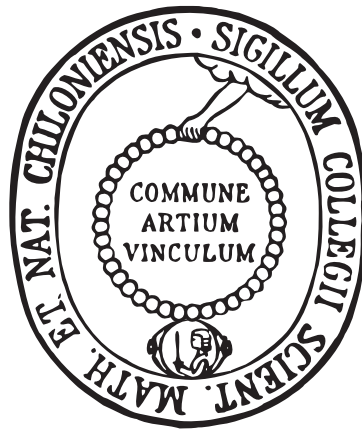


Comparisons and asymptotics
in the theory of portfolio optimization
under fixed and proportional transaction costs

Dissertation

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Abstract

In this thesis we consider two portfolio optimization models under fixed and proportional transaction costs and investigate their asymptotic behavior.

In the Black-Scholes framework of one bond and $d \geq 1$ stocks the problem treated is to maximize the expected asymptotic growth rate with logarithmic utility of wealth, when transaction costs, consisting of a fixed part (proportional to the current portfolio value) and a proportional part (proportional to the trading volume) have to be paid.

The first model, where the transaction costs are paid from the bond, was introduced and treated in [IS05], [IS06] and [IP09]. An optimal strategy for the one-dimensional case was found in the class of constant boundary strategies given by four boundary constants $a < \alpha \leq \beta < b \in]0, 1[$, where $]a, b[$ is the continuation region for the risky fraction process and α, β denote the states it is traded back to at each transaction. For the second model, where the fixed part of the costs is paid proportionally from each asset, the existence of a general optimal strategy was shown in [Tam06] and [Tam08] via a weak solution to a quasi-variational inequality.

By establishing a maximum principle for this solution and additional information on the corresponding continuation region in the general case $d \geq 1$, we first obtain the existence of an optimal constant boundary strategy for the second model in the one-dimensional case. The difference of the two models in terms of optimal boundaries and optimal asymptotic growth rates is then analyzed numerically.

In the second part we investigate in the one-dimensional case the asymptotic behavior of these two models when the proportional part of the transaction costs tends to zero. The convergence to the model with only fixed costs from [MP95] is established with regard to the optimal boundaries and asymptotic growth rates. Moreover the uniqueness of the optimal boundaries for the limit model is obtained.

Finally, the convergence of the models, when the fixed costs tend to zero, is treated. A suitable model with purely proportional costs similar to that in [TKA88] and [AST01] is introduced and an optimal strategy is shown to consist of keeping the risky fraction process in a unique interval $[A, B] \subseteq]0, 1[$ with minimal effort. Furthermore, it is proved to be the limit model in terms of optimal boundaries, asymptotic growth rates and optimal risky fraction processes.

Zusammenfassung

Gegenstand dieser Arbeit sind zwei Portfoliooptimierungsmodelle unter fixen und proportionalen Transaktionskosten und die Untersuchung ihres asymptotischen Verhaltens.

Das behandelte Optimierungsproblem besteht darin, im Black-Scholes-Modell mit einem Bond und $d \geq 1$ Aktien die erwartete asymptotische Rendite bei logarithmischem Nutzen und Vorliegen von fixen (proportional zum Portfoliowert) und proportionalen (proportional zum Transaktionsvolumen) Transaktionskosten zu maximieren.

Im ersten, bereits in [IS05], [IS06] und [IP09] vorgestellten und untersuchten Modell werden die Transaktionskosten aus dem Bond bezahlt. Eine optimale Strategie konnte im eindimensionalen Fall in der Klasse der Konstante-Grenzen-Strategien gefunden werden, die durch vier Parameter $a < \alpha \leq \beta < b \in]0, 1[$ beschrieben werden. Hierbei stellt $]a, b[$ das Fortsetzungsgebiet des Wertanteilsprozesses dar und α , bzw. β , den Wertanteil, zu dem bei Erreichen von a , bzw. b , zurückgehandelt wird. Für das zweite Modell, in dem die fixen Transaktionskosten proportional von jedem Anlageposten bezahlt werden, wurde in [Tam06] und [Tam08] die Existenz einer allgemeinen optimalen Strategie mit Hilfe einer schwachen Lösung einer Quasivariationsungleichung nachgewiesen.

Für diese Lösung zeigen wir zunächst im allgemeinen Fall $d \geq 1$ ein Maximumsprinzip und untersuchen das Fortsetzungsgebiet, was im eindimensionalen Fall den Nachweis der Existenz einer optimalen Konstante-Grenzen-Strategie für das zweite Modell ermöglicht. Der Unterschied der Modelle bezüglich der optimalen Grenzen und der asymptotischen Rendite wird dann numerisch behandelt.

Im zweiten Teil wird für den eindimensionalen Fall das asymptotische Verhalten beider Modelle bei verschwindenden proportionalen Kosten untersucht. Die Konvergenz der beiden Modelle bezüglich der optimalen Grenzen und der asymptotischen Rendite gegen das in [MP95] untersuchte Modell mit rein fixen Transaktionskosten wird gezeigt und die Eindeutigkeit der optimalen Grenzen im Grenzfall nachgewiesen.

Abschließend beschäftigen wir uns mit dem Grenzverhalten der Modelle bei verschwindenden fixen Kosten. Wir führen in Anlehnung an [TKA88] und [AST01] ein geeignetes Grenzmodell mit rein proportionalen Transaktionskosten ein und weisen nach, dass eine optimale Strategie darin besteht, den Wertanteilsprozess durch infinitesimales Handeln in einem eindeutig bestimmten Intervall $[A, B] \subseteq]0, 1[$ zu halten. Dabei wird auch die Konvergenz der optimalen Grenzen, der asymptotischen Renditen und der Wertanteilsprozesse gezeigt.

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Introduction

One of the most important questions an investor in a financial market faces is that of how to invest his money to get the optimal return. Striving for an answer, he has to clarify several decisive factors that will affect his decision such as

- his preferences concerning time horizon, risk, the amount to be invested, the selection of assets and the time for his investment activities he has at his disposal,
- the nature and number of the assets he can invest in and the information he has about the history and trend of their prices,
- the investment modalities of the financial market such as trading hours, costs, partners, availability and many others.

If he then decides to benefit from the achievements of modern Mathematical Finance on the field of portfolio optimization, he has to choose a probabilistic financial market model that suits the factors above and in particular decide on

- the nature of the stochastic processes that model the assets
- in discrete time or continuous time
- on a finite time horizon or an infinite time horizon,
- the structure of transaction costs and
- a utility function and maximization criterion that model his own preferences.

In our work, we focus on the continuous-time framework of the Black-Scholes model consisting of one risk-free asset B , called the bond, and d stocks S^1, \dots, S^d satisfying

$$dB_t = rB_t dt, \quad B_0 = b_0,$$
$$dS_t^i = \mu^i S_t^i dt + \sum_{j=1}^d \sigma^{ij} S_t^i dW_t^j, \quad S_0^i = s_0^i,$$

with trend μ , volatility σ , interest rate r and a d -dimensional Brownian motion W adapted to the standard filtration $(\mathcal{F}_t)_{t \geq 0}$ modeling the information available at the time t .

The wealth of the investor at the time t is then denoted by V_t and is the sum of his positions in the stocks and the bond.

To keep the investor's preferences tractable their modeling is usually twofold. The benefit he derives from capital is modeled via some *utility function* $u : \mathbb{R}_{\geq 0} \rightarrow [-\infty, 0[$ that satisfies different conditions such as concavity and monotonicity, while his time preferences and investment objectives are contained in the so-called *maximization criterion*.

One possible criterion can be the maximization of the *expected discounted utility from consumption*

$$E \left(\int_0^T e^{-\delta t} u(c(t)) dt \right)$$

on a time horizon $T \in]0, \infty]$, where $c(t)$ denotes the money the investor takes out from his wealth for consumption and δ is the discount factor.

Yet another problem is maximizing the *expected utility of terminal wealth* for a finite time horizon $T > 0$, i.e.

$$E(u(V_T)).$$

The criterion we will concentrate on in this work is the so-called *Kelly-criterion* of maximizing the asymptotic growth rate with logarithmic utility

$$\liminf_{T \rightarrow \infty} \frac{1}{T} E(\log V_T),$$

going back to Kelly in [Kel56].

Without transaction costs Merton considered in [Mer69] the logarithmic and power utility and solved the discounted consumption problem on finite and infinite time horizons. The other optimization criterions can be treated via simple pointwise maximization and it can essentially be obtained that the optimal strategy consist of keeping the fraction of total wealth invested in each asset constant. In Merton's honour this fraction is called the *Merton ratio* or *fraction*.

Unfortunately, it is more likely that our investor will face different types of costs such as brokerage and management fees, search and information costs, commissions and many others. For these kind of costs the notion *transaction costs* shall be used.

There are basically three different approaches to model transaction costs. The earliest approach goes back to Magill and Constantinides in [MC76] and considers *proportional transaction costs*, i.e. the investor has to pay a fixed proportion of each trading volume. For the discounted consumption criterion they conjectured for the case of one stock that the optimal strategy is to keep the risky fraction process in a certain interval $[A, B] \subset]0, 1[$ with minimal effort. The resulting process is a reflected diffusion with infinitesimal trading at the boundaries A, B . This was then proved by Davis and Norman for the logarithmic and power utility in [DN90]. Shreve and Soner analyzed rigorously the optimal strategy in

[SS94] and established the value function as the unique solution to the Hamilton-Jacobi-Bellman equation, using viscosity solution techniques. The general case $d \geq 1$ was then treated in [AMS96].

The problem of maximizing the asymptotic growth rate under proportional transaction costs was solved by Taksar, Klass, and Assaf in [TKA88] for one stock, where only selling stocks was punished. They derived the same structure of the optimal strategy as Davis and Norman. Using viscosity techniques Akian, Sulem, and Taksar proved existence of a solution to the HJB-equation, which is of variational form, in the d -stock case in [AST01].

Considering *fixed transaction costs*, i.e. at every transaction a fixed proportion of the investor's wealth has to be paid, solves the problem of the rather unfeasible infinitesimal trading of purely proportional costs and represents the second approach.

Here, the so-called *impulse control theory* is applied and the strategies are of impulsive form, i.e. they are given via a sequence $((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ consisting of stopping times τ_n with respect to $(\mathcal{F}_t)_{t \geq 0}$ that denote the trading times and satisfy

(i) $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ with $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$,

(ii) $P\left(\lim_{n \rightarrow \infty} \tau_n = \infty\right) = 1$

and \mathcal{F}_{τ_n} -measurable \mathbb{R}^d -valued random variables η_n , describing the trading volume at τ_n . The optimal strategy in the one stock case is to wait until the risky fraction process reaches the boundary of some interval $[A, B] \subset]0, 1[$ containing the Merton fraction and then trade back to some fixed fraction in $]A, B[$ near the Merton fraction and restart the process. This optimal behavior was described for the Kelly criterion in [MP95]. Bielecki and Pliska then generalized the results from [MP95] in several ways by characterizing the optimal strategies in terms of solutions to quasi-variational inequalities in [BP00], while existence and uniqueness results for solutions to these HJB-equations in quasi-variational form were established by Nagai in [Nag04] by applying a coordinate transformation to avoid degeneracy.

Despite of the feasibility of the optimal trading strategies under fixed transaction costs the cost structure seems rather unrealistic from the practitioner's point of view. To overcome this problem a combination of fixed and proportional transaction costs was suggested. In some cases as in [EH88] and [Kor98] the fixed component of the transaction costs is a constant amount not depending on the wealth. Here, the authors derive solutions for the discounted consumption criterion for the linear utility and asymptotically for the exponential utility.

However, our attention in the present work will be focused on the cases, where the fixed component of the transaction costs is a fixed proportion of the investor's wealth, as described above, under the maximization of the asymptotic growth rate. The trading strategies are therefore of impulsive form and the costs paid by the investor at time τ_n are of the size

$$c(V_{\tau_n}, \eta_n) = \delta V_{\tau_n} + \sum_{i=1}^d \gamma^i |\eta_n^i|,$$

where $\delta \in]0, 1[$ denotes the fixed part of the costs, while the proportional part is described by $\gamma \in \mathbb{R}_{\geq 0}^d$ with $\sum_{i=1}^d \gamma^i < 1 - \delta$.

Inspired by the results from [MP95] for purely fixed costs Irlle and Sass introduced in [IS05] the class of the so-called *constant boundary strategies* for the one-dimensional case and proved their optimality via a solution to the HJB-equation in quasi-variational form, rigorously constructed in [IS06]. These strategies can be described via four constants $a < \alpha \leq \beta < b \in]0, 1[$ such that $[a, b]$ is the continuation region, or the no-trade region, of the risky fraction process that is then restarted via trading in α , respectively in β , when it reaches the boundary in a , respectively in b .

Though this result itself is appealing, there is little hope of solving the general case $d \geq 1$ by their method of proof. Here a more general solution approach is needed as it is used by Tamura. He adapted the methods from [Nag04] to the one-dimensional case in [Tam06] and to the general case in [Tam08] and derived an optimal strategy via a solution to the corresponding HJB-equation in quasi-variational form. The solution itself, however, is derived via perturbation methods and results on quasi-variational inequalities with discount factors from [BL82] and [BL84] and it infers little structure on the no-trade region in general. Furthermore, the fixed costs in his model were paid from the stocks and the bond, while in [IS06] they were paid from the bond, leaving Tamura's model less realistic.

The contribution of the present work is therefore twofold. On the one hand we establish more information on Tamura's solution and on the continuation region in the general case $d \geq 1$ and show how the existence of an optimal constant boundary strategy in the one-dimensional case can be derived. Using numerical results, the difference to the model from [IS06] can be shown in terms of the optimal boundaries and optimal asymptotic growth rates.

On the other hand we give a connecting link between the different transaction cost models. We prove here, that letting the proportional costs given by γ tend to zero, the models from [IS06] and [Tam08] (for $d=1$) converge to the model from [MP95] with regard to the optimal boundaries and asymptotic growth rates. Furthermore, the convergence of these models, when the fixed costs δ tend to zero, to a model with only proportional costs corresponding to that from [TKA88] is established in terms of optimal boundaries, asymptotic growth rates and optimal risky fraction processes.

The thesis is organized in the following manner. We introduce in Chapter 1 the two models with fixed and proportional costs from [IS06] and [Tam08] in the general setting of $d \geq 1$ stocks and combine the results derived in [IP09] and [Tam08] related to the representation of the maximization problem via risky fraction processes and the subsequent application of coordinate transformation to the whole space \mathbb{R}^d .

In Chapter 2 basic properties of the solution to the HJB-equation from [IS06] are stated in their original form on the interval $[0, 1]$ and are then translated to \mathbb{R} via the coordinate transformation. Furthermore, an optimal stopping result is established.

Tamura's solution from [Tam08] is then studied in Chapter 3. We first introduce the standard HJB-approach and show how under rather strong assumptions on the smoothness of a solution to the corresponding quasi-variational inequality an optimal strategy can be derived. Then a version of Itô's formula for expectations is established for the only weakly differentiable solution from [Tam08]. This allows us to correct some inconsistencies and infer an optimal strategy the same way as Tamura. Finally, we derive in Section 3.3 several results concerning the smoothness and continuation region of Tamura's solution in the general case $d \geq 1$ and prove the existence of an optimal constant boundary strategy in the one-dimensional case in Section 3.4.

We compare both models in Chapter 4 and prove differences in the one-dimensional case concerning the optimal asymptotic growth rates and the boundaries of the optimal constant boundary strategies by conducting numerical calculations.

The convergence of the models, when the proportional costs tend to zero, to the model of [MP95] is investigated in Chapter 5. If we denote by $a(\delta, \gamma)$, $\alpha(\delta, \gamma)$, $\beta(\delta, \gamma)$, $b(\delta, \gamma)$ the boundaries of the optimal constant boundary strategies and by $\rho(\delta, \gamma)$ the optimal asymptotic growth rate depending on the transaction costs given by δ, γ , then the convergence

$$\lim_{\gamma \rightarrow 0} a(\delta, \gamma) = a(\delta, 0), \lim_{\gamma \rightarrow 0} \alpha(\delta, \gamma) = \alpha(\delta, 0) = \beta(\delta, 0) = \lim_{\gamma \rightarrow 0} \beta(\delta, \gamma), \lim_{\gamma \rightarrow 0} b(\delta, \gamma) = b(\delta, 0)$$

and also $\lim_{\gamma \rightarrow 0} \rho(\delta, \gamma) = \rho(\delta, 0)$ is obtained. This is done for the models from [IS06] and [Tam08] separately. Moreover, the uniqueness of the optimal boundaries $a(\delta, 0)$, $\alpha(\delta, 0)$, $\beta(\delta, 0)$, $b(\delta, 0)$ is established.

Finally, we treat the convergence in case of vanishing fixed costs δ in Chapter 6. We introduce a suitable model with only proportional costs similar to that of [TKA88] and obtain a verification theorem for the corresponding HJB-approach in Section 6.1. Then we establish the convergence

$$\lim_{\delta \rightarrow 0} a(\delta, \gamma) = A = \lim_{\delta \rightarrow 0} \alpha(\delta, \gamma), \lim_{\delta \rightarrow 0} \beta(\delta, \gamma) = B = \lim_{\delta \rightarrow 0} b(\delta, \gamma) \text{ and } \lim_{\delta \rightarrow 0} \rho(\delta, \gamma) = \rho(0, \gamma)$$

for the model from [IS06] and obtain the optimality of the reflected risky fraction process on $[A, B]$ for the model with $\delta = 0$ in Section 6.2. This is derived from the uniform convergence of the solutions to the corresponding HJB-equations. Furthermore, the uniqueness of the optimal boundaries A, B is obtained and the weak convergence of the risky fraction processes to the optimal reflected risky process is inferred.

In the last section the model from [Tam08] is considered, where a similar convergence of the boundaries can be obtained.

Chapter 1

Portfolio models with fixed and proportional transaction costs

In this chapter we introduce two portfolio models with fixed and proportional transaction costs. In Section 1.1 we set up common notations and terminology in a general portfolio model. Model A from [IS05], [IS06] and [IP09] is then presented in Section 1.2, whereas Model B from [Tam06] and [Tam08] is introduced in Section 1.3. Finally, a transformation to \mathbb{R} of the corresponding optimization problem is established in Section 1.4.

1.1 A general portfolio model

We consider a financial market model with one bond B and d stocks S^1, \dots, S^d satisfying

$$\begin{aligned} B_t &= b_0 \exp\left(\int_0^t r_s ds\right), \\ S_t^i &= s_0^i \exp\left(\int_0^t \mu_s^i ds + \sum_{j=1}^d \left(\int_0^t \sigma_s^{ij} dW_s^j - \frac{1}{2} \int_0^t (\sigma_s^{ij})^2 ds\right)\right) \end{aligned} \tag{1.1}$$

and hence

$$\begin{aligned} dB_t &= r_t B_t dt, \quad B_0 = b_0, \\ dS_t^i &= \mu_t^i S_t^i dt + \sum_{j=1}^d \sigma_t^{ij} S_t^i dW_t^j, \quad S_0^i = s_0^i \end{aligned} \tag{1.2}$$

for every $i \in \{1, \dots, d\}$ and $t \in [0, \infty[$, with constant starting values s_0^i , b_0 and a d -dimensional Brownian motion $W = (W^1, \dots, W^d)$ adapted to the standard filtration $(\mathcal{F}_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$.

The nonnegative *interest rate process* $(r_t)_{t \geq 0}$, the \mathbb{R}^d -valued *trend process*

$(\mu_t)_{t \geq 0} = ((\mu_t^1, \dots, \mu_t^d))_{t \geq 0}$ and the $\mathbb{R}^{d \times d}$ -valued, *positive definite volatility matrix process* $(\sigma_t)_{t \geq 0} = ((\sigma_t^{ij})_{i,j \in \{1, \dots, d\}})_{t \geq 0}$ are supposed to be progressively measurable and satisfy

$$\int_0^t (r_s + \|\mu_s\| + \|\sigma_s\|^2) ds < \infty \text{ a.s. for every } t \geq 0. \quad (1.3)$$

By X_t , respectively Y_t^i , we denote the amount of money the investor has invested in the bond, respectively the i -th stock, at time t and define $V_t := X_t + \sum_{i=1}^d Y_t^i$ to be the *wealth* or *portfolio value*.

We do not allow short selling or borrowing and therefore have

$$X_t \geq 0 \text{ and } Y_t^i \geq 0 \text{ for all } i \in \{1, \dots, d\}. \quad (1.4)$$

Assuming $V_t > 0$, we can define the *risky fraction process* $h_t = (h_t^1, \dots, h_t^d)$ by

$$h_t^i := \frac{Y_t^i}{V_t} \text{ for every } i \in \{1, \dots, d\} \text{ and } h_t^0 := \frac{X_t}{V_t} \text{ for every } t \geq 0. \quad (1.5)$$

The assumption $V_t > 0$ above is not really restrictive, since for all trading strategies considered in the following it will be a consequence of (1.4) (cf. Lemmas 1.4 and 1.14).

We proceed by defining the simplex

$$\Delta := \left\{ h \in \mathbb{R}^d : \sum_{i=1}^d h^i \leq 1, h^i \in [0, 1] \text{ for every } i \in \{1, \dots, d\} \right\} \quad (1.6)$$

consisting of all risky fractions and suppose further a constant starting wealth $v_0 > 0$ and a constant starting fraction $h_0 = (h_0^1, \dots, h_0^d) \in \Delta$.

In our future models the investor will face fixed and proportional transaction costs, therefore sensible trading can only occur at discrete times τ_n , $n \in \mathbb{N}_0$, and we denote by η_n^i the transaction volume in the i -th stock at time τ_n , $n \in \mathbb{N}_0$, $i \in \{1, \dots, d\}$. Hence the natural class of trading strategies are *impulse control strategies*.

Definition 1.1. An *impulse control strategy* is a sequence $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ consisting of stopping times τ_n , $n \in \mathbb{N}_0$, with respect to $(\mathcal{F}_t)_{t \geq 0}$, satisfying

- (i) $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ with $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$,
- (ii) $P\left(\lim_{n \rightarrow \infty} \tau_n = \infty\right) = 1$

and \mathcal{F}_{τ_n} -measurable \mathbb{R}^d -valued random variables $\eta_n = (\eta_n^1, \dots, \eta_n^d)$, $n \in \mathbb{N}_0$, satisfying

$$\eta_n = 0 \text{ on } \{\tau_n = \infty\} \text{ for all } n \in \mathbb{N}. \quad (1.7)$$

Here, the condition $\tau_0 = 0$ is just for technical reasons and since trading in ∞ will have no effect on the growth rate in (1.9), we restrict ourselves to (1.7) for simplicity.

We will later have to define the class of *admissible* trading strategies

$$\mathcal{A} := \{K : K \text{ is an admissible impulse control strategy}\}, \quad (1.8)$$

which of course will depend on the transaction costs and how they are paid. For every $K \in \mathcal{A}$ we consider the corresponding wealth process $(V_t^K)_{t \geq 0}$ and the expected growth rate

$$J(K) := \liminf_{T \rightarrow \infty} \frac{1}{T} E(\log V_T^K). \quad (1.9)$$

The general aim is to maximize J over all trading strategies, i.e. to find the value

$$\rho := \sup_{K \in \mathcal{A}} J(K) \quad (1.10)$$

and the corresponding maximizing trading strategy K^* , if existing, satisfying

$$\rho = J(K^*). \quad (1.11)$$

Given an admissible trading strategy $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$, we can denote in Sections 1.2 and 1.3 by g_t^i , $i \in \{0, 1, \dots, d\}$, the number of stocks i or bonds held by the investor at time t . Then the assumptions of the following lemma will be satisfied and we can use its representations for the process $\log V^K$ from above and the corresponding risky fraction process h^K .

Lemma 1.2. *Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times satisfying*

(i) $0 \leq \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ with $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$,

(ii) $P\left(\lim_{n \rightarrow \infty} \tau_n = \infty\right) = 1$.

Let further for every $i \in \{0, 1, \dots, d\}$ $(g_t^i)_{t \geq 0}$ be a piecewise constant, $\mathbb{R}_{\geq 0}$ -valued, adapted càdlàg (continue à droite, limitée à gauche, i.e. right continuous with left limits) process with jumps only at τ_n on $\{\tau_n < \infty\}$, $n \in \mathbb{N}_0$, and $\sum_{i=0}^d g_t^i > 0$ for every $t \geq 0$.

We define

$$X_t := g_t^0 B_t, \quad Y_t^i := g_t^i S_t^i, \quad i \in \{1, \dots, d\}, \quad (1.12)$$

and set

$$V_t := X_t + \sum_{i=1}^d Y_t^i \quad \text{and} \quad h_t^i := \frac{Y_t^i}{V_t}, \quad i \in \{1, \dots, d\}, \quad (1.13)$$

for every $t \geq 0$. Then we have for every $t \geq 0$

$$\begin{aligned} \log V_t - \log V_0 &= \int_0^t h_{s-}^T \sigma_s dW_s + \int_0^t r_s - \frac{1}{2} h_{s-}^T \sigma_s \sigma_s^T h_{s-} + (\mu_s - r_s \mathbb{1})^T h_{s-} ds \\ &\quad + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_k-}) \mathbb{1}_{\{\tau_k \leq t\}}, \end{aligned} \quad (1.14)$$

where $\mathbb{1} = (1, \dots, 1)^T \in \mathbb{R}^d$, and for $i \in \{1, \dots, d\}$ also

$$\begin{aligned}
h_t^i - h_0^i &= \int_0^t h_{s-}^i (e_i^T - h_{s-}^T) \sigma_s dW_s + \int_0^t h_{s-}^i (e_i^T - h_{s-}^T) (\mu_s - r_s \mathbb{1} - \sigma_s \sigma_s^T h_{s-}) ds \\
&\quad + \sum_{k=0}^{\infty} \Delta h_{\tau_k}^i \mathbb{1}_{\{\tau_k \leq t\}},
\end{aligned} \tag{1.15}$$

where e_i is the i -th unit vector in \mathbb{R}^d . If we further define $h_t^0 := 1 - \sum_{i=1}^d h_t^i$, then we get in the case of $h_t \in \Delta \setminus \partial\Delta$ for all $t \geq 0$

$$\log \left(\frac{h_t^0}{h_0^0} \right) = - \int_0^t h_{s-}^T \sigma_s dW_s + \int_0^t \frac{1}{2} h_{s-}^T \sigma_s \sigma_s^T h_{s-} - (\mu_s - r_s \mathbb{1})^T h_{s-} ds + \sum_{k=0}^{\infty} \log \left(\frac{h_{\tau_k}^0}{h_{\tau_k-}^0} \right) \mathbb{1}_{\{\tau_k \leq t\}}.$$

Proof. a) Since all the processes considered here are (càdlàg) semimartingales, we get by setting $S_t^0 := B_t$ for all $t \geq 0$ by the generalized Itô's formula (cf. [JS03], Theorem I.4.57) for every $t \geq 0$ and $i \in \{0, 1, \dots, d\}$

$$\begin{aligned}
g_t^i S_t^i - g_0^i S_0^i &= \int_0^t S_s^i dg_s^i + \int_0^t g_{s-}^i dS_s^i + \sum_{s \leq t} (\Delta(g_s^i S_s^i) - S_s^i \Delta g_s^i) \\
&= \int_0^t S_s^i dg_s^i + \int_0^t g_{s-}^i dS_s^i
\end{aligned} \tag{1.16}$$

and hence

$$dV_t = d \left(\sum_{i=0}^d S_t^i g_t^i \right) = \sum_{i=0}^d S_t^i dg_t^i + \sum_{i=0}^d g_{t-}^i dS_t^i. \tag{1.17}$$

Now we get from (1.17) and the continuity of S^i

$$\begin{aligned}
[V^c, V^c]_t &= \left[\sum_{i=0}^d \int_0^t g_{s-}^i dS_s^i \right] = \sum_{i,j=0}^d \int_0^t g_{s-}^i g_{s-}^j d[S^i, S^j]_s \\
&= \sum_{i,j,l=1}^d \int_0^t h_{s-}^i h_{s-}^j (V_{s-})^2 \sigma_s^{il} \sigma_s^{jl} ds.
\end{aligned} \tag{1.18}$$

Itô's formula therefore implies

$$\begin{aligned}
\log V_t - \log V_0 &= \int_0^t \frac{1}{V_{s-}} dV_s - \frac{1}{2} \int_0^t \frac{1}{(V_{s-})^2} d[V^c, V^c]_s + \sum_{s \leq t} \left(\Delta \log V_s - \frac{1}{V_{s-}} \Delta V_s \right) \\
&\stackrel{(1.18)}{=} \int_0^t \frac{1}{V_{s-}} dV_s - \frac{1}{2} \int_0^t h_{s-}^T \sigma_s \sigma_s^T h_{s-} ds + \sum_{s \leq t} \left(\Delta \log V_s - \frac{1}{V_{s-}} \Delta V_s \right)
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(1.17)}{=} \sum_{k=0}^{\infty} \frac{1}{V_{\tau_k^-}} \Delta V_{\tau_k} \mathbb{1}_{\{\tau_k \leq t\}} + \int_0^t h_{s-}^T \sigma_s dW_s + \int_0^t r_s + (\mu_s - r_s \mathbb{1})^T h_{s-} ds \\
& \quad - \frac{1}{2} \int_0^t h_{s-}^T \sigma_s \sigma_s^T h_{s-} ds + \sum_{s \leq t} \left(\Delta \log V_s - \frac{1}{V_{s-}} \Delta V_s \right) \\
& = \int_0^t h_{s-}^T \sigma_s dW_s + \int_0^t r_s - \frac{1}{2} h_{s-}^T \sigma_s \sigma_s^T h_{s-} + (\mu_s - r_s \mathbb{1})^T h_{s-} ds \\
& \quad + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_k^-}) \mathbb{1}_{\{\tau_k \leq t\}}.
\end{aligned}$$

b) For the second assertion we first conclude from (1.16) and (1.17)

$$[Y^{i,c}, V^c]_t = \sum_{j,l=1}^d \int_0^t Y_{s-}^i Y_{s-}^j \sigma_s^{il} \sigma_s^{jl} ds, \quad i \in \{1, \dots, d\}. \quad (1.19)$$

For $i \in \{1, \dots, d\}$ we then get, again using Itô's formula and the definition of h_t^i in (1.13)

$$\begin{aligned}
h_t^i - h_0^i &= \int_0^t \frac{1}{V_{s-}} dY_s^i - \int_0^t \frac{Y_{s-}^i}{(V_{s-})^2} dV_s - \int_0^t \frac{1}{(V_{s-})^2} d[Y^{i,c}, V^c]_s + \int_0^t \frac{Y_{s-}^i}{(V_{s-})^3} d[V^c, V^c]_s \\
& \quad + \sum_{k=0}^{\infty} \left(\Delta \frac{Y_{\tau_k}^i}{V_{\tau_k}} - \frac{\Delta Y_{\tau_k}^i}{V_{\tau_k^-}} + \frac{Y_{\tau_k^-}^i}{(V_{\tau_k^-})^2} \Delta V_{\tau_k} \right) \mathbb{1}_{\{\tau_k \leq t\}} \\
& \stackrel{(1.19)}{=} \int_0^t \frac{1}{V_{s-}} dY_s^i - \int_0^t \frac{Y_{s-}^i}{(V_{s-})^2} dV_s - \int_0^t h_{s-}^i e_i^T \sigma_s \sigma_s^T h_{s-} ds + \int_0^t \frac{Y_{s-}^i}{(V_{s-})^3} d[V^c, V^c]_s \\
& \quad + \sum_{k=0}^{\infty} \left(\Delta \frac{Y_{\tau_k}^i}{V_{\tau_k}} - \frac{\Delta Y_{\tau_k}^i}{V_{\tau_k^-}} + \frac{Y_{\tau_k^-}^i}{(V_{\tau_k^-})^2} \Delta V_{\tau_k} \right) \mathbb{1}_{\{\tau_k \leq t\}} \\
& \stackrel{(1.18)}{=} \int_0^t \frac{1}{V_{s-}} dY_s^i - \int_0^t \frac{Y_{s-}^i}{(V_{s-})^2} dV_s - \int_0^t h_{s-}^i (e_i^T - h_{s-}^T) \sigma_s \sigma_s^T h_{s-} ds \\
& \quad + \sum_{k=0}^{\infty} \left(\Delta \frac{Y_{\tau_k}^i}{V_{\tau_k}} - \frac{\Delta Y_{\tau_k}^i}{V_{\tau_k^-}} + \frac{Y_{\tau_k^-}^i}{(V_{\tau_k^-})^2} \Delta V_{\tau_k} \right) \mathbb{1}_{\{\tau_k \leq t\}} \\
& \stackrel{(1.17)}{=} \int_0^t \frac{1}{V_{s-}} S_s^i dg_s^i + \int_0^t \frac{1}{V_{s-}} g_s^i dS_s^i - \sum_{j=0}^d \int_0^t \frac{Y_{s-}^i}{(V_{s-})^2} S_s^j dg_s^j - \sum_{j=0}^d \int_0^t \frac{Y_{s-}^i}{(V_{s-})^2} g_{s-}^j dS_s^j \\
& \quad - \int_0^t h_{s-}^i (e_i^T - h_{s-}^T) \sigma_s \sigma_s^T h_{s-} ds + \sum_{k=0}^{\infty} \left(\Delta \frac{Y_{\tau_k}^i}{V_{\tau_k}} - \frac{\Delta Y_{\tau_k}^i}{V_{\tau_k^-}} + \frac{Y_{\tau_k^-}^i}{(V_{\tau_k^-})^2} \Delta V_{\tau_k} \right) \mathbb{1}_{\{\tau_k \leq t\}}
\end{aligned}$$

$$\begin{aligned}
&= \int_0^t h_{s-}^i (e_i^T - h_{s-}^T) \sigma_s dW_s + \int_0^t h_{s-}^i (e_i^T - h_{s-}^T) (\mu_s - r_s \mathbb{1} - \sigma_s \sigma_s^T h_{s-}) ds \\
&\quad + \sum_{k=0}^{\infty} \Delta h_{\tau_k}^i \mathbb{1}_{\{\tau_k \leq t\}}.
\end{aligned}$$

c) Now we additionally assume $h_t \in \Delta \setminus \partial\Delta$ for all $t \geq 0$. Then Itô's formula yields

$$\begin{aligned}
\log \frac{h_t^0}{h_0^0} &= - \sum_{i=1}^d \int_0^t \frac{1}{h_{s-}^0} dh_s^i - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{1}{(h_{s-}^0)^2} d[h^{i,c}, h^{j,c}]_s + \sum_{s \leq t} \left(\log \left(\frac{h_s^0}{h_{s-}^0} \right) + \sum_{i=1}^d \frac{\Delta h_s^i}{h_{s-}^0} \right) \\
&\stackrel{b)}{=} - \sum_{i=1}^d \int_0^t \frac{h_{s-}^i}{h_{s-}^0} (e_i^T - h_{s-}^T) \sigma_s dW_s - \sum_{i=1}^d \int_0^t \frac{h_{s-}^i}{h_{s-}^0} (e_i^T - h_{s-}^T) (\mu_s - r_s \mathbb{1} - \sigma_s \sigma_s^T h_{s-}) ds \\
&\quad - \frac{1}{2} \sum_{i,j,k=1}^d \int_0^t \frac{1}{(h_{s-}^0)^2} \left(h_{s-}^i (e_i^T - h_{s-}^T) \sigma_s \right)_k \left(h_{s-}^j (e_j^T - h_{s-}^T) \sigma_s \right)_k ds + \sum_{s \leq t} \log \left(\frac{h_s^0}{h_{s-}^0} \right) \\
&= - \int_0^t h_{s-}^T \sigma_s dW_s - \int_0^t (\mu_s - r_s \mathbb{1})^T h_{s-} ds + \int_0^t h_{s-}^T \sigma_s \sigma_s^T h_{s-} ds \\
&\quad - \int_0^t \frac{1}{2} h_{s-}^T \sigma_s \sigma_s^T h_{s-} ds + \sum_{k=0}^{\infty} \log \left(\frac{h_{\tau_k}^0}{h_{\tau_k-}^0} \right) \mathbb{1}_{\{\tau_k \leq t\}}.
\end{aligned}$$

□

1.2 Model A

In this section we introduce Model A, which is the càdlàg version of the models from [IS05], [IS06] and [IP09]. Furthermore, we state first results from [IP09], where the d -dimensional case is treated, and translate additional achievements from [Tam08] to this model.

We consider the Black-Scholes model described in Section 1.1, but with *deterministic* and *time-constant* interest rate $r \in \mathbb{R}_{\geq 0}$, trend $\mu = (\mu^1, \dots, \mu^d) \in \mathbb{R}^d$ and positive definite volatility matrix $\sigma = (\sigma^{ij})_{i,j \in \{1, \dots, d\}} \in \mathbb{R}^{d \times d}$.

Let V, X, Y and h denote the processes describing the portfolio from Section 1.1 and suppose fixed starting values $v_0 > 0$, $h_0 \in \Delta$ are given.

We assume the investor faces investment fees, given by the cost function

$$c : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}, (x, \eta) \mapsto \delta x + \sum_{i=1}^d \gamma^i |\eta^i|, \quad (1.20)$$

where $\delta \in]0, 1[$ denotes the fraction of the portfolio value (fixed costs) and $\gamma^1, \dots, \gamma^d \in [0, 1 - \delta[$ are the fractions of the transaction volume η^i in the i -th stock that have to be paid for every transaction η (proportional costs).

As described in Section 1.1 trading strategies are impulse control strategies $((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ from Definition 1.1, and we define that after the n -th trading the assets become

$$\begin{aligned} X_{\tau_n} &= X_{\tau_n-} - \sum_{i=1}^d \eta_n^i - c(V_{\tau_n-}, \eta_n), \\ Y_{\tau_n}^i &= Y_{\tau_n-}^i + \eta_n^i, \quad i \in \{1, \dots, d\}, \end{aligned} \quad (1.21)$$

on $\{\tau_n < \infty\}$ and hence

$$\begin{aligned} V_{\tau_n} &= V_{\tau_n-} - c(V_{\tau_n-}, \eta_n), \\ h_{\tau_n} &= V_{\tau_n}^{-1} Y_{\tau_n} \in \Delta. \end{aligned} \quad (1.22)$$

Between the trading times the processes are supposed to evolve according to (1.2), i.e. the number of bonds or stocks i held by the investor has to be constant. Since we do not allow short selling or borrowing we can now define admissible trading strategies.

Definition 1.3. An impulse control strategy $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ is an *admissible monetary strategy* if the corresponding processes from (1.21) satisfy

$$\begin{aligned} X_{\tau_n} &\geq 0 \quad \text{on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0, \\ Y_{\tau_n}^i &\geq 0 \quad \text{on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0, i \in \{1, \dots, d\}. \end{aligned} \quad (1.23)$$

Since admissibility clearly depends on the starting values v_0, h_0 , we define

$$\mathcal{A}_{v_0, h_0} := \{K : K \text{ is an admissible monetary strategy for } v_0, h_0\} \quad (1.24)$$

and we will sometimes write $X^{K, v_0, h_0}, Y^{K, v_0, h_0}$ and V^{K, v_0, h_0} for the processes if needed.

Lemma 1.4. *The condition (1.23) is equivalent to*

$$\begin{aligned} V_{\tau_n} &> 0 \quad \text{on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0, \\ h_{\tau_n} &\in \Delta \quad \text{on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0. \end{aligned} \quad (1.25)$$

Proof. We only have to show that (1.23) implies $V_{\tau_n} > 0$ for all $n \in \mathbb{N}_0$. This is carried out by induction.

Suppose we have $V_{\tau_n} > 0$ on $\{\tau_{n+1} < \infty\}$ and since we do not trade until τ_{n+1} we get $V_{\tau_{n+1}-} > 0$ on $\{\tau_{n+1} < \infty\}$. The interesting case on $\{\tau_{n+1} < \infty\}$ is $Y_{\tau_{n+1}}^i = 0$ and hence $-\eta_{n+1}^i = Y_{\tau_{n+1}-}^i$ by (1.21) for all $i \in \{1, \dots, d\}$. Now we have on $\{\tau_{n+1} < \infty\}$

$$X_{\tau_{n+1}} = X_{\tau_{n+1}-} - \sum_{i=1}^d \eta_{n+1}^i - c(V_{\tau_{n+1}-}, \eta_{n+1}) = (1 - \delta)V_{\tau_{n+1}-} + \sum_{i=1}^d \gamma^i \eta_{n+1}^i > 0,$$

since $\gamma^i \in [0, 1 - \delta[$ for every $i \in \{1, \dots, d\}$.

□

Given an admissible monetary strategy $((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ and the corresponding risky fraction process $(h_t)_{t \geq 0}$ we get from Lemma 1.4

$$\xi_n := h_{\tau_n} \in \Delta \text{ on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0. \quad (1.26)$$

We therefore define *admissible proportional strategies* (or *New Risky Fraction strategies* in [IP09]), which do not depend on the starting values v_0, h_0 .

Definition 1.5. An impulse control strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is an *admissible proportional strategy* if $\xi_n \in \Delta$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$. In analogy to Definition 1.3 we define

$$\tilde{\mathcal{A}} := \left\{ \tilde{K} : \tilde{K} \text{ is an admissible proportional strategy} \right\}. \quad (1.27)$$

If we define $\xi_n \equiv 0$ on $\{\tau_n = \infty\}$ in (1.26), we can see that an admissible proportional strategy can be deduced from an admissible monetary strategy. In fact, the following theorem shows the one-to-one relationship between these two kinds of strategies.

Theorem 1.6. For every $v > 0$ and $h \in \Delta$ let

$$A_{v,h} := \left\{ \eta \in \mathbb{R}^d : h^0 v - c(v, \eta) - \sum_{i=1}^d \eta^i \geq 0 \text{ and } h^i v + \eta^i \geq 0, i \in \{1, \dots, d\} \right\},$$

where $h^0 := 1 - \sum_{i=1}^d h^i$ and define

$$f_{v,h} : A_{v,h} \rightarrow \Delta, \eta \mapsto \left(\frac{h^1 v + \eta^1}{v - c(v, \eta)}, \dots, \frac{h^d v + \eta^d}{v - c(v, \eta)} \right). \quad (1.28)$$

Then $f_{v,h}$ is bijective and given an admissible proportional strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ we get the corresponding admissible monetary strategy $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0} \in \mathcal{A}_{v_0, h_0}$ for fixed starting values v_0, h_0 through the inductive relationship

$$\eta_n = f_{V_{\tau_n-}, h_{\tau_n-}}^{-1}(\xi_n) \text{ on } \{\tau_n < \infty\} \text{ and } \eta_n \equiv 0 \text{ on } \{\tau_n = \infty\}, n \in \mathbb{N}_0.$$

Therefore the function $\tilde{\cdot} : \mathcal{A}_{v_0, h_0} \rightarrow \tilde{\mathcal{A}}, K \mapsto \tilde{K}$ is bijective.

Proof. Cf. Lemma 2.3 and Theorem 2.4 in [IP09]. □

We can also reformulate the cost function to risky fractions in the following theorem. This representation is due to [Tam08] and the proof is an adaption to this model.

Theorem 1.7. There exists a constant $\kappa \in]0, 1[$ and a Lipschitz continuous function

$$C : \Delta^2 \rightarrow [0, \kappa], \quad (1.29)$$

such that for every admissible proportional strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$

$$V_{\tau_n} = (1 - \delta)(1 - C(h_{\tau_n-}, \xi_n))V_{\tau_n-} \quad (1.30)$$

holds on $\{\tau_n < \infty\}$ for every $n \in \mathbb{N}_0$.

Proof. We adapt the ideas of Theorem 2.1 in [Tam08], where some parts of the proof were omitted, to our situation.

a) Let for every $v > 0$ and $h \in \Delta$ $f_{v,h} : A_{v,h} \rightarrow \Delta$ be the bijective function introduced in Theorem 1.6. In a first step we give a useful transformation of $f_{v,h}$ in terms of matrices.

Rewriting the value process in (1.22), we get after the n -th trading on $\{\tau_n < \infty\}$

$$V_{\tau_n} = (1 - \delta) \left(V_{\tau_{n-}} - \sum_{i=1}^d \frac{\gamma^i}{1 - \delta} |\eta_n^i| \right). \quad (1.31)$$

Therefore we define the auxiliary cost function

$$\tilde{c} : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}, \quad \eta \mapsto \sum_{i=1}^d \frac{\gamma^i}{1 - \delta} |\eta^i|$$

and define further for every $\pi \in \{-1, 0, 1\}^d$ the matrix $K^{h,\pi} \in \mathbb{R}^{d \times d}$ by

$$K_{ij}^{h,\pi} := \frac{\gamma^j}{1 - \delta} h^i \pi_j \text{ for all } i, j \in \{1, \dots, d\}. \quad (1.32)$$

Now if we define the sign function $\pi : \mathbb{R}^d \rightarrow \{-1, 0, 1\}^d, \eta \mapsto (\text{sign}(\eta^1), \dots, \text{sign}(\eta^d))$, we get in (1.32) for every $\eta \in \mathbb{R}^d$ and $h \in \Delta$

$$(\text{Id} + K^{h,\pi(\eta)}) \eta = \eta + h \tilde{c}(\eta). \quad (1.33)$$

From the definition of $f_{v,h}$ we have for every $v > 0, h \in \Delta$ and $\eta \in A_{v,h}$

$$v (vh^i + \eta^i) = (\eta^i + (1 - \delta) f_{v,h}(\eta) \tilde{c}(\eta) + vh^i) (v - \tilde{c}(\eta))$$

and hence

$$(f_{v,h}(\eta))^i = \frac{vh^i + \eta^i}{(1 - \delta)(v - \tilde{c}(\eta))} \stackrel{(1.33)}{=} \frac{1}{1 - \delta} \left(\frac{1}{v} (\text{Id} + K^{(1-\delta)f_{v,h}(\eta), \pi(\eta)}) \eta + h \right). \quad (1.34)$$

b) As a second step we derive the new cost function C from (1.34).

Calculating for every $h \in \Delta$ and $\pi \in \{-1, 0, 1\}^d$ the matrix norm

$$\|K^{(1-\delta)h,\pi}\| \stackrel{(1.32)}{\leq} \max_j \gamma^j \sum_{i=1}^d h^i \leq \max_j \gamma^j \stackrel{(1.20)}{<} 1 - \delta \quad (1.35)$$

we get the regularity of $\text{Id} + K^{(1-\delta)h,\pi}$ and can write in (1.34)

$$\eta = v \left(\text{Id} + K^{(1-\delta)f_{v,h}(\eta), \pi(\eta)} \right)^{-1} \left((1 - \delta) f_{v,h}(\eta) - h \right).$$

Therefore we can rewrite the sign function by

$$\tilde{\pi} : \Delta^2 \rightarrow \{-1, 0, 1\}^d, \quad (h, \tilde{h}) \mapsto \pi \left(f_{1,h}^{-1}(\tilde{h}) \right)$$

and define for $v = 1$

$$F : \Delta^2 \rightarrow [-1, 1]^d, (h, \tilde{h}) \mapsto \left(\text{Id} + K^{(1-\delta)\tilde{h}, \tilde{\pi}(h, \tilde{h})} \right)^{-1} \left((1-\delta)\tilde{h} - h \right).$$

This establishes the relationship

$$f_{1,h} \left(F(h, \tilde{h}) \right) = \tilde{h} \quad \text{and} \quad F(h, f_{1,h}(\eta)) = \eta \quad (1.36)$$

and allows us to write in (1.30)

$$V_{\tau_n} = (1-\delta) \left(1 - \tilde{c}(F(h_{\tau_n-}, \xi_n)) \right) V_{\tau_n-},$$

so the function

$$C : \Delta^2 \rightarrow R_{\geq 0}, (h, \tilde{h}) \mapsto \tilde{c} \left(F(h, \tilde{h}) \right) \quad (1.37)$$

then satisfies (1.30).

c) As a last step it remains to show the boundedness and Lipschitz continuity of C . If we can show the Lipschitz continuity of F , then of course C will be Lipschitz continuous by the definition of \tilde{c} . But since we do not go bankrupt by condition (1.25) and have chosen $v = 1$ in the definition of F , we have $C(\Delta^2) \subseteq [0, 1[$ and then get the claimed κ in (1.29) from the compactness of Δ^2 .

c1) First we show the continuity of F .

Let $((h_n, \tilde{h}_n))_{n \in \mathbb{N}} \in \Delta^2$ be a convergent sequence with $\lim_{n \rightarrow \infty} (h_n, \tilde{h}_n) = (h, \tilde{h}) \in \Delta$. By the definition of F we have $\eta_n := F((h_n, \tilde{h}_n)) \in [-1, 1]^d$ for all $n \in \mathbb{N}$. Now given a subsequence $(\eta_{n_k})_{k \in \mathbb{N}}$ we can find a sub-subsequence $(\eta_{n_{k_l}})_{l \in \mathbb{N}}$ converging to some $\eta_0 \in [-1, 1]^d$. From (1.28) and $\lim_{l \rightarrow \infty} (\eta_{n_{k_l}}, h_{n_{k_l}}) = (\eta_0, h)$ we then can deduce using (1.36)

$$\tilde{h}_{n_{k_l}} = f_{1, h_{n_{k_l}}}(\eta_{n_{k_l}}) \xrightarrow{l \rightarrow \infty} f_{1,h}(\eta_0),$$

and therefore $\tilde{h} = f_{1,h}(\eta_0)$ and hence $F(h, \tilde{h}) = \eta_0 = \lim_{l \rightarrow \infty} \eta_{n_{k_l}} = \lim_{l \rightarrow \infty} F((h_{n_{k_l}}, \tilde{h}_{n_{k_l}}))$ holds.

c2) Now given a $\pi \in \{-1, 0, 1\}^d$ we show the Lipschitz continuity of

$$F^\pi : \Delta^2 \rightarrow [-1, 1]^d, (h, \tilde{h}) \mapsto \left(\text{Id} + K^{(1-\delta)\tilde{h}, \pi} \right)^{-1} \left((1-\delta)\tilde{h} - h \right).$$

The function

$$I : \Delta \rightarrow GL(\mathbb{R}^d), h \mapsto \left(\text{Id} + K^{(1-\delta)h, \pi} \right)^{-1}$$

is differentiable on $\text{int}(\Delta)$ with a bounded derivative and continuous on Δ and is therefore Lipschitz continuous. Thus F^π is also Lipschitz continuous as a combination of Lipschitz continuous functions.

c3) To show the Lipschitz continuity of F we can now conclude as follows. Let π be the sign function from above. For every $\lambda \in \{-1, 1\}^d$ the set $\pi^{-1}(\lambda)$ is open in \mathbb{R}^d and hence so is $O_\lambda := F^{-1}(\pi^{-1}(\lambda) \cap [-1, 1]^d)$. Now we have, using c2), that $F|_{O_\lambda} = F^\lambda|_{O_\lambda}$ is

Lipschitz continuous on O_λ . But since F is continuous, we get the Lipschitz continuity of F on Δ^2 if we can show

$$\Delta^2 = \overline{\cup\{O_\lambda : \lambda \in \{-1, 1\}^d\}}. \quad (1.38)$$

Let therefore $(h, \tilde{h}) \in \Delta^2 \setminus (\cup\{O_\lambda : \lambda \in \{-1, 1\}^d\})$ and U be an open neighbourhood of (h, \tilde{h}) in Δ^2 . Then there exists an open neighbourhood \tilde{U} of \tilde{h} in Δ such that $\tilde{U} \times \{h\} \subseteq U$. But since $f_{1,h}$ is continuous we have that $f_{1,h}^{-1}(\tilde{U})$ is open and hence

$$f_{1,h}^{-1}(\tilde{U}) \cap (\cup\{\pi^{-1}(\lambda) : \lambda \in \{-1, 1\}^d\}) \neq \emptyset.$$

Using $F(\tilde{U} \times \{h\}) \stackrel{(1.36)}{=} f_{1,h}^{-1}(\tilde{U})$ this implies

$$U \cap (\cup\{O_\lambda : \lambda \in \{-1, 1\}^d\}) \neq \emptyset$$

and thus (1.38). □

In [Tam08] it is shown that in Model B (cf. Theorem 1.16) the equivalent of the cost function above satisfies a certain triangle inequality, which is later used to find an optimal strategy. Unfortunately, the cost function from above does not satisfy the same inequality.

Example 1.8. Let C denote the cost function introduced in (1.29) for the case $d = 1$. Then for $\delta = 0.1$, $\gamma = 0.09$, $h = 0$, $g = 0.1$ and $f = 1$ the triangle inequality

$$(1 - C(h, g))(1 - C(g, f)) \leq (1 - C(h, f)) \quad (1.39)$$

does not hold.

Proof. We first calculate $C(h, g)$ for all $h, g \in \Delta$.

For every $\eta \in \mathbb{R}^d$ and $v > 0$ equation (1.28) implies

$$g = f_{v,h}(\eta) = \frac{vh + \eta}{(1 - \delta)(v - \lambda\pi(\eta))} \iff \eta = f_{v,h}^{-1}(g) = \frac{v((1 - \delta)g - h)}{1 + (1 - \delta)\lambda g\pi(\eta)}.$$

Hence, $\pi(\eta) = \pi((1 - \delta)g - h)$ holds due to $1 + (1 - \delta)\lambda g\pi(\eta) > 0$. From (1.36) and (1.37) we therefore get for $\lambda := \frac{\gamma}{1 - \delta}$

$$C(h, g) = \tilde{c}(f_{1,h}^{-1}(g)) = \lambda \left| \frac{(1 - \delta)g - h}{1 + (1 - \delta)\lambda g\pi((1 - \delta)g - h)} \right|.$$

Now taking $\delta := 0.1$, $\gamma := 0.09$, $h := 0$, $g := 0.1$ and $f := 1$ we get

$$(1 - C(h, g))(1 - C(g, f)) = \frac{1000}{1009} \cdot \frac{101}{109} > \frac{100}{109} = (1 - C(h, f)).$$

□

Remark 1.9. Since the function C in (1.39) is continuous, there are of course examples for $f, g, h \in \text{int}(\Delta)$ contradicting the triangle inequality above.

We conjecture that for every $d \in \mathbb{N}$, $\delta \in]0, 1[$ and $\gamma \in]0, 1 - \delta[$ there is a small $g \in]0, 1[$ that contradicts (1.39) for $h := 0$ and $f := 1$.

Now we state a more explicit representation from [IP09] for the cost function C above.

Theorem 1.10. *Let C be the cost function introduced in (1.29). If we define*

$$F^i : \Delta \times (\Delta \setminus \partial\Delta) \rightarrow \{-1, 0, 1\}, (h, \xi) \mapsto \text{sign} \left(\xi^i \left(1 - \delta - \sum_{j=1}^d \gamma^j \left| \frac{\xi^j}{\xi^i} h^i - h^j \right| \right) - h^i \right)$$

for every $i \in \{1, \dots, d\}$, then the function

$$\widehat{C} : \Delta \times (\Delta \setminus \partial\Delta) \rightarrow]0, 1[, (h, \xi) \mapsto \frac{1 - \delta + \sum_{i=1}^d \gamma^i F^i(h, \xi) h^i}{1 + \sum_{i=1}^d \gamma^i F^i(h, \xi) \xi^i} \quad (1.40)$$

satisfies $\widehat{C}(h, \xi) = (1 - \delta)(1 - C(h, \xi))$ for all $(h, \xi) \in \Delta \times (\Delta \setminus \partial\Delta)$.

Proof. Cf. Theorem 2.5 in [IP09]. An adaption of this proof to Model B can be found in Theorem 1.17. □

As described in Section 1.1, the main objective is to estimate the optimal growth rate

$$\rho_{v_0, h_0} = \sup_{K \in \mathcal{A}_{v_0, h_0}} J_{v_0, h_0}(K) = \sup_{K \in \mathcal{A}_{v_0, h_0}} \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{K, v_0, h_0} \right), \quad (1.41)$$

which we later prove to be independent of v_0, h_0 , and a corresponding maximizing monetary strategy K_{v_0, h_0}^* , if existing, satisfying

$$\rho_{v_0, h_0} = J_{v_0, h_0}(K_{v_0, h_0}^*). \quad (1.42)$$

Theorem 1.6 shows that, by setting $\tilde{J}_{v_0, h_0}(\tilde{K}) := J_{v_0, h_0}(K)$, we can do this by calculating

$$\rho_{v_0, h_0} = \sup_{\tilde{K} \in \tilde{\mathcal{A}}} \tilde{J}_{v_0, h_0}(\tilde{K}) = \sup_{\tilde{K} \in \tilde{\mathcal{A}}} \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{\tilde{K}, v_0, h_0} \right) \quad (1.43)$$

and finding the corresponding proportional strategy \tilde{K}_{v_0, h_0}^* . Hence we are naturally interested in representations of the process $(\log(V_t))_{t \geq 0}$.

The following result from [IP09] allows an easy computation of the optimal growth rate ρ in the one-dimensional case (cf. [IS05] or Theorem 2.6) and enables us to prove the uniqueness of the optimal boundaries in the purely fixed costs case in Proposition 5.7.

Lemma 1.11. *Let $v_0 > 0$ and $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be an admissible proportional strategy with corresponding processes $(V_t)_{t \geq 0} = (V_t^{\tilde{K}, v_0, h_0})_{t \geq 0}$ and $(h_t)_{t \geq 0} = (h_t^{\tilde{K}, h_0})_{t \geq 0}$ satisfying $h_0, \xi_n \in \Delta \setminus \partial\Delta$ for every $n \in \mathbb{N}_0$. Then the function*

$$g : (\Delta \setminus \partial\Delta)^2 \rightarrow \mathbb{R}, (h, \xi) \mapsto \log \left(\frac{\xi^0}{h^0} \right) + \log \left(\widehat{C}(h, \xi) \right), \quad (1.44)$$

where \widehat{C} is the cost function from (1.40) and $h^0 := 1 - \sum_{i=1}^d h^i$ for every $h \in \Delta$, satisfies

$$\log(V_t) = \log(V_0) + rt + \log\left(\frac{h_t^0}{h_0^0}\right) + \sum_{k=0}^{\infty} g(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq t\}} \quad (1.45)$$

and we get in case of $\limsup_{T \rightarrow \infty} \frac{1}{T} E(\log(h_T^0)) = 0$

$$\widetilde{J}_{v_0, h_0}(\widetilde{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E\left(\sum_{k=0}^{\infty} g(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}}\right). \quad (1.46)$$

Proof. In Theorem 2.5 and Remark 2.7 in [IP09] (1.45) is shown by induction. Here, we give a proof using Itô's formula. From Lemma 1.2 we get for every $t \geq 0$

$$\begin{aligned} \log V_t &= \log V_0 + \int_0^t h_{s-}^T \sigma dW_s + \int_0^t r - \frac{1}{2} h_{s-}^T \sigma \sigma^T h_{s-} + (\mu - r \mathbb{1})^T h_{s-} ds \\ &\quad + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_k-}) \mathbb{1}_{\{\tau_k \leq t\}} \end{aligned}$$

and for the risky fraction process also

$$\log\left(\frac{h_t^0}{h_0^0}\right) = - \int_0^t h_{s-}^T \sigma dW_s + \int_0^t \frac{1}{2} h_{s-}^T \sigma \sigma^T h_{s-} - (\mu - r \mathbb{1})^T h_{s-} ds + \sum_{k=0}^{\infty} \log\left(\frac{h_{\tau_k}^0}{h_{\tau_k-}^0}\right) \mathbb{1}_{\{\tau_k \leq t\}},$$

so (1.45) then directly follows from $\xi_k = h_{\tau_k}$ for every $k \in \mathbb{N}_0$ together with (1.30) and Theorem 1.10, whereas (1.46) is evident. □

We close this section by proving a lemma, which relies on another representation of $(\log(V_t))_{t \geq 0}$. It is used in [IS06] for the one-dimensional case to solve the main problem described above. A further reformulation of this representation will be presented in Section 1.4 and is used in [Tam08] to find the optimal growth rate together with the corresponding optimal trading strategy.

Lemma 1.12. *Let $\widetilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be an admissible proportional strategy with corresponding processes $(V_t)_{t \geq 0} = (V_t^{\widetilde{K}, v_0, h_0})_{t \geq 0}$ and $(h_t)_{t \geq 0} = (h_t^{\widetilde{K}, h_0})_{t \geq 0}$ and*

$$f : \Delta \rightarrow \mathbb{R}, \quad h \mapsto -\frac{1}{2} h^T \sigma \sigma^T h + (\mu - r \mathbb{1})^T h. \quad (1.47)$$

If we adjust the cost function in (1.30) to the logarithm on Δ^2 by

$$\widetilde{C}(h, \xi) := \log(1 - \delta) + \log(1 - C(h, \xi)), \quad (1.48)$$

then we obtain the representation

$$\begin{aligned} \tilde{J}_{v_0, h_0}(\tilde{K}) &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_{k-}}) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} \tilde{C}(h_{\tau_{k-}}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right). \end{aligned} \quad (1.49)$$

Proof. Using Theorem 1.6 we can conclude that the assumptions of Lemma 1.2 are satisfied. Considering that $\left(\int_0^t h_{s-}^T \sigma dW_s \right)_{t \geq 0}$ is a martingale, we therefore get for $t \geq 0$

$$E(\log V_t) = \log v_0 + E \left(\int_0^t r + f(h_{s-}) ds + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_{k-}}) \mathbb{1}_{\{\tau_k \leq t\}} \right),$$

implying the first equation in (1.49). The second one is then easily inferred from (1.30). \square

1.3 Model B

Tamura introduced in [Tam06] for the one-dimensional case and in [Tam08] for the general case a slightly different model. We present this model, which we henceforth call Model B, in this section and state first results from [Tam08]. Furthermore, additional achievements from [IP09] are translated to Model B.

We take the same model as in Section 1.2 but with a slightly different cost function and an adapted definition of admissible strategies.

Trading strategies are again impulse control strategies $((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ from Definition 1.1, and we define that after the n -th trading the assets become

$$\begin{aligned} X_{\tau_n} &= (1 - \delta) \left(X_{\tau_{n-}} - \sum_{i=1}^d \eta_n^i - c(\eta_n) \right), \\ Y_{\tau_n}^i &= (1 - \delta) (Y_{\tau_{n-}}^i + \eta_n^i), \quad i \in \{1, \dots, d\}, \end{aligned} \quad (1.50)$$

on $\{\tau_n < \infty\}$, where the cost function c is defined to be

$$c : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}, \quad \eta \mapsto \sum_{i=1}^d (\lambda^i \eta^{i,+} + \nu^i \eta^{i,-}), \quad (1.51)$$

for fixed $0 \leq \nu^i, \lambda^i < 1$ and $0 < \delta < 1$ and hence on $\{\tau_n < \infty\}$

$$\begin{aligned} V_{\tau_n} &= (1 - \delta)(V_{\tau_{n-}} - c(\eta_n)), \\ h_{\tau_n} &= V_{\tau_n}^{-1} Y_{\tau_n} \in \Delta. \end{aligned} \quad (1.52)$$

As in Model A we do not allow short selling or borrowing and define the admissibility of trading strategies.

Definition 1.13. An impulse control strategy $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ is an *admissible monetary strategy* if the corresponding processes from (1.50) satisfy

$$\begin{aligned} X_{\tau_n} &\geq 0 \quad \text{on } \{\tau_n < \infty\}, n \in \mathbb{N}_0, \\ Y_{\tau_n}^i &\geq 0 \quad \text{on } \{\tau_n < \infty\}, n \in \mathbb{N}_0, i \in \{1, \dots, d\}. \end{aligned} \quad (1.53)$$

We also define for fixed starting values $v_0 > 0$, $h_0 \in \Delta$

$$\mathcal{A}_{v_0, h_0} := \{K : K \text{ is an admissible monetary strategy}\}. \quad (1.54)$$

An analogon to Lemma 1.4 shows that we cannot go bankrupt.

Lemma 1.14. *The condition (1.53) is equivalent to*

$$\begin{aligned} V_{\tau_n} &> 0 \quad \text{on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0, \\ h_{\tau_n} &\in \Delta \quad \text{on } \{\tau_n < \infty\} \text{ for all } n \in \mathbb{N}_0. \end{aligned} \quad (1.55)$$

Proof. We show that (1.53) implies $V_{\tau_n} > 0$ for all $n \in \mathbb{N}_0$ by induction.

On $\{\tau_{n+1} < \infty\}$ we can conclude as follows.

Suppose we have $V_{\tau_n} > 0$ and hence get $V_{\tau_{n+1}-} > 0$. The only nontrivial case is $Y_{\tau_{n+1}}^i = 0$ and thus $-\eta_{n+1}^i = Y_{\tau_{n+1}-}^i$ by (1.50) for all $i \in \{1, \dots, d\}$. Now we have by (1.50)

$$X_{\tau_{n+1}} = (1 - \delta) \left(X_{\tau_{n+1}-} - \sum_{i=1}^d (1 - \nu^i) \eta_{n+1}^i \right).$$

But since we have $X_{\tau_{n+1}-} > 0$ or $-\eta_{n+1}^i = Y_{\tau_{n+1}-}^i > 0$ for some $i \in \{1, \dots, d\}$, we therefore get $X_{\tau_{n+1}} > 0$ and hence $V_{\tau_{n+1}} > 0$. □

For every admissible monetary strategy $((\tau_n, \eta_n))_{n \in \mathbb{N}_0}$ we have from Lemma 1.14

$$\xi_n := h_{\tau_n} \in \Delta \quad \text{on } \{\tau_n < \infty\} \text{ for every } n \in \mathbb{N}_0. \quad (1.56)$$

This determines an *admissible proportional strategy* from the following definition.

Definition 1.15. An impulse control strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is an *admissible proportional strategy* if $\xi_n \in \Delta$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$. We define

$$\tilde{\mathcal{A}} := \left\{ \tilde{K} : \tilde{K} \text{ is an admissible proportional strategy} \right\}. \quad (1.57)$$

Though the definition of the cost function and hence that of monetary trading strategies is slightly different from Section 1.2 while at the same time proportional strategies are equal, the following theorem from [Tam08] establishes the one-to-one relationship between these two kinds of trading strategies and reformulates the cost function.

Theorem 1.16. For every $v > 0$ and $h \in \Delta$ let

$$A_{v,h} := \left\{ \eta \in \mathbb{R}^d : h^0 v - c(v, \eta) - \sum_{i=1}^d \eta^i \geq 0 \text{ and } h^i v + \eta^i \geq 0, i \in \{1, \dots, d\} \right\},$$

where $h^0 := 1 - \sum_{i=1}^d h^i$ and define

$$f_{v,h} : A_{v,h} \rightarrow \Delta, \eta \mapsto \left(\frac{h^1 v + \eta^1}{v - c(\eta)}, \dots, \frac{h^d v + \eta^d}{v - c(\eta)} \right). \quad (1.58)$$

Then $f_{v,h}$ is bijective and given an admissible proportional strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ we get the corresponding admissible monetary strategy $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0} \in \mathcal{A}_{v_0, h_0}$ for fixed starting values $v_0 > 0$, $h_0 \in \Delta$ through the inductive relationship

$$\eta_n = f_{V_{\tau_n-}, h_{\tau_n-}}^{-1}(\xi_n) \text{ on } \{\tau_n < \infty\} \text{ and } \eta_n \equiv 0 \text{ on } \{\tau_n = \infty\}, n \in \mathbb{N}_0. \quad (1.59)$$

Therefore the function $\tilde{\cdot} : \mathcal{A}_{v_0, h_0} \rightarrow \tilde{\mathcal{A}}, K \mapsto \tilde{K}$ is bijective.

Moreover, there exists a constant $\kappa \in]0, 1[$ and a Lipschitz continuous function

$$C : \{(h, \xi) \in \Delta \times \mathbb{R}^d : h + \xi \in \Delta\} \rightarrow [0, \kappa] \quad (1.60)$$

satisfying

$$V_{\tau_n} = (1 - \delta)(1 - C(h_{\tau_n-}, \xi_n - h_{\tau_n-}))V_{\tau_n-} \text{ on } \{\tau_n < \infty\}, n \in \mathbb{N}_0 \quad (1.61)$$

and the triangle inequality

$$(1 - C(h, g - h))(1 - C(g, f - g)) \leq (1 - C(h, f - h)) \text{ for all } f, g, h \in \Delta. \quad (1.62)$$

Proof. Cf. Theorem 2.4 and Remark 2.1 in [Tam08]. □

We adapt the idea of the proof of Theorem 2.5 in [IP09] to Model B and can formulate an explicit representation for the cost function above.

Theorem 1.17. Let C be the cost function introduced in (1.60). If we define

$$F^i : \Delta \times (\Delta \setminus \partial\Delta) \rightarrow \{-1, 0, 1\},$$

$$(h, \xi) \mapsto \text{sign} \left(\xi^i \left(1 - \sum_{j=1}^d \left(\lambda^j \left(\frac{\xi^j}{\xi^i} h^i - h^j \right)^+ + \nu^j \left(\frac{\xi^j}{\xi^i} h^i - h^j \right)^- \right) \right) - h^i \right)$$

for every $i \in \{1, \dots, d\}$, then the function

$$\widehat{C} : \Delta \times (\Delta \setminus \partial\Delta) \rightarrow [0, 1[, (h, \xi) \mapsto \frac{(1 - \delta) \left(1 + \sum_{i=1}^d (\lambda^i F^{i,+}(h, \xi) + \nu^i F^{i,-}(h, \xi)) h^i \right)}{1 + \sum_{i=1}^d (\lambda^i F^{i,+}(h, \xi) + \nu^i F^{i,-}(h, \xi)) \xi^i}$$

satisfies

$$\widehat{C}(h, \xi) = (1 - \delta)(1 - C(h, \xi - h)) \text{ for all } (h, \xi) \in \Delta \times (\Delta \setminus \partial\Delta). \quad (1.63)$$

Proof. a) Let $v > 0$ and $h \in \Delta$ represent the portfolio value and the risky fraction before the trading occurs. Then for every $\xi \in \Delta \setminus \partial\Delta$, representing the risky fraction after the trading, we can apply Theorem 1.16 and take $\eta = \eta(v) = f_{v,h}^{-1}(\xi) \in \mathbb{R}^d$ satisfying $\xi = (\frac{y^1}{v_a}, \dots, \frac{y^d}{v_a})$, where $y^i := (1 - \delta)(h^i v + \eta^i)$ is the value invested in the i -th stock after the trading for all $i \in \{1, \dots, d\}$ and $v_a := (1 - \delta)(v - c(\eta))$ is the corresponding new portfolio value. We therefore have

$$\begin{aligned} \eta^j &= (1 - \delta)^{-1} (\xi^j v_a - (1 - \delta)h^j v) \\ &= (1 - \delta)^{-1} \left(\xi^j \frac{y^j}{\xi^j} - (1 - \delta)h^j v \right) \\ &= \frac{\xi^j}{\xi^i} (h^i v + \eta^i) - h^j v \end{aligned}$$

and hence

$$\begin{aligned} &\text{sign}(\eta^i) = \text{sign}((1 - \delta)^{-1} (\xi^i v_a - (1 - \delta)h^i v)) \\ &= \text{sign} \left(\xi^i \left(v - \sum_{j=1}^d \left(\lambda^j \left(\frac{\xi^j}{\xi^i} (h^i v + \eta^i) - h^j v \right)^+ + \nu^j \left(\frac{\xi^j}{\xi^i} (h^i v + \eta^i) - h^j v \right)^- \right) \right) - h^i v \right), \end{aligned}$$

which is independent of v .

b) Now, given an $i \in \{1, \dots, d\}$, we consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ with

$$g(x) = \xi^i \left(1 - \sum_{j=1}^d \left(\lambda^j \left(\frac{\xi^j}{\xi^i} (h^i + x) - h^j \right)^+ + \nu^j \left(\frac{\xi^j}{\xi^i} (h^i + x) - h^j \right)^- \right) \right) - h^i$$

of which $\eta^i = \eta^i(1)$ is a fixed point. We have $g(-h^i) > -h^i$ and since g is continuous and not increasing from some point onward, η^i is the unique fixed point of g on $] - h^i, \infty[$. This yields

$$\text{sign}(\eta^i) = \text{sign}(g(\eta^i)) = \text{sign}(g(0)) = F^i(h, \xi).$$

c) From the definition above we get

$$\begin{aligned} v_a &= (1 - \delta)(v - c(\eta)) \\ &= (1 - \delta) \left(v - \sum_{i=1}^d (\lambda^i F^{i,+}(h, \xi) + \nu^i F^{i,-}(h, \xi)) \eta^i \right) \\ &= (1 - \delta)v - \sum_{i=0}^d (\lambda^i F^{i,+}(h, \xi) + \nu^i F^{i,-}(h, \xi)) (\xi^i v_a - (1 - \delta)h^i v). \end{aligned}$$

Solving this equation for v_a yields $v_a = \widehat{C}(h, \xi) \cdot v$ and so the assertion now follows from (1.61) using the one-to-one relationship in (1.58).

□

To solve the problem of maximizing the expected growth rate

$$\rho_{v_0, h_0} = \sup_{K \in \mathcal{A}_{v_0, h_0}} J_{v_0, h_0}(K) = \sup_{K \in \mathcal{A}_{v_0, h_0}} \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{K, v_0, h_0} \right), \quad (1.64)$$

as described in Section 1.1, and finding a corresponding maximizing monetary strategy K_{v_0, h_0}^* , if existing, satisfying

$$\rho_{v_0, h_0} = J_{v_0, h_0}(K_{v_0, h_0}^*), \quad (1.65)$$

we now can, using Theorem 1.16, determine the value

$$\rho_{v_0, h_0} = \sup_{\tilde{K} \in \tilde{\mathcal{A}}} \tilde{J}_{v_0, h_0}(\tilde{K}) = \sup_{\tilde{K} \in \tilde{\mathcal{A}}} \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{\tilde{K}, v_0, h_0} \right) \quad (1.66)$$

and find the corresponding proportional strategy \tilde{K}_{v_0, h_0}^* . As in Section 1.2 we are interested in representations of the process $(\log(V_t))_{t \geq 0}$ and state here the two analogous results.

Lemma 1.18. *Let $v_0 > 0$ and $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be an admissible proportional strategy with corresponding processes $(V_t)_{t \geq 0} = (V_t^{\tilde{K}, v_0, h_0})_{t \geq 0}$ and $(h_t)_{t \geq 0} = (h_t^{\tilde{K}, h_0})_{t \geq 0}$ satisfying $h_0, \xi_n \in \Delta \setminus \partial\Delta$ for every $n \in \mathbb{N}_0$. Then the function*

$$g : (\Delta \setminus \partial\Delta)^2 \rightarrow \mathbb{R}, (h, \xi) \mapsto \log \left(\frac{\xi^0}{h^0} \right) + \log \left(\widehat{C}(h, \xi) \right), \quad (1.67)$$

where \widehat{C} is the cost function from Theorem 1.17 and $h^0 := 1 - \sum_{i=1}^d h^i$ for every $h \in \Delta$, satisfies

$$\log(V_t) = \log(V_0) + rt + \log \left(\frac{h_0^0}{h_t^0} \right) + \sum_{k=0}^{\infty} g(h_{\tau_{k-}}, \xi_k) \mathbb{1}_{\{\tau_k \leq t\}} \quad (1.68)$$

and we get in case of $\limsup_{T \rightarrow \infty} \frac{1}{T} E(\log(h_T^0)) = 0$

$$\tilde{J}_{v_0, h_0}(\tilde{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\sum_{k=0}^{\infty} g(h_{\tau_{k-}}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right). \quad (1.69)$$

Proof. Here, the result follows by using the corresponding arguments as in Section 1.2. \square

Now we state the analogon of Lemma 1.12 from Section 1.2, which is a first reformulation of the problem described above. In [Tam08] an optimal strategy is found by using a transformation of the representation in (1.72), which shall be presented in the next section.

Lemma 1.19. *Let $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be an admissible proportional strategy with corresponding processes $(V_t)_{t \geq 0} = (V_t^{\tilde{K}, v_0, h_0})_{t \geq 0}$ and $(h_t)_{t \geq 0} = (h_t^{\tilde{K}, h_0})_{t \geq 0}$. If we define the function*

$$f : \Delta \rightarrow \mathbb{R}, h \mapsto -\frac{1}{2} h^T \sigma \sigma^T h + (\mu - r \mathbb{1})^T h \quad (1.70)$$

and adjust the cost function to the logarithm in (1.61) by

$$\tilde{C}(h, \xi) := \log(1 - \delta) + \log(1 - C(h, \xi)) \quad (1.71)$$

on the domain of the function C of Theorem 1.16, then we have

$$\begin{aligned} \tilde{J}_{v_0, h_0}(\tilde{K}) &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_k-}) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} \tilde{C}(h_{\tau_k-}, \xi_k - h_{\tau_k-}) \mathbb{1}_{\{\tau_k \leq T\}} \right). \end{aligned} \quad (1.72)$$

Proof. The proof of the first equality is the same as in Lemma 1.12, whereas the second one follows directly from (1.61). □

1.4 A transformation of the problem

In [Tam08] a coordinate transformation ψ , already introduced in [Nag04] for a model with only fixed transaction costs, is applied to the risky fraction process h in order to avoid degeneracy at the boundary of Δ . The resulting diffusion $\psi(h)$ is then of a much easier structure and the transformed problem can be solved via the Hamilton-Jacobi-Bellman approach (cf. Chapter 3). In this section we establish this coordinate transformation and the corresponding results from Section 2.3 in [Tam08] and give the rigorous proofs needed due to the brevity of the presentation in [Tam08].

Though everything we show in this section also applies to Model A, we use Model B from Section 1.3. We know from Lemma 1.19 that we want to maximize

$$\tilde{J}_{v_0, h_0}(\tilde{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} \tilde{C}(h_{\tau_k-}, \xi_k - h_{\tau_k-}) \mathbb{1}_{\{\tau_k \leq T\}} \right) \quad (1.73)$$

over all admissible proportional strategies $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0} \in \tilde{\mathcal{A}}$ for fixed starting values $v_0 > 0$, $h_0 \in \Delta$, where

$$\tilde{C}(h, \xi) = \log(1 - \delta) + \log(1 - C(h, \xi)) \quad (1.74)$$

is defined on the domain of the function C of Theorem 1.16. We introduce the bijective coordinate transformation

$$\psi : \Delta \setminus \partial\Delta \rightarrow \mathbb{R}^d, h \mapsto \left(\log(h^1) - \log\left(1 - \sum_{i=1}^d h^i\right), \dots, \log(h^d) - \log\left(1 - \sum_{i=1}^d h^i\right) \right) \quad (1.75)$$

with the inverse

$$\varphi := \psi^{-1} : \mathbb{R}^d \rightarrow \Delta \setminus \partial\Delta, \quad y \mapsto \left(\frac{\exp y^1}{1 + \sum_{i=1}^d \exp y^i}, \dots, \frac{\exp y^d}{1 + \sum_{i=1}^d \exp y^i} \right). \quad (1.76)$$

In order to transform the representation in (1.73) via ψ , we define

$$\bar{f} : \mathbb{R}^d \rightarrow \mathbb{R}, \quad y \mapsto f(\varphi(y)) \quad (1.77)$$

and the cost function

$$\bar{C} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_{<0}, \quad (y, \zeta) \mapsto \tilde{C}(\varphi(y), \varphi(y + \zeta) - \varphi(y)). \quad (1.78)$$

Remark 1.20. In the case of Model A we would define the cost function above by

$$\tilde{C}(h, \xi) := \log(1 - \delta) + \log(1 - C(h, h + \xi)) \quad (1.79)$$

and get exactly the same formulas that are given in this section.

Trading strategies are now just common impulse control strategies and hence independent of the initial values v_0, h_0 .

Definition 1.21. The set of *admissible strategies* is defined to be the collection

$$\bar{\mathcal{A}} := \{((\tau_n, \zeta_n))_{n \in \mathbb{N}_0} : ((\tau_n, \zeta_n))_{n \in \mathbb{N}_0} \text{ is an impulse control strategy}\} \quad (1.80)$$

of all impulse control strategies.

The connection to admissible proportional strategies is given through the next two lemmas.

Lemma 1.22. *Let $v_0 > 0$ and $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be an admissible proportional strategy satisfying $\xi_n \in \Delta \setminus \partial\Delta$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$ with fixed starting value $h_0 \in \Delta \setminus \partial\Delta$. Then the corresponding risky fraction process $(h_t)_{t \geq 0} = (h_t^{\tilde{K}, h_0})_{t \geq 0}$ has its values in $\Delta \setminus \partial\Delta$ and the process $(y_t)_{t \geq 0} := (\psi(h_t))_{t \geq 0}$ satisfies*

$$y_t^i = y_0^i + \left(\mu_i - r - \frac{1}{2}(\sigma\sigma^T)_{ii} \right) \cdot t + \sum_{j=1}^d \sigma_{ij} W_t^j + \sum_{k=0}^{\infty} \zeta_k^i \mathbb{1}_{\{\tau_k \leq t\}}, \quad i \in \{1, \dots, d\}, \quad t \geq 0, \quad (1.81)$$

where $\bar{K} = ((\tau_n, \zeta_n))_{n \in \mathbb{N}_0} \in \bar{\mathcal{A}}$ is defined by $\zeta_n := \psi(h_{\tau_n}) - \psi(h_{\tau_n-})$ on $\{\tau_n < \infty\}$ and $\zeta_n \equiv 0$ on $\{\tau_n = \infty\}$ for all $n \in \mathbb{N}_0$.

Furthermore, we get by setting

$$\bar{J}_{v_0, y_0}(\bar{K}) := \tilde{J}_{v_0, h_0}(\tilde{K}) \quad (1.82)$$

the representation

$$\bar{J}_{v_0, y_0}(\bar{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \bar{f}(y_{s-}) ds + \sum_{k=0}^{\infty} \bar{C}(y_{\tau_k-}, \zeta_k) \mathbb{1}_{\{\tau_k \leq T\}} \right). \quad (1.83)$$

Proof. Since $\xi_n = h_{\tau_n}$ on $\{\tau_n < \infty\}$ for every $n \in \mathbb{N}_0$, we get from $h_0 \in \Delta \setminus \partial\Delta$ inductively $h_t \in \Delta \setminus \partial\Delta$ for every $t \in [\tau_n, \tau_{n+1}[$ and hence for every $t \geq 0$.

(1.16) yields for every $t \geq 0$ and $i \in \{1, \dots, d\}$

$$[Y^{i,c}]_t = \int_0^t (Y_{s-}^i)^2 (\sigma\sigma^T)_{ii} ds \text{ and } [X^c]_t = 0. \quad (1.84)$$

Using

$$y_t^i = \log(h_t^i) - \log\left(1 - \sum_{i=1}^d h_t^i\right) = \log \frac{Y_t^i}{X_t}$$

we therefore get from the generalized Itô's formula (cf. [JS03], Theorem I.4.57)

$$\begin{aligned} y_t^i - y_0^i &= \int_0^t \frac{1}{Y_{s-}^i} dY_s^i - \int_0^t \frac{1}{X_{s-}} dX_s - \frac{1}{2} \int_0^t \frac{1}{(Y_{s-}^i)^2} d[Y^{i,c}]_s + \frac{1}{2} \int_0^t \frac{1}{(X_{s-})^2} d[X^c]_s \\ &\quad + \sum_{s \leq t} \left(\Delta y_s^i - \frac{1}{Y_{s-}^i} \Delta Y_s^i + \frac{1}{X_{s-}} \Delta X_s \right) \\ &\stackrel{(1.84)}{=} \int_0^t \left(\mu_i - r - \frac{1}{2} (\sigma\sigma^T)_{ii} \right) ds + \sum_{j=1}^d \int_0^t \sigma_{ij} dW_s^j + \sum_{k=0}^{\infty} \Delta y_{\tau_k}^i \mathbb{1}_{\{\tau_k \leq t\}}. \end{aligned}$$

Now (1.81) follows from

$$\zeta_k^i = \psi(h_{\tau_k}) - \psi(h_{\tau_k-}) = \Delta y_{\tau_k}^i \text{ on } \{\tau_k < \infty\}, \quad i \in \{1, \dots, d\}, \quad k \in \mathbb{N}_0,$$

and (1.83) is a direct consequence of (1.77), (1.78) and (1.73). □

Lemma 1.23. *Let $((\tau_n, \zeta_n))_{n \in \mathbb{N}_0}$ be an admissible trading strategy, $y_0 \in \mathbb{R}^d$ and*

$$y_t^i := y_0^i + \left(\mu_i - r - \frac{1}{2} (\sigma\sigma^T)_{ii} \right) \cdot t + \sum_{j=1}^d \sigma_{ij} W_t^j + \sum_{k=0}^{\infty} \zeta_k^i \mathbb{1}_{\{\tau_k \leq t\}}, \quad i \in \{1, \dots, d\}, \quad t \geq 0.$$

Then, by setting $h_0 := \varphi(y_0)$ and $\xi_n := \varphi(y_{\tau_n})$ on $\{\tau_n < \infty\}$, $\xi_n \equiv 0$ on $\{\tau_n = \infty\}$ for all $n \in \mathbb{N}_0$, the admissible proportional strategy $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ together with the corresponding risky fraction process $(h_t)_{t \geq 0}$ satisfies the assumptions of Lemma 1.22 for every $v_0 > 0$ and we have $(y_t)_{t \geq 0} = (\psi(h_t))_{t \geq 0}$.

Proof. We have for every $n \in \mathbb{N}_0$ and $t \geq 0$

$$y_{\tau_n} = \psi(\varphi(y_{\tau_n})) \stackrel{(1.56)}{=} \psi(h_{\tau_n}) \text{ on } \{\tau_n < \infty\}.$$

Since $(\psi(h_t))_{t \geq 0}$ meets (1.81), this yields $y_t = \psi(h_t)$ for every $t \in [\tau_n, \tau_{n+1}[$ and hence for every $t \geq 0$. □

Now we define for every $v_0 > 0$, $h_0 \in \Delta \setminus \partial\Delta$ and $y_0 := \psi(h_0)$

$$\bar{\rho}_{v_0, y_0} := \sup_{\bar{K} \in \bar{\mathcal{A}}} \bar{J}_{v_0, y_0}(\bar{K}), \quad (1.85)$$

where \bar{J} is from Lemma 1.22, and get by using Lemma 1.22 and 1.23

$$\bar{\rho}_{v_0, y_0} = \sup_{\bar{K} \in \bar{\mathcal{A}}} \bar{J}_{v_0, y_0}(\bar{K}) \leq \sup_{\tilde{K} \in \tilde{\mathcal{A}}} \tilde{J}_{v_0, h_0}(\tilde{K}) = \rho_{v_0, h_0}. \quad (1.86)$$

It is of course essential to see that we even have equality in (1.86) and that there is no dependence on v_0, h_0 .

In order to show that, we first have to estimate the influence of the initial values v_0, h_0 . To clarify the dependence on v_0, h_0 , and in the case of $h_0 \in \Delta \setminus \partial\Delta$ also on $y_0 = \psi(h_0)$, we use the notation $\rho_{v_0, h_0}, \bar{\rho}_{v_0, y_0}, \mathcal{A}_{v_0, h_0}, \tilde{\mathcal{A}}_{v_0, h_0}, \bar{\mathcal{A}}_{v_0, y_0}$, etc.

Lemma 1.24. *Let $v_0 > 0$, $h_0 \in \Delta$ and $\varepsilon > 0$. Then we have*

$$\rho_{v_0, h_0} = \rho_{\varepsilon v_0, h_0}.$$

A maximizing proportional strategy for ρ_{v_0, h_0} exists iff one exists for $\rho_{\varepsilon v_0, h_0}$ and they are equal in the case of existence. The corresponding monetary strategies differ by the factor ε .

Proof. From the definition of admissible proportional strategies we have $\tilde{\mathcal{A}}_{v_0, h_0} = \tilde{\mathcal{A}}_{\varepsilon v_0, h_0}$.

Let $\tilde{K} \in \tilde{\mathcal{A}}_{v_0, h_0}$ and consider the corresponding monetary strategy

$K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0} \in \mathcal{A}_{v_0, h_0}$ from Theorem 1.16. Then $K_\varepsilon := ((\tau_n, \varepsilon \eta_n))_{n \in \mathbb{N}_0} \in \mathcal{A}_{\varepsilon v_0, h_0}$ and the following equations hold

$$\begin{aligned} V_t^{K_\varepsilon, \varepsilon v_0, h_0} &= \varepsilon V_t^{K, v_0, h_0}, \quad t \geq 0, \\ Y_t^{K_\varepsilon, \varepsilon v_0, h_0, i} &= \varepsilon Y_t^{K, v_0, h_0, i}, \quad t \geq 0, \quad i \in \{1, \dots, d\}, \\ h_t^{K_\varepsilon, \varepsilon v_0, h_0} &= h_t^{K, v_0, h_0}, \quad t \geq 0. \end{aligned}$$

Now the assertion follows from

$$\liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log \varepsilon V_T^{K, v_0, h_0} \right) = \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{K, v_0, h_0} \right).$$

□

Remark 1.25. Using Lemma 1.22 and 1.23, the proof of Lemma 1.24 shows that for every $v_0, \varepsilon > 0$ and $h_0 \in \Delta \setminus \partial\Delta$ we have

$$\bar{\rho}_{\varepsilon v_0, \psi(h_0)} = \bar{\rho}_{v_0, \psi(h_0)}.$$

Having seen that the initial wealth v_0 has no influence on the outcome, we can prove the equality in (1.86) to hold in the following lemma, which is due to [Tam08].

Lemma 1.26. *Let $v_0 > 0$, $h_0 \in \Delta \setminus \partial\Delta$ and $y_0 := \psi(h_0)$. Then $\rho_{v_0, h_0} = \bar{\rho}_{v_0, y_0}$ holds.*

Proof. Take $K \in \mathcal{A}_{v_0, h_0}$ and let $\varepsilon > 1$. Since K is an admissible monetary strategy for v_0 and h_0 , it is of course an admissible monetary strategy for εv_0 and h_0 , i.e. $K \in \mathcal{A}_{\varepsilon v_0, h_0}$, because we have a greater starting wealth. But since the absolute transaction volumes stay the same as before, we even stay out of the boundary of Δ with the risky fraction process $(h_t^{K, \varepsilon v_0, h_0})_{t \geq 0}$. Moreover, we get $V_t^{K, v_0, h_0} \leq V_t^{K, \varepsilon v_0, h_0}$ for every $t \geq 0$ and hence

$$\liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{K, v_0, h_0} \right) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log V_T^{K, \varepsilon v_0, h_0} \right) \stackrel{1.22}{\leq} \bar{\rho}_{\varepsilon v_0, y_0}. \quad (1.87)$$

This implies

$$\rho_{v_0, h_0} \stackrel{(1.87)}{\leq} \bar{\rho}_{\varepsilon v_0, y_0} \stackrel{1.25}{=} \bar{\rho}_{v_0, y_0} \stackrel{(1.86)}{\leq} \rho_{v_0, h_0}.$$

□

Now it remains to show that the initial risky fraction h_0 has no influence. We will prove that in Lemma 1.28 in general, where we use our technical condition $\tau_0 = 0$ in our definition of trading strategies but here we give a proof for a special case being true even without that condition.

Lemma 1.27. *Let $v_0 > 0$, $h_0, \tilde{h}_0 \in \Delta$ and assume $\tilde{h}_0 \in \Delta \setminus \partial\Delta$. Then it holds*

$$\rho_{v_0, h_0} \leq \rho_{v_0, \tilde{h}_0}.$$

Proof. Since $\tilde{h}_0 \in \Delta \setminus \partial\Delta$ we can take $\varepsilon > 1$ large enough to satisfy

$$\begin{aligned} \tilde{h}_0^i \varepsilon v_0 &\geq h_0^i v_0 \text{ for all } i \in \{1, \dots, d\}, \\ \left(1 - \sum_{i=1}^d \tilde{h}_0^i\right) \varepsilon v_0 &\geq \left(1 - \sum_{i=1}^d h_0^i\right) v_0. \end{aligned} \quad (1.88)$$

For every monetary strategy $K \in \mathcal{A}_{v_0, h_0}$ we get from (1.88) $K \in \mathcal{A}_{\varepsilon v_0, h_0}$ and

$$V_t^{K, \varepsilon v_0, \tilde{h}_0} \geq V_t^{K, v_0, h_0}. \quad (1.89)$$

This implies

$$\rho_{v_0, h_0} \stackrel{(1.89)}{\leq} \rho_{\varepsilon v_0, \tilde{h}_0} \stackrel{1.24}{=} \rho_{v_0, \tilde{h}_0}.$$

□

Lemma 1.28. *Let $v_0 > 0$ and $h_0, \tilde{h}_0 \in \Delta$. Then we have $\rho_{v_0, h_0} = \rho_{v_0, \tilde{h}_0}$.*

Proof. Let $K = ((\tau_n, \eta_n))_{n \in \mathbb{N}_0} \in \mathcal{A}_{v_0, h_0}$ be a monetary strategy and h_{τ_0} be the risky fraction after the transaction in $\tau_0 = 0$ (starting from v_0, h_0). Now for every $\varepsilon > 0$ we can get using (1.58) an \mathcal{F}_0 measurable η^ε that attains h_{τ_0} starting from εv_0 and \tilde{h}_0 . Since the cost function is bounded by (1.60) we can take an $\varepsilon > 0$ large enough to satisfy

$$(1 - \delta)(\varepsilon v_0 - c(\eta^\varepsilon)) \geq (1 - \delta)(v_0 - c(\eta_0)), \quad (1.90)$$

which just means a bigger portfolio value after τ_0 . Now if we define \widehat{K} by replacing η_0 in K by η^ε , we get by (1.90)

$$\widehat{K} \in \mathcal{A}_{\varepsilon v_0, \tilde{h}_0} \text{ and } J_{\varepsilon v_0, \tilde{h}_0}(\widehat{K}) \geq J_{v_0, h_0}(K)$$

and hence

$$\rho_{v_0, h_0} \stackrel{1.24}{=} \rho_{\varepsilon v_0, \tilde{h}_0} \geq \rho_{v_0, h_0}.$$

□

To close the section we combine the results stated in the foregoing lemmas.

Theorem 1.29. *Let $v_0 > 0$, $\tilde{v}_0 > 0$, $h_0 \in \Delta$ and $y_0 \in \mathbb{R}^d$. Then the optimal growth rates ρ_{v_0, h_0} from (1.64) and $\bar{\rho}_{\tilde{v}_0, y_0}$ from (1.85) are independent of the initial values $v_0, \tilde{v}_0, h_0, y_0$ and satisfy*

$$\rho_{v_0, h_0} = \bar{\rho}_{\tilde{v}_0, y_0}. \quad (1.91)$$

Proof. This is a direct consequence of Lemma 1.24 and 1.28 together with Lemma 1.26.

□

Remark 1.30. We have seen that it suffices to maximize

$$\bar{J}_{v_0, y_0}(\bar{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \bar{f}(y_{s-}) ds + \sum_{k=0}^{\infty} \bar{C}(y_{\tau_k-}, \zeta_k) \mathbb{1}_{\{\tau_k \leq T\}} \right)$$

for some $v_0 > 0$, $y_0 \in \mathbb{R}^d$ over all admissible strategies $\bar{K} = ((\tau_n, \zeta_n))_{n \in \mathbb{N}_0}$. But using (1.68) we could also maximize

$$\bar{J}_{v_0, y_0}(\bar{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log \left(\frac{\varphi(y_0)^0}{\varphi(y_T)^0} \right) + \sum_{k=0}^{\infty} \left(\log \left(\frac{\varphi(y_{\tau_k})^0}{\varphi(y_{\tau_k-})^0} \right) + \bar{C}(y_{\tau_k-}, \zeta_k) \right) \mathbb{1}_{\{\tau_k \leq T\}} \right)$$

over all admissible strategies $\bar{K} = ((\tau_n, \zeta_n))_{n \in \mathbb{N}_0}$.

Chapter 2

Optimal strategies for Model A

The maximization problem described in Section 1.2 for Model A was solved by Irle and Sass in [IS06] for the one-dimensional case. They proved that an optimal proportional trading strategy can be found in the class of the so-called constant boundary strategies, which are given by four parameters $a < \alpha \leq \beta < b \in]0, 1[$. The parameters of this optimal strategy are given via a solution u to a certain quasi-variational inequality.

Our main purpose in this chapter is a brief introduction of the approach and the results from [IS06] that are needed later. Furthermore, we prove additional results on the solution u and state relevant achievements from [IS05] about constant boundary strategies. This is done in the original version on $[0, 1]$ in Section 2.1 and is then translated via the coordinate transformation ψ to \mathbb{R} in Section 2.2.

2.1 Constant boundary strategies on $[0, 1]$

We begin by recalling some of the results from Section 1.2 for the one-dimensional case. We denote by \tilde{C} the cost function from (1.48) and

$$f : [0, 1] \rightarrow \mathbb{R}, \quad h \mapsto -\frac{1}{2}h^2\sigma^2 + (\mu - r)h. \quad (2.1)$$

For every admissible proportional strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0} \in \tilde{\mathcal{A}}$ we have by Lemma 1.12 for the corresponding risky fraction process h

$$\tilde{J}(\tilde{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} \tilde{C}(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \quad (2.2)$$

and by Lemma 1.2 also

$$h_t = h_0 + \int_0^t h_{s-} (1 - h_{s-}) \sigma dW_s + \int_0^t h_{s-} (1 - h_{s-}) (\mu - r - \sigma^2 h_{s-}) ds + \sum_{k=0}^{\infty} \Delta h_{\tau_k} \mathbb{1}_{\{\tau_k \leq t\}}, \quad (2.3)$$

where $\xi_k = h_{\tau_k}$ and thus $\Delta h_{\tau_k} = \xi_k - h_{\tau_k^-}$. The objective now is to find an optimal strategy \tilde{K}^* and the corresponding value $\tilde{J}(\tilde{K}^*)$ such that

$$\tilde{J}(\tilde{K}^*) = \rho := \sup_{\tilde{K} \in \tilde{\mathcal{A}}} \tilde{J}(\tilde{K}). \quad (2.4)$$

This task is solved in [IS06] by the so-called Hamilton-Jacobi-Bellman approach, which in our situation can be described as follows.

We define for every bounded $u : [0, 1] \rightarrow \mathbb{R}$ the function Mu by

$$Mu(x) := \sup_{y \in [0, 1]} u(y) + \tilde{C}(x, y) \text{ for all } x \in [0, 1] \quad (2.5)$$

and let the operator

$$D := x(1-x)(\mu - r - \sigma^2 x) \frac{d}{dx} + \frac{1}{2} \sigma^2 x^2 (1-x)^2 \frac{d^2}{dx^2} \quad (2.6)$$

denote the generator of the uncontrolled diffusion

$$\tilde{h}_t = \tilde{h}_0 + \int_0^t \sigma(1 - \tilde{h}_s) \tilde{h}_s dW_s + \int_0^t \tilde{h}_s(1 - \tilde{h}_s)(\mu - r - \sigma^2 \tilde{h}_s) ds.$$

Now the task is to find a constant $l \in \mathbb{R}$ and a function $u : [0, 1] \rightarrow \mathbb{R}$, which is smooth enough that Itô's formula can be applied and that is a solution to the quasi-variational inequality (QVI)

$$\max\{Du + f - l, Mu - u\} = 0 \quad (2.7)$$

in the classical sense or in the sense of weak derivatives (cf. also Appendix A). Then a verification theorem is needed that guarantees the admissibility, existence and optimality of the following proportional strategy.

Define the continuation region $A := \{x \in [0, 1] : Mu(x) > u(x)\}$ and for some $h_0 \in A$

$$\tau_0 := 0, \quad \xi_0 := h_0 \quad (2.8)$$

and inductively

$$\tau_n := \inf\{t > \tau_{n-1} : u(h_t^{n-1}) \leq Mu(h_t^{n-1})\}, \quad (2.9)$$

where $\xi_n \in A$ attains

$$u(\xi_n) + \tilde{C}(h_{\tau_n}^{n-1}, \xi_n) = Mu(h_{\tau_n}^{n-1}) = u(h_{\tau_n}^{n-1}) \quad (2.10)$$

and h^{n-1} is the risky fraction process corresponding to the proportional strategy $((\sigma_k, \zeta_k))_{k \in \mathbb{N}}$ given by

$$(\sigma_k, \zeta_k) = \begin{cases} (\tau_k, \xi_k), & k \leq n-1, \\ (\infty, 0), & k \geq n. \end{cases}$$

Such a verification is done in Theorems 4.1 and 7.1 in [IS06] for a constant $l > 0$ and a continuously differentiable function u that is also piecewise twice continuously differentiable.

Our main interest is the structure of this function u and the constant boundary strategy it induces. We begin with a definition of this important class of strategies. In fact, since these strategies are quite easy to handle for an investor, one of the main achievements of [IS06] is the proof that an optimal strategy can be found among them.

Definition 2.1. An impulse control strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is a (*proportional*) *constant boundary strategy* if there exist $a < \alpha \leq \beta < b \in]0, 1[$ such that $\xi_0 \in]a, b[$ and

$$\tau_n = \inf \{t > \tau_{n-1} : h_t^{n-1} \notin]a, b[\}, \quad \xi_n = \begin{cases} \alpha, & h_{\tau_n}^{n-1} = a, \\ \beta, & h_{\tau_n}^{n-1} = b, \end{cases}$$

where h^{n-1} is as in (2.9).

We will refer to \tilde{K} as a *proportional constant boundary strategy given by* (a, α, β, b) .

We will also need a better representation of the cost function \tilde{C} .

Lemma 2.2. We have $\tilde{\Gamma}(x, y) = \tilde{C}(x, y)$ for every $x, y \in]0, 1[$ if we define

$$\tilde{\Gamma}(x, y) := \begin{cases} \log \left(\frac{1-\delta+\gamma x}{1+\gamma y} \right), & y \geq x(1-\delta)^{-1}, \\ \log \left(\frac{1-\delta-\gamma x}{1-\gamma y} \right), & y < x(1-\delta)^{-1}. \end{cases} \quad (2.11)$$

Proof. Cf. p. 926 in [IS06] or apply Theorem 1.10. □

In order to find the solution u , a modified version of the cost function is used in [IS06].

Definition and Lemma 2.3. We define the modified cost function Γ for all $x, y \in]0, 1[$ by

$$\Gamma(x, y) := \begin{cases} \log \left(\frac{1-\delta+\gamma x}{1+\gamma y} \right), & y > x, \\ \log \left(\frac{1-\delta-\gamma x}{1-\gamma y} \right), & y \leq x, \end{cases} \quad (2.12)$$

and get for $x < y$

$$\frac{\partial}{\partial x} \Gamma(x, y) = \frac{\gamma}{1-\delta+\gamma x} > 0 \quad \text{and} \quad \frac{\partial}{\partial y} \Gamma(x, y) = \frac{-\gamma}{1+\gamma y} < 0, \quad (2.13)$$

whereas for $x > y$

$$\frac{\partial}{\partial x} \Gamma(x, y) = \frac{-\gamma}{1-\delta-\gamma x} < 0 \quad \text{and} \quad \frac{\partial}{\partial y} \Gamma(x, y) = \frac{\gamma}{1-\gamma y} > 0. \quad (2.14)$$

Proof. This assertion follows by taking derivatives in Lemma 2.2 and is also stated in [IS06], p. 927. □

Now we are able to state a collection of the main results from [IS06], which we will use as a reference.

Theorem 2.4. *Let $\delta \in]0, 1[$, $\gamma \in [0, 1 - \delta[$ as in Section 1.2, D as in (2.6), f as in (2.1), Γ as in (2.12) and let the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$.*

Then there exist constants $l > 0$ and $a < \alpha \leq x_0 \leq \beta < b \in]0, 1[$ such that the function $u \in C^1([0, 1], \mathbb{R})$ defined by

$$u(x) := \begin{cases} \Gamma(x, \alpha), & x \leq a, \\ u(a) + \int_a^x g(y, x_0, l) dy, & a < x \leq b, \\ u(\beta) + \Gamma(x, \beta), & x > b, \end{cases} \quad (2.15)$$

where

$$g(x, x_0, l) := \begin{cases} \frac{\left(\frac{1-x}{x}\right)^{2\hat{\eta}-1}}{x(1-x)f(1)} \left((l - xf(1)) \left(\frac{x}{1-x}\right)^{2\hat{\eta}-1} - (l - x_0f(1)) \left(\frac{x_0}{1-x_0}\right)^{2\hat{\eta}-1} \right), & \hat{\eta} \neq \frac{1}{2}, \\ \frac{1}{x(1-x)} \int_{x_0}^x \left(\frac{l}{\sigma^2} - \frac{y}{2}(1-y) \right) \frac{2}{y(1-y)} dy, & \hat{\eta} = \frac{1}{2}, \end{cases} \quad (2.16)$$

has the following properties:

- (i) *Every constant boundary strategy \tilde{K} given by the constants (a, α, β, b) in the sense of Definition 2.1 is optimal for the modified cost function Γ with the optimal value $\tilde{J}(\tilde{K}) = r + l = \rho$.*
- (ii) *If $a \leq \alpha(1 - \delta)$, then \tilde{K} is also optimal for the original cost function $\tilde{\Gamma}$ in (2.11).*
- (iii) *$Du(x) + f(x) - l \leq 0$ and $u(y) - u(x) + \Gamma(x, y) \leq 0$ for all $x, y \in [0, 1]$.*
- (iv) *$Du(x) + f(x) - l = 0$ for all $x \in]a, b[$.*
- (v) *$u'(\beta) = -\frac{\partial}{\partial y}\Gamma(b, y)|_{y=\beta} = -\frac{\gamma}{1-\gamma\beta}$ and $u'(\alpha) = -\frac{\partial}{\partial y}\Gamma(a, y)|_{y=\alpha} = \frac{\gamma}{1+\gamma\alpha}$.*
- (vi) *$u'(b) = \frac{\partial}{\partial x}\Gamma(x, \beta)|_{x=b} = -\frac{\gamma}{1-\delta-\gamma b}$ and $u'(a) = \frac{\partial}{\partial x}\Gamma(x, \alpha)|_{x=a} = \frac{\gamma}{1-\delta+\gamma a}$.*
- (vii) *$\frac{\partial}{\partial x}g(x, x_0, l)|_{x=\beta} < -\frac{\gamma^2}{(1-\gamma\beta)^2}$ and $\frac{\partial}{\partial x}g(x, x_0, l)|_{x=b} > -\frac{\gamma^2}{(1-\delta-\gamma b)^2}$.*
- (viii) *$\frac{\partial}{\partial x}g(x, x_0, l)|_{x=\alpha} < -\frac{\gamma^2}{(1+\gamma\alpha)^2}$ and $\frac{\partial}{\partial x}g(x, x_0, l)|_{x=a} > -\frac{\gamma^2}{(1-\delta+\gamma a)^2}$.*
- (ix) *$g(x, x_0, l) < \frac{\gamma}{1-\delta+\gamma x}$ on $]0, a[$ and $g(x, x_0, l) > -\frac{\gamma}{1-\delta-\gamma x}$ on $]b, 1[$.*
- (x) *$\alpha = \beta$ for $\gamma = 0$ and $\alpha < x_0 < \beta$ for $\gamma > 0$.*

Proof. Cf. [IS06], pp. 929-936 and Remark 9.1. □

We close the section with two results from [IS05] that give additional information about arbitrary constant boundary strategies.

Lemma 2.5. *Let the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be a constant boundary strategy given by some constants $a < \alpha \leq \beta < b \in]0, 1[$. Then it holds*

$$E(\tau_1) = \frac{p_0(\xi_0) - p_0(a)}{p_0(b) - p_0(a)}(p_1(b) - p_1(a)) + p_1(a) - p_1(\xi_0) \text{ and } E(\tau_n) < \infty \text{ in general,}$$

where

$$p_0(x) := \begin{cases} -\left(\frac{1-x}{x}\right)^{2\hat{h}-1}, & \hat{h} \neq \frac{1}{2}, \\ \log \frac{1-x}{x}, & \hat{h} = \frac{1}{2}, \end{cases} \quad (2.17)$$

$$p_1(x) := \begin{cases} -\frac{2 \log \frac{1-x}{x}}{\sigma^2(2\hat{h}-1)}, & \hat{h} \neq \frac{1}{2}, \\ \frac{1}{\sigma^2} \left(\log \frac{1-x}{x}\right)^2, & \hat{h} = \frac{1}{2}. \end{cases} \quad (2.18)$$

Proof. Cf. Lemma 3 and Lemma 4 in [IS05]. □

Theorem 2.6. *Let the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and $\tilde{K}(a, \alpha, \beta, b)$ be a proportional constant boundary strategy given by some constants $a < \alpha \leq \beta < b \in]0, 1[$. Then the expected growth rate $\tilde{J}(\tilde{K}(a, \alpha, \beta, b))$ satisfies*

$$\tilde{J}(\tilde{K}(a, \alpha, \beta, b)) = \frac{p(a, \alpha, \beta, b)g(b, \beta) + (1 - p(a, \alpha, \beta, b))g(a, \alpha)}{p(a, \alpha, \beta, b)(p_1(b) - p_1(\beta)) + (1 - p(a, \alpha, \beta, b))(p_1(a) - p_1(\alpha))},$$

where g is from (1.44), p_0 and p_1 are as in Lemma 2.5 and

$$p(a, \alpha, \beta, b) := \frac{p_0(\alpha) - p_0(a)}{p_0(b) - p_0(\beta) + p_0(\alpha) - p_0(a)}.$$

Proof. Cf. Corollary 1 in [IS05]. □

2.2 Constant boundary strategies on \mathbb{R}

Some of the results of Chapters 5 and 6 will be formulated and proved on \mathbb{R} instead of $[0, 1]$. In this section we want to establish the corresponding statements from Section 2.1 for \mathbb{R} , which are needed later, and prove some additional results.

The connecting link between $]0, 1[$ and \mathbb{R} is the transformation functions ψ and its inverse φ from Section 1.4.

If we define $\bar{f} := f \circ \varphi$ and the cost function \bar{C} as in (1.78) (see also Remark 1.20), then every admissible strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0} \in \bar{\mathcal{A}}$ yields by Lemma 1.22 the expected growth rate

$$\bar{J}(\bar{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \bar{f}(Y_{s-}) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right), \quad (2.19)$$

where $Y = \psi(h)$ and h is the corresponding risky fraction process. Furthermore, Y satisfies

$$Y_t = y_0 + \left(\mu - r - \frac{1}{2}\sigma^2\right) \cdot t + \sigma W_t + \sum_{k=0}^{\infty} \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \quad (2.20)$$

and the uncontrolled part of Y , namely the diffusion

$$X_t = y_0 + \left(\mu - r - \frac{1}{2}\sigma^2\right) \cdot t + \sigma W_t, \quad (2.21)$$

has the generator

$$L := \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} + \left(\mu - r - \frac{1}{2}\sigma^2\right) \frac{d}{dx}. \quad (2.22)$$

Now by taking derivatives we have for smooth functions $u : [0, 1] \rightarrow \mathbb{R}$

$$L(u \circ \varphi)(x) = Du(\varphi(x)) \text{ and also } \tilde{C}(\varphi(x), \varphi(x+y)) = \bar{C}(x, y), \quad (2.23)$$

where D and \tilde{C} are from Section 2.1. Hence we can define for bounded functions $v : \mathbb{R} \rightarrow \mathbb{R}$

$$\bar{M}v(x) := \sup_{y \in \mathbb{R}} v(y) + \bar{C}(x, y - x) \quad (2.24)$$

and get the following connection: If a smooth enough function $u : [0, 1] \rightarrow \mathbb{R}$ is a solution to the QVI (2.7) on $[0, 1]$, then $u \circ \varphi$ is a solution to the QVI

$$\max\{Lv + \bar{f} - l, \bar{M}v - v\} = 0 \quad (2.25)$$

on \mathbb{R} . Using that we can translate the results from [IS06] to \mathbb{R} . But before doing so, we need the transformed versions of Lemma 2.2 and 2.3.

Lemma 2.7. *For every $x, y \in \mathbb{R}$ we have*

$$\bar{\Gamma}(x, y) := \bar{C}(x, y - x) = \begin{cases} \log\left(\frac{1-\delta+\gamma\varphi(x)}{1+\gamma\varphi(y)}\right), & \varphi(y) \geq \varphi(x)(1-\delta)^{-1}, \\ \log\left(\frac{1-\delta-\gamma\varphi(x)}{1-\gamma\varphi(y)}\right), & \varphi(y) < \varphi(x)(1-\delta)^{-1}. \end{cases} \quad (2.26)$$

Proof. This is a consequence of (2.23) and Lemma 2.2. □

The modified cost function from Lemma 2.3 is then of the following form.

Lemma 2.8. *Let for every $x, y \in \mathbb{R}$*

$$\Gamma(x, y) := \begin{cases} \log\left(\frac{1-\delta+\gamma\varphi(x)}{1+\gamma\varphi(y)}\right), & y > x, \\ \log\left(\frac{1-\delta-\gamma\varphi(x)}{1-\gamma\varphi(y)}\right), & y \leq x. \end{cases} \quad (2.27)$$

Then we have for $x < y$

$$\frac{\partial}{\partial x}\Gamma(x, y) = \frac{\gamma\varphi'(x)}{1-\delta+\gamma\varphi(x)} > 0 \text{ and } \frac{\partial}{\partial y}\Gamma(x, y) = \frac{-\gamma\varphi'(y)}{1+\gamma\varphi(y)} < 0, \quad (2.28)$$

whereas for $x > y$

$$\frac{\partial}{\partial x}\Gamma(x, y) = \frac{-\gamma\varphi'(x)}{1-\delta-\gamma\varphi(x)} < 0 \text{ and } \frac{\partial}{\partial y}\Gamma(x, y) = \frac{\gamma\varphi'(y)}{1-\gamma\varphi(y)} > 0. \quad (2.29)$$

Proof. This follows directly by taking derivatives in Lemma 2.7. □

Definition 2.9. An impulse control strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is a *constant boundary strategy* if there exist $a < \alpha \leq \beta < b \in \mathbb{R}$ such that $\xi_0 + y_0 \in]a, b[$ and inductively

$$\tau_n = \inf\{t > \tau_{n-1} : Y_t^{n-1} \notin]a, b[\}, \quad \xi_n = \begin{cases} \alpha - a, & Y_{\tau_n}^{n-1} = a, \\ \beta - b, & Y_{\tau_n}^{n-1} = b, \end{cases}$$

where

$$Y_t^{n-1} = y_0 + \left(\mu - r - \frac{1}{2}\sigma^2\right) \cdot t + \sigma W_t + \sum_{k=0}^{n-1} \xi_k \mathbb{1}_{\{\tau_k \leq t\}}.$$

We will later refer to \bar{K} as a *constant boundary strategy given by* (a, α, β, b) .

For a later reference we summarize some of the main results from [IS06] in their transformed form, i.e. after the transformation from $]0, 1[$ to \mathbb{R} via $\psi = \varphi^{-1} : \Delta \setminus \partial\Delta \rightarrow \mathbb{R}$ from Section 1.4. This transformed form is more convenient for us, since we want to use similar proofs as for Model B. The next Theorem is mainly the counterpart of Theorem 2.4.

Theorem 2.10. *Let $\delta \in]0, 1[$, $\gamma \in [0, 1 - \delta[$ as in Section 1.2, L as in (2.22), $\bar{f} = f \circ \varphi$ as in (1.77), Γ as in (2.27) and let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$. Then there exist constants $l > 0$ and $a < \alpha \leq \beta < b \in \mathbb{R}$ and a bounded continuous function $u : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ satisfying*

- (i) $\alpha = \beta \Leftrightarrow \gamma = 0$.
- (ii) Every constant boundary strategy $\bar{K} = \bar{K}(a, \alpha, \beta, b) = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ as in Definition 2.9 is optimal for the modified cost function Γ with optimal value $\bar{J}(\bar{K}) = r + l = \rho$.
- (iii) If $\varphi(a) \leq \varphi(\alpha)(1 - \delta)$, then \bar{K} is also optimal for the original cost function $\bar{\Gamma}$.
- (iv) $\max\{f(0), f(1)\} < l < f(\hat{h})$.
- (v) For every $y \in]a, b[$ and $\sigma \leq \tau^*(y) := \inf\{t > 0 : X_t^y \notin]a, b[\}$, where X^y is the diffusion from (2.21) starting in y , it holds $u(y) = E \left(\int_0^\sigma (\bar{f}(X_s^y) - l) ds + u(X_\sigma^y) \right)$.
- (vi) $\|u\|_\infty \leq \|\Gamma\|_\infty < \infty$, $u|_{[a, b]} \in C^2([a, b])$, $u|_{\mathbb{R} \setminus [a, b]} \in C^2(\mathbb{R} \setminus [a, b])$.
- (vii) There is an $x_0 \in \{y \in \mathbb{R} : \bar{f}(y) - l \geq 0\} \cap [\alpha, \beta]$ such that $u(x_0) = \|u\|_\infty$ and $\alpha < x_0 < \beta \Leftrightarrow \gamma > 0$.
- (viii) $Lu(x) + \bar{f}(x) - l \leq 0$ and $u(y) - u(x) + \Gamma(x, y) \leq 0$ for all $x, y \in \mathbb{R}$.
- (ix) $Lu(x) + \bar{f}(x) - l = 0$ for all $x \in]a, b[$.

- (x) $u(\beta) - u(x) + \Gamma(x, \beta) = 0$ for all $x \geq b$ and $u(\alpha) - u(x) + \Gamma(x, \alpha) = 0$ for all $x \leq a$.
- (xi) $u'(\beta) = -\frac{\partial}{\partial y}\Gamma(b, y)|_{y=\beta} = \frac{-\gamma\varphi'(\beta)}{1-\gamma\varphi(\beta)}$ and $u'(\alpha) = -\frac{\partial}{\partial y}\Gamma(a, y)|_{y=\alpha} = \frac{\gamma\varphi'(\alpha)}{1+\gamma\varphi(\alpha)}$.
- (xii) $u'(b) = \frac{\partial}{\partial x}\Gamma(x, \beta)|_{x=b} = \frac{-\gamma\varphi'(b)}{1-\delta-\gamma\varphi(b)}$ and $u'(a) = \frac{\partial}{\partial x}\Gamma(x, \alpha)|_{x=a} = \frac{\gamma\varphi'(a)}{1-\delta+\gamma\varphi(a)}$.
- (xiii) For $\gamma > 0$ u is strictly increasing on $\mathbb{R}_{<x_0}$ and strictly decreasing on $\mathbb{R}_{>x_0}$.
- (xiv) For $\gamma = 0$ u is strictly increasing on $[a, \alpha]$, strictly decreasing on $[\alpha, b]$ and $u \equiv 0$ on $\mathbb{R} \setminus [a, b]$.
- (xv) $u \circ \psi$ differs only by a constant from its counterpart in Theorem 2.4 while $l, \varphi(x_0), \varphi(a), \varphi(\alpha), \varphi(\beta)$ and $\varphi(b)$ are equal.

Proof. We have added the constant $k = -\inf_{x \in [0,1]} \tilde{u}(x)$ to the function $\tilde{u} \circ \varphi$, where \tilde{u} denotes the solution from Theorem 2.4. Thus we have $\inf_{x \in \mathbb{R}} u(x) = 0$ and we only have to show (v),(vi) and (vii), since the other assertions are shown in [IS06] in their untransformed form on $[0, 1]$.

As described in Theorem 4.1 of [IS06], we can use Itô's formula on $\tilde{u}(h) = \tilde{u}(\varphi(X^y))$, where X^y is from (2.21) and h is the corresponding uncontrolled risky fraction process, and get

$$\tilde{u}(h_t) = \tilde{u}(h_0) + \int_0^t D\tilde{u}(h_s)ds + \int_0^t \sigma(1-h_s)h_s\tilde{u}'(h_s)dW_s.$$

Since \tilde{u}' is bounded and $E(\sigma) \leq E(\tau^*(y)) < \infty$ by Lemma 2.5, we then get

$$\tilde{u}(h_0) = E\left(-\int_0^\sigma D\tilde{u}(h_s)ds + \tilde{u}(h_\sigma)\right)$$

and therefore due to $u = \tilde{u} \circ \varphi - \inf_{x \in [0,1]} \tilde{u}(x)$ also

$$\begin{aligned} u(y) &= E\left(-\int_0^\sigma Lu(X_s^y)ds + u(X_\sigma^y)\right) \\ &= E\left(\int_0^\sigma (\bar{f}(X_s^y) - l)ds + u(X_\sigma^y)\right) \end{aligned}$$

where the last equality is due to (ix). The construction of \tilde{u} in [IS06] guarantees the differentiability assertion in (vi) as well as the existence of an $x \in]\alpha, \beta[$ in the case $\gamma > 0$ and $x = \alpha = \beta$ in the case $\gamma = 0$, where \tilde{u} attains its maximum. By (ix) this x satisfies $\frac{\sigma^2}{2}u''(x) = -\bar{f}(x) + l$ and hence $\bar{f}(x) - l \geq 0$ due to the maximality. Now (viii) yields the inequality $u(x) \leq u(y) - \Gamma(y, x)$ for all $y \in \mathbb{R}$ and $\inf_{y \in \mathbb{R}} u(y) = 0$ then implies $\|u\|_\infty \leq \|\Gamma\|_\infty$. Finally, $\|\Gamma\|_\infty < \infty$ can easily be seen from its definition. □

Chapter 3

Optimal strategies for Model B

The transformed problem of maximizing the expected growth rate from Section 1.4 in Model B was solved by Tamura in [Tam06] for the one-dimensional case and in [Tam08] for the general case. Since the assumption on the transaction costs to be sufficiently small in [Tam06] was dropped in [Tam08], the object of consideration in this chapter is only the solution obtained in [Tam08].

We begin by showing in Section 3.1, under rather strong assumptions, how a solution to a certain quasi-variational inequality can be used to derive an optimal strategy, using the so-called Hamilton-Jacobi-Bellman approach on \mathbb{R} . Section 3.2 is then devoted to the optimal strategy gained in [Tam08], which is induced by some solution u to a quasi-variational inequality in a weak sense (cf. also Appendix A). Finally, we establish several properties of the solution u itself in Section 3.3 and use these results in Section 3.4 to prove that in the one-dimensional case u admits a constant boundary strategy.

3.1 A Hamilton-Jacobi-Bellman approach

In Section 1.4 the problem of maximizing the expected growth rate in (1.64) was transformed from the simplex Δ into the whole space \mathbb{R}^d . We begin by recalling some results on the transformed problem.

To simplify the notation, we denote here by $(X_t)_{t \geq 0} = (X_t^1, \dots, X_t^d)_{t \geq 0}$ the uncontrolled part of the diffusion in (1.81) with starting value y_0 , i.e. we have for every $i = 1, \dots, d$

$$X_t^i = y_0^i + \left(\mu_i - r - \frac{1}{2}(\sigma\sigma^T)_{ii} \right) \cdot t + \sum_{j=1}^d \sigma_{ij} W_t^j. \quad (3.1)$$

The operator associated with $(X_t)_{t \geq 0}$ (or its generator) is therefore

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(\mu_i - r - \frac{1}{2}(\sigma\sigma^T)_{ii} \right) \frac{\partial}{\partial x_i}. \quad (3.2)$$

Given a trading strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0} \in \bar{\mathcal{A}}$ we define the controlled càdlàg process

$$(Y_t)_{t \geq 0} := (Y_t^{\bar{K}})_{t \geq 0} := \left(X_t + \sum_{k=0}^{\infty} \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0}, \quad (3.3)$$

which is just the diffusion in (1.81). Furthermore, let \bar{C} denote the cost function in (1.78) and $\bar{f} = f \circ \varphi$ be as in (1.77). Then by Lemma 1.22 the expected growth rate satisfies

$$\bar{J}(\bar{K}) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \bar{f}(Y_{s-}) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \quad (3.4)$$

and the main task to achieve is finding an optimal strategy \bar{K}^* such that

$$\bar{J}(\bar{K}^*) = \rho := \sup_{\bar{K} \in \bar{\mathcal{A}}} \bar{J}(\bar{K})$$

and estimating the optimal growth rate ρ .

Analogously to Section 2.1 the Hamilton-Jacobi-Bellman approach consists of finding a constant $l \in \mathbb{R}$ and a solution u on \mathbb{R} to the quasi-variational inequality (QVI)

$$\max\{Lu + \bar{f} - l, Mu - u\} = 0, \quad (3.5)$$

where for every bounded function $v : \mathbb{R} \rightarrow \mathbb{R}$ the function Mv is defined on \mathbb{R} by

$$Mv(x) := \sup_{y \in \mathbb{R}} v(y) + \bar{C}(x, y - x).$$

The notion "solution" to (3.5) can mean a solution in the classical sense or in the sense of weak derivatives (see also Appendix A).

If u is smooth enough that Itô's formula is applicable, it remains to check the existence and optimality of the trading strategy $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$, which inductively is given by

$$\tau_0 := 0, \quad \xi_0 \in A + \{-y_0\}, \quad \tau_n := \inf\{t > \tau_{n-1} : u(Y_t^{n-1}) \leq Mu(Y_t^{n-1})\}, \quad n \in \mathbb{N}, \quad (3.6)$$

where $A := \{x \in \mathbb{R} : Mu(x) > u(x)\}$ is the continuation set, $\xi_n \in A + \{-Y_{\tau_n}^{n-1}\}$ attains

$$u(\xi_n) + \bar{C}(Y_{\tau_n}^{n-1}, \xi_n) = Mu(Y_{\tau_n}^{n-1}) = u(Y_{\tau_n}^{n-1}) \quad (3.7)$$

and

$$(Y_t^n)_{t \geq 0} := \left(X_t + \sum_{k=0}^n \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0}.$$

We shall give such a verification theorem under rather strong assumptions to get a better understanding.

Lemma 3.1. *Let $u \in C^2(\mathbb{R}^d, \mathbb{R})$ and $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be a trading strategy with the corresponding controlled process $(Y_t)_{t \geq 0}$ from (3.3). Then we have for every $t \geq 0$*

$$u(Y_t) = u(y_0) + \sum_{i,j=1}^d \int_0^t \frac{\partial}{\partial x_i} u(Y_{s-}) \sigma_{ij} dW_s^j + \int_0^t Lu(Y_{s-}) ds + \sum_{k=0}^{\infty} (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq t\}}.$$

Proof. Since $(Y_t)_{t \geq 0}$ is a càdlàg semimartingale, we get from Itô's formula (cf. [JS03], Theorem I.4.57) for every $t \geq 0$

$$\begin{aligned} u(Y_t) &= u(y_0) + \sum_{i,j=1}^d \int_0^t \frac{\partial}{\partial x_i} u(Y_{s-}) \sigma_{ij} dW_s^j \\ &\quad + \sum_{i=1}^d \left(\int_0^t \frac{\partial}{\partial x_i} u(Y_{s-}) \left(\mu_i - r - \frac{1}{2} (\sigma \sigma^T)_{ii} \right) ds + \sum_{k=0}^{\infty} \frac{\partial}{\partial x_i} u(Y_{\tau_k-}) \xi_k^i \mathbb{1}_{\{\tau_k \leq t\}} \right) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} u(Y_{s-}) d[Y^{i,c}, Y^{j,c}]_s \\ &\quad + \sum_{k=0}^{\infty} \left(u(Y_{\tau_k}) - u(Y_{\tau_k-}) - \sum_{i=1}^d \frac{\partial}{\partial x_i} u(Y_{\tau_k-}) \xi_k^i \right) \mathbb{1}_{\{\tau_k \leq t\}}. \end{aligned}$$

For every $i = 1, \dots, d$ the process $\left(\sum_{k=0}^{\infty} \xi_k^i \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0}$ is of locally finite variation, and thus we get

$$[Y^{i,c}, Y^{j,c}]_t = (\sigma \sigma^T)_{ij} \cdot t.$$

The assertion follows now from observing that

$$\sum_{s \leq t} \Delta Y_s^i \Delta Y_s^j = \sum_{k=0}^{\infty} \xi_{\tau_k}^i \xi_{\tau_k}^j \mathbb{1}_{\{\tau_k \leq t\}}.$$

□

With Lemma 3.1 at hand, we can prove the verification theorem in two propositions.

Proposition 3.2. *Let $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be a trading strategy with the corresponding controlled process $(Y_t)_{t \geq 0}$ from (3.3) and suppose there exists a bounded function $u \in C^2(\mathbb{R}^d, \mathbb{R})$ with bounded first partial derivatives and a constant $l \in \mathbb{R}$ satisfying the QVI (3.5). Then we get for the constant $\bar{J}(\bar{K})$ in (3.4) the upper bound $\bar{J}(\bar{K}) \leq r + l$.*

Proof. From Lemma 3.1 and since $\left(\sum_{i,j=1}^d \int_0^t \frac{\partial}{\partial x_i} u(Y_{s-}) \sigma_{ij} dW_s^j \right)_{t \geq 0}$ is a martingale, we get for every $T \geq 0$

$$E(u(y_0) - u(Y_T)) = E \left(- \int_0^T Lu(Y_{s-}) ds - \sum_{k=0}^{\infty} (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq T\}} \right). \quad (3.8)$$

From $u \geq Mu$ we then deduce for every $k \in \mathbb{N}_0$ on $\{\tau_k < \infty\}$

$$(u(Y_{\tau_k}) - u(Y_{\tau_k-})) \leq -\bar{C}(Y_{\tau_k-}, \xi_k) \quad (3.9)$$

and using $Lu \leq -\bar{f} + l$ we get in (3.8)

$$E(u(y_0) - u(Y_T)) \geq E \left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right). \quad (3.10)$$

With this we can finally conclude

$$\begin{aligned}\bar{J}(\bar{K}) &= r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &\leq r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} (E(|u(Y_T)|) + u(y_0)) \\ &= r + l.\end{aligned}$$

□

Proposition 3.3. *Let u, l satisfy the conditions of Proposition 3.2 and let X denote the uncontrolled diffusion in (3.1). Suppose the set $A := \{x \in \mathbb{R} : u(x) > Mu(x)\}$ is nonempty and there exists a measurable function $\pi : \partial A \rightarrow A$ such that for all $x \in \partial A$*

$$u(x) = u(\pi(x)) + \bar{C}(x, \pi(x) - x).$$

If the trading strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$, which is inductively given by

$$\tau_0 := 0, \xi_0 + y_0 \in A, \tau_n := \inf\{t > \tau_{n-1} : u(Y_t^{n-1}) \notin A\}, \xi_n = \pi(Y_{\tau_n}^{n-1}) - Y_{\tau_n}^{n-1} \quad (3.11)$$

and

$$(Y_t^n)_{t \geq 0} := \left(X_t + \sum_{k=0}^n \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0},$$

satisfies $\tau_n < \infty$ for all $n \in \mathbb{N}$, it holds $\bar{J}(\bar{K}) = r + l$.

Proof. Observing carefully the proof of Proposition 3.2, we have equality in (3.9) and (3.10) except for the case $n = 0$, which has no influence on the expected growth rate due to the boundedness of \bar{C} .

□

3.2 An optimal strategy

Unfortunately, the objective to find a function u satisfying the conditions of Proposition 3.3 is very difficult to achieve. Even in the one-dimensional case, rigorously constructed in [IS06] for the slightly different cost function in Model A, the solution is only piecewise twice continuously differentiable (cf. Chapter 2).

Apart from direct construction, in [Tam06] and [Tam08] the mode of operation is different. Tamura uses variational techniques from [BL82] and [BL84] and perturbation methods. He establishes a solution u_α to the QVI with a discount factor $\alpha > 0$

$$\max\{Lu - \alpha u + \bar{f} - l, Mu - u\} = 0$$

in its weak form (cf. A.11) and by letting α tend to zero obtains a solution u to the weak form of

$$\max\{Lu + \bar{f} - l, Mu - u\} = 0 \quad (3.12)$$

(cf. Theorem 4.2 in [Tam08]).

Furthermore, for any two stopping times $\tau \leq \sigma$ with $E(\tau) \leq E(\sigma) < \infty$ he validates the equation

$$u(Y_\tau) = E\left(\int_\tau^{\tau^*} (\bar{f}(X_s) - l) ds + u(X_{\tau^*}) \middle| \mathcal{F}_\tau\right) \geq E\left(\int_\tau^\sigma (\bar{f}(X_s) - l) ds + u(X_\sigma) \middle| \mathcal{F}_\tau\right), \quad (3.13)$$

where $\tau^* := \inf\{t > \tau : Mu(X_t) \geq u(X_t)\}$ and X is from (3.1), and proves $E(\tau^*) < \infty$ (cf. Lemma 3.7 in [Tam08]). Finally, he uses (3.13) for the verification theorem mentioned above.

We see two problems in this approach. The first one is the reference Theorem 8.5 of [BL82], which is used for the generalized Itô's formula. It considers expectation in contrast to the conditioned expectation in (3.13).

Secondly, the uncontrolled process X is used in both the reference above and (3.13). Instead we need a statement like (3.8) about the controlled process Y from (3.3).

The main objective of this section is therefore the replacement of (3.13) in the proof of a verification theorem.

To the best of the author's knowledge, the conditions our problem poses are nowhere fully covered. These include the weighted Sobolev spaces $W^{1,p,\gamma}(\mathbb{R}^d)$, $W^{2,p,\gamma}(\mathbb{R}^d)$ and also the controlled càdlàg process Y from (3.3).

The version of Itô's formula from Theorem 8.5 in [BL82] is proved via approximation by smooth functions, which the statement is known for. The same is true for Theorem 10.1 in Chapter 2 of [Kry80]. In the following we apply this approach to the function u obtained in [Tam08]. We begin with smooth functions.

Lemma 3.4. *Let $u \in C^2(\mathbb{R}^d, \mathbb{R})$ and $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be a trading strategy with the corresponding controlled process $(Y_t)_{t \geq 0}$ from (3.3). Let L be the generator from (3.2) and assume u and Lu to be bounded in \mathbb{R}^d . If $\sigma \leq \sigma'$ are stopping times with $E(\sigma') < \infty$ and $\sigma' \leq \tau_n$ for some $n \in \mathbb{N}_0$, then we have*

$$E(u(Y_\sigma)) = E\left(-\int_\sigma^{\sigma'} Lu(Y_{s-}) ds + u(Y_{\sigma'}) - \sum_{k=0}^n (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\sigma < \tau_k \leq \sigma'\}}\right). \quad (3.14)$$

Proof. Let $n \in \mathbb{N}_0$ and suppose $\sigma \leq \sigma'$ are stopping times with $E(\sigma) \leq E(\sigma') < \infty$ and $\sigma' \leq \tau_n$. Since $u \in C^2(\mathbb{R}^d, \mathbb{R})$ we know that

$$\left(\int_0^t \frac{\partial}{\partial x_i} u(Y_{s-}) \sigma_{ij} dW_s^j\right)_{t \geq 0} \quad \text{is a local martingale for every } i, j \in \{1, \dots, d\}.$$

Hence we can find a common localizing sequence, i.e. a sequence of stopping times $\sigma_1 \leq \sigma_2 \leq \dots \nearrow \infty$ such that for every $i, j \in \{1, \dots, d\}$ and $m \in \mathbb{N}$

$$\left(\int_0^{t \wedge \sigma_m} \frac{\partial}{\partial x_i} u(Y_{s-}) \sigma_{ij} dW_s^j \right)_{t \geq 0} \text{ is a martingale.}$$

Using the optional sampling theorem we now get from Lemma 3.1 for $t > 0$ and $m \in \mathbb{N}$

$$\begin{aligned} E(u(Y_{\sigma \wedge t \wedge \sigma_m})) &= E(u(Y_{\sigma \wedge t \wedge \sigma_m}) - u(Y_{\sigma' \wedge t \wedge \sigma_m}) + u(Y_{\sigma' \wedge t \wedge \sigma_m})) \\ &= E\left(- \int_{\sigma \wedge t \wedge \sigma_m}^{\sigma' \wedge t \wedge \sigma_m} Lu(Y_{s-}) ds + u(Y_{\sigma' \wedge t \wedge \sigma_m}) - \sum_{k=0}^n (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\sigma \wedge t \wedge \sigma_m < \tau_k \leq \sigma' \wedge t \wedge \sigma_m\}} \right). \end{aligned}$$

From the assumptions we can use Lebesgue's convergence theorem and get by letting $t, m \rightarrow \infty$ the desired conclusion. \square

With the last lemma at hand, we are able to prove the version of Itô's formula with expectations for functions that are only weakly differentiable.

Proposition 3.5. *Let $u \in C_b(\mathbb{R}^d, \mathbb{R})$ be a function with existing weak derivatives $D^\alpha u$ for every multi-index α with $|\alpha| \leq 2$ and assume $Lu \in C_b(\mathbb{R}^d, \mathbb{R})$ (in the sense of Definition A.1) for the generator L defined in (3.2).*

Let $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be a trading strategy with the corresponding controlled process $(Y_t)_{t \geq 0}$ from (3.3). If $\sigma \leq \sigma'$ are stopping times with $E(\sigma') < \infty$ and $\sigma' \leq \tau_n$ for some $n \in \mathbb{N}_0$, then we have

$$E(u(Y_\sigma)) = E\left(- \int_\sigma^{\sigma'} Lu(Y_{s-}) ds + u(Y_{\sigma'}) - \sum_{k=0}^n (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\sigma < \tau_k \leq \sigma'\}} \right). \quad (3.15)$$

Proof. From Proposition A.12 we can take an approximating sequence

$(\varphi_m)_{m \in \mathbb{N}} \in (C^\infty(\mathbb{R}^d))^{\mathbb{N}}$ for u that satisfies

- (i) $\varphi_m \xrightarrow{m \rightarrow \infty} u$ and $L\varphi_m \xrightarrow{m \rightarrow \infty} Lu$ uniformly on every compact set of \mathbb{R}^d ,
- (ii) $\sup_{m \in \mathbb{N}} \|\varphi_m\|_\infty \leq \|u\|_\infty$ and $\sup_{m \in \mathbb{N}} \|L\varphi_m\|_\infty \leq \|Lu\|_\infty$.

If we take $n \in \mathbb{N}_0$ and stopping times $\sigma \leq \sigma'$ with $E(\sigma') < \infty$ and $\sigma' \leq \tau_n$, then we have by (ii) that for every $m \in \mathbb{N}$ φ_m satisfies (3.15) by means of Lemma 3.4, i.e.

$$E(\varphi_m(Y_\sigma)) = E\left(- \int_\sigma^{\sigma'} L\varphi_m(Y_{s-}) ds + \varphi_m(Y_{\sigma'}) - \sum_{k=0}^n (\varphi_m(Y_{\tau_k}) - \varphi_m(Y_{\tau_k-})) \mathbb{1}_{\{\sigma < \tau_k \leq \sigma'\}} \right).$$

We now get the assertion from (i) and (ii) by using Lebesgue's convergence theorem. \square

It remains to prove a verification theorem for the solution u to (3.12) found by Tamura in [Tam08]. Since we want to replace Lemma 3.7 of [Tam08], or equivalently (3.13), we have of course to be careful not to use results relying on this Lemma.

Theorem 4.1 in [Tam08] yields a solution u_α to the weak form of (3.12) with $l = 0$. The constant

$$l := \limsup_{\alpha \rightarrow 0} \left(\alpha \cdot \inf_{x \in \mathbb{R}^d} u_\alpha(x) \right) \quad (3.16)$$

then satisfies $\sup_{x \in \mathbb{R}^d} \bar{f}(x) \geq l \geq \max\{f(0), f(e_1), \dots, f(e_d)\}$, where $f(0) = 0$, e_i is the i -th unit vector in \mathbb{R}^d and $f(e_i) = \mu_i - r - \frac{1}{2}(\sigma\sigma^T)_{ii}$ for every $i \in \{1, \dots, d\}$ (cf. Lemma 4.1 in [Tam08]).

For our verification theorem we state the existence result for the solution u together with an approximation statement.

Theorem 3.6. *Let l be as in (3.16) and assume further $f(e_i) \neq f(0)$ and $f(e_i) \neq f(e_j)$ for all $i, j \in \{1, \dots, d\}$. Then if $l > \max\{f(0), f(e_1), \dots, f(e_d)\}$, there exists some $u \in W^{1,p,\gamma}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d, \mathbb{R})$ with $u \geq Mu$ for some $p \geq 1$. Furthermore, u is Lipschitz continuous and $u \circ \psi$ is uniformly continuous on $\Delta \setminus \partial\Delta$ and there exists a sequence $(u_n)_{n \in \mathbb{N}}$ satisfying*

$$(i) \quad u_n \in W^{2,p,\gamma}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d, \mathbb{R}) \text{ and } u_n \leq u \text{ for every } n \in \mathbb{N},$$

$$(ii) \quad -Lu_n - n(u_n - Mu)^- = \bar{f} - l,$$

$$(iii) \quad \lim_{n \rightarrow \infty} u_n = u \text{ in } C(\mathbb{R}^d, \mathbb{R}).$$

Proof. The existence result is due to Theorems 3.3 and 4.2 in [Tam08]. In the proof of Theorem 3.3 in [Tam08] the function u is approximated by functions u_α for $\alpha \rightarrow 0$ that have a common Lipschitz constant, yielding the Lipschitz continuity of u .

Furthermore, u is also approximated by functions v_α for $\alpha \rightarrow 0$ in the proof of Theorem 4.2 in [Tam08] and the functions $v_\alpha \circ \psi$ are shown to be uniformly equicontinuous, whence it is inferred that $u \circ \psi$ is also uniformly continuous.

Finally, the approximation statements (i)-(iii) can be found in the proof of Lemma 3.7 in [Tam08].

□

Now we can proceed to the verification theorem. We begin with an upper bound for the optimal expected growth rate.

Theorem 3.7. *Let $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ be a trading strategy with the corresponding controlled process $(Y_t)_{t \geq 0}$ from (3.3) and assume the conditions of Theorem 3.6 to hold. Then we get for the constant $\bar{J}(\bar{K})$ in (3.4) the upper bound $\bar{J}(\bar{K}) \leq r + l$ with l is from (3.16).*

Proof. Let u , Mu and $(u_n)_{n \in \mathbb{N}}$ be the functions obtained in Theorem 3.6 and let L denote the generator defined in (3.2).

Since $L^{p,\gamma}(\mathbb{R}^d) \subseteq L^1_{loc}(\mathbb{R}^d)$, we can apply Proposition 3.5 and get for every $T > 0$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$

$$\begin{aligned} E(u_m(y_0)) &= E\left(-\int_0^{\tau_n \wedge T} Lu_m(Y_{s-})ds + u_m(Y_{\tau_n \wedge T}) - \sum_{k=0}^n (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq \tau_n \wedge T\}}\right) \\ &\stackrel{3.6(ii)}{=} E\left(\int_0^{\tau_n \wedge T} (\bar{f}(Y_{s-}) - l)ds + m \int_0^{\tau_n \wedge T} (u_m(Y_{s-}) - Mu(Y_{s-}))^- ds \right. \\ &\quad \left. + u_m(Y_{\tau_n \wedge T}) - \sum_{k=0}^n (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq \tau_n \wedge T\}}\right) \\ &\geq E\left(\int_0^{\tau_n \wedge T} (\bar{f}(Y_{s-}) - l)ds + u_m(Y_{\tau_n \wedge T}) - \sum_{k=0}^n (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq \tau_n \wedge T\}}\right). \end{aligned}$$

By letting $m \rightarrow \infty$ we thus get by (iii) of Theorem 3.6 and Lebesgue's convergence theorem

$$E(u(y_0)) \geq E\left(\int_0^{\tau_n \wedge T} (\bar{f}(Y_{s-}) - l)ds + u(Y_{\tau_n \wedge T}) - \sum_{k=0}^n (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq \tau_n \wedge T\}}\right). \quad (3.17)$$

From $u \geq Mu$ we then conclude

$$u(Y_{\tau_k}) - u(Y_{\tau_k-}) \leq -\bar{C}(Y_{\tau_k-}, \xi_k) \text{ on } \{\tau_k < \infty\}$$

and by plugging it into (3.17) we get

$$\begin{aligned} E(u(y_0)) &\geq E\left(\int_0^{\tau_n \wedge T} (\bar{f}(Y_{s-}) - l)ds + u(Y_{\tau_n \wedge T}) + \sum_{k=0}^n \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq \tau_n \wedge T\}}\right) \\ &= E\left(\int_0^{\tau_n \wedge T} (\bar{f}(Y_{s-}) - l)ds + u(Y_{\tau_n \wedge T}) + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq \tau_n \wedge T\}}\right). \end{aligned}$$

Using Lebesgue's convergence theorem for $n \rightarrow \infty$ we can therefore conclude

$$E(u(y_0)) \geq E\left(\int_0^T (\bar{f}(Y_{s-}) - l)ds + u(Y_T) + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}}\right),$$

which finally implies

$$\begin{aligned} \bar{J}(\bar{K}) &= r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E\left(\int_0^T (\bar{f}(Y_{s-}) - l)ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &\leq r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} (E(u(y_0)) - u(Y_T)) \\ &= r + l, \end{aligned}$$

since u is bounded. □

Before we give an optimal strategy for the problem of maximizing $\bar{J}(\bar{K})$ in (3.4) over all trading strategies \bar{K} , we need some results on approximation of the corresponding stopping times and the behavior of the functions Mu and u .

Lemma 3.8. *Let $u, u_n, \psi \in C(\mathbb{R}^d, \mathbb{R})$, $n \in \mathbb{N}$, satisfy $u \geq \psi$, $\lim_{n \rightarrow \infty} \|u_n - u\|_\infty = 0$ and $u_n \leq u$ for all $n \in \mathbb{N}$. Then for every stopping time τ and every càdlàg process $(Y_t)_{t \geq 0}$, which is continuous on $[\tau, \infty[$, the stopping times*

$$\hat{\tau} := \inf \{t \geq \tau : u(Y_t) \leq \psi(Y_t)\} \text{ and } \tau_n := \inf \{t \geq \tau : u_n(Y_t) \leq \psi(Y_t)\},$$

defined for every $n \in \mathbb{N}$, satisfy

$$(i) \quad \tau_n \leq \hat{\tau},$$

$$(ii) \quad \lim_{n \rightarrow \infty} \tau_n = \hat{\tau}.$$

Proof. (i) This is a direct consequence of $u_n \leq u$ for all $n \in \mathbb{N}$.

(ii) Let $\omega \in \Omega$. From (i) we can assume without loss of generality $\hat{\tau}(\omega) > \tau(\omega)$. Let $t_0 \in]\tau(\omega), \hat{\tau}(\omega)[$. Using $u(Y_t(\omega)) > \psi(Y_t(\omega))$ for every $t \in [\tau(\omega), t_0]$ and $\lim_{n \rightarrow \infty} \|u_n - u\|_\infty = 0$ we get an $n_0 \in \mathbb{N}$ satisfying

$$u_n(Y_t(\omega)) > \psi(Y_t(\omega)) \text{ for every } t \in [\tau(\omega), t_0] \text{ and } n \geq n_0.$$

This implies $\tau_n(\omega) > t_0$ for every $n \geq n_0$ and now (ii) follows from (i). □

Lemma 3.9. *Let u and Mu denote the functions from Theorem 3.6. Then*

$$\{x \in \mathbb{R}^d : u(x) > Mu(x)\} \neq \emptyset.$$

Proof. For an $y \in \mathbb{R}^d$ satisfying $u(y) > \sup_{x \in \mathbb{R}^d} u(x) - \frac{1}{2} |\log(1 - \delta)|$ we have

$$u(y) + \frac{1}{2} \log(1 - \delta) > \sup_{x \in \mathbb{R}^d} u(x) + \log(1 - \delta) \geq \sup_{x \in \mathbb{R}^d} u(x) + \bar{C}(y, x - y) = Mu(y)$$

and hence $y \in \{x \in \mathbb{R}^d : u(x) > Mu(x)\}$. □

The cost function \bar{C} inherits the triangle inequality and the boundedness of the cost function C from Theorem 1.16 in the following form.

Lemma 3.10. *The cost function \bar{C} from (1.78) satisfies*

$$(i) \quad \|\bar{C}\|_\infty < \infty,$$

$$(ii) \quad 0 > \bar{C}(y, z - y) \geq \bar{C}(y, x - y) + \bar{C}(x, z - x) + |\log(1 - \delta)| \text{ for any } x, y, z \in \mathbb{R}^d.$$

Proof. This is easily inferred from Theorem 1.16 by the definition of \bar{C} .

□

Lemma 3.11. *Let u and Mu be from Theorem 3.6 and \bar{C} be as in (1.78). Then for every $x \in \mathbb{R}^d$ we have*

$$Mu(x) \geq u(x) + \log(1 - \delta). \quad (3.18)$$

Furthermore, we also have

$$u(y) \geq Mu(y) - r + |\log(1 - \delta)| \quad (3.19)$$

for every $y \in \mathbb{R}^d$ and $r > 0$ satisfying

$$u(y) + \bar{C}(x, y - x) \geq Mu(x) - r. \quad (3.20)$$

Proof. Let $x \in \mathbb{R}^d$. By the definition of \bar{C} we have

$$\begin{aligned} Mu(x) &\geq u(x) + \bar{C}(x, x - x) \stackrel{(1.78)}{=} u(x) + \tilde{C}(\varphi(x), 0) \\ &\stackrel{(1.74)}{=} u(x) + \log((1 - \delta)(1 - C(\varphi(x), 0))) \\ &\stackrel{(1.63)}{=} u(x) + \log(\hat{C}(\varphi(x), \varphi(x))) \\ &\stackrel{1.17}{=} u(x) + \log(1 - \delta). \end{aligned}$$

Now for every $r > 0$ and $y \in \mathbb{R}^d$ with

$$u(y) + \bar{C}(x, y - x) + r \geq Mu(x)$$

we can use the triangle inequality of the cost function \bar{C} from Lemma 3.10 (ii) to get for every $z \in \mathbb{R}^d$

$$\begin{aligned} u(y) &\geq Mu(x) - r - \bar{C}(x, y - x) \\ &\geq u(x + (y - x) + z) + \bar{C}(x, y - x + z) - r - \bar{C}(x, y - x) \\ &\geq u(x + (y - x) + z) + \bar{C}(x + (y - x), z) - r + |\log(1 - \delta)| \\ &= u(y + z) - r + \bar{C}(y, z) + |\log(1 - \delta)| \end{aligned}$$

and hence

$$u(y) \geq \sup_{z \in \mathbb{R}^d} u(y + z) + \bar{C}(y, z) - r + |\log(1 - \delta)| = Mu(y) - r + |\log(1 - \delta)|.$$

□

Finally, we can proceed to the second part of the verification theorem.

Theorem 3.12. *Let the conditions of Theorem 3.6 be satisfied and let u and Mu denote the functions given by this theorem. We take $y \in \{x \in \mathbb{R}^d : u(x) > Mu(x)\}$ and $\varepsilon > 0$ satisfying $\varepsilon < |\log(1 - \delta)|$ and denote by $(X_t)_{t \geq 0}$ the uncontrolled process from (3.1) with starting value $y_0 \in \mathbb{R}^d$. If we define*

$$\tau_0 := 0, \quad \xi_0 := y - y_0 \quad (3.21)$$

and inductively

$$\tau_n := \inf\{t > \tau_{n-1} : u(Y_t^{n-1}) \leq Mu(Y_t^{n-1})\} \text{ with } \inf \emptyset = \infty, \quad (3.22)$$

where

$$(Y_t^n)_{t \geq 0} := \left(X_t + \sum_{k=0}^n \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0},$$

taking ξ_n that attains

$$u(Y_{\tau_n}^{n-1} + \xi_n) + \bar{C}(Y_{\tau_n}^{n-1}, \xi_n) + \frac{\varepsilon}{2^n} \geq Mu(Y_{\tau_n}^{n-1}) = u(Y_{\tau_n}^{n-1}) \quad (3.23)$$

on $\{\tau_n < \infty\}$ and $\xi_n \equiv 0$ on $\{\tau_n = \infty\}$, then the strategy $\bar{K} := ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is a trading strategy and the corresponding controlled process $(Y_t)_{t \geq 0}$ from (3.3) satisfies $\bar{J}(\bar{K}) = r + l$ for the constants $\bar{J}(\bar{K})$ in (3.4) and l from Theorem 3.6 and hence \bar{K} is optimal.

Proof. a) In order to show that $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is a trading strategy, we have to prove

(i) $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ with $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}_0$ and

(ii) $P\left(\lim_{n \rightarrow \infty} \tau_n = \infty\right) = 1$.

(i) Let Y denote the process from (3.3). We have by Lemma 3.11 on $\{\tau_n < \infty\}$

$$u(Y_{\tau_n}) > Mu(Y_{\tau_n}) \text{ and hence } \tau_n < \tau_{n+1} \text{ on } \{\tau_n < \infty\}$$

for every $n \in \mathbb{N}$. Since $y \in \{x \in \mathbb{R}^d : u(x) > Mu(x)\}$ the same is true for $n = 0$.

(ii) will be shown at the end of the proof by using the optimality of the strategy $((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$, for which we have to be careful not to use (ii).

b) With regard to Theorem 3.7 we only have to show $\bar{J}(\bar{K}) \geq r + l$ for the optimality.

Let $(u_m)_{m \in \mathbb{N}}$ denote the approximating functions from Theorem 3.6. Let further $T > 0$ and $n \in \mathbb{N}_0$. For every $m \in \mathbb{N}$ we define

$$\tau_{m,n} := \inf\{t > \tau_n : u_m(Y_t^n) \leq Mu(Y_t^n)\}. \quad (3.24)$$

We therefore have by Lemma 3.8

$$\tau_{m,n} \leq \tau_{n+1} \text{ and } \lim_{m \rightarrow \infty} \tau_{m,n} = \tau_{n+1}. \quad (3.25)$$

If we set $\sigma_k := \tau_k$, $\zeta_k := \xi_k$ for $k \in \{0, \dots, n+1\}$ and $\sigma_k := \infty$, $\zeta_k := 0$ for $k > n+1$, then by (i) $((\sigma_k, \zeta_k))_{k \in \mathbb{N}_0}$ is a trading strategy with the corresponding process $(Y_t^{n+1})_{t \geq 0}$, which is a semimartingale.

Since $(Y_t^{n+1})_{t \geq 0}$ coincides with $(Y_t)_{t \geq 0}$ on $[0, \tau_{n+1}]$ and $L^{p,\gamma}(\mathbb{R}^d) \subseteq L_{loc}^1(\mathbb{R}^d)$, we can apply Proposition 3.5 on $(Y_t^{n+1})_{t \geq 0}$ and the function u_m and get

$$\begin{aligned}
& E(u_m(Y_{\tau_n \wedge T})) \\
&= E\left(-\int_{\tau_n \wedge T}^{\tau_{m,n} \wedge T} Lu_m(Y_{s-}) ds + u_m(Y_{\tau_{m,n} \wedge T}) - \sum_{k=1}^{n+1} (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_k \leq \tau_{m,n} \wedge T\}}\right) \\
&\stackrel{3.6(ii)}{=} E\left(\int_{\tau_n \wedge T}^{\tau_{m,n} \wedge T} (\bar{f}(Y_{s-}) - l) ds + m \int_{\tau_n \wedge T}^{\tau_{m,n} \wedge T} (u_m(Y_{s-}) - Mu(Y_{s-}))^- ds \right. \\
&\quad \left. + u_m(Y_{\tau_{m,n} \wedge T}) - \sum_{k=1}^{n+1} (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_k \leq \tau_{m,n} \wedge T\}}\right) \\
&\stackrel{(3.24)}{=} E\left(\int_{\tau_n \wedge T}^{\tau_{m,n} \wedge T} (\bar{f}(Y_{s-}) - l) ds + u_m(Y_{\tau_{m,n} \wedge T}) - \sum_{k=1}^{n+1} (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_k \leq \tau_{m,n} \wedge T\}}\right) \\
&\stackrel{3.6(i)}{\leq} E\left(\int_{\tau_n \wedge T}^{\tau_{m,n} \wedge T} (\bar{f}(Y_{s-}) - l) ds + u(Y_{\tau_{m,n} \wedge T}) - \sum_{k=1}^{n+1} (u_m(Y_{\tau_k}) - u_m(Y_{\tau_k-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_k \leq \tau_{m,n} \wedge T\}}\right).
\end{aligned}$$

Now noting that

$$u(Y_{\tau_{m,n} \wedge T}) - (u_m(Y_{\tau_{n+1}}) - u_m(Y_{\tau_{n+1}-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_{n+1} \leq \tau_{m,n} \wedge T\}}$$

converges for $m \rightarrow \infty$ a.s. to

$$u(Y_{\tau_{n+1} \wedge T}) - (u(Y_{\tau_{n+1}}) - u(Y_{\tau_{n+1}-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_{n+1} \leq \tau_{n+1} \wedge T\}},$$

we get in the estimation above by (3.25) and Lebesgue's convergence theorem

$$\begin{aligned}
E(u(Y_{\tau_n \wedge T})) &\leq E\left(\int_{\tau_n \wedge T}^{\tau_{n+1} \wedge T} (\bar{f}(Y_{s-}) - l) ds \right. \\
&\quad \left. + u(Y_{\tau_{n+1} \wedge T}) - \sum_{k=1}^{n+1} (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_k \leq \tau_{n+1} \wedge T\}}\right),
\end{aligned}$$

or equivalently

$$\begin{aligned}
E(u(Y_{\tau_n \wedge T}) - u(Y_{\tau_{n+1} \wedge T})) &\leq E\left(\int_{\tau_n \wedge T}^{\tau_{n+1} \wedge T} (\bar{f}(Y_{s-}) - l) ds \right. \\
&\quad \left. - \sum_{k=1}^{n+1} (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\tau_n \wedge T < \tau_k \leq \tau_{n+1} \wedge T\}}\right).
\end{aligned}$$

Summing up both sides over $n \in \{0, 1, \dots, m-1\}$ we get

$$\begin{aligned} E(u(y) - u(Y_{\tau_m \wedge T})) &\leq E\left(\int_0^{\tau_m \wedge T} (\bar{f}(Y_{s-}) - l) ds - \sum_{k=1}^m (u(Y_{\tau_k}) - u(Y_{\tau_{k-}})) \mathbb{1}_{\{\tau_k \leq \tau_m \wedge T\}}\right) \\ &= E\left(\int_0^{\tau_m \wedge T} (\bar{f}(Y_{s-}) - l) ds - \sum_{k=1}^{\infty} (u(Y_{\tau_k}) - u(Y_{\tau_{k-}})) \mathbb{1}_{\{\tau_k \leq \tau_m \wedge T\}}\right). \end{aligned}$$

Considering (3.23) this implies

$$E(u(y) - u(Y_{\tau_m \wedge T})) \leq E\left(\int_0^{\tau_m \wedge T} (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq \tau_m \wedge T\}} + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} + \|\bar{C}\|_{\infty}\right).$$

Now using Lebesgue's convergence theorem for letting $m \rightarrow \infty$ it yields

$$E(u(y) - u(Y_T)) \leq E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} + \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} + \|\bar{C}\|_{\infty}\right)$$

and hence

$$\begin{aligned} \bar{J}(\bar{K}) &= r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &\geq r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} (E(u(y) - u(Y_T))) \\ &= r + l. \end{aligned}$$

c) Now having proved the optimality of the strategy, it remains to show (ii).

(ii) Let $T_0 > 0$ and define $A_{T_0} := \left\{ \lim_{n \rightarrow \infty} \tau_n < T_0 \right\}$. On A_{T_0} we have for every $T \geq T_0$

$$\begin{aligned} \int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} &= \int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \\ &\leq T \|f - l\|_{\infty} + \sum_{k=0}^{\infty} \log(1 - \delta) \\ &= -\infty \end{aligned}$$

and hence

$$\bar{J}(\bar{K}) = r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}}\right) = -\infty$$

if $P(A_{T_0}) > 0$. But since we have already shown $\bar{J}(\bar{K}) = r + l$, we therefore must have $P\left(\lim_{n \rightarrow \infty} \tau_n = \infty\right) = 1$.

□

Remark 3.13. Equation (3.23) is due to the fact that it is still unsolved that for every $y \in \mathbb{R}^d$ there exists always some $\xi(y) \in \mathbb{R}^d$ that attains

$$u(y + \xi(y)) + \bar{C}(y, \xi(y)) = Mu(y) = u(y).$$

If we assume this to be true, then the optimal strategy described in Theorem 3.12 is inductively given by

$$\tau_n := \inf\{t > \tau_{n-1} : u(Y_t^{n-1}) \leq Mu(Y_t^{n-1})\} \text{ with } \inf \emptyset = \infty, \quad (3.26)$$

where

$$(Y_t^n)_{t \geq 0} := \left(X_t + \sum_{k=0}^n \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0}$$

and

$$\xi_n := \xi(Y_{\tau_n}^{n-1}). \quad (3.27)$$

Now we can formulate the solution to the problem of maximizing the expected growth rate from Section 1.4.

Corollary 3.14. *Let the conditions of Theorem 3.6 be satisfied and let l be the constant given by this theorem. Then we have for every $v_0 > 0$, $h_0 \in \Delta$*

$$\rho_{v_0, h_0} = r + l.$$

Proof. Let u and Mu be the functions given by Theorem 3.6. Now take in (3.1) any initial value $y_0 \in \mathbb{R}^d$. Using Theorems 3.7 and 3.12 we get

$$\rho_{v_0, h_0} \stackrel{1.29}{=} \bar{\rho}_{v_0, y_0} = r + l.$$

□

As it is already mentioned in [Tam08] (cf. Proposition 4.1 in [Tam08]), the assumption $l > \max\{f(0), f(e_1), \dots, f(e_d)\}$ in Theorem 3.6 is not necessary for the existence of an optimal strategy, since in the case $l = \max\{f(0), f(e_1), \dots, f(e_d)\}$ a buy-and-hold strategy is optimal. For the sake of completeness we will give a proof for that case in the following theorem, which summarizes the results from [Tam08] on the optimal growth rate.

Theorem 3.15. *Let $v_0 > 0$, $h_0 \in \Delta$ and let l be as in (3.16). Then under the assumption*

$$f(e_i) \neq 0, f(e_i) \neq f(e_j) \text{ for all } i, j \in \{1, \dots, d\},$$

where e_i denotes the i -th unit vector in \mathbb{R}^d and f is from (1.70), we get

$$\rho_{v_0, h_0} = r + l.$$

Proof. By Lemma 4.1 in [Tam08] we have for every $y_0 \in \mathbb{R}^d$

$$l \geq \bar{\rho}_{v_0, y_0} - r \geq \max\{f(0), f(e_1), \dots, f(e_d)\}.$$

The case $l > \max\{f(0), f(e_1), \dots, f(e_d)\}$ is treated in Corollary 3.14, so we assume

$$l = \bar{\rho}_{v_0, y_0} - r = \max\{f(0), f(e_1), \dots, f(e_d)\}. \quad (3.28)$$

Let $e^0 := (0, \dots, 0) \in \mathbb{R}^d$. We define $\tau_0 := 0$ and $\tau_n := \infty$ for all $n \in \mathbb{N}$.

For every $i \in \{0, 1, \dots, d\}$ we have $\tilde{K} := ((\tau_n, e_i))_{n \in \mathbb{N}_0} \in \tilde{\mathcal{A}}_{v_0, h_0}$, where the transaction volumes are zero, except for τ_0 . This implies

$$h_t = e_i \text{ for all } t \geq 0$$

and we therefore get

$$\begin{aligned} \tilde{J}_{v_0, h_0}(\tilde{K}) &\stackrel{(1.72)}{=} r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} (\log V_{\tau_k} - \log V_{\tau_{k-}}) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &= r + f(e_i). \end{aligned}$$

Now we take $j \in \{0, 1, \dots, d\}$ satisfying $f(e_j) = \max\{f(0), f(e_1), \dots, f(e_d)\}$ and get

$$\tilde{J}_{v_0, h_0}(\tilde{K}) = r + f(e_j) = \max\{f(0), f(e_1), \dots, f(e_d)\}. \quad (3.29)$$

Finally, we have for every $y_0 \in \mathbb{R}^d$

$$r + f(e_j) \stackrel{(3.29)}{\leq} \rho_{v_0, h_0} \stackrel{1.29}{=} \bar{\rho}_{v_0, y_0} \stackrel{(3.28)}{=} r + f(e_j) \stackrel{(3.28)}{=} r + l.$$

□

To close this section we establish a result similar to (3.13). The proof is based on ideas from [Tam08] but differs in the fact that we use our version of Itô's formula for expectations.

Proposition 3.16. *Let X be as in (3.1) with starting value $x_0 \in \mathbb{R}^d$. Then under the conditions of Theorem 3.6 we have*

$$u(x_0) = \sup_{\tau} E \left(\int_0^{\tau} (\bar{f}(X_s) - l) ds + u(X_{\tau}) \right) = E \left(\int_0^{\sigma} (\bar{f}(X_s) - l) ds + u(X_{\sigma}) \right) \quad (3.30)$$

for all $\sigma \leq \tau^*(x_0) := \inf\{t > 0 : u(X_t) = Mu(X_t)\}$. Furthermore, $E(\tau^*(x_0)) < \infty$ holds.

Proof. a) First we take an arbitrary stopping time τ . We can find some $\varepsilon > 0$ with $l - \varepsilon > \max\{f(0), f(e_1), \dots, f(e_d)\}$ due to $l > \max\{f(0), f(e_1), \dots, f(e_d)\}$. It holds

$$\begin{aligned} \int_0^{\tau} (\bar{f}(X_s) - l) ds + u(X_{\tau}) &\leq \int_0^{\tau} ((\bar{f}(X_s) - l + \varepsilon)^+ - \varepsilon) ds + \|u\|_{\infty} \\ &\leq \int_0^{\infty} (\bar{f}(X_s) - l + \varepsilon)^+ ds - \varepsilon\tau + \|u\|_{\infty}. \end{aligned}$$

Since by Lemma 3.2 in [Tam08] we have

$$E\left(\int_0^\infty (\bar{f}(X_s) - l + \varepsilon)^+ ds\right) < \infty,$$

the inequality above together with the boundedness of \bar{f} shows

$$E\left(\int_0^\tau (\bar{f}(X_s) - l) ds + u(X_\tau)\right) > -\infty \iff E(\tau) < \infty.$$

b) Now we consider a stopping time τ with $E(\tau) < \infty$. By observing carefully the proof of Theorem 3.7 in (3.17) we have with $\tau_1 := \tau$ and $X_t = Y_t$ for $t < \tau$

$$u(x_0) \geq E\left(\int_0^{\tau \wedge T} (\bar{f}(X_s) - l) ds + u(X_{T \wedge \tau-})\right).$$

Lebesgue's convergence theorem therefore yields together with the continuity of X

$$u(x_0) \geq E\left(\int_0^\tau (\bar{f}(X_s) - l) ds + u(X_\tau)\right).$$

c) Here, we treat the case $\sigma \leq \tau^*(x_0)$. We observe the proof of Theorem 3.12 and get as in b) above with $\tau_1 = \sigma$

$$u_m(x_0) \leq E\left(\int_0^{\sigma \wedge T \wedge \tau_{m,0}} (\bar{f}(X_s) - l) ds + u(X_{\sigma \wedge T \wedge \tau_{m,0}-})\right).$$

Now we let $m \rightarrow \infty$ and get

$$u(x_0) \leq E\left(\int_0^{\sigma \wedge T} (\bar{f}(X_s) - l) ds + u(X_{\sigma \wedge T})\right).$$

Using the results of a) we have $E(\sigma) = \lim_{T \rightarrow \infty} E(\sigma \wedge T) < \infty$ and hence by letting $T \rightarrow \infty$

$$u(x_0) \leq E\left(\int_0^\sigma (\bar{f}(X_s) - l) ds + u(X_\sigma)\right).$$

□

3.3 Properties of the QVI-solution

The main objective of this section is to establish properties of the function u from Theorem 3.6, which at least in the one-dimensional case can be used to derive the optimality of some constant boundary strategy in the next section.

We begin with the functions φ and M .

Lemma 3.17. *The transformation function φ from (1.76) is Lipschitz continuous.*

Proof. For every $i \neq j \in \{1, \dots, d\}$ we have

$$\frac{\partial \varphi^i}{\partial y_i}(y) = \frac{e^{y_i} \left(1 + \sum_{k=1}^d e^{y_k}\right) - e^{y_i} e^{y_i}}{\left(1 + \sum_{k=1}^d e^{y_k}\right)^2} = \frac{e^{y_i}}{\left(1 + \sum_{k=1}^d e^{y_k}\right)} - \frac{e^{2y_i}}{\left(1 + \sum_{k=1}^d e^{y_k}\right)^2}$$

and similarly

$$\frac{\partial \varphi^i}{\partial y_j}(y) = -\frac{e^{y_i} e^{y_j}}{\left(1 + \sum_{k=1}^d e^{y_k}\right)^2},$$

and so the derivative of φ is bounded in $y \in \mathbb{R}^d$.

□

Lemma 3.18. *Let \bar{C} denote the cost function from Section 1.4 and define M on the Banach space $E = (C_b(\mathbb{R}^d, \mathbb{R}), \|\cdot\|_\infty)$ of all bounded and continuous functions $v : \mathbb{R}^d \rightarrow \mathbb{R}$ by*

$$Mv(x) := \sup_{y \in \mathbb{R}^d} v(y) + \bar{C}(x, y - x). \quad (3.31)$$

Then it holds for M and $\psi = \varphi^{-1}$:

- (i) M is monotone.
- (ii) M is Lipschitz continuous on E .
- (iii) Mv is Lipschitz continuous on \mathbb{R}^d with a common Lipschitz constant for all $v \in E$.
- (iv) $(Mv) \circ \psi$ is Lipschitz continuous on $\Delta \setminus \partial\Delta$ with a common Lipschitz constant for all $v \in E$.

Proof. Since (i) is obvious, we only show (ii), (iii) and (iv). Let κ be the constant from (1.61) and $L_{\log} := \frac{1}{1-\kappa}$ be the Lipschitz constant of the (natural) logarithm on $\mathbb{R}_{\geq 1-\kappa}$. We further denote by L_C and L_φ the Lipschitz constants of the function C from (1.60) and the function φ from Lemma 3.17 respectively.

Now let $v_1, v_2 \in C_b(\mathbb{R}^d, \mathbb{R})$ and $y, y' \in \mathbb{R}^d$. Define $h := \varphi(y)$ and $h' := \varphi(y')$. By the definition of the function M above and \bar{C} in (1.78) we have

$$\begin{aligned} Mv_1(y) &= \sup_{x \in \mathbb{R}^d} v_1(x) + \bar{C}(y, x - y) \\ &= \sup_{x \in \mathbb{R}^d} v_1(x) + \log(1 - \delta) + \log(1 - C(\varphi(y), \varphi(x) - \varphi(y))). \end{aligned}$$

In the case of $Mv_1(y) \geq Mv_2(y')$ we take for $\varepsilon > 0$ some $x \in \mathbb{R}^d$ satisfying $v_1(x) + \overline{C}(y, x - y) \geq Mv_2(y) - \varepsilon$ and get

$$\begin{aligned}
& |Mv_1(y) - Mv_2(y')| = Mv_1(y) - Mv_2(y') \leq v_1(x) + \overline{C}(y, x - y) - Mv_2(y') + \varepsilon \\
& \leq v_1(x) + \overline{C}(y, x - y) - v_2(x) - \overline{C}(y', x - y') + \varepsilon \\
& \leq |v_1(x) + \overline{C}(y, x - y) - v_2(x) - \overline{C}(y', x - y')| + \varepsilon \\
& \leq \|v_1 - v_2\|_\infty + |\log(1 - C(\varphi(y), \varphi(x) - \varphi(y))) - \log(1 - C(\varphi(y'), \varphi(x) - \varphi(y')))| + \varepsilon \\
& \leq \|v_1 - v_2\|_\infty + L_{\log} \cdot |C(\varphi(y), \varphi(x) - \varphi(y)) - C(\varphi(y'), \varphi(x) - \varphi(y'))| + \varepsilon \\
& \leq \|v_1 - v_2\|_\infty + L_{\log} \cdot L_C \cdot 2 \cdot |h - h'| + \varepsilon \\
& \leq \|v_1 - v_2\|_\infty + L_{\log} \cdot L_C \cdot 2 \cdot L_\varphi \cdot |y - y'| + \varepsilon.
\end{aligned}$$

By treating the case $Mv_1(y) \leq Mv_2(y')$ the same way and letting $\varepsilon \rightarrow 0$ we get

$$|Mv_1(y) - Mv_2(y')| \leq \|v_1 - v_2\|_\infty + L_{\log} \cdot L_C \cdot 2 \cdot L_\varphi \cdot |y - y'| \quad (3.32)$$

and

$$|Mv_1(\psi(h)) - Mv_2(\psi(h'))| \leq \|v_1 - v_2\|_\infty + L_{\log} \cdot L_C \cdot 2 \cdot |h - h'|. \quad (3.33)$$

Now (ii) follows from (3.32) with $y = y'$. Taking $v_1 = v_2$ in (3.32) and (3.33) implies (iii) and (iv). □

Remark 3.19. The function u from Theorem 3.6 is continuous and bounded. Therefore $Mu \circ \psi : \Delta \setminus \partial\Delta \rightarrow \mathbb{R}$ is Lipschitz continuous by Lemma 3.18 and admits a unique Lipschitz continuous continuation to Δ , which we also will denote by $Mu \circ \psi$. The same is true with the uniform continuity of $u \circ \psi$ by Theorem 3.6, whose unique continuation to Δ will also be denoted by $u \circ \psi$ in the following.

The following proposition is crucial for the existence of an optimal constant boundary strategy. See also Remark 3.21.

Proposition 3.20. *Under the conditions of Theorem 3.6 the function $Mu \circ \psi : \Delta \rightarrow \mathbb{R}$ from Remark 3.19 satisfies*

$$Mu \circ \psi(h) = u \circ \psi(h)$$

for all $h \in \{0\} \cup \{e_i \mid i \in \{1, \dots, d\}\}$.

Proof. Let at first $i \in \{1, \dots, d\}$. Since $l > \max\{f(0), f(e_1), \dots, f(e_d)\}$, we can find some $\varepsilon > 0$ and an open neighbourhood $U(e_i)$ of e_i in Δ such that

$$f(h) - l < -\varepsilon \text{ for all } h \in U(e_i). \quad (3.34)$$

Without loss of generality we assume $i = 1$ and define the sequences $(y_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ by setting

$$y_n := (n, -n, \dots, -n) \in \mathbb{R}^d \text{ for all } n \in \mathbb{N}$$

and

$$h_n := \varphi(y_n) = \left(\frac{e^{y_n^i}}{1 + \sum_{j=1}^d e^{y_n^j}} \right)_{i \in \{1, \dots, d\}}. \quad (3.35)$$

Now let $R > 0$ and for any $y \in \mathbb{R}^d$ let $(X_t^y)_{t \geq 0}$ be the diffusion introduced in (3.1) with starting value y . We set $\tau_R(y) := \inf\{t > 0 : X_t^y \notin B_{\mathbb{R}^d}(y, R)\}$, which is independent of y by the definition of $(X_t^y)_{t \geq 0}$, and take $\delta > 0$ satisfying $\delta < \varepsilon \cdot E(\tau_R)$.

Suppose $u(\psi(e_1)) > Mu(\psi(e_1))$. Then by the continuity of $u \circ \psi$ and $Mu \circ \psi$ on Δ in Remark 3.19 there exists an open neighbourhood $U \subseteq U(e_1)$ of e_1 such that

$$U \subseteq \{x \in \Delta : u(\psi(x)) > Mu(\psi(x))\} \text{ and } |u(\psi(h)) - u(\psi(h'))| < \delta \text{ f.a. } h, h' \in U. \quad (3.36)$$

From (3.35) we can deduce $\lim_{n \rightarrow \infty} h_n = e_1$ and get for all large $n \in \mathbb{N}$

$$[n, \infty[\times \prod_{i=2}^d]-\infty, -n] \subseteq U.$$

Hence we can take an $n_0 \in \mathbb{N}$ such that

$$B_{\mathbb{R}^d}(y_n, R) \subseteq U \text{ and therefore } \tau_R(y_n) \leq \tau^*(y_n) \text{ for all } n \geq n_0, \quad (3.37)$$

where $\tau^*(y_n)$ is the optimal stopping time from Proposition 3.16. For such an n we obtain

$$\begin{aligned} u(y_n) &\stackrel{(3.30), (3.37)}{=} E \left(\int_0^{\tau_R} (\bar{f}(X_s^{y_n}) - l) ds + u(X_{\tau_R}^{y_n}) \right) \\ &\stackrel{(3.36)}{\leq} E \left(\int_0^{\tau_R} (\bar{f}(X_s^{y_n}) - l) ds \right) + u(y_n) + \delta \\ &\stackrel{(3.34)}{\leq} (-\varepsilon) \cdot E(\tau_R) + u(y_n) + \delta, \end{aligned}$$

which is a contradiction to our choice of δ .

For $x = 0$ we can argue analogously using the sequence

$$y_n = (-n, \dots, -n) \in \mathbb{R}^d, \quad n \in \mathbb{N}.$$

□

Remark 3.21. Following the optimal strategy described in Theorem 3.12, the new fraction h_{τ_n} satisfies as in (3.19)

$$u(\psi(h_{\tau_n})) \geq Mu(\psi(h_{\tau_n})) + |\log(1 - \delta)| + \frac{\varepsilon}{2^n} > Mu(\psi(h_{\tau_n}))$$

after the n -th trading. Since by Proposition 3.20 we have

$$Mu(\psi(h)) = u(\psi(h)) \text{ for all } h \in \{0\} \cup \{e_i \mid i \in \{1, \dots, d\}\},$$

the new fraction h_{τ_n} does not lie within a certain neighbourhood of the points $\{0\} \cup \{e_i \mid i \in \{1, \dots, d\}\}$. If it were possible to show that we even have

$$Mu(\psi(h)) = u(\psi(h)) \text{ for all } h \in \partial\Delta,$$

then the set $A_\varepsilon := \{y \in \mathbb{R}^d \mid \exists x \in \mathbb{R}^d : u(y) + \overline{C}(x, y-x) \geq u(x) - \frac{\varepsilon}{2}\}$ would be compact and since $\psi(h_{\tau_n}) \in A_\varepsilon$ for all $n \in \mathbb{N}$, we would stay away from $\partial\Delta$ with every h_{τ_n} , $n \in \mathbb{N}$.

In the next lemma, equation (3.38) with $\sup_{h \in \Delta} f(h) \geq l$ is already known in [Tam08]. We show that this inequality is strict. Furthermore, we prove a statement concerning the location of the maximum points of $u \circ \psi$.

Lemma 3.22. *Let u , f , \overline{C} and l be from Theorem 3.6. Then we have*

$$0 = \inf_{x \in \mathbb{R}^d} u(x), \quad \sup_{x \in \mathbb{R}^d} u(x) = \|u\|_\infty \leq \|\overline{C}\|_\infty, \quad \sup_{h \in \Delta} f(h) > l \quad (3.38)$$

and $u \circ \psi$ attains its maximum in $\{h \in \Delta : f(h) \geq l\}$.

Proof. a) From Theorem 4.2 in [Tam08] we have $\lim_{k \rightarrow \infty} \|v_{\alpha_k} - u\|_\infty = 0$ for certain bounded and continuous solutions v_{α_k} , $k \in \mathbb{N}$. Since these functions satisfy $0 = \inf_{x \in \mathbb{R}^d} v_{\alpha_k}(x)$ for every $k \in \mathbb{N}$ (cf. p. 689 in [Tam08]), we deduce $0 = \inf_{x \in \mathbb{R}^d} u(x)$.

b) Using that and the fact $u \geq Mu$ and hence $u(x) \geq u(y) + \overline{C}(x, y-x)$ for all $x, y \in \mathbb{R}^d$ we then get $u(y) \leq \|\overline{C}\|_\infty$ for all $y \in \mathbb{R}^d$.

c) Now take a sequence $(y_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d satisfying $\lim_{n \rightarrow \infty} u(y_n) = \|u\|_\infty$ and let $(X_t^{y_n})_{t \geq 0}$ be the diffusion introduced in (3.1) with starting value y_n . For every $n \in \mathbb{N}$ we have by (3.30)

$$u(y_n) = E \left(\int_0^{\tau^*(y_n)} (\bar{f}(X_s^{y_n}) - l) ds + Mu(X_{\tau^*(y_n)}^{y_n}) \right), \quad (3.39)$$

which in case of $\sup_{h \in \Delta} f(h) \leq l$ would imply

$$u(y_n) \leq \sup_{x \in \mathbb{R}^d} Mu(x) \leq \|u\|_\infty + \log(1 - \delta) \text{ for all } n \in \mathbb{N}$$

and therefore a contradiction to the choice of y_n , $n \in \mathbb{N}$.

Let us then additionally take without loss of generality $h_0 := \lim_{n \in \mathbb{N}} \varphi(y_n)$.

Suppose $h_0 \in \{h \in \Delta : f(h) < l\}$. By the definition of Mu in (3.31) together with (3.18) we have $u(\psi(h_0)) = Mu(\psi(h_0)) + |\log(1 - \delta)|$ and hence $u \circ \psi(h_0) > Mu \circ \psi(h_0)$.

Since $\varphi = \psi^{-1}$, u and Mu are Lipschitz continuous, we can find $R, \varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\bar{f}(y) - l \leq -\varepsilon$ and $y \in \{x \in \mathbb{R}^d : u(x) > Mu(x)\}$ for all $y \in B_{\mathbb{R}^d}(y_n, R)$ and $n \geq n_0$.

By defining $\tilde{\sigma}_R(y_n) := \inf\{t > 0 : X_t^{y_n} \notin B_{\mathbb{R}^d}(y_n, R)\}$, which is independent of y_n , and then applying (3.30) we get

$$u(y_n) = E\left(\int_0^{\tilde{\sigma}_R(y_n)} (\bar{f}(X_s^{y_n}) - l) ds + u(X_{\tilde{\sigma}_R(y_n)}^{y_n})\right) \leq (-\varepsilon) \cdot E(\sigma_R) + \|u\|_\infty$$

for all $n \geq n_0$ and therefore a contradiction to the choice of y_n , $n \in \mathbb{N}$. □

Now we want to prove a lemma, which gives only a hint about the size of every connected component in $\{x \in \mathbb{R}^d : \bar{f}(x) < l\}$ in the case $d > 1$, but is crucial for the case $d = 1$, where it shows the boundedness of every such a set.

Lemma 3.23. *Let u, Mu, ψ be from Theorem 3.6. Then there exist neighbourhoods U_0, U_1, \dots, U_d of $0, e_1, \dots, e_d$ in Δ and an $R > 0$ such that for every $i \in \{0, 1, \dots, d\}$ and every $y \in A := \{x \in \mathbb{R}^d : u(x) > Mu(x)\} \cap \psi(U_i \cap (\Delta \setminus \partial\Delta))$ the ball $B_{\mathbb{R}^d}(y, R)$ is not contained in A , i.e. $B_{\mathbb{R}^d}(y, R) \setminus A \neq \emptyset$.*

Proof. By $l > \max\{f(0), f(e_1), \dots, f(e_d)\}$ we can find some $\varepsilon > 0$ and neighbourhoods $\tilde{U}_0, \tilde{U}_1, \dots, \tilde{U}_d$ of $e_0 := 0, e_1, \dots, e_d$ in Δ satisfying

$$\tilde{U}_i \subseteq \{h \in \Delta : f(h) < l - \varepsilon\} \text{ for all } i \in \{0, 1, \dots, d\}.$$

We then take by Proposition 3.20 neighbourhoods $U_i \subseteq \tilde{U}_i$ of e_i satisfying

$$\sup_{h_1, h_2 \in U_i} |u(\psi(h_1)) - u(\psi(h_2))| < \varepsilon \text{ for all } i \in \{0, 1, \dots, d\}.$$

Furthermore, we can find an $R > 0$ such that for every $y \in \mathbb{R}^d$ and the diffusion X^y from (3.1) starting in y the stopping time $\sigma_R := \inf\{t > 0 : X_t^y \notin B_{\mathbb{R}^d}(y, R)\}$, which is independent of the starting point y , satisfies $E(\sigma_R) > 1$.

Now if $B_{\mathbb{R}^d}(y, R) \subseteq \{x \in \mathbb{R}^d : u(x) > Mu(x)\} \cap \psi(U_i \cap (\Delta \setminus \partial\Delta))$ for some $i \in \{0, 1, \dots, d\}$, then by (3.30) we would get

$$u(y) = E\left(\int_0^{\sigma_R} (\bar{f}(X_s^y) - l) ds + u(X_{\sigma_R}^y)\right) \leq E(\sigma_R)(-\varepsilon) + u(y) + \varepsilon < u(y)$$

and hence a contradiction. □

With Lemma 3.23 at hand, we can prove an even stronger version.

Proposition 3.24. *Let u and Mu be from Theorem 3.6. Then there exists some $R > 0$ such that for every $y \in A := \{x \in \mathbb{R}^d : u(x) > Mu(x)\}$ the ball $B_{\mathbb{R}^d}(y, R)$ is not contained in A , i.e. $B_{\mathbb{R}^d}(y, R) \setminus A \neq \emptyset$.*

Proof. Let U_0, U_1, \dots, U_d denote the neighbourhoods of $0, e_1, \dots, e_d$ in Δ given by Lemma 3.23. We define $O_i := \psi(U_i \cap (\Delta \setminus \partial\Delta))$ for every $i = 0, \dots, d$. Similarly to Lemma 3.1 in [Tam08] we can find some $C > 0$ such that

$$\mathbb{R}^d \setminus \bigcup_{i=0}^d O_i \subseteq \bigcup_{i=1}^d D_i \cup \bigcup_{i \neq j=1}^d D_{ij}, \quad (3.40)$$

where $D_i = \{x \in \mathbb{R}^d : |x_i| < C\}$ and $D_{ij} = \{x \in \mathbb{R}^d : |x_i - x_j| < C\}$.

Now suppose we can find some sequence $(y_n)_{n \in \mathbb{N}} \in A^{\mathbb{N}}$ such that $B_{\mathbb{R}^d}(y_n, n) \subseteq A$ for all $n \in \mathbb{N}$. From the definition of the sets D_i, D_{ij} above we can find an $n_0 \in \mathbb{N}$ and for every $n \geq n_0$ also some $x_n \in B_{\mathbb{R}^d}(y_n, n)$ such that

$$B_{\mathbb{R}^d}\left(x_n, \frac{n}{4}\right) \subseteq B_{\mathbb{R}^d}(y_n, n) \setminus \left(\bigcup_{i=1}^d D_i \cup \bigcup_{i \neq j=1}^d D_{ij}\right).$$

For all $n \geq n_0$ (3.40) therefore implies

$$B_{\mathbb{R}^d}\left(x_n, \frac{n}{4}\right) \subseteq A \cap \left(\bigcup_{i=0}^d O_i\right). \quad (3.41)$$

Since the neighbourhoods $O_i, i = 0, \dots, d$, can be chosen to be disjoint, (3.41) imposes a contradiction to Lemma 3.23. □

In order to proceed further, we need the *strict* or *uniform strong* ellipticity of the generator L from (3.2), which is shown in the following lemma.

Lemma 3.25. *The generator L from (3.2) is uniformly strongly elliptic, i.e.*

$$\exists c > 0 \forall x \in \mathbb{R}^d : \frac{1}{c} \|x\|^2 \leq \left| \frac{1}{2} x^T \sigma \sigma^T x \right| \leq c \|x\|^2.$$

Proof. Since σ is positive definite, $\sqrt{\langle \sigma^T \cdot | \sigma^T \cdot \rangle}$ is a norm on \mathbb{R}^d . □

Until now the function u from Theorem 3.6 is only known to be weakly differentiable. We close this section by establishing the C^∞ -property of u on the set $\{x \in \mathbb{R}^d : u(x) > Mu(x)\}$ together with a maximum principle. This result will be crucial for the existence of a constant boundary strategy in the following section and also for the results concerning the convergence of the boundaries in Chapters 5 and 6.

Proposition 3.26. *In the situation of Theorem 3.6 let $D := \{x \in \mathbb{R}^d : u(x) > Mu(x)\}$. Then we have $u \in C^\infty(D)$ and $Lu = -(\bar{f} - l)$ on D . Furthermore, we have for every bounded domain $\Omega_1 \subseteq D \cap \{x \in \mathbb{R}^d : \bar{f}(x) < l\}$, $\Omega_2 \subseteq D \cap \{x \in \mathbb{R}^d : \bar{f}(x) > l\}$ and every $y_1 \in \Omega_1, y_2 \in \Omega_2$*

$$u(y_1) < \sup_{x \in \Omega_1} u(x) = \max_{x \in \partial\Omega_1} u(x), \quad u(y_2) > \inf_{x \in \Omega_2} u(x) = \min_{x \in \partial\Omega_2} u(x). \quad (3.42)$$

Proof. a) Since D is open, we can find a ball $B \subseteq D$. From the definition of f in (1.70) and Lemma 3.17 $\bar{f} = f \circ \varphi$ is Lipschitz continuous and hence uniformly Hölder continuous with exponent $\alpha = 1$. Therefore there exists a (unique) solution $v \in C(\bar{B}) \cap C^2(B)$ to the Dirichlet problem

$$Lv = -(\bar{f} - l) \text{ in } B, \quad v = u \text{ on } \partial B$$

(cf. Theorem 6.13 in [GT01]). Since L is uniformly elliptic in B by Lemma 3.25, Proposition 5.7.2 in [KS88] yields

$$v(x) = E \left(\int_0^{\tau_B(x)} (\bar{f}(Y_s^x) - l) ds + u(X_{\tau_B(x)}^x) \right) \text{ for all } x \in \bar{B},$$

where $(X_t^x)_{t \geq 0}$ is the diffusion from (3.1) starting in x and $\tau_B(x) := \inf\{t \geq 0 : X_t^x \notin B\}$ its exit time from B . Now, since by Proposition 3.16 u satisfies the same equation, we have

$$u(x) = v(x) \text{ for all } x \in \bar{B}.$$

b) We have shown $u \in C^2(D)$ is a solution to $Lu = -(\bar{f} - l)$ on D . From the proof of Lemma A.15 we can easily deduce $\bar{f} \in C^\infty(\mathbb{R}^d)$ and so Theorem 6.17 in [GT01] implies $u \in C^\infty(D)$.

c) From a) we can apply the strong maximum principle from Theorem 3.5 in [GT01] and get (3.42). On the other hand this can also be seen by applying (3.30) above on $\sigma := \tau_{\Omega_1}$, the exit time from Ω_1 , in the following way.

Since $\Omega_1 \subseteq D \cap \{x \in \mathbb{R}^d : \bar{f}(x) < l\}$ is open and bounded and u is continuous, we have by (3.30) for every $y \in \Omega_1$

$$u(y) = E \left(\int_0^{\tau_{\Omega_1}(y)} (\bar{f}(X_s^y) - l) ds + u(X_{\tau_{\Omega_1}(y)}^y) \right) < \max_{x \in \partial\Omega_1} u(x) = \sup_{x \in \Omega_1} u(x).$$

Applying (3.30) analogously to Ω_2 , (3.42) follows. □

3.4 A constant boundary strategy on \mathbb{R}

Here in this section, we consider the case $d = 1$, i.e. our portfolio consists of shares of only one stock and one bond. For Model A from Section 1.2 an optimal strategy can be found in the class of so-called *constant boundary strategies* in the case $d = 1$ as we have already stated in Chapter 2. Using the results from the previous sections we show here that this is also true for Model B from Section 1.3, even with the slightly more general cost structure from (1.51) with $\lambda \neq \nu$. To be more precise, we show that the optimal

strategy from Theorem 3.12 can be forced to be an optimal constant boundary strategy by taking $\varepsilon = 0$ in (3.23).

As we have seen in Theorem 3.15, the optimal value ρ of the portfolio optimization problem described in Chapter 1 and solved in Section 3.2 is

$$\rho = r + l \text{ for some } \sup_{h \in [0,1]} f(h) \geq l \geq \max\{f(0), f(1)\}. \quad (3.43)$$

Here, the case $l = \max\{f(0), f(1)\}$ implies the buy-and-hold strategy of having invested everything in the stock ($l = f(0)$) or the bond ($l = f(1)$) to be optimal (cf. Remark 4.1).

If for example the optimal risky fraction (Merton fraction) $\hat{h} := \frac{\mu-r}{\sigma^2}$, which satisfies $f(\hat{h}) = \sup_{h \in [0,1]} f(h)$, does not lie in $]0, 1[$, then $l = \max\{f(0), f(1)\}$ follows from (3.43) and the optimal strategy (the buy-and-hold strategy from above) is the same as in the portfolio optimization problem without transaction costs.

It remains to show that in the case of $l > \max\{f(0), f(1)\}$ a constant boundary strategy is optimal for the portfolio optimization problem. But before doing so we give the definition, which equals its counterpart from Chapter 2.

Definition 3.27. An impulse control strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ is a *constant boundary strategy* if there exist $a < \alpha \leq \beta < b \in \mathbb{R}$ such that $\xi_0 + y_0 \in]a, b[$ and

$$\tau_n = \inf\{t > \tau_{n-1} : Y_t^{n-1} \notin]a, b[\}, \quad \xi_n = \begin{cases} \alpha - a, & Y_{\tau_n}^{n-1} = a, \\ \beta - b, & Y_{\tau_n}^{n-1} = b, \end{cases}$$

where

$$(Y_t^n)_{t \geq 0} := \left(X_t + \sum_{k=0}^n \xi_k \mathbb{1}_{\{\tau_k \leq t\}} \right)_{t \geq 0}$$

and $(X_t)_{t \geq 0}$ is the uncontrolled process from (3.1) with starting value y_0 .

We will later refer to \bar{K} as a *constant boundary strategy given by (a, α, β, b)* .

Remark 3.28. As it is stated in Lemma 2.5, the stopping time τ_n in Definition 3.27 always satisfies $E(\tau_n) < \infty$, $n \in \mathbb{N}_0$.

Now we have to investigate the cost function and start with a description.

Lemma 3.29. Let \bar{C} be the cost function from Section 1.4 with the corresponding constants δ , λ and ν from (1.50) and (1.51). We denote further by φ the transformation function from Section 1.4. Then we get for every $x, y \in \mathbb{R}$

$$\Gamma(x, y) := \bar{C}(x, y - x) = \begin{cases} \log(1 - \delta) + \log\left(\frac{1 + \lambda\varphi(x)}{1 + \lambda\varphi(y)}\right), & x \leq y, \\ \log(1 - \delta) + \log\left(\frac{1 - \nu\varphi(x)}{1 - \nu\varphi(y)}\right), & x > y. \end{cases} \quad (3.44)$$

Proof. For $x, y \in \mathbb{R}$ we have

$$\begin{aligned} \overline{C}(x, y - x) &\stackrel{(1.78)}{=} \tilde{C}(\varphi(x), \varphi(y) - \varphi(x)) \\ &\stackrel{(1.71)}{=} \log((1 - \delta)(1 - C(\varphi(x), \varphi(y) - \varphi(x)))) \\ &\stackrel{1.17}{=} \log(\widehat{C}(\varphi(x), \varphi(y))). \end{aligned}$$

So it remains to consider the function \widehat{C} from Theorem 1.17, which satisfies

$$\widehat{C}(h, \xi) = (1 - \delta) \frac{1 + (\lambda F^+(h, \xi) + \nu F^-(h, \xi))h}{1 + (\lambda F^+(h, \xi) + \nu F^-(h, \xi))\xi}$$

for $h, \xi \in \Delta \setminus \partial\Delta$. By Theorem 1.17 we calculate $F(h, \xi) = \text{sign}(\xi - h)$ and thus have proved the assertion, since φ is strictly increasing and bijective. □

Remark 3.30. In the case of Model A in Chapter 2 the cost function needed to be modified because of discontinuities of the partial derivatives and the justification for this modification relied on numerical examples (cf. Section 5 in [IS06]). However, the cost structure of Model B, which is used here, does not impose such difficulties as we can see in Lemma 3.31.

For a later reference we state the following

Lemma 3.31. *For the cost function Γ from Lemma 3.29 we have for $x < y$*

$$\frac{\partial}{\partial x} \Gamma(x, y) = \frac{\lambda \varphi'(x)}{1 + \lambda \varphi(x)} > 0 \text{ and } \frac{\partial}{\partial y} \Gamma(x, y) = \frac{-\lambda \varphi'(y)}{1 + \lambda \varphi(y)} < 0, \quad (3.45)$$

whereas for $x > y$

$$\frac{\partial}{\partial x} \Gamma(x, y) = \frac{-\nu \varphi'(x)}{1 - \nu \varphi(x)} < 0 \text{ and } \frac{\partial}{\partial y} \Gamma(x, y) = \frac{\nu \varphi'(y)}{1 - \nu \varphi(y)} > 0. \quad (3.46)$$

Proof. The assertion follows by taking derivatives. □

Corollary 3.32. *For every $x_1 \leq x_2 \leq y_1 \leq y_2 \in \mathbb{R}$ the cost function Γ from Lemma 3.29 satisfies*

$$\Gamma(x_2, y_1) - \Gamma(x_1, y_1) = \Gamma(x_2, y_2) - \Gamma(x_1, y_2) \quad (3.47)$$

and

$$\Gamma(y_2, x_1) - \Gamma(y_1, x_1) = \Gamma(y_2, x_2) - \Gamma(y_1, x_2). \quad (3.48)$$

Proof. This is a consequence of the fundamental theorem of calculus applied to the function $x \mapsto \Gamma(x, y)$ together with (3.45) and (3.46). □

The optimal strategy gained in Theorem 3.12 can roughly be described as follows. We start our process in the set $\{z \in \mathbb{R}^d : u(z) > Mu(z)\}$ and wait until it reaches its boundary, say at x . Then we trade to a portfolio y , satisfying

$$u(y) + \bar{C}(x, y - x) \approx \sup_{\tilde{y} \in \mathbb{R}^d} u(\tilde{y}) + \bar{C}(x, \tilde{y} - x) = Mu(x) = u(x), \quad (3.49)$$

which then lies within $\{z \in \mathbb{R}^d : u(z) > Mu(z)\}$ due to Lemma 3.11 and the procedure starts again.

Though it remains unsolved for a general dimension d , the following lemma states that the supremum in (3.49) is always a maximum in the case $d = 1$ considered in this section. This result is of course crucial for the existence of a constant boundary strategy.

Lemma 3.33. *Let u , Mu and \bar{C} be from Theorem 3.6. Then for every $x \in \mathbb{R}$ there exists some $y \in \mathbb{R}$ satisfying*

$$u(y) + \bar{C}(x, y - x) = Mu(x). \quad (3.50)$$

Such an y then always satisfies

$$u(y) = Mu(y) + |\log(1 - \delta)|. \quad (3.51)$$

Proof. Let $x \in \mathbb{R}$. For every $\varepsilon \in]0, \frac{1}{2} \cdot |\log(1 - \delta)|[$ we can find some $y_\varepsilon \in \mathbb{R}$ such that

$$u(y_\varepsilon) + \bar{C}(x, y_\varepsilon - x) + \varepsilon \geq Mu(x). \quad (3.52)$$

Now using Lemma 3.11 we get

$$u(y_\varepsilon) \geq Mu(y_\varepsilon) - \varepsilon + |\log(1 - \delta)|. \quad (3.53)$$

Proposition 3.20 therefore implies that there exists a compact set $K \subseteq \mathbb{R}$ such that every such y_ε satisfying (3.52) lies in K , and hence (3.50) and (3.51) now follow from the continuity of u and $\bar{C}(x, \cdot - x)$, by letting $\varepsilon \rightarrow 0$ in (3.52) and (3.53) together with (3.18). \square

We are now able to prove a proposition, which is the key for the existence of a constant boundary strategy.

Proposition 3.34. *Let u , Mu , \bar{C} , \bar{f} and l be from Theorem 3.6 and $l > \max\{f(0), f(1)\}$ with $f(0) \neq f(1)$. If $\lambda, \nu > 0$ in Lemma 3.29, then there exist $a, \alpha, \beta, b \in \mathbb{R}$ satisfying*

- (i) $a < \alpha < \beta < b$,
- (ii) $u > Mu$ on $]a, b[$,
- (iii) $u(a) = Mu(a)$ and $u(b) = Mu(b)$,
- (iv) $u(a) = u(\alpha) + \bar{C}(a, \alpha - a)$ and $u(b) = u(\beta) + \bar{C}(b, \beta - b)$,
- (v) u attains its maximum on $[a, b]$ in some point $x \in \{z \in \mathbb{R} : \bar{f}(z) \geq l\} \cap]\alpha, \beta[$.

Proof. a) The set $A := [z_1, z_2] := \{x \in \mathbb{R} : \bar{f}(x) \geq l\}$ is bounded due to the assumption $l > \max\{f(0), f(1)\}$. Therefore u attains its maximum, i.e. $u(y) = \|u\|_\infty$, in some $y \in A$ by Lemma 3.22. By the definition of Mu in (3.31) we have $u(y) = Mu(y) + |\log(1 - \delta)|$ and hence

$$y \in B := \{x \in \mathbb{R} : u(x) > Mu(x)\}.$$

What we first need is a connected component $[a, b] \subseteq B$ and some $\alpha < \beta \in]a, b[$ satisfying (iii) and (iv).

b) Now let D_0 denote the closure of the connected component in B that contains y . By Lemma 3.23 D_0 is bounded and we can write $[x_0, x_1] := D_0$. Since B is open we then deduce $x_1 \notin B$ and find by Lemma 3.33 some $y_1 \in \mathbb{R}$ satisfying

$$u(y_1) + \bar{C}(x_1, y_1 - x_1) = Mu(x_1) = u(x_1). \quad (3.54)$$

From Lemma 3.33 we also conclude

$$u(y_1) = Mu(y_1) + |\log(1 - \delta)|. \quad (3.55)$$

What we show now is that either $y_1 \in D_0$ or we have some $D \subseteq \mathbb{R}$ satisfying

$$\text{there exist some } a, \alpha, \beta, b \in \mathbb{R} \text{ meeting (i)-(iv), } D = [a, b] \text{ and } x_1 \leq a. \quad (3.56)$$

c) We assume $y_1 \notin D_0$ and take \tilde{D}_1 to be the closure of the connected component in B that contains y_1 . Since $u(y) = \|u\|_\infty$, we cannot have $y_1 < x_0$, because it would imply

$$Mu(x_1) \stackrel{(3.54)}{=} u(y_1) + \Gamma(x_1, y_1) \stackrel{(3.46)}{<} u(y) + \Gamma(x_1, y) \leq Mu(x_1),$$

where Γ is taken from Lemma 3.29, yielding a contradiction. From $y_1 \notin D_0$ we then conclude $y_1 > x_1$.

Now \tilde{D}_1 is bounded by Lemma 3.23 and for $[x_2, x_3] := \tilde{D}_1$ we have $x_2 \geq x_1$ and can take, using Lemma 3.33, some $y_2, y_3 \in \mathbb{R}$ satisfying

$$u(y_2) + \bar{C}(x_2, y_2 - x_2) = Mu(x_2) = u(x_2) \text{ and } u(y_2) = Mu(y_2) + |\log(1 - \delta)| \quad (3.57)$$

as well as

$$u(y_3) + \bar{C}(x_3, y_3 - x_3) = Mu(x_3) = u(x_3) \text{ and } u(y_3) = Mu(y_3) + |\log(1 - \delta)|. \quad (3.58)$$

Due to (3.54), (3.55), (3.57) and (3.58) we always have $y_i \in B$ and also $x_j \notin B$ yielding

$$y_i \neq x_j \text{ for all } i, j = 1, 2, 3.$$

c1) We first show that y_2 can be taken from \tilde{D}_1 .

In the case $x_1 = x_2$ we can take $y_2 := y_1 \in \tilde{D}_1$ since y_1 meets (3.57) instead of y_2 due to (3.54) and (3.55).

We can henceforth assume $x_1 < x_2$. It holds

$$\begin{aligned} u(x_2) &= Mu(x_2) \geq u(y_1) + \bar{C}(x_2, y_1 - x_2) \\ &= u(y_1) + \Gamma(x_2, y_1) \stackrel{(3.45)}{>} u(y_1) + \Gamma(x_1, y_1) \\ &\stackrel{(3.54)}{=} Mu(x_1) = u(x_1). \end{aligned} \tag{3.59}$$

We go through the possible positions of y_2 in the following.

c1.1) The assumption $y_2 < x_1$ implies

$$\begin{aligned} u(x_1) &= Mu(x_1) \geq u(y_2) + \Gamma(x_1, y_2) \\ &\stackrel{(3.46)}{>} u(y_2) + \Gamma(x_2, y_2) \stackrel{(3.57)}{=} Mu(x_2) = u(x_2) \end{aligned}$$

and hence poses a contradiction to (3.59).

c1.2) The assumption $x_1 < y_2 < x_2$ on the one hand yields

$$u(x_2) \stackrel{(3.57)}{=} u(y_2) + \Gamma(x_2, y_2) \stackrel{(3.46)}{<} u(y_2) + \Gamma(y_2, y_2) = u(y_2) + \log(1 - \delta). \tag{3.60}$$

On the other hand we have

$$\Gamma(x_1, y_2) - \Gamma(x_1, y_1) \stackrel{(3.47)}{=} \Gamma(y_2, y_2) - \Gamma(y_2, y_1)$$

and therefore

$$u(y_1) + \Gamma(x_1, y_1) \stackrel{(3.54)}{=} Mu(x_1) \geq u(y_2) + \Gamma(x_1, y_2)$$

implies

$$u(y_1) + \Gamma(y_2, y_1) \geq u(y_2) + \Gamma(y_2, y_2). \tag{3.61}$$

Now it follows

$$\begin{aligned} u(y_2) + \log(1 - \delta) &= u(y_2) + \Gamma(y_2, y_2) \stackrel{(3.61)}{\leq} u(y_1) + \Gamma(y_2, y_1) \\ &\stackrel{(3.45)}{<} u(y_1) + \Gamma(x_2, y_1) \leq Mu(x_2) \stackrel{(3.57)}{=} u(x_2) \end{aligned}$$

yielding a contradiction to (3.60).

c1.3) Finally, we assume $y_2 > x_3$ and $u(y_1) + \Gamma(x_2, y_1) < u(y_2) + \Gamma(x_2, y_2)$. (3.47) then implies

$$Mu(x_1) \stackrel{(3.54)}{=} u(y_1) + \Gamma(x_1, y_1) < u(y_2) + \Gamma(x_1, y_2) \leq Mu(x_1),$$

which is a contradiction.

Together with c1.1) and c1.2) this yields either $y_2 \in [x_2, x_3] = \tilde{D}_1$ or $u(y_1) + \Gamma(x_2, y_1) = u(y_2) + \Gamma(x_2, y_2)$ and hence y_2 can always be taken from \tilde{D}_1 .

c2) Here, we show $y_3 > x_2 = \inf \tilde{D}_1$ by going through the possible positions of y_3 .

c2.1) We assume $x_1 < y_3 < x_2$. We have

$$u(y_1) + \Gamma(x_1, y_1) \stackrel{(3.54)}{=} Mu(x_1) \geq u(y_3) + \Gamma(x_1, y_3).$$

Now we use this in (**) together with (3.47) and get

$$\begin{aligned}
u(y_1) + \log(1 - \delta) &= u(y_1) + \Gamma(y_1, y_1) \\
&\stackrel{y_3 < y_1}{>} u(y_1) + \Gamma(y_3, y_1) \\
&\stackrel{(**)}{\geq} u(y_3) + \Gamma(y_3, y_3) \\
&= u(y_3) + \log(1 - \delta)
\end{aligned} \tag{3.62}$$

and the contradiction

$$Mu(x_3) \geq u(y_1) + \Gamma(x_3, y_1) \stackrel{y_3 < y_1}{>} u(y_1) + \Gamma(x_3, y_3) \stackrel{(3.62)}{>} u(y_3) + \Gamma(x_3, y_3) \stackrel{(3.58)}{=} Mu(x_3)$$

follows.

c2.2) We assume $y_3 < x_1$ and $u(y_3) + \Gamma(x_3, y_3) > u(y_1) + \Gamma(x_3, y_1)$. (3.48) then implies

$$u(y_3) + \Gamma(y_1, y_3) > u(y_1) + \Gamma(y_1, y_1), \tag{3.63}$$

and thus we get the contradiction through

$$\begin{aligned}
u(y_1) &\stackrel{(3.55)}{=} Mu(y_1) + |\log(1 - \delta)| \\
&\geq u(y_3) + \Gamma(y_1, y_3) + |\log(1 - \delta)| \\
&\stackrel{(3.63)}{>} u(y_1) + \Gamma(y_1, y_1) + |\log(1 - \delta)| \\
&= u(y_1).
\end{aligned}$$

This implies together with c2.1) that either $y_3 > x_2$ or

$$Mu(x_3) \stackrel{(3.58)}{=} u(y_3) + \Gamma(x_3, y_3) = u(y_1) + \Gamma(x_3, y_1)$$

and hence (by $y_3 := y_1 > x_2$ in the latter case) y_3 can always be taken to satisfy $y_3 > x_2$.

c3) Now if $y_3 \in \tilde{D}_1$, then $D_1 := \tilde{D}_1$ meets (3.56) by c1) and c2) and hence we assume $y_3 \notin \tilde{D}_1$ and get with \tilde{D}_2 the closure of a new bounded connected component in B containing y_3 with $\inf \tilde{D}_2 \geq \sup \tilde{D}_1$.

We can repeat the procedure described in c1) and c2) with \tilde{D}_2 instead of \tilde{D}_1 and inductively with \tilde{D}_n , $n \geq 3$, if \tilde{D}_{n-1} does not meet (3.56). But since we would get

$$\sup_{x \in \tilde{D}_n} |u(x) - Mu(x)| = |\log(1 - \delta)| \text{ for all } n \in \mathbb{N} \tag{3.64}$$

by (3.55), (3.58) and the corresponding equations for \tilde{D}_n , $n \geq 3$, the Lipschitz continuity of u and Mu (cf. Theorem 3.6 and Lemma 3.18) would imply

$$\text{diam}(\tilde{D}_n) \geq K$$

for all $n \in \mathbb{N}$ and some common constant $K > 0$ and hence

$$\limsup_{n \rightarrow \infty} \tilde{D}_n = \infty$$

would follow, which together with (3.64) would contradict Proposition 3.20. Therefore eventually one \tilde{D}_n must meet (3.56), which we then call D_1 .

d) In the case $y_1 \in D_0$ we can do the same with x_0 and take by Lemma 3.33 some $y_0 \in \mathbb{R}$ satisfying

$$u(y_0) + \bar{C}(x_0, y_0 - x_0) = Mu(x_0) = u(x_0) \text{ and } u(y_0) = Mu(y_0) + |\log(1 - \delta)|.$$

By the same argument as above, we then get either $y_0 \in D_0$ or the closure D_2 of some bounded connected component in B that meets $\sup D_2 \leq x_0$ and (3.56). In either case we can proceed to e).

e) We have gained so far a bounded connected set $[a, b] := D \in \{D_0, D_1, D_2\}$ satisfying (ii) and (iii) and some $\alpha, \beta \in [a, b]$ satisfying (iv) and hence

$$u(\alpha) = Mu(\alpha) + |\log(1 - \delta)| \text{ and } u(\beta) = Mu(\beta) + |\log(1 - \delta)| \quad (3.65)$$

by Lemma 3.33. Since (3.65) then implies $a < \alpha$ and $b > \beta$, it remains to show $\alpha < \beta$ and (v).

f) From (iv) we know $u(a) \leq u(\alpha) - |\log(1 - \delta)|$ and $u(b) \leq u(\beta) - |\log(1 - \delta)|$ and hence u attains its maximum on $[a, b]$ in some point $x \in]a, b[$. By Proposition 3.26 we further have $u|_{]a, b[} \in C^\infty(]a, b[)$ and $Lu = -(\bar{f} - l)$, and thus u cannot be constant on some interval $I \subseteq]a, b[$ and $x \notin \{z \in \mathbb{R} : \bar{f}(z) \geq l\}$ would contradict the maximum principle from Proposition 3.26. Therefore we get $x \in \{z \in \mathbb{R} : \bar{f}(z) \geq l\}$ and have finished the proof if we show $\alpha < x < \beta$.

g) We only show $x < \beta$, since $\alpha < x$ can be shown in the same way.

Suppose $\beta < x < b$ and let Γ be from Lemma 3.29. Then we have

$$u(b) \stackrel{(iv)}{=} u(\beta) + \Gamma(b, \beta) \leq u(x) + \Gamma(b, \beta) \stackrel{(3.46)}{<} u(x) + \Gamma(b, x) \leq Mu(b) \stackrel{(iii)}{=} u(b),$$

which is a contradiction, and thus $x \leq \beta$ follows.

But x cannot satisfy $u(b) = u(x) + \Gamma(b, x) = Mu(b)$ as it is done by β , since $u'(x) = 0$ and therefore $Mu(b) \geq u(x + \varepsilon) + \Gamma(b, x + \varepsilon) > u(x) + \Gamma(b, x)$ by (3.46) for some small $\varepsilon > 0$.

□

Remark 3.35. The assumption $0 = f(0) \neq f(1) = \mu - r - \frac{\sigma^2}{2}$ in Proposition 3.34 that is necessary for the existence of the function u in Theorem 3.6 prohibits the Merton fraction $\hat{h} = \frac{\mu - r}{\sigma^2} = \frac{1}{2}$, and thus Theorem 3.6 is only valid for $\hat{h} \in]0, 1[\setminus \{\frac{1}{2}\}$.

With Proposition 3.34 at hand, we can finally prove the main result of this chapter. It shows the existence of an optimal constant boundary strategy, which is equal to the optimal strategy described in Theorem 3.12 if $\varepsilon = 0$ is taken in (3.23).

Theorem 3.36. *Let u and Mu be from Theorem 3.6. Then every constant boundary strategy given by (a, α, β, b) from Proposition 3.34 in the sense of Definition 3.27 is optimal for the portfolio optimization problem from Chapter 3.*

Proof. Let (a, α, β, b) be given by Proposition 3.34. Every constant boundary strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ given by (a, α, β, b) in the sense of Definition 3.27 satisfies (3.22) and (3.23) of Theorem 3.12 for every $\varepsilon > 0$ by the properties (i)-(iv) of Proposition 3.34. Therefore it is optimal, i.e. $\bar{J}(\bar{K}) = r + l$.

□

Due to Proposition 3.34, Theorem 3.36 is valid only for positive proportional costs, i.e. $\lambda, \nu > 0$, and $\hat{h} \in]0, 1[\setminus \{\frac{1}{2}\}$. The case $\lambda = \nu = 0$ is solved in the following theorem.

Theorem 3.37. *Let $\hat{h} \in]0, 1[\setminus \{\frac{1}{2}\}$ and $\lambda = \nu = 0$ define the proportional costs in Model B in the sense of (1.51). Then the conditions of Theorem 3.6 are satisfied and the function u admits a constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$, which is optimal.*

Proof. In the case $\lambda = \nu = 0$ the cost function \bar{C} from Model B is identical to that from Model A with no proportional costs and therefore, due to the representation (1.83), the expected growth rates are also equal. Let $\rho = r + l$ denote the common optimal expected growth rate. We further note that due to $\gamma = 0$ the modified cost function Γ from Lemma 2.8 is equal to the original cost function $\bar{\Gamma}$ from Lemma 2.7.

From Theorem 2.10 we know $l > \max\{f(0), f(1)\}$ and hence the conditions of Theorem 3.6 are satisfied and $\{x \in \mathbb{R} : \bar{f}(x) \geq l\}$ is bounded. Thus u from Theorem 3.6 attains its maximum in some $x_0 \in \mathbb{R}$ by Lemma 3.22. Therefore

$$Mu(x) = u(x_0) + \log(1 - \delta) \text{ for all } x \in \mathbb{R}$$

follows from $\lambda = \nu = 0$.

Let $]a, b[$ denote the connected component of $\{x \in \mathbb{R} : u(x) > Mu(x)\}$ that contains x_0 and is bounded by Lemma 3.23. Now since $0 \leq u(x) \leq |\log(1 - \delta)|$ by Lemma 3.22 for all $x \in \mathbb{R}$, Lemma 3.33 implies $u(a) = Mu(a) = 0 = Mu(b) = u(b)$ and $u(x_0) = |\log(1 - \delta)|$. Then the constants $\alpha := \beta := x_0$ satisfy $a < \alpha = \beta < b$ and (ii)-(iv) of Proposition 3.34, and so as in the proof of Theorem 3.36 the optimality of every constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$ follows.

□

Chapter 4

A comparison of the models

While it is evident that Model A differs from Model B in view of the cost structure in (1.21) and (1.50), Tamura claims in [Tam08] to have studied in [Tam06] exactly the same problem as in [IS06] (cf. p. 674 in [Tam08]) and it is at least not easily seen how the corresponding optimal growth rates behave. The main objective of this chapter is therefore to establish similarities between the models and at the same time show the difference in growth rates by numerical evidence.

As already mentioned in [IP09], p. 215, Model A seems closer to practice from an investor's point of view, since the fixed costs are paid from the bond and not from the bond and the stocks as in Model B. Apart from that both models have fixed and proportional transaction costs and the optimization criterion, i.e. optimizing the asymptotic growth rate in (1.10), is the same.

For the time being we consider for fixed starting values v_0, h_0 the processes, functions and sets

$$(X_t)_{t \geq 0}, (Y_t)_{t \geq 0}, (V_t)_{t \geq 0}, (h_t)_{t \geq 0}, c \text{ and } \mathcal{A}_{v_0, h_0}$$

in Model A from Section 1.2 and denote their counterparts in Model B from Section 1.3 by

$$(\bar{X}_t)_{t \geq 0}, (\bar{Y}_t)_{t \geq 0}, (\bar{V}_t)_{t \geq 0}, (\bar{h}_t)_{t \geq 0}, \bar{c} \text{ and } \bar{\mathcal{A}}_{v_0, h_0}.$$

Rewriting the value processes, we get after the n -th trading on $\{\tau_n < \infty\}$

$$\begin{aligned} V_{\tau_n} &= (1 - \delta) \left(V_{\tau_n-} - \sum_{i=1}^d \frac{\gamma^i}{1 - \delta} |\eta_n^i| \right), \\ \bar{V}_{\tau_n} &= (1 - \delta) \left(\bar{V}_{\tau_n-} - \sum_{i=1}^d (\lambda^i \eta_n^{i,+} + \nu^i \eta_n^{i,-}) \right). \end{aligned} \tag{4.1}$$

Thus we have on the one hand the same cost structure considering the wealth processes and the same amount to pay if we set $\lambda^i := \nu^i := \frac{\gamma^i}{1 - \delta}$ for all $i \in \{1, \dots, d\}$, given $V_{\tau_n-} = \bar{V}_{\tau_n-}$ and the same transaction η_n above.

On the other hand the corresponding new risky fractions $f_{V_{\tau_n-}, h_{\tau_n-}}(\eta_n), \bar{f}_{\bar{V}_{\tau_n-}, \bar{h}_{\tau_n-}}(\eta_n)$

from Theorems 1.6 and 1.16 then clearly satisfy

$$f_{V_{\tau_n-}, h_{\tau_n-}}(\eta_n) \neq \bar{f}_{\bar{V}_{\tau_n-}, \bar{h}_{\tau_n-}}(\eta_n),$$

yielding $V_{\tau_{(n+1)-}} \neq \bar{V}_{\tau_{(n+1)-}}$ on $\{\tau_{n+1} < \infty\}$ in general. Starting with a common portfolio value v_0 we therefore will get $V_{\tau_n-} \neq \bar{V}_{\tau_n-}$ for $n \geq 1$ in general. The numerical results in Proposition 4.6 will even show $\rho \neq \bar{\rho}$ in general.

We now take a look at the admissibility of trading strategies.

While after the n -th trading on $\{\tau_n < \infty\}$ the value invested in the i -th stock becomes

$$\begin{aligned} Y_{\tau_n}^i &= Y_{\tau_n-}^i + \eta_n^i, \\ \bar{Y}_{\tau_n}^i &= (1 - \delta)(\bar{Y}_{\tau_n-}^i + \eta_n^i), \end{aligned} \quad (4.2)$$

the value invested in the bond satisfies

$$\begin{aligned} X_{\tau_n} &= X_{\tau_n-} - \sum_{i=1}^d \eta_n^i - \delta V_{\tau_n-} - \sum_{i=1}^d \gamma^i |\eta_n^i|, \\ \bar{X}_{\tau_n} &= (1 - \delta) \left(\bar{X}_{\tau_n-} - \sum_{i=1}^d \eta_n^i - \sum_{i=1}^d (\lambda^i \eta_n^{i,+} + \nu^i \eta_n^{i,-}) \right). \end{aligned} \quad (4.3)$$

We rewrite it as

$$\begin{aligned} X_{\tau_n} &= (1 - \delta) \left(X_{\tau_n-} - \sum_{i=1}^d \eta_n^i - \sum_{i=1}^d \frac{\gamma^i}{1 - \delta} |\eta_n^i| \right) - \delta \sum_{i=1}^d (Y_{\tau_n-}^i + \eta_n^i), \\ \bar{X}_{\tau_n} &= (1 - \delta) \left(\bar{X}_{\tau_n-} - \sum_{i=1}^d \eta_n^i - \sum_{i=1}^d \frac{\gamma^i}{1 - \delta} |\eta_n^i| \right), \end{aligned} \quad (4.4)$$

where

$$\lambda^i = \nu^i = \frac{\gamma^i}{1 - \delta} \text{ for all } i \in \{1, \dots, d\}. \quad (4.5)$$

So let us assume (4.5) from now on to hold. From (4.2) we can deduce $\mathcal{A}_{v_0, h_0} \not\subseteq \bar{\mathcal{A}}_{v_0, h_0}$ because $Y_{\tau_n}^i \geq 0$ does not imply $\bar{Y}_{\tau_n}^i \geq 0$ on $\{\tau_n < \infty\}$, while on the other hand (4.4) yields $\bar{\mathcal{A}}_{v_0, h_0} \not\subseteq \mathcal{A}_{v_0, h_0}$. So there is a clear difference between the two models in terms of the admissibility of monetary trading strategies, because the fixed transaction costs in Model A are paid from the bond while in Model B they are paid proportionately by all assets. This will also lead to different maximizing strategies in Proposition 4.6.

Remark 4.1. Since the cost function is negative, we can see from (1.49) and the corresponding formula (1.72) by a simple pointwise maximization

$$\max\{\rho, \bar{\rho}\} \leq \sup_{h \in \Delta} f(h) + r.$$

On the contrary, we can conclude by using the same formulas and the buy-and-hold strategies of having invested everything in the i -th stock or the bond that we have

$$\min\{\rho, \bar{\rho}\} \geq \max\{f(0), f(e_1), \dots, f(e_d)\} + r.$$

Thus in the special case that $\max\{f(0), f(e_1), \dots, f(e_d)\} = \sup_{h \in \Delta} f(h)$ is satisfied, it holds $\rho = \bar{\rho}$.

We have seen that the notion of admissible proportional strategies is the same in both models. Using the one-to-one relationship with monetary strategies, the next lemma proves that given any proportional strategy the risky fraction processes are equal.

Lemma 4.2. *Let $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0} \in \tilde{\mathcal{A}}$ and denote by K_{v_0, h_0} , respectively \bar{K}_{v_0, h_0} the corresponding monetary strategy in Model A, respectively Model B, for fixed starting values v_0, h_0 . Then we have*

$$K_{v_0, h_0} \neq \bar{K}_{v_0, h_0} \quad (4.6)$$

and the corresponding risky fraction processes satisfy

$$h_t^{K_{v_0, h_0}} = \bar{h}_t^{\bar{K}_{v_0, h_0}} \text{ for all } t \geq 0. \quad (4.7)$$

Proof. Since we do not trade between τ_n and τ_{n+1} and have $h_{\tau_n}^{K_{v_0, h_0}} = \bar{h}_{\tau_n}^{\bar{K}_{v_0, h_0}} = \xi_n$ on $\{\tau_n < \infty\}$, we can conclude that $h_t^{K_{v_0, h_0}} = \bar{h}_t^{\bar{K}_{v_0, h_0}}$ for all $t \in [\tau_n, \tau_{n+1}[$, which implies (4.7) by induction.

To prove (4.6) we use (4.1) and (4.2), whence we even get $\eta_0 \neq \bar{\eta}_0$, where η_0 , respectively $\bar{\eta}_0$, denotes the transaction at time τ_0 .

□

Now we can prove for a certain class of strategies, which in general will not contain an optimal strategy, the equality of the expected growth rates $\tilde{J}(\tilde{K}), \tilde{\tilde{J}}(\tilde{K})$.

Lemma 4.3. *Let $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0} \in \tilde{\mathcal{A}}$ be a proportional strategy satisfying*

$$\limsup_{T \rightarrow \infty} E \left(\frac{1}{T} |\{n \in \mathbb{N}_0 : \tau_n \leq T\}| \right) \in \{0, \infty\}.$$

Then the expected growth rate $\tilde{J}_{v_0, h_0}(\tilde{K})$ from (1.49) and its counterpart $\tilde{\tilde{J}}_{v_0, h_0}(\tilde{K})$ from (1.72) satisfy

$$\tilde{J}_{v_0, h_0}(\tilde{K}) = \tilde{\tilde{J}}_{v_0, h_0}(\tilde{K}).$$

Proof. a) Using the boundedness of the function \tilde{C} from (1.48) we get in the case of $\lim_{T \rightarrow \infty} E \left(\frac{1}{T} |\{n \in \mathbb{N}_0 : \tau_n \leq T\}| \right) = 0$ from (1.49)

$$\begin{aligned} \tilde{J}_{v_0, h_0}(\tilde{K}) &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} \tilde{C}(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds \right) \end{aligned}$$

and hence $\tilde{J}_{v_0, h_0}(\tilde{K}) = \bar{\tilde{J}}_{v_0, h_0}(\tilde{K})$ since the same holds for $\bar{\tilde{J}}_{v_0, h_0}(\tilde{K})$ using the corresponding formulas.

b) Now in the case of $\limsup_{T \rightarrow \infty} E\left(\frac{1}{T}|\{n \in \mathbb{N}_0 : \tau_n \leq T\}|\right) = \infty$ we get

$$\begin{aligned} \tilde{J}_{v_0, h_0}(\tilde{K}) &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T f(h_{s-}) ds + \sum_{k=0}^{\infty} \tilde{C}(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &\leq r + \|f\|_{\infty} + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log(1 - \delta) \cdot |\{n \in \mathbb{N}_0 : \tau_n \leq T\}| \right) \\ &= -\infty \end{aligned}$$

and therefore $\tilde{J}_{v_0, h_0}(\tilde{K}) = \bar{\tilde{J}}_{v_0, h_0}(\tilde{K})$ as above. □

Remark 4.4. a) The assumption $\lim_{T \rightarrow \infty} E\left(\frac{1}{T}|\{n \in \mathbb{N}_0 : \tau_n \leq T\}|\right) = 0$ in Lemma 4.3 is of course satisfied if we have $P(\tau_n = \infty) = 1$ for some $n \in \mathbb{N}$.

b) The easiest way of getting $\limsup_{T \rightarrow \infty} E\left(\frac{1}{T}|\{n \in \mathbb{N}_0 : \tau_n \leq T\}|\right) = \infty$ would be to have

$P\left(\lim_{n \rightarrow \infty} \tau_n < T_0\right) > 0$ for some $T_0 > 0$. But this is excluded by the admissibility of a trading strategy. As we can see from the proof above, we have $\tilde{J}_{v_0, h_0}(\tilde{K}) = -\infty$ in that case and hence this exclusion does not influence the optimal growth rate ρ .

Apart from these more or less trivial situations the assumptions of Lemma 4.3 are usually not satisfied for optimal strategies, which we prove in the following proposition for each of the models in the one-dimensional case. In Lemma 5.6 a more precise result will be derived for all constant boundary strategies using renewal theory.

Proposition 4.5. *If in the case $d = 1$ a proportional strategy $\tilde{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ satisfies $\tilde{J}(\tilde{K}) > r$ in Model A (respectively $\bar{\tilde{J}}(\tilde{K}) > r$ in Model B) and the corresponding risky fraction process stays in a compact interval $I \subseteq]0, 1[$, then it holds*

$$\limsup_{T \rightarrow \infty} E\left(\frac{1}{T}|\{n \in \mathbb{N}_0 : \tau_n \leq T\}|\right) \in]0, \infty[.$$

Proof. We can use (1.46) (respectively (1.69)) to obtain

$$0 < \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\sum_{k=0}^{\infty} g(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right).$$

Now since the risky fraction process stays in the compact interval $I \subseteq]0, 1[$, we have $|g(h_{\tau_k-}, \xi_k)| \leq C$ for all $k \in \mathbb{N}_0$ and some common $C > 0$ due to the definition of g . Therefore

$$0 < \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\sum_{k=0}^{\infty} g(h_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \leq C \cdot \limsup_{T \rightarrow \infty} E \left(\frac{1}{T} |\{n \in \mathbb{N}_0 : \tau_n \leq T\}| \right).$$

□

Due to the existence of optimal constant boundary strategies for both models, we can use the computation method from Theorem 2.6 and formulate the following statement.

Proposition 4.6. *In the case $d = 1$ and $0 < \gamma = \lambda(1 - \delta) = \nu(1 - \delta)$ we have $\rho \neq \bar{\rho}$ in general and the optimal constant boundary strategies are different.*

Proof. We can only give numerical evidence. We know from Theorem 2.4 that in the case of $\hat{h} = \frac{\mu-r}{\sigma^2} \in]0, 1[$ an optimal proportional constant boundary strategy $\tilde{K}(a, \alpha, \beta, b)$ exists for the modified cost function Γ , which is also optimal for the original cost function $\tilde{\Gamma}$ if $a \leq \alpha(1 - \delta)$.

For Model B we also obtain from Theorem 3.36 in the case of $\hat{h} = \frac{\mu-r}{\sigma^2} \in]0, 1[\setminus \{\frac{1}{2}\}$ and $\rho > r + \max\{f(0), f(1)\}$ an optimal constant boundary strategy on \mathbb{R} . By transforming it to $]0, 1[$ via φ , we get an optimal proportional constant boundary strategy.

Let g denote the function from Lemma 1.11 and let \bar{g} be its counterpart from Lemma 1.18. Then we have for a general proportional constant boundary strategy $\tilde{K}(a, \alpha, \beta, b)$

$$\bar{g}(a, \alpha) = \log\left(\frac{1-\alpha}{1-a}\right) + \log\left(\frac{1-\delta+\gamma a}{1+\frac{\gamma}{(1-\delta)}\alpha}\right), \quad \bar{g}(b, \beta) = \log\left(\frac{1-\beta}{1-b}\right) + \log\left(\frac{1-\delta-\gamma b}{1-\frac{\gamma}{(1-\delta)}\beta}\right),$$

and in the case $a \leq \alpha(1 - \delta)$, which will be fulfilled in our numerical examples, also

$$g(a, \alpha) = \log\left(\frac{1-\alpha}{1-a}\right) + \log\left(\frac{1-\delta+\gamma a}{1+\gamma\alpha}\right), \quad g(b, \beta) = \log\left(\frac{1-\beta}{1-b}\right) + \log\left(\frac{1-\delta-\gamma b}{1-\gamma\beta}\right).$$

We take as a counterexample a scenario from [IS06], where $r := 0$, $\sigma := 0.4$ and $\mu := 0.096$. From Theorem 2.6 we then have

$$\begin{aligned} \tilde{J}(\tilde{K}(a, \alpha, \beta, b)) &= \frac{p(a, \alpha, \beta, b)g(b, \beta) + (1 - p(a, \alpha, \beta, b))g(a, \alpha)}{p(a, \alpha, \beta, b)(p_1(b) - p_1(\beta)) + (1 - p(a, \alpha, \beta, b))(p_1(a) - p_1(\alpha))}, \\ \bar{\tilde{J}}(\tilde{K}(a, \alpha, \beta, b)) &= \frac{p(a, \alpha, \beta, b)\bar{g}(b, \beta) + (1 - p(a, \alpha, \beta, b))\bar{g}(a, \alpha)}{p(a, \alpha, \beta, b)(p_1(b) - p_1(\beta)) + (1 - p(a, \alpha, \beta, b))(p_1(a) - p_1(\alpha))}, \end{aligned}$$

where

$$\begin{aligned} p(a, \alpha, \beta, b) &= \frac{p_0(\alpha) - p_0(a)}{p_0(b) - p_0(\beta) + p_0(\alpha) - p_0(a)}, \\ p_0(x) &= -\left(\frac{1-x}{x}\right)^{0.2}, \\ p_1(x) &= -62.5 \cdot \log\left(\frac{1-x}{x}\right). \end{aligned}$$

Now, maximizing numerically $\tilde{J}(\tilde{K}(a, \alpha, \beta, b))$ and $\bar{\tilde{J}}(\tilde{K}(a, \alpha, \beta, b))$ over all proportional constant boundary strategies

$$\tilde{K}(a, \alpha, \beta, b) \text{ with } 0 < a \leq \alpha(1 - \delta) < \alpha < \beta < b < 1,$$

we compute the difference of the optimal asymptotic growth rates $\rho(B) - \rho(A)$ from Model B and Model A. In Figure 4.1 this difference is illustrated to be dependent on the cost parameters δ, γ . As it can be seen, the difference $\rho(B) - \rho(A)$ is always positive but the

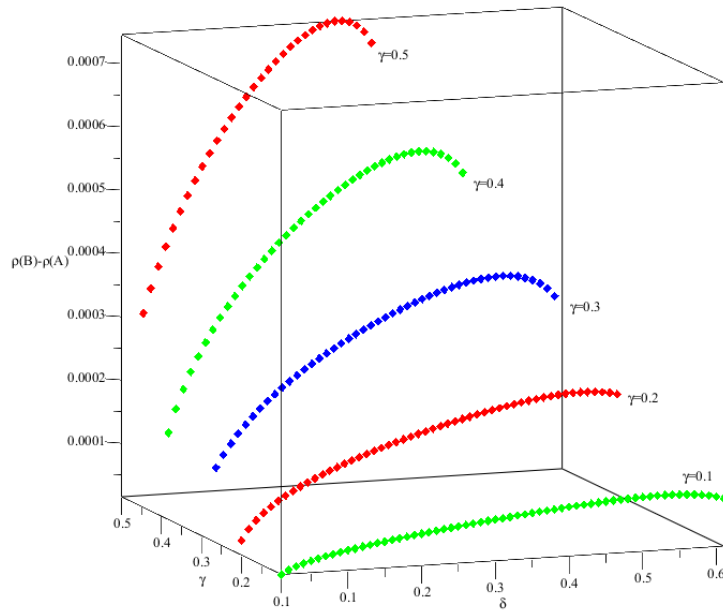


Figure 4.1: Model-Difference $\rho(B) - \rho(A)$ depending on δ, γ

dependence on δ and γ is not that simple. Furthermore, Figure 4.1 already indicates the results of Chapters 5 and 6. The achievements from Chapter 5 will yield

$$\lim_{\gamma \rightarrow 0} \rho(A)(\gamma, \delta) = \lim_{\lambda \rightarrow 0} \rho(B)(\lambda, \delta) \text{ with } \lambda = \nu,$$

while from Chapter 6

$$\lim_{\delta \rightarrow 0} \rho(A)(\gamma, \delta) = \lim_{\delta \rightarrow 0} \rho(B)(\lambda, \delta),$$

where $\gamma = \lambda = \nu$, can only be conjectured due to the limited information in case of Model B on the solution u from Theorem 3.6.

Now we illustrate two different cost-scenarios to analyze the dependencies of the optimal boundaries.

a) In the first case we use considerable small transaction costs, given by $\delta = 0.0001$ and $\gamma = 0.003$. By maximizing numerically as described above, we calculate the optimal constants $\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{b}$ and $\rho = \tilde{J}(\tilde{K}(\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{b}))$ for each model and get

	Model A	Model B
\hat{a}	0.487596394463552	0.487577737373786
$\hat{\alpha}$	0.567969768477474	0.567994551200223
$\hat{\beta}$	0.633830353906469	0.633862764635549
\hat{b}	0.708144466693057	0.708124652982303
ρ	0.028354518116319	0.028354531891455

As expected from Figure 4.1 the difference $\rho(B) - \rho(A)$ is quite small. Furthermore, by using $\hat{\beta}(A), \hat{b}(A)$ for Model A and $\hat{\beta}(B), \hat{b}(B)$ for Model B, we have on the one hand $[\hat{a}(A), \hat{\alpha}(A)] \subset [\hat{a}(B), \hat{\alpha}(B)]$, while on the other hand $[\hat{\beta}(B), \hat{b}(B)] \subset [\hat{\beta}(A), \hat{b}(A)]$.

Now we calculate the dependence of $J(a, \alpha) := \tilde{J}(\tilde{K}(a, \alpha, \hat{\beta}, \hat{b}))$ on a and α in each model, which is illustrated in Figure 4.2.

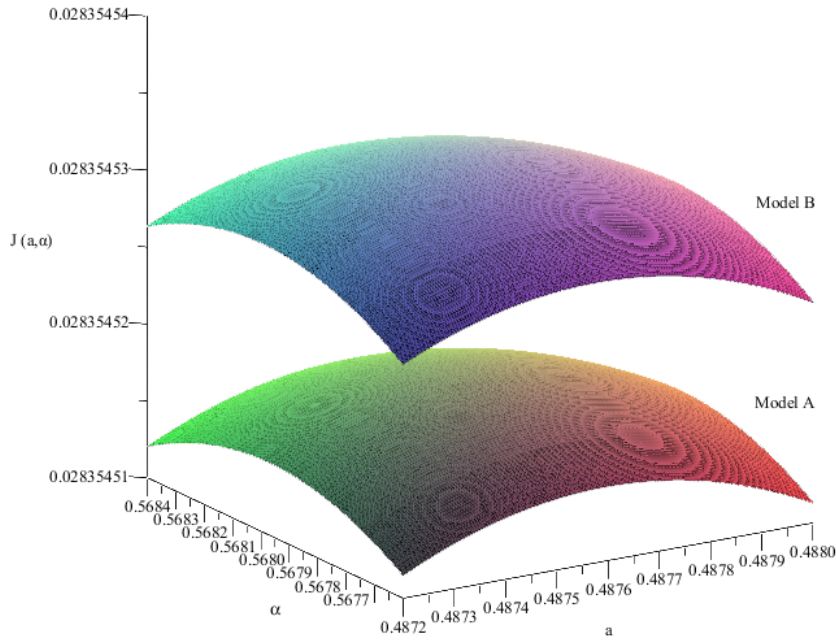


Figure 4.2: Model-difference with fixed optimal $\hat{\beta}, \hat{b}$ and $\delta = 0.0001, \gamma = 0.003$

b) Here, we do the same for considerable big transaction costs, given by $\delta = 0.25$ and $\gamma = 0.5$. Again by maximizing numerically we get

	Model A	Model B
\hat{a}	0.030076333590062	0.028628019243415
$\hat{\alpha}$	0.417534424013946	0.416063779437837
$\hat{\beta}$	0.808259031354207	0.910660849006953
\hat{b}	0.999984588843471	0.999732057212333
ρ	0.016875026800884	0.017594017693905

As indicated by Figure 4.1 the difference $\rho(B) - \rho(A)$ is now much bigger than in scenario a) and the optimal values $\rho(A), \rho(B)$ themselves are smaller due to the bigger transaction costs. Considering the boundaries we now have only the inclusion $[\hat{\beta}(B), \hat{b}(B)] \subset [\hat{\beta}(A), \hat{b}(A)]$.

The dependence of $J(a, \alpha)$ on a and α in each model is now illustrated in Figure 4.3.

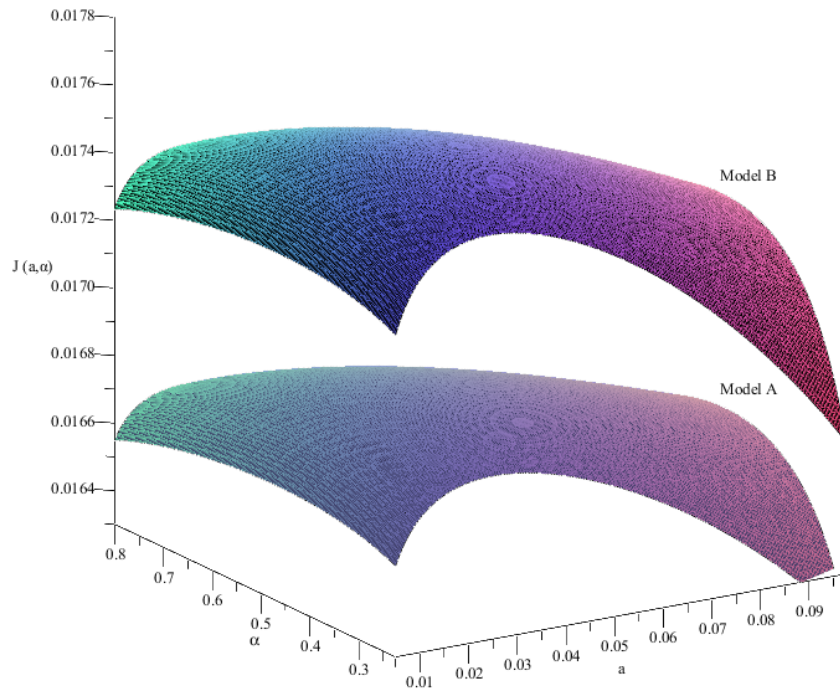


Figure 4.3: Model-difference with fixed optimal $\hat{\beta}, \hat{b}$ and $\delta = 0.25, \gamma = 0.5$

□

Remark 4.7. In the case $\gamma = \lambda = \nu = 0$ the cost function \bar{C} is equal in both models and therefore we not only have $\rho = \bar{\rho}$ due to the representation in (1.83), but we even will show $u = \bar{u}$ in Proposition 5.15, where u is from Theorem 2.10 and \bar{u} its counterpart from Theorem 3.6. Then of course even the optimal boundaries are equal.

Chapter 5

Convergence in case of vanishing proportional costs

Model A and Model B are models with fixed and proportional transactions costs, where the fixed part of the costs is given by the constant δ , whereas γ , or λ, ν in the case of Model B, describes the proportional part. The model with only fixed costs from [MP95] is therefore contained in both models. But even in [IS06] no analytical results concerning the continuity of the optimal boundaries with respect to γ and thus no convergence results are given. However, numerical calculations (cf. Figure 11.2 in [IS05] for Model A) indicate that if we fix δ and let γ , or equivalently λ, ν , tend to zero, then the optimal boundaries obtained in Chapters 2 and 3 should converge to those of the model with only fixed transaction costs. The same should apply to the optimal expected growth rates.

Treating the models separately we establish the described convergences in this chapter for both models and prove additionally the uniqueness of the optimal boundaries in the case with no proportional costs.

5.1 Landau-type inequalities on bounded intervals

To establish the convergence of the boundaries in this chapter and also in Chapter 6 we will use the following argument. If a sequence of functions $f_n : I_n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, on bounded intervals I_n , $n \in \mathbb{N}$, satisfies

$$\lim_{n \rightarrow \infty} \|f_n\|_{I_n, \infty} = 0 \text{ and } \sup_{n \in \mathbb{N}} \|f_n''\|_{I_n, \infty} < \infty,$$

then we have $\lim_{n \rightarrow \infty} \|f_n'\|_{I_n, \infty} = 0$, where $\|f\|_{I, \infty} := \sup_{x \in I} |f(x)|$ for every $f : I \rightarrow \mathbb{R}$. Such a result is easily inferred from Landau-type inequalities such as

$$\|f'\|_{\infty} \leq 2\sqrt{\|f\|_{\infty}\|f''\|_{\infty}}$$

going back to Landau in [Lan14]. Since then a large variety of results concerning the estimation of $\|f'\|$ via $\|f\|$ and the derivatives $\|f^{(k)}\|$ of higher order or even for generators

of semi-groups has been established on different spaces, halfplanes and intervals. We refer the interested reader to [Sch73] and [MPF91] for more detailed information.

Here, we will only need a version on bounded intervals such as the following.

Theorem 5.1. *On the interval $I = [0, 1]$ let $\|f\|_{I,\infty} \leq 1$ and $\|f''\|_{I,\infty} \leq A$. Then we have the sharp inequalities*

$$\|f'\|_{I,\infty} \leq \begin{cases} \frac{1}{2}(4 + A), & 0 < A \leq 4, \\ 2\sqrt{A}, & A > 4. \end{cases}$$

Proof. Cf. Theorem 1 in [CS75]. □

For the sake of completeness we shall use only the following result that is still sharp in some cases considering Theorem 5.1 but allows for an easy proof.

Lemma 5.2. *Let $a < b \in \mathbb{R}$ and $I :=]a, b[$. Then for every $f \in C^2(I, \mathbb{R})$ it holds*

$$\|f'\|_{I,\infty} \leq \sqrt{\|f\|_{I,\infty}\|f''\|_{I,\infty}} + \max\left\{\sqrt{\|f\|_{I,\infty}\|f''\|_{I,\infty}}, \frac{4}{b-a}\|f\|_{I,\infty}\right\}.$$

Proof. We only have to consider the case $\|f\|_{I,\infty}, \|f''\|_{I,\infty} < \infty$. For every $x \in]a, b[$ with $x \neq a + \frac{b-a}{2}$ and every $0 < t \leq \frac{b-a}{2}$ we can take $z \in \{-t, t\}$ satisfying $x + z \in]a, b[$. Then Taylor's formula yields

$$f'(x) = \frac{1}{z}(f(x+z) - f(x)) - \frac{z}{2}f''(\xi)$$

for some $\xi \in]a, b[$ and hence

$$\|f'\|_{I,\infty} \leq \frac{2}{t}\|f\|_{I,\infty} + \frac{t}{2}\|f''\|_{I,\infty} \quad \text{for every } 0 < t \leq \frac{b-a}{2}. \quad (5.1)$$

Since the case $\|f''\|_{I,\infty} = 0$ is now trivial, we can assume $\|f''\|_{I,\infty} \neq 0$. We define $c := 2\sqrt{\|f\|_{I,\infty}\|f''\|_{I,\infty}^{-1}}$ and take $t := c$ in (5.1) if $c \leq \frac{b-a}{2}$, getting

$$\|f'\|_{I,\infty} \leq 2\sqrt{\|f\|_{I,\infty}\|f''\|_{I,\infty}}.$$

If $c > \frac{b-a}{2}$, we take $k := \frac{b-a}{2}c^{-1} < 1$ and $t := c \cdot k = \frac{b-a}{2}$ in (5.1) and get

$$\|f'\|_{I,\infty} \leq \sqrt{\|f\|_{I,\infty}\|f''\|_{I,\infty}} \left(\frac{1}{k} + 1\right) = \sqrt{\|f\|_{I,\infty}\|f''\|_{I,\infty}} + \frac{4}{b-a}\|f\|_{I,\infty}.$$

□

5.2 Model A

With the results from Section 5.1 at hand, we are now able to prove a proposition, which shows the convergence of the trading states α, β from Theorem 2.10 to a common trading state, when the proportional costs given by γ tend to zero.

Proposition 5.3. *Let $d := 1$ and $(\gamma_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the proportional costs in Model A in the sense of (1.20) for a fixed common $0 < \delta < 1$ with $\sup_{n \in \mathbb{N}} \gamma_n < 1 - \delta$. Let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and suppose $\lim_{n \rightarrow \infty} \gamma_n = 0$.*

Then the corresponding constants α_n, β_n given by Theorem 2.10 for every $n \in \mathbb{N}$ satisfy

$$\lim_{n \rightarrow \infty} |\alpha_n - \beta_n| = 0.$$

Proof. We take for each $n \in \mathbb{N}$ from Theorem 2.10 the constant l_n , the function u_n and the state $x_n \in]\alpha_n, \beta_n[$, where u_n attains its maximum.

The idea of the proof can be described as follows: We show $\lim_{n \rightarrow \infty} u_n(x_n) - u_n(\beta_n) = 0$ and since u_n is strictly decreasing on $[x_n, \beta_n]$, the assumption $\inf_{n \in \mathbb{N}} |x_n - \beta_n| > 0$ implies $u'_n \approx 0$ on $[x_n, \beta_n]$ for large n , which contradicts $Lu_n = -(\bar{f} - l_n)$.

a) We first show that there is a constant $c = c(\mu, \sigma^2, r)$ such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in [x_n, \beta_n]} |u''_n(x)| \leq c. \quad (5.2)$$

Let $n \in \mathbb{N}$. By Theorem 2.10 (ix) we have for all $x \in [x_n, \beta_n]$

$$u''_n(x) = \frac{2}{\sigma^2} (-\bar{f}(x) + l_n) - \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2} \sigma^2 \right) u'_n(x). \quad (5.3)$$

In the case of $\mu - r - \frac{1}{2} \sigma^2 = 0$ we get (5.2) directly from (5.3) and the boundedness of \bar{f} . Therefore we assume here $\mu - r - \frac{1}{2} \sigma^2 \neq 0$.

By the choice of x_n we have $u'_n(x_n) = 0$ and u'_n is strictly decreasing on $\mathbb{R}_{\geq x_n}$ by Theorem 2.10. If u'_n attains its minimum in $[x_n, \beta_n]$ in some $x \in]x_n, \beta_n[$, then $u''_n(x) = 0$ and hence by (5.3)

$$\sup_{y \in [x_n, \beta_n]} |u'_n(y)| = |u'_n(x)| \leq \left| \frac{-\bar{f}(x) + l_n}{\mu - r - \frac{1}{2} \sigma^2} \right|. \quad (5.4)$$

Hence we can consider the case that the minimum is attained in β_n . From Theorem 2.10 we have

$$\sup_{y \in [x_n, \beta_n]} |u'_n(y)| = |u'_n(\beta_n)| = \left| \frac{\partial}{\partial y} \Gamma_n(b_n, \beta_n) \right| = \frac{\gamma_n \varphi'(\beta_n)}{1 - \gamma_n \varphi(\beta_n)}. \quad (5.5)$$

(5.2) now follows if we plug (5.4) and (5.5), respectively, into (5.3), where in the latter case we additionally have to use

$$\sup_{x \in \mathbb{R}, n \in \mathbb{N}} \frac{\gamma_n \varphi'(x)}{1 - \gamma_n \varphi(x)} < \infty,$$

which is a consequence of $\varphi(x), \varphi'(x) \in]0, 1[$ from Lemma 3.17 and $\lim_{n \rightarrow \infty} \gamma_n = 0$.

b) Now let $\varepsilon > 0$. Suppose $|x_n - \beta_n| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$ and without loss of generality for all $n \in \mathbb{N}$. We have by Theorem 2.10

$$u_n(x_n) + \Gamma_n(b_n, x_n) \leq u_n(\beta_n) + \Gamma_n(b_n, \beta_n)$$

and hence

$$u(x_n) - u(\beta_n) \leq \Gamma_n(b_n, \beta_n) - \Gamma_n(b_n, x_n) \stackrel{(2.27)}{<} \left| \log \left(\frac{1 - \frac{\gamma_n}{1-\delta} \varphi(b_n)}{1 - \gamma_n \varphi(x_n)} \right) \right|. \quad (5.6)$$

Since u_n is strictly decreasing on $[x_n, \beta_n]$ equation (5.6) implies for the function $\tilde{g}_n : I_n := [x_n, \beta_n] \rightarrow \mathbb{R}$, $x \mapsto u_n(x) - u_n(\beta_n)$

$$\|\tilde{g}_n\|_{I_n, \infty} < \left| \log \left(\frac{1 - \frac{\gamma_n}{1-\delta} \varphi(b_n)}{1 - \gamma_n \varphi(x_n)} \right) \right|$$

and hence $\lim_{n \rightarrow \infty} \|\tilde{g}_n\|_{I_n, \infty} = 0$, which, together with a) and the assumption $|x_n - \beta_n| \geq \varepsilon$, can be used in Lemma 5.2 to get

$$\lim_{n \rightarrow \infty} \|u'_n\|_{I_n, \infty} = \lim_{n \rightarrow \infty} \|\tilde{g}'_n\|_{I_n, \infty} = 0. \quad (5.7)$$

Now Theorem 2.10 yields

$$f(\hat{h}) > \inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$$

and hence $[x_n, x_n + \varepsilon] \subseteq I$ for a compact interval I and all $n \in \mathbb{N}$. Therefore

$$\inf_{n \in \mathbb{N}} \|\bar{f}'\|_{I_n, \infty} > 0$$

and we can get a $\tilde{c} > 0$ and an $\tilde{\varepsilon} \in]0, \varepsilon[$ such that for every $n \in \mathbb{N}$ there are $z_{1,n} < z_{2,n} \in I_n$ satisfying

$$z_{2,n} - z_{1,n} = \tilde{\varepsilon} \quad \text{and} \quad \inf_{x \in [z_{1,n}, z_{2,n}]} |\bar{f}(x) - l_n| \geq \tilde{c}. \quad (5.8)$$

Using (5.7) in (5.3) we therefore can find an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\inf_{x \in [z_{1,n}, z_{2,n}]} |u''_n(x)| \stackrel{(5.8)}{\geq} \frac{\tilde{c}}{\sigma^2}.$$

This implies for all $n \geq n_0$

$$|u'_n(z_{1,n}) - u'_n(z_{2,n})| \geq \frac{\tilde{c}}{\sigma^2} \tilde{\varepsilon}$$

by the mean value theorem, which is a contradiction to (5.7).

□

For a first illustration, we take the same scenario from [IS06] as in Proposition 4.6, where $r := 0$, $\sigma := 0.4$ and $\mu := 0.096$. We calculate by the same method the dependence of the constants a, α, β, b on the proportional costs γ after a transformation of these boundaries to $]0, 1[$ via φ . Figure 5.1 illustrates this for i) $\delta = 0.0001$ and ii) $\delta = 0.1$.

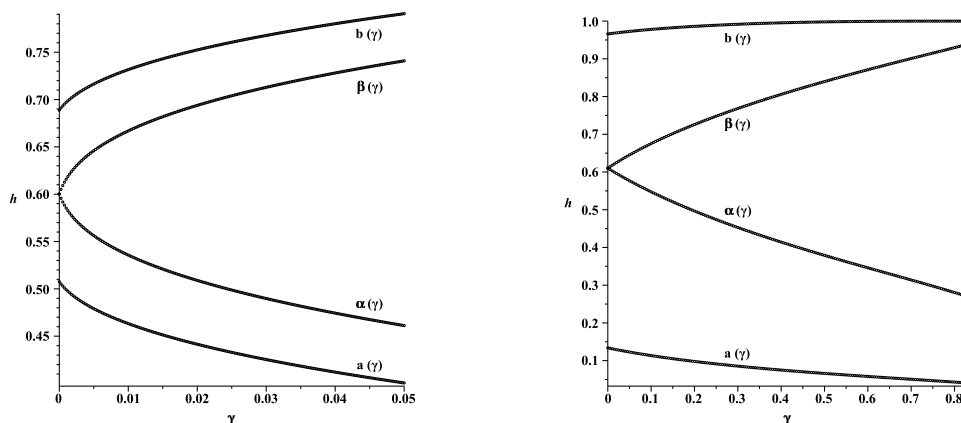


Figure 5.1: Boundary-dependence in Model A on γ for i) $\delta = 0.0001$ and ii) $\delta = 0.1$

We know from Theorem 2.10 that even in the case $\gamma = 0$ an optimal strategy can be found in the class of constant boundary strategies. Using Proposition 5.3 we can derive the same result.

Proposition 5.4. *Let $d := 1$ and $(\gamma_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the proportional costs in Model A in the sense of (1.20) for a fixed common $0 < \delta < 1$ with $\sup_{n \in \mathbb{N}} \gamma_n < 1 - \delta$. Let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and suppose $\lim_{n \rightarrow \infty} \gamma_n = 0$. Let further φ and l_n be from Theorem 2.10 for each $n \in \mathbb{N}$. Then there exists a subsequence $(\gamma_{n_k})_{k \in \mathbb{N}}$ and constants a, α, β, b such that*

$$\lim_{k \rightarrow \infty} a_{n_k} = a < \alpha = \lim_{k \rightarrow \infty} \alpha_{n_k} = \lim_{k \rightarrow \infty} \beta_{n_k} = \beta < b = \lim_{k \rightarrow \infty} b_{n_k}$$

and every constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$ given by these constants defines an optimal strategy for the case $\gamma = 0$. Furthermore, we get for the corresponding optimal growth rate ρ_0

$$\rho_0 - r = \sup_{n \in \mathbb{N}} l_n = \lim_{n \rightarrow \infty} l_n = \lim_{n \rightarrow \infty} \rho_n - r$$

and the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ lies within $[\varphi(a), \varphi(b)]$.

Proof. a) We first show that $l_0 := \rho_0 - r = \lim_{n \rightarrow \infty} l_n$. For any $\varepsilon > 0$ we can take an ε -optimal strategy $\bar{K} = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$, i.e. $\rho_0 - \varepsilon \leq \bar{J}_0(\bar{K})$. Let $(Y_t)_{t \geq 0}$ be a corresponding diffusion from Lemma 1.22, implying

$$\rho_0 - \varepsilon \leq \bar{J}_0(\bar{K}) = \rho_0 + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (\bar{f}(Y_{s-}) - l_0) ds + \sum_{k=0}^{\infty} \bar{C}_0(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right)$$

and hence

$$\liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (\bar{f}(Y_{s-}) - l_0) ds + \sum_{k=0}^{\infty} \bar{C}_0(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \geq -\varepsilon. \quad (5.9)$$

Using $\gamma = 0$ and hence $\bar{C}_0 \equiv \log(1 - \delta)$, this implies

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\}} \right) \leq \frac{-\varepsilon - (f(\hat{h}) - l_0)}{\log(1 - \delta)}. \quad (5.10)$$

Therefore, it follows

$$\begin{aligned} \rho_n &= r + l_n \geq \bar{J}_n(\bar{K}) \\ &= r + l_0 + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (\bar{f}(Y_{s-}) - l_0) ds + \sum_{k=0}^{\infty} \bar{C}_n(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &\stackrel{(5.9)}{\geq} r + l_0 - \varepsilon + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\sum_{k=0}^{\infty} (\bar{C}_n(Y_{\tau_k-}, \xi_k) - \log(1 - \delta)) \mathbb{1}_{\{\tau_k \leq T\}} \right) \\ &\xrightarrow{n \rightarrow \infty} r + l_0 - \varepsilon = \rho_0 - \varepsilon, \end{aligned}$$

where the convergence is due to (5.10) and Lemma 2.8. Now the smaller transaction costs $0 > \bar{C}_0 > \bar{C}_n$ obviously imply $r + l_0 = \rho_0 \geq \rho_n = r + l_n$ for all $n \in \mathbb{N}$ by considering the optimal strategy for ρ_n and hence $l_0 = \lim_{n \rightarrow \infty} l_n$ follows.

b) We need to show that the sequence of the boundaries lies within some compact interval. Let for every $n \in \mathbb{N}$ and $y \in]a_n, b_n[$ $\tau^n(y) := \inf\{t > 0 : X_t^y \notin]a_n, b_n[\}$, where X^y is the diffusion from (2.21) starting in y .

We denote by $x_n \in \{z \in \mathbb{R} : \bar{f}(z) \geq l_n\} \cap]\alpha_n, \beta_n[$ the state, where u_n attains its maximum on $[a_n, b_n]$ by Theorem 2.10, and write (v) of Theorem 2.10 in the form

$$u_n(x_n) = E \left(\int_0^{\tau^n(x_n)} (\bar{f}(X_s^{x_n}) - l_n) ds + u_n(X_{\tau^n(x_n)}^{x_n}) \right). \quad (5.11)$$

Now we can use $u_n(x_n) - u_n(b_n) > |\log(1 - \delta)|$ and $u_n(x_n) - u_n(a_n) > |\log(1 - \delta)|$ in (5.11) and get $\inf_{n \in \mathbb{N}} E(\tau^n(x_n)) > 0$. Since $\tau^n(x_n)$ is the exit time from $]a_n, b_n[$, we also deduce

$$\inf_{n \in \mathbb{N}} b_n - a_n > 0 \quad (5.12)$$

from $\inf_{n \in \mathbb{N}} E(\tau^n(x_n)) > 0$.

Furthermore, we have by Theorem 2.10 $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$, whence it is inferred that $\lim_{y \rightarrow \infty} \bar{f}_n(y) - l_n < 0$. So for some small ε and $y_n := \inf\{x > x_n : \bar{f}(x) - l_n \leq -\varepsilon\}$

we define in the case $b_n > y_n$ the exit time $\sigma_n := \inf\{t > 0 : X_t^{z_n} \notin]y_n, b_n[\}$, where $z_n := \frac{b_n + y_n}{2}$ and X^{z_n} is the diffusion from (2.21) starting in z_n . Then (v) of Theorem 2.10 yields

$$0 \leq u(z_n) = E \left(\int_0^{\sigma_n} (\bar{f}(X_s^{z_n}) - l_n) ds + u(X_{\sigma_n}^{z_n}) \right) \leq (-\varepsilon) \cdot E(\sigma_n) + \|u_n\|_\infty, \quad (5.13)$$

which together with $\sup_{n \in \mathbb{N}} \|u_n\|_{n \in \mathbb{N}} \leq \sup_{n \in \mathbb{N}} \|\bar{C}_n\|_{n \in \mathbb{N}} < \infty$ can be applied to get

$$\sup_{n \in \mathbb{N}} |a_n - x_n| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} |b_n - x_n| < \infty. \quad (5.14)$$

Now

$$\lim_{n \rightarrow \infty} |\alpha_n - \beta_n| = 0 \quad \text{and} \quad u_n(\alpha_n) \geq u_n(a_n) + |\log(1 - \delta)|$$

together with

$$u_n(\beta_n) \geq u_n(b_n) + |\log(1 - \delta)| \quad \text{and} \quad \lim_{n \rightarrow \infty} \|u_n\|_{n \in \mathbb{N}} = \lim_{n \rightarrow \infty} \|\bar{C}_n\|_{n \in \mathbb{N}} = |\log(1 - \delta)|.$$

yield

$$\lim_{n \rightarrow \infty} u_n(\alpha_n) = \lim_{n \rightarrow \infty} u_n(x_n) = |\log(1 - \delta)| = \lim_{n \rightarrow \infty} u_n(\beta_n). \quad (5.15)$$

We write (v) of Theorem 2.10 in the form

$$u_n(\alpha_n) = E \left(\int_0^{\tau^n(\alpha_n)} (\bar{f}(X_s^{\alpha_n}) - l_n) ds + u_n(X_{\tau^n(\alpha_n)}^{\alpha_n}) \right) \quad (5.16)$$

and obtain $\inf_{n \in \mathbb{N}} E(\tau^n(\alpha_n)) > 0$ similar to (5.11) from (5.16) and (5.15). By using (5.14) and (5.12) in Lemma 2.5 this then implies

$$\inf_{n \in \mathbb{N}} \alpha_n - a_n > 0.$$

The same consideration for β_n finally establishes

$$\inf_{n \in \mathbb{N}} b_n - \beta_n > 0.$$

Therefore by Proposition 5.3 there exists a subsequence $(\gamma_{n_k})_{k \in \mathbb{N}}$ such that the claimed limits a, α, β, b exist and satisfy $a < \alpha = \beta < b$.

c) Let $\bar{K}(a_{n_k}, \alpha_{n_k}, \beta_{n_k}, b_{n_k})$, $k \in \mathbb{N}$, be constant boundary strategies given by the constants $(a_{n_k}, \alpha_{n_k}, \beta_{n_k}, b_{n_k})$, $k \in \mathbb{N}$. The representation from Theorem 2.6 implies a continuity in the parameters and hence

$$l_0 = \bar{J}_0(\bar{K}(a, \alpha, \beta, b)) = \lim_{k \rightarrow \infty} \bar{J}_{n_k}(\bar{K}(a_{n_k}, \alpha_{n_k}, \beta_{n_k}, b_{n_k})) = \lim_{k \rightarrow \infty} l_{n_k}.$$

d) Let $(Y_t^y)_{t \geq 0}$ be a diffusion induced by the constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$ and starting in $y \in \mathbb{R}$ as in Lemma 1.22. From $\gamma = 0$ we then have

$$\rho_0 = r + l_0 + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (\bar{f}(Y_{s-}^y) - l_0) ds + \sum_{k=0}^{\infty} \log(1 - \delta) \mathbb{1}_{\{\tau_k \leq T\}} \right). \quad (5.17)$$

Now since the cost function \bar{C}_0 is constant and \bar{f} attains its maximum in $\psi(\hat{h})$, the assumption $\psi(\hat{h}) < a$ would pose a contradiction to the optimality of the strategy $\bar{K}(a, \alpha, \beta, b)$ by shifting the boundaries a little in the direction of $\psi(\hat{h})$, say to $(a - \varepsilon, \alpha - \varepsilon, \beta - \varepsilon, b - \varepsilon)$, and thus increasing the part

$$\frac{1}{T} \int_0^T (\bar{f}(Y_{s-}^y) - l_0) ds$$

in (5.17). Note that the Wiener process representation of Y in (1.81) implies

$$\bar{K}(a, \alpha, \beta, b) = \bar{K}(a - \varepsilon, \alpha - \varepsilon, \beta - \varepsilon, b - \varepsilon)$$

if the starting value y is shifted by ε as well. $\psi(\hat{h}) > b$ can then be treated the same way. \square

Remark 5.5. In part a) of the proof of Proposition 5.4 we could have used an optimal constant boundary strategy instead of an ε -optimal arbitrary one for the case $\gamma = 0$. Therefore Proposition 5.4 is another way of proving the existence of an optimal strategy, which additionally is in the class of constant boundary strategies, and thus reproducing the results from [MP95] and [IS06] for $\gamma = 0$.

To establish the uniqueness of the limit constants from Proposition 5.4 we need the following notions. Given a constant boundary strategy $\bar{K}(a, \alpha, \beta, b) = ((\tau_n, \xi_n))_{n \in \mathbb{N}_0}$ we consider the corresponding diffusion $(Y_t)_{t \geq 0}$ from (2.20) with starting value $y_0 \in \mathbb{R}$ and define for $T > 0$

$$M_{T,a} := \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\} \cap \{Y_{\tau_k-} = a\}} \quad \text{and} \quad M_{T,b} := \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\} \cap \{Y_{\tau_k-} = b\}}.$$

Now it is crucial to determine the expected average number of renewals per unit time in a and b , i.e. the values

$$\lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T,a}) \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T,b}).$$

For that purpose we define for the case that we always trade in $\tau_0 = 0$ to α , i.e. $y_0 + \xi_0 \equiv \alpha$

$$\tau^\alpha := \inf \{t > 0 : Y_{t-} = a\}$$

and analogously

$$\tau^\beta := \inf \{t > 0 : Y_{t-} = b\}$$

for the case $y_0 + \xi_0 \equiv \beta$. The following result is essentially derived in [IS05].

Lemma 5.6. *It holds*

$$\lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T,a}) = \frac{1}{E(\tau^\alpha)} \in]0, \infty[\quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T,b}) = \frac{1}{E(\tau^\beta)} \in]0, \infty[.$$

Proof. We only show $\lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T,a}) = \frac{1}{E(\tau^\alpha)} \in]0, \infty[$ here due to the analogous proof. As in [IS05] we define $N_0 := 0$ and inductively

$$N_k := \inf\{n > N_{k-1} : Y_{\tau_n-} = a\}, \quad k \in \mathbb{N}.$$

Then by Proposition 2 in [IS06] $\tau_{N_k} - \tau_{N_{k-1}}, k \geq 2$, are i.i.d. random variables with distribution $P(\tau^\alpha \in \cdot)$ and expectation $E(\tau^\alpha) < \infty$.

If we define $S_0 := 0$ and $S_n := \sum_{k=1}^n \tau_{N_k} - \tau_{N_{k-1}}$, then

$$M_{T,a} = N(T) := \sup\{n \in \mathbb{N}_0 : S_n \leq T\}.$$

In the case $y_0 + \xi_0 \equiv \alpha$ we have $\tau_{N_1} = \tau^\alpha$ and can use as in the proof of Theorem C.2 in [IS05] the elementary renewal theorem (cf. Theorem 3.8 in [Ros70]) to get

$$\lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T,a}) = \frac{1}{E(\tau^\alpha)}.$$

In the general case $y_0 + \xi_0 \equiv y \in]a, b[$ we can use the same result for delayed renewal processes (c.f. Proposition 3.14 in [Ros70]).

□

The next proposition shows the uniqueness of the optimal boundaries in the case with no proportional costs. In Proposition 7.1 of [IS06] the uniqueness of the boundaries that satisfy certain conditions is established for general γ . However, it is not proved, apart from those of Theorem 2.10, that all other possible optimal boundaries also satisfy these conditions. Furthermore, in usual uniqueness results as in [Nag04], Theorem 5.2, only the uniqueness of the solution u is obtained and to the best of our knowledge no such uniqueness statements for the boundaries exist so far.

Proposition 5.7. *Let $d := 1$ and $\gamma = 0$ defining the proportional costs in the sense of (1.20) for a fixed common $0 < \delta < 1$. Let the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$. Then the constant boundary strategy $\bar{K} = \bar{K}(a, \alpha, \beta, b)$ given by the constants $a < \alpha = \beta < b$ from Theorem 2.10 is optimal, i.e. it satisfies $\rho = \bar{J}(\bar{K}) = r + l$ with $\max\{f(0), f(1)\} < l < f(\hat{h})$. Furthermore, every set of constants $\tilde{a} < \tilde{\alpha} \leq \tilde{\beta} < \tilde{b}$ in \mathbb{R} such that the constant boundary strategy $\tilde{K} = \bar{K}(\tilde{a}, \tilde{\alpha}, \tilde{\beta}, \tilde{b})$ is optimal necessarily satisfies*

$$\tilde{a} = a, \tilde{\alpha} = \alpha = \beta = \tilde{\beta}, \tilde{b} = b.$$

Proof. a) From $\gamma = 0$ we have $\Gamma = \bar{\Gamma}$ in Theorem 2.10, which therefore guarantees the optimality of \bar{K} .

b) Let \bar{f} , φ , l , L and u be from Theorem 2.10 for $\gamma = 0$. We denote by $(Y_t)_{t \geq 0}$ the diffusion induced by $((\tau_n, \xi_n))_{n \in \mathbb{N}_0} := \bar{K}(\tilde{a}, \tilde{\alpha}, \tilde{\beta}, \tilde{b})$ as in (2.20) with a fixed starting value y_0 .

For every $T \geq 0$ we define $M_{T, \tilde{a}} := \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\} \cap \{Y_{\tau_k-} = \tilde{a}\}}$ and $M_{T, \tilde{b}} := \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\} \cap \{Y_{\tau_k-} = \tilde{b}\}}$.

Then by Lemma 5.6 we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T, \tilde{a}}) > 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{T} E(M_{T, \tilde{b}}) > 0. \quad (5.18)$$

c) Here, we prove $\tilde{\alpha} = \alpha = \beta = \tilde{\beta}$, $\tilde{a} \leq a$, $b \leq \tilde{b}$. By Theorem 2.10 we have $Lu + \bar{f} - l \leq 0$ and $\bar{\Gamma} \equiv \log(1 - \delta)$ as well as $u(a) - u(\alpha) = \log(1 - \delta) = u(b) - u(\beta)$. Furthermore, we can use Itô's formula as in (3.8) to get

$$\begin{aligned} & E(u(y_0) - u(Y_T)) \\ &= E\left(-\int_0^T Lu(Y_{s-}) ds - \sum_{k=0}^{\infty} (u(Y_{\tau_k}) - u(Y_{\tau_k-})) \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &\geq E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} (u(Y_{\tau_k-}) - u(Y_{\tau_k})) \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &= E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + (u(\tilde{a}) - u(a) + u(a) - u(\alpha) + u(\alpha) - u(\tilde{\alpha})) \cdot M_{T, \tilde{a}} \right. \\ &\quad \left. + (u(\tilde{b}) - u(b) + u(b) - u(\beta) + u(\beta) - u(\tilde{\beta})) \cdot M_{T, \tilde{b}}\right) \\ &= E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \log(1 - \delta) \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &\quad + (u(\tilde{a}) - u(a) + u(\alpha) - u(\tilde{\alpha})) \cdot E(M_{T, \tilde{a}}) + (u(\tilde{b}) - u(b) + u(\beta) - u(\tilde{\beta})) \cdot E(M_{T, \tilde{b}}). \end{aligned}$$

By (1.83) we then obtain from the boundedness of u

$$\begin{aligned} \bar{J}(\bar{K}) - r - l &= \liminf_{T \rightarrow \infty} \frac{1}{T} E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} \bar{C}(Y_{\tau_k-}, \xi_k) \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \log(1 - \delta) \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \left(- (u(\tilde{a}) - u(a) + u(\alpha) - u(\tilde{\alpha})) \cdot E(M_{T, \tilde{a}}) \right. \\ &\quad \left. - (u(\tilde{b}) - u(b) + u(\beta) - u(\tilde{\beta})) \cdot E(M_{T, \tilde{b}}) \right) \\ &= - \lim_{T \rightarrow \infty} \frac{1}{T} \left((u(\tilde{a}) - u(a) + u(\alpha) - u(\tilde{\alpha})) \cdot E(M_{T, \tilde{a}}) \right. \\ &\quad \left. + (u(\tilde{b}) - u(b) + u(\beta) - u(\tilde{\beta})) \cdot E(M_{T, \tilde{b}}) \right). \quad (5.19) \end{aligned}$$

By Theorem 2.10 u is positive and attains its unique maximum in $\alpha = \beta$ with $u(\alpha) = |\log(1 - \delta)|$. Furthermore, we have $u \equiv 0$ on $\mathbb{R} \setminus]a, b[$. Since $\bar{J}(\bar{K})$ is optimal, it holds $\bar{J}(\bar{K}) = r + l$, and thus (5.19) together with (5.18) yields

$$\tilde{\alpha} = \alpha = \beta = \tilde{\beta}, \quad \tilde{a} \leq a, \quad b \leq \tilde{b}.$$

d) It remains to show $\tilde{a} \geq a$ and $b \geq \tilde{b}$. We first take a closer look at the function u . We can translate the proceeding in [IS06] from the fraction space $]0, 1[$ to \mathbb{R} via the transformation function ψ as follows.

As a first step a solution $\tilde{u} \in C^2(\mathbb{R}, \mathbb{R})$ to $L\tilde{u} = -(\bar{f} - l)$, that satisfies certain conditions, is found by variation of constants and its derivative is the function g from Theorem 2.4. Then u can be gained by setting $u|_{[a,b]} := \tilde{u}|_{[a,b]} + k$ and $u|_{\mathbb{R} \setminus [a,b]} := 0$, where k is a suitable constant.

From Theorem 2.10 $v := \tilde{u} + k$ then satisfies

$$v'(a) = u'(a) = 0, \quad v'(b) = u'(b) = 0.$$

Furthermore, Remark 9.1 and Lemma 7.2 in [IS06] imply $v''(a) > 0$ and $v''(b) > 0$ and that v' has no other roots than a, α, b . Therefore we get

$$v \text{ is strictly decreasing on }]-\infty, a] \text{ and strictly increasing on } [b, \infty[. \quad (5.20)$$

Similarly to c), we can use Itô's formula as in (3.8) and get

$$\begin{aligned} E(v(y_0) - v(Y_T)) &= E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \sum_{k=0}^{\infty} (v(Y_{\tau_k-}) - v(Y_{\tau_k})) \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &= E\left(\int_0^T (\bar{f}(Y_{s-}) - l) ds + \log(1 - \delta) \sum_{k=0}^{\infty} \mathbb{1}_{\{\tau_k \leq T\}}\right) \\ &\quad + (v(\tilde{a}) - v(a)) \cdot E(M_{T,\tilde{a}}) + (v(\tilde{b}) - v(b)) \cdot E(M_{T,\tilde{b}}) \end{aligned}$$

Now v is bounded on $[\tilde{a}, \tilde{b}]$ and, as above, we conclude

$$\bar{J}(\bar{K}) - r - l = - \lim_{T \rightarrow \infty} \frac{1}{T} \left((v(\tilde{a}) - v(a)) \cdot E(M_{T,\tilde{a}}) + (v(\tilde{b}) - v(b)) \cdot E(M_{T,\tilde{b}}) \right),$$

and so $\bar{J}(\bar{K}) = r + l$ together with (5.18) and (5.20) then finally implies

$$\tilde{a} \geq a, \quad b \geq \tilde{b}.$$

□

Using the uniqueness of the optimal boundaries we can now show the convergence for the whole sequence in Proposition 5.4.

Theorem 5.8. *Let $d := 1$ and $0 < \delta < 1$. Let the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$. Let further φ and $l(\gamma)$ be from Theorem 2.10 for each $\gamma < 1 - \delta$ defining the proportional costs in the sense of (1.20). Then we have for a, α, β, b from Proposition 5.4*

$$\lim_{\gamma \rightarrow 0} a(\gamma) = a < \alpha = \lim_{\gamma \rightarrow 0} \alpha(\gamma) = \lim_{\gamma \rightarrow 0} \beta(\gamma) = \beta < b = \lim_{\gamma \rightarrow 0} b(\gamma)$$

and every constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$ defines an optimal strategy for the case $\gamma = 0$. Furthermore, the corresponding optimal growth rate ρ_0 satisfies

$$\rho_0 - r = \sup_{0 < \gamma < 1 - \delta} l(\gamma) = \lim_{\gamma \rightarrow 0} l(\gamma) = \lim_{\gamma \rightarrow 0} \rho(\gamma) - r$$

and the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ lies within $[\varphi(a), \varphi(b)]$.

Proof. Due to Proposition 5.7 every subsequence mentioned in Proposition 5.4 converges to the same limit, whence the assertion is inferred. □

For an illustration, we take here the same data as in figure 5.1 but add the Merton fraction $\eta := \hat{h} = \frac{\mu-r}{\sigma^2} = 0.6$ as well as the states h that satisfy $f(h) = l(\gamma)$, where $l(\gamma) = \rho(\gamma) - r = \rho(\gamma)$ is the asymptotic growth rate given by the optimal proportional constant boundary strategy $\tilde{K}(a(\gamma), \alpha(\gamma), \beta(\gamma), b(\gamma))$. Furthermore, the optimal growth rate without transaction costs $\rho = f(\eta) + r = 0.0288$ is also added and we note that the value of the minimal growth rate $\rho = \max\{f(0), f(1)\} + r$ for all transaction costs is here 0.016. Figure 5.2 illustrates these dependencies on γ .

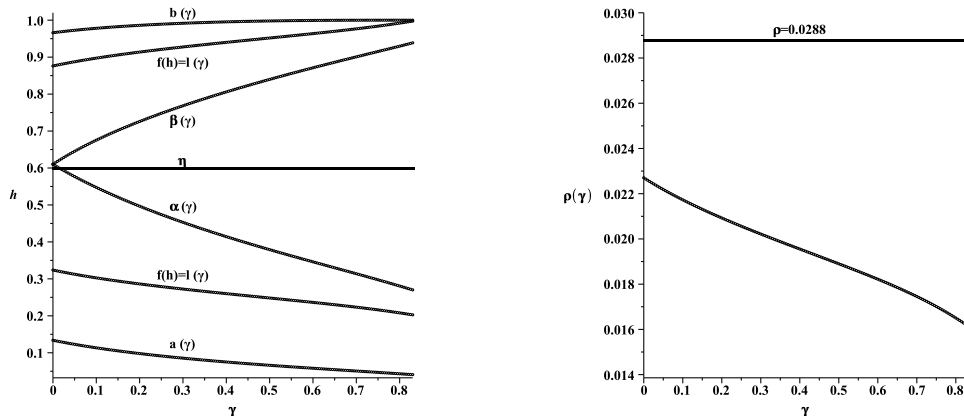


Figure 5.2: Dependence in Model A on γ for $\delta = 0.1$

As indicated by Theorem 5.8 the Merton fraction η satisfies $\eta \in [a(\gamma), b(\gamma)]$ but $\eta \in [\alpha(\gamma), \beta(\gamma)]$ is not always true, which is already mentioned in [MP95].

5.3 Model B

For the analogon of Proposition 5.3 we need the following monotonicity property of the function u from Proposition 3.34.

Lemma 5.9. *In the situation of Proposition 3.34 the function u is strictly increasing on $[a, x]$ and strictly decreasing on $[x, b]$.*

Proof. We give the proof for $[x, b]$. For every $y \in]\beta, b[$ we have

$$u(b) \stackrel{3.34}{=} u(\beta) + \bar{C}(b, \beta - b) \stackrel{(3.46)}{<} u(y) + \bar{C}(y, \beta - y) \leq Mu(y) \leq u(y). \quad (5.21)$$

Now suppose there are $y_1 < y_2 \in [x, b]$ with $u(y_1) \leq u(y_2)$. Since $u \in C^\infty(]a, b[)$ and $Lu = -(\bar{f} - l)$ by Proposition 3.26, u is not constant on $[y_1, y_2]$. Using (5.21) we therefore can find $z_1 < z_2 \in]x, b[$ such that u has a local minimum in z_1 and a local maximum in z_2 . The maximum principle from Proposition 3.26 then yields $\bar{f}(x), \bar{f}(z_2) \geq l_n \geq \bar{f}(z_1)$, which is a contradiction, since f is strictly concave from its definition and φ is strictly monotone.

□

Proposition 5.10. *Let $d := 1$ and $((\lambda_n, \nu_n))_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the proportional costs in Model B in the sense of (1.51) for a fixed common $0 < \delta < 1$. Let $u_n, Mu_n, \bar{C}_n, \bar{f}$ and l_n be the corresponding functions and constants from Theorem 3.6 under the assumption $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$ and $f(0) \neq f(1)$. Then the corresponding constants $a_n, \alpha_n, x_n, \beta_n$ and b_n given by Proposition 3.34 for every $n \in \mathbb{N}$ satisfy*

$$(i) \quad \lim_{n \rightarrow \infty} \nu_n = 0 \text{ implies } \lim_{n \rightarrow \infty} |x_n - \beta_n| = 0,$$

$$(ii) \quad \lim_{n \rightarrow \infty} \lambda_n = 0 \text{ implies } \lim_{n \rightarrow \infty} |x_n - \alpha_n| = 0.$$

Proof. The proof is an adaption to Model B of the proof of Proposition 5.3.

Since (ii) can be treated the same way, we only show (i) and hence assume $\lim_{n \rightarrow \infty} \nu_n = 0$.

a) We first show the existence of some constant $c = c(\mu, \sigma^2, r)$ such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in [x_n, \beta_n]} |u_n''(x)| \leq c. \quad (5.22)$$

We fix $n \in \mathbb{N}$ and have by Proposition 3.26 for all $x \in [x_n, b_n[$

$$u_n''(x) = \frac{2}{\sigma^2} (-\bar{f}(x) + l_n) - \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2} \sigma^2 \right) u_n'(x). \quad (5.23)$$

We note that $\mu - r - \frac{1}{2} \sigma^2 = 0$ is excluded due to $f(0) \neq f(1)$ and thus $\mu - r - \frac{1}{2} \sigma^2 \neq 0$. Now by Lemma 5.9 we have $u_n' \leq 0$ on $]x_n, \beta_n[$ and since u_n attains its maximum in x_n

we also have $u'_n(x_n) = 0$. In the case that u'_n attains its minimum in $[x_n, \beta_n]$ in some $x \in]x_n, \beta_n[$ we have $u''_n(x) = 0$ and by (5.23) therefore

$$\sup_{y \in [x_n, \beta_n]} |u'_n(y)| = |u'_n(x)| \leq \left| \frac{-\bar{f}(x) + l_n}{\mu - r - \frac{1}{2}\sigma^2} \right|. \quad (5.24)$$

In the case that the minimum is attained in β_n we can conclude as follows. From Lemma 3.29 we have $u_n(b_n) = Mu_n(b_n) = u_n(\beta_n) + \Gamma_n(b_n, \beta_n)$. Therefore the function $g : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto u_n(x) + \Gamma_n(b_n, x)$ attains its maximum in β_n and we get

$$\sup_{y \in [x_n, \beta_n]} |u'_n(y)| = |u'_n(\beta_n)| = \left| \frac{\partial}{\partial y} \Gamma_n(b_n, \beta_n) \right| \stackrel{(3.46)}{=} \frac{\nu_n \varphi'(\beta_n)}{1 - \nu_n \varphi(\beta_n)}. \quad (5.25)$$

Now if we plug (5.24) and (5.25), respectively, into (5.23), then (5.22) follows, since in the latter case we have

$$\sup_{x \in \mathbb{R}, n \in \mathbb{N}} \frac{\nu_n \varphi'(x)}{1 - \nu_n \varphi(x)} < \infty,$$

which is due to $\lim_{n \rightarrow \infty} \nu_n = 0$ and $\varphi(x), \varphi'(x) \in]0, 1[$ in Lemma 3.17.

b) To prove (ii) we take $\varepsilon > 0$ and suppose $|x_n - \beta_n| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$ and without loss of generality

$$|x_n - \beta_n| \geq \varepsilon \text{ for all } n \in \mathbb{N}. \quad (5.26)$$

From the definition of Mu_n it holds

$$u_n(x_n) + \Gamma_n(b_n, x_n) \leq Mu_n(b_n) = u_n(\beta_n) + \Gamma_n(b_n, \beta_n)$$

yielding

$$u(x_n) - u(\beta_n) \leq \Gamma_n(b_n, \beta_n) - \Gamma_n(b_n, x_n) \stackrel{(3.44)}{<} \left| \log \left(\frac{1 - \nu_n \varphi(b_n)}{1 - \nu_n \varphi(x_n)} \right) \right|. \quad (5.27)$$

We define $\tilde{g}_n : I_n := [x_n, \beta_n] \rightarrow \mathbb{R}$, $x \mapsto u_n(x) - u_n(\beta_n)$ and get by Lemma 5.9 and (5.27)

$$\|\tilde{g}_n\|_{I_n, \infty} < \left| \log \left(\frac{1 - \nu_n \varphi(b_n)}{1 - \nu_n \varphi(x_n)} \right) \right|$$

and therefore $\lim_{n \rightarrow \infty} \|\tilde{g}_n\|_{I_n, \infty} = 0$. Applying Lemma 5.2 we get due to a) and (5.26)

$$\lim_{n \rightarrow \infty} \|u'_n\|_{I_n, \infty} = \lim_{n \rightarrow \infty} \|\tilde{g}'_n\|_{I_n, \infty} = 0. \quad (5.28)$$

We can use $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$ as in the proof of Proposition 5.3 to get a $\tilde{c} > 0$ and some $\tilde{\varepsilon} \in]0, \varepsilon[$ such that for every $n \in \mathbb{N}$ there are $z_{1,n} < z_{2,n} \in I_n$ satisfying

$$z_{2,n} - z_{1,n} = \tilde{\varepsilon} \quad \text{and} \quad \inf_{x \in [z_{1,n}, z_{2,n}]} |\bar{f}(x) - l_n| \geq \tilde{c}. \quad (5.29)$$

Therefore we can find by using (5.28) with (5.23) some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\inf_{x \in [z_{1,n}, z_{2,n}]} |u''_n(x)| \stackrel{(5.29)}{\geq} \frac{\tilde{c}}{\sigma^2}.$$

This implies for all $n \geq n_0$

$$|u'_n(z_{1,n}) - u'_n(z_{2,n})| \geq \frac{\tilde{c}}{\sigma^2} \tilde{\varepsilon}$$

by the mean value theorem, which is a contradiction to (5.28). □

To illustrate this we take as in Figure 5.1 the scenario where $r := 0$, $\sigma := 0.4$ and $\mu := 0.096$. But in order to compare the results to those of Model A, we take in consideration that the correspondence between Model A and Model B is given in case of

$$\lambda = \nu = \frac{\gamma}{1 - \delta}$$

as described in Chapter 4. Thus we can calculate by the method described in Proposition 4.6 the dependence of the transformed constants a, α, β, b in $]0, 1[$ on the proportional costs γ instead of λ, ν . Figure 5.3 illustrates this for i) $\delta = 0.0001$ and ii) $\delta = 0.1$.

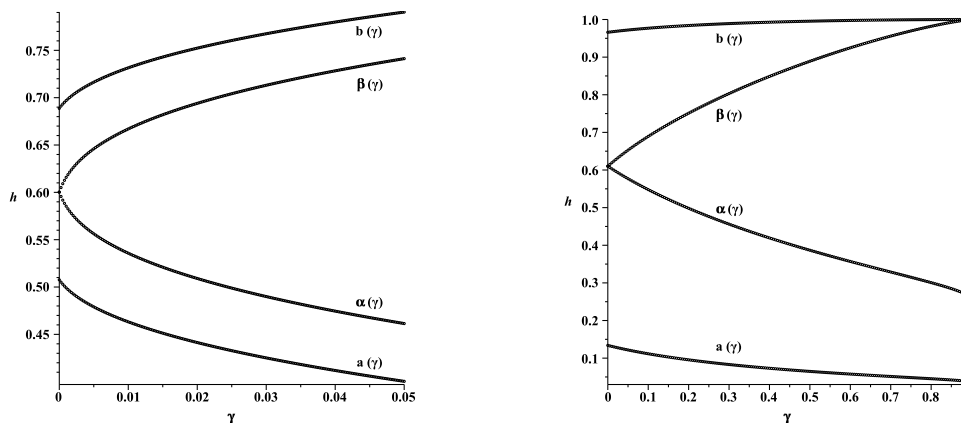


Figure 5.3: Boundary-dependence in Model B on γ for i) $\delta = 0.0001$ and ii) $\delta = 0.1$

Now we are able to prove without using Theorem 3.37 that in the case $\lambda, \nu = 0$, which was still excluded in Proposition 3.34, Model B admits a constant boundary strategy, which is optimal in the class of all admissible strategies.

Proposition 5.11. *Let $d := 1$ and $((\lambda_n, \nu_n))_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions satisfying $\lim_{n \rightarrow \infty} \lambda_n, \nu_n = 0$ and defining the proportional costs in Model B in the sense of (1.51) for a fixed common $0 < \delta < 1$. Let \bar{f} , φ and l_n be from Theorem 3.6 for every $n \in \mathbb{N}$ under the assumption $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$ and $f(0) \neq f(1)$. Then there exists a subsequence $((\lambda_{n_k}, \nu_{n_k}))_{k \in \mathbb{N}}$ and constants a, α, β, b such that*

$$\lim_{k \rightarrow \infty} a_{n_k} = a < \alpha = \lim_{k \rightarrow \infty} \alpha_{n_k} = \lim_{k \rightarrow \infty} \beta_{n_k} = \beta < b = \lim_{k \rightarrow \infty} b_{n_k}$$

and every constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$ given by these constants defines an optimal strategy for the case $\lambda, \nu = 0$. Furthermore, the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ lies within $[\varphi(a), \varphi(b)]$ and we get for the corresponding optimal growth rate ρ_0

$$\rho_0 - r = \sup_{n \in \mathbb{N}} l_n = \lim_{n \rightarrow \infty} l_n = \lim_{n \rightarrow \infty} \rho_n - r.$$

Proof. Here, the proof is again an adaption of the proof of Proposition 5.4 to Model B.

a) Let $l_0 := \rho_0 - r$. Let \bar{C}_0, \bar{C}_n denote the cost functions from (1.78) for the proportional costs given by $\lambda = \nu = 0, \lambda = \lambda_n, \nu = \nu_n$ respectively. As a consequence of $\lambda = \nu = 0$ we have $\bar{C}_0 \equiv \log(1 - \delta)$ and $\lim_{n \rightarrow \infty} \|\bar{C}_n - \bar{C}_0\|_\infty = 0$ by Lemma 3.29. Thus the same argument as in Proposition 5.4 is applicable and yields $\lim_{n \rightarrow \infty} l_n = \sup_{n \rightarrow \infty} l_n = l_0$.

b) We follow part b) of the proof of Proposition 5.4. Instead of Theorem 2.10 (v) we can use (3.30) the same way to get

$$\inf_{n \in \mathbb{N}} b_n - a_n > 0. \quad (5.30)$$

Then our assumption $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$ can be used to validate $\lim_{y \rightarrow \infty} \bar{f}_n(y) - l_n < 0$ as well as

$$\sup_{n \in \mathbb{N}} |a_n - x_n| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} |b_n - x_n| < \infty. \quad (5.31)$$

Finally Lemma 2.5 can be used with (5.30) and (5.31) together with (3.30) as in the proof of Proposition 5.4 to get

$$\inf_{n \in \mathbb{N}} b_n - \beta_n > 0 \quad \text{and} \quad \inf_{n \in \mathbb{N}} \alpha_n - a_n > 0.$$

Therefore by Proposition 5.10 there exists a subsequence $((\lambda_{n_k}, \nu_{n_k}))_{k \in \mathbb{N}}$ such that the claimed limits a, α, β, b exist and satisfy $a < \alpha = \beta < b$.

c) We can use the continuity in the parameters in Theorem 2.6 and get

$$l_0 = \bar{J}_0(\bar{K}(a, \alpha, \beta, b)) = \lim_{k \rightarrow \infty} \bar{J}_{n_k}(\bar{K}(a_{n_k}, \alpha_{n_k}, \beta_{n_k}, b_{n_k})) = \lim_{k \rightarrow \infty} l_{n_k},$$

where $\bar{K}(a_{n_k}, \alpha_{n_k}, \beta_{n_k}, b_{n_k}), k \in \mathbb{N}$, are constant boundary strategies given by the constants $(a_{n_k}, \alpha_{n_k}, \beta_{n_k}, b_{n_k}), k \in \mathbb{N}$.

d) We could use the same argument as in Proposition 5.4 to show $\psi(\hat{h}) \in [a, b]$. But since the optimal growth rates for Model A and Model B are identical for $\gamma = \lambda = \nu = 0$ due to the representation (1.83), we already know this from Proposition 5.7 and Theorem 5.8. \square

Remark 5.12. In part a) of the proof of Proposition 5.11 we could have used an optimal strategy instead of an ε -optimal one for the case $\lambda, \nu = 0$, whose existence is guaranteed by Theorem 3.12 since $\rho_0 - r \geq \rho_n - r = l_n > \max\{f(0), f(1)\}$. Since we do not use this result, Proposition 5.11 is another way to prove that an optimal strategy exists and can be found in the class of constant boundary strategies.

Remark 5.13. The advantage of Model A here is the fact that $l > \max\{f(0), f(1)\}$ is a consequence of $\hat{h} \in]0, 1[$ in Theorem 2.10 and has not to be assumed as in Model B. Furthermore, we have already mentioned in Remark 3.35 that for the case $\hat{h} = \frac{1}{2}$ no solution for Model B is guaranteed by Theorem 3.6.

Theorem 5.14. *Let $d := 1$ and λ, ν define the proportional costs in Model B in the sense of (1.51) for a fixed common $0 < \delta < 1$. Let \bar{f} , φ and $l(\lambda, \nu)$ be from Theorem 3.6 for every $\lambda, \nu \in]0, 1[$ and suppose $f(0) \neq f(1)$ and $l(\lambda_0, \nu_0) > \max\{f(0), f(1)\}$ for some $\lambda_0, \nu_0 \in]0, 1[$. Then we have for a, α, β, b from Proposition 5.11 and $\eta = \max\{\lambda, \nu\}$*

$$\lim_{\eta \rightarrow 0} a(\lambda, \nu) = a < \alpha = \lim_{\eta \rightarrow 0} \alpha(\lambda, \nu) = \lim_{\eta \rightarrow 0} \beta(\lambda, \nu) = \beta < b = \lim_{\eta \rightarrow 0} b(\lambda, \nu)$$

and every constant boundary strategy $\bar{K}(a, \alpha, \beta, b)$ given by these constants defines an optimal strategy for the case $\lambda = \nu = 0$.

Furthermore, we get for the corresponding optimal growth rate ρ_0

$$\rho_0 - r = \sup_{\eta > 0} l(\lambda, \nu) = \lim_{\eta \rightarrow 0} l(\lambda, \nu) = \lim_{\eta \rightarrow 0} \rho(\lambda, \nu) - r$$

and the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ lies within $[\varphi(a), \varphi(b)]$.

Proof. This is a direct consequence of Proposition 5.7 together with Proposition 5.11. □

For an illustration, we add to figure 5.3 the Merton fraction $\eta = \hat{h} = \frac{\mu - r}{\sigma^2} = 0.6$ as well as the states h that satisfy $f(h) = l(\gamma)$, where $l(\gamma) = \rho(\gamma) - r = \rho(\gamma)$ is the asymptotic growth rate given by the optimal constant boundary strategy $\tilde{K}(a(\gamma), \alpha(\gamma), \beta(\gamma), b(\gamma))$. $\rho(\gamma)$ is optimal for the optimization problem in Model B, where the proportional transaction costs are given by $\lambda = \nu = \frac{\gamma}{1 - \delta}$, if $\rho(\gamma) > \max\{f(0), f(1)\} = 0.016$, where $f(1) = \mu - r - \frac{\sigma^2}{2}$, i.e. when an optimal constant boundary strategy performs better than the buy-and-hold strategy of having invested everything in the stock or the bond. Furthermore, we have added the optimal growth rate without transaction costs $\rho = f(\eta) + r = 0.0288$.

Figure 5.4 illustrates these dependencies on γ .

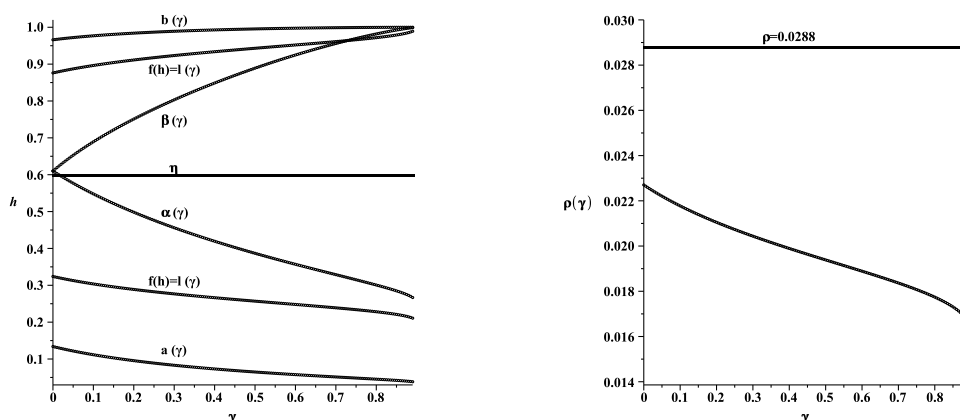


Figure 5.4: Dependence in Model B on γ for $\delta = 0.1$

Here, we can see that $[\alpha(\gamma), \beta(\gamma)] \subseteq \{h : f(h) \geq l(\gamma)\}$ is mostly true but fails to be true in general in Model B.

Theorem 2.10 yields the existence of a solution u to the QVI (2.25) with $\gamma = 0$ and some $l > \max\{f(0), f(1)\}$ given that the Merton fraction \hat{h} satisfies $\hat{h} \in]0, 1[$. This solution u was rigorously constructed in [IS06] and solves the optimization problem for Model A with optimal growth rate $r + l$. Due to the representation (1.83) the same growth rate applies for Model B, since the cost functions \bar{C} are identical for both models without transaction costs. In the case $\hat{h} \neq \frac{1}{2}$ we have $\mu - r - \frac{1}{2}\sigma^2 \neq 0$, and thus the constant l from (3.16) coincides with the constant l from Theorem 2.10 due to Theorem 3.15. Therefore the conditions of Theorem 3.6 are satisfied and it guarantees the existence of a function \tilde{u} , which is the solution to the QVI (3.5) in its weak form with $\gamma = 0$ and l from above, gained by Tamura in [Tam08] by using perturbation methods and variational techniques. As a final result of this chapter we prove that these two a priori different solutions to the same QVI indeed coincide without using the uniqueness result in Theorem 5.2 of [Nag04].

Proposition 5.15. *Let $d := 1$, $0 < \delta < 1$ and the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[\setminus\{\frac{1}{2}\}$. Let further \tilde{u} denote the solution to the QVI (2.25) given by Theorem 3.6 for Model B in the case $\lambda = \nu = 0$ and let u be its counterpart from Theorem 2.10 for Model A in the case $\gamma = 0$. Then we have*

$$u = \tilde{u}.$$

Proof. a) From Theorem 3.37 we know that \tilde{u} admits a constant boundary strategy $\bar{K}(\tilde{a}, \tilde{\alpha}, \tilde{\beta}, \tilde{b})$, which is optimal.

b) Now the constants a, α, β, b from Theorem 2.10 satisfy

$$a = \tilde{a}, \alpha = \tilde{\alpha}, \beta = \tilde{\beta}, b = \tilde{b}$$

due to Proposition 5.7. Therefore we have to show $u = \tilde{u}$ on $[a, b]$ and $\tilde{u} = 0$ on $\mathbb{R} \setminus [a, b]$.

b1) Let $\tau := \tau^*(y) := \inf\{t > 0 : X_t^y \notin]a, b[\}$ be the exit time from $]a, b[$ defined in (v) of Theorem 2.10 for the diffusion $(X_t^y)_{t \geq 0}$ in (2.21) with starting value $y \in]a, b[$. Then by (v) of Theorem 2.10 we have

$$u(y) = E \left(\int_0^\tau (\bar{f}(X_s^y) - l) ds + u(X_\tau^y) \right). \quad (5.32)$$

Let further $\tilde{\tau} := \tilde{\tau}^*(y) := \inf\{t > 0 : \tilde{u}(X_t^y) = M\tilde{u}(X_t^y)\}$ denote its counterpart for \tilde{u} from Proposition 3.16. Then (3.30) yields

$$\tilde{u}(y) = E \left(\int_0^{\tilde{\tau}} (\bar{f}(X_s^y) - l) ds + \tilde{u}(X_{\tilde{\tau}}^y) \right). \quad (5.33)$$

In the proof of Theorem 3.37 we have seen $M\tilde{u} \equiv 0$ and $\tilde{\tau} = \inf\{t > 0 : X_t^y \notin]a, b[\}$. Now $Mu \equiv 0$ follows the same way from Theorem 2.10 due to $\gamma = 0$ and therefore

$$u(a) = \tilde{u}(a) = \tilde{u}(b) = u(b) = 0 \quad \text{and} \quad \tau = \tilde{\tau}.$$

Using this in (5.32) and (5.33), we conclude $u(y) = \tilde{u}(y)$ for every $y \in]a, b[$.

b2) Theorem 2.10 implies

$$u'(a) = u'(\alpha) = u'(b) = 0.$$

Furthermore, Remark 9.1 and Lemma 7.2 in [IS06] yield

$$u''(a) > 0, \quad u''(\alpha) < 0, \quad u''(b) > 0$$

and hence

$$Lu(a) > 0, \quad Lu(\alpha) < 0, \quad Lu(b) > 0.$$

Therefore, $L\tilde{u} = -(\bar{f} - l)$ on $[a, b]$ then induces $(\mathbb{R} \setminus [a, b]) \subseteq \{x \in \mathbb{R} : \bar{f}(x) < l\}$, and thus $\tilde{u} = M\tilde{u} = 0$ on $\mathbb{R} \setminus [a, b]$ can be obtained from the maximum principle in Proposition 3.26.

□

Chapter 6

Convergence in case of vanishing fixed costs

On the one hand assuming the excluded case $\delta = 0$ in Model A and Model B makes these models identical but on the other hand we also leave the scope of these models, since frequent trading is not punished anymore leaving the impulse control strategies too restrictive. For $\delta = 0$ continuous trading has to be considered and the optimal strategy is to keep the risky fraction process in some bounded interval $[A, B] \subset]0, 1[$ with minimal effort, i.e. the risky fraction process is reflected at the boundaries. This behavior seems to be optimal even for different kinds of maximization criterions as in [DN90].

Now if we fix the proportional costs given by γ in Model A, or λ, ν in Model B, and let the fixed costs represented by δ tend to zero, then we should expect convergence of the optimal boundaries $a(\delta), \alpha(\delta), \beta(\delta), b(\delta)$ to A, B as indicated in Figure 11.2 of [IS05], i.e.

$$\lim_{\delta \rightarrow 0} a(\delta) = A = \lim_{\delta \rightarrow 0} \alpha(\delta) \quad \text{and} \quad \lim_{\delta \rightarrow 0} \beta(\delta) = B = \lim_{\delta \rightarrow 0} b(\delta).$$

Moreover, the convergence of the optimal expected growth rates is of interest and since the jump size of the optimal risky fraction processes tends to zero, the limiting process is likely to be a reflected diffusion that is optimal for the case $\delta = 0$ described above.

To establish these convergences in this chapter, we begin by introducing the model with only proportional transaction costs, where we restrict ourselves to the case $\gamma = \lambda = \nu$, and how the corresponding Hamilton-Jacobi-Bellman approach can be used to solve the optimization problem in Section 6.1. The most similar but not identical models are that of [TKA88], where transaction costs are only to be paid for selling stocks, and that of [Sul97] or [AST01], where the investor's wealth is considered after liquidation.

Using the results of [IS06] from Chapter 2, we then obtain the convergence of the boundaries and that of the optimal expected growth rates for Model A in Section 6.2. Moreover, the convergence of the corresponding QVI-solutions is proved, yielding the uniqueness and optimality of the limiting constants A, B from above. Finally, despite of the limited information on the QVI-solutions from Model B, we can show the convergence of the corresponding boundaries to some constants A, B for a subsequence in Section 6.3.

6.1 A model with only proportional transaction costs

As described above, the strategies introduced in Model A, Section 1.2, are too restrictive in the case of $\delta = 0$. We therefore adjust the model and define the class of suitable strategies as in [AST01].

As in Model A we consider the Black-Scholes model described in Section 1.1 with *deterministic* and *time-constant* interest rate $r \in \mathbb{R}_{\geq 0}$, trend $\mu \in \mathbb{R}$ and volatility $\sigma \in \mathbb{R}_{> 0}$.

Let V, X, Y and h denote the processes describing the portfolio in Model A and suppose fixed starting values $(x_0, y_0) \in \mathbb{R}_{\geq 0}^2$ with $v_0 = x_0 + y_0 > 0$, $h_0 = \frac{y_0}{v_0} \in \Delta$ are given.

The transaction costs are now given by a constant $\gamma \in]0, 1[$ denoting the fraction of the transaction volume that has to be paid for every transaction. These transaction fees correspond to the proportional part of the transaction costs in Model A. Since without fixed transaction costs continuous trading is allowed, we no longer consider only impulse control strategies as trading strategies but càdlàg policies.

Definition 6.1. An *investment policy* is a pair (L, M) of nonnegative, nondecreasing and càdlàg processes $L = (L_t)_{t \geq 0}$, $M = (M_t)_{t \geq 0}$. Here, the cumulative amount of money transferred from bond to stock up to the time t is denoted by L_t , whereas M_t is the cumulative amount of money that stocks are sold for up to time t .

For an investment policy (L, M) the portfolio becomes

$$\begin{aligned} X_t &= x_0 + \int_0^t r X_{s-} ds + (1 - \gamma)M_t - (1 + \gamma)L_t, \\ Y_t &= y_0 + \int_0^t \mu Y_{s-} ds + \int_0^t \sigma Y_{s-} dW_s - M_t + L_t, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} X_0 &= x_0 + (1 - \gamma)M_0 - (1 + \gamma)L_0, \\ Y_0 &= y_0 - M_0 + L_0. \end{aligned} \tag{6.2}$$

We therefore call an investment policy (L, M) *admissible for* (v_0, h_0) if there exist unique processes $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$ that satisfy (6.1), $X_t, Y_t \geq 0$ and $V_t = X_t + Y_t > 0$ for all $t \geq 0$. To clarify the dependence we will write $X^{(L, M)}$, $Y^{(L, M)}$ and $V^{(L, M)}$ if needed.

Furthermore, we denote the set of all admissible investment policies as

$$\mathcal{A}_{v_0, h_0} := \{(L, M) : (L, M) \text{ is an admissible investment policy for } (v_0, h_0)\}. \tag{6.3}$$

As for Model A the objective is to find an optimal admissible policy (L^*, M^*) that maximizes the expected growth rate

$$\rho_{v_0, h_0} := \sup_{(L, M) \in \mathcal{A}_{v_0, h_0}} J_{v_0, h_0}(L, M) := \sup_{(L, M) \in \mathcal{A}_{v_0, h_0}} \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log(V_T^{(L, M)}) \right). \tag{6.4}$$

- Remark 6.2.** 1) Like in the Lemmas 1.24 and 1.28 the optimal growth rate ρ_{v_0, h_0} is independent of the starting values v_0, h_0 . Nevertheless we will maintain the notation.
- 2) The definition of an admissible investment policy is slightly different from that in [AST01], where only $V_t \geq 0$ is required. But we cannot trade anymore after $V_t = 0$ and V_s stays at 0 for all $s \geq t$. Therefore $P(V_t = 0) > 0$ implies $J_{v_0, h_0}(L, M) = -\infty$ and we can exclude such strategies.

In [TKA88] the inverse $\frac{1}{h}$ of the risky fraction process is used. But since our aim is the approximation via Model A, we need to obtain a representation via the risky fraction process as in Chapter 1 and use the Hamilton-Jacobi-Bellman approach similar to the d -dimensional version in [AST01]. Therefore the results from [TKA88] are not applicable here. We also cannot use the achievements from [AST01] directly, since the value process and the risky fraction process considered there are adjusted to liquidation costs.

The corresponding representation of $\log V$ in the following lemma can also be found in [AST01].

Lemma 6.3. *Let $(L, M) \in \mathcal{A}_{v_0, h_0}$ and X, Y be as in Definition 6.1 and f be as in Lemma 1.12. Let further $L_t^c := L_t - \sum_{s \leq t} \Delta L_s$ and $M_t^c := M_t - \sum_{s \leq t} \Delta M_s$ denote the continuous parts of L and M . Then the processes $V = X + Y$ and $h := \frac{Y}{V}$ satisfy*

$$\log V_t = \log v_0 + r \cdot t - \int_0^t \frac{\gamma}{V_{s-}} dL_s^c - \int_0^t \frac{\gamma}{V_{s-}} dM_s^c + \int_0^t f(h_{s-}) ds + \int_0^t \sigma h_{s-} dW_s + \sum_{s \leq t} \Delta \log V_s$$

and

$$\begin{aligned} h_t &= h_0 + \int_0^t \sigma(1 - h_{s-}) h_{s-} dW_s + \int_0^t h_{s-}(1 - h_{s-})(\mu - r - \sigma^2 h_{s-}) ds \\ &\quad + \int_0^t \frac{1}{V_{s-}} (1 + h_{s-} \gamma) dL_s^c - \int_0^t \frac{1}{V_{s-}} (1 - h_{s-} \gamma) dM_s^c + \sum_{s \leq t} \Delta h_s. \end{aligned}$$

Proof. Itô's formula for càdlàg semimartingales (cf. [JS03], Theorem I.4.57) yields

$$\log V_t = \log v_0 + \int_0^t \frac{1}{V_{s-}} dV_s - \frac{1}{2} \int_0^t \frac{1}{V_{s-}^2} d[V^c, V^c]_s + \sum_{s \leq t} \left(\Delta \log V_s - \frac{1}{V_{s-}} \Delta V_s \right)$$

and

$$\begin{aligned} h_t &= h_0 + \int_0^t \frac{1}{V_{s-}} dY_s - \int_0^t \frac{Y_{s-}}{V_{s-}^2} dV_s - \int_0^t \frac{1}{V_{s-}^2} d[Y^c, V^c]_s + \int_0^t \frac{Y_{s-}}{V_{s-}^3} d[V^c, V^c]_s \\ &\quad + \sum_{s \leq t} \left(\Delta h_s - \frac{\Delta Y_s}{V_{s-}} + \frac{Y_{s-} \Delta V_s}{V_{s-}^2} \right). \end{aligned}$$

The assertion then follows from

$$dV_t = (rX_{t-} + \mu Y_{t-})dt + \sigma Y_{t-}dW_t - \gamma dL_t - \gamma dM_t$$

together with

$$[V^c, V^c]_t = [Y^c, Y^c]_t = \int_0^t \sigma^2 Y_{s-}^2 ds, \quad \Delta Y_t = \Delta L_t - \Delta M_t \quad \text{and} \quad \Delta V_t = -\gamma(\Delta L_t + \Delta M_t).$$

□

Lemma 6.4. *Let L, M and h be as in Lemma 6.3. Then for every $\varphi \in C^2([0, 1], \mathbb{R})$ we have*

$$\begin{aligned} \varphi(h_t) &= \varphi(h_0) + \int_0^t \varphi'(h_{s-})\sigma(1-h_{s-})h_{s-}dW_s + \int_0^t D\varphi(h_{s-})ds \\ &\quad + \int_0^t \varphi'(h_{s-})(1+h_{s-}\gamma)\frac{1}{V_{s-}}dL_s^c - \int_0^t \varphi'(h_{s-})(1-h_{s-}\gamma)\frac{1}{V_{s-}}dM_s^c + \sum_{s \leq t} \Delta\varphi(h_s), \end{aligned}$$

where

$$D := x(1-x)(\mu - r - \sigma^2 x)\frac{d}{dx} + \frac{1}{2}\sigma^2 x^2(1-x)^2\frac{d^2}{dx^2} \quad (6.5)$$

is the generator of the diffusion

$$\tilde{h}_t = h_0 + \int_0^t \sigma(1-\tilde{h}_s)\tilde{h}_s dW_s + \int_0^t \tilde{h}_s(1-\tilde{h}_s)(\mu - r - \sigma^2 \tilde{h}_s)ds$$

in the uncontrolled case $L = M = 0$.

Proof. We can apply Itô's formula for càdlàg semimartingales and get

$$\varphi(h_t) = \varphi(h_0) + \int_0^t \varphi'(h_{s-})dh_s + \frac{1}{2} \int_0^t \varphi''(h_{s-})d[h^c, h^c]_s + \sum_{s \leq t} (\Delta\varphi(h_s) - \varphi'(h_{s-})\Delta h_s).$$

Now Lemma 6.3 yields

$$[h^c, h^c]_t = \int_0^t \sigma^2(1-h_{s-})^2 h_{s-}^2 ds$$

and then the assertion. □

Now we want to give an upper bound for the optimal expected growth rate from (6.4), following the Hamilton-Jacobi-Bellman approach. The HJB-equation

$$\max \left\{ Dv(x) + f(x) - l, v'(x) - \frac{\gamma}{1+x\gamma}, -v'(x) - \frac{\gamma}{1-x\gamma} \right\} = 0$$

in the following lemma, which is the first part of the verification theorem, is already mentioned in Remark 9.2 of [IS06]. It is also an adaption of the HJB-equations from [Sul97] or [AST01] to our situation.

Lemma 6.5. *Let f be as in Lemma 1.12 and let D denote the generator from (6.5). If there exists some $v \in C^2([0, 1], \mathbb{R})$ and $l \in \mathbb{R}$ such that for all $x \in [0, 1]$*

$$(i) \quad Dv(x) \leq -f(x) + l,$$

$$(ii) \quad v'(x) \leq \frac{\gamma}{1+x\gamma},$$

$$(iii) \quad v'(x) \geq -\frac{\gamma}{1-x\gamma},$$

then the optimal growth rate ρ_{v_0, h_0} from (6.4) satisfies $\rho_{v_0, h_0} \leq r + l$.

Proof. For every policy $(L, M) \in \mathcal{A}_{v_0, h_0}$ we have by Lemma 6.3

$$\begin{aligned} & J_{v_0, h_0}(L, M) - r \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log(V_T^{(L, M)}) \right) - r \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{\gamma}{V_{s-}} dL_s^c - \int_0^T \frac{\gamma}{V_{s-}} dM_s^c + \int_0^T f(h_{s-}) ds + \int_0^T \sigma h_{s-} dW_s + \sum_{s \leq T} \Delta \log V_s \right) \\ &= l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{\gamma}{V_{s-}} dL_s^c - \int_0^T \frac{\gamma}{V_{s-}} dM_s^c + \int_0^T (f(h_{s-}) - l) ds + \sum_{s \leq T} \Delta \log V_s \right), \end{aligned}$$

since $(\sigma h_{t-})_{t \geq 0}$ is bounded. It remains to show

$$\liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{\gamma}{V_{s-}} dL_s^c - \int_0^T \frac{\gamma}{V_{s-}} dM_s^c + \int_0^T (f(h_{s-}) - l) ds + \sum_{s \leq T} \Delta \log V_s \right) \leq 0.$$

By applying Lemma 6.4 to v we get

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{\gamma}{V_{s-}} dL_s^c - \int_0^T \frac{\gamma}{V_{s-}} dM_s^c + \int_0^T (f(h_{s-}) - l) ds + \sum_{s \leq T} \Delta \log V_s \right) \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (v'(h_{s-})(1 + h_{s-}\gamma) - \gamma) \frac{1}{V_{s-}} dL_s^c - \int_0^T (v'(h_{s-})(1 - h_{s-}\gamma) + \gamma) \frac{1}{V_{s-}} dM_s^c \right. \\ & \quad + \int_0^T (Dv(h_{s-}) + f(h_{s-}) - l) ds + \sum_{s \leq T} (\Delta \log V_s + \Delta v(h_s)) + v(h_0) - v(h_T) \\ & \quad \left. + \int_0^T v'(h_{s-}) \sigma (1 - h_{s-}) h_{s-} dW_s \right) \end{aligned}$$

$$\begin{aligned}
&= \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (v'(h_{s-})(1 + h_{s-}\gamma) - \gamma) \frac{1}{V_{s-}} dL_s^c - \int_0^T (v'(h_{s-})(1 - h_{s-}\gamma) + \gamma) \frac{1}{V_{s-}} dM_s^c \right. \\
&\quad \left. + \int_0^T (Dv(h_{s-}) + f(h_{s-}) - l) ds + \sum_{s \leq T} (\Delta \log V_s + \Delta v(h_s)) \right) \\
&\leq \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\sum_{s \leq T} (\Delta \log V_s + \Delta v(h_s)) \right),
\end{aligned}$$

by the assumption on v .

Now we take a closer look at $\Delta \log V_t$ and $\Delta v(h_t)$ by treating the cases $\Delta h_t \geq 0$ and $\Delta h_t < 0$ separately. We note that it holds $\Delta V_t = -\gamma(\Delta L_t + \Delta M_t)$ and $\Delta Y_t = \Delta L_t - \Delta M_t$.

a) In the case $\Delta h_t \geq 0$ we have

$$V_t = V_{t-} - \gamma(\Delta L_t + \Delta M_t) \leq V_{t-} - \gamma(\Delta L_t - \Delta M_t) = V_{t-} - \gamma(Y_t - Y_{t-}) = V_{t-} - \gamma(h_t V_t - h_{t-} V_{t-}).$$

Solving for V_t this yields

$$V_t \leq V_{t-} \left(\frac{1 + \gamma h_{t-}}{1 + \gamma h_t} \right) \text{ and hence } \Delta \log V_t = \log \left(\frac{V_t}{V_{t-}} \right) \leq \log \left(\frac{1 + \gamma h_{t-}}{1 + \gamma h_t} \right),$$

which then implies

$$\Delta \log V_t + \Delta v(h_t) \leq v(h_t) - v(h_{t-}) + \log \left(\frac{1 + \gamma h_{t-}}{1 + \gamma h_t} \right) \leq 0,$$

since for a fixed $y \in [0, 1[$ the function

$$g : [y, 1] \rightarrow \mathbb{R}, x \mapsto v(x) - v(y) + \log \left(\frac{1 + \gamma y}{1 + \gamma x} \right)$$

is decreasing by the assumption (ii) and therefore nonpositive.

b) In the case $\Delta h_t < 0$ we conclude

$$V_t = V_{t-} - \gamma(\Delta L_t + \Delta M_t) \leq V_{t-} + \gamma(\Delta L_t - \Delta M_t) = V_{t-} + \gamma \Delta Y_t = V_{t-} + \gamma(h_t V_t - h_{t-} V_{t-}),$$

and thus solving for V_t then implies

$$V_t \leq V_{t-} \left(\frac{1 - \gamma h_{t-}}{1 - \gamma h_t} \right) \text{ and hence } \Delta \log V_t = \log \left(\frac{V_t}{V_{t-}} \right) \leq \log \left(\frac{1 - \gamma h_{t-}}{1 - \gamma h_t} \right).$$

Now for a fixed $y \in]0, 1]$ the function

$$\tilde{g} : [0, y] \rightarrow \mathbb{R}, x \mapsto v(x) - v(y) + \log \left(\frac{1 - \gamma y}{1 - \gamma x} \right)$$

satisfies

$$\tilde{g}'(x) = v'(x) + \frac{\gamma}{1-\gamma x} \stackrel{(iii)}{\geq} 0$$

and is therefore nonpositive, implying

$$\Delta \log V_t + \Delta v(h_t) \leq v(h_t) - v(h_{t-}) + \log \left(\frac{1-\gamma h_{t-}}{1-\gamma h_t} \right) \leq 0.$$

□

Now we want to introduce a class of investment policies that is later shown to contain an optimal policy. The notion *control limit policy* in the following definition is taken from [TKA88], where the inverse $\frac{1}{h}$ of the risky fraction process was used.

Definition 6.6. A continuous investment policy $(L, M) \in \mathcal{A}_{v_0, h_0}$ with $L_0 = M_0 = 0$ is a *control limit policy* for the limits $A < B \in]0, 1[$ if the corresponding risky fraction process h satisfies

$$L_t = \int_0^t \mathbb{1}_{\{h_s=A\}} dL_s, \quad M_t = \int_0^t \mathbb{1}_{\{h_s=B\}} dM_s, \quad h_t \in [A, B] \text{ for all } t \geq 0. \quad (6.6)$$

The existence and uniqueness of control limit policies was proved in Theorem 9.2 in [SS94].

Finally, we can establish the second part of our verification theorem.

Theorem 6.7. *Let v, l, f and D be as in Lemma 6.5. Let further $A < B \in]0, 1[$ and $h_0 \in [A, B]$. Suppose v additionally satisfies*

- (i) $Dv(x) = -f(x) + l$ for all $x \in [A, B]$,
- (ii) $v'(x) = \frac{\gamma}{1+x\gamma}$ for all $x \in [0, A]$,
- (iii) $v'(x) = -\frac{\gamma}{1-x\gamma}$ for all $x \in [B, 1]$.

Then the control limit policy (L, M) for the limits A, B is optimal.

Proof. Since L, M are continuous, we can calculate as in the proof of Lemma 6.5

$$\begin{aligned} J_{v_0, h_0}(L, M) &= r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T (v'(h_s)(1 + h_s\gamma) - \gamma) \frac{1}{V_s} dL_s \right. \\ &\quad \left. - \int_0^T (v'(h_s)(1 - h_s\gamma) + \gamma) \frac{1}{V_s} dM_s + \int_0^T (Dv(h_s) + f(h_s) - l) ds \right). \end{aligned}$$

Now (6.6) together with (ii) and (iii) yields

$$\int_0^T (v'(h_s)(1 + h_s\gamma) - \gamma) \frac{1}{V_s} dL_s = 0 \quad \text{and} \quad \int_0^T (v'(h_s)(1 - h_s\gamma) + \gamma) \frac{1}{V_s} dM_s = 0,$$

whereas (i) implies

$$\int_0^T (Dv(h_s) + f(h_s) - l) ds = 0.$$

Therefore $J_{v_0, h_0}(L, M) = r + l = \rho_{v_0, h_0}$ follows. □

Now it remains to prove the existence and uniqueness of constants $A < B \in]0, 1[$ such that the control limit policy for the limits A, B is optimal. Their existence will be established in Proposition 6.16 of the next section as the limit of the boundaries $(a_n, \alpha_n, \beta_n, b_n)$ of the constant boundary strategies in Model A for fixed costs $\delta_n \rightarrow 0$. In the rest of this section we present some results, which are needed for the uniqueness of A, B , that will be proved in Theorem 6.17 by also using the achievements of the next section.

We begin with a representation of the expected growth rate similar to (1.46). We restrict ourselves to the continuous case, which is needed later.

Lemma 6.8. *Let $(L, M) \in \mathcal{A}_{v_0, h_0}$ a continuous investment policy and h be the corresponding risky fraction process. Then the process $h^0 := 1 - h$ satisfies*

$$J_{v_0, h_0}(L, M) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\log \left(\frac{h_0^0}{h_T^0} \right) - \int_0^T \frac{1 + \gamma}{1 - h_s} \frac{1}{V_s} dL_s + \int_0^T \frac{1 - \gamma}{1 - h_s} \frac{1}{V_s} dM_s \right) \quad (6.7)$$

and we get in case of $\limsup_{T \rightarrow \infty} \frac{1}{T} E(\log(h_T^0)) = 0$

$$J_{v_0, h_0}(L, M) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{1 + \gamma}{1 - h_s} \frac{1}{V_s} dL_s + \int_0^T \frac{1 - \gamma}{1 - h_s} \frac{1}{V_s} dM_s \right). \quad (6.8)$$

Proof. Since L, M are continuous, Lemma 6.3 yields for the value process V and $t \geq 0$

$$\log V_t = \log v_0 + r \cdot t - \int_0^t \frac{\gamma}{V_s} dL_s - \int_0^t \frac{\gamma}{V_s} dM_s + \int_0^t f(h_s) ds + \int_0^t \sigma h_s dW_s. \quad (6.9)$$

Using Itô's formula we further get

$$\log h_t^0 = \log h_0^0 - \int_0^t \sigma h_s dW_s - \int_0^t h_s \left(\mu - r - \frac{\sigma^2}{2} h_s \right) ds - \int_0^t \frac{1 + h_s \gamma}{1 - h_s} \frac{1}{V_s} dL_s + \int_0^t \frac{1 - h_s \gamma}{1 - h_s} \frac{1}{V_s} dM_s$$

and by plugging it into (6.9) we directly get (6.7), from which (6.8) is evident. □

We also establish a result similar to Lemma 1.22 yielding a representation for the transformed risky fraction process.

Lemma 6.9. *Let $A < B \in]0, 1[$, $h_0 \in [A, B]$ and $(L, M) \in \mathcal{A}_{v_0, h_0}$ be the control limit policy for the limits A, B and h the corresponding risky fraction process. Then the process $y := \psi(h) = \log \frac{h}{1-h}$, where ψ is the transformation function from (1.75), satisfies*

$$y_t = y_0 + \left(\mu - r - \frac{\sigma^2}{2} \right) \cdot t + \sigma W_t + \frac{1 + \gamma A}{A(1-A)} \int_0^t \frac{1}{V_s} dL_s - \frac{1 - \gamma B}{B(1-B)} \int_0^t \frac{1}{V_s} dM_s. \quad (6.10)$$

Proof. We use Itô's formula on $y = \log \frac{h}{1-h}$ together with the representation for h in Lemma 6.3 and get for $t \geq 0$

$$\begin{aligned} y_t &= \log y_0 + \int_0^t \frac{1}{h_s(1-h_s)} dh_s - \frac{1}{2} \int_0^t \frac{1-2h_s}{h_s^2(1-h_s^2)} d[h]_s \\ &= \log y_0 + \int_0^t \left(\mu - r - \frac{\sigma^2}{2} \right) ds + \sigma W_t + \int_0^t \frac{1 + \gamma h_s}{h_s(1-h_s)} \frac{1}{V_s} dL_s - \int_0^t \frac{1 - \gamma h_s}{h_s(1-h_s)} \frac{1}{V_s} dM_s, \end{aligned}$$

since h, L, M are continuous. Now (6.10) is a consequence of (6.6). \square

The following theorem gives an explicit representation for reflected processes as a solution to the so-called *Skorokhod problem*.

Theorem 6.10. *For every $\psi \in C([0, \infty), \mathbb{R})$ there exists a unique nondecreasing $\eta(\psi) \in C([0, \infty), \mathbb{R})$ such that*

$$(i) \quad \psi + \eta(\psi) \geq 0,$$

$$(ii) \quad \eta(\psi)(0) = 0 \text{ and}$$

$$\int_0^\infty \mathbb{1}_{\{\psi(s) + \eta(\psi)(s) > 0\}} d\eta(\psi)(s) = 0.$$

The function $\Psi : C([0, \infty), \mathbb{R}) \rightarrow C([0, \infty), \mathbb{R})$, $\psi \mapsto \psi + \eta(\psi)$ is referred to as Skorokhod map or one-sided reflection map and the function η has the explicit representation $\eta(\psi)(t) = \sup_{s \in [0, t]} (-\psi(s))^+$.

Furthermore, for every $a > 0$ and every $\psi \in C([0, \infty), \mathbb{R})$ there exists a unique $\eta_{0,a}(\psi) \in C([0, \infty), \mathbb{R})$ of bounded variation such that

$$(1) \quad \psi + \eta_{0,a}(\psi) \in [0, a],$$

(2) $\eta_{0,a}(\psi)(0) = 0$ and $\eta_{0,a}(\psi)$ has the decomposition $\eta_{0,a}(\psi) = \eta_0 - \eta_a$ as the difference of nondecreasing continuous functions η_0, η_a satisfying

$$\int_0^\infty \mathbb{1}_{\{\psi(s) + \eta_{0,a}(\psi)(s) > 0\}} d\eta_0(s) = 0 \quad \text{and} \quad \int_0^\infty \mathbb{1}_{\{\psi(s) + \eta_{0,a}(\psi)(s) < a\}} d\eta_a(s) = 0.$$

The function $\Psi_{0,a} : C([0, \infty), \mathbb{R}) \rightarrow C([0, \infty), \mathbb{R})$, $\psi \mapsto \psi + \eta_{0,a}(\psi)$ is called Skorokhod map on $[0, a]$ or two-sided reflection map and has the explicit representation

$$\Psi_{0,a} = \Lambda_a \circ \Psi,$$

where $\Lambda_a : C([0, \infty), \mathbb{R}) \rightarrow C([0, \infty), \mathbb{R})$ is defined by

$$\Lambda_a(\psi)(t) := \psi(t) - \sup_{s \in [0, t]} \left\{ \min \left\{ (\psi(s) - a)^+, \inf_{u \in [s, t]} \psi(u) \right\} \right\}.$$

Proof. Cf. Proposition 1.3 and Theorem 1.4 in [KLR07]. □

Remark 6.11. The process in (6.10) is a Brownian motion with drift on $[\psi(A), \psi(B)]$, where the paths are reflected in the sense of Theorem 6.10 at the boundaries $\psi(A), \psi(B)$ due to Definition 6.6. Therefore the nondecreasing processes from Theorem 6.10 (2) are

$$\eta_A = \frac{1 + \gamma A}{A(1 - A)} \int_0^t \frac{1}{V_s} dL_s, \quad \eta_B = \frac{1 - \gamma B}{B(1 - B)} \int_0^t \frac{1}{V_s} dM_s.$$

As the last result of this section we prove a lower bound for the growth of these processes. It is essentially used in the next section to establish the uniqueness of the optimal A, B .

Lemma 6.12. *Let $A < B \in]0, 1[$ and $h_0 \in [A, B]$. Suppose the control limit policy (L, M) for the limits A, B is optimal with the optimal value $\rho_{v_0, h_0} > r$. Then the corresponding value process V satisfies*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{V_s} dL_s \right) > 0, \quad \limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{V_s} dM_s \right) > 0.$$

Proof. By Lemma 6.8 we have

$$\rho_{v_0, h_0} = J_{v_0, h_0}(L, M) = r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{1 + \gamma}{1 - h_s} \frac{1}{V_s} dL_s + \int_0^T \frac{1 - \gamma}{1 - h_s} \frac{1}{V_s} dM_s \right)$$

and thus for $l := \rho_{v_0, h_0} - r$

$$\begin{aligned} 0 < l &= \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{1 + \gamma}{1 - h_s} \frac{1}{V_s} dL_s + \int_0^T \frac{1 - \gamma}{1 - h_s} \frac{1}{V_s} dM_s \right) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1 - \gamma}{1 - h_s} \frac{1}{V_s} dM_s \right) \\ &= \frac{1 - \gamma}{1 - B} \cdot \limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{V_s} dM_s \right). \end{aligned}$$

Now we assume

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{V_s} dL_s \right) = 0 \quad (6.11)$$

and consider new limits $\tilde{A} < \tilde{B} \in]0, 1[$ and a new starting value $\tilde{h}_0 \in [\tilde{A}, \tilde{B}]$ such that after the transformation from $]0, 1[$ to \mathbb{R} via $\psi(x) := \log \frac{x}{1-x}$ the distances remain the same, i. e.

$$\psi(B) - \psi(A) = \psi(\tilde{B}) - \psi(\tilde{A}), \quad \psi(B) - \psi(h_0) = \psi(\tilde{B}) - \psi(\tilde{h}_0) \quad (6.12)$$

and $\tilde{B} > B$. Furthermore, we denote by (\tilde{L}, \tilde{M}) the control limit policy for the new limits \tilde{A}, \tilde{B} . By Lemma 6.9 the processes $y := \psi(h)$, $\tilde{y} := \psi(\tilde{h})$, where h, \tilde{h} are the risky fraction processes corresponding to $(L, M), (\tilde{L}, \tilde{M})$ respectively, satisfy

$$y_t = y_0 + \left(\mu - r - \frac{\sigma^2}{2} \right) \cdot t + \sigma W_t + \frac{1 + \gamma A}{A(1 - A)} \int_0^t \frac{1}{V_s} dL_s - \frac{1 - \gamma B}{B(1 - B)} \int_0^t \frac{1}{V_s} dM_s, \quad (6.13)$$

$$\tilde{y}_t = \tilde{y}_0 + \left(\mu - r - \frac{\sigma^2}{2} \right) \cdot t + \sigma W_t + \frac{1 + \gamma \tilde{A}}{\tilde{A}(1 - \tilde{A})} \int_0^t \frac{1}{\tilde{V}_s} d\tilde{L}_s - \frac{1 - \gamma \tilde{B}}{\tilde{B}(1 - \tilde{B})} \int_0^t \frac{1}{\tilde{V}_s} d\tilde{M}_s. \quad (6.14)$$

These are both instantaneous reflections of the same diffusion with different boundaries and different starting values. But since the processes involved in an instantaneous reflection are unique (cf. [GS72], p. 185 or also Theorem 6.10 above), (6.12) yields

$$\frac{1 + \gamma A}{A(1 - A)} \int_0^t \frac{1}{V_s} dL_s = \frac{1 + \gamma \tilde{A}}{\tilde{A}(1 - \tilde{A})} \int_0^t \frac{1}{\tilde{V}_s} d\tilde{L}_s, \quad \frac{1 - \gamma B}{B(1 - B)} \int_0^t \frac{1}{V_s} dM_s = \frac{1 - \gamma \tilde{B}}{\tilde{B}(1 - \tilde{B})} \int_0^t \frac{1}{\tilde{V}_s} d\tilde{M}_s.$$

Thus (6.11) implies

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{\tilde{V}_s} d\tilde{L}_s \right) = 0$$

and we therefore get by using Lemma 6.8 the contradiction

$$\begin{aligned} J_{v_0, \tilde{h}_0}(\tilde{L}, \tilde{M}) &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \int_0^T \frac{1 + \gamma}{1 - \tilde{h}_s} \frac{1}{\tilde{V}_s} d\tilde{L}_s + \int_0^T \frac{1 - \gamma}{1 - \tilde{h}_s} \frac{1}{\tilde{V}_s} d\tilde{M}_s \right) \\ &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(- \frac{1 + \gamma}{1 - \tilde{A}} \int_0^T \frac{1}{\tilde{V}_s} d\tilde{L}_s + \frac{1 - \gamma}{1 - \tilde{B}} \int_0^T \frac{1}{\tilde{V}_s} d\tilde{M}_s \right) \\ &= r + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\frac{1 - \gamma}{1 - \tilde{B}} \int_0^T \frac{1}{V_s} dM_s \right) \frac{1 - \gamma B}{B} \frac{\tilde{B}}{1 - \gamma \tilde{B}} \\ &= r + l \cdot \frac{1 - \gamma B}{B} \frac{\tilde{B}}{1 - \gamma \tilde{B}} \\ &> r + l = \rho_{v_0, h_0}. \end{aligned}$$

□

6.2 Model A

6.2.1 Convergence of boundaries and optimality

In this subsection, we first establish the convergence described at the beginning of the chapter, i.e.

$$\lim_{\delta \rightarrow 0} a(\delta) = A = \lim_{\delta \rightarrow 0} \alpha(\delta) \quad \text{and} \quad \lim_{\delta \rightarrow 0} \beta(\delta) = B = \lim_{\delta \rightarrow 0} b(\delta)$$

to some constants $A, B \in]0, 1[$ in the case of Model A. This is achieved on \mathbb{R} via the coordinate transformation ψ . Then we prove the convergence of the corresponding optimal growth rates and that of the QVI-solutions $u(\delta)$ from Theorem 2.4 to some solution u to the HJB-equation from Theorem 6.7 with the above A, B . Using that, we finally obtain uniqueness and optimality of A, B .

Proposition 6.13. *Let $d := 1$ and $(\delta_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the fixed costs in Model A in the sense of (1.20) for a fixed common $\gamma > 0$ defining the proportional costs and satisfying $\gamma < 1 - \sup_{n \in \mathbb{N}} \delta_n$. Let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and suppose $\lim_{n \rightarrow \infty} \delta_n = 0$. Then the corresponding constants a_n, α_n, β_n and b_n given by Theorem 2.10 for every $n \in \mathbb{N}$ satisfy*

$$\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0.$$

Proof. Since $\lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0$ can be shown the same way, we restrict ourselves to proving only $\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0$ here. For that purpose we define for every $n \in \mathbb{N}$ the function

$$g_n : I_n := [\beta_n, b_n] \rightarrow \mathbb{R}, \quad x \mapsto u_n(x) - u_n(\beta_n) - \Gamma_n(x, \beta_n),$$

where u_n is the solution from Theorem 2.10 and Γ_n is the cost function from Lemma 2.8 with the corresponding constants γ, δ_n and the transformation function φ .

The proof is based on the following idea: The assumption $\inf_{n \in \mathbb{N}} |\beta_n - b_n| > 0$ leads to $\lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} g'_n = \lim_{n \rightarrow \infty} g''_n = 0$ uniformly on a common interval $I \subseteq]\beta_n, b_n[$, $n \in \mathbb{N}$. But since we have $Lu_n = -(\bar{f} - l_n)$ on I for every $n \in \mathbb{N}$, the limit cost function $v(x) = -\Gamma_0(x, y)$ with $\delta = 0$ has to satisfy $Lv = -(\bar{f} - l_0)$ on I for some fixed $y \in \mathbb{R}$, which is a contradiction.

a) Let $n \in \mathbb{N}$. What we show here, is that for all $x \in [\beta_n, b_n]$

$$0 \leq g_n(x) \leq \left| \log \left(\frac{1 - \delta_n - \gamma}{1 - \gamma} \right) \right|. \quad (6.15)$$

Since for every $x \in]\beta_n, b_n]$ the inequality

$$u_n(\beta_n) + \Gamma_n(x, \beta_n) < u_n(x) + \Gamma_n(x, x)$$

would imply

$$u_n(b_n) \stackrel{2.10(x)}{=} u_n(\beta_n) + \Gamma_n(b_n, \beta_n) \stackrel{(2.29)}{<} u_n(x) + \Gamma_n(b_n, x) \stackrel{2.10(viii)}{\leq} u_n(b_n)$$

and hence a contradiction, we get

$$u_n(x) \stackrel{2.10(viii)}{\geq} u_n(\beta_n) + \Gamma_n(x, \beta_n) \geq u_n(x) + \Gamma_n(x, x) = u_n(x) + \log \left(\frac{1 - \delta_n - \gamma\varphi(x)}{1 - \gamma\varphi(x)} \right).$$

But since $\Gamma_n(x, x)$ is strictly decreasing in x on \mathbb{R} , (6.15) follows.

b) Now let $\varepsilon > 0$ and suppose $|\beta_n - b_n| \geq 4\varepsilon$ for infinitely many $n \in \mathbb{N}$ and without loss of generality for all $n \in \mathbb{N}$.

b1) We first show that there is a constant $c = c(\mu, \sigma^2, r, \gamma)$ such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in [\beta_n, \beta_n + 2\varepsilon]} |g_n''(x)| \leq c. \quad (6.16)$$

By (2.29) we have for $n \in \mathbb{N}$ and $x \in]\beta_n, b_n[$

$$g_n'(x) = u_n'(x) + \frac{\gamma\varphi'(x)}{1 - \delta_n - \gamma\varphi(x)} \quad (6.17)$$

and hence

$$g_n''(x) = u_n''(x) + \frac{\gamma\varphi''(x)}{1 - \delta_n - \gamma\varphi(x)} + \left(\frac{\gamma\varphi'(x)}{1 - \delta_n - \gamma\varphi(x)} \right)^2, \quad (6.18)$$

where by the proof of Lemma 3.17 and taking derivatives

$$\varphi'(x) = \varphi(x) - \varphi^2(x) \text{ and } \varphi''(x) = \varphi'(x)(1 - 2\varphi(x)) = \varphi(x) - 3\varphi^2(x) + 2\varphi^3(x). \quad (6.19)$$

Since $\varphi(x) \in]0, 1[$ for all $x \in \mathbb{R}$ and $\gamma \in]0, 1 - \sup_{m \in \mathbb{N}} \delta_m[$, it remains by (6.18) and (6.19) to prove the boundedness of u_n'' .

Since $\mu - r - \frac{1}{2}\sigma^2 = 0$ together with (5.23) directly implies the boundedness of u'' , we only consider the case $\mu - r - \frac{1}{2}\sigma^2 \neq 0$ here. We follow the proof of Proposition 5.10, where in view of (5.23) it was enough to see the boundedness of u_n' in $n \in \mathbb{N}$ on $[\beta_n, \beta_n + 2\varepsilon]$.

By Theorem 2.10 u_n is strictly decreasing on $[\beta_n, b_n]$ and we can find an $x_n < \beta_n$, where u_n attains its maximum on $[a_n, b_n]$.

In view of (6.15), (6.17) and the mean value theorem applied to g_n , we further get for some $\xi \in]\beta_n + 2\varepsilon, \beta_n + 3\varepsilon[$

$$|u_n'(\xi)| \leq \frac{1}{\varepsilon} \left| \log \left(\frac{1 - \delta_n - \gamma}{1 - \gamma} \right) \right| + \frac{\gamma\varphi'(\xi)}{1 - \delta_n - \gamma\varphi(\xi)}. \quad (6.20)$$

Now in the case, where u'_n attains its minimum on $[x_n, \xi]$ in some $x \in]x_n, \xi[$, we have $u''_n(x) = 0$ and hence as in (5.24)

$$\sup_{y \in [\beta_n, \beta_n + 2\varepsilon]} |u'_n(y)| \leq |u'_n(x)| \leq \left| \frac{-\bar{f}(x) + l_n}{\mu - r - \frac{1}{2}\sigma^2} \right|.$$

If on the contrary the minimum is attained in ξ , we have by (6.20)

$$\sup_{y \in [\beta_n, \beta_n + 2\varepsilon]} |u'_n(y)| \leq \frac{1}{\varepsilon} \left| \log \left(\frac{1 - \delta_n - \gamma}{1 - \gamma} \right) \right| + \frac{\gamma\varphi'(\xi)}{1 - \delta_n - \gamma\varphi(\xi)}.$$

b2) Here, we show that there exist $\beta < b \in \mathbb{R}$ such that $I := [\beta, b] \subseteq]\beta_n, b_n[$ for infinitely many $n \in \mathbb{N}$.

Theorem 2.10 yields for the constant $\bar{l}_1 := \inf_{n \in \mathbb{N}} l_n$ the boundedness of the set $\{x \in \mathbb{R} : \bar{f}(x) \geq \bar{l}_1\}$ in \mathbb{R} and for every $n \in \mathbb{N}$ also

$$x_n \in \{x \in \mathbb{R} : \bar{f}(x) \geq \bar{l}_1\} =: [y_0, y_1].$$

Since for all $x, y \in \mathbb{R}$ we have $|u_n(x) - u_n(y)| \stackrel{2.10(vi)}{\leq} \sup_{n \in \mathbb{N}} \|\Gamma_n\|_\infty < \infty$, we can use Theorem 2.10 (v) in the same way as in (5.13) to get

$$y_2 := \sup_{n \in \mathbb{N}} |y_1 - b_n| < \infty$$

and thus $[\beta_n, b_n] \subseteq [y_0, y_1 + y_2]$ for all $n \in \mathbb{N}$. Hence we can find convergent subsequences $(\beta_{n_k})_{k \in \mathbb{N}}, (b_{n_k})_{k \in \mathbb{N}}$, which satisfy $|\beta_{n_k} - b_{n_k}| \geq 4\varepsilon$ from our previous assumption.

b3) Now we assume by b2) without loss of generality $I \subseteq]\beta_n, b_n[$ for all $n \in \mathbb{N}$. We then can use a) and b1) together with Lemma 5.2 to get

$$\lim_{n \rightarrow \infty} \|g'_n\|_{I, \infty} = 0. \quad (6.21)$$

We define $l_0 := \sup_{n \in \mathbb{N}} l_n$ and get from (6.21), (6.17) and (5.23)

$$\lim_{n \rightarrow \infty} u''_n(x) = \frac{2}{\sigma^2} (-\bar{f}(x) + l_0) + \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2}\sigma^2 \right) \frac{\gamma\varphi'(x)}{1 - \gamma\varphi(x)}, \quad (6.22)$$

uniformly in $x \in I$, and by (6.18) we therefore have uniformly in $x \in I$

$$\lim_{n \rightarrow \infty} g''_n(x) = \frac{2}{\sigma^2} (-\bar{f}(x) + l_0) + \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2}\sigma^2 \right) \frac{\gamma\varphi'(x)}{1 - \gamma\varphi(x)} + \frac{\gamma\varphi''(x)}{1 - \gamma\varphi(x)} + \left(\frac{\gamma\varphi'(x)}{1 - \gamma\varphi(x)} \right)^2.$$

This implies together with (6.21) for all $x \in I$

$$0 = \frac{2}{\sigma^2} (-\bar{f}(x) + l_0) + \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2}\sigma^2 \right) \frac{\gamma\varphi'(x)}{1 - \gamma\varphi(x)} + \frac{\gamma\varphi''(x)}{1 - \gamma\varphi(x)} + \left(\frac{\gamma\varphi'(x)}{1 - \gamma\varphi(x)} \right)^2. \quad (6.23)$$

b4) Here, we show that (6.23) is not possible.

We replace in (6.23) the functions φ' and φ'' by (6.19) and the function \bar{f} by its definition and convert the fraction, so that it has the common denominator $(1 - \gamma\varphi(x))^2$. (6.23) is then of the form

$$\frac{\sum_{i=0}^4 \lambda_i \varphi^i(x)}{(1 - \gamma\varphi(x))^2} = 0 \quad (6.24)$$

for some $\lambda_i \in \mathbb{R}$, $i \in \{0, 1, 2, 3, 4\}$, where $\lambda_0 = l_0$. Now since the denominator is positive, we consider only the numerator and convert (6.24) by using the definition of φ for all $x \in I$ to the form

$$\sum_{i=0}^4 \mu_i (e^x)^i = 0$$

for some $\mu_i \in \mathbb{R}$, $i \in \{0, 1, 2, 3, 4\}$, where $\mu_0 = \lambda_0 = l_0$. Therefore we get for all $x \in [e^\beta, e^b]$

$$\sum_{i=0}^4 \mu_i x^i = 0.$$

Finally, we deduce by taking derivatives $0 = \mu_0 = l_0 \geq \inf_{n \in \mathbb{N}} l_n > f(0) = 0$ and hence a contradiction.

□

Using the result above we can prove the convergence of the boundaries at least for some subsequence.

Proposition 6.14. *Let $d := 1$ and $(\delta_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the fixed costs in Model A in the sense of (1.20) for a fixed common $\gamma > 0$ defining the proportional costs and satisfying $\gamma < 1 - \sup_{n \in \mathbb{N}} \delta_n$. Let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and suppose $\lim_{n \rightarrow \infty} \delta_n = 0$. Then there exist constants $a_0 < x_0 < b_0 \in \mathbb{R}$ and a subsequence $(\delta_{n_k})_{k \in \mathbb{N}}$ such that the corresponding constants $x_{n_k}, a_{n_k}, \alpha_{n_k}, \beta_{n_k}$ and b_{n_k} given by Theorem 2.10 for every $k \in \mathbb{N}$ satisfy*

$$\lim_{k \rightarrow \infty} x_{n_k} = x_0, \quad \lim_{k \rightarrow \infty} a_{n_k} = a_0 = \lim_{k \rightarrow \infty} \alpha_{n_k}, \quad \lim_{k \rightarrow \infty} \beta_{n_k} = b_0 = \lim_{k \rightarrow \infty} b_{n_k}.$$

Proof. Theorem 2.10 yields a function u_n and a constant l_n such that

$$x_n \in \{y \in \mathbb{R} : \bar{f}(y) - l_n \geq 0\} \cap [\alpha_n, \beta_n]$$

and

$$\sup_{n \in \mathbb{N}} \|u_n\|_\infty \leq \sup_{n \in \mathbb{N}} \|\Gamma_n\|_\infty < \infty.$$

Therefore we can use Theorem 2.10 (v) in the same way as in (5.13) to get

$$\sup_{n \in \mathbb{N}} b_n < \infty, \quad \inf_{n \in \mathbb{N}} a_n > -\infty. \quad (6.25)$$

Now as in the proof of Proposition 6.13 we have

$$\sup_{n \in \mathbb{N}, x \in [\alpha_n, \beta_n]} |u'_n(x)| < \infty$$

and due to Theorem 2.10 (ix) as in (5.23) for all $x \in]a_n, b_n[$

$$u''_n(x) = \frac{2}{\sigma^2} (-\bar{f}(x) + l_n) - \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2} \sigma^2 \right) u'_n(x)$$

and therefore

$$\sup_{n \in \mathbb{N}, x \in [\alpha_n, \beta_n]} |u''_n(x)| < \infty. \quad (6.26)$$

Furthermore, since $u'_n(x_n) = 0$ and $u'_n(\beta_n) = \frac{-\gamma}{1-\gamma\beta_n}$ by Theorem 2.10, the mean value theorem together with (6.26) and (6.25) yields

$$\inf_{n \in \mathbb{N}} |x_n - \beta_n| > 0. \quad (6.27)$$

The corresponding argument implies

$$\inf_{n \in \mathbb{N}} |x_n - \alpha_n| > 0. \quad (6.28)$$

Finally, Proposition 6.13 implies

$$\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0 \text{ and } \lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0$$

and therefore the existence of the desired constants $a_0 < x_0 < b_0 \in \mathbb{R}$ and of the subsequence $(\delta_{n_k})_{k \in \mathbb{N}}$ follows. □

Remark 6.15. The convergence $\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0$ and $\lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0$ shown above is a convergence on \mathbb{R} for the transformed boundaries and is therefore a priori stronger than a convergence on $]0, 1[$ before the transformation. It is equivalent if these sequences stay away from the boundary, since the transformation function ψ is Lipschitz continuous on every compact subset of $]0, 1[$, which is an easy consequence of $\psi \in C^\infty(]0, 1[, \mathbb{R})$, and its inverse φ is also Lipschitz continuous by Lemma 3.17. That these sequences in fact stay away from the boundary, can be seen in (6.25).

The next proposition is crucial. It guarantees the existence of a solution u to the HJB-equation described in Theorem 6.7, which is the limit of the solutions u_n for vanishing fixed costs δ_n . It will also be used for the uniqueness and optimality of the constants a_0, b_0 of Proposition 6.14.

We use the results from Theorem 2.4 for the proof and therefore have to come back from \mathbb{R} to $[0, 1]$ via the transformation function φ from Section 1.4.

Proposition 6.16. *Let $d := 1$ and $(\delta_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the fixed costs in Model A in the sense of (1.20) for a fixed common $\gamma > 0$ defining the proportional costs and satisfying $\gamma < 1 - \sup_{n \in \mathbb{N}} \delta_n$. Let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and suppose $\lim_{n \rightarrow \infty} \delta_n = 0$. Let further $a_0 < b_0 \in \mathbb{R}$ and $(\delta_{n_k})_{k \in \mathbb{N}}$ be from Proposition 6.14 and u_{n_k}, ρ_{n_k} be the corresponding functions and optimal growth rates from Theorem 2.4. Then there exists a function $u \in C^2([0, 1], \mathbb{R})$ given by $u(x) = \lim_{k \rightarrow \infty} u_{n_k}(x)$ for all $x \in [0, 1]$ that satisfies the conditions of Theorem 6.7 for the constants $A := \varphi(a_0) < B := \varphi(b_0)$ and $l = \lim_{k \rightarrow \infty} \rho_{n_k} - r$.*

Proof. Let $l_n, a_n, \alpha_n, x_{0,n}, \beta_n$ and b_n denote the constants and u_n, Γ_n denote the functions given by Theorem 2.4 for every $n \in \mathbb{N}$ and δ_n . Without loss of generality we have

$$\lim_{n \rightarrow \infty} x_{0,n} = x_0, \quad \lim_{n \rightarrow \infty} a_n = A = \lim_{n \rightarrow \infty} \alpha_n, \quad \lim_{n \rightarrow \infty} \beta_n = B = \lim_{n \rightarrow \infty} b_n \quad (6.29)$$

for some $0 < A < x_0 < B < 1$ by Proposition 6.14. Since for decreasing δ_n the corresponding optimal growth rates $\rho_n = r + l_n$ are increasing, we define

$$l_0 := \lim_{n \rightarrow \infty} l_n = \sup_{n \rightarrow \infty} l_n.$$

a) Now we define the function u on $[0, 1]$ by

$$u(x) := \begin{cases} \Gamma_0(x, A), & x \leq A, \\ u(A) + \int_A^x g(y, x_0, l_0) dy, & A < x \leq B, \\ u(B) + \Gamma_0(x, B), & x > B, \end{cases} \quad (6.30)$$

where Γ_0 is the cost function from (2.11) for $\delta = 0$ and g is as in (2.16). By the definition of g and (6.29) we have

$$\lim_{n \rightarrow \infty} g(x, x_{0,n}, l_n) = g(x, x_0, l_0) \text{ uniformly on } [\varepsilon, 1 - \varepsilon]$$

for some small $\varepsilon > 0$ with $[A, B] \subseteq]\varepsilon, 1 - \varepsilon[$. Together with the definition of Γ_n, Γ_0 and (6.29) this then implies

$$\lim_{n \rightarrow \infty} u_n(x) = u(x) \text{ uniformly on } [0, 1]. \quad (6.31)$$

b) We now have to consider the first derivatives. We have by Theorem 2.4

$$u'_n(x) = \begin{cases} \frac{\gamma}{1 - \delta_n + \gamma x}, & x \leq a_n, \\ g(x, x_{0,n}, l_n), & a_n \leq x \leq b_n, \\ -\frac{\gamma}{1 - \delta_n - \gamma x}, & x \geq b_n, \end{cases}$$

since $u_n \in C^1([0, 1], \mathbb{R})$, and define the limit derivative by

$$\tilde{u}(x) := \begin{cases} \frac{\gamma}{1 + \gamma x}, & x \leq A, \\ g(x, x_0, l_0), & A < x \leq B, \\ -\frac{\gamma}{1 - \gamma x}, & x > B. \end{cases}$$

By (6.29) and the continuity of u'_n we get

$$\frac{\gamma}{1 + \gamma A} = \lim_{n \rightarrow \infty} u'_n(a_n) = g(A, x_0, l_0)$$

and also

$$-\frac{\gamma}{1 - \gamma B} = \lim_{n \rightarrow \infty} u'_n(b_n) = g(B, x_0, l_0).$$

Hence \tilde{u} is continuous on $[0, 1]$ and it follows

$$\lim_{n \rightarrow \infty} u'_n(x) = \tilde{u}(x) \text{ uniformly on } [0, 1]. \quad (6.32)$$

(6.31) together with (6.32) allows us to conclude $\tilde{u} = u'$.

c) Here, we want to use the notation $g'(x, y, l)$ instead of $\frac{\partial}{\partial x}g(x, y, l)$. Since $u' = \tilde{u}$ is clearly differentiable in $[0, 1] \setminus \{A, B\}$, we need to show the differentiability of u' in A, B and calculate u'' . For the second derivatives we have

$$u''_n(x) = \begin{cases} -\frac{\gamma^2}{(1 - \delta_n + \gamma x)^2}, & x < a_n, \\ g'(x, x_{0,n}, l_n), & a_n < x < b_n, \\ -\frac{\gamma^2}{(1 - \delta_n - \gamma x)^2}, & x > b_n, \end{cases}$$

where we have discontinuities in a_n, b_n due to (vii) and (viii) of Theorem 2.4. But at the same time (vii) and (viii) of Theorem 2.4 imply

$$-\frac{\gamma^2}{(1 - \gamma B)^2} \leq \lim_{n \rightarrow \infty} g'(b_n, x_{0,n}, l_n) = g'(B, x_0, l_0) = \lim_{n \rightarrow \infty} g'(\beta_n, x_{0,n}, l_n) \leq -\frac{\gamma^2}{(1 - \gamma B)^2}$$

and

$$-\frac{\gamma^2}{(1 + \gamma A)^2} \leq \lim_{n \rightarrow \infty} g'(a_n, x_{0,n}, l_n) = g'(A, x_0, l_0) = \lim_{n \rightarrow \infty} g'(\alpha_n, x_{0,n}, l_n) \leq -\frac{\gamma^2}{(1 + \gamma A)^2}.$$

Therefore the function

$$\tilde{u}(x) := \begin{cases} -\frac{\gamma^2}{(1 + \gamma x)^2}, & x \leq A, \\ g'(x, x_0, l_0), & A < x \leq B, \\ -\frac{\gamma^2}{(1 - \gamma x)^2}, & x > B, \end{cases}$$

is continuous. Furthermore, it holds $\tilde{u}(x) = u''(x)$ for every $x \in [0, 1] \setminus \{A, B\}$. But since u' has left-hand and right-hand derivatives in A, B , which are equal due to the continuity of \tilde{u} , u' is differentiable on $[0, 1]$ with derivative \tilde{u} and therefore $u \in C^2([0, 1], \mathbb{R})$.

For every $x \in [0, 1] \setminus \{A, B\}$ almost all $n \in \mathbb{N}$ satisfy $x \notin \{a_n, b_n\}$ and we therefore have

$$\lim_{n \rightarrow \infty} u''_n(x) = u''(x). \quad (6.33)$$

d) Now (iii) and (iv) of Theorem 2.4 together with (6.32) and (6.33) imply

$$Du(x) + f(x) - l \leq 0 \text{ for all } x \in [0, 1]$$

and $Du(x) + f(x) - l = 0$ for all $x \in]A, B[$.

$$Du(x) + f(x) - l = 0 \text{ for all } x \in [A, B]$$

then follows from the continuity of u' and u'' . It remains to show for all $x \in [0, 1]$

$$u'(x) \leq \frac{\gamma}{1+x\gamma}, \quad u'(x) \geq -\frac{\gamma}{1-x\gamma}.$$

We only show the latter, due to the analogous proof.

e) Here, we show $u'(y) \geq -\frac{\gamma}{1-\gamma y}$ for all $y \in [0, 1]$. By Theorem 2.4 we have

$$u(y) - u(x) + \Gamma_0(x, y) \leq 0 \text{ for all } x, y \in [0, 1]. \quad (6.34)$$

Furthermore, we have $\frac{\partial}{\partial y}\Gamma_0(x, y) = \frac{\gamma}{1-\gamma y}$ for all $x > y \in]0, 1[$. The assumption

$$u'(y_0) < -\frac{\gamma}{1-\gamma y_0} \text{ for some } y_0 \in]0, 1[$$

implies

$$u'(y) < \frac{-\gamma}{1-\gamma y} \text{ for all } y \in]y_0 - \varepsilon, y_0 + \varepsilon[\text{ for some suitable small } \varepsilon > 0$$

and therefore

$$u'(y) + \frac{\partial}{\partial y}\Gamma_0(y_0, y) < 0 \text{ for all } y \in]y_0 - \varepsilon, y_0[. \quad (6.35)$$

Now the function $h : [0, 1] \rightarrow \mathbb{R}, y \mapsto u(y) - u(y_0) + \Gamma_0(y_0, y)$ satisfies

$$h(y_0) = 0 \text{ and } h(y) \leq 0 \text{ for all } y \in [0, 1] \quad (6.36)$$

due to (6.34). But (6.35) implies that h is strictly decreasing on $]y_0 - \varepsilon, y_0]$, which contradicts (6.36). □

Now we are able to prove a result, which was already announced in the last section. To the best of our knowledge there is no uniqueness result for the constants A, B in the model from Section 6.1, not even for the similar models in [TKA88] and [AST01].

Theorem 6.17. *Let the Merton fraction $\hat{h} := \frac{\mu-r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$. Then there exist unique $A < B \in]0, 1[$ such that the control limit policy for the limits A, B is optimal.*

Proof. Proposition 6.16 guarantees the existence of some $A < B \in]0, 1[$ and a function u defined by (6.30) that satisfies the conditions of Theorem 6.7 for the constants A, B . Hence the control limit policy (L, M) for the limits A, B is optimal.

Now we take arbitrary $\tilde{A} < \tilde{B} \in]0, 1[$ such that the control limit policy (\tilde{L}, \tilde{M}) for the limits \tilde{A}, \tilde{B} is also optimal.

a) We first show $\tilde{A} \leq A$ and $\tilde{B} \geq B$ but we will omit the proof for $\tilde{B} \geq B$ due to similarity.

Let $l := l_0$ and $g(x) := g(x, x_0, l_0)$ denote the constant and function introduced in the proof of Proposition 6.16, yielding $\rho = r + l$ and $u'(x) = g(x)$ on $[A, B]$. Let f be as in Lemma 1.12 and let $D = x(1-x)(\mu - r - \sigma^2 x) \frac{d}{dx} + \frac{1}{2} \sigma^2 x^2 (1-x)^2 \frac{d^2}{dx^2}$ denote the generator from (6.5). We have

$$Du(x) = -f(x) + l \text{ on } [A, B], \quad g(A) = \frac{\gamma}{1 + A\gamma}, \quad g'(A) = -\frac{\gamma^2}{(1 + A\gamma)^2}. \quad (6.37)$$

What we need to show is

$$u'(x) < \frac{\gamma}{1 + \gamma x} \text{ for all } A < x \in]0, 1[. \quad (6.38)$$

We note that from Proposition 6.16 we already know

$$u'(x) \leq \frac{\gamma}{1 + \gamma x} \text{ for all } x \in [0, 1] \quad (6.39)$$

and $u'(x) < 0$ for all $x \geq B$, and so we have to consider u' and hence g on $[A, B]$. We further note that $\tilde{u}(x) := \int_0^x g(y, x_0, l) dy$ is a classical solution to $D\tilde{u}(x) = -f(x) + l$ on $]0, 1[$ (cf. [IS06]). Due to (6.39) every $x \in [A, B]$ with $u'(x) = \frac{\gamma}{1 + \gamma x}$ satisfies $u''(x) = -\frac{\gamma^2}{(1 + \gamma x)^2}$. Now (6.37) yields

$$u''(x) = \frac{1}{\sigma^2 x^2 (1-x)^2} (2l - 2(\mu - r)x + \sigma^2 x^2 - 2x(1-x)(\mu - r - \sigma^2 x)u'(x)).$$

To use the same argument as in [IS06], p. 932, we introduce the function

$$v(x) := \frac{1}{\sigma^2 x^2 (1-x)^2} \left(2l - 2(\mu - r)x + \sigma^2 x^2 - 2x(1-x)(\mu - r - \sigma^2 x) \left(\frac{\gamma}{1 + \gamma x} \right) \right) + \frac{\gamma^2}{(1 + \gamma x)^2},$$

which coincides with the derivative of $x \mapsto g(x) - \frac{\gamma}{1 + \gamma x}$ in every $x \in [A, B]$ that satisfies $g(x) = \frac{\gamma}{1 + \gamma x}$. We then calculate that

$$v(x) = \frac{p(x)}{\sigma^2 x^2 (1-x)^2 (1 + \gamma x)^2},$$

where p is a polynomial of degree two, since the terms of degree three and four all cancel. Therefore $v(x) = 0$ and hence $g(x) = \frac{\gamma}{1 + \gamma x}$ has at most two solutions on $[A, B]$. But a

maximum of $x \mapsto g(x) - \frac{\gamma}{1+\gamma x}$ on $[A, B]$ other than A would imply at least three roots of its derivative on $[A, B]$ and hence three roots of v and therefore (6.38) follows.

Using u we can calculate as in the proof of Theorem 6.7

$$\begin{aligned}
 J_{v_0, \tilde{h}_0}(\tilde{L}, \tilde{M}) &= r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \left(u'(\tilde{h}_s)(1 + \tilde{h}_s \gamma) - \gamma \right) \frac{1}{\tilde{V}_s} d\tilde{L}_s \right. \\
 &\quad \left. - \int_0^T \left(u'(\tilde{h}_s)(1 - \tilde{h}_s \gamma) + \gamma \right) \frac{1}{\tilde{V}_s} d\tilde{M}_s + \int_0^T \left(Du(\tilde{h}_s) + f(\tilde{h}_s) - l \right) ds \right) \\
 &\leq r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \left(u'(\tilde{h}_s)(1 + \tilde{h}_s \gamma) - \gamma \right) \frac{1}{\tilde{V}_s} d\tilde{L}_s \right) \\
 &= r + l + \left(u'(\tilde{A})(1 + \gamma \tilde{A}) - \gamma \right) \limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{\tilde{V}_s} d\tilde{L}_s \right).
 \end{aligned}$$

Therefore $\tilde{A} > A$ would imply $J_{v_0, \tilde{h}_0}(\tilde{L}, \tilde{M}) < r + l = \rho$ due to (6.38) and Lemma 6.12, which would pose a contradiction to the optimality of (\tilde{L}, \tilde{M}) .

b) It remains to prove $\tilde{A} \geq A$ and $\tilde{B} \leq B$. Here again, we restrict ourselves to showing $\tilde{A} \geq A$. Instead of u we want to use the function

$$\tilde{u}(x) := \begin{cases} u(A) + \int_A^x g(y, x_0, l) dy, & 0 < x \leq B, \\ u(B) + \Gamma_0(x, B), & x > B, \end{cases} \quad (6.40)$$

on $]0, 1[$, which is a classical solution to $D\tilde{u}(x) = -f(x) + l$ due to the proof of Proposition 6.16. Furthermore, since g is a limit of functions g_n that satisfy (ix) of Theorem 2.4, we have

$$\tilde{u}'(x) = g(x, x_0, l) \leq \frac{\gamma}{1 + \gamma x} \text{ for all } A \neq x \in]0, 1[. \quad (6.41)$$

Now the same argument as in a) applies here and so the inequality in (6.41) is strict. We use \tilde{u} instead of u to calculate as in the proof of Theorem 6.7

$$\begin{aligned}
 J_{v_0, \tilde{h}_0}(\tilde{L}, \tilde{M}) &= r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \left(\tilde{u}'(\tilde{h}_s)(1 + \tilde{h}_s \gamma) - \gamma \right) \frac{1}{\tilde{V}_s} d\tilde{L}_s \right. \\
 &\quad \left. - \int_0^T \left(\tilde{u}'(\tilde{h}_s)(1 - \tilde{h}_s \gamma) + \gamma \right) \frac{1}{\tilde{V}_s} d\tilde{M}_s + \int_0^T \left(D\tilde{u}(\tilde{h}_s) + f(\tilde{h}_s) - l \right) ds \right) \\
 &\leq r + l + \liminf_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \left(\tilde{u}'(\tilde{h}_s)(1 + \tilde{h}_s \gamma) - \gamma \right) \frac{1}{\tilde{V}_s} d\tilde{L}_s \right)
 \end{aligned}$$

$$= r + l + \left(\tilde{u}'(\tilde{A})(1 + \gamma\tilde{A}) - \gamma \right) \limsup_{T \rightarrow \infty} \frac{1}{T} E \left(\int_0^T \frac{1}{\tilde{V}_s} d\tilde{L}_s \right),$$

where we have used $\tilde{B} \geq B > A$ and hence $\tilde{u}'(\tilde{B})(1 - \tilde{B}\gamma) + \gamma = u'(\tilde{B})(1 - \tilde{B}\gamma) + \gamma \geq 0$. The strictness of the inequality (6.41) and Lemma 6.12 therefore yield $\tilde{A} = A$. \square

The following theorem is a main result of this Chapter.

Theorem 6.18. *Let $d := 1$ and $(\delta_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the fixed costs in Model A in the sense of (1.20) for a fixed common $\gamma > 0$ defining the proportional costs and satisfying $\gamma < 1 - \sup_{n \in \mathbb{N}} \delta_n$. Let the Merton fraction $\hat{h} := \frac{\mu - r}{\sigma^2}$ satisfy $\hat{h} \in]0, 1[$ and suppose $\lim_{n \rightarrow \infty} \delta_n = 0$. Then there exist constants $a_0 < b_0 \in \mathbb{R}$ and $\rho_0 > 0$ such that the corresponding constants a_n, α_n, β_n and b_n given by Theorem 2.10 for every $n \in \mathbb{N}$ together with the optimal growth rates ρ_n satisfy*

$$\lim_{n \rightarrow \infty} a_n = a_0 = \lim_{n \rightarrow \infty} \alpha_n, \quad \lim_{n \rightarrow \infty} \beta_n = b_0 = \lim_{n \rightarrow \infty} b_n, \quad \lim_{n \rightarrow \infty} \rho_n = \rho_0$$

and the control limit policy for the limits $A := \varphi(a_0), B := \varphi(b_0)$ is optimal for the optimization problem in the portfolio model with only proportional costs described in Section 6.1 with the optimal growth rate ρ_0 .

Proof. This is now a direct consequence of Proposition 6.14 and 6.16 together with Theorem 6.17. \square

For illustration, we take the scenario from [IS06], where $r := 0$, $\sigma := 0.4$ and $\mu := 0.096$. We calculate by the method described in Proposition 4.6 the dependence of the constants a, α, β, b in $]0, 1[$ and the optimal growth rate ρ on the fixed costs given by δ . We note that $\rho := 0.0288$ is the optimal growth rate without transaction costs. Figure 6.1 illustrates this for $\gamma = 0.003$ whereas Figure 6.2 shows the same dependence for $\gamma = 0.1$.

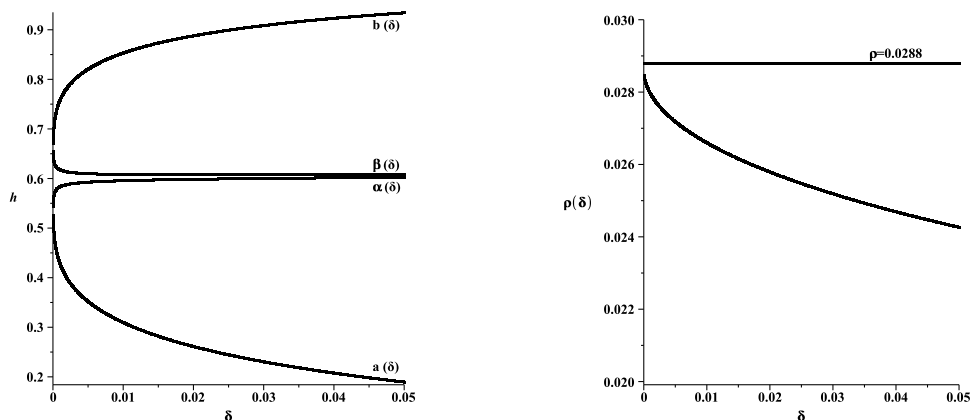
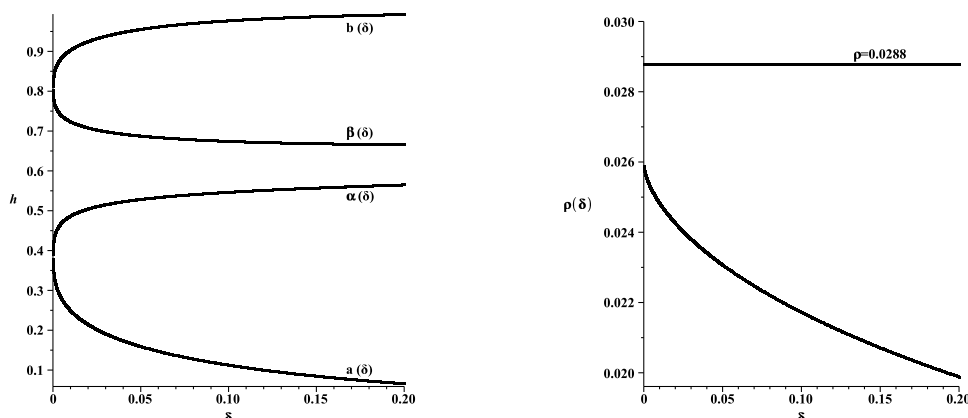


Figure 6.1: Dependence on δ in Model A for $\gamma = 0.003$

Figure 6.2: Dependence on δ in Model A for $\gamma = 0.1$

6.2.2 Weak convergence of processes

We have proved so far the convergence of the boundaries $a_n, \alpha_n, \beta_n, b_n$ to the limits a_0, b_0 and the convergence of the corresponding optimal growth rates ρ_n to ρ_0 . Our aim in this subsection is the weak convergence of the corresponding risky fraction processes induced by the proportional constant boundary strategies $\tilde{K}(a_n, \alpha_n, \beta_n, b_n)$, $n \in \mathbb{N}$, to the risky fraction process induced by the control limit policy for the limits a_0, b_0 , which is a diffusion with instantaneous reflection on $[a_0, b_0]$.

We begin with a characterization of the weak convergence via tightness and convergence in finite-dimensional distributions. For a brief review of the theory we refer to Appendix B or also to [EK86] and [JS03].

Theorem 6.19. *Let $(X^n)_{n \in \mathbb{N}_0}$ be a sequence of càdlàg processes defined on probability spaces $(\Omega_n, \mathcal{A}_n, P_n)$, $n \in \mathbb{N}_0$. If X^0 is continuous, then we have for every dense subset $D \subseteq \mathbb{R}_{\geq 0}$*

$$\left(X^n \xrightarrow{\mathcal{L}(D)} X^0 \text{ and } (X^n)_{n \in \mathbb{N}} \text{ tight} \right) \iff \left(X^n \xrightarrow{\mathcal{L}} X^0 \right).$$

Proof. This is a direct consequence of Theorem B.5 4. and Theorem B.11. □

Since the limiting risky fraction process is a reflected diffusion on $[a_0, b_0]$, it is also a continuous process. Instead of proving the tightness of the sequence of our processes we will prove their C -tightness, which ensures tightness of the sequence and the continuity of the limiting process. We will not use this continuity but it's not a greater task to prove.

Definition 6.20. A sequence $(X^n)_{n \in \mathbb{N}}$ of processes defined on probability spaces $(\Omega_n, \mathcal{A}_n, P_n)$, $n \in \mathbb{N}$, is called C -tight if it is tight and if for every $P \in \mathcal{P}(D_{\mathbb{R}}[0, \infty))$ and every subsequence $(\mathcal{L}(X^{n_k}))$ that weakly converges to P , we necessarily have $P(C([0, \infty), \mathbb{R})) = 1$.

Proposition 6.21. *For a sequence of processes $(X^n)_{n \in \mathbb{N}}$ defined on probability spaces $(\Omega_n, \mathcal{A}_n, P_n)$, $n \in \mathbb{N}$, there is equivalence between*

(i) $(X^n)_{n \in \mathbb{N}}$ is C -tight.

(ii) For all $N \in \mathbb{N}$, $\varepsilon > 0$ and $\eta > 0$ there are $n_0 \in \mathbb{N}$ and $\theta > 0$ such that for all $n \geq n_0$

$$P_n \left(\sup \{ |X_s^n - X_t^n| : s, t \in [0, N], |s - t| \leq \theta \} > \eta \right) \leq \varepsilon.$$

Proof. Cf. Proposition VI.3.26 in [JS03]. □

Using this characterization, we can prove the C -tightness of our sequence of transformed risky fraction processes on \mathbb{R} . But since this is only a consequence of the convergence of the boundaries, we formulate it therefore independently from the optimality of the involved constant boundary strategies.

Lemma 6.22. *Let $a_n < \alpha_n \leq \beta_n < b_n$, $n \in \mathbb{N}_0$, be a sequence of boundaries in \mathbb{R} and suppose there exist $a_0 < b_0 \in \mathbb{R}$ satisfying*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \alpha_n = a_0 < b_0 = \lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} b_n. \quad (6.42)$$

We denote by Y^n the controlled diffusion corresponding to the constant boundary strategy $\bar{K}(a_n, \alpha_n, \beta_n, b_n) = ((\tau_k^n, \xi_k^n))_{k \in \mathbb{N}_0}$ in the sense of Definition 2.9 with $\xi_0^n = 0$ and starting value $y_{0,n} \in]a_n, b_n[$ for every $n \in \mathbb{N}$. Then the sequence $(Y^n)_{n \in \mathbb{N}}$ is C -tight.

Proof. From the definition of a constant boundary strategy we have

$$Y_t^n = y_{0,n} + \left(\mu - r - \frac{\sigma^2}{2} \right) \cdot t + \sigma W_t + \sum_{k=0}^{\infty} \xi_k^n \mathbb{1}_{\{\tau_k^n \leq t\}},$$

where $\xi_0^n = 0$ and for every $k \in \mathbb{N}$

$$\xi_k^n = \begin{cases} \alpha_n - a_n, & Y_{\tau_k^n}^n = a_n, \\ \beta_n - b_n, & Y_{\tau_k^n}^n = b_n. \end{cases}$$

To prove (ii) of Proposition 6.21 we take $N \in \mathbb{N}$, $\varepsilon > 0$, $\eta > 0$ and define $c := \mu - r - \frac{\sigma^2}{2}$. From (6.42) we can assume without loss of generality $\inf_{n \in \mathbb{N}} \beta_n - \alpha_n > 0$ and take for simplicity $\eta < \inf_{n \in \mathbb{N}} \frac{\beta_n - \alpha_n}{2}$. From the continuity of the paths $t \mapsto \sigma W_t + ct$ we can find some $\theta > 0$ such that

$$P \left(\sup \{ |\sigma(W_t - W_s) + c(t - s)| : s, t \in [0, N], |s - t| \leq \theta \} < \frac{\eta}{2} \right) \geq 1 - \varepsilon.$$

Furthermore, we can take by (6.42) some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\max \{ b_n - \beta_n, \alpha_n - a_n \} \leq \frac{\eta}{2}.$$

For every $\omega \in A := \left\{ \omega : \sup \{ |\sigma(W_t(\omega) - W_s(\omega)) + c(t-s)| : s, t \in [0, N], |s-t| \leq \theta \} < \frac{\eta}{2} \right\}$ and every $n \geq n_0$ and $s, t \in [0, N]$ with $|s-t| \leq \theta$ we then have

$$|Y_t^n(\omega) - Y_s^n(\omega)| \leq \max\{b_n - \beta_n, \alpha_n - a_n\} + |\sigma(W_t(\omega) - W_s(\omega)) + c(t-s)| \leq \eta,$$

since there are only jumps in the same direction due to $\eta < \inf_{n \in \mathbb{N}} \frac{\beta_n - \alpha_n}{2}$ and more than one jump then necessarily cancels some of the distance covered by the process $(\sigma W_t + ct)_{t \geq 0}$. Therefore it holds for all $n \geq n_0$

$$P\left(\sup\{|Y_t^n - Y_s^n| : s, t \in [0, N], |s-t| \leq \theta\} > \eta\right) \leq 1 - P(A) \leq \varepsilon.$$

□

Now we turn our attention to the convergence in finite-dimensional distributions. Here, it is again only a consequence of the convergence of the boundaries and no optimality is needed. What we actually will show is $\lim_{n \rightarrow \infty} Y_t^n = Y_t$ for the following limit process Y .

We take $A < B \in]0, 1[$ and the corresponding control limit policy $(L, M) \in \mathcal{A}_{v_0, h_0}$ for the limits A, B from Definition 6.6. By Lemma 6.9 we have

$$Y_t = y_0 + \left(\mu - r - \frac{\sigma^2}{2}\right) \cdot t + \sigma W_t + \frac{1 + \gamma A}{A(1 - A)} \int_0^t \frac{1}{V_s} dL_s - \frac{1 - \gamma B}{B(1 - B)} \int_0^t \frac{1}{V_s} dM_s, \quad (6.43)$$

where $Y = \psi(h)$, ψ is the transformation function from Section 1.4 and h is the corresponding risky fraction process. Now we define $c := \mu - r - \frac{\sigma^2}{2}$ and

$$Z_t^1 := \frac{1 + \gamma A}{A(1 - A)} \int_0^t \frac{1}{V_s} dL_s, \quad Z_t^2 := \frac{1 - \gamma B}{B(1 - B)} \int_0^t \frac{1}{V_s} dM_s,$$

and thus (6.43) becomes

$$Y_t = y_0 + ct + \sigma W_t + Z_t^1 - Z_t^2. \quad (6.44)$$

The processes Z^1, Z^2 are nondecreasing and we deduce for $a_0 := \psi(A)$ and $b_0 := \psi(B)$

$$\int_0^t \mathbb{1}_{\{Y_s > a_0\}} dZ_s^1 = \frac{1 + \gamma A}{A(1 - A)} \int_0^t \mathbb{1}_{\{h_s > A\}} \frac{1}{V_s} dL_s = 0, \quad (6.45)$$

since $\int_0^t \mathbb{1}_{\{h_s > A\}} dL_s = 0$ by Definition 6.6, and analogously

$$\int_0^t \mathbb{1}_{\{Y_s < b_0\}} dZ_s^2 = 0 \text{ and } Y_t \in [a_0, b_0] \text{ for all } t \geq 0. \quad (6.46)$$

Therefore we are in the framework of Theorem 6.10 and can use its explicit representations for the paths of Z^1, Z^2 in the following lemma.

Lemma 6.23. *In the situation of Lemma 6.22 let Y^n , $n \in \mathbb{N}$, be the sequence of controlled processes with starting values $y_{0,n}$, $n \in \mathbb{N}$, and let Y denote the process from (6.44) with starting value $y_0 \in]a_0, b_0[$. We assume additionally $y_{0,n} = y_0 \in]a_n, b_n[$ for every $n \in \mathbb{N}$. Then we have for almost all $\omega \in \Omega$*

$$\lim_{n \rightarrow \infty} Y_t^n(\omega) = Y_t(\omega) \text{ for all } t \geq 0 \text{ and hence } Y^n \xrightarrow{\mathcal{L}(R_{\geq 0})} Y.$$

Proof. From the definition of a constant boundary strategy we have

$$Y_t^n = y_0 + ct + \sigma W_t + \sum_{k=0}^{\infty} \xi_k^n \mathbb{1}_{\{\tau_k^n \leq t\}}, \quad (6.47)$$

where $\xi_0^n = 0$ and for every $k \in \mathbb{N}$

$$\xi_k^n = \begin{cases} \alpha_n - a_n, & Y_{\tau_k^n}^n = a_n, \\ \beta_n - b_n, & Y_{\tau_k^n}^n = b_n. \end{cases} \quad (6.48)$$

Now we can take a set D of probability 1 such that the representations in (6.44), (6.47) and (6.48) hold pathwise for all $\omega \in D$ with continuity of the involved processes W, Z^1, Z^2 and all $n \in \mathbb{N}$.

We define $\tau_0 := 0$, $\tau_1 := \inf\{t > 0 : Y_t \notin]a_0, b_0[\}$ and inductively with $\inf \emptyset = \infty$

$$\tau_n := \inf\{t > \tau_{n-1} : Y_t \in \{a_0, b_0\} \setminus \{Y_{\tau_{n-1}}\}\} \text{ on } \{\tau_{n-1} < \infty\} \text{ for all } n \geq 2.$$

Comparing (6.47) and (6.44) it suffices to show for a fixed $\omega \in D$ and all $n \in \mathbb{N}_0$

$$\lim_{m \rightarrow \infty} \sum_{k=1}^{\infty} \xi_k^m(\omega) \mathbb{1}_{\{\tau_k^m(\omega) \leq t\}} = (Z_t^1 - Z_t^2)(\omega) \text{ for all } t \in [\tau_n(\omega), \tau_{n+1}(\omega)[, \tau_n(\omega) < \infty. \quad (6.49)$$

We define $Z_t := Z_t^1 - Z_t^2$ and prove (6.49) by induction on n for this fixed $\omega \in D$ but to simplify matters we will suppress it in the following. We also note here that since the following arguments are pathwise, they are just about (deterministic) continuous functions and the (deterministic) jump-representation in (6.47) and (6.48).

a) For $n = 0$ and $t < \tau_1$ we have $Y_s \in]a_0, b_0[$ and hence $Z_s^1 = Z_s^2 = 0$ for all $s \leq t$ due to (6.45) and (6.46). Furthermore, it holds $\lim_{m \rightarrow \infty} a_m = a_0$ and $\lim_{m \rightarrow \infty} b_m = b_0$ and we can therefore find some $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$

$$y_0 + cs + \sigma W_s \stackrel{(6.44)}{=} Y_s \in]a_m, b_m[\text{ for all } s \leq t \text{ and hence } \sum_{k=1}^{\infty} \xi_k^m(\omega) \mathbb{1}_{\{\tau_k^m(\omega) \leq t\}} \stackrel{(6.48)}{=} 0 = Z_t.$$

b) Now we take $n \in \mathbb{N}$ and assume (6.49) to hold for $n - 1$ and also $\tau_n < \infty$. Without loss of generality we further assume $Y_{\tau_n} = a_0$ and begin with $t = \tau_n$.

b1) Using the convergence of the boundaries, we can find some $\tilde{\varepsilon}$ satisfying

$$0 < \tilde{\varepsilon} < \inf_{m \geq \tilde{m}_0} \frac{b_m - a_m}{4} \text{ for some } \tilde{m}_0 \in \mathbb{N}. \quad (6.50)$$

Since the path $s \mapsto Y_s$ is continuous, we can then take $\tilde{u} < \tau_n$ such that $Y_s \in]a_0, a_0 + \tilde{\varepsilon}[$ for all $s \in]\tilde{u}, \tau_n[$. Let then $u := \arg \max_{s \in]\tilde{u}, \tau_n[} Y_s$, i.e. we have $Y_s \in]a_0, Y_u]$ for all $s \in [u, \tau_n[$.

Now we take $\varepsilon > 0$ such that $\varepsilon < \frac{Y_u - a_0}{4}$ and also $\tilde{m}_0 \leq m_0 \in \mathbb{N}$ such that

$$|a_0 - \alpha_m|, |\alpha_m - a_m|, |Y_u^m - Y_u| < \varepsilon \text{ for all } m \geq m_0. \quad (6.51)$$

We fix $m \geq m_0$ and obtain from (6.50) and (6.51)

$$|Y_u^m - \alpha_m| \leq |Y_u^m - Y_u| + |Y_u - a_0| + |a_0 - \alpha_m| < \frac{b_m - a_m}{2}. \quad (6.52)$$

By the definition of u , the path of $(cs + \sigma W_s)_{s \in [u, \tau_n]}$ covers a distance of at most $|Y_u - a_0| < \tilde{\varepsilon}$ in the upward direction starting at any time between u and τ_n . In the case that there is no jump of Y^m in a_m on $[u, \tau_n]$, we deduce $b_m - Y_u^m > |Y_u - a_0|$ from (6.52) and therefore Y^m cannot reach b_m on $[u, \tau_n]$. In the case of jumps in a_m to α_m however, we obtain $b_m - \alpha_m > |Y_u - a_0|$ from (6.52) and Y^m still cannot reach b_m on $[u, \tau_n]$.

Now we need to count the jumps of Y^m on $[u, \tau_n]$. Due to (6.51) it holds

$$|Y_u^m - a_m| \geq |Y_u - a_0| - 2\varepsilon.$$

But starting from any new minimum point $s \in [u, \tau_n]$ of Y , e.g. each time Y^m reaches a_m , the path of $(ct + \sigma W_t)_{t \in [s, \tau_n]}$ covers a distance of at most $|Y_u - a_0| - |Y_u - Y_s|$ in the downward direction on $[s, \tau_n]$ by the definition of u . Therefore, after the first jump in a_m , we only have a distance of at most 2ε left for jumping to α_m and then returning to a_m again. This way we obtain for the number of possible jumps at a_m of Y^m

$$|\{k : u < \tau_k^m \leq \tau_n\}| \in \left[0, \frac{2\varepsilon}{\alpha_m - a_m} + 1\right]. \quad (6.53)$$

Finally, we have

$$Z_u = Z_{\tau_n} \text{ and } \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq \tau_n\}} = \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq u\}} + \sum_{k=1}^{\infty} (\alpha_m - a_m) \mathbb{1}_{\{u < \tau_k^m \leq \tau_n\}}$$

and we therefore get from (6.53)

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq \tau_n\}} - Z_{\tau_n} \right| &\leq \left| \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq u\}} - Z_u \right| + \left| \sum_{k=1}^{\infty} (\alpha_m - a_m) \mathbb{1}_{\{u < \tau_k^m \leq \tau_n\}} \right| \\ &\leq |Y_u^m - Y_u| + 2\varepsilon + \alpha_m - a_m \leq 4\varepsilon. \end{aligned}$$

b2) Now let $t \in]\tau_n, \tau_{n+1}[$. We note that our path satisfies $Z_s^2 = Z_{\tau_n}^2$ for all $s \in [\tau_n, \tau_{n+1}]$ due to $Y_{\tau_n} = a_0$. By (6.45) and (6.46) we are in the framework of instantaneous reflection of

Theorem 6.10 and obtain therefore the explicit representation for the path of the process Z , whose growth from τ_n onward is given via

$$Z_t - Z_{\tau_n} = - \inf_{s \in [\tau_n, t]} (y_0 + cs + \sigma W_s + Z_{\tau_n} - a_0). \quad (6.54)$$

Let $\varepsilon > 0$ such that $\varepsilon < \frac{b_0 - a_0}{4}$. Since $Y_{\tau_n} = a_0$ and Y does not reach b_0 on $] \tau_n, \tau_{n+1}[$ we can find by b1) and the convergence of the boundaries some $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the path of Y^m does not reach b_m on $] \tau_n, \tau_{n+1}[$ and therefore has no jumps at b_m and additionally satisfies

$$\left| \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq \tau_n\}} - Z_{\tau_n} \right| = |Y_{\tau_n}^m - Y_{\tau_n}| < \varepsilon \quad \text{and} \quad |Y_{\tau_n}^m - a_m|, |\alpha_m - a_m| < \varepsilon. \quad (6.55)$$

Between τ_n and t the path of $(cs + \sigma W_s)_{s \in [\tau_n, t]}$ covers a distance of $|Z_t - Z_{\tau_{n+1}}|$ in the downward direction by (6.54) and from any new minimum point $s \in [t, \tau_{n+1}[$ of Y , e.g. each time Y^m reaches a_m , also $|Z_s - Z_{\tau_{n+1}}|$. Hence, we can calculate the number of possible jumps of Y^m at a_m by the same method as in b1) and get due to $|Y_{\tau_n}^m - a_m| < \varepsilon$

$$|\{k : \tau_n < \tau_k^m \leq t\}| \in \left[\frac{Z_t - Z_{\tau_n} - \varepsilon}{\alpha_m - a_m}, \frac{Z_t - Z_{\tau_n}}{\alpha_m - a_m} + 1 \right].$$

Therefore (6.55) then implies

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq t\}} - Z_t \right| &\leq \left| \sum_{k=1}^{\infty} \xi_k^m \mathbb{1}_{\{\tau_k^m \leq \tau_n\}} - Z_{\tau_n} \right| + \left| \sum_{k=1}^{\infty} (\alpha_m - a_m) \mathbb{1}_{\{\tau_n < \tau_k^m \leq t\}} + Z_{\tau_n} - Z_t \right| \\ &\leq |Y_{\tau_n}^m - Y_{\tau_n}| + \varepsilon \leq 2\varepsilon. \end{aligned}$$

for all $m \geq m_0$. □

Now we are able to prove one of the main theorems of this Chapter. It applies especially for the transformed risky fraction processes in the situation of Theorem 6.18.

Theorem 6.24. *In the situation of Lemma 6.23 we have*

$$Y^n \xrightarrow{\mathcal{L}} Y.$$

Proof. The assertion follows directly from Theorem 6.19, Lemma 6.22 and Lemma 6.23. □

The same result is also true for the risky fraction processes on $[0, 1]$. We use the transformation function $\varphi = \psi^{-1} : \mathbb{R} \rightarrow]0, 1[$ from Section 1.4.

Theorem 6.25. *In the situation of Lemma 6.23 we have*

$$h^n \xrightarrow{\mathcal{L}} h$$

for the corresponding risky fraction processes $h_t^n = \varphi(Y_t^n)$, $n \in \mathbb{N}$, and $h_t = \varphi(Y_t)$.

Proof. By Lemma 3.17 the function φ is Lipschitz continuous, and thus Lemmas 6.22 and 6.23 are also true for $(h^n)_{n \in \mathbb{N}}$ and h . Theorem 6.19 then yields the assertion. □

6.3 Model B

In the case of Model B we have less information about the solution u to the QVI in Theorem 3.6. Nevertheless we are still able to prove the convergence of the trading states α, β to the boundary points a, b , respectively, when the fixed costs given by δ tend to zero. Though this is only achieved for a subsequence, the convergence of the whole sequence can be conjectured from the corresponding results from Model A.

The next proposition is the direct counterpart of Proposition 6.13 for Model A.

Proposition 6.26. *Let $d := 1$ and $(\delta_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the fixed costs in Model B in the sense of (1.51) for a fixed common $0 < \lambda, \nu < 1$. Let $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$, $f(1) \neq f(0)$ and suppose $\lim_{n \rightarrow \infty} \delta_n = 0$. Then the corresponding constants a_n, α_n, β_n and b_n given by Proposition 3.34 for every $n \in \mathbb{N}$ satisfy*

$$\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0 \text{ and } \lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0.$$

Proof. The proof is an adaption of the proof of Proposition 3.34 to Model B.

We will only prove $\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0$, since $\lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0$ can be shown the same way. For that purpose let $u_n, Mu_n, \bar{C}_n, \bar{f}$ and l_n be the corresponding functions and constants from Theorem 3.6 and define for every $n \in \mathbb{N}$ the function

$$g_n : I_n := [\beta_n, b_n] \rightarrow \mathbb{R}, \quad x \mapsto u_n(x) - u_n(\beta_n) - \Gamma_n(x, \beta_n),$$

where Γ_n is the cost function from Lemma 3.29 with the corresponding constants λ, ν, δ_n .

a) Let $n \in \mathbb{N}$. We prove here for all $x \in [\beta_n, b_n]$

$$0 \leq g_n(x) \leq |\log(1 - \delta_n)|. \quad (6.56)$$

On the one hand we have for every $x \in [\beta_n, b_n]$

$$u_n(x) \geq Mu_n(x) \geq u_n(\beta_n) + \Gamma_n(x, \beta_n).$$

On the other hand $u_n(\beta_n) + \Gamma_n(x, \beta_n) < u_n(x) + \Gamma_n(x, x)$ would imply

$$Mu_n(b_n) \stackrel{3.34}{=} u_n(\beta_n) + \Gamma_n(b_n, \beta_n) \stackrel{(3.46)}{<} u_n(x) + \Gamma_n(b_n, x) \leq Mu_n(b_n)$$

and hence a contradiction, yielding

$$u_n(x) \geq u_n(\beta_n) + \Gamma_n(x, \beta_n) \geq u_n(x) + \Gamma_n(x, x) = u_n(x) + \log(1 - \delta_n)$$

and hence (6.56).

b) Now let $\varepsilon > 0$ and suppose $|\beta_n - b_n| \geq 4\varepsilon$ for infinitely many $n \in \mathbb{N}$ and without loss of generality for all $n \in \mathbb{N}$.

b1) We begin by showing the existence of some constant $c = c(\mu, \sigma^2, r, \nu)$ such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in [\beta_n, \beta_n + 2\varepsilon]} |g_n''(x)| \leq c. \quad (6.57)$$

Let $n \in \mathbb{N}$ and φ be from (1.76). Then by (3.46) we have for $x \in]\beta_n, b_n[$

$$g_n'(x) = u_n'(x) + \frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} \quad (6.58)$$

and therefore

$$g_n''(x) = u_n''(x) + \frac{\nu\varphi''(x)}{1 - \nu\varphi(x)} + \left(\frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} \right)^2, \quad (6.59)$$

where by the proof of Lemma 3.17 and taking derivatives

$$\varphi'(x) = \varphi(x) - \varphi^2(x) \text{ and } \varphi''(x) = \varphi(x) - 3\varphi^2(x) + 2\varphi^3(x). \quad (6.60)$$

Since $\varphi(x), \nu \in]0, 1[$ it remains by (6.59) and (6.60) to prove the boundedness of u_n'' .

We proceed as in the proof of Proposition 5.10, where in view of (5.23) it was enough to see the boundedness of u_n' in $n \in \mathbb{N}$ on $[\beta_n, \beta_n + 2\varepsilon]$.

We have Lemma 5.9 that u_n is strictly decreasing on $[x_n, b_n]$, where $x_n < \beta_n$ is the state, where u_n attains its maximum on $[a_n, b_n]$ by Proposition 3.34.

In view of (6.56), (6.58) and the mean value theorem applied to g_n , we get for some $\xi \in]\beta_n + 2\varepsilon, \beta_n + 3\varepsilon[$

$$|u_n'(\xi)| \leq \frac{|\log(1 - \delta_n)|}{\varepsilon} + \frac{\nu\varphi'(\xi)}{1 - \nu\varphi(\xi)}. \quad (6.61)$$

In the case, where u_n' attains its minimum on $[x_n, \xi]$ in some $x \in]x_n, \xi[$, we have $u_n''(x) = 0$ and hence as in (5.24)

$$\sup_{y \in [\beta_n, \beta_n + 2\varepsilon]} |u_n'(y)| \leq |u_n'(x)| \leq \left| \frac{-\bar{f}(x) + l_n}{\mu - r - \frac{1}{2}\sigma^2} \right|.$$

We note that $\mu - r - \frac{1}{2}\sigma^2 \neq 0$ holds due to the assumption $f(1) \neq 0$.

If on the contrary the minimum is attained in ξ , we have by (6.61)

$$\sup_{y \in [\beta_n, \beta_n + 2\varepsilon]} |u_n'(y)| \leq |u_n'(\xi)| \leq \frac{|\log(1 - \delta_n)|}{\varepsilon} + \frac{\nu\varphi'(\xi)}{1 - \nu\varphi(\xi)}.$$

b2) Here, we prove the existence of some $\beta < b \in \mathbb{R}$ such that $I := [\beta, b] \subseteq]\beta_n, b_n[$ for infinitely many $n \in \mathbb{N}$.

By Proposition 3.34 (v) we have for $\bar{l}_1 := \inf_{n \in \mathbb{N}} l_n$ and every $n \in \mathbb{N}$

$$x_n \in \{x \in \mathbb{R} : \bar{f}(x) \geq \bar{l}_1\} =: [y_0, y_1].$$

Furthermore, we have $|u_n(x) - u_n(y)| \stackrel{3.22}{\leq} \sup_{n \in \mathbb{N}} \|\bar{C}_n\|_\infty < \infty$ for all $x, y \in \mathbb{R}$. We can use (3.30) in the same way as Theorem 2.10 (v) in (5.13) to get

$$y_2 := \sup_{n \in \mathbb{N}} |y_1 - b_n| < \infty$$

and thus $[\beta_n, b_n] \subseteq [y_0, y_1 + y_2]$ for all $n \in \mathbb{N}$. From our previous assumption we can hence find convergent subsequences $(\beta_{n_k})_{k \in \mathbb{N}}, (b_{n_k})_{k \in \mathbb{N}}$ satisfying $|\beta_{n_k} - b_{n_k}| \geq 4\varepsilon$.

b3) Without loss of generality we can assume $I \subseteq]\beta_n, b_n[$ for all $n \in \mathbb{N}$ by b2).

Lemma 5.2 together with a) and b1) yields

$$\lim_{n \rightarrow \infty} \|g'_n\|_{I, \infty} = 0. \quad (6.62)$$

We define $l_0 := \sup_{n \in \mathbb{N}} l_n$. Then (6.62), (6.58) and (5.23) imply

$$\lim_{n \rightarrow \infty} u''_n(x) = \frac{2}{\sigma^2}(-\bar{f}(x) + l_0) + \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2}\sigma^2 \right) \frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} \quad (6.63)$$

uniformly in $x \in I$, and so by (6.59) we have uniformly in $x \in I$

$$\lim_{n \rightarrow \infty} g''_n(x) = \frac{2}{\sigma^2}(-\bar{f}(x) + l_0) + \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2}\sigma^2 \right) \frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} + \frac{\nu\varphi''(x)}{1 - \nu\varphi(x)} + \left(\frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} \right)^2.$$

This implies together with (6.62) for all $x \in I$

$$0 = \frac{2}{\sigma^2}(-\bar{f}(x) + l_0) + \frac{2}{\sigma^2} \left(\mu - r - \frac{1}{2}\sigma^2 \right) \frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} + \frac{\nu\varphi''(x)}{1 - \nu\varphi(x)} + \left(\frac{\nu\varphi'(x)}{1 - \nu\varphi(x)} \right)^2. \quad (6.64)$$

b4) From (6.64) we can deduce in the same way as in the part b4) of the proof of Proposition 6.13

$$0 = l_0 \geq \inf_{n \in \mathbb{N}} l_n > f(0) = 0$$

and hence a contradiction. □

Similarly to Proposition 6.14, we can now prove the convergence of the boundaries for some subsequence.

Proposition 6.27. *Let $d := 1$ and $(\delta_n)_{n \in \mathbb{N}} \in]0, 1[^\mathbb{N}$ be a sequence of trading proportions defining the fixed costs in Model B in the sense of (1.51) for a fixed common $0 < \lambda, \nu < 1$.*

Let $\inf_{n \in \mathbb{N}} l_n > \max\{f(0), f(1)\}$, $f(1) \neq f(0)$ and suppose $\lim_{n \rightarrow \infty} \delta_n = 0$.

Then there exist constants $a_0 < x_0 < b_0 \in \mathbb{R}$ and a subsequence $(\delta_{n_k})_{k \in \mathbb{N}}$ such that the corresponding constants $x_{n_k}, a_{n_k}, \alpha_{n_k}, \beta_{n_k}$ and b_{n_k} given by Proposition 3.34 for every $k \in \mathbb{N}$ satisfy

$$\lim_{k \rightarrow \infty} x_{n_k} = x_0, \quad \lim_{k \rightarrow \infty} a_{n_k} = a_0 = \lim_{k \rightarrow \infty} \alpha_{n_k}, \quad \lim_{k \rightarrow \infty} \beta_{n_k} = b_0 = \lim_{k \rightarrow \infty} b_{n_k}.$$

Proof. By Proposition 3.34 we have for all $n \in \mathbb{N}$

$$x_n \in \{y \in \mathbb{R} : \bar{f}(y) - l_n \geq 0\} \cap [\alpha_n, \beta_n]$$

and by Lemma 3.22 also

$$\sup_{n \in \mathbb{N}} \|u_n\|_\infty \leq \sup_{n \in \mathbb{N}} \|\Gamma_n\|_\infty < \infty.$$

Since $\lim_{y \rightarrow \infty} \bar{f}(y) - \inf_{n \in \mathbb{N}} l_n < 0$ and $\lim_{y \rightarrow -\infty} \bar{f}(y) - \inf_{n \in \mathbb{N}} l_n < 0$ equation (3.30), used in the same way as Theorem 2.10 (v) in (5.13), implies

$$\sup_{n \in \mathbb{N}} b_n < \infty, \quad \inf_{n \in \mathbb{N}} a_n > -\infty. \quad (6.65)$$

Now similarly to part b) of the proof of Proposition 5.10, we have

$$\sup_{n \in \mathbb{N}, x \in [\alpha_n, \beta_n]} |u_n''(x)| < \infty. \quad (6.66)$$

Furthermore, we have seen

$$|u_n'(\beta_n)| = \frac{\nu \varphi'(\beta_n)}{1 - \nu \varphi(\beta_n)} \quad (6.67)$$

in equation (5.25). Now since u_n attains its maximum in x_n , we have $u_n'(x_n) = 0$ and the mean value theorem together with (6.65), (6.66) and (6.67) yields

$$\inf_{n \in \mathbb{N}} |x_n - \beta_n| > 0.$$

The corresponding argument implies

$$\inf_{n \in \mathbb{N}} |x_n - \alpha_n| > 0.$$

Finally, Proposition 6.26 yields

$$\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0$$

and therefore the existence of the desired constants $a_0 < x_0 < b_0 \in \mathbb{R}$ and the subsequence $(\delta_{n_k})_{k \in \mathbb{N}}$ follows. □

Remark 6.28. 1) The Propositions 6.26 and 6.13 consider a fixed proportional cost structure given by γ or λ and ν , respectively, where the fixed costs given by δ tend to zero. But by comparing the models in Chapter 4 we have seen the relationship $\lambda = \nu = \frac{\gamma}{1-\delta}$, which was needed to get a similarity of the models. When δ tends to zero this relationship cannot be maintained.

2) Due to the lack of information about u_n we cannot prove that the control limit policy for the limiting boundaries $A := \varphi(a_0), B := \varphi(b_0)$ is optimal for the transaction cost model in Section 6.1 as we have done in Theorems 6.17 and 6.18. Therefore we cannot use their uniqueness to have the convergence in Proposition 6.27 for the whole sequence. Furthermore, Theorems 6.24 and 6.25 are still valid for this subsequence due to Proposition 6.27.

To illustrate the convergence we take again the scenario from [IS06], where $r := 0$, $\sigma := 0.4$ and $\mu := 0.096$. We are no longer able to take

$$\lambda = \nu = \frac{\gamma}{1 - \delta}$$

as described in Chapter 4, since the dependence on δ for fixed λ, ν is of interest. Thus we calculate by the method described in Proposition 4.6 the dependence of the constants a, α, β, b in $]0, 1[$ and the optimal growth rate ρ on the fixed costs δ , where the proportional costs are given by $\lambda = \nu$. We note that $\rho := 0.0288$ is the optimal growth rate without transaction costs. Figure 6.3 illustrates this for $\lambda = 0.003$ whereas Figure 6.4 shows the same dependence for $\lambda = 0.1$.

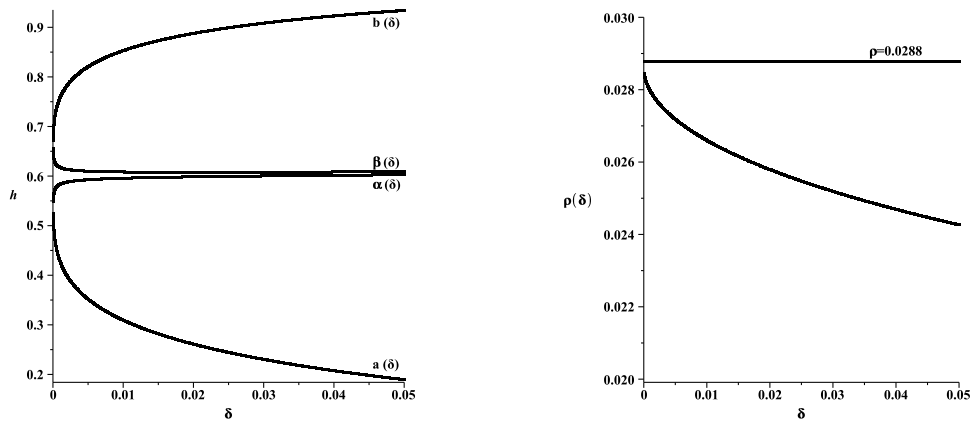


Figure 6.3: Dependence on δ in Model B for $\lambda = 0.003$

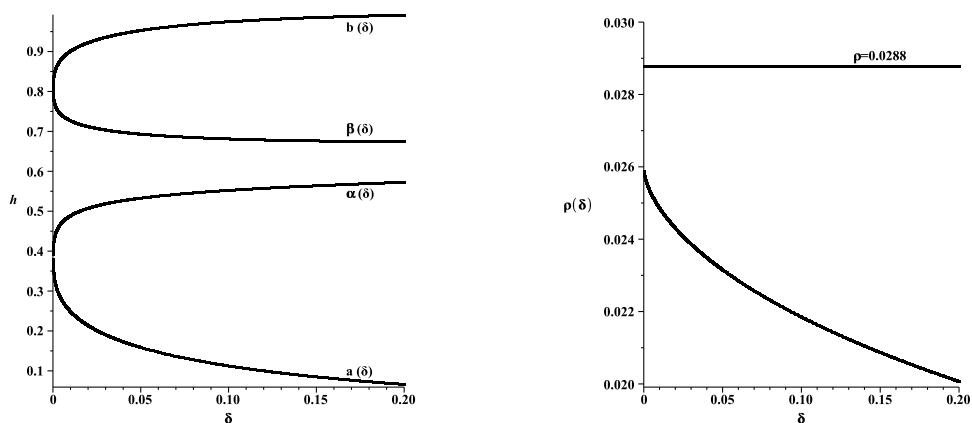


Figure 6.4: Dependence on δ in Model B for $\lambda = 0.1$

Appendix A

Sobolev spaces and variational inequalities

In [Nag04], [Tam06] and [Tam08] certain weighted Sobolev spaces and variational inequalities on them are used. In Section A.1 we therefore give a brief introduction to weak derivatives and these weighted Sobolev spaces. Then using mollification techniques we show how functions from these spaces can be approximated by smooth functions in Section A.2, which is used in Chapter 3 to make Itô's formula applicable to the solution obtained in [Tam08]. Finally, we establish for the sake of completeness in Section A.3 the notions of strong and weak solution to the variational inequality introduced in [Tam08] and the connection between them.

A.1 Weak derivatives and weighted Sobolev spaces

General Assumptions and Notations. Throughout this chapter n will be a positive integer and if not stated otherwise $\Omega \subseteq \mathbb{R}^n$ will denote a *domain*, i.e. a nonempty open set.

For a function $u : \Omega \rightarrow \mathbb{R}$ we define the *support* of u to be the set

$$\text{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

Let then $C^\infty(\Omega)$ denote the space of infinitely differentiable functions on Ω and

$$C_0^\infty(\Omega) := \{u \in C^\infty(\Omega) : \text{supp}(u) \text{ is compact}\}$$

the space of the so-called *test functions*.

For every *multi-index* α , i.e. $\alpha \in \mathbb{N}_0^n$ we define the differential operator

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \dots \cdot \partial x_n^{\alpha_n}} \text{ of order } |\alpha| := \sum_{i=1}^n \alpha_i.$$

We now want to introduce the concept of *weak derivatives* and a certain *weighted Sobolev space* of our interest stating some well-known properties we need. For more detailed information we refer the interested reader to [AF03], [Maz85], [KJF77] and [Tri83] and the references therein.

Definition and Lemma A.1. *Let α be a multi-index and $u \in L^1_{loc}(\Omega)$. If there exists a function $v_\alpha \in L^1_{loc}(\Omega)$ such that*

$$\int_{\Omega} u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v_\alpha(x) \varphi(x) dx \text{ for every } \varphi \in C_0^\infty(\Omega), \quad (\text{A.1})$$

then v_α is unique up to sets of measure zero and is called the α^{th} weak derivative of u , also denoted by $D^\alpha u$. If u is $|\alpha|$ -times continuously differentiable, then $D^\alpha u$ (in the classical sense) satisfies (A.1) by integration by parts and hence coincides a.e. on Ω with the α^{th} weak derivative of u .

Proof. Cf. pp. 21-22 in [AF03]. □

Proposition A.2. *Let $\Omega' \subseteq \Omega$ be a subdomain, α a multi-index and $u \in L^1_{loc}(\Omega)$ be α -times weak differentiable on Ω . Then $u|_{\Omega'}$ is α -times weak differentiable on Ω' with the α^{th} weak derivative $D^\alpha(u|_{\Omega'}) = (D^\alpha u)|_{\Omega'}$.*

Proof. Since $u|_{\Omega'}$ and $(D^\alpha u)|_{\Omega'}$ are elements of $L^1_{loc}(\Omega')$ we just consider for a given $\varphi \in C_0^\infty(\Omega')$ its zero extension $\widehat{\varphi}$ to Ω and observe that (A.1) is valid for $\widehat{\varphi}$ on Ω . Hence (A.1) is still valid for φ and Ω' . □

Definition and Lemma A.3. *Let $\gamma \geq 0$ and consider the continuous weight-function*

$$m_\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}, x \mapsto e^{-\gamma(1+x^T(\sigma\sigma^T)^{-1}x)}.$$

We then define for every $1 \leq p < \infty$ the weighted space

$$L^{p,\gamma}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \mid u^p m_\gamma^2 \in L^1(\Omega) \right\}, \quad \|u\|_{p,\gamma} := \left(\int_{\Omega} |u(x)|^p m_\gamma^2(x) dx \right)^{\frac{1}{p}}.$$

Then $L^{p,\gamma}(\Omega)$ is a separable Banach space and $L^p(\Omega) = L^{p,0}(\Omega)$.

By observing $L^{p,\gamma}(\Omega) \subseteq L^1_{loc}(\Omega)$ we proceed for $m \in \mathbb{N}$ with the weighted Sobolev space of weak differentiable functions in the sense of Definition A.1

$$W^{m,p,\gamma}(\Omega) := \left\{ u \in L^{p,\gamma}(\Omega) : D^\alpha u \in L^{p,\gamma}(\Omega) \text{ for all } 0 \leq |\alpha| \leq m \right\}.$$

Equipped with the norm

$$\|u\|_{W^{m,p,\gamma}} := \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x)|^p m_\gamma^2(x) dx \right)^{\frac{1}{p}}$$

it is also a separable Banach space and by setting $W^{m,p}(\Omega) := W^{m,p,0}(\Omega)$ we get the common Sobolev space over Ω .

In a last step we introduce the Sobolev space

$$W^{m,\infty}(\Omega) := \{u \in L^\infty(\Omega) : D^\alpha u \in L^\infty(\Omega) \text{ for all } 0 \leq |\alpha| \leq m\}$$

with the norm

$$\|u\|_{W^{m,\infty}} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty,$$

which is a Banach space.

Proof. Cf. Subsection 2.17.14 and Theorem 8.10.2 in [KJF77] for the weighted spaces and Theorem 3.3 in [AF03] for $W^{m,\infty}(\Omega)$. □

We state a characterization theorem for weak derivatives.

Theorem A.4. *Let Ω be open and bounded with $\partial\Omega$ of class C^1 . Then a function $u : \Omega \rightarrow \mathbb{R}$ is Lipschitz continuous iff $u \in W^{1,\infty}(\Omega)$.*

Proof. Cf. Theorem 5.8.4 in [Eva98]. □

Using this we can prove the following result.

Proposition A.5. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded and Lipschitz continuous. Then $u \in W^{1,\infty}(\mathbb{R}^n)$.*

Proof. We take domains Ω_N for each $N \in \mathbb{N}$ satisfying the assumptions of Theorem A.4 with $\Omega_N \nearrow \mathbb{R}^n$. Now using Theorem A.4 and Proposition A.2 we get that for every multi-index α with $|\alpha| = 1$ the weak derivative $D^\alpha u$ exists and is bounded by the Lipschitz constant of u . Since u is bounded itself, we thus get the assertion. (See also Rademacher's Theorem, Theorem 5.8.6 in [Eva98].) □

The following lemma then directly follows from the definition of m_γ .

Lemma A.6. *Let $1 \leq p < \infty$ and $\gamma > 0$. Then $L^\infty(\mathbb{R}^n) \subseteq L^{p,\gamma}(\mathbb{R}^n)$.*

Now the following result can be applied on the bounded and Lipschitz continuous functions u and Mu from Chapter 3.

Corollary A.7. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded and Lipschitz continuous. Then for every $1 \leq p < \infty$ and $\gamma > 0$ we have $u \in W^{1,p,\gamma}(\mathbb{R}^n)$.*

Proof. This is a direct consequence of Lemma A.6 and Proposition A.5. □

A.2 Approximation by smooth functions

There are of course many results concerning approximation by smooth functions in numerous Sobolev spaces (cf. [AF03], [Kry08] or [Maz85]). But since we are only interested in approximation of a certain type by functions in $C^\infty(\mathbb{R}^n)$ we establish an approximation result using the standard technique of the so-called *mollifiers*.

Definition and Lemma A.8. A C^∞ -function $J : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ satisfying

$$\text{supp}(J) \subseteq \overline{B_{\mathbb{R}^n}(0, 1)} \text{ and } \int_{\mathbb{R}^n} J(x) dx = 1 \quad (\text{A.2})$$

is called a mollifier. A standard-example would be the zero extension of the function

$$J : B_{\mathbb{R}^n}(0, 1) \rightarrow \mathbb{R}_{\geq 0}, \quad x \mapsto c \cdot e^{-\frac{1}{1-|x|^2}}$$

on the complement of $B_{\mathbb{R}^n}(0, 1)$ in \mathbb{R}^n , where the constant c is suitable chosen.

For such a J and every $h > 0$ we get the function

$$J_h : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}, \quad x \mapsto \frac{1}{h^n} J\left(\frac{x}{h}\right) \quad (\text{A.3})$$

satisfying

$$J_h \in C_0^\infty(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} J_h(x) dx = 1. \quad (\text{A.4})$$

For $u \in L_{loc}^1(\mathbb{R}^n)$ we then define the mollification (or regularization) of u

$$J_h * u : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto \int_{\mathbb{R}^n} J_h(x - y) u(y) dy. \quad (\text{A.5})$$

Proof. Cf. p. 36 in [AF03].

□

Proposition A.9. Let $u \in L_{loc}^1(\mathbb{R}^n)$, $h > 0$, $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then the function $J_h * u$ defined in (A.5) satisfies

- (i) $J_h * u \in C^\infty(\mathbb{R}^n)$ and $D^\alpha (J_h * u) = u * D^\alpha J_h$ for every multi-index α .
- (ii) If α is a multi-index and $D^\alpha u$ exists (weakly), then $D^\alpha (J_h * u) = J_h * D^\alpha u$ holds.
- (iii) If $\text{supp}(u)$ is bounded, then $J_h * u \in C_0^\infty(\mathbb{R}^n)$.
- (iv) If $x \in \mathbb{R}^n$ and u is bounded on $B_{\mathbb{R}^n}(x, h)$, then

$$|J_h * u(x)| \leq \sup_{y \in B_{\mathbb{R}^n}(x, h)} |u(y)| \quad \text{and} \quad |J_h * u(x) - u(x)| \leq \sup_{y \in B_{\mathbb{R}^n}(x, h)} |u(x) - u(y)|.$$

(v) If $u|_{\Omega} \in W^{m,p}(\Omega)$ and Ω' is a subdomain with compact closure in Ω , then

$$\lim_{h \searrow 0} (J_h * u)|_{\Omega'} = u|_{\Omega'} \text{ in } W^{m,p}(\Omega').$$

Proof. (i) and (ii) can be found in Lemma 1.8.2 in [Kry08] by using definition A.1 and then (iii) follows directly from (i) and the definition of $J_h * u$.

For (iv) we note that, since $J_h \equiv 0$ on the complement of $\overline{B_{\mathbb{R}^n}(0, h)}$, we have

$$|J_h * u(x)| = \left| \int_{y \in B_{\mathbb{R}^n}(x, h)} J_h(x-y)u(y)dy \right| \leq \sup_{y \in B_{\mathbb{R}^n}(x, h)} |u(y)|$$

and similarly

$$|J_h * u(x) - u(x)| = \left| \int_{y \in B_{\mathbb{R}^n}(x, h)} J_h(x-y)(u(y) - u(x))dy \right| \leq \sup_{y \in B_{\mathbb{R}^n}(x, h)} |u(x) - u(y)|.$$

(v) can finally be found as Lemma 3.16 in [AF03].

□

Now the standard result for common Sobolev spaces is also true for our weighted case.

Theorem A.10. *Let $m \in \mathbb{N}$, $\gamma \geq 0$ and $1 \leq p < \infty$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p,\gamma}(\mathbb{R}^n)$.*

Proof. Imitating the proof of Theorem 3.22 in [AF03], we proceed as follows.

Let $u \in W^{m,p,\gamma}(\mathbb{R}^n)$, $M > 0$ and f be a function in $C_0^\infty(\mathbb{R}^n)$ satisfying

$$(i) \quad f|_{\overline{B_{\mathbb{R}^n}(0,1)}} \equiv 1,$$

$$(ii) \quad f|_{\mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,2)} \equiv 0,$$

$$(iii) \quad \|D^\alpha f\|_\infty \leq M \text{ for every multi-index } 0 \leq |\alpha| \leq m.$$

For every $\varepsilon > 0$ we then define

$$f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto f(\varepsilon x)$$

yielding

$$f_\varepsilon|_{\overline{B_{\mathbb{R}^n}(0, \frac{1}{\varepsilon})}} \equiv 1, \quad f_\varepsilon|_{\mathbb{R}^n \setminus B_{\mathbb{R}^n}(0, \frac{2}{\varepsilon})} \equiv 0 \text{ and } \|D^\alpha f_\varepsilon\|_\infty \leq M\varepsilon^{|\alpha|}.$$

Setting $u_\varepsilon := f_\varepsilon u$ for every $\varepsilon \in]0, 1]$ we therefore get for $|\alpha| \leq m$ and $x \in \mathbb{R}^n$

$$|D^\alpha u_\varepsilon(x)| = \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} f_\varepsilon(x) \right| \leq M \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta u(x)| \quad (\text{A.6})$$

and hence $u_\varepsilon \in W^{m,p,\gamma}(\mathbb{R}^n)$ has compact support.

Now we get from (A.6) a $C > 0$ such that for every $\varepsilon \in]0, 1]$ and $\Omega_\varepsilon := \mathbb{R}^n \setminus \overline{B_{\mathbb{R}^n}}(0, \frac{1}{\varepsilon})$

$$\begin{aligned} \|u - u_\varepsilon\|_{W^{m,p,\gamma}(\mathbb{R}^n)} &= \|u - u_\varepsilon\|_{W^{m,p,\gamma}(\Omega_\varepsilon)} \\ &\leq \|u\|_{W^{m,p,\gamma}(\Omega_\varepsilon)} + \|u_\varepsilon\|_{W^{m,p,\gamma}(\Omega_\varepsilon)} \\ &\leq C\|u\|_{W^{m,p,\gamma}(\Omega_\varepsilon)} \end{aligned}$$

holds, yielding

$$\lim_{\varepsilon \searrow 0} \|u - u_\varepsilon\|_{W^{m,p,\gamma}(\mathbb{R}^n)} = 0$$

by Lebesgue's convergence theorem.

Finally, let $\delta > 0$ and take $\varepsilon > 0$ satisfying $\|u - u_\varepsilon\|_{W^{m,p,\gamma}(\mathbb{R}^n)} < \frac{\delta}{2}$.

We define $\Omega := B_{\mathbb{R}^n}(0, \frac{4}{\varepsilon})$ and $\Omega' := B_{\mathbb{R}^n}(0, \frac{3}{\varepsilon})$. Since $u_\varepsilon \in W^{m,p,\gamma}(\mathbb{R}^n)$ has its support in $B_{\mathbb{R}^n}(0, \frac{2}{\varepsilon})$, we get $u_\varepsilon \in W^{m,p}(B_{\mathbb{R}^n}(0, \frac{4}{\varepsilon}))$ and can use Proposition A.9 (v) to get

$$\lim_{h \searrow 0} (J_h * u_\varepsilon)|_{\Omega'} = u_\varepsilon|_{\Omega'} \text{ in } W^{m,p}(\Omega').$$

Now by observing that for $h < 1$ we have $(J_h * u_\varepsilon)|_{\mathbb{R}^n \setminus \Omega'} \equiv 0$ by the definition of $J_h * u_\varepsilon$ and hence

$$\lim_{h \searrow 0} J_h * u_\varepsilon = u_\varepsilon \text{ in } W^{m,p,\gamma}(\mathbb{R}^n),$$

we finally can choose $h > 0$ satisfying

$$\|u - J_h * u_\varepsilon\|_{W^{m,p,\gamma}(\mathbb{R}^n)} \leq \|u - u_\varepsilon\|_{W^{m,p,\gamma}(\mathbb{R}^n)} + \|u_\varepsilon - J_h * u_\varepsilon\|_{W^{m,p,\gamma}(\mathbb{R}^n)} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

□

Remark A.11. Theorem 3.33 in [AF03] shows that if $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$, then we have $\lambda(\mathbb{R}^n \setminus \Omega) = 0$, and thus Ω cannot be bounded.

We close this section by proving an approximation result for the generator of the diffusion in Chapter 3 that is needed to establish Itô's formula for expectations in Section 3.2.

Proposition A.12. *Let $u \in C_b(\mathbb{R}^n, \mathbb{R})$ and A be a finite set of multi-indices α such that $D^\alpha u$ exists in the sense of Definition A.1. Let $(a_\alpha)_{\alpha \in A} \in \mathbb{R}^A$ and $L := \sum_{\alpha \in A} a_\alpha D^\alpha$ be a partial differential operator such that $Lu \in C_b(\mathbb{R}^n, \mathbb{R})$. Then there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \in (C^\infty(\mathbb{R}^n))^{\mathbb{N}}$ that satisfies*

$$(i) \quad \varphi_n \xrightarrow[n \rightarrow \infty]{} u \text{ and } L\varphi_n \xrightarrow[n \rightarrow \infty]{} Lu \text{ uniformly on every compact set of } \mathbb{R}^n,$$

$$(ii) \quad \sup_{n \in \mathbb{N}} \|\varphi_n\|_\infty \leq \|u\|_\infty \text{ and } \sup_{n \in \mathbb{N}} \|L\varphi_n\|_\infty \leq \|Lu\|_\infty.$$

Proof. For every $h \in]0, 1[$ and $\alpha \in A$ we have by Proposition A.9 (ii)

$$D^\alpha (J_h * u) = J_h * D^\alpha u$$

and hence

$$L (J_h * u) = J_h * Lu. \quad (\text{A.7})$$

For $n \in \mathbb{N}$ we define $\Omega_n := \overline{B_{\mathbb{R}^n}}(0, n)$. By means of Proposition A.9 (iv) we have

$$\|J_h * u\|_\infty \leq \|u\|_\infty \text{ and } \|L (J_h * u)\|_\infty = \|J_h * Lu\|_\infty \leq \|Lu\|_\infty$$

as well as for every $x \in \mathbb{R}^n$

$$|J_h * u(x) - u(x)| \leq \sup_{y \in B_{\mathbb{R}^n}(x, h)} |u(x) - u(y)|.$$

Now using (A.7) we similarly obtain

$$|L (J_h * u) (x) - Lu(x)| \leq \sup_{y \in B_{\mathbb{R}^n}(x, h)} |Lu(x) - Lu(y)|.$$

Since u and Lu are uniformly continuous on Ω_n we get by setting $\varphi_n := J_{h_n} * u$ after choosing $h_n \in]0, 1[$ sufficiently small

$$\sup_{x \in \Omega_n} |\varphi_n(x) - u(x)| < \frac{1}{n} \text{ and } \sup_{x \in \Omega_n} |L\varphi_n(x) - Lu(x)| < \frac{1}{n}.$$

□

A.3 Variational inequalities

In this section we consider variational inequalities in their strong and weak forms as introduced in [Tam08] and establish the connection between them, which is well-known in the unweighted case. We refer to [BL82] and [BL84] for more detailed information.

For the generator of our diffusion $(X_t)_{t \geq 0}$ from Chapter 3

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(\mu_i - r - \frac{1}{2} (\sigma \sigma^T)_{ii} \right) \frac{\partial}{\partial x_i}$$

we consider for every $\alpha \geq 0$ the variational inequality (in its so-called *strong form*)

$$\max\{Lu - \alpha u + \bar{f} - l, \Psi - u\} = 0, \quad (\text{A.8})$$

where α is the discount factor, $\bar{f}, \Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ are bounded and Lipschitz continuous and l is a constant. A solution to (A.8) is defined to be a function $u \in W^{2,2,\gamma}(\mathbb{R}^n)$ satisfying (A.8).

Now we define a bilinear form, which is later proved to be *associated with L*.

Let $b \in \mathbb{R}^n$ defined by $b_i := \mu_i - r - (\sigma\sigma^T)_{ii}$ for every $i \in \{1, \dots, n\}$.

For every $\alpha \geq 0$ and $u, v \in W^{1,2,\gamma}(\mathbb{R}^n)$ we define

$$\begin{aligned} a_\alpha(u, v) := & \frac{1}{2} \int_{\mathbb{R}^n} (\nabla u(y))^T \sigma \sigma^T \left(\nabla v(y) - \frac{2\gamma(\sigma\sigma^T)^{-1}yv(y)}{\sqrt{1 + y^T(\sigma\sigma^T)^{-1}y}} \right) m_\gamma^2(y) dy \\ & - \int_{\mathbb{R}^n} b^T \nabla u(y)v(y)m_\gamma^2(y) dy + \alpha \int_{\mathbb{R}^n} u(y)v(y)m_\gamma^2(y) dy. \end{aligned} \quad (\text{A.9})$$

Let further $(\cdot, \cdot)_\gamma$ denote the common scalar product on $L^{2,\gamma}(\mathbb{R}^n)$, i.e. we have

$$(u, v)_\gamma = \int_{\mathbb{R}^n} u(x)v(x)m_\gamma^2(x) dx \text{ for every } u, v \in L^{2,\gamma}(\mathbb{R}^n). \quad (\text{A.10})$$

We are now in a state to consider (A.8) in its *weak form*:

$$\begin{aligned} u & \geq \Psi, \\ a_\alpha(u, v - u) & \geq (\bar{f} - l, v - u)_\gamma \text{ for all } v \in W^{1,2,\gamma}(\mathbb{R}^n) \text{ with } v \geq \Psi, \end{aligned} \quad (\text{A.11})$$

where $u \in W^{1,2,\gamma}(\mathbb{R}^n)$ satisfying (A.11) is defined to be a solution.

The connection between the two inequalities (A.8) and (A.11) is the following theorem.

Theorem A.13. *For $\alpha \geq 0$ and $u \in W^{2,2,\gamma}(\mathbb{R}^n)$ the following holds.*

(i) *For every $v \in W^{1,2,\gamma}(\mathbb{R}^n)$ we have $a_\alpha(u, v) = ((-L + \alpha)u, v)_\gamma$.*

(ii) *If u is a solution to (A.11), then u is a solution to (A.8).*

Proof. We adapt the standard method of proof in the unweighted case to our situation.

(i) Since for every $i \in \{1, \dots, n\}$ and $x \in \mathbb{R}^n$ we have

$$\frac{\partial}{\partial x_i} m_\gamma^2(x) = \frac{-\gamma m_\gamma^2(x)}{\sqrt{1 + x^T(\sigma\sigma^T)^{-1}x}} \left(2 \sum_{j=1}^n x_j (\sigma\sigma^T)^{-1}_{ij} \right),$$

we get for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$ by Definition A.1

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} u(x) \varphi(x) m_\gamma^2(x) dx \\ &= \int_{\mathbb{R}^n} -\frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} (\varphi(x) m_\gamma^2(x)) dx \\ &= \int_{\mathbb{R}^n} -\frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} \varphi(x) m_\gamma^2(x) dx - \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) \varphi(x) \frac{\partial}{\partial x_i} m_\gamma^2(x) dx \\ &= \int_{\mathbb{R}^n} -\frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} \varphi(x) m_\gamma^2(x) dx - \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) \varphi(x) \frac{-2\gamma((\sigma\sigma^T)^{-1}x)_i}{\sqrt{1 + x^T(\sigma\sigma^T)^{-1}x}} m_\gamma^2(x) dx. \end{aligned} \quad (\text{A.12})$$

Let $v \in W^{1,2,\gamma}(\mathbb{R}^n)$. We want to prove (A.12) for v instead of φ by approximation. According to Theorem A.10 we can take $\varphi_n \in C_0^\infty(\mathbb{R}^n)$ for every $n \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} \|v - \varphi_n\|_{W^{1,2,\gamma}(\mathbb{R}^n)} = 0. \quad (\text{A.13})$$

We therefore have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_j \partial x_i} u(x) (v(x) - \varphi_n(x)) m_\gamma^2(x) dx \right| &= \left| \left(\frac{\partial^2}{\partial x_j \partial x_i} u, v - \varphi_n \right)_\gamma \right| \\ &\leq \left\| \frac{\partial^2}{\partial x_j \partial x_i} u \right\|_{L^{2,\gamma}(\mathbb{R}^n)} \|v - \varphi_n\|_{L^{2,\gamma}(\mathbb{R}^n)} \\ &\leq \left\| \frac{\partial^2}{\partial x_j \partial x_i} u \right\|_{L^{2,\gamma}(\mathbb{R}^n)} \|v - \varphi_n\|_{W^{1,2,\gamma}(\mathbb{R}^n)}, \end{aligned}$$

and hence

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_j \partial x_i} u(x) \varphi_n(x) m_\gamma^2(x) dx = \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_j \partial x_i} u(x) v(x) m_\gamma^2(x) dx. \quad (\text{A.14})$$

Now using $\left\| \frac{\partial}{\partial x_i} (v - \varphi_n) \right\|_{L^{2,\gamma}(\mathbb{R}^n)} \leq \left\| \frac{\partial}{\partial x_i} (v - \varphi_n) \right\|_{W^{1,2,\gamma}(\mathbb{R}^n)}$ we can prove

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} \varphi_n(x) m_\gamma^2(x) dx = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} v(x) m_\gamma^2(x) dx$$

the same way as (A.14). To verify (A.12) for v , the final task to achieve is showing

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) (v(x) - \varphi_n(x)) \frac{-2\gamma((\sigma\sigma^T)^{-1}x)_i}{\sqrt{1 + x^T(\sigma\sigma^T)^{-1}x}} m_\gamma^2(x) dx = 0,$$

which readily follows as (A.14) if we can prove that

$$g_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto \frac{((\sigma\sigma^T)^{-1}x)_i}{\sqrt{1 + x^T(\sigma\sigma^T)^{-1}x}}$$

is bounded on \mathbb{R}^n .

Let $A := (\sigma\sigma^T)^{-1}$, then A is positive definite, and thus $x \mapsto \sqrt{x^T A x}$ is a norm on \mathbb{R}^n .

Hence we can take $c > 0$ such that

$$|g_i(x)| \leq \frac{|(Ax)_i|}{\sqrt{1 + x^T A x}} \leq \frac{|(Ax)_i|}{\sqrt{x^T A x}} \leq \frac{|(Ax)_i|}{c\|x\|} \leq \frac{\|A\|}{c}$$

and we have proved (A.12) for v , from which we finally conclude

$$\begin{aligned}
(-Lu + \alpha u, v)_\gamma &= - \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{1}{2} (\sigma \sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u(x) v(x) m_\gamma^2(x) dx \\
&\quad - \int_{\mathbb{R}^n} b^T \nabla u(x) v(x) m_\gamma^2(x) dx + \alpha \int_{\mathbb{R}^n} u(x) v(x) m_\gamma^2(x) dx \\
&\stackrel{(A.12)}{=} \sum_{i,j=1}^n \frac{1}{2} (\sigma \sigma^T)_{ij} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} v(x) m_\gamma^2(x) dx \\
&\quad + \sum_{i,j=1}^n \frac{1}{2} (\sigma \sigma^T)_{ij} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} u(x) v(x) \frac{-2\gamma((\sigma \sigma^T)^{-1}x)_i}{\sqrt{1 + x^T(\sigma \sigma^T)^{-1}x}} m_\gamma^2(x) dx \\
&\quad - \int_{\mathbb{R}^n} b^T \nabla u(x) v(x) m_\gamma^2(x) dx + \alpha \int_{\mathbb{R}^n} u(x) v(x) m_\gamma^2(x) dx \\
&= a_\alpha(u, v).
\end{aligned}$$

(ii) Suppose u is a solution to (A.11), i.e. $u \in W^{2,2,\gamma}(\mathbb{R}^n)$ and satisfies $u \geq \Psi$ and

$$a_\alpha(u, v - u) \geq (\bar{f} - l, v - u)_\gamma \text{ for all } v \in W^{1,2,\gamma}(\mathbb{R}^n) \text{ with } v \geq \Psi.$$

Now using (i) we obtain

$$((-L + \alpha)u, v - u)_\gamma \geq (\bar{f} - l, v - u)_\gamma \text{ for all } v \geq \Psi, v \in W^{1,2,\gamma}(\mathbb{R}^n). \quad (\text{A.15})$$

By setting $v := u + \varphi$ in (A.15) we get

$$((\bar{f} - l) + (L - \alpha)u, \varphi)_\gamma \leq 0 \text{ for every nonnegative } \varphi \in C_0^\infty(\mathbb{R}^n)$$

and hence

$$(L - \alpha)u + \bar{f} - l \leq 0 \text{ a.e.} \quad (\text{A.16})$$

Now Ψ is bounded and Lipschitz continuous and so Corollary A.7 allows us to take $v := \Psi$ in (A.15) and get

$$(-(L - \alpha)u - \bar{f} + l, \Psi - u)_\gamma \geq 0$$

yielding

$$(-Lu + \alpha u - \bar{f} + l)(\Psi - u) = 0,$$

which implies (A.8) in view of (A.16).

□

Remark A.14. Lemma A.13 allows us to call $a_\alpha(\cdot, \cdot)$ the bilinear form associated with $-L + \alpha$ and illustrates the terms weak form and strong form.

The advantage of (A.11) over (A.8) is that the solution $u \in W^{1,2,\gamma}(\mathbb{R}^n)$ does not need to be smooth enough to be in $W^{2,2,\gamma}(\mathbb{R}^n)$.

A.4 A classical solution to $Lv = -(\bar{f} - l)$

We consider here the generator

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(\mu_i - r - \frac{1}{2} (\sigma\sigma^T)_{ii} \right) \frac{\partial}{\partial x_i}$$

of the diffusion introduced in Chapter 3 and define as in Section 3.3

$$f : \Delta \rightarrow \mathbb{R}, \quad h \mapsto -\frac{1}{2} h^T \sigma \sigma^T h + (\mu - r \mathbb{1})^T h \quad \text{and} \quad \bar{f} := f \circ \varphi,$$

where

$$\varphi = \psi^{-1} : \mathbb{R}^d \rightarrow \Delta \setminus \partial\Delta, \quad x \mapsto \left(\frac{e^{x_1}}{1 + \sum_{i=1}^d e^{x_i}}, \dots, \frac{e^{x_d}}{1 + \sum_{i=1}^d e^{x_i}} \right).$$

As shown in Chapter 3 we are naturally interested in solutions on \mathbb{R}^d to the differential equation

$$Lv + \bar{f} - l = 0.$$

We shall give a classical solution and solve $Lu = \bar{f}$ as a first step.

Lemma A.15. *The function $u : \mathbb{R}^d \rightarrow \mathbb{R}$, $x \mapsto \log \left(1 + \sum_{i=1}^d e^{x_i} \right)$ is a (classical) solution to $Lu = \bar{f}$ on \mathbb{R}^d .*

Proof. For every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we have

$$\begin{aligned} \bar{f}(x) &= f \left(\left(\frac{e^{x_1}}{1 + \sum_{i=1}^d e^{y_i}}, \dots, \frac{e^{x_d}}{1 + \sum_{i=1}^d e^{x_i}} \right) \right) \\ &= \sum_{i=1}^d (\mu^i - r) \left(\frac{e^{x_i}}{1 + \sum_{j=1}^d e^{x_j}} \right) - \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^T)_{ij} \left(\frac{e^{x_i+x_j}}{\left(1 + \sum_{k=1}^d e^{x_k} \right)^2} \right) \\ &= -\frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^T)_{ij} \left(\frac{e^{x_i+x_j}}{\left(1 + \sum_{k=1}^d e^{x_k} \right)^2} \right) + \frac{1}{2} \sum_{i=1}^d (\sigma\sigma^T)_{ii} \left(\frac{e^{x_i}}{1 + \sum_{j=1}^d e^{x_j}} \right) \\ &\quad + \sum_{i=1}^d \left(\mu^i - r - \frac{1}{2} (\sigma\sigma^T)_{ii} \right) \left(\frac{e^{x_i}}{1 + \sum_{j=1}^d e^{x_j}} \right) \\ &= Lu(x). \end{aligned}$$

□

The next lemma solves $Lv = l$ for all $d \geq 2$ and $l \in \mathbb{R}$.

Lemma A.16. *Let $b_i := \mu_i - r - \frac{1}{2}(\sigma\sigma^T)_{ii}$ for all $i \in \{1, \dots, d\}$ and $l \in \mathbb{R}$. If there are $i \neq j \in \{1, \dots, d\}$ satisfying*

$$b_i, b_j \neq 0 \text{ and } c := (\sigma\sigma^T)_{i,j} + (\sigma\sigma^T)_{j,i} - \frac{b_i}{b_j}(\sigma\sigma^T)_{jj} - \frac{b_j}{b_i}(\sigma\sigma^T)_{ii} \neq 0,$$

then the function

$$v : \mathbb{R}^d \rightarrow \mathbb{R}, (x_1, \dots, x_d) \mapsto -\frac{lb_j}{cb_i}x_i^2 - \frac{lb_i}{cb_j}x_j^2 + \frac{2l}{c}x_ix_j \quad (\text{A.17})$$

solves $Lv(x) = l$ for all $x \in \mathbb{R}^d$.

Proof. The assertion follows by taking derivatives. □

Corollary A.17. *Let $l \in \mathbb{R}$, $d \geq 2$ and u denote the solution from Lemma A.15. Then under the assumptions of Lemma A.16 the function v from (A.17) satisfies*

$$L(-u + v) + \bar{f} - l = 0.$$

Proof. This is a direct consequence of the linearity of L together with Lemma A.15 and A.16. □

Remark A.18. For $d = 1$ a general solution to $Lv = \bar{f} - l$ can be found by variation of constants as it is done in [IS06].

Appendix B

Skorokhod topology and weak convergence of processes

In this chapter we give a brief survey on the weak convergence of processes considered as random elements taking their values in the Polish space $D_E[0, \infty)$ endowed with the Skorokhod topology. For more detailed information we refer to [EK86] and [JS03].

Definition B.1. A *Polish space* is a topological space, which is metrizable, separable and complete.

Now we introduce the metric space $(D_E[0, \infty), d)$.

Definition B.2. Let (E, r) be a metric space. We define the *Skorokhod space* $D_E[0, \infty)$ as the collection of all functions $f : [0, \infty) \rightarrow E$ that are càdlàg (continue à droite, limitée à gauche), i.e. right continuous with left limits.

Let Λ be the set of all bijective increasing functions $\lambda : [0, \infty) \rightarrow [0, \infty)$ which are Lipschitz continuous and satisfy

$$\gamma(\lambda) := \sup_{s>t\geq 0} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| < \infty.$$

Then the *Skorokhod metric* d on $D_E[0, \infty)$ is defined by

$$d(x, y) := \inf_{\lambda \in \Lambda} \max \left\{ \gamma(\lambda), \int_0^\infty e^{-u} d(x, y, \lambda, u) du \right\},$$

where

$$d(x, y, \lambda, u) := \sup_{t \geq 0} q(x(t \wedge u), y(\lambda(t) \wedge u))$$

and q is the metric $q(a, b) := r(a, b) \wedge 1$ on E .

Theorem B.3. *If (E, r) is a Polish space, then $(D_E[0, \infty), d)$ from Definition B.2 is also a Polish space.*

Proof. Cf. Theorem 5.6 in Chapter 3 of [EK86].

□

As a σ -algebra on $D_E[0, \infty)$ we take the Borel σ -algebra $\mathcal{B}_E := \mathcal{B}(D_E[0, \infty))$ generated by the Skorokhod metric. \mathcal{B}_E has then the following useful representation.

Proposition B.4. *If (E, r) is separable, then*

$$\mathcal{B}_E = \sigma(\pi_t : t \in D),$$

where D is any dense subset of $[0, \infty)$ and $\pi_t : D_E[0, \infty) \rightarrow E, \alpha \mapsto \alpha(t)$.

Proof. Cf. Proposition 7.1 in Chapter 3 of [EK86].

□

Now we introduce the notions of tightness and weak convergence for finite measures and state a relation between them.

Definition and Theorem B.5. *Let (E, r) be a Polish space and $\mathcal{B}(E)$ its Borel σ -algebra. Let $\mathcal{P}(E)$ denote the collection of all positive finite measures on $(E, \mathcal{B}(E))$.*

1. *A sequence $(\mu_n)_{n \in \mathbb{N}} \in \mathcal{P}(E)^{\mathbb{N}}$ converges weakly to $\mu \in \mathcal{P}(E)$ if for all bounded continuous functions $f : E \rightarrow \mathbb{R}$ we have*

$$\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu.$$

2. *$\mathcal{P}(E)$ endowed with the weak topology, which is the coarsest topology for which the mappings $\mu \mapsto \int f d\mu$ from 1. are all continuous, is again a Polish space.*
3. *We call a subset $A \subseteq \mathcal{P}(E)$ tight if $\sup_{\mu \in A} \mu(E) < \infty$ and for all $\varepsilon > 0$ there exists a compact set $K \subseteq E$ such that*

$$\mu(E \setminus K) \leq \varepsilon \text{ for all } \mu \in A.$$

4. *A subset $A \subseteq \mathcal{P}(E)$ is relatively compact if and only if it is tight.*

Proof. Cf. §3a in Chapter VI of [JS03].

□

Definition B.6. Let $X = (X_t)_{t \geq 0}$ be a càdlàg process with values in a Polish space (E, r) defined on some probability space (Ω, \mathcal{A}, P) . Then $X : \Omega \rightarrow D_E[0, \infty), \omega \mapsto X(\omega)$ is a random variable with values in $(D_E[0, \infty), \mathcal{B}_E)$ due to Proposition B.4. We define the law $\mathcal{L}(X)$ of X by

$$\mathcal{L}(X)(A) := P(\{\omega : X(\omega) \in A\}) \text{ for all } A \in \mathcal{B}_E.$$

For the rest of the chapter let $(X^n)_{n \in \mathbb{N}_0}$ be a sequence of càdlàg processes defined on probability spaces $(\Omega_n, \mathcal{A}_n, P_n)$, $n \in \mathbb{N}_0$.

Definition B.7. The sequence $(X^n)_{n \in \mathbb{N}}$ converges *weakly* or *in law* to X^0 if the laws $\mathcal{L}(X^n)$ converge weakly to $\mathcal{L}(X^0)$ in $\mathcal{P}(D_E[0, \infty))$ and we write

$$X^n \xrightarrow{\mathcal{L}} X^0.$$

Definition B.8. The sequence $(X^n)_{n \in \mathbb{N}}$ is called *tight* if the sequence of laws $(\mathcal{L}(X^n))_{n \in \mathbb{N}}$ is tight.

Definition B.9. Let $D \subseteq [0, \infty)$. X^n converges to X^0 in *finite-dimensional distributions along D* and we then write

$$X^n \xrightarrow{\mathcal{L}(D)} X^0$$

if it holds

$$\lim_{n \rightarrow \infty} (X^n_{t_1}, \dots, X^n_{t_k}) = (X^0_{t_1}, \dots, X^0_{t_k}) \text{ in distribution for all } t_i \in D, k \in \mathbb{N}.$$

Proposition B.10. *The càdlàg property uniquely determines the process X^0 in Definition B.9 if the set D is dense.*

Proof. Cf. Lemma VI.3.19 in [JS03].

□

Theorem B.11. *Let $D \subseteq [0, \infty)$ be a dense subset.*

(i) *Under tightness of $(X^n)_{n \in \mathbb{N}}$ it holds*

$$(X^n \xrightarrow{\mathcal{L}(D)} X^0) \implies (X^n \xrightarrow{\mathcal{L}} X^0).$$

(ii) *For $\tilde{D} := \{t \geq 0 : P_0(\Delta X_t^0 \neq 0) = 0\}$ we have*

$$(X^n \xrightarrow{\mathcal{L}} X^0) \implies (X^n \xrightarrow{\mathcal{L}(\tilde{D})} X^0).$$

Proof. Cf. Theorem 7.8 in Chapter 3 of [EK86].

□

Notations

\mathbb{N}	$\{1, 2, 3, \dots\}$
\mathbb{N}_0	$\{0, 1, 2, 3, \dots\}$
$\mathbb{R}, \mathbb{R}_{\geq 0}$	set of real numbers, real numbers greater or equal zero
$f _A$	restriction of a function f to A
$\ f\ _\infty, \ f\ _{A,\infty}$	supremum norm on the domain, resp. on a subset A , of a function f
$\mathbf{1}_A$	indicator function of a set A
$x \wedge y$	minimum of x and y
x^+, x^-	maximum of x and 0, maximum of $-x$ and 0
$f(t-), f(t+)$	left-hand, right-hand limit of a function f in t
$\Delta f(t)$	$= f(t+) - f(t-)$, jump size of a function f in t
$B_\Omega(x, r), \overline{B}_\Omega(x, r)$	open, closed ball in Ω with radius r and center x
∂A	boundary of a set A
$\text{diam}(A)$	diameter of a set A
\log	natural logarithm
$C(X, Y)$	space of all continuous functions $f : X \rightarrow Y$
$C(X)$	$:= C(X, \mathbb{R})$
$C^k(X, Y)$	space of all k -times continuously differentiable functions $f : X \rightarrow Y$
$C^k(X)$	$:= C^k(X, \mathbb{R})$
$C_b(X, Y)$	space of all bounded continuous functions $f : X \rightarrow Y$
$L^1_{loc}(\Omega)$	space of all real-valued, locally integrable functions on Ω
$L^p(\Omega), p \in [1, \infty)$	Banach space of all real-valued, p -integrable functions on Ω
$L^\infty(\Omega)$	Banach space of all real-valued, essentially bounded functions on Ω
X^c	continuous martingale part of a semimartingale X
$[X, Y]$	quadratic co-variation of two semimartingales X, Y
$[X]$	quadratic variation of a semimartingale X

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Erklärung

Ich habe die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt und keine anderen als die angegebenen Hilfsmittel verwendet. Diese Arbeit ist bisher weder veröffentlicht noch zur Veröffentlichung eingereicht worden und hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Des Weiteren habe ich noch keinen Promotionsversuch unternommen.

Kiel, den 14. Dezember 2011

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