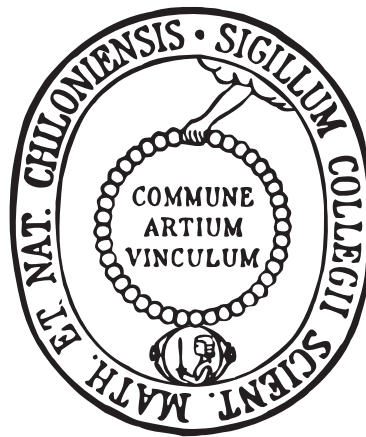


# The Heath-Jarrow-Morton approach for modelling stock options

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## Abstract

The traditional approach for modelling financial markets is to specify the dynamics of an underlying, e.g. a stock, and derive the prices or the dynamics of derivatives from it. The fundamental theorem of asset pricing states that discounted price processes are martingales, or more precisely  $\sigma$ -martingales, under an equivalent measure. In modern stock markets both underlying and many derivatives are traded liquidly. If one starts with a model for the stock, then the corresponding option prices are given by conditional expectation under the pricing measure. However, these model prices often disagree with observed quotes. This is unacceptable from both a theoretical and practical point of view. A way out is *calibration*, i.e. to choose a parametric class of models and find parameters such that model prices match real prices. However, some time later the calibrated parameters usually do not explain the observed option prices any more. Since recalibration, i.e. redefining parameters such that option prices match again, is unsatisfactory from a theoretical point of view, one needs another way out. We propose to specify the dynamics of the options rather than deriving them from the underlying. However, the possible surfaces of arbitrage-free option prices can be rather restrictive and mathematically difficult to handle. It is therefore useful to parametrise the option prices within a more tractable set like a convex cone or a linear space. The aim of this thesis is to develop a method for finding convenient parametrisations of the options, to analyse the implications of the option dynamics for the structure of the stock and to explicitly construct *dynamic models*.

The construction of dynamic models relies partly on the theory of martingale problems. Unlike existence, proving uniqueness of solutions to martingale problems is more involved. In the third part of this thesis, we consider the question of uniqueness in detail.





## Zusammenfassung

Der klassische Ansatz zur Modellierung von Finanzmärkten besteht darin, dass zunächst die Dynamik, d.h. das stochastische Verhalten, des zugrunde liegenden Wertpapiers festgelegt wird und man daraus Preise und Dynamik aufbauender Wertpapiere bestimmt. Der Fundamentalsatz der Preistheorie besagt, dass alle diskontierten Preisprozesse Martingale unter einem äquivalentem Wahrscheinlichkeitsmaß sind. Heutzutage werden an Börsen sowohl Aktien als auch darauf basierende Derivate liquide gehandelt. Wenn man nun ein Modell für die Aktie spezifiziert, dann sind dadurch die Preise der abgeleiteten Wertpapiere festgelegt. Solche Modellpreise weichen in der Praxis häufig von den beobachteten Kursen an der Börse ab. Dies ist natürlich aus theoretischer wie praktischer Sicht inakzeptabel. Eine ad-hoc Methode, dieses Problem zu umgehen, ist das *Kalibrieren*, d.h. man wählt Modellparameter so, dass Modell- und beobachtete Preise übereinstimmen. Nach einiger Zeit weichen die theoretischen Preise üblicherweise von den tatsächlichen Börsenkursen ab. Rekalibrieren, d.h. das erneute Anpassen der Parameter, ist aus theoretischen Standpunkten nicht zu vertreten. Wir schlagen daher vor, die Dynamik der liquiden Derivate zu modellieren, anstatt diese vom zugrunde liegenden Wertpapier abzuleiten. Dabei muss beachtet werden, dass die möglichen arbitragefreien Optionspreise einschränkenden Bedingungen unterliegen, was das direkte Modellieren erschwert. Es ist daher nützlich, die Optionspreise mittels einer Parametrisierung in einer einfacher strukturierten Menge zu beschreiben. Ziel dieser Arbeit ist es, solche Darstellungen methodisch zu finden, die Auswirkungen der Optionsdynamiken auf die Aktie zu studieren und arbitragefreie Modelle explizit zu konstruieren.

Dabei wird unter anderem die Theorie der Martingalprobleme zur Modellkonstruktion herangezogen. Im Gegensatz zur Existenz ist die Eindeutigkeit der Lösung von Martingalproblemen viel schwieriger nachzuweisen. Der dritte Teil dieser Arbeit beschäftigt sich im Detail mit dieser Eindeutigkeitsfrage.



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## Notation

$\operatorname{Re}$  and  $\operatorname{Im}$  denote the real resp. imaginary part of a complex vector in  $\mathbb{C}^d$ . We denote the positive real numbers including 0 by  $\mathbb{R}_+$  and the positive real numbers without 0 by  $\mathbb{R}_{++}$ . We write  $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\}$ . For  $a, b \in \mathbb{R}$  we denote the closed interval  $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ , which is empty if  $a > b$ . For a topological space  $E$  and a Banach space  $Y$  we will use the function spaces  $B(E, Y), C(E, Y), C_0(E, Y), C_b(E, Y)$  resp.  $C_K(E, Y)$  which contain all bounded and Borel-measurable functions, continuous functions, continuous functions vanishing at infinity, continuous and bounded functions resp. continuous functions with compact support from  $E$  to  $Y$ . In the case  $Y = \mathbb{C}$  we write  $B(E)$  etc. instead of  $B(E, \mathbb{C})$  etc. We endow all these spaces except for  $C(E, Y)$  with a norm, namely the uniform norm  $\|f\| := \sup\{|f(x)| : x \in E\}$  for  $f \in B(E, Y)$ . A number  $n \in \mathbb{N}$  in the superscript of one of these sets means the  $n$  times partially differentiable functions in the set such that the first  $n$  partial derivatives are again in the respective function space, e.g.  $C_b^2(\mathbb{R}, \mathbb{R})$  denotes the set of twice continuously differentiable and bounded functions where the first two derivatives are in  $C_b(\mathbb{R}, \mathbb{R})$ .

We use the notations  $\partial_u$  and  $D$  for partial and total derivatives, respectively. We often write  $\beta \bullet X_t = \int_0^t \beta_s dX_s$  for stochastic integrals.  $L(X)$  denotes the set of  $X$ -integrable predictable processes for a semimartingale  $X$ . If we talk about an  $m + n$ -dimensional semimartingale  $(X, Y)$ , we mean that  $X$  is an  $\mathbb{R}^m$ -valued semimartingale and  $Y$  is an  $\mathbb{R}^n$ -valued semimartingale. For  $u, v \in \mathbb{C}^d$  we denote the scalar product of  $u$  and  $v$  by  $uv^\top := uv := \sum_{k=1}^d u_k \bar{v}_k$ . Finally,  $I$  denotes the identity process, i.e.  $I_t = t$ . Throughout this thesis, whenever we say filtration we automatically mean a right-continuous filtration. We will usually suppress the notion of a filtered (right-continuous) probability space and usually every process is allowed to 'live' on a different probability space unless otherwise noted. The abbreviation *PII* stands for processes with independent increments in the sense of [25]. The abbreviations *SBM* resp. *SPP* stand for standard Brownian motion resp. standard Poisson process, i.e. its intensity and jump height is 1. Further unexplained notation are used as in [25].

## Introduction

In the traditional approach for modelling option markets, one specifies the dynamics of the stock and derives the dynamics of the options via conditional expectation under the pricing measure. When options are liquidly traded in the real market, then it is reasonable to require that the model option prices and the observed prices coincide. This can be achieved by calibration, i.e. using a parametric class of models and choose the parameters in such a way that the model option prices are (almost) the same as the real option prices. However, some time later the calibrated parameters usually do not explain the observed option prices any more. That means that the parametric model class fails to explain the joint dynamics of the stock and the options.

A possible way out is to model the liquidly traded options instead of the underlying. This idea goes back to Heath-Jarrow-Morton (HJM, see [21]) approach in interest rate theory. In zero coupon bond markets a bank account can be seen as an underlying and the zero coupon bonds as derivatives on this underlying. *Short rate models* try to model the bank account directly and derive the dynamics and prices of the zero coupon bonds via conditional expectation under the pricing measure. Contrary to the classical approach HJM model the forward rates instead. The whole forward rate curve is a simple reparametrisation of the bond price curve and hence any dynamics for the forward rate curve yield dynamics for the bond price curve and vice versa.

The approach of modelling European call options directly or, more precisely, a reparametrisation of the European call options, has been considered by several authors. [39] and [37] consider the case where European call options for all maturities but a single strike are traded liquidly, whereas [16, 10, 12] allow for all strikes and maturities. Further references in this context include [24, 38]. The HJM approach has been transferred to other option classes including credit models [5, 37, 40] and variance swaps [8], cf. [9, 11] for an overview and further references. Notably, [44] developed a framework where finitely many European call options are considered to be traded liquidly.

A typical situation in financial modelling is to specify a stock or other process by their local dynamics as e.g. a stochastic differential equation or infinitesimal generator. This leads to the question whether the local description determine the stock resp. law of the process uniquely. In many cases this can be rewritten as a martingale problem so that the question of existence and uniqueness can be investigated in a unified and well-studied framework. In practise it is common that the linear operator  $\mathcal{A}$  occurring in a martingale problem is of the form

$$\mathcal{A}f(x) = \int_{\mathbb{R}^d} \check{f}(u)q(x, u)e^{iux} du$$

where  $q$  is the so called *symbol*, see Definition III.1.1 below, and  $\check{f}(u) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x)e^{-iux}$ . Indeed, if  $\mathcal{A}$  has symbol representation, then continuity and some boundedness condition suffices to ensure existence of solutions. This well-known fact is a generalisation of the Peano existence theorem for ODE's (ordinary differential equations). The Picard-Lindelöf theorem yields uniqueness for ODE's under Lipschitz conditions. For martingale problems Lipschitz continuity of the symbol is generally not sufficient for uniqueness, cf. [43, Example]. On the other hand, uniqueness holds if the symbol  $q(x, u)$  is a quadratic function in  $u$  and twice continuously differentiable in  $x$ . This no longer holds for more general dependence of  $u$ . Surprisingly, there are even analytic symbols where the martingale problem has multiple solutions, cf. Example III.2.4 below.

Martingale problems were first introduced in [42] and have been studied intensively afterwards. The book [18] contains an impressive treatment of the theory. For an overview of more recent results and further references see [23].

This thesis is structured as follows. In the first chapter we develop a HJM approach where European call options for all strikes and maturities are assumed to be traded liquidly. Subsequently, we develop a similar approach for a finite set of liquidly traded European options. The third chapter of this thesis is devoted to uniqueness results for the Martingale problem.

### **The Heath-Jarrow-Morton philosophy**

The fundamental theorem of asset pricing states that there exists a pricing measure, i.e. an equivalent measure which turns the discounted prices into martingales or, more precisely, into  $\sigma$ -martingales. For simplicity we will work with discounted price processes unless otherwise mentioned, i.e. we assume that there is a constant bank account.

Before we turn to concrete setups, we want to highlight key features of the HJM approach in general. For more background and examples we refer the reader to the exposition [9].

We state seven informal steps which lead to the development of a HJM framework and illustrate the mathematical steps alongside.

- (1) In HJM setups there typically exists a (possibly untraded) canonical underlying. This can be a bank account in interest rate theory, a log-return of a stock in stock markets or the temperature for a market where weather-related options are traded. The objects of interests are a fixed set  $\mathcal{S}$  of derivatives (i.e. options) on the underlying. The first HJM axiom states that the derivatives instead of the underlying should be treated as primary objects.

EXAMPLE 1. Let us assume that the options are European style with a family of payoff functions  $(f_i)_{i \in I}$  and maturities  $(T_i)_{i \in I}$  respectively, i.e. the  $i$ -th option yields a payoff  $f(S(T_i))$  at time  $T_i$  where  $S(T_i)$  is the value of the underlying at time  $T_i$ . In particular the price of the  $i$ -th option is given by  $C_i(t) = E(f(S(T_i)) | \mathcal{F}_t)$  where  $\mathcal{F}_t$  is the  $\sigma$ -algebra of information at time  $t$  and the expectation is to be taken under the pricing measure. The first HJM axiom states that the dynamics of the process  $C_i$  should be modelled in the first place.

- (2) The first axiom has an immediate consequence, namely one should not model the dynamics of the underlying in detail under the pricing measure. Otherwise the dynamics of the options are specified in detail.

EXAMPLE 2. Let us continue our example from above. If we know the dynamics of  $S$ , then  $E(f(S(T_i)) | \mathcal{F}_t)$  is specified and hence the dynamics of the option prices are no more free to be modelled arbitrarily.

- (3) At each time  $t$ , we cannot allow any static arbitrage for the prices  $(C_i(t))_{i \in I}$ . This can be a rather difficult task for call options, because they have awkward constraints, see [17]. Consequently, special care has to be taken especially if we want to model the dynamics of the processes  $C_i$ . As in the original HJM setup we should parametrise our primary objects, the derivatives, and study them in a more simple space. We adapt the term *code-book* from [9] for the parameter set. A crucial step

in the HJM methodology is to find such a parametrisation, or code-book. The next step proposes a method for finding a convenient parametrisation.

- (4) Choose a parametric class of underlying models such that every admissible state of the options can be uniquely explained by one of the underlying models. In other words, the parametric class of underlying models should allow for perfect calibration to the prices of the derivatives. However, these underlying models will **not** be the dynamic models, because they typically fail to describe the real market's joint dynamics of the stock and the options.

EXAMPLE 3. Let us assume that there is a family of Markov processes  $(Y_k)_{k \in K}$  such that  $K$  is 'simple' (e.g. a cone) and that there is a one-to-one mapping  $G(t, x, \cdot)$  from  $K$  to the set of possible option prices at time  $t$  if the value of the underlying is  $x$ , i.e.  $C_i(t) = G_i(t, Y_k(t), k)$  in the underlying model  $Y_k$ . The Markovian structure is needed to ensure the existence of such a function  $G_i$ , indeed the function is given by

$$G_i(t, x, k) := E(f_i(Y_k(T_i)) | Y_k(t) = x).$$

- (5) Model a new process (the so-called *code-book process*) on the parameter space  $K$  of the class of underlying-models and define the options via the parametrisation one obtained from the underlying-models. The new model is the so-called *dynamic model*. The dynamic model will only be one of the underlying models if the code-book process is constant. However, the option prices at one time  $t$  look exactly like the option prices of one of the 'old' underlying-models. In particular, the options in the dynamic model will not allow for static arbitrage, cf. Remark I.2.2.

EXAMPLE 4. Let us model a process  $dZ(t) = \alpha(t)dt + \beta(t)dW(t)$  with values in  $K$  and define the option prices by  $C_i(t) := G_i(t, S(t), Z(t))$ , where  $S(t)$  denotes the value of the underlying at time  $t$ . For  $(\omega, t)$  fixed we see that with  $k := Z(\omega, t)$ ,  $x := S(\omega, t)$  we have

$$C_i(\omega, t) = G_i(t, S(\omega, t), Z(\omega, t)) = C_i^k(t),$$

where  $C_i^k(t)$  denotes the price of the  $i$ -th option in the underlying-model  $Y_k$  given  $Y_k(t) = x$ . If the underlying-model  $Y_k$  does not allow for arbitrage, then  $(C_i^k(t))_{i \in I}$  does not allow for static arbitrage.



- (6) However, absence of static arbitrage does not imply absence of arbitrage all together. Hence we must derive conditions which imply absence of dynamic arbitrage. This typically leads to drift conditions for the code-book process and to further restrictions for the dynamics of the underlying.

EXAMPLE 5. If we assume that  $G$  is smooth enough, then Itô's formula for the local characteristics yields

$$b_i(t) = \partial_3 G_i(t, S(t), Z(t))\alpha(t) + (\dots)$$

where  $b_i$  is the drift coefficient of the option price process  $C_i$ . However, under the pricing measure the drift rate  $b_i$  must equal 0 and hence the equation reads as

$$\alpha(t) = -(\partial_3 G_i(t, S(t), Z(t)))^{-1}(\dots).$$

However the right-hand side involves the dynamics of the underlying. Thus the drift condition will only be an explicit condition if we know the dynamics of the underlying. However, modelling the dynamics of the underlying determines the dynamics of the code-book process  $Z$  completely. That is the dynamics of the underlying have to be chosen in such a way that the process  $Z$  has the desired diffusion  $\beta$ . An underlying dynamics will be called *weakly consistent* with a given code-book dynamics if no arbitrage is implied. At this point it is not clear if, for given diffusion coefficient  $\beta$ , weakly consistent dynamics for the underlying exist at all.

- (7) In infinite-dimensional setups (i.e.  $K$  is infinite dimensional) it is typical that choosing a martingale part of  $Z$  completely determines the distribution of the underlying. It turns out that the underlying 'locally' has the same dynamics as the underlying-model  $Y_k$  where  $k = Z(\omega, t)$ . This will be referred to *consistency*, cf. Theorem I.2.6.

Contrary to infinite-dimensional setups, in the finite-dimensional setup weakly consistent dynamic models do not necessarily have to be consistent. However, if we try to build a consistent underlying process, then it turns out that the drift restriction simplifies drastically, cf. Theorem II.1.15. Moreover, Theorem II.1.21 yields that consistent dynamic models can be found as solutions of martingale problems.

## CHAPTER I

### The HJM approach for infinitely many traded stock options

An earlier version of this chapter has been submitted for publication, cf. [28, Section 2.2 - 4.4].

#### 1. Introduction

**1.1. Time-inhomogeneous Lévy models.** According to the general interpretation of HJM-type modelling, the approach of [10] to option surface modelling relies on the family of Dupire's local volatility models. Similarly as the independent study [12], we suggest another family of simple models for the stock, also relying on a two-parameter manifold. To this end, suppose that the discounted stock is a martingale of the form  $S = e^X$ , where the *return process*  $X$  denotes a process with independent increments (or time-inhomogeneous Lévy process, henceforth PII) on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$  and  $P$  denotes a risk-neutral measure. More specifically, the characteristic function of  $X$  is assumed to be absolutely continuous in time, i.e.

$$E(e^{iuX_t}) = \exp \left( iuX_0 + \int_0^t \Psi(s, u) ds \right) \quad (\text{I.1.1})$$

with some function  $\Psi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$ .

We assume call options of all strikes and maturities to be liquidly traded. Specifically, we write  $C_t(T, K)$  for the discounted price at time  $t$  of a call which expires at  $T$  with discounted strike  $K$ . A slight extension of [4, Proposition 1] shows that option prices can be expressed in terms of  $\Psi$ . To this end, we define *modified option prices*

$$\mathcal{O}_t(T, x) := e^{-(x+X_t)} C_t(T, e^{x+X_t}) - (e^{-x} - 1)^+.$$

Since call option prices are obtained from  $C_t(T, K) = E((S_T - K)^+ | \mathcal{F}_t)$ , by call-put parity, and by  $E(S_T | \mathcal{F}_t) = S_t$ , we have

$$\mathcal{O}_t(T, x) = \begin{cases} E((e^{(X_T - X_t) - x} - 1)^+ | \mathcal{F}_t) & \text{if } x \geq 0, \\ E((1 - e^{(X_T - X_t) - x})^+ | \mathcal{F}_t) & \text{if } x < 0. \end{cases}$$

Proposition A.4.4 yields

$$\mathcal{O}_t(T, x) = \mathcal{F}^{-1} \left\{ u \mapsto \frac{1 - E(e^{iu(X_T - X_t)} | \mathcal{F}_t)}{u^2 + iu} \right\} (x), \quad (\text{I.1.2})$$

$$\mathcal{F} \{ x \mapsto \mathcal{O}_t(T, x) \} (u) = \frac{1 - E(e^{iu(X_T - X_t)} | \mathcal{F}_t)}{u^2 + iu} \quad (\text{I.1.3})$$

where  $\mathcal{F}^{-1}$  and  $\mathcal{F}$  denote the improper inverse Fourier transform and the improper Fourier transform, respectively, in the sense of (A.4.1, A.4.2) in Section 4.1 in the appendix. Since

$$C_t(T, K) = (S_t - K)^+ + K \mathcal{O}_t \left( T, \log \frac{K}{S_t} \right) \quad (\text{I.1.4})$$

and

$$E(e^{iu(X_T - X_t)} | \mathcal{F}_t) = \exp \left( \int_t^T \Psi(s, u) ds \right), \quad (\text{I.1.5})$$

we can compute option prices according to the following diagram:

$$\Psi \rightarrow \exp \left( \int_t^T \Psi(s, \cdot) ds \right) \rightarrow \mathcal{O}_t(T, \cdot) \rightarrow C_t(T, \cdot).$$

For the last step we also need the current stock price  $S_t$ . Under sufficient smoothness we can invert all transformations. Indeed, we have

$$\Psi(T, u) = \partial_T \log \left( 1 - (u^2 + iu) \mathcal{F} \{ x \mapsto \mathcal{O}_t(T, x) \} (u) \right). \quad (\text{I.1.6})$$

Hence we obtain option prices from  $\Psi$  and vice versa as long as we know the current stock price.

**1.2. Setting Lévy in motion.** Generally we do not assume that the *return process*

$$X := \log(S) \quad (\text{I.1.7})$$

follows a time-inhomogeneous Lévy process. Hence the right-hand side of Equation (I.1.6) will typically change randomly over time. In line with Step 4 above, we define modified option prices

$$\mathcal{O}_t(T, x) := e^{-(x+X_t)} C_t(T, e^{x+X_t}) - (e^{-x} - 1)^+ \quad (\text{I.1.8})$$

as before and

$$\Psi_t(T, u) := \partial_T \log \left( 1 - (u^2 + iu) \mathcal{F} \{ x \mapsto \mathcal{O}_t(T, x) \} (u) \right). \quad (\text{I.1.9})$$

This constitutes our *code-book process* for the surface of discounted option prices. As in Section 1.1 the asset price processes  $S$  and  $C(T, K)$  can be recovered from  $X$  and  $\Psi(T, u)$  via

$$\begin{aligned} S &= \exp(X), \\ \mathcal{O}_t(T, x) &= \mathcal{F}^{-1} \left\{ u \mapsto \frac{1 - \exp(\int_t^T \Psi_t(s, u) ds)}{u^2 + iu} \right\} (x), \\ C_t(T, K) &= (S_t - K)^+ + K \mathcal{O}_t \left( T, \log \frac{K}{S_t} \right). \end{aligned}$$

In the remainder of this chapter we assume that the infinite-dimensional code-book process satisfies an equation of the form

$$d\Psi_t(T, u) = \alpha_t(T, u)dt + \beta_t(T, u)dM_t, \quad (\text{I.1.10})$$

driven by some finite-dimensional semimartingale  $M$ .

## 2. Model setup and risk-neutrality

As before we fix a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$  with trivial initial  $\sigma$ -algebra  $\mathcal{F}_0$ . In this section we single out conditions such that a given pair  $(X, \Psi)$  corresponds via (I.1.7 – I.1.9) to a risk-neutral model for the stock and its call options.

**2.1. Option surface models.** We denote by  $\Pi$  the set of characteristic exponents of Lévy processes  $L$  such that  $E(e^{L_1}) = 1$ . More precisely,  $\Pi$  contains all functions  $\psi : \mathbb{R} \rightarrow \mathbb{C}$  of the form

$$\psi(u) = -\frac{u^2 + iu}{2}c + \int (e^{iux} - 1 - iu(e^x - 1))F(dx),$$

where  $c \in \mathbb{R}_+$  and  $F$  denotes a Lévy measure on  $\mathbb{R}$  satisfying  $\int_{\{x>1\}} e^x F(dx) < \infty$ , cf. [36].

**DEFINITION I.2.1.** A quintuple  $(X, \Psi_0, \alpha, \beta, M)$  is an *option surface model* if

- $(X, M)$  is a  $1 + d$ -dimensional semimartingale that allows for local characteristics in the sense of Section 1,
- $\Psi_0 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$  with  $\int_0^T |\Psi_0(r, u)| dr < \infty$  for any  $T \in \mathbb{R}_+, u \in \mathbb{R}$ ,
- $\alpha(T, u), \beta(T, u)$  are  $\mathbb{R}$ - resp.  $\mathbb{C}^d$ -valued predictable processes for any  $T \in \mathbb{R}_+, u \in \mathbb{R}$ ,
- $(\omega, t, T, u) \mapsto \alpha_t(T, u)(\omega), \beta_t(T, u)(\omega)$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$ -measurable, where  $\mathcal{P}$  denotes the predictable  $\sigma$ -algebra on  $\Omega \times \mathbb{R}_+$ ,

- $\int_0^t \int_0^T |\alpha_s(r, u)| dr ds < \infty$  for any  $t, T \in \mathbb{R}_+, u \in \mathbb{R}$ ,
- $\int_0^T (\beta_t(r, u))^2 dr < \infty$  for any  $t, T \in \mathbb{R}_+, u \in \mathbb{R}$ ,
- $((\int_0^T (\beta_t(r, u))^2 dr)^{\frac{1}{2}})_{t \in \mathbb{R}_+} \in L(M)$  for any fixed  $T \in \mathbb{R}_+, u \in \mathbb{R}$ ,
- the corresponding *code-book process*

$$\Psi_t(T, u) := \Psi_0(T, u) + \int_0^{t \wedge T} \alpha_s(T, u) ds + \int_0^{t \wedge T} \beta_s(T, u) dM_s \quad (\text{I.2.1})$$

has the following properties:

- (1)  $(\omega, t, T, u) \mapsto \Psi_t(T, u)(\omega)$  is  $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$ -measurable, where  $\mathcal{O}$  denotes the optional  $\sigma$ -algebra on  $\Omega \times \mathbb{R}_+$ ,
- (2)  $u \mapsto \int_t^T \Psi_s(r, u) dr$  is in  $\Pi$  a.s. for any  $T \in \mathbb{R}_+, t \in [0, T], s \in [0, t], \omega \in \Omega$ .

In line with Section 1.2, the discounted stock and call price processes associated with an option surface model are defined by

$$S_t := \exp(X_t), \quad (\text{I.2.2})$$

$$\mathcal{O}_t(T, x) := \mathcal{F}^{-1} \left\{ u \mapsto \frac{1 - \exp\left(\int_t^T \Psi_t(r, u) dr\right)}{u^2 + iu} \right\} (x), \quad (\text{I.2.3})$$

$$C_t(T, K) := (S_t - K)^+ + K \mathcal{O}_t\left(T, \log \frac{K}{S_t}\right) \quad (\text{I.2.4})$$

for any  $T \in \mathbb{R}_+, t \in [0, T], x \in \mathbb{R}, K \in \mathbb{R}_+$ , where  $\mathcal{F}^{-1}$  denotes the improper inverse Fourier transform in the sense of Section 4.1. We denote the local exponents of  $(X, M)$ ,  $X$  by  $\psi^{(X, M)}, \psi^X$  and their domains by  $\mathcal{U}^{(X, M)}, \mathcal{U}^X$ , cf. Definitions A.2.2 and A.2.4.

REMARK I.2.2. The existence of these processes is implied by the assumptions above. Indeed, by Fubini's theorem for ordinary and stochastic integrals [33, Theorem IV.65], we have

$$\int_0^T |\Psi_t(r, u)| dr < \infty.$$

Since  $u \mapsto \int_t^T \Psi_t(r, u) dr(\omega) \in \Pi$ , there is a measure  $Q$  on  $(\mathbb{R}, \mathcal{B})$  such that

$$\int e^{iux} Q(dx) = \exp\left(\int_t^T \Psi_t(r, u) dr(\omega)\right),$$

where  $\omega \in \Omega$  is fixed. The random variable  $Y : \mathbb{R} \rightarrow \mathbb{R}, w \mapsto w$  has the property that  $e^Y$  is  $Q$ -integrable with expectation 1 because the characteristic function of  $Y$  is in  $\Pi$ . Thus Proposition A.4.4 yields the existence of the inverse Fourier transform in Equation (I.2.3). Moreover, it implies  $C_t(T, K)(\omega) = E_Q((S_t(\omega)e^Y - K)^+)$  and thus we have

$0 \leq C_t(T, K)(\omega) \leq S_t(\omega)$  and  $0 \leq P_t(T, K)(\omega) \leq K$  for any  $K \in \mathbb{R}_+$ ,  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ , where  $P_t(T, K) := C_t(T, K) + K - S_t$  for any  $K \in \mathbb{R}_+$ ,  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ .

As noted above, we model asset prices under a risk-neutral measure for the whole market. Put differently, we are interested in risk-neutral option surface models in the following sense.

**DEFINITION I.2.3.** An option surface model  $(X, \Psi_0, \alpha, \beta, M)$  is called *risk-neutral* if the corresponding stock  $S$  and all European call options  $C(T, K)$ ,  $T \in \mathbb{R}_+$ ,  $K > 0$  are  $\sigma$ -martingales or, equivalently, local martingales (cf. [26, Proposition 3.1 and Corollary 3.1]). It is called *strongly risk-neutral* if  $S$  and all  $C(T, K)$  are martingales.

Below, risk-neutral option surface models are characterised in terms of the following properties.

**DEFINITION I.2.4.** An option surface model  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the *consistency condition* if

$$\psi_t^X(u) = \Psi_{t-}(t, u), \quad u \in \mathbb{R}$$

outside some  $dP \otimes dt$ -null set. Moreover, it satisfies the *drift condition* if

$$\left( u, -i \int_t^T \beta_t(r, u) dr \right)_{t \in \mathbb{R}_+} \in \mathcal{U}^{(X, M)}$$

and

$$\int_t^T \alpha_t(r, u) dr = \psi_t^X(u) - \psi_t^{(X, M)} \left( u, -i \int_t^T \beta_t(r, u) dr \right) \quad (\text{I.2.5})$$

outside some  $dP \otimes dt$  null set for any  $T \in \mathbb{R}_+$ ,  $u \in \mathbb{R}$ . Finally, the option surface model satisfies the *conditional expectation condition* if

$$\exp \left( \int_t^T \Psi_t(r, u) dr \right) = E(e^{iu(X_T - X_t)} | \mathcal{F}_t)$$

for any  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ ,  $u \in \mathbb{R}$ .

**REMARK I.2.5.** The drift condition can be rewritten as

$$\alpha_t(T, u) = -\partial_T \left( \psi_t^{(X, M)} \left( u, -i \int_t^T \beta_t(r, u) dr \right) \right)$$

for almost all  $T \in \mathbb{R}_+$ . It gets even simpler if  $X$  and  $M$  are assumed to be locally independent in the sense of Definition A.2.8:

$$\alpha_t(T, u) = -\partial_T \left( \psi_t^M \left( -i \int_t^T \beta_t(r, u) dr \right) \right).$$

If the derivative  $\psi'_t(u) := \partial_u \psi_t^M(u)$  exists as well, the drift condition simplifies once more and turns into

$$\alpha_t(T, u) = i\psi'_t \left( -i \int_t^T \beta_t(r, u) dr \right) \beta_t(T, u).$$

Now consider the situation that  $M$  is a one-dimensional Brownian motion which is locally independent of the return process  $X$ . Then  $\psi^M(u) = -\frac{u^2}{2}$  and the drift condition reads as

$$\alpha_t(T, u) = -\beta_t(T, u) \int_t^T \beta_t(r, u) dr.$$

Thus the drift condition for option surface models is similar to the HJM drift condition (cf. [21]).

**2.2. Necessary and sufficient conditions.** The goal of this section is to prove the following characterisation of risk-neutral option surface models.

**THEOREM I.2.6.** *For any option surface model  $(X, \Psi_0, \alpha, \beta, M)$  the following statements are equivalent.*

- (1) *It is strongly risk-neutral.*
- (2) *It is risk-neutral.*
- (3) *It satisfies the conditional expectation condition.*
- (4) *It satisfies the consistency and drift conditions.*

The remainder of this section is devoted to the proof of Theorem I.2.6. We proceed according to the following scheme

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1).$$

We use the notation

$$\begin{aligned} \delta_t(T, u) &:= \int_t^T \alpha_t(r, u) dr - \psi_t^X(u), \\ \sigma_t(T, u) &:= \int_t^T \beta_t(r, u) dr, \\ \Gamma_t(T, u) &:= \int_0^T \Psi_0(r, u) dr + \int_0^t \delta_s(T, u) ds + \int_0^t \sigma_s(T, u) dM_s. \end{aligned}$$

The existence of the integrals above are implied by the condition for option surface models. Observe that  $\Gamma(T, u)$  is a semimartingale.

LEMMA I.2.7. *For all  $u \in \mathbb{R}, T \in \mathbb{R}_+, t \in [0, T]$  we have*

$$\Gamma_t(T, u) - \Gamma_t(t, u) = \int_t^T \Psi_t(r, u) dr.$$

PROOF. Using the definition of  $\Gamma, \delta, \sigma$  and applying Fubini's theorem as in [33, Theorem IV.65] yields

$$\begin{aligned} \Gamma_t(T, u) - \Gamma_t(t, u) &= \int_t^T \Psi_0(r, u) dr + \int_t^T \int_0^t \alpha_s(r, u) ds dr + \int_t^T \int_0^t \beta_s(r, u) dM_s dr \\ &= \int_t^T \Psi_t(r, u) dr. \end{aligned}$$

□

LEMMA I.2.8. *Let  $u \in \mathbb{R}, t \in \mathbb{R}_+$  and  $Y_t := \Gamma_t(t, u)$ . Then*

$$Y_t = \int_0^t (\Psi_{s-}(s, u) - \psi_s^X(u)) ds.$$

PROOF. Observe by Fubini's theorem that

$$\begin{aligned} Y_t &= \int_0^t \Psi_0(r, u) dr + \int_0^t \delta_s(t, u) ds + \int_0^t \sigma_s(t, u) dM_s \\ &= \int_0^t \left( \Psi_0(r, u) + \int_0^r \alpha_s(r, u) ds - \psi_r^X(u) \right) dr + \int_0^t \int_s^t \beta_s(r, u) dr dM_s. \\ &= \int_0^t \left( \Psi_0(r, u) + \int_0^r \alpha_s(r, u) ds - \psi_r^X(u) \right) dr + \int_0^t \int_0^r \beta_s(r, u) dM_s dr. \end{aligned}$$

This yields the claim. □

LEMMA I.2.9. *If  $(X, \Psi_0, \alpha, \beta, M)$  is risk-neutral, then it satisfies the conditional expectation condition.*

PROOF. Let  $T \in \mathbb{R}_+$ . We define

$$O_t(T, x) := \begin{cases} e^{-x} C_t(T, e^x) & \text{if } x \geq 0, \\ e^{-x} P_t(T, e^x) & \text{if } x < 0, \end{cases}$$

where  $P_t(T, K) := C_t(T, K) + K - S_t$  for any  $K \in \mathbb{R}_+, t \in [0, T], x \in \mathbb{R}$ . Then we have

$$O_t(T, x) = \begin{cases} (e^{X_t-x} - 1)^+ + \mathcal{O}_t(T, x - X_t) & \text{if } x \geq 0, \\ (1 - e^{X_t-x})^+ + \mathcal{O}_t(T, x - X_t) & \text{if } x < 0. \end{cases}$$



We calculate the Fourier transform of  $O_t(T, x)$  in two steps by considering the summands separately. The improper Fourier transform of the second summand  $\mathcal{O}_t(T, x - X_t)$  exists and satisfies

$$\begin{aligned} \mathcal{F}\{x \mapsto \mathcal{O}_t(T, x - X_t)\}(u) &= \mathcal{F}\{x \mapsto \mathcal{O}_t(T, x)\}(u)e^{iuX_t} \\ &= \frac{1 - \exp\left(\int_t^T \Psi_t(r, u)dr\right)}{u^2 + iu}e^{iuX_t} \end{aligned}$$

for any  $u \in \mathbb{R} \setminus \{0\}$  by Remark I.2.2, Proposition A.4.4 and the translation property for the Fourier transform, which holds for the improper Fourier transform as well. The Fourier transform of the first summand  $A_t(T, x) := O_t(T, x) - \mathcal{O}_t(T, x - X_t)$  exists and equals

$$\mathcal{F}\{x \mapsto A_t(T, x)\}(u) = \frac{1}{iu} - \frac{e^{-X_t}}{iu - 1} - \frac{e^{iuX_t}}{u^2 + iu}$$

for any  $u \in \mathbb{R} \setminus \{0\}$ . Therefore the improper Fourier transform of  $x \mapsto O_t(T, x)$  exists and is given by

$$\mathcal{F}\{x \mapsto O_t(T, x)\}(u) = \frac{1}{iu} - \frac{e^{-X_t}}{iu - 1} - \frac{\exp(iuX_t + \int_t^T \Psi_t(r, u)dr)}{u^2 + iu}, \quad (\text{I.2.6})$$

for any  $u \in \mathbb{R} \setminus \{0\}$ . By Lemmas I.2.7 and I.2.8 we have that the right-hand side of (I.2.6) is a semimartingale, in particular it has càdlàg paths. Remark I.2.2 yields that  $0 \leq P_t(T, K) \leq K$ . Hence  $(P_t(T, K))_{t \in [0, T]}$  is a martingale because it is a bounded local martingale. Let  $(\tau_n)_{n \in \mathbb{N}}$  denote a common localising sequence for  $(C_t(T, 1))_{t \in [0, T]}$  and  $S$ , i.e.  $S^{\tau_n}, C^{\tau_n}(T, 1)$  are uniformly integrable martingales for any  $n \in \mathbb{N}$ . Since  $C_t^{\tau_n}(T, K) \leq C_t^{\tau_n}(T, 1)$  for  $K \in [1, \infty)$ , we have that  $(\tau_n)_{n \in \mathbb{N}}$  is a common localising sequence for all European calls with maturity  $T$  and strike  $K \geq 1$ . The definition of  $O_t(T, x)$  yields that it is a local martingale for any  $x \in \mathbb{R}$  and  $(\tau_n)_{n \in \mathbb{N}}$  is a common localising sequence for  $(O_t(T, x))_{t \in [0, T]}$ ,  $x \in \mathbb{R}$ .

Fix  $\omega \in \Omega$ . Since  $u \mapsto \int_t^T \Psi_t(r, u)(\omega)dr$  is in  $\Pi$  for any  $t \in [0, T]$ , there is an infinitely divisible distribution  $Q$  on  $(\mathbb{R}, \mathcal{B})$  such that

$$\int e^{iux}Q(dx) = \exp\left(\int_t^T \Psi_t(r, u)(\omega)dr\right).$$

Then  $\int e^yQ(dy) = 1$  and

$$O_t(T, x)(\omega) = \begin{cases} \int (S_t(\omega)e^{y-x} - 1)^+ Q(dy) & \text{if } x \geq 0, \\ \int (1 - S_t(\omega)e^{y-x})^+ Q(dy) & \text{if } x < 0, \end{cases}$$

cf. Remark I.2.2. By Corollary A.4.3 we have  $|\int_{-C}^{\infty} e^{iux} O_t(T, x) dx| \leq S_t(\omega) + \frac{1+2|u|}{u^2}$ .

Proposition A.4.5 yields that

$$(\mathcal{F}\{x \mapsto O_t(T, x)\}(u))_{t \in [0, T]}$$

and hence  $(\Phi_t(u))_{t \in [0, T]}$  given by

$$\Phi : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{C}, (\omega, t, u) \mapsto \Phi_t(u)(\omega) := \exp\left(iuX_t(\omega) + \int_t^T \Psi_t(r, u)(\omega) dr\right)$$

are local martingales for any  $u \in \mathbb{R} \setminus \{0\}$ . Since  $u \mapsto \int_t^T \Psi_t(r, u) dr(\omega)$  is in  $\Pi$  for any  $t \in [0, T]$ ,  $\omega \in \Omega$ , its real part is bounded by 0 from above cf. Lemma A.2.12. Hence  $|\Phi_t(u)| \leq 1$  and thus  $(\omega, t) \mapsto \Phi_t(u)(\omega)$  is a true martingale for any  $u \in \mathbb{R} \setminus \{0\}$ . By  $\Phi_t(0) = 1$  it is a martingale for  $u = 0$  as well. Since  $\Phi_T(u) = \exp(iuX_T)$ , the two martingales  $(\Phi_t(u))_{t \in [0, T]}$  and  $(E(\exp(iuX_T) | \mathcal{F}_t))_{t \in [0, T]}$  coincide for any  $u \in \mathbb{R}$ . Thus we have

$$\exp\left(\int_t^T \Psi_t(r, u) dr\right) = \exp(-iuX_t) \Phi_t(u) = E(e^{iu(X_T - X_t)} | \mathcal{F}_t)$$

for any  $u \in \mathbb{R}, t \in [0, T]$ .  $\square$

LEMMA I.2.10 (Drift condition in terms of  $\delta$  and  $\sigma$ ). *If  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the conditional expectation condition, we have the drift condition*

$$\delta_t(T, u) = \Psi_{t-}(t, u) - \psi_t^X(u) - \psi_t^{(X, M)}(u, -i\sigma_t(T, u))$$

outside some  $dP \otimes dt$ -null set for  $T \in \mathbb{R}_+, u \in \mathbb{R}$ . In particular,  $(u, -i\sigma(T, u)) \in \mathcal{U}^{(X, M)}$ .

PROOF. For  $u \in \mathbb{R}$  and  $T \in \mathbb{R}_+$  define the process  $Z_t := iuX_t + \int_t^T \Psi_t(r, u) dr$ . The conditional expectation condition yields that  $\exp(Z_t) = E(e^{iuX_T} | \mathcal{F}_t)$  is a martingale. Hence  $-i \in \mathcal{U}^Z$  and  $\psi_t^Z(-i) = 0$  by Proposition A.2.14. With  $Y_t := \Gamma_t(t, u)$  we obtain

$$\begin{aligned} 0 &= \psi_t^Z(-i) \\ &= \psi_t^{iuX + \Gamma(T, u) - Y}(-i) \\ &= \psi_t^{iuX + \Gamma(T, u)}(-i) - (\Psi_{t-}(t, u) - \psi_t^X(u)) \\ &= \psi_t^{(iuX, \Gamma(T, u))}(-i, -i) - \Psi_{t-}(t, u) + \psi_t^X(u) \\ &= \psi_t^{(X, M)}(u, -i\sigma_t(T, u)) + \delta_t(T, u) - \Psi_{t-}(t, u) + \psi_t^X(u), \end{aligned}$$

where the second equation follows from Lemma I.2.7, the third from Lemmas I.2.8 and A.2.17, the fourth from Lemma A.2.16 and the last from Lemmas A.2.15 and A.2.17.  $\square$

COROLLARY I.2.11 (Consistency condition). *If  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the conditional expectation condition, then it satisfies the consistency condition.*

PROOF. Lemma I.2.10 and the definition of  $\delta$  yield

$$\Psi_{t-}(t, u) = \delta_t(t, u) + \psi_t^X(u) + \psi_t^{(X, M)}(u, 0) = \psi_t^X(u).$$

□

COROLLARY I.2.12 (Drift condition). *If  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the conditional expectation condition, then it satisfies the drift condition.*

PROOF. This follows from Lemma I.2.10 and Corollary I.2.11. □

LEMMA I.2.13. *If the option surface model satisfies the consistency condition, then*

$$\Gamma_t(T, u) = \int_t^T \Psi_t(r, u) dr$$

for any  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ ,  $u \in \mathbb{R}$ .

PROOF. This is a direct consequence of Lemmas I.2.7 and I.2.8. □

LEMMA I.2.14. *If the option surface model satisfies the drift condition, then*

$$\Phi_t(T, u) := \exp(iuX_t + \Gamma_t(T, u))$$

is a local martingale for any  $u \in \mathbb{R}$ ,  $T \in \mathbb{R}_+$ .

PROOF. Fix  $T, u$  and define  $Z_t := iuX_t + \Gamma_t(T, u)$ . By the drift condition and Lemmas A.2.15 – A.2.17 we have

$$\begin{aligned} 0 &= \psi_t^{(X, M)}(u, -i\sigma_t(T, u)) + \delta_t(T, u) \\ &= \psi_t^{(X, \sigma(T, u) \cdot M)}(u, -i) + \delta_t(T, u) \\ &= \psi_t^{(iuX, \Gamma(T, u))}(-i, -i) \\ &= \psi_t^{iuX + \Gamma(T, u)}(-i) \\ &= \psi_t^Z(-i). \end{aligned}$$

Hence  $\exp(Z)$  is a local martingale by Proposition A.2.14. □

LEMMA I.2.15.  *$(X, \Psi_0, \alpha, \beta, M)$  satisfies the drift and consistency conditions if and only if it satisfies the conditional expectation condition.*

PROOF.  $\Leftarrow$ : This is a restatement of Corollaries I.2.12 and I.2.11.

$\Rightarrow$ : Fix  $u \in \mathbb{R}$ ,  $T \in \mathbb{R}_+$ . Lemma A.2.12 implies that the absolute value of

$$\Phi_t(T, u) := \exp\left(iuX_t + \int_t^T \Psi_t(r, u)dr\right)$$

is bounded by 1. By Lemmas I.2.13 and I.2.14,  $\Phi(T, u)$  is a local martingale and hence a martingale. This yields

$$\Phi_t(T, u) = E(\Phi_T(T, u)|\mathcal{F}_t) = E(e^{iuX_T}|\mathcal{F}_t).$$

□

LEMMA I.2.16. *If the option surface model  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the conditional expectation condition, then  $S = e^X$  is a martingale.*

PROOF. For  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$  define  $Y := X_T - X_t$ . Let  $P^{Y|\mathcal{F}_t}$  denote the conditional distribution of  $Y$  given  $\mathcal{F}_t$ . Moreover, let

$$\Phi(u) := E(e^{iuY}|\mathcal{F}_t) = \exp\left(\int_t^T \Psi_t(r, u)dr\right).$$

Fix  $\omega \in \Omega$ . Since  $u \mapsto \int_t^T \Psi_t(r, u)dr(\omega)$  is in  $\Pi$ , we have

$$E_{\tilde{P}}(e^{iuL_1}) = \exp\left(\int_t^T \Psi_t(r, u)dr\right)(\omega)$$

for some Lévy process  $L$  on some filtered probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in \mathbb{R}_+}, \tilde{P})$ . Consequently, we have  $P^{Y|\mathcal{F}_t}(\omega) = \tilde{P}$ , which in turn implies

$$E(e^Y|\mathcal{F}_t)(\omega) = E_{\tilde{P}}(e^{L_1}) = 1.$$

□

LEMMA I.2.17. *If the option surface model  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the conditional expectation condition, it is strongly risk-neutral.*

PROOF. Lemma I.2.16 implies that  $e^X$  is a martingale and in particular that  $e^{X_t}$  is integrable for any  $t \in \mathbb{R}_+$ . Let  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ . We define

$$\begin{aligned} \tilde{C}(K) &:= E((e^{X_T} - K)^+|\mathcal{F}_t), \\ \tilde{\Theta}(x) &:= e^{-(x+X_t)}\tilde{C}(e^{x+X_t}) - (e^{-x} - 1)^+, \\ Y &:= X_T - X_t \end{aligned}$$

for any  $K \in \mathbb{R}_+$ . Obviously we have

$$\tilde{\mathcal{O}}(x) = \begin{cases} E((e^{Y-x} - 1)^+ | \mathcal{F}_t) & \text{if } x \geq 0, \\ E((1 - e^{Y-x})^+ | \mathcal{F}_t) & \text{if } x < 0 \end{cases}$$

and  $E(e^Y | \mathcal{F}_t) = 1$ . Hence Corollary A.4.4, the conditional expectation condition, and the definition of  $\mathcal{O}$  yield

$$\begin{aligned} \mathcal{F}\{x \mapsto \tilde{\mathcal{O}}(x)\}(u) &= \frac{1 - E(e^{iuY} | \mathcal{F}_t)}{u^2 + iu} \\ &= \frac{1 - \exp\left(\int_t^T \Psi_t(r, u) dr\right)}{u^2 + iu} \end{aligned}$$

as well as

$$\begin{aligned} \tilde{\mathcal{O}}(x) &= \mathcal{F}^{-1} \left\{ u \mapsto \frac{1 - \exp\left(\int_t^T \Psi_t(r, u) dr\right)}{u^2 + iu} \right\} (x) \\ &= \mathcal{O}_t(T, x). \end{aligned}$$

Thus we have

$$\tilde{C}(K) = C_t(T, K)$$

for any  $K \in \mathbb{R}_+$ . Consequently, the option surface model  $(X, \Psi_0, \alpha, \beta, M)$  is strongly risk-neutral.  $\square$

PROOF OF THEOREM I.2.6. (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (3) has been shown in Lemma I.2.9.

(3)  $\Leftrightarrow$  (4) is the conclusion of Lemma I.2.15.

(3)  $\Rightarrow$  (1) has been shown in Lemma I.2.17.  $\square$

### 3. Examples and existence results

**3.1. Building blocks.** The goal in this section is to construct risk-neutral option surface models from basic building blocks. In order to model the forward rate

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \beta(s, T) dW_s \quad (\text{I.3.1})$$

in the original HJM setup, one specifies the initial state  $f(0, T)$ ,  $T \in \mathbb{R}_+$ , the driving Brownian motion (or a more general process)  $W$ , and the volatility processes  $(\beta(t, T))_{t \in [0, T]}$ ,

$T \in \mathbb{R}_+$  of the forward rate curve. The drift process, however, is determined by the HJM drift condition

$$\alpha(t, T) = \beta(t, T) \int_t^T \beta(t, r) dr,$$

cf. [21]. Therefore, it should not be fixed beforehand. Moreover, the short rate is determined by the present forward rate through the consistency condition

$$r(t) = f(t, t).$$

Put differently, the initial forward rate curve  $f(0, \cdot)$ , the volatility process  $\beta$  and the driving process  $W$  constitute the “right” building blocks for a HJM model. In line with the original HJM approach, we suggest that the building blocks in our setup should include the initial state of the code-book  $\Psi_0(T, u)$ ,  $T \in \mathbb{R}_+$ ,  $u \in \mathbb{R}$ , a driving Lévy process  $M$ , and the volatility processes  $(\beta_t(T, u))_{t \in [0, T]}$ ,  $T \in \mathbb{R}_+$ ,  $u \in \mathbb{R}$ . Moreover, we fix the initial stock price  $X_0$ , which has no counterpart in interest rate theory. However,  $X_0, \Psi_0, \beta, M$  do not fully determine the model unless the local dependency of  $X$  and  $M$  is specified as well. This is done by also providing the dependent part  $X^\parallel$  of  $X$  relative to  $M$  as defined in Section 3.

The drift and consistency conditions in terms of  $X_0, \Psi_0, \beta, M, X^\parallel$  read as

$$\begin{aligned} \alpha_t(T, u) &= \partial_T \left( \psi^{(X^\parallel, M)} \left( u, -i \int_t^T \beta_t(r, u) dr \right) \right), \\ \psi_t^X(u) &= \Psi_0(t-, u) + \int_0^t \alpha_s(t, u) ds + \int_0^{t-} \beta_s(t, u) dM_s, \end{aligned} \quad (\text{I.3.2})$$

if the derivation of the right hand side of the first equation exists, cf. Remark I.2.5 and Lemma I.3.2 below. Hence we propose  $X_0, \Psi_0, \beta, M, X^\parallel$  as building blocks for our model.

**DEFINITION I.3.1.** We call a risk-neutral option surface model  $(X, \Psi_0, \alpha, \beta, M)$  *consistent* with  $X_0, \Psi_0, \beta, M, X^\parallel$  if  $X_0$  is the initial value of  $X$  and  $X^\parallel$  equals the dependent part of  $X$  relative to  $M$ .

**LEMMA I.3.2 (Modified drift condition).** *The drift condition (I.2.5) is equivalent to*

$$\left( u, -i \int_t^T \beta_t(r, u) dr \right)_{t \in [0, T]} \in \mathcal{U}^{(X^\parallel, M)}$$

where  $\mathcal{U}^{(X^\parallel, M)}$  is defined as in Definition A.2.4 and

$$\int_t^T \alpha_t(r, u) dr = \psi_t^X(u) - \psi_t^{(X^\parallel, M)} \left( u, -i \int_t^T \beta_t(r, u) dr \right) \quad (\text{I.3.3})$$

outside some  $dP \otimes dt$ -null set for any  $T \in \mathbb{R}_+$ ,  $u \in \mathbb{R}$ .

PROOF. Let  $X^\perp$  denote the independent part of  $X$  relative to  $M$  in the sense of Section 3. Lemmas A.2.16 and A.3.2 yield

$$\begin{aligned} & \psi_t^{(X,M)}\left(u, -i \int_t^T \beta_t(r, u) dr\right) \\ &= \psi_t^{(X^\parallel, M, X^\perp, 0)}\left(u, -i \int_t^T \beta_t(r, u) dr, u, -i \int_t^T \beta_t(r, u) dr\right) \\ &= \psi_t^{(X^\parallel, M, X^\perp)}\left(u, -i \int_t^T \beta_t(r, u) dr, u\right) \\ &= \psi_t^{(X^\parallel, M)}\left(u, -i \int_t^T \beta_t(r, u) dr\right) + \psi_t^{X^\perp}(u) \end{aligned}$$

and similarly

$$\psi_t^X(u) = \psi_t^{(X^\parallel, X^\perp)}(u, u) = \psi_t^{X^\parallel}(u) + \psi_t^{X^\perp}(u).$$

This implies

$$\psi_t^X(u) - \psi_t^{(X,M)}\left(u, -i \int_t^T \beta_t(r, u) dr\right) = \psi_t^{X^\parallel}(u) - \psi_t^{(X^\parallel, M)}\left(u, -i \int_t^T \beta_t(r, u) dr\right)$$

for any  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ ,  $u \in \mathbb{R}$ .  $\square$

**3.2. Existence and uniqueness result.** With the general programme in mind – i.e. constructing a risk-neutral model from the building blocks – we aim at deriving existence and uniqueness results under simple conditions. In this section we will assume that the driving process  $M$  and the dependent part  $X^\parallel$  are Lévy processes. Then the law of  $(X^\parallel, M)$  is fully determined by its Lévy exponent  $\gamma := \psi^{(X^\parallel, M)}$  which we will use as building block instead of the processes itself. In addition we will assume that the drift coefficient  $\beta$  depends only on time  $t$  and on the state of the code-book, i.e.

$$\beta_t(T, u)(\omega) = b(t, \Psi_t(T, u)(\omega))$$

where  $\omega \in \Omega$ ,  $t \in \mathbb{R}_+$ ,  $T \in (t, \infty)$ ,  $u \in \mathbb{R}$ . The functions  $\gamma, b$  together with the initial states  $x_0$  for the return and  $\psi_0$  for the code-book process serve as building blocks, cf. Definition I.3.3. For ease of notation, we focus on one-dimensional driving processes  $M$ . However, the finite-dimensional case can be treated along the same lines.

Let  $E$  be the set of functions  $\psi$  such that

$$\|\psi\| := \sup \left\{ \frac{|\psi(u)|}{1 + |u|^3} : u \in \mathbb{R} \right\} < \infty$$

and  $u \mapsto \frac{\psi(u)}{1 + |u|^3}$  vanishes at infinity. Then  $(E, \|\cdot\|)$  is a separable Banach-space and  $\Pi$  is a closed subset. Indeed, if  $(\phi_n)_{n \in \mathbb{N}}$  is a sequence in  $\Pi$  which converges to some  $\psi \in E$ , then it

converges locally uniformly. Thus the continuity theorem [31, Theorem 3.6.1] yields that it converges to the exponent of a characteristic function and [31, Theorem 5.3.3] implies that it converges to an element in  $\Pi$ . Let  $L^1(E)$  be the set of continuous functions

$$\psi : \mathbb{R}_+ \rightarrow E$$

such that  $\int_0^t \|\psi_s\| ds < \infty$  for all  $t \in \mathbb{R}_+$  and

$$L^1(\Pi) := \{\phi \in L^1(E) : \forall t \in \mathbb{R}_+ \phi_t \in \Pi\}.$$

Note that if  $\psi \in L^1(\Pi)$ , then

$$\int_t^T \psi_s ds \in \bar{\Pi}^E = \Pi$$

for all  $t, T \in \mathbb{R}_+, t \leq T$ . For  $\psi \in L^1(E), T \in \mathbb{R}_+$  we define the complete semi-norm

$$\|\psi\|_T := \int_0^T \|\psi(s)\| ds.$$

**DEFINITION I.3.3.** We call a quadruple  $(x_0, \psi_0, f, \gamma)$  *building blocks* if

- (1)  $x_0 \in \mathbb{R}$ ,
- (2)  $\psi_0 \in L^1(\Pi)$ ,
- (3)  $f : \mathbb{R}_+ \times L^1(E) \rightarrow L^1(\Pi)$  is measurable and for any  $T \in \mathbb{R}_+$  there is a constant  $c_T \in \mathbb{R}_+$  such that we have

$$\|f(t, \psi_1) - f(t, \psi_2)\|_T \leq c_T \|\psi_1 - \psi_2\|_T$$

for any  $t \in [0, T], \psi_1, \psi_2 \in L^1(E)$ ,

- (4)  $f(t, \psi)(T, u) = 0$  for all  $t, T \in \mathbb{R}_+, T < t, u \in \mathbb{R}, \psi \in L^1(E)$ ,
- (5)  $\gamma : \mathbb{R}^2 \rightarrow \mathbb{C}$  is a differentiable Lévy exponent of a 1 + 1-dimensional process  $(X^\parallel, M)$  whose semimartingale projection of  $X^\parallel$  with respect to  $M$  is  $X^\parallel$ ,  $M$  is a pure-jump subordinator and  $\partial_2 \gamma$  is bounded.

**DEFINITION I.3.4.** An option surface model  $(X, \Psi_0, \alpha, \beta, M)$  is said to be *compatible* with building blocks  $(x_0, \psi_0, f, \gamma)$  if

- $X_0 = x_0$ ,
- $\Psi_0 = \psi_0$ ,
- $\beta_t(T, u) = f(t, \Psi_{t-})(T, u)$  for any  $t, T \in \mathbb{R}_+, u \in \mathbb{R}$ ,
- $\psi^{(X^\parallel, M)} = \gamma$ ,

where  $X^\parallel$  is the semimartingale projection of  $X$  relative to  $M$ .



REMARK I.3.5. Note that no requirements are made for the drift coefficient  $\alpha$ . However, in risk-neutral models  $\alpha$  is specified by the drift condition.

We will state an existence and uniqueness result provided that  $M$  is a subordinator. For the statement of Theorem I.3.8 we need some lemmas.

LEMMA I.3.6. *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks. Then*

$$g(t, \psi)(T, u) := -\partial_T \left( \gamma \left( u, -i \int_t^T f(t, \psi)(r, u) dr \right) \right)$$

defines a measurable function  $g : \mathbb{R}_+ \times L^1(E) \rightarrow -L^1(\Pi)$ . Moreover, for all  $T \in \mathbb{R}_+$  there is a constant  $c_T$  such that for any  $t \in [0, T]$ ,  $\psi_1, \psi_2 \in L^1(E)$  we have

$$\|g(t, \psi_1) - g(t, \psi_2)\|_T \leq c_T \|\psi_1 - \psi_2\|_T$$

and  $g(t, \psi)(T, u) = 0$  for all  $t, T \in \mathbb{R}_+, T < t, u \in \mathbb{R}^d, \psi \in L^1(E)$ .

PROOF. Let  $t, T \in \mathbb{R}_+, t \leq T, \psi \in L^1(E)$  and  $\phi := f(t, \psi)$ . Then

$$\lambda(u) := g(t, \psi)(T, u) = i(f(t, \psi)(T, u)) \partial_2 \gamma \left( u, -i \int_t^T f(t, \psi)(r, u) dr \right)$$

thus the Lipschitz-condition for  $g$  holds and  $g(t, \psi)(T, u) = 0$  for all  $t, T \in \mathbb{R}_+, T < t, u \in \mathbb{R}^d, \psi \in L^1(E)$ . Moreover,  $\phi \in L^1(\Pi)$ . Thus

$$\int_t^T f(t, \psi)(r, \cdot) dr = \int_t^T \phi_r dr \in \Pi.$$

Let  $\chi$  be a truncation function on  $\mathbb{R}$  and  $(b, c, F)$  be the Lévy-Khintchine triplet such that

$$\gamma(u) = i(u_1 b_1 + u_2 b_2) - \frac{1}{2} u_1 c_{11} u_1 + \int (e^{i(u_1 h_1 + u_2 h_2)} - 1 - i u_1 \chi(h_1)) F(dh)$$

for all  $u \in \mathbb{R}^2$ . Such a triplet exists due to [36, Theorem 21.5] and  $b_2 \geq 0$  and  $F$  is concentrated on  $\mathbb{R} \times \mathbb{R}_+$ . For all  $h \in \mathbb{R} \times \mathbb{R}_+$  define

$$\eta_h : \mathbb{R} \rightarrow \mathbb{C}, u \mapsto (e^{i u h_1 + \lambda(u) h_2} - 1 - i u \chi(h_1)).$$

[31, Lemma 5.4.1] yields  $\eta_h \in \Pi$  for all  $h \in \mathbb{R} \times \mathbb{R}_+$ . Moreover,  $h \mapsto \eta_h$  is a continuous function with values in  $\Pi$ , thus  $(u \mapsto \int (e^{i(u h_1 + \lambda(u) h_2)} - 1 - i u \chi(h_1)) F(dh)) \in \Pi$ . Hence  $(u \mapsto \gamma(u, -i \lambda(u))) \in \Pi$ . Now we can apply [31, Theorem 5.3.3] to conclude that  $\lambda \in -\Pi$ .  $\square$

PROPOSITION I.3.7. *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks,  $g$  as in Lemma I.3.6 and  $M$  be a subordinator with characteristic exponent  $\gamma(0, \cdot)$ . Then the SDE*

$$d\Theta_t = g(t, \Theta_t)dt + f(t, \Theta_{t-})dN_t, \quad \Theta_0 = \psi_0$$

*has an a.s. unique  $\text{span}(L^1(\Pi))$ -valued solution.*

The proof is a simple modification of the proof of [34, Theorem IX.2.1]:

PROOF. Let  $T \in \mathbb{R}_+$ . For any  $L^1(E)$ -valued process  $\Theta$  let

$$S(\Theta)_t := \psi_0 + \int_0^t g(s, \Theta_s)ds + \int_0^t f(s, \Theta_{s-})dN_s$$

and for any process  $\Psi$  we define

$$\phi(\Theta, \Psi)_t := \|\Theta_t - \Psi_t\|_T$$

$t \in [0, T]$ , hence  $\phi(\Theta, \Psi)_t$  is an  $\mathbb{R}_+$ -valued random variable. The assumed Lipschitz conditions yield

$$\phi(S(\Theta), S(\Psi))_t \leq c_T \left( \int_0^t \phi(\Theta, \Psi)_s d(I + N)_s \right).$$

Now we define recursively  $\Theta_0 : \mathbb{R}_+ \rightarrow E, t \mapsto \psi_0$  and  $\Theta_{n+1} := S(\Theta_n)$  for all  $n \in \mathbb{N}$ . Then a simple inductive argument shows

$$\phi(\Theta_n, \Theta_{n+1})_t \leq Dc_T^n T^n / n!,$$

where  $D := \phi(\Theta_0, \Theta_{n+1})$ . Thus  $(\Theta_n(t))_{n \in \mathbb{N}}$  has a limit  $\Theta(t)$  for all  $t \in [0, T]$ . Since this holds for arbitrary  $T$  it converges for all  $t \in \mathbb{R}_+$  and the limit is a fixed point for the mapping  $S$ . Hence  $\Theta$  is a solution to the SDE above. For any other solution and hence fixed point  $\Psi$  we have

$$\phi(\Psi, \Theta)_t \leq c_T \left( \int_0^t \phi(\Theta, \Psi)_s d(I + N)_s \right)$$

for any  $t \in [0, T]$ . Thus Grönwall yields  $\Psi_t = \Theta_t$  for all  $t \in [0, T]$ . Again this is true for arbitrary  $T$ , hence  $\Psi = \Theta$ .

Finally observe that  $\int_0^t f(s, \Theta_{s-})dN_s$  is  $L^1(\Pi)$ -valued and  $\int_0^t g(s, \Theta_s)ds$  is  $-L^1(\Pi)$ -valued since  $\Pi$  is closed in  $E$ . Hence  $\Theta$  is  $\text{span}(L^1(\Pi))$ -valued.  $\square$

THEOREM I.3.8. *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks and  $g$  be the function given in Lemma I.3.6.*

- (1) *Then any two risk-neutral option surface models  $(X, \Psi_0, \alpha, \beta, M)$ ,  $(\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$  which are compatible with the building blocks have the same law, i.e.  $(X, \Psi, M)$  and  $(\tilde{X}, \tilde{\Psi}, \tilde{M})$  have the same law.*
- (2) *Assume that there is a Lévy process  $N$  with Lévy exponent  $\gamma(0, \cdot)$  such that the  $L^1(E)$ -valued solution to the SDE*

$$d\Theta_t = g(t, \Theta_t)dt + f(t, \Theta_{t-})dN_t, \quad \Theta_0 = \psi_0 \quad (\text{I.3.4})$$

*has values in  $\gamma(\cdot, 0) + L^1(\Pi)$ . Then there is a risk-neutral option surface model which is compatible with the building blocks.*

REMARK I.3.9. The uniqueness statement (1) is based on simple conditions. However, the existence part (2) requires that the solutions of the SDE (I.3.4) stay in the convex cone  $\gamma(\cdot, 0) + L^1(\Pi)$  which is, of course, not necessarily fulfilled.

PROOF. We first start to construct the code-book process together with its driving process on a probability space  $(\Omega^{(1)}, \mathcal{F}^{(1)}, P^{(1)})$ . The return process will be the coordinate process on the path space  $(\mathbb{D}, \mathcal{D}, (\mathcal{D}_t)_{t \in \mathbb{R}_+})$  and the final measurable space will be the product space. In order to construct a probability measure on the product space we make use of the conditional expectation condition and find a measure such that the coordinate process is a conditional PII (cf. [25, Section II.6] for the definition of conditional PII).

*Construction of the code-book process:* By [25, Theorem III.2.16] there is a PII  $(X^\parallel, M)$  on a filtered probability space  $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t \in \mathbb{R}_+}, P^{(1)})$  such that its local exponent is  $\gamma$ . Let  $\Psi$  be the solution of the SDE

$$d\Psi_t = f(t, \Psi_{t-})dt + g(t, \Psi_{t-})dM_t, \quad \Psi_0 = \psi_0$$

given by Proposition I.3.7. We define

$$\begin{aligned} \alpha_t(T, u) &:= g(t, \Psi_{t-})(T, u), \\ \beta_t(T, u) &:= f(t, \Psi_{t-})(T, u) \end{aligned}$$

where  $t, T \in \mathbb{R}_+$ ,  $u \in \mathbb{R}$ . Then we have

$$\Psi_t(T, u) = \Psi_0(T, u) + \int_0^t \alpha_s(T, u)ds + \int_0^t \beta_s(T, u)dM_s$$

for  $t, T \in \mathbb{R}_+$ ,  $t \leq T$ ,  $u \in \mathbb{R}$ .

*Construction of the return process:* Let  $(\mathbb{D}, \mathcal{D}, (\mathcal{D}_t)_{t \in \mathbb{R}_+})$  be the Skorokhod space of real valued càdlàg functions starting in  $x_0$ . Let  $X^\perp$  be the coordinate process and define

$$\begin{aligned}\Omega &:= \Omega_1 \times \mathcal{D}, \\ \mathcal{F} &:= \mathcal{F}^{(1)} \otimes \mathcal{D}, \quad \text{and} \\ \mathcal{F}_t &:= \bigcap_{s>t} \mathcal{F}_s^{(1)} \otimes \mathcal{D}_s.\end{aligned}$$

For  $t \in \mathbb{R}_+, u \in \mathbb{R}$  let

$$\eta(t, u) : \Omega \rightarrow \mathbb{C}, \omega \mapsto (\Psi_{t-}(t, u) - \psi^{X^\parallel}(u))(\omega).$$

By assumption  $\eta$  is an  $L^1(\Pi)$ -valued random variable. Let  $\omega \in \Omega^{(1)}$ . [25, Theorem III.2.16, Theorem II.4.15] yields that there is a probability measure  $P_\omega^{(2)}$  such that  $X^\perp - X^\perp(0)$  is a PII such that  $X^\perp(t) - X^\perp(0)$  has characteristic function  $u \mapsto \exp\left(\int_0^t \eta(s, u)(\omega) ds\right)$ . Let  $A \in \mathcal{D}$ . Then

$$\Omega_1 \rightarrow \mathbb{R}_+, \omega \mapsto P_\omega^{(2)}(A)$$

is measurable because  $\eta$  is measurable. Hence  $P^{(2)}$  is a transition kernel from  $\Omega_1$  to  $\Omega$ . We define  $P(d\omega_1, d\omega_2) := P^{(1)}(d\omega_1)P_{\omega_1}^{(2)}(d\omega_2)$ .

By abuse of notation we will use the same letters for the process  $M, \Psi, X^\parallel, X^\perp$  embedded to the filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P),$$

i.e. we denote the process

$$\widetilde{M} : (\omega, t) \mapsto M_t(\omega_1)$$

also by  $M$ , etc. Let  $X := X^\parallel + X^\perp$ .

Observe that  $X^\perp$  is a  $\mathcal{F} \otimes \{\emptyset, \mathbb{D}\}$ -conditional PII and let  $(b, c, F)$  be its local characteristics relative to some truncation function  $\chi$ . Then [25, Theorem II.6.6] yields

$$\eta(t, u)(\omega_1) = i u b_t(\omega) - \frac{u^2}{2} c_t(\omega) + \int (e^{iuh} - 1 - iu\chi(h)) F(dh)$$

for almost all  $\omega \in \Omega, t \in \mathbb{R}_+, u \in \mathbb{R}$ .  $X^\perp$  and  $(b, c, F)$  are  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$  adapted. Hence Stricker's Theorem [33, Theorem II.2.4] yields that  $X^\perp$  is a semimartingale with respect to the filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ . Its local characteristics is given by  $(b, c, F)$ .

Let  $r, s, t \in \mathbb{R}_+, r < s \leq t, u \in \mathbb{R}^2$ . Then  $Z_1 := \exp(iu(X^\parallel, M)_t - (X^\parallel, M)_s)$  is independent of the filtration  $\mathcal{F}_r^{(1)} \otimes \{\emptyset, \mathbb{D}\}$ . Let  $Z_2$  be  $\mathcal{A} := \mathcal{F}_r^{(1)} \otimes \{\emptyset, \mathbb{D}\}$ -measurable and  $Z_3$  be  $\mathcal{B} := \{\emptyset, \Omega^{(1)}\} \otimes \mathcal{D}_r$ -measurable. Then

$$E(Z_3 | \mathcal{F}^{(1)} \otimes \{\emptyset, \mathbb{D}\}) = E(Z_3 | \mathcal{A}).$$

Thus we have

$$\begin{aligned} E(Z_1 Z_2 Z_3) &= E(E(Z_3 | \mathcal{F}^{(1)} \otimes \{\emptyset, \mathbb{D}\}) Z_1 Z_2) \\ &= E(E(Z_3 | \mathcal{A}) Z_1 Z_2) \\ &= E(E(E(Z_3 | \mathcal{A}) Z_1 Z_2 | \mathcal{A})) \\ &= E(E(Z_1 | \mathcal{A}) Z_2 E(Z_3 | \mathcal{A})) \\ &= E(Z_1) E(Z_2 E(Z_3 | \mathcal{A})) \\ &= E(Z_1) E(Z_2 Z_3). \end{aligned}$$

Thus  $E(Z_1)$  is conditional expectation of  $Z_1$  given  $\sigma(\mathcal{A}, \mathcal{B}) = \mathcal{F}_r^{(1)} \otimes \mathcal{D}_r$ . However, this is true for all  $r < s$  and hence  $E(Z_1)$  is conditional expectation of  $Z_1$  given  $\mathcal{F}_r$ . Finally, right continuity of  $(X^\parallel, M)$  yields

$$E(\exp(iu(X^\parallel, M)_t - (X^\parallel, M)_r) | \mathcal{F}_r) = E(\exp(iu(X^\parallel, M)_t - (X^\parallel, M)_r)).$$

In particular,  $(X^\parallel, M)$  is a Lévy process with characteristic exponent  $\gamma$  on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ .

*Compatibility of the constructed model:* We have  $\beta_t = f(t, \Psi_{t-})$ ,  $X_0 = x_0$ ,  $\Psi_0 = \psi_0$  and  $\gamma = \psi^{(X^\parallel, M)}$ . We have to show that the semimartingale projection of  $X$  relative to  $M$  is  $X^\parallel$ . Since  $X^\parallel$  is the semimartingale projection of  $X^\parallel$  relative to  $M$ , we have to show that  $M$  and  $X^\perp$  are locally independent. Since  $M$  is a subordinator it is sufficient to show that

$$P(\exists t \in \mathbb{R}_+ : \Delta M_t \neq 0, \Delta X_t \neq 0) = 0.$$

Let  $J := \{t : \Delta M_t \neq 0\}$  the set of jump times of  $M$ . Then  $J$  is a.s. countable and

$$\begin{aligned} P(\exists t \in \mathbb{R}_+ : \Delta M_t \neq 0, \Delta X_t \neq 0) &\leq E\left(\sum_{s \in J} 1_{\{\Delta X_s \neq 0\}}\right) \\ &= E\left(\sum_{s \in J} P(\Delta X_s \neq 0 | M)\right) \\ &= 0 \end{aligned}$$

because

$$E(\exp(iu\Delta X_s)|\mathcal{F}^{(1)} \otimes \{\emptyset, \mathbb{D}\}) = \exp\left(\int_{s-}^s \eta(r, u)dr\right) = 0$$

and hence  $P(\Delta X_s \neq 0|M) = 0$ .

*Risk neutrality of the constructed model:* The constructed model satisfies the consistency and the drift condition and hence Theorem I.2.6 yields risk-neutrality.

*Uniqueness:* Let  $(X, \Psi_0, \alpha, \beta, M), (\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$  be risk-neutral option surface models which are compatible with the building blocks. Let  $X^\parallel$  be the semimartingale projection of  $X$  relative to  $M$ . By compatibility we have  $\psi^{(X^\parallel, M)} = \gamma$ . Thus  $(X^\parallel, M)$  is a Lévy process. Let  $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$  be the filtration generated by  $(X^\parallel, M)$ , i.e.  $\mathcal{G}_t$  is the completion of

$$\bigcap_{s>t} \sigma((X^\parallel, M)_r : r \leq s).$$

Since the option surface model  $(X, \Psi_0, \alpha, \beta, M)$  is risk-neutral Theorem I.2.6 yields it satisfies the drift condition, the consistency condition and the conditional expectation condition. Since the model is compatible, we have

$$\begin{aligned} \beta_t(T, u) &:= f(t, \Psi_{t-})(T, u) \quad \text{and} \\ \alpha_t(T, u) &:= g(t, \Psi_{t-})(T, u) \end{aligned}$$

a.s for all  $T \in \mathbb{R}_+, u \in \mathbb{R}$  and almost all  $t \in [0, T]$ . Since  $\Psi_t(T, \cdot) \in \Pi$  a.s. we have  $\Psi_t \in L^1(\Psi)$  a.s. Hence  $\Psi$  is a solution to the SDE

$$d\Psi_t = f(t, \Psi_{t-})dt + g(t, \Psi_{t-})dM_t, \quad \Psi_0 = \psi_0.$$

By Proposition I.3.7  $\Psi$  is the unique pathwise solution to the SDE. Thus  $\Psi_t$  is adapted to the filtration  $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$ .

We denote the  $P$ -completion of the  $\sigma$ -algebra generated by  $\mathcal{E} \subset \mathcal{F}$  as  $\sigma_P(\mathcal{E})$ . Let  $\mathcal{H}_t := \sigma_P(\mathcal{F}_t, \mathcal{G}_\infty)$  for any  $t \in \mathbb{R}_+$  where  $\mathcal{G}_\infty := \sigma_P(\mathcal{G}_t : t \in \mathbb{R}_+)$ . We now observe that the filtration  $(\mathcal{H}_t)_{t \in \mathbb{R}_+}$  is right-continuous. Indeed, let  $t \in \mathbb{R}_+$  and define the  $\sigma$ -algebra of the  $(X^\parallel, M)_t$ -future by  $\mathcal{G}_t^+ := \sigma_P((X^\parallel, M)_s - (X^\parallel, M)_t : s \geq t)$ .  $\mathcal{G}_t^+$  is independent of  $\mathcal{F}_t$  because  $(X^\parallel, M)$  is a Lévy process with respect to the filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ . We have

$$\mathcal{G}_\infty = \sigma_P(M_s : s \in \mathbb{R}_+) = \sigma_P(\sigma(M_s : s \leq t), \sigma(M_s - M_t : s \geq t)) = \sigma_P(\mathcal{G}_t, \mathcal{G}_t^+).$$

Hence  $\mathcal{H}_t = \sigma_P(\mathcal{F}_t, \mathcal{G}_t^+)$  and thus every element in  $L^2(\mathcal{H}_t)$  is the orthogonal sum of a random variable in  $L^2(\mathcal{F}_t)$  and a centred random variable in  $L^2(\mathcal{G}_t^+)$  where those random

variables are given by the Hilbert space projection, i.e. by conditional expectation. In the same way we have

$$Y = E(Y) + E(Y - E(Y)|\mathcal{G}_t) + E(Y - E(Y)|\mathcal{G}_t^+)$$

for all  $Y \in L^2(\mathcal{G}_\infty)$ . Let  $A \in \bigcap_{s>t} \mathcal{H}_s$ . Then  $A_s := E(1_A|\mathcal{F}_s)$  is a martingale and for all  $s \in (t, \infty)$  we have

$$\begin{aligned} 1_A &= E(1_A|\mathcal{H}_s) \\ &= E(1_A|\mathcal{F}_s) + E(1_A - P(A)|\mathcal{G}_s^+) \\ &= A_s + E(E(1_A - P(A)|\mathcal{G}_\infty)|\mathcal{G}_s^+) \\ &= A_s + E(1_A - P(A)|\mathcal{G}_\infty) - E(1_A - P(A)|\mathcal{G}_s). \end{aligned}$$

By taking the limit  $s \rightarrow t$  we have

$$\begin{aligned} 1_A &= A_t + E(1_A - P(A)|\mathcal{G}_\infty) - E(1_A - P(A)|\mathcal{G}_t) \\ &= A_t + E(1_A - P(A)|\mathcal{G}_t^+) \\ &= E(1_A|\mathcal{H}_t). \end{aligned}$$

Thus  $A \in \mathcal{H}_t$ .

Now we show that  $X^\perp$  is a process with  $\mathcal{G}_\infty$ -conditionally independent increments with respect to the right-continuous filtration  $(\mathcal{H}_t)_{t \in \mathbb{R}_+}$  in the sense of [25, Section II.6]. It is sufficient to show that [25, Section II.6.1] holds for the functions

$$(\mathbb{R} \rightarrow \mathbb{C}, x \mapsto \exp(iux))_{u \in \mathbb{R}}.$$

Let  $u \in \mathbb{R}$  and  $Z$  an  $\mathcal{H}_t$ -measurable and bounded random variable and define

$$Y := \exp(iu(X_T^\perp - X_t^\perp)).$$

Then  $Y = E(Y) + Y_1 + Y_2 + Y_3$  where

$$\begin{aligned} Y_1 &:= E(Y - E(Y)|\mathcal{F}_t), \\ Y_2 &:= E(Y - E(Y)|\mathcal{G}_t^+) \quad \text{and} \\ Y_3 &:= Y - E(Y|\mathcal{H}_t) \end{aligned}$$

and this is true because  $\mathcal{F}_t, \mathcal{G}_t^+$  are independent and  $\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{G}_t^+)$ . The conditional expectation condition yields  $Y_1$  is  $\mathcal{G}_t$ -measurable and thus  $Y_1 = E(Y - E(Y)|\mathcal{G}_t)$ . In particular,

$Y_1, Y_2$  are  $\mathcal{G}_\infty$ -measurable and  $Y_1 + Y_2 = (Y - E(Y)|\mathcal{G}_\infty)$ . Thus

$$\begin{aligned} E(YZ|\mathcal{G}_\infty) &= E(Y)E(Z|\mathcal{G}) + E(Y_1Z|\mathcal{G}_\infty) + E(Y_2Z|\mathcal{G}_\infty) + E(Y_3Z|\mathcal{G}_\infty) \\ &= E(Y)E(Z|\mathcal{G}) + Y_1E(Z|\mathcal{G}_\infty) + Y_2E(Z|\mathcal{G}_\infty) + E(E(Y_3|\mathcal{H}_t)Z|\mathcal{G}_\infty) \\ &= (E(Y) + Y_1 + Y_2)E(Z|\mathcal{G}_\infty) \\ &= E(Y|\mathcal{G}_\infty)E(Z|\mathcal{G}_\infty). \end{aligned}$$

Finally [25, Theorem III.6.6] yields the claim because the same arguments apply to the option surface model  $(\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$ .  $\square$

However, in general there is no reason for the solution of the SDE (I.3.4) to stay in the convex cone  $\gamma(\cdot, 0) + L^1(\Pi)$ . In that case, there is no option surface model compatible with the building blocks. However, we introduce a weaker notion of compatibility, namely  $\tau$ -weakly compatibility (see below) which means that the diffusion coefficient is of the form  $\beta_t = f(t, \Psi_t)$  only until some stopping time  $\tau$ . Then we want to prove that there is always a  $\tau$ -weakly compatible model such that the stopping time  $\tau$  is maximal in some sense.

DEFINITION I.3.10. An option surface model  $(X, \Psi_0, \alpha, \beta, M)$  is said to be  $\tau$ -weakly compatible with building blocks  $(x_0, \psi_0, f, \gamma)$  if

- $\tau$  is a stopping time,
- $X_0 = x_0$ ,
- $\Psi_0 = \psi_0$ ,
- $\beta_t(T, u)(\omega) = f(t, \Psi_{t-}(\omega))(T, u)$  for any  $t, T \in \mathbb{R}_+$ ,  $u \in \mathbb{R}$  with  $t \leq \tau(\omega)$  and
- $\psi^{(X^\parallel, M)} = \gamma$ .

where  $X^\parallel$  is the semimartingale projection of  $X$  relative to  $M$ . Let  $g$  be as in Lemma I.3.6 and  $\Theta$  be the solution of the SDE

$$d\Theta_t = g(t, \Theta_t)dt + f(t, \Theta_{t-})dN_t, \quad \Theta_0 = \psi_0.$$

The option surface model  $(X, \Psi_0, \alpha, \beta, M)$  is said to be *maximal weakly compatible* if it is  $\tau$ -weakly compatible and

$$\tau = \inf\{t \in \mathbb{R}_+ : \Theta_t \notin \gamma(\cdot, 0) + L^1(\Pi)\}.$$

PROPOSITION I.3.11. *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks. Then there is a risk-neutral and maximal weakly compatible option surface model.*



PROOF. Let  $(X^\parallel, M)$  be a Lévy process with characteristic exponent  $\gamma$  and  $g$  be as in Lemma I.3.6. Let  $\Theta$  be the solution of the SDE (I.3.4) and

$$\tau := \inf\{t : \Theta_t \notin \gamma(\cdot, 0) + L^1(\Pi)\}.$$

Let  $\Psi_t := \Theta_{t \wedge \tau}$ . Along the same lines as in the proof of I.3.8 we can now construct a risk-neutral model  $(X, \Psi_0, \alpha, \beta, M)$ . Then  $(X, \Psi_0, \alpha, \beta, M)$  is  $\tau$ -compatible and risk-neutral and hence maximal weakly compatible.  $\square$

**THEOREM I.3.12.** *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks. Then there is a maximal weakly compatible and risk-neutral option surface model. Moreover, any two risk-neutral option surface models  $(X, \Psi_0, \alpha, \beta, M)$ ,  $(\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$  which are maximal weakly compatible with the building blocks have the same law, i.e.  $(X, \Psi, M)$  and  $(\tilde{X}, \tilde{\Psi}, \tilde{M})$  have the same law. If there is a risk-neutral and compatible option surface model, then any risk-neutral option surface model which is maximal weakly compatible is compatible.*

PROOF. The first part is a restatement of Proposition I.3.11. Let  $g$  be given as in Lemma I.3.6. Let  $S$  be the set of  $(t, \psi) \in \mathbb{R}_+ \times L^1(E)$  such that for all  $\varepsilon > t$  there is  $\delta \in [t, \infty)$  such that the solution  $\Sigma : [t, \infty) \rightarrow L^1(E)$  of the initial value problem

$$d\Sigma_s = g(s, \Sigma_s)ds, \quad \Sigma_t = \psi$$

has the property  $\Sigma_\delta \notin L^1(\Pi)$ . Then  $\mathbb{R}_+ \times (L^1(E) \setminus L^1(\Pi)) \subset S$ . Let

$$\begin{aligned} \sigma &:= \inf\{t : (t, \Theta_t) \in S\} \quad \text{and} \\ \tau &:= \inf\{t : \Theta_t \notin \gamma(\cdot, 0) + L^1(\Pi)\}, \end{aligned}$$

where  $\Theta$  is the solution of the SDE

$$d\Theta_t = g(t, \Theta_t)dt + f(t, \Theta_t)dM_t, \quad \Theta_0 = \psi.$$

Then  $\tau = \sigma$  because  $f$  is  $L^1(\Pi)$  valued and  $M$  is a subordinator. Hence  $\tau$  is an entrance time depending only on the subordinator  $M$ . Thus the distribution of  $(\tau, X^\parallel, M)$  is determined by the building blocks. Since  $\Psi_t = \Theta_{t \wedge \tau}$  this is also true for  $(\Psi, X^\parallel, M)$ . Since  $X^\perp$  is a  $\sigma(X^\parallel, M)$ -conditional PII this is true for  $(X^\perp, \Psi, X^\parallel, M)$  and in particular for  $(X, \Psi, M)$ .

Now assume that  $(\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$  is a risk-neutral and compatible model and define the stopping time

$$\tilde{\sigma} := \inf\{t : (t, \tilde{\Theta}_t) \in S\}$$

where  $\tilde{\Theta}$  is the solution of the SDE

$$d\tilde{\Theta}_t = g(t, \tilde{\Theta}_t)dt + f(t, \tilde{\Theta}_t)d\tilde{M}_t, \quad \tilde{\Theta}_0 = \psi_0.$$

Then  $\tilde{\sigma} = \infty$  and hence  $(\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$  is a maximal weakly compatible model and as above we conclude  $\tau = \infty$ . Hence  $\Theta = \Psi$  and thus  $(X, \Psi_0, \alpha, \beta, M)$  is compatible as well.  $\square$

**3.3. Vanishing coefficient process  $\beta$ .** The simplest conceivable code-book model (I.1.10) is obtained for building blocks  $(x_0, \psi_0, f, \gamma)$  where  $f = 0$  or equivalently  $\gamma = 0$ . Not surprisingly, it leads to constant code-book processes and hence to the simple model class that we used to motivate option surface models in Section 1.1.

**THEOREM I.3.13.** *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks with  $\gamma = 0$  and  $f = 0$ . Then there is a compatible risk-neutral option surface model  $(X, \Psi_0, \alpha, \beta, M)$ . Moreover,  $X - X_0$  is a PII with characteristic function*

$$E(\exp(iu(X_t - X_0))) = \exp\left(\int_0^t \psi_0(s, u)ds\right).$$

**PROOF.** Define  $g$  as in Lemma I.3.6, i.e.  $g = 0$ . Hence, Theorem I.3.8 yields existence of a compatible risk-neutral option surface model  $(X, \Psi_0, \alpha, \beta, M)$ . We have

$$\Psi_t(T, u) = \psi_0(T, u)$$

for all  $t, T \in \mathbb{R}_+, u \in \mathbb{R}$  because  $f = 0, g = 0$ . Theorem I.2.6 yields that  $(X, \Psi_0, \alpha, \beta, M)$  satisfies the conditional expectation condition. Hence

$$E\left(e^{iu(X(T) - X(t))} | \mathcal{F}_t\right) = \exp\left(\int_t^T \psi_0(r, u)dr\right).$$

In particular,  $X - X(0)$  is a PII by [25, Theorem II.4.15].  $\square$

The Black-Scholes model is obtained for a particular choice of the initial state of the code-book.

**EXAMPLE I.3.14 (Black-Scholes model).** If we choose  $\psi_0(T, u) := -\frac{iu+u^2}{2}\sigma^2$  in Theorem I.3.13 for some  $\sigma > 0$ , we obtain  $E(e^{iuX_T}) = \exp(iuX_0 - iu\frac{\sigma^2}{2}T - \frac{u^2}{2}\sigma^2T)$ , which means  $X_T \sim N(X_0 - \frac{\sigma^2}{2}T, \sigma^2T)$ ,  $T \in \mathbb{R}_+$ . Put differently, the return process  $X$  is Brownian motion with drift rate  $-\frac{\sigma^2}{2}$  and volatility  $\sigma$ .

**3.4. Deterministic coefficient process  $\beta$ .** In this section we consider building blocks  $(x_0, \psi_0, f, \gamma)$  where  $f$  depends on the time parameter only. Then  $g$  defined as in Lemma I.3.6 also depends only on time. Thus the SDE (I.3.4) has trivial solutions.

**COROLLARY I.3.15.** *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks such that  $f$  is constant in its second variable. Let  $g$  be defined as in Lemma I.3.6 and assume that*

$$\left( (T, u) \mapsto \psi_0(T, u) - \int_0^t g(s)(T, u) ds - \gamma(u, 0) \right) \in L^1(\Pi)$$

for all  $t \in \mathbb{R}_+$ . Then there is a compatible risk-neutral option surface model  $(X, \Psi_0, \alpha, \beta, M)$ . If  $(\tilde{X}, \tilde{\Psi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{M})$  is another compatible risk-neutral option surface model, then it has the same law as  $(X, \Psi_0, \alpha, \beta, M)$ .

**PROOF.** In order to apply Theorem I.3.8 we have to show that the solutions of the SDE I.3.4 stay in  $\gamma(0, u) + L^1(\Pi)$ . This is trivial since the  $M$ -driven part is in the convex cone  $L^1(\Pi)$  and the other part is in  $\gamma(\cdot, 0) + L^1(\Pi)$  by assumption.  $\square$

**REMARK I.3.16.** In other words, the initial code-book state  $\psi_0$  has to be greater or equal than

$$(T, u) \mapsto \int_0^T a(s)(T, u) ds + \gamma(u, 0)$$

in the sense that the difference is in  $L^1(\Pi)$ .

The meaning is that code-book uncertainty implies that initial option prices cannot be arbitrary but that there is some lower bound.

**COROLLARY I.3.17.** *Let  $(x_0, \psi_0, f, \gamma)$  be building blocks such that  $f$  is constant in its second variable. Assume additionally that*

$$f_t(T, u) = \phi(u) \exp \left( \int_{t \wedge T}^T \lambda(s) ds \right)$$

for a continuous function  $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}$  and  $\phi \in \Pi$ . Define

$$Z_t := \int_0^t \exp \left( \int_s^t \lambda(s) ds \right) dM_s,$$

where  $(X, \Psi_0, \alpha, \beta, M)$  denotes a compatible risk-neutral option surface model. Then  $(X, Z)$  is a time-inhomogeneous affine process in the sense of [19].

PROOF. Let  $X^\parallel$  be the semimartingale projection of  $X$  relative to  $M$  and  $X^\perp := X - X^\parallel$ . The processes  $X^\perp, Z$  are locally independent because  $X^\perp$  is locally independent of  $M$ . The local exponent of  $X^\perp$  satisfies

$$\begin{aligned} \psi_t^{X^\perp}(u) &= \psi_t^X(u) - \psi^{X^\parallel}(u) \\ &= \Psi_0(t, u) + \int_0^t \alpha_s(t, u) ds + \int_0^{t-} \beta_s(t, u) dM_s - \psi^{X^\parallel}(u) \\ &= \Psi_0(t, u) + \int_0^t \alpha_s(t, u) ds + f(u)Z_{t-} - \psi^{X^\parallel}(u) \\ &= \Psi_0(t, u) - \int_0^t \partial_t \left( \psi^{(X^\parallel, M)} \left( u, -i \int_s^t \beta_s(r, u) dr \right) \right) ds \\ &\quad + f(u)Z_{t-} - \psi^{X^\parallel}(u), \end{aligned}$$

where we used the consistency condition in the first and the drift condition (I.3.2) in the last step. We obtain for the local exponent of  $(X, M, Z)$

$$\begin{aligned} \psi_t^{(X, M, Z)}(u, v, w) &= \psi_t^{(X, M)}(u, v + w) - iw\lambda(t)Z_{t-} \\ &= \psi^{(X^\parallel, M)}(u, v + w) - iw\lambda(t)Z_{t-} + \psi_t^{X^\perp}(u) \\ &= \Phi_0(t; u, v, w) + \Phi_1(t; u, v, w)Z_{t-} \end{aligned}$$

with

$$\begin{aligned} \Phi_0(t; u, v, w) &:= \psi^{(X^\parallel, M)}(u, v + w) - \psi^{X^\parallel}(u) \\ &\quad + \Psi_0(t, u) - \int_0^t \partial_t \left( \psi^{(X^\parallel, M)} \left( u, -i \int_s^t \beta_s(r, u) dr \right) \right) ds, \end{aligned} \quad (\text{I.3.5})$$

$$\Phi_1(t; u, v, w) := f(u) - iw\lambda(t). \quad (\text{I.3.6})$$

which implies that  $(u, v, w) \mapsto \Phi_0(t; u, v, w) + f(u)Z_{t-}(u)$  is a Lévy exponent on  $\mathbb{R}^3$ . Since  $\text{ess inf}(M) = 0$  for all  $k \in \{1, \dots, d\}$  we have  $\text{ess inf}(Z) = 0$  by Corollary A.4.9. Similar arguments yield that  $(u, v, w) \mapsto \Phi_0(t; u, v, w)$  is a Lévy exponent for fixed  $t$ . The same is true for  $\Phi_1$ . Relative to some truncation function  $\chi$ , denote by  $(\beta_t^{(i)}, \gamma_t^{(i)}, \kappa_t^{(i)})$ ,  $i = 0, 1$  Lévy Khintchine triplets on  $\mathbb{R}^3$  which correspond to  $\Phi_0(t; \cdot), \Phi_1(t; \cdot)$  respectively. Observe that  $\Phi_0(t; u, v, w)$  is continuous in  $t$  for fixed  $(u, v, w)$  and likewise  $\Phi_1$ . Lévy continuity theorem and [25, VII.2.9] imply that  $(\beta_t^{(i)}, \gamma_t^{(i)}, \kappa_t^{(i)})$ ,  $i = 0, 1$  are continuous in  $t$  in the sense of  $[\beta_1], [\gamma_1]$  and  $[\delta_{1,3}]$  as in [25, VII.2.9]. A detailed inspection of the arguments shows this weaker continuity suffices for the proof of [19, Proposition 4.1]. The assertion follows now from [19, Proposition 5.4].  $\square$

COROLLARY I.3.18. *Assume that the requirements of Corollary I.3.17 are satisfied and assume that there is  $\eta \in \Pi$ ,  $\lambda, \delta \in \mathbb{R}$  such that*

$$f : \mathbb{R}_+ \rightarrow L^1(\Pi), t \mapsto ((T, u) \mapsto \phi(u)e^{\lambda(T-t)})$$

and  $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}, (u, v) \mapsto \eta(\delta u + v)$  where  $\phi(u) := -\frac{u^2 + iu}{2}$  for all  $u \in \mathbb{R}$ . Assume that

$$\psi^L(T, u) := \left( (T, u) \mapsto \psi_0(T, u) + \int_0^T g(s)(T, u) ds - \eta(\delta u) \right) \in L^1(\Pi)$$

where  $g$  is defined as in Lemma I.3.6.

Then there is a risk-neutral and compatible option surface model  $(X, \Psi_0, \alpha, \beta, M)$ . Moreover, it can be chosen such that there is a standard Wiener process  $W$  and a time-inhomogeneous Lévy process  $L$  with characteristic function

$$E(e^{iuL_T}) = \exp \left( \int_0^T (\psi^L(T, u)) dr \right), \quad T \in \mathbb{R}_+, u \in \mathbb{R}$$

such that  $W, L, M$  are independent and

$$dX_t = dL_t - \frac{1}{2}Z_t dt + \sqrt{Z_t} dW_t + \delta dM_t, \quad (\text{I.3.7})$$

$$dZ_t = -\lambda Z_t dt + dM_t \quad (\text{I.3.8})$$

with  $Z_0 = 0$ .

PROOF. Let  $W, L, M$  be independent Lévy processes on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$  such that  $W$  is a Brownian motion,  $L$  is a PII with characteristic function

$$E(e^{iuL_T}) = \exp \left( \int_0^T \psi^L(r, u) dr \right), \quad T \in \mathbb{R}_+, u \in \mathbb{R}$$

and  $M$  a Lévy process with characteristic exponent  $\eta$ . Let  $(X, Z)$  be a solution to the SDE system (I.3.7, I.3.8). The dependent part of  $X$  relative to  $M$  is  $X^\parallel := \delta M$  because  $W, L$  are independent of  $M$ . Thus  $(X^\parallel, M)$  has characteristic exponent  $\gamma$ . Define

$$\begin{aligned} \Psi_t(T, u) &:= \psi_0(T, u) + \int_0^t g(s)(T, u) ds + \phi(u)Z_s \\ &= \psi_0(T, u) + \int_0^t g(s)(T, u) ds + \int_0^t f(s)(T, u) dM_s. \end{aligned}$$

Then  $\Psi_{t-}(t, u) = \psi_0(t, u) + \int_0^t g(s)(t, u)ds + \phi(u)Z_{t-}$ . Moreover,

$$\begin{aligned}\psi_t^X(u) &= \psi_t^L(u) + \phi(u)Z_{t-} + \eta(\delta u) \\ &= \psi_0(t, u) + \int_0^t g(s)(t, u)ds + \phi(u)Z_{t-} \\ &= \Psi_{t-}(t, u).\end{aligned}$$

Thus  $(X, \Psi_0, \alpha, \beta, M)$  with  $\alpha_t(T, u) := g(t)(T, u), \beta_t(T, u) := f(t)(T, u)$  is an option surface model which is consistent and drift neutral. Theorem I.2.6 yields it is risk-neutral. Hence it is a compatible and risk-neutral option surface model.  $\square$

REMARK I.3.19. Up to the additional time-inhomogeneous Lévy process  $L$ , the stock price model in (I.3.7, I.3.8) is a special case of the so-called BNS model of [2]. If we consider more general functions  $f$ , then, again up to the additional PII  $L$ , we end up with the CGMY extension of the BNS model from [13], cf. also [27].

#### 4. Carmona & Nadtochiy's 'Tangent Lévy market models'

In [12] and its extension [11, Section 5], Carmona and Nadtochiy (CN) developed independently a HJM-type approach for option prices with overlap to the present chapter. Their simple model class in the sense of Step (4) in the section above is based on time-inhomogeneous Lévy processes. These can be described uniquely by their Lévy density and the diffusion coefficient because the drift is determined by the martingale condition for the stock under the risk neutral measure. Instead of the characteristic exponent from (I.1.1) CN use this Lévy density together with the diffusion coefficient as the code-book  $(\kappa_t(T, x), \Sigma_t(T))$ . Since we basically allow for the same class of simple models, their framework can be embedded into ours. Indeed, there is a transformation  $A$  that converts their code-book into ours, given by

$$A(\kappa_t(T, x), \Sigma_t(T)) := -\frac{u^2 + iu}{2}\Sigma_t^2(T) + \int (e^{iux} - 1 - iu(e^x - 1))\kappa_t(T, x)dx.$$

Since the simple models are parametrised differently, the drift condition in the two approaches differ. The condition in the CN framework looks a little more complex because it involves differential operators of second order.

CN focus on Itô processes for modelling the code-book process, which roughly corresponds to choosing  $M$  as Brownian motion in our setup, see below. Surprisingly, their approach leads to a constant diffusion coefficient  $\Sigma_t(T) = \Sigma_0(T)$ , cf. [11, Section 5]. This

does not occur in our setup, cf. Corollary I.3.18. The constant diffusion coefficient implies that the continuous martingale part of the stock price process is a time-inhomogeneous Brownian motion  $B$ , rather than a more complicated continuous semimartingale.

With regards existence and uniqueness of models given basic building blocks, CN and we provide different answers. Our results Theorem I.3.8 and Theorem I.3.12 imply existence and uniqueness given  $\beta_t = b(t, \Psi_t)$  as a deterministic function  $b$  of time and the state of the code-book. By contrast, CN consider a different situation in their [10, Theorem 16] where they assume that the process  $\beta$  in their code-book dynamics

$$d\kappa_t = \alpha_t dt + \beta_t dB_t,$$

is given beforehand. This does not allow for the natural situation that  $\beta$  depends on the current state  $\kappa$  of the code-book itself, which occurs e.g. in the example in Section 6 of [12] and is treated separately.

Both CN and we provide basically one non-trivial example, based on more or less deterministic  $\beta$ . In order to ensure existence of compatible option surface model we consider any sufficiently large initial code-book whereas CN slow down the code-book process when necessary.

## CHAPTER II

### The Heath-Jarrow-Morton approach for finitely many traded derivatives

In this chapter we focus on to a finite set of liquidly traded European options rather than an idealised infinite set. This seeming simplification differs strongly from the infinite case. In the infinite-dimensional case, we have seen that risk-neutrality is equivalent to the drift condition together with the consistency condition, cf. Theorem I.2.6 . This is due to the fact that (complex) prices for options with payoff  $\exp(iuX(T))$  can be found for arbitrary  $u$  and any time to maturity, where  $X$  denotes the log-return of the stock. These options determine the structure of the law of the increments of the underlying. Letting the time to maturity converge to 0 yields the dynamics of the underlying. In the finite-dimensional case both steps are no more possible, i.e option prices do not determine the distribution of the increments of the underlying and the time to maturity cannot be sent to 0.

#### 1. Abstract Framework

**1.1. Simple models.** Given a Markov process  $Y$  with transition semigroup  $U$ , the pricing formula for a European option with payoff function  $f$  is given by  $U_T f$  where  $T$  is the time to maturity and  $Y$  is the underlying, e.g. the log-return of a stock. *Calibrating* a parametric class of Markov models  $(Y_k)_{k \in K}$  to several options means finding a parameter  $k$  such that the real option prices are given by the pricing function in the model  $Y_k$ . In this section we want to develop a framework where frequent recalibration is made rigorous, that is where we have a process  $Z$  in the set  $K$  which 'locally chooses' one of the Markov models  $Y_k$ . This leads to the *dynamic model*  $X := Y_Z$  which in general will *not* be a Markov model. We also characterise absence of arbitrage for dynamic models – in terms of a drift condition for the code-book process  $Z$ .

Throughout this section let  $n \in \mathbb{N}$ ,  $f_1, \dots, f_n$  be real functions and  $0 =: T_0 < T_1 \leq \dots \leq T_n$  real numbers. The functions  $f_1, \dots, f_n$  serve as payoff functions of European options with maturities  $T_1, \dots, T_n$ .



DEFINITION II.1.1. Let  $Y$  be a strong time-inhomogeneous Markov process on some filtered probability space  $(\Omega, \mathcal{F}, P)$  with state space  $\mathbb{R}$  and transition measure  $p$ , i.e.  $p(t, x, T, \cdot)$  is a probability measure for all  $(t, x, T) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$  and

$$E(f(Y(T))|\mathcal{F}_t) = \int_{\mathbb{R}} f(x)p(t, Y(t), T, dx)$$

for all bounded measurable functions  $f$  and all  $x \in \mathbb{R}, t < T$ . We define the *transition semigroup*  $U$  of  $Y$  for all measurable functions, such that  $f(Y(T))$  is integrable by

$$U_{t,T}f(x) := \int_{\mathbb{R}} f(y)p(t, x, T, dy).$$

where  $0 \leq t \leq T, x \in \mathbb{R}$ . The *generator*  $\mathcal{G}$  of  $Y$  is the set of all pairs of measurable real functions  $(f, g)$  such that

$$f(Y(t)) - \int_0^t g(Y(s))ds$$

is a martingale.

REMARK II.1.2. Note that it is usual to define the transition semigroup and the generator only for bounded measurable functions. If  $f$  is a measurable function such that  $f(Y)$  is integrable, then

$$E(f(Y(T))|\mathcal{F}_t) = U_{t,T}f(Y_t) = \int_{\mathbb{R}} f(x)p(t, Y(t), T, dx)$$

where  $0 \leq t \leq T$ .

We now want to introduce the key concept in this chapter, namely simple models. While the simple models define a parametric class of models which are possible models for a market, we usually do not think that a real market behaves as one of them. But they should be chosen in such a way that calibration to the options is quite simple.

DEFINITION II.1.3. A *simple model class* for the options  $f_1, \dots, f_n$  is a family of strong time-inhomogeneous  $\mathbb{R}$ -valued Markov processes  $(Y_k)_{k \in K}$  where  $K$  is a convex cone in  $\mathbb{R}^n$  with non-empty interior such that for each  $k \in K$  the process  $Y_k$  has transition semigroup  $U^k$ , initial value 0,

- $f_j(Y_k(T_j))$  is integrable for all  $j \in \{1, \dots, n\}$ ,
- the functions

$$G_j : [0, T_j] \times \mathbb{R} \times K \rightarrow \mathbb{R}, (t, x, k) \mapsto U_{t,T_j}^k f_j(x)$$

are twice continuously differentiable for  $t < T_j$  and have the property

$$\partial_{k_i} G_j(t, x, k) = 0$$

if  $T_i \leq t$ .

The functions  $G_j$  are the *pricing functions* for the European options.

REMARK II.1.4. If  $g$  is a one-to-one  $C^2$ -function, then  $(g(Y_k))_{k \in K}$  could be called a simple model class as well. However, in the sequel it will lead to the same models and conclusions.

If we assume that for  $k \in K$  the process  $Y_k$  is the return of a stock under the pricing measure, then the price of a European option with payoff function  $f_j$  and maturity  $T_j$  is given by

$$O_j(t) = E(f_j(Y_k(T_j)) | Y_k(t)) = G_j(t, Y_k(t), k).$$

Hence  $G_j$  can be understood as a universal pricing function for the European option with payoff  $f_j$ . Moreover, we choose the dimension of the parameter space  $K$  to be equal to the number of options in the hope that the function  $(G_1, \dots, G_n)(t, x, \cdot)$  is (locally) invertible. However, after  $T_1$  some options disappear from the market and consequently the function  $(G_1, \dots, G_n)$  cannot be locally invertible anymore. However, the second assumption for simple model classes above implies that after  $T_{m-1}$  the parameters  $k_1, \dots, k_{m-1}$  have no effect on the option prices anymore, so we can hope that

$$(k_m, \dots, k_n) \rightarrow (G_1, \dots, G_n)(t, x, k_1, \dots, k_n)$$

is locally invertible. Loosely speaking, whenever options mature, then as many parameters loose their effect. So there are 'always' as many parameters as options.

Sometimes it is enough to show invertibility for only a few points in order to show invertibility for a particularly large set.

PROPOSITION II.1.5. *Let  $(Y_k)_{k \in K}$  be a simple model class and assume that the pricing functions  $G_j, \dots, G_n$  are analytic on  $(T_{j-1}, T_j) \times \mathbb{R} \times \text{int}(K)$  and that there is  $(t, x, k) \in (T_{j-1}, T_j) \times \mathbb{R} \times \text{int}(K)$  such that*

$$M(t, x, k) := (\partial_{k_a} G_b(t, x, k))_{a, b \in \{j, \dots, n\}} \in \mathbb{R}^{(n-j+1) \times (n-j+1)}$$

is invertible. Then there is an open and dense subset  $V \subset ]T_{j-1}, T_j) \times \mathbb{R} \times \text{int}(K)$  such that the matrix  $M(t, x, k)$  is invertible for all  $(t, x, k) \in V$ . Moreover,

$$(t, x, k) \mapsto (M(t, x, k))^{-1}$$

is meromorphic.

PROOF. This is an application of [30, Proposition 7.3.1]. □

The following lemma gives sufficient conditions for  $G_j$  to be analytic.

LEMMA II.1.6. *Let  $(Y_k)_{k \in K}$  be a simple model class and assume*

- $K = \mathbb{R}_+ \times \hat{K}$  for some convex cone  $\hat{K} \subset \mathbb{R}^{n-1}$ ,
- For all  $(c, \hat{k}) = k \in K$  we have  $Y_k(t) = dct + B(ct) + \hat{Y}_{\hat{k}}$  for some SBM  $B$ , some  $d \in \mathbb{R}$  and a Markov process  $\hat{Y}_{\hat{k}}$  independent of  $B$ ,
- $\hat{Y}_0 = 0$ ,
- $f_j$  is exponentially bounded for all  $j \in \{1, \dots, n\}$ , i.e. there are  $a, b \in \mathbb{R}_+$  such that  $|f_j(x)| \leq a \exp(b|x|)$  and
- $(t, x, \hat{k}) \mapsto U_{t, T_j}^{(0, \hat{k})} g(x)$  is analytic in the interior of its domain if  $g$  is analytic and exponentially bounded.

Then  $G_j$  is analytic in the interior of its domain for all  $j \in \{1, \dots, n\}$ .

PROOF. Let  $j \in \{1, \dots, n\}$  and define

$$h : [0, T_j] \times \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}, (t, x, c) \mapsto E(f_j(dct + B(c(T_j - t)) + x)) = U_{t, T_j}^{(c, 0)} f_j(x).$$

Then  $h(t, x, c) = \frac{1}{\sqrt{2\pi c(T_j - t)}} \int_{\mathbb{R}} f(y) e^{-\frac{(y - x - dc(T_j - t))^2}{2c(T_j - t)}} dy$ . The integrand in this integral representation is clearly analytic in  $t, x$  and  $c$ , integral and differentiation with respect to any of these variables are interchangeable. Hence  $h$  is analytic in each of its variables in the interior of its domain. Denote the transition semigroup of the Brownian motion with drift rate  $d$  and with volatility  $c$  by  $W^c$ . Since  $h(t, x, c) = W_{t, T_j}^c f(x)$  is exponentially bounded in  $x$  we have

$$\begin{aligned} G_j(t, x, k) &:= U_{t, T_j}^k f(x) \\ &= U_{t, T_j}^{(0, \hat{k})} W_{t, T_j}^c f(x) \\ &= U_{t, T_j}^{(0, \hat{k})} h(t, \cdot, c)(x) \end{aligned}$$

for all  $(c, \hat{k}) = k \in K, x \in \mathbb{R}, t \in ]0, T_j[$ .  $G_j$  is analytic in the variable  $\hat{k}$  in the interior of  $\hat{K}$  and in  $x$ , by assumption and because  $h$  is an analytic function. By changing the role of  $U^{(0, \hat{k})}$  and  $W^c$  we see that  $G_j$  is analytic in  $c$  for all  $c \in \mathbb{R}_{++}$ . For the same reason

$$[0, T_j]^2 \rightarrow \mathbb{R}, (s, t) \mapsto \hat{U}_{s, T_j}^{(0, \hat{k})} W_{t, T}^c f(x)$$

is analytic on  $(0, T_j)^2$  and hence  $G_j$  is analytic in all of its variables.  $\square$

**1.2. Code-book Process.** Given a simple model class  $(Y_k)_{k \in K}$ , a code-book process is a process with values in  $K$  which 'chooses' a simple model. In this section we want to discuss processes  $(X, Z_1, \dots, Z_n)$  such that  $X$  may serve as the return process of an underlying under the martingale measure and option price processes are given by

$$O_j(t) := G_j(t, X(t), Z_1(t), \dots, Z_n(t)). \quad (\text{II.1.1})$$

That means that the model  $(X, Z)$  uses the same pricing formula as one of the simple models. In practise, the state of  $Z$  is observable via calibration and  $X$  is simply the log-underlying and hence observable as well. More difficult is the question: "How can we observe/estimate the dynamics of  $(X, Z)$ ?"

While we do not want to make any assumptions on the dynamics of  $X$  yet, we want to assume for a moment that  $Z$  is a continuous semimartingale. Then of course, it is possible to estimate its dynamics or more precisely features of its dynamics under the physical measure (also real-world measure). However, we work under the pricing measure in this thesis. Due to Girsanov's theorem (see [25, Section III.3d]) the diffusion coefficient is the same under the physical measure and the pricing measure. Our next step is to see that the drift coefficient under the pricing measure is determined by the diffusion coefficient which will be referred to as 'drift condition'.

Given a process  $Z$ , Equation (II.1.1) imposes some restrictions on the process  $X$ . We want to think of Equation (II.1.1) in the following way:

- The process  $Z$  chooses one of our simple models in the simple model class (cf. Definition II.1.3),
- $X$  behaves 'locally' as the process  $Y_Z$  (this will lead to the consistency condition) and
- the right hand side of Equation (II.1.1) is a martingale (this will lead to the drift condition).

REMARK II.1.7. This setup is in the spirit of the tangent models introduced in [11] with finite dimensional parameter space  $K$  where the models  $(Y_k)_{k \in K}$  are the tangent models. Due to our Markovian setup and the European-style payoff functions  $f$ , the option prices are not path dependent, but only depend on the current price for the tangent models. As a side remark the author wants to note that this is also true for other options, e.g. American-style options in Markovian models and consequently it seems possible to develop a framework for them.

We want to quote Carmona and Nadtochiy from [11, Section 2.5]:

"We want to think of the [...] notion of tangent model as an analog of the notion of tangent vector in classical differential geometry: the two models are tangent in the sense that, locally, at a fixed point in time, they produce the same prices of derivatives in a chosen family. Recall that tangent vectors in differential geometry are often used as a convenient way to describe the time dynamics. In the same way, we hope that the tangent models introduced [...] will help in a better understanding of market models."

Since the processes  $Z_j$  in Equation II.1.1 is only needed up to the time  $T_j$ , we will assume that they are constant after  $T_j$ . This is only one out of many possible continuations which do not have any effect on the model.

DEFINITION II.1.8. Let  $(Y_k)_{k \in K}$  be a simple model class with  $K \subset \mathbb{R}^n$ . A *dynamic model* is a pair  $(X, Z)$  such that  $X$  is an  $\mathbb{R}$ -valued semimartingale,  $Z$  is a  $K$ -valued semimartingale such that its components  $Z_j$  are constant after  $T_j$  and  $(X, Z)$  allows for local characteristics. The process  $Z$  is called *code-book process*.  $(X, Z)$  is *risk-neutral* if the process

$$O_j(t) := G_j(t, X(t), Z(t))$$

is a  $\sigma$ -martingale for any  $j \in \{1, \dots, n\}$ .  $(X, Z)$  is *strongly risk-neutral* if the processes  $(O_j)_{j \in \{1, \dots, n\}}$  are martingales.

REMARK II.1.9. First of all, the process  $X$  is not necessarily a Markov process. In particular it does not have to be one of our parametrised Markov models. We just use the same pricing formula for the options  $O_j$ , which again do not have to be of the form

$$O_j(t) = E(f_j(X(T_j)) | \mathcal{F}_t).$$

But observe that  $G_j(T_j, x, k) = f_j(x)$  and hence we always have  $O_j(T_j) = f_j(X(T_j))$ . Thus  $O_j$  is a stochastic process whose value at maturity is given by the payoff function. In particular, if  $(X, Z)$  is risk-neutral and if  $O$  is of class (D), then  $O_j$  is a true martingale and hence the prices of  $O_j$  are given by conditional expectation, i.e.  $O_j(t) = E(f_j(X(T_j))|\mathcal{F}_t)$ . Since  $G_j$  is a  $C^2$ -function, the options  $O_j$  are semimartingales and the Itô-formula yields that the drift coefficient  $b^{O_j}$  of an option  $O_j$  is given by

$$b^{O_j}(t) = (DG(t, X(t-), \cdot)(Z(t-))) b^Z(t) + r(t)$$

for some (stochastic) function  $r$ , where  $r$  does not depend on  $b^Z(t)$ . Moreover,  $O_j$  is a martingale only if its drift coefficient is 0. If this equation can be solved for  $b^Z$ , then  $O_j$  is a martingale only if

$$b^Z(t) = -(DG(t, X(t-), \cdot)(Z(t-)))^{-1}r(t).$$

This leads to the next

**DEFINITION II.1.10.** Let  $(Y_k)_{k \in K}$  be a simple model class with  $K \subset \mathbb{R}^n$ . A dynamic model  $(X, Z)$  on a filtered probability space  $(\Omega, \mathcal{A}, \mathcal{F}, P)$  is said to satisfy the *drift condition* if for all  $j \in \{1, \dots, n\}$  and  $P \otimes \lambda$ -almost all  $(\omega, t) \in \Omega \times [0, T_j]$

$$D_3 G_j(t, X(t-), Z(t-)) b^Z(t) = -r(t) \quad \text{and} \quad (\text{II.1.2})$$

$$\int |G_j(t, X(t-) + x, Z(t-) + z) - G_j(t, X(t-), Z(t-))| F^{(X,Z)}(t, dx, dz) < \infty \quad (\text{II.1.3})$$

where

$$\begin{aligned} r(t) &:= \frac{1}{2} D_3^2 G_j(t, X(t-), Z(t-)) c^Z(t) \\ &+ \int (G_j(t, X(t-) + x, Z(t-) + z) - G_j(t, X(t-), Z(t-))) F^{(X,Z)}(t, dx, dz) \\ &+ D_1 G_j(t, X(t-), Z(t-)) + D_2 G_j(t, X(t-), Z(t-)) b^X(t) \\ &+ \frac{1}{2} D_2^2 G_j(t, X(t-), Z(t-)) c^X(t) + D_2 D_3 G_j(t, X(t-), Z(t-)) c^{XZ}(t) \end{aligned}$$

and

$$\left( \begin{pmatrix} b^X \\ b^Z \end{pmatrix}, \begin{pmatrix} c^X & c^{XZ} \\ c^{XZ} & c^Z \end{pmatrix}, F^{(X,Z)} \right)$$

is a version of the local characteristics of  $(X, Z)$ ,  $D_1$  is the partial derivation with respect to the time index  $t$ ,  $D_2$  the partial derivation with respect to the second variable,  $D_3$  is the total derivation with respect to the remaining parameters, i.e.

$$D_3 G_j(t, x, k) = (\partial_{k_1} G(t, x, k), \dots, \partial_{k_n} G(t, x, k))$$

and

$$D_3^2 G_j(t, x, k) c^Z(t) := \sum_{a,b=1}^n \partial_{k_a} \partial_{k_b} G_j(t, x, k) (c^Z(t))_{a,b}.$$

**PROPOSITION II.1.11.** *Let  $(X, Z)$  be a dynamic model defined on the simple model class  $(Y_k)_{k \in K}$  with  $K \subset \mathbb{R}^n$ . Then  $(X, Z)$  is risk-neutral if and only if it satisfies the drift condition.*

**PROOF.** This is an application of Propositions A.1.1 and A.1.3. □

**COROLLARY II.1.12.** *Let  $(X, Z)$  be a dynamic model and assume that the payoff functions  $f_1, \dots, f_n$  are bounded. Then the following statements are equivalent:*

- $(X, Z)$  is strongly risk-neutral.
- $(X, Z)$  is risk-neutral.
- $(X, Z)$  satisfies the drift condition.

**PROOF.** Proposition II.1.11 yields equivalence of the last two statements and the first statement obviously implies the second statement.

If the pricing function  $G_j$  is bounded, then the options are bounded and consequently the second statement implies the first one. Let  $b$  be a bound for  $f_j$ , then

$$|G_j(t, x, k)| \leq \int_{\mathbb{R}} |f_j(y)| p(t, x, T_j, dy) \leq b$$

for all  $t \in [0, T_j], x \in \mathbb{R}, k \in K$ . Hence the options are bounded  $\sigma$ -martingales and [26, Corollary 3.1] yields they are martingales. □

**1.3. The consistency condition.** In the original Heath-Jarrow-Morton framework [21] one specifies the martingale part of the code-book process (the forward rate). By contrast, the drift under the pricing measure is given by the Heath-Jarrow-Morton drift condition. The consistency condition identifies the in some sense unique short rate model which is consistent with the HJM model, i.e. the discounted option prices for the short rate model and the HJM model are the same.

If we have a simple model class  $(Y_k)_{k \in K}$ , payoff functions  $f_1, \dots, f_n$  and if we want to set up a risk-neutral dynamic model, then we start to specify the martingale part of the code-book process  $Z$ . We want to obtain the drift from the drift condition, but the drift condition involves the state and the dynamics of the underlying process  $X$  which is not specified yet. In fact, the process  $X$  is in no sense uniquely determined yet. If we start with an arbitrary process  $X$ , then the drift of  $Z$  obtained from the drift condition can (and will often) have a singularity at maturity. Even though this is not shown here, this will prevent  $Z$  and the options  $O_j$  from being semimartingales. In particular, no option is a  $\sigma$ -martingale, i.e. the model is not risk-neutral.

One possible approach to overcome such problems will be a consistency condition which will remove the dynamics of  $X$  from the drift condition. If additionally the code-book process is an Itô-process, i.e.  $dZ(t) = b(t)dt + s(t)dW(t)$ , its drift coefficient  $b$  will depend linearly on  $s^2$  and explosion of the drift near maturity can be prevented. Moreover, the consistency condition formalises the statement that  $X$  locally behaves as  $Y_Z$ .

DEFINITION II.1.13. Let  $(X, Z)$  be a dynamic model defined on the simple model class  $(Y_k)_{k \in K}$  with  $K \subset \mathbb{R}^n$ .  $(X, Z)$  is *consistent* or said to satisfy the *consistency condition* if  $Y_k$  is a Markov process and a semimartingale for all  $k \in K$  (cf. Theorem III.2.8) and

$$\begin{aligned} b^X(\omega, t) &:= b^{Z(\omega, t)}(t, X(\omega, t-)), \\ c^X(\omega, t) &:= c^{Z(\omega, t)}(t, X(\omega, t-)) \quad \text{and} \\ F^X(\omega, t, A) &:= F^{Z(\omega, t)}(t, X(\omega, t-), A) \end{aligned}$$

is a version of the local characteristics of  $X$ , where  $(b^k, c^k, F^k)$  denotes a state-dependent version of the local characteristics of  $Y_k$ , i.e.

$$\begin{aligned} b^{Y_k}(\omega, t) &= b^k(t, Y(\omega, t-)), \\ c^{Y_k}(\omega, t) &= c^k(t, Y(\omega, t-)) \quad \text{and} \\ F^{Y_k}(\omega, t, A) &= F^k(t, Y(\omega, t-), A) \end{aligned}$$

For a dynamic model  $(X, Z)$  where  $X$  and  $Z$  are locally independent (see A.1.4) the consistency condition has two important consequences. First the drift condition simplifies to a much easier condition and second martingale property for  $\exp(X)$  if required can be studied in the simple model class instead. We begin with the first one.



LEMMA II.1.14. *Let  $(X, Z)$  be a consistent dynamic model,  $j \in \{1, \dots, n\}$ ,  $t \in [0, T_j)$  and define*

$$\begin{aligned} R(t) &:= \int (G_j(t, X(t-) + x, Z(t-)) - G_j(t, X(t-), Z(t-))) F^X(t, dx) \\ &\quad + D_1 G_j(t, X(t-), Z(t-)) + D_2 G_j(t, X(t-), Z(t-)) b^X(t) \\ &\quad + \frac{1}{2} D_2^2 G_j(t, X(t-), Z(t-)) c^X(t), \end{aligned}$$

where  $(b^X, c^X, F^X)$  is a version of the local characteristics of  $X$ . Then  $R = 0$  a.s.

PROOF. For any  $k \in K$  let  $\hat{b}^k, \hat{c}^k, \hat{F}^k$  be the functions defined as in Definition II.1.13 for the Markov semimartingale  $Y_k$ . The consistency condition and A.1.1 yield  $R = 0$  a.s.  $\square$

THEOREM II.1.15. *Let  $(X, Z)$  be a consistent dynamic model based on a simple model class  $(Y_k)_{k \in K}$  with  $K \subset \mathbb{R}^n$  and such that  $X$  and  $Z$  are locally independent (see A.1.4). Then  $(X, Z)$  satisfies the drift condition if and only if for all  $j \in \{1, \dots, n\}$  and almost all  $t \in [0, T_j)$*

$$\begin{aligned} &\int |G_j(t, X(t-), Z(t) + z) - G_j(t, X(t-), Z(t))| F^Z(t, dz) < \infty \quad \text{and} \\ &D_3 G_j(t, X(t-), Z(t-)) b^Z(t) \\ &= -\frac{1}{2} D_3^2 G_j(t, X(t-), Z(t-)) c^Z(t) \\ &\quad - \int (G_j(t, X(t-), Z(t) + z) - G_j(t, X(t-), Z(t))) F^Z(t, dz), \end{aligned} \quad (\text{II.1.4})$$

where  $(b^Z, c^Z, F^Z)$  is a version of the local characteristics of  $Z$ . If  $Z$  has continuous paths, then Equation (II.1.4) reduces to

$$D_3 G_j(t, X(t-), Z(t-)) b^Z(t) = -\frac{1}{2} D_3^2 G_j(t, X(t-), Z(t-)) c^Z(t).$$

PROOF. This is a direct consequence of Equation (II.1.3) local independence and Lemma II.1.14.  $\square$

Now we turn to the second consequence, which is particularly interesting for  $f = \exp$ .

THEOREM II.1.16. *Let  $(X, Z)$  be a consistent dynamic model defined on the simple model class  $(Y_k)_{k \in K}$  with  $K \subset \mathbb{R}^n$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^2$ -function such that  $f(Y_k)$  is integrable for all  $k \in K$ . Suppose that  $f(Y_k)$  is a  $\sigma$ -martingale for all  $k \in K$ . Then  $f(X)$  is a  $\sigma$ -martingale.*

PROOF. For all  $k \in K$  let  $\hat{b}^k, \hat{c}^k, \hat{F}^k$  be the functions defined as in Definition II.1.13 for the Markov semimartingale  $Y_k$ . Define

$$\delta(t, x, k) := f'(x)b^k(t, x) + \frac{1}{2}f''(x)c^k(t, x) + \int_{\mathbb{R}} (f(x+h) - f(x) - f'(x)h)F^k(t, dh).$$

Proposition A.1.1 yields  $\delta(t, Y_k(t), k)$  is the drift coefficient of the semimartingale  $f(Y_k)$ . Thus Proposition A.1.3 yields  $\delta(t, x, k) = 0$  for almost all  $t \geq 0, x \in \mathbb{R}, k \in K$ . In particular,  $\delta(t, X(t-), Z(t-)) = 0$ . Proposition A.1.1 implies that it is the drift coefficient of the semimartingale  $f(X)$  and thus  $f(X)$  is a  $\sigma$ -martingale by Proposition A.1.3.  $\square$

Given a simple model class  $(Y_k)_{k \in K}$  we can now start to construct a risk-neutral consistent dynamic model according to the following steps:

- (1) Choose the martingale part  $M$  of  $Z$ .
- (2) Find the drift part for  $Z$  according to the (simplified) drift condition (see Theorem II.1.15).
- (3) The process  $X$  is now determined by the consistency condition.

If all steps are done, then  $(X, Z)$  is a risk-neutral consistent dynamic model where the dynamics are given under the pricing measure. Step 2 is the hard part. While the general drift condition II.1.10 also depends on both the dynamics and the state of the process  $X$ , the simplified drift condition II.1.15 depends not on the dynamics of  $X$ . Due to this state dependence we want to study uniqueness and existence of consistent dynamic models  $(X, Z)$  in a Markovian setup.

Usually there are many risk-neutral non-consistent dynamic models. But it is much more difficult to construct a risk-neutral non-consistent dynamic model given the martingale part of the code-book process than constructing risk-neutral consistent dynamic models. The following example shows that the consistency condition can be very restrictive.

EXAMPLE II.1.17 (Black-Scholes model with one call option). Let  $n = 1, f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto (e^x - 1)^+, K := \mathbb{R}_+$  and  $T > 0$ . Let  $W$  be an SBM and define

$$Y_k(t) := \frac{-k^2}{2}t + kW(t).$$

Then  $\exp(Y_k)$  is a martingale for all  $k \in K$  and the option prices are given by the Black-Scholes formula

$$G(t, x, k) = e^x \Phi\left(\frac{x + tk^2/2}{\sqrt{tk^2}}\right) - \Phi\left(\frac{x - tk^2/2}{\sqrt{tk^2}}\right).$$

In this setup we investigate dynamic models  $(X, Z)$  such that the martingale part of  $Z$  is 0. Assume that  $(X, Z)$  is a risk-neutral dynamic model such that  $Z(t) = Z(0) + \int_0^t \beta(s)ds$  is deterministic and  $X(t) = \int_0^t b(s)ds + \int_0^t a(s)dW(s)$  has the property that  $\exp(X)$  is a martingale. Then  $b(s) = \frac{-a^2(s)}{2}$ . Since

$$G(t, X(t), Z(t)) = E(f(X(T))|\mathcal{F}_t) = G\left(t, X(t), \frac{(\int_t^T a(s)ds)^2}{T-t}\right),$$

we have  $Z(t) = \left(\int_t^T a(s)ds\right)^2 / (T-t)$ . The local characteristics of  $X$  are  $(b, a^2, 0)$  and those of  $Y_k$  are  $(\frac{-k^2}{2}, k, 0)$ .

We conclude that any time-inhomogeneous Black-Scholes model leads to a code-book process with vanishing martingale part.

If  $(X, Z)$  is a risk-neutral *consistent* dynamic model such that  $Z$  is of finite variation, then the simplified drift condition (see Theorem II.1.15) yields that  $Z$  is constant. From the definition of the consistency condition we can see that  $X$  has the same distribution as  $X(0) + Y_{Z(0)}$ , i.e.  $X$  is Brownian motion with constant volatility.

**1.4. Admissible code-book models.** In this section we want to connect the uniqueness and existence problem for risk-neutral consistent dynamic models with a uniqueness and existence problem for a martingale problem. To motivate the Markovian setup let us assume that we have a risk-neutral consistent dynamic model  $(X, Z)$  where  $X$  and  $Z$  are locally independent and  $dZ(t) = b(t)dt + \sqrt{c(t)}dW(t)$  for an SBM  $W$ . If we assume that  $\sqrt{c(t)} = \sigma(t, X(t), Z(t))$  with some deterministic function  $\sigma$ , then the simplified drift condition (see Theorem II.1.15) yields

$$\begin{aligned} b(t) &= \beta(t, X(t), Z(t)) \\ &:= -\frac{1}{2}(D_3G(t, X(t), Z(t)))^{-1} \left( \sum_{a,b=1}^n \partial_{k_a} \partial_{k_b} G_j(t, X(t), Z(t)) [(\sigma\sigma^\top)(t, X(t), Z(t))]_{a,b} \right) \end{aligned}$$

where  $\beta$  is a deterministic function. In other words risk-neutrality implies that  $\beta$  is determined by  $\sigma$  in all relevant states.

**DEFINITION II.1.18.** We call a pair of measurable functions

$$\begin{aligned} \beta &: \mathbb{R}_+ \times \mathbb{R} \times K \rightarrow \mathbb{R}^n \\ \sigma &: \mathbb{R}_+ \times \mathbb{R} \times K \rightarrow \mathbb{R}^{n \times l} \end{aligned}$$

*code-book dynamics*. Code-book dynamics are called *admissible* if

$$D_3 G_j(t, x, k) \beta(t, x, k) = -\frac{1}{2} \left( \sum_{a,b=1}^n \partial_{k_a} \partial_{k_b} G_j(t, x, k) [(\sigma \sigma^\top)(t, x, k)]_{a,b} \right) \quad (\text{II.1.5})$$

for all  $j \in \{1, \dots, n\}$ ,  $t \in [0, T_j]$ ,  $x \in \mathbb{R}$ ,  $k \in \text{int}(K)$  and if

$$\beta_j(t, x, k) = 0, \quad \sigma_j(t, x, k) = 0$$

for  $(t, k) \in ([0, T_j] \times \text{int}(K))^c$ .

First we show that the admissibility condition implies risk-neutrality under some restrictions. Above we have seen that in some sense the converse is also true.

**COROLLARY II.1.19.** *Let  $(X, Z)$  be a consistent dynamic model such that  $X$  and  $Z$  are locally independent,  $Z$  is continuous and its local characteristics  $(b^Z, c^Z, 0)$  are given by*

$$b^Z(t) = \beta_Z(t, X(t-), Z(t)), \quad (\text{II.1.6})$$

$$c^Z(t) = (\sigma_Z \sigma_Z^\top)(t, X(t-), Z(t)) \quad (\text{II.1.7})$$

for some admissible code-book dynamics  $(\beta_Z, \sigma_Z)$ . Then  $(X, Z)$  is risk-neutral.

**PROOF.** Let  $j \in \{1, \dots, n\}$ ,  $t \in [0, T_j]$ . Due to the admissibility condition we have

$$D_3 G_j(t, X(t-), Z(t)) b^Z(t) = -\frac{1}{2} D_3^2 G_j(t, X(t-), Z(t)) c^Z(t).$$

Theorem II.1.15 yields the claim.  $\square$

We now turn to the question whether a consistent model  $(X, Z)$  as in Corollary II.1.19 exists for given admissible code-book dynamics  $(\beta_Z, \sigma_Z)$ .

**LEMMA II.1.20.** *Let  $(\beta_Z, \sigma_Z)$  be code-book dynamics,  $C_0 := C_0(\mathbb{R}_+ \times \mathbb{R} \times K)$  and define the operator*

$$\begin{aligned} \mathcal{E} &: D(\mathcal{E}) \subset C_0 \rightarrow C_0, \\ \mathcal{E}g(t, x, k) &= \partial_t g(t, x, k) + \mathcal{G}^k g(t, x, k) + D_k g(t, x, k) \beta_Z(t, x, k) \\ &\quad + \sum_{i,j=1}^n \frac{1}{2} \partial_{k_i} \partial_{k_j} g(t, x, k) (\sigma_Z \sigma_Z^\top)_{i,j}(t, x, k). \end{aligned}$$

Let  $x \in \mathbb{R}$ ,  $k \in K$  and define the measure  $\mu := \delta_0 \otimes \delta_x \otimes \delta_k$  on  $\mathbb{R}^+ \times \mathbb{R} \times K$ . Let  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $t \mapsto t$ , let  $X$  be a semimartingale with values in  $\mathbb{R}$  and  $Z$  a semimartingale with values in  $K$  such that  $Z_j$  is constant after  $T_j$  for all  $j \in \{1, \dots, n\}$ . Then there is equivalence between

- (1)  $(\Theta, X, Z)$  is a solution to the martingale problem  $(\mathcal{E}, \mu)$  in the sense of [18, Chapter 4.3], i.e.  $P^{(\Theta(0), X(0), Z(0))} = \mu$  and

$$M^g(t) := g(\Theta(t), X(t), Z(t)) - \int_0^t f(\Theta(s), X(s), Z(s)) ds$$

is a martingale for all  $(g, f) \in \mathcal{E}$ ,

- (2)  $(X, Z)$  is a consistent dynamic model such that  $X(0) = x$ ,  $Z(0) = k$ ,  $X$  and  $Z$  are locally independent and  $(b^Z, c^Z, 0)$  is a version of the local characteristics of  $Z$ , where

$$b^Z(t) = \beta_Z(t, X(t-), Z(t)) \quad \text{and} \quad (\text{II.1.8})$$

$$c^Z(t) = \sigma_Z \sigma_Z^\top(t, X(t-), Z(t)). \quad (\text{II.1.9})$$

PROOF. Observe that the domain of  $\mathcal{E}$  contains all infinitely differentiable functions with compact support, due to [15, Theorem 7.16(ii)].

(1)  $\Rightarrow$  (2) : Since

$$P((\Theta, X, Z)(0) = (0, x, k)) = \mu(\{(0, x, k)\}) = 1$$

we have  $X(0) = x$  and  $Z(0) = k$  a.s. Theorem III.2.8 implies that  $(\Theta, X, Z)$  is a semimartingale which allows for local characteristics and the characteristics of  $Z$  are given as asserted. Moreover, Lemma A.2.9 yields  $X$  and  $Z$  are locally independent.

(2)  $\Rightarrow$  (1) : The initial distribution of  $(\Theta, X, Z)$  is given by  $\mu$ . Let  $g \in D(\mathcal{E})$  denote a  $C^2$  function. We have to show that

$$M^g(t) := g(\Theta(t), X(t), Z(t)) - \int_0^t \mathcal{E}g(\Theta(s), X(s-), Z(s)) ds$$

exists and that it is a local martingale. Since  $g \in D(\mathcal{E}) \subset C$ , the function  $g$  is bounded and hence  $g(\Theta, X, Z)$  is a special semimartingale. Propositions A.1.1, A.1.3, the consistency condition and the local independence of  $X$  and  $Z$  yield that the drift coefficient of  $g(\Theta, X, Z)$  is given by  $\mathcal{E}g(\Theta, X, Z)$ . Hence  $M^g$  is a bounded  $\sigma$ -martingale and [26, Lemma 3.1] yields that  $M^g$  is a martingale.  $\square$

**THEOREM II.1.21.** *Let  $(\beta_Z, \sigma_Z)$  be admissible code-book dynamics,  $x \in \mathbb{R}$ ,  $k \in K$  and assume that existence for the martingale problem  $(\mathcal{E}, \delta_0 \otimes \delta_x \otimes \delta_k)$  in Lemma II.1.20 holds in the sense of [18, Section 4.5]. Then there is a consistent dynamic model  $(X, Z)$  such that*

- (1)  $X(0) = x$  and  $Z(0) = k$ ,
- (2)  $X$  and  $Z$  are locally independent semimartingales,

(3)  $Z$  is a continuous semimartingale such that a version of its local characteristics is given by Equations (II.1.8, II.1.9).

Moreover, any consistent dynamic model satisfying (1) to (3) is risk-neutral.

If in addition uniqueness holds for the martingale problem  $(\mathcal{E}, \delta_0 \otimes \delta_x \otimes \delta_k)$ , then the distributions of any two consistent dynamic models  $(X, Z)$  and  $(X', Z')$  satisfying (1) to (3) coincide. Finally, if uniqueness holds for the martingale problem  $\mathcal{E}$ , i.e. uniqueness holds for the martingale problem  $(\mathcal{E}, \mu)$  for any probability measure  $\mu$  on  $\mathbb{R}_+ \times \mathbb{R} \times E$ , then any dynamic model satisfying (1) to (3) is a time-inhomogeneous Markov process.

PROOF. By assumption there is a solution  $(X, Z)$  to the martingale problem. Following the proof (1)  $\Rightarrow$  (2) of Lemma II.1.20 we can see that (3) holds. Hence the admissibility condition implies

$$\begin{aligned} b^{Z_j}(t) &= \beta_j(t, X(t-), Z(t)) = 0, \\ c^{Z_j}(t) &= (\sigma\sigma^T)_{j,j}(t, X(t-), Z(t)) = 0 \end{aligned}$$

for  $t > T_j$ . Consequently,  $Z_j$  is almost constant after  $T_j$ . Hence Lemma II.1.20 yields that  $(X, Z)$  is a consistent dynamic model satisfying (1) to (3). Now assume that  $(X', Z')$  is any such model. Then Corollary II.1.19 yields risk-neutrality. Moreover, by Lemma II.1.20  $(X', Z')$  is solution to the martingale problem as well.

The last statement is a consequence of [18, Theorem 4.4.2].  $\square$

We now combine the results of the abstract framework in some more specific setup.

COROLLARY II.1.22. Assume that  $f_j(x) = (e^x - e^{y_j})^+$  for  $y_1, \dots, y_n \in \mathbb{R}$  and assume that  $\exp(Y_k)$  is a martingale for any  $k \in K$ . Let  $(\beta, \sigma)$  be admissible code-book dynamics. Let  $\mathcal{E}$  be the martingale problem given in Lemma II.1.20 and let  $x \in \mathbb{R}, k \in K$ . If  $(\Theta, X, Z)$  is a solution to the martingale problem  $(\mathcal{E}, \delta_0 \otimes \delta_x \otimes \delta_k)$  with  $\Theta(t) = t$ , then  $(X, Z)$  is a consistent dynamic model and

- (1)  $\exp(X)$  is a local martingale,
- (2)  $O_j(t) := (G(t, X(t), Z(t)))_j$  is a positive local martingale for any  $j = 1, \dots, n, t \in [0, T_j]$ ,
- (3)  $X(0) = x$  and  $Z(0) = k$ ,
- (4)  $X$  and  $Z$  are locally independent semimartingales and
- (5)  $Z$  is a continuous semimartingale such that a version of its local characteristic is given by Equations (II.1.8, II.1.9).

If  $(X', Z')$  is another consistent dynamic model which satisfies (3) to (5) and uniqueness holds for  $(\mathcal{E}, \delta_0 \otimes \delta_x \otimes \delta_k)$ , then the distributions of  $(X', Z')$  and  $(X, Z)$  coincide. If  $\exp(X)$  is a martingale, then  $(X, Z)$  is strongly risk-neutral.

PROOF. Except for the (local) martingale properties, this is a restatement of Theorem II.1.21. Theorem II.1.16 yields  $\exp(X)$  is a local martingale. By Theorem II.1.21  $O_j$  is a  $\sigma$ -martingale and hence a local martingale because it is positive. Let  $j \in \{1, \dots, n\}$ ,  $t \in [0, T_j]$ ,  $k \in K$ ,  $x \in \mathbb{R}$  and denote the transition function of  $Y_k$  by  $p$ . Then

$$G_j(t, x, k)_j = \int_{\mathbb{R}} f_j(y) p(t, x, T_j, dy) \in [0, \exp(x)].$$

In particular,  $0 \leq O_j \leq \exp(X)$ . If  $\exp(X)$  is a martingale, then  $O_j$  is bounded by a martingale and hence it is a martingale.  $\square$

The two most crucial assumptions of Corollary II.1.22 are the admissibility of  $(\beta, \sigma)$  and existence and uniqueness for the martingale problem. However, due to Lemma II.1.6 there is a good chance of  $G_j$  to be analytic. Together with Proposition II.1.5 we can see that there is (up to some regularity) only one choice for  $\beta$  except for a particular small set (invertibility in at least one point assumed).

In the specific framework in Section 3 below we show existence and uniqueness for the martingale problem given admissible and Lipschitz-continuous code-book dynamics. However, Lipschitz-continuity of  $\sigma$  will, in general, not imply Lipschitz-continuity of  $\beta$ . Consequently, we need a condition that ensures Lipschitz-continuity of  $\beta$ .

LEMMA II.1.23. *Let  $\sigma : \mathbb{R}_+ \times \mathbb{R} \times K \rightarrow \mathbb{R}^{n \times l}$  be Lipschitz-continuous with compact support such that it is absolutely bounded by a constant multiple of*

$$\lambda(t, x, k) := \begin{cases} \exp\left(\frac{-1}{|\det(D_3 G(t, x, k))|}\right) & \text{if } D_3 G(t, x, k) \text{ is invertible,} \\ 0 & \text{otherwise,} \end{cases}$$

for  $t \in [0, T_1)$ ,  $x \in \mathbb{R}$ ,  $k \in \text{int}(K)$ . Assume that  $\lambda \neq 0$  and that  $G_j(\cdot, x, k)$  is analytic on the interval  $[0, T_1)$  for all  $j \in \{1, \dots, n\}$ ,  $x \in \mathbb{R}$ ,  $k \in K$ . Then there is a unique Lipschitz-continuous function

$$\beta : [0, T_1] \times \mathbb{R} \times K \rightarrow \mathbb{R}$$

such that

$$D_3 G_j(t, x, k) \beta(t, x, k) = -\frac{1}{2} \left( \sum_{a,b=1}^n \partial_{k_a} \partial_{k_b} G_j(t, x, k) [(\sigma \sigma^\top)(t, x, k)]_{a,b} \right)$$

for any  $j \in \{1, \dots, n\}$ ,  $t \in [0, T_1)$ ,  $x \in \mathbb{R}$ ,  $k \in \text{int}(K)$  satisfying  $\{\beta_j = 0\} \subset \{\lambda = 0\}$ .

PROOF. Let  $d(t, x, k) := \det(D_3 G(t, x, k))$  for all  $(t, x, k) \in \mathbb{R}_+ \times \mathbb{R} \times K$  and let  $\beta$  be defined as in Equation (II.1.5) where  $\lambda \neq 0$  and 0 otherwise. Then  $(\beta, \sigma)$  is admissible by definition. Moreover, the inverse Function theorem yields that  $\beta$  is locally Lipschitz-continuous on the set  $E$  where  $\lambda$  is non-zero. A simple estimation yields that

$$\|\beta(t, x, k)\| \leq \|(D_3 G(t, x, k))^{-1}\| b(t, x, k) \lambda^2(t, x, k)$$

for some bounded function  $b$ . Moreover,

$$\|(D_3 G(t, x, k))^{-1}\| \leq C_n \|D_3 G(t, x, k)\|^{n-1} / |d(t, x, k)|$$

for some constant  $C_n$  only depending on  $n$ . Hence

$$\|\beta(t, x, k)\| \leq \hat{b}(t, x, k) \lambda^2(t, x, k) / |d(t, x, k)|$$

for some bounded function  $\hat{b}$ . On  $E$  we have  $\lambda^2/|d| = \exp(-2/(|d|))/|d|$ . Since  $d$  is holomorphic  $\lambda/|d|$  vanishes on the set of zeros of  $d$ . Hence  $\beta$  has a unique locally Lipschitz-continuous extension to the set of zeros of  $d$  which is zero. Since  $\sigma$  has compact support, the same is true for  $\beta$ . Hence  $\beta$  is Lipschitz-continuous.  $\square$

## 2. A review of the LIBOR model

In this section we want to demonstrate that our framework applied to interest rate markets leads, in some sense, to the LIBOR model as presented in [32, section 14.3]. The securities of primary interest are the zero coupon bonds which guarantee the holder a payoff of 1 at a fixed maturity  $T$ . Usually zero coupon bonds with several maturities are traded liquidly and we denote the price process of a zero coupon bond at time  $t$  with maturity  $T$  by  $B(t, T)$ . In particular  $B(T, T) = 1$ . The zero coupon bonds can be seen as derivatives on the numeraire  $S = e^X$  with discounted payoff function  $f(x) = e^{-x}$ .

For the entire section let  $T_j := j\delta$  for  $j \in \{0, \dots, n\}$ ,  $\delta > 0$  denote the maturities of the traded zero coupon bonds.



**2.1. Simple models.** We want to introduce a dynamic model, thus we need a simple model class. Very simple models in interest rate theory are time-inhomogeneous deterministic models with a bank account where the short rate changes only at dates  $T_1, \dots, T_n$ . These models can be identified by the  $n$ -tuple of their LIBOR rates for the respective  $n$  time intervals. Set  $K := \mathbb{R}_+^n$  for  $k \in K$  we define a deterministic bank account by

$$\begin{aligned} \exp(Y_k(t)) &:= \prod_{l=1}^n (1 + \delta k_l)^{\frac{(T_l \wedge t) - (T_{l-1} \wedge t)}{\delta}} \\ &= \exp \left( \sum_{l=1}^n \frac{(T_l \wedge t) - (T_{l-1} \wedge t)}{\delta} \log(1 + \delta k_l) \right). \end{aligned}$$

In this model the prices of arbitrage-free derivatives are constant multiples of  $\exp(Y_k)$ . In particular, the only arbitrage-free prices for the zero coupon bonds are given by

$$B(t, T_j) = \exp(Y_k(t) - Y_k(T_j))$$

for  $j \in \{1, \dots, n\}$ . Using the bank account as a numeraire, the discounted zero coupon bond prices are given by

$$\hat{B}(t, T_j) = \exp(-Y_k(T_j)) = \prod_{l=1}^j \frac{1}{1 + \delta k_l}$$

for  $j \in \{1, \dots, n\}$ . If we interpret  $Y_k$  as underlying, then the payoff function for the discounted bonds is given by  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^{-x}$ . Hence the pricing function for the discounted zero coupon bonds in this simple model class is given by

$$G_j(t, x, k) = e^{-x} \prod_{l=1}^j (1 + \delta k_l)^{\frac{(T_l \wedge t) - (T_{l-1} \wedge t)}{\delta} - 1},$$

which is clearly analytic in all of its variables outside the maturities. Observe that  $D_3 G(t, x, k)$  is a triangular matrix and the entries on the diagonal are given by

$$\partial_{k_j} G_j(t, x, k) = \frac{(T_j \wedge t) - (T_{j-1} \wedge t) - \delta}{\delta(1 + \delta k_j)} G_j(t, x, k).$$

Finally we calculate a version of the local characteristics of  $Y_k$ , which is given by  $(b^k, 0, 0)$  where

$$b^k(t) = \frac{1}{\delta} \log(1 + \delta k_j)$$

for  $t \in [T_{j-1}, T_j), j \in \{1, \dots, n\}$ .

**2.2. Dynamic model.** Using the models  $(Y_k)_{k \in K}$  from the previous section as simple models, we discuss and try to construct risk-neutral dynamic models for the discounted market.

Assume that we have a strongly risk-neutral dynamic model  $(X, Z)$  where  $\exp(X)$  is the numeraire process. The prices of the discounted zero coupon bonds in the dynamic model are defined by

$$\hat{B}(t, T_j) := G_j(t, X(t), Z(t)).$$

In particular,  $\hat{B}(T_j, T_j) = G_j(T_j, X(T_j), Z(T_j)) = \exp(-X(T_j))$ . Moreover, the drift rate  $b$  of  $Z$  is given by the drift condition II.1.10, which is explicit because  $D_3G(t, x, k)$  is invertible. However, as long as the local dynamics of  $X$  are not known, nothing specific can be said about  $b$ . The LIBOR rates (cf. [32, section 14.3]) in the dynamic model are

$$\begin{aligned} L(t, T_j) &:= \frac{1}{\delta}(B(t, T_j)/B(t, T_{j+1}) - 1) \\ &= Z_{j+1}(t) \end{aligned}$$

for all  $t \in [0, T_j]$ . Not surprisingly the code-book in the dynamic model consists of the LIBOR rates. This is also true for the simple models above by definition.

Now assume that

$$dZ_j(t) = b_j(t)dt + Z_j(t)a_j(t)dW(t) \tag{II.2.1}$$

for some 1-dimensional Brownian motion  $W$ , where  $a$  is deterministic and  $b$  is predictable. Let  $j \in \{2, \dots, n\}$ . Define the martingale  $D(t) := \frac{\hat{B}(t, T_j)}{\hat{B}(0, T_j)}$ . The  $T_j$ -forward measure is the measure  $Q$  given by  $\frac{dQ}{dP} = D$ . By Girsanov's theorem (see [25, Theorem III.3.24]) we have

$$dL(t, T_{j-1}) = dZ_j(t) = b_j^Q(t)dt + Z_j(t)a_j(t)dW^Q(t)$$

for some stochastic process  $b^Q$  and some  $Q$ -Brownian motion  $W^Q$ . The process  $Z_j$  is a  $Q$ -martingale since  $Z_j(t) = \frac{1}{\delta}(\hat{B}(t, T_{j-1})/\hat{B}(t, T_j) - 1)$  and since the discounted process  $\hat{B}(t, T_{j-1})/\hat{B}(t, T_j)$  is a  $Q$ -martingale. Consequently  $b^Q = 0$ . Hence

$$dL(t, T_{j-1}) = L(t, T_{j-1})a_j(t)dW^Q(t)$$

under  $Q$ . Thus we recover a standard log-normal LIBOR model, cf. [32, section 14.3].

Now we want to discuss the meaning of the consistency condition for the model. We now assume that  $(X, Z)$  is a risk-neutral and consistent dynamic model satisfying Equation

(II.2.1). Then a version of the local characteristics of  $X$  is given by  $(b^X, 0, 0)$ , where

$$b^X(t) = \frac{1}{\delta} \log(1 + \delta Z_j(t))$$

for  $t \in [T_{j-1}, T_j), j \in \{1, \dots, n\}$ . Then  $X$  can be seen as a bank account with stochastic short rate  $b^X$ , where the short rate coincides with one of the code-book components.

Let us now address construction and uniqueness of such models. Assume that  $a_j$  is continuous and  $a_j(t) \leq c(T_j - (T_j \wedge t))$  for some constant  $c \in \mathbb{R}_+$ . Theorem II.1.15 yields that there is a function  $\beta$  such that  $(\beta, a)$  is admissible code-book dynamics and  $\beta$  is unique up to a set of Lebesgue measure 0. This function  $\beta$  has a bounded and continuous version. [35, Theorem 11.2] yields uniqueness and existence for the martingale problem  $\mathcal{E}$  where  $\mathcal{E}$  is defined as in Lemma II.1.20. Lemma II.1.20 yields that there is a unique consistent dynamic model  $(X, Z)$  associated with  $(\beta, a)$ . By Corollary II.1.19 this model is risk-neutral.

### 3. Modelling stock options

In this section we want to focus on a concrete simple model class, which will consist of time-inhomogeneous Lévy processes. The Lévy-Khintchine formula yields that the class of all Lévy models is naturally parametrised by their Lévy-Khintchine triplet. Since we are only interested in consistent dynamic models  $(X, Z)$  where  $\exp(X)$  is a martingale, we will only use Lévy processes which satisfy the second assumption of Theorem II.1.16, i.e. their exponentials are martingales. In this special setup the drift coefficients of the Lévy processes are determined by the quadratic variation and the Lévy measure. Since the set of Lévy measures is infinite-dimensional, we focus on Lévy measures which are concentrated on a given finite set. These Lévy-measures can be parametrised by their intensities  $\lambda_1, \dots, \lambda_n$  at the given jump heights.

Throughout this section we will only work with Lévy measures which have no mass in a neighbourhood of zero. Thus we can choose a truncation function which is the identity in this neighbourhood and zero otherwise. This removes the truncation function from all formulas where a truncation function is usually needed. Let  $n \in \mathbb{N}$  and  $f_0, f_1, \dots, f_n$  real-valued measurable functions which are exponentially bounded and  $0 < T_0 \leq T_1 \leq \dots \leq T_n$ .

On  $C := C(\mathbb{R}, \mathbb{R})$  we define the operators

- $\mathcal{C} : C^2(\mathbb{R}, \mathbb{R}) \rightarrow C, f \mapsto \frac{1}{2}(f'' - f')$ ,
- $\mathcal{A}_h : C^1(\mathbb{R}, \mathbb{R}) \rightarrow C, f \mapsto (x \mapsto (f(x+h) - f(x) - (e^h - 1)f'(x)))$

for all  $h \in \mathbb{R} \setminus \{0\}$  and  $\mathcal{A}_0 := 0$ . Note that the restriction of  $\mathcal{C}$  and  $\mathcal{A}_h$  to  $C_0 := C_0(\mathbb{R}, \mathbb{R})$  are generators of Feller semigroups and the restriction of  $\mathcal{A}_h$  to  $C_0$  is the generator of a  $C_0$ -group, i.e. a mapping  $U : \mathbb{R} \rightarrow L(C_0)$  which is a strongly continuous group-homomorphism from  $(\mathbb{R}, +)$  to  $(L(C_0), \circ)$ . Moreover, by [18, Theorem 8.3.4] they generate Lévy processes, i.e. any solution to the martingale problem  $\mathcal{C}$  resp.  $\mathcal{A}^h$  is a Lévy process. The transition semigroup generated by  $\mathcal{C}$  will be denoted by  $V$  and the transition semigroup generated by  $\mathcal{A}_h$  will be denoted by  $W^h$ .

REMARK II.3.1. A Lévy process  $L$  with generator  $\mathcal{C}$  admits the Itô decomposition

$$dL(t) = -\frac{1}{2}dt + dB(t)$$

for some standard Brownian motion  $B$ . In particular,  $d \exp(L)(t) = \exp(L)dB(t)$  by Itô's formula and hence  $\exp(L)$  is a martingale. Likewise, a Lévy process  $L$  with generator  $\mathcal{A}_h$  can be written as

$$dL(t) = (e^h - 1)dt + h dN(t)$$

where  $N$  is a standard Poisson process. Itô's formula yields again that  $\exp(L)$  is a martingale.

A Lévy process  $L$  with characteristics  $(b, c, F)$ , the property that  $\exp(L)$  is a martingale and the property that the Lévy measure is of the form  $F = \sum_{i=1}^n \lambda_i \delta_{h_i}$  for some  $n \in \mathbb{N}$  has the generator

$$c\mathcal{C} + \sum_{i=1}^n \lambda_i \mathcal{A}_{h_i}, \quad (\text{II.3.1})$$

i.e.  $\mathcal{C}$ ,  $\mathcal{A}_{h_i}$  are the generators of exponentially compensated Brownian motion, resp. exponentially compensated Poisson process.

**3.1. Simple model setup.** In this section we will use the simple models

$$dY_k(t) = W(ct) + \sum_{j=1}^n h_j N_j(\lambda_j t) \quad (\text{II.3.2})$$

for the log-return of the stock where  $W$  is an exponentially compensated standard Brownian motion, i.e.  $W$  is a Brownian motion with volatility 1 such that  $\exp(W)$  is a martingale,  $N_1, \dots, N_n$  are independent exponentially compensated SPP, i.e.  $N_1$  is a Poisson process with intensity 1, jump height 1 and drift rate in such a way that  $\exp(N_1)$  is a martingale,  $h_1 < \dots < h_n$  are real numbers and

$$k := (c, \lambda_1, \dots, \lambda_n) \in K := \{(k_0, \dots, k_n) : k_0 \neq 0 \text{ and } k_j \in \mathbb{R}_+ \text{ for } j = 1, \dots, n\}.$$

Since the simple models are Lévy models there are several pricing methods for the European call options, e.g. pricing via Fourier transform, see Proposition II.3.10, or via power-series, see Corollary II.3.8. We aim at constructing consistent and risk-neutral dynamic models where the code-book process is an Itô-process in the form

$$dZ(t) = b^Z(t)dt + \sqrt{c^Z(t)}dB(t)$$

where  $B$  is an SBM independent of  $W, N_1, \dots, N_n$ . The consistent dynamic models  $(X, Z)$  coincide in law with process of the form

$$dX(t) = W \left( \int_0^t Z_0(s)ds \right) + \sum_{j=1}^n h_j N_j \left( \int_0^t Z_j(s)ds \right),$$

Such a model can be easily approximated by an Euler scheme which converges weakly to the process, see [7, Theorem].

The generator for the simple model  $Y_k$  is given by

$$\mathcal{G}^k := c\mathcal{C} + \sum_{j=1}^n \lambda_j \mathcal{A}_{h_j}$$

where  $k = (c, \lambda_1, \dots, \lambda_n) \in K$ . We also define  $\mathcal{G}_s := \sum_{j=1}^{\ell} \mathcal{G}^{s_j} \mathcal{G}^{s_j}$  for all  $s \in K^{\ell}$  which will be needed later. Moreover, the transition semigroup associated with the generator  $\mathcal{G}^k$  is given by

$$U_t^k := W_{\lambda_1 t}^{h_1} \circ \dots \circ W_{\lambda_n t}^{h_n} \circ V_{ct},$$

where  $V$  is the transition semigroup generated by  $W$  and  $W^{h_j}$  is the transition semigroup generated by  $N_j$ . The transition semigroups  $W^{h_j}, W^{h_i}, V$  commute, since the resolvents of the generators  $\mathcal{A}_{h_j}, \mathcal{C}$  commute. In this section we work with a single maturity  $T := T_1$  and assume  $T = T_j$  for all  $j = 0, \dots, n$ . More maturities can later be added to the theory, but for that we will have to change the simple models to piecewise Lévy processes which change their triplets at the maturities (see Section 3.4). As usual let

$$G_j : [0, T] \times \mathbb{R} \times K \rightarrow \mathbb{R}, (t, x, k) \mapsto U_{T-t}^k f_j(x).$$

The following programme is to apply the general theory to this particular example and to provide computer applicable formulas. We start here with an informal step by step instruction. The readers that are interested in the theoretical part only can skip this instruction.

- (1) First we choose jump heights  $h_1, \dots, h_n$ . Then we have  $n + 1$  parameters, namely the diffusion coefficient  $c$  and the intensities  $\lambda_1, \dots, \lambda_n$  of jumps with heights  $h_1, \dots, h_n$  respectively. We also choose a maturity  $T$  and  $n + 1$  payoff functions  $f_0, \dots, f_n$  which determine the type of European option, e.g. call/put. Then the number of parameters and the numbers of options coincide, hence perfect calibration may be possible. We assume that the market behaves like a consistent dynamic model  $(X, Z)$  which is risk-neutral under the pricing measure, that  $X$  and  $Z$  are locally independent and that  $Z$  is continuous. Calibrated parameters are understood as states of the process  $Z$  which are observable from present market data.
- (2) As usual we assume that

$$dZ(t) = b(t)dt + \sqrt{c(t)}dB(t)$$

for some SBB  $B$ . We have to specify the process  $c$  in some way. Here we will assume that

$$\sqrt{c(t)} = \sigma(t, X(t), Z(t)).$$

Then Girsanov's theorem, see [25, Theorem III.3.24] yields that under the physical measure  $Q$  there is a Brownian motion  $B^Q$  and some drift process  $b^Q$  such that

$$dZ_t = b^Q(t)dt + \sigma(t, X(t), Z(t))dB^Q(t).$$

Thus one could use observed states from a time series to estimate the diffusion coefficient  $\sigma$  of the process  $Z$ .

- (3) Since  $(X, Z)$  is risk-neutral we need admissible code-book dynamics  $(\beta, \sigma)$  where  $\beta$  is given by II.1.18. However, in order to calculate  $\beta$  we need explicit expressions for  $G_j$  and its first two derivatives. If the payoff functions are in the form  $f_j(x) = (e^x - e^{y_j})^+$ , i.e. if we are talking about European call options with log-strikes  $y_j$ , we can use the formulas in Proposition II.3.10 below, noting that  $G_j(t, x, k) = U_{T-t}^k f_j(x)$ . In any case the derivatives are given in Proposition II.3.7 below. The expressions occurring in Proposition II.3.7 can be computed using power series, see Corollary II.3.8 and Theorem II.3.9 below.
- (4) Now, we can compute the local characteristics of  $Z$  given the state of  $X$  and  $Z$ . The volatility of  $X$  is simply  $Z_0$  and the intensities of  $X$  are  $Z_1, \dots, Z_n$  of jumps with heights  $h_1, \dots, h_n$  respectively. We can approximate the resulting models with an

Euler-scheme and expectations can be calculated by Monte-Carlo simulation, cf. [7, Theorem].

We start with some technical lemmas.

LEMMA II.3.2.  $\exp(Y_k)$  is integrable for all  $k \in K$ .

PROOF. Let  $k \in K$ . Then  $\exp(Y_k)$  is a martingale and hence it is integrable.  $\square$

LEMMA II.3.3. Let  $(c, \lambda_1, \dots, \lambda_n) = k \in K$ . The conditional characteristic function of  $Y_k(T) - Y_k(t)$  is given by

$$\begin{aligned} & E \left( \exp \left( iuY_k(T) - iuY_k(t) \right) | Y_k(t) \right) \\ &= \exp \left( (T-t) \left( -\frac{c}{2}(u^2 + iu) + \sum_{j=1}^n \lambda_j (e^{iuh_j} - 1 - iuh_j) \right) \right) \end{aligned}$$

for any  $t, T \in \mathbb{R}_+$  with  $t \leq T$  and any  $u \in \mathbb{R}$ .

PROOF. This follows directly from [25, Theorem II.4.15].  $\square$

COROLLARY AND DEFINITION II.3.4. The conditional characteristic function of  $Y_k(T) - Y_k(t)$  has an entire extension and only depends on  $(T-t)k$ . Indeed, it is equal to the characteristic function of  $Y_{(T-t)k}(1) - Y_{(T-t)k}(0)$  and its entire extension will be denoted by  $\hat{Y}_{(T-t)k}$ . Moreover, we have

$$U_{T-t}^k = U_1^{(T-t)k}.$$

The remaining part of this chapter aims at providing computer applicable formulas for  $G_j$  and its first two derivations. We introduce two approaches. One relies on the fact that the option prices for European call options in the Black-Scholes model are known in closed form. The other one is a Fourier analytic approach. However, first of all we have to prove that  $G_j$  is a  $C^2$  function. Indeed, it is analytic.

LEMMA II.3.5. Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be an exponentially bounded function, i.e.  $|f(x)| \leq ae^{b|x|}$  for some  $a, b \in \mathbb{R}_+$ . Then  $f(Y_k)$  is integrable for all  $k \in K$  and  $(t, x) \mapsto V_t f(x)$  is analytic for any  $t > 0, x \in \mathbb{R}$ . If  $f$  is analytic as well, then  $(t, x) \mapsto W_t^h f(x)$  is analytic in all variables for all  $h \in \mathbb{R}$ . In the special case  $f = \exp$  we have  $W_t^h f(x) = f(x)$  and  $V_t f(x) = f(x)$  for any  $x \in \mathbb{R}$ .

In particular, the pricing function

$$G_j : [0, T_j] \times \mathbb{R} \times K \rightarrow \mathbb{C}, (t, x, k) \mapsto U_t^k f_j(x)$$

is analytic on its domain outside  $t = T_j$ , i.e. it can be developed locally into a power series in all points  $(t, x, k)$  with  $t \in [0, T_j)$ .

REMARK II.3.6. Even though we do not prove it here,  $G_j$  has a unique analytic extension to  $[0, T_j) \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{C}^n$ .

PROOF. Let  $k \in K$  and  $t \in \mathbb{R}$ . Then Corollary II.3.4 yields

$$E|f(Y_k(t))| \leq aE|e^{b|Y_{tk}|}| \leq a\hat{Y}_{tk}(ib) + \hat{Y}_{tk}(-ib) < \infty$$

Lemma II.1.6 yields  $(t, x) \mapsto V_t f(x)$  is analytic.

Now assume that  $f$  is analytic and let  $N$  be a standard Poisson process. Then

$$W_t^h f(x) = E(f(hN(t) + x)) = \exp(-t) \sum_{\alpha=0}^{\infty} \frac{t^\alpha}{\alpha!} f(h\alpha + x).$$

The right-hand side is analytic with respect to  $t$  and to  $x$  because

$$\begin{aligned} \left| \exp(-t) \sum_{\alpha=0}^{\infty} \frac{t^\alpha}{\alpha!} f(h\alpha + x) \right| &\leq \exp(-t) \sum_{\alpha=0}^{\infty} \frac{t^\alpha}{\alpha!} a e^{b|h\alpha+x|} \\ &= a e^{b|x|} e^{t(e^{b|h|}-1)}. \end{aligned}$$

We have shown that  $G_j$  is separately holomorphic. [30, Section 1.2] yields it is analytic.  $\square$

3.1.1. *Option prices as power series.* The partial derivatives of  $G_j$  can be expressed in terms of the semigroup  $U$  and the generators  $\mathcal{C}, \mathcal{A}_{h_j}$ . In view of semigroups, in some sense we have the equation  $U_\tau^k = \exp(\tau \mathcal{G}^k)$  and the naive  $k$ -derivation on the right hand side yields  $\partial_{k_j} U_\tau^k = \tau \mathcal{G}^{e_j} U_\tau^k$  where  $e_j$  is the  $j$ th unit vector.

PROPOSITION II.3.7. *Let  $(c, \lambda_1, \dots, \lambda_n) = k \in \text{int}(K)$ ,  $j, \ell, m \in \{1, \dots, n\}$ ,  $x \in \mathbb{R}$  and  $t < T_j$ . Then the first two partial derivations of  $G_j$  are given by:*

- $G_j(t, x, k) = U_{T-t}^k f_j(x)$ ,
- $\partial_c G_j(t, x, k) = (T-t) \mathcal{C} U_{T-t}^k f_j(x)$ ,
- $\partial_{\lambda_\ell} G_j(t, x, k) = (T-t) \mathcal{A}_{h_\ell} U_{T-t}^k f_j(x)$ ,
- $\partial_c \partial_c G_j(t, x, k) = (T-t)^2 \mathcal{C} \mathcal{C} U_{T-t}^k f_j(x)$ ,
- $\partial_c \partial_{\lambda_\ell} G_j(t, x, k) = (T-t)^2 \mathcal{A}_{h_\ell} \mathcal{C} U_{T-t}^k f_j(x)$ ,
- $\partial_{\lambda_\ell} \partial_{\lambda_m} G_j(t, x, k) = (T-t)^2 \mathcal{A}_{h_\ell} \mathcal{A}_{h_m} U_{T-t}^k f_j(x)$ .

PROOF. Lemma II.3.5 implies that  $G_j$  is analytic in the interior of its domain, so the partial derivations exist.



Let  $s \in \mathbb{R}_+$ ,  $c(T-t) > s$  and  $\phi \in C$  infinitely differentiable with compact support such that  $\phi$  is constant 1 in a neighbourhood of 0. Then  $\phi_n : x \mapsto \phi(x/n)$  approximates 1 locally uniformly and its derivations approximate 0 locally uniformly. Define  $A := W_{\lambda_1(T-t)}^{h_1} \circ \dots \circ W_{\lambda_n(T-t)}^{h_n}$ ,  $g := V_s f_j$ . Then  $\phi_n g \in C_0(\mathbb{R}, \mathbb{R})$  and it is infinitely differentiable. Thus

$$\begin{aligned}
\partial_c AV_{c(T-t)-s}(\phi_n g) &= A \partial_c V_{c(T-t)-s}(\phi_n g) \\
&= (T-t) A \mathcal{C} V_{c(T-t)-s}(\phi_n g) \\
&= (T-t) AV_{c(T-t)-s} \mathcal{C}(\phi_n g) \\
&\rightarrow (T-t) AV_{c(T-t)-s} \mathcal{C} g \\
&= (T-t) \mathcal{C} U_{(T-t)}^k f_j,
\end{aligned}$$

where the convergence is locally uniformly. Since  $\partial_c$  is a closed operator, by convergence of  $\partial_c(AV_{c(T-t)-s}(\phi_n g))$  and since  $AV_{c(T-t)-s}(\phi_n g)$  converges to  $AV_{c(T-t)-s}g$  we conclude

$$\begin{aligned}
(T-t) \mathcal{C} U_{(T-t)}^k f_j &= \lim_{n \rightarrow \infty} \partial_c(AV_{c(T-t)-s}(\phi_n g)) \\
&= \partial_c(AV_{c(T-t)-s}g). \\
&= \partial_c U_{(T-t)}^k f_j.
\end{aligned}$$

The same arguments apply to the other derivatives. □

Since we now have  $\partial_c G_j(x, t, k) = (T-t) \mathcal{C} U_{T-t}^k f_j(x)$ , we would need  $G_j(t, x, k) = U_{T-t}^k f_j(x)$  in closed form if we want to apply the differential operator  $\mathcal{C}$ . However,  $U^k$  is the composition of simpler semigroups which commute. Since  $V_t f$  is analytic we have  $V_t f \in \text{dom}(\mathcal{G}^{e_j})$ . Consequently, we have  $\mathcal{C} U_{T-t}^k f = U_{T-t}^{(0, k_1, \dots, k_n)} \mathcal{C} V_{k_0 t} f$  and  $\mathcal{C} V_t f$  is known in closed form if  $f(x) = (e^x - e^y)^+$ . Indeed,  $V_{k_0 t} f$  is the price of a European call option in the Black-Scholes model with volatility  $k_0$ .

**COROLLARY II.3.8.** *Let  $y \in \mathbb{R}$ ,*

$$f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto (e^x - e^y)^+,$$

$(c, (\lambda_1, \dots, \lambda_n)) = k \in \text{int}(K)$ ,  $j, l \in \{1, \dots, n\}$ ,  $x \in \mathbb{R}$  and  $t < T$ . Then

$$\begin{aligned} \partial_c U_{T-t}^k f(x) &= (T-t) W_{\lambda_1 t}^{h_1} \circ \dots \circ W_{\lambda_n t}^{h_n} \mathcal{C} V_{tc} f(x), \\ \partial_{\lambda_j} U_{T-t}^k f(x) &= (T-t) W_{\lambda_1 t}^{h_1} \circ \dots \circ W_{\lambda_n t}^{h_n} \mathcal{A}_{h_j} V_{tc} f(x), \\ V_{tc} f(x) &= e^x \Phi\left(\frac{x-y+ct/2}{\sqrt{ct}}\right) - e^y \Phi\left(\frac{x-y-ct/2}{\sqrt{ct}}\right), \\ \mathcal{C} V_{tc} f(x) &= \frac{1}{2} \phi\left(\frac{x-y-ct/2}{\sqrt{ct}}\right) \text{ and} \\ \mathcal{A}_{h_j} V_{tc} f(x) &= e^{x+h_j} \left( \Phi\left(\frac{x+h_j-y+ct/2}{\sqrt{ct}}\right) - \Phi\left(\frac{x-y+ct/2}{\sqrt{ct}}\right) \right) \\ &\quad + e^{y+h_j} \Phi\left(\frac{x-y-ct/2}{\sqrt{ct}}\right) - e^y \Phi\left(\frac{x+h_j-y-ct/2}{\sqrt{ct}}\right). \end{aligned}$$

Moreover,  $\mathcal{C} V_{tc} f(Y_k)$  and  $\mathcal{A}_{h_j} V_{tc} f(Y_k)$  are integrable functions for all  $k \in K$ .

Finally we have to know how to apply the operator  $U_\tau^{(0, \lambda_1, \dots, \lambda_n)} = W_{\lambda_1 \tau}^{h_1} \circ \dots \circ W_{\lambda_n \tau}^{h_n}$  on  $V_{tc} f$ ,  $\mathcal{C} V_{tc} f$  and on  $\mathcal{A}_{h_j} V_{tc} f$ . Indeed,  $U_\tau^{(0, k_1, \dots, k_n)}$  can be expressed as a power series.

**THEOREM II.3.9.** *Let  $y \in \mathbb{R}$ ,  $(c, \lambda) \in K$ ,  $\tau \in \mathbb{R}_+$  and  $f$  a function such that  $f(Y_k)$  is integrable for all  $k \in K$ . Then*

$$U_\tau^k f(x) = e^{-\tau \lambda} \sum_{\alpha \in \mathbb{N}_0^n} \frac{(\tau \lambda)^\alpha}{\alpha!} V_{c\tau} f(x + \alpha h - \tau \lambda (e^h - 1))$$

where we used multi-index notations, i.e.  $e^\lambda := \prod_{j=1}^n e^{\lambda_j}$ ,  $\lambda^\alpha = \prod_{j=1}^n \lambda_j^{\alpha_j}$ ,  $\alpha! = \prod_{j=1}^n \alpha_j!$ ,  $\alpha h = \sum_{j=1}^n \alpha_j h_j$  and  $\lambda(e^h - 1) = \sum_{j=1}^n \lambda_j (e^{h_j} - 1)$ .

**PROOF.** Let  $j \in \{1, \dots, n\}$  and  $L$  a Poisson process with jump height  $h_j$ . Then

$$U_\tau^{\lambda_j e_j} f(x) = E(f(L(\lambda_j \tau) + x - \tau \lambda_j (e^{h_j} - 1))) = e^{-\tau \lambda_j} \sum_{\alpha=0}^{\infty} \frac{(\tau \lambda_j)^\alpha}{\alpha!} f(x + \alpha h_j - \tau \lambda_j (e^{h_j} - 1)).$$

A simple inductive argument yields the claim.  $\square$

**3.1.2. Option prices via Fourier transform.** As in [14] it is possible to derive option prices and its derivatives via Fourier representation. Moreover, the Fourier transform of  $Y_k$  is known in closed form, see Lemma II.3.3.

**PROPOSITION II.3.10.** *Let  $(c, \lambda_1, \dots, \lambda_n) = k \in K$ ,  $y \in \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto (e^x - e^y)^+$  and*

$$g_k : \mathbb{C} \rightarrow \mathbb{C}, u \mapsto \exp\left((T-t) \left(-c \frac{u^2 + iu}{2} + \sum_{j=1}^n \lambda_j (e^{iuh_j} - 1 - iu(\exp(h_j) - 1))\right)\right)$$

the entire extension of the Fourier transform of  $Y_k(t)$ . Then for any  $v \in \mathbb{R}$  we have

$$U_{T-t}^k f(x) = E(f(x + Y_k(T-t))) = f(x) + \frac{e^y}{2\pi} \int_{\mathbb{R}+iv} \frac{1 - g_k(u)}{u^2 + iu} e^{iu(x-y)} du.$$

The differential operators  $\partial_c$  and  $\partial_{\lambda_j}$  applied to  $U_{T-t}^k f(x)$  can be exchanged with the integral in the Fourier representation, e.g.

$$\partial_c U_{T-t}^k f(x) = \frac{e^y}{2\pi} \int_{\mathbb{R}+iv} \frac{-\partial_c g_k(u)}{u^2 + iu} e^{iu(x-y)} du = \frac{(T-t)e^y}{4\pi} \int_{\mathbb{R}+iv} g_k(u) e^{iu(x-y)} du$$

where  $\partial_c g_k(u) = -(T-t) \frac{u^2+iu}{2} g_k(u)$  etc.

Of course this formula extends to a much more general class of Lévy processes, see [14, 4] or Proposition A.4.4.

PROOF. Proposition A.4.4 yields the first part of the proof.  $g_k$  is a Schwartz-function since  $c > 0$  and hence we can exchange the partial derivatives with the integral.  $\square$

**3.2. Existence result.** In the previous section we prepared the formulas which we will need to set up a risk-neutral and consistent dynamic model. Our aim is to apply Corollary II.1.22. In order to do so we have to prove existence and uniqueness for the martingale problem in Lemma II.1.20. In this section we work again with the simple model class introduced in the previous section, see Equation (II.3.2).

LEMMA II.3.11. *Let  $(X, Z)$  be a consistent dynamic model. Then  $X$  admits the local characteristics  $(b, c, F)$  with*

$$b(t) = -\frac{1}{2} Z_0(t) + \sum_{j=1}^n (e^{h_j} - 1) Z_j(t) \quad (\text{II.3.3})$$

$$c(t) = Z_0(t) \quad (\text{II.3.4})$$

$$F(t, \cdot) = \sum_{j=1}^n Z_j(t) \delta_{h_j} \quad (\text{II.3.5})$$

for any  $t \in \mathbb{R}_+$  where  $\delta_h$  is the Dirac measure in  $h$ . In particular,  $X$  is a special semi-martingale which allows for local characteristics. A version of the local characteristics  $(b^S, c^S, F^S)$  for  $S := \exp(X)$  is given by

$$b^S(t) = 0 \quad (\text{II.3.6})$$

$$c^S(t) = S^2(t-) Z_0(t) \quad (\text{II.3.7})$$

$$F^S(t, A) = \sum_{j=1}^n 1_A(S(t-)(e^{h_j} - 1)) Z_j(t). \quad (\text{II.3.8})$$

PROOF. The local characteristics of  $X$  can be read directly from the consistency condition (see Definition II.1.13). The local characteristics of  $S$  is given by Proposition A.1.1.  $\square$

COROLLARY II.3.12. *Let  $(X, Z)$  be a consistent dynamic model such that  $Z$  is continuous and bounded and  $X, Z$  are locally independent. Then  $\exp(X)$  is a martingale.*

PROOF. This is a simple application of Lemma II.3.11 and [29, Theorem 3.2].  $\square$

LEMMA II.3.13. *Let  $(\beta, \sigma)$  be Lipschitz-continuous and admissible code-book dynamics,  $C := C_0(\mathbb{R}_+ \times \mathbb{R} \times K)$ ,  $\mu$  be a Borel probability measure on  $\mathbb{R}_+ \times \mathbb{R} \times K$  and define the operator*

$$\begin{aligned} \mathcal{E} &: D(\mathcal{E}) \subset C_0 \rightarrow C, \\ \mathcal{E}g(t, x, k) &= \partial_t g(t, x, k) + \mathcal{G}^k g(t, x, k) + D_k g(t, x, k) \beta_Z(t, x, k) \\ &\quad + \sum_{j=1}^l \frac{1}{2} \sigma_{Z,j}(t, x, k)^\top D_k^2 g(t, x, k) \sigma_{Z,j}(t, x, k). \end{aligned}$$

*Then existence and uniqueness for the martingale problem  $(\mathcal{E}, \mu)$  in the sense of [18, Section 4.3] hold.*

PROOF. Define for all  $b \in \mathbb{R}_+ \cup \{\infty\}$

$$\mathcal{T}_b : C \rightarrow C, \quad \mathcal{T}_b g(t, x, k) = \sum_{j=1}^n a_b(\lambda_j) (g(t, x + h_j, k) - g(t, x, k))$$

where  $a_b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is any continuous function bounded by  $b$  and  $a_b(x) = x$  if  $|x| \leq b$ . Then  $\mathcal{T}_b$  is a linear and dissipative operator which is bounded if  $b \neq \infty$ . Define  $\mathcal{E}' := \mathcal{E} - \mathcal{T}_\infty$ , then

$$\begin{aligned} \mathcal{E}' &: D(\mathcal{E}') \subset C_0 \rightarrow C, \\ \mathcal{E}'g(t, x, k) &= \partial_t g(t, x, k) + k_1 \mathcal{C}g(t, x, k) + D_k g(t, x, k) \beta_Z(t, x, k) \\ &\quad + \sum_{j=1}^l \frac{1}{2} \sigma_{Z,j}(t, x, k)^\top D_k^2 g(t, x, k) \sigma_{Z,j}(t, x, k). \end{aligned}$$

$\mathcal{E}'$  is a second order partial differential operator which is dissipative. By [35, Theorem 22.12]  $\mathcal{E}'$  is the generator of a Feller semigroup. Let  $b \in \mathbb{R}_+$ . Then [18, Theorem 1.7.1] yields that  $\mathcal{E}_b := \mathcal{E}' + \mathcal{T}_b$  is also the generator of a Feller semigroup. Thus [18, Theorem 4.2.7] implies existence of a solution for the martingale problem  $(\mathcal{E}_b, \mu)$  and [18, Theorem 4.4.1] ensures uniqueness. [18, Theorem 4.6.1] yields uniqueness for the stopped martingale problem  $(\mathcal{E}_b, \mu, U)$  for any open subset  $U \subset K$ . If  $U$  is bounded by  $b$  then the stopped

martingale problems  $(\mathcal{E}_b, \mu, b)$  and  $(\mathcal{E}, \mu, U)$  have the same solutions and hence we have existence and uniqueness for  $(\mathcal{E}, \mu, U)$ . [18, Theorem 4.5.4, Theorem 4.6.2] yield uniqueness and existence for the martingale problem  $(\mathcal{E}, \mu)$ .  $\square$

Since we have now existence and uniqueness at hand, we can turn to the announced uniqueness and existence theorem.

**THEOREM II.3.14 (Existence and uniqueness).** *Let  $(\beta_Z, \sigma_Z)$  be Lipschitz continuous and admissible code-book dynamics and  $x \in \mathbb{R}, k \in K$ . Then there is a consistent dynamic model  $(X, Z)$  such that*

- (1)  $X(0) = x$  and  $Z(0) = k$ ,
- (2)  $X$  and  $Z$  are locally independent semimartingales and
- (3)  $Z$  is a continuous semimartingale whose characteristics  $(b^Z, c^Z, 0)$  are given by Equations (II.1.8, II.1.9).

Moreover for any consistent dynamic model  $(X', Z')$  satisfying the conditions (1) to (3) we have

- The distributions of  $(X', Z')$  and  $(X, Z)$  coincide,
- $(\Theta, X', Z')$  is a Markov process where  $\Theta(t) := t$ ,
- $\exp(X')$  is a local martingale and
- the dynamic model  $(X', Z')$  is risk-neutral.

Now assume  $f_j(x) = (e^x - e^{y_j})^+$  for some  $y_j \in \mathbb{R}$  and all  $j \in \{1, \dots, n\}$ . Then  $O_j(t) := G_j(t, X(t), Z(t))$  for all  $t \in [0, T_j]$  is a local martingale. If the support of  $(\beta_Z, \sigma_Z)$  is compact, then both  $\exp(X)$  and  $O_j$  are martingales.

**PROOF.** Lemma II.3.13 yields existence and uniqueness for the martingale problem  $\mathcal{E}$  of Lemma II.1.20. Hence Lemma II.1.20 yields the existence of a consistent dynamic model satisfying (1) to (3).

Let  $(X, Z)$  be any such model. Then Lemma II.1.20 implies uniqueness of the law of  $(X, Z)$ . [18, Theorem 4.4.2] yields that  $(\Theta, X, Z)$  is a Markov process, where  $\Theta(t) := t$ . Theorem II.1.16 states  $\exp(X)$  is a local martingale and Theorem II.1.15 yields that  $(X, Z)$  is risk-neutral.

Now assume the additional assumption. Then Corollary II.1.22 together with Corollary II.3.12 yield the claim.  $\square$

In order to apply Theorem II.3.14 we need to construct Lipschitz-continuous and admissible code-book dynamics from a given function  $\sigma$ . Of course we want to apply Lemma II.1.23 which requires some regularity on  $\sigma$ .

COROLLARY II.3.15. *Assume that  $f_j(x) = (e^x - e^{y_j})^+$  for real numbers  $y_1 < \dots < y_{n+1}$ .*

Set

$$\lambda(t, x, k) := \begin{cases} \exp\left(\frac{-1}{|\det(D_3 G(t, x, k))|}\right) & \text{if } D_3 G(t, x, k) \text{ is invertible,} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\sigma : \mathbb{R}_+ \times \mathbb{R} \times K \rightarrow \mathbb{R}^{n \times l}$$

be Lipschitz-continuous, with compact support in  $[0, T] \times \mathbb{R} \times \text{int}(K)$  and bounded by a multiple of  $\lambda$ . Then there is a Lipschitz-continuous function

$$\beta : \mathbb{R}_+ \times \mathbb{R} \times K \rightarrow \mathbb{R}^n$$

such that  $(\beta, \sigma)$  are admissible code-book dynamics.

Assume additionally that  $\lambda$  is not constant zero. Then the only continuous choice for  $\beta$  such that  $(\beta, \sigma)$  are admissible code-book dynamics is given by Equation (II.1.5) on the set  $\{d \neq 0\}$  and  $\beta$  is zero on the set  $\{d = 0\}$ .

REMARK II.3.16. If  $\lambda$  is constant zero, then  $\sigma$  is constant zero and  $\beta$  can be chosen to be zero.  $(0, 0)$  are of course a Lipschitz-continuous and admissible code-book dynamics.

PROOF. By Lemma II.3.5 the functions  $G_j$  are analytic. Hence Lemma II.1.23 yields the claim.  $\square$

COROLLARY II.3.17. *Assume that  $f_j(x) = (e^x - e^{y_j})^+$  for real numbers  $y_1 < \dots < y_{n+1}$ . Let  $x \in \mathbb{R}, k \in K$  and let  $\sigma$  fulfil the requirements of Corollary II.3.15. Then there is a risk-neutral and consistent dynamic model  $(X, Z)$  such that*

- (1)  $X(0) = x$  and  $Z(0) = k$ ,
- (2)  $X$  and  $Z$  are locally independent semimartingales,
- (3)  $Z$  is a continuous semimartingale whose characteristics  $(b^Z, c^Z, 0)$  are given by Equations (II.1.8, II.1.9).

Moreover for any risk-neutral and consistent dynamic model  $(X', Z')$  satisfying the conditions (1) to (3) we have

- *the distributions of  $(X', Z')$  and  $(X, Z)$  coincide,*
- *$(\Theta, X', Z')$  is a Markov process where  $\Theta(t) = t$ ,*
- *$\exp(X')$  is a martingale and*
- *$(X', Z')$  is strongly risk-neutral.*

PROOF. This is the joint conclusion of Corollary II.3.15 and Theorem II.3.14.  $\square$

**3.3. Concrete example with simulation.** We use the same simple model class as before with  $n = 2$ ,  $h_1 = -0.1$  and  $h_2 = -0.2$ . Then the simple model  $Y^{(c, \lambda_1, \lambda_2)}$  is of the form

$$W(ct) + h_1 N_1(\lambda_1 t) + h_2 N_2(\lambda_2 t)$$

where  $W, N_1, N_2$  are independent exponentially compensated Brownian motion, resp. exponentially compensated standard Poisson processes, cf. 3.1. Moreover we choose  $\exp(x) = 100$ ,  $\exp(y_1) = 98$ ,  $\exp(y_2) = 100$ ,  $\exp(y_3) = 102$ ,  $k = (0.2, 0.2, 0.2)$  and maturity  $T = 1/12$  where  $y_1, y_2, y_3$  denote the log-strikes for European call options with maturity  $T$ . We want to drive the process with the constant diffusion parameter

$$\sigma_w := \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e. we model the martingale part of the volatility parameter  $k_0$  by an SBM and ask for vanishing martingale part for the intensity parameters. However, we will have to modify the choice of  $\sigma_w$  in order to apply Corollary II.3.17. For this simulation we chose

$$\sigma(t, x, k) := \sigma_w \min \{1, 100\lambda(t, x, k), (10 - k_0)^+, (10 - k_1)^+, (10 - k_2)^+\}$$

where  $\lambda$  is as in Corollary II.3.15. Of course, we prefer that the minimum takes the values 1 which always happened in all our simulations. Theorem II.3.14 guarantees a consistent and risk-neutral dynamic model which is unique in law. Figures 1-4 result from an Euler-scheme computer simulation with the above parameter choice and daily step sizes. The simulation is based on the algorithm given in [7, Theorem]. All the figures are generated from the same simulated path of  $(X, Z)$ . Figure 1 shows the code-book process. In Figure 2 we see the drift coefficients of  $Z$ . Figure 3 indicates the prices of the stock and the 3 simulated options. Finally Figure 4 depicts the implied Black-Scholes volatility of the options.

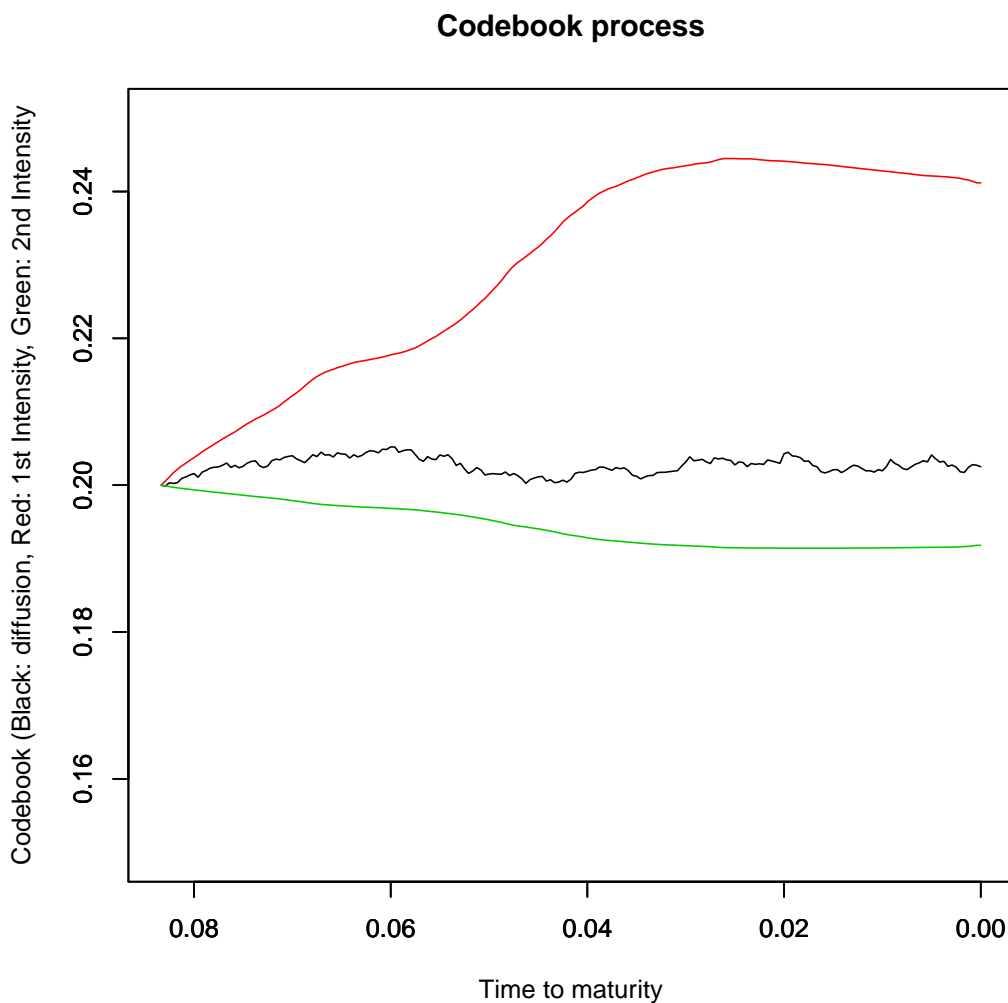


Figure 1: Simulated code-book process  $Z$ .  $Z_0$  is black,  $Z_1$  is red and  $Z_2$  is green.

**3.4. An extension to multiple maturities.** In this section we want to propose an extended simple model class in order to work with multiple maturities. We consider piecewise Lévy processes, in the sense that they are PII's whose local characteristics are constant on intervals  $[T_{j-1}, T_j)$  where  $0 =: T_0 < T_1 < \dots < T_m$  are real numbers. For each time interval we will use the same simple models as in the previous section. In this setup the admissibility condition turns out to be a transformed version of the previous admissibility condition and the construction of a dynamic model actually will be made piecewise by using Corollary II.3.17.

To this end, let  $n, m \in \mathbb{N}$ ,  $0 =: T_0 < T_1 < T_2 < \dots < T_m$  and  $y_0 < \dots < y_n$ . We consider European call options with payoff function  $f_j(x) := (e^x - e^{y_j})^+$ ,  $j \in \{0, \dots, n\}$ .  $T_1, \dots, T_m$  will serve as maturities, while for any maturity  $T_r$  we have  $n + 1$  options with



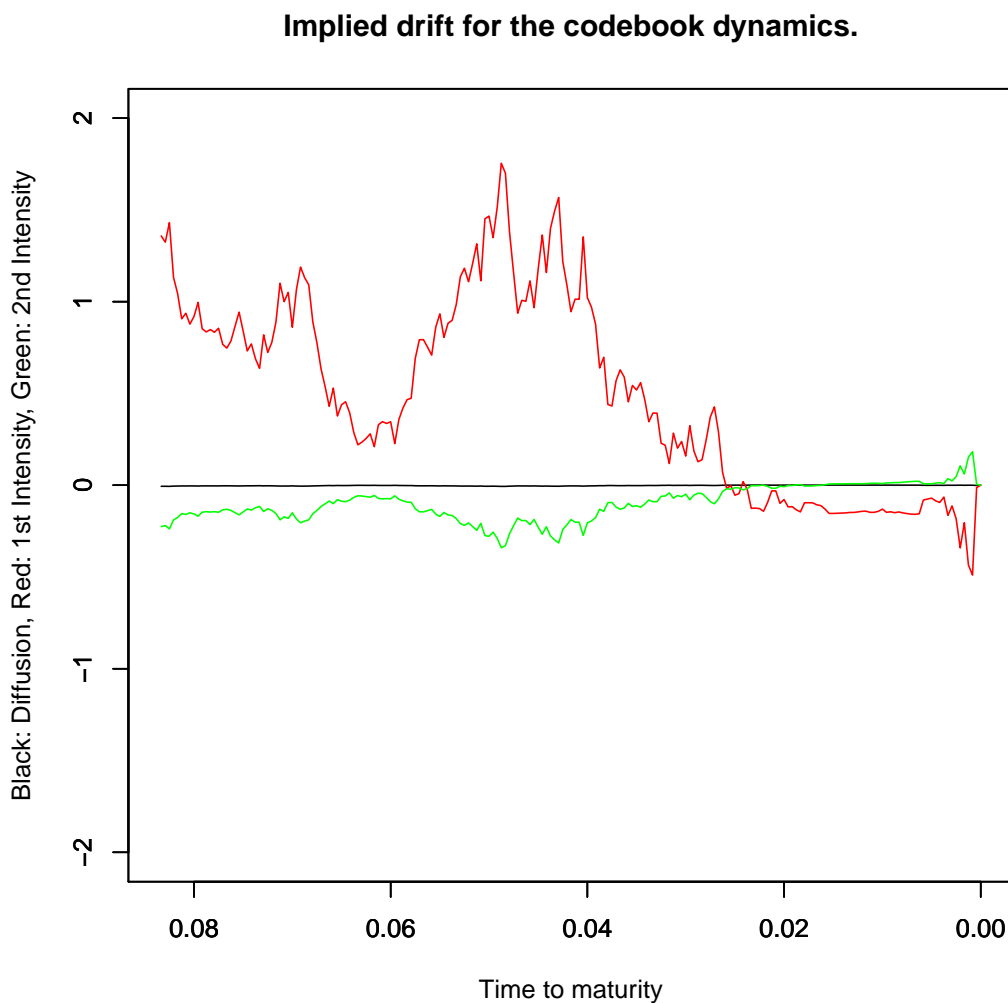


Figure 2: The implied drift from the drift condition

payoff functions  $f_0$  to  $f_n$ . Moreover, we define  $\iota_r := 1_{(T_{r-1}, T_r]}$ ,  $\tau_r(t) := \int_0^t \iota_r(s) ds$  and let  $N_1, \dots, N_n$  denote independent exponentially compensated SPP,  $h_1 < \dots < h_n$  and  $B$  an exponentially compensated SBM. Define

$$Y_k(t) := \sum_{r=1}^m \left( B(k_{(r,0)} \tau_r(t)) + \sum_{j=1}^n h_j N_j(k_{(r,j)} \tau_r(t)) \right),$$

where  $k \in K := \{k \in (\mathbb{R}_+)^{m \times (n+1)} : k_{(r,0)} > 0 \text{ for any } r = 1, \dots, m\}$ . For  $k \in K$  let  $U^k$  be the transition semigroup generated by  $Y_k$ , i.e.

$$U_{(t,T)}^k f(x) := E(f(Y_k(T) - Y_k(t) + x))$$

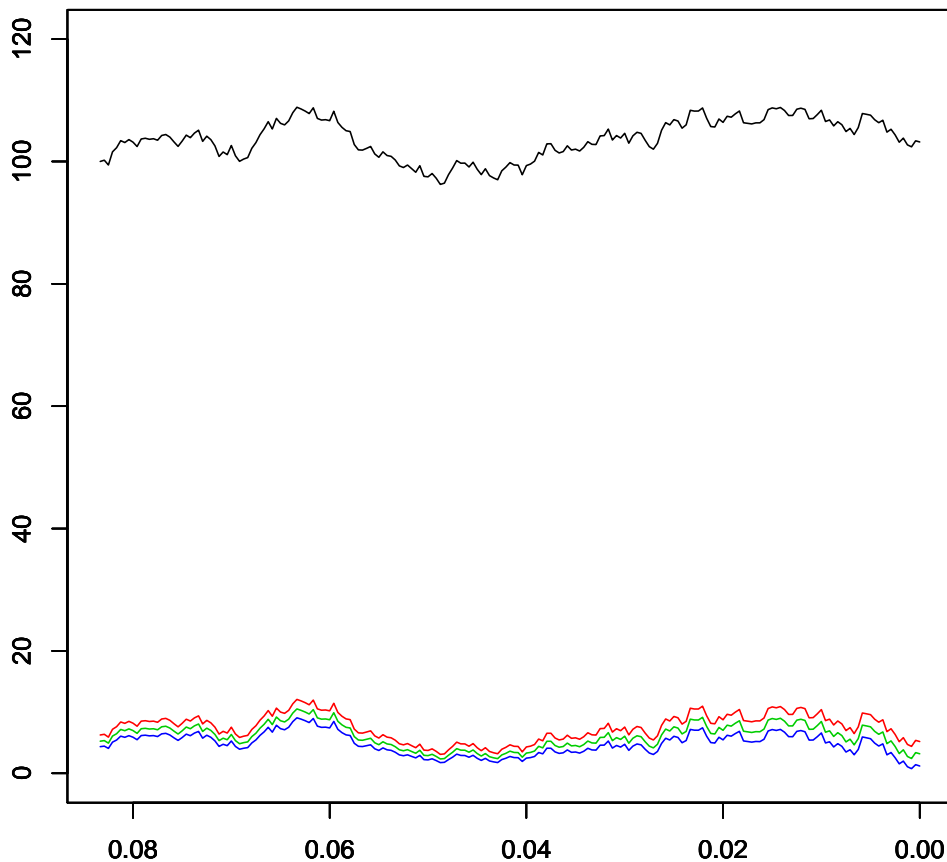


Figure 3: The simulated stock (black) and the three options (red for strike 98, green for strike 100, blue for strike 102).

for any  $0 < t < T$ , any  $x \in \mathbb{R}$  and any function  $f$  such that  $f(Y_k)$  is integrable. Finally denote by  $\mathcal{G}^k$  the generator associated with the process  $Y_k$ , i.e.

$$\mathcal{G}^k f(t, x) = \sum_{r=1}^m \left( \frac{k_{(r,0)} l_r(t)}{2} (\partial_x^2 - \partial_x) f(t, x) + \sum_{j=1}^n k_{(r,j)} l_r(t) (f(t, x + h_j) - f(t, x) - (e^{h_j} - 1) Df(t, x)) \right).$$

The following theorem is an extension of Corollary II.3.17.

**THEOREM II.3.18.** *Let  $x \in \mathbb{R}, k \in K$  and let*

$$\sigma : \mathbb{R}_+ \times \mathbb{R} \times K \rightarrow \mathbb{R}^n$$

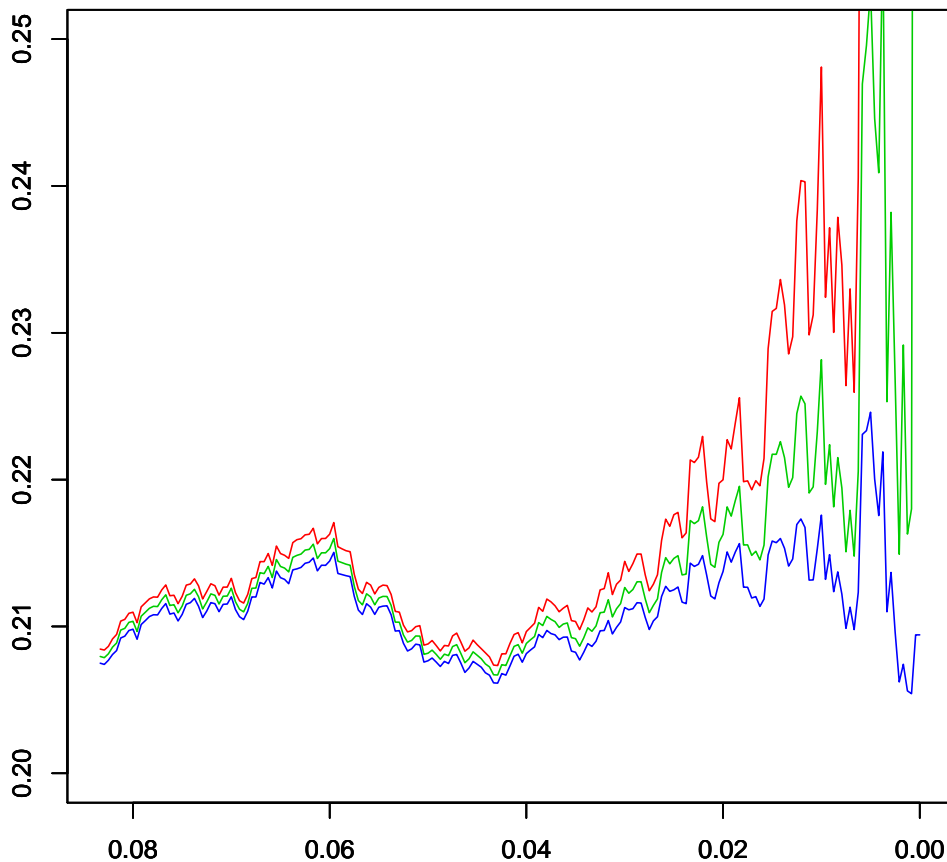


Figure 4: Implied Black-Scholes volatility for the three options (red for strike 98, green for strike 100, blue for strike 102). None of these graphs exceeds 0.3.

be such that

- $\sigma$  is Lipschitz-continuous,
- $\sigma$  has compact support contained in  $(\bigcup_{r=1}^m (T_{r-1}, T_r)) \times \mathbb{R} \times \text{int}(K)$  and
- for

$$d(t, x, k) := |\det((\partial_{k_{(r',j')}} G_{(r,j)}(t, x, k))_{(r',j'),(r,j) \in \{s,\dots,m\} \times \{1,\dots,n\}})|,$$

$s \in \{1, \dots, r\}, t \in (T_{s-1}, T_s), x \in \mathbb{R}, k \in K, a \in \mathbb{R}_+$  we have

$$\sigma(t, x, k) \leq a \exp\left(\frac{-1}{d(t, x, k)}\right)$$

where we set  $\exp(\frac{-1}{0}) := 0$ .

Then there is a risk-neutral and consistent dynamic model  $(X, Z)$  such that

- (1)  $X(0) = x$  and  $Z(0) = k$ ,
- (2)  $X$  and  $Z$  are locally independent semimartingales and
- (3)  $Z$  is a continuous semimartingale allowing for local characteristics such that its diffusion coefficient  $c^Z$  is given by Equation (II.1.9) and its drift coefficient is continuous.

Moreover, for any risk-neutral and consistent dynamic model  $(X', Z')$  satisfying Conditions (1) to (3) we have:

- the finite dimensional distributions of  $(X', Z')$  and  $(X, Z)$  coincide,
- $(\Theta, X', Z')$  is a Markov process for  $\Theta(t) = t$ ,
- $\exp(X')$  is a martingale and
- $(X, Z)$  is strongly risk-neutral.

PROOF. This is a piecewise application of Corollary II.3.17. □

In order to calculate the derivatives of  $G_j$  the same approaches as in the single maturity case are possible. Here we present the Fourier representation approach.

PROPOSITION II.3.19. Let  $k \in K$ ,  $y \in \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto (e^x - e^y)^+$  and

$$g_k : \mathbb{C} \rightarrow \mathbb{C}, u \mapsto \exp \left( \sum_{r=1}^m \left( -k_{(r,0)} \tau_r(t) \frac{u^2 + iu}{2} + \sum_{j=1}^n k_{(r,j)} \tau_r(t) (e^{iuh_j} - 1 - iu(\exp(h_j) - 1)) \right) \right)$$

the entire extension of the Fourier transform of  $Y_k(t)$ . Then we have

$$U_{T-t}^k f(x) = E(f(x + Y_k(T) - Y_k(t))) = f(x) + \frac{e^y}{2\pi} \int_{\mathbb{R}+iv} \frac{1 - g_k(u)}{u^2 + iu} e^{iu(x-y)} du$$

for any  $v \in \mathbb{R}$  The differential operators  $\partial_{k_j}$  and  $\partial_{k_j k_i}$  applied to  $U_{T-t}^k f(x)$  can be applied to the integrand in the Fourier representation, i.e.

$$\partial_* U_{T-t}^k f(x) = \frac{e^y}{2\pi} \int_{\mathbb{R}+iv} \frac{-\partial_* g_k(u)}{u^2 + iu} e^{iu(x-y)} du$$

where  $\partial_*$  is either  $\partial_{k_j}$  or  $\partial_{k_j k_i}$  where  $i, j \in \{1, \dots, n\}$ .

PROOF. Proposition A.4.4 yields the first part of the proof.  $g_k$  is a Schwartz function since  $c > 0$  and hence we can exchange the partial derivatives with the integral. □

REMARK II.3.20. For ease of notation we assumed that the number of traded options at any maturity is the same. However, a close inspection of the proofs shows that this is not necessary.

## CHAPTER III

### Martingale problem

In this chapter we focus on existence and uniqueness for the martingale problem. It is divided into two sections. In the first section we state a known and quite general existence result, see Theorem III.1.4 below or see [18, Theorem 4.5.4] for an even more general case. This theorem can be generalised by localisation, i.e. by applying [18, Theorem 4.6.3]. The second section is devoted to uniqueness for martingale problems. The most commonly used techniques to show uniqueness are

- SDE methods where one often applies fixed-point arguments and
- construction of solution for the backward equation, i.e. construction of solutions for the associated abstract Cauchy problem.

Our approach is based on showing uniqueness for the 1-dimensional laws of solutions to the forward equation, see [18, Theorem 4.4.2].

Our main result in this context is Theorem III.2.6, which is similar to [6, Theorem 2.8] and to [41, Theorem 4.3] and its variants. [41, Theorem 4.3] requires the diffusion coefficient to be positive. [6, Theorem 2.8] allows for a more general kind of ellipticity. This result requires a certain boundedness condition for all derivatives, while we only need this condition for finitely many derivatives.

The idea of the proof of Theorem III.2.6 is as follows. We aim at proving uniqueness for the 1-dimensional laws of solutions to the forward equation in the Fourier domain, i.e. we show that for two solutions  $X, Y$  the characteristic functions  $\phi_{X(t)}, \phi_{Y(t)}$  coincide. This will be done by a typical Grönwall argument. We proceed in two steps. First we show that any solution can be approximated by a piecewise conditional Lévy process, cf. Proposition III.2.14. Secondly we try to find bounds for the deviation rate of two piecewise Lévy processes. These requirements are first shown for simple symbols and then we will see that the set of symbols which do fulfil the requirements is a convex cone which is closed in some sense, cf. Proposition III.2.21 and Proposition III.2.23. Then we can deduce Theorem III.2.5 which states that uniqueness holds if the symbol can be locally approximated with a Fourier

series which have to fulfil a positivity condition. Finally, we construct such Fourier series for elliptic symbols, cf. Theorem III.2.6.

### 1. The symbol and the existence theorem

In this section we introduce the symbol and its associated martingale problem similar to [23]. We state an existence theorem, cf. III.1.4, and some basic properties of solution processes.

We denote by  $\Pi$  the set of characteristic exponents of infinitely divisible distributions, henceforth *Lévy exponents*. Throughout this section let  $d \in \mathbb{N}$ , which denotes the dimension of the state space. We write  $S^d$  for the set of positive semi-definite matrices on  $\mathbb{R}^d$ . Throughout this section we fix a truncation function  $\chi$ , i.e.  $\chi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable, bounded and is equals the identity in a neighbourhood of zero.

**DEFINITION III.1.1.** A measurable function  $q : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$  is a *symbol* if  $q(x, \cdot)$  is a Lévy exponent for all  $x \in \mathbb{R}^d$ . A measurable function  $q : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$  is a *pseudo-symbol* if there is a continuous function  $f$  such that

$$|q(x, u) - q(y, u)| \leq |f(x) - f(y)|(1 + |u|^2)$$

for all  $u \in \mathbb{R}^d, x, y \in \mathbb{R}^d$ . Let  $q$  be a symbol or a pseudo-symbol. A càdlàg  $\mathbb{R}^d$ -valued stochastic process  $X$  is a *solution to the  $q$ -martingale problem* if the process

$$M_u(t) := \exp(iuX(t)) - \int_0^t q(X(s), u) \exp(iuX(s)) ds$$

is a martingale for any  $u \in \mathbb{R}^d$ . We say *uniqueness* holds for the  $q$ -martingale problem if any two solutions  $X, Y$  to the  $q$ -martingale problem and the same initial law (i.e.  $X(0)$  has the same law as  $Y(0)$ ) have the same distribution. We say *existence* holds for the  $q$ -martingale problem if for any probability measure  $\mu$  there is a solution  $X$  to the  $q$ -martingale problem with  $P^{X(0)} = \mu$ .

**REMARK III.1.2.** A symbol  $q$  admits the Lévy-Khintchine representation

$$q(x, u) = iub(x) - \frac{1}{2}uc(x)u + \int (e^{iuh} - 1 - iu\chi(h)) F(x, dh) \quad (\text{III.1.1})$$

where

$$\begin{aligned} b &: \mathbb{R}^d \rightarrow \mathbb{R}^d, \\ c &: \mathbb{R}^d \rightarrow S^d, \end{aligned}$$

and  $F$  is a transition kernel from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  and only  $b$  depends on the choice of  $\chi$ , cf. [36, Theorem 8.1].  $(b(x), c(x), F(x, \cdot))$  is the so called Lévy-Khintchine triplet of the characteristic exponent  $q(x, \cdot)$ .

DEFINITION III.1.3. Let  $q$  be a symbol or a pseudo-symbol. The operator associated with  $q$  is defined as

$$\mathcal{A}_q f(x) := \int_{\mathbb{R}^d} q(x, u) \check{f}(u) e^{iux} du$$

where  $x \in \mathbb{R}^d$ ,  $f$  is a real-valued Schwartz function and

$$\check{f} : \mathbb{R}^d \rightarrow \mathbb{C}, \quad u \mapsto \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) e^{-iux} dx,$$

is the inverse Fourier transform of  $f$ .

We now state a well-known existence result.

THEOREM III.1.4. *Let  $q$  be a continuous symbol. Let  $(b, c, F)$  the triplet associated with  $q$  and assume that*

$$g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}, \quad x \mapsto |b(x)| + |c(x)| + \int_{\mathbb{R}^d} |h|^2 F(x, dh)$$

*is finite and bounded. Then for any probability measure  $\mu$  there is a solution  $X$  to the  $q$ -martingale problem with  $P^{X(0)} = \mu$ .*

PROOF. Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a Schwartz function which is constant 1 on the unit ball in  $\mathbb{R}^d$ . For any  $n \in \mathbb{N}$  define the Schwartz function

$$f_n : \mathbb{R}^d \rightarrow \mathbb{R}, \quad x \mapsto \phi(x/n).$$

Then  $f_n \rightarrow 1$  pointwise and the second derivatives of  $f_n$  are bounded by  $\frac{k}{n^2}$  where  $k$  is a common bound for the first two partial derivatives of  $\phi$ . Moreover, for  $x \in \mathbb{R}^d$  and  $n > |x|$  we have

$$\mathcal{A} f_n(x) = \sum_{i,j=1}^n \partial_i \partial_j f_n(x) c_{ij}(x) + \int_{\mathbb{R}^d} (f_n(x+h) - f_n(x) - f'_n(x)\chi(h)) F(x, dh).$$

The first summand is zero because  $f_n$  is constant 1 in a neighbourhood of  $x$ . For the same reason and due to a remainder estimate for the Taylor series we have

$$|f_n(x+h) - f_n(x) - f'_n(x)\chi(h)| \leq \frac{kd^2h^2}{2n^2}.$$

Thus

$$|\mathcal{A}f_n(x)| \leq \int_{\mathbb{R}^d} \frac{cd^2h^2}{2n^2} F(x, dh) \rightarrow 0.$$

Thus  $\mathcal{A}f_n(x) \rightarrow 0$  pointwise. We also have

$$|\mathcal{A}f_n(x)| \leq Kg(x)$$

for some constant  $K > 0$ .

Let  $\mathcal{A}_q$  be defined as in Definition III.1.3. Lemma III.1.6 below yields the requirements of [18, Theorem 4.5.4] for the martingale problem  $(\mathcal{A}_q, \mu)$  in the sense of [18, Section 4.3]. Thus [18, Theorem 4.5.4, Remark 4.5.5] yield existence of a solution  $X$  to the martingale problem  $(\mathcal{A}_q, \mu)$  with values in  $\mathbb{R}^d$ .  $X$  is a solution to the  $q$ -martingale problem.  $\square$

REMARK III.1.5. The boundedness condition in Theorem III.1.4 can be relaxed by localisation, i.e. by applying [18, Theorem 4.6.2] or [18, Theorem 4.6.3].

LEMMA III.1.6. *Let  $q$  be a continuous symbol. Let  $(b, c, F)$  the triplet associated with  $q$ . Assume that*

$$g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}, \quad x \mapsto |b(x)| + |c(x)| + \int_{\mathbb{R}^d} |h|^2 F(x, dh)$$

*is finite and bounded. Then the range of the linear operator  $\mathcal{A}_q$  associated with  $q$  is contained in the set  $C_0(\mathbb{R}^d)$  of continuous functions vanishing at infinity. In other words  $\mathcal{A}_q$  is a linear operator on  $C_0(\mathbb{R}^d)$  which is densely defined. Moreover,  $\mathcal{A}_q$  satisfies the positive maximum principle in the sense of [18, p.165].*

PROOF. Let  $f$  be a real-valued Schwartz function. The dominated convergence theorem yields  $\mathcal{A}_q f$  is bounded and continuous. The proof of the Riemann-Lebesgue lemma shows that  $\mathcal{A}_q f$  vanishes at infinity, see [20, Proposition 2.2.17].  $\mathcal{A}_q$  is obviously densely defined because its domain contains all Schwartz functions. Let  $x_0$  be a maximum of  $f$ , i.e.  $f(x_0) = \sup_{x \in \mathbb{R}^d} f(x)$ . Denoting the Fourier transform by  $\mathcal{F}$ , [20, Proposition 2.2.11] yields

$$\mathcal{F} \{ib(x_0)\text{id}\check{f}\}(x_0) = ib(x_0)\mathcal{F} \{\text{id}_{\mathbb{R}^d}\check{f}\}(x_0) = b(x_0)f'(x_0) = 0.$$



In the same way we get

$$\mathcal{F} \left\{ -\frac{1}{2}(\text{id})c(x_0)(\text{id})\check{f} \right\} (x_0) \leq 0 \quad \text{and}$$

$$\mathcal{F} \left\{ u \mapsto \int (e^{iuh} - 1 - iu\chi(h)) F(x_0, dh)\check{f}(u) \right\} (x_0) \leq 0.$$

Hence  $\mathcal{A}_q f(x_0) \leq 0$ . □

**PROPOSITION III.1.7.** *Let  $q \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$  be a symbol such that  $(x, u) \mapsto \frac{\nabla_1 q(x, u)}{1+|u|^2}$  is bounded where  $\nabla_1 q(x, u) := (\partial_{x_1} q(x, u), \dots, \partial_{x_d} q(x, u))$  for  $x, u \in \mathbb{R}^d$ . Then it is a pseudo-symbol.*

**PROOF.** This is a simple application of the fundamental theorem of calculus. □

## 2. The symbol and the uniqueness problem

The obvious question is to ask what conditions are needed to ensure uniqueness of the  $q$ -martingale problem when  $q$  is a symbol. The answer is not as simple as it was for existence. The three most important types of processes obtained from general results are

- (1) solutions to stochastic differential equations,
- (2) Markov jump-processes which can be easily obtained from the Hille-Yosida-Ray theorem and
- (3) processes with a non-degenerate diffusion coefficient which can be obtained from the Stroock-Varadhan existence and uniqueness result.

The powerful method of stochastic differential equations has been studied intensively. It allows to show that uniqueness holds for  $C^2$ -symbols which are quadratic functions in their second component, see Theorem III.2.1. Now one might hope that smoothness of the symbol is also sufficient in the general case. However, Example III.2.4 shows that there are analytic symbols where uniqueness fails. On the other hand modern versions of the Stroock-Varadhan existence and uniqueness theorem show that smoothness together with an ellipticity condition is sufficient, see [6, Theorem 2.8] and Theorem III.2.6 below.

**THEOREM III.2.1.** *Let  $q \in C^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$  be a symbol which has the representation  $q(x, u) = iub(x) - \frac{1}{2}uc(x)u$  for functions  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d, c : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ . Then uniqueness holds for the  $q$ -martingale problem.*

**PROOF.** Observe that  $c$  takes actually values in the set of positive semi-definite matrices. Let  $\sigma(x)$  be the positive square root of  $c(x)$  for all  $x \in \mathbb{R}^d$ . Then [35, Theorem V.12.12]

yields that  $\sigma$  is Lipschitz-continuous. [35, Theorem V.22.12] applied to  $(b, \sigma)$  yields the claim.  $\square$

We now state the Hille-Yosida-Ray theorem.

**THEOREM III.2.2.** *Let  $q$  be a symbol satisfying the requirements of the the existence Theorem III.1.4. Let  $\mathcal{A}_q$  the linear operator associated with  $q$ . Then the following statements are equivalent*

- (1) *uniqueness holds for the  $q$ -martingale problem,*
- (2) *the closure  $\overline{\mathcal{A}_q}$  in  $C_0(\mathbb{R}^d)$  is the generator of a positivity preserving contraction semi-group and*
- (3) *there is  $\lambda > 0$  such that the range of  $\lambda - \mathcal{A}_q$  is dense in  $C_0(\mathbb{R}^d)$ .*

**REMARK III.2.3.** Condition (3) seems to be the weakest condition, but the denseness of the range of  $\lambda - \mathcal{A}_q$  is quite often very difficult to show. However, if  $\mathcal{A}_q$  is a bounded operator in  $C_0(\mathbb{R}^d)$ , then condition (3) is satisfied for any  $\lambda$  greater than the bound of  $\mathcal{A}_q$ .

**PROOF.** (2) $\Leftrightarrow$ (3) is the statement of the Hille-Yosida theorem, see [18, Theorem 4.2.2]. (2) $\Rightarrow$ (1) is a special case of [18, Theorem 4.4.1]. Now assume (2). Then [23, Theorem 3.8] yields (3).  $\square$

**EXAMPLE III.2.4.** There is an analytic symbol  $q$  satisfying the requirements of Theorem III.1.4 which has an entire extension to  $\mathbb{C} \times \mathbb{C}$ , where, however, uniqueness does not hold for the  $q$ -martingale problem.

**PROOF.** W.l.o.g. we may assume that the truncation function  $\chi$  is symmetric. Define  $q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by  $q(x, u) := \frac{\cos(ux)-1}{x^2}$  if  $x \neq 0$  and  $q(x, u) := \frac{-u^2}{2}$  if  $x = 0$ .  $q$  has an obvious entire extension. Define

$$\begin{aligned} b(x) &= 0, \\ c(x) &= 1_{\{x=0\}}, \\ F(x, \cdot) &:= 1_{\{x \neq 0\}} \frac{\delta_x + \delta_{-x}}{2x^2}. \end{aligned}$$

Then  $(b, c, F)$  is the corresponding triplet in the sense of Remark III.1.2. For any  $n \in \mathbb{N}$ ,  $c \in \mathbb{R}^+$  we also define

$$q_{c,n}(x, u) := \begin{cases} q(x, u) & \text{if } |x| \geq c2^{-n}, \\ 4^n \frac{\cos(uc2^{-n})-1}{c^2} & \text{otherwise.} \end{cases}$$

Then  $q_{c,n}$  is a continuous symbol and the associated linear operator

$$\mathcal{A}_{c,n}f(x) = \begin{cases} \frac{f(2x)-2f(x)+f(0)}{x^2} & \text{if } |x| \geq c2^{-n}, \\ 4^n \frac{f(x+c2^{-n})-2f(x)+f(x-c2^{-n})}{c^2} & \text{otherwise,} \end{cases}$$

is bounded by  $2^{2n+1}/c^2$ . Hence it satisfies the requirements of Theorem III.2.2 and thus there is a solution  $X_{c,n}$  for the  $(\mathcal{A}_{c,n}, \delta_0)$ -martingale problem. It is easy to see that  $X_{c,n}$  takes values only in  $M_c := c\{2^z, -2^z : z \in \mathbb{Z}\} \cup \{0\}$ . However, for any Schwartz function  $f$  we have

$$\mathcal{A}_{c,n}f \rightarrow \mathcal{A}f$$

and hence [18, Lemma 4.5.1, Remark 4.5.2] yield that a subsequence of  $(X_{c,n})_{n \in \mathbb{N}}$  weakly converges to a solution  $X_c$  of the martingale problem  $\mathcal{A}$ . Then  $X_c$  takes values only in  $M_c$ . Thus  $X_1, X_{\sqrt{2}}$  are solutions to the martingale problem  $(\mathcal{A}, \delta_0)$  where  $X_1$  takes values only in  $M_1$  and  $X_{\sqrt{2}}$  takes values only in  $M_{\sqrt{2}}$ . Since they are clearly non-constant they cannot have the same distribution.  $\square$

We now state our main result and one of its consequences. The techniques needed for its proof are developed in the next two sections.

**THEOREM III.2.5.** *Let  $q$  be a symbol such that for all  $u \in \mathbb{R}^d$*

- (1)  *$q$  satisfies the requirements of the existence theorem, i.e. of Theorem III.1.4,*
- (2) *there is a constant  $c > 0$  such that  $|\nabla_x q(x, u)| \leq c(1 + |u|^2)$  for all  $x, u \in \mathbb{R}^d$ ,*
- (3)  *$q(\cdot, u)$  has local Fourier series representation, i.e. for all  $x_0 \in \mathbb{R}^d$  there is a neighbourhood  $U$  of  $x_0$ ,  $a_n(u), b_n(u) \in \mathbb{R}^d$  for all  $n \in \mathbb{Z}^d$  and a constant  $k > 0$  such that for all  $x \in U$  we have*

$$q(x, u) = \sum_{n \in \mathbb{Z}^d} (a_n(u) \cos(kn \cdot x) + b_n(u) \sin(kn \cdot x))$$

*and the sequence  $(a_n, b_n)_{n \in \mathbb{N}}$  satisfies*

- a) *the real part of  $-a_0(u)$  dominates the absolute sum of the other coefficients, i.e.*

$$-\operatorname{Re}(a_0(u)) \geq \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |a_n(u)| + |b_n(u)|$$

*for all  $u \in \mathbb{R}^d$  and*

- b)  $\sup \left\{ \sum_{n \in \mathbb{Z}^d} |n|^2 \left( \frac{|a_n(u)|}{1+|u|^2} + \frac{|b_n(u)|}{1+|u|^2} \right) : u \in \mathbb{R}^d \right\} < \infty.$

*Then existence and uniqueness hold for the  $q$ -martingale problem.*

PROOF. Theorem III.1.4 yields existence for the  $q$ -martingale problem. By localisation, see Theorem III.2.9, we may assume w.l.o.g. that (3) holds with  $U = \mathbb{R}^d$ . Let  $u \in \mathbb{R}^d, t \in [0, 1]$ . Then the Fourier series can be rewritten as

$$q(x, u) = \tilde{a}_0 + \sum_{n \in \mathbb{Z}^d} (a_n \cos(knx) - |a_n|) + (b_n \sin(knx) - |b_n|)$$

where  $\operatorname{Re}(\tilde{a}_0) \leq 0$ . By Lemma III.2.28 below there are complex measures  $P_n, Q_n$  such that

- (1)  $\widehat{P}_n(x) = \exp(t((a_n \cos(knx) - |a_n|)))$ ,
- (2)  $|P_n|(\mathbb{R}^d) \leq 1$ ,
- (3)  $\int_{\mathbb{R}^d} |v| |P_n|(dv) \leq 0$  and
- (4)  $\int_{\mathbb{R}^d} |v|^2 |P_n|(dv) \leq t|a_n| |kn|^2$ .

and

- (1)  $\widehat{Q}_n(x) = \exp(t(b_n \sin(knx) - |b_n|))$ ,
- (2)  $|Q_n|(\mathbb{R}^d) \leq 1$ ,
- (3)  $\int_{\mathbb{R}^d} |v| |Q_n|(dv) \leq 0$  and
- (4)  $\int_{\mathbb{R}^d} |v|^2 |Q_n|(dv) \leq t|b_n| |kn|^2$ .

for all  $n \in \mathbb{Z}^d \setminus \{0\}$ . Moreover, Proposition III.2.25 below yields the existence of a measure  $P_0$  such that

- (1)  $\widehat{P}_0(x) = \exp(t\tilde{a}_0)$ ,
- (2)  $|P_0|(\mathbb{R}^d) \leq 1$ ,
- (3)  $\int_{\mathbb{R}^d} |v| |P_0|(dv) \leq 0$  and
- (4)  $\int_{\mathbb{R}^d} |v|^2 |P_0|(dv) \leq 0$ .

The assumptions and Proposition III.2.23 below yield that  $q$  is a regular symbol in the sense of Definition III.2.19 below. Proposition III.1.7 below yields  $q$  is a pseudo-symbol. Thus Corollary III.2.20 below yields existence and uniqueness for the  $q$ -martingale problem.  $\square$

The Fourier conditions might seem to be hard to satisfy. However, under some regularity and ellipticity conditions it is always possible to fulfil them.

**THEOREM III.2.6.** *Let  $m \in \mathbb{N}$  with  $m > \frac{d}{2}$  and  $q$  be a continuous symbol such that  $q(\cdot, u) \in C^{m+2}(\mathbb{R}^d, \mathbb{C})$  for all  $u \in \mathbb{R}^d$  and that*

- (1)  *$q$  satisfies the requirements of the existence theorem, i.e. of Theorem III.1.4 and*

(2) for every  $x_0$  there is a neighbourhood  $V$  of  $x_0$  and  $L > 0$  such that  $\psi := q(x_0, \cdot)$  satisfies

$$|\partial_x^\beta q(x, u)| \leq L |\operatorname{Re}(\psi(u))|$$

$$|\partial_x^\alpha q(x, u)| \leq L(1 + |u|^2)$$

for all  $x \in V, u \in \mathbb{R}^d, \beta, \alpha \in \mathbb{N}^d$  with  $|\beta| \leq m$  and  $|\alpha| \leq m + 2$ .

Then existence and uniqueness hold for the  $q$ -martingale problem.

PROOF. Lemma III.2.29 below yields the requirements for Theorem III.2.5 which completes the proof.  $\square$

REMARK III.2.7. Let  $q \in C^{d+2}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$  be a symbol such that there is a characteristic exponent  $\phi$  satisfying the following *ellipticity condition*

$$|\phi(u)g_1(x)| \leq |\partial_x^\alpha q(x, u)| \leq |\phi(u)g_2(x)| \quad (\text{III.2.1})$$

for some bounded and continuous functions  $g_1, g_2 : \mathbb{R}^d \rightarrow \mathbb{R}_{++}$  and any  $\alpha \in \mathbb{N}^d$  with  $|\alpha| \leq d + 2$ . Then satisfies the requirements of Theorem III.2.6.

Such an ellipticity condition occurs in the Stroock-Varadhan existence and uniqueness result, cf. [41, Theorem 4.3]. The big advantage of the result is that continuity suffices and no extra smoothness is needed. Moreover, the drift only needs to be measurable. However, Stroock requires an ellipticity condition with respect to the explicit symbol  $\phi(u) = -\frac{1}{2}u^2$ . His proof also needs some extra regularity for the jump-measure which, however, could be relaxed.

Since Stroock and Varadhan have published their result, some variants of the Stroock-Varadhan Theorem with a more general ellipticity condition have been derived, i.e. with a more general function  $\phi$ . See [23] for an overview. A recent result is due to Böttcher, cf. [6, Theorem 2.8], who requires Equation (III.2.1) for arbitrary  $\alpha$  and, moreover, a certain boundedness for the derivatives with respect to  $u$ . Theorem III.2.6 may be easier to apply in practice, because it involves only finitely many derivatives and no smoothness in  $u$ .

**2.1. Local behaviour of solutions.** The remainder of this chapter is devoted to the proof of Theorem III.2.5. We start with some regularity condition for solutions to martingale problems which are also interesting by themselves.

**THEOREM III.2.8.** *Let  $q$  be a symbol which satisfies the requirements of Theorem III.1.4. Let  $X$  be a solution to the  $q$ -martingale problem. Then  $X$  is a semimartingale which allows for local characteristics. Moreover,  $X$  is stochastically continuous.*

*Let  $\chi$  a truncation function and  $(b, c, F)$  the representing triplet for  $q$ , i.e.*

$$q(x, u) = iub(x) - \frac{1}{2}uc(x)u + \int (e^{iuh} - 1 - iu\chi(h)) F(x, dh)$$

*for all  $x, u \in \mathbb{R}^d$ . Then  $(b(X(\cdot-)), c(X(\cdot-)), F(X(\cdot-), dx))$  is a version of the local characteristics of  $X$ .*

**PROOF.** Since  $q$  is bounded as a function of its first variable we can define the process  $A(u) := q(X(\cdot-), u) \bullet I$  where  $I(t) := t$  for all  $t \in \mathbb{R}_+, u \in \mathbb{R}^d$ . Then

$$\begin{aligned} M_u(t) &:= \exp(iuX(t)) - \int_0^t q(X(s), u) e^{iuX(s)} ds \\ &= \exp(iuX(t)) - \int_0^t q(X(s-), u) e^{iuX(s-)} ds \\ &= \exp(iuX) - e^{iuX(\cdot-)} \bullet A(u) \end{aligned}$$

is a martingale since  $X$  is a solution of the  $q$ -martingale problem. [25, Theorem II.2.42] yields  $X$  is a semimartingale, and it admits the characteristics

$$(b(X(\cdot-)) \bullet I(t), c(X(\cdot-)) \bullet I(t), F(X(\cdot-), dx) dI(t)).$$

In particular,  $(b(X(\cdot-)), c(X(\cdot-)), F(X(\cdot-), dx))$  is a version of the local characteristics of  $X$ .  $\square$

The localisation procedure for martingale problems reveals that uniqueness is a local property, cf. [18, Section 4.6]. Here we state a localisation theorem in such a way that we can apply it easily.

**THEOREM III.2.9.** *Let  $q$  be a symbol such that existence holds for the  $q$ -martingale problem. Let  $\mathcal{U}$  be an open covering for  $\mathbb{R}^d$  and for all  $U \in \mathcal{U}$  let  $q_U$  be a symbol such that*

- (1)  $q(x, u) = q_U(x, u)$  for any  $x \in U, u \in \mathbb{R}^d$  and
- (2) *existence and uniqueness hold for the  $q_U$ -martingale problem.*

*Then existence and uniqueness hold for the  $q$ -martingale problem.*

**PROOF.** W.l.o.g. we may assume that  $\mathcal{U}$  is countable. Let  $U \in \mathcal{U}$  and  $\mu$  be a probability measure on  $\mathbb{R}^d$ . [18, Theorem 4.6.1] yields a solution of the stopped martingale problem  $(\mathcal{A}_{q_U}, \mu, U)$  in the sense of [18, Section 4.6] where  $\mathcal{A}_{q_U}$  is the linear operator associated

with  $q_U$ . Since (1)  $X$  is a solution to the stopped martingale problem  $(\mathcal{A}_q, \mu, U)$  where  $\mathcal{A}_q$  is the linear operator associated with  $q$ . Hence, [18, Theorem 4.6.2] yields that the martingale problem  $(\mathcal{A}_q, \mu)$  has unique solutions.  $\square$

We now state a Grönwall-type theorem with perturbation which will be useful later. The proof is given for completeness.

**PROPOSITION III.2.10.** *Let  $I = [0, T]$ ,  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{t \rightarrow 0} \frac{\beta(t)}{t} = 0$  and  $c \in (0, \infty)$ . Let  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all  $s, t \in I$  with  $s < t$  we have*

$$\phi(t) \leq (1 + (t - s)c)\phi(s) + \beta(t - s).$$

*Then  $\phi(t) \leq \phi(0)e^{ct}$  for all  $t \in I$ . In particular, if  $\phi(0) = 0$ , then  $\phi = 0$ .*

**PROOF.** Let  $t \in \mathbb{R}_+$  and  $N \in \mathbb{N}$ . The inequality above yields

$$\phi\left(t\frac{n+1}{N}\right) \leq (1 + tc/N)\phi\left(t\frac{n}{N}\right) + \beta(t/N)$$

for all  $n = 0, \dots, N - 1$ . Hence

$$\phi(t) \leq (1 + tc/N)^N \phi(0) + \sum_{k=0}^{N-1} (1 + tc/N)^k \beta(t/N).$$

Since  $(1 + tc/N)^N \leq \exp(tc)$  we have

$$\phi(t) \leq \exp(tc)\phi(0) + \sum_{k=0}^{N-1} (1 + tc/N)^k \beta(t/N).$$

The geometric series sums up to

$$\sum_{k=0}^{N-1} (1 + tc/N)^k = \frac{(1 + tc/N)^N - 1}{tc/N} \leq N \frac{e^{tc} - 1}{tc}.$$

However,  $N\beta(t/N)$  converges to 0 for  $N \rightarrow \infty$ . Hence

$$\sum_{k=0}^{N-1} (1 + tc/N)^k \beta(t/N) \rightarrow 0.$$

We conclude

$$\phi(t) \leq \exp(tc)\phi(0)$$

$\square$

In the sequel we will work with the norm

$$\|\cdot\| : C_b(\mathbb{R}^d, \mathbb{C}) \rightarrow \mathbb{R}_+, \quad \phi \mapsto \sup \left\{ \frac{|\phi(u)|}{1 + |u|^2} : u \in \mathbb{R}^d \right\}.$$

REMARK III.2.11. A sequence of characteristic functions which converges with respect to  $\|\cdot\|$ , converges uniformly on compact sets. The continuity theorem [3, Satz 48.7] yields weak convergence of associated random variables.

We now prepare the proof of Theorem III.2.17 below.

LEMMA III.2.12. *Let  $q$  be a continuous pseudo-symbol such that  $q(\cdot, u)$  is bounded for all  $u \in \mathbb{R}^d$  and  $X$  a solution to the  $q$ -martingale problem. Moreover, define*

$$\Gamma(t, s, u) := \int_s^t E \left( e^{(t-r)q(X(s), u)} E(e^{iuX(r)}(q(X(r), u) - q(X(s), u)) | \mathcal{F}_s) \right) dr$$

for  $s, t \in \mathbb{R}_+, u \in \mathbb{R}^d$ . Then for any  $t, s \in \mathbb{R}_+, u \in \mathbb{R}^d$  with  $s < t$  we have

$$\phi_X(t, u) = E \left( \exp((t-s)q(X(s), u)) \exp(iuX(s)) \right) + \Gamma(s, t, u),$$

where  $\phi_X(t, u) = E(e^{iuX(t)})$ . In particular,

$$|\Gamma(t, s, u)| \leq \int_s^t E |q(X(r), u) - q(X(0), u)| dr.$$

PROOF. Let  $s \in \mathbb{R}_+, u \in \mathbb{R}^d$ . For all  $t \in [s, \infty)$  we have

$$\begin{aligned} \phi_s(t, u) &:= E(e^{iuX(t)} | \mathcal{F}_s) \\ &= E \left( M_u(t) + \int_0^t q(X(r), u) \exp(iuX(r)) dr \middle| \mathcal{F}_s \right) \\ &= M_u(s) + \int_0^s q(X(r), u) \exp(iuX(r)) dr \\ &\quad + \int_s^t E(q(X(r), u) \exp(iuX(r)) | \mathcal{F}_s) dr \\ &= e^{iuX(s)} + \int_s^t E(q(X(r), u) \exp(iuX(r)) | \mathcal{F}_s) dr. \end{aligned}$$

Theorem III.2.8 yields  $X$  is stochastically continuous. Since  $q$  is continuous, the process

$$r \mapsto E(q(X(r), u) \exp(iuX(r)) | \mathcal{F}_s)$$

has a.s. continuous paths. The fundamental theorem of calculus yields for all  $t > s$

$$\begin{aligned} \partial_t \phi_s(t, u) &= E(q(X(t), u) \exp(iuX(t)) | \mathcal{F}_s) \\ &= \phi_s(t, u) q(X(s), u) + E(\exp(iuX(t))(q(X(t), u) - q(X(s), u)) | \mathcal{F}_s). \end{aligned}$$

and we have  $\phi_s(s, u) = e^{iuX(s)}$ . The variation of constants formula [22, Satz 98.5] implies

$$\phi_s(t, u) = e^{(t-s)q(X(s), u) + iuX(s)} + \Gamma(t, s, u).$$



Taking expectations yields the claim.  $\square$

LEMMA III.2.13. *Let  $q$  be a pseudo-symbol such that  $q(\cdot, u)$  is bounded for all  $u \in \mathbb{R}^d$ . Let  $I \subset \mathbb{R}_+$  be a bounded interval and  $X$  be a solution to the  $q$ -martingale problem. Then there is a function  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{t \rightarrow 0} \beta(t)/t = 0$  and*

$$\int_s^t \frac{E|q(X(r), u) - q(X(s), u)|}{1 + |u|^2} dr \leq \beta(t - s)$$

for all  $s, t \in I, u \in \mathbb{R}^d$  with  $s < t$ .

PROOF. Let  $f$  be a bounded and continuous function such that

$$|q(x, u) - q(y, u)| \leq |f(x) - f(y)|(1 + |u|^2).$$

Then

$$\frac{E|q(X(r), u) - q(X(s), u)|}{1 + |u|^2} \leq E|f(X(r)) - f(X(s))|.$$

Theorem III.2.8 states that  $X$  is stochastically continuous. Hence

$$H : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+, (r, s) \rightarrow E|f(X(r)) - f(X(s))|$$

is continuous and  $H(s, s) = 0$  for all  $s \in \mathbb{R}_+$ . Thus the fundamental theorem of calculus yields the claim.  $\square$

PROPOSITION III.2.14. *Let  $q$  be a continuous pseudo-symbol such that  $q(\cdot, u)$  is bounded for all  $u \in \mathbb{R}^d$ . Let  $X$  be a solution to the  $q$ -martingale problem and  $L$  a conditional Lévy process with the same initial law as  $X$ , i.e.  $X(0), L(0)$  have the same law, and conditional characteristic exponent  $\psi_L = q(L(0), \cdot)$ . Then there is a function  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{t \rightarrow 0} \beta(t)/t = 0$  and for all  $t \in [0, 1], u \in \mathbb{R}^d$  we have*

$$\frac{|\phi_{X(t)}(u) - \phi_{L(t)}(u)|}{1 + |u|^2} \leq \beta(t)$$

where  $\phi_{X(t)}, \phi_{L(t)}$  denote the characteristic functions of  $X(t)$  resp.  $L(t)$ .

PROOF. Define  $I := [0, 1]$ . Let  $\beta$  be the function given by Lemma III.2.13. Then Lemma III.2.12 applied with  $s = 0$  yields the claim.  $\square$

We can now show that under certain conditions the 1-dimensional laws of solutions are uniquely determined.

PROPOSITION III.2.15. *Let  $I \subset \mathbb{R}_+$  be an interval containing 0,  $c > 0$  and  $q$  a continuous pseudo-symbol such that  $q(\cdot, u)$  is bounded for all  $u \in \mathbb{R}^d$ . Assume that for any  $t \in I, u \in \mathbb{R}^d$  there is a complex measure  $P_{t,u}$  such that*

- (1)  $\widehat{P_{t,u}}(x) := \int_{\mathbb{R}^d} e^{ivx} P_{t,u}(dv) = e^{tq(x,u)}$  for all  $x \in \mathbb{R}^d$  and
- (2)  $\int_{\mathbb{R}^d} \frac{1+|u+v|^2}{1+|u|^2} |P_{t,u}|(dv) \leq 1 + ct$ .

*Let  $X, Y$  be solutions to the  $q$ -martingale problem with the same initial distribution. Then  $X(t)$  and  $Y(t)$  have the same distribution for all  $t \in I$ .*

REMARK III.2.16. The critical assumption is the strict bound (2) for the non-convolution operator  $f \mapsto (u \mapsto \int f P_{t,u})$  in the set of characteristic functions.

PROOF. W.l.o.g. there is  $T \in \mathbb{R}_+$  such that  $I = [0, T]$ . Define  $d(t) := \|\phi_X(t, \cdot) - \phi_Y(t, \cdot)\|$  for all  $t \in \mathbb{R}_+$  where  $\phi_X(t, \cdot)$  and  $\phi_Y(t, \cdot)$  denote the characteristic function of  $X(t)$  resp.  $Y(t)$ . Let  $g_{t,u}(x) := e^{tq(x,u)+iux}$ . Lemma III.2.12 together with Lemma III.2.13 yield

$$d(t) \leq \sup \left\{ \frac{|E(g_{t-s,u}(X(s))) - E(g_{t-s,u}(Y(s)))|}{1 + |u|^2} : u \in \mathbb{R}^d \right\} + \beta(t-s)$$

for all  $s, t \in I$  with  $s < t$ , where  $\beta$  is a function as in Lemma III.2.13 with  $\lim_{t \rightarrow 0} \beta(t)/t = 0$ . Moreover, (1) and Fubini's theorem yield

$$E(g_{t-s,u}(X(s))) = \int_{\mathbb{R}^d} \phi_X(s, u+v) P_{t-s,u}(dv)$$

and hence Hölder's inequality implies

$$d(t) \leq d(s) \int_{\mathbb{R}^d} \frac{1 + |u+v|^2}{1 + |u|^2} |P_{t,u}|(dv) + \beta(t-s).$$

Thus the assumptions of Proposition III.2.10 are met and consequently we have  $d(t) = 0$  for all  $t \in I$ . Thus the characteristic functions of  $X(t)$  and  $Y(t)$  coincide, whence we know they have the same distribution.  $\square$

THEOREM III.2.17. *Assume that the requirements of Proposition III.2.15 are fulfilled and that existence holds for the  $q$ -martingale problem. Then uniqueness holds for the  $q$ -martingale problem.*

PROOF. Due to Proposition III.2.15 the one dimensional distributions are determined. Hence [18, Theorem 4.4.2] yields the claim.  $\square$

REMARK III.2.18. Existence for the  $q$ -martingale problem can be shown with Theorem III.1.4. As long as the pseudo-symbol is smooth, the requirement (1) in Proposition III.2.15 can always be achieved by modifying the symbol with the localisation procedure, cf. [18, Theorem 4.6.2, Theorem 4.6.3].

**2.2. Bounds for the non-convolution operator.** In this section we try to find sufficient conditions for the requirements in Proposition III.2.15 which are also needed for Theorem III.2.17. We call symbols that fulfil the assumptions of Proposition III.2.15 *regular*, see Definition III.2.19 and Corollary III.2.20 below. We start to show that the set of regular symbols is a convex cone which is closed in some sense, cf. Propositions III.2.21, III.2.22 and III.2.23 below. Then we find some simple regular symbols which we want to use to construct more complicated symbols, see Propositions III.2.24, III.2.25 and Lemma III.2.28. Finally, we state the technical Lemma III.2.29 below which is needed for the proof of Theorem III.2.6.

DEFINITION III.2.19. Let  $q \in C(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$  be a symbol or a pseudo-symbol. We say that  $q$  is *regular* if there are  $c, k \in \mathbb{R}_+$  and complex measures  $(P_{t,u})_{t \in [0,1], u \in \mathbb{R}^d}$  such that

- (1)  $\widehat{P_{t,u}}(x) := \int_{\mathbb{R}^d} e^{ivx} P_{t,u}(dv) = e^{tq(x,u)}$  for all  $x \in \mathbb{R}^d$ ,
- (2)  $|P_{t,u}|(\mathbb{R}^d) \leq 1$  and
- (3)  $|\int_{\mathbb{R}^d} v |P_{t,u}|(dv)| \leq ct(1 + |u|)$
- (4)  $\int_{\mathbb{R}^d} |v|^2 |P_{t,u}|(dv) \leq k^2 t(1 + |u|^2)$

for all  $t \in [0, 1], u \in \mathbb{R}^d$ . We call the pair  $(c, k)$  a *bound* for  $q$ .

COROLLARY III.2.20. Let  $q \in C^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$  be a regular pseudo-symbol that satisfies the requirements of Theorem III.1.4. Then we have existence and uniqueness for the  $q$ -martingale problem.

PROOF. Theorem III.1.4 implies existence and Theorem III.2.17 yields uniqueness.  $\square$

PROPOSITION III.2.21. The set of regular symbols on  $\mathbb{R}^d$  is a convex cone. Moreover, let  $q_1, q_2$  be regular symbols and  $(c_1, k_1)$  resp.  $(c_2, k_2)$  be bounds for  $q_1$  resp.  $q_2$ . Then  $(c_1 + c_2, k_1 + k_2)$  is a bound for  $q_1 + q_2$ . If  $s \in [0, 1]$  then  $(sc_1, \sqrt{s}k_1)$  is a bound for  $sq_1$ .

PROOF. Let  $u \in \mathbb{R}^d$ . For all  $t \in [0, 1]$  let  $P_{t,u}^{(1)}, P_{t,u}^{(2)}$  be the complex measures satisfying (1) to (4) as in Definition III.2.19 with  $q_1$  resp.  $q_2$ . Then  $P_{st,u}^{(1)}$  satisfies the requirements of Definition III.2.19 for the symbol  $sq_1$  with bounds  $(sc_1, \sqrt{s}k_1)$ . Let  $t \in [0, 1]$  and define

$$P_{t,u} := P_{t,u}^{(1)} * P_{t,u}^{(2)}.$$

Then  $|P_{t,u}|(\mathbb{R}^d) \leq |P_{t,u}^{(1)}|(\mathbb{R}^d)|P_{t,u}^{(2)}|(\mathbb{R}^d) \leq 1$ , i.e.  $P_{t,u}$  satisfies (2) in Definition III.2.19. Hence it satisfies (3). Cauchy-Schwartz inequality yields (4). And for  $x \in \mathbb{R}^d$  we have

$$\widehat{P_{t,u}}(x) = e^{tq_1(x,u)} e^{tq_2(x,u)} = e^{t(q_1+q_2)(x,u)}.$$

□

**PROPOSITION III.2.22.** *Let  $(q_n)_{n \in \mathbb{N}}$  be a sequence of regular symbols with bounds  $(c_n, k_n)_{n \in \mathbb{N}}$  such that  $c := \sum_{n \in \mathbb{N}} c_n < \infty$  and  $k := \sum_{n \in \mathbb{N}} k_n < \infty$ . Assume that  $\sum_{n=1}^l q_n$  converges pointwise to a continuous function  $q$ . Then  $q$  is a regular symbol with bound  $(c, k)$ .*

**PROOF.** By [31, Theorem 5.3.3, Theorem 3.6.1]  $q$  is a symbol. Let  $t, u \in \mathbb{R}_+$ . For all  $n \in \mathbb{N}$  let  $P_{t,u}^{(n)}$  be the complex measure given in Definition III.2.19. Let  $\Omega := (\mathbb{R}^d)^{\mathbb{N}}$ ,  $\mathcal{A}$  be the product  $\sigma$ -algebra and  $P := \otimes_{n=1}^{\infty} P_{t,u}^{(n)}$ . Then  $P$  is a complex measure with total variation less or equal 1 since all  $P_{t,u}^{(n)}$  have this property. For all  $n \in \mathbb{N}$  let  $X_n : \Omega \rightarrow \mathbb{R}^d, x \mapsto x_n$  the projection on the  $n$ -th coordinate. Then

$$\int_{\Omega} |X_n|^2 dP = \int_{\mathbb{R}^d} |v|^2 P_{t,u}^n \leq k_n^2 t(1 + |u|^2).$$

Hence  $(X_n)_{n \in \mathbb{N}}$  is a convergent series in  $L^2(\Omega, P)$  with limit

$$X := \sum_{n \in \mathbb{N}} X_n.$$

Let  $P_{t,u}$  be the pushforward measure of  $X$ . Then  $P_{t,u}$  has the required properties. □

**PROPOSITION III.2.23.** *Let  $(q_n)_{n \in \mathbb{N}}$  be a sequence of regular symbols with bounds  $(0, k_n)_{n \in \mathbb{N}}$  such that  $k := (\sum_{n \in \mathbb{N}} k_n^2)^{1/2} < \infty$ . Assume that  $\sum_{n=1}^l q_n$  converges pointwise to a continuous function  $q$ . Then  $q$  is a regular symbol with bound  $(0, k)$ .*

**PROOF.** Let all variables be as in the proof of Proposition III.2.22. Then  $\int_{\Omega} X_n X_m dP = 0$  and thus  $(X_n)_{n \in \mathbb{N}}$  are orthogonal in  $L^2(\Omega, P)$ . Thus

$$\int_{\Omega} \left( \sum_{n=1}^l X_n \right)^2 dP = \sum_{n=1}^l \int_{\Omega} |X_n|^2 dP \leq k_n^2 t(1 + |u|^2) \leq k^2 t(1 + |u|^2).$$

Hence  $(X_n)_{n \in \mathbb{N}}$  is a convergent series in  $L^2(\Omega, P)$  with limit

$$X := \sum_{n \in \mathbb{N}} X_n.$$

Let  $P_{t,u}$  be the pushforward measure of  $X$ . Then  $P_{t,u}$  has the required properties. □

Until now we have only analysed the structure of the set of regular symbols. We now develop some criterion for a symbol to be regular.

PROPOSITION III.2.24. *Let  $q \in C^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$  be a symbol such that  $q(\cdot, u)$  is a Lévy exponent for all  $u \in \mathbb{R}^d$ . Then for all  $t \in \mathbb{R}_+, u \in \mathbb{R}^d$  there is a probability measure  $P_{t,u}$  such that*

- $\widehat{P_{t,u}}(x) = \exp(tq(x, u)),$
- $|P_{t,u}|(\mathbb{R}^d) = P_{t,u}(\mathbb{R}^d) = 1,$
- $\left| \int_{\mathbb{R}^d} v |P_{t,u}|(dv) \right| = t |\nabla_1 q(0, u)|$
- $\int_{\mathbb{R}^d} |v|^2 |P_{t,u}|(dv) = t |\Delta_1 q(0, u)|$

where  $\nabla_1, \Delta_1$  denote the gradient, resp. Laplace operator with respect to the first variable. In particular, if

$$c := \sup \left\{ \frac{|\nabla_1 q(0, u)|}{1 + |u|} : u \in \mathbb{R}^d \right\} < \infty \quad \text{and}$$

$$k^2 := \sup \left\{ \frac{|\Delta_1 q(0, u)|}{1 + |u|^2} : u \in \mathbb{R}^d \right\} < \infty,$$

then  $q$  is a regular symbol with bounds  $(c, k)$ .

PROOF. Let  $u \in \mathbb{R}^d, t \in [0, 1]$ . There is a probability measure  $P_{t,u}$  with characteristic function  $\exp(tq(\cdot, u))$ , i.e. it satisfies (1). Since it is a probability measure, we have  $|P_{t,u}| = P_{t,u}$  and thus it satisfies (2). (3) and (4) are usual relations between characteristic functions and the corresponding measures.  $\square$

PROPOSITION III.2.25. *Let  $q$  be a symbol such that  $q(\cdot, u) = \psi(u)$  is constant for all  $u \in \mathbb{R}^d$ . Then  $q$  is a continuous regular symbol with bound  $(0, 0)$ .*

REMARK III.2.26. A symbol which is constant in the space coordinate is just a characteristic exponent. And a solution  $X$  to the  $q$ -martingale problem has the property that  $L := X(t) - X(0)$  is a Lévy process with characteristic exponent  $\psi$ .

PROOF. Let  $u \in \mathbb{R}^d, t \in [0, 1]$ . Define  $P_{t,u} := \exp(t\psi(u))\delta_0$ . Then for  $x \in \mathbb{R}^d$  we have

$$\widehat{P_{t,u}}(x) = \exp(tq(x, u)) = \exp(t\psi(u)).$$

This measure obviously satisfies (3) and (4) in Definition III.2.19 with  $c = 0, k = 0$ . Finally

$$|P_{t,u}| = \exp(t\operatorname{Re}(\psi(u)))\delta_0.$$

Hence it also satisfies (2) in Definition III.2.19.  $\square$

PROPOSITION III.2.27. *Let  $q$  be a regular symbol with bound  $(c, k)$  and let  $y \in \mathbb{R}^d$ .*

*Then*

$$\tilde{q} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}, (x, u) \mapsto q(x + y, u)$$

*is a symbol with bound  $(c, k)$ .*

PROOF. Let  $t \in [0, 1], u \in \mathbb{R}^d$ . Let  $P_{t,u}$  be the complex measure given in Definition III.2.19 for  $q$ . Define  $\tilde{P}_{t,u}(dv) := e^{iyv} P_{t,u}$ . This complex measures satisfies the requirements of Definition III.2.19 for  $\tilde{q}$ .  $\square$

LEMMA III.2.28. *Let  $n \in \mathbb{R}^d, a, b, k \in \mathbb{R}$  such that  $\operatorname{Re}(a) \geq |b|$ . Then for all  $t \in [0, 1]$  there is a complex measure  $P_t$  such that*

- $\widehat{P}_t(x) = \exp(t(b \cos(kn \cdot x) - a))$ ,
- $|P_t|(\mathbb{R}^d) \leq 1$ ,
- $\left| \int_{\mathbb{R}^d} v |P_t|(dv) \right| = 0$  and
- $\int_{\mathbb{R}^d} |v|^2 |P_t|(dv) \leq t|b||kn|^2$ .

*Moreover, there is a complex measure  $Q_t$  such that*

- a)  $\widehat{Q}_t(x) = \exp(t(b \sin(kn \cdot x) - a))$ ,
- b)  $|Q_t|(\mathbb{R}^d) \leq 1$ ,
- c)  $\left| \int_{\mathbb{R}^d} v |Q_t|(dv) \right| = 0$  and
- d)  $\int_{\mathbb{R}^d} |v|^2 |Q_t|(dv) = t|b||kn|^2$ .

PROOF. Let  $\mu$  be a complex measure on  $\mathbb{R}^d$  with total variation less or equal 1. Let  $P_t := \exp(-ta) \exp(tb\mu)$ . Then  $P_t$  is a complex measure and

$$|P_t| \leq \exp(-t\operatorname{Re}(a)) \exp(t|b||\mu|).$$

Hence we have for all  $x \in \mathbb{R}^d$

$$\begin{aligned} \widehat{P}_t(x) &= \exp(-ta) \exp(tb\hat{\mu}(x)), \\ |P_t|(\mathbb{R}^d) &\leq \exp(t(|b| - \operatorname{Re}(a))) \leq 1, \\ \left| \int_{\mathbb{R}^d} v |P_{t,u}|(dv) \right| &\leq t|b| \left| \int_{\mathbb{R}^d} v |\mu|(dv) \right| \quad \text{and} \\ \int_{\mathbb{R}^d} |v|^2 |P_{t,u}|(dv) &\leq t|b| \int_{\mathbb{R}^d} |v|^2 |\mu|(dv). \end{aligned}$$

For the specific choices  $\mu = \frac{1}{2}(\delta_{nk} + \delta_{-nk})$  we get the first part and for  $\mu = \frac{1}{2i}(\delta_{nk} - \delta_{-nk})$  we get the second part.  $\square$

LEMMA III.2.29. *Let  $m \in \mathbb{N}$  such that  $m > \frac{d}{2}$  and  $q$  a continuous symbol such that  $q(\cdot, u) \in C^{m+2}(\mathbb{R}^d, \mathbb{C})$  for any  $u \in \mathbb{R}^d$ . Let  $x_0 \in \mathbb{R}^d$  and define  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}, u \mapsto q(x_0, u)$ . Assume that there is a neighbourhood  $V$  of  $x_0$  and  $c, L > 0$  such that*

$$\begin{aligned} |\partial_x^\beta q(x, u)| &\leq L|\operatorname{Re}(\psi(u))| \\ |\partial_x^\alpha q(x, u)| &\leq L(1 + |u|^2) \end{aligned}$$

for all  $x \in V, u \in \mathbb{R}^d, \beta, \alpha \in \mathbb{N}^d$  with  $|\beta| \leq m$  and  $|\alpha| \leq m + 2$ . Then there is  $k > 0$ , a neighbourhood  $U$  of  $x_0$  and a sequence  $(a_n, b_n)_{n \in \mathbb{N}^d}$  where  $a_n, b_n : \mathbb{R}^d \rightarrow \mathbb{C}$  such that

- (1)  $\sum_{n \in \mathbb{Z}^d} a_n(u) \cos(kn \cdot x) + b_n(u) \sin(kn \cdot x)$  for all  $u \in \mathbb{R}^d, x \in U$ ,
- (2)  $-\operatorname{Re}(a_0(u)) \geq \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |a_n(u)| + |b_n(u)|$  for all  $u \in \mathbb{R}^d$  and
- (3)  $\sup \left\{ \sum_{n \in \mathbb{Z}^d} |n|^2 \left( \frac{|a_n(u)|}{1+|u|^2} + \frac{|b_n(u)|}{1+|u|^2} \right) : u \in \mathbb{R}^d \right\} < \infty$ .

PROOF. Let  $u \in \mathbb{R}^d$ . W.l.o.g. we may assume that  $V$  is convex and bounded because otherwise we just work with an open, convex and bounded subset. The fundamental theorem of calculus implies

$$|q(x, u) - q(x_0, u)| \leq L|x - x_0| |\operatorname{Re}(q(x_0, u))|$$

for all  $x \in V$ . Let  $\phi \in C^\infty([0, 1]^d, [0, 1])$  such that it is constant 1 on  $[0.25, 0.75]^d$  and such that its support is contained in  $(0, 1)^d$ . Let

$$C_\phi := \sup\{|\partial^\alpha \phi(x)| : \alpha \in \mathbb{N}^d, |\alpha| \leq m + 2, x \in \mathbb{R}^d\}.$$

Let  $\ell \geq 1$  such that the cube centred at  $x_0$  of radius  $1/\ell$  is contained in  $V$ , and define

$$q_\ell : [0, 1]^d \times \mathbb{R}^d \rightarrow \mathbb{C}, (y, u) \mapsto \phi(y)(q(\gamma_\ell(y), u) - \psi(u)) + \psi(u)$$

where  $\gamma_\ell : [0, 1]^d \rightarrow V, y \mapsto \frac{y-h}{\ell} + x_0$  and  $h := (\frac{1}{2}, \dots, \frac{1}{2})$ . For  $s \in \{1, \dots, m\}$  we have

$$\begin{aligned} &\sup\{|\partial_y^\beta q_\ell(y, u)| : y \in [0, 1]^d, \beta \in \mathbb{N}^d, |\beta| = s\} \\ &\leq \sup\{|q(\gamma_\ell(y), u) - \psi(u)| |\partial_y^\beta \phi(y)| : y \in [0, 1]^d, \beta \in \mathbb{N}^d, |\beta| = s\} \\ &\quad + C_\phi \sup\{|\partial_y^\beta (q(\gamma_\ell(y), u) - \psi(u))| : y \in [0, 1]^d, \beta \in \mathbb{N}^d, 1 \leq |\beta| \leq s\} \\ &\leq LC_\phi \sup\{|\gamma_\ell(y) - x_0| : y \in [0, 1]^d\} |\operatorname{Re}(\psi(u))| \\ &\quad + C_\phi \frac{1}{\ell} \sup\{|\partial_x^\beta (q(x, u) - \psi(u))| : x \in V, \beta \in \mathbb{N}^d, 1 \leq |\beta| \leq s\} \\ &\leq K_1 \frac{|\operatorname{Re}(\psi(u))|}{\ell} \end{aligned}$$

for some constant  $K_1 > 0$ . By [20, Theorem 3.2.16] there is an other constant  $K_2$  such that

$$\|q_\ell(\cdot, u)\|_{A(T)} \leq K_2 \frac{|\operatorname{Re}(\psi(u))|}{\ell}$$

where  $\|q_\ell(\cdot, u)\|_{A(T)}$  is the absolute sum of the Fourier coefficients of  $q_\ell(\cdot, u)$  except for the coefficient  $a_0(u) := c_0(u) := \int_{[0,1]^d} q_\ell(x, u) dx$  which appears to be missing in the statement of [20, Theorem 3.2.16]. Thus there is  $\ell \geq 2L$  such that

$$\|q_\ell(\cdot, u)\|_{A(T)} \leq \frac{|\operatorname{Re}(\psi(u))|}{4}.$$

We also have

$$\begin{aligned} |a_0(u) - \psi(u)| &\leq \int_{[0,1]^d} |q_\ell(y, u) - \psi(u)| dy \\ &\leq L |\operatorname{Re}(\psi(u))| \int_{[0,1]^d} |\gamma_\ell(y) - x_0| dy \\ &\leq L \frac{|\operatorname{Re}(\psi(u))|}{\ell} \\ &\leq \frac{|\operatorname{Re}(\psi(u))|}{2}. \end{aligned}$$

Hence  $|\operatorname{Re}(\psi(u))| \leq 2|\operatorname{Re}(a_0(u))|$ . For  $n \in \mathbb{Z}^d \setminus \{0\}$  we define

$$\begin{aligned} a_n &:= e^{2\pi i n(h - \ell x_0)} c_n \quad \text{and} \\ b_n &:= i e^{2\pi i n(h - \ell x_0)} c_n \end{aligned}$$

where  $c_n(u) := \int_{[0,1]^d} q_\ell(x, u) e^{-2\pi i n x} dx$  for  $u \in \mathbb{R}^d$ . Then

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |a_n(u)| + |b_n(u)| &\leq 2 \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |c_n(u)| \\ &= 2 \|q_\ell(\cdot, u)\|_{A(T)} \\ &\leq \frac{|\operatorname{Re}(\psi(u))|}{2} \\ &\leq |\operatorname{Re}(a_0(u))| \\ &= -\operatorname{Re}(a_0(u)). \end{aligned}$$

That is (2) holds. Let  $U$  be the cube centred at  $x_0$  with radius  $1/(4\ell)$ . Then for  $x \in U$  with  $y := \gamma_\ell^{-1}(x)$  we have  $y \in [0.25, 0.75]^d$  and

$$q_\ell(y, u) = \phi(y)(q(\gamma_\ell(y), u) - \psi(u)) + \psi(u) = q(x, u).$$



Moreover, the inversion formula, see [20, Proposition 3.1.14], yields

$$\begin{aligned}
q(x, u) &= q_\ell(y, u) \\
&= \sum_{n \in \mathbb{Z}^d} c_n(u) e^{2\pi i n y} \\
&= \sum_{n \in \mathbb{Z}^d} c_n(u) e^{2\pi i n (h - \ell x_0)} e^{2\pi i \ell n x} \\
&= \sum_{n \in \mathbb{Z}^d} a_n(u) \cos(2\pi \ell n x) + b_n(u) \sin(2\pi \ell n x) \\
&= \sum_{n \in \mathbb{Z}^d} a_n(u) \cos(k n x) + b_n(u) \sin(k n x)
\end{aligned}$$

where  $k := 2\pi\ell$ . Thus (1) holds. The statement (3) can be deduced by applying [20, Proposition 3.1.2] and the same arguments as before to  $\Delta_1 q$  and using the second inequality instead of the first inequality.  $\square$

## APPENDIX A

### 1. Local characteristics

Let  $X$  be an  $\mathbb{R}^d$ -valued semimartingale with integral characteristics  $(B, C, \nu)$  in the sense of [25] relative to some fixed truncation function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . By [25, I.2.9] there exist a predictable  $\mathbb{R}^d$ -valued process  $b$ , a predictable  $\mathbb{R}^{d \times d}$ -valued process  $c$ , a kernel  $F$  from  $(\Omega \times \mathbb{R}, \mathcal{P})$  to  $(\mathbb{R}^d, \mathcal{B})$ , and a predictable increasing process  $A$  such that

$$dB_t = b_t dA_t, \quad dC_t = c_t dA_t, \quad \nu(dt, dx) = F_t(dx) dA_t.$$

If  $A_t = t$ , we call the triplet  $(b, c, F)$  *local* or *differential characteristics* of  $X$  relative to truncation function  $h$ . Most processes in applications as e.g. diffusions, Lévy processes etc. allow for local characteristics. In this case  $b$  stands for a drift rate,  $c$  for a diffusion coefficient, and  $F$  for a local Lévy measure representing jump activity. If they exist, the local characteristics are unique up to a  $dP \otimes dt$ -null set on  $\Omega \times \mathbb{R}_+$ .

**PROPOSITION A.1.1** (Itô's formula for local characteristics). *Let  $X$  be an  $\mathbb{R}^d$ -valued semimartingale with local characteristics  $(b, c, F)$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$  a  $C^2$ -function. Then the triplet  $(\tilde{b}, \tilde{c}, \tilde{F})$  defined by*

$$\begin{aligned} \tilde{b}_t &= Df(X_{t-})b_t + \frac{1}{2} \sum_{j,k=1}^n \partial_j \partial_k f(X_{t-}) c_t^{jk} \\ &\quad + \int \left( \tilde{h}(f(X_{t-} + x) - f(X_{t-})) - Df(X_{t-})h(x) \right) F_t(dx), \\ \tilde{c}_t &= Df(X_{t-})c_t(Df(X_{t-}))^\top, \\ \tilde{F}_t(A) &= \int 1_A(f(X_{t-} + x) - f(X_{t-}))F_t(dx), \quad A \in \mathcal{B}^n \text{ with } 0 \notin A, \end{aligned}$$

*is a version of the local characteristics of  $f(X)$  with respect to a truncation function  $\tilde{h}$  on  $\mathbb{R}^n$ . Here,  $\partial_j$  etc. denote partial derivatives relative to the  $j$ th argument.*

**PROOF.** See [27, Proposition 2.5]. □

**PROPOSITION A.1.2.** *Let  $X$  be an  $\mathbb{R}^d$ -valued semimartingale with local characteristics  $(b, c, F)$  and let  $\beta = (\beta^{ij})_{i \in \{1, \dots, n\}, j \in \{1, \dots, d\}}$  be a  $\mathbb{R}^{n \times d}$ -valued predictable process such that  $\beta^i \in L(X)$  for  $i \in \{1, \dots, n\}$ . Then the triplet  $(\tilde{b}, \tilde{c}, \tilde{F})$  defined by*

$$\begin{aligned}\tilde{b}_t &= \beta b_t + \int \left( \tilde{h}(\beta_t x) - \beta_t h(x) \right) F_t(dx), \\ \tilde{c}_t &= \beta_t c_t \beta_t^\top, \\ \tilde{F}_t(A) &= \int 1_A(\beta_t x) F_t(dx), \quad A \in \mathcal{B}^n \text{ with } 0 \notin A,\end{aligned}$$

*is a version of the local characteristics of the  $\mathbb{R}^n$ -valued semimartingale  $\beta \cdot X := (\beta^1 \cdot X, \dots, \beta^n \cdot X)$  with respect to the truncation function  $\tilde{h}$  on  $\mathbb{R}^n$ ,*

**PROOF.** See [27, Proposition 2.4]. □

The  $\sigma$  (resp. local) martingale property can be read from the local characteristics of a semimartingale. Indeed, [26, Lemma 3.1] proves

**PROPOSITION A.1.3.** *Let  $X$  be an  $\mathbb{R}^d$ -valued semimartingale with local characteristics  $(b, c, F)$ . Then  $X$  is a  $\sigma$ -martingale if and only if  $\int_{|x| \geq 1} |x| F(t, dx) < \infty$  and*

$$b(t) - \int (x - h(x)) F(t, dx) = 0 \tag{A.1.1}$$

*holds  $P \otimes \lambda$  everywhere.  $X$  is a local martingale, if and only if  $\int_0^t \int_{|x| \geq 1} |x| F(s, dx) dx < \infty$  and Equation (A.1.1) holds  $P \otimes \lambda$  everywhere.  $X$  is a martingale if and only if  $X$  is of class (D) and Equation (A.1.1) holds  $P \otimes \lambda$  everywhere.*

**PROOF.** See [26, Lemma 3.1]. □

**DEFINITION A.1.4.** Let  $X$  and  $Y$  be semimartingales and

$$\left( \left( \begin{array}{c} B^X \\ B^Y \end{array} \right), \left( \begin{array}{cc} C^X & C^{XY} \\ C^{XY} & C^Y \end{array} \right), \nu^{(X,Y)} \right)$$

their joint integral characteristics with respect to some truncation function  $h$ . Denote the integral characteristics of  $X$  by  $(B^X, C^X, \nu^X)$  and the integral characteristics of  $Y$  by  $(B^Y, C^Y, \nu^Y)$ .  $X$  and  $Y$  are *locally independent* if

$$\begin{aligned}C^{XY} &= 0 \quad \text{and} \\ \nu^{(X,Y)}(I \times A) &= \nu^X(I \times \{x : (x, 0) \in A\}) + \nu^Y(I \times \{y : (0, y) \in A\})\end{aligned}$$

where  $I \in \mathcal{B}(\mathbb{R})$ ,  $A \in \mathcal{B}$  outside some  $dP \otimes dt$ -null set. A family of semimartingales is *locally independent* if it is pairwise locally independent.

REMARK A.1.5. The definition is independent from the choice of the truncation function, since the covariation matrix  $C$  and the compensated jump measure  $\nu$  do not depend on the truncation function. Moreover if  $(X, Y, Z)$  are locally independent semimartingales, then  $\nu^{(X,Y,Z)} = \nu^X + \nu^Y + \nu^Z$  in the additive sense as above. That means that they are 'jointly' locally independent.

COROLLARY A.1.6. *Let  $X^{(1)}, \dots, X^{(n)}$  be locally independent semimartingales and  $Q \stackrel{\text{loc}}{\ll} P$  another probability measure. Then  $X^{(1)}, \dots, X^{(n)}$  are locally independent semimartingales relative to  $Q$ .*

PROOF. This follows from [25, III.3.24]. □

COROLLARY A.1.7. *If  $(X^{(1)}, \dots, X^{(n)})$  is a Lévy process or, more generally, a PII allowing for local characteristics, then  $X^{(1)}, \dots, X^{(n)}$  are independent if and only if they are locally independent.*

PROOF. This follows directly from the generalised Lévy-Khintchine formula, see [25, II.4.16]. □

## 2. Local exponents

DEFINITION A.2.1. Let  $(b, c, F)$  be a Lévy-Khintchine triplet on  $\mathbb{R}^d$  relative to some truncation function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . We call the mapping  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ ,

$$\psi(u) := iub - \frac{1}{2}ucu^\top + \int (e^{iux} - 1 - iuh(x))F(dx)$$

Lévy exponent corresponding to  $(b, c, F)$ . By [25, II.2.44], the Lévy exponent determines the triplet  $(b, c, F)$  uniquely. If  $X$  is a Lévy process with Lévy-Khintchine triplet  $(b, c, F)$ , we call  $\psi$  the *characteristic* or *Lévy exponent* of  $X$ .

In the same vein, local characteristics naturally lead to local exponents.

DEFINITION A.2.2. If  $X$  is an  $\mathbb{R}^d$ -valued semimartingale with local characteristics  $(b, c, F)$ , we write

$$\psi_t^X(u) := iub_t - \frac{1}{2}uc_tu^\top + \int (e^{iux} - 1 - iuh(x))F_t(dx), \quad u \in \mathbb{R}^d \quad (\text{A.2.1})$$

for the Lévy exponent corresponding to  $(b_t, c_t, F_t)$ . We call the family of predictable processes  $\psi^X(u) := (\psi_t^X(u))_{t \in \mathbb{R}_+}$ ,  $u \in \mathbb{R}^d$  *local exponent* of  $X$ . (A.2.1) implies that  $u \mapsto \psi_t^X(u)$  is the characteristic exponent of a Lévy process.

The name *exponent* is of course motivated by the following fact.

REMARK A.2.3. If  $X$  is a semimartingale with deterministic local characteristics  $(b, c, F)$ , it is a PII and we have

$$E(e^{iu(X_T - X_t)} | \mathcal{F}_t) = E(e^{iu(X_T - X_t)}) = \exp\left(\int_t^T \psi_s^X(u) ds\right)$$

for any  $T \in \mathbb{R}_+$ ,  $t \in [0, T]$ ,  $u \in \mathbb{R}^d$ , cf. [25, II.4.15].

We now generalise the notion of local exponents to complex-valued semimartingales and more general arguments.

DEFINITION A.2.4. Let  $X$  be a  $\mathbb{C}^d$ -valued semimartingale and  $\beta$  a  $\mathbb{C}^d$ -valued  $X$ -integrable process. We call a predictable  $\mathbb{C}$ -valued process  $\psi^X(\beta) = (\psi_t^X(\beta))_{t \in \mathbb{R}_+}$  *local exponent* of  $X$  at  $\beta$  if  $\psi^X(\beta) \in L(I)$  and  $(\exp(i\beta \cdot X_t - \int_0^t \psi_s^X(\beta) ds))_{t \in \mathbb{R}_+}$  is a complex-valued local martingale. We denote by  $\mathcal{U}^X$  the set of processes  $\beta$  such that the local exponent  $\psi^X(\beta)$  exists.

From the following lemma it follows that  $\psi^X(\beta)$  is unique up to a  $dP \otimes dt$ -null set.

LEMMA A.2.5. *Let  $X$  be a complex-valued semimartingale and  $A, B$  complex-valued predictable processes of finite variation with  $A_0 = 0 = B_0$  and such that  $\exp(X - A)$  and  $\exp(X - B)$  are local martingales. Then  $A = B$  up to indistinguishability.*

PROOF. Set  $M := e^{X-A}$ ,  $N := e^{X-B}$ ,  $V := e^{A-B}$ . Integration by parts yields that

$$M_- \cdot V = MV - V \cdot M - M_0 V_0 = N - V \cdot M - M_0$$

is a local martingale. Therefore  $V = 1 + \frac{1}{M_-} \cdot (M_- \cdot V)$  is a predictable local martingale with  $V_0 = 1$  and hence  $V = 1$ , cf. [25, I.3.16].  $\square$

The following result shows that Definition A.2.4 truly generalises Definition A.2.2.

PROPOSITION A.2.6. *Let  $X$  be an  $\mathbb{R}^d$ -valued semimartingale with local characteristics  $(b, c, F)$ . Suppose that  $\beta$  is a  $\mathbb{C}^d$ -valued predictable and  $X$ -integrable process. If  $\beta$  is  $\mathbb{R}^d$ -valued for any  $t \in \mathbb{R}_+$ , then  $\beta \in \mathcal{U}^X$ . Moreover there is equivalence between*

(1)  $\beta \in \mathcal{U}^X$ ,

(2)  $\int_0^t \int 1_{\{-\operatorname{Im}(\beta_s x) > 1\}} e^{-\operatorname{Im}(\beta_s x)} F_s(dx) ds < \infty$  almost surely for any  $t \in \mathbb{R}_+$ .

In this case we have

$$\psi_t^X(\beta) = i\beta_t b_t - \frac{1}{2} \beta_t c_t \beta_t^\top + \int (e^{i\beta_t x} - 1 - i\beta_t h(x)) F_t(dx) \quad (\text{A.2.2})$$

outside some  $dP \otimes dt$ -null set.

PROOF. If  $\beta$  is  $\mathbb{R}^d$ -valued, then Statement (2) is obviously true. Thus we only need to prove the equivalence and (A.2.2). For real-valued  $i\beta$  the equivalence follows from [29, Lemma 2.13]. The complex-valued case is derived similarly. For real-valued  $i\beta$  (A.2.2) is shown in [29, Theorems 2.18(1,6) and 2.19]. The general case follows along the same lines.  $\square$

(A.2.2) implies that the local exponent of  $X$  at any  $\beta \in \mathcal{U}^X$  is determined by the triplet  $(b, c, F)$  and hence by the local exponent of  $X$  in the sense of Definition A.2.2.

COROLLARY A.2.7. *Let  $(X, M)$  be a  $1 + d$ -dimensional semimartingale with local exponent  $\psi^{(X, M)}$  such that  $M$  is a Lévy process whose components are subordinators. Then  $\beta \in \mathcal{U}^{(X, M)}$  for any  $\mathbb{R} \times (\mathbb{R} + i\mathbb{R}_+)^d$ -valued  $(X, M)$ -integrable process  $\beta$ .*

PROOF. This follows immediately from Proposition A.2.6.  $\square$

DEFINITION A.2.8. Let  $X^{(1)}, \dots, X^{(n)}$  be semimartingales which allow for local characteristics. We call them  $X^{(1)}, \dots, X^{(n)}$  locally independent if

$$\mathcal{U}^{(X^{(1)}, \dots, X^{(n)})} \cap (L(X^{(1)}) \times \dots \times L(X^{(n)})) = \mathcal{U}^{X^{(1)}} \times \dots \times \mathcal{U}^{X^{(n)}}$$

for

$$L(X^{(1)}) \times \dots \times L(X^{(n)}) := \{\beta = (\beta^{(1)}, \dots, \beta^{(n)}) \text{ complex-valued :} \\ \beta^{(i)} X^{(i)}\text{-integrable for } i = 1, \dots, n\}$$

and

$$\psi^{(X^{(1)}, \dots, X^{(n)})}(\beta) = \sum_{j=1}^n \psi^{X^{(j)}}(\beta^{(j)})$$

outside some  $dP \otimes dt$ -null set for any  $\beta = (\beta^{(1)}, \dots, \beta^{(n)}) \in \mathcal{U}^{(X^{(1)}, \dots, X^{(n)})}$ .

The following lemma provides alternative characterisations of local independence. For ease of notation we consider two semimartingales but the extension to arbitrary finite numbers is straightforward.

LEMMA A.2.9. *Let  $(X, Y)$  be an  $\mathbb{R}^{m+n}$ -valued semimartingale with local characteristics  $(b, c, F)$  and denote by  $(b^X, c^X, F^X)$  resp.  $(b^Y, c^Y, F^Y)$  local characteristics of  $X$  resp.  $Y$ . We have equivalence between*

(1)  *$X$  and  $Y$  are locally independent,*

(2)

$$\psi^{(X,Y)}(u, v) = \psi^X(u) + \psi^Y(v), \quad (u, v) \in \mathbb{R}^{m+n} \quad (\text{A.2.3})$$

*outside some  $dP \otimes dt$ -null set,*

(3)

$$c = \begin{pmatrix} c^X & 0 \\ 0 & c^Y \end{pmatrix}$$

*and*

$$F(A) = F^X(\{x : (x, 0) \in A\}) + F^Y(\{y : (0, y) \in A\}), \quad A \in \mathcal{B}^{m+n}$$

*outside some  $dP \otimes dt$ -null set.*

PROOF. (1) $\Rightarrow$ (2): This is obvious by Proposition A.2.6.

(2) $\Rightarrow$ (3): Both sides of (A.2.3) are Lévy exponents for fixed  $(\omega, t) \in \Omega \times \mathbb{R}_+$ . Indeed, the triplet corresponding to  $(u, v) \mapsto (\psi_t^X(u) + \psi_t^Y(v))(\omega)$  is  $(b_t, \tilde{c}_t, \tilde{F}_t)(\omega)$  with

$$\tilde{c}_t = \begin{pmatrix} c_t^X & 0 \\ 0 & c_t^Y \end{pmatrix}$$

and

$$\tilde{F}_t(A) = F_t^X(\{x : (x, 0) \in A\}) + F_t^Y(\{y : (0, y) \in A\}), \quad A \in \mathcal{B}^{m+n}.$$

Since the Lévy exponent determines the triplet uniquely (cf. [25, II.2.44]), the assertion follows.

(3) $\Rightarrow$ (1): If  $\beta^X$  is  $X$ -integrable and  $\beta^Y$  is  $Y$ -integrable, then  $\beta = (\beta^X, \beta^Y)$  is  $(X, Y)$ -integrable and  $\beta \bullet (X, Y) = \beta^X \bullet X + \beta^Y \bullet Y$ . The characterisation in Proposition A.2.6 yields  $\beta \in \mathcal{U}^{(X,Y)}$  for such  $\beta = (\beta^X, \beta^Y)$  if and only if  $\beta^X \in \mathcal{U}^X, \beta^Y \in \mathcal{U}^Y$ . In addition,  $\psi^{(X,Y)}(\beta) = \psi^X(\beta^X) + \psi^Y(\beta^Y)$  follows from (A.2.2)  $\square$

COROLLARY A.2.10. *Let  $X^{(1)}, \dots, X^{(n)}$  be locally independent semimartingales and  $Q \stackrel{\text{loc}}{\ll} P$  another probability measure. Then  $X^{(1)}, \dots, X^{(n)}$  are locally independent semimartingales relative to  $Q$ .*

PROOF. This follows from Lemma A.2.9 and [25, III.3.24].  $\square$

COROLLARY A.2.11. *If  $(X^{(1)}, \dots, X^{(n)})$  is a Lévy process or, more generally, a PII allowing for local characteristics, then  $X^{(1)}, \dots, X^{(n)}$  are independent if and only if they are locally independent.*

PROOF. By Remark A.2.3 the characteristic function  $\varphi_{X_t}$  of  $X_t := (X_t^{(1)}, \dots, X_t^{(n)})$  is given by

$$\varphi_{X_t}(u^1, \dots, u^n) = \exp\left(\int_0^t \psi_s^{(X^{(1)}, \dots, X^{(n)})}(u^1, \dots, u^n) ds\right).$$

Thus independence of  $X_t^{(1)}, \dots, X_t^{(n)}$  is equivalent to

$$\psi_t^{(X^{(1)}, \dots, X^{(n)})}(u^1, \dots, u^n) = \sum_{k=1}^n \psi_t^{X^{(k)}}(u^k),$$

for Lebesgue-almost any  $t \in \mathbb{R}_+$  and any  $(u^1, \dots, u^n) \in \mathbb{R}^n$ . By Lemma A.2.9 this in turn is equivalent to local independence of  $X_t^{(1)}, \dots, X_t^{(n)}$ .  $\square$

LEMMA A.2.12. *If  $f \in \Pi$ , then  $\operatorname{Re}(f(u)) \leq 0$  for any  $u \in \mathbb{R}$ , where  $\Pi$  is defined in Section 2.1.*

PROOF. For  $f \in \Pi$  we have

$$f(u) = -\frac{u^2 + iu}{2}c + \int (e^{iux} - 1 - iu(e^x - 1))F(dx) \quad (\text{A.2.4})$$

with some Lévy measure  $F$  and some  $c \in \mathbb{R}_+$ . The real part of the first term is obviously negative and the real part of the integrand is negative as well.  $\square$

REMARK A.2.13. If we extend the domain of  $f$  to  $\mathbb{R} + i[-1, 0]$  by keeping the representation (A.2.4), then the conclusion of Lemma A.2.12 is still correct. However, this fact is not used in this thesis.

The following four statements follow immediately from the definition of local exponents.

PROPOSITION A.2.14. *Let  $X$  be a  $\mathbb{C}$ -valued semimartingale that allows for local characteristics. Then there is equivalence between*

- (1)  $\exp(X)$  is a local martingale,
- (2)  $-i \in \mathcal{U}^X$  and  $\psi^X(-i) = 0$  outside some  $dP \otimes dt$ -null set,

LEMMA A.2.15. *Let  $X$  be a  $\mathbb{C}^d$ -valued semimartingale,  $\beta$  a  $\mathbb{C}^d$ -valued and  $X$ -integrable process and  $u \in \mathbb{C}$ . Then  $u\beta \in \mathcal{U}^X$  if and only if  $u \in \mathcal{U}^{\beta \cdot X}$ . In that case we have*

$$\psi^X(u\beta) = \psi^{\beta \cdot X}(u).$$



LEMMA A.2.16. *Let  $X, Y$  be  $\mathbb{C}^d$ -valued semimartingales and  $u \in \mathbb{C}^d$ . Then  $u \in \mathcal{U}^{X+Y}$  if and only if  $(u, u) \in \mathcal{U}^{(X,Y)}$ . In this case we have*

$$\psi^{X+Y}(u) = \psi^{(X,Y)}(u, u)$$

outside some  $dP \otimes dt$ -null set.

LEMMA A.2.17. *Let  $X, Z$  be  $\mathbb{C}^d$ -valued semimartingales and  $\beta, \gamma$  predictable  $\mathbb{C}^d$ -valued processes such that*

- (1)  $\gamma$  has I-integrable components,
- (2)  $\beta\gamma$  is I-integrable,
- (3)  $Z_t = Z_0 + \int_0^t \gamma_s ds + X_t$ .

Then  $\beta \in \mathcal{U}^Z$  if and only if  $\beta \in \mathcal{U}^X$ . In this case  $\psi^Z(\beta) = \psi^X(\beta) + i\beta\gamma$  outside some  $dP \otimes dt$ -null set.

### 3. Semimartingale decomposition relative to a semimartingale

Let  $(X, Y)$  be an  $\mathbb{R}^{1+d}$ -valued semimartingale with local characteristics  $(b, c, K)$ , written here in the form

$$b = \begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \quad c := \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}. \quad (\text{A.3.1})$$

Suppose that  $\int_0^t \int_{(1,\infty)} e^x K_s(dx) ds < \infty$  for any  $t \in \mathbb{R}_+$  or, equivalently,  $e^X$  is a special semimartingale. We set

$$X_t^\parallel := \log \mathcal{E}((c^{X,Y}(c^Y)^{-1}) \bullet Y_t^c + f * (\mu^{(X,Y)} - \nu^{(X,Y)})_t)$$

for any  $t \in \mathbb{R}_+$ , where  $c^-$  denotes the pseudoinverse of a matrix  $c$  in the sense of [1],  $Y^c$  is the continuous local martingale part of  $Y$ ,  $\mu^{(X,Y)}$  resp.  $\nu^{(X,Y)}$  are the random measure of jumps of  $(X, Y)$  and its compensator, and  $f : \mathbb{R}^{1+d} \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto (e^x - 1)1_{\{y \neq 0\}}$ . We call  $X^\parallel$  and  $X^\perp := X - X^\parallel$  the *dependent* resp. *independent part* of  $X$  relative to  $Y$ .

LEMMA A.3.1.  $X \mapsto X^\parallel$  is a projection in the sense that  $(X^\parallel)^\parallel = X^\parallel$ . Moreover, we have  $(X + Z)^\parallel = X^\parallel$  if  $Z$  is a semimartingale such that  $Z, Y$  are locally independent.

PROOF. Observe that  $(X^\parallel)^c = (c^{X,Y}(c^Y)^{-1}) \bullet Y^c$  by [29, Lemma 2.6(2)]. Defining  $c^{X^\parallel, Y}$  similarly as  $c^{X,Y}$  in (A.3.1), we have  $c^{X^\parallel, Y} = c^{X,Y}(c^Y)^{-1}c^Y = c^{X,Y}$ . Moreover,  $f(\Delta X_t^\parallel, \Delta Y_t) = f(\Delta X_t, \Delta Y_t)$  for any  $t \geq 0$ , which implies  $f * (\mu^{(X^\parallel, Y)} - \nu^{(X^\parallel, Y)}) =$

$f * (\mu^{(X,Y)} - \nu^{(X,Y)})$  by definition of stochastic integration relative to compensated random measures. Together, the first assertion follows.

Using the notation of (A.3.1), note that  $c^{(X+Z),Y} = c^{X,Y} + c^{Z,Y} = c^{X,Y}$  by Lemma A.2.9. Lemma A.2.9 also implies that  $Z$  and  $Y$  do not jump together (outside some evanescent set) and hence  $f(\Delta(X+Z)_t, \Delta Y_t) = f(\Delta X_t, \Delta Y_t)$ . This implies  $f * (\mu^{(X+Z),Y} - \nu^{(X+Z),Y}) = f * (\mu^{(X,Y)} - \nu^{(X,Y)})$  and hence  $(X+Z)^\parallel = X^\parallel$ .  $\square$

LEMMA A.3.2.  $e^{X^\parallel}$  is a local martingale. Moreover,  $X^\perp$  and  $(X^\parallel, Y)$  are locally independent semimartingales. Finally,  $e^{X^\perp}$  is a local martingale if and only if  $e^X$  is a local martingale.

PROOF. The first statement is obvious. The last statement follows from the first two and from Proposition A.2.14. It remains to prove local independence of  $X^\perp$  and  $(X^\parallel, Y)$ . Denote the local characteristics of  $(X^\perp, X^\parallel, Y)$  by  $(b^{(X^\perp, X^\parallel, Y)}, c^{(X^\perp, X^\parallel, Y)}, K^{(X^\perp, X^\parallel, Y)})$  and accordingly for  $(X, X^\parallel, Y)$ ,  $X^\perp$  etc. Set  $\bar{c} := c^{X,Y}(c^Y)^{-1}c^{Y,X}$ . Since

$$(X^\parallel)^c = (c^{X,Y}(c^Y)^{-1}) \bullet Y^c,$$

we have

$$c^{(X, X^\parallel, Y)} = \begin{pmatrix} c^X & \bar{c} & c^{X,Y} \\ \bar{c} & \bar{c} & c^{X,Y} \\ c^{Y,X} & c^{Y,X} & c^Y \end{pmatrix}$$

and hence

$$c^{(X^\perp, X^\parallel, Y)} = \begin{pmatrix} c^X - \bar{c} & 0 & 0 \\ 0 & \bar{c} & c^{X,Y} \\ 0 & c^{Y,X} & c^Y \end{pmatrix}$$

e.g. by Proposition A.1.1. Moreover,

$$\begin{aligned} \Delta(X^\perp, X^\parallel, Y)_t &= 1_{\{\Delta Y_t=0\}}(\Delta X_t, 0, 0) + 1_{\{\Delta Y_t \neq 0\}}(0, \Delta X_t, \Delta Y_t) \\ &= \begin{cases} (\Delta X_t^\perp, 0, 0) & \text{if } \Delta X_t^\perp \neq 0, \\ (0, \Delta(X^\parallel, Y)_t) & \text{if } \Delta(X^\parallel, Y)_t \neq 0, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

yields

$$K^{(X^\perp, X^\parallel, Y)}(A) = K^{X^\perp}(\{x : (x, 0, 0) \in A\}) + K^{(X^\parallel, Y)}(\{(x, z) : (0, x, z) \in A\})$$

for  $A \in \mathcal{B}^{2+d}$ . Lemma A.2.9 completes the proof.  $\square$

## 4. Technical Proofs

### 4.1. Option pricing by Fourier transform. By

$$\mathcal{F}f(u) := \lim_{C \rightarrow \infty} \int_{-C}^{\infty} f(x)e^{iux} dx \quad (\text{A.4.1})$$

we denote the (left-)improper Fourier transform of a measurable function  $f : \mathbb{R} \rightarrow \mathbb{C}$  for any  $u \in \mathbb{R}$  such that the expression exists. If  $f$  is Lebesgue integrable, then the improper Fourier transform and the ordinary Fourier transform (i.e.  $u \mapsto \int f(x)e^{iux} dx$ ) coincide. In our application in Section 2 the improper Fourier transform exists for any  $u \in \mathbb{R} \setminus \{0\}$ .

Moreover, we denote by

$$\mathcal{F}^{-1}g(x) := \frac{1}{2\pi} \left( \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} e^{-iux} g(u) du + \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{-\varepsilon} e^{-iux} g(u) du \right) \quad (\text{A.4.2})$$

an improper inverse Fourier transform, which will be suitable to our application in Section 2.

LEMMA A.4.1. *Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\mathcal{G} \subset \mathcal{F}$  a sub- $\sigma$ -algebra. Furthermore suppose that  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is  $\mathcal{F} \otimes \mathcal{B}$ -measurable,  $m : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}$ -measurable and*

$$H(x) := 1_{[0,m]}(x)f(x) - 1_{[m,0)}(x)f(x)$$

*is non-negative with  $E(\int_0^m f(x) dx) < \infty$ . (Note that  $\int_0^m f(x) dx$  is always non-negative.)*

*Then we have*

$$\mathcal{F}\{x \mapsto E(H(x)|\mathcal{G})\}(u) = E\left(\int_0^m f(x)e^{iux} dx \middle| \mathcal{G}\right), \quad u \in \mathbb{R} \quad (\text{A.4.3})$$

*where the improper Fourier transform coincides with the ordinary Fourier transform.*

PROOF. Let  $u \in \mathbb{R}$ . From

$$\begin{aligned} \int_0^{\infty} H(x)e^{iux} dx &= 1_{\{m \geq 0\}} \int_0^m f(x)e^{iux} dx, \\ \int_{-\infty}^0 H(x)e^{iux} dx &= 1_{\{m < 0\}} \int_0^m f(x)e^{iux} dx \end{aligned}$$

it follows that  $\int_{-\infty}^{\infty} H(x)e^{iux} dx = \int_0^m f(x)e^{iux} dx$ . This implies

$$E\left(E\left(\int_{-\infty}^{\infty} H(x) dx \middle| \mathcal{G}\right)\right) = E\left(\int_{-\infty}^{\infty} H(x) dx\right) = E\left(\int_0^m f(x) dx\right) < \infty$$

and hence

$$\int_{-\infty}^{\infty} E(H(x)|\mathcal{G}) dx = E\left(\int_{-\infty}^{\infty} H(x) dx \middle| \mathcal{G}\right) < \infty.$$

Now we can apply Fubini's theorem and we get

$$\mathcal{F}\{x \mapsto E(H(x)|\mathcal{G})\}(u) = E\left(\int_{-\infty}^{\infty} H(x)e^{iux} dx \middle| \mathcal{G}\right) = E\left(\int_0^m f(x)e^{iux} dx \middle| \mathcal{G}\right).$$

□

The next proposition is a modification of [4, Proposition 1], cf. also [14].

LEMMA A.4.2. *Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\mathcal{G} \subset \mathcal{F}$  a sub- $\sigma$ -algebra. Let  $Y$  be a random variable such that  $E(e^Y) < \infty$  and consider*

$$\mathcal{O}(x) := \begin{cases} E((e^{Y-x} - 1)^+ | \mathcal{G}) & \text{if } x \geq 0, \\ E((1 - e^{Y-x})^+ | \mathcal{G}) & \text{if } x < 0. \end{cases}$$

Then we have

$$\mathcal{F}\{x \mapsto \mathcal{O}(x)\}(u) = \frac{1}{iu} - \frac{E(e^Y | \mathcal{G})}{iu - 1} - \frac{E(e^{iuY} | \mathcal{G})}{u^2 + iu}$$

and

$$\begin{aligned} & \mathcal{F}\{x \mapsto 1_{\{x \geq -C\}} \mathcal{O}(x)\}(u) \\ &= \frac{1}{iu} - \frac{E(e^y | \mathcal{G})}{iu - 1} - \frac{1 - E(e^{iu(Y \vee -C)} (1 + iu(e^{0 \wedge (Y+C)} - 1)) | \mathcal{G})}{u^2 + iu} \end{aligned}$$

for any  $C \in \mathbb{R}_+$ ,  $u \in \mathbb{R} \setminus \{0\}$ . If  $E(e^Y | \mathcal{G}) = 1$ , then in particular

$$\mathcal{F}\{x \mapsto \mathcal{O}(x)\}(u) = \frac{1 - E(e^{iuY} | \mathcal{G})}{u^2 + iu}$$

for any  $u \in \mathbb{R} \setminus \{0\}$ .

PROOF. Let  $C \in \mathbb{R}_+$ ,  $u \in \mathbb{R} \setminus \{0\}$ . We define  $m := (Y \vee -C)$ ,  $f(x) := e^{Y-x} - 1$ , and  $H(x) := 1_{[0,m]}(x)f(x) - 1_{[m,0)}(x)f(x)$ . Then we have  $1_{\{x \geq -C\}} \mathcal{O}(x) = E(H(x) | \mathcal{G})$ ,  $H \geq 0$ , and

$$E\left(\int_0^m f(x) dx\right) = E(e^Y - m - e^{Y-m}) < \infty.$$

Hence Lemma A.4.1 yields

$$\begin{aligned}
\int_{-C}^{\infty} \mathcal{O}(x) e^{iux} dx &= \mathcal{F}\{x \mapsto E(H(x)|\mathcal{G})\}(u) \\
&= E\left(\int_0^m (e^{Y-x} - 1) e^{iux} dx \middle| \mathcal{G}\right) \\
&= E\left(\left[\frac{e^{Y+(iu-1)x}}{iu-1} - \frac{e^{iux}}{iu}\right]_{x=0}^m \middle| \mathcal{G}\right) \\
&= \frac{1}{iu} - \frac{E(e^Y|\mathcal{G})}{iu-1} - \frac{E(e^{ium}(1+iu(e^{Y-m}-1))|\mathcal{G})}{u^2+iu}.
\end{aligned}$$

Since  $|e^{ium}(1+iu(e^{Y-m}-1))| \leq 1+|u|$ , we can apply Lebesgue's theorem and get

$$E(e^{ium}(1+iu(e^{Y-m}-1))|\mathcal{G}) \xrightarrow{C \rightarrow \infty} E(e^{iuY}|\mathcal{G}).$$

□

**COROLLARY A.4.3.** *Let  $Q$  be a probability measure on  $\mathbb{R}$  satisfying  $K := \int e^y Q(dy) < \infty$ . Define*

$$O(x) := \begin{cases} \int (e^{y-x} - 1)^+ Q(dy) & \text{if } x \geq 0, \\ \int (1 - e^{y-x})^+ Q(dy) & \text{if } x < 0. \end{cases}$$

Then we have

$$\left| \int_{-C}^{\infty} O(x) e^{iux} dx \right| \leq K + \frac{1+2|u|}{u^2}$$

for any  $C \in \mathbb{R}_+$ ,  $u \in \mathbb{R} \setminus \{0\}$ .

**PROOF.** Apply Lemma A.4.2 with  $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}, Q)$ ,  $\mathcal{G} = \{\emptyset, \mathbb{R}\}$ ,  $Y = \text{id}$ . □

**PROPOSITION A.4.4.** *Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\mathcal{G} \subset \mathcal{F}$  a sub- $\sigma$ -algebra. Let  $Y$  be a random variable with  $E(e^Y|\mathcal{G}) = 1$  and define*

$$\mathcal{O}(x) := \begin{cases} E((e^{Y-x} - 1)^+|\mathcal{G}) & \text{if } x \geq 0, \\ E((1 - e^{Y-x})^+|\mathcal{G}) & \text{if } x < 0. \end{cases}$$

Then we have

$$\begin{aligned}
\mathcal{O}(x) &= \mathcal{F}^{-1}\left\{u \mapsto \frac{1 - E(e^{iuY}|\mathcal{G})}{u^2 + iu}\right\}(x), \\
E(e^{iuY}|\mathcal{G}) &= 1 - (u^2 + iu)\mathcal{F}\{x \mapsto \mathcal{O}(x)\}(u)
\end{aligned}$$

for any  $u, x \in \mathbb{R}$ .

PROOF. The second equation is a restatement of Lemma A.4.2. Let  $0 < \alpha < 1$  and define  $O(x) := e^{\alpha x} \mathcal{O}(x)$ ,  $m := Y$ ,  $f(x) := e^{\alpha x}(e^{Y-x} - 1)$ , and  $H(x) := 1_{[0,m]}(x)f(x) - 1_{[m,0)}(x)f(x)$  for any  $x \in \mathbb{R}$ . Then  $H$  is non-negative,  $O(x) = E(H(x)|\mathcal{G})$ , and

$$E\left(\int_0^m f(x)dx\right) = E\left(\frac{e^{\alpha Y}}{\alpha^2 - \alpha} + \frac{e^Y}{1 - \alpha} + \frac{1}{\alpha}\right) < \infty.$$

Lemma A.4.1 yields

$$\begin{aligned} \mathcal{F}\{x \mapsto O(x)\}(u) &= E\left(\int_0^m f(x)e^{iux}dx \middle| \mathcal{G}\right) \\ &= \frac{E(e^{(\alpha+iu)Y}|\mathcal{G}) - 1}{(\alpha+iu)^2 - (\alpha+iu)}. \end{aligned}$$

We have  $E(|e^{(\alpha+iu)Y}|) \leq E(1 + e^Y) = 2$  and thus  $u \mapsto \mathcal{F}\{x \mapsto O(x)\}(u)$  is integrable. The Fourier inversion theorem yields

$$O(x) = \mathcal{F}^{-1}\{u \mapsto \mathcal{F}\{\tilde{x} \mapsto O(\tilde{x})\}(u)\}(x)$$

because the ordinary inverse Fourier transform coincides with the improper inverse Fourier transform for Lebesgue-integrable functions. Define

$$g : \{z \in \mathbb{C} \setminus \{0\} : -1 < \operatorname{Re}(z) \leq 0\} \rightarrow \mathbb{C}, \quad z \mapsto \frac{E(e^{-zY}|\mathcal{G}) - 1}{z^2 + z}.$$

$g$  is continuous and holomorphic in the interior of its domain. Let  $0 < \varepsilon < \frac{1}{2} =: \alpha$  and define

$$\begin{aligned} \gamma_{(1,\varepsilon)} : [-1, 1] &\rightarrow \mathbb{C}, & t &\mapsto i\frac{t}{\varepsilon} - \frac{1}{2}, \\ \gamma_{(2,\varepsilon)} : [0, 1] &\rightarrow \mathbb{C}, & t &\mapsto i\frac{1}{\varepsilon} - \frac{1-t}{2}, \\ \gamma_{(3,\varepsilon)} : [0, 1] &\rightarrow \mathbb{C}, & t &\mapsto i(1-t)\left(\frac{1}{\varepsilon} - \varepsilon\right) + i\varepsilon, \\ \gamma_{(4,\varepsilon)} : [0, \pi] &\rightarrow \mathbb{C}, & t &\mapsto i\varepsilon e^{it}, \\ \gamma_{(5,\varepsilon)} : [0, 1] &\rightarrow \mathbb{C}, & t &\mapsto it\left(\varepsilon - \frac{1}{\varepsilon}\right) - i\varepsilon, \\ \gamma_{(6,\varepsilon)} : [0, 1] &\rightarrow \mathbb{C}, & t &\mapsto -i\frac{1}{\varepsilon} - \frac{t}{2} \end{aligned}$$

as well as  $\Gamma_\varepsilon := \sum_{k=1}^6 \gamma_{(k,\varepsilon)}$ . Cauchy's integral theorem yields

$$\int_{\Gamma_\varepsilon} g(z)e^{xz}dz = 0.$$

Moreover we have

$$\frac{1}{2\pi i} \int_{\gamma_{(1,\varepsilon)}} g(z)e^{xz} dz \xrightarrow{\varepsilon \rightarrow 0} O(x)e^{-\frac{1}{2}x} = \mathcal{O}(x)$$

and

$$\int_{\gamma_{(k,\varepsilon)}} g(z)e^{xz} dz \xrightarrow{\varepsilon \rightarrow 0} 0$$

for  $k \in \{2, 6\}$  and even for  $k = 4$  because  $zg(z) \rightarrow 0$  for  $z \rightarrow 0$ . Thus we conclude

$$\begin{aligned} & \frac{1}{2\pi} \left( \int_{-1/\varepsilon}^{-\varepsilon} g(-iu)e^{-iux} du + \int_{\varepsilon}^{1/\varepsilon} g(-iu)e^{-iux} du \right) \\ &= \frac{1}{2\pi i} \int_{-\gamma_{(3,\varepsilon)} - \gamma_{(5,\varepsilon)}} g(z)e^{xz} dz \\ &= \frac{1}{2\pi i} \int_{\gamma_{(1,\varepsilon)} - \Gamma_{\varepsilon} + \gamma_{(2,\varepsilon)} + \gamma_{(4,\varepsilon)} + \gamma_{(6,\varepsilon)}} g(z)e^{xz} dz \\ &\xrightarrow{\varepsilon \rightarrow 0} \mathcal{O}(x). \end{aligned}$$

Since

$$\int_{-\infty}^{-1/\varepsilon} g(-iu)e^{-iux} du + \int_{1/\varepsilon}^{\infty} g(-iu)e^{-iux} du \xrightarrow{\varepsilon \rightarrow 0} 0,$$

we have

$$\frac{1}{2\pi} \left( \int_{-\infty}^{-\varepsilon} g(-iu)e^{-iux} du + \int_{\varepsilon}^{\infty} g(-iu)e^{-iux} du \right) \xrightarrow{\varepsilon \rightarrow 0} \mathcal{O}(x)$$

and hence

$$\mathcal{F}^{-1} \left\{ u \mapsto \frac{1 - E(e^{iuY} | \mathcal{G})}{u^2 + iu} \right\} (x) = \mathcal{O}(x).$$

□

**PROPOSITION A.4.5.** *Let  $(N(x))_{x \in \mathbb{R}}$  be a family of non-negative local martingales, and  $(\tau_n)_{n \in \mathbb{N}}$  a common localising sequence for all  $N(x)$  such that*

- (1)  $(\omega, x) \mapsto N_t(x)(\omega)$  is  $\mathcal{F} \otimes \mathcal{B}$ -measurable for all  $t \in \mathbb{R}_+$ ,
- (2)  $x \mapsto N_t(x)(\omega)$  is right-continuous and  $\int_{-C}^{\infty} N_t(x)(\omega) dx < \infty$  for all  $t \in \mathbb{R}_+$ ,
- (3)  $\lim_{C \rightarrow \infty} \int_{-C}^{\infty} e^{iux} N_t(x)(\omega) dx$  exists for all  $\omega \in \Omega$ ,  $t \in \mathbb{R}_+$ ,  $u \in \mathbb{R} \setminus \{0\}$ ,
- (4) for any  $n \in \mathbb{N}$ ,  $t \in \mathbb{R}$ ,  $u \in \mathbb{R} \setminus \{0\}$  there is an integrable random variable  $Z$  such that  $|\int_{-C}^{\infty} e^{iux} N_t^{\tau_n}(x)(\omega) dx| \leq Z$  for any  $C \in \mathbb{R}_+$ .

Define the (improper, cf. (A.4.1)) Fourier transform of  $N$  by

$$X_t(u) := \mathcal{F} \{ x \mapsto N_t(x) \} (u).$$

If  $X(u)$  has càdlàg paths, then it is a local martingale for all  $u \in \mathbb{R} \setminus \{0\}$  with common localising sequence  $(\tau_n)_{n \in \mathbb{N}}$ .

PROOF. For any  $C \in \mathbb{R}_+$ ,  $u \in \mathbb{R} \setminus \{0\}$ ,  $\omega \in \Omega$ ,  $t \in \mathbb{R}_+$  define

$$X_t^C(u)(\omega) := \int_{-C}^{\infty} e^{iux} N_t(x)(\omega) dx.$$

Fix  $n \in \mathbb{N}$ ,  $C \in \mathbb{R}_+$ ,  $u \in \mathbb{R} \setminus \{0\}$ . Then  $X_{t \wedge \tau_n}^C(u)(\omega) = \int_{-C}^{\infty} e^{iux} N_{t \wedge \tau_n}(x)(\omega) dx$  for any  $t \in \mathbb{R}_+$ ,  $\omega \in \Omega$ . Setting

$$c(k, x) := \begin{cases} 1_{\cos(x) > 0} \cos(x) & \text{for } k = 0, \\ 1_{\sin(x) > 0} \sin(x) & \text{for } k = 1, \\ -1_{\cos(x) < 0} \cos(x) & \text{for } k = 2, \\ -1_{\sin(x) < 0} \sin(x) & \text{for } k = 3, \end{cases}$$

$$I_t^C(k) := \int_{-C}^{\infty} c(k, ux) N_{t \wedge \tau_n}(x) dx$$

yields

$$X_{t \wedge \tau_n}^C(u) = \sum_{k=0}^3 i^k I_t^C(k).$$

Since  $c(k, \cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$  and hence  $I^C(k)$  are positive, we can apply Tonelli's theorem and conclude that  $I^C(k)$  is a martingale up to the càdlàg property for all  $k \in \{0, 1, 2, 3\}$ . Thus  $(t, \omega) \mapsto X_{t \wedge \tau_n}^C(u)(\omega)$  is a martingale up to the càdlàg property as well. By the definitions of  $X^C$  and  $X$  we have  $X_t(u)(\omega) = \lim_{C \rightarrow \infty} X_t^C(u)(\omega)$  and thus we get

$$X_{t \wedge \tau_n}(u)(\omega) = \lim_{C \rightarrow \infty} X_{t \wedge \tau_n}^C(u)(\omega).$$

The fourth assumption on  $N$  and Lebesgue's theorem yield

$$\begin{aligned} E(X_{t \wedge \tau_n}(u) | \mathcal{F}_s) &= E\left(\lim_{C \rightarrow \infty} X_{t \wedge \tau_n}^C(u) | \mathcal{F}_s\right) \\ &= \lim_{C \rightarrow \infty} E(X_{t \wedge \tau_n}^C(u) | \mathcal{F}_s) \\ &= X_{s \wedge \tau_n}(u) \end{aligned}$$

for  $s \leq t$ . Thus  $(t, \omega) \rightarrow X_t(u)(\omega)$  is a local martingale.  $\square$



## 4.2. Essential infimum of a subordinator.

DEFINITION A.4.6. Let  $\psi$  be a deterministic local exponent (of some semimartingale  $X$ ). We define the *infimum process*  $E$  of  $\psi$  by  $E_t := \text{ess inf } X_t$  for any  $t \in \mathbb{R}_+$ .

By [25, III.2.16],  $X$  in the previous definition is a PII whose law is determined by  $\psi$ . Since  $E_t$  is in turn determined by the law of  $X_t$ , the infimum process  $E$  does not depend on the particular choice of  $X$ .

LEMMA A.4.7. *Let  $X, Y$  be independent random variables. Then*

$$\text{ess inf } (X + Y) = \text{ess inf } X + \text{ess inf } Y.$$

PROOF. Let  $x := \text{ess inf } X$  and  $y := \text{ess inf } Y$ . We obviously have  $\text{ess inf } (X + Y) \geq x + y$  because  $X + Y \geq x + y$  almost surely. Independence yields

$$\begin{aligned} P(X + Y \leq x + y + \varepsilon) &\geq P\left(X \leq x + \frac{\varepsilon}{2}, Y \leq y + \frac{\varepsilon}{2}\right) \\ &= P\left(X \leq x + \frac{\varepsilon}{2}\right) P\left(Y \leq y + \frac{\varepsilon}{2}\right) \\ &> 0 \end{aligned}$$

for any  $\varepsilon > 0$ . □

PROPOSITION A.4.8. *Let  $X$  be a subordinator (i.e. an increasing Lévy process) and  $E_t := \text{ess inf } X_t$  for any  $t \in \mathbb{R}_+$ . Then  $E_t = tE_1 \geq 0$  for any  $t \geq 0$  and  $E$  is the drift part of  $X$  relative to the truncation function  $h = 0$ . Moreover,  $X - E$  is a subordinator.*

PROOF. Since  $X$  is a subordinator, we have  $E_t \geq 0$  for any  $t \in \mathbb{R}_+$ . Moreover,  $\text{ess inf } (X_t - X_s) = E_{t-s}$  because  $X_{t-s}$  has the same distribution as  $X_t - X_s$ . Since  $X_s$  and  $X_t - X_s$  are independent, Lemma A.4.7 yields  $E_s + E_{t-s} = E_t$ . The mapping  $t \mapsto E_t$  is increasing because  $X$  is a subordinator. Together we conclude  $E_t = tE_1$ . This implies that  $X - E$  is a positive Lévy process and hence a subordinator. By [36, Theorem 21.5] the Lévy-Khintchine triplet  $(b, c, K)$  relative to truncation function  $h = 0$  exists and satisfies  $c = 0$ . Moreover,  $K$  and the random measure of jumps  $\mu^X$  of  $X$  are concentrated on  $\mathbb{R}_+$ . In view of [25, II.2.34] we have  $X_t = x * \mu_t^X + bt$  and thus we get  $E_t = \text{ess inf } X_t \geq bt$ . According to [36, Theorem 21.5],  $X - E$  is a subordinator only if its drift rate  $\tilde{b}$  relative to  $h = 0$  is greater or equal 0. Hence  $bt - E_t = \tilde{b}t \geq 0$ , which implies  $bt = E_t$ . □

COROLLARY A.4.9. *Let  $X$  be a  $d$ -dimensional semimartingale whose components  $X^k$  are subordinators with essential infimum  $E_t^k = \text{ess inf } X_t^k$  for  $k = 1, \dots, d$ . For predictable processes  $\varphi$  whose components are non-negative and bounded we have*

$$\text{ess inf } (\varphi \bullet (X - E)_t) = 0.$$

*Moreover, for bounded predictable  $\mathbb{C}^d$ -valued processes  $\varphi$  we have*

$$\text{ess inf } |\varphi \bullet (X - E)_t| = 0$$

*for any  $t \in \mathbb{R}_+$ .*

PROOF. The second statement is an application of the first statement because

$$0 \leq |\varphi \bullet (X - E)_t| \leq (|\varphi^1|, \dots, |\varphi^d|) \bullet (X - E)_t.$$

Suppose that  $\varphi$  is a predictable process with non-negative and bounded components. Proposition A.4.8 yields that  $X^k - E^k$  is a subordinator with essential infimum 0. Hence we may assume w.l.o.g. that  $E^k = 0$ . Since  $\varphi$  is bounded, there is a constant  $c \in \mathbb{R}_+$  such that  $\varphi^k \leq c$  for  $k = 1, \dots, d$ . Hence  $\varphi \bullet X_t \leq c \sum_{k=1}^d X_t^k$  for any  $t \in \mathbb{R}_+$ . By Proposition A.4.8 the drift part of  $X^k$  is 0 relative to the truncation function  $h = 0$ . Consequently, the drift part of the subordinator  $L := c \sum_{k=1}^d X^k$  is also 0 relative to the truncation function  $h = 0$ . Proposition A.4.8 yields  $\text{ess inf } L_t = 0$  for any  $t \in \mathbb{R}_+$ . Thus we conclude

$$0 \leq \text{ess inf } (\varphi \bullet X_t) \leq \text{ess inf } L_t = 0.$$

□

## Bibliography

1. A. Albert, *Regression and the Moore-Penrose pseudoinverse*, Academic Press, New York, 1972.
2. O. Barndorff-Nielsen and N. Shephard, *Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics*, Journal of the Royal Statistical Society, Series B **63** (2001), 167–241.
3. H. Bauer, *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*, third ed., de Gruyter, Berlin, 1978.
4. D. Belomestny and M. Reiß, *Spectral calibration of exponential Lévy models*, Finance & Stochastics **10** (2006), 449–474.
5. N. Bennani, *The forward loss model: a dynamic term structure approach for the pricing of portfolios of credit derivatives*, Technical report, 2005.
6. B. Böttcher, *A parametrix construction for the fundamental solution of the evolution equation associated with a pseudo-differential operator generating a markov process*, Mathematische Nachrichten **278** (2005), no. 11, 1235–1241.
7. B. Böttcher and A. Schnurr, *The euler scheme for feller processes*, arXiv (2010).
8. H. Bühler, *Consistent variance curve models*, Finance & Stochastics **10** (2006), 178–203.
9. R. Carmona, *HJM: A unified approach to dynamic models for fixed income, credit and equity markets*, Paris-Princeton Lectures in Mathematical Finance, 2005 (R. Carmona et al., ed.), Lecture Notes in Mathematics, vol. 1919, Springer, Berlin, 2009, pp. 3–45.
10. R. Carmona and S. Nadtochiy, *Local volatility dynamic models*, Finance & Stochastics **13** (2009), 1–48.
11. ———, *Tangent models as a mathematical framework for dynamic calibration.*, International Journal of Theoretical and Applied Finance **14** (2011), no. 1.
12. ———, *Tangent Lévy market models*, Finance & Stochastics **16** (2012), no. 1, 63–104, 10.1007/s00780-011-0158-8.
13. P. Carr, H. Geman, D. Madan, and M. Yor, *Stochastic volatility for Lévy processes*, Mathematical Finance **13** (2003), 345–382.
14. P. Carr and D. Madan, *Option valuation using the fast Fourier transform*, The Journal of Computational Finance **2** (1999), 61–73.
15. E. Cinlar, J. Jacod, P. Protter, and M.J. Sharpe, *Semimartingales and Markov processes*, Probability Theory and Related Fields **54** (1980), no. 2, 161–219.
16. R. Cont, V. Durrleman, and I. da Fonseca, *Stochastic models of implied volatility surfaces*, Economic Notes **31** (2002).
17. M. Davis and D. Hobson, *The range of traded option prices*, Mathematical Finance **17** (2007), 1–14.
18. S. Ethier and T. Kurtz, *Markov processes. characterization and convergence*, Wiley, New York, 1986.

19. D. Filipović, *Time-inhomogeneous affine processes*, Stochastic Processes and their Applications **115** (2005), 639–659.
20. L. Grafakos, *Classical fourier analysis*, second ed., Springer, New York, 2008.
21. D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation*, Econometrica **60** (1992), 77–105.
22. H. Heuser, *Lehrbuch der analysis - teil 1*, B.G. Teubner, Stuttgart, 1990.
23. N. Jacob and R.L. Schilling, *Lévy-type processes and pseudo-differential operators*, Lévy processes: theory and applications (2001), 139–167.
24. J. Jacod and P. Protter, *Risk neutral compatibility with option prices*, Finance & Stochastics (2006), To appear.
25. J. Jacod and A. Shiryaev, *Limit theorems for stochastic processes*, second ed., Springer, Berlin, 2003.
26. J. Kallsen,  *$\sigma$ -localization and  $\sigma$ -martingales*, Theory of Probability and Its Applications **48** (2004), 152–163.
27. ———, *A didactic note on affine stochastic volatility models*, From Stochastic Calculus to Mathematical Finance (Yu. Kabanov, R. Liptser, and J. Stoyanov, eds.), Springer, Berlin, 2006, pp. 343–368.
28. J. Kallsen and P. Krühner, *On a Heath-Jarrow-Morton approach for stock options*, Preprint, 2010.
29. J. Kallsen and A. Shiryaev, *The cumulant process and Esscher's change of measure*, Finance & Stochastics **6** (2002), 397–428.
30. S.G. Krantz, *Function theory of several complex variables*, American Mathematical Society, 2001.
31. E. Lukacs, *Characteristic Functions*, Griffin, London, 1970.
32. M. Musiela and M. Rutkowski, *Martingale methods in financial modelling*, Springer, Berlin Heidelberg, 1997.
33. P. Protter, *Stochastic integration and differential equations*, second ed., Springer, Berlin, 2004.
34. D. Revuz and M. Yor, *Continuous martingales and brownian motion*, third ed., Springer, Berlin, 1999.
35. L.C.G. Rogers and D. Williams, *Diffusions, Markov processes and martingales: Volume 2, Itô calculus*, Cambridge University Press, 2000.
36. K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.
37. P. Schönbucher, *Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives*, Technical report, 2005.
38. M. Schweizer and J. Wissel, *Arbitrage-free market models for option prices: The multi-strike case*, Finance & Stochastics **12** (2008), 469–505.
39. ———, *Term structures of implied volatilities: Absence of arbitrage and existence results*, Mathematical Finance **18** (2008), 77–114.
40. J. Sidenius, V. Piterbarg, and L. Andersen, *A new framework for dynamic credit portfolio loss modelling*, International Journal of Theoretical and Applied Finance **11** (2008), 163–197.
41. D. Stroock, *Diffusion processes associated with lévy generators*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **32** (1975), 209–244.

42. D. Stroock and S. Varadhan, *Diffusion processes with continuous coefficients I,II*, Comm. Pure Appl. Math. **22** (1969), 345–400,497–530.
43. David Williams, *Application of the dual-process method to the study of a certain singular diffusion*, Transactions of the American Mathematical Society **237** (1978), 101–110.
44. J. Wissel, *Arbitrage-free market models for liquid options*, Ph.D. thesis, ETH Zürich, 2008.

# Lebenslauf

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Geburtsdatum und -ort	02. Mai 1983, Eckernförde
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## Schulbildung

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08/1989 - 06/1993	Besuch der Hans-Christian-Andersen Grundschule in Kiel
08/1993 - 06/2002	Besuch des Hans-Geiger Gymnasium in Kiel
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## Hochschullaufbahn

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10/2003 - 11/2008	Studium der Mathematik auf Diplom an der Christian-Albrechts-Universität zu Kiel
04/2006	Vordiplom im Fach Mathematik
10/2005 - 09/2008	Studentische Hilfskraft am Mathematischen Seminar der CAU Kiel
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Kiel, den 06. Dezember 2011

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## **Eidesstattliche Erklärung**

Ich erkläre hiermit, dass die von mir vorgelegte Arbeit – abgesehen von der Beratung durch den Betreuer – in eigener Arbeit entstanden ist. Das erste Kapitel der Arbeit wurde zur Veröffentlichung eingereicht. Abgesehen davon ist die Arbeit weder bei einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt worden, noch ist sie veröffentlicht worden, noch zur Veröffentlichung eingereicht worden. Ich erkläre außerdem, dass die Arbeit unter Einhaltung guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft entstanden ist.

Kiel, den 06. Dezember 2011

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(Paul Krühner)