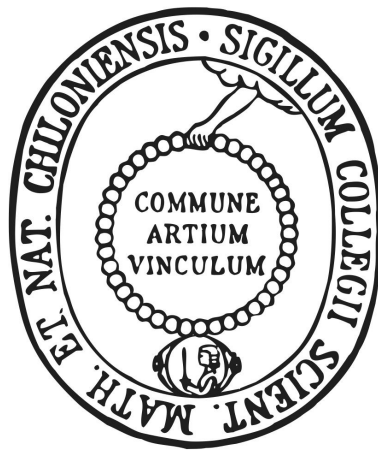


Representation type and Auslander-Reiten theory of Frobenius-Lusztig kernels



Dissertation zur Erlangung des Doktorgrades der
Mathematisch-Naturwissenschaftlichen Fakultät der
Christian-Albrechts-Universität zu Kiel

eingereicht von
Julian Külshammer

Kiel
März 2012

Referent: Prof. Dr. Rolf Farnsteiner

Koreferent: Prof. Dr. Hartmut Laue

Tag der mündlichen Prüfung: 11.05.2012

Zum Druck genehmigt: 15.05.2012

Der Dekan

Abstract

In 1990 George Lusztig introduced a quantum analogue of restricted enveloping algebras of Lie algebras, the small quantum groups. In his PhD thesis Christopher M. Drupieski generalized these to analogues of Frobenius kernels, the Frobenius-Lusztig kernels. They depend on three parameters, the characteristic p of the field, the order ℓ of a root of unity and the height r . The case $r = 0$ gives the small quantum groups defined by George Lusztig. Our thesis concerns basic classification questions from classical representation theory of these algebras.

The determination of the representation type of an algebra answers the question if it is (at least theoretically) possible to classify the indecomposable modules of a given algebra up to isomorphism. If the small quantum group is not attached to an \mathfrak{sl}_2 , we show that all blocks different from the simple Steinberg block are of wild representation type. The \mathfrak{sl}_2 -case was known before: the small half-quantum groups are tame, their half-quantum groups are representation-finite. The higher Frobenius-Lusztig kernels are all wild with the exception of explicitly known $SL(2)$ -blocks in height one and the simple Steinberg blocks. For this result we have to assume certain finiteness assumptions on cohomology which are known in the height zero case, but not proven in general. The Borel and nilpotent parts of the higher Frobenius-Lusztig kernels are wild.

Auslander-Reiten theory is a tool to understand the module category of a finite dimensional algebra by describing the irreducible homomorphisms. The isomorphism classes of indecomposable modules and the irreducible maps are arranged in an oriented graph, the Auslander-Reiten quiver. We are interested in the shape of the connected components of this quiver. In the tame and representation-finite cases, the Auslander-Reiten quiver of the Frobenius-Lusztig kernels is completely understood. For the wild cases we assume $G \neq SL(2)$. It was known before that the connected components of the Auslander-Reiten quiver of a Frobenius-Lusztig kernel are attached to a (finite or infinite) Dynkin diagram or a Euclidean diagram. For the Frobenius-Lusztig kernels we show that there are no components attached to finite Dynkin or Euclidean diagrams. In the height zero case we show that the components containing the restriction of a module for the infinite dimensional quantum group $U_\zeta(\mathfrak{g})$ are attached to \mathbb{A}_∞ . For the small half-quantum groups we show that the components containing compatibly graded modules, i.e. modules for $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$, are also attached to \mathbb{A}_∞ .

Zusammenfassung

George Lusztig definierte 1990 Analoga zu eingeschränkten Einhüllenden von Liealgebren im Quantenfall, die kleinen Quantengruppen. Christopher M. Drupieski verallgemeinerte Lusztigs Definition auf Quanten-Analoga von Frobenius-Kernen, die Frobenius-Lusztig-Kerne. Diese hängen von drei Parametern ab, der Charakteristik p des zugrunde liegenden Körpers, der Ordnung ℓ einer Einheitswurzel und einer Höhe r . Im Fall der Höhe null gelangt man zu den von George Lusztig definierten kleinen Quantengruppen. In dieser Arbeit geht es um grundlegende Fragen klassischer Darstellungstheorie dieser Algebren.

Der Darstellungstyp einer Algebra gibt an, ob es (zumindest theoretisch) möglich ist die unzerlegbaren Moduln einer gegebenen Algebra bis auf Isomorphie zu klassifizieren. Kommt eine kleine Quantengruppe nicht von der Lie-Algebra \mathfrak{sl}_2 , so zeigen wir, dass all ihre Blöcke mit Ausnahme des einfachen Steinberg-Blocks wilden Darstellungstyp haben. Der Fall \mathfrak{sl}_2 war schon vorher bekannt: In diesem Fall sind die kleinen Quantengruppen zahm, die halben kleinen Quantengruppen sogar darstellungsendlich. Auch die höheren Frobenius-Lusztig-Kerne sind - mit Ausnahme explizit bekannter $SL(2)$ -Blöcke in Höhe eins und der einfachen Steinberg-Blöcke - alle wild. Wir beweisen dieses Resultat unter der Annahme bestimmter Endlichkeitsbedingungen an die Kohomologie. Diese sind für Höhe null bekannt, im Allgemeinen jedoch nicht bewiesen. Der Borel- und der nilpotente Teil der höheren Frobenius-Lusztig-Kerne sind wild.

Auslander-Reiten-Theorie ist ein Werkzeug um die Modul-Kategorie einer endlich-dimensionalen Algebra zu verstehen, indem man irreduzible Homomorphismen beschreibt. Man bildet aus den Isomorphieklassen der unzerlegbaren Moduln und diesen irreduziblen Abbildungen einen orientierten Graphen, den Auslander-Reiten-Köcher. In dieser Arbeit interessieren wir uns für die Form der Zusammenhangskomponenten dieses Köchers. In den zahmen und darstellungsendlichen Fällen versteht man den Auslander-Reiten-Köcher der Frobenius-Lusztig-Kerne vollständig. Im wilden Fall nehmen wir $G \neq SL(2)$ an. Es war bekannt, dass die Zusammenhangskomponenten des Auslander-Reiten-Köchers eines Frobenius-Lusztig-Kerns immer aus einem (endlichen oder unendlichen) Dynkin-Diagramm oder einem Euklidischen Diagramm konstruiert werden können. Im Fall der Frobenius-Lusztig-Kerne zeigen wir, dass der Fall endlicher Dynkin-Diagramme und der Fall Euklidischer Diagramme nicht eintreten können. Im Fall $r = 0$ können wir sogar zeigen, dass, wenn die Komponente die Einschränkung eines Moduls für die unendlich-dimensionale Quantengruppe $U_\zeta(\mathfrak{g})$ enthält, sie aus dem unendlichen Dynkin-Diagramm A_∞ entstanden ist. Für die halben kleinen Quantengruppen zeigen wir, dass die Komponenten, die verträglich graduierte Moduln, also Moduln für $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$, enthalten, auch von A_∞ kommen müssen.

Acknowledgement

First of all I want to thank my advisor Rolf Farnsteiner for stimulating discussions, his "open door policy" and for explaining the classical theory of group schemes.

I also thank Chris Drupieski for answering several question on his work on Frobenius-Lusztig kernels, especially my questions on restrictions of the parameters.

Furthermore my thanks go to my working group for a fruitful working atmosphere and for proof reading, especially to Jean-Marie Bois for proof reading early versions of some chapters and to Julia Worch for proof reading the final draft.

I also would like to thank Henning Haahr Andersen and Yorck Sommerhäuser for answering minor questions concerning this thesis.

Contents

Abstract	v
Zusammenfassung	vii
Acknowledgement	ix
Contents	xi
1. Introduction, notation and prerequisites	1
1.1. Introduction	1
1.2. Outline	3
1.3. Notation	3
1.4. Prerequisites on Lie algebras	4
1.5. Prerequisites on finite dimensional algebras	8
1.6. Prerequisites on homological algebra	11
1.7. Prerequisites on algebraic geometry	15
1.8. Prerequisites on algebraic groups	17
1.9. Prerequisites on Auslander-Reiten theory	19
2. Hopf algebras	25
2.1. Definitions	25
2.2. Basic examples	27
2.3. Extensions of Hopf algebras	28
3. Cohomology and support varieties	35
3.1. Cup product and the even cohomology ring	35
3.2. Support varieties for (fg)-Hopf algebras	43
3.3. Carlson modules	50
3.4. Support varieties via Hochschild cohomology	55
3.5. (Fg)-algebras have infinitely many components	56
4. Quantized enveloping algebras	61
4.1. Definitions of several quantum groups	61
4.2. Integrable modules for quantum groups	68
4.3. Simple modules for the small quantum group	68
4.4. Complexity of modules for the small quantum group	70

CONTENTS

5. Frobenius extensions	75
5.1. General theory	75
5.2. Root subalgebras and the small quantum group	76
6. Graded modules for quantum groups	81
6.1. Modules graded by an infinite torus	81
6.2. Highest weight categories	83
7. Exceptional cases for Auslander-Reiten theory	85
7.1. The representation-finite parts of the small quantum group	85
7.2. The tame small quantum group	86
8. Representation type of small quantum groups	91
8.1. Definitions and general results	91
8.2. Representation type of half-quantum groups	93
8.3. Representation type of blocks	97
9. Auslander-Reiten theory of wild algebras	101
9.1. Webb's Theorem for the small quantum group	101
9.2. Periodic components	104
9.3. Components of Kronecker type	108
9.4. Simply-laced Euclidean components	113
9.5. Components of infinite Dynkin type	119
9.6. The one and only component	125
A. List of Dynkin diagrams	129
A.1. Finite Dynkin diagrams	129
A.2. Simply-laced Euclidean diagrams	129
A.3. Infinite Dynkin diagrams	130
Nomenclature	131
Index	137
Bibliography	141
Erklärung	153

1. Introduction, notation and prerequisites

1.1. Introduction

The basic question in classical representation theory is to understand the module category of a given (finite dimensional) associative algebra, i.e. the classification of all the (finite dimensional) modules and homomorphisms. By the Theorem of Krull-Remak-Schmidt, this classification can be reduced to the case of indecomposable modules. However it turned out that such a classification is not always possible. A celebrated result by Yuri A. Drozd [Dro80] distinguishes the classes of algebras according to whether a classification is (at least theoretically) possible and those where it is not. Thus one divides finite dimensional algebras into three classes, the representation-finite algebras, i.e. the algebras which have only finitely many indecomposable modules up to isomorphism, the tame algebras, where all indecomposable modules of a given dimension can be parameterized by finitely many one-parameter families, and wild algebras, in which case the module category of any (finite dimensional) algebra embeds into this algebra. The class an algebra belongs to is called its representation type. So one of the first questions when given an algebra should be to determine its representation type. If the algebra is tame or representation-finite one can continue by trying to classify all the indecomposable modules, in the other case there is no hope to answer this question.

The other main question that we are concerned with in this thesis is the determination of the Auslander-Reiten quiver of an algebra. The idea behind this is to factor homomorphisms into irreducible ones and to describe how they relate. In contrast to the theory of factorial rings, some elements can be factored infinitely many times, they form the infinite radical of the module category. The Auslander-Reiten quiver is a combinatorial tool to describe the module category up to the infinite radical. It is a (usually infinite) directed graph with the set of isomorphism classes of indecomposable modules as vertices and irreducible maps as arrows. Usually it splits into different components and the question we are concerned with is what their shapes are. For tame algebras (or at least for special classes, most prominently special biserial algebras [WW85]), there are methods to determine the Auslander-Reiten quiver. For wild algebras, the task is much more difficult since a complete classification of the indecomposable modules is impossible.

Two classes of algebras have served as paradigms for understanding representation theory: hereditary algebras and group algebras. For these classes our two main questions were answered.

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

For hereditary algebras Pierre Gabriel [Gab72], Peter W. Donovan and Mary-Ruth Freislich [DF73], and Liudmila A. Nazarova [Naz73] showed that the blocks of representation-finite (respectively tame) algebras are associated to finite Dynkin (respectively Euclidean) diagrams.

For group algebras Donald G. Higman [Hig54], and Vitaly M. Bondarenko and Yurii A. Drozd [BD77] proved that the representation type for blocks of group algebras is controlled by their defect group.

The shape of the components of the Auslander-Reiten quiver was determined by Claus M. Ringel in [Rin78] for hereditary algebras and by Karin Erdmann in [Erd95] for group algebras. Building on these two techniques, the shape of the Auslander-Reiten components is known for canonical algebras and coherent sheaves on a weighted projective line by work of Helmut Lenzing and José A. de la Peña [LdlP97], for local restricted enveloping algebras by work of Karin Erdmann in [Erd96] and for quantum complete intersections by work of Petter Andreas Bergh and Karin Erdmann in [BE11]. Strikingly, in all these cases there is essentially only one type of component that occurs.

In this thesis we want to study the two questions for the Frobenius-Lusztig kernels. These quantum analogues of Frobenius kernels of algebraic groups were introduced by George Lusztig around 1990 [Lus90b] and generalized by Christopher M. Drupieski in 2009 [Dru09]. The interesting point is that these algebras behave very much like restricted enveloping algebras in prime characteristic although they were first defined in characteristic zero. The representation type of the zeroth Frobenius-Lusztig kernel was determined by Jörg Feldvoss and Sarah Witherspoon in [FW09]. In this thesis we will generalize this and prove (under appropriate restrictions on the parameters):

Theorem 1.1.1. *There are only four types of tame or representation-finite blocks of the Frobenius-Lusztig kernels. All other blocks are wild.*

For these four types the classification of the indecomposable modules and the Auslander-Reiten quiver was already determined by various authors. For the other wild algebras we prove partial results that look very much like the results in the case of hereditary algebras and group algebras:

Theorem 1.1.2. *The non-periodic components of the zeroth Frobenius-Lusztig kernels of wild representation type that contain modules for certain infinite quantum groups have the shape $\mathbb{Z}[\mathbb{A}_\infty]$.*

The results build on the case of restricted enveloping algebras, where partial results for the shape of the components were mostly obtained by Karin Erdmann and Rolf Farnsteiner. In contrast to the representation theory of finite groups, for Lie algebras there is no theory of vertices, sources and defect groups. In this case the theory of rank and support varieties proved to be a valuable tool for studying Auslander-Reiten theory. In the quantum case, which we consider, there is a theory of support varieties, but it is difficult to compute them since a theory of rank varieties is missing. The most promising approach towards rank varieties is a homogeneous form of Dade's Lemma proved by Christopher M. Drupieski, whose finite generation results in cohomology were also crucial for our proofs.

1.2. Outline

In Chapter 1 we introduce our basic notation and explain the prerequisites that one needs to know to understand this thesis. Chapter 2 gives the prerequisites on the subject of Hopf algebras. It also includes a section where we prove new results for subalgebras of Hopf algebras. With the exception of Section 5, Chapter 3 is well-known but has never been written up properly except for group algebras. Since some statements and proofs need to be changed, the author felt the need for writing up detailed proofs. Section 5 proves that there are infinitely many Auslander-Reiten components for an algebra of infinite representation type for some class of selfinjective algebras, which is a general conjecture for finite dimensional algebras. Chapter 4 introduces our main class of examples, the quantum groups. All results are well-known and we therefore omit the proofs except for Section 4, where we deal with the support varieties for quantum groups. Section 5.1 introduces the prerequisites to find new examples of Frobenius extensions for quantum groups in Section 5.2. Section 6 deals with graded modules. We prove that the category of compatibly graded objects for the quantum groups admits Auslander-Reiten sequences. In Section 6.2 we recall the known fact that the quantum groups we consider give rise to highest weight categories. The results of Section 7 describing the tame and representation-finite Frobenius-Lusztig kernels are well-known. We include them for the sake of completeness and slightly modify the existing proofs. Chapter 8 and 9 contain the main results of this thesis. In Chapter 8 we determine the representation type of all Frobenius-Lusztig kernels. In Chapter 9 we partially answer the question which types of components occur for them. The Appendix includes a list of the Dynkin diagrams.

1.3. Notation

In this section we fix our basic notation that will be in force throughout this thesis. By k we denote an algebraically closed field. This is not always necessary, but our main results require this assumption because we will make use of Schur's Lemma for algebraically closed fields, we will also apply special statements of algebraic geometry over algebraically closed fields and it simplifies the normal forms that are stated in some places. The field can be of almost arbitrary characteristic p , we just exclude 2 if there is a Lie algebra involved in the statement and 3 if the Lie algebra is of type \mathbb{G}_2 .

For the definition of the quantum groups that we want to study it is useful to first introduce a \mathbb{Z} -form of these algebras. For quantum groups we need the ring of Laurent polynomials that we denote by $\mathbf{A} := \mathbb{Z}[q, q^{-1}]$.

Quantum groups at non-roots of unity are semisimple and therefore not of interest for our questions. Thus we fix here a primitive ℓ -th root of unity ζ , where $\ell > 1$ is an odd integer, not divisible by 3 if the root system mentioned in the corresponding statement is of type \mathbb{G}_2 . This restriction of ℓ is necessary, because the representation theory of quantum groups at even roots of unity is much less understood and one runs into troubles comparable with the case of characteristic two (or three for \mathbb{G}_2) for restricted Lie algebras.

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

We denote by \mathbb{N} , \mathbb{N}_1 , \mathbb{Z} , \mathbb{Q} and \mathbb{C} the set of natural numbers (including zero), natural numbers (excluding zero), integers, rational numbers and complex numbers, respectively. We will write " \subseteq " for an inclusion, " \subsetneq " for a strict inclusion, where the strictness is relevant for the statement and " \subset " for a strict inclusion, such that the strictness is not really necessary or follows from the axioms. Furthermore we write $r \gg 0$ as an abbreviation for the existence of an $r_0 \geq 0$, such that a statements holds for all $r \geq r_0$.

For a k -vector space V , we write $\dim V = \dim_k V$ for its k -dimension. If V is a vector space with basis $\{b_1, \dots, b_n\}$ we denote its k -dual $\text{Hom}_k(V, k)$ by V^* . It has a dual basis $\{b_1^*, \dots, b_n^*\}$. A tensor product \otimes without subscript will always refer to the tensor product \otimes_k over the field k . For a vector space M by abuse of notation we denote by ψ_M the two isomorphisms $M \otimes k \rightarrow M$ and $k \otimes M \rightarrow M$ induced by multiplication.

For quickly looking up notation there is a nomenclature at the end of this thesis.

We assume that the reader is familiar with the basic notions of category theory as can be found e.g. in [Mac98].

1.4. Prerequisites on Lie algebras

We will only give a brief summary of the definitions and main results (without proofs) in the theory of Lie algebras that we need for this thesis. For a detailed account we refer to [EW06], [Hum72] or [Bou02]. We consider only Lie algebras over the field of complex numbers \mathbb{C} .

Definition 1.4.1. A **complex Lie algebra** is a \mathbb{C} -vector space L together with a bilinear map, the **Lie bracket**, $L \times L \rightarrow L$, $(x, y) \mapsto [x, y]$ with the following two properties:

(L1) $[x, x] = 0$ for all $x \in L$

(L2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$

A complex Lie algebra is called **finite dimensional** if the underlying vector space is finite dimensional.

Definition 1.4.2. (i) Let L be a Lie algebra. A subspace I of L is called an **ideal** if $[x, y] \in I$ for all $x \in L, y \in I$.

(ii) A Lie algebra L is called **abelian** if $[x, y] = 0$ for all $x, y \in L$.

(iii) A Lie algebra L is called **simple** if the only ideals in L are 0 and L , and L is not abelian.

(iv) A Lie algebra L is called **semisimple** if it is the direct sum of ideals, which are simple as Lie algebras.

If we are dealing with a finite dimensional semisimple complex Lie algebra (which will always be the case in this thesis) we will use the letter \mathfrak{g} instead of L . This is because there is a deep relationship between semisimple Lie algebras and reductive algebraic groups G (compare Section 1.8).

1.4. PREREQUISITES ON LIE ALGEBRAS

Definition 1.4.3. For each $x \in \mathfrak{g}$ define a linear map $\text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]$, the **adjoint map**.

Definition 1.4.4. Let \mathfrak{g} be a finite dimensional semisimple complex Lie algebra.

- (i) A **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, such that $\text{ad } x$ is diagonalizable for all $x \in \mathfrak{h}$, \mathfrak{h} is abelian and \mathfrak{h} is maximal with this property.
- (ii) Fix a Cartan subalgebra \mathfrak{h} . For each $\alpha \in \mathfrak{h}^*$ we define

$$\mathfrak{g}_\alpha := \{x \in L \mid \text{ad } h(x) = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

- (iii) Let $\Phi := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ be the **set of roots** for \mathfrak{g} .

Lemma 1.4.5. Let \mathfrak{g} be a finite dimensional semisimple complex Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Lemma 1.4.6. Let \mathfrak{g} be finite dimensional semisimple complex Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then the corresponding set of roots Φ spans \mathfrak{h}^* and $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$.

Definition 1.4.7. The **Killing form** $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is defined by

$$\kappa(x, y) = \text{Tr}((\text{ad } x) \cdot (\text{ad } y)),$$

where Tr denotes the trace.

Proposition 1.4.8. The Killing form on a finite dimensional semisimple complex Lie algebra \mathfrak{g} is non-degenerate and even non-degenerate when restricted to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Remark 1.4.9. This proposition provides us with an isomorphism between a Cartan subalgebra \mathfrak{h} and its dual space \mathfrak{h}^* : For $\phi \in \mathfrak{h}^*$ define $t_\phi \in \mathfrak{h}$ via $\kappa(t_\phi, h) = \phi(h)$ for all $h \in \mathfrak{h}$. This way also the Killing form can be transferred to \mathfrak{h}^* by defining $\langle \phi, \psi \rangle_{\mathfrak{h}^*} = \kappa(t_\phi, t_\psi)$.

Proposition 1.4.10. If $E = \mathbb{R}\Phi \subseteq \mathfrak{h}^*$, then $(-, -)_{\mathfrak{h}^*}|_{E \times E}$ is a real-valued inner product on E . This way E becomes a Euclidean space.

Definition 1.4.11. Let $(E, (-, -)_E)$ be a Euclidean vector space. A subset $\Phi \subset E$ is called a **root system** if the following axioms are satisfied:

- (R1) Φ is finite, spans E and does not contain 0.
- (R2) If $\alpha \in \Phi$ and $\lambda \in k$ is such that $\lambda\alpha \in \Phi$ then $\lambda = \pm 1$.
- (R3) If $\alpha \in \Phi$, then the reflection $s_\alpha : E \rightarrow E, \beta \mapsto \beta - \frac{2(\beta, \alpha)_E}{(\alpha, \alpha)_E} \alpha$ leaves Φ invariant.
- (R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle_E := \frac{2(\beta, \alpha)_E}{(\alpha, \alpha)_E} \in \mathbb{Z}$.

A root system Φ is called **irreducible** if whenever $\Phi = \Phi_1 \cup \Phi_2$ with $\langle \alpha, \beta \rangle_E = 0$ for all $\alpha \in \Phi_1, \beta \in \Phi_2$, then one of the subsets is empty.

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

Proposition 1.4.12. *Let \mathfrak{g} be a finite dimensional semisimple complex Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then the set of roots Φ forms a root system for $E := \mathbb{R}\Phi$ with the bilinear form $(-, -)_{\mathfrak{h}^*}|_{E \times E}$. It is irreducible iff the corresponding Lie algebra is simple.*

Lemma 1.4.13. *For any irreducible root system Φ there are at most two root lengths, i.e. values for $(\beta, \beta)_E$ with $\beta \in \Phi$. Furthermore $\frac{(\beta, \beta)_E}{(\alpha, \alpha)_E} \in \{1, 2, 3, \frac{1}{2}, \frac{1}{3}\}$ for all roots $\alpha, \beta \in \Phi$.*

Remark 1.4.14. (i) The values 3 and $\frac{1}{3}$ only occur for the root system of a simple Lie algebra if it is of type G_2 (see Theorem 1.4.21).

(ii) From now on we will work with the **normalized form** $(-, -) := \frac{2(-, -)_E}{l}$, where l is the length of a short root. This way we get a new inner product for E (for which Φ is a root system), such that $(\alpha, \alpha) = 2$ for any short root α . The form $\langle -, - \rangle := \langle -, - \rangle_E$ is independent of the normalization.

Definition 1.4.15. Let Φ be a root system. A subset $\Pi \subset \Phi$ is called a **base** for Φ if

(B1) Π is a vector space basis for E , and

(B2) every $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Pi} k_{\alpha} \alpha$ with $k_{\alpha} \in \mathbb{Z}$ all nonnegative or all nonpositive.

Theorem 1.4.16. *Every root system has a base.*

Definition 1.4.17. Let Φ be a root system with base Π , we call the elements of:

- Φ **roots**
- Π **simple roots**,
- $\Phi^+ := \{\beta \in \Phi | \beta = \sum_{\alpha \in \Pi} k_{\alpha} \alpha, k_{\alpha} \in \mathbb{N}\}$ **positive roots**,
- $\Phi^- := \{\beta \in \Phi | \beta = \sum_{\alpha \in \Pi} k_{\alpha} \alpha, k_{\alpha} \in -\mathbb{N}\}$ **negative roots**,
- $\Phi^{\vee} = \{\beta^{\vee} | \beta \in \Phi\}$ **coroots**, where $\beta^{\vee} := \frac{\beta}{(\beta, \beta)}$ is the coroot corresponding to β .

Lemma 1.4.18. *Let $\alpha, \alpha' \in \Pi$ be nonproportional simple roots, then $(\alpha, \alpha') \leq 0$. If $(\alpha, \alpha') < 0$, then $\alpha + \alpha'$ is also a root.*

Definition 1.4.19. Let Φ be a root system with base $\Pi = \{\alpha_1, \dots, \alpha_n\}$ for some ordering of the simple roots.

- (i) The matrix A defined by $a_{ij} := \langle \alpha_i, \alpha_j \rangle$ is called the **Cartan matrix** of the root system.
- (ii) The **Dynkin diagram** associated to Φ is defined as the partially oriented graph whose vertices are labeled by the simple roots and for two simple roots $\alpha \neq \beta$ there are $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges between α and β . Furthermore if α and β have different length, then the edges are oriented to point at the shorter root.

Remark 1.4.20. The Cartan matrix does not depend on the choice of Π .

Theorem 1.4.21. *The assignment of the Dynkin diagram to the set of roots of a simple Lie algebra sets up a one-to-one correspondence between simple Lie algebras up to isomorphism and the list of graphs in Appendix A.1.*

So for studying semisimple Lie algebras it is sufficient to work with the combinatorial datum given by a Dynkin diagram. For the representation theory of semisimple Lie algebras (and also quantized enveloping algebras, which is what we are interested in) it is often useful to work with some other combinatorial data, that we will introduce in the remainder of this section. Some of it cannot be found in the literature given as a reference for complex Lie algebras. We refer the reader to [Hum90], [Jan03] and [NPV02].

Definition 1.4.22. Let Φ be a root system of E with base $\Pi = \{\alpha_1, \dots, \alpha_n\}$.

- (i) The dual basis vectors to Π^\vee with respect to $(-, -)$, i.e. the vectors $\lambda_i \in E$ with $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$, are called the **fundamental dominant weights**.
- (ii) The set $X := \sum_{i=1}^n \mathbb{Z}\lambda_i$ is called the **weight lattice**, the elements of the subset $X^+ := \sum_{i=1}^n \mathbb{N}\lambda_i$ are called **dominant weights**.
- (iii) We define $\rho := \sum_{i=1}^n \lambda_i \in E$.
- (iv) The subgroup of $W \subset \text{GL}(E)$ generated by the reflections $s_\alpha, \alpha \in \Phi$ is called the **Weyl group**.
- (v) The element $s_{\alpha_1} \cdots s_{\alpha_n}$ is called the **Coxeter element**, its order is denoted by h , the **Coxeter number**.

Remark 1.4.23. (i) The element ρ can also be expressed as $\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

- (ii) The Coxeter element depends on the chosen base and the ordering of the simple roots, but the Coxeter number does not.

For small quantum groups (or also for restricted enveloping algebras) the following sets will also turn out to be useful:

Definition 1.4.24. Let $\ell \in \mathbb{Z}$ and Φ be a root system with base Π .

- (i) The set $X_\ell := \{\lambda \in X \mid (\lambda, \alpha^\vee) < \ell \ \forall \alpha \in \Pi\}$ is called the set of **ℓ -restricted weights**.
- (ii) For $\lambda \in X$ we define the following sets: $\Phi_\lambda := \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha \rangle \in \ell\mathbb{Z}\}$ and $\Phi_\lambda^+ = \Phi_\lambda \cap \Phi^+$.

Remark 1.4.25. We will sometimes identify X_ℓ with $X/\ell X$, especially when we regard X_ℓ as an abelian group. This is done in the following way: Via the canonical projection $X \rightarrow X/\ell X$ the subset X_ℓ is in bijection with $X/\ell X$. This way the addition in $X/\ell X$ can be transferred to X_ℓ .

1.5. Prerequisites on finite dimensional algebras

In this thesis we are mostly concerned with finite dimensional algebras and their representation theory, so in this section we want to state the main definitions and statements that we want to use and fix our notation. For an introduction to the subject we refer the reader to [ARS95], [ASS06], [Ben95] or [EGH⁺11]. The results on Frobenius algebras can be found in [Gab80] and [Lam99].

An algebra will in our context always be an associative k -algebra with a unit element. We will consider left modules and just call them modules. The category of all modules over A will be denoted by $\text{Mod } A$, the full subcategory of all finite dimensional modules by $\text{mod } A$. We denote the zero vector space by 0 . For rings R and S , we understand an (R, S) -bimodule to be an R -left module and an S -right module such that the actions from left and right commute.

Definition 1.5.1. Let A be an algebra.

- (i) An A -module S is called **simple**, if it has no proper submodules except 0 . A complete set of representatives for the isomorphism classes of the simple A -modules will be denoted by \mathcal{S}_A .
- (ii) An A -module M is called **semisimple**, if it is the direct sum of simple modules.
- (iii) An algebra A is called **semisimple**, if every A -module is semisimple.
- (iv) An A -module P is called **projective**, if $\text{Hom}_A(P, -)$ is right exact.
- (v) If M is an A -module, an epimorphism $p : P \rightarrow M$ is called the **projective cover** of M if P is projective and for every homomorphism $f : N \rightarrow P$, such that pf is surjective, also f is surjective. In this case we also call $P(M) := P$ the projective cover of M .
- (vi) Let M be an A -module. A **projective resolution** P_* of M is an exact sequence $\dots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$, where the P_i are projective. Such a sequence is called **minimal** if $d_i : P_i \rightarrow \ker d_{i-1}$ and $d_0 : P_0 \rightarrow M$ are projective covers for all $i \geq 1$.
- (vii) An A -module I is called **injective**, if $\text{Hom}_A(-, I)$ is right exact.
- (viii) If M is an A -module, a monomorphism $i : M \rightarrow I$ is called the **injective hull** of M if I is injective and for every homomorphism $g : I \rightarrow N$, such that gi is injective, also g is injective. In this case we also call $I(M) := I$ the injective hull of M .
- (ix) Let M be an A -module. An **injective resolution** I_* of M is an exact sequence $M \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \rightarrow \dots$, where the I_i are injective. Such a sequence is called **minimal** if $d_i : \text{Coker } d_i \rightarrow I_{i+1}$ and $d_0 : M \rightarrow I_0$ are injective hulls for all $i \geq 0$.

1.5. PREREQUISITES ON FINITE DIMENSIONAL ALGEBRAS

- (x) Let M be an A -module. Then the **socle** of M is the sum of all simple submodules of M , denote it by $\text{soc}(M)$. The preimage of the socle of $M/\text{soc}(M)$ is denoted by $\text{soc}^2(M)$.
- (xi) Let M be an A -module. Then the **radical** of M is the intersection of all maximal submodules of M , denote it by $\text{rad}(M)$. The radical of the radical of M is abbreviated by $\text{rad}^2(M)$.
- (xii) Let M be an A -module. Then the **top** of M is $\text{top}(M) := M/\text{rad}(M)$.
- (xiii) For a non-simple projective-injective A -module P denote the **heart** of P by

$$\text{ht}(P) := \text{rad}(P)/\text{soc}(P).$$

Lemma 1.5.2 (Schur's Lemma). *Let A be a finite dimensional algebra. If S and S' are simple A -modules with $S \not\cong S'$, then $\text{Hom}_A(S, S') = 0$ and $\text{End } S \cong k$.*

Lemma 1.5.3 (Schanuel's Lemma). *Let A be a finite dimensional algebra. If P_1, P_2 are projective modules with submodules M_1 and M_2 respectively, such that $P_1/M_1 \cong P_2/M_2$, then $P_1 \oplus M_2 \cong P_2 \oplus M_1$.*

Definition 1.5.4. Let A and A' be finite dimensional algebras. Then A and A' are called **Morita equivalent** if the corresponding categories of finite dimensional modules are equivalent.

Definition 1.5.5. A finite dimensional algebra is called **basic** if every simple module is one-dimensional.

Basic algebras can be defined purely combinatorially by the following construction:

Definition 1.5.6. (i) A **quiver** Q is a quadruple (Q_0, Q_1, s, t) , where Q_0 is a set, the **vertices**, Q_1 is a set, the **arrows** and $s, t : Q_1 \rightarrow Q_0$ are functions, the **starting** resp. **terminal point**.

(ii) Given a quiver Q , the **path algebra** kQ is defined as the vector space with basis all paths in Q , including a path of length zero for each vertex $i \in Q_0$, denoted by e_i . Multiplication is given by concatenation of paths if this is possible and zero otherwise.

(iii) The ideal generated by all paths of strictly positive length will be denoted by kQ^+ .

(iv) An ideal $I \subseteq kQ$ is called **admissible** if there exists some $m \in \mathbb{N}$, such that $(kQ^+)^m \subseteq I \subseteq (kQ^+)^2$.

The following theorem due to Pierre Gabriel explains why path algebras are especially interesting for our questions:

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

Theorem 1.5.7. *Every finite dimensional algebra A is Morita equivalent to a basic algebra A' . Every basic algebra A' is isomorphic to a quotient of a path algebra kQ by an admissible ideal. In this case the quiver Q is uniquely determined (up to isomorphism of quivers) and called the **(ordinary) quiver** of A .*

The ordinary quiver can be computed explicitly in terms of the algebra.

Definition 1.5.8. Let A be a finite dimensional algebra. A set $\{e_1, \dots, e_n\}$ is called a **complete set of primitive orthogonal idempotents** if the e_i are **idempotents**, i.e. $e_i^2 = e_i$, they are **pairwise orthogonal**, i.e. $e_i e_j = 0$ for $i \neq j$, and **primitive**, i.e. each e_i cannot be written as a sum of two nonzero orthogonal idempotents of A and if in addition $1 = \sum_i e_i$.

Lemma 1.5.9. *Let A be a basic finite dimensional algebra. Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents. Then the ordinary quiver can be defined to have vertices $\{1, \dots, n\}$ and $\dim e_j \text{rad}(A) / \text{rad}^2(A) e_i$ arrows from i to j .*

Almost all algebras that we will encounter in this thesis will be of the following type:

Definition 1.5.10. A finite dimensional algebra A is called **selfinjective** if every projective A -module is also injective.

Definition 1.5.11. For a selfinjective algebra A we define the stable module category $\underline{\text{mod}}A$ as the category with the same objects as $\text{mod}A$ and for A -modules M and N the homomorphism space $\underline{\text{Hom}}(M, N) := \text{Hom}(M, N) / \mathcal{P}(M, N)$, where $\mathcal{P}(M, N)$ are those homomorphisms $f : M \rightarrow N$ that factor through projective modules.

There are several subclasses of the class of selfinjective algebras, that we want to define next:

Definition 1.5.12. Let A be a finite dimensional algebra.

- (i) A is called a **Frobenius algebra** if there exists a k -linear map $\pi : A \rightarrow k$, such that $\ker \pi$ does not contain a non-zero right or left ideal. In this case π is called a **Frobenius homomorphism** for A .
- (ii) For a Frobenius algebra A with Frobenius homomorphism π , an automorphism $\nu_A \in \text{Aut}(A)$ is called a **Nakayama automorphism** if $\pi(xy) = \pi(y\nu_A(x))$.
- (iii) A Frobenius algebra A is called **symmetric** if the identity is a Nakayama automorphism.
- (iv) A selfinjective algebra A is called **weakly symmetric** if $\text{soc} P(S) \cong S$ for every simple A -module S .

Remark 1.5.13. The Frobenius homomorphism is not unique for an algebra A . The Nakayama automorphism is only unique up to inner automorphisms of A .

Lemma 1.5.14.

- (i) Any Frobenius algebra is selfinjective.
- (ii) Any basic selfinjective algebra is Frobenius.
- (iii) Any symmetric algebra is weakly symmetric.

Definition 1.5.15. Let A be a finite dimensional algebra. Then the **Nakayama functor** is defined to be $\mathcal{N}_A := \text{Hom}_A(-, A)^*$.

For Frobenius algebras there is a way to compute the Nakayama functor in terms of the Nakayama automorphism.

Lemma 1.5.16. Let A be a Frobenius algebra. Then we have $\mathcal{N}_A(M) \cong M^{(\nu)}$. Here $M^{(\nu)}$ is the A -module with underlying vector space M and A -module action defined via $a \cdot m := \nu(a)m$ for $a \in A$ and $m \in M$.

For a left A -module M , M^* is a right A -module by defining $(fa)(m) := f(am)$. Conversely if M is a right A -module, then M^* is a left A -module by defining $(af)(m) := f(ma)$ for $f \in M^*$, $a \in A$ and $m \in M$. Using this definition there is an easy procedure to get a symmetric algebra out of every algebra:

Definition 1.5.17. Let A be a finite dimensional algebra. Then the symmetric algebra $T(A) := A \oplus A^*$ with multiplication given by $(a, f)(b, g) = (ab, ag + fb)$ is called the **trivial extension** of A .

1.6. Prerequisites on homological algebra

In this section we want to recall definitions and fundamental statements from homological algebra. We assume the reader is familiar with the basic notions of homological algebra like derived functors. They can be found in [Wei94], [Rot09] or [GM03], where also most of the results stated in this section can be found. Those which are not mentioned there are contained in [Ben95]. For a functor F its i -th right (resp. left) derived functor will be denoted by R^iF (resp. L^iF). We will mainly be concerned with the left exact functors $\text{Hom}_A(M, -)$ (which are covariant) and $\text{Hom}_A(-, M)$ (which are contravariant) for a finite dimensional algebra A and an A -module M . Furthermore we will consider the covariant right exact functors $- \otimes_A M$ and $M \otimes_A -$.

Definition 1.6.1. Let A be a finite dimensional algebra. Then the right derived functors of the functor $\text{Hom}_A(-, M)$ are called **Ext-groups** and denoted $\text{Ext}_A^i(-, M)$.

Lemma 1.6.2. Let A be a finite dimensional algebra. Then the vector spaces $\text{Ext}_A^i(N, M)$ can also be computed as the right derived functors of the functor $\text{Hom}_A(N, -)$.

We will now give the reason for calling these groups groups of extensions:

Definition 1.6.3. An *extension* of M by M' is an exact sequence

$$0 \rightarrow M' \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$$

There is an equivalence relation \sim on the set of extensions induced by $\xi \sim \xi'$, where

$$\xi : M' \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$$

and

$$\xi' : M' \rightarrow M'_{n-1} \rightarrow \cdots \rightarrow M'_0 \rightarrow M \rightarrow 0$$

are extensions, if there exist A -homomorphisms $f_i : M_i \rightarrow M'_i$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc} \xi : & 0 & \longrightarrow & M' & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \downarrow f_{n-1} & & & & \downarrow f_0 & & \parallel & & \\ \xi' : & 0 & \longrightarrow & M' & \longrightarrow & M'_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Denote the set of equivalence classes by $e^n(M, M')$.

Proposition 1.6.4. *There is a bijection Ψ from $e^n(M, M')$ to $\text{Ext}^n(M, M')$. It is constructed as follows: Let $\xi : 0 \rightarrow M' \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$ be an extension and let P_* be a projective resolution of M . Then there exist $\alpha_i : P_i \rightarrow M_i, \alpha_n : P_n \rightarrow M'$ such that the following diagram commutes:*

$$\begin{array}{ccccccccccc} & \cdots & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow \alpha_n & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_0 & & \parallel & & \\ \xi : & 0 & \longrightarrow & M' & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Define $\Psi(\xi) := [\alpha_n] \in \text{Ext}^n(M, M')$.

We now recall some technical lemmas about 1-extensions, i.e. exact sequences:

Lemma 1.6.5. *A 1-extension of finite dimensional modules splits iff its middle term is isomorphic to the direct sum of its end terms.*

Lemma 1.6.6 (Snake lemma). *Let A be an algebra. Let the following be a commutative diagram with exact rows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow h & & \downarrow g & & \\ 0 & \longrightarrow & M' & \longrightarrow & E' & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

Then there is an exact sequence $0 \rightarrow \ker f \rightarrow \ker h \rightarrow \ker g \rightarrow \text{Coker } f \rightarrow \text{Coker } h \rightarrow \text{Coker } g \rightarrow 0$.

1.6. PREREQUISITES ON HOMOLOGICAL ALGEBRA

Lemma 1.6.7 (Horseshoe lemma). *Let A be a finite dimensional algebra. Let $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ be a short exact sequence of A -modules. Let P_* and Q_* be projective resolutions of N and M , respectively. Then there exist homomorphisms such that the given maps fit in the following commutative diagram:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P_* & \longrightarrow & P_* \oplus Q_* & \longrightarrow & Q_* \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

We are now able to define the syzygy or Heller functor Ω_A on modules, homomorphisms and exact sequences:

Definition 1.6.8. Let A be a finite dimensional algebra.

- (i) Let M be an A -module. Then $\Omega_A M$ is defined as the kernel of a projective cover of M . It is uniquely determined up to isomorphism. Dually $\Omega_A^{-1} M$ is defined as the cokernel of an injective hull of M .
- (ii) Let M and N be A -modules, $f : M \rightarrow N$ be a homomorphism. Then the universal properties of projective modules and kernels induce a map $\Omega_A f : \Omega_A M \rightarrow \Omega_A N$. It is uniquely determined in the stable module category. Dually the universal properties of injective modules and cokernels induce a map $\Omega_A^{-1} f : \Omega_A^{-1} M \rightarrow \Omega_A^{-1} N$, uniquely determined in the stable module category.
- (iii) Let M, E, N be A -modules, $\xi : 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ be a short exact sequence. Then an application of the horseshoe and the snake lemma induces a short exact sequence $\Omega_A \xi : 0 \rightarrow \Omega_A M \rightarrow \Omega_A E \oplus P \rightarrow \Omega_A N \rightarrow 0$, where P is a projective module.

For selfinjective algebras this functor will be extremely useful:

Proposition 1.6.9. *Let A be a selfinjective algebra. Then Ω_A and Ω_A^{-1} are inverse autoequivalences of the stable module category.*

For all derived functors there is a long exact sequence, that we recall for Ext-groups in the next proposition:

Proposition 1.6.10. *Let A be a finite dimensional algebra. Let X be an A -module and $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ be a short exact sequence of A -modules.*

(i) *The following sequence of vector spaces is exact:*

$$\dots \rightarrow \text{Ext}_A^i(X, M) \rightarrow \text{Ext}_A^i(X, E) \rightarrow \text{Ext}_A^i(X, N) \rightarrow \text{Ext}_A^{i+1}(X, M) \rightarrow \dots$$

(ii) *Dually the following sequence of vector spaces is exact:*

$$\dots \rightarrow \text{Ext}_A^i(N, X) \rightarrow \text{Ext}_A^i(E, X) \rightarrow \text{Ext}_A^i(M, X) \rightarrow \text{Ext}_A^{i+1}(N, X) \rightarrow \dots$$

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

A useful corollary is the following:

Corollary 1.6.11. *Let A be a finite dimensional algebra. Let M and N be A -modules. Then $\text{Ext}_A^n(M, N) \cong \text{Ext}_A^{n-1}(\Omega_A M, N)$ for all $n > 1$. Furthermore if A is a selfinjective algebra we have $\text{Ext}_A^1(M, N) \cong \underline{\text{Hom}}_A(\Omega_A M, N) \cong \underline{\text{Hom}}_A(M, \Omega_A^{-1} N)$.*

Ext-groups also describe the multiplicity of indecomposable projectives in a projective resolution:

Lemma 1.6.12. *Let A be a finite dimensional algebra. Let S be a simple module. Let M be an A -module and P_* be a minimal projective resolution of M . Then*

$$[P_n : P(S)] = \dim \text{Hom}_A(P_n, S) = \dim \text{Ext}_A^n(M, S),$$

where $[P_n : P(S)]$ denotes the multiplicity of $P(S)$ in P_n .

Lemma 1.6.13. *Let A be an algebra. Let M be a module. Then the isomorphism*

$$\text{Hom}_A(\Omega_A M, S) \cong \text{Ext}_A^1(M, S)$$

can be described as for $0 \neq \phi \in \text{Hom}_A(\Omega_A M, S)$ applying the snake lemma to:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \phi & \longrightarrow & \Omega_A M & \xrightarrow{\phi} & S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_M & \longrightarrow & P_M & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

where the other vertical maps are just the canonical inclusions (and zero).

We now state some statements about spectral sequences. They are a tool for computing derived functors of a composition of functors. We will only apply the corollaries in such a way that one does not need to know what a spectral sequence is. As the reader does not need the theory of spectral sequences to understand the proofs of the results of this thesis, we refrain from stating them here. Again the reader is referred to the above mentioned text books and furthermore to [McC01] and [Jan03].

Proposition 1.6.14 (Grothendieck spectral sequence). *Let \mathcal{C} , \mathcal{D} and \mathcal{E} be abelian categories containing enough injectives. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors, such that G is left exact and F maps injectives in \mathcal{C} to acyclic ones for G . Then for each object M in \mathcal{C} there is a spectral sequence with differentials d_r of bidegree $(r, 1 - r)$, such that*

$$E_2^{s,t} = (R^s G)(R^t F)(M) \Rightarrow R^{s+t}(G \circ F)(M)$$

Corollary 1.6.15. *Let \mathcal{C} , \mathcal{D} and \mathcal{E} be abelian categories containing enough injectives. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors, such that F is exact and sends injective modules to acyclics for G . Then $R^n(G \circ F)(M) = R^n G(F(M))$.*

Corollary 1.6.16. *Let \mathcal{C} , \mathcal{D} and \mathcal{E} be abelian categories containing enough injectives. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors, such that G is exact. Then $R^n(G \circ F)(M) = G \circ R^n F(M)$.*

Proposition 1.6.17. *Let $E_r^{s,t}$ be a spectral sequence. Then there is an exact sequence:*

$$0 \rightarrow E_2^{1,0} \rightarrow E^1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow E^2.$$

For Hopf algebras (that we will define in Chapter 2) tensor products of modules will be of much use. Therefore we will use the remaining part of this section to recall some information about them.

Definition 1.6.18. If $\mathbf{C} = (C_n, f_n)$ and $\mathbf{D} = (D_n, g_n)$ are complexes of vector spaces, then $\mathbf{C} \otimes \mathbf{D}$ is defined as the following complex: $(\mathbf{C} \otimes \mathbf{D})_n := \bigoplus_{s+t=n} C_s \otimes D_t$ with differentials ∂_n given by $c_s \otimes d_t \mapsto f_s(c_s) \otimes d_t + (-1)^t c_s \otimes g_t(d_t)$.

Theorem 1.6.19 (Künneth formula). *Let P and Q be complexes of vector spaces. Then we have: $H_n(P \otimes Q) \cong \bigoplus_{i+j=n} H_i(P) \otimes H_j(Q)$ defined via $[f \otimes g] \mapsto [f] \otimes [g]$. Here $H_n(C_*)$ denotes the n -th homology of the complex C_* .*

1.7. Prerequisites on algebraic geometry

In this section we recall some statements from commutative algebra and algebraic geometry that are necessary to understand the results about support varieties in Chapter 3. Our main sources will be [Eis95] for commutative algebra, and [Har77] for algebraic geometry. One may also consult [AM69] for commutative algebra and [GW10] for algebraic geometry. We suppose that the reader is familiar with the basic concepts like localization and integral extensions. We will denote the localization of a module M by a multiplicatively closed subset U of a commutative ring R by $U^{-1}M$. Furthermore we will denote the set of prime ideals (resp. maximal ideals) of a ring R by $\text{Spec}(R)$ (resp. $\text{maxSpec}(R)$).

Proposition 1.7.1. *Let U be a multiplicatively closed set of a commutative ring R and let M be an R -module. Then an element m goes to 0 in $U^{-1}M$ iff m is annihilated by an element $u \in U$. In particular, if M is finitely generated, then $U^{-1}M = 0$ iff M is annihilated by an element of U .*

Lemma 1.7.2 (Nakayama's Lemma). *Let I be an ideal contained in the Jacobson radical of a ring R and let M be a finitely generated R -module. Then $IM = M$ implies $M = 0$.*

We will now define the basic geometric objects that we will work with, the affine varieties. The most important example for us is the support variety defined in Section 3.2 and 3.4.

Definition 1.7.3. (i) Let $I \subseteq k[X_1, \dots, X_n]$ be an ideal. Then the **vanishing set** for I is defined as $Z(I) := \{x \in k^n \mid f(x) = 0 \forall f \in I\}$.

(ii) The topology $\mathcal{O} := \{k^n \setminus Z(I) \mid I \subseteq k[X_1, \dots, X_n]\}$ is called **Zariski topology**. The set k^n is denoted by \mathbb{A}^n if we view it as a topological space with Zariski topology.

(iii) A subset $V \subseteq \mathbb{A}^n$ is called **affine variety**, if there exists an ideal $I \subseteq k[X_1, \dots, X_n]$, such that $V = Z(I)$. The **morphisms of affine varieties** are the restrictions of the maps, that are polynomial in each coordinate.

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

- (iv) If $V \subseteq \mathbb{A}^n$ is an affine variety, then $I(V) := \{f \in k[X_1, \dots, X_n] \mid f(x) = 0 \forall x \in V\}$ is called the **vanishing ideal** of V .
- (v) If $V \subseteq \mathbb{A}^n$ is an affine variety, then $k[V] := k[X_1, \dots, X_n]/I(V)$ is called the **coordinate ring** of V .

Lemma 1.7.4. *We have $Z(I \cup J) = Z(I) \cap Z(J)$ and $Z(IJ) = Z(I) \cup Z(J)$.*

Lemma 1.7.5. *The natural inclusions $R, S \rightarrow R \otimes S$ induce an isomorphism $\max\text{Spec}(R \otimes S) \cong \max\text{Spec}(R) \times \max\text{Spec}(S)$.*

We proceed with some dimension theory.

- Definition 1.7.6.** (i) Let R be a ring. Then $\text{Kdim } R := \sup\{n \in \mathbb{N} \mid \exists \mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n \in \text{Spec}(R)\}$ is called the **Krull dimension** of R .
- (ii) A topological space X is called **irreducible** if $X = X_1 \cup X_2$ with closed sets X_1, X_2 implies $X_1 = X$ or $X_2 = X$.
 - (iii) Let X be a topological space. Define $\text{Kdim } \emptyset = -1$ and for $X \neq \emptyset$ define $\text{Kdim } X := \sup\{n \in \mathbb{N} \mid \exists \emptyset \neq X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n \subsetneq X \text{ with } X_i \text{ closed and irreducible}\}$. Then $\text{Kdim } X$ is called the **Krull dimension** of X .

Proposition 1.7.7. *Let $V \subseteq k^n$ be an affine variety. Then $\text{Kdim } V = \text{Kdim } k[V]$.*

Sometimes it will be easier to work with projective varieties instead of affine varieties. These correspond to homogeneous polynomials. We will work with the projectivized support varieties.

- Definition 1.7.8.** (i) Let G be a semigroup. A ring R is called **G -graded**, if there are subgroups $R_n, n \in G$, such that $R = \bigoplus_{n \in G} R_n$ and $R_n R_m \subseteq R_{n+m}$ for all $n, m \in G$.
- (ii) Let G be a partially ordered semigroup. Then a ring R is called **G -filtered** if $R_i \subseteq R_j$ for $i \leq j$, $R_i R_j \subseteq R_{i+j}$ for all $i, j \in G$ and $\bigcup_{i \in G} R_i = R$.
 - (iii) Let G be a partially ordered semigroup. Let R be a G -filtered ring. Then the **associated graded** is defined to be the G -graded ring $\text{gr}(R) := \bigoplus_{g \in G} \text{gr}(R)_g$, where $\text{gr}(R)_g := R_g / \left(\sum_{h \in G, h < g} R_h \right)$.
 - (iv) Let R be an \mathbb{N} -graded ring. An ideal $I \subseteq R$ is called **homogeneous** if we have $I = \bigoplus_{n \geq 0} I \cap R_n$.
 - (v) The set $\mathbb{P}^n := k^{n+1} \setminus \{0\}/k^\times$ is called **n -dimensional projective space**. Its elements are denoted $[x_0, \dots, x_n]$.
 - (vi) Let $I \subseteq k[X_0, \dots, X_n]$ be a homogeneous ideal. Then $Z(I) := \{p \in \mathbb{P}^n \mid f(p) = 0 \forall f \in I\}$ is called the **vanishing set** of I .
 - (vii) A subset $V \subseteq \mathbb{P}^n$ is called **projective variety** if there exists a homogeneous ideal $I \subseteq k[X_0, \dots, X_n]$, such that $V = Z(I)$.

(viii) The topology $\mathcal{O} := \{\mathbb{P}^n \setminus V \mid V \subseteq \mathbb{P}^n \text{ projective variety}\}$ is called the **Zariski topology** on \mathbb{P}^n .

Certain affine varieties correspond to projective varieties:

Definition 1.7.9. Let $\pi : \mathbb{A}^{n+1} \rightarrow \mathbb{P}^n$ be the canonical projection. An affine variety $V \subseteq \mathbb{A}^{n+1}$ is called **conical** if $\pi^{-1}\pi V = V$.

Lemma 1.7.10. A variety $V \subseteq \mathbb{A}^{n+1}$ is conical iff $I(V)$ is homogeneous.

1.8. Prerequisites on algebraic groups

In this section we recall the basic statements about algebraic groups. Our main sources are [Spr98], [Hum75] and [Bor91]. For our purposes there are two important topics, the study of the action of algebraic groups on varieties and the theory of semisimple algebraic groups, that correspond to semisimple Lie algebras. We suppose that the reader is familiar with the basic (ordinary) group theory.

Definition 1.8.1. An **algebraic group** is a group which in addition is a variety, such that multiplication and inverse are given by morphisms of varieties.

We start with some theorems about the action of groups on varieties:

Theorem 1.8.2 (Borel's fixed point theorem). *Let G be a connected solvable algebraic group and let X be a (nonempty) projective variety on which G acts. Then G has a fixed point in X .*

Definition 1.8.3. Let G be an algebraic group. Let X be a variety on which G acts. Then the **stabilizer** of $x \in X$ is the closed subgroup $G_x := \{g \in G \mid g \cdot x = x\}$.

Definition 1.8.4. Let G be a connected algebraic group. The maximal closed, connected, normal, solvable subgroup of G is called the **radical** of G . An algebraic group is called **semisimple** if its radical is trivial.

Theorem 1.8.5. *Let G be a connected semisimple algebraic group. Then $G = G_1 \cdots G_r$, where the G_i are the minimal non-zero connected, closed, normal subgroups of G of positive dimension. Furthermore we have $(G_i, G_j) = \{e\}$.*

Definition 1.8.6. Let G be an algebraic group.

- (i) A closed, connected, solvable subgroup of G , which is maximal for these properties is called a **Borel subgroup**.
- (ii) A closed subgroup of G is called **parabolic** iff it includes a Borel subgroup.

Proposition 1.8.7. *Let G be an algebraic group acting on a variety X . Then the orbits of minimal dimensions are closed.*

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

Proposition 1.8.8 (Orbit formula). *Let G be an algebraic group acting on a variety X . Let $x \in X$. Then we have*

$$\mathrm{Kdim} G = \mathrm{Kdim} G \cdot x + \mathrm{Kdim} G_x.$$

We now turn to small examples of algebraic groups that correspond to the simple Lie algebras of type \mathbb{A}_n . Here we refer the reader additionally to [Jan03] and especially for the representation theory of SL_2 to [Fis82]:

Definition 1.8.9. The general linear group GL_n and the special linear group SL_n are algebraic groups. The quotient of the general linear group GL_n by its center is denoted PGL_n .

Lemma 1.8.10. *The group PGL_n is simple (as an abstract group).*

Proposition 1.8.11 ([Far00b, Proof of Proposition 5.1]). *Let P be a parabolic subgroup of an algebraic group G , such that $\mathrm{Kdim} G/P = 1$, then there is a surjective morphism $\varphi : G \rightarrow \mathrm{PGL}_2$.*

Definition 1.8.12. Let G be an algebraic group.

- (i) A **G -representation** is a morphism of algebraic groups $G \rightarrow \mathrm{GL}_n$.
- (ii) Let $\mathrm{char} k = p > 0$. Define $G^{(r)}$ to be the algebraic group whose coordinate ring is the same as that of $k[G]$ but with scalar multiplication given by $\lambda \cdot x := \lambda^{p^{-r}} x$. The **Frobenius homomorphism** is the map $G^{(r)} \rightarrow G$ corresponding to the comorphism $k[G] \rightarrow k[G^{(r)}]$ given by $x \mapsto x^{p^r}$. The **r -th Frobenius kernel** G_r is defined to be the scheme theoretic kernel of the Frobenius homomorphism (for details see e.g. [Jan03] or [Wat79]).

Proposition 1.8.13. *For $G = \mathrm{SL}_2$ and $\mathrm{char} k = p \geq 3$ there are p simple G_1 -representations $L(0), \dots, L(p-1)$, such that $L(0)$ is the trivial module, $L(p-1)$ is projective and $\dim L(i) = i + 1$. The full subcategory of representations with composition factors $L(i)$ and $L(p-i-2)$ is equivalent to the category of modules for the trivial extension of the Kronecker algebra for $0 \leq i < p-1$.*

Lemma 1.8.14. *If $G = \mathrm{SL}_2$ and $\mathrm{char} k = p \geq 3$, then we have $L(1) \otimes L(j) \cong L(j-1) \oplus L(j+1)$ as G_1 -modules for $j < p-1$.*

The last part of this section concerns the relationship between algebraic groups and Lie algebras.

Definition 1.8.15. Let G be an algebraic group. Denote by λ_g the action of G on $k[G]$ defined by $\lambda_g f(x) := f(g^{-1}x)$. Then $\mathrm{Lie}(G) := \{D \in \mathrm{Der}_k(k[G], k[G]) \mid \lambda_g \circ D = D \circ \lambda_g \forall g \in G\}$ is called the **Lie algebra** of G . Here Der_k denotes the set of k -linear derivations.

Lemma 1.8.16. *The Lie algebra of an algebraic group is a Lie algebra. Lie can be extended to a functor from algebraic groups to Lie algebras.*

Lemma 1.8.17. *Let G be a closed subgroup of $\mathrm{GL}(n, k)$. If G leaves stable a subspace W of k^n , then $\mathrm{Lie}(G) \subseteq \mathfrak{gl}(n, k)$ also does.*

Lemma 1.8.18. *Let H be a closed subgroup of an algebraic group G . Let $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\mathrm{Lie}(C_G(H)) \subseteq C_{\mathfrak{g}}(\mathrm{Lie}(H))$, where C_G and $C_{\mathfrak{g}}$ denote the centralizers.*

1.9. Prerequisites on Auslander-Reiten theory

Auslander-Reiten theory is a tool to classify modules and morphisms of modules by decomposing them into certain minimal ones. For a detailed introduction to this topic we refer to [ARS95], [ASS06] or [Ben95].

The following classical theorem gives the decomposition of modules into indecomposable ones:

Theorem 1.9.1 (Krull-Remak-Schmidt). *Let A be a finite dimensional algebra. Let M be a finite dimensional module. Then $M \cong \bigoplus_{i=1}^t M_i$ with M_i indecomposable and this decomposition is unique up to renumbering and isomorphism.*

The notion of a minimal morphism is given by the following definition:

Definition 1.9.2. Let A be a finite dimensional algebra. A morphism $f : V \rightarrow W$ of A -modules is called *irreducible* if:

- f is not a split monomorphism,
- f is not a split epimorphism,
- for any factorization $f = f_2 f_1$, f_1 is a split monomorphism or f_2 is a split epimorphism.

The irreducible maps can be identified with a vector space:

Definition 1.9.3. Let A be a finite dimensional algebra. For X, Y indecomposable finite dimensional A -modules define

$$\mathrm{rad}(X, Y) := \{f : X \rightarrow Y \mid f \text{ not invertible}\}$$

and for $X, Y \in \mathrm{mod} A$ with $X = \bigoplus X_i$ and $Y = \bigoplus Y_j$ with X_i, Y_j indecomposable

$$\mathrm{rad}(X, Y) = (\mathrm{rad}(X_i, Y_j))_{i,j}.$$

Then if $\mathrm{rad}^2(X, Y) = \{f \mid \exists g \in \mathrm{rad}(X, M), h \in \mathrm{rad}(M, Y) \text{ with } f = hg\}$ the space

$$\mathrm{Irr}(X, Y) := \mathrm{rad}(X, Y) / \mathrm{rad}^2(X, Y)$$

is called the *bimodule of irreducible maps*.

Lemma 1.9.4. *Let A be a finite dimensional algebra. If $X, Y \in \mathrm{mod} A$, then $f : X \rightarrow Y$ is irreducible iff $f \in \mathrm{rad}(X, Y) \setminus \mathrm{rad}^2(X, Y)$.*

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

Definition 1.9.5. Let A be a finite dimensional algebra. A short exact sequence $0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} N \rightarrow 0$ is called **Auslander-Reiten sequence** or **almost split sequence** if f and g are irreducible morphisms.

Example 1.9.6. Let A be a finite dimensional algebra. For every non-simple projective-injective module P there is the following almost split sequence:

$$0 \rightarrow \text{rad}(P) \rightarrow P \oplus \text{rad}(P)/\text{soc } P \rightarrow P/\text{soc } P \rightarrow 0.$$

Theorem 1.9.7. Let A be a finite dimensional algebra. For every indecomposable non-projective module M there exists an Auslander-Reiten-sequence $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$. The module N is uniquely determined by M up to isomorphism. We denote it by $\tau_A(M)$. Dually for each non-injective module N there exists an Auslander-Reiten sequence $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$. Here the module M is uniquely determined by N up to isomorphism. We denote it by $\tau_A^{-1}(N)$.

This can be proven by the following theorem, which we only formulate in the selfinjective context:

Theorem 1.9.8 (Auslander-Reiten formula). Let A be a finite dimensional selfinjective algebra. Let M, N be finite dimensional nonprojective modules. Then there exist natural isomorphisms

$$\text{Ext}_A^1(M, N) \cong D\text{Hom}(\tau_A^{-1}N, M) \cong D\text{Hom}_A(N, \tau M),$$

Lemma 1.9.9. Let A be a finite dimensional algebra. Let $f : X \rightarrow Y$ be either part of an Auslander-Reiten sequence $0 \rightarrow X \xrightarrow{f} Y \rightarrow \text{Coker } f \rightarrow 0$ or the map $I \rightarrow I/\text{soc } I$ and $f' : X \rightarrow Y'$ be arbitrary. Then the following are equivalent:

(1) $\exists f'' : X \rightarrow Y''$ and an isomorphism $h : Y \rightarrow Y' \oplus Y''$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \left(\begin{array}{c} f' \\ f'' \end{array} \right) \downarrow & & \swarrow h \\ Y' \oplus Y'' & & \end{array}$$

(2) f' is irreducible or $Y' = 0$.

Remark 1.9.10. This lemma and its dual essentially say that any irreducible map is part of an Auslander-Reiten sequence. This answers the question how the Auslander-Reiten quiver locally looks like.

We now combine the information given by the indecomposable modules and irreducible morphisms into an oriented graph, the so called Auslander-Reiten quiver:

Definition 1.9.11. (i) Let (Q_0, Q_1, s_1, t_1) and (Q_0, Q_2, s_2, t_2) be quivers. Then the 7-tuple $(Q_0, Q_1, Q_2, s_1, t_1, s_2, t_2)$ is called **biquiver**.

1.9. PREREQUISITES ON AUSLANDER-REITEN THEORY

- (ii) Let A be a finite dimensional algebra. The **Auslander-Reiten quiver** $\Gamma(A)$ is defined as follows. The set of vertices is

$$\Gamma(A)_0 := \{[M] \mid M \text{ indecomposable } A\text{-module}\}.$$

The set of 1-arrows is

$$\Gamma(A)_1 := \{ [X] \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} [Y] \mid \dim \text{Irr}(X, Y) = n \}.$$

The set of 2-arrows is

$$\Gamma(A)_2 := \{ [\tau_A X] \leftarrow - - [X] \}.$$

- (iii) If A is a selfinjective algebra, then the **stable Auslander-Reiten quiver** $\Gamma_s(A)$ is obtained from the Auslander-Reiten quiver by removing all projective vertices (and all arrows adjacent to them).

For selfinjective algebras, the syzygy is of great use, since it provides a method to compute the Auslander-Reiten translation and to switch between different components:

Proposition 1.9.12. *Let A be a selfinjective algebra. Then Ω_A is a stable equivalence. In particular it induces an isomorphism $\text{Irr}(M, N) \cong \text{Irr}(\Omega_A M, \Omega_A N)$ for non-projective indecomposable A -modules M, N . In particular if $0 \rightarrow M \rightarrow E \oplus P \rightarrow N \rightarrow 0$ is an Auslander-Reiten sequence with M, N non-projective, P projective and E not containing any projective direct summands, then there is a projective module P' such that $0 \rightarrow \Omega_A M \rightarrow \Omega_A E \oplus P' \rightarrow \Omega_A N \rightarrow 0$ is an Auslander-Reiten sequence.*

Lemma 1.9.13. *If A is a selfinjective algebra, then $\tau_A(M) \cong \Omega_A^2 \circ \mathcal{N}_A(M) \cong \mathcal{N}_A \circ \Omega_A^2(M)$ for all modules M , that have no nonzero projective direct summand.*

For the next theorem we need the following definition:

Definition 1.9.14. (i) Let T be a tree or an oriented cycle with set of vertices I . Then $\mathbb{Z}[T]$ is defined as the following biquiver: The set of vertices is $\mathbb{Z} \times I$. The set of 1-arrows is given by first choosing an orientation of T (not the cyclic for the cycle) and then adding arrows $(z, i) \rightarrow (z, j)$ and $(z, j) \rightarrow (z + 1, i)$ in $\mathbb{Z}[T]$ for every arrow $i \rightarrow j$ in T and all $z \in \mathbb{Z}$. The set of 2-arrows is given by $(z + 1, i) \dashrightarrow (z, i)$ for all $z \in \mathbb{Z}$ and $i \in I$.

- (ii) Let Γ be a biquiver. A group $G < \text{Aut}(\Gamma)$ is called **admissible** if no orbit of G on Γ_0 intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point.

The question of what form the Auslander-Reiten quiver can be, was answered by Christine Riedtmann:

Theorem 1.9.15 (Riedtmann Struktursatz). *Every component of the stable Auslander-Reiten quiver is of the form $\mathbb{Z}[T]/G$ where T is a tree and G is an admissible group. Moreover T is uniquely determined by the component.*

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

Definition 1.9.16. For a component Θ of the stable Auslander-Reiten quiver, such that $\Theta \cong \mathbb{Z}[T]/G$ for some admissible group action G , we call T the **tree class** of Θ .

In 1974 Maurice Auslander answered the question whether finite components can occur. His question implies the famous Brauer-Thrall conjecture which stated that if all indecomposable modules over an algebra have bounded length, then there are only finitely many indecomposable modules up to isomorphism.

Theorem 1.9.17. *Let A be a finite dimensional connected (i.e. the ordinary quiver of A is connected) algebra and let Θ be a component of the Auslander-Reiten quiver of A , such that all indecomposable modules of Θ have length at most b . Then Θ is finite and is the whole Auslander-Reiten quiver of A . In particular, A is representation-finite (see Section 8.1 for the definition).*

The second Brauer-Thrall conjecture states as follows. For algebras over algebraically closed fields it was first proven by Liudmila A. Nazarova and Andrei V. Roiter in [NR73]. Later there were different proofs from various authors in this case. For non-algebraically closed fields the conjecture is open in general.

Theorem 1.9.18. *Let A be a finite dimensional algebra, which is not representation-finite (see Section 8.1 for the definition). Then there exists an infinite sequence of numbers $d_i \in \mathbb{N}$ such that for each i there exists an infinite number of nonisomorphic indecomposable A -modules with dimension d_i .*

The simplest non-semisimple algebras are the following:

Definition 1.9.19. An algebra A is called **Nakayama** if all projective and injective indecomposable A -modules are uniserial, i.e. have a unique composition series.

Theorem 1.9.20. *A connected algebra A is Nakayama iff its ordinary quiver is either a linearly oriented \mathbb{A}_n or a linearly oriented $\tilde{\mathbb{A}}_n$. If A is selfinjective, then only the second case appears. Every Nakayama algebra is representation-finite.*

The only algebras considered in this thesis where one can classify all indecomposable modules up to isomorphism, i.e. the algebra is representation-finite or tame (see Section 8.1 for the definition), will be special biserial. For them the classification was done by Burkhard Wald and Josef Waschbüsch in 1985. As a reference on special biserial algebras we refer the reader to [WW85] and [BR87].

Definition 1.9.21. A finite dimensional algebra A is called **special biserial** if it is Morita equivalent to a path algebra with relations kQ/I , such that the following properties are satisfied:

- (SB1) In each point of the quiver there start at most two arrows.
- (SB1') In each point of the quiver there end at most two arrows.
- (SB2) Whenever there are arrows $\alpha \neq \beta \in Q_1$ and $\gamma \in Q_1$, such that $s(\alpha) = s(\beta) = t(\gamma)$ we have that $\alpha\gamma \in I$ or $\beta\gamma \in I$.

1.9. PREREQUISITES ON AUSLANDER-REITEN THEORY

(SB2') Whenever there are arrows $\alpha \neq \beta \in Q_1$ and $\gamma \in Q_1$, such that $s(\gamma) = t(\alpha) = t(\beta)$ we have that $\gamma\alpha \in I$ or $\gamma\beta \in I$.

Definition 1.9.22. A special biserial algebra is called a **string algebra** if I is generated by paths.

Theorem 1.9.23. *If A is a special biserial algebra, then $A/\bigoplus_i \text{soc}(P_i)$ is a string algebra, where the sum is taken over all non-uniserial projective-injective modules. The indecomposable modules for A agree with the modules for $A/\bigoplus_i \text{soc}(P_i)$ with the exception of the non-uniserial projective-injective modules.*

Definition 1.9.24. Let kQ/I be a string algebra. To each vertex $u \in Q_0$ we associate two trivial words, $1_{(u,1)}$ and $1_{(u,-1)}$.

- (i) A word $C = c_1 \cdots c_n$ in $Q_1 \cup Q_1^{-1}$ is called a **string** if $c_i \neq c_{i+1}^{-1}$ for all i and no subword or inverse subword is in I or $C = 1_{(u,t)}$, where $u \in Q_0$ and $t = \pm 1$. There is an equivalence relation on the set of all strings induced by $C \sim C^{-1}$, where $1_{(u,t)}^{-1} = 1_{(u,-t)}$. A complete set of representatives for the equivalence classes will be denoted by **String**.
- (ii) A string B is called a **band** if B^n is a string for all n and $B \neq (B')^m$ for all $m > 1$ and all words B' . There is an equivalence relation on the set of all bands induced by $B \sim B^{-1}$ and $b_1 \cdots b_n \sim b_2 \cdots b_n b_1$. A complete set of representatives for the equivalence classes will be denoted by **Band**.

Definition 1.9.25. Let kQ/I be a string algebra.

- (i) Let $C = c_1 \cdots c_n$ be a string. Then we define a module $M(C) = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ via $V_i := \bigoplus_{j: s(c_j)=i \text{ or } j=0 \text{ and } t(c_1)=i} \langle z_j \rangle_k$. We define V_α via $V_\alpha(z_i) = \begin{cases} z_{i-1} & \text{if } c_i = \alpha \\ z_{i+1} & \text{if } c_{i+1} = \alpha^{-1} \\ 0 & \text{else} \end{cases}$.
- (ii) Let $B = b_1 \cdots b_n$ be a band. Then we define a module $M(B, n, \lambda) = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ via $W_j := k^n$ and $V_i := \bigoplus_{s(b_j)=i} W_j$ and maps V_α defined for $j \leq n-1$ via $V_\alpha|_{W_j} = \text{id}_{k^n} : W_j \rightarrow W_{j-1}$ if $b_j = \alpha$, $V_\alpha|_{W_j} = \text{id}_{k^n} : W_j \rightarrow W_{j+1}$ if $b_{j+1} = \alpha^{-1}$ and for $j = n$ by $V_\alpha|_{W_n} = J(n, \lambda) : W_n \rightarrow W_{n-1}$ if $b_n = \alpha$ and $V_\alpha|_{W_{n-1}} = J(n, \lambda) : W_{n-1} \rightarrow W_n$ if $b_n = \alpha$.

Theorem 1.9.26. *Let A be a string algebra. Then the $M(C)$ and $M(B, n, \lambda)$ for $C \in \mathbf{String}$, $B \in \mathbf{Band}$, $\lambda \in k$, $n \in \mathbb{N}$ form a complete set of representatives for the isomorphism classes of indecomposable modules and the following are all Auslander-Reiten sequences in $\text{mod } A$:*

- (i) $0 \rightarrow M(B, 1, \lambda) \rightarrow M(B, 2, \lambda) \rightarrow M(B, 1, \lambda) \rightarrow 0$ for every band B and every $\lambda \neq 0$.
- (ii) $0 \rightarrow M(B, n, \lambda) \rightarrow M(B, n-1, \lambda) \oplus M(B, n+1, \lambda) \rightarrow M(B, n, \lambda)$ for every band B , every $\lambda \neq 0$ and every $n > 1$.

CHAPTER 1. INTRODUCTION, NOTATION AND PREREQUISITES

(iii) $0 \rightarrow M(\gamma_r^{-1} \cdots \gamma_1^{-1}) \rightarrow M(\gamma_r^{-1} \cdots \gamma_1^{-1} \beta \delta_1^{-1} \cdots \delta_s^{-1}) \rightarrow M(\delta_1^{-1} \cdots \delta_s^{-1}) \rightarrow 0$ for every arrow $\beta \in Q_1$ and r, s maximal, such that $\gamma_r^{-1} \cdots \gamma_1^{-1} \beta \delta_1^{-1} \cdots \delta_s^{-1}$ is a string. Denote the starting term of this sequence by $U(\beta)$, its end term by $V(\beta)$.

(iv) For every string C , such that $M(C)$ is not injective and not isomorphic to some $U(\beta)$ for some $\beta \in Q_1$, such that there is an arrow $\beta_0 \in Q_1$ with $\beta_0^{-1}C$ a string and there is an arrow γ_0 with $C\gamma_0$ a string:

$$0 \rightarrow M(C) \rightarrow M(\beta_r \cdots \beta_1 \beta_0^{-1}C) \oplus M(C\gamma_0\gamma_1^{-1} \cdots \gamma_s^{-1}) \rightarrow M({}_h C_h) \rightarrow 0,$$

where r, s are maximal such that ${}_h C_h := \beta_r \cdots \beta_1 \beta_0^{-1}C\gamma_0\gamma_1^{-1} \cdots \gamma_s^{-1}$ is a string.

(v) For every string C , such that $M(C)$ is not injective and not isomorphic to one of the $U(\beta)$ for some $\beta \in Q_1$, and such that there is an arrow γ_0 with $C\gamma_0$ a string, but no arrow β_0 with $\beta_0^{-1}C$ a string, we have $C = \delta_r^{-1} \cdots \delta_1^{-1} \delta_0 D$ and an Auslander-Reiten sequence:

$$0 \rightarrow M(C) \rightarrow M(D) \oplus M(C\gamma_0\gamma_1^{-1} \cdots \gamma_s^{-1}) \rightarrow M(D\gamma_0\gamma_1^{-1} \cdots \gamma_s^{-1}) \rightarrow 0,$$

where s is maximal, such that $C\gamma_0\gamma_1^{-1} \cdots \gamma_s^{-1}$ is a string.

(vi) For every string C , such that $M(C)$ is not injective and not isomorphic to some $U(\beta)$ for some $\beta \in Q_1$ and there is no arrow γ_0 with $C\gamma_0$ a string, but there is an arrow β_0 with $\beta_0^{-1}C$ a string, we have $C = D\delta_0^{-1}\delta_1 \cdots \delta_r$ and there is the following Auslander-Reiten sequence:

$$0 \rightarrow M(C) \rightarrow M(\beta_s \cdots \beta_1 \beta_0^{-1}C) \oplus M(D) \rightarrow M(\beta_s \cdots \beta_1 \beta_0^{-1}D) \rightarrow 0,$$

where s is maximal, such that $\beta_s \cdots \beta_1 \beta_0^{-1}C$ is a string. For every string C , such that $M(C)$ is not injective and not isomorphic to one of $U(\beta)$ for some $\beta \in Q_1$ and no other of the cases (iv)-(vi) occurs, there have to exist $r, s \in \mathbb{N}$, such that $C = \gamma_r^{-1} \cdots \gamma_1^{-1} \gamma_0 D \beta_0^{-1} \beta_1 \cdots \beta_s$ and the following is an Auslander-Reiten sequence:

$$0 \rightarrow M(C) \rightarrow M(D\beta_0^{-1}\beta_1 \cdots \beta_s) \oplus M(\gamma_r^{-1} \cdots \gamma_1^{-1} \gamma_0 D) \rightarrow M(D) \rightarrow 0.$$

If A is a special biserial algebra. Then a complete set of isomorphism classes of indecomposable modules is given by the $M(C)$ and the $M(B, n, \lambda)$ for the algebra $A_{str} := A / \text{soc} \bigoplus P_i$, where the sum is taken over all non-uniserial projective-injective indecomposable modules, together with the P_i . For the Auslander-Reiten quiver of A one has to insert the sequences $0 \rightarrow \text{rad}(P_i) \rightarrow P_i \oplus \text{rad}(P_i) / \text{soc}(P_i) \rightarrow P_i / \text{soc}(P_i) \rightarrow 0$ into the Auslander-Reiten quiver of A_{str} .

2. Hopf algebras

In this section we provide an introduction into the theory of Hopf algebras. They form the class of algebras we will be concerned with in the main part of this thesis. Most of the examples of algebras we deal with will be Hopf algebras. The basic examples the reader might already know of are group algebras, universal enveloping algebras and algebras of distributions. For an introduction to the general theory of Hopf algebra we refer to [Kas95] or [Mon93]. In Section 2.3 we will prove some new results about subalgebras of Hopf algebras.

2.1. Definitions

For an algebra A we have a multiplication map $\mu : A \otimes A \rightarrow A$ and a unit $\eta : k \rightarrow A$.

Definition 2.1.1. An algebra A is called *augmented* if there is an algebra map $\varepsilon : A \rightarrow k$.

Definition 2.1.2. An algebra A is called *Hopf algebra* if there exists a morphism of algebras $\Delta : A \rightarrow A \otimes A$ called the *coproduct* and a morphism of algebras $\varepsilon_A : A \rightarrow k$ called the *counit* and a k -linear map $S : A \rightarrow A$, the *antipode*, such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow A \otimes \Delta \\ A \otimes A & \xrightarrow{\Delta \otimes A} & A \otimes A \otimes A \end{array}$$

the so called *coassociativity*,

$$\begin{array}{ccccc} k \otimes A & \xleftarrow{\varepsilon \otimes A} & A \otimes A & \xrightarrow{A \otimes \varepsilon} & A \otimes k \\ & \searrow \psi_A & \uparrow \Delta & \nearrow \psi_A & \\ & & A & & \end{array}$$

and

$$\begin{array}{ccccc} A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow A \otimes S & & \downarrow \varepsilon & & \downarrow S \otimes A \\ A \otimes A & \xrightarrow{\mu} & A & \xleftarrow{\mu} & A \otimes A \\ & & \downarrow \eta & & \end{array}$$

CHAPTER 2. HOPF ALGEBRAS

Remark 2.1.3. Let A be a Hopf algebra.

- (i) We will occasionally use **Sweedler notation** of writing $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and similarly $(\Delta \otimes 1)\Delta(a) = (1 \otimes \Delta)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$.
- (ii) The ground field k becomes an A -module by defining $a \cdot \lambda := \varepsilon(a)\lambda$ for $a \in A$ and $\lambda \in k$. It is called the **trivial A -module**.
- (iii) If M and N are A -modules then the tensor product $M \otimes N$ becomes an A -module via $a \cdot m \otimes n := \sum a_{(1)}m \otimes a_{(2)}n$.
- (iv) If M and N are A -modules then $\text{Hom}_k(M, N)$ becomes an A -module via $(af)(v) := \sum a_{(1)}f(S(a_2)v)$. In particular the dual module $M^* := \text{Hom}_k(M, k)$ is an A -module.
- (v) For an A -module N the k -linear maps $\psi_N : N \otimes k \rightarrow N$ and $\psi_N : k \otimes N \rightarrow N$ are isomorphisms of A -modules.
- (vi) For A -modules M and N the map $\sigma_{M,N} : N^* \otimes M^* \rightarrow (M \otimes N)^*$, $(\sigma_{M,N}(g \otimes f))(m \otimes n) = g(n)f(m)$ is an isomorphism of A -modules.
- (vii) The modules k^* and k are isomorphic.
- (viii) For A -modules M, M' and M'' the map $\alpha_{M,M',M''} : (M \otimes M') \otimes M'' \rightarrow M \otimes (M' \otimes M'')$, $(m \otimes m') \otimes m'' \mapsto m \otimes (m' \otimes m'')$ is an isomorphism of A -modules.
- (ix) Let M be an A -module. Then the **invariants** of A on M are the set $M^A := \{m \in M \mid am = \varepsilon(a)m \ \forall a \in A\}$. If M and N are A -modules, then we have $\text{Hom}_A(M, N) = (\text{Hom}_k(M, N))^A$. This way $\text{Hom}_A(M, N)$ gets the structure of an A -module.

Remark 2.1.4. (i) If A is a finite dimensional Hopf algebra. Then A^* becomes a Hopf algebra if we define the multiplication by $\mu_{A^*} := \Delta_A^* \circ \sigma_{A,A} \circ \tau_{A^*,A^*}$, where $\tau_{X,Y} : X \otimes Y \rightarrow Y \otimes X, x \otimes y \mapsto y \otimes x$ is the **flip**. This is called the **convolution product**, the unit by $\eta_{A^*} := \varepsilon_A^*$, the comultiplication via $\Delta_{A^*} := \tau_{C^*,C^*} \circ \sigma_{A,A}^{-1} \circ \mu_A^*$, the counit as $\varepsilon_{A^*} := \eta^*$ and the antipode is S^* .

- (ii) If A and B are Hopf algebras then $A \otimes B$ becomes a Hopf algebra if we define the product via $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2 \otimes b_1b_2)$, the coproduct via $\Delta_{A \otimes B}(a \otimes b) = \sum a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}$, the counit via $\varepsilon_{A \otimes B}(a \otimes b) = \varepsilon_A(a)\varepsilon_B(b)$ and the antipode via $S_{A \otimes B} = S_A \otimes S_B$.

Definition 2.1.5. Let A be a finite dimensional Hopf algebra.

- (i) The **space of left integrals** is defined as the set $\int_A^l := \{x \in A \mid ax = \varepsilon(a)x \ \forall a \in A\}$.
- (ii) A **left modular function** $\zeta_l : A \rightarrow k$ is defined via $xa = \zeta_l(a)x$ for all $a \in A$ and $x \in \int_A^l$. The element ζ_l is also called **distinguished group-like element** of A^* .

Proposition 2.1.6 ([FMS97, Lemma 1.5]). *Any finite dimensional Hopf algebra A is Frobenius and therefore selfinjective. Furthermore the Nakayama automorphism is of finite order and is given by $S^{-2} \circ (\text{id}_A * \zeta_l)$, where $*$ is the convolution product on A^* . Furthermore the left modular function is equal to $\varepsilon \circ \nu_A$ for any Nakayama automorphism ν_A of A .*

The remaining statements of this section can be found in [Ben95] and [Wei94].

Lemma 2.1.7. *Let A be a finite dimensional Hopf algebra, P be a projective A -module and M be an A -module. Then $P \otimes M$ and $M \otimes P$ are projective A -modules.*

Proposition 2.1.8. *If A is a finite dimensional Hopf algebra and M and N are A -modules, then there is a projective module P such that $\Omega_A(M \otimes N) \oplus P \cong \Omega_A M \otimes N$ and a projective A -module P' such that $\Omega_A(M \otimes N) \oplus P' \cong M \otimes \Omega_A N$.*

Lemma 2.1.9. *Let A be a finite dimensional Hopf algebra. For finite dimensional modules M_1, M_2 and M_3 there is an isomorphism of vector spaces*

$$\psi : \text{Hom}_A(M_1, \text{Hom}_k(M_2, M_3)) \rightarrow \text{Hom}_A(M_1 \otimes M_2, M_3)$$

defined via $\psi(f)(m \otimes p) = f(m)(p)$ for $f \in \text{Hom}_A(M_1, \text{Hom}_k(M_2, M_3))$, $m \in M_1$ and $p \in M_3$. It induces isomorphisms of vector spaces

$$\text{Ext}_A^n(M_1 \otimes M_2, M_3) \cong \text{Ext}_A^n(M_1, \text{Hom}_k(M_2, M_3)) \cong \text{Ext}_A^n(M_1, M_3 \otimes M_2^*).$$

2.2. Basic examples

In this section we study the basic examples of Hopf algebras: group algebras, enveloping algebras and algebras of distributions. We refer the reader to [Ben95] for the statements on group algebras and to the references stated in the section about Lie algebras for the other two cases.

Definition 2.2.1. Let G be a finite group. Then the **group algebra** kG is defined as the vector space with basis G and multiplication defined as the bilinear extension of the group law.

Lemma 2.2.2. *The group algebra kG is a Hopf algebra. The comultiplication is given on basis vectors by $\Delta(g) = g \otimes g$, the counit by $\varepsilon(g) = 1$ and the antipode by $S(g) = g^{-1}$ for all $g \in G$.*

For group algebras, the representation theory heavily depends on the characteristic of the underlying field. In this thesis the group algebras studied will be semisimple. They are then used to extend certain algebras, such that one has a good control over the representation theory.

Theorem 2.2.3 (Maschke's Theorem). *Let G be a finite group. If $\text{char } k$ does not divide the order of G , then kG is semisimple.*

Definition 2.2.4. Let G be a finite group. A **representation** of G is a group homomorphism $G \rightarrow \text{GL}(V)$ for some vector space V .

CHAPTER 2. HOPF ALGEBRAS

Lemma 2.2.5. *The category of G -representations is equivalent to the category of kG -modules.*

Definition 2.2.6. Let G be a finite group. Let $\varphi : G \rightarrow \mathrm{GL}(V)$ be a representation of G . The **character** of V is the map $G \rightarrow k$ associating to every element $g \in G$ the trace of $\varphi(g)$. A character is called **irreducible** if V is simple as a kG -module.

Lemma 2.2.7. *If G is a finite group such that kG is semisimple, then the irreducible characters of G are in one-to-one-correspondence with the simple kG -modules.*

Definition 2.2.8. Let \mathfrak{g} be a Lie algebra. Then there exists a (up to isomorphism) unique algebra $U(\mathfrak{g})$ and a homomorphism of Lie algebras $\mathfrak{g} \rightarrow U(\mathfrak{g})$, such that for all algebras A and all Lie algebra homomorphisms $\mathfrak{g} \rightarrow A$, there is a unique algebra homomorphism $U(\mathfrak{g}) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad} & A \\ & \searrow & \nearrow \\ & U(\mathfrak{g}) & \end{array}$$

In this definition the algebras are regarded as Lie algebras via the commutator.

Lemma 2.2.9. *Let \mathfrak{g} be a Lie algebra.*

- (i) *The map $\mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective and $U(\mathfrak{g})$ is generated by \mathfrak{g} .*
- (ii) *The algebra $U(\mathfrak{g})$ is a Hopf algebra with comultiplication defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, counit defined by $\varepsilon(x) = 0$ and antipode defined by $S(x) = -x$ for all $x \in \mathfrak{g}$.*
- (iii) *The category of representations of \mathfrak{g} and modules over $U(\mathfrak{g})$ are equivalent.*

Definition 2.2.10. Let G be an algebraic group (or a finite group scheme). Let I be the ideal of $k[G]$ corresponding to the identity element. The **algebra of distributions** of G is defined as $\mathrm{Dist}(G) := \{f \in k[G]^* \mid \exists n \in \mathbb{N} : f(I^n) = 0\}$.

Lemma 2.2.11. *Let G be an algebraic group.*

- (i) *The algebra $\mathrm{Dist}(G)$ is a Hopf algebra.*
- (ii) *If G is a semisimple algebraic group, then $\mathrm{Dist}(G)$ has a triangular decomposition $\mathrm{Dist}(G) \cong \mathrm{Dist}(N) \otimes \mathrm{Dist}(T) \otimes \mathrm{Dist}(N^+)$ and $\mathrm{Dist}(N^+)$ and $\mathrm{Dist}(N^-)$ have a basis, called the Poincaré-Birkhoff-Witt basis whose elements will be denoted by X_γ and Y_γ for $\gamma \in \Phi^+$, respectively.*

2.3. Extensions of Hopf algebras

The notion of a normal Hopf subalgebra and a Hopf quotient in the following definition are standard (cf. [Dru09]), the other two definitions are ad hoc for the situations encountered in this thesis.

Definition 2.3.1. Let A be a Hopf algebra.

- (i) A Hopf subalgebra $B \subseteq A$ is called **normal** if the left ideal generated by $\ker \varepsilon|_B$ is also a right ideal.
- (ii) If $B \subseteq A$ is a normal Hopf subalgebra, then the **Hopf quotient** of A by B is defined to be $A//B := A/(\ker \varepsilon_B \cdot A)$.
- (iii) If A , B and C are Hopf algebras we say that there is a **short exact sequence of Hopf algebras** $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ if B is a normal Hopf subalgebra of A , such that $A//B \cong C$.
- (iv) Let A , B and C be augmented algebras. Let A be a Hopf algebra. Then $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ is called a **weak Hopf sequence** if B is an augmented subalgebra of A , such that the left ideal generated by $\ker \varepsilon|_B$ is also a right ideal and $C \cong A//B := A/(\ker \varepsilon_B \cdot A)$. Furthermore we assume that B is selfinjective and that A is B -projective.

Remark 2.3.2. For a weak Hopf sequence $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ there is an equivalence of the category of C -modules and the full subcategory of A -modules on which B acts trivially. Denote this equivalence by $M \mapsto M^{[1]}$ for a C -module M and its inverse by $N \mapsto N^{[-1]}$ for an A -module, such that B acts trivially. This notation comes from algebraic groups. Furthermore if M and N are A -modules, then $\text{Hom}_B(M, N)$ gets the structure of an A -module by first recognizing that it has the structure of a C -module (see e.g. [Bar85, Lemma 4.5]) and then pulling back via $(-)^{[1]}$.

The following results about the relationship between the corresponding module categories for a weak Hopf sequence were first proven in the case of short exact sequences of cocommutative Hopf algebras by Rolf Farnsteiner in [Far06]. For the non-cocommutative case one has to be careful about the order of the tensor products in each step.

Lemma 2.3.3. *Let $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ be a weak Hopf sequence, such that A is finite dimensional. If L_1, \dots, L_n are simple A -modules, such that $\{L_1|_B, \dots, L_n|_B\}$ is a complete set of representatives for the isomorphism classes of simple B -modules, then every simple A -module S is of the form $S \cong M^{[1]} \otimes L_i$ for a unique $i \in \{1, \dots, n\}$ and a (up to isomorphism) unique simple C -module M .*

Proof. Let S be a simple A -module. Then by assumption there exists $i \in \{1, \dots, n\}$, such that $L_i|_B \hookrightarrow S|_B$. The map $\varphi : \text{Hom}_B(L_i, S) \otimes L_i \rightarrow S, f \otimes x \mapsto f(x)$ is A -linear:

$$\begin{aligned} \varphi(a(f \otimes x)) &= \sum \varphi(a_{(1)}f \otimes a_{(2)}x) = \sum a_{(1)}f(S(a_{(2)})a_{(3)}x) \\ &= \sum a_{(1)}f(\varepsilon(a_{(2)})x) = \sum a_{(1)}\varepsilon(a_{(2)})f(x) \\ &= af(x) = a\varphi(f \otimes x) \end{aligned}$$

CHAPTER 2. HOPF ALGEBRAS

By the stated embedding this map is non-zero and hence surjective. The image is the L_i -isotypical component of $\text{soc}(S|_B)$. By Schur's Lemma (Lemma 1.5.2) the two modules also have the same dimensions since

$$\begin{aligned} \dim \text{Hom}_B(L_i, S) \otimes L_i &= \dim \text{Hom}_B(L_i, \text{soc}(S|_B)) \otimes L_i \\ &= [\text{soc}(S|_B) : L_i] \dim L_i \leq \dim S. \end{aligned}$$

Therefore, the map is an isomorphism. Thus $S \cong \text{Hom}_B(L_i, S) \otimes L_i$ and $\text{Hom}_B(L_i, S)$ is a simple C -module: Suppose there was a proper C -submodule $N \hookrightarrow \text{Hom}_B(L_i, S)$. Then this is also an A -submodule and tensoring with L_i we get a monomorphism $N \otimes L_i \hookrightarrow \text{Hom}_B(L_i, S) \otimes L_i \cong S$, a contradiction to the fact that S was simple.

The unicity is also proven: If a simple module S is isomorphic to $M^{[1]} \otimes N$ with M a simple C -module and N a simple B -module, then N is uniquely determined by $S|_B \cong N^{\dim M}$ and $M \cong \text{Hom}_B(N, S)$ is also uniquely determined. \square

Lemma 2.3.4. *Let $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ be a weak Hopf sequence, such that A is finite dimensional. Suppose $\text{Ext}_B^1(L, L) = 0$ for every simple B -module L . If L_1, L_2 are simple A -modules and M_1, M_2 are simple C -modules such that $L_i|_B$ is simple, then we have*

$$\text{Ext}_A^1(M_1^{[1]} \otimes L_1, M_2^{[1]} \otimes L_2) \cong \begin{cases} \text{Ext}_C^1(M_1, M_2) & \text{if } L_1 \cong L_2 \\ \text{Hom}_C(M_1, M_2 \otimes \text{Ext}_B^1(L_1, L_2)^{[-1]}) & \text{if } L_1 \not\cong L_2. \end{cases}$$

Proof. Let M be a C -module and let E and V be A -modules. Then we have isomorphisms of A -modules

$$\begin{aligned} \text{Hom}_k(M^{[1]} \otimes E, V) &\cong V \otimes (M^{[1]} \otimes E)^* \cong V \otimes (E^* \otimes (M^{[1]})^*) \\ &\cong (V \otimes E^*) \otimes (M^{[1]})^* \cong \text{Hom}_k(M^{[1]}, V \otimes E^*). \end{aligned}$$

Taking A -invariants one gets:

$$\begin{aligned} \text{Hom}_A(M^{[1]} \otimes E, V) &\cong (\text{Hom}_k(M^{[1]} \otimes E, V))^A \cong ((\text{Hom}_k(M^{[1]}, V \otimes E^*))^B)^C \\ &\cong (\text{Hom}_k(M^{[1]}, (V \otimes E^*)^B))^C \cong \text{Hom}_C(M, \text{Hom}_B(E, V)^{[-1]}). \end{aligned}$$

This shows that $\text{Hom}_B(E, -)$ is right adjoint to $- \otimes E$ as a functor $\text{mod } A \rightarrow \text{mod } C$. Therefore [Wei94, Proposition 2.3.10] tells us that it sends injective modules to injectives, as $- \otimes E$ is exact.

Now $\text{Hom}_C(M, -)$ is left exact and $\text{Hom}_B(E, -)$ maps injectives to injectives, in particular to acyclic ones for $\text{Hom}_C(M, -)$. Therefore by Proposition 1.6.14 there is a Grothendieck spectral sequence $\text{Ext}_C^n(M, \text{Ext}_B^m(E, V)^{[-1]}) \Rightarrow \text{Ext}_A^{n+m}(M^{[1]} \otimes E, V)$. This gives rise to a 5-term exact sequence (Proposition 1.6.17), for which we only display the first four terms:

$$\begin{aligned} 0 \rightarrow \text{Ext}_C^1(M, \text{Hom}_B(E, V)^{[-1]}) &\rightarrow \text{Ext}_A^1(M^{[1]} \otimes E, V) \\ &\rightarrow \text{Hom}_C(M, \text{Ext}_B^1(E, V)^{[-1]}) \rightarrow \text{Ext}_C^2(M, \text{Hom}_B(E, V)^{[-1]}). \end{aligned}$$

2.3. EXTENSIONS OF HOPF ALGEBRAS

Now we set $M := M_1$, $E := L_1$ and $V := M_2^{[1]} \otimes L_2$. Furthermore we recognize that there are natural isomorphisms $\text{Hom}_k(L_1, M_2^{[1]} \otimes -) \cong (M_2^{[1]} \otimes -) \otimes L_1^* \cong M_2^{[1]} \otimes (- \otimes L_1^*) \cong M_2^{[1]} \otimes \text{Hom}_k(L_1, -)$. Again we take B -invariants to arrive at $\text{Hom}_B(L_1, M_2^{[1]} \otimes -) \cong M_2^{[1]} \otimes \text{Hom}_B(L_1, -)$. Thus we have a natural equivalence $M_2^{[1]} \otimes - \circ \text{Hom}_B(L_1, -) \circ \text{res}_B^A \cong \text{Hom}_B(L_1, -) \circ \text{res}_B^A \circ M_2^{[1]} \otimes -$ of functors $\text{mod } A \rightarrow \text{mod } C$. Now we have that $M_2^{[1]} \otimes -$ is exact, so by Corollary 1.6.15 and Corollary 1.6.16 this implies $M_2^{[1]} \otimes R^n(\text{Hom}_B(L_1, -) \circ \text{res}_B^A) \cong R^n(\text{Hom}_B(L_1, -) \circ \text{res}_B^A) \circ M_2^{[1]} \otimes -$. Since A and B are selfinjective and res_B^A maps projectives to projectives we get $\text{Ext}_B^1(L_1, M_2^{[1]} \otimes L_2) \cong M_2^{[1]} \otimes \text{Ext}_B^1(L_1, L_2)$. Thus the 5-term sequence reads as:

$$\begin{aligned} 0 \rightarrow \text{Ext}_C^1(M_1, M_2 \otimes \text{Hom}_B(L_1, L_2)^{[-1]}) &\rightarrow \text{Ext}_A^1(M_1^{[1]} \otimes L_1, M_2^{[1]} \otimes L_2) \\ &\rightarrow \text{Hom}_C(M_1, M_2 \otimes \text{Ext}_B^1(L_1, L_2)^{[-1]}) \rightarrow \text{Ext}_C^2(M_1, M_2 \otimes \text{Hom}_B(L_1, L_2)^{[-1]}). \end{aligned}$$

Therefore if $L_1 \not\cong L_2$, then $\text{Ext}_A^1(M_1^{[1]} \otimes L_1, M_2^{[1]} \otimes L_2) \cong \text{Hom}_C(M_1, M_2 \otimes \text{Ext}_B^1(L_1, L_2)^{[-1]})$ as $M_2 \otimes \text{Hom}_B(L_1, L_2) = 0$ by Schur's Lemma (Lemma 1.5.2).

If on the other hand $L_1 \cong L_2$, then by Schur's Lemma (Lemma 1.5.2) $\text{Hom}_B(L_i|_B, L_i|_B)$ and $\text{Hom}_A(L_i, L_i)$ are both 1-dimensional. Thus they are equal and hence C acts trivially on this space. Therefore $\text{Ext}_A^1(M_1^{[1]} \otimes L_1, M_2^{[1]} \otimes L_2) \cong \text{Ext}_C^1(M_1, M_2 \otimes \text{Hom}_B(L_1, L_2)^{[-1]}) \cong \text{Ext}_C^1(M_1, M_2)$ as additionally by assumption $M_2 \otimes \text{Ext}_B^1(L_1, L_1) = 0$. \square

The following proposition is an analogue of a statement for quantum category \mathcal{O} in characteristic zero, proven by Henning H. Andersen and Volodymyr Mazorchuk in [AM11]. We borrow their notation and speak of the special block (ideal).

Proposition 2.3.5. *Let $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ be a weak Hopf sequence, such that A is finite dimensional. Let L_1, \dots, L_n be simple A -modules, such that $\{L_1|_B, \dots, L_n|_B\}$ is a complete set of representatives for the simple B -modules. Furthermore let $L_n|_B$ be projective. Then the simple modules of the form $M^{[1]} \otimes L_n$ form a block ideal $\mathcal{B}^{\text{spec}}$ in $\text{mod } A$ (i.e. a direct sum of blocks), that is equivalent to $\text{mod } C$.*

Proof. We define two functors $F : \text{mod } A \rightarrow \text{mod } C, V \mapsto \text{Hom}_B(L_n, V)^{[-1]}$ and $G : \text{mod } C \rightarrow \text{mod } A, M \mapsto M^{[1]} \otimes L_n$. Both functors are exact since $L_n|_B$ is projective and k -flat. Furthermore they are adjoint since $\text{Hom}_A(M^{[1]} \otimes E, V) \cong \text{Hom}_C(M, \text{Hom}_B(E, V)^{[-1]})$, compare the proof of the foregoing lemma.

We have $\text{Hom}_B(L_n, M^{[1]} \otimes L_n) \cong M^{[1]} \otimes \text{End}_B(L_n) \cong M^{[1]}$, i.e. $F \circ G \cong \text{Id}$. Also $G \circ F \cong \text{Id}$, since this holds for the simple modules in the special block by the proof of Lemma 2.3.3. Now apply the adjunction to get a map in $\text{Hom}_A(G \circ FM, M)$. Induction using the 5-Lemma tells us that this map is an isomorphism. Thus we have an equivalence of categories.

To prove that it is a block ideal apply the foregoing lemma to get:

$$\text{Ext}_A^1(M^{[1]}, L_n, N^{[1]} \otimes L_r) \cong \text{Hom}_C(M, N \otimes \text{Ext}_B^1(L_n, L_r)^{[-1]}) = 0$$

for $r \neq n$. \square

CHAPTER 2. HOPF ALGEBRAS

Having proven some statements about simple modules for extensions of Hopf algebra, we now proceed with the projectives:

Proposition 2.3.6. *Let $k \rightarrow B \rightarrow A \rightarrow C \rightarrow k$ be a weak Hopf sequence, such that A is finite dimensional.*

- (i) *For A -modules X, Y and a projective C -module Q , one has isomorphisms $\text{Ext}_A^j(Q^{[1]} \otimes X, N^{[1]} \otimes Y) \cong \text{Hom}_C(Q, N \otimes \text{Ext}_B^j(X, Y)^{[-1]})$ for any C -module N and all $j \in \mathbb{N}$.*
- (ii) *Suppose that every simple B -module is the restriction of an A -module. If P_1, \dots, P_n are A -modules, such that $\{P_1|_B, \dots, P_n|_B\}$ is a complete set of representatives of projective indecomposable B -modules and $\{Q_1, \dots, Q_m\}$ is a complete set of representatives of the isomorphism classes of the projective indecomposable C -modules, then the A -modules $Q_r^{[1]} \otimes P_i$ form a complete system of projective indecomposable A -modules.*

Proof. (i) Since Q is projective for C , we have that $\text{Hom}_C(Q, -)$ is exact and therefore by Corollary 1.6.16 isomorphisms $\text{Ext}_A^j(Q^{[1]} \otimes X, V) \cong \text{Hom}_C(Q, \text{Ext}_B^j(X, V))$ for all A -modules X and V . As in the proof of the foregoing lemma we have $\text{Ext}_B^j(X, N^{[1]} \otimes Y) \cong N^{[1]} \otimes \text{Ext}_B^j(X, Y)$. Setting $V := N^{[1]} \otimes Y$ for a C -module N and an A -module Y we get $\text{Ext}_A^j(Q^{[1]} \otimes X, N^{[1]} \otimes Y) \cong \text{Hom}_C(Q, N \otimes \text{Ext}_B^j(X, Y)^{[-1]})$.

- (ii) Setting $Q = Q_r$, $X = P_i$ and $N = k$ in (i) we get the projectivity of the A -module $Q_r^{[1]} \otimes P_i$. By assumption we have that there exist simple A -modules L_1, \dots, L_n , such that $\text{top}(P_i|_B) \cong L_i|_B$. Setting $M_r := \text{top}_C(Q_r)$ we obtain a complete set of representatives of simple C -modules $\{M_1, \dots, M_m\}$. Thanks to Lemma 2.3.3 the modules $M_r^{[1]} \otimes L_i$ form a complete set of representatives of simple A -modules. The A -module P_i has a non-zero top. Therefore there exists an epimorphism $P_i \rightarrow M_r^{[1]} \otimes L_j$ for some r, j . This restricts to a surjection $P_i|_B \rightarrow L_j^{\dim M_r}$. Since $\text{top}(P_i|_B) = L_i|_B$, this can only happen if $i = j$ and $\dim M_r = 1$. Without loss of generality assume that $M_r = k$, otherwise take a different choice of the L_i . This yields $\text{Hom}_A(P_i, L_i) \neq 0$. Again from (i) we can derive that $\text{Hom}_A(Q_r^{[1]} \otimes P_i, M_s^{[1]} \otimes L_j) \cong \text{Hom}_C(Q_r, M_s \otimes \text{Hom}_B(P_i, L_j))$. Since $\text{top}(P_i|_B) = L_i$, the right-hand space is zero unless $i = j$. In that case $\text{Hom}_A(P_i, L_i) \subseteq \text{Hom}_B(P_i, L_i)$, which is one-dimensional and hence the trivial module. Therefore $\dim \text{Hom}_C(Q_r, M_s \otimes \text{Hom}_B(P_i, L_i)) = \delta_{ij} \delta_{rs}$. Thus $Q_r^{[1]} \otimes P_i$ is the projective cover of the simple A -module $M_r^{[1]} \otimes L_i$. □

The remainder of this section will be concerned with a special kind of "extension". They are an analogue of the semidirect product of groups for Hopf algebras. As a reference we again refer to [Mon93].

Definition 2.3.7. Let A be an algebra and H be a Hopf algebra. Then A is called a **(left) H -module algebra** if

- (i) A is a left H -module via $h \otimes a \mapsto h(a)$,

2.3. EXTENSIONS OF HOPF ALGEBRAS

$$(ii) \quad h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b),$$

$$(iii) \quad h \cdot 1_A = \varepsilon_H(h)1_A.$$

for all $h \in H$, $a, b \in A$.

Definition 2.3.8. Let H be a Hopf algebra, A be an H -module algebra. For $a \in A$, $h \in H$ denote the action by $h(a)$. Then define the **smash product** $A\#H$ of A by H via: As a vector space it is just $A \otimes H$ with multiplication given by $(a\#h)(a'\#h') := \sum_{(h)} ah_{(1)}(a')\#h_{(2)}h'$ for all $a, a' \in A, h, h' \in H$.

If $G < \text{Aut}(A)$ is a finite group, then A is an kG -module algebra and we call $A*G := A\#kG$ the **skew group algebra** of A by G .

CHAPTER 2. HOPF ALGEBRAS

3. Cohomology and support varieties

In this chapter we will introduce one of our main tools in studying representation type and Auslander-Reiten theory of Hopf algebras, the so called support varieties. They were first introduced for group algebras. In this case there are good references such as [Ben91] and [Car96]. For the more general case of Hopf algebras most of the statements generalize and the proofs can be applied verbatim. This was known to experts, see e.g. [Bro98]. However there are some subtle problems. For convenience of the reader we therefore provide complete proofs of all the statements. Some of the proofs were already given with much detail in [FW09].

3.1. Cup product and the even cohomology ring

Definition 3.1.1. Let A be a finite dimensional Hopf algebra. Let M , M' and M'' be A -modules. For $n, n' \in \mathbb{N}$ we will define the *Yoneda product*

$$-Y- : \text{Ext}_A^{n'}(M', M'') \times \text{Ext}_A^n(M, M') \rightarrow \text{Ext}_A^{n+n'}(M, M'').$$

If $n, n' > 0$ and we have extensions $\xi' : 0 \rightarrow M'' \xrightarrow{f'_{n'-1}} M'_{n'-1} \rightarrow \dots \xrightarrow{f'_0} M_0 \xrightarrow{f'} M' \rightarrow 0$ and $\xi : 0 \rightarrow M' \xrightarrow{f_{n-1}} M_{n-1} \rightarrow \dots \xrightarrow{f_0} M_0 \xrightarrow{f} M \rightarrow 0$, then the Yoneda product $[\xi']Y[\xi]$ is defined as the equivalence class of the following extension in $\text{Ext}_A^{n+n'}(M, M'')$:

$$0 \rightarrow M'' \xrightarrow{f'_{n'-1}} M'_{n'-1} \rightarrow \dots \xrightarrow{f'_0} M_0 \xrightarrow{f_{n-1} \circ f'} M_{n-1} \rightarrow \dots \xrightarrow{f_0} M_0 \xrightarrow{f} M \rightarrow 0.$$

If $n' = 0$, $f \in \text{Ext}_A^0(M', M'') = \text{Hom}_A(M', M'')$ and $[\xi] \in \text{Ext}_A^n(M, M')$, then $fY[\xi] := \text{Ext}_A^n(M, f)([\xi])$. Dually if $n = 0$, $g \in \text{Hom}_A(M, M')$ and $[\xi'] \in \text{Ext}_A^{n'}(M', M'')$, then $[\xi']Yg := \text{Ext}_A^{n'}(g, M'')([\xi'])$.

Remark 3.1.2. (i) This operation is well defined because if we have $n, n' > 0$ and n -extensions ξ, ς of M by M' and n' -extensions ξ', ς' of M' by M'' with $\xi \sim \varsigma$, $\xi' \sim \varsigma'$, i.e. we have maps $\alpha_i : M_i \rightarrow N_i$ and $\alpha'_i : M'_i \rightarrow N'_i$ such that the following diagrams commute:

$$\begin{array}{ccccccccccc} \xi : & 0 & \longrightarrow & M' & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_0 & & \parallel & & \\ \varsigma : & 0 & \longrightarrow & M' & \longrightarrow & N_{n-1} & \longrightarrow & \cdots & \longrightarrow & N_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccccc} \xi' : & 0 & \longrightarrow & M'' & \longrightarrow & M'_{n'-1} & \longrightarrow & \cdots & \longrightarrow & M'_0 & \longrightarrow & M' & \longrightarrow & 0 \\ & & & \parallel & & \downarrow \alpha'_{n'-1} & & & & \downarrow \alpha'_0 & & \parallel & & \\ \zeta' : & 0 & \longrightarrow & M'' & \longrightarrow & N'_{n'-1} & \longrightarrow & \cdots & \longrightarrow & N'_0 & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

then we also have an equivalence between the resulting Yoneda products:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & M'' & \longrightarrow & M'_{n'-1} & \longrightarrow & \cdots & \longrightarrow & M'_0 & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \parallel & & \downarrow \alpha'_{n'-1} & & & & \downarrow \alpha'_0 & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_0 & & \parallel & & \\ 0 & \longrightarrow & M'' & \longrightarrow & N'_{n'-1} & \longrightarrow & \cdots & \longrightarrow & N'_0 & \longrightarrow & N_{n-1} & \longrightarrow & \cdots & \longrightarrow & N_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

- (ii) If the equivalence class of $\xi : 0 \rightarrow M' \rightarrow M_{n-1} \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$ is represented by a map $\tilde{\xi} : \Omega_A^n M \rightarrow M'$ and the equivalence class of $\xi' : 0 \rightarrow M'' \rightarrow M'_{n'-1} \rightarrow \cdots \rightarrow M'_0 \rightarrow M' \rightarrow 0$ is represented by a map $\tilde{\xi}' : \Omega_A^{n'} M' \rightarrow M''$ then the following commutative diagram shows that $\tilde{\xi}' \circ \Omega_A^{n'} \tilde{\xi}$ is a representative for $[\xi]Y[\xi']$:

$$\begin{array}{ccccccccccccccc} P_{n+n'} & \longrightarrow & P_{n+n'-1} & \longrightarrow & \cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \\ \downarrow \gamma_{n'} & & \downarrow \gamma_{n'-1} & & & & \downarrow \gamma_1 & & \downarrow \alpha_n & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_0 & & \parallel & \\ P'_{n'} & \longrightarrow & P'_{n'-1} & \longrightarrow & \cdots & \longrightarrow & P'_0 & \longrightarrow & M' & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \\ \downarrow \beta_{n'} & & \downarrow \beta_{n'-1} & & & & \downarrow \beta_1 & & \parallel & & \parallel & & & & \parallel & & \parallel & \\ M'' & \longrightarrow & M'_{n'-1} & \longrightarrow & \cdots & \longrightarrow & M'_0 & \longrightarrow & M' & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M. \end{array}$$

Here α_i resp. β_i are the morphisms from the bijection between extensions and Ext-groups, which induce $\tilde{\xi}$ resp. $\tilde{\xi}'$ (see Proposition 1.6.4), whereas γ_i are the morphisms from the definition of $\Omega_A^{n'} \tilde{\xi}$ (see Definition 1.6.8).

We will now define another product, the cup product, which is closely related to the Yoneda product.

Definition 3.1.3. We define the *cup product*

$$- \cup - : \text{Ext}_A^m(M, M') \times \text{Ext}_A^n(N, N') \rightarrow \text{Ext}_A^{m+n}(M \otimes N, M' \otimes N')$$

as follows: For $m, n > 0$ let $\xi : 0 \rightarrow M' \rightarrow M_{m-1} \rightarrow \cdots \xrightarrow{f_0} M_0 \xrightarrow{f} M \rightarrow 0$ and $\xi' : 0 \rightarrow N' \rightarrow N_{n-1} \rightarrow \cdots \xrightarrow{g_0} N_0 \xrightarrow{g} N \rightarrow 0$ be extensions. Then $[\xi] \cup [\xi']$ is defined as the equivalence class of the following sequence:

$$0 \rightarrow M' \otimes N' \rightarrow \cdots \xrightarrow{h_1} M_0 \otimes N_0 \xrightarrow{f \otimes g} M \otimes N \rightarrow 0 :$$

Here the l -th module ($1 \leq l \leq m+n-1$) is $\bigoplus_{i+j=l} M_i \otimes N_j$ and h_l is the direct sum of maps $h_{i,j} : M_i \otimes N_j \rightarrow M_{i-1} \otimes N_j \oplus M_i \otimes N_{j-1}$ defined by $h_{i,j}(m \otimes n) = f_i(m) \otimes n + m \otimes g_j(n)$ with $i+j=l$ ($1 \leq i \leq m-1$, $1 \leq j \leq n-1$).

3.1. CUP PRODUCT AND THE EVEN COHOMOLOGY RING

If $m = 0$, $f \in \text{Hom}_A(M, M')$ and $[\xi'] \in \text{Ext}_A^n(M', M'')$, we define $f \cup [\xi'] := (f \otimes N')Y[M \otimes \xi'] = \text{Ext}_A^n(f \otimes N, M \otimes N)([\xi'])$, see Section 1.6.

If $n = 0$, $[\xi] \in \text{Ext}_A^n(M, M')$ and $g \in \text{Hom}_A(M', M'')$, we define $[\xi] \cup g := [\xi \otimes N']Y(M \otimes g) = \text{Ext}_A^m(M \otimes g, M' \otimes N')([\xi])$, see Section 1.6.

Remark 3.1.4. We will show in two steps that this product is well-defined. First we show that the resulting sequence is indeed exact and second we show that it is independent of the choice of the extension:

- (i) Taking the tensor product $\mathbf{M} \otimes \mathbf{N}$ of the truncated complexes, i.e. $\mathbf{M} : 0 \rightarrow M' \rightarrow M_{m-1} \rightarrow \cdots \rightarrow M_0$ and $\mathbf{N} : 0 \rightarrow N' \rightarrow N_{n-1} \rightarrow \cdots \rightarrow N_0$ we get the following sequence (see Definition 1.6.18):

$$\mathbf{M} \otimes \mathbf{N} : 0 \rightarrow M' \otimes N' \rightarrow (M_{m-1} \otimes N') \oplus (M' \otimes N_{n-1}) \xrightarrow{h_{m+n-1}} \cdots \xrightarrow{h_1} M_0 \otimes N_0.$$

The Künneth formula (Theorem 1.6.19) now tells us that the homology of this complex is $M \otimes N$ in degree zero and this sequence is exact elsewhere. So we may complete it to the following exact sequence:

$$0 \rightarrow M' \otimes N' \rightarrow \cdots \xrightarrow{h_1} M_0 \otimes N_0 \xrightarrow{f \otimes g} M \otimes N \rightarrow 0 :$$

Obviously $f \otimes g$ is surjective. Choose vector space complements \tilde{M}_0 resp. \tilde{N}_0 of $\ker f$ resp. $\ker g$, i.e. $M_0 = \ker f \oplus \tilde{M}_0$ and $N_0 = \ker g \oplus \tilde{N}_0$. Then

$$M_0 \otimes N_0 = ((\ker f) \otimes N_0 + M_0 \otimes \ker g) \oplus (\tilde{M}_0 \otimes \tilde{N}_0) = \text{Im } h_1 \oplus (\tilde{M}_0 \otimes \tilde{N}_0).$$

We have $\dim \tilde{M}_0 \otimes \tilde{N}_0 = \dim M \otimes N$ and $\text{Im } h_1 \subseteq \ker(f \otimes g)$. The dimension formula yields equality.

- (ii) Let $m, n > 0$ and let ξ, ς and ξ', ς' be m -extensions of M by M' and n -extensions of N by N' , respectively, with $\xi \sim \varsigma$ and $\xi' \sim \varsigma'$, i.e. we have maps $\alpha_i : M_i \rightarrow M'_i$ and $\beta_i : N_i \rightarrow N'_i$ such that the following diagrams commute:

$$\begin{array}{ccccccccccc} \xi : & 0 & \longrightarrow & M' & \longrightarrow & M_{m-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \downarrow \alpha_{m-1} & & & & \downarrow \alpha_0 & & \parallel & & \\ \varsigma : & 0 & \longrightarrow & M' & \longrightarrow & M'_{m-1} & \longrightarrow & \cdots & \longrightarrow & M'_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccccc} \xi' : & 0 & \longrightarrow & N' & \longrightarrow & N_{n-1} & \longrightarrow & \cdots & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & 0 \\ & & & \parallel & & \downarrow \beta_{n-1} & & & & \downarrow \beta_0 & & \parallel & & \\ \varsigma' : & 0 & \longrightarrow & N' & \longrightarrow & N'_{n-1} & \longrightarrow & \cdots & \longrightarrow & N'_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

We also get maps between the two resulting definitions of cup products:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M' \otimes N' & \longrightarrow & (\mathbf{M} \otimes \mathbf{N})_{m+n-1} & \longrightarrow & \cdots & \longrightarrow & (\mathbf{M} \otimes \mathbf{N})_0 & \longrightarrow & M \otimes N & \longrightarrow & 0 \\ & & \parallel & & \downarrow (\alpha_i \otimes \beta_j)_{i+j=m+n-1} & & & & \downarrow (\alpha_i \otimes \beta_j)_{i+j=0} & & \parallel & & \\ 0 & \longrightarrow & M' \otimes N' & \longrightarrow & (\mathbf{M}' \otimes \mathbf{N}')_{m+n-1} & \longrightarrow & \cdots & \longrightarrow & (\mathbf{M}' \otimes \mathbf{N}')_0 & \longrightarrow & M \otimes N & \longrightarrow & 0 \end{array}$$

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

Proposition 3.1.5. *Let M, M', N and N' be A -modules. If ξ and ξ' are extensions with $[\xi] \in \text{Ext}_A^m(M, M')$ and $[\xi'] \in \text{Ext}_A^n(N, N')$, then $[\xi] \cup [\xi'] = [\xi \otimes N']Y[M \otimes \xi']$.*

Proof. For $m = 0$ or $n = 0$ the statement is the definition. Therefore let $m, n > 0$ and let $\xi : 0 \rightarrow M' \xrightarrow{f_{m-1}} M_{m-1} \rightarrow \cdots \xrightarrow{f_0} M_0 \xrightarrow{f} M \rightarrow 0$ and $\xi' : 0 \rightarrow N' \xrightarrow{f'_{n-1}} N_{n-1} \rightarrow \cdots \xrightarrow{f'_0} N_0 \xrightarrow{f'} N \rightarrow 0$ be extensions. Let $\mathbf{M} : 0 \rightarrow M' \xrightarrow{f_{m-1}} M_{m-1} \rightarrow \cdots \xrightarrow{f_0} M_0$ and $\mathbf{N} : 0 \rightarrow N' \xrightarrow{f'_{n-1}} N_{n-1} \rightarrow \cdots \xrightarrow{f'_0} N_0$ be the truncated sequences. Then $[\xi] \cup [\xi']$ is given by the following sequence

$$0 \rightarrow M' \otimes N' \rightarrow (\mathbf{M} \otimes \mathbf{N})_{m+n-1} \rightarrow \cdots \rightarrow (\mathbf{M} \otimes \mathbf{N})_0 \rightarrow M \otimes N \rightarrow 0.$$

The Yoneda product $[\xi \otimes N']Y[M \otimes \xi']$ is given by the following exact sequence

$$0 \rightarrow M' \otimes N' \rightarrow M_{m-1} \otimes N' \rightarrow \cdots \rightarrow M_0 \otimes N' \rightarrow M \otimes N_{n-1} \rightarrow \cdots \rightarrow M \otimes N_0 \rightarrow M \otimes N \rightarrow 0.$$

To prove the statement it suffices to construct morphisms ϕ_i such that the following diagram commutes:

$$\begin{array}{ccccccccccccccc} 0 & \rightarrow & M' \otimes N' & \rightarrow & (\mathbf{M} \otimes \mathbf{N})_{m+n-1} & \rightarrow & \cdots & \rightarrow & (\mathbf{M} \otimes \mathbf{N})_n & \rightarrow & (\mathbf{M} \otimes \mathbf{N})_{n-1} & \rightarrow & \cdots & \rightarrow & (\mathbf{M} \otimes \mathbf{N})_0 & \rightarrow & M \otimes N & \rightarrow & 0 \\ & & \parallel & & \downarrow \phi_{m+n-1} & & & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & & & \downarrow \phi_0 & & \parallel & & & & \\ 0 & \rightarrow & M' \otimes N' & \rightarrow & M_{m-1} \otimes N' & \rightarrow & \cdots & \rightarrow & M_0 \otimes N' & \rightarrow & M \otimes N_{n-1} & \rightarrow & \cdots & \rightarrow & M \otimes N_0 & \rightarrow & M \otimes N & \rightarrow & 0 \end{array}$$

We define the morphisms as follows:

$$\phi_l := \begin{cases} (f \otimes N_l) \circ \pi_{M_0 \otimes N_l} & \text{if } 0 \leq l \leq n, \\ \pi_{M_{l-n} \otimes N'} & \text{if } n+1 \leq l \leq n+m-1. \end{cases}$$

Now we have to prove that the diagram commutes: The leftmost small square reads as

$$\begin{array}{ccc} M' \otimes N' & \xrightarrow{\begin{pmatrix} f_{m-1} \otimes N' \\ (-1)^m M' \otimes f'_{n-1} \end{pmatrix}} & (M_{m-1} \otimes N') \oplus (M' \otimes N_{n-1}) \\ \parallel & & \downarrow (1,0) \\ M' \otimes N' & \xrightarrow{f_{m-1} \otimes N'} & (M_{m-1} \otimes N') \end{array}$$

So it obviously commutes. For $n \leq l \leq m+n-1$ and the square having vertical edges ϕ_l and ϕ_{l-1} we decompose $(\mathbf{M} \otimes \mathbf{N})_l$ into direct summands and look at the diagrams containing one direct summand and the maps starting at it: We have

$$\begin{array}{ccc} M_{l-n} \otimes N' & \xrightarrow{\begin{pmatrix} f_{l-n-1} \otimes N' \\ (-1)^{l-n} M_{l-n} \otimes f'_{n-1} \end{pmatrix}} & (M_{l-n-1} \otimes N') \oplus (M_{l-n} \otimes N_{n-1}) \\ \parallel & & \downarrow (1,0) \\ M_{l-n} \otimes N' & \xrightarrow{f_{l-n-1} \otimes N'} & (M_{l-n-1} \otimes N') \end{array}$$

3.1. CUP PRODUCT AND THE EVEN COHOMOLOGY RING

which commutes and for $l - n + 1 \leq i \leq l$ we have the following diagrams:

$$\begin{array}{ccc} M_i \otimes N_{l-i} & \xrightarrow{\begin{pmatrix} f_{i-1} \otimes N_{l-i} \\ (-1)^i M_i \otimes f'_{l-i} \end{pmatrix}} & (M_{i-1} \otimes N_{l-i}) \oplus (M_i \otimes N_{l-i-1}) \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\quad\quad\quad} & 0 \end{array}$$

For the square containing ϕ_n and ϕ_{n-1} as vertical edges we have the following squares: First we have

$$\begin{array}{ccc} M_0 \otimes N' & \xrightarrow{M_0 \otimes f'_{n-1}} & M_0 \otimes N_{n-1} \\ \parallel & & \downarrow f \otimes N_{n-1} \\ M_0 \otimes N' & \xrightarrow{(M \otimes f'_{n-1}) \circ (f \otimes N')} & M \otimes N_{n-1} \end{array}$$

which commutes because both compositions are equal to $f \otimes f'_{n-1}$. In the remainder of the proof all maps resp. modules should be treated as 0 if their index is negative. Secondly for $2 \leq i \leq \min\{n, m\}$

$$\begin{array}{ccc} M_i \otimes N_{n-i} & \xrightarrow{\begin{pmatrix} f_{i-1} \otimes N_{n-i} \\ (-1)^i M_i \otimes f'_{n-i-1} \end{pmatrix}} & (M_{i-1} \otimes N_{n-i}) \oplus (M_i \otimes N_{n-i-1}) \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\quad\quad\quad} & 0 \end{array}$$

which trivially commute and thirdly we have

$$\begin{array}{ccc} M_1 \otimes N_{n-1} & \xrightarrow{\begin{pmatrix} f_0 \otimes N_{n-1} \\ -M_1 \otimes f'_{n-2} \end{pmatrix}} & (M_0 \otimes N_{n-1}) \oplus (M_1 \otimes N_{n-2}) \\ \downarrow & & \downarrow (f \otimes N_{n-1}, 0) \\ 0 & \xrightarrow{\quad\quad\quad} & M \otimes N_{n-1} \end{array}$$

which commutes as M_\bullet is an exact sequence and therefore $f \circ f_0 = 0$. For $1 \leq l \leq n - 1$ we have the following diagrams containing ϕ_l and ϕ_{l-1} :

$$\begin{array}{ccc} M_0 \otimes N_l & \xrightarrow{M_0 \otimes f_{l-1}} & M_0 \otimes N_{l-1} \\ f \otimes N_l \downarrow & & \downarrow f \otimes N_{l-1} \\ M \otimes N_l & \xrightarrow{M \otimes f_{l-1}} & M \otimes N_{l-1}. \end{array}$$

This diagram commutes since both compositions are equal to $f \otimes f_{l-1}$. For $2 \leq i \leq$

$\min\{l, m\}$ we have the following diagrams

$$\begin{array}{ccc} M_i \otimes N_{l-i} & \xrightarrow{\begin{pmatrix} f_{i-1} \otimes N_{l-i} \\ (-1)^i M_i \otimes f'_{l-i-1} \end{pmatrix}} & (M_{i-1} \otimes N_{l-i}) \oplus (M_i \otimes N_{l-i-1}) \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\quad\quad\quad} & 0 \end{array}$$

which obviously commutes. Furthermore there is the diagram

$$\begin{array}{ccc} M_1 \otimes N_{l-1} & \xrightarrow{\begin{pmatrix} f_0 \otimes N_{l-1} \\ -M_1 \otimes f'_{l-2} \end{pmatrix}} & (M_0 \otimes N_{l-1}) \oplus (M_1 \otimes N_{l-2}) \\ \downarrow & & \downarrow (f \otimes N_{l-1}, 0) \\ 0 & \xrightarrow{\quad\quad\quad} & M \otimes N_{l-1} \end{array}$$

which commutes as M_\bullet is exact. The rightmost square reads as:

$$\begin{array}{ccc} M_0 \otimes N_0 & \xrightarrow{f \otimes f'} & M \otimes N \\ f \otimes N_0 \downarrow & & \parallel \\ M \otimes N_0 & \xrightarrow{M \otimes f'} & M \otimes N \end{array}$$

Obviously this diagram also commutes. □

Definition 3.1.6. If ξ and ξ' are extensions with $[\xi] \in \text{Ext}_A^m(k, k)$ and $[\xi'] \in \text{Ext}_A^n(M, N)$ then $[\xi] \cup [\xi'] \in \text{Ext}_A^{m+n}(k \otimes M, k \otimes N)$. If $m > 0$ or $n > 0$ let $\xi'' : 0 \rightarrow k \otimes N \xrightarrow{f_{m+n-1}} M_{m+n-1} \rightarrow \cdots \xrightarrow{f_0} M_0 \xrightarrow{f} k \otimes M \rightarrow 0$ be a representative of $[\xi] \cup [\xi']$, otherwise let $\xi'' = \xi \otimes \xi' \in \text{Hom}_A(k \otimes M, k \otimes N)$. Then $\hat{\xi}'' : 0 \rightarrow N \xrightarrow{f_{m+n-1} \circ \psi_N^{-1}} M_{m+n-1} \rightarrow \cdots \xrightarrow{f_0} M_0 \xrightarrow{\psi_M \circ f} M \rightarrow 0$ is an extension resp. $\hat{\xi}'' := \psi_N^{-1} \circ \xi'' \circ \psi_M \in \text{Hom}_A(M, N)$. In this case we will define a new product

$$\hat{\cup} : \text{Ext}_A^m(k, k) \times \text{Ext}_A^n(M, N) \rightarrow \text{Ext}_A^{m+n}(M, N)$$

by $[\xi] \hat{\cup} [\xi'] := [\hat{\xi}'']$ resp. $\xi \hat{\cup} \xi' := \hat{\xi}''$ that we will also call the **cup product**.

Similarly if $m > 0$ or $n > 0$ let $\xi''' : 0 \rightarrow N \otimes k \xrightarrow{g_{m+n-1}} M_{m+n-1} \rightarrow \cdots \xrightarrow{g_0} M_0 \xrightarrow{g} M \otimes k \rightarrow 0$ be a representative of $[\xi'] \cup [\xi]$ otherwise let $\xi''' = \xi' \otimes \xi \in \text{Hom}_A(M \otimes k, N \otimes k)$. Then $\hat{\xi}''' : 0 \rightarrow N \xrightarrow{g_{m+n-1} \circ \psi_N^{-1}} M_{m+n-1} \rightarrow \cdots \xrightarrow{g_0} M_0 \xrightarrow{\psi_M \circ g} M \rightarrow 0$ is an extension resp. $\hat{\xi}''' := \psi_N^{-1} \circ \xi''' \circ \psi_M \in \text{Hom}_A(M, N)$. Also in this case we will define a new product that we denote by the same symbol by abuse of notation

$$\hat{\cup} : \text{Ext}_A^n(M, N) \times \text{Ext}_A^m(k, k) \rightarrow \text{Ext}_A^{m+n}(M, N)$$

3.1. CUP PRODUCT AND THE EVEN COHOMOLOGY RING

The next proposition justifies the use of the same symbol. The argument is taken from Mariano Suárez-Álvarez, who generalized an earlier proof of Beno Eckmann and Peter J. Hilton given in [EH62]:

Proposition 3.1.7 ([Sl04, Theorem 1.7]). *Let ξ and ξ' be extensions with $[\xi] \in \text{Ext}_A^n(k, k)$, $[\xi'] \in \text{Ext}_A^m(M, N)$. Then $[\xi] \hat{\cup} [\xi'] = (-1)^{mn} [\xi'] \hat{\cup} [\xi]$.*

Proof. Let $\tilde{\xi} : \Omega_A^n k \rightarrow k$ resp. $\tilde{\xi}' : \Omega_A^m M \rightarrow N$ be representatives of $[\xi]$ resp. $[\xi']$. Consider the following diagram: All the small squares commute in the stable module category except the one with $(-1)^{mn}$, which commutes up to this sign:

$$\begin{array}{ccccccc}
 N & \xleftarrow{\psi_N} & k \otimes N & \xleftarrow{\tilde{\xi} \otimes N} & (\Omega_A^n k) \otimes N & \xleftarrow{\quad} & \Omega_A^n(k \otimes N) & \xleftarrow{\Omega_A^n \psi_N} & \Omega_A^n N \\
 \parallel & & & & & & & & \parallel \\
 N & \xleftarrow{\psi_N} & N \otimes k & \xleftarrow{N \otimes \tilde{\xi}} & N \otimes (\Omega_A^n k) & \xleftarrow{\quad} & \Omega_A^n(N \otimes k) & \xleftarrow{\Omega_A^n \psi_N} & \Omega_A^n N \\
 \uparrow \tilde{\xi}' & & \uparrow \tilde{\xi}' \otimes k & \searrow \tilde{\xi}' \otimes \tilde{\xi} & \uparrow \tilde{\xi}' \otimes \Omega_A^n k & & \uparrow \Omega_A^n(\tilde{\xi}' \otimes k) & & \uparrow \Omega_A^n \tilde{\xi}' \\
 \Omega_A^m M & \xleftarrow{\psi_{\Omega_A^m M}} & (\Omega_A^m M) \otimes k & \xleftarrow{\Omega_A^m M \otimes \tilde{\xi}} & \Omega_A^m M \otimes \Omega_A^n k & \xleftarrow{\quad} & \Omega_A^n(\Omega_A^m M \otimes k) & \xleftarrow{\Omega_A^n \psi_{\Omega_A^m M}} & \Omega_A^{m+n} M \\
 \parallel & & & & & & & & \parallel \\
 \Omega_A^m M & \xleftarrow{\Omega_A^m \psi_M} & \Omega_A^m(M \otimes k) & \xleftarrow{\Omega_A^m(M \otimes \tilde{\xi})} & \Omega_A^m(M \otimes \Omega_A^n k) & \xleftarrow{\quad} & \Omega_A^{m+n}(M \otimes k) & \xleftarrow{\Omega_A^{m+n} \psi_M} & \Omega_A^{m+n} M
 \end{array}$$

$(-1)^{mn}$

For the upper square and the leftmost square between the second and third row this follows from the fact that $N \otimes \Omega_A^n -$ and $\Omega_A^n(N \otimes -)$ commute in the stable module category. The commutativity of the rightmost square at this level now follows from the fact that Ω_A is a functor of the stable module category. The square with the diagonal is obviously commutative. The commutativity of the square next to it to the right as well as the commutativity of the lowest squares follows from the fact that Ω_A is a functor on the stable module category that commutes with the tensor product.

It remains to consider the square with the sign: Therefore let M' and N' be A -modules.

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

Let P_\bullet resp. Q_\bullet be projective resolutions of M' resp. N' . Then the following diagram shows where the sign comes in:

$$\begin{array}{ccccccc}
 P_1 \otimes N & \xrightarrow{p_1 \otimes 1} & P_0 \otimes N & \xrightarrow{p_0 \otimes 1} & M' \otimes N' & \longrightarrow & 0 \\
 \uparrow \begin{pmatrix} 1 \otimes q_0 \\ 0 \end{pmatrix} & & \uparrow 1 \otimes q_0 & & \parallel & & \\
 (P_1 \otimes Q_0) \oplus (P_0 \otimes Q_0) & \xrightarrow{\begin{pmatrix} p_1 \otimes 1 \\ -1 \otimes q_1 \end{pmatrix}} & P_0 \otimes Q_0 & \xrightarrow{p_0 \otimes q_0} & M' \otimes N' & \longrightarrow & 0 \\
 \downarrow \begin{pmatrix} 0 \\ p_0 \otimes 1 \end{pmatrix} & & \downarrow p_0 \otimes 1 & & \parallel & & \\
 M \otimes Q_1 & \xrightarrow{1 \otimes q_1} & M \otimes Q_0 & \xrightarrow{1 \otimes q_0} & M \otimes N & \longrightarrow & 0
 \end{array}$$

An induction gives the commutativity.

Since the lower boundary is a representative for $[\xi] \hat{\cup} [\xi']$ and the upper boundary is a representative for $[\xi'] \hat{\cup} [\xi]$ the statement follows. \square

Definition 3.1.8. Let A be a finite dimensional Hopf algebra. We define $H^i(A, M) := \text{Ext}_A^i(k, M)$. In this case $H^\bullet(A, k)$ is called the **cohomology ring** of A . It is a ring with product given by the cup product $\hat{\cup}$. The **even cohomology ring** is the subring

$$H^{ev}(A, k) := \begin{cases} \bigoplus_{i \geq 0} \text{Ext}_A^{2i}(k, k) & \text{if char } k \neq 2, \\ H^\bullet(A, k) & \text{if char } k = 2. \end{cases}$$

Corollary 3.1.9. *The even cohomology ring $H^{ev}(A, k)$ is commutative.*

Definition 3.1.10. A finite dimensional Hopf algebra **satisfies (fg)** or is an **(fg)-Hopf algebra**, if $H^{ev}(A, k)$ is finitely generated and $H^\bullet(A, M)$ is finitely generated as a module over $H^{ev}(A, k)$ for every finite dimensional A -module M where the action of $H^{ev}(A, k)$ on $H^\bullet(A, M)$ is given by the cup product $\hat{\cup}$.

Proposition 3.1.11. *The isomorphism $\text{Ext}_A^n(k, \text{Hom}(M, N)) \xrightarrow{\psi} \text{Ext}_A^n(M, N)$ from Lemma 2.1.9 is compatible with the cup product. In particular one can replace the second condition on an (fg)-Hopf algebra with the condition that $\text{Ext}_A^\bullet(M, N)$ is finitely generated as a module for $H^{ev}(A, k)$.*

Proof. Let $\tilde{\xi} : \Omega_A^{n'} k \rightarrow k$ and $\tilde{\xi}' : \Omega_A^{n'} k \rightarrow \text{Hom}_k(M, N)$ be representatives of extensions. Then $\xi \hat{\cup} \xi'$ is represented by the map $\tilde{\xi}' \circ \Omega_A^{n'} \tilde{\xi} : \Omega_A^{n+n'} k \rightarrow \text{Hom}_k(M, N)$. Via the isomorphism ψ this is given by $\psi(\xi \hat{\cup} \xi')(m \otimes s) = \tilde{\xi}' \circ \Omega_A^{n'} \tilde{\xi}(s)(m)$. We have $M \otimes \Omega_A^{n'} k \cong \Omega_A^{n'}(M \otimes k) \oplus P$ for some projective module P . We want to compute $\xi \hat{\cup} \psi(\xi')|_{\Omega_A^{n'}(M \otimes k)}$. This is done by the following procedure: First apply the functor

3.2. SUPPORT VARIETIES FOR (FG)-HOPF ALGEBRAS

$M \otimes -$. We get a map $M \otimes \tilde{\xi} : M \otimes \Omega_A^n k \rightarrow M \otimes k$. Now apply $\Omega_A^{n'}$ to arrive at a map $\Omega_A^{n'}(M \otimes \tilde{\xi}) : \Omega_A^{n'}(M \otimes \Omega_A^n k) \rightarrow \Omega_A^{n'}(M \otimes k)$. This is the restriction of the map $M \otimes \Omega_A^{n'} \tilde{\xi} : M \otimes \Omega_A^{n+n'} k \rightarrow M \otimes \Omega_A^{n'} k$. Therefore we arrive at $\psi(\tilde{\xi}') \circ (M \otimes \Omega_A^n \tilde{\xi})(m \otimes s) = \psi(\tilde{\xi}')(m \otimes \Omega_A^n \tilde{\xi}(s)) = (\tilde{\xi}' \circ \Omega_A^n \tilde{\xi})(s)(m)$.

Since the two results coincide, the statement follows. \square

Conjecture 3.1.12 ([EO04, Conjecture 2.18]). Any finite dimensional Hopf algebra satisfies (fg).

Remark 3.1.13. By a result of Eric M. Friedlander and Andrei Suslin ([FS97, Theorem 1.1]) this conjecture holds for all cocommutative Hopf algebras. It also holds for the small quantum groups and their Borel parts under some restrictions on the parameter by generalizations of a result of Victor Ginzburg and Shrawan Kumar, see Theorem 8.2.1.

3.2. Support varieties for (fg)-Hopf algebras

Definition 3.2.1. Let A be an (fg)-Hopf algebra. For $M, M' \in \text{mod } A$ we define $I_A(M', M)$ to be the annihilator of the module $\text{Ext}_A^\bullet(M', M)$ in the ring $H^{ev}(A, k)$. Write $\mathcal{V}_A(k)$ for the maximal spectrum of $H^{ev}(A, k)$ and $\mathcal{V}_A(M', M)$ for the closed subvariety of $\mathcal{V}_A(k)$ associated with $I_A(M', M)$. The variety $\mathcal{V}_A(M) := \mathcal{V}_A(M, M)$ is called the **support variety** for M .

Remark 3.2.2. (i) As the ideals $I_A(M', M)$ are homogeneous, the varieties $\mathcal{V}_A(M', M)$ are conical (Lemma 1.7.10).

(ii) The ideal $I_A(M, M)$ can also be constructed in another way: Define the morphism $\Phi_M : H^{ev}(A, k) \rightarrow \text{Ext}^\bullet(M, M)$ via $[0 \rightarrow k \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_0} M_0 \xrightarrow{f} k \rightarrow 0] \mapsto [0 \rightarrow M \xrightarrow{(f_{n-1} \otimes M) \circ \psi_M^{-1}} M_{n-1} \otimes M \xrightarrow{f_{n-2} \otimes M} \dots \xrightarrow{f_0 \otimes M} M_0 \otimes M \xrightarrow{\psi_M \circ (f \otimes M)} M \rightarrow 0]$ and $\text{Hom}(k, k) \ni f \mapsto \psi_M \circ (f \otimes M) \circ \psi_M^{-1}$. Then $I_A(M, M)$ is equal to the kernel of Φ_M : As $[\xi] \hat{\cup} \text{id}_M = \Phi_M([\xi])$ we have $I_A(M, M) \subseteq \ker \Phi_M$. Conversely we have the factorization of $[\xi] \hat{\cup} [\xi'] = \Phi_M([\xi]) Y[\xi']$ by Proposition 3.1.5, which yields $\ker \Phi_M \subseteq I_A(M, M)$.

(iii) As Φ_k is just the inclusion and Φ_M factors through Φ_k we have that $I_A(k, k) \subseteq I_A(M, M)$.

We will see that the dimension of the support variety can also be defined in a different manner. Therefore recall:

Definition 3.2.3. (i) Let $V = (V_n)_{n \in \mathbb{N}}$ be a family of finite dimensional vector spaces. Then the **growth rate** γ of V is the smallest non-negative integer m such that there exists a constant $C \in \mathbb{N}_1$ such that $\dim V_n \leq Cn^{m-1}$ for all $n \in \mathbb{N}_1$.

(ii) Let M be a finitely generated A -module. Then the **complexity** of M , $\text{cx}_A M$, is the growth rate of a minimal projective resolution $(P_n)_{n \in \mathbb{N}}$ of M

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

We now recall two statements about the rate of growth in certain situations:

Lemma 3.2.4. *Let R be a commutative \mathbb{N} -graded ring, finitely generated such that each R_n is finite-dimensional over k . If $0 \rightarrow V'' \rightarrow V' \rightarrow V \rightarrow 0$ is a short exact sequence of finitely generated \mathbb{N} -graded R -modules, then $\gamma(V') = \max\{\gamma(V), \gamma(V'')\}$. In particular for each finitely generated \mathbb{N} -graded module V , we have that $\gamma(V) \leq \gamma(R)$.*

Proposition 3.2.5. *If R is a finitely generated commutative \mathbb{N} -graded finite-dimensional k -algebra. Then $\text{Kdim } R = \gamma(R)$.*

Furthermore for Hopf algebras, the complexity behaves nicely with respect to subalgebras.

Lemma 3.2.6. *If B is a Hopf subalgebra of a finite dimensional Hopf algebra A and M is a finite dimensional A -module. Then $\text{cx}_B(M) \leq \text{cx}_A(M)$.*

Proposition 3.2.7. *For an (fg)-Hopf algebra A and a finitely generated module M we have $\text{Kdim } \mathcal{V}_A(M) = \gamma(\text{Ext}^\bullet(M, M)) = \max\{\gamma(\text{Ext}^\bullet(M, S)) \mid S \in \mathcal{S}_A\} = \text{cx}_A M$.*

Proof. According to Remark 3.2.2 we have that $I_A(M, M) = \ker(\Phi_M)$. Since $\text{Ext}_A^\bullet(M, M)$ is finitely generated over $H^{ev}(A, k)$ by the (fg)-assumption, it is also finitely generated over $H^{ev}(A, k)/I_A(M)$. Since the dimension of a variety is equal to the Krull dimension of its coordinate ring (Proposition 1.7.7) whose dimension is equal to the dimension of $H^{ev}(A, k)/I_A(M)$ (since they only differ by nilpotent elements, which are contained in any prime ideal) and the Krull dimension of a finitely generated commutative graded algebra coincides with its rate of growth (Proposition 3.2.5), by Lemma 3.2.4 we have:

$$\begin{aligned} \text{Kdim } \mathcal{V}_A(M) &= \text{Kdim } H^{ev}(A, k) / \sqrt{I_A(M, M)} = \text{Kdim } H^{ev}(A, k) / I_A(M, M) \\ &= \gamma(H^{ev}(A, k) / I_A(M, M)) = \gamma(\text{Ext}_A^\bullet(M, M)). \end{aligned}$$

Now suppose that P_\bullet is a minimal projective resolution of M . Then by Lemma 1.6.12 we have that $[P_n : P_A(S)] = \dim \text{Hom}_A(P_n, S) = \dim \text{Ext}_A^n(M, S)$. Summing up over all isomorphism classes of simple A -modules we get:

$$\dim P_n = \sum_{S \in \mathcal{S}_A} \dim P_A(S) \cdot \dim \text{Ext}_A^n(M, S) \leq \dim A \cdot \sum_{S \in \mathcal{S}_A} \dim \text{Ext}_A^n(M, S).$$

This shows that $\text{cx}_A(M) = \gamma(P_\bullet) \leq \max\{\gamma(\text{Ext}_A^\bullet(M, S)) \mid S \in \mathcal{S}_A\}$. By the (fg)-assumption, each $\text{Ext}_A^\bullet(M, S)$ is a finitely generated $H^{ev}(A, k)$ -module. The action of $H^{ev}(A, k)$ on $\text{Ext}_A^\bullet(M, S)$ factors through $\text{Ext}_A^\bullet(M, M)$ by Proposition 3.1.5. Thus $\text{Ext}_A^\bullet(M, S)$ is a finitely generated $\text{Ext}_A^\bullet(M, M)$ -module. This implies $\gamma(\text{Ext}_A^\bullet(M, S)) \leq \gamma(\text{Ext}_A^\bullet(M, M))$ by Lemma 3.2.4. Since $\dim \text{Ext}_A^n(M, M) \leq \dim \text{Hom}_k(P_n, M) = (\dim M)(\dim P_n)$ as $\text{Ext}_A^n(M, M)$ is defined as a subfactor of $\text{Hom}_A(P_n, M)$ we have that $\gamma(\text{Ext}_A^\bullet(M, M)) \leq \gamma(P_\bullet)$. Summarizing we get:

$$\begin{aligned} \max\{\gamma(\text{Ext}_A^\bullet(M, S)) \mid S \in \mathcal{S}_A\} &\leq \gamma(\text{Ext}_A^\bullet(M, M)) \\ &\leq \gamma(P_\bullet) = \text{cx}_A(M) \\ &\leq \max\{\gamma(\text{Ext}_A^\bullet(M, S)) \mid S \in \mathcal{S}_A\}. \end{aligned}$$

3.2. SUPPORT VARIETIES FOR (FG)-HOPF ALGEBRAS

Thus all inequalities are equalities. So we have that

$$\text{cx}_A(M) = \gamma(\text{Ext}_A^\bullet(M, M)) = \max\{\gamma(\text{Ext}_A^\bullet(M, S)) \mid S \in \mathcal{S}_A\} = \text{Kdim } \mathcal{V}_A(M).$$

□

In the remainder we will obtain corollaries on the complexity of a module from statements about support varieties. Often these statements do have easier proofs without using support varieties. We will not include them here. Some of them are also valid for more general classes of algebras, e.g. selfinjective algebras.

Corollary 3.2.8. *Let A and M be as in the foregoing proposition. Then M is projective iff $\mathcal{V}_A(M) = \{0\}$.*

Proof. M is projective iff $0 = \text{cx}_A(M) = \text{Kdim } \mathcal{V}_A(M)$ since A is selfinjective (Proposition 2.1.6). This happens iff $\mathcal{V}_A(M) = \{0\}$ as the support variety is conical. □

Proposition 3.2.9. *Let A be an (fg)-Hopf algebra. Let M and N be finite dimensional A -modules. Then:*

- (i) $\mathcal{V}_A(M \oplus N) = \mathcal{V}_A(M) \cup \mathcal{V}_A(N)$,
- (ii) $\mathcal{V}_A(M, N) \subseteq \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$.
- (iii) *If $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ is a short exact sequence of A -modules, then $\mathcal{V}_A(E, X) \subseteq \mathcal{V}_A(M, X) \cup \mathcal{V}_A(N, X)$ and $\mathcal{V}_A(X, E) \subseteq \mathcal{V}_A(X, M) \cup \mathcal{V}_A(X, N)$ for any A -module X .*
- (iv) $\mathcal{V}_A(M) = \bigcup_{S \in \mathcal{S}_A} \mathcal{V}_A(M, S) = \bigcup_{S \in \mathcal{S}_A} \mathcal{V}_A(S, M)$,
- (v) $\mathcal{V}_A(M \otimes N) \subseteq \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$.

Proof. (i) The functorial isomorphism $- \otimes (M_1 \oplus M_2) \cong (- \otimes M_1) \oplus (- \otimes M_2)$ leads to the following factorization of $\Phi_{M_1 \oplus M_2}$:

$$H^{ev}(A, k) \xrightarrow{(- \otimes M_1, - \otimes M_2)} \text{Ext}_A^\bullet(M_1, M_1) \oplus \text{Ext}_A^\bullet(M_2, M_2) \xrightarrow{- \oplus -} \text{Ext}_A^\bullet(M_1 \oplus M_2, M_1 \oplus M_2).$$

Since the latter morphism is injective, we deduce that $I_A(M_1 \oplus M_2) = I_A(M_1) \cap I_A(M_2)$. It follows that $\mathcal{V}_A(M_1 \oplus M_2) = \mathcal{V}_A(M_1) \cup \mathcal{V}_A(M_2)$ (Lemma 1.7.4).

- (ii) The cup product factors as Φ_M followed by Yoneda composition (Proposition 3.1.5 and Proposition 3.1.7) or as Φ_N followed by Yoneda composition (Proposition 3.1.5), so we get $I_A(M) + I_A(N) \subseteq I_A(M, N)$ which implies $\mathcal{V}_A(M, N) \subseteq \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$.
- (iii) Let $0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} N \rightarrow 0$ be a short exact sequence. We want to show that $I_A(M, X) \hat{\cup} I_A(N, X) \subseteq I_A(E, X)$, which proves the first statement by Lemma 1.7.4. Let $x \in I_A(M, X)$ be homogeneous of degree m and $z \in \text{Ext}^n(E, X)$. Then $\text{Ext}_A^{m+n}(f, X)(x \hat{\cup} z) = (x \hat{\cup} z)Yf = x \hat{\cup} (zYf) = 0$ as the cup product is equal to a Yoneda product (Proposition 3.1.5), which is associative. Therefore by the long exact

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

sequence 1.6.10 (i) there exists $x' \in \text{Ext}_A^{m+n}(N, X)$ with $x\hat{\cup}z = \text{Ext}_A^{m+n}(g, X)(x') = x'Yg$. Now let $y \in I_A(N, X)$ be homogeneous of degree m' . Then $(y\hat{\cup}x)\hat{\cup}z = y\hat{\cup}(x\hat{\cup}z) = y\hat{\cup}(x'Yg) = (y\hat{\cup}x')Yg = 0$. So the claim is established.

Similarly let $x \in I_A(X, M)$ be homogeneous of degree m and $z \in \text{Ext}^n(X, E)$. Then $\text{Ext}_A^{m+n}(X, g)(z\hat{\cup}x) = gY(z\hat{\cup}x) = (gYz)\hat{\cup}x = 0$. Therefore by the long exact sequence there exists $x' \in \text{Ext}_A^{m+n}(X, M)$ such that $z\hat{\cup}x = \text{Ext}_A^{m+n}(X, f)(x') = fYx'$. Now let $y \in I_A(X, N)$ be homogeneous of degree m' . Then by Proposition 3.1.7 $(-1)^{n(m+m')}(x\hat{\cup}y)\hat{\cup}z = z\hat{\cup}(x\hat{\cup}y) = (z\hat{\cup}x)\hat{\cup}y = (fYx')\hat{\cup}y = fY(x'\hat{\cup}y) = 0$.

- (iv) “ \supseteq ” follows from (ii). “ \subseteq ” follows from (iii) by induction over the length of M .
- (v) Since $-\otimes(M \otimes N) \cong (-\otimes M) \otimes N$ we have that the action of $H^{ev}(A, k)$ on $\text{Ext}_A^\bullet(M \otimes N, M \otimes N)$ factors through its action on $\text{Ext}_A^\bullet(M, M)$. Thus $I_A(M, M) \subseteq I_A(M \otimes N, M \otimes N)$, which implies that $\mathcal{V}_A(M \otimes N) \subseteq \mathcal{V}_A(M)$. To show $\mathcal{V}_A(M \otimes N) \subseteq \mathcal{V}_A(N)$ recall that $\text{Ext}_A^\bullet(M \otimes N, M \otimes N) \cong \text{Ext}_A^\bullet(N, M^* \otimes M \otimes N)$ (Proposition 3.1.11). Therefore $\mathcal{V}_A(M \otimes N) = \mathcal{V}_A(N, M^* \otimes M \otimes N) \subseteq \mathcal{V}_A(N)$ by (ii). □

Corollary 3.2.10. *Keep the assumptions of the foregoing proposition. Then:*

- (i) $\text{cx}_A(M \oplus N) = \max\{\text{cx}_A(M), \text{cx}_A(N)\}$,
- (iii) $\text{cx}_A E \leq \max\{\text{cx}_A(M), \text{cx}_A(N)\}$,
- (v) $\text{cx}_A(M \otimes N) \leq \min\{\text{cx}_A(M), \text{cx}_A(N)\}$.

The following result on Auslander-Reiten components was first observed by Rolf Farnsteiner in [Far95].

Proposition 3.2.11. *Let A be an (fg)-Hopf algebra and M and N be finitely generated A -modules. Then:*

- (i) $\mathcal{V}_A(\Omega_A M, \Omega_A M') = \mathcal{V}_A(M, M'), \mathcal{V}_A(M) = \mathcal{V}_A(M^*)$
- (ii) *Let $\xi : A \rightarrow k$ be an algebra morphism. Denote by k_ξ the vector space k together with the action given by $a\lambda = \xi(a)\lambda$. Then $-\otimes k_\xi$ is an autoequivalence of $\text{mod } A$ (with inverse $-\otimes k_{\xi \circ S}$, see Remark 2.1.4 (i)). We have $\mathcal{V}_A(M \otimes k_\xi) = \mathcal{V}_A(M)$. In particular $\mathcal{V}_A(M^{(\nu_A)}) = \mathcal{V}_A(M)$ where ν_A is the Nakayama automorphism of A .*
- (iii) $\mathcal{V}_A(M) = \mathcal{V}_A(N)$ if M and N belong to the same component in the stable Auslander-Reiten quiver of A .

Proof. (i) The result can be seen via the description of the Yoneda product on stable homomorphisms:

$$\begin{array}{ccccc} \underline{\text{Hom}}_A(\Omega_A^n k, k) \times \underline{\text{Hom}}_A(\Omega_A^m M, M') & \longrightarrow & \underline{\text{Hom}}_A(\Omega_A^n M', M') \times \underline{\text{Hom}}_A(\Omega_A^m M, M') & \longrightarrow & \underline{\text{Hom}}_A(\Omega_A^{n+m} M, M') \\ \downarrow & & & & \downarrow \\ \underline{\text{Hom}}_A(\Omega^n k, k) \times \underline{\text{Hom}}_A(\Omega^{m+1} M, \Omega M') & \longrightarrow & \underline{\text{Hom}}_A(\Omega_A^{n+1} M', \Omega_A M') \times \underline{\text{Hom}}_A(\Omega_A^{m+1} M, \Omega_A M') & \longrightarrow & \underline{\text{Hom}}_A(\Omega^{n+m+1} M, \Omega M') \end{array}$$

3.2. SUPPORT VARIETIES FOR (FG)-HOPF ALGEBRAS

Going first to the right and then down a pair of stable homomorphisms (f, g) is mapped to $\Omega_A((f \otimes M') \circ \Omega_A^n g)$. Going first down and then to the right the pair is mapped to $(f \otimes \Omega_A M') \circ (\Omega_A^{n+1} g)$. Thus the diagram commutes since Ω is a functor on the stable module category that commutes with tensor products.

For the dual we have the following commutative diagram

$$\begin{array}{ccc} H^{ev}(A, k) & \xrightarrow{\Phi_M} & \text{Ext}_A^\bullet(M, M) \\ \downarrow (-)^* & & \downarrow (-)^* \\ H^{ev}(A, k) & \xrightarrow{\Phi_{M^*}} & \text{Ext}_A^\bullet(M^*, M^*) \end{array}$$

Here the vertical maps are given by sending the equivalence class of an extension $0 \rightarrow N \xrightarrow{g_{n-1}} N_{n-1} \rightarrow \dots \xrightarrow{g_0} N_0 \xrightarrow{g} N \rightarrow 0$ to the equivalence class of the extension $0 \rightarrow N^* \xrightarrow{g^*} N_0^* \xrightarrow{g_0^*} \dots \rightarrow N_{n-1}^* \xrightarrow{g_{n-1}^*} N^* \rightarrow 0$. For the left hand side note that $k^* \cong k$ by Remark 2.1.3 (vii).

(ii) We have the following commutative diagram

$$\begin{array}{ccc} H^{ev}(A, k) & \xrightarrow{\Phi_M} & \text{Ext}_A^\bullet(M, M) \\ \parallel & & \downarrow -\otimes k_\xi \\ H^{ev}(A, k) & \xrightarrow{\Phi_{M \otimes k_\xi}} & \text{Ext}_A^\bullet(M \otimes k_\xi, M \otimes k_\xi) \end{array}$$

Since the right morphism is an isomorphism it follows that $I_A(M, M) = I_A(M \otimes k_\xi, M \otimes k_\xi)$ and therefore $\mathcal{V}_A(M) = \mathcal{V}_A(M \otimes k_\xi)$. The second statement follows from the fact that the Nakayama automorphism can be expressed via the modular function by Proposition 2.1.6.

(iii) It obviously suffices to prove this statement under the additional assumption that there is an irreducible map from M to N . This implies that there is an almost split sequence $0 \rightarrow \tau_A N \rightarrow M \oplus M' \rightarrow N \rightarrow 0$. It follows that $\mathcal{V}_A(M) \subseteq \mathcal{V}_A(M \oplus M')$ by Proposition 3.2.9 (i) and $\mathcal{V}_A(M \oplus M') \subseteq \mathcal{V}_A(N) \cup \mathcal{V}_A(\tau_A N)$ by Proposition 3.2.9 (iii). Since $\tau_A N \cong \Omega_A^2 N^{(\nu_A)}$ it follows from (i) that $\mathcal{V}_A(\tau_A N) = \mathcal{V}_A(N)$ and therefore $\mathcal{V}_A(M) \subseteq \mathcal{V}_A(N)$. On the other hand we also have an almost split sequence $0 \rightarrow M \rightarrow N \oplus N' \rightarrow \tau_A^{-1} M \rightarrow 0$ which similarly yields $\mathcal{V}_A(N) \subseteq \mathcal{V}_A(N \oplus N') \subseteq \mathcal{V}_A(M) \cup \mathcal{V}_A(\tau_A^{-1} M) = \mathcal{V}_A(M)$.

□

Corollary 3.2.12. *Keep the assumptions of the foregoing proposition. Then:*

(i) $\text{cx}_A M = \text{cx}_A M^* = \text{cx}_A \Omega_A M,$

(ii) $\text{cx}_A(M \otimes k_\xi) = \text{cx}_A M.$

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

(iii) The complexity of a module is an invariant for the Auslander-Reiten components of A .

Corollary 3.2.13. *With the assumptions of the foregoing proposition we have: If $\mathcal{V}_A(M) \cap \mathcal{V}_A(N) = \{0\}$, then $\text{Ext}_A^n(M, N) = 0$ for all $n > 0$.*

Proof. By Proposition 3.2.9 (v) and 3.2.11 (i) we have $\mathcal{V}_A(M^* \otimes N) \subseteq \mathcal{V}_A(M^*) \cap \mathcal{V}_A(N) = \mathcal{V}_A(M) \cap \mathcal{V}_A(N) = \{0\}$. Hence Corollary 3.2.8 implies that $M^* \otimes N$ is projective. This yields $\text{Ext}_A^n(M, N) \cong \text{Ext}_A^n(k, M^* \otimes N) = 0$ by Lemma 2.1.9. \square

Proposition 3.2.14. *Let A and B be (fg)-Hopf algebras. If M is a finitely generated A -module and N is a finitely generated B -module, then $\mathcal{V}_{A \otimes B}(M \otimes M', N \otimes N') \cong \mathcal{V}_A(M, N) \times \mathcal{V}_B(M', N')$.*

Proof. If \mathbf{P} is a projective resolution of M as an A -module and \mathbf{P}' is a projective resolution of M' as a B -module, then the Künneth formula (Theorem 1.6.19) tells us that $\mathbf{P} \otimes \mathbf{P}'$ is a projective resolution of $M \otimes M'$ as an $A \otimes B$ -module. We have $\text{Hom}_{A \otimes B}(\mathbf{P} \otimes \mathbf{P}', N \otimes N') \cong \text{Hom}_A(\mathbf{P}, N) \otimes \text{Hom}_B(\mathbf{P}', N')$. Therefore if we apply the Künneth formula to that complex we get $\text{Ext}_{A \otimes B}^n(M \otimes M', N \otimes N') \cong \bigoplus_{i+j=n} \text{Ext}_A^i(M, N) \otimes \text{Ext}_B^j(N, N')$. We want to show that this is compatible with the cup product, i.e. we want to check that the following diagram with the previously given maps commutes:

$$\begin{array}{ccc} H^{s'}(A, k) \times \text{Ext}^s(M, N) \times H^{r'}(B, k) \times \text{Ext}^r(M', N') & \longrightarrow & H^{s'+r'}(A \otimes B, k \otimes k) \times \text{Ext}^{s+r}(M \otimes M', N \otimes N') \\ \downarrow & & \downarrow \\ \text{Ext}^{s'+s}(k \otimes M, k \otimes N) \times \text{Ext}^{r'+r}(k \otimes M', k \otimes N') & \longrightarrow & \text{Ext}^{s'+s+r'+r}(M \otimes M', N \otimes N'). \end{array}$$

Let $\xi' = [0 \rightarrow k \rightarrow M'_{s'-1} \rightarrow \cdots \rightarrow M'_0 \rightarrow k \rightarrow 0] \in H^{s'}(A, k)$, $\xi = [0 \rightarrow N \rightarrow M_{s-1} \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0] \in \text{Ext}^s(M, N)$, $\zeta' = [0 \rightarrow k \rightarrow N'_{r'-1} \rightarrow \cdots \rightarrow N'_0 \rightarrow k \rightarrow 0] \in H^{r'}(B, k)$, $\zeta = [0 \rightarrow N' \rightarrow N_{r-1} \rightarrow \cdots \rightarrow N_0 \rightarrow M' \rightarrow 0] \in \text{Ext}^r(M', N')$.

Then going first to the right and then down results in the following sequence:

$$0 \rightarrow k \otimes k \otimes M \otimes M' \rightarrow \cdots \rightarrow k \otimes k \otimes N \otimes N' \rightarrow 0,$$

while going first down and then to the right yields the following sequence:

$$0 \rightarrow k \otimes M \otimes k \otimes M' \rightarrow \cdots \rightarrow k \otimes N \otimes k \otimes N' \rightarrow 0.$$

In the middle we have the following parts for the first sequence, where we omit the obvious

3.2. SUPPORT VARIETIES FOR (FG)-HOPF ALGEBRAS

differentials:

$$\begin{array}{c}
 M'_{i'-1} \otimes N'_{j'} \otimes M_i \otimes N_j \\
 \nearrow \\
 M'_{i'} \otimes N'_{j'-1} \otimes M_i \otimes N_j \\
 \nearrow \\
 M'_{i'} \otimes N'_{j'} \otimes M_i \otimes N_j \\
 \searrow \quad \searrow \\
 M'_{i'} \otimes N'_{j'} \otimes M_{i-1} \otimes N_j \\
 \searrow \\
 M'_{i'} \otimes N'_{j'} \otimes M_i \otimes N_j
 \end{array}$$

$(-1)^{i'}$
 $(-1)^{i'+j'}$
 $(-1)^{i'+j'+i}$

and for the second sequence:

$$\begin{array}{c}
 M'_{i'-1} \otimes M_i \otimes N'_{j'} \otimes N_j \\
 \nearrow \\
 M'_{i'} \otimes M_{i-1} \otimes N'_{j'} \otimes N_j \\
 \nearrow \\
 M'_{i'} \otimes N'_{j'} \otimes M_i \otimes N_j \\
 \searrow \quad \searrow \\
 M'_{i'} \otimes M_i \otimes N'_{j'-1} \otimes N_j \\
 \searrow \\
 M'_{i'} \otimes M_i \otimes N'_{j'} \otimes N_j
 \end{array}$$

$(-1)^{i'}$
 $(-1)^{i'+i}$
 $(-1)^{i'+i+j'}$

The flip $\text{id}_k \otimes \tau_{k,M} \otimes \text{id}_k$ now identifies $k \otimes k \otimes M \otimes M'$ and $k \otimes M \otimes k \otimes M'$ yielding a chain map $\sigma_{i'+j'+i+j}(m' \otimes n' \otimes m \otimes n) = (-1)^{j'i} m' \otimes m \otimes n' \otimes n$ for m' of degree i' , n' of degree j' , m of degree i , n of degree j , which has the appropriate sign.

We have that $H^{ev}(A \otimes B, k) \cong H^{ev}(A, k) \otimes H^{ev}(B, k) \oplus H^{odd}(A, k) \otimes H^{odd}(B, k)$. Furthermore $(x \otimes y)^2 = -x^2 \otimes y^2$ for $x \in H^i(A, k)$, $y \in H^j(B, k)$ with $i, j \in \mathbb{N}$ by the above considerations. Therefore the elements in $H^{odd}(A, k) \otimes H^{odd}(B, k)$ are all nilpotent. This implies that the maximal ideal spectra of $H^{ev}(A \otimes B, k)$ and $H^{ev}(A, k) \otimes H^{ev}(B, k)$ coincide. Since the isomorphism is compatible with Yoneda product, the statement follows from the fact that the maximal ideal spectrum of a tensor product of rings is isomorphic to the product of the maximal ideal spectra of the rings (see Lemma 1.7.5). \square

Corollary 3.2.15. *Keep the assumptions of the foregoing proposition, then*

$$\text{cx}_{A \otimes B} M \otimes N = \text{cx}_A M + \text{cx}_B N.$$

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

The following statement is a generalization of [Far95, Lemma 2.1] and [Far11, Theorem 1.1]:

Theorem 3.2.16. *Let A be an (fg)-Hopf algebra. Let M be a module, such that the commutative graded subalgebra $S := \text{Im } \Phi_M \subseteq \text{Ext}^\bullet(M, M)$ is generated by $\bigoplus_{b|a} S_b$ for some $a \in \mathbb{N}$. Then $\text{cx } M \leq \dim \text{Ext}^{an}(M, M)$ for every $n \geq 1$.*

Proof. Denote by $T_{(n)}$ the subalgebra of S generated by the subspace S_{an} of homogeneous elements of degree an . Since S is generated by $\bigoplus_{b|a} S_b$ as an algebra, it follows that it is generated as a $T_{(n)}$ -algebra by finitely many integral elements. By [Eis95, Corollary 4.5] it is therefore finitely generated as a $T_{(n)}$ -module. Since $\text{Ext}^\bullet(M, M)$ is finitely generated as an S -module, we also have that it is finitely generated as a $T_{(n)}$ -module. Hence $\text{cx}_A(M) = \gamma(\text{Ext}_A^*(M, M)) = \gamma(T_{(n)}) \leq \dim S_{an} \leq \dim \text{Ext}_A^{an}(M, M)$, where the equalities follow from Proposition 3.2.7, as $\text{Ext}^*(M, M)$ is a finitely generated module over $T_{(n)}$, the fact that equality holds for the polynomial ring in r variables, and the fact that S_{an} is a subspace of $\text{Ext}_A^{an}(M, M)$, respectively. \square

3.3. Carlson modules

Definition 3.3.1. Let $y \in H^r(A, k)$ be represented by a map $\tilde{y} : \Omega_A^r k \rightarrow k$. Then the *Carlson module* L_y associated with y is defined as the kernel of \tilde{y} .

Normally the Carlson modules are denoted by L_ζ . But since we will use the symbol ζ throughout this thesis to denote a primitive ℓ -th root of unity, we have decided to use the letter y instead.

Theorem 3.3.2. *Let A be an (fg)-Hopf algebra, let M be a finite dimensional A -module, and let y be a non-zero homogeneous element of positive degree in $H^{ev}(A, k)$. Then $\mathcal{V}_A(L_y \otimes M) = \mathcal{V}_A(M) \cap Z(\langle y \rangle)$. In particular $\mathcal{V}_A(L_y) = Z(\langle y \rangle)$.*

Proof. Let N and N' be finite dimensional A -modules and let \mathfrak{m} be a maximal ideal in $H^{ev}(A, k)$. Since $\text{Ext}_A^\bullet(N, N')$ is finitely generated over $H^{ev}(A, k)$ by (fg), we have that $\mathfrak{m} \in \mathcal{V}_A(N, N')$ iff $I_A(N, N') \subseteq \mathfrak{m}$ iff $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}^\bullet(N, N') \neq 0$ (see Proposition 1.7.1). We may assume that \mathfrak{m} is not the maximal ideal spanned by all elements of positive degree. This ideal corresponds to the origin, which is certainly an element of any support variety since the corresponding ideals are homogeneous.

We will first show that $\mathcal{V}_A(M) \cap Z(\langle y \rangle) \subseteq \mathcal{V}_A(L_y \otimes M)$. By applying Proposition 3.2.9 (iv) to M and to $L_y \otimes M$ we have $\mathcal{V}_A(M) = \bigcup_{S \in \mathcal{S}_A} \mathcal{V}_A(M, S)$ and $\mathcal{V}_A(L_y \otimes M) = \bigcup_{S \in \mathcal{S}_A} \mathcal{V}_A(L_y \otimes M, S)$. Thus it suffices to show that $\mathcal{V}_A(M, S) \cap Z(\langle y \rangle) \subseteq \mathcal{V}_A(L_y \otimes M, S)$ for every simple A -module S . Let \mathfrak{m} be a maximal ideal in $\mathcal{V}_A(M, S) \cap Z(\langle y \rangle)$. Then $I_A(M, S) \subseteq \mathfrak{m}$ and $\langle y \rangle \subseteq \mathfrak{m}$ and therefore \mathfrak{m} contains the ideal generated by $I_A(M, S)$ and y . We must show that $\mathfrak{m} \in \mathcal{V}_A(L_y \otimes M, S)$, that is $I_A(L_y \otimes M, S) \subseteq \mathfrak{m}$. Suppose $I_A(L_y \otimes M, S) \not\subseteq \mathfrak{m}$. Then as noted above $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(L_y \otimes M, S) = 0$. Applying $-\otimes M$ to $0 \rightarrow L_y \rightarrow \Omega^n(k) \rightarrow k \rightarrow 0$ we get a short exact sequence $0 \rightarrow L_y \otimes M \rightarrow \Omega^n(k) \otimes M \rightarrow M \rightarrow 0$ as A is

a Hopf algebra. Applying $\text{Hom}_A(-, S)$ we get a long exact Ext-sequence (see Proposition 1.6.10):

$$\dots \rightarrow \text{Ext}_A^i(M, S) \rightarrow \text{Ext}_A^i(\Omega_A^n(k) \otimes M, S) \rightarrow \text{Ext}_A^i(L_y \otimes M, S) \rightarrow \dots$$

Summing up over all i the maps in the sequence are homogeneous $H^{ev}(A, k)$ -module homomorphisms: For the maps of the form $\text{Ext}_A^n(h, S)$, where $h : M'' \rightarrow M'$ is an A -module homomorphism, this follows from the commutativity of the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S & \longrightarrow & (\mathbf{N} \otimes \mathbf{M}) & \longrightarrow & M'' \longrightarrow 0 \\ & & \parallel & & \downarrow \text{id} \otimes h_i & & \downarrow h \\ 0 & \longrightarrow & S & \longrightarrow & (\mathbf{N} \otimes \mathbf{M}') & \longrightarrow & M' \longrightarrow 0 \end{array}$$

where the h_i and the M'_i are induced from h , when forming $\text{Ext}^n(h, S)(0 \rightarrow S \rightarrow \mathbf{M} \rightarrow M'' \rightarrow 0)$. For δ the compatibility follows from naturality of the connecting homomorphism, i.e. by considering the following commutative diagram of complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(L_y \otimes M, I^*) & \longrightarrow & \text{Hom}_A(\Omega_A^n k \otimes M, I^*) & \longrightarrow & \text{Hom}_A(M, I^*) \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_A(L_y \otimes M, I^{*+n}) & \longrightarrow & \text{Hom}_A(\Omega_A^n k \otimes M, I^{*+n}) & \longrightarrow & \text{Hom}_A(M, I^{*+n}) \longrightarrow 0 \end{array}$$

where I^* is an injective resolution of S and the vertical maps are the morphisms that induce multiplication by a homogeneous element of $H^{ev}(A, k)$.

By Proposition 2.1.8 there exists a projective module P such that $\Omega^n(M) \oplus P \cong \Omega^n(k) \otimes M$. Therefore $\text{Ext}_A^i(\Omega^n(k) \otimes M, S) \cong \text{Ext}_A^i(\Omega^n(M) \oplus P, S) = \text{Ext}_A^i(\Omega^n(M), S) \cong \text{Ext}_A^{i+n}(M, S)$ for $i > 0$ by Lemma 1.6.11. The compatibility of the last isomorphism with the action of $H^{ev}(A, k)$ is obvious since this is just a degree shift for the projective resolution. Thus we obtain the following long exact sequence

$$\dots \rightarrow \text{Ext}_A^i(M, S) \xrightarrow{\tilde{y}} \text{Ext}_A^{i+n}(M, S) \xrightarrow{\eta} \text{Ext}_A^i(L_y \otimes M, S) \xrightarrow{\delta} \text{Ext}_A^{i+1}(M, S) \rightarrow \dots$$

The map $\tilde{y} : \text{Ext}_A^i(M, S) \rightarrow \text{Ext}_A^{i+n}(M, S)$ is just the action of $y \in H^n(A, k)$ on $\text{Ext}_A^i(M, S)$ by Proposition 3.1.5. By the previous considerations the sums of the maps are $H^{ev}(A, k)$ -module homomorphisms. Let $z \in \text{Ext}_A^{i+n}(M, S)$ and let x be any homogeneous element of $I_A(L_y \otimes M, S)$ that is not in \mathfrak{m} . So we have $\eta(xz) = x\eta(z) = 0$. Considering the above long exact sequence we therefore have that there exists $y' \in \text{Ext}_A^{i+\deg(x)}(M, S)$ such that $xz = yy'$, implying that $z = yx^{-1}y' \in y(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(M, S)$. Therefore $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^j(M, S) = y(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^{j-n}(M, S)$ for all $j > n$. We will show that this also holds for $j \leq n$. Assume $z \in \text{Ext}_A^j(M, S)$ for some $j \leq n$. Let x' be a homogeneous element of positive degree in $H^{ev}(A, k)$ that is not in \mathfrak{m} . Multiply z by a large enough power of x' such that $\deg((x')^m z) > n$. Then $(x')^m z \in (H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(M, S) = y(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(M, S)$ as above. Now x' is invertible in $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(M, S)$ so we obtain $z \in y(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(M, S)$.

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

Since $y \in \mathfrak{m}$ and $\text{Ext}^\bullet(M, S)$ is finitely generated over $H^{ev}(A, k)$ by (fg), Nakayama's Lemma (Lemma 1.7.2) applied to the local ring $H^{ev}(A, k)$ implies that

$$(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(M, S) = 0.$$

This contradicts our assumption that $I_A(M, S) \subseteq \mathfrak{m}$. Therefore $I_A(M \otimes L_y, S) \subseteq \mathfrak{m}$ and so $\mathcal{V}_A(M, S) \cap Z(\langle y \rangle) \subseteq \mathcal{V}_A(M \otimes L_y)$.

To prove the opposite inclusion $\mathcal{V}_A(L_y \otimes M) \subseteq \mathcal{V}_A(M) \cap Z(\langle y \rangle)$ by Proposition 3.2.9 (v) it suffices to show that $\mathcal{V}_A(L_y) \subseteq Z(\langle y \rangle)$. Applying Proposition 3.2.9 (iv) again, it is enough to show that $\mathcal{V}_A(L_y, S) \subseteq Z(\langle y \rangle)$ for every simple A -module S . Thus we need to show that if \mathfrak{m} is a maximal ideal of $H^{ev}(A, k)$ for which $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(L_y, S) \neq 0$ then $y \in \mathfrak{m}$. Assume to the contrary that $y \notin \mathfrak{m}$. Then multiplication by y induces an automorphism on $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(L_y, S)$, since y is invertible in $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} H^{ev}(A, k)$. As the localization functor is exact, the existence of the short exact sequence defining L_y implies that $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(L_y, S)$ is the kernel of the isomorphism $y : (H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(k, S) \rightarrow (H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^{\bullet+n}(k, S)$ and therefore $(H^{ev}(A, k) \setminus \mathfrak{m})^{-1} \text{Ext}_A^\bullet(L_y, S) = 0$.

The second statement follows by setting $M = k$. □

Lemma 3.3.3. *Let A be an (fg)-Hopf algebra. If $y_1 \in H^r(A, k)$ and $y_2 \in H^s(A, k)$ then there exist a projective module P and a short exact sequence $0 \rightarrow \Omega_A^r L_{y_2} \rightarrow L_{y_1 y_2} \oplus P \rightarrow L_{y_1} \rightarrow 0$.*

Proof. Let $\tilde{y}_2 : \Omega_A^s k \rightarrow k$ be a representative for y_2 . By Proposition 2.1.8 there exist projective modules Q and Q' such that $L_{y_2} \otimes \Omega_A^r k \cong \Omega_A^r(L_{y_2} \otimes k) \oplus Q \xrightarrow{(\Omega_A^r \psi_{L_{y_2}}, \text{id})} \Omega_A^r(L_{y_2}) \oplus Q$ and $\Omega_A^s k \otimes \Omega_A^r k \cong \Omega_A^{r+s}(k \otimes k) \oplus Q' \xrightarrow{(\Omega_A^{r+s} \psi_k, \text{id})} \Omega_A^{r+s}(k) \oplus Q'$. Tensoring the exact sequence $0 \rightarrow L_{y_2} \rightarrow \Omega_A^s k \xrightarrow{\tilde{y}_2} k \rightarrow 0$ with $\Omega_A^r k$ we therefore obtain an exact sequence $0 \rightarrow \Omega_A^r L_{y_2} \oplus Q \rightarrow \Omega_A^{r+s} k \oplus Q' \rightarrow \Omega_A^r k \rightarrow 0$. Since $y_1 \hat{\cup} y_2 = \tilde{y}_1 \circ \Omega_A^r(\tilde{y}_2)$ (Remark 3.1.2 and Proposition 3.1.5), the following commutative diagram, where Q'' is a projective module, follows from an application of the snake lemma, since $L_{y_1 \hat{\cup} y_2}$ is uniquely determined in the

stable module category up to isomorphism.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \Omega_A^r L_{y_2} \oplus Q & \longrightarrow & L_{y_1 \cup y_2} \oplus Q'' & \longrightarrow & L_{y_1} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_A^r L_{y_2} \oplus Q & \longrightarrow & \Omega_A^{r+s} k \oplus Q'_{\Omega_A^r \tilde{y}_2} & \longrightarrow & \Omega_A^r k \longrightarrow 0 \\
 & & & & \downarrow \widetilde{y_1 \cup y_2} & & \downarrow \tilde{y}_1 \\
 & & & & k & \xlongequal{\quad} & k \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

The lemma follows from the fact that projective A -modules are injective. This implies that Q occurs as a direct summand of Q' and we can drop the two copies. \square

Lemma 3.3.4. *Let A be an (fg) -Hopf algebra. Under the isomorphism $H^n(A, k) \cong \text{Hom}_A(\Omega_A^n k, k) \cong \text{Ext}_A^1(\Omega_A^{n-1} k, k)$ an element $0 \neq y \in H^n(A, k)$ such that $L_y \neq 0$, corresponds to an extension of the form $0 \rightarrow k \rightarrow \Omega_A^{-1} L_y \rightarrow \Omega_A^{n-1} k \rightarrow 0$.*

Proof. Let P_{n-1} be the $(n-1)$ st projective module in a minimal projective resolution of k , i.e. $\Omega_A^n k$ is a submodule of P_{n-1} . Choose a representative \tilde{y} of y . Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L_y & \longrightarrow & \Omega_A^n k & \xrightarrow{\tilde{y}} & k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-1} & \longrightarrow & 0 \longrightarrow 0
 \end{array}$$

We want to apply the snake lemma (Lemma 1.6.6) to this diagram to get an exact sequence of the form $0 \rightarrow k \rightarrow \Omega_A^{-1} L_y \rightarrow \Omega_A^{n-1} k \rightarrow 0$: Since $\Omega_A^n k$, the cokernel of $\Omega_A^n k \rightarrow P_{n-1}$, is projective-free, we have that P_{n-1} is the injective hull of $\Omega_A^n k$. Since $L_y \neq 0$ and $\Omega_A^n k$ is non-simple indecomposable we have $\text{soc } L_y = \text{soc } \Omega_A^n k$, which gives that P_{n-1} is also the injective hull of L_y and the claim is established. Thus the statement follows from Lemma 1.6.13. \square

Proposition 3.3.5. *Let A be an (fg) -Hopf algebra and let $y \in H^n(A, k)$ such that $L_y \neq 0$. Then $y \in \ker \Phi_M$ iff there exists a projective module P such that $\Omega_A^{-1} L_y \otimes M \cong M \oplus \Omega_A^{n-1} M \oplus P$.*

Proof. By the previous lemma the image of y under the homomorphism $\text{Ext}_A^n(k, k) \rightarrow \text{Ext}_A^1(\Omega_A^{n-1} M, M)$ is represented by the sequence $0 \rightarrow M \rightarrow \Omega_A^{-1} L_y \otimes M \rightarrow \Omega_A^{n-1} k \otimes M \rightarrow 0$ (see the proof of Theorem 3.3.2 for compatibility with Φ_M). This image is zero iff the

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

sequence splits, iff $\Omega_A^{-1}L_y \otimes M \cong M \oplus (\Omega_A^{n-1}k \otimes M)$ (Lemma 1.6.5). The proposition now follows from Proposition 2.1.8, i.e. that there exists P such that $\Omega_A^{n-1}k \otimes M \cong \Omega_A^{n-1}M \oplus P$. \square

Proposition 3.3.6. *Let A be an (fg)-Hopf algebra. If M is a finitely generated A -module and $\mathcal{V}_A(M) = V_1 \cup V_2$ for conical subvarieties V_1, V_2 , such that $V_1 \cap V_2 = \{0\}$, then $M \cong M_1 \oplus M_2$ with $\mathcal{V}_A(M_1) = V_1$ and $\mathcal{V}_A(M_2) = V_2$. In particular if M is an indecomposable A -module, then $\mathbb{P}\mathcal{V}_A(M)$ is connected.*

Proof. The theorem is proven by induction on $\text{Kdim } V_1 + \text{Kdim } V_2$. If either V_1 or V_2 is $\{0\}$ the statement is obvious. So assume that both are non-zero. We may choose homogeneous elements $y_1, y_2 \in H^{ev}(A, k)$ such that $V_1 \subseteq Z(\langle y_1 \rangle)$, $V_2 \subseteq Z(\langle y_2 \rangle)$, $\text{Kdim}(V_2 \cap Z(\langle y_1 \rangle)) < \text{Kdim } V_2$ and $\text{Kdim}(V_1 \cap Z(\langle y_2 \rangle)) < \text{Kdim } V_1$ (The element y_1 can be chosen as follows. Let U_1, \dots, U_r be the irreducible components of V_2 , and suppose that U_1, \dots, U_q are those whose dimension equals $\text{Kdim } V_2$. Let W_i be the union of V_1 and all U_j , $1 \leq j \leq r, j \neq i$. Choose γ_i to be a homogeneous element in the ideal of W_i but not in the ideal of $\mathcal{V}_A(M)$. Then $y_1 := \sum_i \gamma_i^{a_i}$, where $a_i = a/(\deg a_i)$, a being the least common multiple of $\deg \gamma_1, \dots, \deg \gamma_n$ satisfies the condition. y_2 can be chosen analogously). Since $Z(\langle y_1 \hat{\cup} y_2 \rangle) = Z(y_1) \cup Z(y_2) \supseteq V_1 \cup V_2 = \mathcal{V}_A(M)$, some power of $y_1 \hat{\cup} y_2$ lies in $I_A(M, M)$. By replacing y_1 and y_2 by suitable powers if necessary we get homogeneous elements y_1 and y_2 of degree r resp. s such that $y_1 \hat{\cup} y_2 \in I_A(M, M)$. Thus by Proposition 3.3.5 we have $L_{y_1 \hat{\cup} y_2} \otimes M \cong \Omega_A M \oplus \Omega_A^{r+s} M \oplus P$, where P is projective. Furthermore by Lemma 3.3.3 there is a short exact sequence $0 \rightarrow \Omega_A^r L_{y_2} \rightarrow L_{y_1 \hat{\cup} y_2} \oplus P' \rightarrow L_{y_1} \rightarrow 0$, where P' is projective. Tensoring with M we obtain a short exact sequence of the form $0 \rightarrow \Omega_A^r L_{y_2} \otimes M \rightarrow \Omega_A^{r+s} M \oplus \Omega_A M \oplus (P' \otimes M \oplus P) \rightarrow L_{y_1} \otimes M \rightarrow 0$. We have $\mathcal{V}_A(\Omega_A^r L_{y_2} \otimes M) \stackrel{2.1.8}{=} \mathcal{V}_A(L_{y_2} \otimes \Omega_A^r M) \stackrel{3.3.2}{=} Z(\langle y_2 \rangle) \cap \mathcal{V}_A(\Omega_A^r M) \stackrel{3.2.11(i)}{=} Z(\langle y_2 \rangle) \cap \mathcal{V}_A(M) = (V_1 \cap Z(\langle y_2 \rangle)) \cup V_2$. The induction hypothesis gives $\Omega_A^r L_{y_2} \otimes M \cong N_1 \oplus N_2$ with $\mathcal{V}_A(N_1) = V_1 \cap Z(\langle y_2 \rangle)$ and $\mathcal{V}_A(N_2) = V_2$. Similarly $\mathcal{V}_A(L_{y_1} \otimes M) = Z(\langle y_1 \rangle) \cap \mathcal{V}_A(M) = V_1 \cup (V_2 \cap Z(\langle y_1 \rangle))$ so the induction hypothesis gives $L_{y_1} \otimes M \cong N'_1 \oplus N'_2$ with $\mathcal{V}_A(N'_1) = V_1$ and $\mathcal{V}_A(N'_2) = V_2 \cap Z(\langle y_1 \rangle)$. Since $\mathcal{V}_A(N'_1) \cap \mathcal{V}_A(N_2) = V_1 \cap V_2 = \{0\}$ it follows from Corollary 3.2.13 that $\text{Ext}_A^1(N'_1, N_2) = 0$. Similarly $\mathcal{V}_A(N'_2) \cap \mathcal{V}_A(N_1) \subseteq V_1 \cap V_2 = \{0\}$ so that we have $\text{Ext}_A^1(N'_2, N_1) = 0$. This forces the above exact sequence to decompose as the direct sum of two sequences of the form $0 \rightarrow N_1 \rightarrow N''_1 \rightarrow N'_1 \rightarrow 0$ and $0 \rightarrow N_2 \rightarrow N''_2 \rightarrow N'_2 \rightarrow 0$ with $\mathcal{V}_A(N''_1) \subseteq V_1$ and $\mathcal{V}_A(N''_2) \subseteq V_2$. By the Theorem of Krull-Remak-Schmidt (Theorem 1.9.1) for each direct summand M' of M , $\Omega_A M'$ is isomorphic to a direct summand of N''_1 or N''_2 . Denote by M_1 the direct sum of all indecomposable direct summands M' of M such that $\Omega_A M'$ is a direct summand of N''_1 and let M_2 be a direct complement. Then $\mathcal{V}_A(M_1) = \mathcal{V}_A(\Omega_A M_1) \subseteq \mathcal{V}_A(N''_1) \subseteq V_1$ and $\mathcal{V}_A(M_2) = \mathcal{V}_A(\Omega_A M_2) \subseteq \mathcal{V}_A(N''_2) \subseteq V_2$. Since $\mathcal{V}_A(M) = V_1 \cup V_2$ this forces $\mathcal{V}_A(M_1) = V_1$ and $\mathcal{V}_A(M_2) = V_2$. \square

Corollary 3.3.7. *Let A be an (fg)-Hopf algebra. Let M be an indecomposable A -module with $\text{cx}_A M = 1$. Then $\mathcal{V}_A(M)$ is a line.*

Proof. Let $\text{cx}_A M = 1$. Then $\text{Kdim } \mathbb{P}\mathcal{V}_A(M) = \text{Kdim } \mathcal{V}_A(M) - 1 = \text{cx}_A M - 1 = 0$ by Proposition 3.2.7. Therefore it is a point as it is connected by Proposition 3.3.6. Therefore

the support variety $\mathcal{V}_A(M)$ is a line. \square

3.4. Support varieties via Hochschild cohomology

Although in this thesis we mostly deal with Hopf algebras. We want to present an approach for arbitrary selfinjective algebras again under suitable finiteness assumptions. There are several different approaches by various people using different finiteness assumptions. We will restrict to the approach by Karin Erdmann, Miles Holloway, Rachel Taillefer, Nicole Snashall and Øyvind Solberg in [EHT⁺04]. Necessary prerequisites on Hochschild cohomology can for example be found in [Sol06].

Definition 3.4.1. Let A be a finite dimensional algebra. Then the *enveloping algebra* of A is $A^e := A \otimes A^{op}$. Considering A as an A^e -module one defines $HH^*(A) := \text{Ext}_{A^e}^*(A, A)$, the *Hochschild cohomology ring* of A . Let $y \in HH^n(A)$. Choose a representative $\tilde{y} : \Omega_{A^e} A \rightarrow A$. Define $\Phi_M(y) := \tilde{y} \otimes M : \Omega_{A^e} A \otimes N \rightarrow A \otimes_A N$. The element $\Phi_N(y)$ corresponds to an element n_y of $\text{Ext}_A^n(N, N)$. Then for A -modules M and N and an extension $\xi \in \text{Ext}_A^n(M, N)$ define the *cup product* to be $y \cup \xi := n_y Y \xi$. For a subalgebra H of the Hochschild cohomology denote the annihilator of the action of H on $\text{Ext}_A^*(M, N)$ by $A_H(M, N)$.

Definition 3.4.2. A selfinjective algebra A is said to *satisfy (Fg)* or is an *(Fg)-selfinjective algebra* if the following two conditions are satisfied:

(Fg1) There exists a graded subalgebra H of $HH^*(A)$ such that H is a commutative Noetherian ring and $H^0 = HH^0(A) = Z(A)$.

(Fg2) $\text{Ext}_A^*(A/\text{rad}(A), A/\text{rad}(A))$ is finitely generated as an H -module.

Remark 3.4.3. Note that for a finite dimensional Hopf algebra A Jörg Feldvoss and Sarah Witherspoon have shown in [FW11, Section 7] that the (fg)-property implies the (Fg)-property for the subalgebra $H := HH^0(B_0) \cdot H^{ev}(A, k)$. Here B_0 is the principal block of A , i.e. the block corresponding to the trivial module k . We will therefore in the remainder state our results only for the (Fg)-selfinjective algebras.

Definition 3.4.4. Define the *support variety* of (M, N) by

$$\mathcal{V}_H(M, N) = \max\text{Spec}(H/A_H(M, N)).$$

Define $\mathcal{V}_H(M) := \mathcal{V}_H(M, M)$.

We summarize the similar properties that we need in the following theorem:

Theorem 3.4.5. *Let A be a selfinjective algebra satisfying (Fg).*

- (i) *The variety $\mathcal{V}_H(M)$ is trivial iff M is projective.*
- (ii) *A module M is periodic iff $\text{cx}_A M = 1$ iff the variety $\mathcal{V}_H(M)$ is a line.*
- (iii) *We have $\text{Kdim } \mathcal{V}_H(M) = \text{cx}_A(M) = \gamma(\text{Ext}_A^\bullet(M, M)) < \infty$ for any module M in $\text{mod } A$.*

3.5. (Fg)-algebras have infinitely many components

For a component Θ of the stable Auslander-Reiten quiver and a positive integer d set $\Theta(d) := \{[M] \in \Theta \mid \dim M \leq d\}$. Furthermore in this section let $\mathcal{V}(M)$ be the support variety of M for an (Fg)-selfinjective algebra. If not stated otherwise the results in this section are generalizations of statements in [Far00b] and [Far98], where special cases of cocommutative Hopf algebras were considered. We sometimes use slightly different combinatorics.

Lemma 3.5.1. *Let A be an (Fg)-selfinjective algebra and let $\Theta \subseteq \Gamma_s(A)$ be a component. If $\mathcal{T} \subseteq \Theta$ is a τ -orbit, then $\mathcal{T} \cap \Theta(d)$ is finite for every $d > 0$.*

Proof. Assume there is $d > 0$, such that $\mathcal{T} \cap \Theta(d)$ is infinite. Then Θ is obviously a non-periodic component, i.e. one containing no τ -periodic module.

Since Θ is not periodic, it follows from Theorem 3.4.5 (i) and Theorem 3.4.5 (ii) that $\text{Kdim } \mathcal{V}_H(M) \geq 2$ for all $M \in \Theta$. Hence by Theorem 3.4.5 (iii) and Noether normalization applied to $H/A_H(M, M)$ as in the proof of Theorem 3.2.16 there exists $D \in 2\mathbb{N}_1$ such that $\dim \text{Ext}_A^{Dj}(M, M) \geq j$ for all j .

Since $\mathcal{T} \cap \Theta(d)$ is infinite, there exists $n \in \mathbb{Z}$, $M \in \Theta$ such that $\{\tau_A^{Dj+n}(M) \mid j \in \mathbb{Z}\} \cap \Theta(d)$ is infinite. Replacing M by $\tau^n M$ we can assume that $n = 0$. Hence we have $\dim \tau_A^{Dj}(M) \leq d$ for j belonging to some infinite set J . As $\tau_A = \Omega_A^2 \circ \mathcal{N}_A$ we also have $\dim \Omega_A^{Dj}(M) \leq d'$ for j belonging to J , since \mathcal{N}_A is an exact functor permuting the isomorphism classes of the simples. Without loss of generality assume that $d' = d$ and that $0 \in J$. Consequently

$$\dim \text{Ext}_A^{Dj}(M, M) \leq \dim \text{Hom}_A(\Omega_A^{Dj}(M), M) \leq d^2 \text{ for all } j \in J \cap \mathbb{N}$$

while

$$\dim \text{Ext}_A^{-Dj}(M, M) \leq \dim \text{Hom}_A(M, \Omega_A^{Dj}(M)) \leq d^2 \text{ for all } j \in J \cap -\mathbb{N}.$$

Hence there are infinitely many $j \in \mathbb{N}$, such that $\dim \text{Ext}_A^{Dj}(M, M) \leq d^2$, a contradiction. \square

For proving that there are infinitely many components for (Fg)-selfinjective algebras it is helpful to restrict the possibilities of the shapes of the components. A rough answer to the question how the components of the stable Auslander-Reiten quiver may look like was already mentioned (Theorem 1.9.15). For many components of the stable Auslander-Reiten quiver of a selfinjective algebra there remain only a few possibilities. This was first recognized for group algebras by Peter J. Webb in [Web82, Theorem A]. Therefore we define the following.

Definition 3.5.2. We say that **Webb's Theorem holds** for a Frobenius category \mathcal{C} admitting almost split sequences if all components of the stable Auslander-Reiten quiver of \mathcal{C} have one of the following tree classes: simply laced Dynkin, simply laced Euclidean, \tilde{A}_{12} , infinite Dynkin $A_\infty, A_\infty^\infty, D_\infty$.

3.5. (FG)-ALGEBRAS HAVE INFINITELY MANY COMPONENTS

Furthermore we define the set of Dynkin and extended Dynkin diagrams occurring in Webb's Theorem:

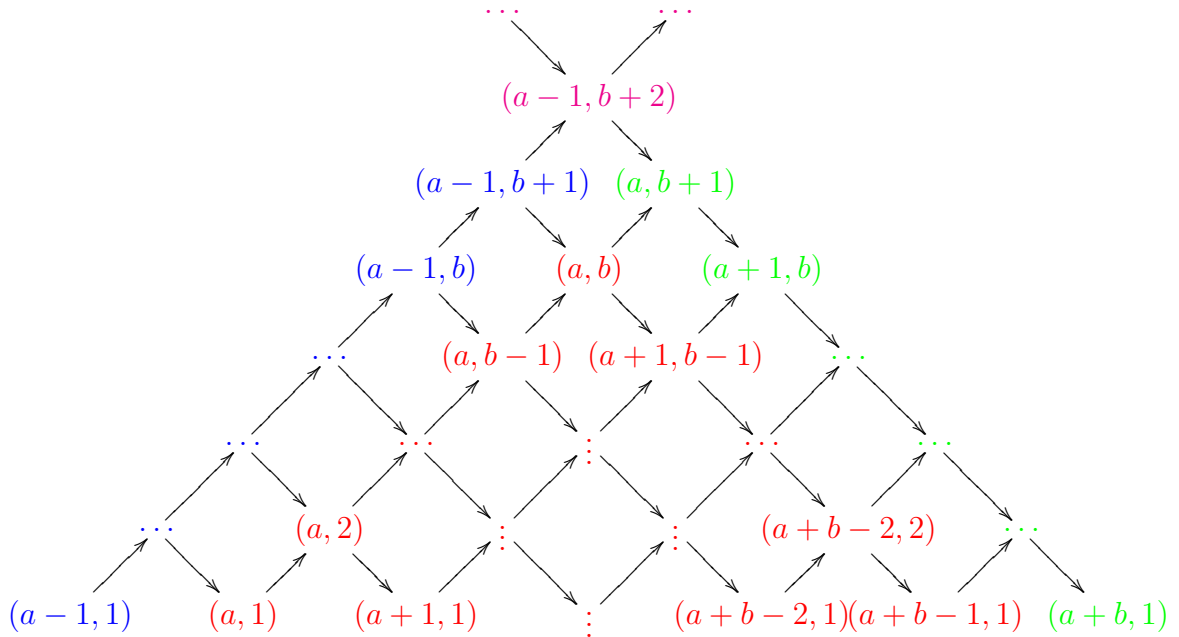
$$\mathcal{D} := \{\tilde{\mathbb{A}}_{12}, \tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_r, \mathbb{A}_\infty, \mathbb{D}_\infty, \mathbb{A}_\infty^\infty | n \in \mathbb{N}, r \in \{6, 7, 8\}\}$$

Theorem 3.5.3 ([KZ11, Main Theorem]). *Webb's Theorem holds for a selfinjective algebra A for which all modules have finite complexity. More precisely every non-periodic component of the Auslander-Reiten quiver of A is of the form $\mathbb{Z}[\Delta]$, where $\Delta \in \mathcal{D}$*

Theorem 3.5.4. *Let A be an (Fg)-selfinjective algebra and let $\Theta \subseteq \Gamma_s(A)$ be a component. Then $\Theta(d)$ is finite for every $d > 0$.*

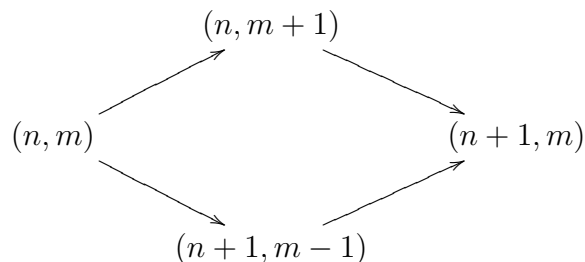
Proof. If Θ is τ -periodic, this result was obtained by Raymundo Bautista and Flavio U. Coelho (cf. [Liu96, Proposition 5.4]). If Θ is non-periodic, then by the version of Webb's Theorem proved by Otto Kerner and Dan Zacharia (Theorem 3.5.3) $\Theta \cong \mathbb{Z}[\Delta]$, where Δ is a Euclidean or infinite Dynkin diagram. If Δ is Euclidean, then $\mathbb{Z}[\Delta]$ possesses only finitely many τ -orbits, and the statement follows immediately from the foregoing lemma. If Δ is \mathbb{A}_∞ or \mathbb{D}_∞ the statement follows from the proof of [MR88, Proposition]. Therefore it suffices to prove the statement for $\Theta \cong \mathbb{Z}[\mathbb{A}_\infty]$. We label the vertices of Θ by (a, b) , where $a \in \mathbb{Z}$ as above and $b \in \mathbb{N}$:

Since there are only finitely many meshes in Θ that are associated with projective modules, there exists a vertex (a, b) , such that all meshes associated with projective modules occur inside the region $\tilde{\Upsilon}(a, b) := \{(n, m) \mid n + m \leq a + b, n \geq a\}$, which is highlighted red in the following diagram:



CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

A mesh outside this region looks as follows:



Since it corresponds to an exact sequence, we have $\dim(n, m + 1) - \dim(n + 1, m) = \dim(n, m) - \dim(n + 1, m - 1)$ and since every irreducible morphism is either injective or surjective the surjectivity of the map $(n, m) \rightarrow (n + 1, m - 1)$ implies the surjectivity of the map $(n, m + 1) \rightarrow (n + 1, m)$. But since the map $(n, 2) \rightarrow (n + 1, 1)$ is surjective for all $n < a$ this implies the surjectivity of all maps $(n, m) \rightarrow (n + 1, m - 1)$ for all $n < a$, i.e. in the blue and magenta region. Dually the injectivity of the map $(n + 1, m - 1) \rightarrow (n + 1, m)$ implies the injectivity of the map $(n, m) \rightarrow (n, m + 1)$ and since the map $(n, 1) \rightarrow (n, 2)$ is injective, for all $n + m > a + b$ we have the injectivity of all maps $(n, m) \rightarrow (n, m + 1)$ for $n + m > a + b$, i.e. in the green and magenta region.

Therefore for (n, m) with $n < a$ and $n + m \leq a + b$ we have that $\dim(n, m) \geq \min\{a - n + 1, m\}$. Now assume that $(n, m) \in \Theta(d)$. Then $d \geq a - n + 1$ together with $n + m \leq a + b$ implies that $m \leq a + b - n \leq b + d - 1$. Thus there are no $(n, m) \in \Theta(d)$ inside the blue region with $m > b + d - 1$.

Dually for (n, m) with $n \geq a$ and $n + m > a + b$ we have that $\dim(n, m) \geq \min\{n + m - (a + b), m\}$. Again assume $(n, m) \in \Theta(d)$. Then $d \geq n + m - (a + b)$ together with $n \geq a$ implies that $m \leq d - n + (a + b) \leq d + b$. Thus there are no $(n, m) \in \Theta(d)$ inside the green region with $m > b + d$.

In the magenta region we have that $\dim(n, m) \geq m - (b + 1)$ and therefore also in this region for $m > d + b - 1$ there are no $(n, m) \in \Theta(d)$.

Consequently $\Theta(d) \subseteq \{(n, m) | m < b + d - 1\}$ is contained in a finite union of τ -orbits. Hence the result follows by applying the foregoing lemma. \square

Example 3.5.5. Example 9.2.5 will show that this theorem is no longer true for arbitrary selfinjective algebras.

The following is conjectured for any finite dimensional algebra (see e.g. [ARS95, p. 409, (3)]):

Corollary 3.5.6. *There are infinitely many components for a block of an (Fg) -selfinjective algebra, which is not of finite representation type (see Section 8.1 for the definition).*

Proof. This follows from the foregoing theorem in conjunction with the second Brauer-Thrall conjecture (Theorem 1.9.18). \square

Even the following weaker problem is open in general (see e.g. [ARS95, p.409, (2)]):

3.5. (FG)-ALGEBRAS HAVE INFINITELY MANY COMPONENTS

Corollary 3.5.7. *Let B be a block of an (Fg)-selfinjective algebra. Then the Auslander-Reiten quiver of B is connected iff B is of finite representation type (see Section 8.1 for the definition).*

For tame algebras the conjecture is known to be true by [CB88, Corollary F].

CHAPTER 3. COHOMOLOGY AND SUPPORT VARIETIES

4. Quantized enveloping algebras

In this chapter we will introduce the main class of examples that we will study in this thesis, for historical reasons called quantum groups. They were first introduced for the \mathfrak{sl}_2 -case by Petr P. Kulish and Nikolai J. Reshetikhin in [KR81] and later independently generalized to arbitrary irreducible simple root systems by Vladimir G. Drinfeld [Dri85] and Michio Jimbo [Jim85]. The name quantum group was given to them by Vladimir G. Drinfeld in [Dri87]. The finite dimensional versions of quantum groups at a root of unity that we will work with were defined by George Lusztig in [Lus90a]. We mostly follow the conventions and notation of [Dru09]. For a general introduction to quantum groups we refer to [Jan96] and [BG02]. Recall the notation introduced in the preliminary section on Lie algebras. In particular \mathfrak{g} is a finite dimensional complex semisimple Lie algebra, Π and Φ^+ the corresponding sets of simple and positive roots, respectively.

4.1. Definitions of several quantum groups

For q an indeterminate over k define $q_\alpha := q^{d_\alpha}$, where $d_\alpha := (\alpha, \alpha)/2$ for $\alpha \in \Pi$. Furthermore for $t \in \mathbb{Z}$, $s \in \mathbb{N}$ define $[t]_\alpha := \frac{q_\alpha^t - q_\alpha^{-t}}{q_\alpha - q_\alpha^{-1}}$, $\begin{bmatrix} t \\ s \end{bmatrix}_\alpha := \frac{[t]_\alpha \cdots [t-s+1]_\alpha}{[s]_\alpha \cdots [1]_\alpha}$ and $[t]_\alpha! := [t]_\alpha \cdots [1]_\alpha$.

Definition 4.1.1. The *quantized enveloping algebra* $\mathbb{U}_Q(\mathfrak{g})$ is defined as the $\mathbb{Q}(q)$ -algebra with generators $\{E_\alpha, F_\alpha, K_\alpha^{\pm 1} | \alpha \in \Pi\}$ and the following relations for all $\alpha, \beta \in \Pi$:

$$(R1) \quad K_\alpha K_\beta = K_\beta K_\alpha \text{ and } K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1$$

$$(R2) \quad K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha, \beta)} E_\beta$$

$$(R3) \quad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha, \beta)} F_\beta$$

$$(R4) \quad E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

$$(R5) \quad \sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-s} E_\beta E_\alpha^s = 0$$

$$(R6) \quad \sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0.$$

CHAPTER 4. QUANTIZED ENVELOPING ALGEBRAS

Remark 4.1.2. (i) This is a non-commutative and non-cocommutative Hopf algebra if we define:

$$\begin{aligned} \Delta(E_\alpha) &= E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, & \varepsilon(E_\alpha) &= 0, & S(E_\alpha) &= -K_\alpha^{-1}E_\alpha, \\ \Delta(F_\alpha) &= F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, & \varepsilon(F_\alpha) &= 0, & S(F_\alpha) &= -F_\alpha K_\alpha, \\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, & \varepsilon(K_\alpha) &= 1, & S(K_\alpha) &= K_\alpha^{-1}. \end{aligned}$$

(ii) Let $\mathbb{U}_\mathbb{Q}(\mathfrak{n}^+)$ denote the subalgebra of $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ generated by the $E_\alpha, \alpha \in \Pi$ and let $\mathbb{U}_\mathbb{Q}(\mathfrak{n})$ denote the subalgebra of $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ generated by the $F_\alpha, \alpha \in \Pi$ and let $\mathbb{U}_\mathbb{Q}^0(\mathfrak{g})$ denote the subalgebra of $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ generated by the $K_\alpha^{\pm 1}, \alpha \in \Pi$. Then we have the usual triangular decomposition, i.e. there are vector space isomorphisms $\mathbb{U}_\mathbb{Q}(\mathfrak{n}^+) \otimes \mathbb{U}_\mathbb{Q}^0(\mathfrak{g}) \otimes \mathbb{U}_\mathbb{Q}(\mathfrak{n}) \cong \mathbb{U}_\mathbb{Q}(\mathfrak{g}) \cong \mathbb{U}_\mathbb{Q}(\mathfrak{n}) \otimes \mathbb{U}_\mathbb{Q}^0(\mathfrak{g}) \otimes \mathbb{U}_\mathbb{Q}(\mathfrak{n}^+)$ induced by multiplication.

(iii) Since the K_α commute we define $K_\mu := \prod (K_\alpha)^{n_\alpha}$ for $\mu = \sum_\alpha n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}$.

To construct a basis of this quantum group, George Lusztig constructed in [Lus90b] the following automorphisms. We follow the notation of [Jan96].

Definition 4.1.3. For each $\alpha \in \Pi$ define the *T-automorphism* T_α of $\mathbb{U}_\mathbb{Q}(\mathfrak{g})$ via

$$T_\alpha(K_\mu) = K_{s_\alpha(\mu)}, \tag{4.1.3.1}$$

$$T_\alpha(E_\alpha) = -F_\alpha K_\alpha, \tag{4.1.3.2}$$

$$T_\alpha(F_\alpha) = -K_\alpha^{-1}E_\alpha, \tag{4.1.3.3}$$

$$T_\alpha(E_\beta) = \sum_{i=0}^r (-1)^i q_\alpha^{-i} E_\alpha^{(r-i)} E_\beta E_\alpha^{(i)}, \tag{4.1.3.4}$$

$$T_\alpha(F_\beta) = \sum_{i=0}^r (-1)^i q_\alpha^{-i} F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)}. \tag{4.1.3.5}$$

for $\beta \in \Pi, \beta \neq \alpha$ and $r := -(\beta, \alpha^\vee)$.

Using this automorphisms it is now possible to define a basis for this quantum group. It is called the *Poincaré-Birkhoff-Witt basis*.

Theorem 4.1.4. Let $w_0 \in W$ denote the longest word in W and fix a reduced expression $w_0 = s_{\beta_1} \cdots s_{\beta_N}$, where $N = |\Phi^+|$. Set $\gamma_1 := \beta_1$ and $\gamma_i := s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$ for $1 < i \leq N$. Then $\Phi^+ = \{\gamma_1, \dots, \gamma_N\}$ and we have the following bases:

(i) We have $E_{\gamma_i} := T_{\beta_1} \cdots T_{\beta_{i-1}}(E_{\beta_i}) \in \mathbb{U}_\mathbb{Q}(\mathfrak{n}^+)$ and if $\gamma_i = \alpha \in \Pi$, then $E_{\gamma_i} = E_\alpha$. Furthermore the elements $E^m := E_{\gamma_1}^{m_1} \cdots E_{\gamma_N}^{m_N}$ for all $m = (m_1, \dots, m_N) \in \mathbb{N}^N$ form a $\mathbb{Q}(q)$ -basis of $\mathbb{U}_\mathbb{Q}(\mathfrak{n}^+)$.

(ii) We have $F_{\gamma_i} := T_{\beta_1} \cdots T_{\beta_{i-1}}(F_{\beta_i}) \in \mathbb{U}_\mathbb{Q}(\mathfrak{n})$ and if $\gamma_i = \alpha \in \Pi$, then $F_{\gamma_i} = F_\alpha$. Furthermore the elements $F^m := F_{\gamma_1}^{m_1} \cdots F_{\gamma_N}^{m_N}$ for all $m = (m_1, \dots, m_N) \in \mathbb{N}^N$ form a $\mathbb{Q}(q)$ -basis of $\mathbb{U}_\mathbb{Q}(\mathfrak{n})$.

4.1. DEFINITIONS OF SEVERAL QUANTUM GROUPS

(iii) The $K_\mu := K_{\alpha_1}^{\mu_1} \cdots K_{\alpha_n}^{\mu_n}$ for all $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ define a vector space basis for $\mathbb{U}_{\mathbb{Q}}^0(\mathfrak{g})$.

For defining a quantum group over a given field k it is necessary to define an integral form first. This was done by George Lusztig.

Definition 4.1.5. Recall that we defined $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$. The *integral (Lusztig) form of the quantized enveloping algebra* $U_{\mathbf{A}}(\mathfrak{g})$ is defined as the \mathbf{A} -subalgebra of $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ generated by $\{E_{\alpha}^{(m)}, F_{\alpha}^{(m)}, K_{\alpha}^{\pm 1}, m \geq 0, \alpha \in \Pi\}$, where the divided powers are defined as $E_{\alpha}^{(m)} := \frac{E_{\alpha}^m}{[m]_{\alpha}!}$ and $F_{\alpha}^{(m)} := \frac{F_{\alpha}^m}{[m]_{\alpha}!}$.

Define $U_k(\mathfrak{g}) := U_{\mathbf{A}}(\mathfrak{g}) \otimes k$ where the \mathbf{A} -module structure of k is defined via $q \mapsto \zeta \in k$ and the following quotient by central elements $U_{\zeta}(\mathfrak{g}) := U_k(\mathfrak{g}) / \langle K_{\alpha}^{\ell} - 1 \rangle, \alpha \in \Pi$.

Define $U_{\mathbf{A}}(\mathfrak{n}) := U_{\mathbf{A}}(\mathfrak{g}) \cap \mathbb{U}_{\mathbb{Q}}(\mathfrak{n})$, $U_{\mathbf{A}}(\mathfrak{n}^+) := U_{\mathbf{A}}(\mathfrak{g}) \cap \mathbb{U}_{\mathbb{Q}}(\mathfrak{n}^+)$ and $U_{\mathbf{A}}^0(\mathfrak{g}) := U_{\mathbf{A}}(\mathfrak{g}) \cap \mathbb{U}_{\mathbb{Q}}^0(\mathfrak{g})$. We define $U_{\mathbf{A}}(\mathfrak{b}) := U_{\mathbf{A}}(\mathfrak{n})U_{\mathbf{A}}^0(\mathfrak{g})$. Similarly define $U_k(\mathfrak{n}) := U_{\mathbf{A}}(\mathfrak{n}) \otimes k$, $U_k(\mathfrak{n}^+) := U_{\mathbf{A}}(\mathfrak{n}^+) \otimes k$ and $U_k^0(\mathfrak{g}) := U_{\mathbf{A}}^0(\mathfrak{g}) \otimes k$. We define $U_k(\mathfrak{b}) := U_k(\mathfrak{n})U_k^0(\mathfrak{g})$. Furthermore the images under the corresponding projections will be denoted by U_{ζ} of the corresponding symbol. In addition by abuse of notation we will denote $E_{\alpha} \otimes 1$, $F_{\alpha} \otimes 1$, $K_{\alpha} \otimes 1$ and their images under the canonical projection again by E_{α} , F_{α} and K_{α} , respectively.

Proposition 4.1.6. *The algebra $U_{\zeta}(\mathfrak{g})$ is a sum of blocks of the algebra $U_k(\mathfrak{g})$.*

Proof. There is an exact sequence of Hopf algebras $k \rightarrow k[(\mathbb{Z}/2)^n] \rightarrow U_k(\mathfrak{g}) \rightarrow U_{\zeta}(\mathfrak{g}) \rightarrow k$, where $k[(\mathbb{Z}/2)^n]$ is seen as a subalgebra of $U_k(\mathfrak{g})$ by sending the i -th generator to $K_{\alpha_i}^{\ell}$. The images of the first map are all central in $U_k(\mathfrak{g})$. The group algebra $k[(\mathbb{Z}/2)^n]$ is semisimple since $\text{char } k \neq 2$ and therefore decomposes as a direct sum of simple blocks. We have that $U_{\zeta}(\mathfrak{g}) = U_k(\mathfrak{g}) / k[(\mathbb{Z}/2)^n]$ and there is a unique idempotent e_0 , corresponding to the principal block, that does not vanish under ε . Therefore $U_k(\mathfrak{g}) = U_k(\mathfrak{g})e_0 \oplus U_k(\mathfrak{g})(1 - e_0)$. We have $U_k(\mathfrak{g})e_0 \cong U_{\zeta}(\mathfrak{g})$, where the isomorphism is given by the canonical projection. \square

Remark 4.1.7 ([APW91, 1.6]). Even more is true: the (integral) representation theory of $U_k(\mathfrak{g})$ decomposes into blocks parameterized by $\sigma \in (\mathbb{Z}/2)^n$ and all these categories are equivalent, where the equivalence is provided by tensoring with a one-dimensional module. Therefore in this thesis we will restrict our attention to $U_{\zeta}(\mathfrak{g})$.

Also this form of the quantized enveloping algebra admits a PBW-type basis:

Theorem 4.1.8. (i) *If we define $E_{\gamma_i}^{(m)} := T_{\beta_1} \cdots T_{\beta_{i-1}}(E_{\beta_i}^{(m)})$ then the divided powers $E^{(m)} := E_{\gamma_1}^{(m_1)} \cdots E_{\gamma_N}^{(m_N)}$, where $m \in \mathbb{N}^N$ form an \mathbf{A} -basis for $U_{\mathbf{A}}(\mathfrak{n}^+)$. The corresponding statements hold true if we base change to k , i.e. also these elements form a k -basis for $U_k(\mathfrak{n}^+)$ and for $U_{\zeta}(\mathfrak{n}^+)$.*

(ii) *The same holds true for the negative part: If we define $F_{\gamma_i}^{(m)} := T_{\beta_1} \cdots T_{\beta_{i-1}}(F_{\beta_i}^{(m)})$ then the divided powers $F^{(m)} := F_{\gamma_1}^{(m_1)} \cdots F_{\gamma_N}^{(m_N)}$, where $m \in \mathbb{N}^N$ form an \mathbf{A} -basis for $U_{\mathbf{A}}(\mathfrak{n})$. The corresponding statements hold true if we base change to k , i.e. also these elements form a k -basis for $U_k(\mathfrak{n})$ and for $U_{\zeta}(\mathfrak{n})$.*

CHAPTER 4. QUANTIZED ENVELOPING ALGEBRAS

(iii) If we define $\begin{bmatrix} K_\alpha \\ n \end{bmatrix} := \prod_{i=1}^n \frac{K_\alpha q_\alpha^{-i+1} - K_\alpha^{-1} q_\alpha^{-(-i+1)}}{q_\alpha^i - q_\alpha^{-i}}$ for $\alpha \in \Pi$ and $n \in \mathbb{N}$. Then $U_{\mathbb{A}}^0(\mathfrak{g})$ admits an \mathbb{A} -basis of all monomials $\prod_{\alpha \in \Pi} (K_\alpha^{\delta_\alpha} \begin{bmatrix} K_\alpha \\ n_\alpha \end{bmatrix})$, where $n_\alpha \in \mathbb{N}$ and $\delta_\alpha \in \{0, 1\}$.

The following definition is a generalization of George Lusztig's small quantum group in positive characteristic. It was introduced by Christopher M. Drupieski in his PhD thesis.

Definition 4.1.9. Let $r \in \mathbb{N}$ and let $\text{char } k = p$, possibly zero. The r -th **Frobenius-Lusztig kernel** $U_\zeta(G_r)$ is the subalgebra of $U_\zeta(\mathfrak{g})$ generated by the elements

$$\{E_\alpha, E_\alpha^{(p^j \ell)}, F_\alpha, F_\alpha^{(p^j \ell)}, K_\alpha^{\pm 1} \mid \alpha \in \Pi, 0 \leq j < r\}.$$

The zeroth Frobenius-Lusztig kernel is also called the **small quantum group** and denoted by $u_\zeta(\mathfrak{g})$.

Remark 4.1.10. (i) The Frobenius-Lusztig kernels for $r > 0$ only exist if the field has prime characteristic, otherwise the set $\{E_\alpha, E_\alpha^{(\ell)}, F_\alpha, F_\alpha^{(\ell)}, K_\alpha^{\pm 1} \mid \alpha \in \Pi\}$ spans $U_\zeta(\mathfrak{g})$.

(ii) One can also define an algebra $u_k(\mathfrak{g}) \subset U_k(\mathfrak{g})$ generated by $E_\alpha, F_\alpha, K_\alpha$. The same argument as in Proposition 4.1.6 shows that $u_\zeta(\mathfrak{g})$ is a sum of blocks of the algebra $u_k(\mathfrak{g})$. Also the representation theory of $u_k(\mathfrak{g})$ is essentially covered by that of $u_\zeta(\mathfrak{g})$. In this thesis we will call $u_k(\mathfrak{g})$ the **restricted quantum group**. Some authors also call the algebra $u_k(\mathfrak{g})$ small quantum group or Frobenius-Lusztig kernel or the algebra $u_\zeta(\mathfrak{g})$ restricted quantum group.

(iii) Also for the Frobenius-Lusztig kernels we can define a positive and negative part that we will denote by u_ζ (resp. U_ζ) of the corresponding symbol.

(iv) The algebra u_ζ also admits a PBW type basis, where the exponents have to be less than ℓ . In particular, we have $\dim u_\zeta(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$.

(v) From the definition of the quantum group and the PBW basis theorem we can conclude that if $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, then the quantum group of \mathfrak{g} is the tensor product of the quantum groups of \mathfrak{g}' and \mathfrak{g}'' , e.g. $u_\zeta(\mathfrak{g}) \cong u_\zeta(\mathfrak{g}') \otimes u_\zeta(\mathfrak{g}'')$. For a version of the big quantum group this is noted in [Tha94].

The next proposition explains the name Frobenius-Lusztig kernel, it is the Hopf algebraic kernel of the quantum Frobenius homomorphism, defined by George Lusztig:

Proposition 4.1.11. *There is a short exact sequence of Hopf algebras*

$$k \rightarrow u_\zeta(\mathfrak{g}) \rightarrow U_\zeta(\mathfrak{g}) \xrightarrow{\text{Fr}_\zeta} \text{Dist}(G) \rightarrow k,$$

where Fr_ζ is the quantum Frobenius morphism defined via $E_\alpha^{(\ell)} \mapsto X_\alpha$ and $F_\alpha^{(\ell)} \mapsto Y_\alpha$. It satisfies $\text{Fr}_\zeta(U_\zeta(G_r)) = \text{Dist}(G_r)$. It restricts to short exact sequences of Hopf algebras

$$k \rightarrow u_\zeta(\mathfrak{g}) \rightarrow U_\zeta(G_r) \rightarrow \text{Dist}(G_r) \rightarrow k,$$

4.1. DEFINITIONS OF SEVERAL QUANTUM GROUPS

and

$$k \rightarrow u_\zeta(\mathfrak{b}) \rightarrow U_\zeta(B_r) \rightarrow \text{Dist}(B_r) \rightarrow k.$$

There is another way to define the small quantum group via another version of a quantum group, the De Concini-Kac form. While the Lusztig form should be seen as a quantum analogue of the algebra of distributions of an algebraic group G , the De Concini-Kac form should be seen as a quantum analogue of the universal enveloping algebra of \mathfrak{g} . We need some localization of the base ring. Let $S \subset \mathbf{A}$ be the multiplicatively closed subset generated by $\{q - q^{-1}\}$ if Φ has type ADE , $\{q - q^{-1}, q^2 - q^{-2}\}$ if Φ has type BCF and $\{q - q^{-1}, q^2 - q^{-2}, q^3 - q^{-3}\}$ if Φ has type G . and $\mathcal{A} := S^{-1}\mathbf{A}$.

Definition 4.1.12. The *De Concini-Kac form* of the quantized enveloping algebra $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ is the \mathcal{A} -subalgebra of $\mathbb{U}_{\mathbb{Q}}(\mathfrak{g})$ generated by the elements $E_\alpha, F_\alpha, K_\alpha^{\pm 1}, \alpha \in \Pi$.

Define the De Concini-Kac form over a field k by $\mathcal{U}_k(\mathfrak{g}) := \mathcal{U}_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} k$ where the \mathcal{A} -module structure is obtained via $q \mapsto \zeta$. Again one can define the quotient by the two-sided ideal $\{K_\alpha^\ell \otimes 1 - 1 \otimes 1 | \alpha \in \Pi\}$, denote this by $\mathcal{U}_\zeta(\mathfrak{g})$.

Denote by \mathcal{Z} the central subalgebra of $\mathcal{U}_k(\mathfrak{g})$ generated by $E_\gamma^\ell, F_\gamma^\ell, K_\alpha^{\pm \ell}$ for $\gamma \in \Phi^+, \alpha \in \Pi$, and by $\overline{\mathcal{Z}}$ its image in $\mathcal{U}_\zeta(\mathfrak{g})$ under the canonical projection.

Also this form of a quantum group admits a PBW-type basis:

Theorem 4.1.13. *The monomials $\{F^r K_\mu E^s | r, s \in \mathbb{N}^N, \mu \in \mathbb{Z}\Phi\}$ form an \mathcal{A} -basis for $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$.*

The following Theorem was proven in characteristic zero by Corrado de Concini and Victor G. Kac in [dCK90] (see also [dCKP93]). It essentially relies on the commutation relation proven for topological quantum groups by I. Z. Levendorskiĭ and Yan S. Soĭbelman in [LS90, Proposition 5.5.2]. Since this is also valid in prime characteristic (see [Dru09, Lemma 3.1]), the statement follows.

Theorem 4.1.14. *The algebra $\mathcal{U}_k(\mathfrak{g})$ admits a filtration induced by the PBW basis if we define the total degree of $F^m K_\mu E^{m'}$ to be*

$$d(F^m K_\mu E^{m'}) := (m'_N, \dots, m'_1, m_1, \dots, m_N, \text{ht}(F^m K_\mu E^{m'})),$$

where $\text{ht}(F^m K_\mu E^{m'}) := \sum_{j=1}^N (m'_j + m_j) \text{ht}(\beta_j)$. Define a total ordering on \mathbb{N}^{2N+1} by using the lexicographical ordering. The associated graded algebra $\text{gr}\mathcal{U}_k(\mathfrak{g})$ has generators $E_\beta, F_\beta, K_\alpha^{\pm 1}, \beta \in \Phi^+, \alpha \in \Pi$ subject to the following relations:

- (i) $K_\alpha K_{\alpha'} = K_{\alpha'} K_\alpha$ and $K_\alpha K_\alpha^{-1} = 1$
- (ii) $K_\alpha E_\beta = \zeta^{(\alpha, \beta)} E_\beta K_\alpha$
- (iii) $K_\alpha F_\beta = \zeta^{-(\alpha, \beta)} F_\beta K_\alpha$
- (iv) $E_\beta F_{\beta'} = F_{\beta'} E_\beta$
- (v) $E_\beta E_{\beta'} = \zeta^{(\beta, \beta')} E_{\beta'} E_\beta$

CHAPTER 4. QUANTIZED ENVELOPING ALGEBRAS

$$(vi) \quad F_\beta F_{\beta'} = \zeta^{(\beta, \beta')} F_{\beta'} F_\beta$$

for all $\alpha, \alpha' \in \Pi$ and all $\beta, \beta' \in \Phi^+$ and $\beta < \beta'$ in the convex ordering for (v) and (vi), i.e. if $\beta = \gamma_i$ and $\beta' = \gamma_j$, then $i < j$.

The following lemma provides another construction of the small quantum group, that will turn out to be useful later:

Lemma 4.1.15. *There is a short exact sequence of Hopf algebras*

$$k \rightarrow \mathcal{Z} \rightarrow \mathcal{U}_k(\mathfrak{g}) \rightarrow u_\zeta(\mathfrak{g}) \rightarrow k$$

induced by the ordinary inclusion. The other map is induced by the inclusion of \mathcal{B} -forms $\mathcal{U}_{\mathcal{B}} \rightarrow U_{\mathcal{B}}$, where $\mathcal{B} := S^{-1}k[q, q^{-1}]$.

Lemma 4.1.16 ([DNN10, Lemma 5.1.1]). *The set $\overline{\mathcal{Z}} \subset \mathcal{U}_\zeta(\mathfrak{g})$ is a normal Hopf subalgebra.*

For studying the representation theory of the half-quantum groups it will be useful, to observe that they are linked via a smash product construction.

Lemma 4.1.17. *For each $r \in \mathbb{N}$ the algebra $U_\zeta(B_r)$ is the smash product of $U_\zeta(N_r)$ by the Hopf algebra $U_\zeta^0(G_r)$. In particular the algebra $u_\zeta(\mathfrak{b})$ is the skew group algebra of $u_\zeta(\mathfrak{n})$ by the group $(\mathbb{Z}/(\ell))^n$.*

Proof. For $U_{\mathbb{Q}}(\mathfrak{b})$ this is obvious from the generators and relations. By restriction this also follows for $U_{\mathbb{A}}(\mathfrak{b})$ because of the triangular decomposition. Tensoring with k we obtain the statement for $U_k(\mathfrak{b})$. It now suffices to note that K_α^ℓ acts trivially on $U_k(\mathfrak{n})$ since it is central in $U_k(\mathfrak{b})$. Therefore $U_k(\mathfrak{n}) = U_\zeta(\mathfrak{n})$ is a module for $U_\zeta^0(\mathfrak{g})$. \square

Corollary 4.1.18 ([Dru09, p. 85]). *All projective modules for $U_\zeta(B_r)$ have the same dimension.*

Remark 4.1.19. For the Borel of the small quantum group one can also use the following result by Sarah Scherotzke, which is a special case of the considerations by Idun Reiten and Christine Riedtmann in [RR85], to deduce the foregoing result: In [Sch09, Lemma 5.1 and the following page] the following is stated:

Let U be a local basic algebra with simple module S and G be a finite group such that $G < \text{Aut}(U)$. Let $\{e_1, \dots, e_m\}$ be a complete set of primitive orthogonal idempotents in kG . Let $P_i := Ae_i$ (resp. $S_i := \text{top } P_i$) be the corresponding projective indecomposables (resp. simples). Then every simple $U * G$ -module is of the form $S \otimes S_i$ for some i , a complete set of primitive orthogonal idempotents is given by $1 * e_i$ and the corresponding projective modules are $U \otimes P_i$.

If G is abelian and $p \nmid |G|$, then the set of characters, i.e. also the set of primitive orthogonal idempotents, can be identified with G and we denote the idempotents by $e_g, g \in G$. If $\alpha_1, \dots, \alpha_m$ are the simultaneous eigenvectors of the action of G on $\text{rad}(U)/\text{rad}^2(U)$, let n_i be the corresponding characters, then the arrows of $U * G$ are given by $\alpha_i * e_g$, where

4.1. DEFINITIONS OF SEVERAL QUANTUM GROUPS

$t(\alpha_i * e_g) = 1 * e_g$ and $s(\alpha_i * e_g) = 1 * e_g$, where $q = gn_i$.

This applies to the Borel of the small quantum group since an embedding of $(\mathbb{Z}/\ell)^n$ into the automorphism group of $u_\zeta(\mathfrak{n})$ is given by sending e_i to conjugation by K_{α_i} . Then the PBW basis theorem together with the commutation relation of the F_α and the K_β gives the desired result. The corollary then follows by applying the cited lemma.

The following statement is well-known, see e.g. [FW09].

Lemma 4.1.20. *The algebra $u_\zeta(\mathfrak{b})$ is basic and connected.*

Proof. That it is basic follows from the Lemma cited in Remark 4.1.19. Identify $(\mathbb{Z}/\ell)^n$ with the set of characters of $(\mathbb{Z}/\ell)^n$ via $e_i \mapsto \chi_i$, where $\chi_i(e_j) := \zeta^{(\alpha_i, \alpha_j)}$. Then the simultaneous Eigenvector $E_i \in u_\zeta(\mathfrak{n})$ corresponds to the character χ_i . Therefore there is a path $1 * g \rightarrow 1 * 0$ which is a concatenation of the arrows $E_i * h : 1 * (h + e_i) \rightarrow 1 * h$. \square

We will use the following statement to deduce this also for the higher Frobenius-Lusztig kernels:

Lemma 4.1.21 ([FV00, Proposition 2.4]). *Let B be a Borel subgroup of a semisimple algebraic group. Then $\text{Dist}(B_r)$ is basic and connected.*

Corollary 4.1.22. *The algebras $U_\zeta(B_r)$ are all basic and connected.*

Proof. We can apply Lemma 2.3.4 to the sequence $k \rightarrow u_\zeta(\mathfrak{b}) \rightarrow U_\zeta(B_r) \rightarrow \text{Dist}(B_r) \rightarrow k$ (see Proposition 4.1.11, the simple $u_\zeta(\mathfrak{b})$ -modules extend to $U_\zeta(B_r)$, see e.g. [Dru09, Appendix B]). By the proof of Lemma 4.1.20 we know that $u_\zeta(\mathfrak{b})$ has no self-extensions. Hence we know that all simple $U_\zeta(B_r)$ -modules are of the form $M^{[1]} \otimes L$, where $L|_{u_\zeta(\mathfrak{b})}$ is simple and M is a simple module for $\text{Dist}(B_r)$. Furthermore all of them are one-dimensional (since by the two previous lemmas $\text{Dist}(B_r)$ and $u_\zeta(\mathfrak{b})$ are basic), hence the algebra is basic (which was known before, see e.g. [Dru09, Appendix B]). By Lemma 2.3.4 we know that all vertices corresponding to the simple modules $M_i^{[1]} \otimes L$ for fixed L and varying M_i are connected by a sequence of paths. By the proof of the previous lemma we also know that $\text{Ext}_{u_\zeta(\mathfrak{b})}^1(L_1, L_2)$ is a one-dimensional $U_\zeta(B_r)$ -module or zero and for each simple $u_\zeta(\mathfrak{b})$ -module L there is a sequence of simple modules S_j with $S_n = L$, $S_0 = k$ and $\dim \text{Ext}_{u_\zeta(\mathfrak{b})}^1(S_j, S_{j-1}) = 1$. Therefore it is given by a character, say λ . Now chose simple $\text{Dist}(B_r)$ -modules M_1 and M_2 given by characters α and β such that $\alpha - \beta = \lambda$. Thus $\text{Ext}_{U_\zeta(B_r)}^1(M_1^{[1]} \otimes S_j, M_2^{[1]} \otimes S_{j-1}) \cong \text{Hom}_{\text{Dist}(B_r)}(k_\alpha, k_\beta \otimes k_\lambda)$, which is non-zero (and one-dimensional). This implies that $U_\zeta(B_r)$ is connected. \square

Remark 4.1.23. There is a left adjoint action of the quantum group $U_{\mathbb{Q}}(\mathfrak{g})$. It satisfies:

$$\mathrm{Ad}(E_{\alpha}^{(r)})(u) = \sum_{s=0}^r (-1)^s q_{\alpha}^{s(r-s)} E_{\alpha}^{(r-s)} K_{\alpha}^s u K_{\alpha}^{-s} E_{\alpha}^{(s)}, \quad (4.1.23.1)$$

$$\mathrm{Ad}(F_{\alpha}^{(r)})(u) = \sum_{s=0}^r (-1)^{r-s} q_{\alpha}^{(r-s)(s-1)} F_{\alpha}^{(s)} u F_{\alpha}^{(r-s)} K_{\alpha}^r, \quad (4.1.23.2)$$

$$\mathrm{Ad}(K_{\alpha}^{\pm 1})(u) = K_{\alpha}^{\pm 1} u K_{\alpha}^{\mp 1}. \quad (4.1.23.3)$$

It induces an action of $U_{\zeta}(\mathfrak{g})$ on itself and $u_{\zeta}(\mathfrak{g})$ is stable under this adjoint action.

4.2. Integrable modules for quantum groups

Studying representations for $u_{\zeta}(\mathfrak{g})$ it is often useful to study representations of $U_{\zeta}(\mathfrak{g})$. Not all representations of $U_{\zeta}(\mathfrak{g})$ are of interest, but only those, that are a direct sum of their weight spaces. They are called integrable representations. Luckily these include all finite dimensional representations. As a reference on the integral representation theory we refer to [APW91], where one can find the statements in this section.

Lemma 4.2.1. *For each $\lambda \in X$ there is an algebra homomorphism, called the **quantum character**, $\chi_{\lambda} : U_k^0(\mathfrak{g}) \rightarrow k$ defined via $\chi_{\lambda}(K_{\alpha}) = \zeta^{d_{\alpha}\lambda_{\alpha}}$ and $\chi_{\lambda}\left(\begin{bmatrix} K_{\alpha} \\ t \end{bmatrix}\right) = \begin{bmatrix} \lambda_{\alpha} \\ t \end{bmatrix}_{d_{\alpha}}$. The set of all characters forms a group with respect to the convolution product. This **group of quantum characters** will be denoted by $\mathrm{ch}(U_k^0(\mathfrak{g}))$. The map $X \rightarrow \mathrm{ch}(U_k^0(\mathfrak{g})), \lambda \mapsto \chi_{\lambda}$ is a group homomorphism.*

Definition 4.2.2. For $\lambda \in X$ and a $U_k^0(\mathfrak{g})$ - (resp. $U_{\zeta}^0(\mathfrak{g})$ -)module M define the **weight space** for λ via $M_{\lambda} := \{m \in M \mid u \cdot m = \chi_{\lambda}(u)m \ \forall u \in U_k^0(\mathfrak{g})\}$ (resp. $M_{\lambda} := \{m \in M \mid u \cdot m = \overline{\chi_{\lambda}}(u)m \ \forall u \in U_{\zeta}^0(\mathfrak{g})\}$).

A module M for $U_{\zeta}^0(\mathfrak{g})$ is called **integrable (of type 1)** if $M = \bigoplus_{\lambda \in X} M_{\lambda}$ and $M = \{m \in M \mid E_{\alpha}^{(r)} m = F_{\alpha}^{(r)} m = 0 \ \forall \alpha \in \Pi, r \gg 0\}$.

Proposition 4.2.3 ([DS94] Appendix 3). *Let M be a finite dimensional $U_{\zeta}^0(\mathfrak{g})$ -module, then M has a weight space decomposition.*

4.3. Simple modules for the small quantum group

In this section we present the classification of the simple modules, they occur as the socles of the so called induced modules. As a reference we again refer to [APW91] and [APW92]. For the corresponding statements in prime characteristic we refer to [AW92] and [Dru09].

Definition 4.3.1. The functor \mathfrak{F} defined by $\mathfrak{F}(M) := \{m \in \bigoplus_{\lambda \in X} M_{\lambda} \mid F_i^{(r)} m = E_i^{(r)} m = 0 \text{ for } r \gg 0\}$ takes $U_{\zeta}(\mathfrak{g})$ -modules to integrable $U_{\zeta}(\mathfrak{g})$ -modules. The **induction functor** is defined as $H^0(M) := \mathfrak{F}(\mathrm{Hom}_{U_{\zeta}(\mathfrak{b})}(U_{\zeta}(\mathfrak{g}), M))$. Set $H^0(\lambda) := H^0(k_{\lambda})$.

4.3. SIMPLE MODULES FOR THE SMALL QUANTUM GROUP

Theorem 4.3.2. *Let $\lambda \in X^+$. Then $H^0(\lambda)$ has a unique simple submodule $L_\zeta(\lambda)$. It has highest weight λ and each simple integrable $U_\zeta(\mathfrak{g})$ -module is of this form. Furthermore if $\lambda \in X_{p^\ell}$, then $L_\zeta(\lambda)$ stays simple when restricted to $U_\zeta(G_r)$ and these $U_\zeta(G_r)$ -module form a complete set of non-isomorphic simple modules.*

Remark 4.3.3. If $\mathfrak{g} \cong \bigoplus_{i=1}^n \mathfrak{g}_i$, then $H^0(\lambda) \cong \bigotimes_{i=1}^n H^0(\lambda_i)$.

Proposition 4.3.4. *Let k be a field with $\text{char } k = 0$. Then $\text{soc}_{U_\zeta(\mathfrak{g})} M = \text{soc}_{u_\zeta(\mathfrak{g})} M$ for any integrable module M . In particular, the quantized coinduced module $H^0(\lambda)$ is indecomposable as a $u_\zeta(\mathfrak{g})$ -module for $\lambda \in X_\ell$.*

Proposition 4.3.5. *Let k be a field with $\text{char } k = p > 0$. If $\lambda \in X_\ell$ then $H^0(\lambda)$ contains a unique simple $u_\zeta(\mathfrak{g})$ -submodule. In particular $H^0(\lambda)$ is indecomposable as a $u_\zeta(\mathfrak{g})$ -module.*

Proof. By the proof of [AW92, Theorem 1.10] we have that $\text{soc}_{u_\zeta(\mathfrak{g})} M = \bigoplus_{\mu \in X_\ell} L_\zeta(\mu) \otimes \text{Hom}_{u_\zeta(\mathfrak{g})}(L_\zeta(\mu), M)$. For $M = H^0(\lambda)$ a non-zero homomorphism $L_\zeta(\mu) \rightarrow H^0(\lambda)$ would be injective as $L_\zeta(\mu)$ is simple. Therefore the image is again a simple $u_\zeta(\mathfrak{g})$ -submodule of $H^0(\lambda)$. But each simple $u_\zeta(\mathfrak{g})$ -module is also a simple $U_\zeta(\mathfrak{g})$ -module. Therefore it has to be $L_\zeta(\lambda)$ by Theorem 4.3.2. Thus we have $\text{soc}_{u_\zeta(\mathfrak{g})} H^0(\lambda) = L_\zeta(\lambda)$. \square

There is a quantum analogue of the Steinberg tensor product theorem. For one order of the simple modules this is from [AW92], for the other one may consult [AM11].

Theorem 4.3.6. *Let $\lambda \in X_\ell$ and $\mu \in X^+$. Then we have an isomorphism of $U_\zeta(\mathfrak{g})$ -modules*

$$L(\lambda + \ell\mu) \cong L(\lambda) \otimes \bar{L}(\mu)^{[1]} \cong \bar{L}(\mu)^{[1]} \otimes L(\lambda),$$

where $\bar{L}(\mu)$ is a simple module for $\text{Dist}(G)$.

Remark 4.3.7. A version of this for $U_\zeta(G_r)$ and $\text{Dist}(G_r)$ can be proven by applying Lemma 2.3.3 to the exact sequence of Hopf algebras $k \rightarrow u_\zeta(\mathfrak{g}) \rightarrow U_\zeta(G_r) \rightarrow \text{Dist}(G_r) \rightarrow k$ since all simple $u_\zeta(\mathfrak{g})$ -modules lift to $U_\zeta(\mathfrak{g})$ and therefore to all $U_\zeta(G_r)$. In the remainder of this section we will give other applications of the existence of this sequence.

The following criterion turns out to be useful for proving the symmetry of the Frobenius-Lusztig kernels $U_\zeta(G_r)$:

Proposition 4.3.8 (cf. [Gor01, Remark 4.6, 2]). *Let A be a Hopf algebra with a unique one-dimensional module, such that the square of the antipode is an inner automorphism. Then A is symmetric.*

The following corollary is well-known for $r = 0$ and has been proven e.g. in [Xi89] (see also [Xi96]) and [Kum94].

Corollary 4.3.9. *The Frobenius-Lusztig kernels $U_\zeta(G_r)$ are symmetric.*

Proof. That the antipode squared is inner is proven in [Jan96, (4.9)]. There is a unique one-dimensional module for $u_\zeta(\mathfrak{g})$ since the Weyl group acts on the set of weight spaces (with action compatible to the Weyl group action on the weights) for a $U_\zeta(\mathfrak{g})$ -module M (see [AW92, 1.7]) with zero as the unique fixed point. The same is known for $\text{Dist}(G_r)$

CHAPTER 4. QUANTIZED ENVELOPING ALGEBRAS

(cf. [Jan03, 1.19 (1)]). Thus there is also a unique one-dimensional module for $U_\zeta(G_r)$ by Lemma 2.3.3. The result follows from the foregoing proposition. \square

Lemma 4.3.10. *The module $\text{St}_\ell := L((\ell - 1)\rho)$ is projective for $u_\zeta(\mathfrak{g})$. It is called the **Steinberg module** for $u_\zeta(\mathfrak{g})$.*

Corollary 4.3.11. *The category $\text{Dist}(G_r)$ is a block ideal in the category of $U_\zeta(G_r)$ -modules.*

Proof. This follows by applying Proposition 2.3.5 to the exact sequence $k \rightarrow u_\zeta(\mathfrak{g}) \rightarrow U_\zeta(G_r) \rightarrow \text{Dist}(G_r) \rightarrow k$, the module St_ℓ is the projective simple. \square

Definition 4.3.12. There is an automorphism ω on $U_{\mathbb{Q}}(\mathfrak{g})$ defined by $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$ and $\omega(K_\alpha) = K_\alpha^{-1}$. It satisfies $\omega^2 = \text{id}$. It induces automorphisms on $U_\zeta(\mathfrak{g})$ and each $U_\zeta(G_r)$.

The following statement is well-known (cf. [AM11, 3.8])

Lemma 4.3.13. *The endofunctor $M \mapsto M^* := (M^*)^{(\omega)}$ satisfies $(L^*)^{(\omega)} \cong L$ for each simple $U_\zeta(\mathfrak{g})$ -module L . Furthermore it is a duality on $\text{mod } A$, i.e. $\text{Hom}_A(M, N) \cong \text{Hom}_A(N^*, M^*)$.*

Definition 4.3.14. The indecomposable module $Z_r(\lambda) := U_\zeta(G_r) \otimes_{U_\zeta(B_r^+)} k_\lambda$ is called the **baby Verma module** with highest weight λ .

Proposition 4.3.15. *The projective cover of the simple $U_\zeta(G_r)$ -module $L(\lambda)$ has a filtration of the form $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = P(L(\lambda))$, such that each factor is of the form $M_i/M_{i-1} \cong Z_r(\lambda_i)$ for some $\lambda_i \in X_{p^r\ell}$. For each $\mu \in X_{p^r\ell}$ the number of i with $\lambda_i = \mu$ equals $[Z_r(\mu) : L(\lambda)]$.*

Corollary 4.3.16. *The following are equivalent for a weight $\lambda \in X_{p^r\ell}$:*

- (1) $Z_r(\lambda)$ is simple.
- (2) $Z_r(\lambda)$ is projective.

Proof. If $Z_r(\lambda)$ is simple, then it is of course isomorphic to $L(\lambda)$. Hence $L(\lambda)$ has dimension equal to $Z_r(\mu)$ for each μ . Thus $L(\lambda)$ cannot occur as a composition factor of $Z_r(\mu)$ for all $\mu \neq \lambda$. Thus $Z_r(\lambda)$ is projective, because $[P(L(\lambda)) : Z_r(\mu)] = [Z_r(\mu) : L(\lambda)]$.

If $Z_r(\lambda)$ is projective, then $L(\lambda)$ is isomorphic to the socle and the top of $Z_r(\lambda)$, hence $[Z_r(\lambda) : L(\lambda)] = 1$ implies $Z_r(\lambda) \cong L(\lambda)$. \square

4.4. Complexity of modules for the small quantum group

As noted in [FW09, page 10], we can compute the complexity of the quantized coinduced modules $H^0(\lambda)$. This is proven analogously to the classical case as can be found e.g. in [NPV02], [VIG07]. Except for the following proposition we include the details of the proof for the convenience of the reader:

4.4. COMPLEXITY OF MODULES FOR THE SMALL QUANTUM GROUP

Proposition 4.4.1 ([NPV02, Proposition 2.1.1]). *Let $p(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}[[t]]$*

- (i) *If $p(t) = \frac{f(t)}{(1-t)^d}$ for some positive integer d and $f(t) \in \mathbb{C}[t]$ with $f(1) \neq 0$, then a_n is a polynomial in n of degree $d - 1$. Hence $\gamma(a_n) = d$.*
- (ii) *Assume the poles of $p(t)$ are roots of unity. If $e^{i\theta}$ is a pole of order γ , then $\gamma \leq \gamma(a_n)$.*
- (iii) *Assume $p(t) = \frac{f(t)}{(1-t^r)^b}$ for positive integers b, r and $f(t) \in \mathbb{C}[t]$ with $f(1) \neq 0$. Fix $i, 0 \leq i < r$. For j sufficiently large, a_{i+rj} is a polynomial in j (say of degree $d_i - 1$).*
- (iv) *In (c), let $d = \max d_i$. Then $d = \gamma(a_n)$. If $S_n = |a_0| + \dots + |a_n|$, then $\gamma(S_n) = d + 1$.*

Definition 4.4.2. Fix a weight λ . Let $M \in \text{mod } u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ be a module such that all weights of M satisfy $\mu \leq \lambda$, then the **graded dimension** of M is defined as $\dim_t M = \sum_{\mu \leq \lambda} (\dim M_\mu) t^{\text{ht}(\lambda - \mu)}$, where $\text{ht}(\gamma) = \sum_{\alpha \in \Pi} n_\alpha$ if $\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha$ for $\gamma \in \Phi^+$ and X^+ is partially ordered via $\mu \leq \lambda$ iff $\lambda - \mu = \sum_{\alpha \in \Pi} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}_0$ for weights $\mu, \lambda \in X^+$.

There is a quantum analogue of Weyl's character formula, that is satisfied for the $H^0(\lambda)$. With the stated generality on the parameters it has been proven in [RH03]. A weaker version can be found in [APW92].

Lemma 4.4.3. *Let $\lambda \in X^+$. Then Weyl's character formula holds for $H^0(\lambda)$, i.e.*

$$\dim_t H^0(\lambda) = \prod_{\gamma \in \Phi^+} \frac{1 - t^{(\lambda + \rho, \gamma)}}{1 - t^{(\rho, \gamma)}}.$$

Lemma 4.4.4. *Let $\mu \leq \lambda$ be a weight, then $\dim_t P(\mu) = t^{\text{ht}(\lambda - \mu)} h(t)$, where $P(\mu)$ is the projective cover of the simple $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -module $L(\mu)$ and $h(t) := \prod_{\gamma \in \Phi^+} \frac{1 - t^{\ell(\rho, \gamma)}}{1 - t^{(\rho, \gamma)}}$.*

Proof. This follows from the fact that $P(\mu)$ is a coinduced module from $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ to $u_\zeta(\mathfrak{g})U_\zeta^0(\mathfrak{g})$, see e.g. [Dru11, Lemma 3.3.1]. \square

Theorem 4.4.5. *Let M be a finite dimensional $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -module with all weights of M being less than or equal to some $\lambda \in X^+$. Let*

$$q(t) := \frac{\dim_t M}{h(t)} = \frac{f(t)}{(1 - t^\ell)^\gamma g(t)},$$

where $f(t), g(t) \in \mathbb{Q}[t]$ and $\Psi_\ell(t) \nmid f(t)g(t)$, where $\Psi_\ell(t)$ is the ℓ -th cyclotomic polynomial in $\mathbb{Q}[t]$. Then $\gamma \leq \text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$.

Proof. We can assume that $\gamma \geq 0$ and that γ is the order of the pole of any primitive ℓ -th root of unity in $q(t)$ otherwise it is smaller than the order of the pole and we can replace γ by that order. Let P_* be a minimal projective resolution of M in the category of $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -modules. Now express $\frac{\dim_t P_n}{h(t)} = \sum_m b(m, n) t^m$, where each $b(m, n)$ is a non-negative integer. We have that $P_n \cong \bigoplus_{\mu' \in X} P(\mu)^{n_{\mu'}}$, where $n_{\mu'} = \dim \text{Hom}_{u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})}(P_n, L(\mu'))$. But

CHAPTER 4. QUANTIZED ENVELOPING ALGEBRAS

$\sum_{\substack{\mu' \in X \\ \mu' = \mu \in X / \ell X = X_\ell}} \text{Hom}_{u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})}(P_n, L(\mu)) = \text{Hom}_{u_\zeta(\mathfrak{b})}(P_n, L(\mu))$ as we can view $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ as graded $u_\zeta(\mathfrak{b})$ -modules (compare Lemma 6.1.2) and we have that projective covers for $u_\zeta(\mathfrak{b})$ and $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ coincide (see e.g. [Dru10, Lemma 3.3.1]). Therefore it follows from Proposition 3.2.7 that

$$\sum_m b(m, n) = \dim \text{Ext}_{u_\zeta(\mathfrak{b})}^n(M, \bigoplus_{\mu \in X_\ell} L(\mu)) \leq Cn^{\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)-1}.$$

Suppose that $b(m, n) \neq 0$. Then for some $\sigma \in X_\ell$ and $\nu \in X$, $P(\sigma + \ell\nu)$ appears as a direct summand of P_n with $m = \text{ht}(\lambda - (\sigma + \ell\nu))$. Set $\tau_n := \sigma + \ell\nu$. Since P_* is a minimal projective resolution it follows that τ_n is a weight in the radical of P_{n-1} . Consequently $\tau_{n-1} := \tau_n + \beta$ appears in the top of P_{n-1} , where β is some non-trivial sum of positive roots, because the projective modules are highest weight modules. We have that $\lambda - \tau_n = \lambda - \tau_{n-1} + \beta$. Applying heights we see that $\text{ht}(\lambda - \tau_n) \geq \text{ht}(\lambda - \tau_{n-1}) + 1$ and iterating this process n times yields $\text{ht}(\lambda - \tau_n) \geq \text{ht}(\lambda - \tau) + n$ where τ is a weight in the top of M . Hence if $b(m, n) \neq 0$, then $m \geq n$. Hence $q(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\dim_t P_n}{h(t)}$. Expand $q(t) = \sum_{i=0}^{\infty} a_i t^i$ into a power series. It follows that for $i > 0$,

$$\begin{aligned} S_i &= \sum_{m=0}^i |a_i| = \sum_{m=0}^i |(-1)^m \sum_{n=0}^{\infty} b(m, n)| \leq \sum_{m=0}^i \sum_{n=0}^{\infty} b(m, n) = \sum_{m=0}^i \sum_{n=0}^m b(m, n) \\ &\leq \sum_{n=0}^i Cn^{\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)-1} \leq D i^{\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)} \end{aligned}$$

for some $D > 0$.

From the definition of $h(t)$ one can deduce that the poles of $q(t)$ are roots of unity. Let b be the least common multiple of the orders of these poles. Then $q(t) = \frac{m(t)}{(1-t^b)^c}$ for some positive integer c and $m(t) \in \mathbb{C}[t]$. According to Proposition 4.4.1 (iii) we have that $q(t) = q_0(t) + \dots + q_{a-1}(t)$ with $q_i(t) = \sum_j b_{ij} t^{i+jb}$ with the property that b_{ij} is a polynomial in j of degree $d_i - 1$. Set $d = \max d_i$. Then by Proposition 4.4.1 (iv), $\gamma(S_i) = d + 1$ and $\gamma(a_i) = d$. Thus it follows that $\gamma(a_i) \leq \text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}$. The statement of the theorem now follows by Proposition 4.4.1 (ii). \square

Lemma 4.4.6. *We have that $\text{cx}_{u_\zeta(\mathfrak{b})}(H^0(\lambda)) \geq |\Phi^+| - |\Phi_\lambda^+|$.*

Proof. For any positive integer n , $t^n - 1 = \prod_{d|n} \Psi_d(t)$. Consider $q(t) = \frac{\dim_t H^0(\lambda)}{h(t)} = \prod_{\alpha \in \Phi^+} \frac{t^{(\lambda+\rho, \alpha^\vee)} - 1}{t^{\ell(\lambda, \alpha^\vee)} - 1}$. Observe that $\Psi_\ell(t)$ divides the numerator exactly $|\Phi_\lambda^+|$ times and $\Psi_\ell(t)$ divides the denominator exactly $|\Phi^+|$ times. Therefore by Theorem 4.4.5 we have

$$\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}(H^0(\lambda)) \geq |\Phi^+| - |\Phi_\lambda^+|.$$

\square

In [FW09] the following lemma, stated in [BNPP11], is also mentioned. For the prime characteristic case we refer to [Dru10]:

4.4. COMPLEXITY OF MODULES FOR THE SMALL QUANTUM GROUP

Corollary 4.4.7. *Assume that ℓ is **good** for Φ (i.e. $p \geq 3$ for type $\mathbb{B}_n, \mathbb{C}_n$ and \mathbb{D}_n , $p \geq 5$ for type $\mathbb{E}_6, \mathbb{E}_7$ and \mathbb{G}_2 and $p \geq 7$ for \mathbb{E}_8) and that $\ell > 3$ if Φ is of type \mathbb{B}_n or \mathbb{C}_n . If $\text{char } k = p > 0$ assume in addition that p is good for Φ and that $\ell > h$. Then we have $\text{cx}_{u_\zeta(\mathfrak{g})}(H^0(\lambda)) \geq 2 \text{cx}_{u_\zeta(\mathfrak{b})}(H^0(\lambda)) \geq |\Phi| - |\Phi_\lambda|$.*

Theorem 4.4.8 ([DNP10, Theorem 3.3]). *Suppose $\ell > h$ is an odd integer and that $\text{char } k = 0$. Then we have $\mathcal{V}_{u_\zeta(\mathfrak{g})}(L(\lambda)) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(H^0(\lambda))$. In particular $\text{cx}_{u_\zeta(\mathfrak{g})} L_\zeta(\lambda) = |\Phi| - |\Phi_\lambda|$.*

CHAPTER 4. QUANTIZED ENVELOPING ALGEBRAS

5. Frobenius extensions

5.1. General theory

In this section we recall the basic definitions and results from the general theory of Frobenius extensions from [NT60] and [BF93]. In the next section we will then apply them to the small quantum group.

Definition 5.1.1. Let R be a ring, $S \subseteq R$ be a subring of R and γ be an automorphism of S . If M is an S -module denote by $M^{(\gamma)}$ the S -module with the new action defined by $s * m = \gamma(s)m$. We say that R is a γ -**Frobenius extension** of S if

- (i) R is a finitely generated projective S -module, and
- (ii) there exists an isomorphism $\varphi : R \rightarrow \text{Hom}_S(R, S^{(\gamma)})$ of (R, S) -bimodules.

It is called **free** if R is a free S -module.

Definition 5.1.2. Let R be a ring, S be a subring of R and γ be an automorphism of S .

- (i) A γ -**associative form** from R to S is a biadditive map $\langle -, - \rangle_\pi : R \times R \rightarrow S$, such that:

- $\langle sx, y \rangle_\pi = s \langle x, y \rangle_\pi$
- $\langle x, ys \rangle_\pi = \langle x, y \rangle_\pi \gamma(s)$
- $\langle xr, y \rangle_\pi = \langle x, ry \rangle_\pi$

for all $s \in S, r, x, y \in R$.

- (ii) Let $\langle -, - \rangle_\pi : R \times R \rightarrow S$ be a γ^{-1} -associative form. Two subsets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ of R are said to form a **dual free pair** relative to $\langle -, - \rangle_\pi$ if

- $R = \sum_{i=1}^n Sx_i = \sum_{i=1}^n y_i S$
- $\langle x_i, y_j \rangle_\pi = \delta_{ij}$ for $1 \leq i, j \leq n$.

Lemma 5.1.3. Let S be a subring of R and let γ be an automorphism of S . Then the following statements are equivalent:

- (1) R is a free γ -Frobenius extension of S .
- (2) There is a γ^{-1} -associative form $\langle -, - \rangle_\pi$ from R to S relative to which a dual free pair $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$ exists.

CHAPTER 5. FROBENIUS EXTENSIONS

More precisely if $\langle -, - \rangle_\pi$ is a γ^{-1} -associative form relative to which a dual free pair exists, then an isomorphism $R \rightarrow \text{Hom}_S(R, S^{(\gamma)})$ is given by $y \mapsto (x \mapsto \gamma(\langle x, y \rangle_\pi))$ with inverse $f \mapsto \sum_{i=1}^n y_i f(x_i)$.

Definition 5.1.4. With the assumptions of the foregoing lemma a map $\pi : R \rightarrow S$ defined by $\pi(x, y) = \langle x, y \rangle_\pi$ is called a **Frobenius homomorphism** of the extension $R : S$.

Remark 5.1.5. The notion of a Frobenius algebra is a special case of a γ -Frobenius extension. Here $S = k$ and $\gamma = \text{id}$. The two notions of a Frobenius homomorphism coincide.

Proposition 5.1.6. Let $T \subseteq S \subseteq R$ be rings such that

- (i) $R : S$ is a γ -Frobenius extension with Frobenius homomorphism π ,
- (ii) $S : T$ is a γ' -Frobenius extension with Frobenius homomorphism π' , and
- (iii) $\gamma(T) = T$.

Then $R : T$ is a $(\gamma \circ \gamma')$ -Frobenius extension with Frobenius homomorphism given by $\pi' \circ \pi$.

Proposition 5.1.7. Let $R : S$ be a γ -Frobenius extension. Then for any R -module N and any S -module M we have $\text{Ext}_R^i(N, R \otimes_S M) \cong \text{Ext}_S^i(N, M^{(\gamma^{-1})})$. Furthermore we have $\text{Ext}_R^i(\text{Hom}_S(A, M), N) \cong \text{Ext}_S^i(M^{(\gamma^{-1})}, N)$ and $R \otimes_S M \cong \text{Hom}_S(A, M^{(\gamma^{-1})})$.

5.2. Root subalgebras and the small quantum group

This section deals with root subalgebras of small quantum groups. They were introduced by Christopher M. Drupieski in [Dru10] as analogues of cyclic shifted subgroups for group algebras. Here we use them to prove that the small quantum groups are symmetric in a similar fashion as Rolf Farnsteiner and Helmut Strade did for modular Lie algebras in [FS91] and Allen D. Bell and Rolf Farnsteiner for Lie superalgebras in [BF93].

Definition 5.2.1. Define the **root subalgebra** of $u_\zeta(\mathfrak{g})$ to be the subalgebra $u_\zeta(f_\alpha)$ generated by F_α for some $\alpha \in \Phi^+$.

Remark 5.2.2. As an algebra we have $u_\zeta(f_\alpha) \cong k[Y]/(Y^\ell)$.

In an analogous fashion as Rolf Farnsteiner and Helmut Strade we will now prove that the small quantum group is a Frobenius extension of each of these algebra. In fact we first prove a similar result for the De Concini-Kac form of the quantized universal enveloping algebra $\mathcal{U}_k(\mathfrak{g})$ and then factor out by the quotient (Recall Lemma 4.1.15).

Definition 5.2.3. Let $\alpha \in \Pi$. Define $\mathcal{O}_k(\mathfrak{g}, \alpha)$ to be the subalgebra of $\mathcal{U}_k(\mathfrak{g})$ generated by F_α and $E_\beta^\ell, F_{\beta'}^\ell, K_{\alpha'}^\ell$ with $\beta, \beta' \in \Phi^+, \alpha' \in \Pi$ and $\beta' \neq \alpha$. The subalgebra generated by $F_\alpha, F_{\beta'}^\ell, K_{\alpha'}^\ell$ with $\beta' \in \Phi^+, \alpha' \in \Pi, \beta' \neq \alpha$ will be denoted by $\mathcal{O}_k(\mathfrak{b}, \alpha)$.

The following is an easy consequence of the Poincaré-Birkhoff-Witt theorem for quantum groups (Theorem 4.1.13)

5.2. ROOT SUBALGEBRAS AND THE SMALL QUANTUM GROUP

Lemma 5.2.4. *The module $\mathcal{U}_k(\mathfrak{g})$ (resp. $\mathcal{U}_k(\mathfrak{b})$) is a free for $\mathcal{O}_k(\mathfrak{g}, \alpha)$ (resp. $\mathcal{O}_k(\mathfrak{b}, \alpha)$) for every $\alpha \in \Phi^+$.*

Lemma 5.2.5. *Let $\alpha \in \Pi$. Let $\mathcal{L}^n \in \mathbb{N}^n$ be the vector with $\mathcal{L}_i^n := \ell - 1$, $\mathcal{L}^N \in \mathbb{N}^N$ be the vector with $\mathcal{L}_i^N := \ell - 1$ and $\mathcal{L}^{N-1} \in \mathbb{N}^N$ be the vector with $\mathcal{L}_i^{N-1} := \delta_{i,\alpha}(\ell - 1)$. Let \mathcal{O}' be the complement of $\mathcal{O}_k(\mathfrak{g}, \alpha)F^{\mathcal{L}^{N-1}}K^{\mathcal{L}^n}E^{\mathcal{L}^N}$ with respect to the PBW basis, i.e. $\mathcal{U}_k(\mathfrak{g}) = \mathcal{O}' \oplus \mathcal{O}_k(\mathfrak{g}, \alpha)F^{\mathcal{L}^{N-1}}K^{\mathcal{L}^n}E^{\mathcal{L}^N}$. Let $\pi : \mathcal{U}_k(\mathfrak{g}) \rightarrow \mathcal{O}_k(\mathfrak{g}, \alpha)$ be the projection onto the second summand. Then the following hold:*

(i) $\pi(xy) = x\pi(y)$ and $\pi(yF_\alpha) = \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \alpha)}\pi(y)F_\alpha$ for all $x \in \mathcal{O}_k(\mathfrak{g}, \alpha), y \in \mathcal{U}_k(\mathfrak{g})$.

(ii) Let $a, a', c, c' \in \mathbb{N}^N, b, b' \in \mathbb{N}^n$. Then we have

$$\pi(F^{\mathcal{L}^{N-1}-a'}K^{\mathcal{L}^n-b'}E^{\mathcal{L}^N-c'}F^aK^bE^c) = \zeta^{c(a,b,c)}\delta_{a,a'}\delta_{b,b'}\delta_{c,c'},$$

where $c(a, b, c)$ is some integer depending only on a, b and c .

Proof. (i) The first statement is obvious. It suffices to show the second statement on the basis vectors $F^aK^bE^c$ for $a \in \{x \in \mathbb{N}^N | x_\alpha = 0\}$, $b \in \mathbb{N}^n$ and $c \in \mathbb{N}^N$ and for the associated graded algebra $\text{gr}\mathcal{U}_k(\mathfrak{g})$, where the reduced expression used to construct the convex ordering is chosen in such a way that α is a minimal root. This algebra is given by generators and relations in Theorem 4.1.14. It follows that $F^aK^bE^cF_\alpha = \zeta^{-(\sum b_i, \alpha) + (\sum a_i, \alpha)}F_\alpha F^aK^bE^c$ in $\text{gr}\mathcal{U}_k(\mathfrak{g})$. So low-order terms will not have an influence on $\text{Im}(\pi)$. And

$$F^{\mathcal{L}^{N-1}}K^{\mathcal{L}^n}E^{\mathcal{L}^N}F_\alpha = \zeta^{(\ell-1)(\sum_{\alpha \neq \alpha' \in \Phi^+}(\alpha', \alpha) - \sum_{\alpha' \in \Pi}(\alpha', \alpha))}F_\alpha F^{\mathcal{L}^{N-1}}K^{\mathcal{L}^n}E^{\mathcal{L}^N}.$$

The coefficient is equal to

$$\zeta^{-(2\rho, \alpha) + (\alpha, \alpha) + \sum_{\alpha' \in \Pi}(\alpha', \alpha)} = \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \alpha)},$$

since α is a simple root.

(ii) This follows as in (i) from the description of the corresponding graded algebra. □

Proposition 5.2.6. *Let $\alpha \in \Pi$. Then $\mathcal{U}_k(\mathfrak{g})$ is a free γ -Frobenius extension of $\mathcal{O}_k(\mathfrak{g}, \alpha)$ where γ is given by $F_\alpha \mapsto \zeta^{-\sum_{\alpha' \in \Pi}(\alpha', \alpha)}F_\alpha$.*

Proof. By Lemma 5.1.3 this is equivalent to Lemma 5.2.5. The γ -associative form is then given by $\langle x, y \rangle_\pi := \pi(xy)$. The dual free pair will be the PBW basis $\{F^aK^bE^c\}$ and the set $\{\zeta^{-c(a,b,c)}F^{\mathcal{L}^{N-1}-a}K^{\mathcal{L}^n-b}E^{\mathcal{L}^N-c}\}$. □

To deduce this also for the small quantum group we use the following theorem:

Theorem 5.2.7 ([Far96, Theorem 2.3]). *Let $R : S$ be a γ -Frobenius extension with Frobenius homomorphism π , $I \subseteq R$ an ideal with $\pi(I) \subseteq I \cap S$ and $\gamma(I \cap S) = I \cap S$. Then $R/I : S/(S \cap I)$ is a $\hat{\gamma}$ -Frobenius extension with Frobenius homomorphism $\hat{\pi}$, where $\hat{\gamma}(s + S \cap I) = \gamma(s) + S \cap I$ for all $s \in S$ and $\hat{\pi}(r + I) = \pi(r) + S \cap I$ for all $r \in R$.*

CHAPTER 5. FROBENIUS EXTENSIONS

Proposition 5.2.8. *Let $\alpha \in \Pi$. Then $u_\zeta(\mathfrak{g})$ is a free γ -Frobenius extension of $u_\zeta(f_\alpha)$ where γ is given by $F_\alpha \mapsto \zeta^{-\sum_{\alpha' \in \Pi}(\alpha', \alpha)} F_\alpha$ for each root subalgebra.*

Proof. The ideal $I = \langle E_\beta^\ell, F_\beta^\ell, K_\alpha^\ell - 1 \mid \beta \in \Phi^+, \alpha \in \Pi \rangle$ satisfies the conditions of the foregoing theorem since $\pi(I) = 0$ and γ is on generators just given by scalar multiplication. Hence the result follows. \square

Corollary 5.2.9. *$u_\zeta(\mathfrak{g})$ is a free γ -Frobenius extension of $u_\zeta(f_\beta)$, where $\beta \in \Phi^+$ and γ as in the foregoing proposition.*

Proof. Let T be one of the T -automorphisms (see Definition 4.1.3). Then there is an isomorphism $\text{Hom}_{u_\zeta(f_\alpha)}(u_\zeta(\mathfrak{g}), u_\zeta(f_\alpha)^{(\gamma)}) \rightarrow \text{Hom}_{u_\zeta(f_\beta)}(u_\zeta(\mathfrak{g}), u_\zeta(f_\beta)^{(\gamma)})$ given by $f \mapsto T f T^{-1}$. Induction on the length of β now gives the result. \square

Corollary 5.2.10. *$u_\zeta(\mathfrak{g})$ is a Frobenius algebra with Frobenius homomorphism induced by $F^{\ell-1} K^{\ell-1} E^{\ell-1} \mapsto 1$.*

Proof. That it is a Frobenius algebra is implied by the fact that it is a Hopf algebra (see Proposition 2.1.6). The formula for a Frobenius homomorphism follows from Proposition 5.1.6: $u_\zeta(\mathfrak{g})$ is a γ -Frobenius extension of $u_\zeta(f_\alpha)$ with Frobenius homomorphism given above and this is a Frobenius algebra with Frobenius homomorphism induced by $F_\alpha^{\ell-1} \mapsto 1$. \square

Proposition 5.2.11. *A Nakayama automorphism of $u_\zeta(\mathfrak{g})$ is given on generators by $K_\alpha \mapsto K_\alpha$, $F_\beta \mapsto \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \beta)} F_\beta$, $E_\beta \mapsto \zeta^{-\sum_{\alpha' \in \Pi}(\alpha', \beta)} E_\beta$.*

Proof. The proof is similar to the proof of Lemma 5.2.5. We have

$$F^{\mathcal{L}^N} K^{\mathcal{L}^N} E^{\mathcal{L}^N} K_\alpha = \zeta^{\sum_{\gamma \in \Phi^+}(\alpha, \gamma)} F^{\mathcal{L}^N} K_\alpha K^{\mathcal{L}^N} E^{\mathcal{L}^N} = K_\alpha F^{\mathcal{L}^N} K^{\mathcal{L}^N} E^{\mathcal{L}^N} \pmod{\ker \pi},$$

so $K_\alpha \mapsto K_\alpha$. We also have

$$\begin{aligned} F^{\mathcal{L}^N} K^{\mathcal{L}^N} E^{\mathcal{L}^N} F_\beta &= \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \beta)} F^{\mathcal{L}^N} F_\beta K^{\mathcal{L}^N} E^{\mathcal{L}^N} \\ &= \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \beta)} F_\beta F^{\mathcal{L}^N} K^{\mathcal{L}^N} E^{\mathcal{L}^N} \pmod{\ker \pi}, \end{aligned}$$

so $F_\beta \mapsto \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \beta)} F_\beta$. Similarly $E_\beta \mapsto \zeta^{-\sum_{\alpha' \in \Pi}(\alpha', \beta)} E_\beta$. \square

Corollary 5.2.12. *The left/right modular functions of $u_\zeta(\mathfrak{g})$ are equal to the counit ε of $u_\zeta(\mathfrak{g})$.*

Proof. According to Proposition 2.1.6, the left modular function ζ_l can be computed by $\zeta_l = \varepsilon \circ \nu$ for any Nakayama automorphism ν . Since the Nakayama automorphism given above is only some scalar multiple of the identity on generators E_β and F_β and the identity on K_α we get that $\zeta_l = \varepsilon$ since $\varepsilon(E_\beta) = \varepsilon(F_\beta) = 0$ and $\varepsilon(K_\alpha) = 1$. The right modular function is then given by $\zeta_l \circ S^{-1}$ so the assertion follows. \square

With this we are now able to reprove a result that we have seen before (Corollary 4.3.9):

Corollary 5.2.13. *The small quantum group $u_\zeta(\mathfrak{g})$ is symmetric.*

5.2. ROOT SUBALGEBRAS AND THE SMALL QUANTUM GROUP

Proof. According to general theory a Nakayama automorphism of a Hopf algebra is given by $S^{-2} \circ (\text{id} * \zeta_l)$. The result now follows from the fact that S^{-2} is an inner automorphism by the element $K_{2\rho}^{-1}$ (see [Jan96, (4.9)]). \square

Analogous results can be proven for $u_\zeta(\mathfrak{b})$ with the appropriate changes, e.g. $u_\zeta(\mathfrak{b})$ is no longer symmetric. We will state the results without giving the proofs which can be adapted from the ones for $u_\zeta(\mathfrak{g})$ by deleting the E 's everywhere.

Lemma 5.2.14. *Let $\alpha \in \Pi$. Let \mathcal{O}' be the complement of $\mathcal{O}_k(\mathfrak{b}, \alpha)F^{\ell-1}K^{\ell-1}$ with respect to the PBW basis, i.e. $\mathcal{U}_k(\mathfrak{b}) = \mathcal{O}' \oplus \mathcal{O}_k(\mathfrak{b}, \alpha)F^{\mathcal{L}^{N-1}}K^{\mathcal{L}^n}$. Let $\mathcal{U}_k(\mathfrak{b}) \rightarrow \mathcal{O}_k(\mathfrak{b}, \alpha)$ be the projection onto the second summand. Then the following hold:*

- (i) $\pi(xy) = x\pi(y)$ and $\pi(yF_\alpha) = \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \alpha)}\pi(y)F_\alpha$ for all $x \in \mathcal{O}_k(\mathfrak{b}, \alpha), y \in \mathcal{U}_k(\mathfrak{b})$.
- (ii) Let $a, a' \in \mathbb{N}^N, b, b' \in \mathbb{N}^n$. Then we have $\pi(F^{\mathcal{L}^{N-1}-a'}K^{\mathcal{L}^n-b'}F^aK^b) = \zeta^{c'(a,b)}\delta_{a,a'}\delta_{b,b'}$, where $c'(a, b)$ is some integer depending only on a and b .

Proposition 5.2.15. *Let $\alpha \in \Pi$. Then $\mathcal{U}_k(\mathfrak{b})$ is a free γ -Frobenius extension of $\mathcal{O}_k(\mathfrak{b}, \alpha)$ where γ is given by $F_\alpha \mapsto \zeta^{-\sum_{\alpha' \in \Pi}(\alpha', \alpha)}F_\alpha$.*

Proposition 5.2.16. *Let $\alpha \in \Pi$. Then $u_\zeta(\mathfrak{b})$ is a free γ -Frobenius extension of $u_\zeta(f_\alpha)$ where γ is given by $F_\alpha \mapsto \zeta^{-\sum_{\alpha' \in \Pi}(\alpha', \alpha)}F_\alpha$ for each root subalgebra.*

Corollary 5.2.17. *We have $u_\zeta(\mathfrak{b})$ is a free γ -Frobenius extension of $u_\zeta(f_\beta)$, where $\beta \in \Phi^+$ and γ as in the foregoing proposition.*

Corollary 5.2.18. *The algebra $u_\zeta(\mathfrak{b})$ is a Frobenius algebra with Frobenius homomorphism induced by $F^{\mathcal{L}^N}K^{\mathcal{L}^n} \mapsto 1$.*

Proposition 5.2.19. *A Nakayama automorphism of $u_\zeta(\mathfrak{b})$, whose order divides ℓ , is given on generators by $K_\alpha \mapsto \zeta^{-\sum_{\gamma \in \Phi^+}(\alpha, \gamma)}K_\alpha$ and $F_\beta \mapsto \zeta^{\sum_{\alpha' \in \Pi}(\alpha', \beta)}F_\beta$*

For the nilpotent part in an analogous fashion one can prove:

Lemma 5.2.20. *We have a γ -Frobenius extension $u_\zeta(\mathfrak{n}) : u_\zeta(f_\beta)$, where $\beta \in \Phi^+$ and γ is given by the usual formula.*

Lemma 5.2.21. *The algebra $u_\zeta(\mathfrak{n})$ is symmetric with Frobenius homomorphism given by the projection onto the summand $F^{\mathcal{L}^N}$ with respect to the PBW basis.*

The following theorem by Christopher M. Drupieski is a first step towards rank varieties. We use it in the sequel to prove some results about Auslander-Reiten components containing $u_\zeta(\mathfrak{g})U_\zeta^0(\mathfrak{g})$ -modules.

Theorem 5.2.22 ([Dru10, Theorem 4.1, Theorem 4.3]). *Let $L \in \{\mathfrak{n}, \mathfrak{b}, \mathfrak{g}\}$. Then a $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -module M is injective for $u_\zeta(L)$ iff it is injective for each of the root subalgebras $u_\zeta(f_\alpha), \alpha \in \Pi$.*

CHAPTER 5. FROBENIUS EXTENSIONS

6. Graded modules for quantum groups

6.1. Modules graded by an infinite torus

In this section we consider modules for the infinite-dimensional algebras $u_\zeta(L)U_\zeta^0(\mathfrak{g})$, where $L \in \{\mathfrak{g}, \mathfrak{b}, \mathfrak{n}\}$. We show that they are the \mathbb{Z}^n -graded modules among the modules for $u_\zeta(L)$. This allows us to use the following result, which was first obtained in the case $n = 1$ by Robert Gordon and Edward L. Green. The stated generalization to arbitrary n uses a result by Pierre Gabriel.

Theorem 6.1.1 ([Far09, Corollary 1.3, Proposition 1.4 (2), Theorem 2.3], cf. [Gab62, Corollaire IV.4.4], cf. [GG82, Theorem 3.5]). *Let A be a \mathbb{Z}^n -graded algebra. Then the category of graded modules admits almost split sequences and the forgetful functor from finite dimensional graded modules to finite dimensional modules sends indecomposables to indecomposables, projectives to projectives and almost split sequences to almost split sequences.*

Using this we now proceed to prove statements about modules for the algebra $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ in a similar way as Rolf Farnsteiner did for restricted enveloping algebras in [Far05].

Lemma 6.1.2. *Let $L \in \{\mathfrak{g}, \mathfrak{b}, \mathfrak{n}\}$.*

- (i) *The category of finite dimensional modules over $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ is a sum of blocks for the category $\text{mod } u_\zeta(L)\#U_\zeta^0(\mathfrak{g})$.*
- (ii) *The category of finite dimensional $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -modules has almost split sequences.*
- (iii) *The canonical restriction functor $\text{mod } u_\zeta(L)U_\zeta^0(\mathfrak{g}) \rightarrow \text{mod } u_\zeta(L)$ sends indecomposables to indecomposables and almost split sequences to almost split sequences.*

Proof. There is an exact sequence of Hopf algebras

$$k \rightarrow u_\zeta(L) \cap U_\zeta^0(\mathfrak{g}) \rightarrow u_\zeta(L)\#U_\zeta^0(\mathfrak{g}) \rightarrow u_\zeta(L)U_\zeta^0(\mathfrak{g}) \rightarrow k,$$

where $K_\alpha \mapsto K_\alpha\#K_\alpha^{-1}$ by the first map and the second map is just given by $u\#v \mapsto uv$. For $u_\zeta(\mathfrak{n})$ we have that $u_\zeta(\mathfrak{n}) \cap U_\zeta^0(\mathfrak{g}) = k$, so we have that $u_\zeta(\mathfrak{n})\#U_\zeta^0(\mathfrak{g}) = u_\zeta(\mathfrak{n}) \times U_\zeta^0(\mathfrak{g}) \cong u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g})$. So the categories of interest are actually isomorphic in this case. For $L \in \{\mathfrak{g}, \mathfrak{b}\}$ the following computation shows that via this embedding $u_\zeta^0(\mathfrak{g}) = u_\zeta(L) \cap U_\zeta^0(\mathfrak{g})$ lies in the center of $u_\zeta(L)\#U_\zeta^0(\mathfrak{g})$:

$$\begin{aligned} (K_\alpha\#K_\alpha^{-1})(F_\beta\#u) &= K_\alpha K_\alpha^{-1}(F_\beta)\#K_\alpha^{-1}u = K_\alpha K_\alpha^{-1}F_\beta K_\alpha\#u K_\alpha^{-1} \\ &= F_\beta u K_\alpha u^{-1}\#u K_\alpha^{-1} = (F_\beta\#u)(K_\alpha\#K_\alpha^{-1}), \end{aligned}$$

CHAPTER 6. GRADED MODULES FOR QUANTUM GROUPS

where $u \in U_\zeta^0(\mathfrak{g})$, similarly for E_β and K_α . Thus $u_\zeta^0(\mathfrak{g})$ -weight spaces are $u_\zeta(L)\#U_\zeta^0(\mathfrak{g})$ -submodules. So indecomposable modules are given by a central character and using Krull-Remak-Schmidt (Theorem 1.9.1) we get

$$\text{mod } u_\zeta(L)\#U_\zeta^0(\mathfrak{g}) = \bigoplus_{\lambda \in \text{ch}(u_\zeta^0(\mathfrak{g}))} (\text{mod}(u_\zeta(L)\#U_\zeta^0(\mathfrak{g})))_\lambda$$

with $\text{mod } u_\zeta(L)U_\zeta^0(\mathfrak{g}) = (\text{mod } u_\zeta(L)\#U_\zeta^0(\mathfrak{g}))_0$. If V, W are finite dimensional simple modules giving rise to different characters $\lambda \neq \mu$, then

$$\text{Ext}_{u_\zeta(L)\#U_\zeta^0(\mathfrak{g})}^1(V, W) \cong H^1(u_\zeta(L)U_\zeta^0(\mathfrak{g}), \text{Hom}_{u_\zeta^0(\mathfrak{g})}(V, W)) \cong 0,$$

by Corollary 1.6.15. Thus they belong to different blocks. This shows (i).

(ii) and (iii) follow from the fact that $\text{mod}(u_\zeta(L)\#U_\zeta^0(\mathfrak{g}))$ coincides with the category of X -graded $u_\zeta(L)$ -modules with homomorphisms in degree 0. The restriction functors coincide. So these statements follow from Theorem 6.1.1. \square

The following statement was first obtained by Christopher M. Drupieski in [Dru10] by imitating the proof for the corresponding statement for algebraic group. For us it is just a corollary of the foregoing two statements:

Corollary 6.1.3. *Let $L \in \{\mathfrak{n}, \mathfrak{b}, \mathfrak{g}\}$, and let M be a finite dimensional $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -module. Then the following statements are equivalent:*

- (1) M is an integrable injective $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -module.
- (2) M is an injective $u_\zeta(L)$ -module.
- (3) M is a projective $u_\zeta(L)$ -module.
- (4) M is an integrable projective $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -module.

Lemma 6.1.4. *Let $L \in \{\mathfrak{n}, \mathfrak{b}, \mathfrak{g}\}$. Then the restriction functor from the last lemma induces a homomorphism $\mathcal{F} : \Gamma_s(u_\zeta(L)U_\zeta^0(\mathfrak{g})) \rightarrow \Gamma_s(u_\zeta(L))$ of stable translation quivers and components are mapped to components via this functor.*

Proof. Thanks to the foregoing corollary a finite dimensional module is projective for $u_\zeta(L)$ if and only if it is projective for $u_\zeta(L)U_\zeta^0(\mathfrak{g})$. The forgetful functor obviously commutes with direct sums so the homomorphism follows from Lemma 6.1.2 and Lemma 1.9.9.

Let Θ be a component of $\Gamma_s(u_\zeta(L)U_\zeta^0(\mathfrak{g}))$. Since \mathcal{F} is a homomorphism of stable translation quivers, there exists a unique component $\Psi \subseteq \Gamma_s(u_\zeta(L))$ with $\mathcal{F}(\Theta) \subseteq \Psi$. As $\mathcal{F}(\Theta)$ is $\tau_{u_\zeta(L)}$ -invariant, we only have to show that each neighbour of an element of $\mathcal{F}(\Theta)$ also belongs to $\mathcal{F}(\Theta)$. To that end, we consider an isomorphism class $[M] \in \Theta$ as well as the almost split sequence $0 \rightarrow \tau_{u_\zeta(L)}(\mathcal{F}(M)) \rightarrow E \rightarrow \mathcal{F}(M) \rightarrow 0$. We decompose $E = \bigoplus_{i=1}^n E_i^{m_i}$ into indecomposable modules, so that the distinct isomorphism classes $[E_i]$ are the predecessors of $[\mathcal{F}(M)] \in \Psi$. We next consider the almost split sequence $0 \rightarrow \tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M) \rightarrow X \rightarrow$

$M \rightarrow 0$ terminating in M . Thanks to Lemma 6.1.2 the almost split sequence terminating in $\mathcal{F}(M)$ is isomorphic to $0 \rightarrow \tau_{u_\zeta(L)}(\mathcal{F}(M)) \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(M) \rightarrow 0$. In particular, if $X = \bigoplus_{j=1}^m X_j^{r_j}$ is the decomposition of X into its indecomposable constituents, then $E \cong \mathcal{F}(X) \cong \bigoplus_{j=1}^m \mathcal{F}(X_j)^{r_j}$ is the corresponding decomposition of $\mathcal{F}(X)$. Thus the Theorem of Krull-Remak-Schmidt (Theorem 1.9.1) implies that for each $i = 1, \dots, n$ there exists $i_j \in \{1, \dots, m\}$, such that $[E_i] = [\mathcal{F}(X_{i_j})]$. Consequently, each isomorphism class $[E_i]$ belongs to $\mathcal{F}(\Theta)$ as desired. Using the bijectivity of $\tau_{u_\zeta(L)}$ one proves the corresponding statement for the successors of vertices belonging to $\mathcal{F}(\Theta)$. \square

6.2. Highest weight categories

The categories $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ and $U_\zeta(\mathfrak{g})$ are examples for a class of categories that generalize the module categories of quasi-hereditary algebras to categories with infinitely many simple objects:

Definition 6.2.1 ([CPS88, Definition 3.1]). A locally artinian category \mathcal{C} over k is called a **highest weight category** if there exists an interval-finite poset Λ satisfying the following conditions:

- (HWC1) There is a complete collection $\{S(\lambda)\}_{\lambda \in \Lambda}$ of non-isomorphic simple objects of \mathcal{C} indexed by the set Λ
- (HWC2) There is a collection $\{\nabla(\lambda)\}_{\lambda \in \Lambda}$ of objects of \mathcal{C} and, for each λ an embedding $S(\lambda) \subseteq \nabla(\lambda)$ such that all composition factors $S(\mu)$ of $\nabla(\lambda)/S(\lambda)$ satisfy $\mu < \lambda$. For $\lambda, \mu \in \Lambda$, we have that $\dim \text{Hom}_{\mathcal{C}}(\nabla(\lambda), \nabla(\mu))$ and $[\nabla(\lambda) : S(\mu)]$ are finite.
- (HWC3) Each simple object $S(\lambda)$ has an injective envelope $I(\lambda)$ in \mathcal{C} . Also $I(\lambda)$ has a (finite or infinite) filtration $0 = F_0(\lambda) \subset F_1(\lambda) \subset \dots \subseteq I(\lambda)$ such that
 - (i) $F_1(\lambda) \cong \nabla(\lambda)$
 - (ii) for $n > 1$, $F_n(\lambda)/F_{n-1}(\lambda) \cong \nabla(\mu)$ for some $\mu = \mu(n) > \lambda$
 - (iii) for a given $\mu \in \Lambda$, $\mu(n) = \mu$ for only finitely many n
 - (iv) $\bigcup_{i=0}^{\infty} F_i(\lambda) = I(\lambda)$

The elements of Λ are called the **weights** of \mathcal{C} .

The following theorem is well-known:

Theorem 6.2.2 (cf. [PS11, p. 24]). *The categories of $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -modules are highest weight categories for $L \in \{\mathfrak{g}, \mathfrak{b}, \mathfrak{n}\}$.*

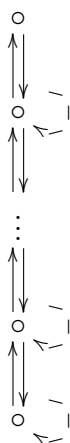
CHAPTER 6. GRADED MODULES FOR QUANTUM GROUPS

7. Exceptional cases for Auslander-Reiten theory

In this chapter we will now study the Frobenius-Lusztig kernels where a classification of indecomposable modules up to isomorphism is known. This is the case if $r = 0$, i.e. we are dealing with the case of a small quantum group, and $\mathfrak{g} = \mathfrak{sl}_2$. We will see later that these are essentially the only cases where a classification is possible.

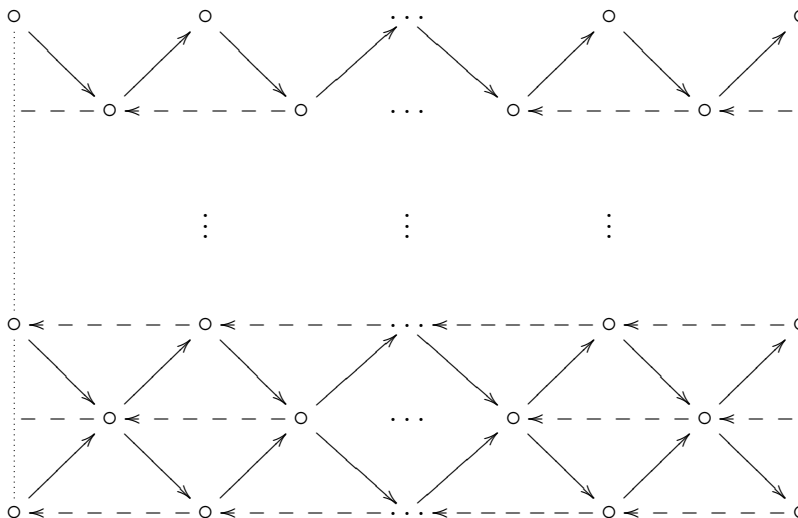
7.1. The representation-finite parts of the small quantum group

We have that $u_\zeta(\mathfrak{n}_{\mathfrak{sl}_2}) \cong k[X]/(X^\ell)$. Therefore it is a Nakayama algebra with exactly ℓ isomorphism classes of indecomposable modules and the Auslander-Reiten quiver looking as follows (e.g. with Theorem 1.9.26):



By Lemma 4.1.17 we have that $u_\zeta(\mathfrak{b}_{\mathfrak{sl}_2}) \cong u_\zeta(\mathfrak{n}_{\mathfrak{sl}_2}) \# u_\zeta^0(\mathfrak{sl}_2) \cong k[X]/(X^\ell) * \mathbb{Z}/\ell$. This algebra is isomorphic to a linearly oriented $\tilde{\mathbb{A}}_\ell$ (see Appendix A.2 with all paths of length ℓ as zero relations as can be checked easily). A very similar case is given in [RR85, 2.4 (i)]. So it is a Nakayama algebra and the Auslander-Reiten quiver looks as follows (e.g. with

Theorem 1.9.26):



7.2. The tame small quantum group

In this section we consider the tame blocks of small quantum groups. As we will prove in Corollary 8.3.2 it suffices to study the blocks of the tame small quantum group $u_\zeta(\mathfrak{sl}_2)$. The classification of the indecomposable modules and the Auslander-Reiten theory were independently obtained by Ruedi Suter in [Sut94] and Xiao Jie in [Xia94] (cf. [Xia97]) for the restricted quantum group $u_k(\mathfrak{g})$. They are in a slightly different situation (e.g. Ruedi Suter studies the case of even roots of unity) but their methods and results carry over. The classification of indecomposables for the small quantum group was also obtained via different methods by Vyjayanthi Chari and Alexander Premet in 1994 ([CP94a]). We will modify their proof by making use of the fact that $u_\zeta(\mathfrak{sl}_2)$ is a symmetric algebra to determine the quiver and relations. Let us mention that Yusuke Arike also has a proof of the classification of indecomposable modules which can be found in [Ari07]. It has the advantage that it determines the block idempotents explicitly.

Remark 7.2.1. It follows from the definition of the small quantum group and from the PBW theorem for quantum groups that $u_\zeta(\mathfrak{sl}_2)$ is the k -algebra generated by $E := E_\alpha, F := F_\alpha$ and $K := K_\alpha$ with relations:

$$(R2) \quad KEK^{-1} = \zeta^2 E$$

$$(R3) \quad KFK^{-1} = \zeta^{-2} F$$

$$(R4) \quad EF - FE = \frac{K - K^{-1}}{\zeta - \zeta^{-1}}$$

$$(small) \quad E^\ell = 0, F^\ell = 0, K^\ell = 1$$

7.2. THE TAME SMALL QUANTUM GROUP

Lemma 7.2.2. *There is an automorphism $T_\mu : u_\zeta(\mathfrak{sl}_2) \rightarrow u_\zeta(\mathfrak{sl}_2)$ for every non-zero complex number μ , defined on generators by $T_\mu(K) = K$, $T_\mu(E) = \mu E$, $T_\mu(F) = \mu^{-1}F$. They satisfy $T_\mu T_\nu = T_{\mu\nu}$ and the action of T on $u_\zeta(\mathfrak{sl}_2)$ defines a \mathbb{Z} -gradation on $u_\zeta(\mathfrak{g})$.*

Theorem 7.2.3. *The simple modules of $u_\zeta(\mathfrak{sl}_2)$ are given by the modules $S(t)$ for $t = 0, \dots, \ell - 1$. The module $S(t)$ has as a vector space a basis e_0^t, \dots, e_t^t and the actions of the generators of $u_\zeta(\mathfrak{sl}_2)$ are given by:*

$$\begin{aligned} Ke_s^t &= \zeta^{t-2s} e_s^t \\ Ee_s^t &= \left(\sum_{i=0}^{s-1} \frac{\zeta^{r-2i} - \zeta^{-r+2i}}{\zeta - \zeta^{-1}} \right) e_{s-1}^t \\ Fe_s^t &= e_{s+1}^t \end{aligned}$$

Lemma 7.2.4. *For $t = 0, \dots, \ell - 1$ there is an action of T on $S(t)$ defined by $T_\mu e_s^t = \mu^{r-1-2s} e_s^t$ for $s = 0, \dots, t$. It satisfies $T_\mu E T_\mu^{-1} = \mu^2 E$, $T_\mu F T_\mu^{-1} = \mu^{-2} F$ and $T_\mu K T_\mu^{-1} = K$. Furthermore there is an antigraded antiinvolution of $u_\zeta(\mathfrak{sl}_2)$ induced by $\sigma(E) = F$, $\sigma(F) = E$ and $\sigma(K) = K$. We have $\lambda^\sigma \cong \lambda$, where λ is the one-dimensional $u_\zeta^0(\mathfrak{g})$ -module of weight λ .*

Definition 7.2.5. Let $0 \leq r \leq \ell - 2$. Define $Z(r)$ to be the **Verma module** over $u_\zeta(\mathfrak{sl}_2)$. This is a quotient of $u_\zeta(\mathfrak{sl}_2)$ generated by E and $K - \zeta^r$. It is ℓ -dimensional and a universal highest weight module with highest weight ζ^r .

Lemma 7.2.6. *For $0 \leq r \leq \ell - 2$ the module $Z(r)$ is an ℓ -dimensional with basis $\overline{1}, \dots, \overline{F^{\ell-1}}$ on which the generators of $u_\zeta(\mathfrak{sl}_2)$ act as:*

$$F \cdot \overline{F^s} = \overline{F^{s+1}}, K \cdot \overline{F^s} = \zeta^{r-2s} \overline{F^s} \text{ and } E \cdot \overline{F^s} = \left(\sum_{i=0}^{s-1} \frac{\zeta^{r-2i} - \zeta^{-r+2i}}{\zeta - \zeta^{-1}} \right) \overline{F^{s-1}},$$

for $s = 0, \dots, \ell - 1$. In particular $E \cdot \overline{F^r} = 0$ in $Z(r)$.

Proof. The dimension of the module follows from the PBW basis. The action of the generators F and K is obvious. For the action of E we use induction on s :

If $s = 1$ then $E \cdot \overline{F^1} = \overline{FE} + \frac{K-K^{-1}}{\zeta-\zeta^{-1}} = \frac{\zeta^r - \zeta^{-r}}{\zeta - \zeta^{-1}}$. The induction step $s \rightarrow s + 1$ is as follows:

$$\begin{aligned} E \cdot \overline{F^{s+1}} &= \left(FE + \frac{K - K^{-1}}{\zeta - \zeta^{-1}} \right) \overline{F^s} = FE \cdot \overline{F^s} + \frac{K \overline{F^s} - K^{-1} \overline{F^s}}{\zeta - \zeta^{-1}} \\ &= F \left(\sum_{i=0}^{s-1} \frac{\zeta^{r-2i} - \zeta^{-r+2i}}{\zeta - \zeta^{-1}} \right) \overline{F^{s-1}} + \frac{\zeta^{r-2s} - \zeta^{-r+2s}}{\zeta - \zeta^{-1}} \overline{F^s} \\ &= \sum_{i=0}^s \frac{\zeta^{r-2i} + \zeta^{-r+2i}}{\zeta - \zeta^{-1}} \overline{F^s} = 0, \end{aligned}$$

since the sum can be split up into two different sums each containing the same summands but with different sign. □

CHAPTER 7. EXCEPTIONAL CASES FOR AUSLANDER-REITEN THEORY

Lemma 7.2.7. *For $0 \leq r \leq \ell - 2$ we have a non-split exact sequence:*

$$0 \rightarrow S(\ell - 2 - r) \rightarrow Z(r) \rightarrow S(r) \rightarrow 0.$$

Proof. Let U_r be the submodule of $Z(r)$ spanned by the $\overline{F^s}$ for $s = r, \dots, \ell - 1$. One checks immediately that the map given by $U_r \rightarrow S(\ell - 2 - r), \overline{F^{r+1+s}} \mapsto e_s^{\ell-r-2}$ for $s = 0, \dots, \ell - r - 1$ defines an isomorphism of $u_\zeta(\mathfrak{sl}_2)$ -modules. Furthermore the map $Z(r) \rightarrow S(r)$ is given by $\overline{F^s} \mapsto e_s^r$ for $s = 0, \dots, r$ and $\overline{F^s} \mapsto 0$ otherwise. One checks that it is an epimorphism of $u_\zeta(\mathfrak{sl}_2)$ -modules with kernel U_r . \square

Definition 7.2.8. For $0 \leq t \leq \ell - 1$ let $P(t)$ be the projective cover of $S(t)$.

We will now use the following restriction of a theorem due to Randall R. Holmes and Daniel K. Nakano:

Theorem 7.2.9 ([HN91, Theorem 5.1]). *Let $A = \sum_{i \in \mathbb{Z}} A_i$ be a finite dimensional graded algebra with graded subalgebras $A^- \subseteq k \cdot 1_A + \sum_{i < 0} A_i, A^0 \subseteq A_0$ and $A^+ \subseteq k \cdot 1_A + \sum_{i > 0} A_i$, such that*

- (i) $A = A^- A^0 A^+$,
- (ii) $\dim A = \dim A^- \dim A^0 \dim A^+$,
- (iii) $A^- A^0 = A^0 A^-$ and $A^0 A^+ = A^+ A^0$.

Assume furthermore that A^0 is semisimple and that A has an antigraded antiautomorphism t of order 2, such that $t(A^+ A^0) = A^- A^0$ and $\lambda^t \cong \lambda$ for $\lambda \in \Lambda_A$, where Λ_A be a fixed set of isomorphism class representatives of simple A_0 -modules. Then for $Z(\lambda) := A \otimes_{\bigoplus_{i \geq 0} A_i} S(\lambda)$ we have that the projective A -modules have a filtration by $Z(\lambda)$ -modules with $[P(S) : Z(\lambda)] = (Z(\lambda) : S)$ for each $\lambda \in \Lambda_A$.

Proposition 7.2.10. (i) *The module $S(\ell - 1)$ is the simple projective-injective Steinberg module. The other projective modules $P(t)$ for $0 \leq t \leq \ell - 2$ satisfy*

$$[P(t) : Z(s)] = \begin{cases} 1 & \text{if } s = t, \ell - 2 - t \\ 0 & \text{otherwise.} \end{cases}$$

- (ii) *For $0 \leq r \leq \ell - 2$ we have $\dim P(r) = 2\ell$.*
- (iii) *For $0 \leq r \leq \ell - 2$ we have $\text{soc } P(r) = S(r)$.*
- (iv) *For $0 \leq r \leq \ell - 2$ there is a non-split exact sequence of $u_\zeta(\mathfrak{sl}_2)$ -modules:*

$$0 \rightarrow Z(\ell - 2 - r) \rightarrow P(r) \rightarrow Z(r) \rightarrow 0$$

7.2. THE TAME SMALL QUANTUM GROUP

Proof. By the PBW theorem for $u_\zeta(\mathfrak{sl}_2)$ the requirements (i)-(iii) of the foregoing theorem are satisfied if one takes $A^+ = u_\zeta(\mathfrak{n}_{\mathfrak{sl}_2}^+)$, $A^0 = u_\zeta^0(\mathfrak{sl}_2)$ and $A^- = u_\zeta(\mathfrak{n}_{\mathfrak{sl}_2})$. As ζ is a primitive ℓ -th root of unity in k , we have that ℓ and $\text{char } k$ are coprime and therefore $\mathbb{Z}/\ell\mathbb{Z}$ is semisimple by Maschke's Theorem (Theorem 2.2.3). The antiautomorphism was given in Lemma 7.2.4. Therefore the first statement follows from the exact sequence in Lemma 7.2.7. The second one is then immediate. The third one follows as $u_\zeta(\mathfrak{sl}_2)$ is symmetric by Corollary 4.3.9. The fourth statement is then again immediate as $P(r)$ is indecomposable. \square

Now we can use the fact that the algebra is symmetric to determine the quiver and relations of this algebra.

Theorem 7.2.11. *The blocks of the small quantum group $u_\zeta(\mathfrak{sl}_2)$ consist of one simple block and $\frac{\ell-1}{2}$ blocks Morita equivalent to the trivial extension of the Kronecker quiver.*

Proof. The simple block is of course the Steinberg block (Lemma 4.3.10). Since $u_\zeta(\mathfrak{sl}_2)$ is symmetric, $\text{ht}(P(t))$ is a module of length two with composition factors $S(\ell - 2 - t)$. Therefore there are two possibilities for that module, either it is decomposable or uniserial. But if it was uniserial it would be a self-extension of $S(\ell - 2 - t)$. Thus $S(\ell - 2 - t)$ would appear as a direct summand of $\text{rad}(P(\ell - 2 - t))/\text{rad}^2(P(\ell - 2 - t))$. But there is no such direct summand, so $\text{ht } P(t)$ is decomposable for every projective indecomposable non-simple module for $u_\zeta(\mathfrak{sl}_2)$. Since $P(t)$ and $P(\ell - 2 - t)$ are non-isomorphic we have the following quiver (see Section 9.3 for more details):

$$1 \begin{array}{c} \xrightarrow{y_1} \\ \xleftarrow{x_2} \\ \xleftarrow{y_2} \end{array} 2$$

With relations $x_1x_2 - y_1y_2$, x_1y_2 and y_1x_2 and one of the following:

- (a) $x_2x_1 - \lambda y_2y_1$, x_2y_1 , y_2x_1 , or
- (b) $x_2y_1 - \mu y_2x_1$, x_2x_1 , y_2y_1 ,

with $\lambda, \mu \in k^\times$. But under these algebras the trivial extension of the Kronecker quiver, i.e. case (a) with $\lambda = 1$, is the only symmetric one. \square

Corollary 7.2.12. *The small quantum group $u_\zeta(\mathfrak{sl}_2)$ is special biserial. In particular it is tame.*

We now provide a description of the projective modules by identifying them as the only non-split extension of Verma modules.

Proposition 7.2.13. *For $1 \leq r \leq \ell - 1$ we have $\dim \text{Ext}^1(Z(r), Z(\ell - 2 - r)) = 1$.*

Proof. According to Proposition 7.2.10 (iv) we have that $\Omega(Z(r)) \cong Z(\ell - 2 - r)$. Therefore by Corollary 1.6.11 we have $\text{Ext}^1(Z(r), Z(\ell - 2 - r)) \cong \underline{\text{Hom}}(\Omega Z(r), Z(\ell - 2 - r)) \cong \underline{\text{End}}(Z(\ell - 2 - r))$. But $Z(\ell - 2 - r)$ is a non-projective module of length two with two different composition factors by 7.2.7 and 7.2.10 (iv). Hence this space is one-dimensional. \square

CHAPTER 7. EXCEPTIONAL CASES FOR AUSLANDER-REITEN THEORY

Corollary 7.2.14. *For $0 \leq r \leq \ell - 2$ the module $P(r)$ is a 2ℓ -dimensional with basis $\{v_0, \dots, v_{\ell-1}, w_0, \dots, w_{\ell-1}\}$ where the generators act as*

$$\begin{aligned} K v_s &= \zeta^{\ell-r-2s-2} v_s, & K w_s &= \zeta^{r-2s} w_s, \\ F v_s &= v_{s+1}, & F w_s &= w_{s+1}, \\ E v_s &= \left(\sum_{i=0}^{s-1} \frac{\zeta^{r-2i} - \zeta^{-r+2i}}{\zeta - \zeta^{-1}} \right) v_{s-1}, & E w_s &= \sum_{i=0}^{s-1} \left(\frac{\zeta^{r-2i} - \zeta^{-r+2i}}{\zeta - \zeta^{-1}} \right) w_{s-1} + v_{\ell-r-2+s} \end{aligned}$$

Proof. Let $0 \rightarrow Z(\ell - 2 - r) \xrightarrow{\alpha} P(r) \xrightarrow{\beta} Z(r) \rightarrow 0$ be a non-split short exact sequence. Define $v_s := \alpha(\overline{F^s})$. Furthermore denote by γ a splitting of β as a $u_\zeta^0(\mathfrak{g})$ -homomorphism (this exists since $u_\zeta^0(\mathfrak{g})$ is semisimple). Define $w_0 := \gamma(\overline{F^0})$. We have $K w_0 = \gamma\beta(K w_0) = \gamma(K\beta(w_0)) = \gamma(\zeta^r \beta(w_0)) = \zeta^r w_0$. Hence $K E w_0 = \zeta^2 E K w_0 = \zeta^{r+2} E w_0$. Since the $u_\zeta^0(\mathfrak{g})$ -weight spaces are one-dimensional we have $E w_0 = \mu_0 v_{\ell-r-2}$ for some $\mu_0 \in k$. If $\mu_0 = 0$, then the subspace spanned by $\{w_j := F^j w_0 | j = 0, \dots, \ell - 1\}$ would be isomorphic to $Z(r)$ and we would have a $u_\zeta(\mathfrak{sl}_2)$ -splitting, a contradiction. Hence $\mu_0 \neq 0$ and we can rescale w_0 to get $\mu_0 = 1$. This yields:

$$K w_j = K F w_{j-1} = \zeta^{-2} F K w_{j-1} = \zeta^{-2+r-2(j-1)} F w_{j-1} = \zeta^{r-2j} w_j$$

and

$$\begin{aligned} E w_j &= E F w_{j-1} = F E w_{j-1} + \frac{K - K^{-1}}{\zeta - \zeta^{-1}} w_{j-1} \\ &= F \left(\sum_{s=0}^{j-2} \left(\frac{\zeta^{r-2s} - \zeta^{-r+2s}}{\zeta - \zeta^{-1}} \right) w_{j-2} + v_{\ell-r-2+j-1} \right) + \frac{\zeta^{r-2(j-1)} - \zeta^{-(r-2(j-1))}}{\zeta - \zeta^{-1}} w_{j-1} \\ &= \sum_{s=0}^{j-1} \left(\frac{\zeta^{r-2s} - \zeta^{-r+2s}}{\zeta - \zeta^{-1}} \right) w_{j-1} + v_{\ell-r-2+j} \end{aligned}$$

Hence the defined v_s, w_s satisfy the claimed action of the generators. \square

In particular it is now possible to classify all the modules and give an explicit description via a basis and the fact how E, F and K act on it using the shape of the Auslander-Reiten components and the explicit description of the simple and the projective modules. One has to form pushouts and pullbacks. A complete list of modules can be found e.g. in [CP94a].

8. Representation type of small quantum groups

In this chapter we will now prove that the by now studied examples of Frobenius-Lusztig kernels are indeed the algebras where a classification of modules is feasible.

8.1. Definitions and general results

Definition 8.1.1. Let A be a finite dimensional algebra. Then A is called

- (i) **representation-finite** if there is only a finite number of indecomposable modules up to isomorphism. Otherwise it is called **representation-infinite**.
- (ii) **tame** if A is representation-infinite and for each dimension d there is a finite number t of A - $k[X]$ -bimodules M_1, \dots, M_t , finitely generated free as right $k[X]$ -modules, such that all but finitely many indecomposable A -modules M of dimension d are of the form $M \cong M_i \otimes_{k[X]} S$ for some i and some simple $k[X]$ -module S . Denote the smallest number t that satisfies this property for modules of dimension d by $\mu(d)$.
- (iii) **domestic** if A is tame and there exists $m \in \mathbb{N}_1$, such that $\mu(d) \leq m$ for all dimensions d .
- (iv) **wild** if there is an A - $k\langle X, Y \rangle$ -bimodule, finitely generated free as a right $k\langle X, Y \rangle$ -module, such that $M \otimes_{k\langle X, Y \rangle} - : \text{mod } k\langle X, Y \rangle \rightarrow \text{mod } A$ preserves isomorphism classes and indecomposability.

The property of being representation-finite, tame or wild is called the **representation type** of A .

Theorem 8.1.2 ([Dro80, Theorem 2], [CB88, Corollary C]). *Any finite-dimensional k -algebra is either representation-finite, tame or wild.*

Loosely speaking (from some perspectives conjecturally) the representation type of an algebra tells us whether it is possible to classify all the modules for this algebra. For example it is conjectured that the distinction between tame and wild is related to the question of decidability, i.e. the representation theory of A -modules is undecidable iff A is wild. For $k\langle X, Y \rangle$ it is known that $\text{mod } k\langle X, Y \rangle$ is undecidable (see e.g. [Ben95, Theorem 4.4.3]). Furthermore it is well-known that for any finitely-generated k -algebra B there is a full and

CHAPTER 8. REPRESENTATION TYPE OF SMALL QUANTUM GROUPS

exact embedding of $\text{mod } B$ into $\text{mod } k\langle X, Y \rangle$ (see e.g. [Rin74, p. 283]).

The following two results give connections between the representation type of an algebra and the complexity of its modules. The second one will turn out to be useful in our determination of representation type of the small quantum groups.

Proposition 8.1.3 ([Hel61, Proposition 2]). *Let A be a selfinjective algebra. If A is representation-finite, then every A -module M is Ω_A -periodic, in particular $\text{cx}_A M \leq 1$.*

For tame algebras the corresponding result was first stated by Jeremy Rickard for self-injective algebras and is therefore known as ‘‘Rickard’s Theorem’’. But Karin Erdmann discovered that his proof was wrong. The statement was then recovered by Rolf Farnsteiner for finite group schemes. J org Feldvoss and Sarah Witherspoon generalized this result to the following:

Theorem 8.1.4 ([FW09, Corollary 3.2]). *Let A be an (fg)-Hopf algebra. If B is a tame block of A , then $\text{cx}_A(M) \leq 2$ for every finite dimensional B -module M .*

Example 8.1.5. The converses of the last two statements do not hold as the following two examples show.

- (i) Let $G = \langle x, y \mid x^2 = y^2, xyx = y \rangle$ be the quaternion group, then kG is tame, but all modules are periodic: We recall the argument from [CE56, Chapter XII.7, p. 253]. The following is a projective resolution of the trivial module k :

$$\dots \rightarrow P_5 \xrightarrow{d_5} P_4 \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} k \rightarrow 0,$$

where $P_{4n} = P_{4n+3} = kG$ and $P_{4n+1} = P_{4n+2} = kG \oplus kG$, ε is the counit of kG and the d_i are defined by: $d_{4n+1}(1, 0) = x-1$, $d_{4n+1}(0, 1) = y-1$, $d_{4n+2}(1, 0) = (x+1, -(y+1))$, $d_{4n+2}(0, 1) = (xy+1, x-1)$, $d_{4n+3}(1) = (x-1, -(xy-1))$ and $d_{4n+4}(1) = \sum_{g \in G} g$. Obviously we have $\varepsilon d_1 = 0$ and $d_{4n+3} d_{4n+4} = 0$. We compute the other compositions: $d_{4n+1} d_{4n+2}(1, 0) = d_{4n+1}(x+1, -(y+1)) = (x+1)(x-1) - (y+1)(y-1) = 0$, $d_{4n+1} d_{4n+2}(0, 1) = d_{4n+1}(xy+1, x-1) = (xy+1)(x-1) + (x-1)(y-1) = 0$ and $d_{4n+2} d_{4n+3}(1) = d_2(x-1, -(xy-1)) = (x-1)(x+1, -(y+1)) - (xy-1)(xy+1, x-1) = 0$. Therefore the foregoing sequence is a complex.

It is obvious that ε is surjective and that $\dim \text{Im } d_{4n+4} = 1$. Therefore $\dim \ker \varepsilon = \dim \ker d_{4n+4} = 7$. Since $x-1 = d_{4n+1}(1, 0)$, $y-1 = d_{4n+1}(0, 1)$, $x^3-1 = d_{4n+1}(-x^3, 0)$, $y^3-1 = d_{4n+1}(0, -y^3)$, $xy-1 = d_{4n+1}(1, x)$, $x^3y-1 = d_{4n+1}(-x^3, x^3)$ and $x^2-1 = d_{4n+1}(x+1, 0)$ we also have that $\dim \text{Im } d_{4n+1} = 7$. Thus $\dim \ker d_{4n+1} = 9$. Assume $g \in \ker d_3$, then $(x-1)g = 0$ and therefore $xg = g$. If $g = a + bx + cx^2 + dx^3 + ey + fy^3 + gxy + hx^3y$ this leads to the system of equations $a = d = c = b$ and $e = g = f = h$. Furthermore $xyg = g$ leads to $a = e$ and therefore $\dim \ker d_{4n+3} = 1$. Thus $\dim \text{Im } d_{4n+3} = 7$.

So we are only left to check that $\dim \ker d_{4n+2} = 7$. This is easily checked (either by hand or with the help of a computer) by taking an arbitrary element $(z, z') \in kG \oplus kG$ expressing it with respect to the basis as in the foregoing step (with coefficients

8.2. REPRESENTATION TYPE OF HALF-QUANTUM GROUPS

$a, \dots, h, a', \dots, h'$). This leads to the system of equations $d + a + h' + a' = 0, a + b + e' + b' = 0, b + c + g' + c' = 0, c + d + f' + d' = 0, g + e + d' + e' = 0, h + f + b' + f' = 0, f + g + a' + g' = 0, e + h + c' + h' = 0, -f - a + d' - a' = 0, -h - b + a' - b' = 0, -e - c + b' - c' = 0, -g - d + c' - d' = 0, -a - e + g' - e' = 0, -c - f + h' - f' = 0, -b - g + f' - g' = 0, -d - h + e' - h' = 0$. The corresponding matrix has rank 9.

- (ii) Let Δ be a Dynkin quiver except \mathbb{A}_n , where $n \leq 5$, and \mathbb{D}_4 . Then it is well-known that the preprojective algebra of Δ , which is selfinjective, is wild (see e.g. [GS05, Proposition 3.1], where further references are given). But it was independently proved by Maurice Auslander and Idun Reiten ([AR96, Corollary 3.4]) and by Aidan H. Schofield ([Sch]) that all modules are τ -periodic and hence all modules have complexity one since the preprojective algebra of Δ is a Frobenius algebra.
- (iii) If $G := \mathbb{Z}/p \times \mathbb{Z}/p$ then if $\text{char } k = p \geq 3$ we have that kG is wild (see e.g. [Ben95, Theorem 4.4.4]), but $\text{cx}_{kG} k = \text{cx}_{k\mathbb{Z}/p} k + \text{cx}_{k\mathbb{Z}/p} k = 2$ by Corollary 3.2.15.

8.2. Representation type of half-quantum groups

We want to apply ‘‘Rickard’s Theorem’’ to the small quantum groups. Therefore we need the following finite generation result first stated by Victor Ginzburg and Shrawan Kumar in [GK93] under weaker assumptions on the parameters (see also [BNPP11] for an intermediate step), for $\text{char } k = 0$ the stated generality was proven by Mitja Mastnak, Julia Pevtsova, Peter Schauenburg and Sarah Witherspoon in [MPSW10]. The prime characteristic case was obtained by Christopher M. Drupieski in [Dru09] (see also [Dru11]).

Theorem 8.2.1. *(i) The Frobenius-Lusztig kernels for the Borel part of the quantum group $U_\zeta(B_r)$ satisfy (fg).*

(ii) Let $\text{char } k = 0$ or let $\text{char } k$ be good for Φ and $\ell > h$. Then the small quantum group $u_\zeta(\mathfrak{g})$ satisfies (fg).

(iii) Let $\ell > h$ and $\text{char } k = p$ be very good for Φ . Furthermore assume that Φ is of type \mathbb{A}_n or \mathbb{D}_n and $n > p + 2$ or Φ is of type \mathbb{A}_n and $\ell \geq n + 4 = h + 3$ or Φ is of type \mathbb{D}_n and $\ell \geq 4n = 2h + 4$. Then $U_\zeta(G_1)$ satisfies (fg).

The following theorem was first proven by Claude Cibils in [Cib97] for the simply-laced case. In their paper [FW09] J org Feldvoss and Sarah Witherspoon used ‘‘Rickard’s Theorem’’ (Theorem 8.1.4) to determine the representation type of the Borel part of the small quantum group. As this algebra is connected (see Lemma 4.1.20), it suffices to calculate a lower bound for the complexity of the trivial module. We include the proof for the sake of completeness, a similar method is used for $u_\zeta(\mathfrak{g})$ below.

Theorem 8.2.2. *The connected algebra $u_\zeta(\mathfrak{b})$ has wild representation type except for $\mathfrak{g} = \mathfrak{sl}_2$.*

CHAPTER 8. REPRESENTATION TYPE OF SMALL QUANTUM GROUPS

Proof. Let $\mathfrak{g} \neq \mathfrak{sl}_2$. First suppose $\ell \geq h$. Then $|\Phi_0^+| = 0$ and Corollary 4.4.7 implies that $\text{cx}_{u_\zeta(\mathfrak{b})} k \geq |\Phi^+| \geq 3$. By Theorem 8.1.4 we have that $u_\zeta(\mathfrak{b})$ is wild.

Now suppose $\ell < h$. If \mathfrak{g} is neither of type \mathbb{B}_2 nor of type \mathbb{G}_2 , then $u_\zeta(\mathfrak{b})$ contains $u_\zeta(\mathfrak{b}_{\mathfrak{sl}_3})$ as a Hopf subalgebra (as the Dynkin diagram of G contains \mathbb{A}_2 as a subquiver). Applying Lemma 3.2.6 we have that $\text{cx}_{u_\zeta(\mathfrak{b})} k \geq \text{cx}_{u_\zeta(\mathfrak{b}_{\mathfrak{sl}_3})} k \geq 3$ by the foregoing case and therefore $u_\zeta(\mathfrak{b})$ is wild.

It remains to consider the cases \mathbb{B}_2 and \mathbb{G}_2 and $\ell < h$. If \mathfrak{g} is of type \mathbb{B}_2 , then $h = 4$ and thus $\ell = 3$, which yields $|\Phi_0^+| = 1$. It follows from Corollary 4.4.7 that $\text{cx}_{u_\zeta(\mathfrak{b})} k \geq 3$, implying again that $u_\zeta(\mathfrak{b})$ is wild. If \mathfrak{g} is of type \mathbb{G}_2 , then $h = 6$ and by hypothesis $\ell = 5$, which yields $|\Phi_0^+| = 1$. It follows that $\text{cx}_{u_\zeta(\mathfrak{b})} k \geq 5$, implying again that $u_\zeta(\mathfrak{b})$ is wild. \square

From this result we are able to prove a first step towards the determination of the representation type of $U_\zeta(B_r)$ for higher r :

Corollary 8.2.3. *If $U_\zeta(B_r)$ is of tame or finite representation type then $\mathfrak{g} = \mathfrak{sl}_2$ and $r \leq 1$.*

Proof. The algebra $U_\zeta(B_r)$ contains $u_\zeta(\mathfrak{b})$ as a Hopf subalgebra. Therefore $\text{cx}_{u_\zeta(\mathfrak{b})} k \leq \text{cx}_{U_\zeta(B_r)} k$ for all $r \geq 0$ (Lemma 3.2.6). The foregoing theorem shows that the former is greater than two. Thus the algebras are wild by Theorem 8.1.4 since $U_\zeta(B_r)$ satisfies (fg) by Theorem 8.2.1. For $\mathfrak{g} = \mathfrak{sl}_2$ note that $U_\zeta(B_r)/u_\zeta(\mathfrak{b}) \cong \text{Dist}(B_r)$. The latter is known to be wild for $r \geq 2$: By [FV03, Corollary 2.6], we have that it is not tame. But the r -th Frobenius kernel of the additive group $G_{a(r)}$ is contained in B_r and $\text{Dist}(G_{a(r)}) \cong k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$. Therefore $\text{cx}_{\text{Dist}(B_r)} k \geq \text{cx}_{\text{Dist}(G_{a(r)})} k \geq 2$. Thus $\text{Dist}(B_r)$ is not representation-finite. \square

For the remaining case the complexity does not provide any information. We therefore use a different tool, we compute the ordinary quiver and relations of that algebra.

Proposition 8.2.4. *For $\mathfrak{g} = \mathfrak{sl}_2$, the algebra $U_\zeta(B_1)$ has the following quiver and relations: It has vertices labeled by (i, j) , where $i \in \mathbb{Z}/\ell\mathbb{Z}$ and $j \in \mathbb{Z}/p\mathbb{Z}$, arrows $(i, j) \xrightarrow{a_{ij}} (i+1, j)$ and $(i, j) \xrightarrow{b_{ij}} (i, j+1)$ and relations $b_{i+1, j} a_{ij} - a_{i, j+1} b_{ij}$ and $a_{i+\ell-1, j} \cdots a_{ij}$ and $b_{i+(p-1), j+p-1} \cdots b_{ij}$.*

Proof. We regard $U_\zeta(N_1)$ as a module for the adjoint action of $U_\zeta(T_1)$. We have the following equalities and use the relations given in [CP94b, Theorem 9.3.4] (the numbers in brackets refer to the number of the equation in this Theorem) and [CP94b, Proof of

8.2. REPRESENTATION TYPE OF HALF-QUANTUM GROUPS

Proposition 11.2.4]: $\text{Ad}(K)(E) \stackrel{(8)}{=} \zeta^2 E$, $\text{Ad}(K)(E^{(\ell)}) \stackrel{(8)}{=} E^{(\ell)}$ and

$$\begin{aligned}
\text{Ad}\left(\begin{bmatrix} K \\ \ell \end{bmatrix}\right)(E^{(\ell)}) &= \sum_{s=0}^{\ell} \begin{bmatrix} K \\ \ell-s \end{bmatrix} K^{-s} E^{(\ell)} S(K^{\ell-s} \begin{bmatrix} K \\ s \end{bmatrix}) \\
&\stackrel{(8+9)}{=} E^{(\ell)} \sum_{s=0}^{\ell} \begin{bmatrix} K; 2\ell \\ \ell-s \end{bmatrix} K^{-s} S(K^{\ell-s} \begin{bmatrix} K \\ s \end{bmatrix}) \\
&\stackrel{(5)}{=} E^{(\ell)} \sum_{s=0}^{\ell} \sum_{t=0}^{\ell-s} \zeta^{2\ell(\ell-s-t)} \begin{bmatrix} 2\ell \\ t \end{bmatrix} K^{-t} \begin{bmatrix} K \\ \ell-s-t \end{bmatrix} K^{-s} S(K^{\ell-s} \begin{bmatrix} K \\ s \end{bmatrix}) \\
&= E^{(\ell)} \sum_{t=0}^{\ell} \sum_{s=0}^{\ell-t} \begin{bmatrix} 2\ell \\ t \end{bmatrix} K^{-t} \begin{bmatrix} K \\ \ell-s-t \end{bmatrix} K^{-s} S(K^{\ell-s} \begin{bmatrix} K \\ s \end{bmatrix}) \\
&= E^{(\ell)} \sum_{t=0}^{\ell} \begin{bmatrix} 2\ell \\ t \end{bmatrix} K^{-t} \varepsilon\left(\begin{bmatrix} K \\ \ell-t \end{bmatrix}\right) = \begin{bmatrix} 2\ell \\ \ell \end{bmatrix} E^{(\ell)}
\end{aligned}$$

The last two equations use the fact that $\varepsilon(a) = \sum_{(a)} a_{(1)} S(a_{(2)})$ and that $\begin{bmatrix} 2\ell \\ t \end{bmatrix} = 0$ for $t < \ell$.

We compute the action of $\begin{bmatrix} K \\ \ell \end{bmatrix}$ on E in the integral form of the quantum group and then specialize:

$$\begin{aligned}
\text{Ad}\left(\begin{bmatrix} K \\ \ell \end{bmatrix}\right)(E) &= \text{Ad}\left(\frac{\prod_{r=1}^{\ell} K q^{1-r} - K^{-1} q^{r-1}}{\prod_{r=1}^{\ell} q^r - q^{-r}}\right)(E) \\
&= \frac{\prod_{r=1}^{\ell} q^2 q^{1-r} - q^{-2} q^{r-1}}{\prod_{r=1}^{\ell} (q^r - q^{-r})} E = 0.
\end{aligned}$$

Therefore $\langle E \rangle_k$ and $\langle E^{(\ell)} \rangle_k$ are simple modules with respect to the action of $U_{\zeta}(T_1)$. Since the character group is $X_{p\ell} \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, we identify the set of characters with pairs (i, j) , where $i \in \mathbb{Z}/\ell\mathbb{Z}$ and $j \in \mathbb{Z}/p\mathbb{Z}$, such that addition with the character corresponding to $E^{(\ell)}$ amounts to addition of 1 in the second component (resp. addition with the character corresponding to E amounts to addition of 1 in the first component). Now decompose $U_{\zeta}(T_1) = \bigoplus k u_{\lambda}$ with primitive orthogonal idempotents u_{λ} . Accordingly we have $au_{\lambda} = \lambda(a)u_{\lambda}$ for all $a \in U_{\zeta}(T_1)$. Hence $\lambda(u_{\lambda'}) = \delta_{\lambda, \lambda'}$ for $\lambda, \lambda' \in X_{p\ell}$.

Since $U_{\zeta}(T_1)$ is semisimple and commutative, any finite dimensional module is the direct sum of one-dimensional simple modules, determined by characters. For a finite dimensional module M , let M_{α} be the sum of all submodules isomorphic to the one-dimensional module corresponding to the character α . Now let $x \in U_{\zeta}(N_1)_{\alpha}$. Then we have:

$$\begin{aligned}
u_{\mu} x u_{\lambda} &= \sum_{(u_{\mu})} \text{Ad}(u_{\mu, (1)})(x) u_{\mu, (2)} u_{\lambda} = \sum_{(u_{\mu})} \alpha(u_{\mu, (1)}) x \lambda(u_{\mu, (2)}) u_{\lambda} = (\lambda + \alpha)(u_{\mu}) x u_{\lambda} \\
&= \delta_{\lambda + \alpha, \mu} x u_{\lambda}.
\end{aligned}$$

CHAPTER 8. REPRESENTATION TYPE OF SMALL QUANTUM GROUPS

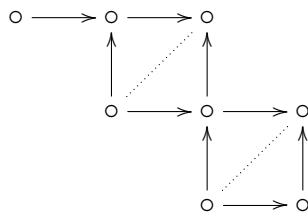
Let Q be the quiver described in the statement of the proposition. Since the simple $U_\zeta(T_1)$ -modules are of form k_λ with $\lambda \in X_{p\ell}$, it follows from the foregoing equation, that there is a unique homomorphism $\Gamma : kQ \rightarrow U_\zeta(B_1)$ satisfying $\Gamma(e_\lambda) = u_\lambda$, $\Gamma(a_{ij}) = Eu_{ij}$ and $\Gamma(b_{ij}) = E^{(\ell)}u_{ij}$. Since $Eu_\lambda, E^{(\ell)}u_\lambda \in \text{Im } \Gamma$ and $E = \sum_\lambda Eu_\lambda$ and $E^{(\ell)} = \sum_\lambda E^{(\ell)}u_\lambda$, we have that Γ is surjective. Furthermore we have $\Gamma(b_{i+1,j}a_{ij} - a_{i,j+1}b_{ij}) = E^{(\ell)}u_{i+1,j}Eu_{ij} - Eu_{i,j+1}E^{(\ell)}u_{i,j} = 0$. Since E has order ℓ (resp. $E^{(\ell)}$ has order p) we as well have that multiplications of Eu_λ of length ℓ (resp. $E^{(\ell)}u_\lambda$ of length p) are zero. Therefore if we let I be the ideal generated by the elements $b_{i+1,j}a_{ij} - a_{i,j+1}b_{ij}$ and $a_{i+\ell-1,j} \cdots a_{ij}$ and $b_{i,j+p-1} \cdots b_{ij}$, then Γ factors through a surjective morphism $\hat{\Gamma} : kQ/I \rightarrow U_\zeta(B_1)$.

Now let $\pi : kQ \rightarrow kQ/I$ be the canonical projection. We have $[\sum_{i,j} a_{ij}, \sum_{i,j} b_{ij}] \in I$ and $(\sum a_{ij})^\ell \in I$ and $(\sum b_{ij})^p \in I$. Therefore there exists a unique algebra homomorphism $\Gamma' : U_\zeta(N_1) \rightarrow kQ/I$, such that $\Gamma'(E) = \pi(\sum a_{ij})$ and $\Gamma'(E^{(\ell)}) = \pi(\sum b_{ij})$. We now extend Γ' to a linear map $\hat{\Gamma}'$ on $U_\zeta(B_1) = U_\zeta(N_1) \# U_\zeta(T_1)$ via $\hat{\Gamma}'(a \# u_\lambda) := \Gamma'(a)\pi(e_\lambda)$ for all $a \in U_\zeta(N_1)$ and all λ . Direct computation shows that this is a homomorphism of k -algebras.

Let J be the Jacobson radical of kQ/I . Then $a_{ij} = e_{i+1,j} \sum_{j'} a_{ij'}$ and $b_{ij} = e_{i,j+1} \sum_{j'} b_{ij'}$ yields $kQ/I = \text{Im } \hat{\Gamma}' + J^2$. Now apply [Ben95, Proposition 1.2.8] to see that $\hat{\Gamma}'$ is surjective. As a result the two algebras have the same dimension. Therefore the two given maps are isomorphisms. \square

This enables us to apply the following lemma by Luise Unger:

Lemma 8.2.5 ([Ung90]). *The algebra with the indicated quiver and relations is wild:*



Theorem 8.2.6. *The connected algebra $U_\zeta(B_r)$ is wild except for $r = 0$ and $\mathfrak{g} = \mathfrak{sl}_2$.*

Proof. If $r = 1$ and $\mathfrak{g} = \mathfrak{sl}_2$, the algebra contains the subquiver with relations from the foregoing lemma. \square

We proceed by determining the representation type of the nilpotent part of the Frobenius-Lusztig kernels. This was done independently by Feldvoss and Witherspoon ([FW11, Theorem 5.4]) for $r = 0$ by using a variant of Farnsteiner's Theorem for Hochschild cohomology ([FW11, Theorem 2.1]). We apply the following result:

Lemma 8.2.7 ([Far06, Lemma 5.2.1]). *Let H be a semisimple finite dimensional Hopf algebra. Let A be a finite dimensional tame H -module algebra. Then $A \# H$ is tame.*

Theorem 8.2.8. *The local algebra $U_\zeta(N_r)$ is wild except for $r = 0$ and $\mathfrak{g} = \mathfrak{sl}_2$.*

8.3. REPRESENTATION TYPE OF BLOCKS

Proof. We have that $U_\zeta(N_r)$ is a finite dimensional algebra, which is a $U_\zeta^0(G_r)$ -module algebra, and $U_\zeta(B_r) = U_\zeta(N_r) \# U_\zeta^0(G_r)$. Therefore the wildness of $U_\zeta(B_r)$ implies the wildness of $U_\zeta(N_r)$ by the foregoing lemma since $U_\zeta^0(G_r)$ is semisimple. \square

Remark 8.2.9. (i) For $r = 0$ there is an alternative proof of this result using that by the proof of Theorem 8.1.4 there are infinitely many indecomposable non-isomorphic $u_\zeta(\mathfrak{b})$ -modules with the same dimension of complexity greater than 1. Using the good control that one has in the situation of $U * G$, where U is local and G is finite, s.t. kG is semisimple, described in [Sch09] one can show that this yields to infinitely many such modules for $u_\zeta(\mathfrak{n})$. But this excludes the possibility of $u_\zeta(\mathfrak{n})$ being tame or representation-finite, since by [CB88, Corollary E] we know that for tame algebras almost all modules lie in homogeneous tubes.

(ii) For $r = 1$ this result follows more direct from the fact that

$$U_\zeta(N_1) \cong k[E, E^{(\ell)}]/(E^\ell, (E^{(\ell)})^p).$$

8.3. Representation type of blocks

As the small quantum groups $u_\zeta(\mathfrak{g})$ are not connected, Jörg Feldvoss and Sarah Witherspoon were only able to determine the representation type of the principal block ([FW09, Theorem 4.3, Theorem 4.6]). The following theorem provides a generalization to arbitrary blocks:

Theorem 8.3.1. *Let \mathfrak{g} be a simple Lie algebra. For $\text{char } k = 0$ assume that ℓ is good for Φ and that $\ell > 3$ if Φ is of type \mathbb{B}_n or \mathbb{C}_n . For $\text{char } k = p > 0$ assume that p is good for Φ and that $\ell > h$. Then any block of $u_\zeta(\mathfrak{g})$ different from the Steinberg block is of wild representation type unless $\mathfrak{g} = \mathfrak{sl}_2$.*

Proof. Assume to the contrary that a block of $u_\zeta(\mathfrak{g})$ is tame. This in particular implies that $\text{cx}_{u_\zeta(\mathfrak{g})} H^0(\lambda) \leq 2$ for all $\lambda \in X_\ell$ corresponding to this block. This implies that $\text{cx}_{u_\zeta^+(\mathfrak{g})} H^0(\lambda) \leq \frac{1}{2} \text{cx}_{u_\zeta(\mathfrak{g})} H^0(\lambda) \leq 1$ by Corollary 4.4.7. Lemma 4.4.6 then leads to $|\Phi^+ \setminus \Phi_\lambda^+| \leq 1$. Since \mathfrak{g} is simple, Φ_λ^+ then spans Φ^+ unless $\mathfrak{g} = \mathfrak{sl}_2$ (see Lemma 1.4.18). The following easy calculation shows that with $\alpha_1, \alpha_2 \in \Phi_\lambda^+$ and $a_1, a_2 \in \mathbb{Z}$ such that $a_1\alpha_1 + a_2\alpha_2 \in \Phi^+$, also $a_1\alpha_1 + a_2\alpha_2 \in \Phi_\lambda^+$: Let $(\lambda + \rho, \alpha_1^\vee) = r\ell$ and $(\lambda + \rho, \alpha_2^\vee) = s\ell$, then

$$\begin{aligned} (\lambda + \rho, (a_1\alpha_1 + a_2\alpha_2)^\vee) &= (\lambda + \rho, \frac{2(a_1\alpha_1 + a_2\alpha_2)}{(a_1\alpha_1 + a_2\alpha_2, a_1\alpha_1 + a_2\alpha_2)}) \\ &= \frac{a_1(\alpha_1, \alpha_1)}{(a_1\alpha_1 + a_2\alpha_2, a_1\alpha_1 + a_2\alpha_2)} r\ell + \frac{a_2(\alpha_2, \alpha_2)}{(a_1\alpha_1 + a_2\alpha_2, a_1\alpha_1 + a_2\alpha_2)} s\ell \\ &= \frac{ra_1 \cdot (\alpha_1, \alpha_1) + sa_2 \cdot (\alpha_2, \alpha_2)}{(a_1\alpha_1 + a_2\alpha_2, a_1\alpha_1 + a_2\alpha_2)} \cdot \ell \equiv 0 \pmod{\ell}, \end{aligned}$$

The last congruence follows from the fact that the whole expression is an integer and ℓ is odd and $3 \nmid \ell$ for \mathbb{G}_2 and there are at most two root length differing by a factor of 2 or 3

CHAPTER 8. REPRESENTATION TYPE OF SMALL QUANTUM GROUPS

for \mathbb{G}_2 (see Lemma 1.4.13 and Remark 1.4.14). Therefore the first factor already has to be an integer. Thus we have $\Phi_\lambda^+ = \Phi^+$. This in particular implies that the simple roots $\alpha_1, \dots, \alpha_n$ are contained in Φ_λ^+ . But $(\rho, \alpha_i^\vee) = 1$ for any simple root α_i and $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. Therefore $\lambda = (\ell - 1)\lambda_i \pmod{\ell\lambda_i, \lambda_j (j \neq i)}$ for any $1 \leq i \leq n$. Thus we have $\lambda = (\ell - 1)\rho$. This means that the block is the Steinberg module. \square

Corollary 8.3.2. *Let B be a block of $u_\zeta(\mathfrak{g})$. Assume the restrictions on the parameters of the foregoing theorem for each direct summand of \mathfrak{g} . Then B is of wild representation type unless B is the Steinberg module or $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{g}'$ for some semisimple Lie algebra \mathfrak{g}' and $B \cong B_1 \otimes B_2$ where B_1 is a block of $u_\zeta(\mathfrak{sl}_2)$ and B_2 is the Steinberg module for $u_\zeta(\mathfrak{g}')$. In particular $B \cong M_{m \times m}(B_1)$ for some $m \in \mathbb{N}$, where $M_{m \times m}(B_1)$ denotes the algebra of $m \times m$ -matrices over B_1 .*

Proof. Let $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ for simple complex Lie algebras \mathfrak{g}_i . According to Remark 4.1.10 (v) we then have $u_\zeta(\mathfrak{g}) \cong \bigotimes_{i=1}^s u_\zeta(\mathfrak{g}_i)$. It follows that the blocks of $u_\zeta(\mathfrak{g})$ are tensor products of blocks for $u_\zeta(\mathfrak{g}_i)$. According to Remark 4.3.3 we also have $H^0(\lambda) = \bigotimes_{i=1}^s H^0(\lambda_i)$, where λ_i is a restricted weight for \mathfrak{g}_i . That implies $\text{cx}_{u_\zeta(\mathfrak{g})} H^0(\lambda) = \sum_{i=1}^s \text{cx}_{u_\zeta(\mathfrak{g}_i)} H^0(\lambda_i)$ by Corollary 3.2.15. Suppose the block B to which $H^0(\lambda)$ belongs is tame, then this sum is less or equal to two. This implies that all the $H^0(\lambda_i)$ are Steinberg modules except for possibly one i , for which it can hold that $\text{cx}_{u_\zeta(\mathfrak{g}_i)} H^0(\lambda_i) = 2$ and $\mathfrak{g}_i = \mathfrak{sl}_2$. \square

Combining these results with the results in Section 7.2 we get the following:

Corollary 8.3.3. *The following are equivalent for a block B of a small quantum group $u_\zeta(\mathfrak{g})$:*

- (1) B is tame.
- (2) B is domestic.
- (3) B is special biserial.
- (4) B is Morita equivalent to a trivial extension of the Kronecker algebra.
- (5) There exists an induced module $H^0(\lambda)$ for B with $\text{cx}_B H^0(\lambda) = 2$.
- (6) There exists a simple B -module S s.t. $\text{cx}_B S = 2$

Proof. This is proven in Theorem 7.2.11 and Corollary 7.2.12. For (6) we use Theorem 9.4.4. \square

Lemma 8.3.4. *For $\lambda \neq \mu \in X_\ell$ let $L(\lambda), L(\mu)$ be two simple $u_\zeta(\mathfrak{sl}_2)$ -modules belonging to the same block. Then $\text{Ext}_{u_\zeta(\mathfrak{sl}_2)}^1(L(\lambda), L(\mu)) \cong L(1)$ as $\text{Dist}(\text{SL}(2)_r)$ -modules.*

8.3. REPRESENTATION TYPE OF BLOCKS

Proof. It is known that the projective $u_\zeta(\mathfrak{sl}_2)$ -module $P(\lambda)$ is a uniserial $U_\zeta(\mathfrak{sl}_2)$ -module with composition factors $L(\lambda), L(2\ell - 2 - \lambda), L(\lambda)$ in that order: Being a tilting module $P(\lambda)$ has a filtration by Weyl modules, where the Weyl module to the highest weight λ occurs exactly once. Furthermore $P(\lambda)|_{u_\zeta(\mathfrak{sl}_2)}$ has composition factors of weight $(\lambda, \ell - 2 - \lambda, \ell - 2 - \lambda, \lambda)$. The Weyl modules corresponding to the restricted weights are known to be simple, the others are uniserial of length 2, see e.g. [CP94b, Example 11.2.7]. But as the Weyl module for λ occurs exactly once, the Weyl filtration has to have factors $W(2\ell - 2 - \lambda)$ (with simple top $L(2\ell - 2 - \lambda)$) and $W(\lambda)$ (see also [CK05, p. 202-203] for a computational argument in characteristic zero). Let $\mu := \ell - 2 - \lambda$. The sequence $0 \rightarrow W(2\ell - 2 - \lambda) \rightarrow P(\lambda) \rightarrow L(\lambda) \rightarrow 0$ induces a long exact Ext-sequence, which is a sequence of $\text{Dist}(\text{SL}(2)_r)$ -modules (the action of $\text{Dist}(\text{SL}(2)_r)$ on the Ext-groups is defined in the proof of Lemma 2.3.4):

$$\begin{aligned} 0 \rightarrow \text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(L(\lambda), L(\mu)) &\rightarrow \text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(P(\lambda), L(\mu)) \rightarrow \text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(W(2\ell - 2 - \lambda), L(\mu)) \\ &\rightarrow \text{Ext}_{u_\zeta(\mathfrak{sl}_2)}^1(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{u_\zeta(\mathfrak{sl}_2)}^1(P(\lambda), L(\mu)). \end{aligned}$$

Since $P(\lambda)$ is the $u_\zeta(\mathfrak{sl}_2)$ -projective cover of $L(\lambda)$ we have that

$$\text{Ext}_{u_\zeta(\mathfrak{sl}_2)}^1(L(\lambda), L(\mu)) \cong \text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(W(2\ell - 2 - \lambda), L(\mu)).$$

But $\text{top}_{u_\zeta(\mathfrak{sl}_2)} W(2\ell - 2 - \lambda) = \text{top}_{U_\zeta(\text{SL}(2)_r)} W(2\ell - 2 - \lambda) = L(1)^{[1]} \otimes L(\mu)$, where $L(1)^{[1]}$ denotes the two-dimensional simple $\text{Dist}(\text{SL}(2)_r)$ -module $L(1)$ viewed as a $U_\zeta(\text{SL}(2)_r)$ -module (since $W(2\ell - 2 - \lambda)$ is not semisimple as a $u_\zeta(\mathfrak{sl}_2)$ -module and we have a short exact sequence of $U_\zeta(\text{SL}(2))$ -modules $0 \rightarrow L(\lambda) \rightarrow W(2\ell - 2 - \lambda) \rightarrow L(2\ell - 2 - \lambda)$). Thus the canonical projection $W(2\ell - 2 - \lambda) \rightarrow \text{top} W(2\ell - 2 - \lambda)$ is a $U_\zeta(\text{SL}(2)_r)$ -module homomorphism. Hence $\text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(\text{top} W(2\ell - 2 - \lambda), L(\mu)) \rightarrow \text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(W(2\ell - 2 - \lambda), L(\mu))$ is a $U_\zeta(\text{SL}(2)_r)$ -module monomorphism on which $u_\zeta(\mathfrak{sl}_2)$ acts trivially, i.e. a $\text{Dist}(\text{SL}(2)_r)$ -module isomorphism, since they have the same dimension. However we have

$$\text{Hom}_{u_\zeta(\mathfrak{sl}_2)}(L(1)^{[1]} \otimes L(\mu), L(\mu)) \cong L(1)$$

by the proof of Lemma 2.3.4. □

The foregoing lemma can be used to compute the quiver of $U_\zeta(\text{SL}(2)_1)$ by applying Lemma 2.3.4. To determine the representation type of the algebra it is only necessary to compute a small part of it, which will be of the following form:

Lemma 8.3.5 ([Erd90, I.10.8 (iii)]). *Let A be an algebra. If the quiver of A contains a subquiver of the form $\circ \rightrightarrows \circ \longleftarrow \circ$, then the algebra is wild.*

Proposition 8.3.6. *Impose the same restriction on the parameters as in the foregoing theorem. The algebra $U_\zeta(G_r)$ is of wild representation type except for $r = 0$ and $G = \text{SL}(2)$.*

Proof. Using the fact that $U_\zeta(G_r)/u_\zeta(\mathfrak{g}) \cong \text{Dist}(G_r)$ one concludes that $G = \text{SL}(2)$ and $r \leq 1$ since if these assumptions do not hold, $\text{Dist}(G_r)$ is known to be wild.

CHAPTER 8. REPRESENTATION TYPE OF SMALL QUANTUM GROUPS

For $r = 1$ we have by Lemma 2.3.4 that if none of the involved simples is the Steinberg module for the corresponding algebra, then for given μ' there exists μ , such that we have that $\dim \text{Ext}_{U_\zeta(\text{SL}(2)_r)}^1(L(\mu)^{[1]} \otimes L(\lambda'), L(\mu')^{[1]} \otimes L(\lambda')) = 2$ for all λ . Furthermore we have

$$\dim \text{Ext}^1(L(\mu)^{[1]} \otimes L(\lambda), L(\mu')^{[1]} \otimes L(\lambda')) = \delta_{\lambda', \ell-2-\lambda}(\delta_{\mu', \mu-1} + \delta_{\mu', \mu+1})$$

for $\mu' \neq p-1$ by Lemma 1.8.14 and Lemma 2.3.4. Thus the quiver of $U_\zeta(\text{SL}(2)_1)$ contains a subquiver of the form given in the previous lemma. Thus the algebra is wild. \square

We can even show more in the case that $U_\zeta(G_r)$ satisfies (fg), which is only proven for $r = 1$ under the restrictions of the parameters in Theorem 8.2.1.

Theorem 8.3.7. *Let G be simple. Assume that $U_\zeta(G_r)$ satisfies (fg). If there is a tame block, then $\mathfrak{g} = \mathfrak{sl}_2$ and either $r = 0$ or it is induced by a block of $\text{Dist}(\text{SL}(2)_1)$, i.e. the corresponding simple modules have weight $(\ell - 1 + p\mu)$.*

Proof. Restricting the induced modules to $u_\zeta(\mathfrak{g})$, we see that their complexity is too big in case $\mathfrak{g} \neq \mathfrak{sl}_2$. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ the wildness of the remaining blocks follows from the proof of the foregoing proposition if we observe that the argument in the foregoing proof already shows that each of the simple modules of the form $L(p-1)^{[1]} \otimes L(\lambda)$ corresponds to one of the blocks, that we have already shown to be wild for $\lambda \neq \ell - 1$. \square

Remark 8.3.8. Recall from Proposition 1.8.13 that the blocks of $\text{Dist}(\text{SL}(2)_1)$ (hence the blocks of $U_\zeta(G_r)$ with weights $(\ell - 1 + p\mu)$) are $\frac{p-1}{2}$ blocks Morita equivalent to the trivial extension of the Kronecker quiver and one simple block.

9. Auslander-Reiten theory of wild algebras

9.1. Webb's Theorem for the small quantum group

For almost all of the algebras considered, finite complexity for the modules follows from finite generation of the even cohomology ring. For $U_\zeta(G_r)$ this result is not yet established, nevertheless finite complexity was proven by Christopher M. Drupieski:

Theorem 9.1.1 ([Dru11, Thm 6.3.1]). *Let M be a finite dimensional $U_\zeta(G_r)$ -module. Then*

$$\text{cx}_{U_\zeta(G_r)} M \leq (r + 1)(\dim \mathfrak{g} - \dim \mathfrak{h}).$$

Theorem 9.1.2. *Webb's Theorem holds for the following algebras: $u_\zeta(\mathfrak{g})$, $u_\zeta(\mathfrak{b})$, $U_\zeta(G_r)$, $U_\zeta(B_r)$. The non-periodic components are of the form $\mathbb{Z}[\Delta]$, where $\Delta \in \mathcal{D}$.*

Proof. For $U_\zeta(B_r)$ this follows from Theorem 8.2.1 (ii). For $U_\zeta(G_r)$ this follows from the foregoing theorem. \square

For the algebras $u_\zeta(\mathfrak{n})$ we use the description of the Borel part $u_\zeta(\mathfrak{b})$ as a skew group algebra of $u_\zeta(\mathfrak{n})$ and the following result by Idun Reiten and Christine Riedtmann:

Proposition 9.1.3 ([RR85, Proposition 1.8 (a)]). *Let A be an algebra and G be a group. If M is an indecomposable A -module, then $(A * G \otimes_A M)|_A \cong M \oplus M^{g_2} \oplus \dots \oplus M^{g_n}$, where $G = \{e, g_2, \dots, g_n\}$.*

Theorem 9.1.4. *Webb's Theorem holds for the algebras $u_\zeta(\mathfrak{n})$. The non-periodic components are of the form $\mathbb{Z}[\Delta]$, where $\Delta \in \mathcal{D}$.*

Proof. As $u_\zeta(\mathfrak{b}) = u_\zeta(\mathfrak{n}) \# u_\zeta^0(\mathfrak{g}) = u_\zeta(\mathfrak{n}) * (\mathbb{Z}/\ell\mathbb{Z})^n$ (Lemma 4.1.17) the results follow from the foregoing proposition: For $M \in \text{mod } u_\zeta(\mathfrak{n})$ take a minimal projective resolution P_* of $u_\zeta(\mathfrak{b}) \otimes_{u_\zeta(\mathfrak{n})} M$. The rate of growth of this projective resolution is finite as $u_\zeta(\mathfrak{b})$ is a finite dimensional Hopf algebra satisfying (fg). If we restrict this resolution to $u_\zeta(\mathfrak{n})$ we will get a projective resolution of M which has finite rate of growth. Therefore the complexity of M is finite. So we can apply Theorem 3.5.3. \square

Up to now we were only concerned with finite dimensional selfinjective algebras and could therefore use the result by Otto Kerner and Dan Zacharia. Another tool to prove Webb's Theorem for some Frobenius category is to construct a special function on the vertices of the stable Auslander-Reiten quiver:

Definition 9.1.5. Let \mathcal{C} be a Frobenius category admitting almost split sequences. Let Θ be a component of the stable Auslander-Reiten quiver of \mathcal{C} . Then a function $d : \Theta \rightarrow \mathbb{N}_0$ is called **subadditive** if for every $[M] \in \Theta$ with predecessors $[E_1], \dots, [E_n]$ we have $\sum_i d([E_i]) \leq d([M]) + d(\tau(M))$.

For future reference we record the following statement:

Lemma 9.1.6 ([Far, Proposition 1.1 (1)]). *Let \mathcal{C} be a Frobenius category admitting almost split sequences. Let Θ be a component of the stable Auslander-Reiten quiver of \mathcal{C} . If there exists a subadditive τ -invariant function $d : \Theta \rightarrow \mathbb{N}_0$. Then $d = 0$ or $d([M]) > 0$ for every $[M] \in \Theta$.*

Proposition 9.1.7 ([HPR80, Theorem, p.286]). *Webb's Theorem holds for a Frobenius category \mathcal{C} admitting almost split sequences if there is a subadditive τ -invariant function on each component of the stable Auslander-Reiten quiver.*

An important class of subadditive functions is given in the following propositions:

Proposition 9.1.8 ([ES92, Lemma 3.2]). *Let A be a selfinjective finite dimensional algebra. Let W be an A -module. Define $d_W : \Gamma_s(A) \rightarrow \mathbb{N}_0$ by $d_W([M]) := \dim \underline{\text{Hom}}_A(W, M)$. Let $0 \rightarrow \tau_A M \rightarrow E \rightarrow M \rightarrow 0$ be an Auslander-Reiten sequence.*

(i) *If M is not a summand of W , then $d_W([M]) + d_W([\tau_A M]) \geq d_W([E])$.*

(ii) *If in addition $\Omega_A M$ is not a summand of W , then equality holds in (i).*

Corollary 9.1.9. *Let A be a selfinjective algebra. Let W be an A -module. Define $d_W^i : \Gamma_s(A) \rightarrow \mathbb{N}_0$ by $d_W^i([M]) := \dim \text{Ext}_A^i(M, W)$. Let $0 \rightarrow \tau_A M \rightarrow E \rightarrow M \rightarrow 0$ be an Auslander-Reiten sequence.*

(i) *If M is not a summand of $\Omega_A^{-(i-1)} \tau_A^{-1} W$, then $d_W^i([M]) + d_W^i([\tau_A M]) \geq d_W^i([E])$.*

(ii) *If in addition $\Omega_A M$ is not a summand of $\Omega_A^{-(i-1)} \tau_A^{-1} W$, then equality holds in (i).*

Proof. We have $\dim \text{Ext}_A^i(M, W) = \dim \text{Ext}_A^1(\Omega_A^{i-1} M, W) = \dim \underline{\text{Hom}}_A(\tau_A^{-1} W, \Omega_A^{i-1} M) = \dim \underline{\text{Hom}}_A(\Omega_A^{-(i-1)} \tau_A^{-1} W, M)$ according to Lemma 1.6.11 and Theorem 1.9.8. Therefore the result follows from the foregoing corollary. \square

As noted in [Far99b] we even have the following:

Corollary 9.1.10 ([Far99b, Proof of Theorem 2.1]). *Let A be a selfinjective algebra. For a simple module S , the functions $d_S^i : \Gamma_s(A) \rightarrow \mathbb{N}$ are subadditive and fail to be additive at only finitely many vertices.*

Proof. If M is not isomorphic to $\Omega_A^{-(i-1)} \tau_A^{-1} S$ it follows from the foregoing corollary that $d_S^i([M]) + d_S^i([\tau_A M]) \geq d_S^i([E])$. In the remaining case the Auslander-Reiten sequence $0 \rightarrow \Omega_A^{-(i-1)} S \rightarrow E \rightarrow \Omega_A^{-(i-1)} \tau_A^{-1} S \rightarrow 0$ induces an exact sequence $0 \rightarrow \Omega_A^i \Omega_A^{-(i-1)} S \rightarrow \Omega_A^i E \rightarrow \Omega_A^i \Omega_A^{-(i-1)} \tau_A^{-1} S \rightarrow 0$. Applying $\text{Hom}_A(-, S)$ to this sequence we get an exact

9.1. WEBB'S THEOREM FOR THE SMALL QUANTUM GROUP

sequence $0 \rightarrow \text{Hom}_A(\Omega_A^i \Omega_A^{-(i-1)} S, S) \rightarrow \text{Hom}_A(\Omega_A^i E, S) \rightarrow \text{Hom}(\Omega_A^i \Omega_A^{-(i-1)} \tau_A^{-1} S, S)$. But since $\text{Hom}(\Omega_A^i N, S) \cong \text{Ext}_A^i(N, S)$ for any module N and any simple module S we get the assertion. \square

We will use this result to prove Webb's Theorem for the $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -modules, the graded modules among the $u_\zeta(L)$ -modules. The proof is similar to [Far05] for restricted enveloping algebras.

Theorem 9.1.11. *Webb's Theorem holds for the category of finite dimensional $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -modules, where $L \in \{\mathfrak{g}, \mathfrak{b}, \mathfrak{n}\}$.*

Proof. Let Θ be a non-periodic component of the stable Auslander-Reiten quiver of the category of finite-dimensional $u_\zeta(L)U^0(\mathfrak{g})$. Let $[M'] \in \Theta$. By Theorem 5.2.22 and Lemma 6.1.2 there exists $\alpha \in \Phi^+$ such that the restriction $\mathcal{F}(M')|_{u_\zeta(f_\alpha)}$ is not injective. Consider the induced module $M_\alpha := u_\zeta(L) \otimes_{u_\zeta(f_\alpha)} k$. The function $d_\alpha : \Theta \rightarrow \mathbb{N}_0, [M] \mapsto \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(M))$ is a subadditive $\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}$ -invariant function on Θ : We have that $\mathcal{F}(\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M)) \cong \tau_{u_\zeta(L)}(\mathcal{F}(M)) \cong \Omega_{u_\zeta(L)}^2(\mathcal{F}(M)^\nu)$ by Lemma 6.1.2, $M_\alpha^{(\nu)} \cong M_\alpha$ since $\nu|_{u_\zeta(f_\alpha)} = \text{id}$ ($\nu \otimes \text{id}$ provides an isomorphism) and $\Omega_{u_\zeta(L)}^2(M_\alpha) \cong M_\alpha \oplus P$ for a projective module P by inducing a projective $u_\zeta(f_\alpha)$ -resolution of k to $u_\zeta(L)$. Therefore we obtain:

$$\begin{aligned} d_\alpha([\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M)]) &= \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M))) \\ &= \dim \text{Ext}_{u_\zeta(L)}^1(\Omega_{u_\zeta(L)}^2(M_\alpha), \Omega_{u_\zeta(L)}^2(\mathcal{F}(M)^\nu)) \\ &= \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(M)^\nu) \\ &= \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha^{(\nu)}, \mathcal{F}(M)^\nu) = d_\alpha([M]). \end{aligned}$$

As Θ is non-periodic we have that the τ -periodic module M_α does not belong to $\Theta \cup \Omega(\Theta)$. Thus we can apply Proposition 9.1.8 (to ΩM_α by using Corollary 1.6.11) to get that d_α is a subadditive τ -periodic function on Θ .

It remains to prove that $d_\alpha \neq 0$. By Lemma 9.1.6 it suffices to prove that d_α is non-zero at one point of Θ . By Proposition 5.2.9 we have that $u_\zeta(L) : u_\zeta(f_\alpha)$ is a Frobenius extension of the second kind. Therefore we can use Frobenius reciprocity (Proposition 5.1.7) to conclude that:

$$d_\alpha([M']) = \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(M')) = \dim \text{Ext}_{u_\zeta(f_\alpha)}^1(k, \mathcal{F}(M')|_{u_\zeta(f_\alpha)}) \neq 0.$$

Thus the statement follows. \square

Remark 9.1.12. This provides a different way to prove the result for components of $u_\zeta(\mathfrak{g})$ or $u_\zeta(\mathfrak{b})$ consisting of gradable modules: The function which is obtained by removing the forgetful functor \mathcal{F} in the foregoing proposition provides a subadditive $\tau_{u_\zeta(L)}$ -invariant function for every component of the stable Auslander-Reiten quiver of $u_\zeta(L)$ containing gradable modules. The foregoing proof applies verbatim (replacing $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ with $u_\zeta(L)$

and removing \mathcal{F} everywhere). Thus this provides us with a subadditive τ -invariant function on Θ , a component of $u_\zeta(L)U_\zeta^0(\mathfrak{g})$, that stays subadditive τ -invariant upon restriction. This way one can get information on Θ or $\mathcal{F}(\Theta)$ by observing the other.

The remainder of this section is devoted to more facts about subadditive functions on special components that we will use in the sequel.

Proposition 9.1.13 ([HPR80, Corollary 1]). *Let A be a finite dimensional algebra of finite representation type. Let Θ be a component of $\Gamma_s(A)$. Then the tree class of Θ is a Dynkin diagram.*

Proposition 9.1.14 ([Web82, Remark p.105]). *Let Δ be a Euclidean diagram. If f is a subadditive function on $\mathbb{Z}[\Delta]$ that fails to be additive in only finitely many places then $f(n, \alpha)$ grows at most linearly in n .*

Proposition 9.1.15 ([Web82, Corollary 4.2]). *Let Δ be a Euclidean diagram. Any non-negative additive function on $\mathbb{Z}[\Delta]$ takes bounded values.*

Corollary 9.1.16. *Every Euclidean component for a selfinjective algebra A is attached to a principal indecomposable module, i.e. contains an Auslander-Reiten sequence of the form $0 \rightarrow \text{rad}(P) \rightarrow P \oplus \text{rad}(P)/\text{soc } P \rightarrow P/\text{soc } P \rightarrow 0$.*

Proof. If this was not the case, the length function would provide a non-negative additive function on the component. This would take bounded values by the foregoing proposition. But then the first Brauer-Thrall conjecture tells us that the component is finite which cannot be the case by Proposition 9.1.13. \square

9.2. Periodic components

Periodic components were the first ones to be considered. Several results about them were obtained in 1980 by Dieter Happel, Udo Preiser and Claus M. Ringel. For the algebras considered in this thesis there remain only a few possibilities.

Definition 9.2.1. For a selfinjective algebra A a module $M \neq 0$ is called Ω_A -**periodic** if $\Omega_A^n M \cong M$ for some $n \in \mathbb{N}_1$. It is called τ_A -**periodic** if $\tau_A^n M \cong M$ for some $n \in \mathbb{N}_1$.

The following two lemmas are well-known:

Lemma 9.2.2. *If A is a finite dimensional Hopf algebra then an A -module M is Ω_A -periodic iff it is τ_A -periodic.*

Proof. The statement follows from the fact that $\tau_A(M) \cong \Omega_A^2(M^{(\nu)})$ (see Lemma 1.9.13) and the fact that the Nakayama automorphism is of finite order for Hopf algebras (see Proposition 2.1.6). \square

Lemma 9.2.3. *If M is an Ω_A -periodic module for a selfinjective algebra A , then $\text{cx}_A M = 1$.*

Proof. As $\Omega_A^n M \cong M$ it follows that

$$\dim \Omega_A^m M \in \{\dim M, \dim \Omega_A M, \dots, \dim \Omega_A^{n-1} M\} < \infty.$$

for $m \in \mathbb{N}$. □

The following proposition is well-known but not explicitly written down somewhere in the stated generality (compare the remarks from the chapter about support varieties).

Proposition 9.2.4. *Let A be an (fg)-Hopf algebra. Then any indecomposable module M with $\text{cx}_A M = 1$ is periodic.*

Proof. Let $\text{cx}_A M = 1$. Then by Corollary 3.3.7, the support variety $\mathcal{V}_A(M) = kx$ is a line. As M is not projective (Corollary 3.2.8), there exists $y \in H^{ev}(A, k)$ homogeneous, such that y does not vanish on x when considered as a polynomial function on $\mathcal{V}_A(M)$. This implies $Z(\langle y \rangle) \cap \mathcal{V}_A(M) = (0)$. Therefore $\mathcal{V}_A(L_y \otimes M) = Z(\langle y \rangle) \cap \mathcal{V}_A(M) = (0)$ by Proposition 3.3.2. So $L_y \otimes M$ is projective and therefore injective as A is selfinjective (Theorem 2.1.6). So if we tensor the defining sequence for L_y with M we get a split exact sequence

$$0 \rightarrow L_y \otimes M \rightarrow \Omega_A^{2n}(k) \otimes M \rightarrow M \rightarrow 0$$

where n is the degree of y . Proposition 2.1.8 now implies that there exists a projective module P with $\Omega_A^{2n} M \oplus P \cong \Omega_A^{2n} k \otimes M \cong L_y \otimes M \oplus M$. Therefore applying Ω_A and noting that Ω_A of a projective module vanishes we get

$$\Omega_A^{2n+1}(M) \cong \Omega_A(\Omega_A^{2n} M \oplus P) \cong \Omega_A(L_y \otimes M \oplus M) \cong \Omega_A M$$

As Ω_A defines a bijection on the set of isomorphism classes of nonprojective indecomposable A -modules it follows that $\Omega_A^{2n} M \cong M$. □

Example 9.2.5. If A is not an (fg)-Hopf algebra, this property fails in general. Rainer Schulz [Sch86] and Shiping Liu and Rainer Schulz [LS94] provided examples for this fact:

- (i) The first example was studied by Rainer Schulz in [Sch86]. It is the truncated quantum plane $A_q := k\langle x, y \rangle / (x^2, y^2, yx - qxy)$ with $q \in k^\times$. This is a 4-dimensional Frobenius algebra with Frobenius homomorphism given by $xy \mapsto 1, x, y, 1 \mapsto 0$. For q not a root of unity the Nakayama automorphism $x \mapsto qx, y \mapsto q^{-1}y$ is not of finite order and therefore it cannot be a Hopf algebra. For a classification of the indecomposable modules and the Auslander-Reiten theory we refer to Theorem 9.3.4. The modules $M(B, 1, \lambda)$ are τ -periodic, have complexity 1, but are not Ω -periodic: Fix a basis $\{a_1, a_2\}$ of $M(B, 1, \lambda)$, such that $ya_1 = a_2$ and $xa_1 = \lambda a_2$. Then a projective cover $A \twoheadrightarrow M(B, 1, \lambda)$ is given by $1 \mapsto a_1, x \mapsto a_2, y \mapsto \lambda a_2, yx \mapsto 0$. The kernel is spanned by the elements $x - \lambda y$ and yx . Since y maps $x - \lambda y$ to yx and x maps $x - \lambda y$ to $-q^{-1}\lambda$ it follows that $\Omega_A(M(B, 1, \lambda)) = M(B, 1, -q^{-1}\lambda)$. Since q is not a root of unity it follows that $M(B, 1, \lambda)$ is not Ω -projective. But since all $M(B, 1, \mu)$ have dimension 2 it is obviously of complexity 1.

- (ii) The next example is a refinement of (i). It is the trivial extension T of the truncated quantum plane and was first studied in [LS94]. Denote the dual basis of $\{1, x, y, xy\}$ by $\{1^*, x^*, y^*, (xy)^*\}$. Then we have the following multiplication table:

\cdot	1	x	y	xy	1^*	x^*	y^*	$(xy)^*$
1	1	x	y	xy	1^*	x^*	y^*	$(xy)^*$
x	x	0	xy	0	0	1^*	0	qy^*
y	y	qxy	0	0	0	0	1^*	x^*
xy	xy	0	0	0	0	0	0	1^*
1^*	1^*	0	0	0	0	0	0	0
x^*	x^*	1^*	0	0	0	0	0	0
y^*	y^*	0	1^*	0	0	0	0	0
$(xy)^*$	$(xy)^*$	y^*	qx^*	1^*	0	0	0	0

In [Rin96] Claus M. Ringel observed that it has a different presentation as

$$k\langle x_0, x_1, x_2 | x_i^2, x_{i+1}x_i - qx_ix_{i+1} \rangle,$$

where the indices are taken modulo 3. The isomorphism is given by $x \mapsto x_0, y \mapsto x_1, (xy)^* \mapsto x_2$. Therefore it is a local symmetric algebra. But for q not a root of unity the modules $T(x + \lambda y)$ have complexity 1, but are neither Ω - nor τ -periodic: A projective cover of $T(x + \lambda y)$ is given by right multiplication in T by $(x + \lambda y)$. The kernel is computed in the following:

$$\begin{aligned} 0 &= (\alpha + \beta x + \gamma y + \delta xy + \alpha' 1^* + \beta' x^* + \gamma' y^* + \delta' (xy)^*)(x + \lambda y) \\ &= \alpha x + \alpha \lambda y + \beta \lambda xy + \gamma y x + \beta' 1^* + \lambda \gamma' 1^* + \delta' y^* + \delta' q x^* \end{aligned}$$

implies that $\alpha = \delta' = 0$, $\beta \lambda = -q\gamma$ and $\beta' = -\lambda \gamma'$. Since $1^* = y(-\lambda x^* + y^*)$, $qxy = y(x - \lambda y)$ and $(xy)^*(qx - \lambda y) = q(-\lambda x^* + y^*)$ we have that $\Omega_T T(x + \lambda y) = T(qx - \lambda y)$. Therefore $\Omega_T^i T(x + \lambda y) = T(q^i x - \lambda y)$. All these modules obviously have dimension 4, hence they have complexity 1. But the following calculation shows that only $T(q^i x - \lambda y)$ is annihilated by $\text{rad}(T)(q^{i+1} + \lambda y)$ and therefore they are not isomorphic:

$$\begin{aligned} \text{rad}(T)(q^{i+1}x + \lambda y)T(q^s x - \lambda y) &= J(T)(-\lambda q^{i+1}xy + \lambda q^s yx) \\ &= \begin{cases} 0 & \text{for } s = i + 1 \\ \neq 0 & \text{else} \end{cases} \end{aligned}$$

Claus M. Ringel [Rin96, Theorem 6.1] observed that the components containing these modules are of the form $\mathbb{Z}[\mathbb{A}_\infty]$.

Question 9.2.6. According to Conjecture 3.1.12, the algebra in the foregoing example (ii) should not appear as a block of a Hopf algebra. Why is this?

We now turn to the small quantum groups and want to study what the Ω - and τ -period of their modules can be. This is done by noting that the even cohomology ring is generated in degree 2.

Proposition 9.2.7 ([BNPP11, Theorem 1.2.3], [Dru11, Theorem 5.1.3], [Dru11, Theorem 5.1.4]). *For char $k = 0$ let ℓ be a good prime for Φ and $\ell > 3$ for types \mathbb{B} and \mathbb{C} and $\ell \nmid n + 1$ for type \mathbb{A}_n and $\ell \neq 9$ for \mathbb{E}_6 . For char $k = p > 0$ let p be good for Φ and $\ell \geq h$. Then $H^{ev}(u_\zeta(\mathfrak{b}), k)$ and $H^{ev}(u_\zeta(\mathfrak{g}), k)$ are generated in degree two.*

Corollary 9.2.8 (of the proof of Proposition 9.2.4). *If the even cohomology ring of an (fg)-Hopf algebra A is generated in degree n then the Ω_A -period of every module is smaller or equal to n . In particular the Ω_A -period of every module for $u_\zeta(\mathfrak{b})$ or $u_\zeta(\mathfrak{g})$ is one or two under the assumptions of the foregoing proposition.*

This has also an analogue for (Fg)-selfinjective algebras:

Proposition 9.2.9 ([EHT⁺04, Proposition 5.4]). *Let A be an (Fg)-selfinjective algebra with respect to the subalgebra H of the Hochschild cohomology ring. Then if H is generated as a subalgebra of HH^* by elements η_1, \dots, η_t in degrees n_1, \dots, n_t respectively and M is an indecomposable periodic module in mod A , then the period of M divides one of the n_i .*

Apart from the counterexamples for non-(fg)-Hopf algebras given above the property of modules of complexity 1 to be periodic is also satisfied for the nilpotent part of the small quantum group $u_\zeta(\mathfrak{n})$ by the following proposition due to Sarah Scherotzke:

Proposition 9.2.10 ([Sch09, Lemma 5.13, Proof of Lemma 5.11]). *Let U be a local basic algebra and $G < \text{Aut}(U)$ be a finite abelian group, such that kG is semisimple, and let $B := U \# kG$. Then every indecomposable U -module of complexity 1 is τ_U -periodic iff every indecomposable B -module of complexity one is τ_B -periodic.*

Combining the last two results we get the following for the nilpotent part of the small quantum group:

Proposition 9.2.11. *For char $k = 0$ let ℓ be a good prime for Φ and $\ell > 3$ for types \mathbb{B} and \mathbb{C} and $\ell \nmid n + 1$ for type \mathbb{A}_n and $\ell \neq 9$ for \mathbb{E}_6 . For char $k = p > 0$ let p be good for Φ and $\ell \geq h$. Then every $u_\zeta(\mathfrak{n})$ -module of complexity one is Ω and τ -periodic. The Ω -period of every periodic module for $u_\zeta(\mathfrak{n})$ is one or two.*

Proof. We apply Proposition 9.2.9 to the subalgebra $HH^0(u_\zeta(\mathfrak{n})) \cdot H^{ev}(u_\zeta(\mathfrak{n}), k)^{\mathbb{Z}/(\ell)^n}$ of the Hochschild cohomology $HH^\bullet(u_\zeta(\mathfrak{n}))$ (compare [FW11, Section 5]). As $H^{ev}(u_\zeta(\mathfrak{n}), k)^{\mathbb{Z}/(\ell)^n} \cong H^{ev}(u_\zeta(\mathfrak{b}), k)$ as algebras, the finite generation in degree two follows. The arguments of [FW11, Section 5] carry through in our slightly more general setting to show the appropriate (Fg)- conditions for Hochschild cohomology. Thus the result follows. \square

Theorem 9.2.12 ([HPR80, Theorem, p.292]). *Let A be a selfinjective algebra. If Θ is a component of $\Gamma_s(A)$, that contains a τ -periodic module, then the tree class of the component is either a finite Dynkin diagram or \mathbb{A}_∞ . If the component is finite, then the tree class is a finite Dynkin diagram. If it is infinite, the tree class is \mathbb{A}_∞ .*

Theorem 9.2.13. *There are no finite components for $U_\zeta(G_r)$, $U_\zeta(B_r)$ and $U_\zeta(N_r)$ except for $u_\zeta(\mathfrak{b}_{\mathfrak{sl}_2})$ and $u_\zeta(\mathfrak{n}_{\mathfrak{sl}_2})$.*

Proof. Theorem 1.9.17 implies that a finite component of the stable Auslander-Reiten quiver corresponds to a non-semisimple block of finite representation type. But by Theorem 8.2.6, Theorem 8.2.8, Theorem 8.3.1 and the proof of Theorem 8.3.7 this is not possible except for $r = 0$ and $\mathfrak{g} = \mathfrak{sl}_2$. In the case of $u_\zeta(\mathfrak{sl}_2)$ we observed in Section 7.2 that there are no finite components. \square

Theorem 9.2.14. *Impose the restrictions of the parameters of Proposition 9.2.7. Then the only tubes existing for $u_\zeta(\mathfrak{g})$ and $u_\zeta(\mathfrak{n})$ are homogeneous. The tubes for $u_\zeta(\mathfrak{b})$ have length dividing ℓ .*

Proof. The Ω -period of a module is two by Corollary 9.2.8 and 9.2.11 and the Nakayama functor is equal to the identity as $u_\zeta(\mathfrak{g})$ and $u_\zeta(\mathfrak{n})$ is symmetric by Corollary 4.3.9 and 5.2.21. For the Borel part it follows from Corollary 9.2.8 and Proposition 5.2.19. \square

Remark 9.2.15. The bounds are sharp, e.g. for $\mathfrak{g} = \mathfrak{sl}_2$ we have seen that the Ω -period is two while the τ -periods are ℓ and one for the Borel and nilpotent part, respectively.

9.3. Components of Kronecker type

In this section we are concerned with the only non-simply laced Euclidean component in the list of Webb's Theorem. We will use results of Karin Erdmann for symmetric algebras having such components from [Erd90]. In [Far99a] Rolf Farnsteiner noted, that most of her proofs also apply to selfinjective algebras. We state the results including the proofs for selfinjective algebras here and give two corollaries. First we classify all selfinjective algebras having components of Kronecker type. As in the symmetric case we get that they are all special biserial. Concerning small quantum groups we use the fact that components of Kronecker type are associated to indecomposable projective modules of length 4 to derive that the only case that a small quantum group has such component is the one we know already, $u_\zeta(\mathfrak{sl}_2)$.

Lemma 9.3.1. *Let A be a selfinjective algebra, such that there is a component of type $\mathbb{Z}[\tilde{A}_{12}]$ and $W \rightarrow X$ is an irreducible map in this component.*

- (i) *If $l(W) > l(X)$, then there is an Auslander-Reiten sequence of the form $0 \rightarrow \tau_A X \rightarrow W \oplus W \rightarrow X \rightarrow 0$ and $l(\tau_A X) > l(W)$.*
- (ii) *If $l(W) < l(X)$, then there is an Auslander-Reiten sequence of the form $0 \rightarrow W \rightarrow X \oplus X \rightarrow \tau_A^{-1} W \rightarrow 0$ and $l(X) < l(\tau_A^{-1} W)$.*

Here $l(M)$ denotes the length of a module M .

9.3. COMPONENTS OF KRONECKER TYPE

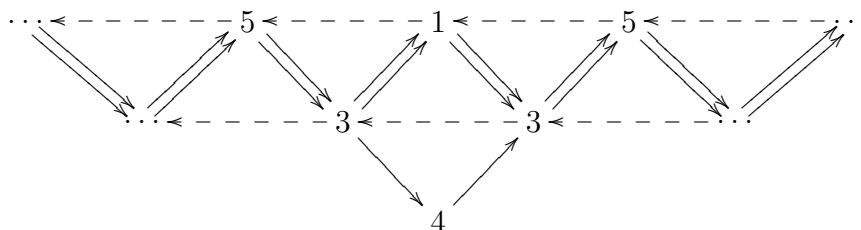
Proof. (i) Let $l(W) > l(X)$. As the component is of Kronecker type there is an Auslander-Reiten sequence of the form $0 \rightarrow \tau_A X \rightarrow W \oplus W \oplus P \rightarrow X \rightarrow 0$, where $P = 0$ or P is a projective indecomposable module. Assume that P is a projective indecomposable module, then $W \oplus W \cong \text{rad}(P)/\text{soc } P$, $\tau_A X \cong \text{rad}(P)$ and $X \cong P/\text{soc } P$. Therefore $l(X) = l(P) - 1 = l(\tau_A X)$. As the sequence is exact we get $2l(W) = 2l(X) - l(P)$ and therefore $l(W) < l(X)$, a contradiction to the original assumption. Therefore the sequence is of the claimed form and we have $l(\tau_A X) = 2l(W) - l(X) = l(W) + (l(W) - l(X)) > l(W)$.

(ii) Let $l(W) < l(X)$. As the component is of Kronecker type there is an Auslander-Reiten sequence of the form $0 \rightarrow W \rightarrow X \oplus X \oplus P \rightarrow \tau_A^{-1} W \rightarrow 0$, where $P = 0$ or P is a projective indecomposable module. Assume that P is a projective indecomposable module, then $X \oplus X \cong \text{rad}(P)/\text{soc } P$, $W \cong \text{rad}(P)$ and $\tau_A^{-1} W \cong P/\text{soc } P$. Therefore $l(W) = l(P) - 1 = l(\tau_A^{-1} W)$. As the sequence is exact we get $2l(X) = 2l(W) - l(P)$ and therefore $l(X) < l(W)$, a contradiction to the original assumption. Therefore the sequence is of the claimed form and we have $l(\tau_A^{-1} W) = 2l(X) - l(W) = l(X) + (l(X) - l(W)) > l(X)$.

□

The following proposition (first proven by Karin Erdmann) implies for example a result by Rolf Farnsteiner and Gerhard Röhrle in the context of Frobenius kernels ([FR11, Lemma 2.5]), stating that every Kronecker type component contains exactly one simple module.

Proposition 9.3.2. *Let A be a selfinjective algebra. If \mathcal{C} is a component of the stable Auslander-Reiten quiver of type $\mathbb{Z}[\tilde{\mathbb{A}}_{12}]$. Then it is attached to a projective indecomposable module of length 4. The lengths of the modules occurring in the non-stable component belonging to \mathcal{C} are as follows:*



Proof. Choose X_0 in the component to be of minimal length. Then all the irreducible morphisms starting in X_0 are monomorphisms, while those ending in X_0 are epimorphisms. Label the modules in the component by integers, increasing from left to right. Then there is a unique projective attached to this component, which lies between X_{-1} and X_1 because by the foregoing lemma it cannot lie anywhere else for length reasons. Let $0 \rightarrow \text{rad}(P) \rightarrow \text{rad}(P)/\text{soc } P \oplus P \rightarrow P/\text{soc } P \rightarrow 0$ be the corresponding sequence. By Proposition 1.9.12 we know that Ω_A induces an equivalence of components and $\Omega_A(P/\text{soc } P) = \text{soc } P$, so there is a simple module $S := \text{soc } P$ attached to this component. This has to be the unique module of minimal length for this component. Therefore there exists a projective

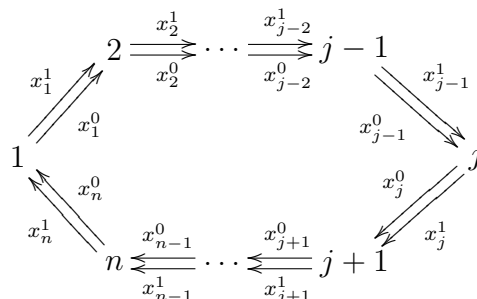
indecomposable module Q and an Auslander-Reiten sequence of the form $0 \rightarrow \text{rad}(Q) \rightarrow S \oplus S \oplus Q \rightarrow Q/\text{soc } Q \rightarrow 0$. This shows that Q has length 4. Applying this argument to $\Omega_A^{-1}\mathcal{C}$ we get that P is also of length 4. The statement about the length of the modules follows immediately. \square

Remark 9.3.3. (i) In fact we can read off more than the length of the modules, also their explicit construction from the projective indecomposable ones can easily be deduced. The next corollary tells us that they are all string modules for special biserial algebras.

(ii) There is a similar theorem for non-selfinjective algebras, but one has to assume that the algebra is tame. This is due to Sheila Brenner and Michael C. R. Butler in [BB98, Theorem 1.1].

The following result classifies all selfinjective algebras having Kronecker type components.

Theorem 9.3.4. (i) *If A is a selfinjective connected algebra with a component of type $\mathbb{Z}[\tilde{A}_{12}]$, then there are as many simple modules as there are components of this shape and the only other components are homogeneous tubes. Up to Morita equivalence the algebras have the following quivers:*



with relations for each j given by either $x_{j+1}^i x_j^i - \lambda_j^i x_{j+1}^{i+1} x_j^{i+1}$ and $x_{j+1}^i x_j^{i+1}$ for any i or by $x_{j+1}^{i+1} x_j^i - \lambda_j^i x_{j+1}^i x_j^{i+1}$ and $x_{j+1}^i x_j^i$ for any i , where $\lambda_j^i \in k^\times$ for all i, j . (Many of these algebras are of course isomorphic.) Here the indices are added modulo 2 and modulo n respectively. In particular all these algebras are special biserial.

(ii) *If A is weakly-symmetric, then there are at most two simple modules. And if there are two of them then the following algebras remain:*

$$1 \begin{array}{c} \xrightarrow{-x_1} \\ \xrightarrow{y_1} \\ \xleftarrow{-y_2} \\ \xleftarrow{x_2} \end{array} 2$$

with relations: $x_2 y_1 = y_2 x_1 = 0$ and $x_2 x_1 - y_2 y_1 = 0$ and one of the following possibilities:

a) $x_1 y_2 = y_1 x_2 = 0$ and $x_1 x_2 - \lambda y_1 y_2 = 0$, where $\lambda \in k^\times$, say $A_2(\lambda)$;

b) $x_1 x_2 = y_1 y_2 = 0$ and $x_1 y_2 - y_1 x_2 = 0$, say $A_2(0)$.

9.3. COMPONENTS OF KRONECKER TYPE

We have that $A_2(\mu) \cong A_2(\mu')$ iff $\mu' = \mu^{\pm 1}$.

If A is local, i.e. there is only one simple module, then the following algebras remain:

$$x \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} 1 \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} y$$

with relations $xy = yx = 0$ and $x^2 = y^2$, say $A_1(0)$, or $x^2 = y^2 = 0$ and $xy = \lambda yx$, where $\lambda \in k^\times$, say $A_1(\lambda)$. For $\text{char } k = 2$ we have $A_1(0) \not\cong A_1(1)$. For $\text{char } k \neq 2$ we have $A_1(\mu) \cong A_1(\mu')$ iff $\mu' = \mu^{\pm 1}$ or $\{\mu, \mu'\} = \{0, 1\}$.

(iii) The only symmetric algebras among these are $A_2(1)$ and $A_1(0)$ and $A_1(1)$.

Proof. Without loss of generality we assume that the algebra is basic as up to Morita equivalence every algebra is basic. The properties of being selfinjective, weakly symmetric or symmetric and the shapes of components are invariant under Morita equivalence.

(i) Let P_1 be the indecomposable projective module of length 4 attached to the component of Kronecker type \mathcal{C} . Let $S_1 := \text{top } P_1$, $S_3 := \text{soc } P_1$, S_2 be the simple module, such that $\text{ht } P_1 \cong S_2 \oplus S_2$. We know that $S_3 = \Omega_A(P_1/\text{soc } P_1) \in \Omega_A \mathcal{C}$ and this is also a component of Kronecker type with a projective module P_2 and $\text{ht } P_2 \cong S_3 \oplus S_3$ by the proof of the foregoing proposition. By the shape of P_1 we conclude that $\text{top } P_2 \cong S_2$ and define $S_4 := \text{soc } P_2$. Now we continue by induction. Let P_1, \dots, P_{k-1} and S_1, \dots, S_{k+1} be defined such that $\text{top } P_i \cong S_i$, $\text{ht } P_i \cong S_{i+1} \oplus S_{i+1}$ and $\text{soc } P_i \cong S_{i+2}$. Furthermore $P_i \in \Omega_A^{i-1} \mathcal{C}$. We want to prove these statements for P_j . As for $j = 2$ we conclude that there is a projective module $P_j \in \Omega_A^{j-1} \mathcal{C}$ with $\text{top } P_j \cong S_j$ and $\text{ht } P_j \cong S_{j+1} \oplus S_{j+1}$. Define $S_{j+2} := \text{soc } P_j$.

Now let l, m be such that $S_l \cong S_m$ and $|l - m|$ is minimal. Such pair of natural numbers exists as there are only finitely many simple modules up to isomorphism. Without loss of generality let $l \geq m$. We conclude that $P_{l-2} \cong P_{m-2}$ as the injective hull of a module is unique up to isomorphism and isomorphic for isomorphic modules. From this we deduce that $S_{l-1} \cong S_{m-1}$. By induction this leads to $S_1 \cong S_{l-m+1}$ and the minimality of $l - m$ gives that $l - m + 1$ is the minimal natural number n , such that $S_1 \cong S_n$.

The modules S_1, \dots, S_n form a complete list of composition factors of all the projective modules in all $\Omega_A^j \mathcal{C}$ for all j . Therefore the corresponding projective modules P_1, \dots, P_n form a block. As the algebra is connected, they are all the simple modules and the number of Kronecker components is also equal to n as P_i is the unique projective module in $\Omega_A^{i-1} \mathcal{C}$. The quiver of A can now be constructed by choosing a basis of $\text{rad}(P_i)/\text{rad}^2(P_i)$ for all i . Lifting this base to P_i the relations can easily be determined.

This tells us that the algebra is special biserial. The shape of the Auslander-Reiten-quiver now follows from Theorem 1.9.26. The string modules and the projective-injective modules lie in the components of Kronecker type. The band modules lie in homogeneous tubes and therefore these are the only components.

- (ii) If the algebra is weakly-symmetric then we have that $S_1 \cong S_3$ and there are two possibilities: Either $S_1 \cong S_2$ or $S_1 \not\cong S_2$. We get the stated quivers and relations by constructing isomorphisms just given by sending the arrows to scalar multiples of themselves. (Here we use that the field is algebraically closed.)

To show that they are indeed non-isomorphic in the stated cases we use the fact that the trace of the Nakayama automorphism provides an invariant for the isomorphism class of an algebra.

By [HZ08, Proposition 3.1] we have that for $A_2(\lambda)$ a Frobenius homomorphism is given on the paths by $x_1x_2 \mapsto \lambda$, $x_2x_1 \mapsto 1$ and all others to 0. Then it is easily checked that a Nakayama automorphism is given on the arrows by $\nu(y_2) = y_2$, $\nu(y_1) = y_1$, $\nu(x_1) = \lambda x_1$ and $\nu(x_2) = \lambda^{-1}x_2$. Therefore the trace of the Nakayama automorphism is $6 + \lambda + \lambda^{-1}$, so we get the claimed non-isomorphism. For $A_2(0)$ a Frobenius homomorphism is given by $x_2x_1 \mapsto 1$ and $x_1y_2 \mapsto 1$, all others to 0. A Nakayama automorphism is then given by $\nu(x_1) = y_1$, $\nu(y_1) = x_1$, $\nu(x_2) = y_2$ and $\nu(y_2) = x_2$. Therefore the trace is 4. We have indeed $A_2(0) \cong A_2(-1)$ for $\text{char } k \neq 2$: In $A_2(-1)$ the elements $u_1 := x_1 + y_1$, $v_1 := -x_1 + y_1$, $u_2 := x - 2 + y_2$, $v_2 := -x_2 + y_2$ satisfy the relations of $A_2(0)$. For $\text{char } k = 2$ we immediately get a contradiction if we want to define these elements.

For the local case again by [HZ08, Proposition 3.1] we get a Frobenius homomorphism for $A_1(\lambda)$, where $\lambda \in k^\times$ defined by $xy \mapsto \lambda$, $x \mapsto 0$, $y \mapsto 0$. A Nakayama automorphism is then $\nu(x) = \lambda x$, $\nu(y) = \lambda^{-1}y$ and identity on all other basis vectors. Then the trace is $2 + \lambda + \lambda^{-1}$.

The stated isomorphism for $\text{char } k \neq 2$ is given by setting $u = x + iy$ and $v = x - iy$ for i a primitive second root of unity. Then $(x + iy)^2 = 0 = (x - iy)^2$ and $(x + iy)(x - iy) \neq 0$. For $\text{char } k = 2$ we can only choose $u = x + y$ since $x^2, y^2 \neq 0$. But then there is no possible choice for v , a contradiction.

- (iii) It is easy to see that the stated algebras are symmetric (compare [HS10, Proposition 3.2]) with symmetrizing form given by 1 on $\text{soc}(A)$ and 0 for all other paths. In the other cases assume for a contradiction that there was a symmetric form for $\lambda \neq 1$. In the case of two simple modules this would lead to

$$\begin{aligned} \lambda(y_1, y_2)_\pi &= \lambda\pi(y_1y_2) = \pi(x_1x_2) = (x_1, x_2) \\ &= (x_2, x_1) = \pi(x_2x_1) = \pi(y_2y_1) = (y_2, y_1)_\pi = (y_1, y_2)_\pi. \end{aligned}$$

As the form is non-degenerate we have $\lambda = 1$, a contradiction. In the case of a local algebra we have similarly

$$\lambda(y, x)_\pi = \pi(\lambda yx) = \pi(x, y) = (y, x)_\pi,$$

a contradiction as $\lambda \neq 1$.

□

Remark 9.3.5. One could ask if these algebras actually appear (up to Morita equivalence) as blocks of a Hopf algebra. The following examples are known:

- $k[x, y]/(x^2, y^2)$, which is the group algebra of the Klein four group

$$D_4 := \langle x, y \mid x^2 = y^2 = 1, xy = yx \rangle = \mathbb{Z}/2 \times \mathbb{Z}/2$$

in characteristic 2.

- (iii) a) is the trivial extension of the Kronecker algebra and is Morita equivalent to the principal block of $u(\mathfrak{sl}_2)$ (Proposition 1.8.13) for $\text{char } k = p > 0$ and to the principal block of $u_\zeta(\mathfrak{sl}_2)$ for $\text{char } k = 0$ as we have seen in Theorem 7.2.11.
- $k[x, y]/(x^2 - y^2, xy)$ is Morita equivalent to a block of the algebra $U_t(\mathfrak{sl}(2))$ as defined in [Xia97].

An obvious restriction for the weakly symmetric case is that the scalar λ is a root of unity, because otherwise the Nakayama automorphism is not of finite order contradicting 2.1.6. As a further negative result we have that except for (iii) a) these algebras do not appear as the principal block of a cocommutative Hopf algebra for $\text{char } k \geq 3$ according to [Far06, Proposition 7.4.3, Remark (2)].

Question 9.3.6. (i) Which of the other algebras can be realized as blocks of Hopf algebras?

(ii) Are there characteristic-independent versions of the aforementioned examples?

(iii) Can the block of $U_t(\mathfrak{sl}(2))$ be realized as a block of a cocommutative Hopf algebra?

Theorem 9.3.7. *The algebras $U_\zeta(N_r)$ and $U_\zeta(B_r)$ do not admit any component of Kronecker type. Assume $U_\zeta(G_r)$ satisfies (fg). Then $U_\zeta(G_r)$ has Kronecker components iff $\mathfrak{g} \cong \mathfrak{sl}_2$ and $r = 0$ or $r = 1$. For $r = 1$ they belong to the image of the block embedding $F : \text{mod Dist}(\text{SL}(2)_1) \rightarrow \text{mod } U_\zeta(\text{SL}(2)_1), V \mapsto V \otimes \text{St}_\ell$ (compare Proposition 2.3.5).*

Proof. The algebras $U_\zeta(N_r)$ and $U_\zeta(B_r)$ are basic by Corollary 4.1.22, so an indecomposable module of length 4 would have dimension 4. But the projective modules have dimension divisible by ℓ by Theorem 5.2.22. So there is no such component. For $U_\zeta(G_r)$ a component of Kronecker type would be attached to a tame block. But such a block does not exist for $\mathfrak{g} \neq \mathfrak{sl}_2$ and $r = 0$ or $r = 1$ by Corollary 8.3.2 and Theorem 8.3.7. \square

9.4. Simply-laced Euclidean components

In this section we exclude the simply-laced Euclidean components for the small quantum groups. Note that in contrast to other authors we distinguish between Euclidean components and components of Euclidean tree class. The latter excludes the case of $\mathbb{Z}[\tilde{A}_n]$ which we also consider in this section. For $u_\zeta(\mathfrak{g})$ we use an action of the algebraic group on the support variety. For a low-dimensional support variety this action has to have a large stabilizer and we can conclude that the Lie algebra has to be \mathfrak{sl}_2 . For $u_\zeta(\mathfrak{b})$ and $u_\zeta(\mathfrak{n})$ we use results of Sarah Scheretzke's, who corrected and generalized results of Rolf Farnsteiner

for restricted enveloping algebras.

We start by stating a result of Otto Kerner and Dan Zacharia about the complexity of modules in Euclidean components:

Theorem 9.4.1 ([KZ11, Proposition 1.1], cf. [Far99b, Theorem 2.1]). *Let A be a selfinjective algebra. If Δ is a Euclidean diagram, such that $\mathbb{Z}[\Delta]$ is isomorphic to a component Θ of $\Gamma_s(A)$ then $\text{cx } \Theta = 2$.*

To illustrate in another case how geometric methods and subadditive functions can be put together, we include a proof in the special case where A is an (fg)-Hopf algebra:

Proof. Let A be an (fg)-Hopf algebra. By Corollary 3.2.8 and Proposition 9.2.4 we have that $\text{Kdim } \mathcal{V}_A(\Theta) \geq 2$. Let $[N]$ be an element of Θ . By Proposition 9.1.10 we have subadditive functions $d_S^i : \Theta \rightarrow \mathbb{N}$, $[M] \mapsto \dim \text{Ext}_A^i(M, S)$ for $1 \leq i \leq 2$ and every simple A -module S that fail to be additive at only finitely many vertices. Proposition 9.1.14 then implies that every d_S^i grows at most linearly for a τ -orbit, i.e. $\gamma(\text{Ext}_A^i(\tau_A^n N, S)) \leq 2$. But since $\tau_A M \cong \Omega_A^2 M^{(\nu_A)}$ by Lemma 1.9.13 and $\text{Ext}_A^m(N, S) \cong \text{Ext}_A^{m-1}(\Omega_A N, S)$ by Lemma 1.6.11 we have $\text{Ext}_A^i(\tau_A^n N, S) \cong \text{Ext}_A^i(\Omega_A^{2n} N^{(\nu_A)^n}, S) \cong \text{Ext}_A^i(\Omega_A^{2n} N, S^{(\nu_A)^{-n}}) \cong \text{Ext}_A^{2n+i}(N, S^{(\nu_A)^{-n}})$ and therefore $\gamma(\text{Ext}_A^n(N, S)) \leq 2$. Proposition 3.2.7 yields

$$\text{Kdim } \mathcal{V}_A(\Theta) = \max\{\gamma(\text{Ext}_A^n(N, S)) \mid S \in \mathcal{S}_A\} \leq 2.$$

□

Theorem 9.4.2 ((i) [Dru09, Corollary 4.23], (ii) [Dru10, Theorem 6.1]). *Let $\text{char } k$ be odd or zero and good for Φ . Assume ℓ is odd, coprime to three if Φ has type \mathbb{G}_2 and $\ell \geq h$. Then the following hold:*

(i) *We have $H^{\text{odd}}(u_\zeta(\mathfrak{g}), k) = 0$. There exists a natural graded G -algebra isomorphism $H(u_\zeta(\mathfrak{g}), k) \cong k[\mathcal{N}]$, where $\mathcal{N} \subset \mathfrak{g}$ is the nullcone (i.e. the set of nilpotent elements in \mathfrak{g}), where the action of G on the cohomology ring $H^\bullet(u_\zeta(\mathfrak{g}), k)$ is induced by the adjoint action of $U_\zeta(\mathfrak{g})$ on $H^\bullet(u_\zeta(\mathfrak{g}), k)$, and the action of G on $k[\mathcal{N}]$ is induced by the usual adjoint action of G on $\mathcal{N} \subset \mathfrak{g}$.*

(ii) *If M is a finite dimensional $U_\zeta(\mathfrak{g})$ -module. Then $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = G \cdot \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$.*

Remark 9.4.3. The stated isomorphism yields a G -equivariant isomorphism

$$\mathcal{N} \cong \text{maxSpec } H^{\text{ev}}(u_\zeta(\mathfrak{g}), k).$$

In the remainder we will therefore identify the support varieties of $u_\zeta(\mathfrak{g})$ -modules with conical subvarieties of $\mathcal{N} \subseteq \mathfrak{g}$.

The remaining results of this chapter are proven similarly to the corresponding results in [Far99b], [Far99a] and [Far00a]. Note also the corrections made in [Sch09].

9.4. SIMPLY-LACED EUCLIDEAN COMPONENTS

Theorem 9.4.4. *Let $\text{char } k$ be zero or odd and good for Φ . Assume $\ell \geq h$. Let M be a $U_\zeta(\mathfrak{g})$ -module, such that $\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = 2$. Then the following statements hold:*

- (i) *There exist normal subgroups $K, H \subset G$ such that*
 - (a) $G = H \cdot K$,
 - (b) $\mathfrak{g} = \text{Lie}(H) \oplus \text{Lie}(K)$, and
 - (c) $\text{Lie}(K) \cong \mathfrak{sl}_2$.
- (ii) *If M is indecomposable, then $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = \mathcal{V}_{u_\zeta(\text{Lie}(K))}(M)$ and M is a projective $u_\zeta(\text{Lie}(H))$ -module.*

Proof. (i) According to Theorem 1.8.5 we have $G = G_1 \cdots G_r$ where the G_i are the closed, connected, normal subgroups of G . Since M is a $U_\zeta(\mathfrak{g})$ -module, the support variety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ and its projectivization $\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ are stable under the adjoint representation by the foregoing theorem. Owing to Borel's Fixed Point Theorem (Theorem 1.8.2) there exists $[x_0] \in \mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ that is fixed by a given Borel subgroup $B \subset G$. Consequently, the stabilizer P_0 of $[x_0]$ is parabolic.

According to the orbit formula (Proposition 1.8.8) we have $\text{Kdim } G - \text{Kdim } P_0 = \text{Kdim } G \cdot [x_0] \leq 1$. Therefore there are two possibilities for this dimension, either $\text{Kdim } G = \text{Kdim } P_0$, which implies $P_0 = G$ or $\text{Kdim } G - \text{Kdim } P_0 = 1$.

The assumption $P_0 = G$ implies that $kx_0 \subseteq \mathfrak{g}$ is invariant under the adjoint representation. Hence by Lemma 1.8.17 we have that kx_0 is a p -unipotent ideal of \mathfrak{g} . As \mathfrak{g} is reductive it follows from [Hum67, (11.8)] that $x_0 = 0$, a contradiction. Thus we have $\text{Kdim } P_0 = \text{Kdim } G - 1$. Using Proposition 1.8.11 we see that the action of G on G/P_0 induces a surjective homomorphism $\varphi : G \rightarrow \text{PGL}_2$ of algebraic groups. As the $\varphi(G_i)$ are normal subgroups of the simple group PGL_2 (Lemma 1.8.10) we have that $\varphi(G_i) = \text{PGL}_2$ or $G_i \subseteq \ker \varphi$. Since $(G_i, G_j) = 1$, we also have that $(\varphi(G_i), \varphi(G_j)) = 1$, so if there were two indices i, j with $\varphi(G_i) = \varphi(G_j) = \text{PGL}_2$, then $\varphi(G_j) \in Z(\text{PGL}_2)$, a contradiction. Thus there exists exactly one index $i_0 \in \{1, \dots, r\}$ such that $\varphi(G_{i_0}) = \text{PGL}_2$. Without loss of generality, we have $i_0 = 1$. Thus setting $K := G_1$ and $H := G_2 \cdots G_r$ we have $\text{Kdim } K = 3$ and $H \subseteq \ker \varphi \subseteq P_0$. By [Hum67, Lemma 11.7, Proposition 12.1] we have that $\text{Lie}(K) \cong \mathfrak{sl}_2$. Since $(G_i, G_j) = 1$ for $i \neq j$ by Lemma 1.8.18 we have $\text{Lie}(H) \subseteq C_{\mathfrak{g}}(\text{Lie}(K))$. Since $\text{Lie}(K)$ is complete, i.e. centerless with all derivations being inner (see e.g. [Jac62, p. 29]), it follows that for $x \in \mathfrak{g}$ the derivation $\text{ad } x|_{\text{Lie}(K)}$ is inner, i.e. there exists $v \in \text{Lie}(K)$, such that $x - v \in C_{\mathfrak{g}}(\text{Lie}(K))$. Therefore since the center of $\text{Lie}(K)$ is trivial, it follows that $\mathfrak{g} = \text{Lie}(K) \oplus C_{\mathfrak{g}}(\text{Lie}(K))$. Moreover, $H \cap K$ is finite, so that

$$\dim \mathfrak{g} = \text{Kdim } G = \text{Kdim } H + \text{Kdim } K = \dim \text{Lie}(H) + \dim \text{Lie}(K).$$

This shows that $\mathfrak{g} = \text{Lie}(K) \oplus \text{Lie}(H)$.

- (ii) Let M be indecomposable. By Proposition 3.3.6 the projective variety $\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ is connected. Since $\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = 2$, we have $\text{Kdim}(G \cdot [x]) \leq 1$ for all $[x] \in$

$\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$. If $\text{Kdim } G \cdot [x] = 0$ for some x , then $G \cdot [x] = [x]$ and $kx \neq 0$ is a p -unipotent ideal of \mathfrak{g} , a contradiction. Hence all orbits have dimension 1, so that each of them is closed by Proposition 1.8.7 and thus an irreducible component of $\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$. Since the orbits do not intersect and $\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ is connected it follows that $\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ is irreducible. In particular we have

$$\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = G \cdot [x_0] = K \cdot H \cdot [x_0] = K \cdot [x_0],$$

since $H \subseteq P_0$. Since H and K commute, this implies that H operates trivially on $\mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$.

Let x be an element of $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) \setminus \{0\}$. By the observation above there exists a character $\alpha_x : H \rightarrow k^\times$ such that $\text{Ad}(h)(x) = \alpha_x(h)x$ for all $h \in H$. Since $G_i = (G_i, G_i)$ for all $2 \leq i \leq r$ and commutators are mapped to commutators via α_x , it readily follows that $\alpha_x(h) = 1$ for all $h \in H$. Consequently $[\text{Lie}(H), x] = 0$. Writing $x = y + z$ where $y \in \text{Lie}(K)$ and $z \in \text{Lie}(H)$, we have $[\mathfrak{g}, z] = [\text{Lie}(H), z] = [\text{Lie}(H), x] = 0$. Hence $z \in C(\mathfrak{g})$ defines a p -unipotent ideal $kz \subset \mathfrak{g}$. Since \mathfrak{g} is reductive we conclude that $z = 0$. Consequently $x = y \in \text{Lie}(K)$. According to Proposition 3.2.14 we now obtain $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) \cap \text{Lie}(K) = \mathcal{V}_{\text{Lie}(K)}(M)$. Thus $\mathcal{V}_{\text{Lie}(H)}(M) = 0$ and M is $u_\zeta(\text{Lie}(H))$ -projective by Corollary 3.2.8. \square

Theorem 9.4.5. *Let B be a block of $u_\zeta(\mathfrak{g})$. The Auslander-Reiten quiver of B possesses a Euclidean component Θ iff B is tame. In this case the component is of type $\mathbb{Z}[\tilde{\mathbb{A}}_{12}]$.*

Proof. By Theorem 9.4.1 we have that $\text{cx } \Theta = 2$. As every Euclidean component is attached to a principal indecomposable module by Corollary 9.1.16 we have that it contains a simple module. But for a simple module of complexity 2, Theorem 9.4.4 together with Corollary 8.3.2 and Corollary 7.2.12 implies that B is tame. \square

So for $u_\zeta(\mathfrak{g})$ there is no component of Euclidean type except for the known case. Our next goal is to show this for $u_\zeta(\mathfrak{b})$ and $u_\zeta(\mathfrak{n})$.

Proposition 9.4.6. *Impose the restrictions of Proposition 9.2.7. If $\mathfrak{b} \neq \mathfrak{b}_{512}$ there is a non-periodic module of length 3 for $u_\zeta(\mathfrak{b})$.*

Proof. As $u_\zeta(\mathfrak{b})$ is not a Nakayama algebra, it follows from Theorem 1.9.20 that there is a simple module S , such that

$$\sum_{[T] \in \mathcal{S}_{u_\zeta(\mathfrak{b})}} \dim \text{Ext}_{u_\zeta(\mathfrak{b})}^1(T, S) \geq 2$$

Let P be the injective hull of S . By the definition of Ext^1 and soc^2 we immediately obtain

$$2 \leq \sum_{[T] \in \mathcal{S}_{u_\zeta(\mathfrak{b})}} \dim \text{Hom}_{u_\zeta(\mathfrak{b})}(T, \text{soc}^2(P)/\text{soc}(P)),$$

9.4. SIMPLY-LACED EUCLIDEAN COMPONENTS

so that $l(\text{soc}^2(P)/\text{soc } P) \geq 2$ and $l(\text{soc}^2(P)) \geq 3$.

Denote by $\pi : P \rightarrow P/\text{soc}(P)$ the natural projection. Then π induces a surjection $\text{soc}^2(P) \twoheadrightarrow \text{soc}(P/\text{soc}(P))$ and $\pi(\text{rad}(u_\zeta(\mathfrak{b}))\text{soc}^2(P)) = 0$. Thus $\text{rad}(u_\zeta(\mathfrak{b}))\text{soc}^2(P) = \text{soc}(P)$. Now let $X \subset \text{soc}^2(P)$ be a submodule of length 3. Then $\text{soc } X = \text{soc } P$ and $\text{rad}(u_\zeta(\mathfrak{b}))X \subseteq \text{rad}(u_\zeta(\mathfrak{b}))\text{soc}^2(P) = \text{soc}(X)$, so that $\text{rad}(u_\zeta(\mathfrak{b}))X = \text{soc } X$. Thus $\text{top } X = X/\text{soc } X$ has length 2. Suppose that X was periodic. Then by Corollary 9.2.8 $\Omega_{u_\zeta(\mathfrak{b})}^2 X \cong X$. As P is the injective hull of $\text{soc } X$ there exists a projective module Q and an exact sequence

$$0 \rightarrow X \rightarrow P \rightarrow Q \rightarrow X \rightarrow 0$$

Therefore $\dim P = \dim Q$, so that Q is indecomposable as all projective indecomposable modules have the same dimension by 4.1.17. But as $u_\zeta(\mathfrak{b})$ is selfinjective this would mean that $\text{top } X$ is irreducible, therefore has length 1, a contradiction. \square

We need the following two results of Sarah Scherotzke for components of Euclidean tree class:

Theorem 9.4.7 ([Sch09, Theorem 3.3, Theorem 3.7, Theorem 3.11]). *Let B be an algebra and $G < \text{Aut}(B)$ be a group, such that G acts transitively on the simple B -modules. Furthermore assume that all modules with complexity 1 are Ω_B and τ_B -periodic.*

- (i) *Every component of $\Gamma_s(B)$ of Euclidean tree class is isomorphic to $\mathbb{Z}[\tilde{\mathbb{A}}_{12}]$ or $\mathbb{Z}[\tilde{\mathbb{D}}_n]$ with n odd.*
- (ii) *$\Gamma_s(B)$ does not have a component with tree class $\tilde{\mathbb{D}}_5$.*
- (iii) *If B contains an indecomposable non-periodic module of length 3 and has a component with tree class $\tilde{\mathbb{D}}_n$, $n > 5$, then $l(P) = 8$ for every indecomposable projective B -module P .*

Theorem 9.4.8 ([Sch09, Theorem 5.15]). *Let U be a local basic algebra. Let $G < \text{Aut}(U)$ be an abelian group, such that kG is semisimple. Let $B := U \# kG$. $\Gamma_s(U)$ has a component of tree class $\tilde{\mathbb{D}}_n$ iff $\Gamma_s(B)$ has a component of tree class $\tilde{\mathbb{D}}_n$.*

For $\tilde{\mathbb{A}}_n$, which is not of Euclidean tree class but has tree class \mathbb{A}_∞ we need the following two results:

Lemma 9.4.9 ([Far99a, Lemma 2.2]). *We have $\text{Aut}(\mathbb{Z}[\tilde{\mathbb{A}}_n]) = \{\tau^s \circ \alpha^r \mid s \in \mathbb{Z}, 0 \leq r \leq n-1\}$, where α is given by $\alpha(j, [i]) = (j, [i+1])$ if we identify $\mathbb{Z}[\tilde{\mathbb{A}}_n] = \mathbb{Z} \times \mathbb{Z}/(n)$.*

Lemma 9.4.10 ([BR87, Proposition, p. 155 for the length function]). *For an algebra A a component of tree class \mathbb{A}_∞ , which is not isomorphic to $\mathbb{Z}[\mathbb{A}_\infty]$, is not regular.*

Theorem 9.4.11. *For the stable Auslander-Reiten quivers of $u_\zeta(\mathfrak{b})$ and $u_\zeta(\mathfrak{n})$ the only Euclidean components can be of the form $\mathbb{Z}[\tilde{\mathbb{D}}_n]$, where $n > 5$. If we further impose that $\ell > h$ then also these components cannot occur.*

Proof. As $(\mathbb{Z}/\ell\mathbb{Z})^n$ is isomorphic to its character group we can define an action of $(\mathbb{Z}/\ell\mathbb{Z})^n$ on $u_\zeta(\mathfrak{b})$ by setting $g(u\#h) = u\#\chi_{g^{-1}}(h)h$ for all $u \in u_\zeta(\mathfrak{b})$ and all $g, h \in (\mathbb{Z}/\ell\mathbb{Z})^n$. With this action $(\mathbb{Z}/\ell\mathbb{Z})^n$ is a subgroup of $\text{Aut}(u_\zeta(\mathfrak{b}))$. We have that $(\mathbb{Z}/\ell\mathbb{Z})^n$ acts transitively on the simple $u_\zeta(\mathfrak{b})$ -modules by Lemma 4.1.17 and all modules of complexity 1 are Ω and τ -periodic by Proposition 9.2.4 and Lemma 9.2.2. By Theorem 9.4.7 all components of Euclidean tree class of $u_\zeta(\mathfrak{b})$ would be of the form $\mathbb{Z}[\tilde{\mathbb{A}}_{12}]$. But this case was already excluded in Theorem 9.3.7.

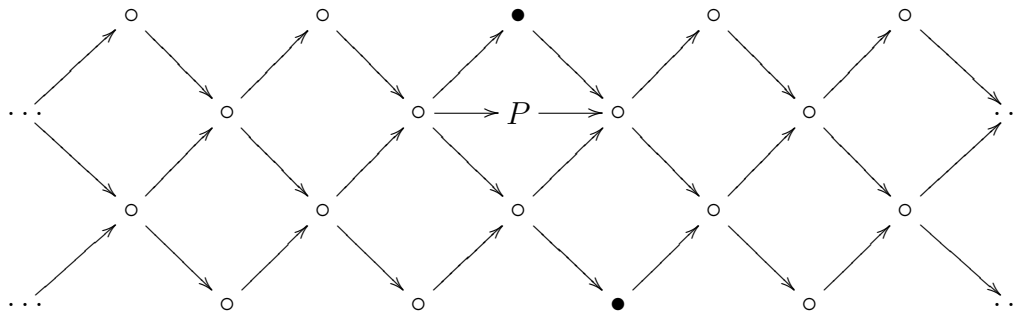
For $u_\zeta(\mathfrak{n})$ note that $u_\zeta(\mathfrak{b}) = u_\zeta(\mathfrak{n})\#u_\zeta^0(\mathfrak{g})$ (Lemma 4.1.17). Therefore also $u_\zeta(\mathfrak{n})$ satisfies the conditions of Theorem 9.4.7 for $G = \{e\}$ by Proposition 9.2.10 and the fact that $u_\zeta(\mathfrak{n})$ has only one simple module. Thus as $u_\zeta(\mathfrak{b})$ has no components of tree class $\tilde{\mathbb{D}}_n$, also $u_\zeta(\mathfrak{n})$ has no components of tree class $\tilde{\mathbb{D}}_n$ by Theorem 9.4.8. But the case of a Kronecker type component was already excluded in Theorem 9.3.7.

It remains to consider the case of a component Θ of type $\mathbb{Z}[\tilde{\mathbb{A}}_m]$. For $u_\zeta(\mathfrak{b})$ let $G = (\mathbb{Z}/\ell\mathbb{Z})^n$ for $u_\zeta(\mathfrak{n})$ let $G = \{e\}$. Let $U = u_\zeta(\mathfrak{b})$, respectively $U = u_\zeta(\mathfrak{n})$. Such a component is attached to a principal indecomposable module by the foregoing lemma. Since the principal indecomposable modules are all related via shifting by some $\lambda \in G$ the same holds true for these components. Since $\Omega\Theta \cong \Theta$ there exists $\lambda \in G$, such that the automorphism $\varphi : \Gamma_s(U) \rightarrow \Gamma_s(U)$, $[M] \mapsto [\Omega(\lambda M)]$ satisfies $\varphi(\Theta) = \Theta$. Note that $(\varphi|_\Theta)^{2\ell} = \Omega^{2\ell} = \tau^\ell$ by Lemma 5.2.21 and Proposition 5.2.19. According to Lemma 9.4.9 we write $\varphi|_\Theta = \tau^j \circ \alpha^r$ and obtain $\tau^\ell = \tau^{2\ell j} \circ \alpha^{2\ell r}$, whence $\tau^{\ell(2j-1)} = \alpha^{-2\ell r}$, a contradiction. \square

Analogously to the foregoing section we now want to deduce properties for a general self-injective algebra just from the appearance of a component with small tree class:

Proposition 9.4.12. *Let A be a selfinjective algebra with a component Θ of the stable Auslander-Reiten quiver isomorphic to $\mathbb{Z}[\tilde{\mathbb{A}}_2]$. Then there is a unique projective module P_Θ attached to the component which satisfies $\text{ht } P_\Theta = S_\Theta \oplus M_\Theta$ for a simple module S_Θ . In addition there is an irreducible map $M_\Theta \rightarrow S_\Theta$ or $S_\Theta \hookrightarrow M_\Theta$. Furthermore $\Omega\Theta \neq \Theta$ and if $M_\Theta \rightarrow S_\Theta$ is irreducible, then there is an irreducible map $S_{\Omega\Theta} \hookrightarrow M_{\Omega\Theta}$, and if $S_\Theta \hookrightarrow M_\Theta$ is irreducible, then there is an irreducible map $M_{\Omega\Theta} \rightarrow S_{\Omega\Theta}$.*

Proof. A component of type $\mathbb{Z}[\tilde{\mathbb{A}}_2]$ looks as follows (since it is necessarily attached to a projective module by Lemma 9.4.10, we have drawn that too):



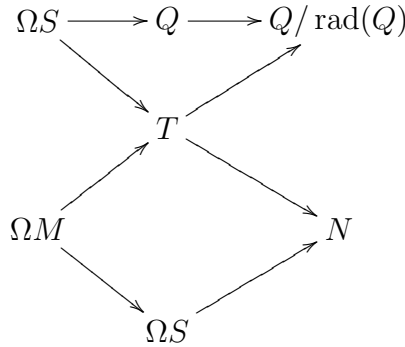
Here the two bullets are identified (resp. their τ -shifts). The standard almost split sequence attached to P is $0 \rightarrow \text{rad}(P) \rightarrow \text{ht } P \oplus P \rightarrow P/\text{soc } P \rightarrow 0$. Therefore the irreducible maps

9.5. COMPONENTS OF INFINITE DYNKIN TYPE

originating in the predecessor of P are surjective (except for the map $\text{rad}(P) \rightarrow P$) while the irreducible maps terminating in the successor of P (except for the map $P \rightarrow P/\text{soc } P$) are injective. Since an irreducible map is either injective or surjective, an induction implies that the other meshes do not contain projective vertices and furthermore the maps "parallel" to these maps are injective (resp. surjective).

As $\Omega^{-1}\Theta$ satisfies the same properties, the component necessarily contains a simple module. Thus we have that this has to be at a vertex belonging to $\text{ht } P$ since in all other vertices there either end injective irreducible maps or there start surjective irreducible maps. Depending on whether the simple module belongs to the bullet or to the other vertex we either have $M \twoheadrightarrow S$ or $S \hookrightarrow M$. Without loss of generality suppose the first possibility, otherwise dual arguments will yield the result.

Suppose that $\Omega\Theta = \Theta$. Then ΩS and ΩM are predecessors of $S \cong \Omega(P/\text{soc } P)$ (since this is the only simple module in this component). Hence there are two possibilities. Either $\Omega M \cong M$, a contradiction since for a Euclidean component Δ , we have that $\text{cx } \mathbb{Z}[\Delta] = 2$ by 9.4.1 or $\Omega M \cong \text{rad}(P)$, a contradiction, since the two modules have different dimension. Therefore $\text{soc } P =: T \not\cong S$. Since $\Omega\Theta$ is again a component isomorphic to $\mathbb{Z}[\tilde{A}_2]$ we have that one of the two predecessors of T , ΩS or ΩM is isomorphic to $\text{rad}(Q)$, where Q is the projective module attached to $\Omega\Theta$. If $\Omega S \cong \text{rad}(Q)$, then Q is the projective cover of S . Thus we have the following part of $\Omega\Theta$:



where N is the other direct summand of $\text{ht } Q$. Thus there is an inclusion $T \hookrightarrow N$. Otherwise $\Omega S \cong N$, i.e. $N \cong \text{rad}(P)(S)$, a contradiction to the fact that $P(S)$ cannot be attached to the component. □

9.5. Components of infinite Dynkin type

Up to here we have excluded all components of Euclidean tree class and all finite components except for the known cases. The case of infinite Dynkin components is more complicated. We mostly use results from Auslander-Reiten theory obtained by Karin Erdmann in her studies of group algebras. We can only prove partial results. One of the main obstructions is that there is no theory of rank varieties for quantum groups up to now.

Proposition 9.5.1 ([Erd95, Proposition 1.5], cf. [Far99b, Proposition 1.1]). *Let $g : B \rightarrow C$ be a surjective properly irreducible homomorphism, i.e. an irreducible map that is not*

CHAPTER 9. AUSLANDER-REITEN THEORY OF WILD ALGEBRAS

almost split, between two indecomposable modules such that $[C]$ belongs to a component of tree class \mathbb{A}_∞^∞ or \mathbb{D}_∞ . If V is an A -module such that $\tau_A(V) \oplus P \cong V$ for some projective modules P , then one of the following holds:

- (i) There is an embedding $\ker g \rightarrow V$, or
- (ii) every map $\phi : \ker g \rightarrow \Omega_A^{-1}(V)$ which does not factor through a projective module is a monomorphism.

For future reference we also record the dual version of this proposition:

Proposition 9.5.2 ([Far99b, Proposition 1.2]). *Let $g : C \rightarrow B$ be an injective properly irreducible homomorphism between two indecomposable modules such that $[C]$ belongs to a component of tree class \mathbb{A}_∞^∞ or \mathbb{D}_∞ . If V is an A -module such that $\tau_A^{-1}(V) \oplus P \cong V$ for some projective module P , then one of the following holds:*

- (i) There is a surjection $V \rightarrow \text{Coker } g$, or
- (ii) every map $\phi : \Omega_A V \rightarrow \text{Coker } g$ which does not factor through a projective module is an epimorphism.

The results in the remainder of this section are proven by similar methods as the corresponding statements for restricted enveloping algebras by Rolf Farnsteiner in [Far99b] and [Far00a].

The following definition has its origin in Dade's Lemma, i.e. 5.2.22 for group algebras. Compare also [ST11, Definition 3.3].

Definition 9.5.3. Let A be a Frobenius algebra. Then A is said to **admit an analogue of Dade's Lemma** for a subcategory \mathcal{M} of the module category if there is a family of subalgebras \mathcal{B} , such that $A : B$ is a β -Frobenius extension for some automorphism β (possibly depending on B) and $B \cong k[X]/(X^m)$ for all $B \in \mathcal{B}$ and $M \in \mathcal{M}$ is projective iff $M|_B$ is projective for all $B \in \mathcal{B}$.

Our basic example of an algebra admitting an analogue of Dade's Lemma is $U_\zeta(G_0)$ by a result of Drupieski ([Dru10, Theorem 4.1]).

Theorem 9.5.4. *Let A be a Frobenius algebra admitting an analogue of Dade's Lemma for a subcategory \mathcal{M} of the module category closed under kernels of irreducible morphisms. Let $\Theta \neq \mathbb{Z}[\mathbb{A}_\infty]$ be a non-periodic component of finite complexity of the stable Auslander-Reiten quiver of A contained in \mathcal{M} . Then the component is of complexity two.*

Proof. We have already seen that a non-periodic component Θ of finite complexity is of the form $\mathbb{Z}[\Delta]$, where Δ is a Euclidean or infinite Dynkin diagram. If Δ is Euclidean the statement was proven in Theorem 9.4.1. So let Δ be of infinite Dynkin type, not \mathbb{A}_∞ . For each $B \in \mathcal{B}$ denote the unique simple B -module by S_B and consider the A -module $M_B := A \otimes_B S_B$. We have $\Omega_B^2 S_B \cong S_B$ by the representation theory of $k[X]/X^n$. Applying the induction functor to a minimal projective resolution we get that there exist projective modules P, Q , such that $\Omega_A^2 M_B \oplus P \cong M_B \cong \Omega_A^{-2} M_B \oplus Q$. As S_B is a factor as well as

9.5. COMPONENTS OF INFINITE DYNKIN TYPE

a submodule of B , we have that M_B is a factor as well as a submodule of A as tensoring with A is an exact functor since A is a projective B -module (because $A : B$ is a Frobenius extension). So $\max\{\dim M_B, \dim \Omega_A M_B, \dim \Omega_A^{-1} M_B\} \leq \dim A$.

Choose $[E] \in \Theta$ to have two successors and two predecessors and $g : E \rightarrow N$ to be irreducible, where N is indecomposable. By choice of $[E]$, g is properly irreducible. Any irreducible map is either injective or surjective. Suppose g is surjective, then $M := \ker g$ is not projective, so there is $B \in \mathcal{B}$, such that $M|_B$ is not projective. Since $A : B$ is a Frobenius extension, we have that $\underline{\text{Hom}}_A(M, \Omega^{-1} M_B) \cong \text{Ext}_A^1(M, M_B) \cong \text{Ext}_B^1(M, S_B)$ by Corollary 1.6.11 and Proposition 5.1.7. But this is non-zero since M is not B -projective. From Proposition 9.5.1 (note that $\Omega^2 M \cong \tau M$ by Lemma 1.9.13) it follows now that $\dim E \leq \dim N + \dim A$. If g is injective, let $M := \text{Coker } g$ and choose $B \in \mathcal{B}$, such that M is not B -projective. As above it follows that the space $\underline{\text{Hom}}_A(\Omega_A M, M_B) \cong \text{Ext}_A^1(M, M_B) \cong \text{Ext}_B^1(M, S_B)$ is non-zero. Therefore we have that $\dim N \leq \dim E + \dim u_\zeta(\mathfrak{g})$.

By choice of $[E]$, there exists a walk $\tau(E) \rightarrow N \rightarrow E$ in Θ given by properly irreducible maps. We therefore have $\dim \Omega_A^2(E) = \dim \tau(E) \leq \dim E + 2 \dim A$. Repeated application of τ gives $\dim \Omega_A^{2n} E \leq \dim E + 2n \dim A$.

By choice of $[E]$, there exists a walk $\tau(E) \rightarrow N \rightarrow E$ in Θ given by properly irreducible maps. We therefore have $\dim \Omega_A^2(E) = \dim \tau(E) \leq \dim E + 2 \dim A$. Repeated application of τ gives $\dim \Omega_A^{2n} E \leq \dim E + 2n \dim A$.

As the module ΩE satisfies the same properties as E , since $\Omega_A \Theta \cong \Theta$, we conclude that there is some $C > 0$, such that $\dim \Omega_A^n E \leq Cn$ for all $n \geq 1$. Therefore $\text{cx } E \leq 2$. \square

In our situation this gives:

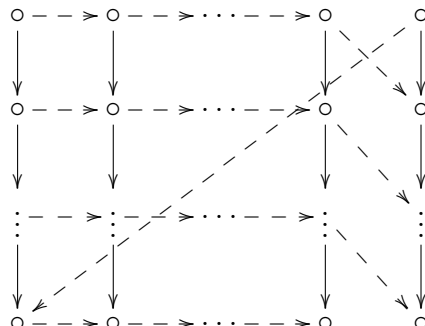
Proposition 9.5.5. *Let \mathfrak{g} be simple. Let $\Theta \neq \mathbb{Z}[\mathbb{A}_\infty]$ be a nonperiodic component of the stable Auslander-Reiten quiver of $u_\zeta(\mathfrak{g})$ containing the restriction of a $u_\zeta(\mathfrak{g})U_\zeta^0(\mathfrak{g})$ -module. Then $\text{cx } \Theta = 2$.*

Proof. By Corollary 5.2.9 the algebras $u_\zeta(f_\beta)$ satisfy $u_\zeta(\mathfrak{g}) : u_\zeta(f_\beta)$ being a Frobenius extension. The subcategory \mathcal{M} of restrictions of $u_\zeta(\mathfrak{g})U_\zeta^0(\mathfrak{g})$ -modules is closed under kernels of irreducible maps as we have proven in Lemma 6.1.2. Therefore Drupieski's Theorem (Theorem 5.2.22) implies the result. \square

For higher r this approach does not seem to work although Drupieski's Theorem holds more general for higher r , but the algebras to which one restricts are no longer Nakayama.

Example 9.5.6. Consider $U_\zeta(N_{\text{SL}(2)_1}) \cong k[X, Y]/(X^p, Y^\ell)$, an algebra to which Theorem 5.2.22 restricts in the case $r = 1$. Then this algebra does not admit an analogue of Dade's

Lemma. Consider the module M indicated by the following picture:



The circles represent the basis vectors of the module. The different arrows stand for the action of X and Y respectively. It is easy to see that this module is projective for $k[X]/(X^p)$ and $k[Y]/(Y^\ell)$, but not projective for $k[X, Y]/(X^p, Y^\ell)$, since it has a two-dimensional socle and is $p\ell$ -dimensional. For the other subalgebras B it can also be checked that either $A : B$ is not a Frobenius extension or M is projective for B .

Theorem 9.5.7. *Let \mathfrak{g} be simple. Let $\text{char } k$ be odd or zero and good for Φ . Assume $\ell \geq h$. Let Θ be a nonperiodic component of $\Gamma_s(u_\zeta(\mathfrak{g}))$ containing a $U_\zeta(\mathfrak{g})$ -module, where $u_\zeta(\mathfrak{g})$ is not tame. Then $\Theta \cong \mathbb{Z}[\mathbb{A}_\infty]$.*

Proof. This follows from the foregoing theorem in combination with Proposition 9.4.4. \square

We now proceed to the case of the small quantum groups associated to the Borel or nilpotent part. Therefore we need the following theorem due to Christopher M. Drupieski, which is a strengthened geometric version of Theorem 5.2.22.

Theorem 9.5.8 ([Dru10, Theorem 5.12]). *Let $\text{char } k$ be odd or zero and good for Φ . Assume $\ell \geq h$. Let M be a finite dimensional $u_\zeta(\mathfrak{b})$ -module and let $\alpha \in \Phi^+$. Then the root vector $f_\alpha \in \mathfrak{n}$ is an element of $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$ if and only if M is not projective for $u_\zeta(f_\alpha)$.*

Proposition 9.5.9. *Let $\text{char } k$ be odd or zero and good for Φ . Assume $\ell \geq h$ and $\mathfrak{n} \neq \mathfrak{n}_{\text{sl}_2}$. If M is a projective or periodic $u_\zeta(\mathfrak{n})$ -module, which is also a $u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g})$ -module, then $\dim M \equiv 0 \pmod{\ell}$.*

Proof. If M is projective, this directly follows from Theorem 5.2.22. If M is periodic, then $\text{Kdim } \mathcal{V}_{u_\zeta(\mathfrak{b})}(M) = 1$, hence $\text{Kdim } \mathbb{P}\mathcal{V}_{u_\zeta(\mathfrak{b})}(M) = 0$. Without loss of generality we may assume that M is indecomposable. Then by Corollary 3.3.7, $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$ is a line. Thus there exists $\alpha \in \Phi^+$, such that $f_\alpha \notin \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$. Hence by the foregoing theorem we have that $M|_{u_\zeta(f_\alpha)}$ is projective, hence $\dim M \equiv 0 \pmod{\ell}$. \square

Lemma 9.5.10 ([Far00a, Lemma 2.5], cf. [Erd96, Proof of Theorem 3]). *Let $\Theta \subseteq \Gamma_s(A)$ be a component of tree class \mathbb{A}_∞^∞ or \mathbb{D}_∞ for a symmetric algebra A . Then there exist vertices $[M], [N] \in \Theta \cup \Omega_A^{-1}\Theta$ and a properly irreducible map $\psi : M \rightarrow N$ such that*

- (i) ψ or $\Omega_A\psi$ are injective, or

9.5. COMPONENTS OF INFINITE DYNKIN TYPE

(ii) ψ or $\Omega_A^{-1}\psi$ are surjective

Proof. Consider the “upper” part of Θ (i.e. for \mathbb{D}_∞ all modules having two predecessors, such that all predecessors also have exactly two predecessors, for \mathbb{A}_∞ just make a horizontal cut somewhere), which “looks like” a component of type $\mathbb{Z}[\mathbb{A}_\infty]$, hence we can speak of the quasi-length of vertices in this part. Since Θ contains only finitely many projective meshes, there exists n_0 such that no module of quasilenhth $\geq n_0$ belongs to a projective mesh. Suppose the lemma fails. Pick a properly irreducible map $f : M \rightarrow E$ with M of quasilenhth $\geq n_0$ and E of quasilenhth one bigger than M , i.e. an arrow pointing toward infinity. Since the lemma fails, either $\tau_A^n f$ is surjective for every $n \in \mathbb{Z}$ or $\tau_A^n f$ is injective for every $n \in \mathbb{Z}$, and the other case holds for $\tau_A^n \Omega_A f$. Hence we may assume that f is surjective. By the mesh relations we now have that the chain of arrows pointing toward infinity starting in M only consists of epimorphisms, a contradiction. \square

The following theorem is proven analogously to the corresponding statement for local restricted enveloping algebras (see [Erd96]), which was proven in a similar way as the corresponding statement for group algebras, which were both proven by Karin Erdmann.

Theorem 9.5.11. *Let $\text{char } k$ be odd or zero and good for Φ . Assume $\ell \geq h$. The category of $u_\zeta(\mathfrak{n})$ -modules, that are also $u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g})$ -modules does not contain any components of type $\mathbb{Z}[\mathbb{A}_\infty]$ or $\mathbb{Z}[\mathbb{D}_\infty]$.*

Proof. The foregoing lemma shows that if we have a component Θ of type \mathbb{A}_∞ or \mathbb{D}_∞ , then there exists an irreducible map ψ corresponding to an arrow in Θ , such that ψ and $\Omega^{-1}\psi$ are surjective or ψ and $\Omega\psi$ are injective.

Consider first the case that they are surjective. Hence there is a non-split sequence which is not almost split $0 \rightarrow M \rightarrow E \xrightarrow{\psi} N \rightarrow 0$. Let α be such that $M|_{u_\zeta(f_\alpha)}$ is not projective (This is possible by Theorem 5.2.22). Let $M|_{u_\zeta(f_\alpha)} \cong M_1^{n_1} \oplus \dots \oplus M_\ell^{n_\ell}$ where M_i is the indecomposable $u_\zeta(f_\alpha)$ -module of dimension i and $n_i \geq 0$. Then $\sum_{i=1}^{\ell-1} n_i \neq 0$ as M is not projective. Let $V_i := u_\zeta(\mathfrak{n}) \otimes_{u_\zeta(f_\alpha)} M_i$. Induction of the sequence $0 \rightarrow M_{\ell-i} \rightarrow u_\zeta(f_\alpha) \rightarrow M_i \rightarrow 0$ yields the sequence $0 \rightarrow V_{\ell-i} \rightarrow u_\zeta(\mathfrak{n}) \rightarrow V_i \rightarrow 0$ since $u_\zeta(\mathfrak{n})$ is free over $u_\zeta(f_\alpha)$. Therefore V_i has simple top and socle (since $u_\zeta(\mathfrak{n})$ is local and selfinjective) and V_i is Ω -periodic.

We will show next that $\text{soc } M$ is also simple. Our assumption shows that for the Ω -periodic module V_1 , one of Proposition 9.5.1 (i) or (ii) must hold. If Proposition 9.5.1 (i) holds, then there is a monomorphism $M \rightarrow V_1$. In this case since $\text{soc } V_1$ is simple, so is $\text{soc } M$. Otherwise Proposition 9.5.1 (ii) holds. By Proposition 5.1.7 we have

$$\underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V_i) \cong \text{Ext}_{u_\zeta(\mathfrak{n})}^1(M, V_i) \cong \text{Ext}_{u_\zeta(f_\alpha)}^1(M, k),$$

which is non-zero since $M|_{u_\zeta(f_\alpha)}$ is non-projective. Hence if we take a non-zero representative this has to be a monomorphism. Moreover $\text{soc } \Omega^{-1}V_i \cong \text{soc } V_{\ell-i}$ is simple and so is $\text{soc } M$. Concluding and taking into account that $u_\zeta(\mathfrak{n})$ is local we have a minimal injective resolution of the form $0 \rightarrow M \rightarrow u_\zeta(\mathfrak{n}) \rightarrow u_\zeta(\mathfrak{n}) \rightarrow \dots$

Hence the complexity of M is ≤ 1 and therefore M is periodic; in particular $\dim M \equiv 0$

mod ℓ by Proposition 9.5.9

Now $\dim M + \dim \Omega^{-1}M = \dim u_\zeta(\mathfrak{n})$ by the minimal injective resolution. We may assume that $\dim M \geq \frac{1}{2} \dim u_\zeta(\mathfrak{n})$, otherwise replace M by $\Omega^{-1}M$, which is possible since by the choice of ψ we can do the same argument with $\Omega^{-1}\psi$. Now

$$\dim V_1 = \frac{1}{\ell} \dim u_\zeta(\mathfrak{n}) < \frac{1}{2} \dim u_\zeta(\mathfrak{n}) \leq \dim M.$$

So there is no monomorphism $M \rightarrow V_1$. Therefore Proposition 9.5.1 (ii) holds. By the above argument we have that $\dim \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V_1) \geq \sum_{i=1}^{\ell-1} n_i$. This must be at least 2; otherwise we would have that $M|_{u_\zeta(f_\alpha)}$ has a unique non-projective summand and the dimension of M would not be divisible by ℓ . Hence there are $\phi_1, \phi_2 \in \text{Hom}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V)$ with $[\phi_1], [\phi_2] \in \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V_1)$ linearly independent. Since Proposition 9.5.1 (ii) holds, ϕ_1 and ϕ_2 must be monomorphisms.

We know that $\text{soc } M$ is simple, hence there is some $c \in k$ such that $\phi_1 - c\phi_2$ is not a monomorphism, and since Proposition 9.5.1 holds, $\phi_1 - c\phi_2$ factors through a projective module, a contradiction.

The case that ψ and $\Omega\psi$ are both injective is dual. Therefore we only sketch the proof, if not given references are the same as above: Take such ψ . Then there is a non-split sequence which is not almost split $0 \rightarrow N \xrightarrow{\psi} E \rightarrow M \rightarrow 0$. Take α , such that $M|_{u_\zeta(f_\alpha)}$ is not projective. If $M|_{u_\zeta(f_\alpha)} \cong M_1^{n_1} \oplus \cdots \oplus M_\ell^{n_\ell}$, then $\sum_{i=1}^{\ell-1} n_i \neq 0$ since M is not projective. Define the Ω -periodic module $V := (u_\zeta(\mathfrak{n}) \otimes_{u_\zeta(f_\alpha)} M_1)^*$ and $W := V$ or $\Omega^{-1}V$. Then for $i = 1, \ell - 1$ respectively we have

$$\underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(W, M) \cong \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M^*, W^*) \cong \underline{\text{Hom}}_{u_\zeta(f_\alpha)}(M^*, M_i) \cong \underline{\text{Hom}}_{u_\zeta(f_\alpha)}(M_i^*, M),$$

which has dimension $\geq \sum_{i=0}^{\ell-1} n_i$. Furthermore $\text{top } W$ is simple as

$$\text{Hom}_{u_\zeta(\mathfrak{n})}(W, k) \cong \text{Hom}_{u_\zeta(\mathfrak{n})}(k^*, W^*) \cong \text{Hom}_{u_\zeta(f_\alpha)}(k, M_1),$$

which is one-dimensional. In particular V is an indecomposable periodic module.

We have that Proposition 9.5.2 (i) or (ii) holds. If (i) holds, then there is an epimorphism $V_1 \rightarrow M$ and therefore $\text{top } V_1$ simple implies that $\text{top } M$ is simple. If (ii) holds, take a non-zero element of $\underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(\Omega V_1, M)$. This has to be an epimorphism and $\text{top } \Omega V_1$ simple implies that $\text{top } M$ is simple.

Again we want to deduce that all $\text{top } \Omega^j M$ are simple. We have a short exact sequence $0 \rightarrow \Omega^j N \rightarrow \Omega^j E \oplus P \rightarrow \Omega^j M \rightarrow 0$, where P is projective. If $P = 0$, then $\Omega^j \psi$ is a monomorphism and the above argument yields $\text{top } \Omega^j M$ simple. If $P \neq 0$, then the sequence is of the form $0 \rightarrow \text{rad}(P) \rightarrow \text{rad}(P)/\text{soc}(P) \oplus P \rightarrow P/\text{soc}(P) \rightarrow 0$. Therefore $\Omega^j \psi$ is an epimorphism and $\text{top } \Omega^j M \cong \text{top } P$, which is simple.

Therefore we have a minimal projective resolution $\cdots \rightarrow u_\zeta(\mathfrak{n}) \rightarrow u_\zeta(\mathfrak{n}) \rightarrow M \rightarrow 0$. Hence M is periodic and therefore $\dim M \equiv 0 \pmod{\ell}$. Now $\dim M + \dim \Omega M = \dim u_\zeta(\mathfrak{n})$, thus assume without loss of generality by the choice of ψ that $\dim M \geq 1/2 \dim u_\zeta(\mathfrak{n})$. Then

$$\dim V_1 = \frac{1}{\ell} \dim u_\zeta(\mathfrak{n}) < \frac{1}{2} \dim u_\zeta(\mathfrak{n}) \leq \dim M.$$

So there is no epimorphism $V_1 \rightarrow M$. Therefore Proposition 9.5.2 (ii) must hold. Since $\dim \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(\Omega V_1, M) \geq 2$ we have ϕ_1, ϕ_2 such that $[\phi_1], [\phi_2] \in \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(\Omega V_1, M)$ are linearly independent. Since (ii) holds, they are epimorphisms. But $\text{top } M$ is simple and therefore there exists $c \in K$ with $\phi_1 - c\phi_2$ not an epimorphism. Therefore $\phi_1 - c\phi_2$ factors through a projective module, a contradiction. \square

As a corollary we also get the corresponding statement for the Borel part:

Theorem 9.5.12. *The category of gradable $u_\zeta(\mathfrak{b})$ -modules does not contain any components of type $\mathbb{Z}[\mathbb{A}_\infty]$ or $\mathbb{Z}[\mathbb{D}_\infty]$.*

Proof. In the following commutative diagram indecomposable modules are mapped to indecomposable modules and Auslander-Reiten sequences are mapped to Auslander-Reiten sequences by the vertical arrows by Lemma 6.1.2. Therefore also by the horizontal arrow.

$$\begin{array}{ccc} \text{mod } u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g}) & \xlongequal{\quad} & \text{mod } u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \text{mod } u_\zeta(\mathfrak{b}) & \xrightarrow{\text{res}} & \text{mod } u_\zeta(\mathfrak{n}) \end{array}$$

Thus the existence of an Auslander-Reiten sequence with three indecomposable direct summands would be preserved and therefore the non-existence of such sequences for $u_\zeta(\mathfrak{n})$ implies the same for $u_\zeta(\mathfrak{b})$, i.e. there are no components of type $\mathbb{Z}[\mathbb{D}_\infty]$. Furthermore as for every component Θ containing a gradable module there is an Auslander-Reiten sequence in Θ with indecomposable middle term for $u_\zeta(\mathfrak{n})$ the same has to hold for $u_\zeta(\mathfrak{b})$ and thus the case of components of type $\mathbb{Z}[\mathbb{A}_\infty]$ can be excluded. \square

Remark 9.5.13. The same argument also works in the case of gradable modules for the restricted enveloping algebra of a Borel subalgebra.

9.6. The one and only component

In this section we want to prove some statements about the most important component $\mathbb{Z}[\mathbb{A}_\infty]$. We have already proven that many fundamental modules, e.g. simple modules for $u_\zeta(\mathfrak{g})$ lie in such components if $\mathfrak{g} \neq \mathfrak{sl}_2$. The question remains where they occur. Our proofs follow the lines of [FR11]. We use the following statement by Shigeto Kawata:

Theorem 9.6.1 ([Kaw97, Theorem 1.5]). *Let A be a symmetric algebra and Θ be an Auslander-Reiten component of A with tree class \mathbb{A}_∞ . If Θ contains a simple module S , which is not **quasi-simple**, i.e. having two predecessors and two successors in this component, then there exist simple modules T_1, \dots, T_n , such that $P(T_i)$ are uniserial and*

have the following Loewy series:

$$P(T_1) : \begin{pmatrix} T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_1 \end{pmatrix}, P(T_2) : \begin{pmatrix} T_2 \\ T_1 \\ S \\ T_n \\ \vdots \\ T_2 \end{pmatrix}, \dots, P(T_n) : \begin{pmatrix} T_n \\ \vdots \\ T_1 \\ S \\ T_n \end{pmatrix}.$$

Theorem 9.6.2. *Let \mathfrak{g} be a simple Lie algebra. If a simple module for $U_\zeta(G_r)$ belongs to a component of type $\mathbb{Z}[\mathbb{A}_\infty]$, then it is quasi-simple.*

Proof. Let S be a simple module contained in some component $\Theta \cong \mathbb{Z}[\mathbb{A}_\infty]$. Suppose S is not quasi-simple. Then since $U_\zeta(G_r)$ is symmetric, by the foregoing theorem there exist simple modules T_0, T_1, \dots, T_n , such that $T_0 \cong S$, the projective cover of T_i is uniserial of length $n + 2$ with $\text{top rad}(P)(T_i) \cong T_{i-1}$. By the same token T_i has multiplicity one in $P(T_i)$ for all i .

If $n \geq 2$, then this yields the contradiction $\dim \text{Ext}_{U_\zeta(G_r)}^1(T_1, T_2) = 0$ while by Lemma 4.3.13 we have $\dim \text{Ext}_{U_\zeta(G_r)}^1(T_2, T_1) = 1$. Therefore $n = 1$, i.e. $P(T_1)$ has length three. By [Dru09, Proposition B.9], the projective cover of S has a filtration by baby Verma modules. Hence one baby Verma module has to be simple or projective. But by the same token this implies that it is simple and projective. Hence S is the Steinberg module for $U_\zeta(G_r)$, a contradiction. \square

Corollary 9.6.3. *Let \mathfrak{g} be a simple Lie algebra. Let S be a simple non-projective $U_\zeta(G_r)$ -module in a component of type $\mathbb{Z}[\mathbb{A}_\infty]$. Then $\text{ht}(P(S))$ is indecomposable.*

Proof. Since $U_\zeta(G_r)$ is symmetric, we have $\text{soc } P(S) \cong S$. Hence the following sequence is the standard almost split sequence originating in $\text{rad}(P(S))$:

$$0 \rightarrow \text{rad}(P(S)) \rightarrow P(S) \oplus \text{ht } P(S) \rightarrow P(S)/S \rightarrow 0.$$

The autoequivalence on the stable module category Ω induces an automorphism on the stable Auslander-Reiten quiver. Hence S and $\Omega^{-1}S$ have the same number of non-projective predecessors. By the foregoing theorem we thus have that $\text{ht } P(S)$ is indecomposable. \square

Theorem 9.6.4. *Let \mathfrak{g} be a simple Lie algebra. Let $\ell > 1$ be an odd integer not divisible by 3 if Φ is of type \mathbb{G}_2 . Furthermore for $\text{char } k = 0$ assume that ℓ is good for Φ and that $\ell > 3$ if Φ is of type \mathbb{B}_n or \mathbb{C}_n . For $\text{char } k = p > 0$ assume that p is good for Φ and that $\ell > h$. Then there is only one simple $U_\zeta(G_r)$ -module in each Auslander-Reiten component of type $\mathbb{Z}[\mathbb{A}_\infty]$.*

Proof. We first prove the case of $r = 0$ and then combine this result with the classical case to get the result for all r . If $r = 0$, suppose there is another simple module T in Θ . Then S and T are both quasi-simple by the foregoing considerations. Hence they lie

9.6. THE ONE AND ONLY COMPONENT

in the same τ -orbit, without loss of generality let $\Omega_{U_\zeta(G_0)}^{2n} T \cong S$. Accordingly we have $\dim \text{Ext}_{U_\zeta(G_0)}^{2n}(T, T) = \dim \text{Hom}_{U_\zeta(G_0)}(S, T) = 0$. Hence T is projective by Proposition 3.2.7 and Theorem 3.2.16. If $r \neq 0$, then by the Steinberg tensor product theorem, simple modules are tensor products of a simple module for $U_\zeta(G_0)$ and a simple module for $\text{Dist}(G_r)$. Let $\bar{T} \otimes T$ and $\bar{S} \otimes S$ be such simple modules. Then $\Omega_{U_\zeta(G_r)}^{2n} \bar{S} \otimes S \cong \bar{T} \otimes T$ implies that $T^{\dim \bar{T}} \cong \Omega_{U_\zeta(G_0)}^{2n} S^{\dim \bar{S}} \oplus P$, where P is projective. In particular, since $\Omega_{U_\zeta(G_0)}^{2n} S$ is indecomposable we have $\Omega_{U_\zeta(G_0)}^{2n} S \cong T$. Hence $S \cong T$ is a projective or a periodic module for $U_\zeta(G_0)$, hence it is the Steinberg module for $U_\zeta(G_0)$. But then the result follows from the classical case considered in [FR11, Theorem 2.6], because these modules form a block ideal equivalent to $\text{mod Dist}(G_r)$. \square

CHAPTER 9. AUSLANDER-REITEN THEORY OF WILD ALGEBRAS

A. List of Dynkin diagrams

A.1. Finite Dynkin diagrams

$$A_n: \circ - \circ - \dots - \circ - \circ, n \geq 1$$

$$B_n: \circ \Leftarrow \circ - \dots - \circ - \circ, n \geq 2$$

$$C_n: \circ \Rightarrow \circ - \dots - \circ - \circ, n \geq 3$$

$$D_n: \begin{array}{c} \circ \\ \diagdown \\ \circ - \dots - \circ - \circ \\ \diagup \\ \circ \end{array}, n \geq 4$$

$$E_6: \begin{array}{c} \circ \\ | \\ \circ - \circ - \circ - \circ - \circ \end{array}$$

$$E_7: \begin{array}{c} \circ \\ | \\ \circ - \circ - \circ - \circ - \circ - \circ \end{array}$$

$$E_8: \begin{array}{c} \circ \\ | \\ \circ - \circ - \circ - \circ - \circ - \circ - \circ \end{array}$$

$$F_4: \circ - \circ \Rightarrow \circ - \circ$$

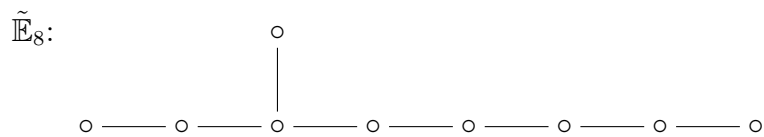
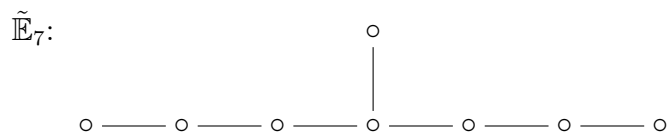
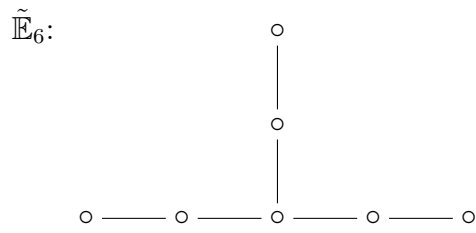
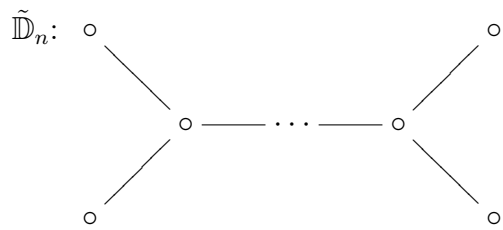
$$G_2: \circ \Rightarrow \circ$$

For each graph the number of vertices is given by the index.

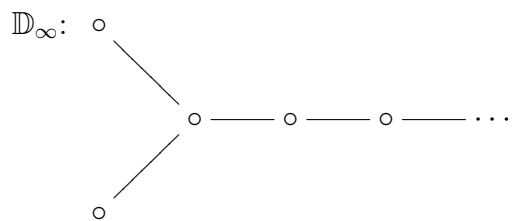
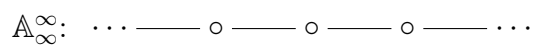
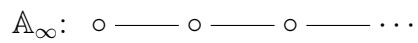
A.2. Simply-laced Euclidean diagrams

$$\tilde{A}_n: \begin{array}{c} \circ - \circ - \dots - \circ - \circ \\ \diagdown \quad \diagup \\ \circ \end{array}$$

CHAPTER A. LIST OF DYNKIN DIAGRAMS



A.3. Infinite Dynkin diagrams



Nomenclature

- $(-, -)$ normalized form, page 6
- $(-, -)_{\mathfrak{g}^*}$ Killing form on dual space, page 5
- $- \cup -$ cup product on Hochschild cohomology, page 54
- $- \cup -$ cup product, page 36
- $-\hat{\cup}-$ modified cup product, page 40
- $-Y-$ Yoneda product, page 35
- $[t]_{\alpha}!$ quantum factorial, page 59
- $[t]_{\alpha}$ quantum integer, page 59
- Ad adjoint action, page 65
- ad adjoint map, page 5
- Band** bands, page 23
- $\begin{bmatrix} t \\ s \end{bmatrix}$ quantum binomial coefficient, page 59
- β^{\vee} coroot corresponding to β , page 6
- χ_{λ} quantum character, page 66
- cx complexity, page 43
- Δ coproduct, page 25
- dim k -dimension of a vector space, page 4
- \dim_t graded dimension, page 68
- $\text{Dist}(G)$ algebra of distributions, page 28
- ℓ odd integer, $3 \nmid \ell$ for \mathbb{G}_2 , page 3
- η unit of an algebra, page 25
- Ext Ext-groups, page 11
- Fr_{ζ} quantum Frobenius homomorphism, page 62
- γ growth rate, page 43
- GL_n general linear group, page 18
- $\text{gr}(R)$ associated graded of R , page 16

NOMENCLATURE

- ht heart, page 9
- ht height, page 68
- \int_A^l space of left integrals, page 27
- $\text{Irr}(X, Y)$ bimodule of irreducible maps, page 19
- κ Killing form, page 5
- $\text{top}(M)$ top of M , page 9
- λ_i fundamental dominant weight, page 7
- $\langle -, - \rangle$ (non-bilinear) form, page 6
- \mathbb{A}^n affine space, page 15
- \mathbb{A}_∞ infinite Dynkin diagram of type A_∞ , page 126
- \mathbb{A}_∞^∞ infinite Dynkin diagram of type A_∞^∞ , page 126
- \mathbb{A}_n Dynkin diagram of type A , page 125
- \mathbb{B}_n Dynkin diagram of type B , page 125
- \mathbb{C} complex numbers, page 4
- \mathbb{C}_n Dynkin diagram of type C , page 125
- \mathbb{D}_∞ infinite Dynkin diagram of type D_∞ , page 126
- \mathbb{D}_n Dynkin diagram of type D , page 125
- \mathbb{E}_r Dynkin diagram of type E , page 125
- \mathbb{F}_4 Dynkin diagram of type F , page 125
- \mathbb{G}_2 Dynkin diagram of type G , page 125
- \mathbb{N} natural numbers (including zero), page 4
- \mathbb{N}_1 natural numbers (excluding zero), page 4
- \mathbb{P}^n projective space, page 16
- \mathbb{Q} rational numbers, page 4
- $\text{U}_\mathbb{Q}(\mathfrak{g})$ quantized enveloping algebra, page 59
- \mathbb{Z} integers, page 4
- $\mathbb{Z}[T]$ biquiver constructed from a tree T , page 21
- \mathcal{D} Dynkin and extended Dynkin diagrams, page 56
- \mathcal{N}_A Nakayama functor, page 11
- $\mathcal{O}_k(\mathfrak{g}, \alpha)$ infinite dimensional root subalgebra, page 72
- \mathcal{S}_A complete set of representatives for the isomorphism classes of simple A -modules, page 8

- $\mathcal{U}_A(\mathfrak{g})$ quantized enveloping algebra over A , De Concini-Kac form, page 63
- $\mathcal{U}_k(\mathfrak{g})$ quantized enveloping algebra over k , De Concini-Kac form, page 63
- $\mathcal{V}_A(M', M)$ relative support variety, page 43
- $\mathcal{V}_A(M)$ support variety, page 43
- $\mathcal{V}_H(M, N)$ support variety defined via Hochschild cohomology, page 55
- \mathcal{Z} ℓ -center of quantized enveloping algebra, page 63
- \mathfrak{g} finite dimensional semisimple complex Lie algebra, page 4
- \mathfrak{h} Cartan subalgebra, page 5
- A ring of Laurent polynomials, page 3
- $\max\text{Spec}(R)$ set of maximal ideals, page 15
- μ multiplication map of an algebra, page 25
- ν Nakayama automorphism, page 10
- $\Omega_A M$ syzygy functor, page 13
- \otimes tensor product over a field k , page 4
- PGL_n quotient of general linear group by its center, page 18
- Φ roots, page 6
- Φ^+ positive roots, page 6
- Φ^- negative roots, page 6
- Φ^\vee coroots, page 6
- Φ_λ roots orthogonal to $\lambda + \rho$ modulo ℓ , page 7
- Φ_M map $H^{ev}(A, k) \rightarrow \text{Ext}^\bullet(M, M)$ induced by tensoring, page 43
- Π simple roots, page 6
- π Frobenius homomorphism, page 10
- ψ_M isomorphism $M \otimes k \cong M$ or $k \otimes M \cong M$, page 4
- $\text{rad}^2(M)$ radical squared of M , page 9
- $\text{rad}(M)$ radical of M , page 9
- $\text{rad}(X, Y)$ homomorphisms in the radical, page 19
- ρ half sum of positive roots, page 7
- $\sigma_{M, N}$ isomorphism $N^* \otimes M^* \cong (M \otimes N)^*$, page 26
- SL_n special linear group, page 18
- $\text{soc}(M)$ socle of M , page 9
- $\text{soc}^2(M)$ socle squared of M , page 9

NOMENCLATURE

- $\text{Spec}(R)$ set of prime ideals, page 15
 St_ℓ Steinberg module for $u_\zeta(\mathfrak{g})$, page 67
String strings, page 23
 \subset strict inclusion, where strictness is not relevant, page 4
 \subseteq inclusion, page 4
 \subsetneq strict inclusion, page 4
 $\tau_A(M)$ Auslander-Reiten translation, page 20
 $\tau_{X,Y}$ flip, page 26
 $\Theta(d)$ modules with dimension smaller or equal d in a component Θ , page 55
 $\tilde{\mathbb{A}}_n$ Euclidean diagram of type A , page 125
 $\tilde{\mathbb{D}}_n$ Euclidean diagram of type D , page 126
 $\tilde{\mathbb{E}}_r$ Euclidean diagram of type E , page 126
 $\underline{\text{Hom}}(M, N)$ stable homomorphism space, page 10
 $\underline{\text{mod}}(A)$ stable module category, page 10
 ε counit, page 25
 ζ primitive ℓ -th root of unity, page 3
 ζ_l left modular function, distinguished group-like element, page 27
 $A * G$ skew group algebra, page 33
 $A//B$ Hopf quotient, page 29
 $A\#H$ smash product, page 33
 $a_{(1)} \otimes a_{(2)}$ Sweedler notation, page 26
 b^* dual basis vector, page 4
 $d_\alpha = (\alpha, \alpha)/2$, page 59
 d_W subadditive $\underline{\text{Hom}}$ -functions, page 98
 d_W^i subadditive $\underline{\text{Ext}}$ -functions, page 98
 $E_\gamma^{(m)}$ divided power of a PBW basis element, page 61
 e_i path of length 0 corresponding to the vertex i , page 9
 E_γ PBW basis element, page 60
 $F_\gamma^{(m)}$ divided power of a PBW basis element, page 61
 F_γ PBW basis element, page 60
 G_r r -th Frobenius kernel of an algebraic group, page 18
 G_x stabilizer of x , page 17

- h Coxeter number, page 7
- $H^0(M)$ induced module, page 66
- $H^\bullet(A, k)$ cohomology ring, page 42
- $H^{ev}(A, k)$ even cohomology ring, page 42
- $HH^*(A)$ Hochschild cohomology, page 54
- $I(-)$ vanishing ideal, page 16
- $I(M)$ injective hull of M , page 8
- k algebraically closed field, page 3
- $k[-]$ coordinate ring, page 16
- K_α generator of the quantized enveloping algebra, page 59
- kG group algebra, page 27
- kQ path algebra, page 9
- kQ^+ ideal generated by the arrows, page 9
- $L_\zeta(\lambda)$ simple module for $U_\zeta(\mathfrak{g})$, page 66
- L_y Carlson module, page 50
- M^A space of invariants, page 26
- p characteristic of k , page 3
- $P(M)$ projective cover of M , page 8
- q an indeterminate, page 59
- $q_\alpha = q^{d_\alpha}$, page 59
- S antipode, page 25
- s starting point function, page 9
- t terminal point function, page 9
- $T(A)$ trivial extension, page 11
- $U_{\mathbf{A}}(\mathfrak{g})$ quantized enveloping algebra over \mathbf{A} , Lusztig form, page 61
- $U_\zeta(\mathfrak{g})$ quantized enveloping algebra, Lusztig form, page 61
- $u_\zeta(\mathfrak{g})$ small quantum group, page 62
- $u_\zeta(f_\alpha)$ root subalgebra, page 72
- $U_\zeta(G_r)$ r -th Frobenius Lusztig kernel, page 62
- $U_k(\mathfrak{g})$ quantized enveloping algebra over k , Lusztig form, page 61
- $u_k(\mathfrak{g})$ restricted quantum group, page 62
- V^* k -dual of a vector space, page 4

NOMENCLATURE

- W Weyl group, page 7
 X weight lattice, page 7
 X^+ dominant weights, page 7
 X_ℓ ℓ -restricted weights, page 7
 $Z(-)$ vanishing set, page 15
 $Z(-)$ vanishing set, page 17
 $Z_r(\lambda)$ baby Verma module, page 68
 $l(M)$ length of a module M , page 104

Index

- T -automorphism, 62
- Ext-groups, 11
- (Fg)-selfinjective algebra, 55
- (fg)-Hopf algebra, 42

- adjoint action, 68
- adjoint map, 5
- admissible group action, 21
- admissible ideal, 9
- affine variety, 15
- algebra, 8
- algebraic group, 17
- almost split sequence, 20
- antipode, 25
- arrow, 9
- associated graded, 16
- augmented algebra, 25
- Auslander-Reiten formula, 20
- Auslander-Reiten quiver, 21
- Auslander-Reiten sequence, 20

- baby Verma module, 70
- band, 23
- base, 6
- basic, 9
- bimodule, 8
- Borel subgroup, 17
- Borel's fixed point theorem, 17
- Brauer-Thrall conjecture I, 22
- Brauer-Thrall conjecture II, 22

- Carlson module, 50
- Cartan matrix, 6
- Cartan subalgebra, 5
- character, 28
- cohomology ring, 42

- complete set of primitive orthogonal idempotents, 10
- complexity, 43
- conical, 17
- convolution product, 26
- coordinate ring, 16
- coproduct, 25
- counit, 25
- Coxeter element, 7
- Coxeter number, 7
- cup product, 36
- cup product on Hochschild cohomology, 55

- distinguished group-like element, 26
- distributions, 28
- domestic, 91
- dominant weights, 7
- Dynkin diagram, 6

- enveloping algebra, 55
- even cohomology ring, 42
- extension, 12

- filtered ring, 16
- flip, 26
- Frobenius algebra, 10
- Frobenius extension, 75
 - free, 75
- Frobenius homomorphism, 10, 18, 76
- Frobenius kernel, 18
- Frobenius-Lusztig kernel, 64
- fundamental dominant weights, 7

- good integer, 73
- graded dimension, 71
- graded ring, 16

INDEX

- Grothendieck spectral sequence, 14
- group algebra, 27
- group of quantum characters, 68
- growth rate, 43

- heart, 9
- highest weight category, 83
- Hochschild cohomology ring, 55
- homogeneous ideal, 16
- Hopf algebra, 25
- Hopf quotient, 29
- Horseshoe lemma, 13

- ideal, 4
- induction functor, 68
- injective, 8
- injective hull, 8
- injective resolution, 8
- integrable, 68
- invariants, 26
- irreducible, 16
- irreducible character, 28
- irreducible morphism, 19

- Künneth formula, 15
- Killing form, 5
- Krull dimension, 16
- Krull-Remak-Schmidt, 19

- left integral, 26
- left modular function, 26
- Lie algebra, 4
 - abelian, 4
 - semisimple, 4
 - simple, 4
- Lie algebra of an algebraic group, 18
- Lie bracket, 4

- Maschke's Theorem, 27
- minimal injective resolution, 8
- minimal projective resolution, 8
- module, 8
- module algebra, 32
- Morita equivalent, 9

- Nakayama algebra, 22
- Nakayama automorphism, 10
- Nakayama functor, 11
- Nakayama's Lemma, 15
- normal Hopf subalgebra, 29
- normalized form, 6

- orbit formula, 18
- ordinary quiver, 10

- parabolic, 17
- path algebra, 9
- periodic, 104
- Poincaré-Birkhoff-Witt basis, 62
- projective, 8
- projective cover, 8
- projective resolution, 8
- projective space, 16
- projective variety, 16

- quantized enveloping algebra, 61
 - De Concini-Kac form, 65
 - Lusztig form, 63
- quantum Frobenius homomorphism, 64
- quasi-simple, 125
- quiver, 9

- radical, 9
- radical of an algebraic group, 17
- representation of a group, 27
- representation of an algebraic group, 18
- representation type, 91
- representation-finite, 91
- restricted quantum group, 64
- restricted weights, 7
- Riedtmann Struktursatz, 21
- root subalgebra, 76
- root system, 5
- roots, 6
 - co-, 6
 - negative, 6
 - positive, 6
 - simple, 6

- Schanuel's Lemma, 9

- Schur's Lemma, 9
- selfinjective, 10
- semisimple, 8
- semisimple algebraic group, 17
- short exact sequence of Hopf algebras, 29
- simple, 8
- skew group algebra, 33
- small quantum group, 64
- smash product, 33
- Snake lemma, 12
- socle, 9
- special biserial, 22
- stabilizer, 17
- stable Auslander-Reiten quiver, 21
- stable module category, 10
- starting point, 9
- Steinberg module, 70
- Steinberg tensor product theorem, 69
- string, 23
- string algebra, 23
- subadditive function, 102
- support variety, 43, 55
- Sweedler notation, 26
- symmetric algebra, 10
- syzygy functor, 13

- tame, 91
- terminal point, 9
- top, 9
- tree class, 22
- trivial extension, 11
- trivial module, 26

- vanishing ideal, 16
- vanishing set, 15, 16
- vertex, 9

- weak Hopf sequence, 29
- weakly symmetric algebra, 10
- Webb's Theorem, 56
- weight lattice, 7
- weight space, 68
- Weyl group, 7
- wild, 91

- Yoneda product, 35
- Zariski topology, 15, 17

INDEX

Bibliography

- [AM69] Michael Francis Atiyah and I. G. MacDonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co. Reading, Mass.-London-Don, Mills, Ont., 1969.
- [AM11] Henning Haahr Andersen and Volodymyr Mazorchuk. Category \mathcal{O} for quantum groups. preprint, June 2011.
- [APW91] Henning Haahr Andersen, Patrick Polo, and Kexin Wen. Representations of quantum algebras. *Inventiones Mathematicae*, 104:1–59, 1991.
- [APW92] Henning Haahr Andersen, Patrick Polo, and Kexin Wen. Injective modules for quantum algebras. *American Journal of Mathematics*, 114(3):571–604, 1992.
- [AR96] Maurice Auslander and Idun Reiten. $D\text{Tr}$ -periodic modules and functors. In *Representation theory of algebras (Cocoyoc, 1994)*, volume 18 of *CMS Conf. Proc.*, pages 39–50. Amer. Math. Soc., Providence, RI, 1996.
- [Ari07] Yusuke Arike. Symmetric linear functions of the restricted quantum group $\overline{U}_q\mathfrak{sl}_2(\mathbb{C})$. preprint, June 2007.
- [ARS95] Maurice Auslander, Idun Reiten, and Sverre Olaf Smalø. *Representation Theory of Artin Algebras*. Cambridge University Press, 1995.
- [ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras, Volume I: Techniques of Representation Theory*. Cambridge University Press, 2006.
- [AW92] Henning Haahr Andersen and Kexin Wen. Representations of quantum algebras. The mixed case. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 427:35–50, 1992.
- [Bar85] Donald W. Barnes. Spectral sequence constructors in algebra and topology. *Memoirs of the American Mathematical Society*, 53(317):viii+174, 1985.
- [BB98] Sheila Brenner and Michael Charles Richard Butler. Wild subquivers of the Auslander-Reiten quiver of a tame algebra. In *Trends in the representation theory of finite dimensional algebras (Seattle, WA, 1997)*, volume 229 of *Contemporary Mathematics*, pages 29–48. American Mathematical Society, Providence, RI, 1998.

BIBLIOGRAPHY

- [BD77] Vitaly M. Bondarenko and Yurii Anatolievich Drozd. The representation type of finite groups. *Zapiski Naučnyh Seminarov Leningradskogo Otdelenija Matematičeskogo Instituta im . V. A. Steklova Akademii Nauk SSSR (LOMI)*, 71:24–41, 282, 1977. Modules and representations.
- [BE11] Petter Andreas Bergh and Karin Erdmann. The stable Auslander-Reiten quiver of a quantum complete intersection. *Bulletin of the London Mathematical Society*, 43(1):79–90, 2011.
- [Ben91] David John Benson. *Representations and cohomology, II: Cohomology of groups and modules*. Cambridge University Press, 1991.
- [Ben95] David John Benson. *Representations and cohomology, I: Basic representation theory of finite groups and associative algebras*. Cambridge University Press, 1995.
- [BF93] Allen Davis Bell and Rolf Farnsteiner. On the theory of Frobenius extensions and its application to Lie superalgebras. *Transactions of the American Mathematical Society*, 335(1):407–424, 1993.
- [BG02] Kenneth Alexander Brown and Kenneth Ralph Goodearl. *Lectures on algebraic quantum groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002.
- [BNPP11] Christopher Paul Bendel, Daniel Ken Nakano, Brian J. Parshall, and Cornelius Pillen. Cohomology for quantum groups via the geometry of the nullcone. preprint, February 2011.
- [Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Text in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [Bou02] Nicolas Bourbaki. *Elements of mathematics*. Springer-Verlag, 2002.
- [BR87] Michael Charles Richard Butler and Claus Michael Ringel. Auslander-Reiten sequences with few middle terms and applications to string algebras. *Communications in Algebra*, 15(15):145–179, 1987.
- [Bro98] Kenneth Alexander Brown. Representation theory of Noetherian Hopf algebras satisfying a polynomial identity. In *Trends in the representation theory of finite dimensional algebras. Proceedings of the AMS-IMS-SIAM joint summer research conference, Seattle, WA, USA, July 20-24, 1997*, volume 229 of *Contemp. Math.*, pages 49–79. American Mathematical Society, Providence, RI, 1998.
- [Car96] Jon F. Carlson. *Modules and group algebras*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Notes by Ruedi Suter.

BIBLIOGRAPHY

- [CB88] William Crawley-Boevey. On tame algebras and bocses. *Proceedings of the London Mathematical Society*, 56:451–483, 1988.
- [CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.
- [Cib97] Claude Cibils. Half-quantum groups at roots of unity, path algebras, and representation type. *International Mathematics Research Notices*, (12):541–553, 1997.
- [CK05] William Chin and Leonid Krop. Quantized hyperalgebras of rank 1. *Israel Journal of Mathematics*, 145:193–219, 2005.
- [CP94a] Vyjayanthi Chari and Alexander Premet. Indecomposable restricted representations of quantum sl_2 . *Publications of the Research Institute for Mathematical Sciences*, 30:335–352, 1994.
- [CP94b] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, 1994.
- [CPS88] Edward Cline, Brian J. Parshall, and Léonard Lewy Scott. Finite-dimensional algebras and highest weight categories. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 391:85–99, 1988.
- [dCK90] Corrado de Concini and Victor G. Kac. Representations of quantum groups at roots of 1. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, volume 92 of *Progress in Mathematics*, pages 471–506. Birkhäuser Boston, Boston, MA, 1990.
- [dCKP93] Corrado de Concini, Victor G. Kac, and Claudio Procesi. Some remarkable degenerations of quantum groups. *Communications in Mathematical Physics*, 157(2):405–427, 1993.
- [DF73] Peter Donovan and Mary-Ruth Freislich. *The representation theory of finite graphs and associated algebras*. Number 5 in Carleton Mathematical Lecture Notes. Carleton University, Ottawa, Ont., 1973.
- [DNN10] Christopher Martin Drupieski, Daniel Ken Nakano, and Nham Vo Ngo. Cohomology for infinitesimal unipotent algebraic and quantum groups. preprint, August 2010.
- [DNP10] Christopher Martin Drupieski, Daniel Ken Nakano, and Brian J. Parshall. Differentiating the Weyl generic dimension formula and support varieties for quantum groups. preprint, May 2010.
- [Dri85] Vladimir Gershonovich Drinfel'd. Hopf algebras and the quantum Yang-Baxter equation. *Doklady Akademii Nauk SSSR*, 283(5):1060–1064, 1985.

BIBLIOGRAPHY

- [Dri87] Vladimir Gershonovich Drinfel'd. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, California, 1986)*, pages 798–820, Providence, RI, 1987.
- [Dro80] Yuri Anatolievich Drozd. Tame and wild matrix problems. In *Representation theory, II (Proceedings of the Second International Conference, Carleton University, Ottawa, ON, 1979)*, volume 832 of *Lecture Notes in Mathematics*, pages 242–258. Springer, Berlin, 1980.
- [Dru09] Christopher Martin Drupieski. *Cohomology of Frobenius-Lusztig Kernels of Quantized Enveloping Algebras*. PhD thesis, University of Virginia, 2009.
- [Dru10] Christopher Martin Drupieski. On injective modules and support varieties for the small quantum group. *International Mathematics Research Notices*, 2011(10):2263–2294, 2010.
- [Dru11] Christopher Martin Drupieski. Representations and cohomology for Frobenius-Lusztig kernels. *Journal of Pure and Applied Algebra*, 215(6):1473–1491, 2011.
- [DS94] Jie Du and Léonard Lewy Scott. Lusztig conjectures, old and new i. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 455:141–182, 1994.
- [EGH⁺11] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. *Introduction to representation theory*, volume 59 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2011. With historical interludes by Slava Gerovitch.
- [EH62] Beno Eckmann and Peter John Hilton. Group-like structures in general categories. I. Multiplications and comultiplications. *Mathematische Annalen*, 145:227–255, 1961/1962.
- [EHT⁺04] Karin Erdmann, Miles Holloway, Rachel Taillefer, Nicole Snashall, and Øyvind Solberg. Support varieties for selfinjective algebras. *K-Theory*, 33(1):67–87, 2004.
- [Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
- [EO04] Pavel Etingof and Viktor Ostrik. Finite tensor categories. *Moscow Mathematical Journal*, 4(3):627–654, 782–783, 2004.
- [Erd90] Karin Erdmann. *Blocks of tame representation type and related algebras*, volume 1428 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990.

- [Erd95] Karin Erdmann. On Auslander-Reiten components for group algebras. *Journal of Pure and Applied Algebra*, 104:149–160, 1995.
- [Erd96] Karin Erdmann. The Auslander-Reiten quiver of restricted enveloping algebras. In *Representation theory of algebras (Cocoyoc, 1994)*, volume 18 of *CMS Conf. Proc.*, pages 201–214. American Mathematical Society, Providence, RI, 1996.
- [ES92] Karin Erdmann and Andrzej Skowroński. On Auslander-Reiten components of blocks and self-injective special biserial algebras. *Transactions of the American Mathematical Society*, 330:165–189, 1992.
- [EW06] Karin Erdmann and Mark J. Wildon. *Introduction to Lie algebras*. Springer-Verlag, 2006.
- [Far] Rolf Farnsteiner. Jordan types for indecomposable modules of finite group schemes. preprint.
- [Far95] Rolf Farnsteiner. Periodicity and representation type of modular Lie algebras. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 464:47–65, 1995.
- [Far96] Rolf Farnsteiner. Note on Frobenius extensions and restricted Lie superalgebras. *Journal of Pure and Applied Algebra*, 108(3):241–256, 1996.
- [Far98] Rolf Farnsteiner. On the distribution of AR-components of restricted Lie algebras. In *Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997)*, volume 229 of *Contemporary Mathematics*, pages 139–157. American Mathematical Society, Providence, RI, 1998.
- [Far99a] Rolf Farnsteiner. On Auslander-Reiten quivers of enveloping algebras of restricted Lie algebras. *Mathematische Nachrichten*, 202:43–66, 1999.
- [Far99b] Rolf Farnsteiner. On support varieties of Auslander-Reiten components. *Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae. New Series*, 10(2):221–234, 1999.
- [Far00a] Rolf Farnsteiner. Auslander-Reiten components for Lie algebras of reductive groups. *Advances in Mathematics*, 155(1):49–83, 2000.
- [Far00b] Rolf Farnsteiner. On the Auslander-Reiten quiver of an infinitesimal group. *Nagoya Mathematical Journal*, 160:103–121, 2000.
- [Far05] Rolf Farnsteiner. Auslander-Reiten components for G_1T -modules. *Journal of algebra and its applications*, 4(6):739–759, 2005.

BIBLIOGRAPHY

- [Far06] Rolf Farnsteiner. Polyhedral groups, McKay quivers, and the finite algebraic groups with tame principal blocks. *Inventiones Mathematicae*, 166(1):27–94, 2006.
- [Far09] Rolf Farnsteiner. Support varieties, AR-components and good filtrations, April 2009.
- [Far11] Rolf Farnsteiner. Complexity, periodicity and one-parameter subgroups. To appear in: Transactions of the American Mathematical Society, July 2011.
- [Fis82] Georg Fischer. *Darstellungstheorie der ersten Frobeniuskerne der SL_2* . PhD thesis, Universität Bielefeld, 1982.
- [FMS97] Davida Debra Fischman, M. Susan Montgomery, and Hans-Jürgen Schneider. Frobenius extensions of subalgebras of Hopf algebras. *Transactions of the American Mathematical Society*, 349(12):4857–4895, 1997.
- [FR11] Rolf Farnsteiner and Gerhard Röhrle. Support varieties, AR-components, and good filtrations. *Mathematische Zeitschrift*, 267:185–219, 2011.
- [FS91] Rolf Farnsteiner and Helmut Strade. Shapiro’s lemma and its consequences in the cohomology theory of modular Lie algebras. *Mathematische Zeitschrift*, 206:153–168, 1991.
- [FS97] Eric Marc Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Inventiones Mathematicae*, 127(2):209–270, 1997.
- [FV00] Rolf Farnsteiner and Detlef Voigt. On cocommutative Hopf algebras of finite representation type. *Advances in Mathematics*, 155(1):1–22, 2000.
- [FV03] Rolf Farnsteiner and Detlef Voigt. On infinitesimal groups of tame representation type. *Mathematische Zeitschrift*, 244(3):479–513, 2003.
- [FW09] Jörg Feldvoss and Sarah Witherspoon. Support varieties and representation type of small quantum groups. *International Mathematics Research Notices*, 2010(7):1346–1362, 2009.
- [FW11] Jörg Feldvoss and Sarah Witherspoon. Support varieties and representation type of self-injective algebras. preprint, May 2011.
- [Gab62] Pierre Gabriel. Des catégories abéliennes. *Bulletin de la Société Mathématique de France*, 90:323–448, 1962.
- [Gab72] Peter Gabriel. Unzerlegbare Darstellungen I. *Manuscripta Mathematica*, 6:71–103, 1972.

- [Gab80] Peter Gabriel. Auslander-Reiten sequences and representation-finite algebras. In *Representation theory I (Proceedings, Workshop, Carleton University, Ottawa, 1979)*, volume 831 of *Lecture Notes in Mathematics*, pages 1–71. Springer, Berlin, 1980.
- [GG82] Robert Gordon and Edward Lewis Green. Representation theory of graded Artin algebras. *Journal of Algebra*, 76:138–152, 1982.
- [GK93] Victor Ginzburg and Shrawan Kumar. Cohomology of quantum groups at roots of unity. *Duke Mathematical Journal*, 69:179–198, 1993.
- [GM03] Sergei I. Gelfand and Yuri Ivanovich Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.
- [Gor01] Iain Grant Gordon. Representations of Lie algebras in positive characteristic and quantised enveloping algebras at roots of unity. In Andrew Pressley, editor, *Quantum groups and Lie theory*, volume 290 of *London Mathematical Society Lecture Note Series*, pages 149–167. Cambridge University Press, 2001.
- [GS05] Christof Geiss and Jan Schröer. Extension-orthogonal components of preprojective varieties. *Transactions of the American Mathematical Society*, 357(5):1953–1962, 2005.
- [GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.
- [Har77] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.
- [Hel61] Alex Heller. Indecomposable representations and the loop-space operation. *Proceedings of the American Mathematical Society*, 12:640–643, 1961.
- [Hig54] Donald Gordon Higman. Indecomposable representations at characteristic p . *Duke Mathematical Journal*, 21:377–381, 1954.
- [HN91] Randall Reed Holmes and Daniel Ken Nakano. Brauer-type reciprocity for a class of graded associative algebras. *Journal of Algebra*, 144:117–126, 1991.
- [HPR80] Dieter Happel, Udo Preiser, and Claus Michael Ringel. Vinberg’s characterization of Dynkin diagrams using subadditive functions with applications to D Tr-periodic modules. In *Representation Theory, II (Proceedings of the Second International Conference on Representations of Algebras, Carleton University, Ottawa, Ont., 1979)*, volume 832 of *Lecture Notes in Mathematics*, pages 280–294. Springer, Berlin, 1980.

BIBLIOGRAPHY

- [HS10] Thorsten Holm and Andrzej Skowroński. Derived equivalence classification of symmetric algebras of polynomial growth. *Glasgow Mathematical Journal*, 2010.
- [Hum67] James Edward Humphreys. *Algebraic groups and modular Lie algebras*. Number 71 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence, R.I., 1967.
- [Hum72] James Edward Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, 1972.
- [Hum75] James Edward Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975.
- [Hum90] James Edward Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [HZ08] Thorsten Holm and Alexander Zimmermann. Generalized Reynolds ideals and derived equivalences for algebras of dihedral and semidihedral type. *Journal of Algebra*, 320(9):3425–3437, 2008.
- [Jac62] Nathan Jacobson. *Lie algebras*. Number 10 in Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962.
- [Jan96] Jens Carsten Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, 1996.
- [Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.
- [Jim85] Michio Jimbo. A q -difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. *Letters in Mathematical Physics. A Journal for the Rapid Dissemination of Short Contributions in the Field of Mathematical Physics*, 10(1):63–69, 1985.
- [Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [Kaw97] Shigeto Kawata. On Auslander-Reiten components and simple modules for finite group algebras. *Osaka Journal of Mathematics*, 34(3):681–688, 1997.
- [KR81] Petr Petrovich Kulish and Nikolai Jurievič Reshetikhin. Quantum linear problems for the sine-Gordon equation and higher representations. *Zapiski Naučnyh Seminarov Leningradskogo Otdelenija Matematičeskogo Instituta im . V. A. Steklova Akademii Nauk SSSR (LOMI)*, 101:101–110, 207, 1981. Questions in quantum field theory and statistical physics, 2.

- [Kum94] Shrawan Kumar. Representations of quantum groups at roots of unity. In David N. Yetter, editor, *Proceedings of the conference on quantum topology*, pages 187–224. Singapore: World Scientific, 1994.
- [KZ11] Otto Kerner and Dan Zacharia. Auslander-Reiten theory for modules of finite complexity over self-injective algebras. *Bulletin of the London Mathematical Society*, 43(1):44–56, 2011.
- [Lam99] Tsit-Yuen Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.
- [LdlP97] Helmut Lenzing and José Antonio de la Peña. Wild canonical algebras. *Mathematische Zeitschrift*, 224(3):403–425, 1997.
- [Liu96] Shiping Liu. Shapes of connected components of Auslander-Reiten quivers of Artin algebras. In *Representation theory of algebras and related topics (Mexico City, 1994)*, volume 19 of *CMS Conference Proceedings*, pages 109–137. American Mathematical Society, Providence, RI, 1996.
- [LS90] S. Z. Levendorskii and Yan S. Soibel'man. Some applications of the quantum Weyl groups. *Journal of Geometry and Physics*, 7(2):241–254, 1990.
- [LS94] Shiping Liu and Rainer Schulz. The existence of bounded infinite $D\text{Tr}$ -orbits. *Proceedings of the American Mathematical Society*, 122(4):1003–1005, 1994.
- [Lus90a] George Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *Journal of the American Mathematical Society*, 3(1):257–296, 1990.
- [Lus90b] George Lusztig. Quantum groups at roots of 1. *Geometriae Dedicata*, 35:89–114, 1990.
- [Mac98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [McC01] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.
- [Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
- [MPSW10] Mitja Mastnak, Julia Pevtsova, Peter Schauenburg, and Sarah Witherspoon. Cohomology of finite-dimensional pointed Hopf algebras. *Proceedings of the London Mathematical Society*, 100:377–404, 2010.

BIBLIOGRAPHY

- [MR88] Eugenia Marmolejo and Claus Michael Ringel. Modules of bounded length in Auslander-Reiten-components. *Archiv der Mathematik*, 50(2):128–133, 1988.
- [Naz73] Liudmila A. Nazarova. Representations of quivers of infinite type. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 37:752–791, 1973.
- [NPV02] Daniel Ken Nakano, Brian J. Parshall, and David Charles Vella. Support varieties for algebraic groups. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 547:15–49, 2002.
- [NR73] Liudmila A. Nazarova and Andrei V. Roiter. *Kategornye matrichnye zadachi i problema Brauera-Trella*. Izdat. "Naukova Dumka", Kiev, 1973.
- [NT60] Tadasi Nakayama and Tosihiro Tsuzuku. On Frobenius extensions I. *Nagoya Mathematical Journal*, 17:89–110, 1960.
- [PS11] Brian J. Parshall and Leonard L. Scott. A new approach to the Koszul property in representation theory using graded subalgebras. preprint, September 2011.
- [RH03] Steen Rym-Hansen. A q -analogue of Kempf's vanishing theorem. *Moscow Mathematical Journal*, 3(1):1609–3321, 2003.
- [Rin74] Claus Michael Ringel. The representation type of local algebras. In *Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974)*, Paper No. 22, pages 24 pp. Carleton Math. Lecture Notes, No. 9, Ottawa, Ont., 1974.
- [Rin78] Claus Michael Ringel. Finite dimensional hereditary algebras of wild representation type. *Mathematische Zeitschrift*, 161(3):235–255, 1978.
- [Rin96] Claus Michael Ringel. The Liu-Schulz example. In *Representation theory of algebras (Cocoyoc, 1994)*, volume 18 of *CMS Conference Proceedings*, pages 587–600. American Mathematical Society, Providence, RI, 1996.
- [Rot09] Joseph Jonah Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.
- [RR85] Idun Reiten and Christine Riedtmann. Skew group algebras in the representation theory of Artin algebras. *Journal of Algebra*, 92(1):224–282, 1985.
- [Sch] Aidan H. Schofield. Wild algebras with periodic Auslander-Reiten translate. preprint.
- [Sch86] Rainer Schulz. Boundedness and periodicity of modules over QF rings. *Journal of Algebra*, 101(2):450–469, 1986.
- [Sch09] Sarah Scherotzke. Euclidean components for a class of self-injective algebras. *Colloquium Mathematicum*, 115(2):219–245, 2009.

- [Sl04] Mariano Suárez-Álvarez. The Hilton-Eckmann argument for cup products. *Proceedings of the American Mathematical Society* 132, 132(8):2241–2246, 2004.
- [Sol06] Øyvind Solberg. Support varieties for modules and complexes. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemporary Mathematics*, pages 239–270. American Mathematical Society, Providence, RI, 2006.
- [Spr98] Tony Albert Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1998.
- [ST11] Sarah Scherotzke and Matthew Towers. Rank varieties for Hopf algebras. *Journal of Pure and Applied Algebra*, 215(5):829–838, 2011.
- [Sut94] Ruedi Suter. Modules over $\mathfrak{U}_q(\mathfrak{sl}_2)$. *Communications in Mathematical Physics*, 163:359–393, 1994.
- [Tha94] Lars Thams. The blocks of a quantum algebra. *Communications in Algebra*, 22(5):1617–1628, 1994.
- [Ung90] Luise Unger. The concealed algebras of the minimal wild, hereditary algebras. *Bayreuther Mathematische Schriften*, (31):145–154, 1990.
- [VIG07] Support varieties for Weyl modules over bad primes. *Journal of Algebra*, 312(2):602–633, 2007. University of Georgia VIGRE Algebra Group.
- [Wat79] William Charles Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.
- [Web82] Peter J. Webb. The Auslander-Reiten quiver of a finite group. *Mathematische Zeitschrift*, 179:97–121, 1982.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [WW85] Burkhard Wald and Josef Waschbüsch. Tame biserial algebras. *Journal of Algebra*, 95:480–500, 1985.
- [Xi89] Nanhua Xi. Representations of finite dimensional Hopf algebras arising from quantum groups. preprint, 1989.
- [Xi96] Nanhua Xi. Irreducible modules of quantized enveloping algebras at roots of 1. *Publications of the Research Institute for Mathematical Sciences*, 32(2):235–276, 1996.

BIBLIOGRAPHY

- [Xia94] Jie Xiao. Restricted representations of $U(\mathfrak{sl}(2))$ -quantizations. *Algebra Colloquium*, 1(1):56–66, 1994.
- [Xia97] Jie Xiao. Finite dimensional representations of $U_t(\mathfrak{sl}(2))$ at roots of unity. *Canadian Journal of Mathematics*, 49(4):772–787, 1997.

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Die Hauptergebnisse der Dissertation sind in folgenden Publikationen enthalten:

- Külshammer, Julian: Representation type of Frobenius-Lusztig kernels. *Accepted by The Quarterly Journal of Mathematics*, doi: 10.1093/qmath/has003 (Preprint: arXiv: 1108.4219), 2011.
- Külshammer, Julian: Auslander-Reiten theory of Frobenius-Lusztig kernels. *Submitted for Publication to Transactions of the American Mathematical Society* (Preprint: arXiv: 1201.5303), 2012.
- Külshammer, Julian: Auslander-Reiten theory of small half quantum groups. *Accepted by the Proceedings of the American Mathematical Society* (Preprint: arXiv: 1202.1714), 2012.

Kiel, den 15.5.2012

(Julian Külshammer)