24 (c), the chartists will always prevail over the fundamentalists in the long run and the state variable  $z_t$  converges to zero, meaning that the total wealth in the market by far exceeds the risky-asset supply (i.e., asset supply/total wealth ratio is equal to zero).<sup>81</sup> Furthermore, this leads to  $r_t^{RP}$  becoming equal to  $r_f$  in the long run. It might be again worth stressing that these findings are in harmony with the findings discussed above. Considering Figure 4.19 (a), which is the corresponding scatter plot to Figure 4.16 (a), it can clearly be detected that

<sup>81</sup>This can easily be understood by using Eq. (4.111). Since the chartists prevail over the fundamentalists in this market scenario, it is  $\bar{\omega}_t \approx -1$ . Furthermore, the simulation data shows that the chartists periodically invest much more of their wealth in the risky asset than the group of fundamentalists. Remember that short selling is not restricted in our model framework. Thus, their growth rate of wealth  $(1 + \beta_{c,t+1})$  is regularly much higher than the numerator  $(1 + n^s + r_{t+1})$  of Eq. (4.111), which eventually leads to  $z_t \rightarrow 0$  for  $t \rightarrow \infty$ .

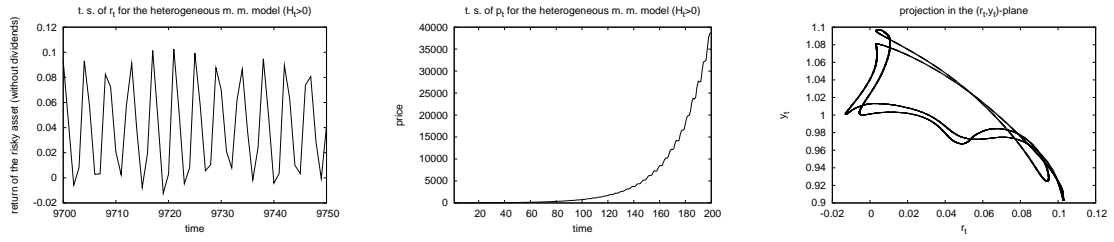


**Figure 4.24:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model ( $H_t > 0$  case) associated to Figure 4.16 (a) with a relatively high value of  $\beta_2$ . Comparing these results with the preceding Figure 4.23, an increase in the chartists' extrapolation parameter  $\beta_2$  obviously leads to a chartist superiority in the long run, i.e., the chartists accumulate all the wealth in the market. Again the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

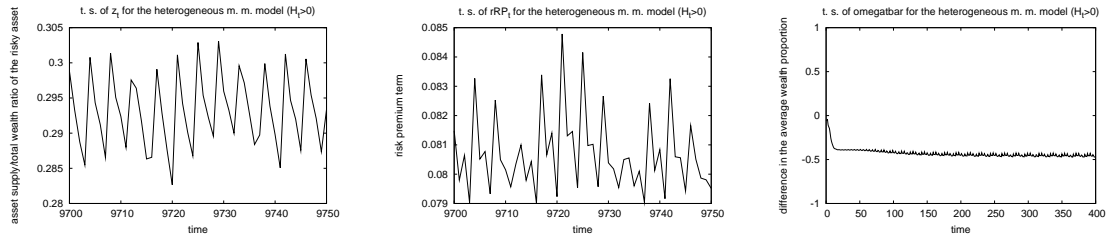
for low values of  $\beta_2$  the average wealth proportion  $\bar{\omega}_t$  mostly remains around its initial value 0.0 (depending on the chosen initial conditions, i.e., presence of SDIC), while increasing  $\beta_2$  leads to chartist superiority.<sup>82</sup>

The corresponding bifurcation diagram and scatter plots to Figures 4.25/4.26 are shown in Figure 4.17 (g)–(i). Here the extrapolation rate of the fundamentalists  $\beta_1$  is slightly increased, the extrapolation rate of the chartists is chosen as  $\beta_2 = 0.4$  and the speed of the price adjustment of the market maker is reduced by 50%, i.e.,  $\lambda = 0.05$ . Especially from the scatter plot in the middle panel (see Figure 4.17 (h)), it can be assumed that the presence of SDIC might be quite pronounced for this parameter set. Indeed, performing only a few identical simulation runs but with randomly initialized state variables (as described in Figures 4.25/4.26) has led to three qualitatively different model outcomes, which could already be conjectured from the corresponding scatter plot. Depending on the concrete initial values of the state variables, there are several possible results. First, the fundamentalists perform poorer than the chartists but survive in the long run (see Figure 4.25 (a)–(f)). In this case the corresponding phase space plot is similar to the phase space plot shown in Figure 4.23. Second, a fundamental steady state is reached (see Figure 4.25 (g)–(l)). In

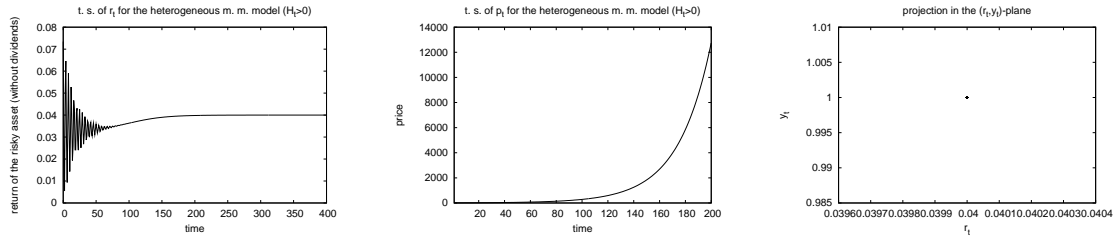
<sup>82</sup>The model outcomes in which the fundamentalists prevail over the chartists (grey colored area in Figure 4.19 (a)) are much rarer than might be expected when looking at the plot. However, a further look at the simulation data would confirm this expectation to be incorrect.



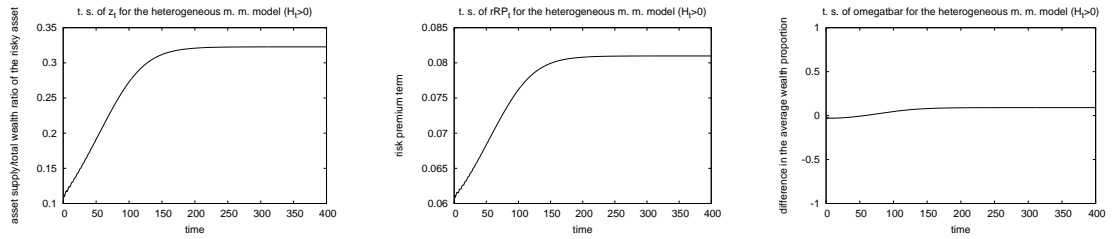
(a) time series of  $r_t$  for  $\lambda = 0.05$ ,  $\beta_1 = 0.25$  and  $\beta_2 = 0.4$ , Shape I (see Figure 4.17 (g)–(i)) (b) corresponding time series of risky-asset price  $p_t$  with  $p_0 = 100$  (c) corresponding phase space plot



(d) corresponding time series of asset supply/total wealth ratio  $z_t$  (e) corresponding time series of risk-premium term  $r_t^{RP}$  (f) corresponding time series of average wealth proportion  $\bar{\omega}_t$



(g) time series of  $r_t$  for  $\lambda = 0.05$ ,  $\beta_1 = 0.25$  and  $\beta_2 = 0.4$ , Shape II (see Figure 4.17 (g)–(i)) (h) corresponding time series of risky-asset price  $p_t$  with  $p_0 = 100$  (i) corresponding phase space plot



(j) corresponding time series of asset supply/total wealth ratio  $z_t$  (k) corresponding time series of risk-premium term  $r_t^{RP}$  (l) corresponding time series of average wealth proportion  $\bar{\omega}_t$

**Figure 4.25:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model ( $H_t > 0$  case) associated to Figure 4.17 (g)–(i). For this parameter set the presence of SDIC can clearly be detected (compare Shape I vs. Shape II vs. Shape III, where Shape III is sourced out to Figure 4.26 for editorial reasons). Again the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

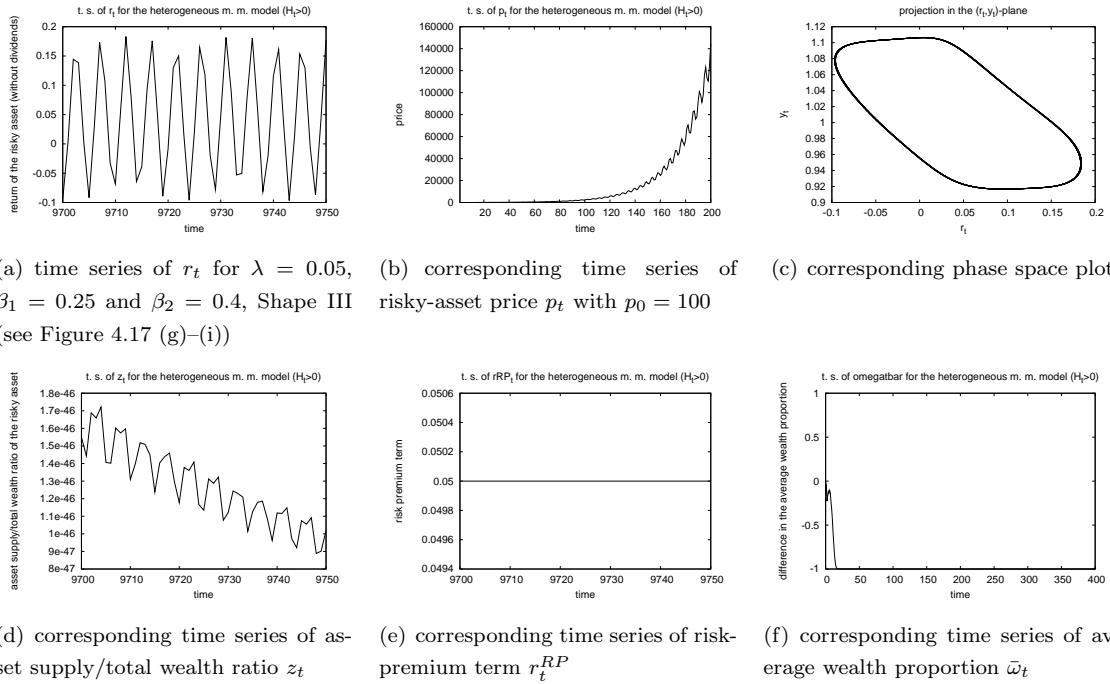


Figure 4.26: Figure 4.25 continued.

this case the fundamentalists might perform even slightly better than their chartist counterparts. The variables  $r_t$ ,  $z_t$  and  $r_t^{RP}$  (as well as  $y_t$ ) converge to their steady-state values and the risky-asset price increases exponentially in an absolutely smooth way. Finally, it is also possible that the model outcome is similar to the simulation result shown in Figure 4.24 (see Figure 4.26 (a)–(f)). In this case only the chartists survive in the long run. Note that because of space constraints the time series plots for  $p_t$ ,  $z_t$  and  $r_t^{RP}$  are omitted. However, they are qualitatively very similar to the corresponding time series shown in Figure 4.24.

Although many other experiments were also performed with parameter sets different to those shown so far, it is impossible to outline all their results in this subsection. Nevertheless, I will here summarize the main findings. Let us now assume the parameter sets used in Figure 4.23 and Figures 4.25/4.26 as a benchmark and call it ‘ $\lambda = 0.1$  parameter set’ and ‘ $\lambda = 0.05$  parameter set’.<sup>83</sup> Then it can be observed that the simulation results for different, relatively low values of  $\beta_1$  are quite comparable in that a variation of the extrapolation rate  $\beta_1$  for the fundamentalists within a range of relatively low values (e.g.,  $0.1 \leq \beta_1 \leq 0.3$ ) does not influence the results significantly. Furthermore, a mutual increase in the risk-aversion coefficient of the fundamentalists and chartists might stabilize the dynamics but does not lead to the underlying dynamic system exhibiting totally different behavior. The same is true for a decrease in the risk-aversion coefficient, although the oscillating behavior of the underlying state variables is enhanced primarily with respect to the amplitude of their fluctuations. But let us start with the  $\lambda = 0.1$  parameter set as a benchmark. A decrease in the ‘stabilizing term’ factor  $\alpha$  leads to  $n$ -cycles or limit cycles as in Figure 4.24 (c),

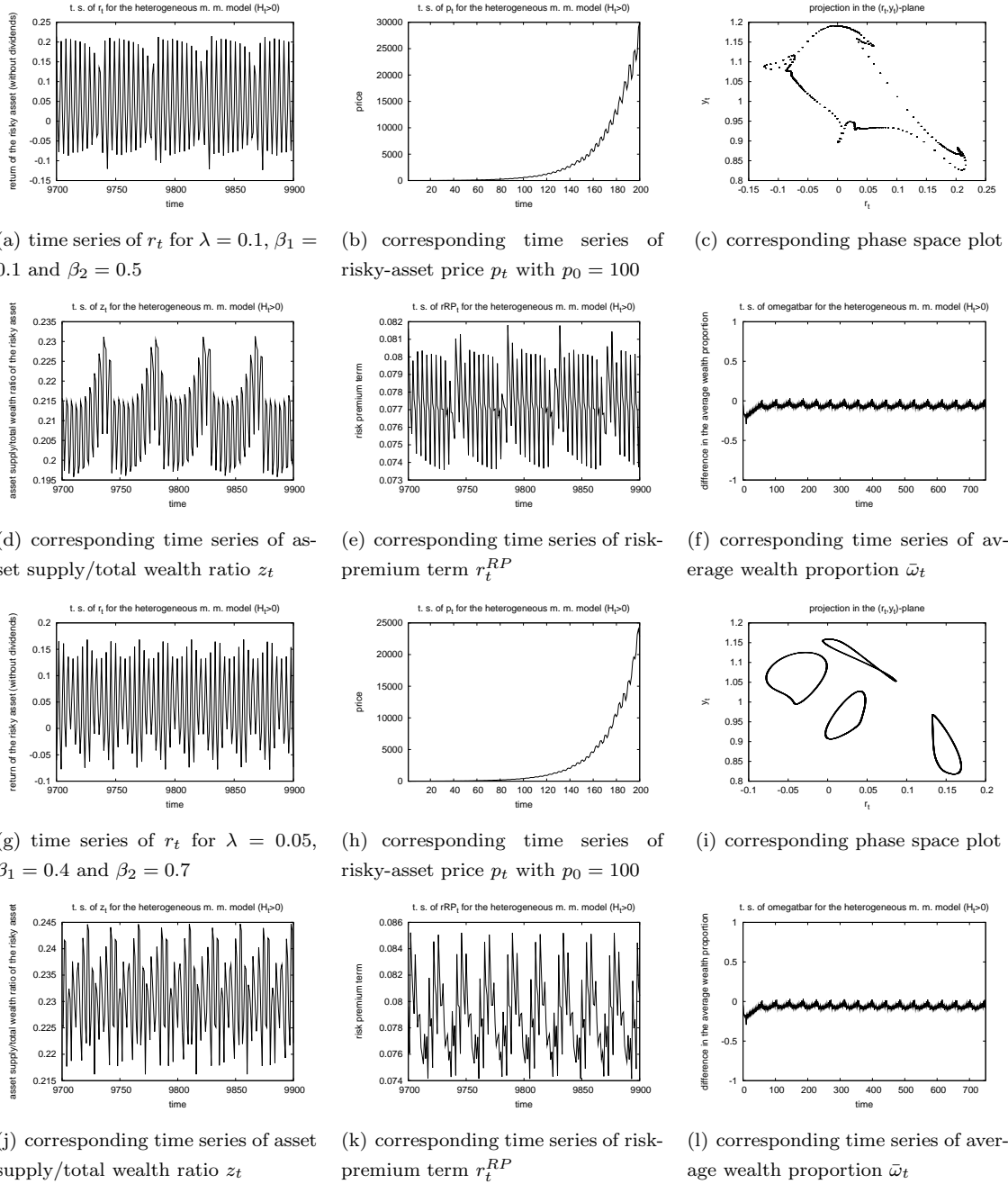
<sup>83</sup>As already mentioned above the parameter sets used for Figure 4.23 and Figure 4.24 are assumed as ‘identical’ since they belong to the same bifurcation diagram and thus to the same value of  $\lambda$ .

in which chartists eventually dominate the market by accumulating the entire wealth share. These results seem independent from the choice of the chartist extrapolation rate  $\beta_2$ . However, for a relatively high value of  $\alpha$ , the dynamic system converges to a fundamental steady state where the fundamentalists do even better than the chartists. This effect is slightly more pronounced if the chartists use a relatively high extrapolation rate  $\beta_2$ . A higher growth rate  $n^s$  of the risky-asset supply makes the occurrence of a phase space plot as shown in Figure 4.23 (c) more likely than for the benchmark model, even for relatively high chartist extrapolation rates  $\beta_2$ . A lower growth rate of the risky-asset supply definitely favors the chartists, who prevail over the fundamentalists more often than for parameter combinations (with respect to the parameter  $\beta_2$ ) of the benchmark set. A relatively low geometric decay rate  $\delta$ , which makes the chartists strongly believe in the most recent risky-asset return, leads to a more pronounced chartist superiority, while a relatively high geometric decay rate does not produce the opposite effect, but actually benefits the chartists also. Finally, a decrease in the parameter  $\beta_3$ , which weights the influence of the sample variance on the conditional variance estimated by the chartists, again benefits the chartists, who then take greater risks in their investment decisions since their conditional expectation of the variance of the risky-asset return  $r_{t+1}$  becomes smaller, thus increasing their optimal investment proportion  $\pi_{c,t}$  and allowing them to prevail clearly over the fundamentalists. Otherwise, a decrease in the  $\beta_3$  value also leads to the underlying dynamic system exhibiting more explosive behavior. Contrastingly, an increase in  $\beta_3$  favors the fundamentalists but only in the sense that they can survive in the market and only if their extrapolation parameter is low enough, e.g.,  $\beta_1 = 0.1$ . For the  $\lambda = 0.05$  parameter set similar observations can be made, although the fundamentalists generally do better than with the previous benchmark parameter set. The reason for this is simply that a fundamental steady state is reached much more often than for the  $\lambda = 0.1$  parameter set. For example, both a decrease as well as an increase in the ‘stabilizing term’ factor  $\alpha$  mostly cause a convergence to a fundamental steady state, regardless of the chartists’ specific choice of the extrapolation parameter  $\beta_2$ . A variation in  $\delta$  (decrease and increase from 0.5) also leads to a wider  $\beta_2$  range, in which the underlying dynamic system converges to one of its fundamental steady states. The same is true for a variation in the parameter  $\beta_3$ .

Most interestingly, so far it has not been possible to find either a parameter set in which the fundamentalists are able to rule the chartists out of the market<sup>84</sup>, or a situation of coexistence where both types of agents survive in the long run with the wealth shares exhibiting oscillatory behavior.<sup>85</sup> Figures 4.27 and 4.28 both give an example for the latter behavior, although the oscillatory behavior of the dynamics, and in particular the oscillatory behavior of the wealth shares, is not very distinctive. Nevertheless, a periodic market behavior can be detected for these parameter sets. While Figure 4.27 (a)–(f) contains a  $\lambda = 0.1$  parameter set in accordance with the corresponding bifurcation plot in Figure 4.16 (a), Figure 4.27 (g)–(l) refers to a  $\lambda = 0.05$  parameter set where both extrapolation parameters  $\beta_1$  and  $\beta_2$  are slightly increased. Figure 4.28 shows a similar market outcome for a  $\lambda = 0.05$  parameter set which leads to a strange attractor in the

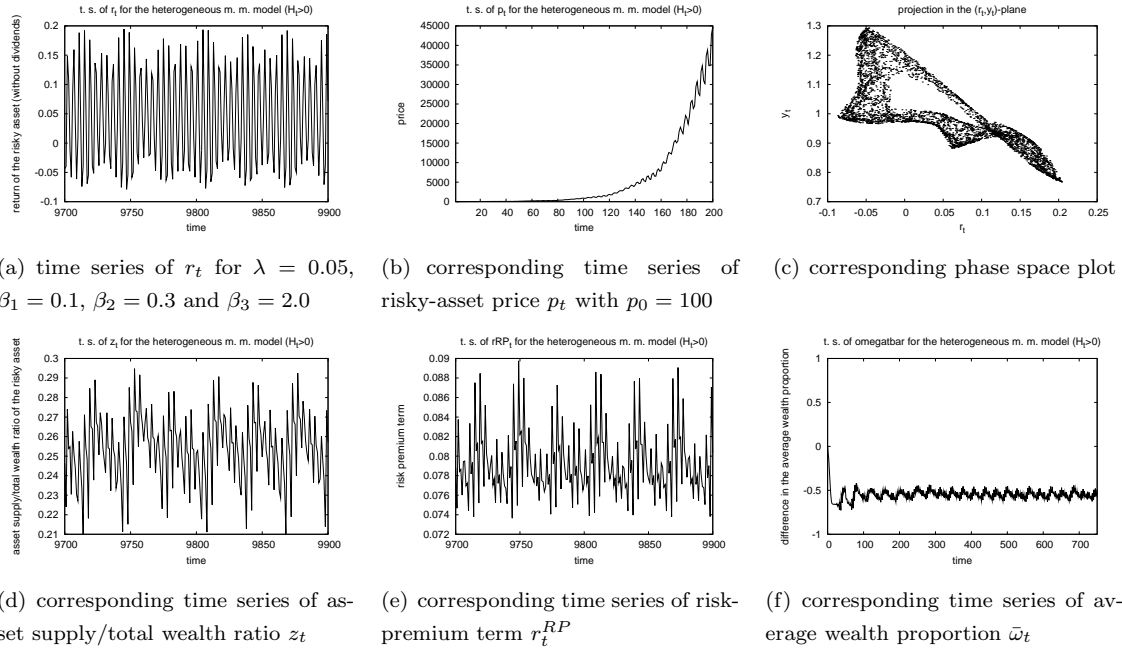
<sup>84</sup>See also Footnote 80 for a more precise statement.

<sup>85</sup>See Chiarella et al. (2006) and Figure 4.9 (d)–(f) in this thesis for such results. However, at least for the model framework outlined in this thesis, such model outcomes are rather hard to detect since only quite specific parameter sets are able to generate this oscillatory behavior in the wealth shares of the investor groups.



**Figure 4.27:** Time series of state variables  $r_t, z_t, \bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model ( $H_t > 0$  case) for a  $\lambda = 0.1$  and a  $\lambda = 0.05$  parameter set. For both parameter combinations cyclical movements of the state variables can be detected where even the wealth shares show this peculiarity of cycles, albeit not so pronounced. Again the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2, r_f = 0.05, g = 0.04, \sigma = 0.2, \delta = 0.5, n^s = 0.03, \alpha = 0.5, \beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

phase space. It should be remarked that for these kind of plots the value of  $\beta_3$  has to be slightly reduced, which means that the chartists are generally willing to invest more of their wealth in the risky asset than by using a higher value of  $\beta_3$  (see Eq. (4.146)). However, similar attractors can be obtained by increasing the extrapolation rate  $\beta_2$  for the chartists (e.g.,  $\beta_2 = 0.5$  or  $\beta_2 = 0.6$ ), so that in this sense these results are not limited to very specific parameter combinations. Finally,



**Figure 4.28:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model ( $H_t > 0$  case) for another  $\lambda = 0.05$  parameter set. This parameter combination, where the parameter  $\beta_3$ , which weights the influence of the sample variance on the conditional variance estimated by the chartists, is slightly decreased, leads to a strange attractor with cyclical movements of risky-asset returns, the asset supply/total wealth ratio, the risk-premium term and, less markedly, the wealth shares. Note that the existence of a strange attractor stands in contrast to the findings shown in Figures 4.25/4.26 and Figure 4.27 (g)–(l). Again, the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

it might be worth remembering that Figures 4.16 (c) and 4.18 (g)–(i) depict bifurcation plots for a similar parameter set, although  $\beta_3$  is chosen as its ‘benchmark value’ 0.5. With this choice of  $\beta_3$  the resulting dynamic system is locally stable (see also Footnote 71) for approximately  $\beta_2 < 0.7$ , which can also be seen in the straight line at the fundamental risky-asset return value 0.04 in the bifurcation plot in Figure 4.18 (h) and even in Figures 4.18 (i) and 4.16 (c), albeit less clearly.<sup>86</sup> It can thus be concluded that (at least) for the  $\lambda = 0.05$  parameter set a decrease in  $\beta_3$  destabilizes the market, although a decrease in  $\beta_2$  does not lead to total superiority for the chartist group.

I now briefly want to summarize how the artificial market will react when further model parameters and initial conditions are changed. As already stated previously, the proposed model

<sup>86</sup>See also the discussion around Figure 4.18 for further details on how the initialization of the underlying dynamic system influences the resulting long-term behavior.

framework does not restrict the optimal investment proportion  $\pi_{i,t}$  derived by the investor groups in general. In other words the investors are allowed to do ‘short selling’, i.e., they are allowed to invest more money in the risky asset than they actually are endowed with ( $\pi_{i,t} > 1$ ). Conversely, the optimal investment proportion can also become negative in our model framework, corresponding to the fact that the investors spend more money on the risk-free asset than they currently have in their pocket ( $\pi_{i,t} < 0$ ). Thus the investors are assumed in both cases to borrow money from someone. Moreover, it should be emphasized that this investment behavior actually occurs very often in the artificial financial market, in particular when the market dynamics does not settle down to a fundamental steady state. Therefore, I have also investigated the market behavior for different parameter sets when short selling is explicitly forbidden by restricting the investors’ optimal investment proportion between  $0.01 \leq \pi_{i,t} \leq 0.99$ . Interestingly, for the parameter sets that have been investigated (e.g., the  $\lambda = 0.1$  and  $\lambda = 0.05$  parameter set for different realizations of the chartists’ extrapolation rate  $\beta_2$ , but also further parameter sets), the market behavior remains qualitatively the same. In particular, the fundamentalists are still unable to gain superiority over the chartists, although the prohibition of short selling seems to stabilize the situation of the fundamentalists for certain parameter combinations in which the fundamentalists performed quite poorly before the restriction was enabled. However, quite often this simply means that the decline in the fundamentalists’ wealth share (see Figures 4.24 and 4.26 (a)–(f) for an appropriate parameter set and investors’ wealth share evolution in the benchmark setup) is slightly delayed but cannot be stopped in the long run.

In the second step, I will investigate whether or not the initial population proportion  $\bar{n}$  and the initial average wealth proportion  $\bar{\omega}_0$  have a crucial impact on the overall market behavior. Again, diverse simulation runs for several parameter sets have been performed. However, to avoid an overly discussion I will omit an explicit visualization of the results. Both a shift of the initial population proportion  $\bar{n}$  from a balanced condition (i.e.,  $\bar{n} = 0.0$ ) towards a numerical majority of the fundamentalists’ investor group (i.e.,  $\bar{n} > 0.0$ ) as well as an improved initial (monetary) endowment of the fundamentalists (i.e.,  $\bar{\omega}_0 > 0.0$ ) should strengthen the fundamentalists’ market power, in general leading to a better overall performance for this investor group. Indeed, numerical simulations confirm this conjecture. For the latter case it can be shown that a sufficiently higher initial wealth level for the fundamentalists (i.e., sufficiently higher than the chartist group’s) can allow the underlying dynamic system converge to a fundamental steady state. This is even the case for parameter sets for which the system has not converged yet, provided that the initial wealth level of both investor groups was assumed to be equally balanced. It might be worth mentioning that the fundamentalists even retain their higher wealth share in the long run in these cases. Of course, the greater the superiority of the fundamentalists over the chartists due to their initial wealth endowment, the higher the chartists’ extrapolation rate  $\beta_2$  can be without the underlying dynamic system not converging to a fundamental steady state.<sup>87</sup> As seen before, however, the fundamentalists cannot completely rule the chartists out of the market even in this case. This is not surprising, because a fundamental steady state means that both investor groups are assumed to form correct (and thus homogeneous) expectations about the fundamental price path of the risky

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<sup>87</sup>Remember that an increase in the extrapolation parameter  $\beta_2$  of the chartists generally leads to limit cycles and a superiority for the chartist investor group (see Figure 4.24).



asset. Consequently, this guarantees the coexistence of both investor groups in the long run in such a market environment. An increase in the investor population of the fundamentalist' group shows similar results as stated above. Again, a shift towards the fundamentalists (i.e., positive  $\bar{n}$ ) leads to a wider range of parameter sets for which a fundamental steady state is reached. Nevertheless, the influence that varying the initial condition for  $\bar{n}$  has on the behavior of the underlying dynamic system seems less strong than in the case of  $\bar{\omega}_0$ . In the case of limit cycles (see Figure 4.24) the chartists need more trading periods to rule the fundamentalists out of the market, which is also not surprising since more fundamentalists are in the market that has to be displaced.

In the final investigation I want to examine whether different risk-aversion coefficients used by the investor groups have a significant influence on the resulting market behavior. Remember from the Technical Remark 4.15 at the very beginning of this subsection that the steady states of the underlying dynamic system cannot easily be determined analytically for these cases. Nevertheless, numerical simulations can be performed. Furthermore, note that the assumption of non-identical risk-aversion coefficients  $\nu_f$  and  $\nu_c$  for both trading groups is qualitatively the same as assuming different  $\sigma^2$ s for the investors, because all relevant model equations see the occurrence of a factor  $\nu_i \sigma_{i,t}^2$ .<sup>88</sup> Assuming a lower risk-aversion coefficient for the fundamentalists than for the chartists dramatically favors the former. In almost all simulation runs they comfortably prevail over the chartists in the long run. While this result is not unsurprising for parameter sets for which a fundamental steady state is reached, simply because the resulting optimal investment proportion in the equilibrium  $\pi_f^*$  becomes higher than for the chartists, it is remarkable for situations in which the dynamic system does not settle down to a fundamental steady state. In these cases only a high extrapolation rate  $\beta_2$  for the chartists might avoid their defeat for some parameter combinations. If  $\nu_f > \nu_c$  is chosen, the chartists collect the entire wealth share for almost all parameter sets quite rapidly. Finally, if  $\nu_f < \nu_c$  is chosen together with a relatively low value  $\beta_3$  for the chartists<sup>89</sup>, which can be seen as compensation for the chartists' more conservative risk-aversion coefficient, the numerical results do not show any new features in the resulting market behavior of the two trading groups. Instead, depending on the concrete combination of these three parameters, either the fundamentalists prevail over the chartists or vice versa. However, for these parameter combinations no cyclical movements of the state variables or even of the wealth shares of investors can be observed as shown, for instance, in Figure 4.28.

So far the main focus of the present chapter has been on the qualitative price, return and wealth behavior that results from the interaction of two different investor types in a model structure which is driven by a nonlinear deterministic dynamics. However, when presenting an artificial financial market model of this type, two further questions typically arise. First, it might be interesting to check whether the model framework is able to generate generally realistic time series of asset prices and asset returns by showing phases of booms and market crashes as observed in real financial data. As seen in the discussions so far, these typical market peculiarities are practically impossible to observable in the purely nonlinear deterministic market maker model; this is not really surprising

<sup>88</sup>This is not completely true for two model equations in which the chartists' beliefs about the conditional variances of returns  $\sigma_{c,t}^2$  is involved (see Eq. (4.146) and Eq. (4.147)).

<sup>89</sup>A  $\beta_3$  value close to zero in combination with a relatively high  $\lambda$  value leads mostly to explosions for the underlying dynamic system.

because it shares this lack of ‘realistic’ market behavior with almost all other prominent model frameworks of this type.<sup>90</sup> To overcome this problem it is an accepted procedure to combine the nonlinear deterministic dynamic system with external noise by introducing noise terms into one or even more model equations. If generally realistic price paths can only be obtained with those model extensions, clearly only the *interaction* of the nonlinear deterministic dynamics (which describes the economic fundamentals of the model) with the external noise (which describes the economic deficiencies in the given model framework) will be able to generate this more realistic market behavior. Second, closely connected with the first point mentioned above, the choice of parameters (and of the noise level) used in the numerical experiments are of crucial interest where the calibration of the model is always naturally related to the question of the time scale used in the model. In the investigations so far I have not calibrated the model parameters in a specific fashion or even calibrated them to real financial data. However, most model parameters are chosen in such a way that they can describe our market maker model on a yearly time scale. Although these kinds of nonlinear structural agents models are less suited to replicating certain financial market patterns (less so than the ‘stylized facts’ of financial markets) or typical characteristics of financial time series, the implicitly assumed relatively large time scale is still somewhat unsatisfying. Therefore, it might be natural to consider whether or not the model will still show similar results if appropriate model parameters are rescaled to a smaller time scale. The next two subsections briefly present simulation results related to these two issues, where the model equations are recalibrated on a daily trading period level followed by some simulation results for a simple stochastic version of our market maker model.

#### *Simulation Results – Daily Trading Period*

In this subsection I want to see if the findings outlined above can be maintained, or if they have to be revised when I scale the exogenous model parameters on a daily level by assuming 250 trading days per year. Technically speaking, the dynamic system Eq. (4.137)–Eq. (4.142) is retained as it is where Eq. (4.143)–Eq. (4.147) still hold but the parameters are downsized (so far they have been annually sized) according to the following transformation rules:

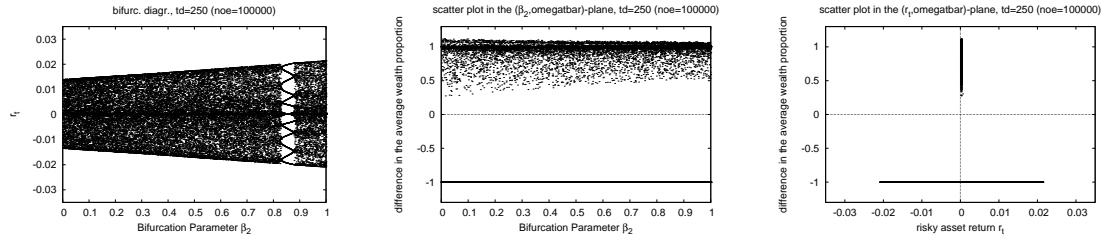
$$r_{f_{\text{daily}}} = r/t_d, \quad g_{\text{daily}} = g/t_d, \quad \sigma_{\text{daily}} = \sigma/\sqrt{t_d}, \quad \lambda_{\text{daily}} = \lambda/t_d, \quad n_{\text{daily}}^s = n^s/t_d, \quad (4.148)$$

where  $t_d$  denotes the trading days per year. Furthermore, I adjust the randomly driven initialization of the dynamic system analogously by reducing the mean of the initial values by a factor  $1/t_d$  and the standard deviation by a factor  $1/\sqrt{t_d}$  to keep consistency in the initial perturbation of the system.<sup>91</sup>

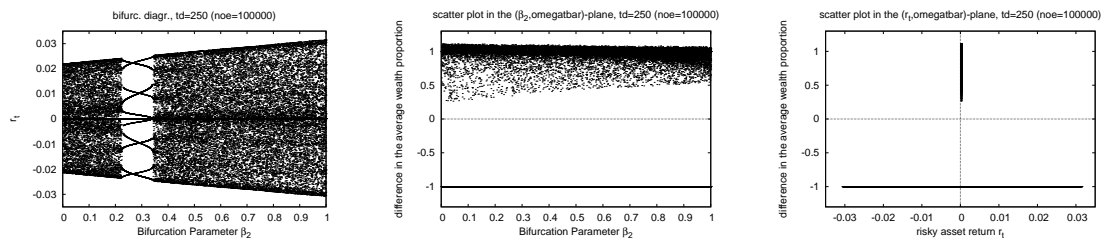
<sup>90</sup>See the discussions on further prominent nonlinear deterministic model frameworks in Chapters 2 and 3 of this thesis for further details.

<sup>91</sup>The initialization of the fundamental/price ratio  $y_t$  and the asset supply/total wealth ration  $z_t$  should not have to be adjusted since the scaling factor  $t_d$  cancels out in the corresponding dynamic equations Eq. (4.138) and Eq. (4.139). Nevertheless, unreckonedly, numerical simulations indicate that this always leads to undesirable results where the system settles down to an unwanted ‘quasi’ steady state, irrespective of the choice of parameter sets (see Figure 4.31 for a similar model outcome). Therefore, in the following discussion we will concentrate on numerical experiments where the variable  $z_t$  is also scaled down. Since  $z_t$  had been hitherto initialized by a uniformly distributed random variable in order to assure the desired positive initial value ( $H_t > 0$  case), I here decided to

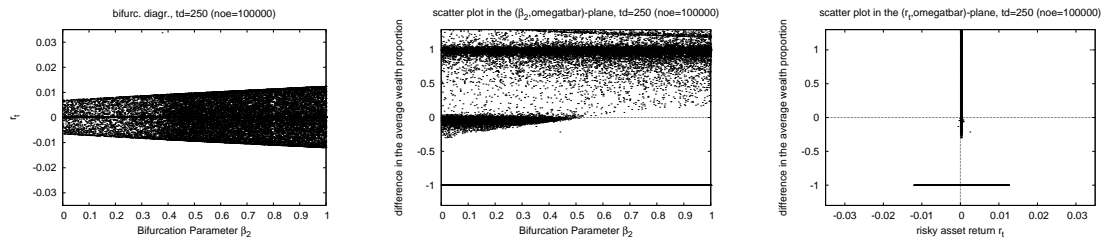
Figures 4.29 and 4.30 show different bifurcation diagrams and corresponding scatter plots for parameter sets already used in Figures 4.16 and 4.17 but on a daily scale (i.e., 250 trading days per year). Although the figures seem to resemble the findings for the annual treatment outlined



(a) bifurcation diagram,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$ ,  $\beta_3 = 5.0$ , daily level (see Figure 4.16 (a) as a benchmark) (b) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane (see Figure 4.19 (a) as a benchmark) (c) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane



(d) bifurcation diagram,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$ ,  $\beta_3 = 2.0$ , daily level (see Figure 4.16 (b) as a benchmark) (e) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane (see Figure 4.19 (b) as a benchmark) (f) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane

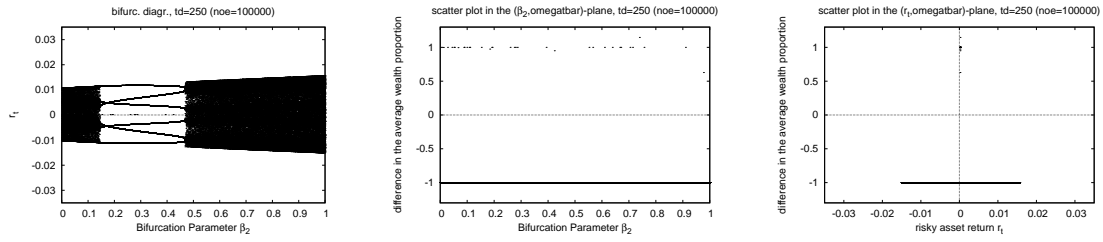


(g) bifurcation diagram,  $\lambda = 0.05$ ,  $\beta_1 = 0.1$ ,  $\beta_3 = 5.0$ , daily level (see Figure 4.16 (c) as a benchmark) (h) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane (see Figure 4.19 (c) as a benchmark) (i) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane

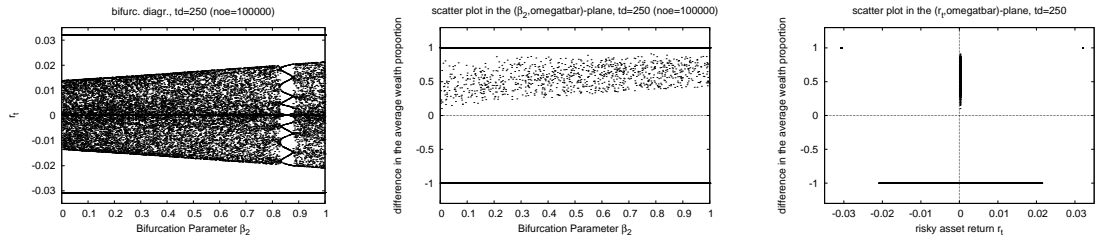
**Figure 4.29:** Bifurcation diagrams with respect to  $\beta_2$  and corresponding scatter plots for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case). Again the resulting dynamics becomes rather complex, although, at least for the upper and lower panels, the risky-asset return seems to reach its fundamental steady-state level much more often than in the corresponding bifurcation plots for the annual treatment (see Figure 4.16). Obviously, risky-asset returns quite far away from the dividend growth rate  $g_{\text{daily}}$  favor the chartist group, while for returns close to  $g_{\text{daily}}$  the fundamentalists have the advantage. Only a relatively low value of  $\lambda$  in combination with a relatively weak extrapolation rate  $\beta_2$  for the chartists leads to a balanced wealth distribution among the two competing investor groups. The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

use the scaling factor  $1/\sqrt{td}$ . Admittedly, this could seem somewhat arbitrary because a factor  $1/td$  might even be more consistent. However, since a uniformly distributed initialization is used, the scaling factor  $1/\sqrt{td}$  ( $=0.06325$  for  $td = 250$ ) also covers the initializations generated by a factor  $1/td$  ( $=0.00400$  for  $td = 250$ ).

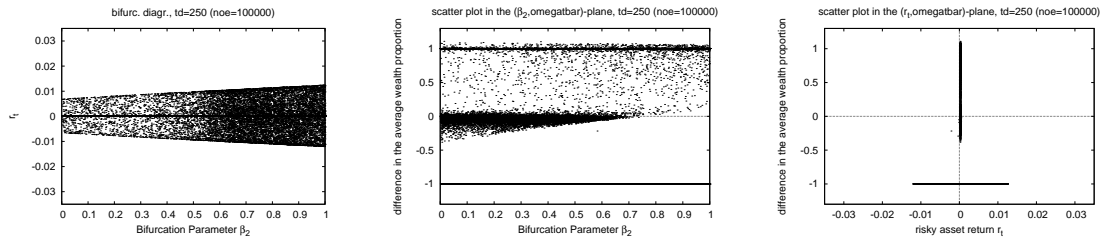
above, the results are in fact rather disappointing. In most of the cases either the chartists or



(a) bifurcation diagram,  $\lambda = 0.1$ ,  $\beta_1 = 0.1$ ,  $\alpha = 0.1$ , daily level (see Figure 4.17 (a) as a benchmark) (b) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane (see Figure 4.17 (b) as a benchmark) (c) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane (see Figure 4.17 (c) as a benchmark)



(d) bifurcation diagram,  $\lambda = 0.1$ ,  $\beta_1 = 0.25$ ,  $\alpha = 0.5$ , daily level (see Figure 4.17 (d) as a benchmark) (e) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane (see Figure 4.17 (e) as a benchmark) (f) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane (see Figure 4.17 (f) as a benchmark)



(g) bifurcation diagram,  $\lambda = 0.05$ ,  $\beta_1 = 0.25$ ,  $\alpha = 0.5$ , daily level (see Figure 4.17 (g) as a benchmark) (h) corresponding scatter plot in the  $(\beta_2, \bar{\omega}_t)$ -plane (see Figure 4.17 (h) as a benchmark) (i) corresponding scatter plot in the  $(r_t, \bar{\omega}_t)$ -plane (see Figure 4.17 (i) as a benchmark)

**Figure 4.30:** More bifurcation diagrams with respect to  $\beta_2$  and corresponding scatter plots for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case). While the middle and lower panels show similar results as for the corresponding parameter sets in the previous figure ( $\beta_1 = 0.25$  in this figure vs.  $\beta_1 = 0.1$  in Figure 4.29), the upper panels show the chartists always dominating the market in the long run. However, note that two further long-run risky-asset returns occur for the parameter set used in the middle Panels (d)–(f), which are rather excessive (or unrealistic) for a daily model treatment (i.e., 2-cycles). Nevertheless, the fundamentalists prevail over the chartist group for these returns. The experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

the fundamentalists accumulate the total wealth in the market and a balanced coexistence of both trading types is a much rarer model outcome than for the market scenarios in the previous discussions.<sup>92</sup> More importantly, the dynamic system most often behaves in a way that can only

<sup>92</sup>This statement has been checked and confirmed through a multitude of simulation runs with different parameter sets that cannot be shown in detail here.

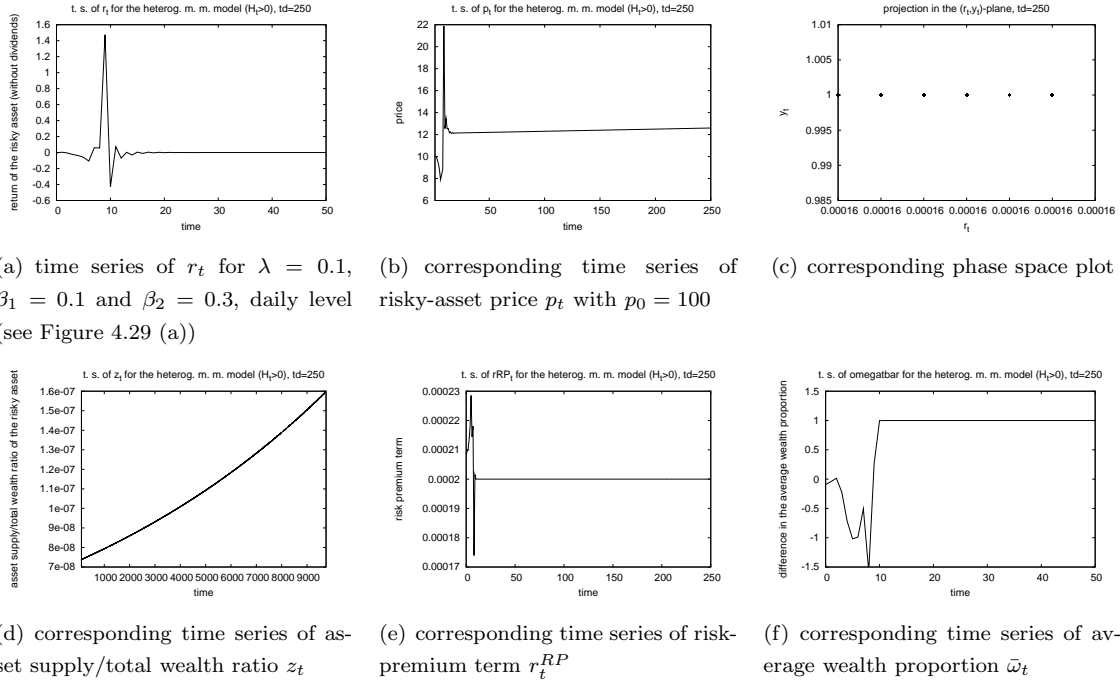
be described as economically unrealistic. For example, the dynamic system scaled on a daily level frequently seems to converge to a fundamental steady state, although checking the corresponding data sets of the simulation shows us that this is not the case. Instead, the optimal investment proportion  $\pi_{i,t}$  of both investor types converges to zero, which implies that the supply/total wealth ratio  $z_t$  also becomes almost zero but increases slightly afterwards. Hence the investors' growth rate of wealth becomes equal to the risk-free rate  $r_{f\text{daily}}$  and the fundamental/price ratio  $y_t$  becomes 1. In such market situations the fundamentalists prevail over the chartists in almost all simulation runs.<sup>93</sup> Figure 4.31 refers to such a market situation. All stated observations can clearly be detected there.

Unfortunately, this model behavior is quite persistent in the sense that neither the restriction of short selling nor an increase in the risk-aversion coefficient  $\nu$  (which should in principle settle the investors' investment decisions) can change these results significantly.<sup>94</sup> Only the combination of a low  $\lambda$  value and a relatively low chartists' extrapolation parameter  $\beta_2$  leads to market situations in which neither the fundamentalists nor the chartists can distance themselves from one another with respect to their wealth levels (see Figures 4.29 (h) and 4.30 (h)). Interestingly, checking the corresponding simulation data, even in such cases, shows us that the results are not very promising since the optimal investment proportion  $\pi_{i,t}$  of both investor groups still remains very low (most of the time less than 5%). However, when initializing the dynamic system very near to a fundamental steady state (which might be a quite unrealistic manipulation with respect to a real financial market situation), the system naturally converges to the fundamental steady state in the long run. Thus the system again shows the presence of SDIC, as already discussed in some detail in the previous section.

Another unwanted feature of the daily scaled market maker model can be observed in the  $(\beta_2, \bar{\omega}_t)$ -plane scatter plots of Figures 4.29 and 4.30. It can clearly be detected here that the difference in the average wealth proportions of investor groups  $\bar{\omega}_t$  quite often exceeds +1. This phenomenon has already been referred to in Footnote 73. Nevertheless, it should be remarked that

<sup>93</sup>More precisely, it can be shown that the dynamic system Eq. (4.137)–Eq. (4.142) seems to have a further steady state, namely  $\phi^* = g_{\text{daily}}$ ,  $v^* = 0$ ,  $r^* = g_{\text{daily}}$ ,  $y^* = 1$ ,  $z^* = 0$  and  $\bar{\omega}^* \in [-1, 1]$ , which leads to  $\beta_f^* = \beta_c^* = r_{f\text{daily}}$ ,  $\pi_f^* = \pi_c^* = 0$ ,  $r^{RP^*} = r_{f\text{daily}}$  and  $q^* = 0$ . Indeed, these values fulfill the steady-state condition. However, this steady state can never be reached because  $z_t$  is assumed to be always positive since it represents the asset supply/total wealth ratio (i.e.,  $H_t > 0$  case) and therefore  $z_t$  should always be initialized as a positive number (see also Eq. (4.104) and the discussion around). Since according to Eq. (4.139) the numerator will always exceed (or equal) its denominator with the steady-state values given above, this leads to a contradiction because then  $z_t$  cannot converge to its steady-state value  $z^* = 0$ . To see this more clearly, note that it is  $\{2(1 + n^s + r^*)\}/\{[1 + \bar{\omega}^*](1 + \beta_f^*) + [1 - \bar{\omega}^*](1 + \beta_c^*)\} = \{1 + n^s + g\}/\{1 + r_f\} \geq 1$  since it is a necessary requirement that  $n^s + g \geq r_f$  according to Eq. (4.56). Note that the scaling factor  $td$  is left out of this calculation for the sake of better readability. Note that this issue is discussed in the proofs of the Theorems 4.11, 4.14 and 4.16, which can be found in the Appendices A 4.12, A 4.13 and A 4.14 respectively. Thus the value set given above is not a steady state at all. Nevertheless, numerical simulations show that the dynamic system can reach *and remain* in a very similar situation where it is  $\phi_t = g_{\text{daily}}$ ,  $v_t = 0$ ,  $r_t = g_{\text{daily}}$ ,  $y_t = 1$ ,  $\beta_{f,t} = \beta_{c,t} = r_{f\text{daily}}$ ,  $\pi_{f,t} = \pi_{c,t} = 0$  and  $z_t$  is almost equal to zero but slowly increases with the growth factor  $\{1 + n_{\text{daily}}^s + g_{\text{daily}}\}/\{1 + r_{f\text{daily}}\}$  as mentioned in the continuous text. Obviously, this growth factor is extremely small, particularly for a parameter set scaled on a daily basis which remains the system in this status (see Figure 4.31).

<sup>94</sup>On the contrary, an exceeding *decrease* in the risk-aversion coefficient  $\nu$  leads to explosive behavior in the underlying system.



**Figure 4.31:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case) associated to Figure 4.29 (a) with a relatively low value of  $\beta_2$ . For this parameter set the fundamentalists dominate the market in the long run. The panels exhibit typical market behavior for the daily scaled model. While in the first trading periods heavy oscillations can occur, the system rapidly settles down to a ‘quasi’ steady state where the risky-asset return is (almost) equal to its fundamental value  $g_{\text{daily}}$ , the risk-premium term  $r_t^{RP}$  is equal to the daily risk-free rate  $r_{f_{\text{daily}}}$  (since  $\pi_{f,t} = \pi_{c,t} = 0$ , not shown here) and the asset supply/total wealth ratio  $z_t$  is almost zero but increases in the long run. Furthermore, the difference in the average wealth proportion  $\bar{\omega}_t$  being interim below -1 indicates that no economically driven restrictions are assumed for the underlying dynamic system but also that, on a daily scale, the system behaves rather sensitively to its initialization and runs out of its inherent (or designed) scale. The phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

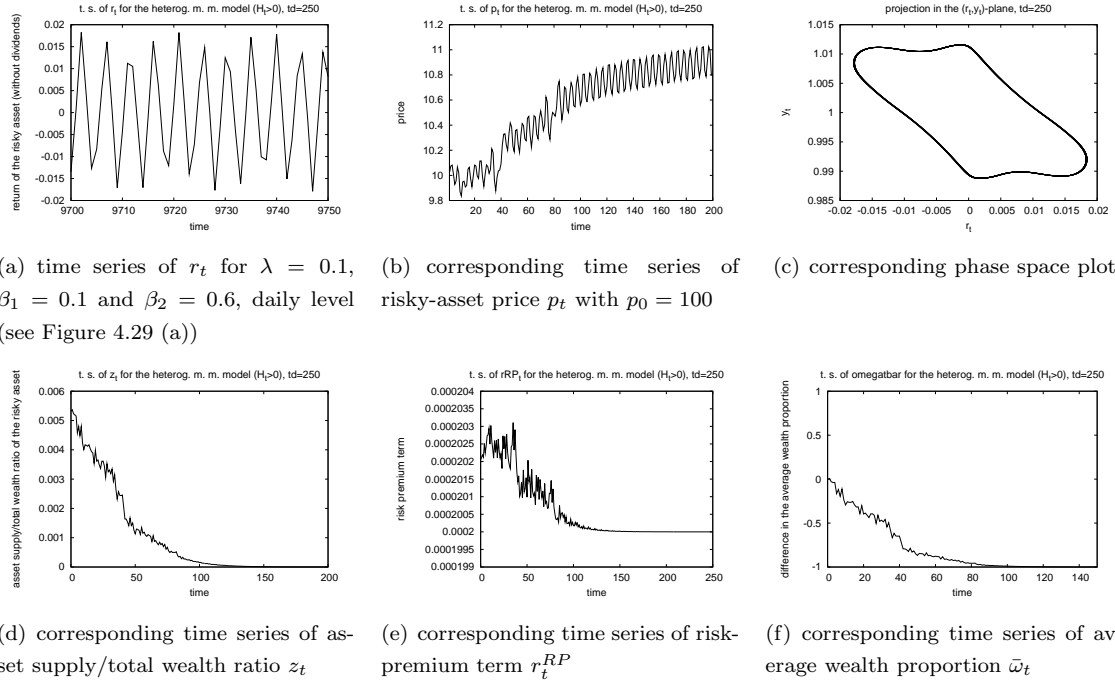
such events occur much more frequently for the  $td = 250$  treatment than in the previous model versions with  $td = 1$  (see also Figure 4.31 (f) for similar market behavior).

Figure 4.32 shows another simulation run in which a limit cycle occurs.<sup>95</sup> As already seen in previous simulations generating convergence to limit cycles, the chartists prevail over the fundamentalists in the long run. While  $r_t^{RP}$  again settles down to the daily risk-free rate  $r_{f_{\text{daily}}}$  and  $z_t$  becomes almost zero, the risky-asset price  $p_t$  evolves exponentially, growing in a regular periodic mode.

Although this discussion concentrates on market situations in which the fundamentalists keep their extrapolation parameter  $\beta_1$  rather small<sup>96</sup> I also want to emphasize that the findings men-

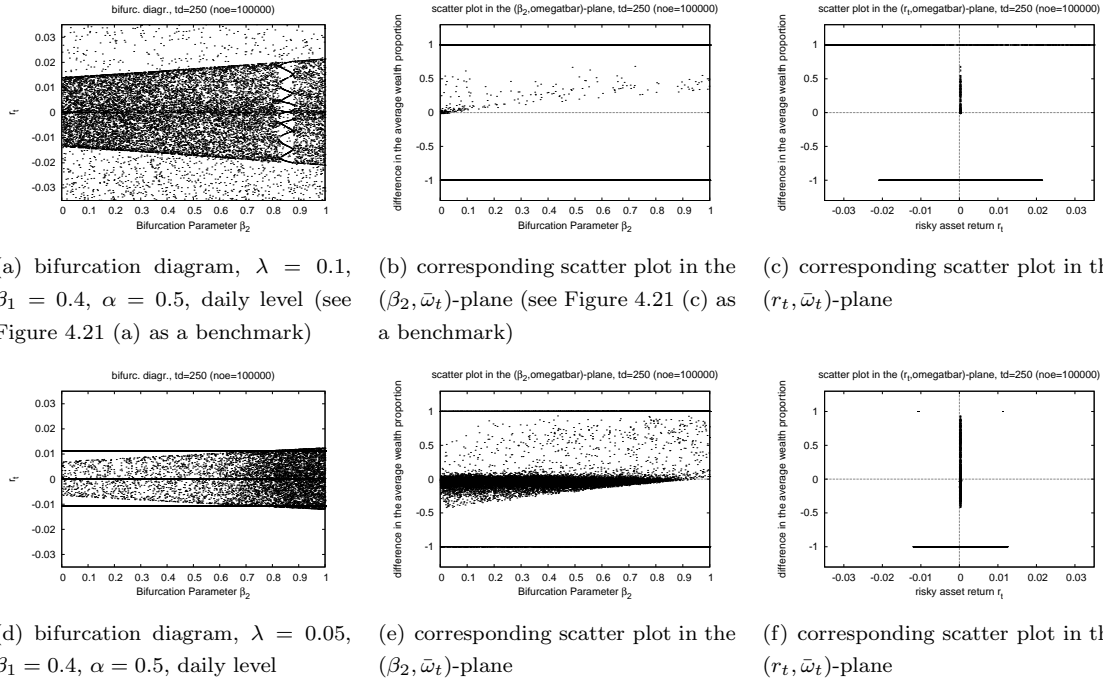
<sup>95</sup>The numerical experiments indicate that the dynamic system primarily generates limit cycles whenever  $z_t$  is initialized very close to zero. See also Footnote 91.

<sup>96</sup>For a motivation for this assumption, remember the discussion below-mentioned at Remark 4.17 in Subsection



**Figure 4.32:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case) associated to Figure 4.29 (a) with a relatively high value of  $\beta_2$ . Comparing these results with the preceding Figure 4.31, an increase in the chartists' extrapolation parameter  $\beta_2$  (combined with a low  $z_t$  initialization) obviously leads to chartist superiority in the long run, i.e., the chartists accumulate all the wealth in the market. Again the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

tioned above will remain qualitatively the same even if the extrapolation parameter  $\beta_1$  is chosen rather high. However, as for the annually scaled model, an increase in the fundamentalists' extrapolation parameter  $\beta_1$  leads to intensified explosive behavior in the dynamic system, where a simultaneous decrease in the price adjustment speed  $\lambda$  has a stabilizing effect on the dynamics. For instance, an increase in  $\beta_1$  to the level of 0.65 (and higher) very often leads to model explosions for  $\lambda = 0.1$  but not for  $\lambda = 0.05$ , while for  $\beta_1 = 0.4$  the dynamic system remains manageable for almost all reasonable parameter combinations. Figure 4.33 refers to two such parameter sets. In the upper left panel it can clearly be detected that the combination of a higher  $\beta_1$  value and a relatively high  $\lambda$  value leads to long-term returns that might become quite large; this can be seen by the more widely scattered realizations of the risky-asset return  $r_t$  (see Figure 4.33 (a) vs. Figures 4.29 (a) and 4.30 (d)). For this parameter combination the fundamentalists can theoretically prevail over the chartists for all possible (and more importantly, for all reasonable) risky-asset return realizations (at least between -0.35 and 0.35 as shown in the plots) while for low values of  $\beta_1$  (i.e., for  $\beta_1 = 0.1$  or  $\beta_1 = 0.25$ ) the risky-asset return has to be near to its fundamental value to show the same result (see Figure 4.33 (c) vs. Figures 4.29 (c) and 4.30 (f)). For  $\lambda = 0.05$  an increase in



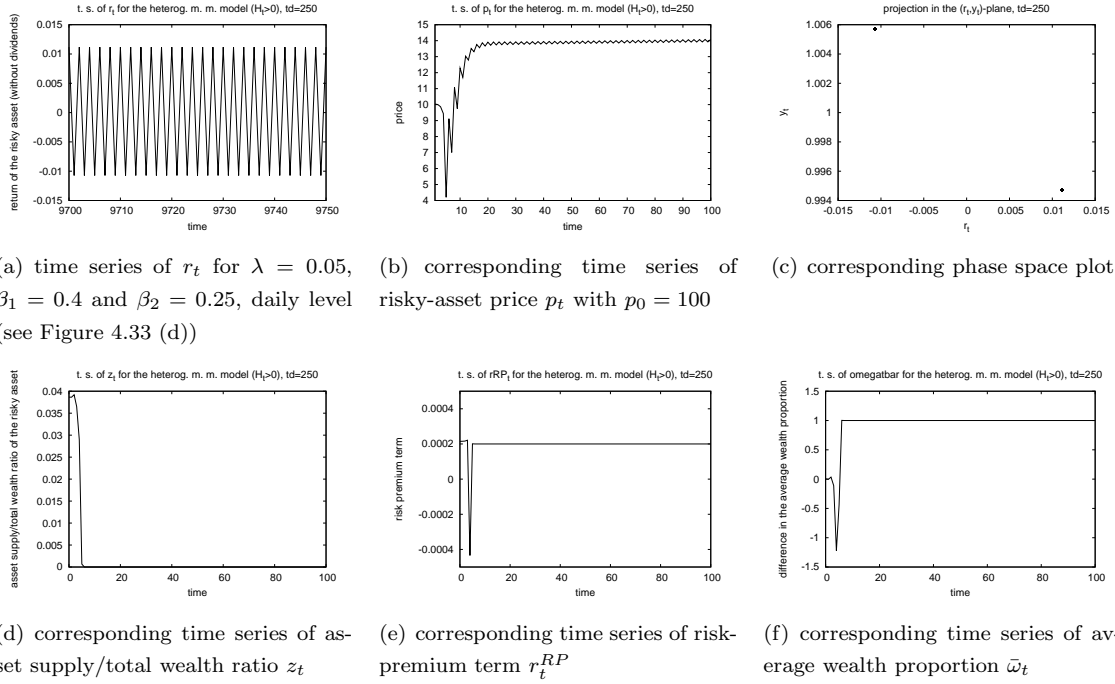
**Figure 4.33:** Bifurcation diagrams with respect to  $\beta_2$  and corresponding scatter plots for a slightly higher value of  $\beta_1$  for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case). While the bifurcation plots look rather similar to the corresponding bifurcation plots shown in Figures 4.29 and 4.30, the scatter plots clearly indicate that the dynamic behavior for an increased value of  $\beta_1$  deviates from the model behavior for a relatively low value of  $\beta_1$  (i.e.,  $\beta_1 = 0.1$  or  $\beta_1 = 0.25$ ). Again the experiments are conducted in exactly the same way as described in Figure 4.2. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

the extrapolation parameter  $\beta_1$  leads to a wider  $\beta_2$  range, in which a balanced wealth distribution among the two competing investor groups can be reached (see Figure 4.33 (e) vs. Figures 4.29 (h) and 4.30 (h)). Interestingly, the risky-asset return here sometimes moves towards a 2-cycle with relatively high and low daily long-term returns. Note that for the corresponding parameter set with  $\beta_1 = 0.25$  no 2-cycles occur, although they do when  $\lambda$  is chosen at the higher level of 0.1 (see Figure 4.33 (f) vs. Figure 4.30 (f)). For  $\beta_1 = 0.1$  2-cycles never occur. For even higher values of  $\beta_1$  the 2-cycle behavior in the  $\lambda = 0.05$  system holds, where the range of the 2-cycle widens significantly.<sup>97</sup> Figure 4.34 shows this 2-cycle behavior for a  $\beta_1 = 0.4$  case. As for all 2-cycle movements that occur for this model variant, the fundamentalists accumulate all the wealth in the market very rapidly.

In summary, one of four model outcomes occurs in the daily scaled model variant for a wide range of meaningful parameter sets in almost all numerical experiments. The dynamic system can converge to its ‘quasi’ steady state (see Figure 4.31) or 2-cycles in the risky-asset return can occur (see Figure 4.34). In both cases the fundamentalists dominate. A third possible model outcome is that the dynamics leads to limit cycles. Here the chartists do better than the fundamentalists (see Figure 4.32). As a final highly probable model result both fundamentalists and chartists coexist

<sup>97</sup>For too high values of  $\beta_1$  (e.g., for  $\beta_1=0.8$ ), however, this 2-cycle behavior disappears again.





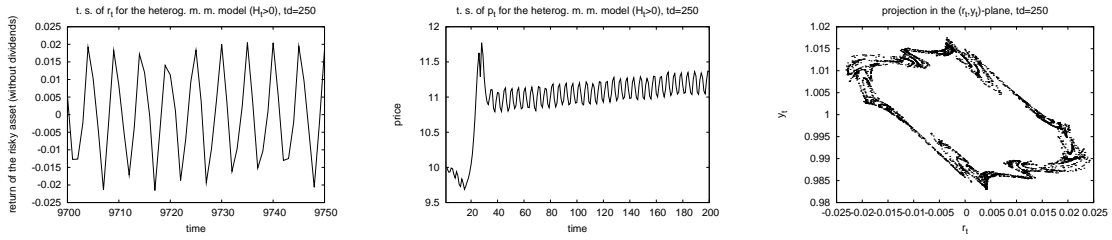
**Figure 4.34:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case) associated to Figure 4.33 (d). The dynamic system stays in a region with 2-cycles, where the fundamentalists dominate the chartists within only a few trading periods. Again the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

in the long run, but almost no risky asset is demanded in the long run (not shown here in detail). Otherwise, a fundamental steady state will only be reached if the dynamic system is initialized almost at this equilibrium.

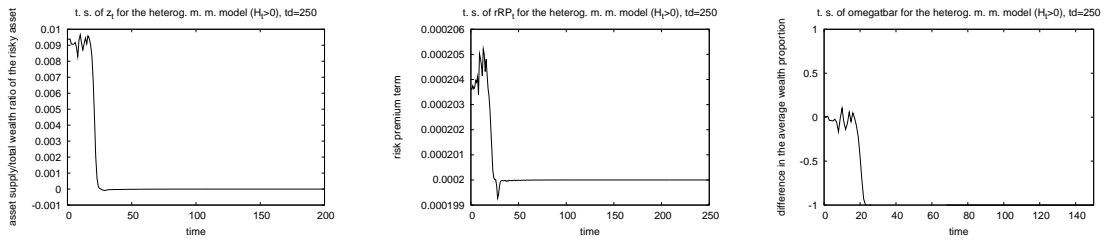
Figure 4.35 exhibits two further experiments for slightly different parameter sets and with slightly different results. While the first experiment assumes a slightly lower decay rate  $\delta$  for the geometric decay process of the sample mean and variance of the risky-asset return used by the chartist group (see Figure 4.35 (a)–(f)), the chartists lower their weighting factor  $\beta_3$  that measures the influence of the sample variance on the conditional variance  $Var_{c,t}(\rho_{t+1})$  of the risky-asset return in the second experiment (see Figure 4.35 (g)–(l)).

Obviously, both parameter sets lead to cyclical movements of risky-asset returns and risky-asset prices that show booms and crashes in rather predictable way. However, the first parameter set provides a more complex dynamics with a strange attractor in the  $(r_t, y_t)$ -plane, while the dynamic system of the second parameter set eventually stays in a region with 5-cycles in the corresponding phase space representation.

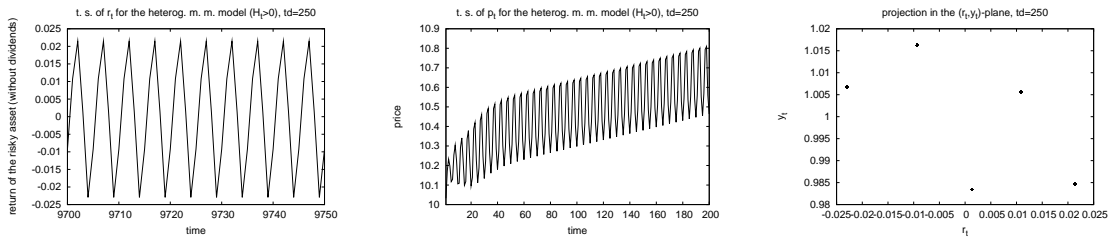
I want to close this subsection with one final remark relating to the optimal investment proportion  $\pi_{i,t}$  in the experiments. Because the evolution of the variables  $\pi_{f,t}$  and  $\pi_{c,t}$  has not been plotted in the figures, it should be stressed that, for the daily treatment of our model framework, the



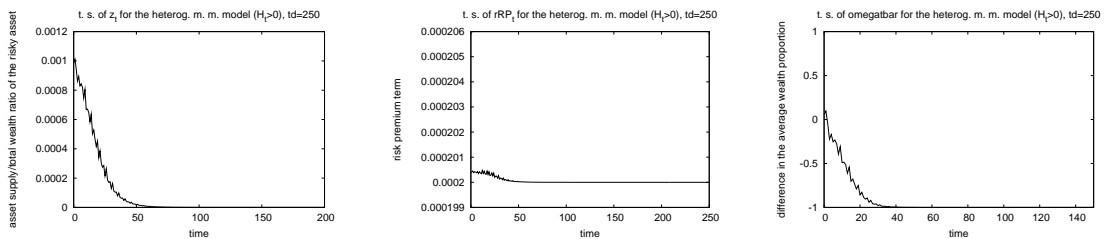
(a) time series of  $r_t$  for  $\lambda = 0.1$ ,  $\delta = 0.25$ ,  $\beta_1 = 0.25$ ,  $\beta_2 = 0.4$  and  $\beta_3 = 5.0$  (daily level) (b) corresponding time series of risky-asset price  $p_t$  with  $p_0 = 100$  (c) corresponding phase space plot



(d) corresponding time series of asset supply/total wealth ratio  $z_t$  (e) corresponding time series of risk-premium term  $r_t^{RP}$  (f) corresponding time series of average wealth proportion  $\bar{\omega}_t$



(g) time series of  $r_t$  for  $\lambda = 0.1$ ,  $\delta = 0.5$ ,  $\beta_1 = 0.25$ ,  $\beta_2 = 0.3$  and  $\beta_3 = 2.0$  (daily level) (h) corresponding time series of risky-asset price  $p_t$  with  $p_0 = 100$  (i) corresponding phase space plot



(j) corresponding time series of asset supply/total wealth ratio  $z_t$  (k) corresponding time series of risk-premium term  $r_t^{RP}$  (l) corresponding time series of average wealth proportion  $\bar{\omega}_t$

**Figure 4.35:** Time series of state variables  $r_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$ , risk-premium term  $r_t^{RP}$  and phase space plot for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case) for two further parameter sets. Both experiments show market outcomes which are rather similar to the limit cycle results depicted in a previous experiment (see Figure 4.32). Again the phase space representations are generated by 10,000 iterations from which the first 5,000 are deleted afterwards. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

realizations of these variables very often reach unrealistically high values (between 5 and 30) which is clearly not sufficiently pronounced for the  $td = 1$  treatment. Furthermore, the magnitudes of  $\pi_{f,t}$  and  $\pi_{c,t}$  differ quite significantly from each other in these simulation runs, where  $\pi_{c,t}$  oscillates with a much larger amplitude.<sup>98</sup> Although I am no expert on the calibration issues for similar model frameworks proposed and investigated by other authors, I might suggest that this problem is somehow a quite general feature of this class of financial market models. For instance, note that a tangens hyperbolicus term is assumed in Chiarella et al. (2006) for the chartist demand function, which keeps the chartist demand in a predefined range. Furthermore, a relatively high risk-aversion coefficient (at least above 6.0) is assumed in this paper, which might also help to restrain the dynamic system and keep it in an appropriate and desired region. Unfortunately, these (sometimes even radically) limiting procedures are not easy to motivate and, in this case, even seem not to work within the model framework. For example, an increase in the risk-aversion coefficient  $\nu$  or an increase in  $\sigma^2$  or  $\beta_3$ , which should lower the demand functions of the investors, is of no advantage in calibrating and smoothing down the model dynamics, but rather leads directly to unrealistic and undesired market behavior as shown in Figure 4.35.

#### *Simulation Results with Noise Terms*

In the final investigation of this thesis a simple stochastic version of our market maker model is considered. At first I will introduce noisy disturbances fourfold in the dynamic equations of our market maker model but afterwards I will switch them on step by step during the investigations of this model variant. More precisely, it is assumed that the dividend process does not grow deterministically but instead randomly, as already indicated by Eq. (4.25). The noise dividend evolves according to

$$d_{t+1} = (1 + g + \sigma_{\epsilon,d}\epsilon_{d,t+1})d_t, \quad (4.149)$$

which means that the fundamental price  $p_{t+1}^*$  evolves according to  $p_{t+1}^* = (1 + g + \sigma_{\epsilon,d}\epsilon_{d,t+1})p_t^*$ , where  $\epsilon_{d,t} \sim \mathcal{N}(0,1)$  are i.i.d. random shocks and  $\sigma_{\epsilon,d} > 0$  represents the standard deviation of the dividend (and fundamental) growth rate. Analogously, we assume that the number of outstanding risky assets grows randomly over time, namely via

$$N_{t+1}^s = (1 + n^s + \sigma_{\epsilon,n}\epsilon_{n,t+1})N_t^s,$$

where again  $\epsilon_{n,t} \sim \mathcal{N}(0,1)$  are i.i.d. random shocks and  $\sigma_{\epsilon,n} > 0$  means the standard deviation of the outstanding risky assets' growth rate (see Eq. (4.48)). Obviously, this directly leads to

$$H_{t+1} \approx (1 + n^s + g + \sigma_{\epsilon,n}\epsilon_{n,t+1} + \sigma_{\epsilon,d}\epsilon_{d,t+1})H_t$$

for the fundamental price path (see Eq. (4.53)) and to

$$H_{t+1} \approx (1 + n^s + r_{t+1} + \sigma_{\epsilon,n}\epsilon_{n,t+1})H_t$$

beyond the fundamental price path respectively (see Eq. (4.106)).

<sup>98</sup>The high values of  $\pi_{f,t}$  and  $\pi_{c,t}$  corresponds to market situations in which 2-cycles or limit cycles occur where for limit cycles, in addition,  $\pi_{c,t} \gg \pi_{f,t}$  is observed.

In addition, two stochastic terms are added to the price setting equation; one of them is already displayed in Eq. (4.58). Remember that the price setting rule is given there by

$$p_{t+1} = p_t [(1 + g) + \lambda(q_t + \epsilon_t)] + \alpha(p_t^* - p_t), \quad (4.150)$$

where  $\epsilon_t$  stands for noisy excess demand in order to include, for instance, the effect of noise traders. For consistency reasons this noise term will here be denoted by  $\sigma_{\epsilon,q}\epsilon_{q,t}$ . In this passage this assumption is shown to be equivalent to

$$r_{t+1} = \frac{p_{t+1} - p_t}{p_t} = g + \lambda(q_t + \sigma_{\epsilon,q}\epsilon_{q,t}) + \alpha(y_t - 1 + \sigma_{\epsilon,y}\epsilon_{y,t}), \quad (4.151)$$

implying that the market maker adjusts the relative return of the risky asset according to the return of dividend  $g$ , the aggregate excess demand proportion  $q_t$ , the noisy demand proportion  $\sigma_{\epsilon,q}\epsilon_{n,t}$  at time  $t$  and a correction term, i.e., the so-called ‘stabilizing term’. However, in contrast to Eq. (4.59), an additional noise term  $\sigma_{\epsilon,y}\epsilon_{y,t}$  is built into the noisy price setting equation (4.151).<sup>99</sup> Again, for both noise terms it is  $\sigma_{\epsilon,q} > 0$ ,  $\sigma_{\epsilon,y} > 0$  and  $\epsilon_{q,t} \sim \mathcal{N}(0, 1)$ ,  $\epsilon_{y,t} \sim \mathcal{N}(0, 1)$  are i.i.d. random disturbances.

To conclude these model modifications, all relevant equations of the noisy dynamic system are collected, resulting in the following nonlinear, stochastic system for the heterogeneous beliefs model with two trading types acting in the model market:

$$r_{t+1} = g + \sigma_{\epsilon,d}\epsilon_{d,t+1} + \lambda(q_t + \sigma_{\epsilon,q}\epsilon_{q,t}) + \alpha(y_t - 1 + \sigma_{\epsilon,y}\epsilon_{y,t}), \quad (4.152)$$

$$y_{t+1} = \frac{(1 + g + \sigma_{\epsilon,d}\epsilon_{d,t+1})(r_t^{RP} - g)}{(1 + r_{t+1})(r_{t+1}^{RP} - g)} y_t, \quad (4.153)$$

$$z_{t+1} = \frac{2(1 + n^s + r_{t+1} + \sigma_{\epsilon,n}\epsilon_{n,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{c,t+1})} z_t, \quad (4.154)$$

$$\phi_{t+1} = \delta\phi_t + (1 - \delta)r_{t+1}, \quad (4.155)$$

$$v_{t+1} = \delta v_t + \delta(1 - \delta)(r_{t+1} - \phi_t)^2, \quad (4.156)$$

$$\bar{\omega}_{t+1} = \frac{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) - [1 - \bar{\omega}_t](1 + \beta_{c,t+1})}{[1 + \bar{\omega}_t](1 + \beta_{f,t+1}) + [1 - \bar{\omega}_t](1 + \beta_{c,t+1})}, \quad (4.157)$$

where

$$q_t = \frac{[(1 + \beta_{f,t})\pi_{f,t} - (1 + r_t)\pi_{f,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} + \frac{[(1 + \beta_{c,t})\pi_{c,t} - (1 + r_t)\pi_{c,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}] - 4(n^s + \sigma_{n,\epsilon}\epsilon_{n,t})z_{t-1}}{(1 + \beta_{f,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{c,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]}$$

and

$$\beta_{i,t+1} = r_f + \pi_{i,t} [r_{t+1} - r_f + (r_t^{RP} - g)y_t], \quad i \in \{f, c\},$$

$$\pi_{f,t} = \frac{1}{\nu\sigma^2} [(g - r_f) - \beta_1(r_t - g) + (r_t^{RP} - g)y_t],$$

$$\pi_{c,t} = \frac{r_t + \beta_2(r_t - \phi_t) + (r_t^{RP} - g)y_t - r_f}{\nu[\sigma^2 + \beta_3 v_t]},$$

$$r_t^{RP} = r_f + \left[ \frac{(1 + \bar{n})[1 + \bar{\omega}_t]}{4\nu\sigma^2} + \frac{(1 - \bar{n})[1 - \bar{\omega}_t]}{4\nu(\sigma^2 + \beta_3 v_t)} \right]^{-1} z_t.$$

<sup>99</sup>The  $\sigma_{\epsilon,y}\epsilon_{y,t}$ -term is not easy to explain in Eq. (4.150) since Eq. (4.150) then has to be written as  $p_{t+1} = p_t[(1 + g) + \lambda(q_t + \sigma_{\epsilon,q}\epsilon_{q,t})] + \alpha(p_t^* - p_t + p_t\sigma_{\epsilon,y}\epsilon_{y,t}) = p_t[(1 + g) + \lambda(q_t + \sigma_{\epsilon,q}\epsilon_{q,t})] + \alpha(p_t^* - p_t(1 - \sigma_{\epsilon,y}\epsilon_{y,t}))$ . However, adding  $\sigma_{\epsilon,y}\epsilon_{y,t}$  in Eq. (4.151) for the stochastic version of the model might be reasonable.

Three further remarks are worth making here. First, we assume that the fundamentalists still believe that the return of the risky asset eventually moves back to its (long-term) average return of dividends  $g$  as formulated in Eq. (4.63)<sup>100</sup> and that  $g$  is known by both trading types. Second, the dividend growth rate  $g$  in the  $(r_t^{RP} - g)$  terms of the dynamic system is left undisturbed. This is done because according to Eq. (4.35) and Eq. (4.43), the fundamental price  $p_t^*$  might easily become negative when the time-dependent dividend return  $g_t$  is assumed to be normally distributed around its long-run value  $g$ . Therefore, we assume that in these corresponding model terms no disturbance of the dividend growth rate is applied, but we do account for the fact that the dividends are not strictly growing deterministically (see Eq. (4.152) and Eq. (4.153)). Third, it is important to remember that, for a daily treatment of the stochastic version of the market maker model, the transformation rules summarized in Eq. (4.148) again have to be applied. Furthermore, the standard deviations of the four newly introduced noise terms have to be scaled appropriately, namely by a factor  $1/\sqrt{td}$ .

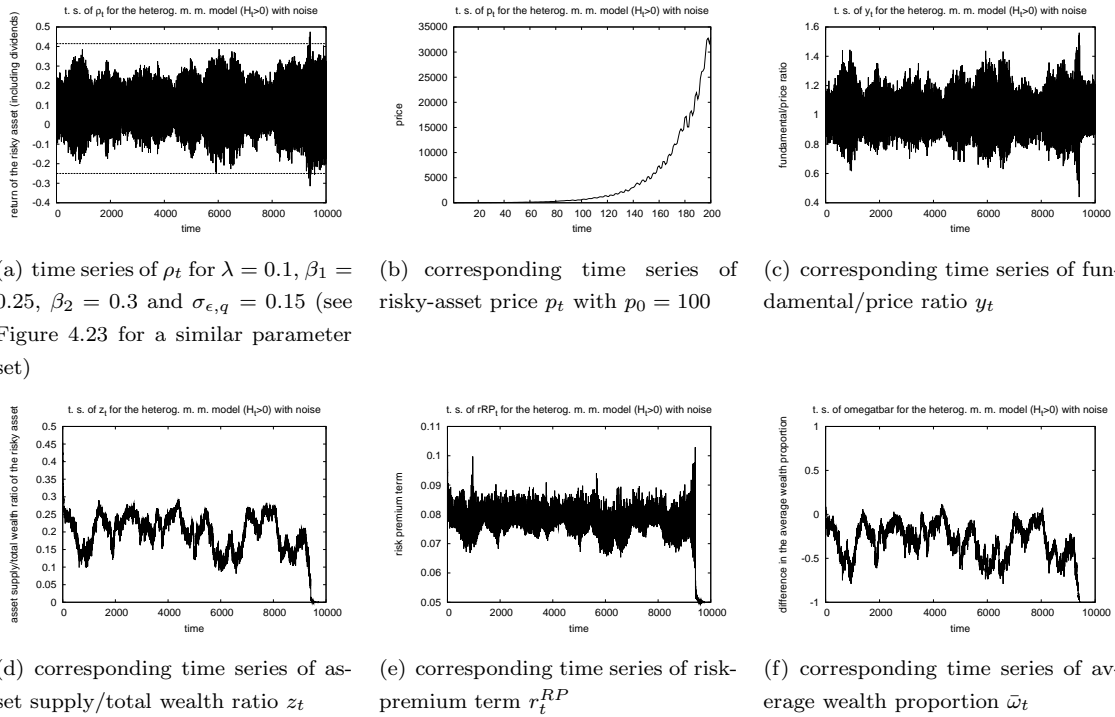
Again, a multitude of experiments for different parameter sets and model specifications has been conducted, the full details of which are not shown here. To summarize the findings, it was found that enabling noise terms leads mostly of the time to model outcomes that significantly deviate from the results obtained for the corresponding deterministic model variant. For instance, if noise is turned on in a (fully deterministic) model version for which a fundamental steady state can be reached and both investor types coexist *pari passu* in the long run with respect to their wealth levels, one group of investors is suddenly favored and they will eventually prevail over their counterparts. In other words there are only a few parameter combinations and model outcomes in which both trading types can survive simultaneously in the market. Interestingly, this result is not influenced by either the specific noise term(s) that is (are) switched on or by the size of the chosen noise level. Neither does disabling short selling have any influence on this result. However, no straightforward rule can be developed with which the long-term outcome of the model can be forecasted.<sup>101</sup> Nevertheless, some overall findings can be stated. On the one hand, the simple stochastic version of our market maker model again shows that, for a huge variety of parameter combinations, no balanced wealth distribution among the two competing investor groups can be obtained. This model outcome is even more pronounced than in the deterministic model versions. On the other hand, whenever the corresponding deterministic model variant leads to a coexistence of both investor types (see Figures 4.23, 4.25 (f) and (l) as well as Figure 4.27), switching on the noise term favors either the fundamentalists or the chartists. Furthermore, for deterministic model variants favoring one group anyway (see for instance limit cycles in Figures 4.24 and 4.26 (a)–(f)), the corresponding noisy model variants do not change this result in almost all cases. In some cases, however, enabling noise can quite clearly slow down the process of one group being ruled out of the market.

Figures 4.36 and 4.37 show two notable model outcomes for two different parameter sets for

<sup>100</sup>These assumptions can be motivated by referring to Eq. (4.36), which remains the same in this more general noisy case. This is due to the fact that it is still  $E_t[d_t + 1] = E_t[(1 + g + \sigma_{\epsilon,d} \epsilon_{d,t+1})d_t] = (1 + g)d_t$ , so that the derivation of Eq. (4.36) can also be taken for this special noisy case. See also Appendix A 4.2 for further details.

<sup>101</sup>Similar to the procedure in the previous simulations, the initialization of the dynamic system is chosen randomly in all experiments. A weaker randomization of the initial conditions of the dynamic system might reduce this effect, but this is not investigated here in further detail.

the  $td = 1$  treatment (i.e., one trading period per year). Figure 4.36 exhibits one of the rather rare situations in which neither fundamentalists nor chartists can dominate the market in the long run. Note that Figure 4.23 uses a similar parameter set to that in Figure 4.36, leading to very



(a) time series of  $\rho_t$  for  $\lambda = 0.1$ ,  $\beta_1 = 0.25$ ,  $\beta_2 = 0.3$  and  $\sigma_{\epsilon,q} = 0.15$  (see Figure 4.23 for a similar parameter set) (b) corresponding time series of risky-asset price  $p_t$  with  $p_0 = 100$  (c) corresponding time series of fundamental/price ratio  $y_t$

(d) corresponding time series of asset supply/total wealth ratio  $z_t$  (e) corresponding time series of risk-premium term  $r_t^{RP}$  (f) corresponding time series of average wealth proportion  $\bar{\omega}_t$

**Figure 4.36:** Time series of state variables  $r_t$ ,  $y_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$  and risk-premium term  $r_t^{RP}$  for the heterogeneous market maker model ( $H_t > 0$  case) with noise. For this parameter set both investor groups coexist in the market although the chartists prevail over the fundamentalists after about 9500 trading periods. Note that, in contrast to the previous figures, the upper left panel exhibits the risky-asset return *including* dividends. The dashed lines refer to the  $\pm 3\sigma$  region. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 1$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

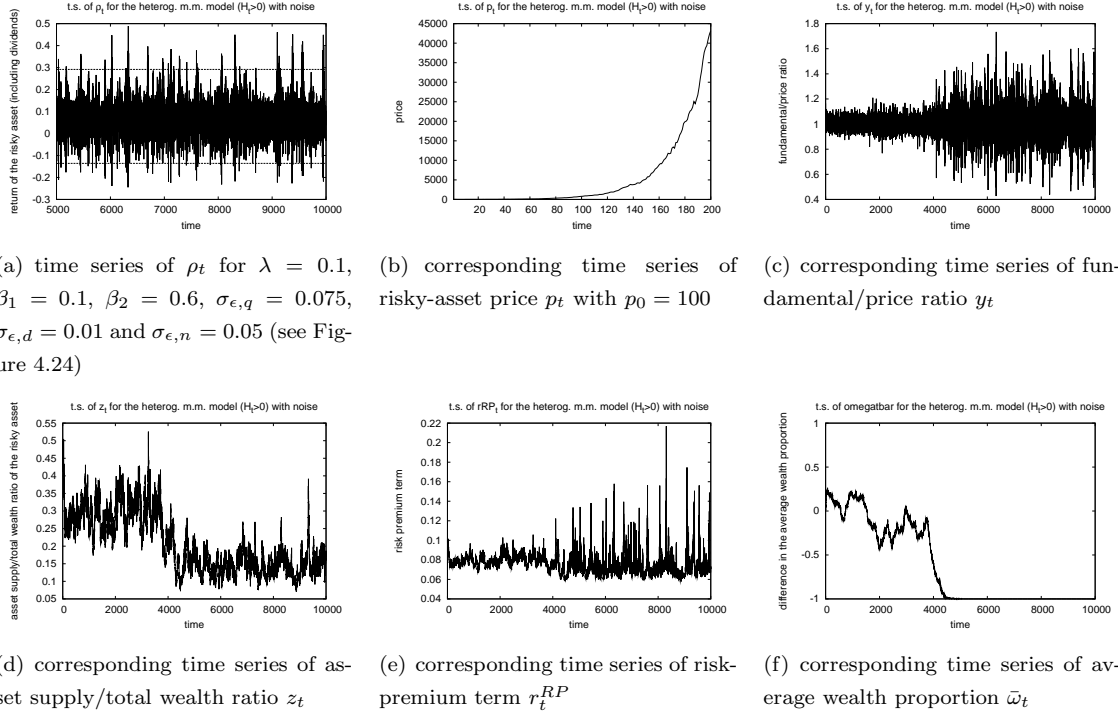
similar time series and phase space plot results as would exist for the dynamic system shown in Figure 4.36 if the noise term was enabled. Hence both plots can be compared anyway although the parameter sets slightly deviate from each other. Interestingly, further experiments indicate<sup>102</sup> that only noise in terms of a noisy excess demand (and/or weak noise in the ‘stabilizing term’ of the price setting equation, i.e.,  $\sigma_{\epsilon,q} > 0$  and/or  $\sigma_{\epsilon,y} > 0$ ) can lead to situations in which both trading types coexist in the market. Contrastingly, enabling noise for the dividend process and/or for the outstanding risky asset’s growth rate always favors one trading group over the other<sup>103</sup>, where the precise model outcome depends on the realized initialization of the underlying dynamic system. As a final remark to Figure 4.36 it is worth mentioning that the time series of the risky-asset returns including dividends, i.e.,  $\rho_t$  (upper left panel), is characterized by rather smooth fluctuations within the  $\pm 3\sigma$  region, neither showing phases of high volatility nor any other ‘anomalies’ which might

<sup>102</sup>Experiments are not shown in detail here.

<sup>103</sup>This result is also independent of the chosen size of the noise level.

give a hint to the ‘stylized facts’ that can be observed in real financial time series.<sup>104</sup>

Figure 4.37 shows another simulation experiment in which three noise terms are enabled, namely the excess demand noise term, the noise term for the dividend process and the noise term for the risky-asset supply. The noise levels are chosen rather low, with  $\sigma_{\epsilon,q} = 0.075$ ,  $\sigma_{\epsilon,d} = 0.01$



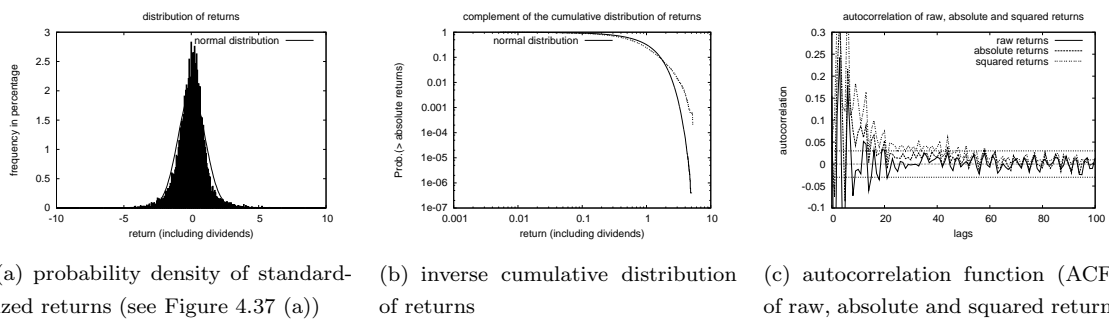
**Figure 4.37:** Time series of state variables  $r_t$ ,  $y$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$  and risk-premium term  $r_t^{RP}$  for the heterogeneous market maker model ( $H_t > 0$  case) with noise associated to Figure 4.24. While for the first 4000 trading periods the fundamentalists can coexist with their competitors, the chartists start to dominate the market in the following periods. Again the risky-asset return *including* dividends is depicted in the upper left panel. The dashed lines refer to the  $\pm 3\sigma$  region. Obviously, the time series of returns shows time-varying fluctuations with phases of low volatility and higher volatility. Note that no short selling has been allowed for this experiment, where the chartists’ investment proportion is even slightly more constrained (see the continuous text for further details). The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 1$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

and  $\sigma_{\epsilon,n} = 0.05$ . Again, the outcome of the noisy dynamic system can be compared with the outcome of the corresponding deterministic model variant which is depicted in Figure 4.24. In both experiments the chartists dominate the market in the long run although the noisy model variant requires many more trading periods for the chartists to achieve the dominance over the fundamentalists. However, this result is enhanced (or manipulated) by the fact that the noisy experiment shown in Figure 4.37 includes a ban on short selling for both trading types, as well as the following assumption for the chartists’ group:<sup>105</sup> in addition to the interdiction of short

<sup>104</sup>See Footnote 39 and the discussion around for further details.

<sup>105</sup>Nevertheless, even without the interdiction of short selling for both trading groups the chartists need more trading periods to dominate the market in the noisy model variant as well as in the corresponding deterministic model version.

selling for both trading types, the chartists' optimal investment proportion  $\pi_{c,t}$  is assumed to be further constrained so that it remains in the interval  $[0.21, 0.89]$ . More precisely, it is assumed that, whenever the chartists derive a higher (lower) optimal investment proportion than 89% (21%) for the current trading period, they adjust their investment proportion for this period to the boundaries by hand mentioned above. Although this assumption seems rather arbitrary, the resulting model outcome is interesting. Comparing the time series of returns of Figures 4.36 and 4.37, it becomes obvious that the risky-asset returns in the latter model version show more clearly time-varying fluctuations with phases of low and high volatility than can be observed in real financial data. It is particularly notable that the  $\pm 3\sigma$  region is breached at regular but not totally predictable intervals. Therefore, Figure 4.38 refers to further investigations of the corresponding return distribution that have already been introduced for an extension of the BH model outlined in Subsection 2.2.3 of Chapter 2. The left panel of Figure 4.38 compares the density function of a standard normal



**Figure 4.38:** Simulation results for returns for the heterogeneous market maker model ( $H_t > 0$  case) with noise associated to Figure 4.37. A deviation from purely Gaussian behavior can clearly be detected (left and middle panel). However, the ACF of absolute and squared returns does not show a significantly lower decay rate than expected from real financial data (right panel, see also Figure 2.13 (a) in Subsection 2.2.3 of Chapter 2). The same data set is used as for the experiment shown in Figure 4.37. The remaining parameters are  $td = 1$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\lambda = 0.1$ ,  $\alpha = 0.5$ ,  $\beta_1 = 0.1$ ,  $\beta_2 = 0.6$ ,  $\beta_3 = 5.0$ ,  $\bar{n} = 0.0$ ,  $\sigma_{\epsilon,q} = 0.075$ ,  $\sigma_{\epsilon,d} = 0.01$  and  $\sigma_{\epsilon,n} = 0.05$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

distribution with the sample probability density function of the normalized returns  $\rho_t$  generated by the simulation experiment shown in Figure 4.37. Obviously, in contrast to a normal distribution which is the selected distribution for the three noise terms in the simple stochastic model version used in this experiment, the return distribution of the simulated model has a higher probability for large events and, consequently, contains more probability mass around the symmetry axis of the distribution. The same result is visualized in a slightly different way in the middle panel of Figure 4.38, where the outer part of the distribution is exhibited. It can clearly be detected that the probability of large returns decays significantly more slowly than under the assumption of a Gaussian distribution with the same standard deviation.<sup>106</sup> Hence the simple stochastic version of the heterogeneous market maker model is able to generate return distributions that share the ‘fat-tail phenomenon’ (or the leptokurtic character) of real financial time series, although these model results are hardly robust against parameter modifications and changes in the size of the chosen

<sup>106</sup>For further details on the description of generating these plots we can refer again to Subsection 2.2.3 of Chapter 2.



noise levels.<sup>107</sup> Furthermore, it must be acknowledged that different ‘stylized facts’ of empirical data from financial markets cannot be reproduced by the artificial market model. For instance, our model does not share the slow decay rate of the autocorrelation function of absolute and squared returns as observed in real financial data (see right panel of Figure 4.38 vs. Figure 2.13 (a) in Subsection 2.2.3 of Chapter 2).

Neither Figure 4.36 nor Figure 4.37 show a time series of risky-asset prices  $p_t$  that looks like price evolutions known from real financial data. Instead, the model price of the risky asset grows exponentially in a very smooth and regular way. To investigate the influence of switching on different noise levels and noise types to the price evolution of the risky asset within the model framework, several experiments are conducted which are summarized in Figures 4.39 and 4.40. Note that for these figures the same parameter set (and the same initialization procedure for the underlying dynamic system) is used as for Figures 4.25/4.26. It comes as no surprise that a higher noise level tends to accompany more realistic price fluctuations showing irregular increases and decreases in the overall model price evolution. It is also quite obvious that adding noise to the dividend process leads to a rather pronounced effect, since the assumption of a growing dividend process is primary responsible for obtaining a growth model in both risky-asset prices and investors’ wealth levels (see Figure 4.39 (a)–(c)).<sup>108</sup> Figure 4.40 depicts the corresponding time series of average wealth proportion  $\bar{\omega}_t$ . Since even for the deterministic model variant the presence of SDIC can clearly be detected (in particular for this parameter set), leading to different model outcomes with respect to the eventual wealth distribution among the two trading types (see Figure 4.25 (f) and (l) as well as Figure 4.26 (f)), the corresponding results for the noisy model variant is shown simply for the sake of completeness here.<sup>109</sup> More precisely, no well-founded rule can be established from these plots (or, more precisely, from these experiments). However, note that enabling noise might favor the fundamentalists slightly more since, in general (i.e., in all experiments that have been conducted during the preparation of this subsection), they seem to dominate the chartists more often than in the corresponding deterministic model variants.

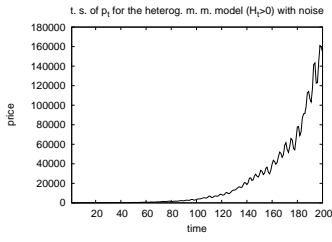
Finally, Figure 4.41 refers to a model variant where 250 trading days per year are assumed. Again, a short selling ban is assumed for both trading types in order to stabilize the simulation results.<sup>110</sup> Nevertheless, as for the deterministic model variant outlined in Subsubsection 4.5.5, the simulation results for the daily scaled market maker model are rather disappointing all in all. For instance, the optimal investment proportion again reaches extremely high values which are very unrealistic and often lead to model outcomes which are unrealistic. Therefore, I do not want to exhibit more than one typical experiment that has been conducted for the daily model treatment here. Again, the risky-asset price is growing in a rather regular way, where both trading types can eventually survive in the market. Furthermore, the risky-asset return (including dividends) is

<sup>107</sup>This unwanted model feature has also been observed in the modified BH model (see Subsection 2.2.3 of Chapter 2).

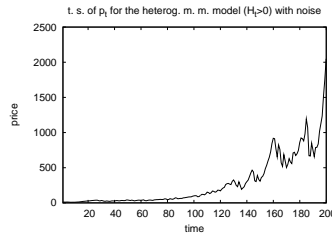
<sup>108</sup>Experiments were also conducted in which no short selling was allowed (not shown in detail here). The findings in these simulation runs indicate that the interdiction of short selling diminishes the occurrence of irregular price fluctuations; this seems a rather obvious result.

<sup>109</sup>Nevertheless, note the different scales used for the  $x$ -axis in Figure 4.40.

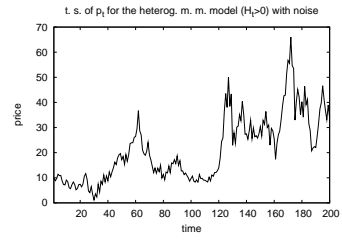
<sup>110</sup>In contrast to the experiment shown in Figure 4.37, no further constraints with respect to the optimal investment proportion are assumed for the chartist group in this experiment.



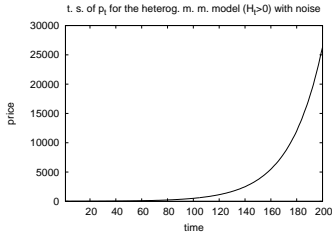
(a) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,d} = 0.03$



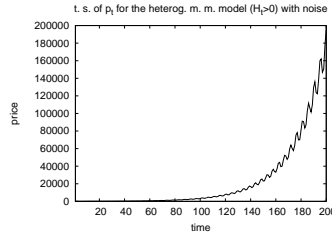
(b) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,d} = 0.07$



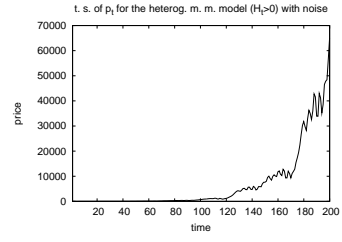
(c) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,d} = 0.15$



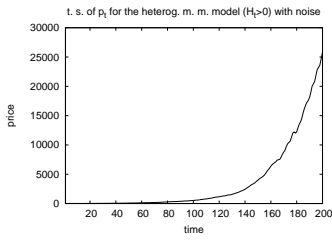
(d) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,q} = 0.03$



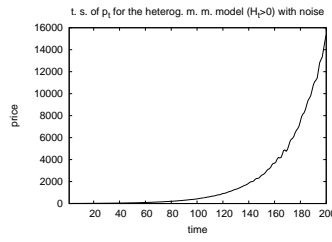
(e) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,q} = 0.07$



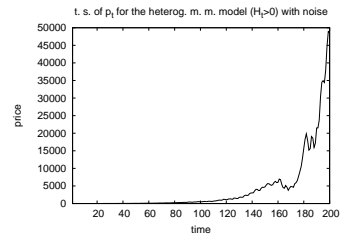
(f) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,q} = 0.15$



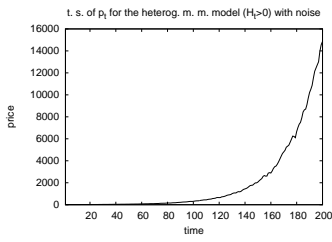
(g) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,n} = 0.03$



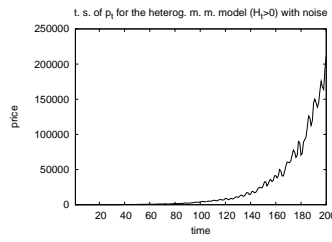
(h) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,n} = 0.07$



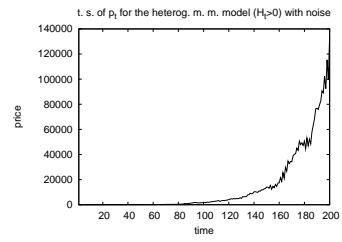
(i) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,n} = 0.15$



(j) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,y} = 0.03$

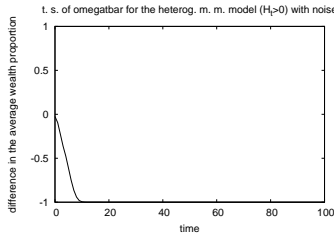


(k) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,y} = 0.07$

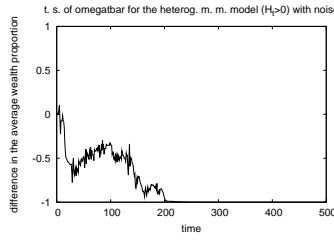


(l) time series of risky-asset price  $p_t$  with  $p_0 = 100$  for  $\sigma_{\epsilon,y} = 0.15$

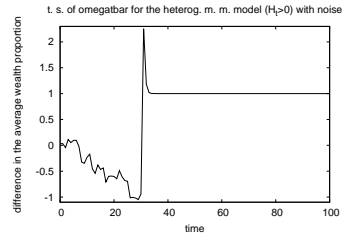
**Figure 4.39:** Time series of risky-asset prices  $p_t$  for different noise types and noise levels for the heterogeneous market maker model ( $H_t > 0$  case). Although the evolution of the risky-asset price  $p_t$  also clearly depends on the precise initialization of the underlying dynamic system (not shown here), an increase in the noise level generally leads to a more unpredictable and more realistic price pattern for all noise types. Obviously, enabling noise for the dividend process seems to affect the time evolution of risky-asset prices quite considerably. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 1$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\lambda = 0.05$ ,  $\alpha = 0.5$ ,  $\beta_1 = 0.25$ ,  $\beta_2 = 0.4$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .



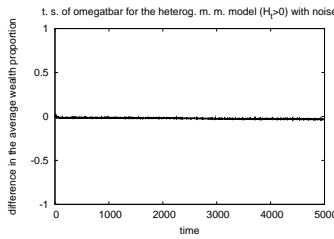
(a) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,d} = 0.03$



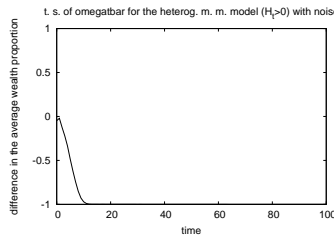
(b) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,d} = 0.07$



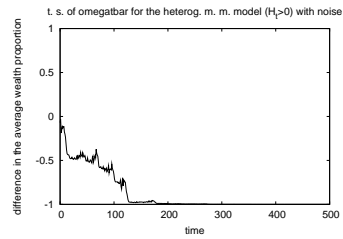
(c) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,d} = 0.15$



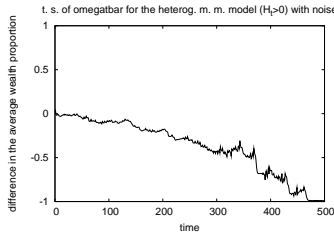
(d) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,q} = 0.03$



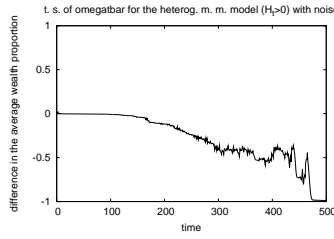
(e) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,q} = 0.07$



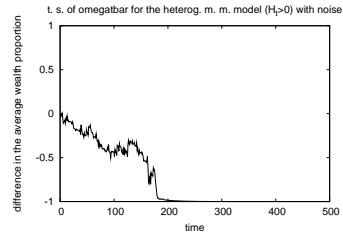
(f) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,q} = 0.15$



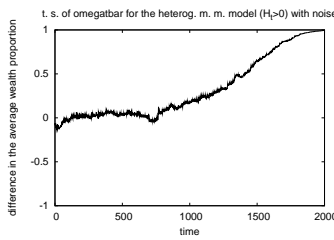
(g) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,n} = 0.03$



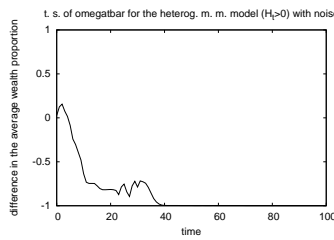
(h) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,n} = 0.07$



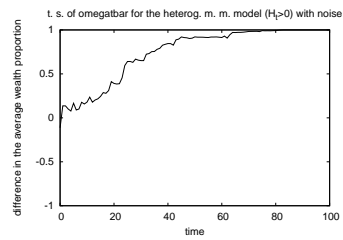
(i) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,n} = 0.15$



(j) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,y} = 0.03$

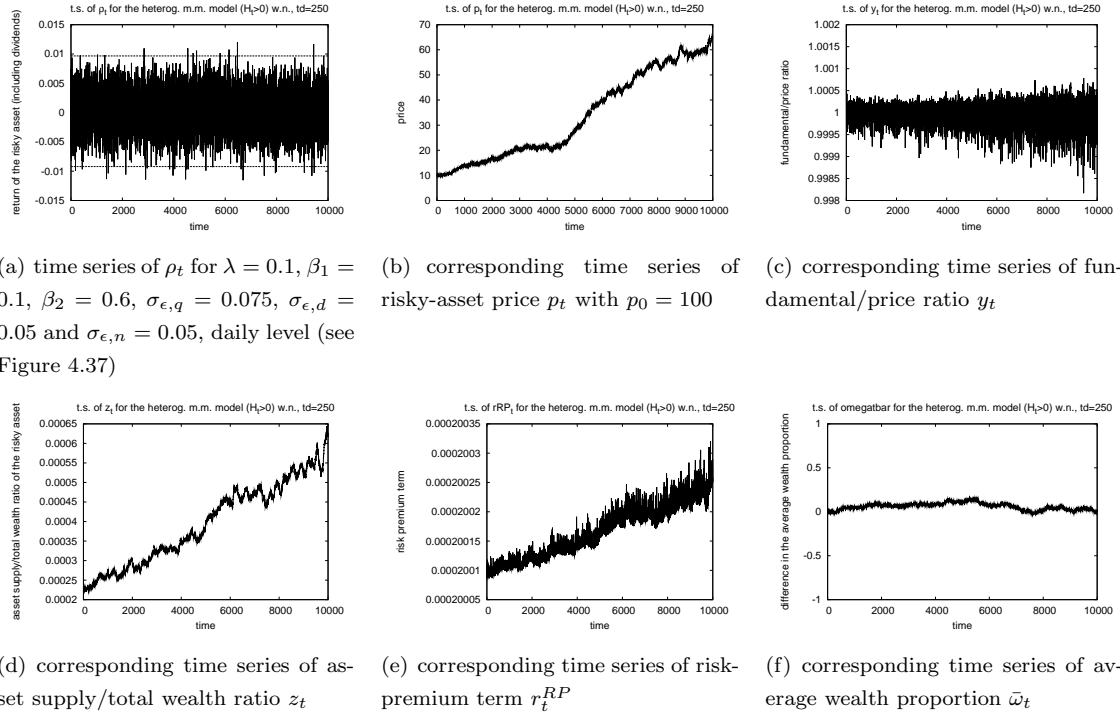


(k) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,y} = 0.07$



(l) time series of average wealth proportion  $\bar{\omega}_t$  for  $\sigma_{\epsilon,y} = 0.15$

**Figure 4.40:** Corresponding time series of average wealth proportion  $\bar{\omega}_t$  for Figure 4.39. Depending on the precise initialization of the underlying dynamic system (not shown here) and the chosen noise type and noise level the chartists (or, less frequently, the fundamentalists) dominate the market in the long run. The same data sets are used as for the experiments shown in Figure 4.39. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 1$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\lambda = 0.05$ ,  $\alpha = 0.5$ ,  $\beta_1 = 0.25$ ,  $\beta_2 = 0.4$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .



**Figure 4.41:** Time series of state variables  $r_t$ ,  $y_t$ ,  $z_t$ ,  $\bar{\omega}_t$  and corresponding risky-asset price  $p_t$  and risk-premium term  $r_t^{RP}$  for the heterogeneous market maker model scaled on a daily basis with 250 trading days per year ( $H_t > 0$  case) with noise associated to Figure 4.37. Fundamentalists and chartists can coexist in the market in the long run. Again the risky-asset return *including* dividends is depicted in the upper left panel. The dashed lines refer to the  $\pm 3\sigma$  region. The time series of returns show no time-varying fluctuations with phases of low volatility and higher volatility. Note that no short selling has been allowed for this experiment. The dynamic variables are randomly initialized in all plots. The remaining parameters are  $td = 250$ ,  $\nu = 1.2$ ,  $r_f = 0.05$ ,  $g = 0.04$ ,  $\sigma = 0.2$ ,  $\delta = 0.5$ ,  $n^s = 0.03$ ,  $\alpha = 0.5$ ,  $\beta_3 = 5.0$  and  $\bar{n} = 0.0$ . The initial condition for  $\bar{\omega}_t$  is  $\bar{\omega}_0 = 0.0$ .

smoothly fluctuating around its mean, hardly ever breaching the  $\pm 3\sigma$  region.

## 4.6 Summary

This chapter presents a structural heterogeneous agents model of an artificial financial market under a market maker scenario in a discrete time setup. Due to a CRRA expected utility optimization framework used by each single investor, the time evolution of the basic state variables of the resulting dynamic system (such as asset prices and total wealth of the market participants) become growing processes over time and thus have to be appropriately transformed to obtain a stationary model. As already shown in Chiarella and He (2001), such a stationary model can be determined in terms of the return of the risky asset, the (average) wealth proportions among different investor groups and, newly introduced here, the ‘fundamental/price ratio’ which measures the deviation of the current price level of the risky asset from its underlying fundamental price (see also Chiarella et al., 2006). Heterogeneity among agents is introduced by assuming them to form different conditional expectations about the mean and the variance of the return of the risky asset. Two different investor ‘prototypes’ are considered in our artificial market: fundamentalists,

who believe in the eventual reversion of the current asset price back to its fundamental value, and chartists, who extrapolate the future return of the risky asset from historical data. In addition, an explicitly modeled supply side of the risky assets is incorporated into the model framework, directly leading to a ‘risk-premium term’ which becomes a non-vanishing and time-dependent quantity in the artificial market. It is endogenously determined by the risky-asset price evolution and the market behavior of the agents. To account for this fact it is necessary to introduce a further state variable called the ‘asset supply/total wealth ratio’. This stationary variable is mainly controlled by the growth rate of the number of outstanding assets, the risky-asset return and the investors’ growth rate of wealth. Furthermore, it itself mainly controls the risk-premium term. Contrastingly, when an asset supply of zero is assumed, the model-inherent risk-premium term vanishes and the model falls back to a version that shares some more similarities with the model framework outlined in Chiarella et al. (2006). A comprehensive analytic and numerical study of the market maker model for a variety of different market situations is performed. In particular, it includes both homogeneous model versions as well as model variants with noise terms. While the first step sees an investigation of a simplified model version with zero supply of outside stocks, the second step considers the fully-developed market maker model with a positive risky-asset supply. The results of this chapter can be summarized as follows:

- The proposed model is able to embed the fundamental pricing scheme as a special case into a more general heterogeneous multi-agent types model framework. In the fundamental steady state all agents use the rational valuation formula for stock prices to derive their demand functions for the risky-asset return. Hence building blocks well known from the classical quantitative pricing literature are combined with more advanced modeling features, such as allowing for interim deviations of the risky-asset price from its fundamental value, the observation of the wealth evolution of the market participants, and an explicit consideration of the market-inherent and endogenously driven risk premium which market participants have to be offered to compensate them for undertaking risky financial prospects.
- For all model variants a ‘quasi’-unique fundamental steady state or even a continuum of fundamental steady states exists where the risky-asset return (ex dividends) equals the growth rate of the assumed dividend process. In such market environments the price of the risky asset increases along its fundamental price path for all time periods.
- For the simplified model version in a homogeneous-beliefs model variant with either only fundamentalists or only chartists in the market, the resulting dynamics is already rather multifaceted. On the one hand, convergence to the fundamental steady state occurs for certain parameter combinations where the system might undergoes flip and Hopf bifurcations if the agents’ extrapolation parameter is changed appropriately. On the other hand, even more complex dynamics is possible, leading to the occurrence of periodic behavior and limit cycles.
- For the simplified model version in a heterogeneous-beliefs model variant with fundamentalists and chartists competing against each other in the market, the investors’ average wealth proportion enters as a new model variable into the focus of investigations. The dynamic

system again converges to its fundamental steady states for a quite large range of parameter combinations. In such situations both investor types always survive in the market in the long run. Furthermore, the heterogeneous beliefs model is also able to exhibit 2-cycle behavior and, albeit less frequently, high-order cycles or limit cycles. Whenever cyclical dynamics occurs, the chartists eventually dominate the market. Parameter sets for which the fundamentalists prevail over the chartists are almost impossible to find. The same is true for the occurrence of cyclical movements of wealth shares between the two investor types.

- An adaptive model version where the chartists' expectation formation is revised does not change the results significantly, although the resulting dynamics becomes even more complex, and convergence to the fundamental steady states can be observed less frequently. Nevertheless, numerical experiments indicate that the adaptive model seems to be more robust with respect to parameter changes than the heterogeneous beliefs model with the original chartists' expectation formation.
- The fully-developed market maker model which assumes a non-zero asset supply in the market shows similar results to the adaptive model version with respect to its robustness of model outcomes and the resulting long-term wealth distribution among fundamentalists and chartists. While long-term dynamics leading to limit cycles again favors the chartists' group in general, the fundamentalists are mostly not able to perform significantly better than their counterparts, as was the case before. Instead, the best result they can achieve is to survive *pari passu* in the market in the long run, or at least to reckon with cyclical movements of their wealth share. As a *new* observation in this revised model variant, the dynamics shows a rather pronounced presence of 'sensitive dependence on initial conditions' (SDIC) and 'sensitive dependence on parameters (SDP)', which is a widely known characteristic of nonlinear chaotic systems. Obviously, this new model behavior is mainly caused by the increased complexity of the underlying nonlinear difference equation system, which describes the market behavior of the market participants in a quite detailed and economically motivated way.
- The attempt to scale the model to daily trading periods (e.g., 250 trading days per day) does not lead to promising and presentable results.<sup>111</sup> For a wide range of reasonable parameter sets the model shows a 'convergence' to market situation in which the investors have almost no demand for the risky asset. However, numerical experiments can be created for the daily model treatment, in which the fundamentalists are able to rule out the chartist group and eventually dominate the market.
- For some simple stochastic versions of the fully-developed market maker model, model outcomes can be generated which resemble, at least visually, real financial time series of risky-asset prices and returns. However, scaling the stochastic model to a scale of daily trading periods leads to disappointing results, as already observed for the deterministic model variant.

In summary, the proposed market maker model in this chapter can be classified in the line of model frameworks that has been presented and reproduced in the preceding two chapters. While it shares

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<sup>111</sup>See also the summarizing Section 3.8 of Chapter 3 and Footnote 51 given therein.

some general ideas of the BH model framework (see Sections 2.1 and 2.2 in Chapter 2) as well as of the CH model (see Chapter 3), it also provides further insights into these kind of financial market models since it explicitly models the wealth evolution of the market participants (like the CH model), but also eliminates some crucial limitations and inconsistencies within the CH model framework. For instance, it assumes model-consistent expectations of all market participants on the fundamental price path of the risky asset. Furthermore, it takes into account the risky-asset supply, which directly leads to the appearance of a risk-premium term in the model framework which is endogenously driven by the market over time. However, the resulting dynamics does not change significantly in terms of possible model outcomes. Moreover, the model presented in this chapter fails to solve any of the frequently asked open issues and unsolved problems for these model types such as the calibration to different time scales in general and the calibration to real financial market data in particular.

## Appendix to Chapter 4

### A 4.1 Derivation of Eq. (4.21) and Eq. (4.22)

From Eq. (4.10) and Eq. (4.20) we have

$$q_t = \frac{\check{\zeta}_t}{W_t} = \frac{\sum_{i=1}^2 W_{i,t-1} [(1 + \beta_{i,t})\pi_{i,t} - (1 + r_t)\pi_{i,t-1}] - [H_t - (1 + r_t)H_{t-1}]}{W_t}. \quad (\text{A } 4.1)$$

Using Eq. (4.14) and Eq. (4.16) it follows that the right-hand side expression of Eq. (A 4.1) can be written as

$$\begin{aligned} & \frac{\sum_{i=1}^2 W_{i,t-1} [(1 + \beta_{i,t})\pi_{i,t} - (1 + r_t)\pi_{i,t-1}] - [H_t - (1 + r_t)H_{t-1}]}{W_t} \\ &= \frac{\sum_{i=1}^2 [(1 + \beta_{i,t})\pi_{i,t} - (1 + r_t)\pi_{i,t-1}]W_{i,t-1} - [H_t - (1 + r_t)H_{t-1}]}{\sum_{i=1}^2 (1 + \beta_{i,t})W_{i,t-1}} \\ &= \frac{\sum_{i=1}^2 [(1 + \beta_{i,t})\pi_{i,t} - (1 + r_t)\pi_{i,t-1}]N_i\bar{W}_{i,t-1} - [H_t - (1 + r_t)H_{t-1}]}{\sum_{i=1}^2 (1 + \beta_{i,t})N_i\bar{W}_{i,t-1}}, \end{aligned}$$

from which Eq. (4.21) can directly be concluded by multiplying nominator and denominator with  $1/I$  and  $1/\bar{W}_{t-1}$ .

The calculation of Eq. (4.22) is very similar to the calculation of Eq. (3.23) in Theorem 3.3, Chapter 3. According to Eq. (4.6) it is

$$\bar{\omega}_{i,t+1} = \frac{\bar{W}_{i,t+1}}{\bar{W}_{t+1}} = \frac{\bar{W}_{i,t}(1 + \beta_{i,t+1})}{\bar{W}_{t+1}} = \frac{\bar{\omega}_{i,t}(1 + \beta_{i,t+1})}{\bar{W}_{t+1}/\bar{W}_t}, \quad (\text{A } 4.2)$$

where the denominator can be written by

$$\frac{\bar{W}_{t+1}}{\bar{W}_t} = \frac{\bar{W}_{1,t}(1 + \beta_{1,t+1}) + \bar{W}_{2,t}(1 + \beta_{2,t+1})}{\bar{W}_t} = \bar{\omega}_{1,t}(1 + \beta_{1,t+1}) + \bar{\omega}_{2,t}(1 + \beta_{2,t+1}). \quad (\text{A } 4.3)$$

Combining Eq. (A 4.2) and Eq. (A 4.3) directly leads Eq. (4.22).

### A 4.2 Derivation of Eq. (4.35)

As already outlined in the discussion on the BH model framework (see Subsection 2.1.1, Chapter 2), the fundamental market equilibrium equation Eq. (4.32) has a unique fundamental solution

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[d_{t+k}]}{(1 + r_f)^k}$$

if the ‘transversality condition’ holds (see Eq. (2.7) for details). Since the dividend growth process in the deterministic skeleton is assumed to evolve according to

$$d_{t+1} = (1 + g)d_t ,$$

we have

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[d_{t+k}]}{(1 + r_f)^k} = \sum_{k=1}^{\infty} \left( \frac{1 + g}{1 + r_f} \right)^k d_t = \left[ \sum_{k=0}^{\infty} \left( \frac{1 + g}{1 + r_f} \right)^k - 1 \right] d_t = \left[ \frac{1}{1 - \frac{1+g}{1+r_f}} - 1 \right] d_t ,$$

which leads to

$$p_t^* = \left[ \frac{1 + r_f}{g - r_f} - 1 \right] d_t ,$$

from which Eq. (4.35) directly follows.

### A 4.3 Proof of Theorem 4.2

Let  $r^* = r_{t+1} = r_t = r_{t-1}$  and  $y^* = y_{t+1} = y_t = y_{t-1}$  be the steady state of the system Eq. (4.66)–Eq. (4.67). Then, a necessary (and sufficient) condition for the steady state to exist is determined by

$$\begin{aligned} r^* &= g + \lambda q^* + \alpha(y^* - 1) , \\ y^* &= \frac{1 + g}{1 + r^*} y^* , \end{aligned} \tag{A 4.4}$$

which implies

$$\left( y^* \neq 0 \wedge y^* \left( 1 - \frac{1 + g}{1 + r^*} \right) = 0 \right) \quad \vee \quad (y^* = 0 \wedge r^* = g + \lambda q^* - \alpha) , \tag{A 4.5}$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^*}{1 + \beta_f^*} , \quad \beta_f^* = r_f + \pi_f^* \{ (r^* - r_f) + (r_f - g)y^* \}$$

and

$$\pi_f^* = \frac{1}{\nu\sigma^2} \{ (r_f - g)(y^* - 1) - \beta_1(r^* - g) \} .$$

#### (i) Analysis of the Left-Hand Side of Eq. (A 4.5)

From the condition on the left-hand side of Eq. (A 4.5), the steady state  $E_1$  can be determined in the following way. Firstly, it is easy to see that  $r^* = g$ , which implies

$$q^* = \frac{(\beta_f^* - g)\pi_f^*}{1 + \beta_f^*} , \quad \beta_f^* = r_f + \pi_f^* \{ (r_f - g)(y^* - 1) \} , \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{ (r_f - g)(y^* - 1) \} .$$

Using the abbreviating notation  $C \equiv \nu\sigma^2$ ,  $\Delta \equiv r_f - g$  and  $X \equiv y^* - 1$ , it follows that

$$\beta_f^* = r_f + \frac{\Delta^2}{C} X^2 \quad \text{and} \quad \pi_f^* = \frac{\Delta}{C} X$$

and thus

$$q^* = \frac{\left( r_f + \frac{\Delta^2}{C} X^2 - g \right) \frac{\Delta}{C} X}{1 + r_f + \frac{\Delta^2}{C} X^2} . \tag{A 4.6}$$

Inserting Eq. (A 4.6) in Eq. (A 4.4) leads to

$$-\frac{\alpha}{\lambda} X = \frac{\left( \Delta + \frac{\Delta^2}{C} X^2 \right) \frac{\Delta}{C} X}{1 + r_f + \frac{\Delta^2}{C} X^2} \tag{A 4.7}$$

which can easily be solved and yields

$$X = \pm \sqrt{\frac{-\{\alpha C^2(1 + r_f) + \Delta^2 C \lambda\}}{\Delta^2 \{\Delta \lambda + \alpha C\}}} .$$



However, note that  $y^* \neq 1$  has to be assumed for this manipulation because Eq. (A 4.7) can be divided by  $X$ , which is defined by  $X \equiv y^* - 1$  as stated already above. Since  $\alpha, \lambda, C, \Delta$  are all positive parameters, the square root has no real solution, which implies that the assumption  $y^* \neq 1$  does not lead to a reasonable result for our economic model. However, the assumption  $y^* = 1$  (which means  $X = 0$ ) does lead to a meaningful result and directly implies that  $q^* = 0$  (i.e., no market excess demand in the steady state),  $\pi_f^* = 0$  and  $\beta_f^* = r_f$ .

(ii) *Analysis of the Right-Hand Side of Eq. (A 4.5)*

Finally, the condition on the right-hand side of Eq. (A 4.5) has to be investigated, i.e., the case  $y^* = 0$ . This directly implies that

$$r^* = g + \lambda q^* - \alpha ,$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^*}{1 + \beta_f^*} , \quad \beta_f^* = r_f + \pi_f^*(r^* - r_f) , \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{(g - r_f) - \beta_1(r^* - g)\} . \quad (\text{A 4.8})$$

Unfortunately, this system of equations is analytically almost intractable. For this reason, the possible steady states for  $y^* = 0$  are numerically investigated for a wide range of reasonable parameter sets with the help of the software package MATHEMATICA 7. The chosen parameter sets enclose the following parameter ranges:

$$\begin{aligned} \nu = 1.2 , & \quad 0.05 \leq \sigma \leq 1.15 , & \quad 0.02 \leq r_f \leq 0.18 , & \quad 0.02 \leq g \leq 0.18 , \\ 0.1 \leq \beta_1 \leq 0.9 , & \quad 0.05 \leq \lambda \leq 0.1 , & \quad 0.1 \leq \alpha \leq 0.9 . \end{aligned}$$

Note that, according to Eq. (A 4.8), the parameters  $\nu$  and  $\sigma$  always occur as a factor  $\nu\sigma^2$ , so that only one of both parameters has to be changed within the parameter sets. An extensive numerical analysis reveals that *almost no* economically reasonable (non-fundamental) steady state exists within these parameter ranges. Only for very low choices of  $\sigma$  does one economically reasonable (non-fundamental) steady state exist which, however, is always accompanied by negative values of  $\pi_f^*$  and a return (ex dividends) of the risky asset slightly higher than the risk-free rate. Note that the following conditions have to be fulfilled to arrive at an economically reasonable (non-fundamental) steady state:

$$\beta_f^* \geq 0 , \quad r_f > g , \quad r^* \geq -1 \text{ (note that } r^* < -1 \text{ leads to negative prices) ,}$$

as well as

$$r^* , q^* , \beta_f^* , \pi_f^* \in \mathbb{R} \quad \text{and} \quad -(g \geq r^* \wedge r^* > -1) .$$

The last condition has to be fulfilled because Eq. (4.67) states that the growth rate of the state variable  $y_t$  is given by  $(1+g)/(1+r_{t+1})$  and that  $y_0 > 0$  is used as an initial condition for the dynamics of  $y_t$  in all the investigations of the underlying dynamic system. This means that a steady state with  $y^* = 0$  cannot be reached as long as  $g \geq r^*$  and  $r^* > -1$  because it is then  $y_{t+1} \geq y_t > 0$  for all  $t$ . This completes the proof.

#### A 4.4 Proof of Theorem 4.4

Since determining analytically the local stability property of the fundamental steady state  $E_1$  is rather complex even in the most simple homogeneous case with only fundamentalists in the market, the analysis

of the stability properties of the homogeneous model is performed with the assistance of the software package MATHEMATICA 4.1, which provides analytical as well as numerical solutions.

In the first step the system Eq. (4.66)–Eq. (4.67) is equivalently transformed into its canonical form of first-order difference equations.<sup>112</sup> By introducing a new set of variables:

$$a_{t+1} = r_{t+1}, \quad b_{t+1} = r_t, \quad c_{t+1} = y_{t+1}, \quad d_{t+1} = y_t,$$

the system Eq. (4.66)–Eq. (4.67) can be rewritten as

$$\begin{aligned} a_{t+1} &= g + \lambda Q(\beta_{f,t}, \pi_{f,t}, \pi_{f,t-1}, a_t) + \alpha(c_t - 1), \\ b_{t+1} &= a_t, \\ c_{t+1} &= \frac{1+g}{1+g + \lambda Q(\beta_{f,t}, \pi_{f,t}, \pi_{f,t-1}, a_t) + \alpha(c_t - 1)} c_t, \\ d_{t+1} &= c_t, \end{aligned}$$

where  $Q(\cdot, \cdot, \cdot, \cdot)$  is defined as

$$Q(u, v, w, x) \equiv \frac{(1+u)v - (1+x)w}{1+u}.$$

Note that  $Q(\beta_{f,t}, \pi_{f,t}, \pi_{f,t-1}, a_t) = \text{fct.}(r_t, r_{t-1}, y_t, y_{t-1})$  or, using the new set of variables,  $Q(\beta_{f,t}, \pi_{f,t}, \pi_{f,t-1}, a_t) = \text{fct.}(a_t, b_t, c_t, d_t)$ .

Now, in the second step, the Jacobian of the fundamental steady state  $E_1$  can easily be determined as

$$J_1 \equiv J_{|(r^*=g, y^*=1)} = \begin{pmatrix} -\frac{\beta_1 \lambda}{\nu \sigma^2} & \frac{\beta_1 \lambda (1+g)}{(1+r_f) \nu \sigma^2} & \alpha + \frac{\lambda(r_f - g)}{\nu \sigma^2} & \frac{\lambda(1+g)(g-r_f)}{(1+r_f) \nu \sigma^2} \\ 1 & 0 & 0 & 0 \\ \frac{\beta_1 \lambda}{(1+g) \nu \sigma^2} & -\frac{\beta_1 \lambda}{(1+r_f) \nu \sigma^2} & 1 - \frac{\alpha + \frac{\lambda(r_f - g)}{\nu \sigma^2}}{1+g} & -\frac{\lambda(g-r_f)}{(1+r_f) \nu \sigma^2} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

from which the characteristic equation can be derived as

$$\begin{aligned} \Gamma_1(\xi) \equiv \det(\xi \mathbb{I} - J_1) &= \frac{\xi}{(1+g)(1+r_f) \nu \sigma^2} \{ \lambda [\xi(1+r_f) - (1+g)] [\beta_1(\xi-1)(1+g) + \xi(r_f-g)] \\ &\quad + \xi^2(1+r_f) [\alpha + (\xi-1)(1+g)] \nu \sigma^2 \} \end{aligned} \quad (\text{A } 4.9)$$

by using the Laplace's expansion rule. Simple transformations show that Eq. (A 4.9) is equivalent to Eq. (4.72). Note that  $\Gamma_1(\xi)$  is a polynomial of order four with one trivial eigenvalue  $\xi_1 = 0$  and three further non-trivial eigenvalues  $\xi_2, \xi_3, \xi_4$  which are generally non-zero and are given by the solution of the expression in the curly brackets of Eq. (4.72). This completes the proof.

#### A 4.5 Proof of Theorem 4.5

According to Eq. (A 4.9) it is

$$\Gamma_1(1) = \frac{\lambda(r_f - g)^2 + \alpha(1+r_f) \nu \sigma^2}{(1+g)(1+r_f) \nu \sigma^2},$$

which is non-zero for  $\alpha \neq 0$ ,  $\lambda \neq 0$  and  $r_f > g$ . Therefore, from

$$\Gamma_1(-1) = \frac{-\lambda(-2-r_f-g) [-2\beta_1(1+g) - (r_f-g)] - (1+r_f) [\alpha - 2(1+g)] \nu \sigma^2}{(1+g)(1+r_f) \nu \sigma^2}$$

it can be concluded that a flip bifurcation occurs at  $\beta_1^*$ , which is given by Eq. (4.73). This completes the proof.

<sup>112</sup>See Medio and Lines (2001, Chapter 1) for further details on this procedure.

## A 4.6 Proof of Theorem 4.6

Let  $r^* = r_{t+1} = r_t = r_{t-1} = r_{t-2}$  and  $y^* = y_{t+1} = y_t = y_{t-1}$  be the steady state of the system Eq. (4.74)–Eq. (4.75). The ‘quasi’ uniqueness of the fundamental steady state  $E_1$  can be proven using a very similar procedure, as already applied in the proof of Theorem 4.2 in Appendix A 4.3. Again, the necessary (and sufficient) condition for the steady state to exist can be expressed by

$$\left( y^* \neq 0 \wedge y^* \left( 1 - \frac{1+g}{1+r^*} \right) = 0 \right) \quad \vee \quad (y^* = 0 \wedge r^* = g + \lambda q^* - \alpha) , \quad (\text{A 4.10})$$

where

$$q^* = \frac{(\beta_c^* - r^*)\pi_c^*}{1 + \beta_c^*} , \quad \beta_c^* = r_f + \pi_c^* \{ (r^* - r_f) + (r_f - g)y^* \}$$

and

$$\pi_c^* = \frac{(r^* - r_f) + (r_f - g)y^*}{\nu [\sigma^2 + \beta_3 \{ (r^* - r_f) + (r_f - g)y^* \}^2]} .$$

*(i) Analysis of the Left-Hand Side of Eq. (A 4.10)*

From the condition on the left-hand side of Eq. (A 4.10) it is again  $r^* = g$  and thus

$$q^* = \frac{(\beta_c^* - g)\pi_c^*}{1 + \beta_c^*} , \quad \beta_c^* = r_f + \pi_c^* \{ (r_f - g)(y^* - 1) \} , \quad \pi_c^* = \frac{(r_f - g)(y^* - 1)}{\nu [\sigma^2 + \beta_3 (r_f - g)^2 (y^* - 1)^2]} . \quad (\text{A 4.11})$$

Using the notation  $C \equiv \nu\sigma^2$ ,  $\Delta \equiv r_f - g$ ,  $X \equiv y^* - 1$  and equating  $q^*$  from Eq. (A 4.11) with  $r^* = g + \lambda q^* + \alpha(y^* - 1)$  (note that it is  $r^* = g$ ) finally leads to a bi-quadratic equation in the variable  $Y \equiv \Delta^2 X^2$  being of the form

$$Y^2 + pY + q = 0 \quad (\text{A 4.12})$$

with  $p, q > 0$ . Note that the condition  $y^* \neq 1$  again has to be fulfilled in order that the algebraic manipulation can be performed because of a division by the variable  $X$  (see also proof of Theorem 4.2 in Appendix A 4.3 for a very similar argument). Note that the transformation is quite laborious and  $p$  and  $q$  are quite unmanageable expressions in the variable  $\nu, \alpha, \lambda, r_f, C$  and  $\Delta$ . However, since  $p, q$  are positive numbers for all economically meaningful choices of values for the underlying parameters, it follows that the real roots of Eq. (A 4.12) are both negative (if existing). This leads to  $X^2 < 0$ . Thus there is no (real) steady state with  $r^* = g$ ,  $y^* \neq 0$  and  $y^* \neq 1$ . Assuming instead  $y^* = 1$  leads to  $\bar{\sigma}_c^2 = q^* = \pi_c^* = 0$  and  $\beta_c^* = r_f$ . Hence  $E_1 = (g, g, g, 1, 1)$  is the fundamental steady state of the system.

*(ii) Analysis of the Right-Hand Side of Eq. (A 4.10)*

Assuming  $y^* = 0$  (see the condition on the right-hand side of Eq. (A 4.10)) directly leads to

$$r^* = g + \lambda q^* - \alpha ,$$

where

$$q^* = \frac{(\beta_c^* - r^*)\pi_c^*}{1 + \beta_c^*} , \quad \beta_c^* = r_f + \pi_c^* (r^* - r_f) , \quad \pi_c^* = \frac{r^* - r_f}{\nu [\sigma^2 + \beta_3 (r^* - r_f)^2]} .$$

This system of equations is again analytically almost intractable. For this reason, the possible steady states for  $y^* = 0$  are numerically investigated for a wide range of reasonable parameter sets with the help of the software package MATHEMATICA 7, as already performed for the proof of Theorem 4.2 in Appendix A 4.3. The chosen parameter sets enclose the following parameter ranges:

$$\begin{aligned} \nu = 1.2 , & \quad 0.05 \leq \sigma \leq 1.15 , & \quad 0.02 \leq r_f \leq 0.18 , & \quad 0.02 \leq g \leq 0.18 , \\ 0.05 \leq \lambda \leq 0.1 , & \quad 0.1 \leq \alpha \leq 0.9 , & \quad 2 \leq \beta_3 \leq 10 . \end{aligned}$$

An extensive numerical analysis reveals that *no* economically reasonable (non-fundamental) steady state exists within these parameter ranges, where the same conditions as in the proof of Theorem 4.2 in Appendix A 4.3 are used to define an economically reasonable (non-fundamental) steady state.

(iii) *Stability of Fundamental Steady State  $E_1$*

Note that the complexity (i.e., the high dimensionality) of the dynamic system means that the analysis of the stability of the steady state  $E_1$  is again conducted with the aid of MATHEMATICA 4.1. Thus only the main steps are outlined here. Very similar to the proceeding used in the proof of Theorem 4.4 in Appendix A 4.4, the system Eq. (4.74)–Eq. (4.75) can easily be reformulated in its canonical form of first-order difference equations. A new set of variables is again introduced:

$$a_{t+1} = r_{t+1}, \quad b_{t+1} = r_t, \quad c_{t+1} = r_{t-1}, \quad d_{t+1} = y_{t+1}, \quad e_{t+1} = y_t.$$

Hence the system Eq. (4.74)–Eq. (4.75) can be rewritten as

$$\begin{aligned} a_{t+1} &= g + \lambda Q(\beta_{c,t}, \pi_{c,t}, \pi_{c,t-1}, a_t) + \alpha(d_t - 1), \\ b_{t+1} &= a_t, \\ c_{t+1} &= b_t, \\ d_{t+1} &= \frac{1 + g}{1 + g + \lambda Q(\beta_{c,t}, \pi_{c,t}, \pi_{c,t-1}, a_t) + \alpha(d_t - 1)} c_t, \\ e_{t+1} &= d_t, \end{aligned}$$

where  $Q(\cdot, \cdot, \cdot, \cdot)$  is again defined as

$$Q(u, v, w, x) \equiv \frac{(1 + u)v - (1 + x)w}{1 + u}.$$

Note that  $Q(\beta_{c,t}, \pi_{c,t}, \pi_{c,t-1}, a_t)$  now also depends on  $r_{t-2}$ . It is  $Q(\beta_{c,t}, \pi_{c,t}, \pi_{c,t-1}, a_t) = \text{fct.}(r_t, r_{t-1}, r_{t-2}, y_t, y_{t-1})$  or, using the new set of variables,  $Q(\beta_{f,t}, \pi_{f,t}, \pi_{f,t-1}, a_t) = \text{fct.}(a_t, b_t, c_t, d_t, e_t)$  respectively. The characteristic polynomial of the Jacobian of the dynamic system evaluated at the fundamental equilibrium point  $E_1$  can be determined by

$$\begin{aligned} \Gamma_1(\xi) &= \frac{1}{(1 + g)(1 + r_f)\nu\sigma^2} \left\{ \xi \left[ -\lambda(-1 + \xi - g + \xi r_f) \left[ -1 - g + \beta_2(\xi - 1)^2(1 + g) \right. \right. \right. \\ &\quad \left. \left. \left. + \xi(1 + g + \xi g - \xi r_f) \right] + \xi^3 [\alpha + (\xi - 1)(1 + g)](1 + r_f)\nu\sigma^2 \right] \right\}, \quad (\text{A } 4.13) \end{aligned}$$

which is a product of an ordinary polynomial of order four and the linear factor  $\xi$ . Therefore,  $E_1$  has one trivial eigenvalue  $\xi_1 = 0$  and four further non-trivial eigenvalues  $\xi_2, \dots, \xi_5$  which are implicitly given by the expression in the big square brackets in Eq. (A 4.13). This completes the proof.

#### A 4.7 Proof of Theorem 4.7

The proof of this theorem can be conducted in exactly the same way as the proofs of the corresponding Theorems 4.2 and 4.6 for the homogeneous beliefs models. However, the resulting equations become even more laborious. Therefore, only the main steps of the proof are outlined here.

##### *Analysis 1*

For  $y^* \neq 0$  it is  $r^* = g$ . For  $y^* \neq 1$  it can be shown that  $y^*$  has to fulfill the equation

$$Y^3 + pY^2 + qY + r = 0$$

with  $p, q, r > 0$ , where  $p, q, r$  are again functions of  $\nu, \alpha, \lambda, r_f, C, \Delta$  (with  $C \equiv \nu\sigma^2$  as before) as well as of  $(1 \pm \bar{n})$  and  $(1 \pm \check{\omega}^*)$  which are all non-negative numbers. Furthermore, it is  $Y \equiv \Delta^2 X^2$  with  $\Delta \equiv r_f - g$  and  $X \equiv y^* - 1$ . Again, there is no (real)  $y^*$  that can satisfy this condition. Otherwise, it can easily be verified that it is  $\bar{\sigma}_c^2 = q^* = 0$  and thus  $\pi_f^* = \pi_c^* = 0$  as well as  $\beta_f^* = \beta_c^* = r_f$  for  $y^* = 1$ . From this it can directly be concluded that at the equilibrium point ( $r^* = g, y^* = 1$ ) the equation (4.89) is fulfilled for any choice of  $\check{\omega}^*$  (see proof of Remark 3.7 in Appendix A 3.6, Chapter 3 for a similar discussion).

### Analysis 2

For  $y^* = 0$  the following equations have to be solved simultaneously:

$$r^* = g + \lambda q^* - \alpha ,$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^*(1 + \bar{n})[1 + \check{\omega}^*] + (\beta_c^* - r^*)\pi_c^*(1 - \bar{n})[1 - \check{\omega}^*]}{(1 + \beta_f^*)(1 + \bar{n})[1 + \check{\omega}^*] + (1 + \beta_c^*)(1 - \bar{n})[1 - \check{\omega}^*]} , \quad \beta_f^* = r_f + \pi_f^*(r^* - r_f)$$

and

$$\beta_c^* = r_f + \pi_c^*(r^* - r_f) , \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{ (g - r_f) - \beta_1(r^* - g) \} , \quad \pi_c^* = \frac{r^* - r_f}{\nu[\sigma^2 + \beta_3(r^* - r_f)^2]} ,$$

as well as

$$\check{\omega}^* = \frac{[1 + \check{\omega}^*](1 + \beta_f^*) - [1 - \check{\omega}^*](1 + \beta_c^*)}{[1 + \check{\omega}^*](1 + \beta_f^*) + [1 - \check{\omega}^*](1 + \beta_c^*)} .$$

As in the case of the homogeneous-beliefs model variants, this system of equations is analytically almost intractable.<sup>113</sup> For this reason, the possible steady states for  $y^* = 0$  are again numerically investigated for a wide range of reasonable parameter sets with the help of the software package MATHEMATICA 7. The chosen parameter sets enclose the following parameter ranges:

$$\begin{aligned} \nu = 1.2 , \quad & 0.05 \leq \sigma \leq 1.15 , \quad & 0.02 \leq r_f \leq 0.18 , \quad & 0.02 \leq g \leq 0.18 , \\ 0.1 \leq \beta_1 \leq 0.9 , \quad & 0.05 \leq \lambda \leq 0.1 , \quad & 0.1 \leq \alpha \leq 0.9 , \quad & 2 \leq \beta_3 \leq 10 \end{aligned}$$

and  $\bar{n} = 0.0$ . An extensive numerical analysis reveals that *almost no* economically reasonable (non-fundamental) steady state exists within these parameter ranges. Similar to the observations for the homogeneous beliefs model with only fundamentalists in the market, only for very low choices of  $\sigma$ , one economically reasonable (non-fundamental) steady state exist which, however, is always accompanied by negative values of  $\pi_f^*$ , positive values of  $\pi_c^*$ ,  $\check{\omega}^* = 1$ , and a return (ex dividends) of the risky asset which is slightly higher than the risk-free rate. Note that the following conditions are used to define an economically reasonable (non-fundamental) steady state:<sup>114</sup>

$$\beta_f^* \geq 0 , \quad \beta_c^* \geq 0 , \quad r_f > g , \quad r^* \geq -1 \text{ (note that } r^* < -1 \text{ leads to negative prices) ,}$$

as well as

$$r^* , q^* , \beta_f^* , \beta_c^* , \pi_f^* , \pi_c^* , \check{\omega}^* \in \mathbb{R} \quad \text{and} \quad \neg(g \geq r^* \wedge r^* > -1) .$$

This completes the proof.

<sup>113</sup> $\pi_f^* = \pi_c^*$  as well as  $\beta_f^* = \beta_c^*$  does not hold for  $y^* = 0$ . This implies that  $\check{\omega}^*$  reaches a unique value when  $y^* = 0$  is assumed instead of a continuum of values, i.e.,  $\check{\omega}^* \in [-1, 1]$ , as was seen for  $y^* \neq 0$ .

<sup>114</sup>See also the proof of Theorem 4.2 in Appendix A 4.3 for further details.

### A 4.8 Geometric Decay Process

In this appendix a generalization of the conventional sample mean and sample variance of the return  $r_t$  is considered in which the current realization of the return and the mean of the former ones can be weighted by a weighting factor. For this, consider the sample mean and the sample variance of the return (ex dividends)  $r_t$  up to the time period  $T$

$$\begin{aligned}\phi_T &= \frac{1}{T+1} \sum_{i=0}^T r_i \\ v_T &= \frac{1}{T+1} \sum_{i=0}^T (r_i - \phi_T)^2.\end{aligned}$$

It can be shown (see Appendix A 4.9 for details), that the sample mean and the sample variance can be written as

$$\phi_T = \delta \phi_{T-1} + (1 - \delta) r_T \quad (\text{A 4.14})$$

$$v_T = \delta v_{T-1} + \delta(1 - \delta)(r_T - \phi_{T-1})^2 \quad (\text{A 4.15})$$

where  $\delta$  is determined by  $\delta \equiv T/(T+1)$ . Now, an obvious generalization of these sample mean and variance is to allow for  $\delta$  to assume any value between 0 and 1. This leads to the following process on mean and variance

$$\begin{aligned}\phi_t &= \delta \phi_{t-1} + (1 - \delta) r_t \\ v_t &= \delta v_{t-1} + \delta(1 - \delta)(r_t - \phi_{t-1})^2, \quad \delta \in [0, 1].\end{aligned}$$

The resulting process can be interpreted as a limiting process of a ‘geometric decay process’ with the decay rate  $\delta$  when the memory lag length tends to infinity. For instance, assume that  $r_t = \bar{r}$  is constant for all  $t$ . Furthermore, exclude the pathological cases and let  $\delta$  be non-zero and smaller than 1. Then, for  $\phi_{t-T} < \infty$  it follows that

$$\phi_t = \bar{r}(1 - \delta) \sum_{i=0}^{T-1} \delta^i + \delta^T \phi_{t-T} \xrightarrow{T \rightarrow \infty} \bar{r}(1 - \delta) \sum_{i=0}^{\infty} \delta^i \equiv \phi_{\infty},$$

from which the geometric decay property of the process becomes obvious. Using the properties of geometric series, it is easy to verify that  $\phi_{\infty} = \bar{r}$  as a definition of a mean should suffice. Furthermore, using the properties of telescopic sums, it is  $\phi_t = \bar{r}$  if  $\phi_{t-T} = \bar{r}$  is assumed. For  $\delta = 0$ , the generalized sample mean  $\phi_t$  is equal to current realized return  $r_t$  for all  $t$ . In this case the generalized sample variance  $v_t$  is zero if and only if  $v_0 = 0$ . Contrastingly, if  $\delta = 1$ ,  $\phi_t$  and  $v_t$  are determined by the first observed return of the process, i.e., both the sample mean  $\phi_t$  and the sample variance  $v_t$  equal their initial value  $r_0$  and  $v_0$  for all  $t$ . Finally, for  $0 < \delta < 1$  the half life  $t^*$  of a shock exposed to the process can be considered. Without loss of generality, assume that the process is exposed to a shock on the scale of  $k$  at time  $t = 0$ , i.e.,  $\phi_0 = k$ . Then the half life of the shock is defined by

$$\frac{k}{2} = \delta^{t^*} k,$$

which leads to

$$t^* = -\frac{\ln 2}{\ln \delta}.$$

Thus for  $\delta = 0.9$  the half life of the shock is 6.5 trading periods, while for  $\delta = 0.999$  the half life is about 692 trading periods.

### A 4.9 Derivation of the Sample Mean and Sample Variance

Consider the sample mean of the returns up to time  $T$  and split the expression into two parts such as

$$\begin{aligned}
 \phi_T &= \frac{1}{T+1} \sum_{i=0}^T r_i \\
 &= \frac{1}{T+1} \sum_{i=0}^{T-1} r_i + \frac{1}{T+1} r_T \\
 &= \frac{T}{T+1} \phi_{T-1} + \frac{1}{T+1} r_T \\
 &= \delta \phi_{T-1} + (1-\delta) r_T
 \end{aligned} \tag{A 4.16}$$

with  $\delta \equiv T/(T+1)$ , as proposed in Eq. (A 4.14). Applying the same idea to the sample variance leads to

$$\begin{aligned}
 v_T &= \frac{1}{T+1} \sum_{i=0}^T (r_i - \phi_T)^2 \\
 &= \frac{1}{T+1} \sum_{i=0}^{T-1} (r_i - \phi_T)^2 + \frac{1}{T+1} (r_T - \phi_T)^2 \\
 &= \frac{1}{T+1} \sum_{i=0}^{T-1} [(r_i - \phi_{T-1}) + (\phi_{T-1} - \phi_T)]^2 + \frac{1}{T+1} (r_T - \phi_T)^2 \\
 &= \frac{1}{T+1} \sum_{i=0}^{T-1} [(r_i - \phi_{T-1})^2 + (\phi_{T-1} - \phi_T)^2] + \\
 &\quad + \left\{ \frac{1}{T+1} \sum_{i=0}^{T-1} 2(r_i - \phi_{T-1})(\phi_{T-1} - \phi_T) \right\} + \frac{1}{T+1} (r_T - \phi_T)^2 \\
 &= \frac{T}{T+1} v_{T-1} + \frac{T}{T+1} (\phi_{T-1} - \phi_T)^2 + \frac{1}{T+1} (r_T - \phi_T)^2 \\
 &= \delta v_{T-1} + \delta (\phi_{T-1} - \phi_T)^2 + (1-\delta) (r_T - \phi_T)^2,
 \end{aligned}$$

where, inter alia, the term in the curly brackets is equal to zero. Note that from Eq. (A 4.16)

$$\phi_{T-1} - \phi_T = \phi_{T-1} - \delta \phi_{T-1} - (1-\delta) r_T = (1-\delta)(\phi_{T-1} - r_T)$$

and

$$r_T - \phi_T = r_T - \delta \phi_{T-1} - (1-\delta) r_T = \delta (r_T - \phi_{T-1}).$$

Thus,

$$\begin{aligned}
 v_T &= \delta v_{T-1} + \delta (1-\delta)^2 (\phi_{T-1} - r_T)^2 + (1-\delta) \delta^2 (r_T - \phi_{T-1})^2 \\
 &= \delta v_{T-1} + (r_T - \phi_{T-1})^2 \cdot \{\delta (1-\delta)^2 + (1-\delta) \delta^2\} \\
 &= \delta v_{T-1} + \delta (1-\delta) (r_T - \phi_{T-1})^2
 \end{aligned}$$

as given in Eq. (A 4.15).

### A 4.10 Proof of Theorem 4.8

The proof proceeds analogous to the proof of Theorem 4.7 in Appendix A 4.7. Thus only the main steps are sketched here. Obviously, it is  $\phi^* = r^*$  as well as  $v^* = 0$ . Following now the considerations outlined in Appendix A 4.7 it can be concluded that the ‘quasi’-unique steady state is given by  $(r^* = g \wedge y^* = 1)$  where it is  $q^* = 0$ . Again, it is  $\pi_f^* = \pi_c^* = 0$  and  $\beta_f^* = \beta_c^* = r_f$  from which  $\tilde{\omega}^* \in [-1, 1]$ ,  $\phi^* = g$  and  $v^* = 0$  can directly be obtained. In the same way very similar results can be obtained for  $y^* = 0$ , as already stated in the proof of Theorem 4.7 in Appendix A 4.7. This completes the proof.

## A 4.11 Derivation of Eq. (4.107)

Again, we use the approximation performed in Subsection 4.2.4 (see Eq. (4.53)), where  $n^s \cdot g \ll 1$  is assumed, thus

$$\begin{aligned}
q_t &= \frac{[(1 + \beta_{1,t})\pi_{1,t} - (1 + r_t)\pi_{1,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} \\
&+ \frac{[(1 + \beta_{2,t})\pi_{2,t} - (1 + r_t)\pi_{2,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}] - 4[H_t - (1 + r_t)H_{t-1}]/(\bar{W}_{t-1}I)}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} \\
&\approx \frac{[(1 + \beta_{1,t})\pi_{1,t} - (1 + r_t)\pi_{1,t-1}](1 + \bar{n})[1 + \bar{\omega}_{t-1}]}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]} \\
&+ \frac{[(1 + \beta_{2,t})\pi_{2,t} - (1 + r_t)\pi_{2,t-1}](1 - \bar{n})[1 - \bar{\omega}_{t-1}] - 4[(1 + n^s + r_t)H_{t-1} - (1 + r_t)H_{t-1}]/(\bar{W}_{t-1}I)}{(1 + \beta_{1,t})(1 + \bar{n})[1 + \bar{\omega}_{t-1}] + (1 + \beta_{2,t})(1 - \bar{n})[1 - \bar{\omega}_{t-1}]},
\end{aligned}$$

from which Eq. (4.107) directly follows. This completes the proof.

## A 4.12 Proof of Theorem 4.11

As before, let  $r^* = r_{t+1} = r_t = r_{t-1}$ ,  $y^* = y_{t+1} = y_t = y_{t-1}$  and  $z^* = z_{t+1} = z_t = z_{t-1}$  be the steady state of the system Eq. (4.113)–Eq. (4.115). From  $H_t > 0$  it follows that it is  $z_t > 0$  for all  $t$  and thus  $r^{RP^*} = r_f + \nu\sigma^2 z^* \geq r_f > 0$ . Assuming that  $z^* \neq 0$ , i.e.,  $\lim_{t \rightarrow \infty} z_t = z^* > 0$ . Then it can directly be concluded that  $\beta_f^* = n^s + r^*$ . Analogous to the proofs for the  $H_t = 0$  case outlined previously, the necessary (and sufficient) condition for the steady state to exist can be expressed by

$$\left( y^* \neq 0 \wedge y^* \left( 1 - \frac{1+g}{1+r^*} \right) = 0 \right) \quad \vee \quad (y^* = 0 \wedge r^* = g + \lambda q^* - \alpha) \quad (\text{A 4.17})$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^* - n^s z^*}{1 + \beta_f^*} = \frac{n^s(\pi_f^* - z^*)}{1 + \beta_f^*}, \quad \beta_f^* = r_f + \pi_f^* \{ (r^* - r_f) + (r_f - g)y^* + \nu\sigma^2 z^* y^* \}$$

and

$$\pi_f^* = \frac{1}{\nu\sigma^2} \{ (r_f - g)(y^* - 1) - \beta_1(r^* - g) + \nu\sigma^2 z^* y^* \}.$$

(i) *Analysis of the Left-Hand Side of Eq. (A 4.17)*

For  $y^* \neq 0$  (see left-hand side of Eq. (A 4.17)) it is obvious that  $r^* = g$  and thus

$$q^* = \frac{n^s(\pi_f^* - z^*)}{1 + \beta_f^*}, \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{ (r_f - g)(y^* - 1) + \nu\sigma^2 z^* y^* \}, \quad \beta_f^* = r_f + \pi_f^{*2} \nu\sigma^2 \quad (\text{A 4.18})$$

as well as  $\beta_f^* = n^s + g$ . As a consequence it is  $\pi_f^* = \pm \sqrt{(n^s + g - r_f)/(\nu\sigma^2)}$ , where the square root has real solutions since  $n^s \geq (r_f - g)$  is assumed in the theorem. The next steps are quite similar to the steps outlined in Appendix A 4.3 for the proof of Theorem 4.2. From the  $\pi_f^*$ -expression in Eq. (A 4.18) it follows that

$$z^* = \frac{1}{y^*} \left\{ \pi_f^* - \frac{1}{\nu\sigma^2} [(r_f - g)(y^* - 1)] \right\}. \quad (\text{A 4.19})$$

Inserting Eq. (A 4.19) into the  $q^*$ -expression of Eq. (A 4.18) leads to

$$q^* = \frac{An^s(y^* - 1)}{By^*}, \quad (\text{A 4.20})$$

where  $A \equiv \pi_f^* + (r_f - g)/(\nu\sigma^2)$  and  $B \equiv 1 + n^s + g$ . Rewriting Eq. (4.113) as  $q^* = -(\alpha/\lambda)(y^* - 1)$  and combining it with Eq. (A 4.20) yields

$$y^* = -\frac{A}{B} \frac{n^s \lambda}{\alpha}, \quad (\text{A 4.21})$$



where  $y^* \neq 1$  has to be assumed in order that the algebraic manipulation leading to this result can be performed. Inserting Eq. (A 4.21) into Eq. (A 4.19) leads to

$$z^* = -\frac{1}{C} - \frac{1}{\nu\sigma^2}(r_f - g)$$

with  $C \equiv n^s \lambda / (B\alpha)$ , which is always a negative expression and thus leads to a contradiction because it is assumed that  $z^* > 0$ . Consequently,  $y^*$  has to be equal to 1, which directly leads to  $z^* = \pi_f^*$  and  $q^* = 0$ . Since  $z^*$  is assumed to be positive as outlined above, only the positive square root of  $\pi_f^*$  is valid and the fundamental steady state  $E_1$  is given by  $r^* = g$ ,  $y^* = 1$  and  $z^* = \sqrt{(n^s + g - r_f)/(\nu\sigma^2)}$ .

Remaining at the assumption  $y^* \neq 0$  (see left-hand side of Eq. (A 4.17)), but assuming that it is  $z^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} z_t = z^* = 0$  directly leads to

$$r^* = g, \quad r^{RP^*} = r_f, \quad q^* = \frac{(\beta_f^* - r^*)\pi_f^*}{1 + \beta_f^*}, \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{(r_f - g)(y^* - 1)\}, \quad \beta_f^* = r_f + \pi_f^{*2} \nu\sigma^2. \quad (\text{A 4.22})$$

Equating the  $q^*$ -expression of Eq. (A 4.22) with  $q^* = -(\alpha/\lambda)(y^* - 1)$  (which can be rewritten from Eq. (4.113) as already done above) by using all other relevant equations yields

$$-\frac{\alpha}{\lambda} = \frac{\frac{1}{\nu\sigma^2}(r_f - g)A(y^*)}{B(y^*)}, \quad (\text{A 4.23})$$

where again  $y^* \neq 1$  has to be assumed and  $A(y^*)$  and  $B(y^*)$  are positive functions of  $y^*$ . Obviously, Eq. (A 4.23) cannot be fulfilled since the left-hand side is a negative expression while the right-hand side is a positive expression. Consequently,  $y^*$  has to be equal to 1, which directly leads to  $\pi_f^* = 0$ ,  $\beta_f^* = r_f$  and  $q^* = 0$ . However, from Eq. (4.115) it follows that

$$z_{t+1} = \frac{1 + n^s + g}{1 + r_f} z_t, \quad (\text{A 4.24})$$

which cannot converge to zero, since it is  $z_t > 0$  for all  $t$  (see the beginning of the proof) and because  $n^s + g \geq r_f$  is assumed in the theorem, which leads to a growth factor of  $\geq 1$  in Eq. (A 4.24). Thus a steady state with  $z^* = 0$  and  $y^* \neq 0$  does not exist.

(ii) *Analysis of the Right-Hand Side of Eq. (A 4.17)*

Finally, the condition on the right-hand side of Eq. (A 4.17) has to be investigated, i.e., the case  $y^* = 0$ . For  $z^* > 0$  this directly leads to

$$r^* = g + \lambda q^* - \alpha,$$

where

$$\beta_f^* = n^s + r^*, \quad q^* = \frac{(\pi_f^* - z^*)n^s}{1 + \beta_f^*}, \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{(g - r_f) - \beta_1(r^* - g)\}, \quad \beta_f^* = r_f + \pi_f^*(r^* - r_f), \quad (\text{A 4.25})$$

while for  $z^* = 0$  this implies

$$r^* = g + \lambda q^* - \alpha,$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^*}{1 + \beta_f^*}, \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{(g - r_f) - \beta_1(r^* - g)\}, \quad \beta_f^* = r_f + \pi_f^*(r^* - r_f). \quad (\text{A 4.26})$$

Unfortunately, both systems of equations are analytically almost intractable. For this reason, the possible steady states for  $y^* = 0 \wedge z^* > 0$  and for  $y^* = 0 \wedge z^* = 0$  are numerically investigated for a wide range of reasonable parameter sets with the help of the software package MATHEMATICA 7. The chosen parameter sets enclose the following parameter ranges:

$$\begin{aligned} \nu = 1.2, & & 0.05 \leq \sigma \leq 1.15, & & 0.02 \leq r_f \leq 0.18, & & 0.02 \leq g \leq 0.18, \\ 0.1 \leq \beta_1 \leq 0.9, & & 0.05 \leq \lambda \leq 0.1, & & 0.1 \leq \alpha \leq 0.9, & & 0.02 \leq n^s \leq 0.18. \end{aligned}$$

Note that according to Eq. (A 4.25) and Eq. (A 4.26) the parameters  $\nu$  and  $\sigma$  always occur as a factor  $\nu\sigma^2$ , so that only one of the two parameters has to be changed within the parameter sets. An extensive numerical analysis for both dynamic systems reveals that *no* economically reasonable (non-fundamental) steady state exists within these parameter ranges. Note that, similar to the proof of Theorem 4.2 in Appendix A 4.3, the following conditions have to be fulfilled to arrive at an economically reasonable (non-fundamental) steady state:

$$\beta_f^* \geq 0, \quad r_f > g, \quad n^s \geq (r_f - g), \quad r^* \geq -1 \text{ (note that } r^* < -1 \text{ leads to negative prices),}$$

as well as

$$r^*, q^*, \beta_f^*, \pi_f^* \in \mathbb{R} \quad \text{and} \quad -(g \geq r^* \wedge r^* > -1).$$

This completes the proof.

#### A 4.13 Proof of Theorem 4.14

Since  $\delta \neq 0$  it is  $\phi^* = r^*$  and  $v^* = 0$  and thus  $r^{RP^*} = r_f + \nu\sigma^2 z^* \geq r_f > 0$  with the same argument as in the preceding proof outlined in Appendix A 4.12.

##### *Analysis 1*

It is easy to see that for  $y^* \neq 0$  (leading to  $r^* = g$ ) the conditional equations for determining the steady states of the system are exactly the same as the corresponding equations in the proof of Theorem 4.11. Therefore it can be concluded that for  $z^* > 0$  only the fundamental steady state exists, where  $z^* = \pi_c^*$ ,  $q^* = 0$  and thus  $r^* = g$ ,  $y^* = 1$  and  $z^* = \sqrt{(n^s + g - r_f)/(\nu\sigma^2)}$  and no steady state exists for  $z^* = 0$ .

##### *Analysis 2*

However, for  $y^* = 0$  the resulting equations differ from the corresponding equations in the proof of Theorem 4.11. For  $y^* = 0$  and  $z^* > 0$  it is

$$r^* = g + \lambda q^* - \alpha,$$

where

$$\beta_c^* = n^s + r^*, \quad q^* = \frac{(\pi_c^* - z^*)n^s}{1 + \beta_c^*}, \quad \beta_c^* = r_f + \pi_c^*(r^* - r_f), \quad \pi_c^* = \frac{r^* - r_f}{\nu\sigma^2},$$

while for  $y^* = 0$  and  $z^* = 0$  it directly follows that

$$r^* = g + \lambda q^* - \alpha,$$

where

$$q^* = \frac{(\beta_c^* - r^*)\pi_c^*}{1 + \beta_c^*}, \quad \beta_c^* = r_f + \pi_c^*(r^* - r_f), \quad \pi_c^* = \frac{r^* - r_f}{\nu\sigma^2}.$$

These systems of equations are again analytically almost intractable. For this reason, the possible steady states for  $y^* = 0 \wedge z^* > 0$  as well as for  $y^* = 0 \wedge z^* = 0$  are numerically investigated for a wide range of reasonable parameter sets with the help of the software package MATHEMATICA 7, as already performed for the proof of Theorem 4.11 in Appendix A 4.12. The chosen parameter sets enclose the following parameter ranges:

$$\begin{aligned} \nu = 1.2, \quad 0.05 \leq \sigma \leq 1.15, \quad 0.02 \leq r_f \leq 0.18, \quad 0.02 \leq g \leq 0.18, \\ 0.05 \leq \lambda \leq 0.1, \quad 0.1 \leq \alpha \leq 0.9, \quad 0.02 \leq n^s \leq 0.18. \end{aligned}$$

An extensive numerical analysis reveals that *no* economically reasonable (non-fundamental) steady state exists for  $y^* = 0 \wedge z^* > 0$  within these parameter ranges but that economically reasonable (non-fundamental) steady states *do exist* for  $y^* = 0 \wedge z^* = 0$  within these parameter ranges. Note that the same conditions as in the proof of Theorem 4.11 in Appendix A 4.12 are used to define an economically reasonable (non-fundamental) steady state. Further numerical simulations confirm that these (non-fundamental) steady states are not stable. Furthermore, these (non-fundamental) steady states are accompanied by quite unrealistically high values for  $r^*$ ,  $q^*$ ,  $\beta_c^*$ , and  $\pi_c^*$ . This completes the proof.

#### A 4.14 Proof of Theorem 4.16

Again it is  $\phi^* = r^*$  and  $v^* = 0$  since it is  $\delta \neq 0$ . Furthermore it is  $r^{RP^*} \geq r_f > 0$  since it is  $z_t > 0$  for all  $t$ . Therefore the necessary (and sufficient) condition for the steady state to exist can once more be expressed by

$$\left( y^* \neq 0 \wedge y^* \left( 1 - \frac{1+g}{1+r^*} \right) = 0 \right) \quad \vee \quad (y^* = 0 \wedge r^* = g + \lambda q^* - \alpha), \quad (\text{A } 4.27)$$

where the explicit expressions for  $q^*$ ,  $\beta_f^*$ ,  $\beta_c^*$ ,  $\pi_f^*$  and  $\pi_c^*$  are not displayed here in order to avoid an overly long presentation of the proof.

##### (i) Analysis of the Left-Hand Side of Eq. (A 4.27)

Knowing already that it is  $r^* = g$  for  $y^* \neq 0$  (see left-hand side of Eq. (A 4.27)), it also becomes obvious that it is  $\pi_f^* = \pi_c^*$  in this case (denoted by  $\pi^*$  in the following) and as a direct consequence that it is  $\beta_f^* = \beta_c^*$  (denoted by  $\beta^*$  in the following). The first equality implies that  $\beta^* = r_f + \nu\sigma^2\pi^{*2}$  while from the latter equality it follows that  $\beta^* = n^s + g$  when  $z^* \neq 0$ , i.e.,  $\lim_{t \rightarrow \infty} z_t = z^* > 0$  is assumed in addition. Both equations together lead to  $\pi^* = \pm \sqrt{(n^s + g - r_f)/(\nu\sigma^2)}$ . Furthermore, it is

$$r^{RP^*} = r_f + \frac{2\nu\sigma^2}{1 + \bar{n}\bar{\omega}^*} z^*.$$

The following steps of the proof proceed in a very similar way to that in the proof of Theorem 4.11. It can easily be calculated that  $z^*$  can be written as

$$z^* = \frac{\left\{ \pi^* - \frac{1}{\nu\sigma^2} [(r_f - g)(y^* - 1)] \right\}}{\frac{2}{1 + \bar{n}\bar{\omega}^*} y^*}, \quad (\text{A } 4.28)$$

while  $q^*$  is again given by

$$q^* = \frac{An^s(y^* - 1)}{By^*},$$

where  $A \equiv \pi^* + (r_f - g)/(\nu\sigma^2)$  and  $B \equiv 1 + n^s + g$  (see Eq. (A 4.20)). Using the same manipulations as in the proof of Theorem 4.11 finally leads to

$$y^* = -\frac{A}{B} \frac{n^s \lambda}{\alpha}, \quad (\text{A } 4.29)$$

where again  $y^* \neq 1$  has to be assumed in order that the algebraic manipulation leading to this result can be performed. Inserting Eq. (A 4.29) into Eq. (A 4.28) yields

$$z^* = -\left\{ \frac{1 + \bar{n}\bar{\omega}^*}{2} \left[ \frac{1}{C} + \frac{1}{\nu\sigma^2} (r_f - g) \right] \right\}$$

with  $C \equiv n^s \lambda / (B\alpha)$ , which is always a negative expression and thus leads to a contradiction because it is assumed that  $z^* > 0$ . Hence, instead,  $y^*$  has to be equal to 1, which directly leads to  $z^* = (1 + \bar{n}\bar{\omega}^*)\pi^*/2$  and  $q^* = 0$ . Since  $z^*$  is assumed to be positive as outlined above, only the positive square root of  $\pi^*$  is valid and the fundamental steady state  $E_1$  is given by  $r^* = g$ ,  $y^* = 1$ ,  $z^* = [(1 + \bar{n}\bar{\omega}^*)/2]\sqrt{(n^s + g - r_f)/(\nu\sigma^2)}$  and  $\bar{\omega}^* \in [-1, 1]$ .

Again, for the assumption  $y^* \neq 0$  (see left-hand side of Eq. (A 4.27)), the case  $z^* = 0$ , i.e.,  $\lim_{t \rightarrow \infty} z_t = z^* = 0$  has to be considered separately. Fortunately, it can easily be seen that even for this case it is  $\phi^* = r^*$ ,  $v^* = 0$ ,  $\pi_f^* = \pi_c^*$  and  $\beta_f^* = \beta_c^*$  and that the resulting equations for  $r^*$ ,  $r^{RP^*}$ ,  $q^*$ ,  $\pi_f^*$  and  $\beta_f^*$  are identical to the corresponding equations of the homogeneous beliefs model with only fundamentalists in the market, i.e., identical to Eq. (A 4.22). From following the lines in this proof (see Appendix A 4.12 for details), it can be concluded that no steady state exists for  $z^* = 0$  and  $y^* \neq 0$ .

(ii) Analysis of the Right-Hand Side of Eq. (A 4.27)

For  $y^* = 0$  (see right-hand side of Eq. (A 4.27)) the following two systems of equations have to be solved simultaneously:

$$r^* = g + \lambda q^* - \alpha,$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^*(1 + \bar{n})[1 + \check{\omega}^*] + (\beta_c^* - r^*)\pi_c^*(1 - \bar{n})[1 - \check{\omega}^*] - 4n^s z^*}{(1 + \beta_f^*)(1 + \bar{n})[1 + \check{\omega}^*] + (1 + \beta_c^*)(1 - \bar{n})[1 - \check{\omega}^*]}, \quad \beta_f^* = r_f + \pi_f^*(r^* - r_f)$$

and

$$\beta_c^* = r_f + \pi_c^*(r^* - r_f), \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{(g - r_f) - \beta_1(r^* - g)\}, \quad \pi_c^* = \frac{r^* - r_f}{\nu\sigma^2},$$

as well as

$$\check{\omega}^* = \frac{[1 + \check{\omega}^*](1 + \beta_f^*) - [1 - \check{\omega}^*](1 + \beta_c^*)}{[1 + \check{\omega}^*](1 + \beta_f^*) + [1 - \check{\omega}^*](1 + \beta_c^*)} \quad \text{and} \quad r^* = \frac{[1 + \bar{\omega}^*](1 + \beta_f^*) + [1 - \bar{\omega}^*](1 + \beta_c^*)}{2} - (1 + n^s)$$

for  $z^* > 0$ , and

$$r^* = g + \lambda q^* - \alpha,$$

where

$$q^* = \frac{(\beta_f^* - r^*)\pi_f^*(1 + \bar{n})[1 + \check{\omega}^*] + (\beta_c^* - r^*)\pi_c^*(1 - \bar{n})[1 - \check{\omega}^*]}{(1 + \beta_f^*)(1 + \bar{n})[1 + \check{\omega}^*] + (1 + \beta_c^*)(1 - \bar{n})[1 - \check{\omega}^*]}, \quad \beta_f^* = r_f + \pi_f^*(r^* - r_f)$$

and

$$\beta_c^* = r_f + \pi_c^*(r^* - r_f), \quad \pi_f^* = \frac{1}{\nu\sigma^2} \{(g - r_f) - \beta_1(r^* - g)\}, \quad \pi_c^* = \frac{r^* - r_f}{\nu\sigma^2},$$

as well as

$$\check{\omega}^* = \frac{[1 + \check{\omega}^*](1 + \beta_f^*) - [1 - \check{\omega}^*](1 + \beta_c^*)}{[1 + \check{\omega}^*](1 + \beta_f^*) + [1 - \check{\omega}^*](1 + \beta_c^*)}$$

for  $z^* = 0$ . As in the case of the homogeneous-beliefs model variants, these two systems of equations are analytically almost intractable.<sup>115</sup> For this reason, the possible steady states for  $y^* = 0 \wedge z^* > 0$  as well as for  $y^* = 0 \wedge z^* = 0$  are again numerically investigated for a wide range of reasonable parameter sets with the help of the software package MATHEMATICA 7. The chosen parameter sets enclose the following parameter ranges:

<sup>115</sup> $\pi_f^* = \pi_c^*$  as well as  $\beta_f^* = \beta_c^*$  does not hold for  $y^* = 0$ . This implies that  $\check{\omega}^*$  reach a unique value when  $y^* = 0$  is assumed instead of a continuum of values, i.e.,  $\check{\omega}^* \in [-1, 1]$ , as we have seen for  $y^* \neq 0$ .

$$\begin{aligned} \nu = 1.2, & & 0.05 \leq \sigma \leq 1.15, & & 0.02 \leq r_f \leq 0.18, & & 0.02 \leq g \leq 0.18, \\ 0.1 \leq \beta_1 \leq 0.9, & & 0.05 \leq \lambda \leq 0.1, & & 0.1 \leq \alpha \leq 0.9, & & 0.02 \leq n^s \leq 0.18 \end{aligned}$$

and  $\bar{n} = 0.0$ . An extensive numerical analysis reveals that *no* economically reasonable (non-fundamental) steady state exists within these parameter ranges. Note that the following conditions are used to define an economically reasonable (non-fundamental) steady state:<sup>116</sup>

$$\beta_f^* \geq 0, \quad \beta_c^* \geq 0, \quad r_f > g, \quad n^s \geq (r_f - g), \quad r^* \geq -1 \text{ (note that } r^* < -1 \text{ leads to negative prices),}$$

as well as

$$r^*, q^*, \beta_f^*, \beta_c^*, \pi_f^*, \pi_c^*, \check{\omega}^* \in \mathbb{R} \quad \text{and} \quad \neg(g \geq r^* \wedge r^* > -1).$$

This completes the proof.

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<sup>116</sup>See also the proof of Theorem 4.2 in Appendix A 4.3 for further details.

# Conclusion and Future Research

Price fluctuations observed in financial markets seem too volatile to be explained by standard asset pricing models mainly based on the geometric Brownian motion price process. Furthermore, traditional economic theories in finance cannot explain some of the typical market behavior observed in financial markets, such as volatility clustering and fat-tail phenomena, nowadays also known as the ‘stylized facts’ of asset returns in academic literature. Interestingly, these models and theories are almost all built on the assumption of economic agents having homogeneous and ‘correct’ beliefs, and are based on three building blocks (or ‘paradigms’) of economic modeling: the ‘Rational Expectation Hypothesis’ (REH), the ‘Efficient Market Hypothesis’ (EMH) and the ‘Representative Agent Hypothesis’ (RAH). However, it is widely held that trade among individuals is mainly based on their specific tastes such as their risk attitude or other more general preferences, their current endowments and their individual beliefs, which will differ among the individuals and which might even change over time. For this reason, introducing heterogeneity among individuals into a financial market model might be a first step in bringing the model to a more realistic level, where it might be able to explain some of the typical market behavior observed in financial markets and to reproduce realistic asset price dynamics. This is the aim of the thesis and thus it follows a broad strand of literature that has emerged in the past few years. More precisely, heterogeneity has been introduced threefold in this thesis, namely by risk aversion, by wealth and by expectation. Asset price and wealth dynamics have been investigated with two different market clearing scenarios, i.e., Walrasian scenario and market maker scenario. In particular, different types of market participants have been by defining stylized representations, fundamentalists and chartists, whose heterogeneity seems to be one of the driving economic mechanisms of price dynamics in financial markets. The proposed model framework uses a deterministic nonlinear model approach and investigates whether nonlinear economic laws can explain a significant part of observed (irregular) fluctuations in economic and financial markets.

**Remark:** Since a comprehensive summary of each chapter can be found at the end of each of them, the following two subsections will almost exclusively concentrate on the last chapter of this thesis because this contains my own model framework and thus the main results I want to contribute to the field of research.

## Conclusion

My deterministic nonlinear structural agent model is largely motivated by three asset pricing frameworks proposed by Brock and Hommes, Levy, Levy and Solomon, and Chiarella and He, and

is examined in Brock and Hommes (1997*b,a*, 1998, 1999); Levy et al. (1994, 1995); Levy and Levy (1996); Levy et al. (1996); Chiarella and He (2001) and Chiarella et al. (2006). After recapitulating these model frameworks in Chapter 2 and Chapter 3, where I also generalized a modified version of the Brock-Hommes model proposed by Gaunersdorfer (2000) to give a more realistic and more complex model variant, and did the same for the Chiarella-He model by introducing the risk-aversion coefficient as an exogenous parameter, I proposed my own model framework in Chapter 4. The model presented in this chapter is similar to the asset pricing models mentioned above, but differs in several crucial aspects. It is similar because my model is based on a nonlinear, evolutionary system with two competing trading strategies under a utility maximization scheme using CRRA utility preferences. As in the Chiarella-He model, a stationary model in terms of return and wealth proportion among heterogeneous investors can be established, leading to a dynamic system for which steady states can be derived whose local stability can be determined by a combination of an analytical and a numerical treatment. The crucial features distinguishing my model from the others are the following:

- Although my model shares the assumption of a CRRA utility function for the agents leading to a growth model in both risky-asset prices and agents' wealth, it differs from Chiarella and He (2001) by using expectation formation rules for the future return on the risky asset. These rules are consistent with the existing (fundamental) steady states in the model, meaning that all agents form 'correct' expectations of the fundamental price path.
- In harmony with the agents' growing demand for risky assets, I model the risky-asset supply explicitly by letting it also grow over time.
- Due to the growing risky-asset supply, the risk premium demanded by the market participants for taking the market risk can be determined endogenously within my model framework.

As mentioned earlier, the resulting dynamic model can be formulated as a stationary system by expressing the law of motion in terms of risky-asset price changes, fundamental/price ratio, asset supply/total wealth ratio and wealth proportions of the two types of agents. It was found that the presence of fundamentalists and chartists in the market leads the stationary model to have a continuum of fundamental steady states, where the price is at its fundamental price level, and that the steady-state wealth level distribution the system eventually reaches depends on the initial condition. As expected, the local stability of the fundamental steady states depends crucially on the chosen parameters. In this respect the extrapolation parameters used by the two trading types as well as the price adjustment speed used by the market maker play an important role. More precisely, choosing sufficiently high values of these parameters or an appropriate combination of them destabilizes the dynamic system. Thus the model is able to show the dynamic system displaying rather complex behavior and with a wide range of dynamic scenarios. These are local stability, bifurcation routes to complicated asset price dynamics, limit cycles, periodic orbits, and evidence of strange chaotic attractors. There are also two additional important features. On the one hand, the model shows a strong dependence on small changes in the parameters and in the initial conditions. On the other hand, the model allows us to investigate the interrelation of the risky asset's price fluctuations and the evolution of the wealth distribution among the two trading types.

The model suggests that, whenever the risky-asset price converges to a fundamental steady state, neither of the two trading types, i.e., neither fundamentalist nor chartist, is able to prevail over the other group. Instead, both types of agents survive in the long run and keep (more or less) their initial wealth proportion. However, when the dynamic system runs into a limit cycle, the chartists are able to control the market and the fundamentalists cannot accumulate as much wealth as the chartists. Moreover, numerical simulations indicate that specific parameter combinations produce market situations in which both trading types can eventually coexist in the market with an equally distributed wealth share, although the dynamic system does not settle down to a fundamental equilibrium. At any rate, the fundamentalists generally do not accumulate more wealth than the chartists. From a qualitative point of view, these results are in good accordance with the findings published in Chiarella et al. (2006), although our model framework might be even more complex, but also more realistic. In sum, I have shown how the risk attitudes and rather simple expectation formation schemes leading to competing trading strategies among the investor groups can be used to generate and explain a significant proportion of observed fluctuations in financial markets and how these can lead to the emergence of complicated and complex dynamics for asset prices.

### Future Research

Although my proposed model framework already results in a rather complex and highly nonlinear dynamic system, the analysis in this thesis is still based on a simplified model and some extensions might be economically significant in developing an even more realistic framework. I would thus propose six extensions.

First, as in most of the papers that motivated this thesis, especially those studying the Brock-Hommes model and the Chiarella-He model, my analysis has mainly focused on a deterministic dynamic model commonly seen in this strand of literature as the deterministic skeleton of a stochastic model with noisy dividend process (see Chiarella et al. (2006)). Although I have also investigated a stochastic version of the model, this analysis might be worth intensifying by implementing more sophisticated and more realistic stochastic components into the deterministic skeleton and by analyzing the resulting dynamics in greater detail. Second, my analysis has shown that the calibration of this type of model (e.g., a transformation of the model parameters on a daily level) is still an open issue. Although results for a rescaled model version are shown in this thesis, my findings are somewhat unsatisfactory and this issue remains an unsolved problem. It might be worth stating that some of the ‘stylized facts’ mentioned above can only be detected on a daily or even intra-daily scale, meaning that a calibration of this type of model highly desirable if it should then be able to generate such phenomena observed in real financial time series. Third, it has been found that for certain parameter combinations the optimal investment proportion of the chartists derived by a standard CRRA utility maximization scheme clearly exceeds the optimal investment proportion of the fundamentalists, which might be a rather unrealistic assumption. Although Chiarella et al. (2006) solve this problem by assuming a tanh-function for the chartists’ demand function, I would suggest there is a better approach needed that helps to reduce the relatively high risky-asset demand of the chartists that results from the given model framework. Fourth, the proportions of agents using a certain trading strategy is fixed in my model. As the market envi-



ronment might change over time, the agents should be allowed to switch between certain trading strategies using the information about their realized profits or wealth level as a decision criterion (for instance, according to the 'Adaptive Belief System' (ABS) introduced by Brock and Hommes (1997*a*, 1998)). However, although the introduction of a switching mechanism and time-varying proportions of agents might be an interesting extension of the existing model variants, it has to be analyzed whether such a model variant could simply lead to a market situation in which all agents accumulate the same wealth on average through displaying a kind of 'herding' behavior by using the same trading strategies in most of the trading periods. My fifth suggestion is more a better alternative to the model framework, which refers to a rather concrete modification of the model design. Eq. (4.41) shows us that the pricing formula for the risky asset is given by an analytically intractable expression in the model. To simplify the problem, an idea could be adopted from Poterba and Summers (1986), which also appears in Manzan (2003). They approximate the pricing formula given in Eq. (4.41) by a first-order Taylor expansion around the mean of the required one-period holding return. Checking whether this modification leads to significantly different results might be worth investigating. Sixth, a financial market consists of hundreds of heterogeneous agents and the inclusion of more than two stylized representations of agents would certainly be interesting.

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## Eidesstattliche Erklärung

Ich erkläre hiermit an Eides Statt, dass ich meine Doktorarbeit “Asset Price and Wealth Dynamics with Heterogeneous Expectations – A Deterministic Nonlinear Structural Model Approach” selbständig und ohne fremde Hilfe angefertigt habe und dass ich alle von anderen Autoren wörtlich übernommenen Stellen, wie auch die sich an die Gedanken anderer Autoren eng anlehenden Ausführungen meiner Arbeit, besonders gekennzeichnet und die Quellen nach den mir angegebenen Richtlinien zitiert habe.

Kiel, 27. Mai 2010

(Florian Heitger)

