

RESTRICTION THEOREMS FOR THE FOURIER TRANSFORM

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Stefan Buschenhenke

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Erster Gutacher: Prof. Dr. Detlef Müller

Zweiter Gutachter: Prof. Dr. Hermann König

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Chapter 1

An introduction to Fourier restriction

1.1 The restriction operator

Let

$$\hat{g}(x) = \int_{\mathbb{R}^n} g(\xi) e^{-ix \cdot \xi} d\xi, \quad g \in L^1(\mathbb{R}^n), x \in \mathbb{R}^n, \quad (1.1.1)$$

denote the Fourier transform of an L^1 -function. For any smooth subsurface S of \mathbb{R}^n , consider the Fourier restriction operator

$$R(g) = R_S(g) = \hat{g}|_S. \quad (1.1.2)$$

Restriction of an L^p -function to a subset of \mathbb{R}^n with Lebesgue measure zero is à priori not well defined. However, the operator R is well defined, for instance, on the subset $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of Schwartz functions on \mathbb{R}^n . The Schwartz functions are not only dense in every $L^p(\mathbb{R}^n)$, $p < \infty$, the space \mathcal{S} is also mapped to itself under Fourier transform.

It is often useful to study the adjoint operator

$$R^* f(x) = \widehat{f d\sigma}(x) = \int_S f(\xi) e^{-ix \cdot \xi} d\sigma(\xi), \quad (1.1.3)$$

$x \in \mathbb{R}^n$, where σ is the surface measure of S . One would like to know for which range of $p, q \in [1, \infty]$, depending on the surface S , this operator R^* is bounded from $L^q(S, \sigma)$ to $L^p(\mathbb{R}^n)$. In other words: for which $p, q \in [1, \infty]$ does there exist a constant $C(p, q, S)$ such that for all $f \in L^q(S, \sigma)$ we have

$$\|\widehat{f d\sigma}\|_{L^p(\mathbb{R}^n)} \leq C(p, q, S) \|f\|_{L^q(S, \sigma)} ? \quad (1.1.4)$$

We shall concentrate on the case of compact hypersurfaces. Locally, a hypersurface is given as the graph of some function (at least after a permutation of coordinates); due to the compactness, we may decompose S in finitely many subsets given as a graph. From now on we will thus assume without loss of generality that S is a compact hypersurface, given as a graph.

In that case, we always have the trivial restriction estimate

$$\|\widehat{f d\sigma}\|_{\infty} \leq \|f\|_{L^1(S, \sigma)}. \quad (1.1.5)$$

Moreover, if $R^* : L^{q_0}(S) \rightarrow L^p(\mathbb{R}^n)$ is bounded for some p and q_0 , we observe that for any $q \geq q_0$ by Hölder's inequality we have

$$\|R^* f\|_p \leq C(q_0, p, S) \|f\|_{L^{q_0}(S, \sigma)} \leq C(q_0, p, S) \sigma(S)^{\frac{1}{q_0} - \frac{1}{q}} \|f\|_{L^q(S, \sigma)} = C(q', p, S) \|f\|_{L^q(S, \sigma)}, \quad (1.1.6)$$

i.e. $R^* : L^q(S) \rightarrow L^p(\mathbb{R}^n)$ is bounded as well. Thus every point $(\frac{1}{q}, \frac{1}{p})$ where such an estimate holds provides us with a certain area where it is valid as well (cf. Figure 1.1).

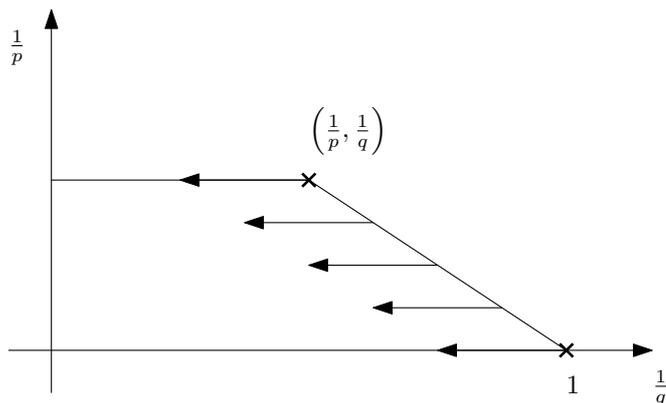


Figure 1.1: Schematic picture of the area of p and q for an adjoint restriction operator

1.2 First results

A very fundamental achievement concerning the above question was the result of Stein and Tomas for the unit sphere S^{n-1} . They proved that $R : L^{p'}(\mathbb{R}^n) \rightarrow L^2(S^{n-1})$ is bounded if and only if $1 \leq p' \leq \frac{2n+2}{n+3}$ [St]. The proof uses estimates on oscillatory integrals, which in turn rely on the non vanishing Gaussian curvature of the sphere. Therefore the proof remains valid for every compact hypersurface with non vanishing Gaussian curvature. Another important example is any compact part of the paraboloid. Greenleaf gave a further generalisation for surfaces satisfying some uniform decay estimates, with the range of p depending on the degree of decay [Gre]. Thus the "analytic" question of Fourier restriction has a "geometric" answer in terms of the Gaussian curvature. This philosophy is also reflected in the fact that for a "flat" hypersurface, which has vanishing curvature everywhere, all restriction estimates beside the trivial ones fail. Indeed the geometry of the surface plays a very important role not only in the Stein-Tomas result but in the whole theory.

Interpolation between the Stein-Tomas Theorem and trivial results immediately gives even some (p', q') -results, but not the sharp range, which is generally believed to be the following:

CONJECTURE

Let $S \subset \mathbb{R}^n$ be a compact hypersurface with positive principle curvatures, and let $p > \frac{2n}{n-1}$ and $\frac{1}{q'} \geq \frac{n+1}{(n-1)p}$. Then $R^* : L^q(S) \rightarrow L^p(\mathbb{R}^n)$ is bounded.

The conditions on p and q are necessary. The first one can be obtained by a careful analysis of the oscillatory integral given by $\widehat{d\sigma}$, the second one is due to a so-called Knapp-Box-example. That means testing the adjoint restriction operator on a small box, adapted to the geometric shape of the surface. In dimension $n = 2$, the question whether the conditions on p and q are sufficient as well was answered positively by Zygmund in 1974 [Z]. Observe that for $n = 2$, we have $\frac{2n}{n-1} = 4$ and Plancherel's formula implies

$$\|\widehat{f d\sigma}\|_4^2 = \|\widehat{f d\sigma} \widehat{f d\sigma}\|_2 = \|f d\sigma * f d\sigma\|_2. \quad (1.2.1)$$

The general case $n \neq 2$ does not come with such an advantage and requires further, more complicated techniques. Despite the work of many outstanding mathematicians in this field, including Thomas Wolff, and Fields medalists Jean Bourgain and Terence Tao, and a lot of progress over the recent decades, the complete answer still remains open.

1.3 The bilinear method

For a long period of time, the state-of-the-art technique for this problem was the so-called bilinear method. One studies the bilinear operator $(f_1, f_2) \rightarrow R_{S_1}^*(f_1)R_{S_2}^*(f_2)$, where the subsurfaces $S_1, S_2 \subset S$ fulfill some separation assumption, while their normal fields obey a certain "transversality" condition. The goal is to establish a local bilinear estimate of the form

$$\|R_{S_1}^*(f_1)R_{S_2}^*(f_2)\|_{L^p(Q(R))} \leq C_\varepsilon R^\varepsilon \|f_1\|_2 \|f_2\|_2, \quad (1.3.1)$$

for all $\varepsilon > 0$, where R is a large parameter and $Q(R)$ for instance a cube of side length R . If R grows to infinity, the left-hand-side converges to the integral over the whole space, but the right-hand-side would blow up to infinity. However, there is some " ε -removal" technique available, meaning that one has to pay in the range of p for lowering ε . Additional work is required to deduce a linear estimate.

One ingredient of the local bilinear estimate is a method called "induction on scales" invented by Wolff [W]. Starting with a large power of R obtained by a trivial estimate, ε is reduced step by step by an inductive argument: very roughly speaking, on $Q(R^{1-\delta})$, R comes with the power $\alpha' = (1-\delta)\alpha < \alpha$ in the bilinear estimate due to the induction hypothesis. However, the challenge is to deal with the remaining part $Q(R) \setminus Q(R^{1-\delta})$. This involves a kind of discretisation, a decomposition into "wave packets", which are functions well localised both in position and momentum space. If one interprets the last coordinate direction as time variable, the closest analogon to a wave packet from physics is a particle of a laser beam: a light ray of highly concentrated frequencies, stable over a long period of time. The wave packets were developed essentially by Bourgain [Bo1], although some rudimentary versions were used by Fefferman [F] and A. Córdoba [C]. A detailed application of the whole method will be carried out in Chapter 3 of my thesis. Unfortunately, even a sharp bilinear estimate implies some, but in general not the full linear estimate.

Other related surfaces were also studied, using and developing the described techniques, for instance the truncated cone $\{(x, t) \in \mathbb{R}^n \times [1, 2] : t = |x|\}$. On the one hand, for a fixed t , the corresponding section of the cone is a dilated sphere S^{n-1} . On the other hand, for a fixed x , variation of t gives a line segment without any curvature. Therefore the restriction conjecture for the n -dimensional cone is the same as for the $(n-1)$ -dimensional sphere. If one takes a cylinder instead of a cone, this behaviour is even more obvious. Wolff proved the sharp bilinear estimate and the consequent linear estimate for the cone [W].

Lee and Vargas independently studied an example for a surface where the principle curvatures have different signs, namely the graph of the function $\phi(x, y) = x^2 - y^2$, see [L1]

and [V]. Their approach does not cover perturbation terms since it uses the change of coordinates $x' = x + y, y' = x - y$, i.e. the surface transforms to the graph of $x' \cdot y'$. Later, they joined in working on surfaces with $n - k$ non vanishing, positive principle curvatures [LV], generalising Wolff's work on the cone, which has $n - 1$ non vanishing principle curvatures.

New progress in this area was recently performed by Bourgain and Guth 2011 [BG], further extending the range of p and q for the paraboloid and the sphere closer to the conjectured one. They make use of a trilinear approach instead of a bilinear one.

1.4 Surfaces of finite type

There also exists some intermediate situation between the case of non vanishing principle curvature and the "flat" case of identically vanishing curvature: What about a surface, where the principle curvature vanishes at some point, but not identically? An example is the graph S of the function $\phi(x) = x^m, x \in [0, 1], m \geq 2$. For $m = 2$ this is a part of the parabola we already discussed, but for $m > 2$ the second derivative of ϕ and therefore the Gaussian curvature vanishes at the origin. Therefore one would expect a smaller range than $\frac{1}{q'} \geq \frac{3}{p}, p > 4$ such as for the parabola. However, the surface is not completely "flat" at the origin in the sense that not all derivatives are vanishing at the origin (in this example, the m -th derivative does not vanish); we say that ϕ is of *finite type* m at 0. Since the origin is the only point with vanishing curvature, a first approach would be a dyadic decomposition $(0, 1] = \bigcup_{j \in \mathbb{N}} [2^{-j-1}, 2^{-j}]$. On each piece, the curvature does not vanish, but becomes small for $j \rightarrow \infty$. A certain rescaling allows to apply the already mentioned results for curves where the curvature is comparable to 1. The different contributions turn out to sum to a restriction estimate

$$\|\widehat{f d\sigma}\|_{L^p(\mathbb{R}^2)} \leq C(p, q, S) \|f\|_{L^q(S)} \quad (1.4.1)$$

provided that $\frac{1}{q'} > \frac{m+1}{p}$. Another Knapp-Box-example shows that at least the condition $\frac{1}{q'} \geq \frac{m+1}{p}$ is *necessary*, thus our somewhat simple approach is not far from the optimal result, which is as follows: $R^* : L^q(\{(x, x^m) : x \in [0, 1]\}) \rightarrow L^p(\mathbb{R}^2)$ is bounded if and only if $\frac{1}{q'} \geq \frac{m+1}{p}, p > 4$ [ACK].

1.5 A brief description of the thesis

Whereas the setting of curves is quite well understood (see for instance [ACK] or [Ba2]), less is known about surfaces with principle curvatures vanishing at a certain point, which will be the objects of interest in my thesis.

One of these surfaces is the 2-dimensional truncated cone in \mathbb{R}^3 , but with the underlying curve, the circle, replaced by a curve of finite type like $\phi(x) = x^m$. Like in the situation of the classical cone, the behaviour should be determined by this curve, and the conjectured range for p and q is the same. This is reflected in the proof as well, which makes use of adapting several classical techniques for curves instead of the described bilinear method. After obtaining a sharp solution of this problem, I learned that there already exists a

partial result by Barcelo [Ba2]. More details, as well as the complete proof, can be found in Chapter 2 of my thesis.

A very natural question is to ask whether the estimate can be extended to higher dimensions, for instance if one replaces the underlying curve by the surface $S = \{(x, \phi(x)) : x \in U\}$, where U is some compact neighbourhood of the origin in \mathbb{R}^2 and for instance $\phi(x_1, x_2) = x_1^{m_1} + x_2^{m_2}$. It is important however to first understand the restriction theory of S . Some results are known: The L^p - L^2 -restriction was handled even in a far more general setting by Ikromov, Kempe and Müller [IKM]. Similar to the Stein-Tomas argument, they estimate the corresponding oscillatory integral. Another partial result is due to Ferreyra and Urciuolo [FU]. After a certain decomposition and an appropriate rescaling, they exploit curvature in only one direction, therefore being able to apply previous results for curves. The disadvantage is that the results for curves require $p > 4$. For comparison, take $m_1 = 2 = m_2$, meaning S is a compact part of the 2-dimensional paraboloid. The conjectured range then is $p > 3$, and the best known result is $p > \frac{10}{3}$ [T1].

Part II of my thesis will therefore cover a bilinear approach to the problem, developed in joint work with Prof. Dr. Detlef Müller and Prof. Dr. Ana Vargas. It includes all the techniques mentioned above in some modified version; an induction on scales argument, wave packets adapted to the local curvature and a reformulation of the " ε -removal" lemma. We obtain an essentially sharp bilinear estimate and also the subsequent linear estimate.

It is remarkable that the approaches from Chapter 2 and Chapter 3, although they use very different methods, at the end both reduce to very concrete geometric questions. This indicates once more that the question of Fourier restriction is dictated by the geometry of the specific surface.

1.6 Connections with other topics in analysis

A related field are Bochner-Riesz-multipliers: for any $\delta > 0$, the Bochner-Riesz-multiplier is defined as $m^\delta(\xi) = (1 - |\xi|^2)_+^\delta$, $\xi \in \mathbb{R}^n$, where f_+ is the positive part of a function f . The support of m^δ is the set $|\xi| \leq 1$, i.e. the unit ball, and m^δ is smooth at any point outside the sphere S^{n-1} . Of course one can construct similar examples for other surfaces, but for simplicity we shall discuss this example.

The Bochner-Riesz-operator S^δ is the operator associated to the Fourier multiplier m^δ , i.e. $\widehat{S^\delta f}(\xi) = m^\delta(\xi)\hat{f}(\xi)$. Due to fundamental work by Fefferman, we know that S^0 is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $p = 2$ [F]. For $\delta > 0$, the conjecture states that S^δ is bounded if and only if

$$n \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2} < \delta. \quad (1.6.1)$$

As for the restriction problem, the condition is known to be necessary, but for the inverse implication, only partial results are known.

Fefferman also discovered a close connection between Bochner-Riesz-multipliers and Fourier restriction for the sphere. If for some p the restriction estimate for the sphere $R_{S^{n-1}} : L^{p'}(\mathbb{R}^n) \rightarrow L^2(S^{n-1})$ holds true, then the Bochner-Riesz conjecture holds true for that p .

Apart from that link to $L^{p'}-L^2$ restriction theory, Lee established a similar connection to the already mentioned bilinear $L^2(S_1) \times L^2(S_2) \rightarrow L^{2p}(\mathbb{R}^n)$ adjoint restriction estimates, which in turn are connected to linear $L^q(S) \rightarrow L^p(\mathbb{R}^n)$ estimates [L2].

Fourier restriction theory is also connected to the Kakeya needle problem. The question formulated by Kakeya was: How large does an area have to be to be able to rotate a "needle" (more formally, a line segment) of length one? Surprisingly, Besikovitch discovered that one can construct arbitrary small areas with this property. The proof uses the construction of null sets in \mathbb{R}^2 which at least contain a "needle" (or line segment of length one) in every direction. These so-called Besikovitch sets can also be studied in higher dimensions. A kind of discrete version of this is to take "needles" of length one and thickness δ , where the set of directions is a δ -grid in S^{n-1} . The problem is then to give a lower bound in terms of δ for the size of the set formed by all needles.

This somehow resembles the wave packet decomposition, since the wave packets are essentially supported in "needles" of certain thickness and different directions. A crucial point is then to understand the interaction and overlap. A more thorough description of the connection can be found in [T3].

There also exist relations to further problems, for instance the so-called local smoothing problem or Strichartz estimates in partial differential equations, but we will not go into details here and refer to the survey article by Tao [T2].

Chapter 2

A sharp Fourier restriction theorem for
a conical surface of finite type

2.1 Introduction and first considerations

2.1.1 Introduction

Let S be a compact hypersurface in \mathbb{R}^n (or more general a smooth submanifold) with surface measure σ . We say that the $L^p(\mathbb{R}^n)$ - $L^q(S)$ *Restriction Theorem* holds if there exists a constant $C > 0$ such that

$$\left(\int_S |\hat{f}(\xi)|^q d\sigma(\xi) \right)^{\frac{1}{q}} \leq C \cdot \|f\|_p \quad (2.1.1)$$

holds for every Schwartz function f . The object of my investigations is a surface of so-called finite type, where the tangent plane has finite order of contact. This means - describing the surface locally as a graph of a function - that the second derivative of the function with respect to some direction vanishes at some points (and therefore the curvature as well). One just demands not all derivatives to vanish. A simple example is the curve x^m , $m \geq 3$. However, the restriction estimates for curves are known since 1979 [ACK]. But we can construct a cone-like surface from the curve, like the "classical" cone arises from a circle. A sharp restriction theorem for the "classical" cone in \mathbb{R}^3 was proven by Barcelo [Ba1], remarkably without bilinear estimates. The reason for this is that the cone somehow behaves like a curve, and in the one-dimensional situation, you may use other methods.

Just when I finished my work, it came to my knowledge that Barcelo himself already obtained a partial result for the same problem in another paper [Ba2], which I was not aware of. The basic structure of both Barcelo's and mine approach might appear similar at the end, since both strategies rely on Barcelo's first paper [Ba1]. Nevertheless, my work was independently of [Ba2], and the details and techniques are quite different. This can be seen clearest in the different choice of decompositions.

Another point is that Barcelo always considered the full cone (equipped with a modified measure to compensate scaling) and not just a compact part. But, at least away from a sharp line, both approaches are equivalent. Furthermore, as the "conical direction" gives neither curvature nor a finite type condition, one would expect no impact from here, and my notation allows to verify this. Besides obtaining a similar result independently, the most striking point certainly is that unlike Barcelo, I was able to get the sharp range of p and q .

There is a technical obstacle, preventing Barcelo from obtaining the optimal result. When I tried to adopt Barcelo's arguments, I got into trouble with the same obstacle quickly. But eventually, I discovered a way to outmaneuver it. The "correct" formulation of the problem I found, and which I could prove, can be seen in terms of the affine arclength measure, which is constructed in order to compensate the decreasing curvature. Such variations of restriction theorems can be found in Drury and Marshall [DM], who also considered curves of finite type. This is of course a different problem. But like in [DM], we can derive the original restriction estimate from the estimate with affine arclength measure, with a different range of p and q .

The development of my proof might be of interest for the reader in so far, as I first developed the basic ideas for curves and secondly adopted it for the generalised cone. Hence,

the ideas and pictures will often be just in x - y -space.

Following the exact formulation of the problem and reduction to the essential core of it, in the second part the mentioned step with the affine arclength measure will be explained. A further reduction will be achieved by a certain approximation, simultaneously both of the function and of the surface measure, like in [Ba1]. The precise arguments are given in the following section. For this approximation we decompose the surface into certain pieces, on each of which the curvature will be almost constant. Furthermore, we need to estimate the overlap of the sets we obtain by taking the sum of two sets of the original decomposition. In section 5, we want to investigate on the convolution of the characteristic functions (or smooth versions) of two sets of our decomposition.

Well prepared by these instruments, we are ready to conclude the proof. The last section contains some technical details and auxiliary lemmas.

2.1.2 The main theorem and necessary conditions

Let γ be a compact curve of finite type, i.e. for all $p \in \gamma$ exists a local parametrisation Φ , $\Phi(x) = p$, and $m = m_p \in \mathbb{N}_{\geq 2}$ such that Φ is m -times differentiable at x and $\Phi^{(m)}(x) \neq 0$. By compactness, we may assume that there exist a universal M such that we always can find such a m_p bounded by M . Moreover, we choose M minimal with this property.

THEOREM 2.1.1

Let $1 \leq p, q, r \leq \infty$. We consider the generalised truncated cone

$\Gamma = \{(\xi, z) \in \mathbb{R}^2 \times \mathbb{R} \mid 1 \leq z \leq 2, \frac{\xi}{z} \in \gamma\}$ with surface measure σ . Then the Fourier restriction theorem

$$\|\hat{f}|_{\Gamma}\|_{L^{q,r}(\Gamma,\sigma)} \leq C \|f\|_{L^{p,r}(\mathbb{R}^3)} \quad \forall f \in \mathcal{S}(\mathbb{R}^3), \quad (2.1.2)$$

holds if $1 \leq p < \frac{M+1}{M}$ and $\frac{1}{q} \geq \frac{M+1}{p'}$.

If furthermore $p \leq q$ or $\frac{1}{q} > \frac{M+1}{p'}$, then we have

$$\|\hat{f}|_{\Gamma}\|_{L^q(\Gamma,\sigma)} \leq C \|f\|_{L^p(\mathbb{R}^3)} \quad \forall f \in \mathcal{S}(\mathbb{R}^3). \quad (2.1.3)$$

The theorem is sharp in the sense that there exists surfaces where (2.1.2) is not valid if the conditions on p' and q are violated. For curves of finite type, it is known that the strong $L^p \rightarrow L^q$ -estimate fails for $p > q$ or $\frac{1}{q} = \frac{M+1}{p'}$ [S]. Notice that (2.1.2) implies (2.1.3). For $p \leq q$, this is a consequence of the Marcinkiewicz interpolation theorem, whereas for $\frac{1}{q} > \frac{M+1}{p'}$, we use the fact that for $q < \tilde{q}$ we have $\|\hat{f}|_{\Gamma}\|_{L^{q,1}(\Gamma,\sigma)} \leq \|\hat{f}|_{\Gamma}\|_{L^{\tilde{q},\infty}(\Gamma,\sigma)}$ since $\sigma(\Gamma) < \infty$.

First we deal with the necessity of the conditions. Instead of doing so in full generality, to convince us of this necessity, we will discuss an example in \mathbb{R}^2 , and for simplicity omit the cone. This might be more helpful in order to just understand the idea behind it. We consider the curve $\gamma = \{(x, x^m) \mid x \in [0, 1]\}$ in \mathbb{R}^2 : Assume the restriction estimate

$$\|\hat{f}|_{\gamma}\|_{L^q(\gamma,\sigma)} \leq C \|f\|_{L^p(\mathbb{R}^2)} \quad \forall f \in \mathcal{S}(\mathbb{R}^2) \quad (2.1.4)$$

holds for some $q > \frac{p'}{m+1}$. The following is a variation of the classical Knapp box example. We define the anisotropic dilation of a function $f \in \mathcal{S}$ with respect to $r > 0$ by $f_r(x, y) =$

$f(rx, r^m y)$, such that $\hat{f}_r(\xi, \eta) = r^{-(m+1)} \hat{f}\left(\frac{\xi}{r}, \frac{\eta}{r^m}\right)$. We apply (2.1.4) for f_r , $r \leq 1$: on the left-hand side we get

$$\begin{aligned} \|\hat{f}_r|_\gamma\|_{L^q(\gamma, \sigma)} &= \left(\int_0^1 |r^{-(m+1)} \hat{f}\left(\frac{\xi}{r}, \frac{\xi^m}{r^m}\right)|^q d\xi \right)^{\frac{1}{q}} \\ &= r^{-(m+1)} \left(\int_0^{\frac{1}{r}} |\hat{f}(\xi, \xi^m)|^q r d\xi \right)^{\frac{1}{q}} \\ &\stackrel{r \leq 1}{\geq} r^{\frac{1}{q} - (m+1)} \left(\int_0^1 |\hat{f}(\xi, \xi^m)|^q d\xi \right)^{\frac{1}{q}} \\ &= r^{\frac{1}{q} - (m+1)} \|\hat{f}\|_{L^q(\gamma, \sigma)}, \end{aligned}$$

and therefore

$$r^{\frac{1}{q} - (m+1)} \|\hat{f}\|_{L^q(\gamma, \sigma)} \leq \|\hat{f}_r\|_{L^q(\gamma, \sigma)} \leq C \|f_r\|_p = C r^{-\frac{1+m}{p}} \|f\|_p.$$

We conclude that

$$r^{\frac{1}{q} - (m+1) + \frac{1+m}{p}} \|\hat{f}\|_{L^q(\gamma, \sigma)} \leq C \|f\|_p,$$

where the exponent of r is negative: $\frac{1}{q} - (m+1) + \frac{1+m}{p} = \frac{1}{q} - (m+1) \left(1 - \frac{1}{p}\right) = \frac{1}{q} - \frac{m+1}{p'} < 0$. We get a contradiction by choosing r small enough, and therefore $\frac{1}{q} \geq \frac{m+1}{p'}$. Since $q \geq 1$, we may already conclude that $p' \geq m+1$. That the case of equality is excluded can be proven by results of the theory of oscillatory integrals:

Again consider $\gamma = \{(x, x^m) | x \in [0, 1]\}$ and assume that (2.1.4) and respectively

$$\|\widehat{f d\sigma}\|_{L^{p'}(\mathbb{R}^2)} \lesssim \|f\|_{L^{q'}(\gamma)} \tag{2.1.5}$$

holds for $p' = m+1$. A careful study of the proof of proposition 3, chapter VIII, §1 in [St] shows that

$$\left| \int_0^1 e^{-i(ax - bx^m)} dx \right| \gtrsim b^{-\frac{1}{m}} \tag{2.1.6}$$

for $1 \leq b < \infty$, $1 \ll a^m \ll b$. We obtain for $f \equiv 1$

$$\begin{aligned} \|\widehat{f d\sigma}\|_{p'}^{p'} &\gtrsim \int_1^\infty b^{-\frac{p'}{m}} \int_{1 \ll a^m \ll b} da db \\ &\approx \int_1^\infty b^{\frac{1-p'}{m}} db \\ &= \infty \end{aligned}$$

for $p' = m+1$. However, even in the case $p' = m+1$ a weak-type estimate might still hold. This is beyond the methods of this paper.

2.1.3 Reduction

Let γ be a compact curve of finite type and M the maximal order of contact like above. Obviously, it suffices to decompose γ into finitely many parts and prove the theorem for each part. At first, we cut the curve into finitely many parts, where the curve (after a permutation of coordinates, if necessary) can be described as the graph of some function $\Phi : [a, b] \rightarrow \mathbb{R}$. Then we decompose each such part again: For every $x_0 \in [a, b]$ with $\Phi''(x_0) = 0$, we will prove the restriction theorem for $\Phi|_{[x_0-\varepsilon, x_0+\varepsilon]}$ for some $\varepsilon > 0$ (depending on x_0). The compact set $\{\Phi'' = 0\}$ is covered by finitely many of these intervals. But outside an arbitrary open neighbourhood of the set $\{\Phi'' = 0\}$, the curvature is bounded from above and from below, and in this case results are known [Ba1]. So, how to determine ε ?

Given $\Phi''(x_0) = 0$, choose $m \geq 2$ with $\Phi^{(m)}(x_0) \neq 0$ minimal, thus $m \geq 3$ and $\Phi''(x_0) = \dots = \Phi^{(m-1)}(x_0) = 0$. We claim that we may assume without loss of generality

$$x_0 = 0 \quad \text{and} \quad \Phi(0) = \Phi'(0) = 0. \quad (2.1.7)$$

For this purpose we introduce the affine transformation

$$\tilde{\Phi}(x - x_0) = \Phi(x) - \Phi(x_0) - (x - x_0)\Phi'(x_0), \quad (2.1.8)$$

which provides $\tilde{\Phi}(0) = 0 = \tilde{\Phi}'(0)$, whereas $\tilde{\Phi}^{(k)}(x - x_0) = \Phi^{(k)}(x)$ for $k \geq 2$ (and therefore the finite type condition is preserved).

If we define $\gamma_0 = \{(x, \Phi(x)) | x \in [x_0 - \varepsilon, x_0 + \varepsilon]\}$ and $\Gamma_0 = \{(\xi, z) \in \mathbb{R}^2 \times \mathbb{R} | 1 \leq z \leq 2, \frac{\xi}{z} \in \gamma_0\}$, and correspondingly $\tilde{\gamma}_0 = \{(x, \tilde{\Phi}(x)) | x \in [-\varepsilon, \varepsilon]\}$ and $\tilde{\Gamma}_0$, then we compute

$$\tilde{\Gamma}_0 = A\Gamma_0$$

with the volume preserving transformation

$$A = \begin{pmatrix} 1 & 0 & -x_0 \\ -\Phi'(x_0) & 1 & x_0\Phi'(x_0) - \Phi(x_0) \\ 0 & 0 & 1 \end{pmatrix}.$$

This justifies (2.1.7).

Additionally, we split $\Phi|_{(-\varepsilon, \varepsilon)}$ into $\Phi|_{(-\varepsilon, 0)}$ and $\Phi|_{(0, \varepsilon)}$. On $(-\varepsilon, 0)$, we switch to the function $\Phi^- : (0, \varepsilon) \rightarrow \mathbb{R}$, $\Phi^-(x) = \Phi(-x)$, and hence, we need just to consider parts like $\Phi|_{(0, \varepsilon)}$. Eventually, without loss of generality, we may assume that $\Phi^{(m)}(0) > 0$. To summarize, we have

$$\Phi^{(m)}(0) > 0 \quad \text{and} \quad \Phi(0) = \Phi'(0) = \dots = \Phi^{(m-1)}(0) = 0. \quad (2.1.9)$$

By choosing ε small enough we may ensure that $\Phi^{(m)}|_{[0, \varepsilon]} > 0$. The next step is to establish the following:

$$\forall k \leq m \quad \forall 0 < x \leq \varepsilon : \Phi^{(k)}(x) > 0. \quad (2.1.10)$$

If this does not hold, we would find $x_k > 0$ such that $\Phi^{(k)}(x_k) \leq 0$. Since $\Phi^{(k)}(0) = 0$, the mean value theorem would provide a $x_{k+1} \in (0, x_k)$ such that $\Phi^{(k+1)}(x_{k+1}) \leq 0$, and by

induction eventually a $x_m > 0$ such that $\Phi^{(m)}(x_m) \leq 0$ in contradiction to our assumption. Furthermore $\Phi(x) \approx x^m$. To be more precise, Taylor expansion provides

$$\frac{1}{m!} \min_{t \in [0, \varepsilon]} \Phi^{(m)}(t) x^m \leq \Phi(x) \leq \frac{1}{m!} \max_{t \in [0, \varepsilon]} \Phi^{(m)}(t) x^m,$$

where the minimum on the left-hand-side does not vanish, since (2.1.9) and (2.1.10) imply $\Phi^{(m)}(t) > 0$ for all $t \in [0, \varepsilon]$. In other words, we find a $\chi \in C^2([0, \varepsilon], (0, \infty))$ fulfilling $\Phi(x) = x^m \chi(x)$. This implies

$$\Phi'(x) = mx^{m-1}\chi(x) + x^m\chi'(x) = x^{m-1}[m\chi(x) + x\chi'(x)].$$

By reducing ε if necessary, we may assume $\chi^1 := m\chi(x) + x\chi'(x) \in C^1([0, \varepsilon], (0, \infty))$. In the same manner, we find $\chi^2 \in C([0, \varepsilon], (0, \infty))$ fulfilling

$$\Phi''(x) = x^{m-2}\chi^2(x).$$

By further reduction of ε , we may also find an open neighbourhood of the origin, where Φ has the desired properties. Eventually, we may rescale to $\varepsilon = 1$.

2.2 Reduction of the problem

Altogether, it remains to discuss the following problem:

DEFINITION

Let $m \in \{3, 4, \dots\}$ and let $\Phi : [0, 1] \rightarrow \mathbb{R}$ satisfy the following conditions:

- (i) $\exists \chi \in C^2[0, 1], \chi > 0 : \Phi(x) = x^m \chi(x),$
 $\exists \chi^1 \in C^1[0, 1], \chi^1 > 0 : \Phi'(x) = x^{m-1} \chi^1(x),$
 $\exists \chi^2 \in C[0, 1], \chi^2 > 0 : \Phi''(x) = x^{m-2} \chi^2(x),$

(ii) $\Phi^{(k)}(x) > 0$ for $x > 0, k \leq m$; especially, Φ and Φ' are convex.

We define the generalised cone (more exactly, a section of a cone)

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 1 \leq z \leq 2, \Phi\left(\frac{x}{z}\right) = \frac{y}{z}\}.$$

The associated surface measure will be denoted by σ .

THEOREM 2.2.1

For $1 \leq p < \frac{m+1}{m}$ and $\frac{1}{q} \geq \frac{m+1}{p'}$ holds

$$\|\hat{f}|_{\Gamma}\|_{L^q(\Gamma, \sigma)} \leq C \|f\|_{L^p(\mathbb{R}^3)} \quad \forall f \in \mathcal{S}(\mathbb{R}^3). \quad (2.2.1)$$

2.2.1 Decomposition

The critical part of Γ is the line $x = 0$, where the curvature vanishes. To take this fact into account, we decompose Γ into dyadic pieces, becoming smaller near the line $x = 0$. By rescaling to the case of (almost) constant curvature, we would be able to make use of already known estimates. Unfortunately, summation is only possible if $\frac{1}{q} > \frac{m+1}{p'}$. In the limit case $\frac{1}{q} = \frac{m+1}{p'}$, we need to deal with the problem as a whole. For this, we will further decompose each dyadic piece, depending on the curvature.

Let $\Gamma^\delta = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x < 1, 1 \leq z \leq 2, \Phi\left(\frac{x}{z}\right) \leq \frac{y}{z} \leq \Phi\left(\frac{x}{z}\right) + \delta\}$ be the thickening of Γ by $\delta > 0$ (we drop the points with $x = 1$ for technical purposes). Moving on the x -axis from the origin by length $\sqrt[m]{\delta}$ corresponds to Φ changing by δ . In other words, this part of Γ is contained in a box of width δ . We thus define $\gamma = \sqrt[m]{\delta}$ and

$$\Gamma_k^\delta = \{(x, y, z) \in \Gamma^\delta \mid (2^k-1)\gamma \leq x < (2^{k+1}-1)\gamma\}, \quad k = 0, \dots, \left\lceil \log_2 \frac{1}{\gamma} \right\rceil - 1.$$

Now how to determine the finer decomposition? We change coordinates, or respectively Φ by affine transformation into

$$\Phi^k(x) = \Phi(x + (2^k-1)\gamma) - \Phi((2^k-1)\gamma) - x\Phi'((2^k-1)\gamma),$$

such that

$$\Phi^k(0) = 0 = (\Phi^k)'(0).$$

According to Taylor, we get

$$\Phi^k(x) \approx \frac{m(m-1)}{2} ((2^k-1)\gamma)^{m-2} x^2 + \mathcal{O}(x^3).$$

On which distance γ_k from the (new) origin does Φ varies at most δ ? We demand ($k \neq 0$)

$$\delta = \Phi^k(\gamma_k) \approx \frac{m(m-1)}{2} ((2^k-1)\gamma)^{m-2} \gamma_k^2,$$

i.e. $\gamma_k \approx \sqrt{\delta((2^k-1)\gamma)^{2-m}} \approx 2^{k(1-\frac{m}{2})}\gamma.$

Concerning the z -coordinate, we decompose equidistantly with width β , where we require $\beta \lesssim \delta$. This ensures that the projection of such a set in x - y -space does not appear to different to a intersection parallel to x - y -space. To choose $\beta = \delta$ would be appropriate and you might assume this. Nevertheless, we will distinct these two quantities to be aware how each of them effects our computations. We will see that all the β 's cancels at the end of the proof, reflecting the fact that there is no impact from the z -direction. We obtain a decomposition of Γ^δ as follows:

DEFINITION (DECOMPOSITION)

Let $\delta > 0$, $\gamma > 0$ and $\gamma^m = \delta$, let $\beta < \delta$ with $\frac{1}{\beta} \in \mathbb{N}$. For $k = 0, \dots, \lceil \log_2 \frac{1}{\gamma} \rceil - 1$, $j \in I_k = \{0, \dots, 2^{k\frac{m}{2}} - 1\}$ and $n = \frac{1}{\beta}, \dots, \frac{2}{\beta} - 1$ define $\gamma_k = 2^{k(1-\frac{m}{2})}\gamma$, $x_{kj} = (2^k-1)\gamma + j\gamma_k$, $x_k = x_{k,0}$ and

$$\Gamma_{kjn}^\delta = \{(x, y, z) \in \mathbb{R}^3 \mid n\beta \leq z \leq (n+1)\beta, \Phi\left(\frac{x}{z}\right) \leq \frac{y}{z} \leq \Phi\left(\frac{x}{z}\right) + \delta, x_{kj} \leq x < x_{k,j+1}\}. \quad (2.2.2)$$

Furthermore let ϕ_{kjn} be a bump function adapted to Γ_{kjn} . To be more precise, if $\eta \in C_0^\infty(\mathbb{R})$, $\chi_{[-1,1]} \leq \eta \leq \chi_{[-\frac{5}{4}, \frac{5}{4}]}$, let

$$\phi_{kjn}(x, y, z) = \eta\left(\frac{x - x_{kj}}{\gamma_k}\right) \eta\left(\frac{y - z\Phi(x/z)}{\delta}\right) \eta\left(\frac{z - n\beta}{\beta}\right).$$

This means that ϕ_{kjn} is to some extend supported in an anisotropic thickening of Γ_{kjn}^δ , precisely in the set

$$\begin{aligned} \tilde{\Gamma}_{kjn}^\delta &= \{(x, y, z) \in \mathbb{R}^3 \mid |z - n\beta| \leq \frac{5}{4}\beta, \left|\frac{y}{z} - \Phi\left(\frac{x}{z}\right)\right| \leq \frac{5}{4}\delta, |x - x_{kj}| \leq \frac{5}{4}\gamma_k\} \\ &\subset \bigcup_{u,v,w \in \{-1,0,1\}} (u\gamma_k, v\delta, w\beta) + \Gamma_{kjn}^\delta. \end{aligned}$$

For the further proceeding, we will always denote by α the triple (k, j, n) , and, if required, by μ a second triple (l, i, p) .

For simplification, we write \sum_α instead of $\sum_{k=0}^{\lceil \log_2 \gamma \rceil - 1} \sum_{j=0}^{2^{k\frac{m}{2}} - 1} \sum_{n=\frac{1}{\beta}}^{\frac{2}{\beta} - 1}$.

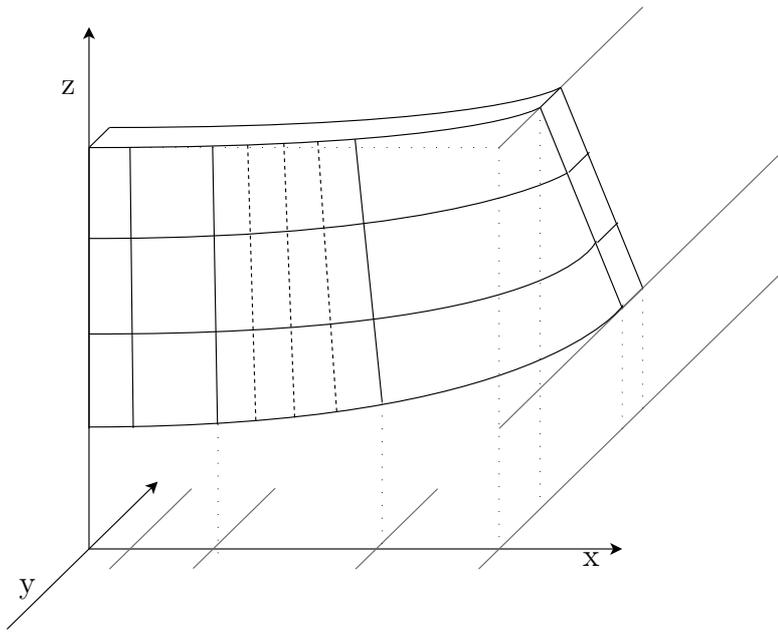


Figure 2.1: Decomposition

2.2.2 The heart of the problem

Of essential impact is the following theorem. It is a weighted, discrete version of the adjoint restriction estimate and, as we will see, already implies the restriction theorem.

THEOREM 2.2.2

Let $p' > m + 1$, $\frac{3}{p'} < \frac{1}{q} < \frac{1}{2} + \frac{1}{p'}$. Then

$$\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'} \lesssim \delta^{\frac{1}{q}} \left\| \sum_{\alpha=(k,j,n)} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{q'}. \tag{2.2.3}$$

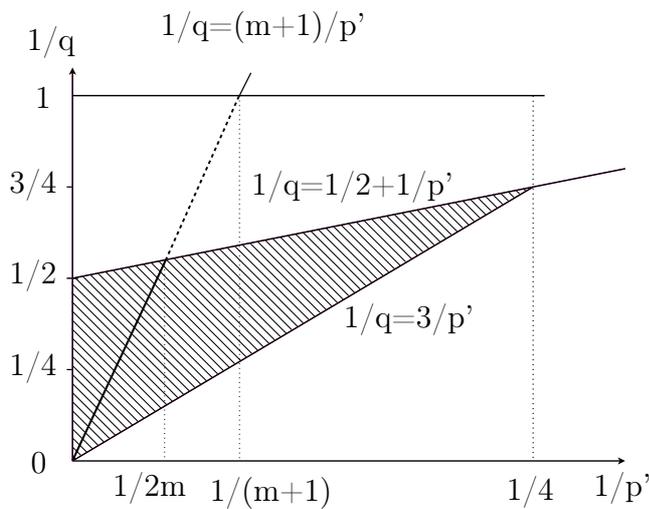


Figure 2.2: Range of p' and q

The proof will be the main work in this paper and be done in the next chapters. First of all, we will derive the restriction theorem.

As an immediate consequence of Theorem 2.2.2, we get as corollary

COROLLARY 2.2.3

Let $p' > m + 1$, $\frac{3}{p'} < \frac{1}{q}$. Then

$$\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'} \lesssim \delta^{\frac{1}{q}} \left\| \sum_{\alpha=(k,j,n)} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{q',1}. \quad (2.2.3)$$

Proof: Actually, we will just use that Theorem 2.2.2 is valid for every $p' > m + 1$ and for some range $\frac{3}{p'} < \frac{1}{q} < \frac{3}{p'} + \varepsilon_p$. According to the assumptions, we have $p' > m + 1 \geq 4$. Hence $\frac{3}{p'} = \frac{2}{p'} + \frac{1}{p'} < \frac{1}{2} + \frac{1}{p'}$. Therefore we find r with $\frac{3}{p'} < \frac{1}{r} < \frac{1}{2} + \frac{1}{p'}$ (cf. Figure 2.2), i.e. satisfying the requirements of Theorem 2.2.2. So it is sufficient to show that if (2.2.3) holds for some (p, r) , then it also holds for (p, q) with $\frac{1}{r} < \frac{1}{q}$. Under this condition, there exists $1 \leq s < \infty$ such that $\frac{1}{r'} = \frac{1}{q'} + \frac{1}{s}$. This means

$$\frac{1}{r} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{r'} = -\frac{1}{s}. \quad (2.2.4)$$

For (x, y, z) in the support of $\phi_{k,j,n}$, it follows $x \approx (2^k-1)\gamma$, so we introduce $g(x, y, z) := x^{-\frac{1}{s}}$. We now may apply Theorem 2.2.2 and use Hölder's inequality for Lorentz spaces (see Lemma 2.7.2):

$$\begin{aligned} \left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'} &\lesssim \delta^{\frac{1}{r}} \left\| \sum_{\alpha} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{r} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{r'} \\ &= \delta^{\frac{1}{r}} \left\| \sum_{\alpha} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'} - \frac{1}{s}} \phi_{\alpha} \right\|_{r'} \\ &\lesssim \delta^{\frac{1}{r}} \left\| \sum_{\alpha} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} g \right\|_{L^{r'}(\Gamma^{\delta})} \\ &\leq \delta^{\frac{1}{r}} \left\| \sum_{\alpha} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{L^{q',r'}(\Gamma^{\delta})} \|g\|_{L^{s,\infty}(\Gamma^{\delta})}. \end{aligned} \quad (2.2.5)$$

A short computation yields

$$\begin{aligned} |\{(x, y, z) \in \Gamma^{\delta} : x^{-\frac{1}{s}} > \lambda\}| &= \int_0^{\lambda^{-s}} \int_1^2 \int_{z\Phi(\frac{x}{z})}^{z\Phi(\frac{x}{z})+z\delta} dy dz dx \\ &= \delta \lambda^{-s} \int_1^2 z dz \approx \delta \lambda^{-s}, \end{aligned}$$

i.e.

$$\|g\|_{L^{s,\infty}(\Gamma^{\delta})} = \sup_{\lambda>0} \lambda |\{(x, y, z) \in \Gamma^{\delta} : x^{-\frac{1}{s}} > \lambda\}|^{\frac{1}{s}} \approx \delta^{\frac{1}{s}}. \quad (2.2.6)$$

Combined with (2.2.5), we end up with

$$\begin{aligned} \left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'} &\lesssim \delta^{\frac{1}{r} + \frac{1}{s}} \left\| \sum_{\alpha} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{q',r'} \\ &\leq \delta^{\frac{1}{q}} \left\| \sum_{\alpha} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{q',1}, \end{aligned}$$

since (2.2.4) means $\frac{1}{r} + \frac{1}{s} = \frac{1}{q}$. □

For $\frac{1}{q} = \frac{m+1}{p'}$ as considered in Theorem 2.2.1, (2.2.3) reads

$$\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'} \lesssim \delta^{\frac{1}{q}} \left\| \sum_{\alpha} a_{\alpha} \phi_{\alpha} \right\|_{(q',1)} \quad (2.2.7)$$

for all sequences a_{α} and every $\delta > 0$. In the next chapter, we will conclude that this statement, at least when $q < \infty$, implies the desired restriction theorem. To be more precise, for $\frac{1}{q} = \frac{m+1}{p'}$ and $q < \infty$ or $p' > m + 1$ respectively, the weak $L^p(\mathbb{R}^3)$ - $L^{q,\infty}(\Gamma)$ estimate

$$\|\hat{f}|_{\Gamma}\|_{L^{q,\infty}(\Gamma)} \lesssim \|f\|_p \quad \forall f \in \mathcal{S}(\mathbb{R}^3)$$

holds. Eventually we finish the proof by using the generalised Marcinkiewicz interpolation theorem (for instance, see Theorem 1.4.19 in [Gra]). In the endpoint $p = 1$, $q = \infty$, where we are not able to interpolate, the estimate is trivial.

2.3 Dense families

2.3.1 Basic properties

DEFINITION (DENSE FAMILY)

Let X be a metric (or even topological) space. A family M_δ , $\delta > 0$, of subsets of X is called a dense family in X , if

$$\forall f \in X \exists \delta_f > 0 \forall 0 < \delta < \delta_f \exists f_\delta \in M_\delta f = \lim_{\delta \rightarrow 0} f_\delta.$$

REMARK

Given a dense family M_δ , $\delta > 0$, the set $\bigcup_{\delta > 0} M_\delta$ is dense in X . Conversely, none of the sets M_δ has to be dense.

DEFINITION

Let (X, d) be a metric space.

- For a given set A with $\text{diam}(A) < \infty$, we define its doubling by $\tilde{A} = \bigcup_{x \in A} B(x, \text{diam } A)$.
Observe that $\text{diam } \tilde{A} \leq 3 \text{ diam } A$.
- A family $\{A_i\}_i$ of subsets of X is said to fulfill the bounded overlap property if for some constant $N \in \mathbb{N}$, every point of X is contained in at most N sets A_i .

We will discuss an example of a dense family:

LEMMA 2.3.1

Let X be a compact metric space and $\delta_0 > 0$. For every $\delta \in (0, \delta_0)$, let $\{A_{\delta,i}\}_i$ be a finite family with the bounded overlap property covering X , and satisfying

$$\limsup_{\delta \rightarrow 0} \sup_i \text{diam}(A_{\delta,i}) = 0. \quad (2.3.1)$$

Let $\{\phi_{\delta,i}\}_i$ be a partition of unity adopted to $\{A_{\delta,i}\}_i$, i.e.

$$\sum_i \phi_{\delta,i} = 1 \text{ on } X, \quad \phi_{\delta,i} \in C(X, [0, \infty)) \quad \text{and} \quad \text{supp } \phi_{\delta,i} \subset \tilde{A}_{\delta,i}.$$

Under these assumptions, the linear hull $M_\delta := LH\{\phi_{\delta,i}\}_i$, $\delta > 0$ is a dense family in $(C(X), \|\cdot\|_\infty)$.

Proof: Let $f \in C(X)$, $\delta_f := \delta_0$ and $x_{\delta,i} \in A_{\delta,i}$. Define

$$f_\delta := \sum_i f(x_{\delta,i}) \phi_{\delta,i} \in M_\delta.$$

We have to show that $\lim_{\delta \rightarrow 0} \|f_\delta - f\|_\infty = 0$, or

$$\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \forall \delta \in (0, \delta(\varepsilon)) \|f_\delta - f\|_\infty < \varepsilon$$

respectively. The compactness of X provides uniform continuity of f . Given $\varepsilon > 0$ there exists $\gamma > 0$, such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in X$ with distance $d(x, y) < \gamma$. Using

condition (2.3.1), we find $\delta(\varepsilon) > 0$ such that $\sup_i \text{diam}(A_{\delta,i}) < \frac{1}{3}\gamma$ for all $0 < \delta < \delta(\varepsilon)$, i.e. $\text{diam}(\tilde{A}_{\delta,i}) < \gamma$, for all i and for all $\delta \in (0, \delta(\varepsilon))$. Therefore, we obtain

$$\forall \delta \in (0, \delta(\varepsilon)) \forall i \forall x, y \in \tilde{A}_{\delta,i} |f(x) - f(y)| < \varepsilon. \quad (2.3.2)$$

Since $\text{supp } \phi_{\delta,i} \subset \tilde{A}_{\delta,i}$, we conclude

$$\forall \delta \in (0, \delta(\varepsilon)) \forall i \forall y \in \tilde{A}_{\delta,i} \forall x \in X |f(x) - f(y)| \phi_{\delta,i}(x) \leq \varepsilon \phi_{\delta,i}(x). \quad (2.3.3)$$

For such a δ and every $x \in X$

$$\begin{aligned} |f_\delta(x) - f(x)| &= \left| \sum_i f(x_{\delta,i}) \phi_{\delta,i}(x) - f(x) \sum_i \phi_{\delta,i}(x) \right| \\ &\leq \sum_i |f(x_{\delta,i}) - f(x)| \phi_{\delta,i}(x) \\ &\stackrel{(2.3.3)}{<} \varepsilon \sum_i \phi_{\delta,i}(x) = \varepsilon \end{aligned}$$

holds, as we claimed. □

EXAMPLE

We consider as metric space the surface

$$X = \Gamma = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 1 \leq z \leq 2, \Phi \left(\frac{x}{z} \right) = \frac{y}{z} \right\}$$

from the previous chapter. The families

$$A_{\delta,kjn} = \Gamma \cap \Gamma_{kjn}^\delta = \{(x, y, z) \in \mathbb{R}^3 \mid n\beta \leq z \leq (n+1)\beta, \Phi \left(\frac{x}{z} \right) = \frac{y}{z}, x_{kj} \leq x < x_{k,j+1}\},$$

$x_{kj} = (2^k - 1)\gamma + j2^{k(1-\frac{m}{2})}\gamma$, are fulfilling the required assumptions of Lemma 2.3.1: For every δ , they present a covering of X (with their overlap having measure zero) and it holds

$$\begin{aligned} \sup_{kjn} \text{diam}(A_{\delta,kjn}) &= \sup_{kjn} \max\{\beta, 2^{k(1-\frac{m}{2})}\gamma\} \\ &= \max\{\beta, \gamma\} \leq \max\{\delta, \delta^{\frac{1}{m}}\} \\ &= \delta^{\frac{1}{m}} \xrightarrow{\delta \rightarrow 0} 0. \end{aligned}$$

The sets Γ_{kjn}^δ can be seen as a certain thickening of $A_{\delta,kjn}$ by δ in the direction of $(0, 1, 0)$, which we are going to formalise soon.

2.3.2 An application

A classical result states that it is enough to know the behaviour of a bounded operator on some dense subset. We will obtain a similar statement concerning dense families.

During the following explanations, let Γ be a compact hypersurface in \mathbb{R}^n and σ the associated surface measure. Moreover, we take the existence of a universal transversal unit vector $N \in \mathbb{R}^n$ for granted, i.e. $\forall p \in \Gamma : N \notin T_p\Gamma$. (This is not a strong assumption, since Γ can at least be decomposed in finitely many subsets with such a universal transversal vector.)

DEFINITION (δ -THICKENING)

Let Γ^δ be the thickening of Γ by $\delta > 0$ in the direction of the universal transversal vector N , i.e. $\Gamma^\delta = \Gamma + [0, \delta]N$.

Given $h \in L^\infty(\Gamma)$, we denote by \tilde{h} the continuation along N on Γ^δ , defined by $\tilde{h}(\xi + tN) = h(\xi)\eta\left(\frac{t}{\delta}\right)$ for all $\xi \in \Gamma$, $t \in \mathbb{R}$. Here $\eta \in C_0^\infty(\mathbb{R})$ with $\chi_{[-1,1]} \leq \eta \leq \chi_{[-2,2]}$.

EXAMPLE

For our example, $N = (0, 1, 0)$ is a universal transversal unit vector. The functions

$$\phi_{kjn}(x, y, z) = \eta\left(\frac{x - x_{kj}}{\gamma_k}\right) \eta\left(\frac{y - z\Phi(x/z)}{\delta}\right) \eta\left(\frac{z - n\beta}{\beta}\right)$$

introduced in Section 2.2.1 can be considered as the continuation of the functions

$$\phi_{kjn}(x, z\Phi(x/z), z) = \eta\left(\frac{x - x_{kj}}{\gamma_k}\right) \eta\left(\frac{z - n\beta}{\beta}\right),$$

which are defined on Γ .

REMARK 2.3.2

Integration on Γ^δ can be done as follows:

There is a constant $A > 0$ and a smooth density $\psi : \Gamma \rightarrow (\frac{1}{A}, A)$ in the way that

$$\int_{\Gamma^\delta} f(x) dx = \int_0^\delta \int_\Gamma f(\xi + tN) \psi(\xi) d\sigma(\xi) dt \quad \forall f \in L^1(\Gamma^\delta).$$

Introducing $d\mu = \psi d\sigma$, the formula gives

$$\int_{\Gamma^\delta} f(x) dx = \int_0^\delta \int_\Gamma f(\xi + tN) d\mu(\xi) dt \quad \forall f \in L^1(\Gamma^\delta). \quad (2.3.4)$$

For our purposes, it's no matter whether to consider σ or μ : if the conjugated restriction theorem

$$\|\widehat{g d\mu}\|_{L^p(\mathbb{R}^n)} \leq C \|g\|_{L^q(\Gamma)} \quad \forall g \in C(\Gamma),$$

holds, then

$$\|\widehat{g d\sigma}\|_{L^p(\mathbb{R}^n)} = \left\| \frac{g}{\psi} d\mu \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \frac{g}{\psi} \right\|_{L^q(\Gamma)} \leq AC \|g\|_{L^q(\Gamma)} \quad \forall g \in C(\Gamma)$$

holds as well.

LEMMA 2.3.3

Let $C > 0$, $\delta_0 > 0$ and $1 \leq p \leq \infty$, $1 < q \leq \infty$.

Furthermore let $\{M_\delta\}_{\delta > 0}$ be a dense family in $C(\Gamma)$ fulfilling

$$\|\hat{\tilde{h}}\|_{L^{p'}(\mathbb{R}^n)} \leq C \delta^{\frac{1}{q}} \|\tilde{h}\|_{L^{q',1}(\Gamma^\delta)} \quad \forall \delta \in (0, \delta_0) \quad \forall h \in M_\delta. \quad (2.3.5)$$

Then

$$\|\widehat{g d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C' \|g\|_{L^{q',1}(\Gamma, \sigma)} \quad \forall g \in C(\Gamma)$$

and

$$\|\hat{f}|_\Gamma\|_{L^{q,\infty}(\Gamma, \sigma)} \leq C' \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in \mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n). \quad (2.3.6)$$

A more canonical version of the lemma would be to test the estimate on a *dense set* like $\bigcup_{\eta>0} M_\eta$. To handle the surface measure, we then may use a δ -thickening of Γ . Thus the canonical version of condition (2.3.5) would be

$$\|\hat{h}\|_{L^{p'}(\mathbb{R}^n)} \leq C\delta^{\frac{1}{q'}} \|\tilde{h}\|_{L^{q',1}(\Gamma^\delta)} \quad \forall \delta \in (0, \delta_0) \quad \forall h \in \bigcup_{\eta>0} M_\eta.$$

The lemma states that it suffices to check the estimate for fewer functions: we are allowed to link up η and δ . These considerations eventually lead to the definition of a dense family.

Proof:

According to Remark 2.3.2, we may use the measure μ instead of σ .

By our assumptions, every $g \in C(\Gamma)$ provides approximations $g_\delta \in M_\delta$, $\delta > 0$ such that $\lim_{\delta \rightarrow 0} g_\delta = g$ in $L^\infty(\Gamma)$.

First of all, we observe that $\lim_{\delta \rightarrow 0} \widehat{g_\delta d\mu} = \widehat{g d\mu}$ uniformly and pointwise, because

$$\|\widehat{g_\delta d\mu} - \widehat{g d\mu}\|_\infty = \sup_{x \in \mathbb{R}^n} \left| \int_\Gamma (g_\delta - g)(\xi) e^{-ix\xi} d\mu(\xi) \right| \leq \mu(\Gamma) \|g_\delta - g\|_\infty \xrightarrow{\delta \rightarrow 0} 0.$$

This implies

$$\begin{aligned} \frac{1}{\delta} \widehat{g_\delta}(x) &\stackrel{(2.3.4)}{=} \frac{1}{\delta} \int_{-2\delta}^{2\delta} \int_\Gamma \tilde{g}_\delta(\xi + tN) e^{-ix \cdot (\xi + tN)} d\mu(\xi) dt \\ &= \int_\Gamma g_\delta(\xi) e^{-ix \cdot \xi} d\mu(\xi) \quad \frac{1}{\delta} \int_{-2\delta}^{2\delta} \eta\left(\frac{t}{\delta}\right) e^{-itx \cdot N} dt \\ &= \underbrace{\widehat{g_\delta d\mu}(x)}_{\xrightarrow{\delta \rightarrow 0} \widehat{g d\mu}(x)} \underbrace{\int_{-2}^2 \eta(t) e^{-itx \cdot N} dt}_{\xrightarrow{\delta \rightarrow 0} \|\eta\|_1} \quad \xrightarrow{\delta \rightarrow 0} \|\eta\|_1 \widehat{g d\mu}(x), \end{aligned}$$

i.e.

$$\widehat{g d\mu}(x) \approx \lim_{\delta \rightarrow 0} \frac{1}{\delta} \widehat{g_\delta}(x). \quad (2.3.7)$$

Furthermore, we compute the distribution function

$$\begin{aligned} d_{\tilde{g}_\delta}(s) &= |\{v \in \Gamma^\delta : |\tilde{g}_\delta(v)| > s\}| \\ &\stackrel{(2.3.4)}{=} (\mu \otimes \lambda)(\{(\xi, t) \in \Gamma \times \mathbb{R} : |g_\delta(\xi)\eta(t/\delta)| > s\}) \\ &\leq 4\delta \mu(\{(\xi \in \Gamma : |g_\delta(\xi)| > s\}) = 4\delta d_{g_\delta}(s) \end{aligned}$$

and thus the decreasing rearrangement is

$$\tilde{g}_\delta^*(t) = \inf\{s > 0 | d_{\tilde{g}_\delta}(s) \leq t\} \leq \inf\left\{s > 0 | d_{g_\delta}(s) \leq \frac{t}{4\delta}\right\} = g_\delta^*\left(\frac{t}{4\delta}\right).$$

These estimates yield

$$\begin{aligned}
\|\tilde{g}_\delta\|_{L^{q',1}(\Gamma^\delta)} &= \int_0^\infty t^{\frac{1}{q'}} \tilde{g}_\delta^*(t) \frac{dt}{t} \\
&\leq \int_0^\infty t^{\frac{1}{q'}} g_\delta^* \left(\frac{t}{4\delta} \right) \frac{dt}{t} \\
&\approx \delta^{\frac{1}{q'}} \int_0^\infty t^{\frac{1}{q'}} g_\delta^*(t) \frac{dt}{t} \\
&= \delta^{\frac{1}{q'}} \|g_\delta\|_{L^{q',1}(\Gamma,\mu)}.
\end{aligned} \tag{2.3.8}$$

Moreover, it is valid for a general $f \in L^\infty(\Gamma)$ that

$$\begin{aligned}
\|f\|_{L^{q',1}(\Gamma,\mu)} &= \int_0^\infty t^{\frac{1}{q'}} f^*(t) \frac{dt}{t} \\
&= \int_0^{\mu(\Gamma)} t^{\frac{1}{q'}-1} f^*(t) dt \\
&\leq f^*(0) \int_0^{\mu(\Gamma)} t^{\frac{1}{q'}-1} dt \\
&\approx \|f\|_\infty,
\end{aligned}$$

since $\frac{1}{q'} > 0$, i.e. $q > 1$, and therefore

$$\|g_\delta - g\|_{L^{q',1}(\Gamma,\mu)} \lesssim \|g_\delta - g\|_\infty \xrightarrow{\delta \rightarrow 0} 0,$$

so especially

$$\lim_{\delta \rightarrow 0} \|g_\delta\|_{L^{q',1}(\Gamma,\mu)} = \|g\|_{L^{q',1}(\Gamma,\mu)}. \tag{2.3.9}$$

Hence by using assumption (2.3.5), we obtain

$$\begin{aligned}
\|\widehat{g d\mu}\|_{L^{p'}(\mathbb{R}^n)} &\stackrel{(2.3.7)}{\approx} \left\| \liminf_{\delta \rightarrow 0} \frac{1}{\delta} |\widehat{g_\delta}| \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\stackrel{Fatou}{\leq} \liminf_{\delta \rightarrow 0} \delta^{-1} \|\widehat{g_\delta}\|_{L^{p'}(\mathbb{R}^n)} \\
&\stackrel{(2.3.5)}{\leq} C \liminf_{\delta \rightarrow 0} \delta^{\frac{1}{q}-1} \|\tilde{g}_\delta\|_{L^{q',1}(\Gamma^\delta)} \\
&\stackrel{(2.3.8)}{\lesssim} C \liminf_{\delta \rightarrow 0} \delta^{\frac{1}{q}-1+\frac{1}{q'}} \|g_\delta\|_{L^{q',1}(\Gamma,\mu)} \\
&\stackrel{(2.3.9)}{=} C \|g\|_{L^{q',1}(\Gamma,\mu)}.
\end{aligned}$$

The second part of our claim is a simple duality argument. □

2.4 Estimation of the overlap

2.4.1 Straightforward results

It will become essential for us to understand the overlap of the sets $\Gamma_{kjn}^\delta + \Gamma_{lip}^\delta$, $k, l = 0, \dots, |\log_2 \gamma| - 1$, $j \in I_k$, $i \in I_l$, $n, p = \frac{1}{\beta}, \dots, \frac{2}{\beta} - 1$, i.e. to examine the maximal number of them containing a single point. However, this number will not be bounded by a absolute constant (which would be optimal). We will get some overlap from the z -direction. Nevertheless, this will not cause us any trouble.

LEMMA 2.4.1

For every $\xi \in \mathbb{R}^3$ we have

$$\#\{(n, p) | \exists k, l, j, i : \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} \leq 3\beta^{-1}. \quad (2.4.1)$$

Proof: If $\xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta$, there exist $\mathbf{x}_1 \in \Gamma_{kjn}^\delta$ and $\mathbf{x}_2 \in \Gamma_{lip}^\delta$ with $\xi = \mathbf{x}_1 + \mathbf{x}_2$. Let z_1, z_2 denote the last component of \mathbf{x}_1 and \mathbf{x}_2 respectively, which means $n\beta \leq z_1 \leq (n+1)\beta$ and $p\beta \leq z_2 \leq (p+1)\beta$. The sum of both inequalities leads to $(n+p)\beta \leq z \leq (n+p+2)\beta$. This implies $\frac{z}{\beta} - 2 \leq n+p \leq \frac{z}{\beta}$ and hence

$$n \in -p + \left[\frac{z}{\beta} - 2, \frac{z}{\beta}\right] \cap \mathbb{N}.$$

We conclude

$$\begin{aligned} \#\{(n, p) | \exists k, l, j, i : \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} &= \sum_{p=\frac{1}{\beta}}^{\frac{2}{\beta}-1} \#\{n | \exists k, l, j, i : \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} \\ &\leq \sum_{p=\frac{1}{\beta}}^{\frac{2}{\beta}-1} \#\left(\left[\frac{z}{\beta} - 2, \frac{z}{\beta}\right] \cap \mathbb{N}\right) \\ &\leq \sum_{p=\frac{1}{\beta}}^{\frac{2}{\beta}-1} 3 = 3\beta^{-1}, \end{aligned}$$

completing the proof. □

Exploiting the dyadic structure of the decomposition, we obtain a further simple result:

LEMMA 2.4.2

For $k = 0, \dots, |\log_2 \gamma| - 1$ let $V_k(n, p) = \bigcup \{\Gamma_{kjn}^\delta + \Gamma_{lip}^\delta | l \leq k, j \in I_k, i \in I_l\}$. Then for all $\xi \in \mathbb{R}^3$, $n, p = \frac{1}{\beta}, \dots, \frac{2}{\beta} - 1$ holds

$$\#\{k | \xi \in V_k(n, p)\} \leq 3.$$

Proof: The procedure is comparable to the previous proof, though we now concentrate on the x -component. If $\xi \in V_k(n, p) = \bigcup\{\Gamma_{kjn}^\delta + \Gamma_{lip}^\delta | l \leq k, j \in I_k, i \in I_l\}$ and x denotes the first component of ξ , there exist l, j, i such that

$$(2^k-1)\gamma + j\gamma_k + (2^l-1)\gamma + i\gamma_l \leq x \leq (2^k-1)\gamma + (j+1)\gamma_k + (2^l-1)\gamma + (i+1)\gamma_l.$$

Since we are just interested in the dyadic size, we estimate quite roughly

$$(2^k-1)\gamma \leq x \leq (2^k-1)\gamma + 2^{k\frac{m}{2}}2^{k(1-\frac{m}{2})}\gamma + (2^l-1)\gamma + 2^{l\frac{m}{2}}2^{l(1-\frac{m}{2})}\gamma = (2^{k+1}-1)\gamma + (2^{l+1}-1)\gamma.$$

and since we assumed $l \leq k$, this reduces to

$$(2^k-1)\gamma \leq x \leq (2^{k+1}-1)\gamma + (2^{l+1}-1)\gamma \leq (2^{k+2}-1)\gamma.$$

Hence

$$2^k \leq \frac{x}{\gamma} + 1 \leq 2^{k+2}$$

and therefore

$$\log_2 \left(\frac{x}{\gamma} + 1 \right) - 2 \leq k \leq \log_2 \left(\frac{x}{\gamma} + 1 \right).$$

Our claim is an immediate consequence of this inequality. \square

The handling of the overlap concerning the remaining parameters is more complicated. Here we need to involve the y -coordinate. Therefore, we need as a start a new coordinate system.

2.4.2 Further results

LEMMA 2.4.3

There exists an absolute constant $C > 0$ (not depending on δ) such that for all $\xi \in \mathbb{R}^3$, $k = 0, \dots, |\log_2 \gamma| - 1$, $n, p = \frac{1}{\beta}, \dots, \frac{2}{\beta} - 1$

$$\#\{(j, l, i) | l \leq k, \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} \leq C. \quad (2.4.2)$$

Proof: We are allowed to restrict ourself to the case $l \ll k$ (more precise: $l < k - 2$), since in the case $l \approx k$ the second derivative of Φ and therefore the Gaussian curvature is comparable on the regions $x \approx 2^k\gamma$ and $x \approx 2^l\gamma$. After rescaling, we are back in the classical case with (almost) constant curvature.

The procedure differs a bit from the previous lemmas: We claim that the number of triples (j, l, i) , $l \leq k$, with the property $\xi \in \Gamma_{kjn}^\delta + \Gamma_{lip}^\delta$ is bounded by a fixed constant. So let

$$\xi \in (\Gamma_{kjn}^\delta + \Gamma_{lip}^\delta) \cap (\Gamma_{k'j'n}^\delta + \Gamma_{l'i'p}^\delta), \quad (2.4.3)$$

$l, l' \leq k$. Without loss of generality, we may assume $l' \leq l$ and in the case $l' = l$ furthermore $i' \leq i$ by interchanging the parameters if necessary.

Additionally, we introduce

$$\xi_{kjn} = (x_{kj}, \Phi_n(x_{kj}), n\beta) \in \Gamma_{kjn}^\delta \quad \xi_{lip} = (x_{li}, \Phi_p(x_{li}), p\beta) \in \Gamma_{lip}^\delta \quad (2.4.6)$$

and analogue $\xi_{kj'n}$, $\xi_{l'i'p}$.

Step 2

$$(i) \quad P(\xi_{l'i'p}) - P(\xi_{lip}) \approx \delta 2^{k(m-1)-l'(\frac{m}{2}-1)} \Delta_{l'i'i'}$$

$$(ii) \quad P(\xi_{kj'n}) - P(\xi_{kjn}) \gtrsim \delta j(j' - j), \quad \text{if } j \leq j' + 1$$

$$\text{with } \Delta_{l'i'i'} = 2^{l'(\frac{m}{2}-1)} \frac{x_{li} - x_{l'i'}}{\gamma} = 2^{l'(\frac{m}{2}-1)} (2^l + i2^{l(1-\frac{m}{2})} - 2^{l'} - i'2^{l'(1-\frac{m}{2})}).$$

The term $\Delta_{l'i'i'}$ looks somehow artificial. However, we will discover that this quantity is the crucial one, expressing (in some sense) the distance between Γ_{lip}^δ and $\Gamma_{l'i'p}^\delta$.

Part (i): The mean value theorem provides the existence of an $\tilde{x} \in (x_{l'i'}, x_{li})$ with

$$\Phi_p(x_{l'i'}) - \Phi_p(x_{li}) = \Phi'_p(\tilde{x})(x_{l'i'} - x_{li}). \quad (2.4.7)$$

Thus especially

$$\tilde{x} \leq x_{li} \leq x_{l+1} \leq_{l < k-2} x_{k-2}$$

and hence

$$\frac{\tilde{x}}{p\beta} \leq \frac{n\beta}{p\beta} \frac{x_{k-2}}{n\beta} \leq \frac{2x_{k-2}}{n\beta} = \frac{2(2^{k-2}-1)\gamma}{n\beta} = \frac{1}{2} \frac{(2^k-4)\gamma}{n\beta} \leq \frac{1}{2} \frac{(2^k-1)\gamma}{n\beta} = \frac{1}{2} \frac{x_k}{n\beta}.$$

Using monotony and convexity of Φ' , as well as $\Phi'(0) = 0$, we obtain

$$\Phi' \left(\frac{\tilde{x}}{p\beta} \right) \leq \Phi' \left(\frac{1}{2} \frac{x_k}{n\beta} \right) \leq \frac{1}{2} \Phi' \left(\frac{x_k}{n\beta} \right). \quad (2.4.8)$$

Hence

$$\begin{aligned} \Phi'_n(x_k) - \Phi'_p(\tilde{x}) &= \Phi' \left(\frac{x_k}{n\beta} \right) - \Phi' \left(\frac{\tilde{x}}{p\beta} \right) \\ &\stackrel{(2.4.8)}{\approx} \Phi' \left(\frac{x_k}{n\beta} \right) \\ &\approx \frac{x_k^{m-1}}{(n\beta)^{m-1}} \\ &\approx \gamma^{m-1} 2^{k(m-1)}. \end{aligned} \quad (2.4.9)$$

Thus we conclude

$$\begin{aligned} P(\xi_{l'i'p}) - P(\xi_{lip}) &= \Phi_p(x_{l'i'}) - \Phi_p(x_{li}) - (x_{l'i'} - x_{li}) \Phi'_n(x_k) \\ &\stackrel{(2.4.7)}{=} (\Phi'_n(x_k) - \Phi'_p(\tilde{x}))(x_{li} - x_{l'i'}) \\ &= (\Phi'_n(x_k) - \Phi'_p(\tilde{x})) 2^{-l'(\frac{m}{2}-1)} \Delta_{l'i'i'} \gamma \\ &\stackrel{(2.4.9)}{\approx} \gamma^m 2^{k(m-1)} 2^{-l'(\frac{m}{2}-1)} \Delta_{l'i'i'} \\ &= \delta 2^{k(m-1)-l'(\frac{m}{2}-1)} \Delta_{l'i'i'} \end{aligned}$$

Part (ii):

Since the projection P depends on k , we are not able to use (i). However, we proceed in a similar manner: like in part (i), we obtain a \tilde{x}_k between x_{kj} and $x_{kj'}$ with

$$\Phi_n(x_{kj'}) - \Phi_n(x_{kj}) = \Phi'_n(\tilde{x}_k)(x_{kj'} - x_{kj}).$$

Furthermore, we obtain a $\tilde{\tilde{x}}_k$ between \tilde{x}_k and x_k with

$$\begin{aligned} P(\xi_{kj'n}) - P(\xi_{kjn}) &= \Phi_n(x_{kj'}) - \Phi_n(x_{kj}) - (x_{kj'} - x_{kj})\Phi'_n(x_k) \\ &= (\Phi'_n(\tilde{x}_k) - \Phi'_n(x_k))(x_{kj'} - x_{kj}) \\ &= \Phi''_n(\tilde{\tilde{x}}_k)(\tilde{x}_k - x_k)(x_{kj'} - x_{kj}). \end{aligned} \quad (2.4.10)$$

$$\text{Especially } \tilde{\tilde{x}}_k \in (x_k, \tilde{x}_k) \subset (x_k, x_{k+1}), \text{ i.e. } \tilde{\tilde{x}}_k \approx 2^k \gamma. \quad (2.4.11)$$

In the case $j' \geq j$ we use

$$\tilde{x}_k > \min\{x_{kj}, x_{kj'}\} = x_{kj} = (2^k - 1)\gamma + j2^{k(1-\frac{m}{2})}\gamma = x_k + j2^{k(1-\frac{m}{2})}\gamma. \quad (2.4.12a)$$

This gives

$$\begin{aligned} P(\xi_{kj'n}) - P(\xi_{kjn}) &\stackrel{(2.4.10)}{=} \Phi''_n(\tilde{\tilde{x}}_k)(\tilde{x}_k - x_k)(x_{kj'} - x_{kj}) \\ &\stackrel{(2.4.11)}{\approx} (2^k \gamma)^{m-2}(\tilde{x}_k - x_k)(x_{kj'} - x_{kj}) \\ &\stackrel{(2.4.12a)}{\geq} (2^k \gamma)^{m-2} j 2^{k(1-\frac{m}{2})}\gamma (j' - j) 2^{k(1-\frac{m}{2})}\gamma \\ &= \gamma^m j(j' - j) \\ &= \delta j(j' - j). \end{aligned}$$

In the other case $j' < j$ the assumption ensures $j = j' + 1$, i.e. $j' - j = -1$. Now we use

$$\tilde{x}_k < \max\{x_{kj}, x_{kj'}\} = x_{kj} = (2^k - 1)\gamma + j2^{k(1-\frac{m}{2})}\gamma = x_k + j2^{k(1-\frac{m}{2})}\gamma. \quad (2.4.12b)$$

In this case we obtain

$$\begin{aligned} P(\xi_{kj'n}) - P(\xi_{kjn}) &\stackrel{(2.4.10)}{=} \Phi''_n(\tilde{\tilde{x}}_k)(\tilde{x}_k - x_k)(x_{kj'} - x_{kj}) \\ &\stackrel{(2.4.11)}{\approx} (2^k \gamma)^{m-2}(\tilde{x}_k - x_k)(-2^{k(1-\frac{m}{2})}\gamma) \\ &\stackrel{(2.4.12b)}{\geq} -(2^k \gamma)^{m-2} j 2^{k(1-\frac{m}{2})}\gamma 2^{k(1-\frac{m}{2})}\gamma \\ &= -\gamma^m j \\ &= -\delta j = \delta j(j' - j), \end{aligned}$$

completing Step 2.

The next task will be to estimate the size of the pieces Γ_{kjn}^δ of our decomposition, with respect to the projection P . The size of a set U with respect to P is measured by $\text{diam}_P(U) = \sup\{|P(u) - P(v)| : u, v \in U\}$.

Step 3

$$(i) \text{ diam}_P(\Gamma_{lip}^\delta) \lesssim \delta 2^{k(m-1)-l(\frac{m}{2}-1)}$$

$$(ii) \text{ diam}_P(\Gamma_{kjn}^\delta) \lesssim \delta 2^{k\frac{m}{2}}$$

Part (i):

$$\begin{aligned} \text{diam}_P(\Gamma_{lip}^\delta) &= P(x_{li}, \Phi_p(x_{li}) + \mathcal{O}(\delta), p\beta) - P(x_{l,i+1}, \Phi_p(x_{l,i+1}), p\beta) \\ &= P(x_{li}, \Phi_p(x_{li}), p\beta) - P(x_{l,i+1}, \Phi_p(x_{l,i+1}), p\beta) + \mathcal{O}(\delta) \\ &= P(\xi_{lip}) - P(\xi_{l,i+1,p}) + \mathcal{O}(\delta) \end{aligned}$$

We apply Step 2(i), replacing l' by l , i by $i+1$ and i' by i . Then $\Delta_{l,i+1,i} = 2^{l(\frac{m}{2}-1)} \left((i+1)2^{l(1-\frac{m}{2})} - i2^{l(1-\frac{m}{2})} \right) = 1$ and

$$\begin{aligned} \text{diam}_P(\Gamma_{lip}^\delta) &= P(\xi_{lip}) - P(\xi_{l,i+1,p}) + \mathcal{O}(\delta) \\ &\approx \delta 2^{k(m-1)-l(\frac{m}{2}-1)} \Delta_{l,i+1,i} + \mathcal{O}(\delta) \\ &= \delta 2^{k(m-1)-l(\frac{m}{2}-1)} + \mathcal{O}(\delta) \\ &\stackrel{l \leq k}{\lesssim} \delta 2^{k(m-1)-l(\frac{m}{2}-1)}. \end{aligned}$$

Part (ii):

$$\begin{aligned} \text{diam}_P(\Gamma_{kjn}^\delta) &= P(x_{k,j+1}, \Phi_n(x_{k,j+1}) + \mathcal{O}(\delta), n\beta) - P(x_{kj}, \Phi_n(x_{kj}), n\beta) \\ &= P(x_{k,j+1}, \Phi_n(x_{k,j+1}), n\beta) - P(x_{kj}, \Phi_n(x_{kj}), n\beta) + \mathcal{O}(\delta) \\ &= P(\xi_{k,j+1,n}) - P(\xi_{kjn}) + \mathcal{O}(\delta) \end{aligned}$$

Step 2(ii) implies

$$P(\xi_{kj'n}) - P(\xi_{kjn}) \gtrsim \delta j(j' - j), \quad \text{if } j \leq j' + 1$$

or, equivalently

$$P(\xi_{kjn}) - P(\xi_{kj'n}) \lesssim \delta j(j - j'), \quad \text{if } j \leq j' + 1,$$

so especially

$$P(\xi_{k,j+1,n}) - P(\xi_{kjn}) \lesssim (j+1)\delta \leq \delta 2^{k\frac{m}{2}}.$$

This leads to

$$\begin{aligned} \text{diam}_P(\Gamma_{kjn}^\delta) &= P(\xi_{k,j+1,n}) - P(\xi_{kjn}) + \mathcal{O}(\delta) \\ &\lesssim \delta 2^{k\frac{m}{2}} + \mathcal{O}(\delta) \\ &\lesssim \delta 2^{k\frac{m}{2}}. \end{aligned}$$

A further step will analyse the quantity $\Delta_{ll'ii'}$. Here we exploit (2.4.3): $\Gamma_{kj'n}^\delta + \Gamma_{l'i'p}^\delta \neq \emptyset$.

Step 4 $\Delta_{ll'ii'} \lesssim 1$

We apply Lemma 2.7.5. Since

$$\xi_\alpha \in \Gamma_\alpha^\delta, \quad \alpha = (k, j, n), (k, j', n), (l, i, p), (l', i', p)$$

the lemma yields

$$\begin{aligned} & P(\xi_{kj'n}) - P(\xi_{kjn}) + P(\xi_{l'i'p}) - P(\xi_{lip}) \\ & \leq \text{diam}_P(\Gamma_{kjn}^\delta) + \text{diam}_P(\Gamma_{kj'n}^\delta) + \text{diam}_P(\Gamma_{lip}^\delta) + \text{diam}_P(\Gamma_{l'i'p}^\delta). \end{aligned} \quad (2.4.13)$$

In Step 1, we established $j \leq j'$ (at least if $(l, i) \neq (l', i')$, but in the case $(l, i) = (l', i')$ we may also assume this). The application of Step 2 and Step 3 results in

$$\begin{aligned} \delta 2^{k(m-1)-l'(\frac{m}{2}-1)} \Delta_{ll'ii'} & \leq \delta 2^{k(m-1)-l'(\frac{m}{2}-1)} \Delta_{ll'ii'} + \delta j(j' - j) \\ & \stackrel{(2.4.13)}{\lesssim} \delta 2^{k\frac{m}{2}} + \delta 2^{k(m-1)-l'(\frac{m}{2}-1)} + \delta 2^{k(m-1)-l'(\frac{m}{2}-1)} \end{aligned} \quad (2.4.14)$$

Now we use $l' \leq l$ and $l' < k$, since the last one implies $k(m-1) - l'(\frac{m}{2}-1) = k\frac{m}{2} + (k-l')(\frac{m}{2}-1) \geq k\frac{m}{2}$. Thus (2.4.14) transforms into

$$2^{k(m-1)-l'(\frac{m}{2}-1)} \Delta_{ll'ii'} \lesssim 2^{k(m-1)-l'(\frac{m}{2}-1)} + 2^{k(m-1)-l'(\frac{m}{2}-1)} + 2^{k\frac{m}{2}} \lesssim 2^{k(m-1)-l'(\frac{m}{2}-1)},$$

i.e. $\Delta_{ll'ii'} \lesssim 1$.

The former results will be merged into the following step, which states that there are not "too many" (l', i') appropriate to a given (l, i) .

Step 5 *One of these three alternatives holds:*

- (i) $l' = l$: Then $|i - i'| \lesssim 1$.
- (ii) $l' = l - 1$: Then $i \lesssim 1$ and $2^{l'\frac{m}{2}} - i' \lesssim 1$.
- (iii) $l' < l - 1$: Then $l', l \lesssim 1$ and especially $i', i \lesssim 1$.

Especially $l' \approx l$ is necessary.

Let me remind you that we assumed $l' \leq l$ and even $i' \leq i$ for $l' = l$. The proof is divided into three cases:

Case 1: $l' = l$

According to Step 4, we know

$$1 \gtrsim \Delta_{llii'} = 2^{l(\frac{m}{2}-1)} (2^l + i2^{l(1-\frac{m}{2})} - 2^l - i'2^{l(1-\frac{m}{2})}) = i - i' \geq 0.$$

Case 2: $l' = l - 1$

Step 4 now reads

$$1 \gtrsim \Delta_{ll'ii'} = 2^{l'(\frac{m}{2}-1)} \left(2^{l'+1} + i2^{(l'+1)(1-\frac{m}{2})} - 2^{l'} - i'2^{l'(1-\frac{m}{2})} \right) = \underbrace{2^{l'\frac{m}{2}} - i'}_{\geq 0} + \underbrace{i2^{1-\frac{m}{2}}}_{\geq 0},$$

hence

$$1 \gtrsim i2^{1-\frac{m}{2}} \approx i \quad \text{and} \quad 1 \gtrsim 2^{l'\frac{m}{2}} - i'.$$

Case 3: $l > l' + 1$

In this case, Step 4 yields

$$\begin{aligned} 1 \gtrsim \Delta_{ll'ii'} & = 2^{l'(\frac{m}{2}-1)} \left(2^l + i2^{l(1-\frac{m}{2})} - 2^{l'} - i'2^{l'(1-\frac{m}{2})} \right) \\ & \gtrsim 2^{l'(\frac{m}{2}-1)} \left(2^l - 2^{l'+1} \right) \gtrsim 2^{l'(\frac{m}{2}-1)} 2^l \geq 2^l, \end{aligned}$$

which already implies $l' \leq l \lesssim 1$ and thus $i \leq 2^{l\frac{m}{2}} \lesssim 1$, as well as $i' \lesssim 1$.

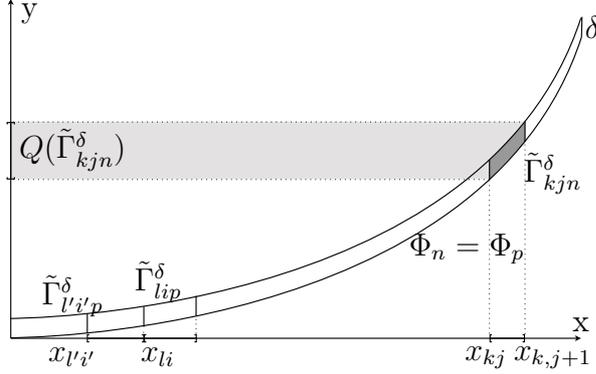


Figure 2.4: Projection on the y-axis in the case $n = p$

To come to a similar conclusion concerning j and j' , we consider the projection on the y -axis, denoted $Q(x, y, z) = y$. Then - cf. (2.4.6) - $Q(\xi_{k'jn}) = \Phi_n(x_{kj})$ and $Q(\xi_{lip}) = \Phi_p(x_{li})$, analogue for the primed coordinates.

Step 6

$$(i) \quad Q(\xi_{lip}) - Q(\xi_{l'i'p}) \lesssim \delta 2^{l\frac{m}{2}}$$

$$(ii) \quad Q(\xi_{k'jn}) - Q(\xi_{kjn}) \approx \delta 2^{k\frac{m}{2}} (j' - j)$$

Part (ii): Again, we use the mean value theorem: there exists a \tilde{x} between x_{kj} and $x_{k'j}$ fulfilling

$$Q(\xi_{k'jn}) - Q(\xi_{kjn}) = \Phi_n(x_{k'j}) - \Phi_n(x_{kj}) = \Phi'_n(\tilde{x})(x_{k'j} - x_{kj}) = \Phi' \left(\frac{\tilde{x}}{n\beta} \right) (x_{k'j} - x_{kj}).$$

Notice that $\tilde{x} \approx x_k \approx 2^k \gamma$, i.e.

$$\begin{aligned} Q(\xi_{k'jn}) - Q(\xi_{kjn}) &\approx (2^k \gamma)^{m-1} (x_{k'j} - x_{kj}) \\ &= 2^{k(m-1)} \gamma^{m-1} (j' - j) 2^{k(1-\frac{m}{2})} \gamma \\ &= \delta 2^{k\frac{m}{2}} (j' - j). \end{aligned}$$

Part (i): In this case, the mean value theorem provides a $\bar{x} \approx x_l \approx 2^l \gamma$ such that

$$\begin{aligned} Q(\xi_{lip}) - Q(\xi_{l'i'p}) &= \Phi_p(x_{li}) - \Phi_p(x_{l'i'}) \\ &= \Phi'_p(\bar{x})(x_{li} - x_{l'i'}) \\ &\approx (2^l \gamma)^{m-1} 2^{-l'(\frac{m}{2}-1)} \gamma \Delta_{ll'i'} \\ &= \delta 2^{l\frac{m}{2}} 2^{(l-l')(\frac{m}{2}-1)} \Delta_{ll'i'}. \end{aligned}$$

Since $\Delta_{ll'i'} \lesssim 1$ by Step 4 and $l' \approx l$ according to Step 5, the equation implies

$$Q(\xi_{lip}) - Q(\xi_{l'i'p}) \lesssim \delta 2^{l\frac{m}{2}},$$

thus completing Step 6.

Step 7 $|j - j'| \lesssim 1$

We have

$$\begin{aligned}
diam_Q(\Gamma_{kjn}^\delta) &= \sup_{\xi, \eta \in \Gamma_{kjn}^\delta} |Q(\xi) - Q(\eta)| \\
&= Q(x_{kj}, \Phi_n(x_{kj}) + \mathcal{O}(\delta), n\beta) - Q(x_{k,j-1}, \Phi_n(x_{k,j-1}), n\beta) \\
&= \Phi_n(x_{kj}) + \mathcal{O}(\delta) - \Phi_n(x_{k,j-1}) \\
&\stackrel{\text{Step 6(ii)}}{\approx} \delta 2^{k\frac{m}{2}} + \mathcal{O}(\delta) \\
&\lesssim \delta 2^{k\frac{m}{2}}.
\end{aligned}$$

Using Step 6(i) we observe $diam_Q(\Gamma_{lip}^\delta) \lesssim \delta 2^{l\frac{m}{2}} \leq \delta 2^{k\frac{m}{2}}$ and $diam_Q(\Gamma_{l'ip}^\delta) \lesssim \delta 2^{k\frac{m}{2}}$. Now we again apply Lemma 2.7.5:

$$\begin{aligned}
Q(\xi_{kj'n}) - Q(\xi_{kjn}) &+ Q(\xi_{l'ip}) - Q(\xi_{lip}) \\
&\leq diam_Q(\Gamma_{kj'n}^\delta) + diam_Q(\Gamma_{kjn}^\delta) + diam_Q(\Gamma_{l'ip}^\delta) + diam_Q(\Gamma_{lip}^\delta) \\
&\lesssim \delta 2^{k\frac{m}{2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\delta 2^{k\frac{m}{2}}(j' - j) &\stackrel{\text{Step 6(ii)}}{\approx} Q(\xi_{kj'n}) - Q(\xi_{kjn}) \\
&\lesssim Q(\xi_{lip}) - Q(\xi_{l'ip}) + \delta 2^{k\frac{m}{2}} \\
&\stackrel{\text{Step 6(i)}}{\lesssim} \delta 2^{l\frac{m}{2}} + \delta 2^{k\frac{m}{2}} \\
&\lesssim \delta 2^{k\frac{m}{2}}, \quad \square
\end{aligned}$$

and thus $j' - j \lesssim 1$. This already implies the desired estimate, since $j \leq j' + 1$ according to Step 1.

Together with Step 5, Step 7 completes the proof of Lemma 2.4.3.

2.4.3 Summary of the results

COROLLARY 2.4.4

For all $\xi \in \mathbb{R}^3$, we have

$$\#\{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} \leq 18C\beta^{-1}, \quad (2.4.15)$$

where C is the constant from Lemma 2.4.3.

Proof: For $\xi \in \mathbb{R}^3$ let

$$\begin{aligned}
M &= \{(n, p) | \exists k, l, j, i : \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\}, \\
M(n, p) &= \{k | \exists l, j, i : \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta, l \leq k\}, \\
M(n, p; k) &= \{(l, j, i) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\}.
\end{aligned}$$

Then the previous Lemma 2.4.1, 2.4.2 and 2.4.3 states

$$\#M \leq 3\beta^{-1} \quad (2.4.16)$$

$$\#M(n, p) \leq 3 \quad \forall n, p \quad (2.4.17)$$

$$\#M(n, p; k) \leq C \quad \forall n, p, k. \quad (2.4.18)$$

We proceed by

$$\begin{aligned} \{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} &= \{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta, l \leq k\} \cup \{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta, l \geq k\} \\ &= \{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta, l \leq k\} \cup \{(\alpha, \mu) | \xi \in \Gamma_\mu^\delta + \Gamma_\alpha^\delta, k \leq l\}, \end{aligned}$$

and

$$\{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta, l \leq k\} = \{(\alpha, \mu) | (n, p) \in M, k \in M(n, p), (l, j, i) \in M(n, p; k)\}.$$

It follows

$$\begin{aligned} \#\{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta\} &\leq 2\#\{(\alpha, \mu) | \xi \in \Gamma_\alpha^\delta + \Gamma_\mu^\delta, l \leq k\} \\ &= 2 \sum_{(n,p) \in M} \sum_{k \in M(n,p)} \#M(n, p; k) \\ &\leq 2 \cdot 3\beta^{-1} \cdot 3 \cdot C = 18C\beta^{-1} \end{aligned}$$

using Fubini's theorem. □

If we enlarge the sets Γ_α^δ in the right manner, the statement essentially remains valid:

COROLLARY 2.4.5

For the sets Γ_{kjn}^δ , we introduce their adjacent sets

$$\Gamma_{kjn}^\delta(u, v, w) = (0, v\delta, 0) + \Gamma_{k, j+u, n+w}^\delta, \quad u, v, w \in \{-1, 0, 1\},$$

as well as their "doubling"

$$G_{kjn}^\delta = \bigcup_{u,v,w \in \{-1,0,1\}} \Gamma_{kjn}^\delta(u, v, w).$$

Then for every $\xi \in \mathbb{R}^3$:

$$\#\{(\alpha, \mu) | \xi \in G_\alpha^\delta + G_\mu^\delta\} \leq 3^3 \cdot 18C\beta^{-1}.$$

The last variation of this lemma is the version we want to apply finally.

COROLLARY 2.4.6

Let $1 \leq s < \infty$ and G_{kjn}^δ like above. Then there exists a constant $C_s > 0$ satisfying the following: Let $f_{\alpha\mu} \in C_0^\infty(\mathbb{R}^3)$, $k, l = 0, \dots, |\log_2 \gamma| - 1$, $j = 0, \dots, 2^{k\frac{m}{2}} - 1$, $i = 0, \dots, 2^{l\frac{m}{2}} - 1$, $n, p = \frac{1}{\beta}, \dots, \frac{2}{\beta} - 1$, be non-negative functions fulfilling

$$\text{supp } f_{\alpha\mu} \subset G_\alpha^\delta + G_\mu^\delta$$

Then for all $\xi \in \mathbb{R}^3$

$$\left(\sum_{\alpha\mu} f_{\alpha\mu}(\xi) \right)^s \leq C_s \beta^{1-s} \sum_{\alpha\mu} |f_{\alpha\mu}(\xi)|^s.$$

2.5 L^p -estimates for convolutions

Consider two cuboids in \mathbb{R}^3 with two short and one long edge. Both cuboids shall lie at parallel planes (we will concretise this soon). If we form the convolution of two functions, each one supported in one of the cuboids, what can we say about the L^p -Norm, depending on size and relative position?

2.5.1 Introduction to the problem and formulation of the result

For $i = 1, 2$ let $A_i = \{(x, y) : |x| \leq \frac{\gamma_i}{2}, |y - m_i x| \leq \frac{\delta}{2}\}$. We assume that $\delta \ll \gamma_2, \gamma_1$ and moreover, that the slope m_i of the boxes is bounded by an absolute constant (i.e. not depending on δ and γ_i). For convenience, let $\gamma_2 \leq \gamma_1$.

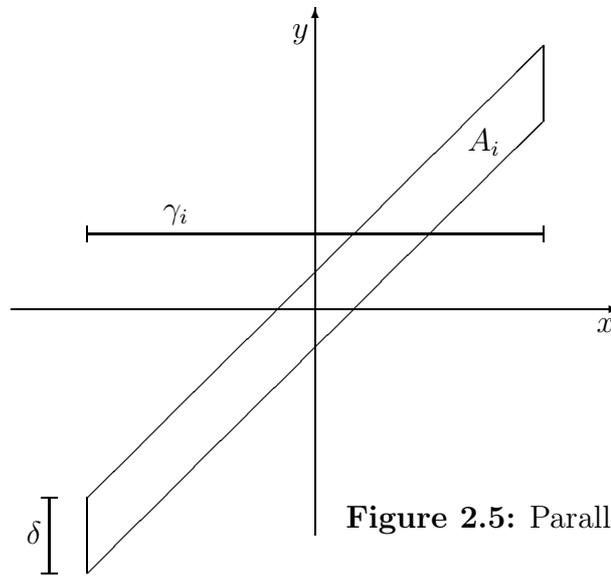


Figure 2.5: Parallelogram A_i

Furthermore, we configure the position of the figures relative to each other. Let $\alpha = \angle(A_1, A_2)$ be the angle between the parallelograms, i.e. between their longer sides. The assumption of the boundedness of the slopes guarantees $\sin \alpha \approx \alpha \approx \tan \alpha$. (2.5.1)

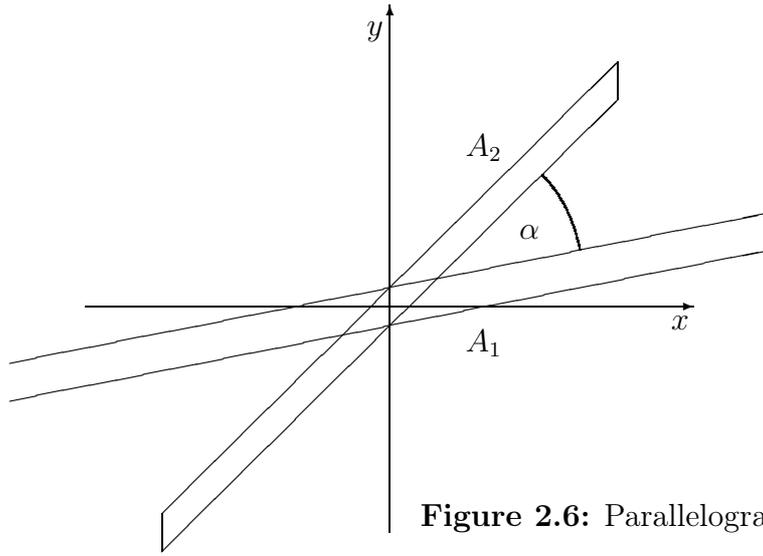


Figure 2.6: Parallelograms A_i

Now we introduce the parallelepipeds $Q_i = A_i \times (0, \beta) \subset \mathbb{R}^3$. Let $\xi_i \in \mathbb{R}^3$ and ϕ_i a bump function adapted to $\xi_i + Q_i$.

The main result of this section is the following:

LEMMA 2.5.1

Let $1 \leq s < \infty$. Then

$$\int_{\mathbb{R}^3} |\phi_1 * \phi_2|^s dx \lesssim \frac{(\beta\delta)^{(s+1)}}{(1 + \frac{\gamma_2}{\delta}\alpha)^{s-1}} \gamma_2^s \gamma_1. \quad (2.5.2)$$

2.5.2 Proof of the result

At the beginning a simple remark:

REMARK 2.5.2

For $C > 0$ and $i = 1, 2$ let B_i be symmetric ($B_i = -B_i$) subsets of \mathbb{R}^n , $x_i \in \mathbb{R}^n$, $\text{supp } \psi_i \subset x_i + B_i$ and $\|\psi_i\|_\infty \leq C$. Then

$$\|\psi_1 * \psi_2\|_\infty \leq C^2 \sup_z |B_1 \cap (z + B_2)|. \quad (2.5.3)$$

Proof:

$$\begin{aligned} |\psi_1 * \psi_2|(x) &= \left| \int_{\mathbb{R}^n} \psi_1(y) \psi_2(x - y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |\psi_1(x_1 + y) \psi_2(x - x_1 - y)| dy \end{aligned}$$

If the integrand does not vanish, we have $x_1 + y \in \text{supp } \psi_1 \subset x_1 + B_1$, i.e. $y \in B_1$. Furthermore must $x - x_1 - y \in \text{supp } \psi_2 \subset x_2 + B_2$ be valid, i.e. $y \in x - x_1 - x_2 - B_2 = x - x_1 - x_2 + B_2$. We conclude

$$\begin{aligned} |\psi_1 * \psi_2|(x) &\leq \int_{B_1 \cap (x - x_1 - x_2 + B_2)} |\psi_1(x_1 + y)\psi_2(x - x_1 - y)| dy \\ &\leq C^2 |B_1 \cap (x - x_1 - x_2 + B_2)| \\ &\leq C^2 \sup_z |B_1 \cap (z + B_2)| \end{aligned}$$

as claimed. □

REMARK

If the sets are like in Figure 2.6, the supremum is achieved (amongst others) in $z = 0$, as we can see clearly in the picture. In this concrete situation, a formal proof can be obtained with some efforts by an elementary computation.

However, it would be more typical for a mathematician to conjecture a more general statement: "Let K and L be two convex bodies in \mathbb{R}^n symmetric with respect to the origin (i.e. $K = -K$). Then $|K \cap (z + L)| \leq |K \cap L|$ for all $z \in \mathbb{R}^n$."

As it is often in convex geometry, the statement is quite easy to understand, whereas a proof is not immediately clear. One might use the inequality of Brunn-Minkowski (cf. Lemma 2.7.4).

Anyway, we now need to estimate $|A_1 \cap A_2|$:

LEMMA 2.5.3

We have

$$|A_1 \cap A_2| \lesssim \frac{\delta^2 \gamma_2}{\max\{\delta, \gamma_2 \alpha\}} \approx \frac{\delta^2 \gamma_2}{\delta + \gamma_2 \alpha}.$$

Proof: Elongating the two parallelograms in direction of their longer edges to infinity, we obtain by a elementary computation (cf. Figure 2.7)

$$|A_1 \cap A_2| \leq \frac{\delta^2}{\sin \alpha} \stackrel{(2.5.1)}{\approx} \frac{\delta^2}{\alpha}. \tag{2.5.4}$$

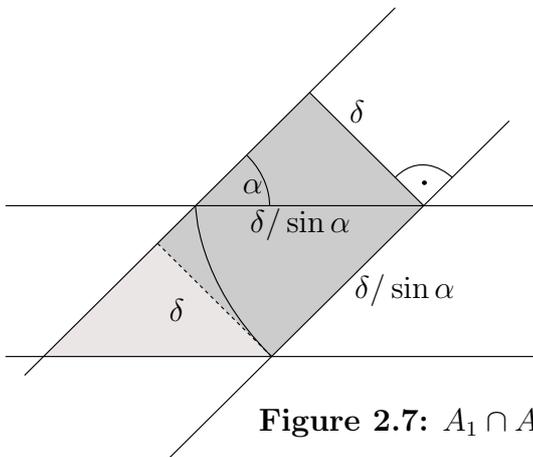


Figure 2.7: $A_1 \cap A_2$

This expression is indeed to large for small angles; in this case we therefore apply the trivial estimate

$$|A_1 \cap A_2| \leq |A_2| = \delta\gamma_2. \quad (2.5.5)$$

The changeover appears if the corners of the small parallelogram get into touch with the large parallelogram, i.e. if $\alpha \approx \frac{\delta}{\gamma_2}$. Given $\gamma_2\alpha \geq \delta$ we thus apply (2.5.4) and find

$$|A_1 \cap A_2| \lesssim \frac{\delta^2}{\alpha} = \frac{\delta^2\gamma_2}{\max\{\delta, \gamma_2\alpha\}}.$$

In the contrary case $\gamma_2\alpha < \delta$ we apply (2.5.5) and find

$$|A_1 \cap A_2| \leq \delta\gamma_2 = \frac{\delta^2\gamma_2}{\max\{\delta, \gamma_2\alpha\}},$$

and thus the desired estimate in both cases. \square

This lemma can be proved in a similar manner, as basic as before:

LEMMA 2.5.4

$$|A_1 + A_2| \lesssim \gamma_1(\delta + \gamma_2\alpha).$$

Now we are able to start with the proof of Lemma 2.5.1:

Proof:

$$\begin{aligned} \int_{\mathbb{R}^3} |\phi_1 * \phi_2|^s dx &\leq \|\phi_1 * \phi_2\|_\infty^s |supp(\phi_1 * \phi_2)| \\ &\stackrel{\text{Lemma 2.5.2}}{\lesssim} \left(\sup_{\eta \in \mathbb{R}^2 \times \mathbb{R}} |Q_1 \cap (\eta + Q_2)| \right)^s |\xi_1 + Q_1 + \xi_2 + Q_2| \\ &= |Q_1 \cap Q_2|^s |Q_1 + Q_2| \\ &= |(A_1 \cap A_2) \times (0, \beta)|^s |(A_1 \times (0, \beta)) + (A_2 \times (0, \beta))| \\ &= |A_1 \cap A_2|^s \beta^s |(A_1 + A_2) \times (0, 2\beta)| \\ &\stackrel{\text{Lemma 2.5.3}}{\lesssim} \left(\frac{\delta^2\gamma_2}{\delta + \gamma_2\alpha} \right)^s |(A_1 + A_2)| \beta^{s+1} \\ &\stackrel{\text{Lemma 2.5.4}}{\lesssim} \left(\frac{\delta^2\gamma_2}{\delta + \gamma_2\alpha} \right)^s \gamma_1(\delta + \gamma_2\alpha) \beta^{s+1} \\ &= \frac{\delta^{2s} \gamma_2^s \gamma_1}{(\delta + \gamma_2\alpha)^{s-1}} \beta^{s+1} \\ &= \frac{(\beta\delta)^{s+1} \gamma_2^s \gamma_1}{(1 + \frac{\gamma_2}{\delta}\alpha)^{s-1}}. \end{aligned} \quad \square$$

2.6 Proof of the main theorem

Before we start to complete the proof of Theorem 2.2.2, we need a further lemma, which concretises the general results from the previous chapter in our special situation.

2.6.1 Application of the results from Chapter 2.5

LEMMA 2.6.1

Let $1 \leq s < \infty$. The functions ϕ_{kjn} introduced in the previous chapter satisfy the following estimates:

(i) If $|k - l| \gg 1$, then

$$\int |\phi_{kjn} * \phi_{lip}|^s dx \lesssim (\beta\delta\gamma)^{s+1} 2^{\frac{k+l}{2}(1+s-sm)} 2^{-\frac{|k-l|}{2}(m-1)(s-1)}. \quad (2.6.1)$$

(ii) If $|k - l| \lesssim 1$, we find either a function $f : I_k \rightarrow I_l$ with $\forall i \in I_l : |\{j : f(j) = i\}| \lesssim 1$ or a function $g : I_l \rightarrow I_k$ with $\forall j \in I_k : |\{i : g(i) = j\}| \lesssim 1$ such that

$$\int |\phi_{kjn} * \phi_{lip}|^s dx \lesssim (\beta\delta\gamma)^{s+1} 2^{\frac{k+l}{2}(1+s-sm)} \left[\frac{2^{\frac{k+l}{2} \frac{m}{2}}}{1 + h(i, j)} \right]^{s-1}, \quad (2.6.2)$$

where

$$h(i, j) = |f(j) - i| \quad \text{or} \quad h(i, j) = |j - g(i)|. \quad (2.6.3)$$

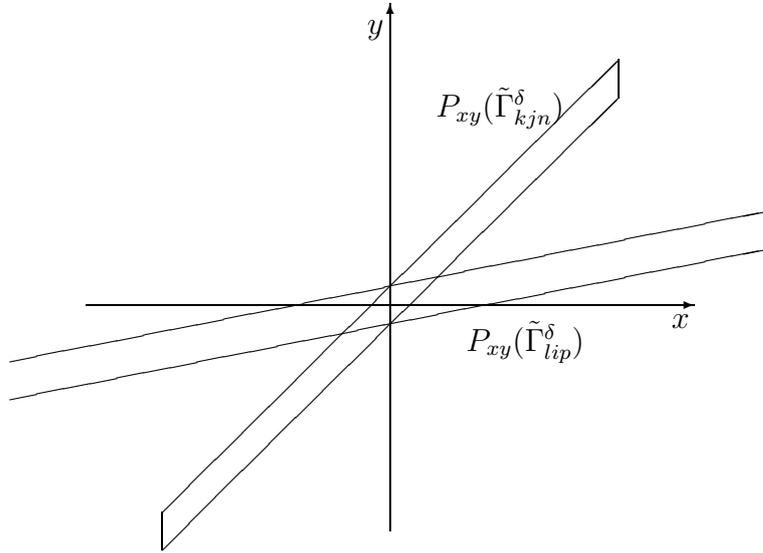


Figure 2.8: Angle between the translations to the origin of the projections

Proof: Recall the definition of $\tilde{\Gamma}_{kjn}^\delta = \{(x, y, z) \in \mathbb{R}^3 \mid n\beta \leq z \leq (n+1)\beta, |y - \Phi_n(x)| \leq 10m\delta, x_{kj} \leq x \leq x_{k,j+1}\}$. The projections of these sets on x - y -space are contained in parallelograms with slope $\Phi'_n(x_{kj})$, thickness $\mathcal{O}(\delta)$ and width γ_k . When we shift their centers to the origin, they intersect with angle

$$\alpha = \angle(\tilde{\Gamma}_{kjn}^\delta, \tilde{\Gamma}_{lip}^\delta) = |\arctan \Phi'_n(x_{kj}) - \arctan \Phi'_p(x_{li})|. \quad (2.6.4)$$

Since Φ' and therefore as well $\Phi'_n = \Phi' \left(\frac{\cdot}{n\beta} \right)$ and Φ'_p are bounded from above and from below independently of n and p , we have

$$\alpha \approx |\Phi'_n(x_{kj}) - \Phi'_p(x_{li})|. \quad (2.6.5)$$

At first we consider the case (i) $|k - l| \gg 1$ and assume without loss of generality $l \leq k$, we obtain due to the dyadic nature of the construction

$$\begin{aligned} \alpha &\approx \Phi'_n(x_{kj}) \approx (2^k \gamma)^{m-1} \\ &= \frac{\gamma^m}{2^{k(1-\frac{m}{2})} \gamma} 2^{k\frac{m}{2}} \\ &= \frac{\delta}{\gamma_k} 2^{k\frac{m}{2}}. \end{aligned} \quad (2.6.6)$$

The case (ii) $|k - l| \lesssim 1$ is a bit more complicated. This is due to the fact that we decomposed orthogonal to the x -axis, regardless the cone-like shape of the surface. Formula (2.6.5) illustrates the difficulties: in the special case $n = p$ we get rid of the difference by the fundamental theorem of calculus, but unfortunately, the general case appears much harder.

Instead of this, we choose an other approach: let k, l, n, p be fixed and $a_i = \Phi'_p(x_{li})$, $b_j = \Phi'_n(x_{kj})$. Then for all j, j'

$$\begin{aligned} |b_j - b_{j'}| &= |\Phi'_n(x_{kj}) - \Phi'_n(x_{kj'})| \\ &\approx \Phi''(x_k) |x_{kj} - x_{kj'}| \\ &\approx (2^k \gamma)^{m-2} \gamma_k |j - j'| \\ &= 2^{k(\frac{m}{2}-1)} \gamma^{m-1} |j - j'| \\ &= \frac{\delta}{\gamma_k} |j - j'|. \end{aligned} \quad (2.6.7)$$

In the same manner we obtain

$$|a_i - a_{i'}| \approx \frac{\delta}{\gamma_l} |i - i'| \stackrel{|k-l| \lesssim 1}{\approx} \frac{\delta}{\gamma_k} |i - i'|. \quad (2.6.8)$$

This basically means that we have a good idea how to compare the a_i with each other, the same with the b_j 's. But we lack control in comparing some a_i with a b_j . Therefore we apply the abstract result of Lemma 2.7.6. At first we check the preconditions:

The sequences $a_i, i \in I_l$ and $b_j, j \in I_k$ are increasing, since (for instance) $x_{li} < x_{li} + \gamma_l = x_{l,i+1}$ and Φ'_p is monotonously increasing. Thus we may apply Lemma 2.7.6 to the renormed sequences $\frac{\gamma_k}{\delta} a_i$ and $\frac{\gamma_k}{\delta} b_j$. Provided $b_{2^{km/2-1}} - b_0 \leq a_{2^{lm/2-1}} - a_0$ we get a function $f: I_k \rightarrow I_l$ almost injective, i.e. $\forall i \in I_l: |\{j \in I_k: f(j) = i\}| \lesssim 1$, and fulfilling

$$|a_i - b_j| \geq \frac{1}{2} |a_i - a_{f(j)}| \quad \forall i \in I_l \forall j \in I_k. \quad (2.6.9)$$

Consequently

$$\begin{aligned} \alpha &\approx |\Phi'_n(x_{kj}) - \Phi'_p(x_{li})| \\ &= |b_j - a_i| \\ &\stackrel{(2.6.9)}{\gtrsim} |a_i - a_{f(j)}| \\ &\stackrel{(2.6.8)}{\approx} \frac{\delta}{\gamma_k} |i - f(j)|. \end{aligned}$$

Provided $b_{2^{km/2-1}} - b_0 > a_{2^{lm/2-1}} - a_0$ we switch the roles of a_i and b_j and obtain a correspondingly result; in any case, we obtain a function h as desired and fulfilling

$$\alpha \gtrsim \frac{\delta}{\gamma_k} h(i, j). \quad (2.6.10)$$

Combining (2.6.6) and (2.6.10) we see that

$$\frac{\gamma_k}{\delta} \alpha = \begin{cases} 2^{k\frac{m}{2}}, & \text{if } l \ll k \\ h(i, j), & \text{if } |k - l| \lesssim 1. \end{cases} \quad (2.6.11)$$

Using Lemma 2.5.1 and taking into account $l \leq k$, i.e. $\gamma_k \leq \gamma_l$, we conclude

$$\begin{aligned} \int |\phi_{kjn} * \phi_{lip}|^s dx &\lesssim \frac{(\beta\delta)^{s+1}}{(1 + \frac{\gamma_k}{\delta} \alpha)^{s-1}} \gamma_k^s \gamma_l \\ &= \frac{(\beta\delta\gamma)^{s+1}}{(1 + \frac{\gamma_k}{\delta} \alpha)^{s-1}} 2^{k(1-\frac{m}{2})s} 2^{l(1-\frac{m}{2})} \\ &= \frac{(\beta\delta\gamma)^{s+1}}{(1 + \frac{\gamma_k}{\delta} \alpha)^{s-1}} 2^{\frac{k}{2}(2s-ms)} 2^{\frac{l}{2}(2-m)} \\ &= (\beta\delta\gamma)^{s+1} 2^{\frac{k+l}{2}(1+s-sm)} \cdot \frac{2^{\frac{k}{2}(s-1) + \frac{l}{2}(ms-m-s+1)}}{(1 + \frac{\gamma_k}{\delta} \alpha)^{s-1}}. \end{aligned}$$

It remains to consider the second expression. In the case $l \ll k$, it is transformed by (2.6.11) into

$$\begin{aligned} \frac{2^{\frac{k}{2}(s-1) + \frac{l}{2}(ms-m-s+1)}}{(1 + \frac{\gamma_k}{\delta} \alpha)^{s-1}} &\approx \frac{2^{\frac{k}{2}(s-1) + \frac{l}{2}(ms-m-s+1)}}{2^{k\frac{m}{2}(s-1)}} \\ &= 2^{\frac{k}{2}(s-1-ms+m) + \frac{l}{2}(ms-m-s+1)} \\ &= 2^{-\frac{k-l}{2}(m-1)(s-1)}, \end{aligned}$$

whereas in the case $|k - l| \lesssim 1$, it gives

$$\begin{aligned} \frac{2^{\frac{k}{2}(s-1) + \frac{l}{2}(ms-m-s+1)}}{(1 + \frac{\gamma_k}{\delta} \alpha)^{s-1}} &= \frac{2^{\frac{k}{2}(s-1) + \frac{l}{2}(ms-m-s+1)}}{(1 + h(i, j))^{s-1}} \\ &\stackrel{k \approx l}{\approx} \frac{2^{\frac{k+l}{4}(s-1) + \frac{k+l}{4}(ms-m-s+1)}}{(1 + h(i, j))^{s-1}} \\ &= \frac{2^{\frac{k+l}{4}(ms-m)}}{(1 + h(i, j))^{s-1}} \\ &= \left(\frac{2^{\frac{k+l}{2} \frac{m}{2}}}{1 + h(i, j)} \right)^{s-1}, \end{aligned}$$

whereby the claim is verified. \square

2.6.2 Completing the proof

One further intermediate step will be helpful since we get rid of the split-up in the two different cases from the previous lemma.

LEMMA 2.6.2

Let $\frac{3}{p'} < \frac{1}{q} < \frac{1}{2} + \frac{1}{p'}$ and $\frac{1}{s} + \frac{2}{p'} = 1$. (2.6.12)

Then for all k, l, n, p and for all finite sequences $a \in \mathbb{R}^{I_k}, b \in \mathbb{R}^{I_l}$ we have

$$\begin{aligned} & \sum_{ij} |a_j|^s |b_i|^s \int |\phi_{kjn} * \phi_{lip}|^s dx \\ & \lesssim (\beta\delta\gamma)^{s+1} \|a\|_{q'}^s \|b\|_{q'}^s 2^{\frac{k+l}{2}(1+s-sm+m(1-\frac{s}{q'}))} 2^{-\frac{|k-l|}{2}(m-1)(s-1)}. \end{aligned}$$

Proof: At first we consider the case $|k-l| \gg 1$. Hölder's inequality implies

$$\begin{aligned} \sum_j |a_j|^s & \leq \left(\sum_j |a_j|^{q'} \right)^{\frac{s}{q'}} \left(\sum_{j=0}^{2^k \frac{m}{2} - 1} 1 \right)^{1-\frac{s}{q'}} \\ & = \|a\|_{q'}^s 2^{\frac{k}{2}m(1-\frac{s}{q'})}, \end{aligned}$$

which, together with Lemma 2.6.1, results in the desired estimate.

In the case $|k-l| \leq 1$ we apply Lemma 2.7.1 with parameter $r = \frac{q'}{s}$. (2.6.13)

Since we assumed $\frac{1}{q} < \frac{1}{2} + \frac{1}{p'}$, it follows

$$\frac{1}{q'} = 1 - \frac{1}{q} > \frac{1}{2} - \frac{1}{p'} = \frac{1}{2} \left(1 - \frac{2}{p'} \right) \stackrel{(2.6.12)}{=} \frac{1}{2s}, \quad \text{thus } r = \frac{q'}{s} < 2. \quad (2.6.14)$$

Furthermore, we have

$$\begin{aligned} \frac{1}{r'} & = 1 - \frac{s}{q'} = 1 - s + \frac{s}{q} \\ & > 1 - s + s \frac{3}{p'} = 1 - s + \frac{3s}{2} \frac{2}{p'} \\ & \stackrel{(2.6.12)}{=} 1 - s + \frac{3s}{2} \left(1 - \frac{1}{s} \right) = -(s-1) + \frac{3}{2}(s-1) \\ & = \frac{s-1}{2}, \end{aligned}$$

i.e.

$$Q := (s-1) \frac{r'}{2} < 1. \quad (2.6.15)$$

Assume for simplicity once more that $l \leq k$. To apply the lemma, we have to analyse the kernel $G_{\frac{k+l}{2}}(x) = \frac{2^{\frac{k+l}{2} \frac{m}{2}}}{1+x}$ from (2.6.2), $x \in [0, 2^{k \frac{m}{2}} - 1]$. It fulfills

$$\begin{aligned} \int_0^{2^{k \frac{m}{2}} - 1} G_{\frac{k+l}{2}}^Q dx & \leq \int_1^{2^{k \frac{m}{2}}} \left(\frac{2^{k \frac{m}{2}}}{x} \right)^Q dx = 2^{k \frac{m}{2}} \int_{2^{-k \frac{m}{2}}}^1 x^{-Q} dx \\ & \leq 2^{k \frac{m}{2}} \left[\frac{x^{1-Q}}{1-Q} \right]_0^1 \approx 2^{k \frac{m}{2}} \frac{1}{|1-Q|} \stackrel{|k-l| \lesssim 1}{\approx} 2^{\frac{k+l}{2} \frac{m}{2}}, \end{aligned}$$

i.e.

$$\|G_{\frac{k+l}{2}}^{s-1}\|_{\frac{r'}{2}} = \|G_{\frac{k+l}{2}}\|_Q^{s-1} \lesssim 2^{\frac{k+l}{2} \frac{m}{2} \frac{s-1}{Q}} \stackrel{(2.6.15)}{\leq} 2^{\frac{k+l}{2} \frac{m}{r'}} \stackrel{(2.6.13)}{=} 2^{\frac{k+l}{2} m \left(1 - \frac{s}{q'}\right)}. \quad (2.6.16)$$

Next we again apply Lemma 2.6.1, but this time part (ii), though we just discuss the first case $h(i, j) = |f(j) - i|$. The lemma states

$$\begin{aligned} & \sum_{ij} |a_j|^s |b_i|^s \int |\phi_{kjn} * \phi_{lip}|^s dx \\ & \lesssim (\beta\delta\gamma)^{s+1} \sum_{ij} |a_j|^s |b_i|^s 2^{\frac{k+l}{2}(1+s-sm)} \left[\frac{2^{\frac{k+l}{2} \frac{m}{2}}}{1 + |f(j) - i|} \right]^{s-1} \end{aligned} \quad (2.6.17)$$

and according to Lemma 2.7.1

$$\begin{aligned} \sum_{ij} |a_j|^s |b_i|^s \left[\frac{2^{\frac{k+l}{2} \frac{m}{2}}}{1 + |f(j) - i|} \right]^{s-1} & \lesssim \|a^s\|_r \|b^s\|_r \|G_{\frac{k+l}{2}}^{s-1}\|_{\frac{r'}{2}} \\ & \stackrel{(2.6.16)}{\lesssim} \|a^s\|_r \|b^s\|_r 2^{\frac{k+l}{2} m \left(1 - \frac{s}{q'}\right)} \\ & \stackrel{(2.6.13)}{=} \|a\|_{q'}^s \|b\|_{q'}^s 2^{\frac{k+l}{2} m \left(1 - \frac{s}{q'}\right)}. \end{aligned}$$

Putting this into (2.6.17) yields the claim, when we insert the expression $2^{-\frac{|k-l|}{2}(m-1)(s-1)}$, which is of constant order in case of $|k-l| \lesssim 1$. \square

Eventually we can complete the proof of Theorem 2.2.2. Let us recall the statement:

THEOREM

Let $p' > m + 1$, $\frac{3}{p'} < \frac{1}{q} < \frac{1}{2} + \frac{1}{p'}$, then

$$\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'} \lesssim \delta^{\frac{1}{q}} \left\| \sum_{\alpha=(k,j,n)} a_{\alpha} ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_{\alpha} \right\|_{q'} \quad (2.6.18)$$

holds for all $\delta > 0$ and for all sequences a_{α} .

Proof: Since $2 < p' < \infty$, we find $s \in (1, \infty)$ such that $\frac{1}{s} + \frac{2}{p'} = 1$. $(2.6.19)$

Since even $4 < p'$, $\frac{p'}{2} > 2$ holds, thus we can apply Youngs inequality:

$$\begin{aligned} \left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'}^{2s} & = \left\| \sum_{\alpha, \mu} a_{\alpha} a_{\mu} \hat{\phi}_{\alpha} \hat{\phi}_{\mu} \right\|_{\frac{p'}{2}}^s \\ & = \left\| \mathfrak{F} \left(\sum_{\alpha, \mu} a_{\alpha} a_{\mu} \phi_{\alpha} * \phi_{\mu} \right) \right\|_{\frac{p'}{2}}^s \\ & \stackrel{Young}{\leq} \left\| \sum_{\alpha, \mu} a_{\alpha} a_{\mu} \phi_{\alpha} * \phi_{\mu} \right\|_s^s \\ & = \int \left| \sum_{\alpha, \mu} a_{\alpha} a_{\mu} \phi_{\alpha} * \phi_{\mu} \right|^s dx. \end{aligned}$$

Now we exploit the estimate from Chapter 2.4 of the overlap of the supports of the functions $\phi_\alpha * \phi_\mu$:

$$\begin{aligned}
\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'}^{2s} &\leq \int \left| \sum_{\alpha, \mu} a_{\alpha} a_{\mu} \phi_{\alpha} * \phi_{\mu} \right|^s dx. \\
&\stackrel{\text{Corollary 2.4.6}}{\lesssim} \beta^{1-s} \sum_{\alpha, \mu} |a_{\alpha}|^s |a_{\mu}|^s \int |\phi_{\alpha} * \phi_{\mu}|^s dx \\
&\stackrel{\text{Lemma 2.6.2}}{\lesssim} \beta^2 (\delta \gamma)^{s+1} \sum_{klnp} \|a_{kn}\|_{q'}^s \|a_{lp}\|_{q'}^s 2^{\frac{k+l}{2}(1+s-sm+m(1-\frac{s}{q}))} 2^{-\frac{|k-l|}{2}(m-1)(s-1)},
\end{aligned}$$

where $a_{kn} = (a_{kjn})_{j \in I_k}$, $a_{lp} = (a_{lip})_{i \in I_l}$. Notice that for fixed k Hölder's inequality implies

$$\begin{aligned}
\sum_n \|a_{kn}\|_{q'}^s &\leq \left(\sum_n \|a_{kn}\|_{q'}^{q'} \right)^{\frac{s}{q'}} \left(\sum_{n=\frac{1}{\beta}} 1 \right)^{1-\frac{s}{q'}} \\
&= \|a_k\|_{q'}^s \left(\frac{1}{\beta} \right)^{1-\frac{s}{q'}} = \|a_k\|_{q'}^s \beta^{\frac{s}{q'}-1},
\end{aligned} \tag{2.6.20}$$

where $a_k = (a_{kjn})_{j,n}$. Implementing this in our equations yields

$$\begin{aligned}
\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'}^{2s} &\lesssim \beta^{2(\frac{s}{q'}-1)} \beta^2 (\delta \gamma)^{(s+1)} \sum_{k,l} \|a_k\|_{q'}^s \|a_l\|_{q'}^s 2^{\frac{k+l}{2}(1+s-sm+m(1-\frac{s}{q}))} 2^{-|k-l|\frac{(m-1)(s-1)}{2}} \\
&= \beta^{\frac{2s}{q'}} (\delta \gamma)^{(s+1)} \sum_{k,l} \|a_k\|_{q'}^s \|a_l\|_{q'}^s 2^{\frac{k+l}{2}[s(1-m)+m+1-m\frac{s}{q}]} 2^{-|k-l|\frac{(m-1)(s-1)}{2}} \\
&= \beta^{\frac{2s}{q'}} (\delta \gamma)^{(s+1)} \sum_{k,l} \|a_k\|_{q'}^{2k[\frac{1-m}{2} + \frac{m+1}{2s} - \frac{m}{2q'}]} \|a_l\|_{q'}^{2l[\frac{1-m}{2} + \frac{m+1}{2s} - \frac{m}{2q'}]} 2^{-|k-l|\frac{(m-1)(s-1)}{2}}.
\end{aligned} \tag{2.6.21}$$

We again apply Lemma 2.7.1 for $r = \frac{q'}{s}$ (recall $r < 2$, cf. (2.6.14)):

$$\begin{aligned}
\left\| \sum_{\alpha} a_{\alpha} \hat{\phi}_{\alpha} \right\|_{p'}^{2s} &\lesssim \beta^{\frac{2s}{q'}} (\delta \gamma)^{s+1} \left(\sum_k \|a_k\|_{q'}^{2k[\frac{1-m}{2} + \frac{m+1}{2s} - \frac{m}{2q'}]} \right)^{\frac{2s}{q'}} \\
&= (\delta \gamma)^{s+1-\frac{2s}{q'}} \gamma^{-\left(\frac{1-m}{2} + \frac{m+1}{2s} - \frac{1}{q'}\right)2s} \left(\sum_k \|a_k\|_{q'}^{2k(1-\frac{m}{2})} \gamma \beta \delta \right)^{\frac{2s}{q'}} \\
&\stackrel{\delta \equiv \gamma^m}{=} \gamma^m \gamma^{(m+1)\left(s+1-\frac{2s}{q'}\right) - \left(\frac{1-m}{2} + \frac{m+1}{2s} - \frac{1}{q'}\right)2s} \left(\sum_k \|a_k\|_{q'}^{2k(1-\frac{m}{2})} \gamma \beta \delta \right)^{\frac{2s}{q'}}.
\end{aligned} \tag{2.6.22}$$

We determine the exponents: it holds

$$\begin{aligned}
& (m+1) \left(s+1 - \frac{2s}{q'} \right) - \left(\frac{1-m}{2} + \frac{m+1}{2s} - \frac{1}{q'} \right) 2s \\
&= 2s \left[(m+1) \left(\frac{1}{2} + \frac{1}{2s} - \frac{1}{q'} \right) - \frac{1-m}{2} - \frac{m+1}{2s} + \frac{1}{q'} \right] \\
&= 2s \left[\frac{m+1}{2} - \frac{m}{q'} - \frac{1-m}{2} \right] \\
&= 2s m \left[1 - \frac{1}{q'} \right] = \frac{2sm}{q}, \tag{2.6.23}
\end{aligned}$$

and

$$\frac{1-m}{2} + \frac{m+1}{2s} - \frac{1}{q'} = \frac{1}{q} - \frac{m+1}{2} + \frac{m+1}{2s} = \frac{1}{q} - \frac{m+1}{2} \left(1 - \frac{1}{s} \right) \stackrel{(2.6.19)}{=} \frac{1}{q} - \frac{m+1}{p'}. \tag{2.6.24}$$

A further computation shows

$$\int \phi_\alpha^{q'} dx \approx |\Gamma_\alpha^\delta| = |\Gamma_{kjn}^\delta| \approx \gamma_k \beta \delta = 2^{k(1-\frac{m}{2})} \gamma \beta \delta, \tag{2.6.25}$$

thus (2.6.22) translates into

$$\begin{aligned}
\| \sum_\alpha a_\alpha \hat{\phi}_\alpha \|_{p'}^{2s} &\lesssim \gamma^{m \frac{2s}{q}} \left(\sum_k \| a_k (2^k \gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \|_{q'}^{q'} \int \phi_\alpha^{q'} dx \right)^{\frac{2s}{q'}} \\
&= \delta^{\frac{2s}{q}} \left(\int \sum_\alpha | a_\alpha (2^k \gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_\alpha |^{q'} dx \right)^{\frac{2s}{q'}} \\
&\approx \left(\delta^{\frac{1}{q}} \| \sum_\alpha a_\alpha ((2^k-1)\gamma)^{\frac{1}{q} - \frac{m+1}{p'}} \phi_\alpha \|_{q'} \right)^{2s},
\end{aligned}$$

completing the proof of Theorem 2.2.2. □

2.7 Appendix: auxiliary lemmas

2.7.1 Classical lemmas

LEMMA 2.7.1

Let $1 \leq r < 2$, and $f : \mathbb{N} \rightarrow \mathbb{N}$ "almost" injective, i.e. there exists a constant $C > 0$ such that $\forall l : |\{k \in \mathbb{N} : f(k) = l\}| \leq C$. (2.7.1)

Then

$$\left| \sum_{k,l} a_k G_{f(k)-l} b_l \right| \leq C^{\frac{1}{r'}} \|a\|_r \|G\|_{\frac{r'}{2}, \infty} \|b\|_r$$

holds for all sequences $a, b \in \ell_r$, $G \in \ell_{\frac{r'}{2}, \infty}$.

Proof: At first, Hölder's inequality gives

$$\left| \sum_{k,l} a_k G_{f(k)-l} b_l \right| \leq \|a\|_r \|(G * b)(f(\cdot))\|_{r'}.$$

We further observe

$$\sum_k |G * b|^{r'}(f(k)) \stackrel{(2.7.1)}{\leq} C \sum_l |G * b|^{r'}(l).$$

Since $r < 2$, we have $r' > 2$, i.e. $\frac{r'}{2} > 1$. Hence we apply Young's inequality with parameters $1 + \frac{1}{r'} = \frac{2}{r'} + \frac{1}{r}$ to obtain

$$\|G * b\|_{r'} \leq \|G\|_{\frac{r'}{2}, \infty} \|b\|_r.$$

Altogether, this provides the desired estimate. □

LEMMA 2.7.2 (HÖLDER'S INEQUALITY IN LORENTZ SPACES)

Let $\frac{1}{p_i} = \frac{1}{q_i} + \frac{1}{r_i}$, $i = 1, 2$. Then

$$\|fg\|_{p_1, p_2} \lesssim \|f\|_{q_1, q_2} \|g\|_{r_1, r_2}.$$

Proof: This is a consequence of the classical Hölder inequality and the fact that the decreasing rearrangement satisfies $(fg)^*(2t) \leq f^*(t)g^*(t)$ (see Proposition 1.4.5 No.(7) in [Gra]). □

LEMMA 2.7.3 (BRUNN-MINKOWSKI)

Let K_0, K_1 be compact and convex subsets of \mathbb{R}^n and $0 \leq t \leq 1$. Then

$$(1-t) |K_0|^{\frac{1}{n}} + t |K_1|^{\frac{1}{n}} \leq |(1-t)K_0 + tK_1|^{\frac{1}{n}}.$$

Proof: Compare [GW], theorem 6.2, p. 57. □

2.7.2 More specific lemmas

LEMMA 2.7.4

Let K and L be two convex bodies in \mathbb{R}^n symmetric with respect to the origin (i.e. $K = -K$). Then $\sup_{z \in \mathbb{R}^n} |K \cap (z + L)| = |K \cap L|$ holds for all $z \in \mathbb{R}^n$.

It is a simple observation that both conditions on K and L are crucial.

Proof: Let z be some point in \mathbb{R}^n . We have to show that $|K \cap (z + L)| \leq |K \cap L|$. Therefore we introduce the sets $A = (L + z) \cap K$ and $B = (L - z) \cap K$. The proof will be divided in three steps:

$$(i) \quad \frac{A+B}{2} \subset K \cap L$$

$$(ii) \quad |A| = |B|$$

$$(iii) \quad |A| \leq \left| \frac{A+B}{2} \right|.$$

Notice that (iii) and (i) immediately imply what we claimed.

Proof of (i): On the one hand, since $A \subset K$ and $B \subset K$, we have $\frac{A+B}{2} \subset \frac{K+K}{2} \stackrel{K \text{ convex}}{=} K$.

On the other hand, since $A \subset L + z$, $B \subset L - z$, we have $\frac{A+B}{2} \subset \frac{L+z+L-z}{2} \stackrel{L \text{ convex}}{=} L$.

Proof of (ii): This is clear since $A = -B$ by the symmetry condition on K and L .

Proof of (iii): We apply the inequality of Brunn-Minkowski:

$$\begin{aligned} |A| &= \left(\frac{1}{2}|A|^{\frac{1}{n}} + \frac{1}{2}|A|^{\frac{1}{n}} \right)^n \\ &\stackrel{(ii)}{=} \left(\frac{1}{2}|A|^{\frac{1}{n}} + \frac{1}{2}|B|^{\frac{1}{n}} \right)^n \\ &\stackrel{\text{Brunn-Minkowski}}{\leq} \left| \frac{A+B}{2} \right|. \quad \square \end{aligned}$$

LEMMA 2.7.5

Let X be a normed vector space and $U_1, U_2, V_1, V_2 \subset X$ with the property $(U_1 + V_1) \cap (U_2 + V_2) \neq \emptyset$. Furthermore let P be a linear functional on X and let $x_i \in U_i$, $a = P(x_2) - P(x_1)$ and $y_i \in V_i$, $b = P(y_2) - P(y_1)$. Then we have

$$a + b \leq \sum_{i=1,2} [\text{diam}(P(U_i)) + \text{diam}(P(V_i))]. \quad (2.7.2)$$

Proof: Choose some $\xi \in (U_1 + V_1) \cap (U_2 + V_2)$. There exists $u_i \in U_i$, $v_i \in V_i$, $i = 1, 2$ with $u_1 + v_1 = \xi = u_2 + v_2$. It follows

$$\begin{aligned} a + b &= P(x_2 - x_1) + P(y_2 - y_1) \\ &= P(x_2 - x_1) + P(y_2 - y_1) + P(u_1 + v_1) - P(u_2 + v_2) \\ &= P(x_2 - u_2) + P(y_2 - v_2) + P(u_1 - x_1) + P(v_1 - y_1) \\ &\leq \text{diam}(P(U_2)) + \text{diam}(P(V_2)) + \text{diam}(P(U_1)) + \text{diam}(P(V_1)), \end{aligned}$$

completing the proof. □

LEMMA 2.7.6

Let $I = \{0, \dots, n\}$, $J = \{0, \dots, m\} \subset \mathbb{N}$ and let $a = (a_i)_{i \in I} \in \mathbb{R}^I$, $b = (b_j)_{j \in J} \in \mathbb{R}^J$ be two increasing, finite sequences such that

$$\begin{aligned} C^{-1} &\leq a_{i+1} - a_i \leq C && \text{for all } i \in I \\ C^{-1} &\leq b_{j+1} - b_j \leq C && \text{for all } j \in J. \end{aligned} \quad (2.7.3)$$

Moreover we assume that $b_m - b_0 \leq a_n - a_0$. (2.7.4)

Then there exists a function $f : J \rightarrow I$ such that

$$\begin{aligned} (i) \quad &|a_i - b_j| \geq \frac{1}{2}|a_i - a_{f(j)}| \\ (ii) \quad &\forall i \in I : |\{j : f(j) = i\}| \leq 4C^2 + 2. \end{aligned}$$

The trivial case $I = J$, $a = b$ can of course be solved by $f = id$. In the general setting, we in some sense had to replace every b_j by some $a_{f(j)}$, leaving the distances to other points (almost) unchanged. Condition (ii) can be read as a weakening of injectivity.

Proof:

Without loss of generality, we may assume $b_m \leq a_n$. (2.7.5)

To be more precise, if $b_m > a_n$ would hold, we would consider $\bar{a}_i = -a_{n-i}$ and $\bar{b}_j = -b_{m-j}$, which also fulfill the requirements of the lemma. Then

$$\bar{b}_m = -b_0 \stackrel{(2.7.4)}{\leq} a_n - a_0 - b_m < -a_0 = \bar{a}_n.$$

So, if we would find a function \bar{f} appropriate to \bar{a} , \bar{b} in the sense of (i) and (ii), $f(j) := n - \bar{f}(m - j)$ would be a solution appropriate to a and b since

$$|a_i - b_j| = |\bar{a}_{n-i} - \bar{b}_{m-j}| \geq \frac{1}{2}|\bar{a}_{n-i} - \bar{a}_{\bar{f}(m-j)}| = \frac{1}{2}|a_i - a_{n-\bar{f}(m-j)}|.$$

Furthermore, we may also assume $a_0 \leq b_m$ without loss of generality. (2.7.6)

In the case $a_0 > b_m$, we would introduce $\bar{a}_i = a_i - a_0 + b_m < a_i$, such that $\bar{a}_0 = b_m \leq b_m$. If f is a function associated to \bar{a} and b as required, then

$$|b_j - a_i| \stackrel{b_m < a_0}{=} a_i - b_j > \bar{a}_i - b_j \stackrel{b_m = \bar{a}_0}{=} |b_j - \bar{a}_i| \geq \frac{1}{2}|\bar{a}_{f(j)} - \bar{a}_i| = \frac{1}{2}|a_{f(j)} - a_i|.$$

It would be helpful for the construction of f to assign an i to every given j in a way that b_j is close to a_i . Nevertheless, if the sequences are somehow shifted against each others (for instance, $b_0 \ll a_0$), there might be no a_i "close" to b_j . If we would always choose the closest a_i , this would hurt condition (ii).

Therefore we reflect a_i 's at a_0 , to ensure that for every b_j there is a (maybe reflected) point a_i nearby.

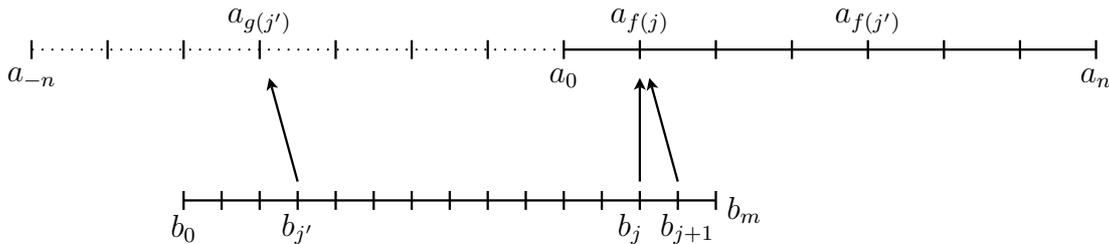


Figure 2.9: Setting of the sequences a and b

Let $\bar{I} = I \cup (-I)$. Define the continuation of a on \bar{I} by $a_{-i} = 2a_0 - a_i$, $i \in I$. We then find a function $g : J \rightarrow \bar{I}$ such that $|a_{g(j)} - b_j| = \min_{i \in \bar{I}} |a_i - b_j|$. The claim is that $f = |g|$ is a solution of our problem.

Checking condition (i):

Case 1: $g(j) \geq 0$. Here

$$|a_i - a_{f(j)}| = |a_i - a_{g(j)}| \leq |a_i - b_j| + |b_j - a_{g(j)}| \stackrel{\text{minimality of } g}{\leq} 2|a_i - b_j|.$$

Case 2: $g(j) < 0$. Here

$$\begin{aligned} |a_i - a_{f(j)}| &= |a_i - a_{-g(j)}| = |a_i - 2a_0 + a_{g(j)}| \\ &\leq |a_i - a_0| + |a_0 - a_{g(j)}| \stackrel{\text{monotony}}{=} a_i - a_0 + a_0 - a_{g(j)} = |a_i - a_{g(j)}|, \end{aligned}$$

and we proceed as in case 1.

Checking condition (ii):

Obviously, it is sufficient to show that $|\{j : g(j) = i\}| \leq 2C^2 + 1$ holds for every $i \in \bar{I}$.

Thus let $g(j) = i$.

We claim that $|a_i - b_j| \leq C$ and check this:

Case 1: $a_i < b_j$. Would $i = n$, then $b_j \leq b_m \stackrel{(2.7.5)}{\leq} a_n = a_i < b_j$, hence $i < n$. Thus a_{i+1}

is well-defined and $b_j < a_{i+1}$ holds according to the minimality in the choice of g . We conclude $|a_i - b_j| = b_j - a_i \leq a_{i+1} - a_i \leq C$.

Case 2: $a_i > b_j$. Would $i = -n$, then

$$b_j < a_i = a_{-n} = 2a_0 - a_n \stackrel{(2.7.6)}{\leq} b_m - (a_n - a_0) \stackrel{(2.7.4)}{\leq} b_0 \leq b_j.$$

Hence we have $i > -n$, thus a_{i-1} is well-defined and $b_j > a_{i-1}$ holds. We finish as in case 1.

Case 3: $a_i = b_j$. This case is trivial.

If additionally $g(j') = i$, then $|a_i - b_j| \leq C$ and $|a_i - b_{j'}| \leq C$. Now we utilise (2.7.3). Therefrom we get by induction

$$\frac{1}{C}|j - j'| \leq |b_j - b_{j'}| \leq |b_j - a_i| + |a_i - b_{j'}| \leq 2C,$$

i.e.

$$|j - j'| \leq 2C^2,$$

and thus condition (ii). □

REMARK

If assumption (2.7.4) is not fulfilled, a comparable result can be obtained by interchanging the roles of I and J or a and b respectively. Actually, the lemma remains also essentially valid if these quantities are not interchanged. Therefore we just need some further mirrorings, if necessary also at the "upper" end a_n . This would enlarge the bound $4C^2 + 2$ from (ii) depending on the relation between n and m . Nevertheless, the notation would quickly become confusing, thus I omit such a version of the lemma.

Chapter 3

A Fourier restriction theorem for a two-dimensional surface of finite type

Joint work with Prof. Dr. Detlef Müller and Prof. Dr. Ana Vargas

3.1 Introduction

We start with a description of the surfaces we want to study. Let $Q = [0, 1] \times [0, 1]$. For a smooth phase function $\phi : Q \rightarrow \mathbb{R}$ define the graph $\Gamma = \Gamma(\phi) = \{(x, \phi(x)) | x \in Q\}$. We equip Γ with the Lebesgue measure¹ σ and study the operator

$$R^*(f)(x) = \widehat{f d\sigma}(x) = \int_{\Gamma} f(\xi) e^{-ix \cdot \xi} d\sigma(\xi), \quad (3.1.1)$$

where $f \in L^q(\Gamma)$.

3.1.1 Outline of the problem

The phase function we have in mind shall obey the following definition:

DEFINITION

Let $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq 2$. We say that a function ϕ is of normalised finite type (m_1, m_2) if there exists $\phi_1, \phi_2 \in C^\infty([0, 1])$ and $a, b > 0$ such that $\phi(x) = \phi_1(x_1) + \phi_2(x_2)$ and

$$at^{m_i-k} \leq \phi_i^{(k)}(t) \leq bt^{m_i-k} \quad (3.1.2)$$

for all $i = 1, 2$, $t \in [0, 1]$, $0 \leq k \leq m_i$.

The definition may surprise as it does not allow perturbation terms depending on both x_1 and x_2 . However, such perturbation terms are not covered by our proof. Although it might be a problem depending on the method, this is totally unclear at the moment. This certainly is an interesting question for future research.

The generic example is $\phi(x) = x_1^{m_1} + x_2^{m_2}$, but even more general functions with finite order of contact can be normalised to our requirements:

REMARK

(i) Let $\varepsilon > 0$ and $\varphi \in C^\infty(-\varepsilon, \varepsilon)$ be of finite type m in 0, i.e. $\varphi(0) = \varphi'(0) = \dots = \varphi^{(m-1)}(0) = 0 \neq \varphi^{(m)}(0)$. Assume $\varphi^{(m)}(0) > 0$. Then there exist $\varepsilon' \in (0, \varepsilon)$ such that

$$\varphi^{(k)}(t) \approx t^{m-k} \quad (3.1.3)$$

for all $0 \leq k \leq m$, $|t| < \varepsilon'$.

(ii) Further let $\phi(x) = \varphi_1(x_1) + \varphi_2(x_2)$, $|x| \leq \varepsilon$, where $\varphi_i \in C^\infty(-\varepsilon, \varepsilon)$ is of finite type m_i in 0 with $\varphi_i^{(m_i)}(0) > 0$. Then there exist an $\bar{\varepsilon} > 0$ such that $y \rightarrow \phi(\bar{\varepsilon}y)$ is of normalised finite type (m_1, m_2) .

Proof: (i) Since φ has a zero of order m at the origin, we find $e_0 > 0$, a smooth function $\chi_0 : (-\varepsilon_0, \varepsilon_0) \rightarrow (0, \infty)$ and a sign $\sigma = \pm 1$ such that

$$\varphi(t) = \sigma t^m \chi_0(t) \quad (3.1.4)$$

¹Since Γ is compact, the surface measure has a density bounded from above and from below; therefore the boundedness of the restriction operator remains the same whatever of the both measures we take.

for all $|t| < \varepsilon_0$. Thus $\varphi'(t) = \sigma t^{m-1}[m\chi_0(t) + t\chi_0'(t)]$. Since χ_0' is bounded on any compact neighborhood of the origin, and χ_0 is bounded from below by a positive constant if the neighborhood is small enough, we have $t\chi_0'(t) \ll m\chi_0(t)$ for t small enough, thus we find $0 < \varepsilon_1 < \varepsilon_0$

$$\varphi'(t) = \sigma t^{m-1}\chi_1(t) \quad (3.1.5)$$

where $\chi_1(t) = m\chi_0(t) + t\chi_0'(t) > 0$ for all $|t| \leq \varepsilon_1$. Iterating the procedure gives $\varepsilon_k > 0$, $\chi_k : (-\varepsilon_k, \varepsilon_k) \rightarrow (0, \infty)$ such that

$$\varphi^{(k)}(t) = \sigma t^{m-k}\chi_k(t) \quad (3.1.6)$$

for any $|t| < \varepsilon_k$. Finally observe that $0 < \varphi^{(m)}(0) = \sigma\chi_m(0)$, i.e. $\sigma = 1$.

(ii) Chose $\bar{\varepsilon} > 0$ such that for both $i = 1, 2$, $0 \leq k \leq m_i$ and all $0 \leq t \leq \bar{\varepsilon}$

$$\varphi_i^{(k)}(t) \approx t^{(m_i-k)}. \quad (3.1.7)$$

Then for all $0 \leq s \leq 1$

$$\frac{d^k}{ds^k}\varphi_i(\bar{\varepsilon}s) \approx \bar{\varepsilon}^k(\bar{\varepsilon}s)^{(m_i-k)} = \bar{\varepsilon}^{m_i} s^{(m_i-k)}. \quad (3.1.8)$$

□

To formulate our main theorem, we need to introduce the height h of the surface, which is given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$. Further define $\bar{m} = m_1 \vee m_2 = \max\{m_1, m_2\}$.

THEOREM 3.1.1

Let $p > p_c = 4\frac{\bar{m}+3}{\bar{m}+4}$ and $\frac{1}{q'} \geq \frac{h+1}{p}$. Additionally we assume $h+1 < p$.

Then R^* is bounded from $L^{q,s}(\Gamma)$ to $L^{p,s}(\mathbb{R}^3)$ for any $1 \leq s \leq \infty$.

If moreover $q \leq p$ or $\frac{1}{q'} > \frac{h+1}{p}$, then R^* is bounded from $L^q(\Gamma)$ to $L^p(\mathbb{R}^3)$.

Unfortunately, the methods we use do not give the strong $L^q(\Gamma)$ to $L^p(\mathbb{R}^3)$ estimate above the diagonal $q = p$.

3.1.2 Necessary conditions

The first condition $p > 4\frac{\bar{m}+3}{\bar{m}+4}$ is not sharp, but we will discuss some examples. For $m_1 = 2$ fixed and $\bar{m} = m_2 \rightarrow \infty$, the contact order in the second coordinate direction degenerates, thus we would expect the same p -range as for a twodimensional cylinder, which is the same range as for a parabola, namely $p > 4$ [Z]. Indeed $4\frac{\bar{m}+3}{\bar{m}+4} \rightarrow 4$ as $\bar{m} \rightarrow \infty$, so in the limit we obtain the correct range. Another interesting case is to take the paraboloid, i.e. $m_1 = 2 = m_2 = \bar{m}$. Then our condition becomes $p > 4\frac{\bar{m}+3}{\bar{m}+4} = \frac{10}{3}$. The conjectured range is $p > 3$. Nevertheless the best known result in this dimension is exactly $p > \frac{10}{3}$ [T1].

The second condition $\frac{1}{q'} \geq \frac{h+1}{p}$ is necessary as one can see by a so-called Knapp box example. That means testing the restriction operator on a function f whose Fourier

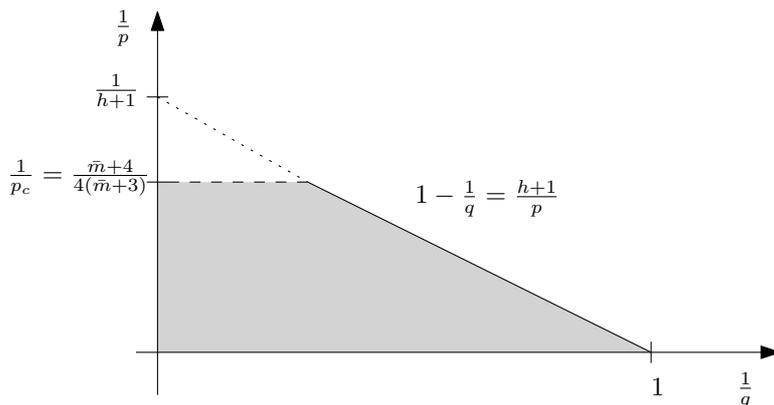


Figure 3.1: Range of p and q in Theorem 3.1.1

transform is a smoothed version of the characteristic function of a box near the origin of size $\varepsilon^{\frac{1}{m_1}} \times \varepsilon^{\frac{1}{m_2}} \times \varepsilon$. Because this box is very well adapted to the surface, we obtain

$$\|\hat{f}|_{\Gamma}\|_{L^{q'}(S)} \approx \varepsilon^{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{1}{q'}} = \varepsilon^{\frac{1}{hq'}},$$

whereas

$$\|f\|_{L^{p'}(\mathbb{R}^3)} \approx \varepsilon^{\left(\frac{1}{m_1} + \frac{1}{m_2} + 1\right)\left(1 - \frac{1}{p'}\right)} = \varepsilon^{\frac{h+1}{hp}}.$$

Dualisation of the boundedness of R^* yields

$$\|Rf\|_{L^{q'}(\Gamma)} = \|\hat{f}|_{\Gamma}\|_{L^{q'}(\Gamma)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^3)}.$$

For $\varepsilon \rightarrow 0$ we obtain $\frac{1}{q'} \geq \frac{h+1}{p}$.

The third condition $p > h + 1$ is in some sense the weakest. The second condition already implies $p \geq h + 1$, for $q' > 1$ even $p > h + 1$. Thus the third condition only plays a role when the sharp line $\frac{1}{q'} = \frac{h+1}{p}$ intersects the axis $\frac{1}{q} = 0$ at a point $p > p_c$ (cf. Figure 3.1). However, the condition $p > h + 1$ is necessary as well, although some kind of weak type estimate might hold true at the endpoint.

This can be shown by analysing the oscillatory integral. A careful study of the proof of proposition 3, chapter VIII, §1 in [St] shows that

$$\left| \int_0^1 e^{-i(x_1\xi - x_3\xi^{m_1})} d\xi \right| \gtrsim x_3^{-\frac{1}{m_1}} \quad (3.1.9)$$

for $1 \leq x_3 < \infty$, $1 \ll x_1^{m_1} \ll x_3$. Therefore

$$\begin{aligned} \|\widehat{d\sigma}\|_p^p &\geq \int_1^\infty \int_{1 \ll x_2 \ll x_3^{\frac{1}{m_2}}} \int_{1 \ll x_1 \ll x_3^{\frac{1}{m_1}}} |\widehat{d\sigma}(x_1, x_2, -x_3)|^p dx_1 dx_2 dx_3 \\ &\gtrsim \int_1^\infty \int_{1 \ll x_2 \ll x_3^{\frac{1}{m_2}}} dx_2 \int_{1 \ll x_1 \ll x_3^{\frac{1}{m_1}}} dx_1 x_3^{-p\left(\frac{1}{m_1} + \frac{1}{m_2}\right)} dx_3 \\ &\gtrsim \int_1^\infty x_3^{(1-p)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)} dx_3 \\ &= \int_1^\infty x_3^{-\frac{p-1}{h}} dx_3. \end{aligned}$$

If we assume the adjoint Fourier restriction operator to be bounded, the integral has to be finite, thus $\frac{p-1}{h} > 1$, i.e. $p > h + 1$.

3.1.3 The structure of the approach

The approach is to study certain bilinear operators. For a suitable pair of surfaces $S_1, S_2 \subset \Gamma$ (we will be more specific on this point later), we seek to establish a bilinear estimate like

$$\|\widehat{f_1 d\sigma_{S_1}} \widehat{f_2 d\sigma_{S_2}}\|_{L^p(\mathbb{R}^3)} \leq C_p C(S_1, S_2) \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)}. \quad (3.1.10)$$

The big difference to the usual bilinear estimates is that we need to know how the constant $C(S_1, S_2)$ depends on S_1 and S_2 to be able to sum all the bilinear estimates to pass to the linear estimate. Classically, this is done by proving a bilinear estimate for a single "generic" case of subsurfaces. If Γ is the paraboloid, other pairs of subsurfaces can be reduced to that case by suitable affine transformations and homogeneous rescaling. General surfaces do not come with this self-similarity under that transformations.

Therefore it is very important to compute the constant $C(S_1, S_2)$ (which will be some expression depending of the side length and local principal curvatures of the surfaces) and keep track of it during the whole proof. In this sense, many of the lemmas are generalised, quantitative versions of well known results from classical bilinear theory.

The pairs of subsurfaces we would like to discuss are parts of the surface sitting over two dyadic rectangles, satisfying certain separation or "transversality" assumptions. However, such a rectangle might touch one of the axis, where some principle curvature is vanishing. In this case we will decompose dyadical. But even on these smaller sets, we do not have the correct "transversality" conditions, we first have to find a proper rescaling such that the scaled subsurfaces allow to run the bilinear machinery.

The following chapter will start with the bilinear argument to provide us with a very general bilinear result for a sufficiently "good" pair of surfaces. In the next chapter, we construct the suitable scaling to apply this general result to our situation. After rescaling and several additional arguments, we pass to a global bilinear estimate and finally proceed to the linear estimate.

Since the bilinear result we obtain is very technical and the constant not obvious at all, the very last chapter discusses the sharpness of the bilinear result. We will estimate the constant at the endpoint from below and see that it is essentially optimal.

A few more remarks on how to read this paper: as mentioned, it is very important to know precisely how the constants depend on the specific choice of subsurfaces. Moreover, there appear other constants maybe depending on m_1, m_2, p, q , or other quantities, but explicitly not on the choice of subsurfaces. We will not keep track of these type of constants, since it would even set a false focus and distract the reader. Instead we will simply use the symbol \lesssim for an inequality involving one of these constants of minor importance. To be more precise on this, later we introduce a family of pairs of subsurfaces \mathcal{S}_0 . Then for quantities $A, B : \mathcal{S}_0 \rightarrow \mathbb{R}$ the inequality $A \lesssim B$ means there exists a constant $C > 0$ such that $A(S) \leq CB(S)$ uniformly for all $S \in \mathcal{S}_0$.

Moreover, we will also use the notation $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We will even use this

notation for vectors, meaning their entries are comparable in each coordinate. Similarly, we write $A \ll B$ if there exists a constant $c > 0$ such that $A(S) \leq cB(S)$ for all $S \in \mathcal{S}_0$ and c is "small enough" for our purposes. This "small enough" depends on the situation and other constants, but is still uniformly in \mathcal{S}_0 .

3.1.4 List of notations

$C^\infty(U)$	set of all smooth functions on $U \subset \mathbb{R}^n$
$C_0^\infty(U)$	set of all smooth functions compactly supported in $U \subset \mathbb{R}^n$
$B(x, r)$	ball with radius r centered at x with respect to the euclidean norm
$B_p(x, r)$	ball with radius r centered at x with respect to the ℓ^p -norm
$\mathcal{S}, \mathcal{S}(\mathbb{R}^n)$	space of Schwartz functions on \mathbb{R}^n
$a \vee b$	maximum of two real numbers a and b
$a \wedge b$	minimum of two real numbers a and b
R^*	adjoint Fourier restriction operator
σ	Lebesgue measure of a surface
T_w, T_{w_j}	wave tubes
p_w, p_{w_j}	wave packets
$\mathcal{V}, \mathcal{V}_j$	set of directions of the wave packets
$\mathcal{Y}, \mathcal{Y}_j$	set of translations of wave packets
$\mathcal{W}, \mathcal{W}_j$	$\mathcal{W} = \mathcal{Y} \times \mathcal{V}$
\mathcal{S}_0	set of pairs of subsurfaces
p'	Hölder exponent conjugate to p , given by $\frac{1}{p} + \frac{1}{p'} = 1$
p^*	the smaller one of the exponents p and p'
Π_{v_1, v_2}	intersection set of $S_1 + v_2$ and $S_2 + v_1$, thickened by $\mathcal{O}(R'^{-1})$
$[W_j]^{\Pi_{v_1, v_2}}$	all $w_j \in W_j$ with directions belonging to Π_{v_1, v_2}
$[V_j]^{\Pi_{v_1, v_2}}$	projection of $[W_j]^{\Pi_{v_1, v_2}}$ to \mathcal{V} -coordinates
ϕ_1, ϕ_2	component functions of ϕ , i.e. $\phi(x) = \phi_1(x_2) + \phi_2(x_2)$
ϕ^s	scaled phase function $\phi^s(\eta_1, \eta_2) = \frac{1}{a_1 a_2} \phi(a_1 \eta_1, a_2 \eta_2)$
a_1, a_2	scaling factors, defined by $a_i = \bar{\kappa}_{i+1 \bmod 2} \bar{d}_{i+1 \bmod 2}$
$q(a, b)$	maximal quotient of two positive numbers a and b , i.e. the larger one of the numbers $\frac{a}{b}$ and $\frac{b}{a}$
x_+	the positive part of x , i.e. $x_+ = x$ if $x \geq 0$, and $x_+ = 0$ if $x < 0$
$x \cdot y, \langle x, y \rangle$	standard scalar product of two vectors $x, y \in \mathbb{R}^n$

More notations, more precisely the general assumptions relevant for a specific section, can be found on pages 62, 84 and 89.

3.2 General bilinear theory

We start with what is basically a well known result, although we need a more quantitative version.

3.2.1 Wave packet decomposition

LEMMA 3.2.1

Let $U \subset \mathbb{R}^d$ compact, $\phi \in C^\infty(U, \mathbb{R})$ with $\|\nabla\phi\|_\infty \lesssim 1$, $\|\phi''|_U\|_\infty = \kappa$. We assume that there exists a constant $D \leq \frac{1}{\kappa}$ such that $\|\partial^\alpha\phi\|_\infty \lesssim \kappa D^{2-|\alpha|}$ for all $|\alpha| \geq 2$. Then for every $R \geq 1$ there exists a so-called wave packet decomposition adapted to ϕ with tubes of size $\frac{R}{D} \times \frac{R}{D} \times \frac{R^2}{D^2\kappa} = R' \times R' \times \frac{R^2}{\kappa}$, $R = R'D$, i.e.: Let $\mathcal{Y} = R'\mathbb{Z}^d$, $\mathcal{V} = R'^{-1}\mathbb{Z}^d \cap U$, and for $w = (y, v) \in \mathcal{Y} \times \mathcal{V} = \mathcal{W}$, define

$$T_w = \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |t| \leq \frac{R'^2}{\kappa}, |x - y + t\nabla\phi(v)| \leq R'\}. \quad (3.2.1)$$

Then there exist functions (wave packets) $\{p_w\}_{w \in \mathcal{W}}$ such that for all $|t| \leq \frac{R'^2}{\kappa}$

(P1) $p_w = R^*(\mathfrak{F}^{-1}p_w(\cdot, 0))$,

(P2) $\text{supp } \mathfrak{F}p_w \subset B((v, \phi(v)), \mathcal{O}(R'^{-1}))$,

(P3) p_w is essentially supported in T_w , i.e. $|p_w(x, t)| \leq C_N R'^{-\frac{d}{2}} \left(1 + \frac{|x-y+t\nabla\phi(v)|}{R'}\right)^{-N}$.
Especially $\|p_w(\cdot, t)\|_2 \lesssim 1$.

(P4) For all $W \subset \mathcal{W}$, we have $\|\sum_{w \in W} p_w(\cdot, t)\|_2 \leq |W|^{\frac{1}{2}}$.

(P5) For all $f \in L^2$ there exists a sequence $c = \{c_w\}_{w \in \mathcal{W}} \in \mathbb{C}^{\mathcal{W}}$ such that $R^*f(x, t) = \sum_{w \in \mathcal{W}} c_w p_w(x, t)$ and

(P6) $\|c\|_{\ell^2} \lesssim \|f\|_{L^2}$.

REMARK

The boundedness of the gradient of ϕ is not needed in the proof explicitly. However, it makes the wave packets having the "correct" length of (roughly) $\frac{R'^2}{\kappa}$, and not to degenerate.

REMARK 3.2.2

i) Let $N(v) = (-\nabla\phi(v), 1)$. We may rewrite $T_w = (y, 0) + \{tN(v) : |t| \leq \frac{R'^2}{\kappa}\} + \mathcal{O}(R')$. Therefore for every $|t| \leq \frac{R'^2}{\kappa}$ we have $(y, 0) + tN(v) \in T_w$ and thus

$$|x - y + t\nabla\phi(v)| = |(x, t) - (y, 0) - tN(v)| \geq \text{dist}((x, t), T_w).$$

This justifies the statement " p_w is essentially supported in T_w "; (P3) can be changed to

$$|p_w(z)| \leq C_N R'^{-\frac{d}{2}} \left(1 + \frac{\text{dist}(z, T_w)}{R'}\right)^{-N} \quad (3.2.2)$$

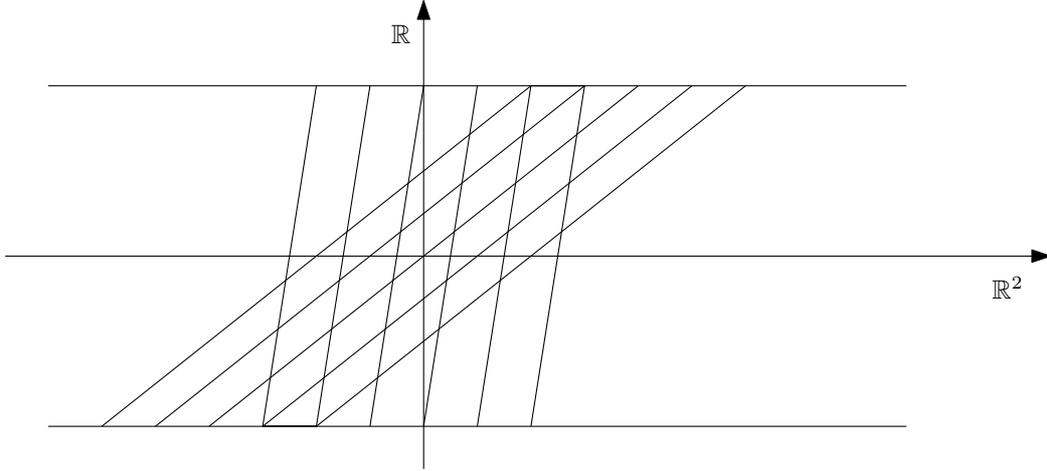


Figure 3.2: The tubes T_w fill a horizontal strip

for all $z \in \mathbb{R}^{d+1}$ with $|\langle z, e_3 \rangle| \leq \frac{R'^2}{\kappa}$.

ii) Notice further that we can reparametrise the wave packets by lifting \mathcal{V} to $\mathcal{V}' = \{(v, \phi(v)) : v \in \mathcal{V}\} \subset S$. Since $\|\nabla\phi\| \lesssim 1$, we have $|(v, \phi(v)) - (v', \phi(v'))| \approx |v - v'|$, thus \mathcal{V}' becomes a R'^{-1} -net in S . Finally, we identify a parameter $y \in \mathbb{R}^d$ with the point $(y, 0)$ in the hyperplane $\mathbb{R}^d \times \{0\}$.

Proof: We will essentially copy the proof by Lee [L1] or refer to it. The only new feature is the role of the constant κ .

Let $\psi, \hat{\eta} \in C_0^\infty(B(0, 1))$ be chosen in a way that for $\eta_y(x) = \eta(\frac{x-y}{R'})$, $\psi_v(\xi) = \psi(R'(\xi - v))$ we have $\sum_{v \in \mathcal{V}} \psi_v = 1$ on U and $\sum_{y \in \mathcal{Y}} \eta_y = 1$. Let $\tilde{\psi} \in C_0^\infty(B(0, 3))$ such that $\tilde{\psi} = 1$ on

$B(0, 2) \supset \text{supp} \psi + \text{supp} \hat{\eta}$. Introduce also $\tilde{\psi}_v(x) = \tilde{\psi}(R'(x - v))$.

Then the functions $F_{y,v} = \mathfrak{F}^{-1}(\widehat{\psi_v f \eta_y}) = (\psi_v f) * \tilde{\eta}_y$ $y \in \mathcal{Y}$, $v \in \mathcal{V}$ are well localised in both position and momentum space. Define $q_w = R^*(F_w)$, $w = (y, v) \in \mathcal{W}$; up to a certain factor c_w which will be determined later, these are already the announced wave packets, i.e. $q_w = c_w p_w$.

Since $f = \sum_{w \in \mathcal{W}} F_w$, we have $R^* f = \sum_{w \in \mathcal{W}} q_w$ (P5). The most difficult property is (P3).

Since $\text{supp} F_{y,v} \subset B(v, \mathcal{O}(R'^{-1}))$, we have for every $w = (y, v) \in \mathcal{W}$

$$\begin{aligned}
 q_w(x, t) &= \int e^{-i(x\xi + t\phi(\xi))} F_w(\xi) d\xi \\
 &= \int e^{-i(x\xi + t\phi(\xi))} F_w(\xi) \tilde{\psi}_v(\xi) d\xi \\
 &= \int \int e^{-i((x-z)\xi + t\phi(\xi))} \tilde{\psi}_v(\xi) d\xi \hat{F}_w(z) dz \\
 &= R'^{-d} \int \int e^{-i((x-z)(\frac{\xi}{R'} + v) + t\phi(\frac{\xi}{R'} + v))} \tilde{\psi}(\xi) d\xi \hat{F}_w(z) dz \\
 &= R'^{-d} \int K(x - z, t) \hat{F}_w(z) dz
 \end{aligned}$$

with the kernel

$$K(x, t) = \int e^{i(x(\frac{\xi}{R'}+v)+t\phi(\frac{\xi}{R'}+v))} \tilde{\psi}(\xi) d\xi. \quad (3.2.3)$$

We claim that

$$|K(x, t)| \lesssim \left(1 + \frac{|x + t\nabla\phi(v)|}{R'}\right)^{-N} \quad (3.2.4)$$

for every $N \in \mathbb{N}$. We have to analyse the oscillatory integral

$$K_\lambda = \int e^{i\lambda\Phi(\xi)} \tilde{\psi}(\xi) d\xi, \quad (3.2.5)$$

with

$$\Phi(\xi) = \frac{x(\frac{\xi}{R'} + v) + t\phi(\frac{\xi}{R'} + v)}{1 + R'^{-1}|x + t\nabla\phi(v)|}, \quad (3.2.6)$$

where the λ we have in mind finally is $\lambda = 1 + R'^{-1}|x + t\nabla\phi(v)|$. In order to apply integration by parts on K_λ , we need to check

$$\|\nabla\Phi\|_\infty \approx 1 \quad (3.2.7)$$

$$\text{and } \|\partial^\alpha\Phi\|_\infty \lesssim 1 \text{ for all } \alpha \geq 2. \quad (3.2.8)$$

To prove (3.2.4), we may assume $|x + t\nabla\phi(v)| \gg R'$. Then

$$\begin{aligned} \frac{|t|\nabla\phi(\frac{\xi}{R'} + v) - \nabla\phi(v)|}{|x + t\nabla\phi(v)|} &\ll \frac{|t|\nabla\phi(\frac{\xi}{R'} + v) - \nabla\phi(v)|}{R'} \\ &\lesssim \frac{|t|}{R'^2} \|\phi''\|_\infty \\ &\leq \frac{1}{\kappa} \|\phi''\|_\infty \leq 1, \end{aligned}$$

for every $\xi \in \text{supp } \tilde{\psi}$, hence

$$|t|\nabla\phi(\frac{\xi}{R'} + v) - \nabla\phi(v)| \ll |x + t\nabla\phi(v)|. \quad (3.2.9)$$

Thus

$$\begin{aligned} |\nabla\Phi(\xi)| &= \frac{|x + t\nabla\phi(\frac{\xi}{R'} + v)|}{R' + |x + t\nabla\phi(v)|} \\ &= \frac{|x + t\nabla\phi(v) - t[\nabla\phi(v) - \nabla\phi(\frac{\xi}{R'} + v)]|}{R' + |x + t\nabla\phi(v)|} \\ &\approx \frac{|x + t\nabla\phi(v)|}{R' + |x + t\nabla\phi(v)|} \approx 1. \end{aligned}$$

For $|\alpha| \geq 2$ we have

$$\begin{aligned} |\partial^\alpha \Phi(\xi)| &\leq |tR'^{-|\alpha|}(\partial^\alpha \phi)\left(\frac{\xi}{R'} + v\right)| \\ &\lesssim \frac{R'^2}{\kappa} R'^{-|\alpha|} \kappa D^{2-|\alpha|} \\ &\leq (DR')^{2-|\alpha|} = R^{2-|\alpha|} \leq 1. \end{aligned}$$

Integrating by parts, we obtain $|K_\lambda| \lesssim \lambda^{-N}$ for all $N \in \mathbb{N}$, i.e. (3.2.4). Following the proof in [L1], we conclude

$$|q_w(x, t)| \leq R'^{-d} \int \left| K(x - z - y, t) \hat{F}_w(z + y) \right| dz \quad (3.2.10)$$

$$= R'^{-d} \int \left| K(x - z - y, t) \eta\left(\frac{z}{R'}\right) \widehat{\psi_v f}(z + y) \right| dz \quad (3.2.11)$$

$$\lesssim \left(1 + \frac{|x - y + t\nabla\phi(v)|}{R'}\right)^{-N} M(\widehat{\psi_v f})(y), \quad (3.2.12)$$

where M is the Hardy-Littlewood maximal operator. Thus the choice

$c_w = c_{y,v} = R'^{\frac{d}{2}} M(\widehat{\psi_v f})(-y)$ is evident, and gives (P3).

Properties (P1) and (P2) follow from the definition of the wave packets. From (P2) and (P3) we can deduce (P4). We already mentioned (P5), and for (P6) we refer to [L1]. \square

The next step will be a slightly reformulation, where we aim for avoiding using coordinates to much. For a hypersurface $H = \{n\}^\perp \subset \mathbb{R}^d$, define the partial Fourier transform $\mathfrak{F}_H f(\xi + tn) = \int_H f(x + tn) e^{ix \cdot \xi} dx$, $\xi \in H$.

COROLLARY 3.2.3 (WAVE PACKET DECOMPOSITION)

Let ϕ to be like in the previous theorem. Denote by S the graph of ϕ and by N_0 its unit normal field and let n be a unit vector transversal to S , i.e. $|\langle n, N_0(x) \rangle| \gtrsim 1$ for all $x \in S$. Then for every $R \geq 1$ there exists a wave packet decomposition adapted to S in coordinates n , $H = \{n\}^\perp$ with tubes of size $\frac{R}{D} \times \frac{R}{D} \times \frac{R^2}{D^2\kappa} = R' \times R' \times \frac{R'^2}{\kappa}$, $R = R'D$, i.e.: There exists a R' -net \mathcal{Y} in H , a R'^{-1} -grid \mathcal{V} in S such that for $w = (y, v) \in \mathcal{Y} \times \mathcal{V} = \mathcal{W}$ and

$$T_w = y + \{tN(v) : |t| \leq \frac{R'^2}{\kappa}\} + \mathcal{O}(R'), \quad (3.2.13)$$

there exist wave packets $\{p_w\}_{w \in \mathcal{W}}$ such that for all $x = x' + tn$, $|\langle x, n \rangle| \leq \frac{R'^2}{\kappa}$, $x' \in H$

(P1) $p_w = R^*(\mathfrak{F}_H p_w)$,

(P2) $\text{supp } \mathfrak{F}_H p_w \subset B(v, R'^{-1})$ and $\text{supp } \mathfrak{F}_H p_w(\cdot + tn) \subset B(v', \mathcal{O}(R'^{-1}))$, where v' is the orthogonal projection of $v \in S$ to H .

(P3) p_w is essentially supported in T_w , i.e. $|p_w(x)| \leq C_N R'^{-1} \left(1 + \frac{\text{dist}(x, T_w)}{R'}\right)^{-N}$.

(P4) For all $W \subset \mathcal{W}$, we have $\left\| \sum_{w \in W} p_w(\cdot + tn) \right\|_{L^2(H)} \leq |W|^{\frac{1}{2}}$.

(P5) For all $f \in L^2$ there exists a sequence $\{c_w\}_{w \in \mathcal{W}} \in \mathbb{C}^{\mathcal{W}}$ such that $R^* f(x) = \sum_{w \in \mathcal{W}} c_w p_w(x)$
and

(P6) $\|c\|_{\ell^2} \lesssim \|f\|_{L^2}$.

Proof: For the case $n = e_3$ and $H = \mathbb{R}^2 \times \{0\}$, this is just a reformulation of Lemma 3.2.1 as indicated in Remark 3.2.2.

To deal with the general case, notice that the crucial condition $|\partial^\alpha \phi(\eta)| \leq \kappa D^{2-\alpha}$ for all $|\alpha| \geq 2$ remains essentially valid (see Lemma 3.8.4 in the appendix) in rotated coordinates such that e_3 becomes n . \square

The following lemma will deal with the separation of tubes along curves.

LEMMA 3.2.4 (TUBE-SEPARATION ALONG INTERSECTION CURVE)

Let $\mathcal{Y}, \mathcal{V}, \mathcal{W}, R, T_w$ be like introduced above. Let $\phi \in C^\infty(U, \mathbb{R})$, $U \subset \mathbb{R}^2$ such that $\partial_i^2 \phi(x) \approx \kappa_i$ for all $x \in U, i = 1, 2$ and $\partial_1 \partial_2 \phi = 0$. Define $\kappa = \kappa_1 \vee \kappa_2$. Let γ be a curve in U with partial derivatives of modulus approximately 1, i.e. $|\dot{\gamma}_i| \approx 1$. Then for all $v_1, v_2 \in \text{Im}(\gamma) + \mathcal{O}(R'^{-1})$, $v_1 - v_2 = \frac{j}{R'}$, $j \in \mathbb{Z}^2$, $|j| \gg 1$ the following separation condition holds true:

$$|\nabla \phi(v_1) - \nabla \phi(v_2)| \approx |j| \frac{R'}{R'^2/\kappa}. \quad (3.2.14)$$

Proof: Find t_1, t_2 with $v_i = \gamma(t_i) + \mathcal{O}(R'^{-1})$. Then

$$|\nabla \phi(v_i) - \nabla \phi(\gamma(t_i))| \leq \|\phi''|_U\|_\infty |v_i - \gamma(t_i)| \lesssim \frac{\kappa}{R'}. \quad (3.2.15)$$

Therefore

$$\begin{aligned} \frac{|j|}{R'} &= |v_1 - v_2| \\ &= |\gamma(t_1) - \gamma(t_2)| + \mathcal{O}(R'^{-1}) \\ &\approx |\dot{\gamma}_1| |t_1 - t_2| + |\dot{\gamma}_2| |t_1 - t_2| + \mathcal{O}(R'^{-1}) \\ &\approx |t_1 - t_2| + \mathcal{O}(R'^{-1}), \end{aligned}$$

and since $|j| \gg 1$, we obtain

$$|t_1 - t_2| \approx \frac{|j|}{R'}. \quad (3.2.16)$$

Moreover, there exist s_1 and s_2 between t_1 and t_2 such that

$$\begin{aligned} |\nabla \phi(v_1) - \nabla \phi(v_2)| &\stackrel{(3.2.15)}{\geq} |\nabla \phi(\gamma(t_1)) - \nabla \phi(\gamma(t_2))| - \kappa \mathcal{O}(R'^{-1}) \\ &\approx |\partial_1^2 \phi(\gamma(s_1)) \dot{\gamma}(s_1)| |t_1 - t_2| + |\partial_2^2 \phi(\gamma(s_2)) \dot{\gamma}(s_2)| |t_1 - t_2| - \kappa \mathcal{O}(R'^{-1}) \\ &\stackrel{(3.2.16)}{\approx} (\kappa_1 + \kappa_2) \frac{|j|}{R'} + \kappa \mathcal{O}(R'^{-1}) \\ &\approx |j| \frac{\kappa}{R'}, \end{aligned}$$

where we used again $|j| \gg 1$. \square

3.2.2 A general bilinear result

GENERAL ASSUMPTIONS

Let $\phi \in C^\infty(\mathbb{R}^2)$ with $\partial_1 \partial_2 \phi = 0$, and for $j = 1, 2$ let

$$S_j = \text{Graph}(\phi|_{U_j}), \quad U_j = r^{(j)} + [0, d_1^{(j)}] \times [0, d_2^{(j)}],$$

and assume that the principal curvatures of ϕ on U in x_1 -direction (and x_2 -direction) are comparable to $\kappa_1^{(j)}$ ($\kappa_2^{(j)} > 0$ respectively). Define $\kappa^{(j)} = \kappa_1^{(j)} \vee \kappa_2^{(j)}$. Let $\bar{d}_i = d_i^{(1)} \vee d_i^{(2)}$, $\bar{\kappa}_i = \kappa_i^{(1)} \vee \kappa_i^{(2)}$, $\bar{\kappa} = \bar{\kappa}_1 \vee \bar{\kappa}_2 = \kappa^{(1)} \vee \kappa^{(2)}$ and $D = \min_{i,j} d_i^{(j)}$.

Let $N = (-\nabla \phi, 1)$ be the normal field on S_1 and S_2 , whereas $N_0 = \frac{N}{|N|}$ denotes the normalised unit normal field.

We assume

$$(i) \text{ For all } i, j = 1, 2 \text{ and all } \eta \in S_j, \text{ we have } |\partial_i \phi(\eta) - \partial_i \phi(r^{(j)})| \lesssim \kappa_i^{(j)} d_i^{(j)} \text{ and } \bar{\kappa}_i \bar{d}_i \lesssim 1. \quad (3.2.17)$$

$$(ii) \text{ For all } \eta \in U_1 \cup U_2 \text{ and for all } \alpha \in \mathbb{N}^2, |\alpha| \geq 2 \text{ we have } |\partial^\alpha \phi(\eta)| \lesssim \kappa D^{2-|\alpha|}.$$

(iii) In both variables $i = 1, 2$, we have the separation condition

$$|\partial_i \phi(\eta_1) - \partial_i \phi(\eta_2)| \approx 1 \quad \text{for all } \eta_j \in S_j, j = 1, 2. \quad (3.2.18)$$

The set of all pairs (S_1, S_2) with such properties will be called \mathcal{S}_0 .

The main goal of this chapter will be to establish a local, bilinear adjoint Fourier restriction estimate on cuboids adapted to the wave packets.

THEOREM 3.2.5

Let $\frac{5}{3} \leq p \leq 2$. Then for every $\alpha > 0$ there exists constants $C_\alpha, \gamma_\alpha > 0$ such that for every pair $S = (S_1, S_2) \in \mathcal{S}_0$, every parameter $R \geq 1$ and all functions $f_j \in L^2(S_j)$, $j = 1, 2$ we have

$$\|R^* f_1 R^* f_2\|_{L^p(Q_{S_1, S_2}^0(R))} \leq C_\alpha R^\alpha (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0(S)) \|f_1\|_2 \|f_2\|_2, \quad (3.2.19)$$

where

$$Q_{S_1, S_2}^0(R) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r) x_3| \leq \frac{R^2}{D^2 \bar{\kappa}}, i = 1, 2, |x_3| \leq \frac{R^2}{D^2 \kappa^{(1)} \wedge \kappa^{(2)}} \right\} \quad (3.2.20)$$

with $r = r^{(j)}$ if $\kappa^{(j)} = \kappa^{(1)} \wedge \kappa^{(2)}$ and $C_0(S) = \frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} (D[\kappa^{(1)} \wedge \kappa^{(2)}])^{-\frac{1}{p}} (D\kappa^{(1)} D\kappa^{(2)})^{-\frac{1}{2}}$.

REMARK

In the case $\kappa^{(1)} = \kappa^{(2)} = \bar{\kappa}$, r is not well defined. But in this case the two sets $Q_{S_1, S_2}^0(R; j) = \{x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r^{(j)}) x_3| \leq \frac{R^2}{D^2 \bar{\kappa}}, i = 1, 2, |x_3| \leq \frac{R^2}{D^2 \bar{\kappa}}\}$, $j = 1, 2$, coincide (up to scaling by a numerical constant): since $|\nabla \phi(r^{(1)}) - \nabla \phi(r^{(2)})| \approx 1$ due to the transversality assumption (iii), an easy geometric consideration shows

$$aQ_{S_1, S_2}^0(R; 1) \subset Q_{S_1, S_2}^0(R; 2) \subset bQ_{S_1, S_2}^0(R; 1) \quad (3.2.21)$$

for some constants a, b independent of R and S .

By applying a harmless affine transformation if necessary, we may assume without loss of generality $r = 0$ and $\nabla\phi(r) = 0$. Notice that conditions (i)-(iv) are invariant under affine transformations.

Furthermore, we will prove the theorem on the even larger cuboids

$$Q_{S_1, S_2}(R) = \left\{ x \in \mathbb{R}^3 : |x_{i_0}| \leq \frac{R^2}{D^2 \bar{\kappa}}, \|x\|_\infty \leq \frac{R^2}{D^2 \kappa^{(1)} \wedge \kappa^{(2)}} \right\}, \quad (3.2.22)$$

for an appropriate choice of the coordinate i_0 for the smaller side length. Later we need to combine different cuboids with possibly having smaller side length in different directions. Thus it will become necessary to restrict to their intersection, which leads to (3.2.20).

To give a bit more information on that, there will be two directions where the side length is determined precisely by the length of wave packets, but more freedom in the remaining third direction.

Moreover, since $\bar{\kappa}_i \bar{d}_i \lesssim 1$, we may even assume without loss of generality that

$$\bar{\kappa}_i \bar{d}_i \ll 1 \quad \text{for all } i = 1, 2. \quad (3.2.23)$$

Otherwise we decompose S_1 and S_2 into a finite number of subsets where the side lengths are small enough.

For $\eta_j \in U_j$ define $\phi_1(\eta) = \phi(\eta - \eta_2) + \phi(\eta_2)$, $\eta \in \eta_2 + U_1$ and $\phi_2(\eta) = \phi(\eta - \eta_1) + \phi(\eta_1)$, $\eta \in \eta_1 + U_2$. The set $((\eta_2, \phi(\eta_2)) + S_1) \cap ((\eta_1, \phi(\eta_1)) + S_2) = \text{Graph}(\phi_1) \cap \text{Graph}(\phi_2)$ will be called an intersection or intersection curve of S_1 and S_2 . It equals the graph of ϕ_1 (or ϕ_2) restricted to the set where $\psi = \phi_1 - \phi_2 = 0$. On this set, the normal field $N_j(\eta) = (-\nabla\phi_j(\eta), 1)$ forms the conical set $\Gamma_j = \{sN_j(\eta) | s \in \mathbb{R}, \psi(\eta) = 0\}$.

LEMMA 3.2.6

Let $(S_1, S_2) \in \mathcal{S}_0$. Assume $\nabla\phi(r) = 0$ for some $r \in S_1 \cup S_2$ and $\bar{\kappa}_i \bar{d}_i \ll 1$. Then the following holds:

(i) $D\kappa_i^{(j)} \ll 1$ for all $i, j = 1, 2$.

(ii) $|\nabla\phi(x)| \lesssim 1$ for all $x \in S_1 \cup S_2$. (3.2.24)

(iii) The unit normal fields on S_1 and S_2 are transversal, i.e.

$$|N_0(\eta^1) - N_0(\eta^2)| \approx 1 \quad \text{for all } \eta^j \in S_j. \quad (3.2.25)$$

(iv) N_j and $\Gamma_{j+1 \bmod 2}$ are transversal for $j = 1, 2$ and for any choice of intersection of S_1 and S_2 . (3.2.26)

(v) If γ is a parametrisation by arclength of the x_1 - x_2 -projection of an intersection curve of S_1 and S_2 , then $|\dot{\gamma}_1| \approx 1 \approx |\dot{\gamma}_2|$.

Proof: (i) is clear since $D = \min_{i,j=1,2} d_i^{(j)}$. To prove (ii), notice that for any $x, x' \in S_1 \cup S_2$ we have $|\nabla\phi(x) - \nabla\phi(x')| \lesssim 1$: if x and x' belong to different sets S_j , we apply condition (iii) from the definition of \mathcal{S}_0 , if x and x' lay in the same set S_j , we use condition (i).

Thus we have $|\nabla\phi(x)| = |\nabla\phi(x) - \nabla\phi(r)| \lesssim 1$ for all $x \in S_1 \cup S_2$.

This gives $|N(x)| = \sqrt{1 + |\nabla\phi(x)|^2} \approx 1$ for all $x \in S_1 \cup S_2$ which already implies the transversality of the normal fields (cf. Remark 3.8.1):

$$|N_0(\eta^1) - N_0(\eta^2)| \approx |N(\eta^1) - N(\eta^2)| = |\nabla\phi(\eta^1) - \nabla\phi(\eta^2)| \approx 1 \quad \text{for all } \eta^j \in S_j, j = 1, 2. \quad (3.2.27)$$

We will prove (v) first, since we need it for the proof of (iii). It suffices to prove $|\partial_i\psi(\eta)| \approx 1$ for all η such that $\eta - \tilde{\eta}_j \in U_{j+1 \bmod 2}$, $\eta_j \in U_j$ since the tangentials of the curve are orthogonal to $\nabla\psi$. But

$$|\partial_i\psi(\eta)| = |\partial_i\phi(\eta - \eta_2) - \partial_i\phi(\eta - \tilde{\eta}_1)| \stackrel{(3.2.18)}{\approx} 1.$$

For the third claim (iii), we will show that N_1 and Γ_2 are transversal; the symmetric statement is similar. Since we have

$$|N_1(\eta) - N_1(\eta')| = |\nabla\phi_1(\eta) - \nabla\phi_1(\eta')| \lesssim \kappa_1^{(1)}d_1^{(1)} + \kappa_2^{(1)}d_2^{(1)} \ll 1 \quad (3.2.28)$$

for all $\eta, \eta' \in U_1 + \eta_2$, whereas $|N_1(\eta)| \approx 1$ for all $\eta \in U_1 + \eta_2$, it is even enough to show that $N_1(\eta)$ and the tangent space $T_\eta\Gamma_2$ of Γ_2 at the point η are transversal. If γ is a parametrisation by arclength of the zero set of ψ , the tangent space of Γ_2 at the point $\eta = \gamma(t)$ is spanned by $N_2(\eta)$ and $s(-Hess(\phi_2)(\eta) \cdot \dot{\gamma}(t), 0)$, or, by the (roughly) normalised vectors $N_2(\eta)$ and $\frac{1}{\kappa^{(2)}}(-Hess(\phi_2)(\eta) \cdot \dot{\gamma}(t), 0)$. The transversality can be checked by estimating the volume of the span of $N_1(\eta)$ and two transversal unit vectors spanning the tangent space $T_\eta\Gamma_2$, i.e.

$$\begin{aligned} & \left| \begin{array}{ccc} -\partial_1\phi_1(\eta) & -\partial_2\phi_1(\eta) & 1 \\ -\partial_1\phi_2(\eta) & -\partial_2\phi_2(\eta) & 1 \\ \frac{1}{\kappa^{(2)}}\partial_1^2\phi_2(\eta)\dot{\gamma}_1(t) & \frac{1}{\kappa^{(2)}}\partial_1^2\phi_2(\eta)\dot{\gamma}_2(t) & 0 \end{array} \right| \\ &= \frac{1}{\kappa^{(2)}} |-\partial_1^2\phi_2(\eta)\dot{\gamma}_1(t)\partial_2\psi(\eta) + \partial_2^2\phi_2(\eta)\dot{\gamma}_2(t)\partial_1\psi(\eta)|. \end{aligned}$$

Since $\psi \circ \gamma = 0$ by definition, we have

$$\begin{aligned} 0 &= \nabla\psi(\eta) \cdot \dot{\gamma}(t) \\ &= \partial_1\psi(\eta)\dot{\gamma}_1(t) + \partial_2\psi(\eta)\dot{\gamma}_2(t), \end{aligned}$$

i.e.

$$\partial_2\psi(\eta) = -\partial_1\psi(\eta) \frac{\dot{\gamma}_1(t)}{\dot{\gamma}_2(t)}.$$

Thus

$$\begin{aligned} & \left| \begin{array}{ccc} -\partial_1\phi_1(\eta) & -\partial_2\phi_1(\eta) & 1 \\ -\partial_1\phi_2(\eta) & -\partial_2\phi_2(\eta) & 1 \\ \frac{1}{\kappa^{(2)}}\partial_1^2\phi_2(\eta)\dot{\gamma}_1(t) & \frac{1}{\kappa^{(2)}}\partial_1^2\phi_2(\eta)\dot{\gamma}_2(t) & 0 \end{array} \right| \\ &= \frac{|\partial_1\psi(\eta)|}{\kappa^{(2)}|\dot{\gamma}_2(t)|} (\partial_1^2\phi_2(\eta)\dot{\gamma}_1^2(t) + \partial_2^2\phi_2(\eta)\dot{\gamma}_2^2(t)) \\ &\approx |\partial_1\phi(\eta - \eta_2) - \partial_1\phi(\eta - \eta_1)| \frac{\kappa_1^{(2)} + \kappa_2^{(2)}}{\kappa^{(2)}} \approx 1, \end{aligned}$$

finishing the proof. \square

We now come to the introduction of the wave packets we want to use in the proof. For simplicity assume for the argument

$$\kappa^{(1)} \leq \kappa^{(2)}, \quad (3.2.29)$$

i.e. $r = r^{(1)}$ and $\nabla\phi(r^{(1)}) = 0$. On S_1 we decompose into wave packets of length $\frac{R^2}{\kappa^{(1)}}$ according to Lemma 3.2.1 without changing coordinates, thereby covering the set $\mathbb{R}^2 \times [-\frac{R^2}{\kappa^{(1)}}, \frac{R^2}{\kappa^{(1)}}]$. In other words, we apply Corollary 3.2.3 with

$$n_1 = (0, 0, 1) \quad \text{and} \quad H_1 = \mathbb{R}^2 \times \{0\}. \quad (3.2.30)$$

If we would keep the same coordinate system for S_2 , we would have to truncate even further in x_3 -direction since $\frac{R^2}{\kappa^{(2)}} \leq \frac{R^2}{\kappa^{(1)}}$. However, we have for $\eta \in U_2$ and both $i = 1, 2$ that

$$|\langle e_i, N(\eta) \rangle| = |\partial_i \phi(\eta)| = |\partial_i \phi(\eta) - \partial_i \phi(r^{(1)})| \stackrel{(3.2.18)}{\approx} 1.$$

This means we are able to apply Corollary 3.2.3 with

$$n_2 = e_{i_0} \quad \text{and} \quad H_2 = \{n_2\}^\perp = \{e_{i_0}\}^\perp, \quad (3.2.31)$$

for $i_0 \in \{1, 2\}$, this time covering the set

$$\left\{ x \in \mathbb{R}^3 : |\langle x, n_2 \rangle| \leq \frac{R^2}{\kappa^{(2)}} \right\} = \left\{ x \in \mathbb{R}^3 : |x_{i_0}| \leq \frac{R^2}{\kappa^{(2)}} \right\}.$$

To decide how to chose i_0 , we observe that for $\eta \in U_1$ we have

$$|\partial_i \phi(\eta) - \partial_i \phi(r^{(1)})| \stackrel{(3.2.17)}{\lesssim} \kappa_i^{(1)} d_i^{(1)} \leq \frac{\kappa_i^{(1)}}{\bar{\kappa}_i} \bar{\kappa}_i \bar{d}_i \stackrel{(3.2.23)}{\ll} \frac{\kappa_i^{(1)}}{\bar{\kappa}_i}. \quad (3.2.32)$$

The direction of the wave packets coming from S_1 are roughly $N(r^{(1)}) = (0, 0, -1)$, but they rotate a bit. The length projected into x_i -direction of the wave packet with direction $N(\eta)$, $\eta \in U_1$, is

$$|\langle e_i, \frac{R^2}{\kappa^{(1)}} N(\eta) \rangle| = \frac{R^2}{\kappa^{(1)}} |\partial_i \phi(\eta)| = \frac{R^2}{\kappa^{(1)}} |\partial_i \phi(\eta) - \partial_i \phi(r^{(1)})|. \quad (3.2.33)$$

If we choose i_0 so that

$$\frac{\kappa_{i_0}^{(1)}}{\bar{\kappa}_{i_0}} = \frac{\kappa_1^{(1)}}{\bar{\kappa}_1} \wedge \frac{\kappa_2^{(1)}}{\bar{\kappa}_2},$$

then we have for $\eta \in U_1$

$$\begin{aligned} \bar{\kappa} |\partial_{i_0} \phi(\eta) - \partial_{i_0} \phi(r^{(1)})| &\lesssim (\bar{\kappa}_1 \vee \bar{\kappa}_2) \left(\frac{\kappa_1^{(1)}}{\bar{\kappa}_1} \wedge \frac{\kappa_2^{(1)}}{\bar{\kappa}_2} \right) \\ &\leq \kappa_1^{(1)} \vee \kappa_2^{(1)} = \kappa^{(1)}, \end{aligned}$$

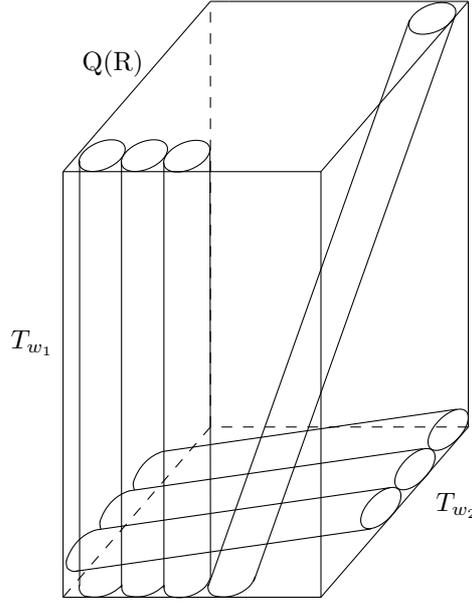


Figure 3.3: The wave packets filling the cuboid $Q_{S_1, S_2}(R)$

i.e.

$$|\langle e_{i_0}, \frac{R'^2}{\kappa^{(1)}} N(\eta) \rangle| \lesssim \frac{R'^2}{\bar{\kappa}} \stackrel{(3.2.29)}{=} \frac{R'^2}{\kappa^{(2)}}. \quad (3.2.34)$$

Thus the geometry fits well: the wave packets of S_1 do not rotate direction to much; their length in x_{i_0} -direction is not larger than the wave packages associated to S_2 (cf. Figure 3.3). (3.2.35)

However, for the remaining coordinate direction x_i , $i \in \{1, 2\} \setminus \{i_0\}$, we cannot guarantee such a behaviour. The intersection set contains

$$\begin{aligned} & \mathbb{R}^2 \times \left[-\frac{R'^2}{\kappa^{(1)}}, \frac{R'^2}{\kappa^{(1)}} \right] \cap \left\{ x \in \mathbb{R}^3 : |x_{i_0}| \leq \frac{R'^2}{\kappa^{(2)}} \right\} \\ &= \left\{ x' \in \mathbb{R}^2 : |x_{i_0}| \leq \frac{R'^2}{\kappa^{(2)}} \right\} \times \left[-\frac{R'^2}{\kappa^{(1)}}, \frac{R'^2}{\kappa^{(1)}} \right] \\ &\supset \left\{ x \in \mathbb{R}^3 : |x_{i_0}| \leq \frac{R'^2}{\bar{\kappa}}, \|x\|_\infty \leq \frac{R'^2}{\kappa^{(1)} \wedge \kappa^{(2)}} \right\} \\ &= Q_{S_1, S_2}(R), \end{aligned}$$

i.e. on the cuboid $Q_{S_1, S_2}(R)$ we are able to apply the developed wave packet theory with both sets of wave packets associated to S_1 or S_2 respectively.

For every $\alpha > 0$, define the statement $E(\alpha)$ by

$$E(\alpha) : \Leftrightarrow \exists C_\alpha > 0 \exists \gamma_\alpha > 0 \forall S = (S_1, S_2) \in \mathcal{S}_0 \forall R \geq 1 \forall f_j \in L^2(S_j), j = 1, 2$$

$$\|R_{S_1}^* f_1 R_{S_2}^* f_2\|_{L^p(Q_{S_1, S_2}(R))} \leq C_\alpha R^\alpha \log^{\gamma_\alpha}(1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0(S)) \|f_1\|_2 \|f_2\|_2. \quad (3.2.36)$$

The goal is to show that $E(\alpha)$ holds true for every $\alpha > 0$, which would prove Theorem 3.2.5. This will be achieved by the method called induction on scales.

Let $\{q\}_{q \in \mathcal{Q}}$ be a family of cubes of the same side length R' , covering $Q_{S_1, S_2}(R)$, and for $\chi \in \mathcal{S}(\mathbb{R}^3)$ with $\text{supp } \hat{\chi} \subset B(0, 1)$ and $\hat{\chi}(0) = 1$, let $\chi_q(x) = \chi\left(\frac{x - c_q}{R'}\right)$ for every cube $q = B_\infty(c_q, R')$. Using the Poisson summation formula, we may assume $\sum \chi_q = 1$ on $Q_{S_1, S_2}(R)$.²

For a given index set $W_j \subset \mathcal{W}_j$ of wave packets, we collect all the tubes passing through (a slightly thickened) q by

$$W_j(q) = \{w_j \in W_j : T_{w_j} \cap R^\delta q \neq \emptyset\}, \quad (3.2.37)$$

where $\delta > 0$ is some small parameter to be chosen later. $R^\delta q$ should be the blow up of q by R^δ around the center of q . We count the wave packets W_j passing through a cube q by

$$Q^\mu = \{q : |W_j(q)| \sim \mu_j, j = 1, 2\}, \quad \mu = (\mu_1, \mu_2), \quad \mu_j \in \mathcal{N} = \{2^n | n \in \mathbb{N}\} \cup \{0\}. \quad (3.2.38)$$

Obviously Q^μ forms a partition of the family of all cubes q . For $w_j \in W_j$, we further introduce the set of all cubes in Q^μ near T_{w_j} :

$$Q^\mu(w_j) = \{q \in Q^\mu : T_{w_j} \cap R^\delta q \neq \emptyset\}. \quad (3.2.39)$$

Finally, we fix the number of such cubes by

$$W_j^{\lambda_j, \mu} = \{w_j \in W_j : |Q^\mu(w_j)| \sim \lambda_j\}. \quad (3.2.40)$$

For every fixed μ , the family $\{W_j^{\lambda_j, \mu}\}_{\lambda_j \in \mathcal{N}}$ forms a partition of W_j .

We are now in a position to reduce the statement $E(\alpha)$ to a formulation in terms of wave packets:

3.2.3 Reduction to a wave packet formulation

LEMMA 3.2.7

Let $\alpha > 0$. Assume there are $C_\alpha, \gamma_\alpha > 0$ such that for all $(S, \tilde{S}) \in \mathcal{S}_0$, $R \geq 1$, $\|f_j\|_{L^2(S_j)} = 1$, $\lambda_j, \mu_j \in \mathcal{N}$ and $W_j \subset \mathcal{W}_j, j = 1, 2$ the wave packets associated to f_1, f_2 fulfill

$$\begin{aligned} & \left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(Q_{S_1, S_2}(R))} \\ & \leq C_\alpha R^\alpha \log^{\gamma_\alpha}(1 + R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0(S)) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \end{aligned} \quad (3.2.41)$$

Then $E(\alpha)$ holds true.

²Notice that unlike other papers using this method, we do not take χ_q to be the characteristic function of q . This makes things a bit more technical, but we will need the compact Fourier support later.

Proof: To show $E(\alpha)$, we may of course assume that $\|f_j\|_2 = 1$, $j = 1, 2$. First observe that for fixed q and v_j the number of y_j such that the tube $T_{(y_j, v_j)}$ passes through $R^\delta q$ is bounded by $R^{c\delta}$, whereas the total number of $v_j \in V_j$ is bounded by

$$|V_j| \approx R'^2 |U_j| \leq R^2 \frac{\bar{d}_1 \bar{d}_2}{D^2}. \quad (3.2.42)$$

Thus we have

$$\begin{aligned} |W_j(q)| &\leq R^{2+c\delta} \frac{\bar{d}_1 \bar{d}_2}{D^2} \\ &\leq R^{2+c} \frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} (D\bar{\kappa})^{-\frac{1}{p}} (D\kappa^{(1)} D\kappa^{(2)})^{-\frac{1}{2}} \\ &= R^{c'} C_0, \end{aligned}$$

where we used condition (i) of Lemma 3.2.6.

Additionally, the number of cubes q of side length R' such that $R^\delta q$ intersects with a tube T_{w_j} of length $\frac{R'^2}{\kappa^{(j)}}$ is bounded by $R^{c\delta} \frac{R'}{\kappa^{(j)}} = R^{1+c\delta} \frac{1}{D\kappa^{(j)}}$. Since $D \leq \bar{d}_1, \bar{d}_2$, we have

$$\begin{aligned} |Q^\mu(w_j)| &\leq R^{1+c\delta} \frac{1}{D\kappa^{(j)}} \\ &\leq R^{c'} \frac{\bar{d}_1^4 \bar{d}_2^4}{D^8} (D\bar{\kappa})^{-\frac{2}{p}} (D\kappa^{(1)} D\kappa^{(2)})^{-1} \\ &= R^{c'} C_0^2. \end{aligned}$$

Let $\mathcal{N}(C) = \{\nu \in \mathcal{N} | \nu \leq C\}$, then

$$\mathcal{Q} = \bigcup_{\mu_1 \in \mathcal{N}(R^{c'} C_0^2)} \bigcup_{\mu_2 \in \mathcal{N}(R^{c'} C_0^2)} Q^\mu \quad \text{and} \quad W_j = \bigcup_{\lambda_j \in \mathcal{N}(R^{c'} C_0)} W_j^{\lambda_j, \mu}$$

holds true for every μ . Applying this decomposition yields

$$\begin{aligned} &\| \prod_{j=1,2} \sum_{w_j \in W_j} p_{w_j} \|_{L^p(Q_{S_1, S_2}(R))} \\ &\leq \sum_{\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathcal{N}(R^{c'} C_0^2)} \| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \|_{L^p(Q_{S_1, S_2}(R))} \\ &\leq C_\alpha R^\alpha \log^4(R^c C_0^2) \log^{\gamma_\alpha} (1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha} (C_0) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}} \\ &\leq (2c)^4 C_\alpha R^\alpha \log^{\gamma_\alpha + 4} (1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha + 4} (C_0) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \end{aligned}$$

So we proved that for all $(S, \tilde{S}) \in \mathcal{S}_0$, $R \geq 1$, $\|f_j\|_{L^2(S_j)} = 1$ and $W_j \subset \mathcal{W}_j$, $j = 1, 2$ the wave packets associated to f_1, f_2 fulfill

$$\| \prod_{j=1,2} \sum_{w_j \in W_j} p_{w_j} \|_{L^p(Q_{S_1, S_2}(R))} \leq C_\alpha R^\alpha \log^{\gamma_\alpha + 4} (1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha + 4} (C_0) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \quad (3.2.43)$$

Recall that the wave packets are linked to the functions f_j by $R^* f_j = \sum_{w_j \in \mathcal{W}_j} c_{w_j} p_{w_j}$.

Introduce $W_j^k = \{w_j \in \mathcal{W}_j : |c_{w_j}| \sim 2^{-k}\}$, such that $\mathcal{W}_j = \bigcup_{k \in \mathbb{N}} W_j^k$. Fix a k_0 , which value will be determined later. Then

$$\left\| \sum_{k > k_0} \sum_{w_1 \in W_1^k} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} \leq |Q_{S_1, S_2}(R)|^{\frac{1}{p}} \sum_{k > k_0} \left\| \sum_{w_1 \in W_1^k} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{\infty}.$$

The wave packets p_{w_j} are well separated in the y -variable, whereas the number of w_j is bound by $\frac{\bar{d}_1 \bar{d}_2}{D^2} R^2$, see (3.2.42). According to (P4), the wave packets are bounded by

$\mathcal{O}(R'^{-1})$. We have $|c_{w_1}| \lesssim 2^{-k}$ for $w_1 \in W_1^k$, and $|c_{w_2}| \leq \|(c_{w_2})_{w_2 \in \mathcal{W}_2}\|_{\ell^2} \stackrel{(P6)}{\lesssim} \|f_2\|_2 = 1$. Putting all these informations in our estimate, we obtain

$$\begin{aligned} & \left\| \sum_{k > k_0} \sum_{w_1 \in W_1^k} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} \\ & \lesssim \left(\frac{R^6}{[\kappa^{(1)} \wedge \kappa^{(2)}]_{\kappa^{(1)} \kappa^{(2)}}} \right)^{\frac{1}{p}} \frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} R^4 R'^{-2} \sum_{k > k_0} 2^{-k} \\ & \approx R^{\frac{6}{p}+2} D^{2-\frac{5}{p}} \frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} (D\bar{\kappa})^{-\frac{1}{p}} (\kappa^{(1)} \kappa^{(2)})^{-\frac{1}{p}} 2^{-k_0}. \end{aligned}$$

If we now choose $k_0 = \log C_0 + \log R^{\frac{6}{p}+2} = \log \left(\frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} (D\bar{\kappa})^{-\frac{1}{p}} (D\kappa^{(1)} D\kappa^{(2)})^{-\frac{1}{2}} \right) + \log R^{\frac{6}{p}+2}$, then

$$\begin{aligned} \left\| \sum_{k > k_0} \sum_{w_1 \in W_1^k} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} & \lesssim D^{2-\frac{5}{p}} (D\kappa^{(1)} D\kappa^{(2)})^{\frac{1}{2}} (\kappa^{(1)} \kappa^{(2)})^{-\frac{1}{p}} \\ & = D^{3-\frac{5}{p}} (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} =: C(S, \tilde{S}). \end{aligned} \quad (3.2.44)$$

It can be shown in a similar manner that

$$\left\| \sum_{k_1 \leq k_0} \sum_{w_1 \in W_1^{k_1}} \sum_{k_2 > k_0} \sum_{w_2 \in W_2^{k_2}} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} \lesssim C(S, \tilde{S}). \quad (3.2.45)$$

It remains to estimate

$$\begin{aligned} & \left\| \sum_{k_1, k_2=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} \\ & \leq \sum_{k_1, k_2=0}^{k_0} 2^{-k_1-k_2} \left\| \sum_{w_1 \in W_1^{k_1}} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} 2^{k_1} p_{w_1} c_{w_2} 2^{k_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))}. \end{aligned}$$

Since $|c_{w_j} 2^{k_j}| \sim 1$ for $w_j \in W_j^{k_j}$, it is appropriate to apply (3.2.43) to the modified wave packets $\tilde{p}_{w_j} = c_{w_j} 2^{k_j} p_{w_j}$:

$$\begin{aligned} & \left\| \sum_{k_1, k_2=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} \sum_{w_2 \in \mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} \\ & \leq C_{\alpha} R^{\alpha} \log^{\gamma_{\alpha}+4}(1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_{\alpha}+4}(C_0) \sum_{k_1, k_2=1}^{k_0} 2^{-k_1-k_2} |W_1^{k_1}|^{\frac{1}{2}} |W_2^{k_2}|^{\frac{1}{2}}. \end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k_1, k_2=1}^{k_0} 2^{-k_1-k_2} |W_1^{k_1}|^{\frac{1}{2}} |W_2^{k_2}|^{\frac{1}{2}} &\leq k_0 \left(\sum_{k_1=1}^{k_0} |W_1^{k_1}| 2^{-2k_1} \sum_{k_2=1}^{k_0} |W_2^{k_2}| 2^{-2k_2} \right)^{\frac{1}{2}} \\
&\lesssim k_0 \left(\sum_{k_1=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} |c_{w_1}| \sum_{k_2=1}^{k_0} \sum_{w_2 \in W_2^{k_2}} |c_{w_2}| \right)^{\frac{1}{2}} \\
&\lesssim k_0 \|f_1\|_2 \|f_2\|_2 = k_0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\| \sum_{k_1, k_2=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} \sum_{w_2 \in W_2^{k_2}} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \right\|_{L^p(Q_{S_1, S_2}(R))} \\
&\lesssim C_\alpha R^\alpha \log^{\gamma_\alpha+5} (1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha+5} (C_0)
\end{aligned} \tag{3.2.46}$$

and combining (3.2.44), (3.2.45) and (3.2.46)

$$\begin{aligned}
\|R^* f_1 R^* f_2\|_{L^p(Q_{S_1, S_2}(R))} &= \left\| \prod_{j=1,2} \sum_{w_j \in W_j} c_{w_j} p_{w_j} \right\|_{L^p(Q_{S_1, S_2}(R))} \\
&\lesssim C_\alpha R^\alpha \log^{\gamma_\alpha+5} (1+R) (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha+5} (C_0).
\end{aligned}$$

But this means $E(\alpha)$ holds true. □

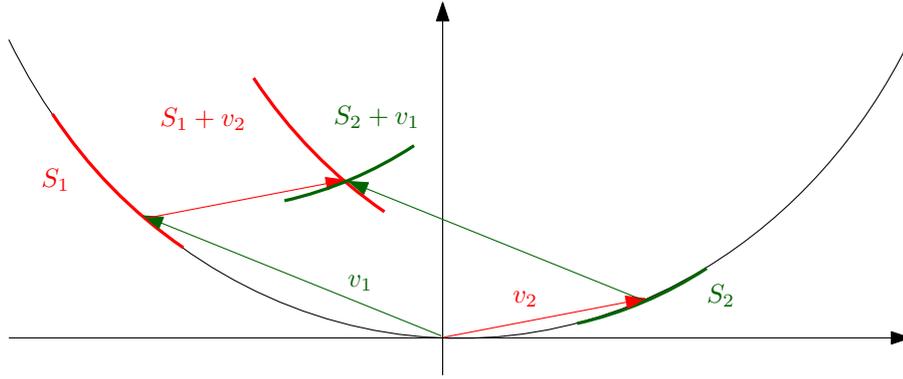


Figure 3.4: Intersection of two translated curves

3.2.4 Estimates for wave packets

Let $v_j \in \mathcal{V}_j$, $j = 1, 2$, and define the (thickened) "intersection" of S_1 and S_2 by

$$\Pi_{v_1, v_2} = (v_1 + S_2) \cap (v_2 + S_1) + \mathcal{O}(R^{-1}). \quad (3.2.47)$$

For any subset $W_j \subset \mathcal{W}_j$, let

$$W_j^{\Pi_{v_1, v_2}} = \{w'_j \in W_j \mid v'_j + v_{j+1} \in \Pi_{v_1, v_2}\} \quad (3.2.48)$$

(where $j + 1$ has to be interpreted as mod 2) and

$$V_j = \{v'_j \in \mathcal{V}_j \mid \exists y'_j \in \mathcal{Y} : (y'_j, v'_j) \in W_j\} \quad (3.2.49)$$

the \mathcal{V} -projection of W_j . Further let

$$\begin{aligned} V_j^{\Pi_{v_1, v_2}} &= \{v'_j \in \mathcal{V}_j \mid \exists y'_j \in \mathcal{Y}_j : (y'_j, v'_j) \in W_j^{\Pi_{v_1, v_2}}\} \\ &= \{v'_j \in \mathcal{V}_j \mid \exists y'_j \in \mathcal{Y}_j : (y'_j, v'_j) \in W_j \text{ and } v'_j + v_{j+1} \in \Pi_{v_1, v_2}\}. \end{aligned} \quad (3.2.50)$$

LEMMA 3.2.8

Let $W_j \subset \mathcal{W}_j$, $j = 1, 2$. Then

$$\left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^1(Q_{S_1, S_2}(R))} \leq \frac{R^2}{\sqrt{\kappa(1)\kappa(2)}} |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}} \quad (3.2.51)$$

$$\left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^2(Q_{S_1, S_2}(R))} \lesssim R^{-\frac{1}{2}} \min_j \sup_{v_1, v_2} |V_j^{\Pi_{v_1, v_2}}|^{\frac{1}{2}} |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \quad (3.2.52)$$

Proof: The proof follows very closely the outline in [LV] lemma 2.2. We only slightly modify the proof of the second estimate.

The first estimate is easy. Using Hölder's inequality, we see that

$$\begin{aligned} \left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^1(Q_{S_1, S_2}(R))} &\leq \prod_{j=1,2} \left\| \sum_{w_j \in W_j} p_{w_j} \right\|_{L^2(Q_{S_1, S_2}(R))} \\ &\leq \prod_{j=1,2} \left(\int_{-R'^2/\kappa^{(j)}}^{R'^2/\kappa^{(j)}} \left\| \sum_{w_j \in W_j} p_{w_j}(\cdot + tn_j) \right\|_{L^2(H_j)}^2 dt \right)^{\frac{1}{2}} \\ &\stackrel{(P4)}{\lesssim} \prod_{j=1,2} \frac{R'}{\sqrt{\kappa^{(j)}}} |W_j|^{\frac{1}{2}}. \end{aligned}$$

The second one is more involved. We write

$$\left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^2(Q_{S_1, S_2}(R))}^2 = \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \sum_{v'_1 \in V_1} \sum_{v'_2 \in V_2} \langle p_{w_1} \sum_{y'_2 \in Y_2(v'_2)} p_{w'_2}, p_{w_2} \sum_{y'_1 \in Y_1(v'_1)} p_{w'_1} \rangle, \quad (3.2.53)$$

where $V_j = \{v_j \in \mathcal{V}_j : \exists y_j \in \mathcal{Y}_j : (y_j, v_j) \in W_j\}$, $Y_j(v_j) = \{y \in \mathcal{Y}_j | (y, v'_j) \in W_j\}$. Since the Fourier transform of $\sum_{y'_{j+1} \in Y_{j+1}(v'_{j+1})} p_{w'_{j+1}} p_{w_j}$ is supported in $v'_{j+1} + v_j + \mathcal{O}(R'^{-1})$, $j = 1, 2$,

we may assume the intersection of these two sets to be not empty and thus

$$v'_1 + v_2 = v'_2 + v_1 + \mathcal{O}(R'^{-1}). \quad (3.2.54)$$

Especially

$$v'_{j+1} + v_j \in \Pi_{v_1, v_2}$$

and

$$v'_j \in V_j^{\Pi_{v_1, v_2}}, \quad j = 1, 2. \quad (3.2.55)$$

Hence

$$\left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^2(Q_{S_1, S_2}(R))}^2 \quad (3.2.56)$$

$$\leq \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \sum_{v'_1 \in V_1^{\Pi_{v_1, v_2}}} \sum_{\substack{v'_2 \\ v'_2 = v'_1 + v_2 - v_1 + \mathcal{O}(R'^{-1})}} \int_{\mathbb{R}^3} |p_{w_1} p_{w_2}| dx \left\| \sum_{y'_2 \in Y(v'_2)} p_{w'_2} \right\|_{\infty} \left\| \sum_{y'_1 \in Y(v'_1)} p_{w'_1} \right\|_{\infty}. \quad (3.2.57)$$

Observe that there are at most $\mathcal{O}(1)$ possible choices for v'_2 such that

$$v'_2 = v'_1 + v_2 - v_1 + \mathcal{O}(R'^{-1}).$$

Since the wave packets p_{w_j} are essentially supported in the tubes T_{w_j} , which are well separated in the y -variable, the sum in y'_j can be replaced by a supremum. Since T_{w_1}

and T_{w_2} fulfill the transversality condition (3.2.25), $p_{w_1}p_{w_2}$ decays rapidly away from the intersection $T_{w_1} \cap T_{w_2}$, i.e.

$$\int_{\mathbb{R}^3} |p_{w_1}p_{w_2}| dx \lesssim \int_{\mathbb{R}^3} R'^{-2} \left(1 + \frac{|x|}{R'}\right)^{-N} dx = R' \int_{\mathbb{R}^3} (1 + |x|)^{-N} dx \approx R'.$$

We obtain

$$\begin{aligned} \left\| \sum_{\substack{w_1 \in W_1 \\ w_2 \in W_2}} p_{w_1}p_{w_2} \right\|_{L^2(Q_{S_1, S_2}(R))}^2 &\lesssim R' |W_1| |W_2| \sup_{v_1, v_2} |V_1^{\Pi_{v_1, v_2}}| \sup_{w'_1 \in W_1, w'_2 \in W_2} \|p_{w'_2}p_{w'_1}\|_\infty \quad (3.2.58) \\ &\lesssim R'^{-1} |W_1| |W_2| \sup_{v_1, v_2} |V_1^{\Pi_{v_1, v_2}}|. \end{aligned}$$

Repeating the same computation with the roles of v'_1 and v'_2 interchanged gives (3.2.52). \square

3.2.5 Basis of the induction on scales argument

REMARK 3.2.9

Let $V_j \subset \mathcal{V}_j$. Then $\min_j \sup_{v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2} |V_j^{\Pi_{v_1, v_2}}| \lesssim R$.

Proof: Define the graph mapping $\Phi : U_1 \cup U_2 \rightarrow S_1 \cup S_2, \Phi(x) = (x, \phi(x))$. If $v'_j = \Phi(x'_j) \in V_j^{\Pi_{v_1, v_2}}$, then $v'_j + v_{j+1} \in \Pi_{v_1, v_2}$ and for $x_{j+1} = \Phi^{-1}(v_{j+1})$ we have $x'_j + x_{j+1} \in \gamma(I) + \mathcal{O}(R'^{-1})$, where $\gamma : I \rightarrow [U_1 + x_2] \cap [U_2 + x_1] \subset \mathbb{R}^2$ is a parametrisation of the intersection curve Π_{v_1, v_2} projected to x_1 - x_2 -space. Due to our assumption (iv) from the beginning, we know that γ is close to a diagonal, i.e. $|\dot{\gamma}_i| \approx 1, i = 1, 2$.

For all $t, t' \in I$, we have $\gamma(t), \gamma(t') \in [U_1 + x_2] \cap [U_2 + x_1]$, hence

$$\min_j d_i^{(j)} \geq |\gamma_i(t) - \gamma_i(t')| \geq \min_{t'' \in I} |\dot{\gamma}_i(t'')| |t - t'| \approx |t - t'|. \quad (3.2.59)$$

Therefore $|I| = \sup_{t, t' \in I} |t - t'| \lesssim \min_{i, j} d_i^{(j)} = D$. We conclude

$$L(\gamma) = \int_I |\dot{\gamma}(t)| dt \lesssim |I| \lesssim D. \quad (3.2.60)$$

Thus

$$\begin{aligned} |V_j^{\Pi_{v_1, v_2}}| &\stackrel{(3.2.24)}{\approx} |\Phi^{-1}(V_j^{\Pi_{v_1, v_2}})| \\ &\leq |\{x'_j \in \Phi^{-1}(\mathcal{V}_j) : x'_j \in \gamma(I) - x_{j+1} + \mathcal{O}(R'^{-1})\}| \\ &\lesssim L(\gamma)R' \\ &\lesssim DR' = R, \end{aligned}$$

since $\Phi^{-1}(\mathcal{V}_j)$ is a R'^{-1} -grid in U_j . \square

COROLLARY 3.2.10

$E(1)$ holds true.

Proof: Due to Lemma 3.2.7, it is enough to show the corresponding estimate for wave packets (3.2.41) with $\alpha = 1$. Putting the bound from Remark 3.2.9 into inequality (3.2.52) from Lemma 3.2.8, we obtain

$$\left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^2(Q_{S_1, S_2}(R))} \lesssim R'^{-\frac{1}{2}} R^{\frac{1}{2}} |W_1|^{\frac{1}{2}} |W_1|^{\frac{1}{2}}. \quad (3.2.61)$$

Interpolation³ with the L^1 -estimate we obtain from (3.2.51) gives

$$\left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(Q_{S_1, S_2}(R))} \lesssim (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} R'^{\frac{5}{p} - 3} R^{1 - \frac{1}{p}} |W_1|^{\frac{1}{2}} |W_1|^{\frac{1}{2}} \quad (3.2.62)$$

$$\leq (\kappa^{(1)} \kappa^{(2)})^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} R |W_1|^{\frac{1}{2}} |W_1|^{\frac{1}{2}}. \quad (3.2.63)$$

for all $\frac{5}{3} \leq p \leq 2$. □

3.2.6 Further decompositions

We shall introduce a further decomposition of $Q_{S_1, S_2}(R)$ into cuboids b which have the size of $Q_{S_1, S_2}(R)$ reduced by $R^{-2\delta}$, i.e. all the b 's are some translation of $Q(R^{1-\delta})$. Here, $\delta > 0$ is some small parameter we will chose later. Thus although not written explicitly, these decomposition depends on δ . Since $\frac{R'^2}{\bar{\kappa}} R^{-2\delta} = \frac{R^{1-2\delta} R'}{D\kappa} \geq R^{1-2\delta} R'$, the smallest side length of b is still much larger than the side length of the thickened cubes $R^\delta q$. Observe further that the number of b 's is of the size⁴ $R^{c\delta}$. (3.2.64)

If μ is a fixed dyadic number, to every w_j we associate a $b(w_j)$ by choosing a b that contains a maximal number of q 's from $Q^\mu(w_j)$. We say that $b \sim w_j$ if b is contained in $10b(w_j)$. If $b \not\sim w_j$, this does not mean that there are "few" cubes q in b (since there can be more than one b with a maximal number of q 's), but it means that there are many cubes q "away" from b . To give a more quantitative version of this, we have

$$|\{q \in Q^\mu(w_j) | q \cap 5b = \emptyset\}| \geq |\{q \in Q^\mu(w_j) | q \subset b(w_j)\}| \gtrsim R^{-c\delta} |Q^\mu(w_j)| \quad \forall b \not\sim w_j. \quad (3.2.65)$$

For a fixed b , we can decompose any given set W_j into $W_j^{\not\sim b} = \{w_j \in W_j : b \not\sim w_j\}$ and $W_j^{\sim b} = \{w_j \in W_j : b \sim w_j\}$. Thus we have

$$\begin{aligned} \left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(Q_{S_1, S_2}(R))} &\leq \sum_b \left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)} \quad (3.2.66) \\ &= I + II + III, \end{aligned}$$

³In fact, it is just a simple application of Hölder's inequality, see Remark 3.8.2 in the appendix.

⁴Here and during the following considerations, c denotes some constant independent of R and S_1, S_2 , but which precise value might differ from line to line.

where

$$I = \sum_b \left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu, \sim b}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)} \quad (3.2.67)$$

$$II = \sum_b \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \not\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)} \quad (3.2.68)$$

$$III = \sum_b \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu, \not\sim b}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)}. \quad (3.2.69)$$

Part II requires very precise geometric estimates. Part III can be treated analogous, whereas part I can be handled by an inductive argument. For that part of the proof, we just need that $w_j \rightarrow b(w_j)$ is some function, and $|\{b : b \sim w_j\}| \lesssim 1$. We will only make use of the particular definition of $b(w_j)$ in the proof of II and III.

LEMMA 3.2.11

Let $\alpha > 0$, assume $E(\alpha)$ holds. Then

$$I \leq C_\alpha R^{\alpha(1-\delta)} \log^{\gamma_\alpha}(1+R) (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \quad (3.2.70)$$

Proof: To shorten notation, write $C_1 = C_\alpha (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0)$. Recall the reproduction formula (P1): $p_{w_j} = R^*(\mathfrak{F}_{H_j} p_{w_j})$. Since b is just a translation of $Q(R^{1-\delta})$ (and translation of $R^* f_j$ can be absorbed into some modulation of f_j), $E(\alpha)$ implies

$$\begin{aligned} I &= \sum_b \left\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu, \sim b}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)} \\ &\leq \sum_b \left\| \prod_{j=1,2} R^* \left(\sum_{w_j \in W_j^{\lambda_j, \mu, \sim b}} \mathfrak{F}_{H_j} p_{w_j} \right) \right\|_{L^p(b)} \\ &\stackrel{E(\alpha)}{\leq} C_1 (R^{1-\delta})^\alpha \log^{\gamma_\alpha}(1+R^{1-\delta}) \sum_b \prod_{j=1,2} \left\| \sum_{w_j \in W_j^{\lambda_j, \mu, \sim b}} \mathfrak{F}_{H_j} p_{w_j} \Big|_{H_j} \right\|_{L^2(H_j)} \\ &\stackrel{(P4)}{\leq} C_1 R^{\alpha(1-\delta)} \log^{\gamma_\alpha}(1+R) \sum_b \prod_{j=1,2} |W_j^{\lambda_j, \mu, \sim b}|^{\frac{1}{2}}. \end{aligned}$$

Using Hölder, we see that

$$\sum_b \prod_{j=1,2} |W_j^{\lambda_j, \mu, \sim b}|^{\frac{1}{2}} \leq \prod_{j=1,2} \left(\sum_b |W_j^{\lambda_j, \mu, \sim b}| \right)^{\frac{1}{2}},$$

where due to Fubini

$$\begin{aligned} \sum_b |W_j^{\lambda_j, \mu, \sim b}| &= \sum_b |\{w_j \in W_j^{\lambda_j, \mu} : w_j \sim b\}| \\ &= \sum_{w_j \in W_j^{\lambda_j, \mu}} |\{b : b \sim w_j\}| \\ &\lesssim |W_j|. \end{aligned}$$

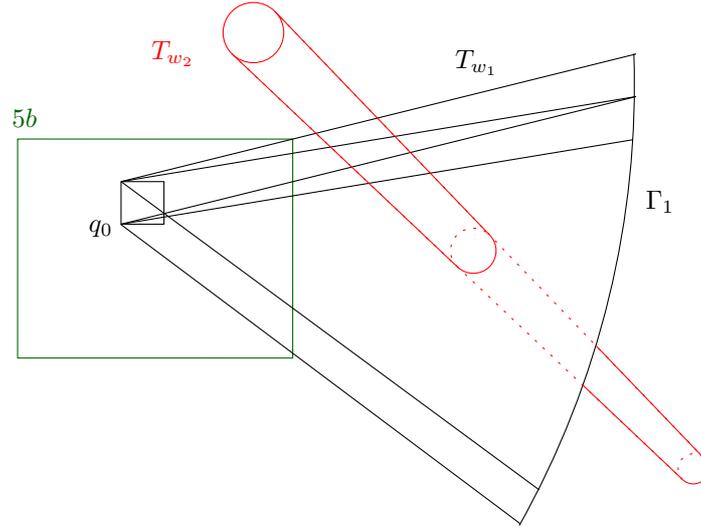


Figure 3.5: Situation in Lemma 3.2.12

To summarize, we have

$$I \leq C_1 R^{\alpha(1-\delta)} \log^{\gamma\alpha} (1+R) \prod_{j=1,2} |W_j|^{\frac{1}{2}},$$

like desired. \square

3.2.7 The geometric argument

LEMMA 3.2.12

Let $\lambda_j, \mu_j \in \mathcal{N}$, $W_j \subset \mathcal{W}_j$, $v_j \in \mathcal{V}_j$, $j = 1, 2$, and let b and q_0 be cuboids from our collections such that $q_0 \cap 2b \neq \emptyset$. If we define $W_j^{\lambda_j, \mu_j, \gamma^b}(q_0) = W_j^{\lambda_j, \mu_j, \gamma^b} \cap W_j(q_0)$, then

$$(i) \quad \lambda_1 \mu_2 \left| \left[W_1^{\lambda_1, \mu_2, \gamma^b}(q_0) \right]^{\Pi_{v_1, v_2}} \right| \lesssim R^{c\delta} |W_2| \quad (3.2.71)$$

$$(ii) \quad \lambda_2 \mu_1 \left| \left[W_2^{\lambda_2, \mu_1, \gamma^b}(q_0) \right]^{\Pi_{v_1, v_2}} \right| \lesssim R^{c\delta} |W_1|. \quad (3.2.72)$$

The lemma is a variation of lemma 2.3 in [LV].

Proof: We show (i), since (ii) is analogous.

Let $\Gamma_1 = \bigcup \{T_{w_1} : w_1 \in [W_1^{\lambda_1, \mu_2, \gamma^b}(q_0)]^{\Pi_{v_1, v_2}}\} \setminus 5b$ and $Q_{\Gamma_1}^\mu = \{q \in Q^\mu \mid R^\delta q \cap \Gamma_1 \neq \emptyset\}$. Since we saw that T_{w_2} is transversal to Γ_1 in (3.2.26), we have $|Q_{\Gamma_1}^\mu \cap Q^\mu(w_2)| \lesssim R^{c\delta}$. (3.2.73) Due to the separation of the tube directions, the sets $T_{w_1} \setminus 5b$ do not overlap to much. To be more precise, we claim that for all cubes $q \in Q_{\Gamma_1}^\mu$

$$|\{w_1 \in [W_1^{\lambda_1, \mu_2, \gamma^b}(q_0)]^{\Pi_{v_1, v_2}} : R^\delta q \cap T_{w_1} \setminus 5b \neq \emptyset\}| \lesssim R^{c\delta}. \quad (3.2.74)$$

Let $w_1, w'_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}}$ and $x \in R^\delta q \cap T_{w_1} \setminus 5b$, $x' \in R^\delta q \cap T_{w'_1} \setminus 5b$. The definition of $W_1(q_0)$ means that we find $x_0 \in R^\delta q_0 \cap T_{w_1}$ and $x'_0 \in R^\delta q_0 \cap T_{w'_1}$; then we may write

$$x = x_0 + |x - x_0|N(v_1) + \mathcal{O}(R') \text{ and } x' = x'_0 + |x' - x'_0|N_0(v'_1) + \mathcal{O}(R'). \quad (3.2.75)$$

Furthermore we have

$$\left| |x - x_0| - |x' - x'_0| \right| \leq |x - x'| + |x_0 - x'_0| = \mathcal{O}(R^{c\delta} R'). \quad (3.2.76)$$

Since T_{w_1} has length $\frac{R'^2}{\kappa^{(1)}}$, i.e. the length of b in the direction of T_{w_1} is $R^{-2\delta} \frac{R'^2}{\kappa^{(1)}}$, cf. (3.2.35). Since $x_0 \in R^\delta q_0 \subset 4b$ but $x \notin 5b$, we conclude

$$R^{-2\delta} \frac{R'^2}{\kappa^{(1)}} \leq |x - x_0|. \quad (3.2.77)$$

Applying Lemma 3.2.4, we obtain

$$\begin{aligned} \left| \frac{v_1 - v'_1}{R'} \right| &\lesssim \frac{R'}{\kappa^{(1)}} |N(v_1) - N(v'_1)| \\ &\stackrel{(3.2.77)}{\lesssim} R^{2\delta} R'^{-1} |x - x_0| |N_0(v_1) - N_0(v'_1)| \\ &\stackrel{(3.2.76)}{\leq} R^{2\delta} R'^{-1} \left(|x - x_0|N_0(v_1) - |x' - x'_0|N_0(v'_1) \right) + \mathcal{O}(R^{c\delta}) \\ &\stackrel{(3.2.75)}{\leq} R^{2\delta} R'^{-1} \left(|x - x'| - |x'_0 - x_0| \right) + \mathcal{O}(R^{c\delta}) \\ &\stackrel{(3.2.76)}{=} \mathcal{O}(R^{c\delta}). \end{aligned}$$

Thus the possible choices of directions v_1 in the set

$$\{w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}} : R^\delta q \cap T_{w_1} \setminus 5b\}$$

is $\mathcal{O}(R^{c\delta})$. But for a fixed direction v_1 , the number of y_1 such that the tube $T_{(y_1, v_1)}$ passes through $R^{c\delta} q_0$ is bounded by $\mathcal{O}(R^{c\delta})$ anyway, thus (3.2.74) holds true.

Recall (3.2.65): for $w_1 \not\sim b$ we have

$$|\{q \in Q^\mu(w_1) : q \cap 5b = \emptyset\}| \gtrsim R^{-c\delta} |Q^\mu(w_1)|. \quad (3.2.78)$$

Recall further that $w_1 \in W_1^{\lambda_1, \mu}$ means $|Q^\mu(w_1)| \sim \lambda_1$. Therefore

$$\begin{aligned}
& R^{-c\delta} \lambda_1 \left| \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}} \right| \\
\lesssim & R^{-c\delta} \sum_{w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}}} |Q^\mu(w_1)| \\
(3.2.65) \quad & \lesssim \sum_{w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}}} |\{q \in Q^\mu(w_j) : q \cap 5b = \emptyset\}| \\
& \leq \sum_{w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}}} |\{q \in Q^\mu : R^\delta q \cap T_{w_1} \neq \emptyset, R^\delta q \cap 5b = \emptyset\}| \\
& \leq \sum_{w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}}} |\{q \in Q^\mu : R^\delta q \cap (T_{w_1} \setminus 5b) \neq \emptyset\}| \\
& = \sum_{q \in Q^\mu} |\{w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}} : R^\delta q \cap (T_{w_1} \setminus 5b) \neq \emptyset\}| \\
(3.2.74) \quad & \lesssim R^{c\delta} |\{q \in Q^\mu : \exists w_1 \in \left[W_1^{\lambda_1, \mu, \not\sim b}(q_0) \right]^{\Pi_{v_1, v_2}} R^\delta q \cap (T_{w_1} \setminus 5b) \neq \emptyset\}| \\
& = R^{c\delta} |\{q \in Q^\mu : R^\delta q \cap \Gamma_1 \neq \emptyset\}| = R^{c\delta} |Q_{\Gamma_1}^\mu|.
\end{aligned}$$

Further do we have

$$\begin{aligned}
|Q_{\Gamma_1}^\mu|_{\mu_2} &= \sum_{q \in Q_{\Gamma_1}^\mu} |W_2(q)| \\
&\leq \sum_{w_2 \in W_2} |Q_{\Gamma_1}^\mu \cap Q^\mu(w_2)| \\
(3.2.73) \quad &\lesssim R^{c\delta} |W_2|.
\end{aligned}$$

Combining the two last inequalities immediately gives the desired estimate. \square

LEMMA 3.2.13

Let $0 < \delta < \frac{1}{4}$. Then

$$II = \sum_b \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \not\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)} \leq C_\alpha R^{c\delta} (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}} \quad (3.2.79)$$

and

$$III = \sum_b \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu, \not\sim b}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)} \leq C_\alpha R^{c\delta} (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \quad (3.2.80)$$

Proof: We will just prove the first inequality, the second one works similarly. Since the summation over b in (3.2.79) gives at most another $R^{c\delta}$, it is enough to estimate for a fixed b

$$\left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \neq b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^p(b)}. \quad (3.2.81)$$

For $p = 1$, we apply (3.2.51) from Lemma 3.2.8:

$$\begin{aligned} & \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \neq b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^1(b)} \\ & \lesssim \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \neq b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \right\|_{L^1(Q_{S_1, S_2}(R))} \\ & \leq \frac{R'^2}{\sqrt{\kappa(1)\kappa(2)}} |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}. \end{aligned}$$

For $p = 2$, we claim that

$$\left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \neq b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \right\|_{L^2(b)}^2 \lesssim C_\alpha R^{c\delta} R'^{-1} |W_1| |W_2|, \quad (3.2.82)$$

which gives the desired inequality (3.2.79) by interpolation with the L^1 -estimate, if we further use that $R^{\frac{5}{p}-3} \leq 1$ since $\frac{5}{3} \leq p$.

The side length of b are of the form $\frac{R'^2}{\kappa(j)} R^{-2\delta} = \frac{R'}{D\kappa(j)} R^{1-2\delta} \geq R' R^{1-2\delta}$, $j \in \{1, 2\}$. If $q \cap 2b = \emptyset$, then for $x \in b$ we have $|x - c_q| \geq \inf_{y \notin 2b} |x - y| = d(x, (2b)^c) \geq R' R^{1-2\delta}$.

Therefore for all $x \in b$

$$\begin{aligned} \left| \sum_{q \in Q^\mu, q \cap 2b = \emptyset} \chi_q(x) \right| & \leq C_N \sum_{\substack{l \in \mathbb{N} \\ 2^l \geq R^{1-2\delta}}} \sum_{\substack{q \in Q^\mu \\ |x - c_q| \sim R' 2^l}} \left(1 + \frac{|x - c_q|}{R'} \right)^{-N-2} \\ & \lesssim C_N \sum_{\substack{l \in \mathbb{N} \\ 2^l \geq R^{1-2\delta}}} |\{q : |x - c_q| \sim R' 2^l\}| 2^{-(N+2)l} \\ & \approx C_N \sum_{\substack{l \in \mathbb{N} \\ 2^l \geq R^{1-2\delta}}} 2^{-Nl} \\ & \approx C_N R^{-(1-2\delta)N} \\ & = C_{\delta, N'} R^{-N'}. \end{aligned} \quad (3.2.83)$$

Here we need that δ is not too large. Choosing N large enough, we see that by Lemma 3.2.8 and Lemma 3.2.9

$$\begin{aligned}
& \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \gamma b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q^\mu, q \cap 2b = \emptyset} \chi_q \right\|_{L^2(b)}^2 \\
& \lesssim \left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \gamma b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \right\|_{L^2}^2 \cdot \left\| \sum_{q \in Q^\mu, q \cap 2b = \emptyset} \chi_q \right\|_{L^\infty(b)}^2 \\
& \lesssim C_{\delta, N'} R'^{-1} |W_1| |W_2| \min_j \sup_{v_1, v_2} |V_j^{\Pi_{v_1, v_2}}| R^{-2N'} \\
& \lesssim C_{\delta, N'} R'^{-1} |W_1| |W_2| R^{1-2N'} \\
& \lesssim C'_{\delta, N''} R'^{-1} |W_1| |W_2| R^{-N''}.
\end{aligned}$$

Thus it is enough to consider the sum over the set $Q_b^\mu = \{q \in Q^\mu : q \cap 2b \neq \emptyset\}$. For fixed w_1, w_2 we split this set again into

$$\begin{aligned}
Q_b^\mu(w_1, w_2) &= Q_b^\mu \cap Q^\mu(w_1) \cap Q^\mu(w_2), \\
Q_b^\mu \cap Q^\mu(w_1) \setminus Q^\mu(w_2) \\
\text{and } Q_b^\mu \setminus Q^\mu(w_1) &= Q_b^\mu \cap Q^\mu(w_2) \setminus Q^\mu(w_1) \cup Q_b^\mu \setminus (Q^\mu(w_2) \cap Q^\mu(w_1)),
\end{aligned}$$

where all except the first part can be treated similar since they fall under the following more general case:

Let $Q_0 = Q_0(w_1, w_2) \subset Q_b^\mu$ such that there exists an $j = 1, 2$ with $R^\delta q \cap T_{w_j} = \emptyset$ for all $q \in Q_0$. Then

$$\left\| \sum_{w_1 \in W_1^{\lambda_1, \mu, \gamma b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2, \mu}} p_{w_2} \sum_{q \in Q_0} \chi_q \right\|_{L^2(b)}^2 \lesssim C_\alpha R^{c\delta} R'^{-1} |W_1| |W_2|. \quad (3.2.84)$$

For the proof assume without loss of generality $j = 1$. Let $q \in Q_0$, then $T_{w_1} \cap R^\delta q = \emptyset$ and for all $x \in \frac{1}{2}R^\delta q$ we have $\frac{1}{2}R^\delta R' \leq \text{dist}(x, (R^\delta q)^c) \leq \text{dist}(x, T_{w_1})$. Thus for every $x \in Q_{S_1, S_2}(R)$, we have $\text{dist}(x, T_{w_1}) \geq \frac{1}{2}R^\delta R'$ or $x \notin \frac{1}{2}R^\delta q$. If the first case, $\text{dist}(x, T_{w_1}) \geq \frac{1}{2}R^\delta R'$, holds true, then

$$|p_{w_1}(x)| \leq C_N R'^{-1} \left(1 + \frac{\text{dist}(x, T_{w_1})}{R'}\right)^{-2N} \quad (3.2.85)$$

$$\leq C'_N R'^{-1} R^{-\delta N} \left(1 + \frac{\text{dist}(x, T_{w_1})}{R'}\right)^{-N}. \quad (3.2.86)$$

On the other hand, if the second case, $x \notin \frac{1}{2}R^\delta q$, holds true, we have $\frac{1}{2}R^\delta R' \leq |x - c_q|$. Using the rapid decay of the Schwartz function ϕ we see that

$$|\chi_q(x)| = \left| \chi \left(\frac{x - c_q}{R'} \right) \right| \leq C_N \left(\frac{|x - c_q|}{R'} \right)^{-N} \leq C'_N R^{-\delta N}. \quad (3.2.87)$$

A similar argument as in (3.2.83) even gives

$$\left| \sum_{q \in Q_0} \chi_q(x) \right| \leq C''_N R^{-\delta N} \quad (3.2.88)$$

for all $x \notin \frac{1}{2}R^\delta q$. To summarize, we obtain for every $x \in Q_{S_1, S_2}(R)$

$$|p_{w_1} \sum_{q \in Q_0(w_1, w_2)} \chi_q|(x) \leq C(N, \delta) R'^{-1} R^{-\delta N} \left(1 + \frac{\text{dist}(x, T_{w_1})}{R'}\right)^{-N}, \quad (3.2.89)$$

This means the expression $p_{w_1} \sum_{q \in Q_0(w_1, w_2)} \chi_q$ behave as good as the original wave packet p_{w_1} itself, but is aided by an additional factor $R^{-\delta N}$. The symmetric term simply gives

$$|p_{w_2} \sum_{q \in Q_0(w_1, w_2)} \chi_q|(x) \lesssim \|p_{w_2}\|_\infty \lesssim R'^{-1} \left(1 + \frac{\text{dist}(x, T_{w_2})}{R'}\right)^{-N}, \quad (3.2.90)$$

without the additional factor.

The next step will follow the proof of Lemma 3.2.8. The crucial argument was the fact that the Fourier transform of $p_{w_1} p_{w_2}$ is supported in $v'_{j+1} + v_j + \mathcal{O}(R'^{-1})$. Since $\text{supp } \hat{\chi}_q = \text{supp } \hat{\chi}(R' \cdot) \subset B(0, R'^{-1})$, the Fourier support of $p_{w_1} p_{w_2} \sum_{q \in Q_0(w_1, w_2)} \chi_q$ remains essentially the same. At this point, we need the compact Fourier support of the functions χ_q . The modified wave packets $p_{w_i} \sum_{q \in Q_0(w_1, w_2)} \chi_q$ are still well separated in the y_i -variable for fixed direction v_i thanks to (3.2.89) and (3.2.90). Thus the argument from Lemma 3.2.8 applies; according to (3.2.58), we obtain

$$\begin{aligned} & \left\| \sum_{\substack{w_1 \in W_1^{\lambda_1, \mu, \neq b} \\ w_2 \in W_2^{\lambda_2, \mu}}} p_{w_1} p_{w_2} \sum_{q \in Q_0(w_1, w_2)} \chi_q \right\|_{L^2(b)}^2 \\ & \lesssim R' |W_1| |W_2| \min_j \sup_{v_1, v_2} |V_j^{\Pi_{v_1, v_2}}| \\ & \quad \sup_{w_1 \in W_1, w'_2 \in W_2} \|p_{w_2} \sum_{q \in Q_0(w_1, w'_2)} \chi_q\|_\infty \sup_{w'_1 \in W_1, w_2 \in W_2} \|p_{w'_1} \sum_{q \in Q_0(w'_1, w_2)} \chi_q\|_\infty \\ (3.2.89), (3.2.90) \quad & \lesssim C_{\delta, N'} R'^{-1} |W_1| |W_2| \min_j \sup_{v_1, v_2} |V_j^{\Pi_{v_1, v_2}}| R^{-N'} \\ \text{Lemma 3.2.9} \quad & \lesssim C_{\delta, N'} R'^{-1} R^{1-N'} |W_1| |W_2|. \end{aligned}$$

Therefore we may restrict ourselves to cubes from $Q_b^\mu(w_1, w_2)$. Notice that the kernel $K(q, q') = \chi_q(x) \chi_{q'}(x)$ fulfills Schur's test condition:

$$\sup_q \sum_{q'} \chi_q(x) \chi_{q'}(x) \lesssim \sum_{q'} \chi_{q'}(x) \lesssim 1. \quad (3.2.91)$$

$$\begin{aligned}
\text{Hence for } f_q &= \sum_{\substack{w_1 \in W_1^{\lambda_1, \mu, \gamma^b}(q) \\ w_2 \in W_2^{\lambda_2, \mu}(q)}} p_{w_1} p_{w_2} \text{ we have} \\
&\| \sum_{\substack{w_1 \in W_1^{\lambda_1, \mu, \gamma^b} \\ w_2 \in W_2^{\lambda_2, \mu}}} p_{w_1} p_{w_2} \sum_{q \in Q_b^\mu} \chi_q \|_{L^2(b)}^2 \\
&= \| \sum_{q \in Q_b^\mu} \chi_q f_q \|_{L^2(b)}^2 \\
&= \int_b \left| \sum_{q, q' \in Q_b^\mu} \chi_q \chi_{q'} f_q f_{q'} \right| dx \\
&\leq \int_b \left(\sum_q |f_q|^2 \right)^{\frac{1}{2}} \left(\sum_q \left| \sum_{q'} K(q, q') f_{q'} \right|^2 \right)^{\frac{1}{2}} dx \\
&\stackrel{\text{Schur}}{\leq} \int_b \sum_q |f_q|^2 dx \\
&= \sum_{q \in Q^\mu, q \cap 2b \neq \emptyset} \left\| \sum_{\substack{w_1 \in W_1^{\lambda_1, \mu, \gamma^b}(q) \\ w_2 \in W_2^{\lambda_2, \mu}(q)}} p_{w_1} p_{w_2} \right\|_{L^2(b)}^2 \\
&\stackrel{\text{Lemma 3.2.8}}{\lesssim} \sum_{q \in Q^\mu, q \cap 2b \neq \emptyset} R'^{-1} |W_1^{\lambda_1, \mu, \gamma^b}(q)| |W_2^{\lambda_2, \mu}(q)| \sup_{v_1, v_2} |[W_1^{\lambda_1, \mu, \gamma^b}(q)]^{\Pi_{v_1, v_2}}| \\
&\stackrel{\text{Lemma 3.2.12}}{\lesssim} R^{c\delta} R'^{-1} \sum_{q \in Q^\mu, q \cap 2b \neq \emptyset} |W_1^{\lambda_1, \mu}(q)| |W_2(q)| \frac{|W_2|}{\lambda_1 \mu_2} \\
&\lesssim R^{c\delta} R'^{-1} \sum_{w_1 \in W_1^{\lambda_1, \mu}} |Q^\mu(w_1)| \frac{|W_2|}{\lambda_1} \\
&\lesssim R^{c\delta} R'^{-1} |W_1| |W_2|.
\end{aligned}$$

□

3.2.8 Induction on scales

COROLLARY 3.2.14

There exist $c, \delta_0 > 0$ such that $c\delta_0 > 1$ with the following property: Let $\alpha > 0$ be such that $E(\alpha)$ holds true. Then for all $0 < \delta < \delta_0$, we have $E(\max\{\alpha(1-\delta), c\delta\})$.

Proof: We proved in the previous Lemmas 3.2.11 and 3.2.13 that

$$\begin{aligned}
&\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j, \mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \|_{L^p(Q_{S_1, S_2}(R))} \\
&\leq I + II + III \\
&\lesssim (C_\alpha R^{\alpha(1-\delta)} \log^{\gamma_\alpha}(1+R) + C_\delta R^{c\delta}) (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}} \\
&\lesssim C_{\alpha, \delta} R^{\alpha(1-\delta) \vee c\delta} \log^{\gamma_\alpha}(1+R) (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0) |W_1|^{\frac{1}{2}} |W_2|^{\frac{1}{2}}
\end{aligned}$$

for all $\delta < \delta_0 = \frac{1}{4}$. By simply increasing the constant c if necessary, we ensure $c\delta_0 > 1$. According to Lemma 3.2.7, this estimate then is sufficient for $E(\alpha)$. \square

COROLLARY 3.2.15

Let $\alpha > 0$. Then $E(\alpha)$ holds true and

$\exists C_\alpha > 0 \exists \gamma_\alpha > 0 \forall (S, \tilde{S}) \in \mathcal{S}_0 \forall R \geq 1 \forall f_j \in L^2(S_j), j = 1, 2$

$$\|R^* f_1 R^* f_2\|_{L^p(Q_{S_1, S_2}(R))} \leq C_\alpha R^\alpha (\kappa^1 \kappa^2)^{\frac{1}{2} - \frac{1}{p}} D^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0) \|f_1\|_2 \|f_2\|_2,$$

where $C_0 = \frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} (D\bar{\kappa})^{-\frac{1}{p}} (D\kappa^1 D\kappa^2)^{-\frac{1}{2}}$.

Proof: Define the sequence $\alpha_0 = 1, \alpha_{j+1} = \frac{c\alpha_j}{c+\alpha_j}$. Since it is decreasing and converges to zero, we are done if we can show that $E(\alpha_j)$ holds true for any $j \in \mathbb{N}$. We know that $E(\alpha_0) = E(1)$ is valid from Lemma 3.2.10. Assume that $E(\alpha_j)$ holds true for some $j \in \mathbb{N}$. To see that $E(\alpha_{j+1})$ holds true as well, we apply Lemma 3.2.14 with $\delta = \frac{\alpha_j}{c+\alpha_j} < \frac{1}{c} < \delta_0$. Since $\alpha_j(1 - \delta) = c\delta = \alpha_j \frac{c}{c+\alpha_j} = \alpha_{j+1}$, we obtain $E(\alpha_{j+1})$. \square

3.3 Scaling

The goal of this chapter is to establish a bilinear estimate for two surfaces with essentially constant curvature, located over boxes with given side length. However, we cannot apply directly the theory we developed previously. Instead, we first have to find an appropriate rescaling.

LEMMA 3.3.1

Let $S_j = \{(\xi, \phi(\xi)) | \xi \in \tau_j\}$, $\tau_j \subset \mathbb{R}^d$, $j = 1, 2$. Let $A \in GL(d)$, $a > 0$, and put $\phi^s(\eta) = \frac{1}{a}\phi(A\eta)$. By σ we denote the surface measure on $\text{Graph}(\phi)$, by σ_s the surface measure on $\text{Graph}(\phi^s)$, and let $S_j^s = \{(\eta, \phi^s(\eta)) | \eta \in \tau_j^s\}$, $\tau_j^s = A^{-1}(\tau_j)$, $j = 1, 2$. For any $Q^s \subset \mathbb{R}^{d+1}$, let $Q = \{x | ({}^tAx', ax_{d+1}) \in Q^s\}$. Assume that

$$\|\widehat{g_1 d\sigma_s g_2 d\sigma_s}\|_{L^p(Q^s)} \leq C_s \|g_1\|_2 \|g_2\|_2 \text{ for all } g_j \in L^2(S_j^s, d\sigma^s). \quad (3.3.1)$$

Then

$$\|\widehat{f_1 d\sigma f_2 d\sigma}\|_{L^p(Q)} \leq C_s |\det A|^{\frac{1}{p}} a^{-\frac{1}{p}} \|f_1\|_2 \|f_2\|_2 \text{ for all } f_j \in L^2(S_j, d\sigma). \quad (3.3.2)$$

Proof: The lemma is proven in the appendix at first. \square

GENERAL ASSUMPTIONS

Let $S = \Gamma(\phi|_U)$, $U = r + [0, d_1] \times [0, d_2]$, and let the principal curvatures on S be comparable to $\kappa_i = r_i^{m_i-2}$, $r_i \geq d_i$. Furthermore let $\kappa = \kappa_1 \vee \kappa_2$.

Let $\tilde{S} = \Gamma(\phi|_{\tilde{U}})$ with corresponding quantities $\tilde{r}_i, \tilde{d}_i, \tilde{\kappa}_i, \tilde{\kappa}$, and let $\bar{d}_i = d_i \vee \tilde{d}_i$, $\bar{\kappa}_i = \kappa_i \vee \tilde{\kappa}_i$, $\bar{\kappa} = \kappa \vee \tilde{\kappa} = \bar{\kappa}_1 \vee \bar{\kappa}_2$ and $\Delta r = r - \tilde{r}$.

We assume that in each direction the set with larger principal curvature has the larger length. This can be expressed in the following formula:

$$(\kappa_i d_i) \vee (\tilde{\kappa}_i \tilde{d}_i) = \bar{\kappa}_i \bar{d}_i. \quad (3.3.3)$$

Further do we assume the separation condition

$$\text{dist}_{x_i}(U, \tilde{U}) = \inf\{|x_i - \tilde{x}_i| : x \in U, \tilde{x} \in \tilde{U}\} \approx |\Delta r_i| \approx \bar{d}_i. \quad (3.3.4)$$

For the normal field N on $S \cup \tilde{S}$ given by $N(\xi, \phi(\xi)) = (-\nabla\phi(\xi), 1)$ let $N_0 = \frac{N}{|N|}$ denote the induced unit normal field.

Define

$$a_1 = \bar{\kappa}_2 \bar{d}_2, \quad a_2 = \bar{\kappa}_1 \bar{d}_1. \quad (3.3.5)$$

In order to apply our bilinear theory, we introduce a rescaling:

Let

$$\phi^s(\eta) = \frac{1}{a}\phi(A\eta) = \frac{1}{a_1 a_2} \phi(a_1 \eta_1, a_2 \eta_2). \quad (3.3.6)$$

We will denote the corresponding quantities under this scaling by $r_i^s = \frac{r_i}{a_i}$, $d_i^s = \frac{d_i}{a_i}$, $\kappa_i^s = \frac{1}{a_1 a_2} a_i^2 \kappa_i = \frac{a_i}{a_{i+1 \bmod 2}} \kappa_i$, $U^s = r^s + [0, d_1^s] \times [0, d_2^s]$, N_0^s, \tilde{r}_i^s , and so on. Then the following lemma holds true:

LEMMA 3.3.2 (SCALING)

(i) For all $i = 1, 2$ and all $\eta \in U^s$, $\tilde{\eta} \in \tilde{U}^s$ we have $|\partial_i \phi^s(\eta) - \partial_i \phi^s(r^s)| \lesssim \kappa_i^s d_i^s \lesssim 1$ and $|\partial_i \phi^s(\tilde{\eta}) - \partial_i \phi^s(\tilde{r}^s)| \lesssim \tilde{\kappa}_i^s \tilde{d}_i^s \lesssim 1$.

(ii) $|\partial^\alpha \phi^s(\eta)| \lesssim \kappa^s \max_i |d_1^s \wedge d_2^s|^{2-|\alpha|}$ for all $|\alpha| \geq 2$, $\eta \in U^s$;
 $|\partial^\alpha \phi^s(\tilde{\eta})| \lesssim \tilde{\kappa}^s \max_i |\tilde{d}_1^s \wedge \tilde{d}_2^s|^{2-|\alpha|}$ for all $|\alpha| \geq 2$, $\tilde{\eta} \in \tilde{U}^s$.

(iii) In both variables $i = 1, 2$, we have the separation condition

$$|\partial_i \phi^s(\eta) - \partial_i \phi^s(\tilde{\eta})| \approx 1 \quad \forall \eta \in S, \forall \tilde{\eta} \in \tilde{S}. \quad (3.3.7)$$

Proof: For $\eta \in U$ we obviously have

$$|\partial_i \phi^s(\eta) - \partial_i \phi^s(r^s)| \leq \sup_{\eta' \in U} |\partial_i^2 \phi^s(\eta')| |\eta_i - r_i^s| \lesssim \kappa_i^s d_i^s, \quad (3.3.8)$$

with

$$\kappa_i^s d_i^s = \frac{a_i}{a_{i+1 \bmod 2}} \kappa_i \frac{d_i}{a_i} = \frac{\kappa_i d_i}{\bar{\kappa}_i \bar{d}_i} \leq 1. \quad (3.3.9)$$

To prove the second statement, we observe that ϕ^s has vanishing mixed derivatives since ϕ does. In the unscaled situation, we have for $k \leq m_i$

$$\partial_i^k \phi(\xi) \approx \xi_i^{m_i - k} \approx \partial_i^2 \phi(\xi) \xi_i^{2-k} \approx \kappa_i \xi_i^{2-k}$$

for $\xi \in U$. Thus for $\eta \in U^s$

$$\begin{aligned} \partial_i^k \phi^s(\eta) &= \frac{1}{a_1 a_2} a_i^k \partial_i^k \phi(A\eta) \\ &\approx \frac{1}{a_1 a_2} a_i^k \kappa_i (a_i \eta_i)^{2-k} \\ &= \frac{a_i^2}{a_1 a_2} \kappa_i \eta_i^{2-k} \\ &= \kappa_i^s \eta_i^{2-k}. \end{aligned}$$

Since we have for $\eta \in U^s$

$$\eta_i \geq r_i^s = \frac{r_i}{a_i} \geq \frac{d_i}{a_i} = d_i^s \geq d_1^s \wedge d_2^s,$$

we conclude

$$|\partial_i^k \phi^s(\eta)| \lesssim \kappa^s (d_1^s \wedge d_2^s)^{2-k} \quad (3.3.10)$$

for $k \geq 2$. The corresponding result for $\eta \in \tilde{U}^s$ is proven the same way. Furthermore for $\xi \in U$, $\tilde{\xi} \in \tilde{U}$ we have

$$\begin{aligned} |\partial_i \phi(\xi) - \partial_i \phi(\tilde{\xi})| &= \left| \int_{\xi_i}^{\tilde{\xi}_i} \phi_i''(t) dt \right| \\ &\approx \left| \int_{\xi_i}^{\tilde{\xi}_i} t^{m_i-2} dt \right| \\ &= |\tilde{\xi}_i^{m_i-1} - \xi_i^{m_i-1}| \\ &= |\tilde{\xi}_i - \xi_i| \sum_{k=0}^{m_i-2} \xi_i^k \tilde{\xi}_i^{m_i-2-k} \\ &\approx |\tilde{\xi}_i - \xi_i| (\xi_i^{m_i-2} + \tilde{\xi}_i^{m_i-2}) \stackrel{(3.3.4)}{\approx} \bar{\kappa}_i \bar{d}_i = a_{i+1 \bmod 2}, \end{aligned}$$

and thus

$$|\partial_i \phi^s(\eta) - \partial_i \phi^s(\tilde{\eta})| = \frac{|\partial_i \phi(A\eta) - \partial_i \phi(A\tilde{\eta})|}{a_{i+1 \bmod 2}} \approx 1. \quad (3.3.11)$$

□

Now we apply Theorem 3.2.5 to the scaled phase function ϕ^s . According to (3.2.20), the scaled cuboids are

$$Q_{S^s, \tilde{S}^s}(R) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi^s(r_0^s)x_3| \leq \frac{R^2}{(D^s)^2 \bar{\kappa}^s}, i = 1, 2, |x_3| \leq \frac{R^2}{(D^s)^2 \kappa^s \wedge \tilde{\kappa}^s} \right\}$$

with $r_0^s = r^s$ if $\kappa^s = \kappa^s \wedge \tilde{\kappa}^s$ or $r_0^s = \tilde{r}^s$ if $\tilde{\kappa}^s = \kappa^s \wedge \tilde{\kappa}^s$. Thus for every $\alpha > 0$, we have for all $R \geq 1$

$$\|R_{S^s, \tilde{S}^s}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S^s, \tilde{S}^s}(R))} \leq (\kappa^s \tilde{\kappa}^s)^{\frac{1}{2} - \frac{1}{p}} (D^s)^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0^s) C_\alpha R^\alpha, \quad (3.3.12)$$

with $C_0^s = \frac{\bar{d}_1^s \bar{d}_2^s}{(D^s)^4} (D^s [\kappa^s \wedge \tilde{\kappa}^s])^{-\frac{1}{p}} (D^s \kappa^s D^s \tilde{\kappa}^s)^{-\frac{1}{2}}$ and $D^s = \min\{d_1^s, d_2^s, \tilde{d}_1^s, \tilde{d}_2^s\}$. Scaling back using Lemma 3.3.1, we obtain

$$\begin{aligned} \|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R))} &\leq (a_1 a_2)^{1 - \frac{2}{p}} (\kappa^s \tilde{\kappa}^s)^{\frac{1}{2} - \frac{1}{p}} (D^s)^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0^s) C_\alpha R^\alpha \\ &= (a_1 a_2 \kappa^s \cdot a_1 a_2 \tilde{\kappa}^s)^{\frac{1}{2} - \frac{1}{p}} (D^s)^{3 - \frac{5}{p}} \log^{\gamma_\alpha}(C_0^s) C_\alpha R^\alpha, \end{aligned} \quad (3.3.13)$$

where

$$\begin{aligned} Q_{S, \tilde{S}}(R) &= \left\{ x \in \mathbb{R}^3 : |a_i x_i + \partial_i \phi^s(r_0^s) a_1 a_2 x_3| \leq \frac{R^2}{(D^s)^2 \bar{\kappa}^s}, i = 1, 2, |a_1 a_2 x_3| \leq \frac{R^2}{(D^s)^2 \kappa^s \wedge \tilde{\kappa}^s} \right\} \\ &= \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r_0)x_3| \leq \frac{R^2}{a_i (D^s)^2 \bar{\kappa}^s}, i = 1, 2, |x_3| \leq \frac{R^2}{a_1 a_2 (D^s)^2 \kappa^s \wedge \tilde{\kappa}^s} \right\}. \end{aligned}$$

We have

$$\bar{\kappa}^s = \bar{\kappa}_1^s \vee \bar{\kappa}_2^s = \frac{a_1}{a_2} \bar{\kappa}_1 \vee \frac{a_2}{a_1} \bar{\kappa}_2 = \frac{a_1}{\bar{d}_1} \vee \frac{a_2}{\bar{d}_2} = \frac{\bar{\kappa}_2 \bar{d}_2^2 \vee \bar{\kappa}_1 \bar{d}_1^2}{\bar{d}_1 \bar{d}_2} \quad (3.3.14)$$

and

$$D^s = \min\{d_1^s, d_2^s, \tilde{d}_1^s, \tilde{d}_2^s\} \leq \min\left\{\frac{\bar{d}_1}{a_1}, \frac{\bar{d}_2}{a_2}\right\} = (\bar{\kappa}^s)^{-1}, \quad (3.3.15)$$

hence

$$a_1(D^s)^2 \bar{\kappa}^s \leq a_1 D^s \leq \bar{d}_1. \quad (3.3.16)$$

A similar computation yields $a_2(D^s)^2 \bar{\kappa}^s \leq \bar{d}_2$. Furthermore

$$(D^s)^2 a_1 a_2 (\kappa^s \wedge \tilde{\kappa}^s) \leq D^s a_1 a_2 \leq a_2 \bar{d}_1 \wedge a_1 \bar{d}_2 = \bar{\kappa}_1 \bar{d}_1^2 \wedge \bar{\kappa}_2 \bar{d}_2^2.$$

Therefore

$$Q_{S, \tilde{S}}(R) \supset Q_{S, \tilde{S}}^0(R) := \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r_0) x_3| \leq \frac{R^2}{d_i}, i = 1, 2, |x_3| \leq \frac{R^2}{\bar{\kappa}_1 \bar{d}_1^2 \wedge \bar{\kappa}_2 \bar{d}_2^2} \right\}. \quad (3.3.17)$$

Moreover we have

$$d_i^s = \frac{d_i}{a_i} = \frac{\bar{\kappa}_i \bar{d}_i}{a_1 a_2} d_i,$$

and

$$\min\{d_i^s, \tilde{d}_i^s\} = \frac{\bar{\kappa}_i}{a_1 a_2} \bar{d}_i \min\{d_i, \tilde{d}_i\} = \frac{\bar{\kappa}_i d_i \tilde{d}_i}{a_1 a_2}.$$

Further

$$a_1 a_2 \kappa^s \approx a_1 a_2 \left(\frac{a_2}{a_1} \kappa_2 + \frac{a_1}{a_2} \kappa_1 \right) = (\bar{\kappa}_1^2 \bar{d}_1^2 \kappa_2 + \bar{\kappa}_2^2 \bar{d}_2^2 \kappa_1) = \bar{\kappa}_1 \bar{\kappa}_2 \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\kappa_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\kappa_1}{\bar{\kappa}_1} \right). \quad (3.3.18)$$

Thus the constant in (3.3.13) (up to $C_\alpha R^\alpha \log^{\gamma_\alpha}(C_0^s)$) transforms like

$$\begin{aligned} & (a_1 a_2 \kappa^s \cdot a_1 a_2 \tilde{\kappa}^s)^{\frac{1}{2} - \frac{1}{p}} (D^s)^{3 - \frac{5}{p}} \\ &= (a_1 a_2)^{\frac{5}{p} - 3} (\bar{\kappa}_1 \bar{\kappa}_2)^{1 - \frac{2}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\kappa_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\kappa_1}{\bar{\kappa}_1} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\tilde{\kappa}_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\tilde{\kappa}_1}{\bar{\kappa}_1} \right)^{\frac{1}{2} - \frac{1}{p}} \min_i (\bar{\kappa}_i d_i \tilde{d}_i)^{3 - \frac{5}{p}} \\ &= (\bar{\kappa}_1 \bar{d}_1 \bar{\kappa}_2 \bar{d}_2)^{\frac{5}{p} - 3} (\bar{\kappa}_1 \bar{\kappa}_2)^{1 - \frac{2}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\kappa_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\kappa_1}{\bar{\kappa}_1} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\tilde{\kappa}_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\tilde{\kappa}_1}{\bar{\kappa}_1} \right)^{\frac{1}{2} - \frac{1}{p}} \min_i (\bar{\kappa}_i d_i \tilde{d}_i)^{3 - \frac{5}{p}} \\ &= (\bar{\kappa}_1 \bar{\kappa}_2)^{\frac{3}{p} - 2} (\bar{d}_1 \bar{d}_2)^{\frac{5}{p} - 3} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\kappa_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\kappa_1}{\bar{\kappa}_1} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\tilde{\kappa}_2}{\bar{\kappa}_2} + \bar{\kappa}_2 \bar{d}_2^2 \frac{\tilde{\kappa}_1}{\bar{\kappa}_1} \right)^{\frac{1}{2} - \frac{1}{p}} \min_i (\bar{\kappa}_i d_i \tilde{d}_i)^{3 - \frac{5}{p}}. \end{aligned}$$

For $a, b \in (0, \infty)$ write $q(a, b) = \frac{a \vee b}{a \wedge b} = \frac{a}{b} \vee \frac{b}{a} \geq 1$. A lower bound for D^s is

$$D^s = \frac{d_1 \wedge \tilde{d}_1}{a_1} \wedge \frac{d_2 \wedge \tilde{d}_2}{a_2} \geq \left(\frac{d_1 \wedge \tilde{d}_1}{d_1} \wedge \frac{d_2 \wedge \tilde{d}_2}{d_2} \right) \left(\frac{\bar{d}_1}{a_1} \wedge \frac{\bar{d}_2}{a_2} \right) \stackrel{(3.3.14)}{\geq} \frac{1}{q(d_1, \tilde{d}_1) q(d_2, \tilde{d}_2)} \frac{1}{\bar{\kappa}^s}. \quad (3.3.19)$$

From formula (3.3.18), we can deduce

$$\kappa^s \gtrsim \frac{\bar{\kappa}_1 \bar{\kappa}_2}{a_1 a_2} (\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2) \frac{\kappa_1 \kappa_2}{\bar{\kappa}_1 \bar{\kappa}_2} \geq \frac{\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2}{\bar{d}_1 \bar{d}_2} \frac{1}{q(\kappa_1, \tilde{\kappa}_1) q(\kappa_2, \tilde{\kappa}_2)} \stackrel{(3.3.14)}{=} \frac{\bar{\kappa}^s}{q(\kappa_1, \tilde{\kappa}_1) q(\kappa_2, \tilde{\kappa}_2)}. \quad (3.3.20)$$

Especially we have that

$$\begin{aligned} (D^s \kappa^s)^{-1} &\lesssim \prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i), \\ (D^s \tilde{\kappa}^s)^{-1} &\lesssim \prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i). \end{aligned} \quad (3.3.21)$$

Thus we conclude (and we can be generous in the exponents since C_0^s only appears logarithmically)

$$\begin{aligned} C_0^s &= \frac{\bar{d}_1^{s^2} \bar{d}_2^{s^2}}{(D^s)^4} (D^s [\kappa^s \wedge \tilde{\kappa}^s])^{-\frac{1}{p}} (D^s \kappa^s D^s \tilde{\kappa}^s)^{-\frac{1}{2}} \\ &\stackrel{(3.3.21)}{\leq} \frac{\bar{d}_1^{s^2} \bar{d}_2^{s^2}}{(D^s)^4} \left[\prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i) \right]^{\frac{1}{p}+1} \\ &\stackrel{(3.3.19)}{\leq} \left[\prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i) \right]^{\frac{1}{p}+5} (\bar{d}_1 \bar{d}_2)^2 (\bar{\kappa}^s)^4 \\ &\leq \left[\prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i) \right]^{\frac{1}{p}+5} \left(\frac{\bar{d}_1 \bar{d}_2}{a_1 a_2} \right)^2 \left(\frac{\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2}{\bar{d}_1 \bar{d}_2} \right)^4 \\ &= \left[\prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i) \right]^{\frac{1}{p}+5} \left(\frac{(\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2)^2}{\bar{\kappa}_1 \bar{d}_1^2 \bar{\kappa}_2 \bar{d}_2^2} \right)^2 \\ &= \left[\prod_{i=1,2} q(\kappa_i, \tilde{\kappa}_i) q(d_i, \tilde{d}_i) \right]^{\frac{1}{p}+5} q(\bar{\kappa}_1 \bar{d}_1^2, \bar{\kappa}_2 \bar{d}_2^2)^2. \end{aligned}$$

We end up with the following result:

LEMMA 3.3.3

For all $\alpha > 0$ there exist $C_\alpha, \gamma_\alpha > 0$ such that for every S, \tilde{S} as described at the beginning of this chapter and every $R > 0$, we have

$$\begin{aligned} \|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R))} &\leq C_\alpha R^\alpha (\bar{\kappa}_1 \bar{\kappa}_2)^{\frac{3}{p}-2} (\bar{d}_1 \bar{d}_2)^{\frac{5}{p}-3} \min_i (\bar{\kappa}_i d_i \tilde{d}_i)^{3-\frac{5}{p}} \\ &\quad \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\kappa_2}{\bar{\kappa}_2} \vee \bar{\kappa}_2 \bar{d}_2^2 \frac{\kappa_1}{\bar{\kappa}_1} \right)^{\frac{1}{2}-\frac{1}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\tilde{\kappa}_2}{\bar{\kappa}_2} \vee \bar{\kappa}_2 \bar{d}_2^2 \frac{\tilde{\kappa}_1}{\bar{\kappa}_1} \right)^{\frac{1}{2}-\frac{1}{p}} \\ &\quad \left[1 + \log^{\gamma_\alpha} \left(q(\bar{\kappa}_1 \bar{d}_1^2, \bar{\kappa}_2 \bar{d}_2^2) \prod_{i=1,2} q(d_i, \tilde{d}_i) q(\kappa_i, \tilde{\kappa}_i) \right) \right]. \end{aligned} \quad (3.3.22)$$

3.4 Dyadic Summation

The next step will be a bilinear result for the surface over a pair of dyadic rectangles with not necessarily constant curvature. The idea is to decompose these rectangles into smaller parts with (roughly) constant curvature where we can apply previous results.

Define a "weighted" maximum

$$M(a, b) = M_{m_1, m_2}(a, b) = \begin{cases} a & \text{if } m_1 > m_2 \\ b & \text{if } m_1 < m_2 \\ a \vee b & \text{if } m_1 = m_2. \end{cases} \quad (3.4.1)$$

Notice that $M_{m_1, m_2}(a, b) = M_{m_2, m_1}(b, a) = \lim_{t \rightarrow 0} \frac{at^{m_2} \vee bt^{m_1}}{t^{m_1 \wedge m_2}}$.

GENERAL ASSUMPTIONS

Let $U, \tilde{U} \subset [0, 1] \times [0, 1]$ be two dyadic rectangles of side length and distance in x_1 -direction (x_2 -direction) equal to ρ_1 (ρ_2 respectively). By \varkappa_i denote the maximal principal curvature in x_i -direction of both $S = \text{Graph}(\phi|_U)$ and $\tilde{S} = \text{Graph}(\phi|_{\tilde{U}})$. Let σ and $\tilde{\sigma}$ be the surface measures of S and \tilde{S} respectively. (3.4.2)

THEOREM 3.4.1

Let $\frac{5}{3} < p \leq 2$, $\bar{m} = m_1 \vee m_2$, $m = m_1 \wedge m_2$, and let

$$Q_{S, \tilde{S}}(R; r) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r)x_3| \leq \frac{R}{\rho_i}, i = 1, 2, |x_3| \leq \frac{R}{\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2} \right\}$$

for $r \in U \cup \tilde{U}$. Then for $p > p_c = 2\frac{\bar{m}+3}{\bar{m}+4}$ and all $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that for all $R \geq 1$ and all $r \in U \cup \tilde{U}$ we have

$$\|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R; r))} \leq C_\alpha R^\alpha [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p} - \frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2} - \frac{5}{2p}} \cdot \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \quad (3.4.3)$$

If only $p > 2\frac{m+3}{m+4}$ is valid, then we still have

$$\|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R; r))} \leq C_\alpha R^\alpha [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p} - \frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2} - \frac{5}{2p}} \cdot \left(\frac{M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{4}{p} - \frac{5}{2}} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right), \quad (3.4.4)$$

with $M = M_{m_1, m_2}$ (see (3.4.1)).

Observe that for $m_1 = m_2$, both parts of the theorem coincide.

In general, the first part of the theorem is the one we are more interested in, since we are able to proceed to a linear estimate. In the second case, where $2\frac{\bar{m}+3}{\bar{m}+4} \geq p > 2\frac{m+3}{m+4}$, unfortunately, we are not able to do so. However, the case is interesting in so far as we will see that our estimate is essentially sharp for p near $2\frac{m+3}{m+4}$. This issue will be discussed in Chapter 3.7.

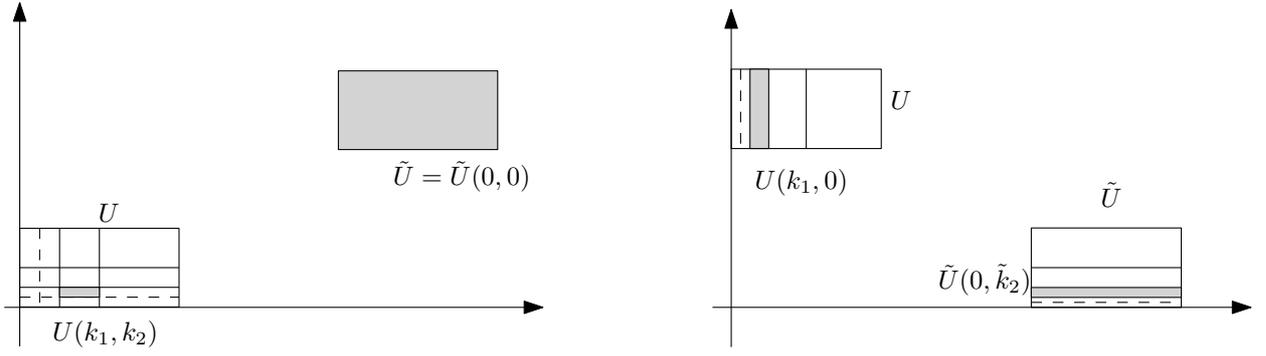


Figure 3.6: Two possibilities for the decomposition into subboxes

Proof: If U does not intersect with the x_i -axis, the curvature on U is indeed comparable to \varkappa_i . Otherwise we decompose U further in sets with (roughly) constant principal curvatures in order to apply the previous results. To each dyadic interval $I = [r2^{-j}, (r+1)2^j]$, $r, j \in \mathbb{N}$ we associate a family of subsets $\{I(k)\}_{k \in \mathcal{N}_0}$ with $\bigcup_{k \in \mathcal{N}_0} I(k) = I$ according to the following two alternatives:

(i) If $r > 0$, choose $\mathcal{N}_0 = \{0\}$ and $I(0) = I$.

(ii) If $r = 0$, choose $\mathcal{N}_0 = \mathbb{N} = \{1, 2, 3, \dots\}$ and $I(k) = [2^{-k}(r+1)2^{-j}, 2^{1-k}(r+1)2^{-j}]$.

If we write $U = I_1 \times I_2$, then denote by $\{I_i(k_i)\}_{k_i \in \mathcal{N}_i}$ their associated family and let $U(k) = I(k_1) \times I(k_2)$, $k = (k_1, k_2) \in \mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$, $S(k) = \text{Graph}(\phi|_{U(k)})$. Introduce $\tilde{\mathcal{N}}$, $\tilde{U}(k)$ and $\tilde{S}(k)$, $k \in \tilde{\mathcal{N}}$ in an analogous manner. Other quantities which will be important are the principal curvatures on $U(k)$, i.e. $\kappa_i(k_i) := 2^{-k_i(m_i-2)}\varkappa_i$, and the side length $d_i(k_i) = 2^{-k_i}\rho_i$ of $U(k)$.

One simple but crucial observation is that since I_i and \tilde{I}_i are separated, for both $i = 1, 2$, we have $\mathcal{N}_i = \{0\}$ or $\tilde{\mathcal{N}}_i = \{0\}$ (cf. Figures 3.6). (3.4.5)

Hence for each pair $(k_i, \tilde{k}_i) \in \mathcal{N}_i \times \tilde{\mathcal{N}}_i$, $k_i = 0$ or $\tilde{k}_i = 0$, and thus

$$\bar{\kappa}_i(k_i, \tilde{k}_i) := \max\{\kappa_i(k_i), \tilde{\kappa}_i(\tilde{k}_i)\} = \max\{2^{-k_i(m_i-2)}, 2^{-\tilde{k}_i(m_i-2)}\}\varkappa_i = \varkappa_i \quad (3.4.6)$$

$$\text{and} \quad \bar{d}_i(k_i, \tilde{k}_i) := \max\{d_i(k_i), \tilde{d}_i(\tilde{k}_i)\} = \max\{2^{-k_i}, 2^{-\tilde{k}_i}\}\rho_i = \rho_i. \quad (3.4.7)$$

We conclude

$$\frac{\kappa_i(k_i)}{\bar{\kappa}_i(k_i, \tilde{k}_i)} = 2^{-k_i(m_i-2)}, \quad (3.4.8)$$

$$\frac{\tilde{\kappa}_i(\tilde{k}_i)}{\bar{\kappa}_i(k_i, \tilde{k}_i)} = 2^{-\tilde{k}_i(m_i-2)} \quad (3.4.9)$$

and

$$\frac{d_i(k_i)}{\bar{d}_i} = 2^{-k_i}, \quad (3.4.10)$$

$$d_i(k_i)\tilde{d}_i(\tilde{k}_i) = 2^{-k_i-\tilde{k}_i}\rho_i^2 = 2^{-k_i \vee \tilde{k}_i}\rho_i^2, \quad (3.4.11)$$

hence

$$\begin{aligned}
& \log(q(\bar{\kappa}_1 \bar{d}_1^2, \bar{\kappa}_2 \bar{d}_2^2) \prod_{i=1,2} q(d_i(k_i), \tilde{d}_i(\tilde{k}_i)) q(\kappa_i(k_i), \tilde{\kappa}_i(\tilde{k}_i))) \\
& \lesssim k_1 + \tilde{k}_1 + k_2 + \tilde{k}_2 + \log(q(\bar{\kappa}_1 \bar{d}_1^2, \bar{\kappa}_2 \bar{d}_2^2)) \\
& \lesssim [k_1 + \tilde{k}_1 + k_2 + \tilde{k}_2] \log \left(\frac{\varkappa_1 \rho_1}{\varkappa_2 \rho_2} + \frac{\varkappa_2 \rho_2}{\varkappa_1 \rho_1} \right).
\end{aligned} \tag{3.4.12}$$

According to (3.3.17), the associated cuboids are

$$Q_{S(k), \tilde{S}(\tilde{k})}(R) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r^{k, \tilde{k}}) x_3| \leq \frac{R^2}{\rho_i}, i = 1, 2, |x_3| \leq \frac{R^2}{\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2} \right\},$$

where $r^{k, \tilde{k}}$ is a certain fixed point⁵ in $S(k)$ or $\tilde{S}(\tilde{k})$. For any $r \in U \cup \tilde{U}$ define the shorter boxes

$$Q_{S, \tilde{S}}(R; r) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r) x_3| \leq \frac{R^2}{\rho_i}, i = 1, 2, |x_3| \leq \frac{R^2}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right\}. \tag{3.4.13}$$

We claim that for all $r \in U \cup \tilde{U}$, $R \geq 1$ and all $(k, \tilde{k}) \in \mathcal{N} \times \tilde{\mathcal{N}}$

$$Q_{S, \tilde{S}}(R; r) \subset Q_{S(k), \tilde{S}(\tilde{k})}(2R). \tag{3.4.14}$$

If $x \in Q_{S, \tilde{S}}(R; r)$, then obviously $|x_3| \leq \frac{R^2}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \leq \frac{(2R)^2}{\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2}$. Furthermore

$$\begin{aligned}
|x_i + \partial_i \phi(r^{k, \tilde{k}}) x_3| & \leq |x_i + \partial_i \phi(r) x_3| + |\partial_i \phi(r) - \partial_i \phi(r^{k, \tilde{k}})| |x_3| \\
& \leq \frac{R^2}{\rho_i} + \|\partial_i^2 \phi\|_\infty |r_i - r_i^{k, \tilde{k}}| \frac{R^2}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \\
& \leq \frac{R^2}{\rho_i} + \varkappa_i \cdot 3 \rho_i \frac{R^2}{\varkappa_i \rho_i^2} \\
& = \frac{4R^2}{\rho_i}.
\end{aligned}$$

Putting (3.4.6)-(3.4.12) into (3.3.22) we conclude

$$\begin{aligned}
& \|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R; r))} \\
& \leq \sum_{k \in \mathcal{N}, \tilde{k} \in \tilde{\mathcal{N}}} \|R_{S(k), \tilde{S}(\tilde{k})}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S(k), \tilde{S}(\tilde{k})}(2R))} \\
& \leq C_\alpha R^\alpha (\varkappa_1 \varkappa_2)^{\frac{3}{p}-2} (\rho_1 \rho_2)^{\frac{5}{p}-3} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\
& \quad \sum_{k \in \mathcal{N}, \tilde{k} \in \tilde{\mathcal{N}}} [k_1 + \tilde{k}_1 + k_2 + \tilde{k}_2]^{\gamma_\alpha} (\varkappa_1 \rho_1^2 2^{-k_1 - \tilde{k}_1} \wedge \varkappa_2 \rho_2^2 2^{-k_2 - \tilde{k}_2})^{3 - \frac{5}{p}} \\
& \quad \cdot \left(\varkappa_1 \rho_1^2 2^{-k_2(m_2-2)} \vee \varkappa_2 \rho_2^2 2^{-k_1(m_1-2)} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\varkappa_1 \rho_1^2 2^{-\tilde{k}_2(m_2-2)} \vee \varkappa_2 \rho_2^2 2^{-\tilde{k}_1(m_1-2)} \right)^{\frac{1}{2} - \frac{1}{p}}.
\end{aligned}$$

⁵We also have some algorithm to determine how to choose $r^{k, \tilde{k}}$, but it is not important at this point.

We claim that for the first part of the theorem where $p > p_c = \frac{2\bar{m}+3}{\bar{m}+4}$

$$\begin{aligned} & \sum_{k \in \mathcal{N}, \tilde{k} \in \tilde{\mathcal{N}}} (\varkappa_1 \rho_1^2 2^{-k_1 - \tilde{k}_1} \wedge \varkappa_2 \rho_2^2 2^{-k_2 - \tilde{k}_2})^{3 - \frac{5}{p}} [k_1 + \tilde{k}_1 + k_2 + \tilde{k}_2]^{\gamma_\alpha} \\ & \quad \cdot \left(\varkappa_1 \rho_1^2 2^{-k_2(m_2-2)} \vee \varkappa_2 \rho_2^2 2^{-k_1(m_1-2)} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\varkappa_1 \rho_1^2 2^{-\tilde{k}_2(m_2-2)} \vee \varkappa_2 \rho_2^2 2^{-\tilde{k}_1(m_1-2)} \right)^{\frac{1}{2} - \frac{1}{p}} \\ & \lesssim (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{3 - \frac{5}{p}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{1 - \frac{2}{p}} \log^{1+\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \end{aligned} \quad (3.4.15)$$

Taking this for granted, we conclude

$$\begin{aligned} & \|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R))} \\ & \lesssim C_\alpha R^\alpha (\varkappa_1 \varkappa_2)^{\frac{3}{p} - 2} (\rho_1 \rho_2)^{\frac{5}{p} - 3} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{3 - \frac{5}{p}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{1 - \frac{2}{p}} \log^{1+2\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ & = C_\alpha R^\alpha (\varkappa_1 \varkappa_2)^{\frac{1}{2p} - \frac{1}{2}} (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{5}{2p} - \frac{3}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{3 - \frac{5}{p}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{1 - \frac{2}{p}} \log^{\gamma'_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ & = C_\alpha R^\alpha [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p} - \frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2} - \frac{5}{2p}} \log^{\gamma'_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \end{aligned}$$

For the second part of the theorem, where we only have $p > \frac{2m+3}{m+4}$, our claim is

$$\begin{aligned} & \sum_{k \in \mathcal{N}, \tilde{k} \in \tilde{\mathcal{N}}} (\varkappa_1 \rho_1^2 2^{-k_1 - \tilde{k}_1} \wedge \varkappa_2 \rho_2^2 2^{-k_2 - \tilde{k}_2})^{3 - \frac{5}{p}} [1 + k_1 + \tilde{k}_1 + k_2 + \tilde{k}_2]^{\gamma_\alpha} \\ & \quad \cdot \left(\varkappa_1 \rho_1^2 2^{-k_2(m_2-2)} \vee \varkappa_2 \rho_2^2 2^{-k_1(m_1-2)} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\varkappa_1 \rho_1^2 2^{-\tilde{k}_2(m_2-2)} \vee \varkappa_2 \rho_2^2 2^{-\tilde{k}_1(m_1-2)} \right)^{\frac{1}{2} - \frac{1}{p}} \\ & \leq (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{3 - \frac{5}{p}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{1 - \frac{2}{p}} \left(\frac{M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{4}{p} - \frac{5}{2}} \log^{1+\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right), \end{aligned} \quad (3.4.16)$$

and leads us to

$$\begin{aligned} & \|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R))} \\ & \lesssim C_\alpha R^\alpha [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p} - \frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2} - \frac{5}{2p}} \left(\frac{M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{4}{p} - \frac{5}{2}} \\ & \quad \cdot \log^{\gamma'_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \end{aligned}$$

It remains to deal with the dyadic sums (3.4.15) and (3.4.16). Let $\mu = \frac{1}{p} - \frac{1}{2} \geq 0$, $\nu = 3 - \frac{5}{p} > 0$ and $c_i = m_i - 2$. Notice that

$$\begin{aligned} & c_i \mu < \nu \\ \Leftrightarrow & (m_i - 2) \left(\frac{1}{p} - \frac{1}{2} \right) < 3 - \frac{5}{p} \\ \Leftrightarrow & \frac{1}{p} (m_i + 3) < \frac{m_i}{2} + 2 = \frac{m_i + 4}{2} \\ \Leftrightarrow & p > 2 \frac{m_i + 3}{m_i + 4}. \end{aligned}$$

We will outsource the proof of (3.4.15) and (3.4.16) into the next lemma. Take into account that due to (3.4.5), we have to sum over at most two of the parameters $k_1, k_2, \tilde{k}_1, \tilde{k}_2$. There are basically four possibilities: if exactly two of the parameters are nonzero, there are two different cases: either these parameters belong to the same surface (i.e. $k_1, k_2 = 0$ or $\tilde{k}_1, \tilde{k}_2 = 0$), which correspond to the left picture in Figure 3.6, or the nonzero parameters belong two different surfaces, like in the "over cross" situation shown in the picture on the right hand side of Figure 3.6. The remaining two possibilities are firstly only one parameter $k_1, k_2, \tilde{k}_1, \tilde{k}_2$ nonzero, which happens if only one of the rectangles U, \tilde{U} touches only one of the axes, and secondly both rectangles sitting away from the axes, where we do not have to sum at all.

Only the first two of the afore mentioned possibilities are dealt with in the next lemma, but the corresponding sums of course dominate the sums over fewer parameters (or even none) as in the other two possibilities. \square

LEMMA 3.4.2

Let $\mu \geq 0, \nu > 0, n, c_1, c_2 \geq 0$ such that $c_1\mu, c_2\mu < \nu$, and let $a, b \in \mathbb{R}_+$. Then

$$\begin{aligned} (i) \quad & \sum_{k_1, k_2 \in \mathbb{N}} (k_1 + k_2)^n (a2^{-k_2 c_2} \vee b)^{-\mu} (a \vee b2^{-k_1 c_1})^{-\mu} (a2^{-k_1} \wedge b2^{-k_2})^\nu \\ & \leq \sum_{k_1, k_2 \in \mathbb{N}} (k_1 + k_2)^n (a \vee b)^{-\mu} (a2^{-k_2 c_2} \vee b2^{-k_1 c_1})^{-\mu} (a2^{-k_1} \wedge b2^{-k_2})^\nu \\ & \lesssim (a \vee b)^{-2\mu} (a \wedge b)^\nu \log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right). \end{aligned}$$

If only $(c_1 \wedge c_2)\mu < \nu$, then

$$\begin{aligned} (ii) \quad & \sum_{k_1, k_2 \in \mathbb{N}} (k_1 + k_2)^n (a2^{-k_2 c_2} \vee b)^{-\mu} (a \vee b2^{-k_1 c_1})^{-\mu} (a2^{-k_1} \wedge b2^{-k_2})^\nu \\ & \leq \sum_{k_1, k_2 \in \mathbb{N}} (k_1 + k_2)^n (a \vee b)^{-\mu} (a2^{-k_2 c_2} \vee b2^{-k_1 c_1})^{-\mu} (a2^{-k_1} \wedge b2^{-k_2})^\nu \\ & \lesssim (a \vee b)^{-2\mu} (a \wedge b)^\nu \left(\frac{a \vee b}{M(a, b)} \right)^{\mu+\nu} \log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right). \end{aligned}$$

In this case, the constant indicated in " \lesssim " only depends on the number n .

Proof: To prove the first inequality in both (i) and (ii), observe that $a2^{-k_2 c_2} \vee b2^{-k_1 c_1}$ is bounded by $a2^{-k_2 c_2} \vee b$, also by $a \vee b2^{-k_1 c_1}$ and hence by their minimum. Thus we have

$$\begin{aligned} (a2^{-k_2 c_2} \vee b) \wedge (a \vee b2^{-k_1 c_1}) & \geq a2^{-k_2 c_2} \vee b2^{-k_1 c_1} \\ (a2^{-k_2 c_2} \vee b) \vee (a \vee b2^{-k_1 c_1}) & = a \vee b, \end{aligned}$$

i.e.

$$(a2^{-k_2 c_2} \vee b)(a \vee b2^{-k_1 c_1}) \geq (a \vee b)(a2^{-k_2 c_2} \vee b2^{-k_1 c_1}).$$

Notice that in the right hand side in (i) and (ii) we may rewrite

$$(a \vee b)^{-\mu} (a \wedge b)^\nu = (ab)^\nu (a \vee b)^{-\mu-\nu} \quad (3.4.17)$$

and

$$(a \vee b)^{-\mu} (a \wedge b)^\nu \left(\frac{a \vee b}{M(a, b)} \right)^{\mu+\nu} = (ab)^\nu M(a, b)^{-\mu-\nu}. \quad (3.4.18)$$

Using the symmetry in the formula, it suffices to estimate

$$S = \sum_{\substack{k_1, k_2 \in \mathbb{N} \\ a2^{-k_1} \leq b2^{-k_2}}} k_1^n k_2^n (a2^{-k_2 c_2} \vee b2^{-k_1 c_1})^{-\mu} a^\nu 2^{-k_1 \nu}.$$

If we have $c_2 \mu < \nu$, then

$$\begin{aligned} S &\leq a^{\nu-\mu} \sum_{k_2} k_2^n 2^{k_2 c_2 \mu} \sum_{k_1: a2^{-k_1} \leq b2^{-k_2}} k_1^n 2^{-k_1 \nu} \\ &\approx a^{\nu-\mu} \log^n \left(\frac{a}{b} + \frac{b}{a} \right) \left(\frac{b}{a} \right)^\nu \sum_{k_2} k_2^{2n} 2^{k_2 (c_2 \mu - \nu)} \\ &\approx a^{-\mu} b^\nu \log^n \left(\frac{a}{b} + \frac{b}{a} \right). \end{aligned}$$

If $c_1 \mu < \nu$ holds true, we have

$$\begin{aligned} S &\leq a^\nu b^{-\mu} \sum_{k_1} k_1^n 2^{k_1 (c_1 \mu - \nu)} \sum_{k_2: a2^{-k_1} \leq b2^{-k_2}} k_2^n \\ &\leq a^\nu b^{-\mu} \log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right) \sum_{k_1} k_1^{2n+1} 2^{k_1 (c_1 \mu - \nu)} \\ &\lesssim a^\nu b^{-\mu} \log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right). \end{aligned}$$

To see (i), assume $c_1 \mu < \nu$ and $c_2 \mu < \nu$. We may then apply both of the previous approaches and obtain

$$\begin{aligned} \frac{S}{\log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right)} &\lesssim a^{-\mu} b^\nu \wedge a^\nu b^{-\mu} \\ &= a^\nu b^\nu (a^{-\mu-\nu} \wedge b^{-\mu-\nu}) \\ &= (ab)^\nu (a \vee b)^{-\mu-\nu} \\ &= (a \vee b)^{-\mu} (a \wedge b)^\nu. \end{aligned}$$

For Part (ii), where only $(c_1 \wedge c_2) \mu < \nu$, consider three cases:

If we have $c_1 > c_2$, i.e. $c_2 \mu < \nu$, then $M(a, b) = a$ and

$$S \lesssim a^{-\mu} b^\nu \log^n \left(\frac{a}{b} + \frac{b}{a} \right) = (ab)^\nu (M(a, b))^{-\mu-\nu} \log^n \left(\frac{a}{b} + \frac{b}{a} \right)$$

If $c_1 < c_2$ holds true, i.e. $c_1 \mu < \nu$, then $M(a, b) = b$ and

$$\begin{aligned} S &\lesssim a^\nu b^{-\mu} \log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right) \\ &= (ab)^\nu (M(a, b))^{-\mu-\nu} \log^{n+1} \left(\frac{a}{b} + \frac{b}{a} \right). \end{aligned}$$

The subcase $c_1 = c_2$ is already included in Part (i), since then both $c_1 \mu, c_2 \mu < \nu$ and $\frac{a \vee b}{M(a, b)} = 1$. \square

3.5 Globalisation

Recall (3.4.2): the surfaces we are interested in are $S = S_1 = \Gamma(\phi|_U)$ and $\tilde{S} = S_2 = \Gamma(\phi|_{\tilde{U}})$, with $U = U_1 = r^{(1)} + [0, \rho_1] \times [0, \rho_2]$ and $\tilde{U} = U_2 = r^{(2)} + [0, \rho_1] \times [0, \rho_2]$ two dyadic rectangles of side length and distance in x_1 -direction (x_2 -direction) equal to ρ_1 (ρ_2 respectively). Let \varkappa_i denote the maximal principal curvature in x_i -direction of both S_1, S_2 , and let σ_j , $j = 1, 2$ be the associated Lebesgue measures of S_1, S_2 respectively.

Our bilinear estimate is still a local result restricted to cuboids

$$Q_{S_1, S_2}(R; r) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r) x_3| \leq \frac{R}{\rho_i}, i = 1, 2, |x_3| \leq \frac{R}{\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2} \right\}, \quad (3.5.1)$$

for any $r \in U_1 \cup U_2$. The r we will choose are $r^{(1)}, r^{(2)}$, and since S_1 and S_2 are fixed during this chapter, we define

$$Q_j(R) = Q_{S_1, S_2}(R; r^{(j)}). \quad (3.5.2)$$

3.5.1 General globalisation results

The next task will be to extend the inequalities from cuboids to the whole space and to get rid of the factor R^α . There is a certain amount of so-called globalisation or ε -removal technique available for this purpose, for instance a lemma from Tao and Vargas [TV1], who themselves followed ideas from Bourgain [Bo2]. By isolating the essential ingredients of the proof and taking them as assumptions, we will give a slightly reformulated version, although the proof basically remains the same.

LEMMA 3.5.1

Let $C, \alpha, s > 0$, $1 \leq p_0 < p < p_1 \leq \infty$, and let S_1, S_2 be hypersurfaces like introduced above, with associated cuboids $Q_j(R)$, $R \geq 1$. Let ν_1, ν_2, μ be measures supported on S_1, S_2, \mathbb{R}^3 respectively, with smooth densities with respect to the respective Lebesgue measures. Assume that for all $R \geq 1$ and all $f_j \in L^2(S_j, \nu_j)$, $j = 1, 2$

$$(i) \quad \|\widehat{f_1 d\nu_1} \widehat{f_2 d\nu_2}\|_{L^{p_0}(Q_j(R), \mu)} \leq CR^\alpha \|f_1\|_{L^2(S_1, \nu_1)} \|f_2\|_{L^2(S_2, \nu_2)},$$

$$(ii) \quad |\widehat{d\nu_j}(x)| \leq CR^{-s} \text{ for all } x \notin Q_j(R),$$

$$(iii) \quad \|\widehat{f_j d\nu_j}\|_{L^{p_1}(\mathbb{R}^n, \mu)} \leq C \|f_j\|_{L^2(S_j, \nu_j)}.$$

Assume $\frac{1}{p} + \frac{2\alpha}{sp} \leq \frac{1}{p_0} + \frac{2\alpha}{sp_1}$. Then

$$\|\widehat{f_1 d\nu_1} \widehat{f_2 d\nu_2}\|_{L^p(\mathbb{R}^n, \mu)} \leq C' \|f_1\|_{L^2(S_1, \nu_1)} \|f_2\|_{L^2(S_2, \nu_2)} \text{ for all } f_j \in L^2(S_j, \nu_j), j = 1, 2, \quad (3.5.3)$$

where C' only depends on $C, \alpha, s, p, p_0, p_1$.

REMARK

(i) In [TV1], the authors assume that the Fourier transform of the measures ν_j has some decay, i.e. $|\widehat{d\nu_j}(x)| \lesssim |x|^{-s}$. Therefrom they deduce (ii) and (iii), with $p_1 = \frac{2s+2}{s}$.

(ii) An important step in the proof is the splitting of the surface measure into $\sigma_1 = \sigma_1^R + \sigma_{1,R}$ where $\widehat{d\sigma_{1,R}}(x) = \phi_R(x)\widehat{d\sigma_1}(x)$ with a bump function ϕ_R adapted to a cube of side length R . The function ϕ_R has to be modified to a function adapted to our cuboid $Q_1(R)$. The argument is repeated for σ_2 , where we take the second cuboid to be $Q_2(R)$.

Although we are provided with this general result, we need to modify it since we seek to keep track of the sharp constants. The following modification takes this into account.

COROLLARY 3.5.2 (GLOBALISATION LEMMA)

Let $C, \alpha, s > 0$, $1 \leq p_0 < p < p_1 < \infty$, and let S_1, S_2 be hypersurfaces like introduced above, with associated cuboids $Q_j(R)$, $R \geq 1$. Let σ_1, σ_2 be the Lebesgue measures of S_1, S_2 . We take the Lebesgue measure on \mathbb{R}^n . Assume that there exist constants $A, B > 0, R_0 > 1$, $A_r = A \cdot B^{\frac{1}{r}}$ for all $r \in \{p, p_0, \frac{p_1}{2}\}$ such that for all $R > R_0$ and all $f_j \in L^2(S_j, \sigma_j)$, $j = 1, 2$,

$$(i) \quad \|\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}\|_{L^{p_0}(Q_j(R))} \leq C A_{p_0} R^\alpha \|f_1\|_{L^2(S_1, \sigma_1)} \|f_2\|_{L^2(S_2, \sigma_2)},$$

$$(ii) \quad |\widehat{d\sigma_j}(x)| \leq C A_\infty R^{-s} \text{ for all } x \notin Q_j(R),$$

$$(iii) \quad \|\widehat{f_j d\sigma_j}\|_{L^{p_1}(\mathbb{R}^n)} \leq C A_{\frac{p_1}{2}}^{1/2} \|f_j\|_{L^2(S_j, \sigma_j)}.$$

Assume $\frac{1}{p} + \frac{2\alpha}{sp} \leq \frac{1}{p_0} + \frac{2\alpha}{sp_1}$. Then

$$\|\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}\|_{L^p(\mathbb{R}^n)} \leq C' A_p \|f_1\|_{L^2(S_1, \sigma_1)} \|f_2\|_{L^2(S_2, \sigma_2)} \text{ for all } f_j \in L^2(S_j, \sigma_j), \quad j = 1, 2, \quad (3.5.4)$$

where C' depends only on $C, \alpha, s, p, p_0, p_1$.

REMARK

We call a constant $A_r = A \cdot B^{\frac{1}{r}}$ log-affine, since $\log A_r = \log A + \frac{1}{r} \log B$ is an affine function in $\frac{1}{r}$. This log-affine shape of the constant is vital for the proof.

Proof: The idea is to absorb the constants into the measures: Define $\nu_i = \frac{1}{A} \sigma_i$, $i = 1, 2$ and $\mu = \frac{1}{B} \lambda$, where λ denotes the Lebesgue measure on \mathbb{R}^n . Then

$$\begin{aligned} \|\widehat{f_1 d\nu_1} \widehat{f_2 d\nu_2}\|_{L^{p_0}(Q(R), \mu)} &= A^{-2} B^{-\frac{1}{p_0}} \|\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}\|_{L^{p_0}(Q(R), \lambda)} \\ &\leq C A^{-2} B^{-\frac{1}{p_0}} A B^{\frac{1}{p_0}} R^\alpha \|f_1\|_{L^2(S_1, \sigma_1)} \|f_2\|_{L^2(S_2, \sigma_2)} \\ &= C A^{-1} R^\alpha \|f_1\|_{L^2(S_1, \sigma_1)} \|f_2\|_{L^2(S_2, \sigma_2)} \\ &= C R^\alpha \|f_1\|_{L^2(S_1, \nu_1)} \|f_2\|_{L^2(S_2, \nu_2)}. \end{aligned}$$

Similarly, we see that the assumptions of Corollary 3.5.2 imply the assumptions in Lemma 3.5.1, and finally that (3.5.3) implies (3.5.4). \square

3.5.2 On the conditions for the Globalisation Lemma

That the two cuboids $Q_j(R)$, $j = 1, 2$ are naturally adapted to the two subsurfaces S_j , $j = 1, 2$ can be seen in the next lemma:

LEMMA 3.5.3

Let $j \in \{1, 2\}$. Then for some $s > 0$ and any $x \in \mathbb{R}^3$ we have

$$|\widehat{d\sigma}_j(x)| \leq C_s \rho_1 \rho_2 (1 + |(\rho_1(x_1 + \partial_1 \phi(r^{(j)} x_3), \rho_2(x_2 + \partial_2 \phi(r^{(j)} x_3), (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2) x_3))|)^{-s}. \quad (3.5.5)$$

If moreover $x \notin Q_j(R)$, then

$$|\widehat{d\sigma}_j(x)| \leq C_s \rho_1 \rho_2 R^{-2s}. \quad (3.5.6)$$

Proof: Recall that ϕ splits into $\phi(x) = \phi_1(x_1) + \phi_2(x_2)$. Fix $j \in \{1, 2\}$ and let $r = r^{(j)}$. Then

$$\begin{aligned} |\widehat{d\sigma}_j(x)| &= \left| \int_{r_1}^{r_1+\rho_1} \int_{r_2}^{r_2+\rho_2} e^{-i(x_1 \xi_1 + x_2 \xi_2 + x_3(\phi_1(\xi_1) + \phi_2(\xi_2)))} d\xi \right| \\ &= \left| \int_{r_1}^{r_1+\rho_1} e^{-i(x_1 \xi_1 + x_3 \phi_1(\xi_1))} d\xi_1 \int_{r_2}^{r_2+\rho_2} e^{-i(x_2 \xi_2 + x_3 \phi_2(\xi_2))} d\xi_2 \right|. \end{aligned}$$

Since both integrals have the same appearance, we use the same estimate. Let $i \in \{1, 2\}$.

Case 1: $r_i > 0$

Since $[r_i, r_i + \rho_i]$ is a dyadic interval, $r_i > 0$ means $r_i \geq \rho_i$, thus $\varkappa_i \approx \phi_i''(r_i) \approx r_i^{m_i-2}$. We obtain

$$\begin{aligned} \left| \int_{r_i}^{r_i+\rho_i} e^{-i(x_i \xi_i + x_3 \phi_i(\xi_i))} d\xi_i \right| &= \left| \int_0^{\rho_i} e^{-i(x_i(r_i+y_i) + x_3 \phi_i(r_i+y_i))} dy_i \right| \\ &= \left| \int_0^{\rho_i} e^{-i((x_i + \phi_i'(r_i) x_3) y_i + x_3(\phi_i(r_i+y_i) - \phi_i(r_i) - \phi_i'(r_i) y_i))} dy_i \right| \\ &= \left| \int_0^{\rho_i} e^{-i((x_i + \phi_i'(r_i) x_3) y_i + x_3 \Psi(y_i))} dy_i \right| \\ &= \rho_i \left| \int_0^1 e^{-i((x_i + \phi_i'(r_i) x_3) \rho_i y_i + x_3 \Psi(\rho_i y_i))} dy_i \right| \\ &\lesssim \rho_i (1 + |(\rho_i(x_i + \phi_i'(r_i) x_3), \varkappa_i \rho_i^2 x_3)|)^{-s}, \end{aligned}$$

where $\Psi(y_i) = \phi_i(r_i + y_i) - \phi_i(r_i) - \phi_i'(r_i) y_i$, such that

$$\frac{d^2}{dy_i^2} \Psi(\rho_i y_i) = \phi_i''(r_i + \rho_i y_i) \rho_i^2 \approx \varkappa_i \rho_i^2.$$

In this case we may choose $s = \frac{1}{2}$.

Case 2: $r_i = 0$

Here,

$$\left| \int_0^{\rho_i} e^{-i(x_i \xi_i + x_3 \phi_i(\xi_i))} d\xi_i \right| = \rho_i \left| \int_0^1 e^{-i(x_i \rho_i y_i + x_3 \phi_i(\rho_i y_i))} dy_i \right| \lesssim \rho_i (1 + |(\rho_i x_i, \rho_i^{m_i} x_3)|)^{-s},$$

with $s = \frac{1}{m_i}$. However, $\phi'_i(r_i) = \phi'_i(0) = 0$ and $\varkappa_i \approx \rho_i^{m_i-2}$ in this case, i.e. $\rho_i^{m_i} \approx \varkappa_i \rho_i^2$. The desired statement follows from the observation that

$$\begin{aligned} & (1 + |(\rho_1(x_1 + \phi'_1(r_1)x_3), \varkappa_1 \rho_1^2 x_3)|)(1 + |(\rho_2(x_2 + \phi'_2(r_2)x_3), \varkappa_2 \rho_2^2 x_3)|) \\ & \gtrsim 1 + |(\rho_1(x_1 + \partial_1 \phi(r)x_3), (\rho_2(x_2 + \partial_2 \phi(r)x_3), \varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)x_3)|. \end{aligned}$$

The second part of the lemma is clear due to the definition of the cuboids (3.5.1). \square

The following lemma is a variation of Greenleaf's theorem that some decay of the Fourier transform of a surface measure implies a Fourier restriction estimate [Gre]. The first version of such a theorem was by Stein and Tomas for the sphere. Our proof will closely follow the proof of the first approach in Theorem 3, Chapter VIII, §4 from [St], although this does not give the optimal range. But since we only seek to obtain some result for the Globalisation Lemma, we do not need the optimal result (which would require a much more complicated proof).

LEMMA 3.5.4

Let S be some hypersurface in \mathbb{R}^n with associated surface measure μ . Assume that

$$|\widehat{d\mu}(x)| \leq C_0(1 + |T(x)|)^{-s} \quad (3.5.7)$$

for some $C_0, s > 0$ and some $T \in GL(\mathbb{R}^n)$. Then there exists a $p_1 \geq 4$ such that

$$\|f \widehat{d\mu}\|_{p_1} \leq C_{p_1} \sqrt{C_0} |\det T|^{-\frac{1}{p_1}} \|f\|_{L^2(S)} \quad (3.5.8)$$

holds true for all $f \in L^2(S)$.

One might be tempted to simply apply a well-known result, like Greenleaf's, to some appropriately transformed measure, i.e. scale to the situation where T is the identity matrix. However, like before, we have to control how the constant depends on the choice of surface, so we will go through the argument.

Proof: Let R be the restriction operator associated to S and $\frac{1}{p_1} + \frac{1}{p'_1} = 1$. It suffices to show for $g \in \mathcal{S}(\mathbb{R}^n)$

$$\|R(g)\|_{L^2(S)} \lesssim \sqrt{C_0} |\det T|^{-\frac{1}{p_1}} \|g\|_{p'_1},$$

or, equivalently

$$\|R^* R(g)\|_{p_1} \lesssim C_0 |\det T|^{-\frac{2}{p_1}} \|g\|_{p'_1}.$$

Notice that $R^* R(g) = g * \widehat{d\mu}$ and

$$\begin{aligned} |(g * \widehat{d\mu})(x)| & \leq \int |g(x-y)| |\widehat{d\mu}(y)| dy \\ & \leq C_0 \int |g(x-y)| |T(y)|^{-s} dy \\ & = C_0 |\det T|^{-1} \int |g(T^{-1}(Tx-z))| |z|^{-s} dz \end{aligned}$$

i.e.

$$\|R^*Rg\|_{p_1} = \|g * \widehat{d\mu}\|_{p_1} \leq C_0 |\det T|^{-1-\frac{1}{p_1}} \| |g \circ T^{-1}| * |\cdot|^{-s} \|_{p_1}.$$

Notice that if (3.5.7) is valid for some $s > 0$, it is also valid for all $s' \in (0, s)$. Therefore we may assume $s < n$ without loss of generality. According to the classical Hardy-Littlewood-Sobolev Inequality (see [St]), for $1 < p_1, q < \infty$, $0 < s < n$ such that $\frac{1}{p_1} = \frac{1}{q} + \frac{s}{n} - 1$ we have

$$\|h * |\cdot|^{-s}\|_{p_1} \leq C_{p_1, q} \|h\|_q.$$

Since we are interested in $q = p'_1$, we have $\frac{2}{p_1} = \frac{s}{n}$, i.e. $p_1 = \frac{2n}{s}$. Because we may even assume $s \leq \frac{n}{2}$ without loss of generality, we can ensure $p_1 = \frac{2n}{s} \geq 4$. We obtain

$$\|R^*Rg\|_{p_1} \lesssim C_0 |\det T|^{-1-\frac{1}{p_1}} \|g \circ T^{-1}\|_{p'_1} = C_0 |\det T|^{-1-\frac{1}{p_1}+\frac{1}{p'_1}} \|g\|_{p'_1} = C_0 |\det T|^{-\frac{2}{p_1}} \|g\|_{p'_1}.$$

□

Combining this result with Lemma 3.5.3, we immediately obtain

COROLLARY 3.5.5

For some value $4 \leq p_1 < \infty$

$$\|\widehat{fd\sigma_j}\|_{p_1} \lesssim \sqrt{\rho_1\rho_2} [\rho_1\rho_2(\varkappa_1\rho_1^2 \vee \varkappa_2\rho_2^2)]^{-\frac{1}{p_1}} \|f\|_{L^2(S_j, \sigma_j)}. \quad (3.5.9)$$

3.5.3 Application

Let us recall that the local estimate we obtained in Theorem 3.4.1 for $p_c = 2\frac{\bar{m}+3}{\bar{m}+4} < p \leq 2$ was

$$\|R_{S_1, S_2}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_j(R))} \leq C_\alpha R^\alpha [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p}-\frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} \quad (3.5.10)$$

$$\begin{aligned} & \cdot \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ & = C_\alpha R^\alpha A_p, \end{aligned} \quad (3.5.11)$$

if we define for $p \leq 2$

$$\begin{aligned} A_p &= [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p}-\frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ &= (\varkappa_1 \varkappa_2)^{\frac{1}{2p}-\frac{1}{2}} [(\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)]^{\frac{1}{2}-\frac{1}{2p}} \frac{(\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{1}{p}-1}}{(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{2}{p}-1}} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ &= (\rho_1 \rho_2)^{1-\frac{1}{p}} \frac{(\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{1}{p}-1}}{(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{2}{p}-1}} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ &= A \cdot B^{\frac{1}{p}}, \end{aligned}$$

where

$$A = \rho_1 \rho_2 \frac{\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \quad (3.5.12)$$

$$B = (\rho_1 \rho_2)^{-1} \frac{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2}{(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^2}. \quad (3.5.13)$$

However this does not cover the region $p \geq 2$. In this area, we interpolate our L^2 -estimate

$$\begin{aligned} \|R_{S_1, S_2}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_j(R))} &\leq C_\alpha R^\alpha A_2 \\ &= C_\alpha R^\alpha (\rho_1 \rho_2)^{\frac{1}{2}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{-\frac{1}{2}} \log^{\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \end{aligned}$$

with the trivial L^∞ -result

$$\begin{aligned} \|R_{S_1, S_2}^*(f, g)\|_{L^\infty(Q_j(R))} &\leq \|\widehat{f d\sigma}\|_\infty \|\widehat{g d\sigma}\|_\infty \\ &\leq \|f\|_{L^1(S_1)} \|g\|_{L^1(S_2)} \\ &\leq \rho_1 \rho_2 \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}. \end{aligned}$$

Let

$$A_\infty = \rho_1 \rho_2, \quad (3.5.14)$$

then we have also for $p \geq 2$

$$\|R_{S_1, S_2}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_j(R))} \leq C_\alpha R^\alpha A_p, \quad (3.5.15)$$

if we define for $p \geq 2$

$$A_p = A_2^{\frac{2}{p}} A_\infty^{1-\frac{2}{p}} = A_2^{\frac{2}{p}} B^{\frac{1}{p}} A_\infty^{1-\frac{2}{p}} = A_\infty \left(B \frac{A_2^2}{A_\infty^2} \right)^{\frac{1}{p}},$$

i.e.

$$A_p = A' B'^{\frac{1}{p}} \quad (p \geq 2) \quad (3.5.16)$$

with

$$A' = A_\infty = \rho_1 \rho_2 \quad (3.5.17)$$

$$B' = B \frac{A_2^2}{A_\infty^2} = [\rho_1 \rho_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{-1} \log^{2\gamma_\alpha} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \quad (3.5.18)$$

We see that for $p, p_0 \geq 2$, $p_1 \geq 4$ the assumptions (i), (ii) and (iii) of Corollary 3.5.2 are fulfilled by Lemma 3.5.3 and Corollary 3.5.5 since

$$\begin{aligned} A_{p_1/2}^{1/2} &= A^{1/2} B'^{\frac{1}{p_1}} \\ &= \sqrt{\rho_1 \rho_2} [\rho_1 \rho_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{-\frac{1}{p_1}} \log^{\frac{2\gamma_\alpha}{p_1}} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ &\gtrsim \sqrt{\rho_1 \rho_2} [\rho_1 \rho_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{-\frac{1}{p_1}}. \end{aligned}$$

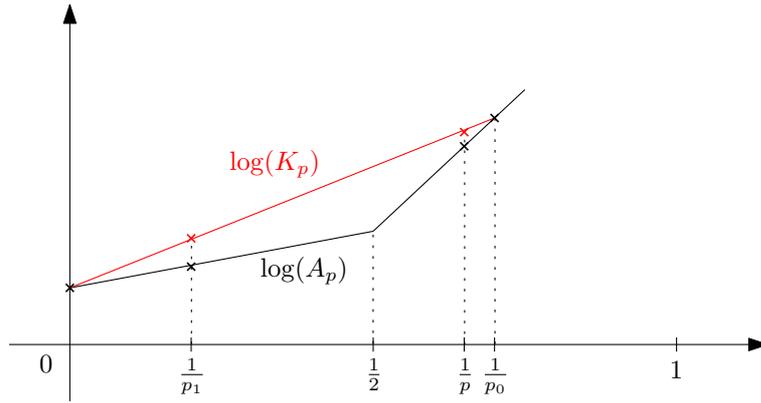


Figure 3.7: The schematic relation of the constants

Unfortunately this does not extend below $p = 2$ since $\log A_p$ is not affine in $\frac{1}{p}$ on the whole interval $[0, 1]$ as we wish, but just piecewise affine (cf. Figure 3.7).

The solution to this problem is to interpolate the local L^{p_0} -result with the trivial L^∞ -result in order to get the log-affine structure in the constant. This will slightly increase the constant for the global bilinear L^p -result. However, since we can choose p and p_0 arbitrarily close together, the impact is negligible, as we shall see. In particular, we will obtain the same linear result as we would have obtained by the conjectured bilinear result, i.e. with the constant A_p from the local bilinear result.

Introduce $\bar{m} = m_1 \vee m_2$ and $p_c = \frac{2\bar{m}+3}{\bar{m}+4}$. The result we obtain is the following. Observe that it includes both cases $p \leq 2$ and $p > 2$:

THEOREM 3.5.6

Let $p_c < p_0 < 2$, $p > p_0$. Then there exists constants $C, \gamma > 0$ such that

$$\begin{aligned} \|R_{S_1, S_2}^*\|_{L^2 \times L^2 \rightarrow L^p(\mathbb{R}^3)} &\leq C \left(\frac{A'}{A}\right)^{1-\frac{p_0}{p}} AB^{\frac{1}{p}} \\ &= C(\rho_1 \rho_2)^{1-\frac{1}{p}} \frac{(\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{1-p_0}{p}}}{(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{2-p_0}{p}}} \log^{\gamma \frac{p_0}{p}} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \end{aligned} \quad (3.5.19)$$

Proof: Define

$$K_p = A_{p_0}^t A_\infty^{1-t}, \quad \frac{1}{p} = \frac{t}{p_0}. \quad (3.5.20)$$

According to Lemma 3.5.3, there is an $s > 0$ such that $|\widehat{d\sigma_j}(x)| \lesssim \rho_1 \rho_2 R^{-s}$ for all $x \notin Q_j(R)$. Chose p_1 according to Corollary 3.5.5, then fix an $\alpha > 0$ small enough such that $\frac{1}{p} + \frac{2\alpha}{sp} \leq \frac{1}{p_0} + \frac{2\alpha}{sp_1}$.

We have to check the assumptions in our Globalisation Lemma, Corollary 3.5.2, with the constants K_p instead of A_p . The first one, (i), is clear since $K_{p_0} = A_{p_0}$. Additionally

$K_\infty = A_\infty = \rho_1 \rho_2$, thus (ii) follows from Lemma 3.5.3.

Condition (iii) requires slightly more work. Corollary 3.5.5 tells us

$$\begin{aligned} \|R_{S_j}^*\|_{L^2 \rightarrow L^{p_1}} &\lesssim (\rho_1 \rho_2)^{\frac{1}{2}} [\rho_1 \rho_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{-\frac{1}{p_1}} \\ &\leq A'^{\frac{1}{2}} B'^{\frac{1}{p_1}} \\ &= A_{p_1/2}^{1/2} \lesssim K_{p_1/2}^{1/2}. \end{aligned}$$

To show the last inequality, observe that

$$\left(\frac{B'}{B}\right)^{\frac{1}{p_0}} \stackrel{(3.5.18)}{=} \left(\frac{A}{A'}\right)^{\frac{2}{p_0}} \lesssim \frac{A}{A'}$$

since $p_0 < 2$, and $A \lesssim A'$. Thus

$$A' B'^{\frac{1}{p_0}} \lesssim A B^{\frac{1}{p_0}} = A_{p_0}. \quad (3.5.21)$$

We write $\frac{2}{p_1} = \frac{t}{p_0}$ and conclude

$$A_{p_1/2} = A' B'^{\frac{2}{p_1}} = A'^{1-t} (A' B'^{\frac{1}{p_0}})^t \lesssim A_\infty^{1-t} A_{p_0}^t = K_{p_1/2}.$$

Applying Corollary 3.5.2 gives ($t = \frac{p_0}{p}$ and $\gamma = \gamma_\alpha$)

$$\begin{aligned} \|R_{S_1, S_2}^*\|_{L^2 \times L^2 \rightarrow L^p} &\lesssim K_p = A_\infty^{1-t} A_{p_0}^t \\ &= A'^{1-\frac{p_0}{p}} A^{\frac{p_0}{p}} B^{\frac{1}{p}} \\ &= \left(\frac{A'}{A}\right)^{1-\frac{p_0}{p}} A B^{\frac{1}{p}} \\ &= \left[\frac{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2} \log^{-\gamma} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \right]^{\frac{p-p_0}{p}} \\ &\quad \cdot (\rho_1 \rho_2)^{1-\frac{1}{p}} \frac{(\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{1}{p}-1}}{(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{2}{p}-1}} \log^\gamma \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right) \\ &= (\rho_1 \rho_2)^{1-\frac{1}{p}} \frac{(\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{1-p_0}{p}}}{(\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{2-p_0}{p}}} \log^{\gamma \frac{p_0}{p}} \left(\frac{\varkappa_1 \rho_1^2}{\varkappa_2 \rho_2^2} + \frac{\varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2} \right). \end{aligned}$$

□

3.6 Passage from the bilinear to the linear estimate

Recall $\bar{m} = m_1 \vee m_2$ and $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$. Recall further our main Theorem 3.1.1 which we seek to prove:

THEOREM

Let $p > p_c = 2\frac{\bar{m}+3}{\bar{m}+4}$ and $\frac{1}{q} \geq \frac{h+1}{2p}$. Additionally we assume $h+1 < 2p$.

Then R^* is bounded from $L^{q,s}(\Gamma)$ to $L^{2p,s}(\mathbb{R}^3)$ for any $1 \leq s \leq \infty$.

If moreover $q \leq 2p$ or $\frac{1}{q} > \frac{h+1}{2p}$, then R^* is bounded from $L^q(\Gamma)$ to $L^{2p}(\mathbb{R}^3)$.

Proof: We claim that it suffices to prove the restricted weak-type estimate

$$\|\widehat{\chi_\Omega d\sigma}\|_{2p} \lesssim |\Omega|^{\frac{1}{q}} \quad (3.6.1)$$

for any measurable set $\Omega \subset Q = [0, 1] \times [0, 1]$. The reason for this is that establishing (3.6.1) for all p, q fulfilling our condition allows to apply an interpolation theorem (for instance theorem 1.4.19 in [Gra]) and obtain that R^* is bounded from $L^{q,s}(\Gamma)$ to $L^{2p,s}(\mathbb{R}^3)$ for any $1 \leq s \leq \infty$. Under the additional assumption $q \leq 2p$, we have

$$\|\widehat{fd\sigma}\|_{2p} = \|\widehat{fd\sigma}\|_{2p,2p} \leq \|\widehat{fd\sigma}\|_{2p,q} \lesssim \|f\|_{q,q} = \|f\|_q. \quad (3.6.2)$$

Another way to see this is to choose $s = \infty$, i.e. use the boundedness of R^* from $L^{q,\infty}(\Gamma)$ to $L^{2p,\infty}(\mathbb{R}^3)$ in order to apply Marcinkiewicz interpolation (although we would lose the endpoint $q = 2p$ of the line $\frac{1}{q'} = \frac{h+1}{2p}$). Unfortunately, this requires $q \leq 2p$ as well. With the techniques we use, we seem to be limited to the restricted estimate above the diagonal $q = 2p$.

Using a bidyadic decomposition $Q \times Q = \bigcup_j \bigcup_{k \sim \tilde{k}} \tau_{jk} \times \tau_{j\tilde{k}}$, where

$$\tau_{jk} = [(k_1 - 1)2^{-j_1}, k_1 2^{-j_1}] \times [(k_2 - 1)2^{-j_2}, k_2 2^{-j_2}]$$

and $k \sim \tilde{k}$ means $2 \leq |k_i - \tilde{k}_i| \leq C$, $i = 1, 2$, we obtain

$$\begin{aligned} \|\widehat{\chi_\Omega d\sigma}\|_{2p}^2 &= \|\widehat{\chi_\Omega d\sigma} \widehat{\chi_\Omega d\sigma}\|_p \\ &\leq \sum_j \left[\sum_{k \sim \tilde{k}} \|\mathfrak{F}(\chi_{\Omega \cap \tau_{jk}} d\sigma) \mathfrak{F}(\chi_{\Omega \cap \tau_{j\tilde{k}}} d\sigma)\|_p^{p^*} \right]^{\frac{1}{p^*}}, \end{aligned} \quad (3.6.3)$$

where $p^* = p \wedge p'$, $\frac{1}{p} + \frac{1}{p'}$.

The last step can be obtained for $1 \leq p \leq 2$ by interpolation of the cases $p = 2$, where one may apply Plancherel and Pythagoras, and $p = 1$ or $p = \infty$ which are simply triangle inequality. More details can be found in Lemma 6.3. in [TVV].

Now apply Theorem 3.5.6 on $S_{jk} = \{(\xi, \phi(\xi)) | \xi \in \tau_{jk}\}$ and $S_{j\tilde{k}}$ with $\rho_i = 2^{-j_i}$, $\varkappa_i \approx (k_i 2^{-j_i})^{m_i-2} \approx (\tilde{k}_i 2^{-j_i})^{m_i-2}$ and $\varkappa_i \rho_i^2 \approx k_i^{m_i-2} 2^{-j_i m_i}$.

Without loss of generality, we may assume

$$k \in I := \{k | k_1^{m_1-2} 2^{-j_1 m_1} \geq k_2^{m_2-2} 2^{-j_2 m_2}\}, \quad (3.6.4)$$

i.e. $\varkappa_1 \rho_1^2 \geq \varkappa_2 \rho_2^2$. Thus

$$\begin{aligned} & \|R_{S_{jk}, S_{j\tilde{k}}}\|_{L^2 \times L^2 \rightarrow L^p} \\ & \lesssim (\rho_1 \rho_2)^{1-\frac{1}{p}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{-\frac{1}{p}} \left(\frac{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2} \right)^{\frac{2-p_0}{p}} \log^{\gamma \frac{p_0}{p}} \left(1 + \frac{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2} \right) \\ & = 2^{-(j_1+j_2)\frac{1}{p'}} k_1^{-(m_1-2)\frac{1}{p}} 2^{j_1 m_1 \frac{1}{p}} \left(\frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-(j_1 m_1 - j_2 m_2)} \right)^{\frac{2-p_0}{p}} \log^{\gamma \frac{p_0}{p}} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-(j_1 m_1 - j_2 m_2)} \right) \\ & = A_j \cdot B_{k,j}^{\frac{1}{p}}, \end{aligned}$$

if we define

$$A_j = 2^{-(j_1+j_2)\frac{1}{p'}} 2^{j_1 m_1 \frac{1}{p}} 2^{-(j_1 m_1 - j_2 m_2)\frac{2-p_0}{p}}, \quad (3.6.5)$$

$$B_{j,\tilde{k}} \approx B_{k,j} = k_1^{-(m_1-2)} \left(\frac{k_1^{m_1-2}}{k_2^{m_2-2}} \right)^{2-p_0} \log^{\gamma p_0} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-(j_1 m_1 - j_2 m_2)} \right). \quad (3.6.6)$$

Since $|\{k|k \sim \tilde{k}\}| \lesssim 1$ for fixed k , we conclude

$$\begin{aligned} \|\widehat{\chi_\Omega d\sigma}\|_{2p}^2 & \lesssim \sum_j A_j \left[\sum_{k \sim \tilde{k}} \left(B_{k,j}^{\frac{1}{p}} |\Omega \cap \tau_{jk}|^{\frac{1}{2}} |\Omega \cap \tau_{j\tilde{k}}|^{\frac{1}{2}} \right)^{p^*} \right]^{\frac{1}{p^*}} \\ & \lesssim \sum_j A_j \left[\sum_k B_{k,j}^{\frac{p^*}{p}} |\Omega \cap \tau_{jk}|^{p^*} \right]^{\frac{1}{p^*}}. \end{aligned}$$

Therefore we have to show that

$$\sum_j A_j \left[\sum_k B_{k,j}^{\frac{p^*}{p}} |\Omega \cap \tau_{jk}|^{p^*} \right]^{\frac{1}{p^*}} \lesssim |\Omega|^{\frac{2}{q}}.$$

3.6.1 Reduction

We decompose $p^* = \frac{\alpha}{r^*} + \frac{1}{r^*}$ and introduce $r = \frac{r^* p^*}{p}$. Apply Hölder's inequality on the summation in k with Hölder exponent $r^* \geq 1$ (which we will determine later):

$$\begin{aligned} \left(\sum_{k \in I} B_{k,j}^{\frac{p^*}{p}} |\Omega \cap \tau_{jk}|^{p^*} \right)^{\frac{1}{p^*}} & \leq \left(\sum_{k \in I} B_{k,j}^{\frac{p^* r^*}{p}} |\Omega \cap \tau_{jk}|^\alpha \right)^{\frac{1}{p^* r^*}} \left(\sum_{k \in I} |\Omega \cap \tau_{jk}| \right)^{\frac{1}{p^* r^*}} \\ & \leq \left(\sum_{k \in I} B_{k,j}^r \right)^{\frac{1}{p^* r}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{\frac{\alpha}{p^* r^*}} |\Omega|^{\frac{1}{p^*} (1-\frac{1}{r^*})} \\ & = \left(\sum_{k \in I} B_{k,j}^r \right)^{\frac{1}{p^* r}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{1-\frac{1}{p^*} + \frac{1}{p^* r}} |\Omega|^{\frac{1}{p^*} - \frac{1}{p^* r}}. \end{aligned}$$

Moreover we have $|\Omega| \leq |Q| = 1$ as well as $\frac{1}{q'} \geq \frac{h+1}{2p}$, i.e. $2 - \frac{h+1}{p} \geq \frac{2}{q}$. Therefore $|\Omega|^{2 - \frac{h+1}{p}} \leq |\Omega|^{\frac{2}{q}}$, and it suffices to show

$$\begin{aligned} |\Omega|^{2 - \frac{h+1}{p} - \frac{1}{p^*} + \frac{1}{pr}} &\gtrsim \sum_j A_j \min\{|\Omega|, 2^{-j_1-j_2}\}^{1 - \frac{1}{p^*} + \frac{1}{pr}} \left[\sum_{k \in I} B_{k,j}^r \right]^{\frac{1}{rp}} \\ &= \sum_j 2^{-(j_1 m_1 - j_2 m_2) \frac{2-p_0}{p}} 2^{-(j_1+j_2) \frac{1}{p'}} 2^{j_1 m_1 \frac{1}{p}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{1 - \frac{1}{p^*} + \frac{1}{pr}} \\ &\quad \cdot \left[\sum_{k \in I} k_1^{(m_1-2)(1-p_0)r} k_2^{(m_2-2)(p_0-2)r} \log^{\gamma p_0 r} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-(j_1 m_1 - j_2 m_2)} \right) \right]^{\frac{1}{rp}} \end{aligned}$$

We apply the change of variables $l = j_1 + j_2 \in \mathbb{N}$, $l' = j_1 m_1 - j_2 m_2 \in \mathbb{Z}$, such that $j_1 = \frac{m_2 l + l'}{m_1 + m_2}$. Then the exponent in j_1, j_2 becomes

$$\begin{aligned} &(j_1 m_1 - j_2 m_2) \left(\frac{p_0 - 2}{p} \right) + (j_1 + j_2) \left(\frac{1}{p} - 1 \right) + j_1 m_1 \frac{1}{p} \\ &= l' \left(\frac{p_0 - 2}{p} \right) + l \left(\frac{1}{p} - 1 \right) + \frac{m_1 m_2 l + m_1 l'}{m_1 + m_2} \frac{1}{p} \\ &= \frac{l'}{p} \left(p_0 - \frac{m_1 + 2m_2}{m_1 + m_2} \right) + l \left(\frac{h+1}{p} - 1 \right). \end{aligned}$$

The summation over $k \in I = \{k_1^{m_1-2} \geq k_2^{m_2-2} 2^{l'}\}$ is independent of l , hence we have reduced to showing that⁶

$$\sum_{l'=-\infty}^{\infty} 2^{\frac{l'}{p} (p_0 - \frac{m_1 + 2m_2}{m_1 + m_2})} \left[\sum_{k \in I} k_1^{(m_1-2)(1-p_0)r} k_2^{(m_2-2)(p_0-2)r} \log^{\gamma p_0 r} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-l'} \right) \right]^{\frac{1}{rp}} < \infty \quad (3.6.7)$$

and

$$\sum_{l=0}^{\infty} 2^{l(\frac{h+1}{p} - 1)} \min\{|\Omega|, 2^{-l}\}^{1 - \frac{1}{p^*} + \frac{1}{pr}} \lesssim |\Omega|^{2 - \frac{h+1}{p} - \frac{1}{p^*} + \frac{1}{pr}}. \quad (3.6.8)$$

3.6.2 The case $m > 2$

Since $m = m_1 \wedge m_2$, the condition $m > 2$ means both $m_1, m_2 > 2$.

Summation in k

We compare the sum over k in (3.6.7) with an integral. We claim that

$$\begin{aligned} &\int_{\substack{k_1, k_2 \geq 1 \\ k_1^{m_1-2} \geq k_2^{m_2-2} 2^{l'}}} k_1^{(m_1-2)(1-p_0)r} k_2^{(m_2-2)(p_0-2)r} \log^{\gamma p_0 r} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-l'} \right) dk \\ &\lesssim \begin{cases} 2^{|l'|(\frac{1}{m_1-2} + (1-p_0)r)}, & l' \geq 0 \\ |l'| \gamma p_0 r 2^{|l'|(\frac{1}{m_2-2} + (p_0-2)r)_+}, & l' < 0 \end{cases}, \end{aligned} \quad (3.6.9)$$

⁶Technically, we only have to sum over the smaller set $l' \in m_1 \mathbb{N} - m_2 \mathbb{N}$.

if

$$a := \boxed{\frac{1}{m_1 - 2} + (1 - p_0)r < 0} \quad (3.6.10)$$

$$b := \frac{1}{m_2 - 2} + (p_0 - 2)r \quad (3.6.11)$$

$$a + b = \boxed{\frac{1}{m_1 - 2} + \frac{1}{m_2 - 2} - r < 0}. \quad (3.6.12)$$

For the moment, we will simply assume that these conditions hold true. We shall collect several such conditions on the exponent r , and verify at the end of the chapter that we find an r such that all conditions are satisfied.

After the transformation $s = k_1^{m_1 - 2}$, $t = k_2^{m_2 - 2}$ (i.e. $dk \approx s^{\frac{1}{m_1 - 2} - 1} t^{\frac{1}{m_2 - 2} - 1} d(s, t)$), this simplifies to

$$J(a, b) = \int \int_{\substack{s, t \geq 1 \\ s \geq t^{2^{l'}}}} s^a t^b \log^{\gamma p_0 r} \left(1 + \frac{s}{t} 2^{-l'}\right) \frac{ds}{s} \frac{dt}{t} \lesssim \begin{cases} 2^{|l'|a} & l' \geq 0 \\ |l'|^{\gamma p_0 r} 2^{|l'|b_+} & l' < 0 \end{cases}, \quad (3.6.13)$$

if $a < 0$, $a + b < 0$. Changing $t' = \frac{s}{t} 2^{-l'}$, the set of integration for the t -variable $\{t : t \geq 1, \frac{s}{t} 2^{-l'} \geq 1\}$ transforms into $\{t' : \frac{s}{t'} 2^{-l'} \geq 1, t' \geq 1\}$. Thus

$$\begin{aligned} J(a, b) &= \int \int_{\substack{s, t' \geq 1 \\ s \geq t' 2^{l'}}} s^a \left(\frac{s}{t'} 2^{-l'}\right)^b \log^{\gamma p_0 r} (1 + t') \frac{ds}{s} \frac{dt'}{t'} \\ &= 2^{-l'b} \int_{t'=1}^{\infty} t'^{-b} \log^{\gamma p_0 r} (1 + t') \int_{s=1 \vee t' 2^{l'}}^{\infty} s^{a+b} \frac{ds}{s} \frac{dt'}{t'} \\ &\stackrel{a+b < 0}{=} 2^{-l'b} \int_1^{\infty} (1 \vee t' 2^{l'})^{a+b} t'^{-b} \log^{\gamma p_0 r} (1 + t') \frac{dt'}{t'}. \end{aligned}$$

For $l' \geq 0$, clearly $1 \vee t' 2^{l'} = t' 2^{l'}$, and hence

$$J(a, b) = 2^{l'a} \int_1^{\infty} t'^a \log^{\gamma p_0 r} (1 + t') \frac{dt'}{t'} \stackrel{a < 0}{\approx} 2^{l'a}.$$

For $l' < 0$, we split into

$$\begin{aligned} J(a, b) &= 2^{-l'b} \int_1^{2^{|l'|}} t'^{-b} \log^{\gamma p_0 r} (1 + t') \frac{dt'}{t'} + 2^{l'a} \int_{2^{|l'|}}^{\infty} t'^a \log^{\gamma p_0 r} (1 + t') \frac{dt'}{t'} \\ &\lesssim 2^{-l'b} |l'|^{\gamma p_0 r} \int_1^{2^{|l'|}} t'^{-b-1} dt' + 2^{l'a} \int_{2^{|l'|}}^{\infty} t'^a \log^{\gamma p_0 r} (2^{-l'} + t') \frac{dt'}{t'} \\ &= 2^{-l'b} |l'|^{\gamma p_0 r} \frac{1 - 2^{l'b}}{b} + \int_1^{\infty} u^a \log^{\gamma p_0 r} (2^{-l'} (1 + u)) \frac{du}{u} \\ &\lesssim 2^{-l'b} |l'|^{\gamma p_0 r} (1 + 2^{l'b}) + |l'|^{\gamma p_0 r} \int_1^{\infty} u^{a-1} \log^{\gamma p_0 r} (1 + u) du \\ &\stackrel{a < 0}{\approx} |l'|^{\gamma p_0 r} (2^{-l'b} + 1) \\ &\approx |l'|^{\gamma p_0 r} 2^{|l'|b_+}, \end{aligned}$$

as we claimed.

Summation in l'

To put (3.6.9) in (3.6.7), we split the sum in (3.6.7) into summation over $l' \geq 0$ and summation over $l' < 0$. In the first case $l' \geq 0$, we obtain

$$\begin{aligned}
& \sum_{l' \geq 0} 2^{\frac{l'}{p}(p_0 - \frac{m_1 + 2m_2}{m_1 + m_2})} \left[\sum_{k \in I} k_1^{(m_1 - 2)(1 - p_0)r} k_2^{(m_2 - 2)(p_0 - 2)r} \log^{\gamma p_0 r} \left(1 + \frac{k_1^{m_1 - 2}}{k_2^{m_2 - 2}} 2^{-l'} \right) \right]^{\frac{1}{rp}} \\
& \lesssim \sum_{l' \geq 0} 2^{\frac{l'}{p}(p_0 - 1 - \frac{m_2}{m_1 + m_2})} 2^{l'(\frac{1}{pr} \frac{1}{m_1 - 2} + \frac{1 - p_0}{p})} \\
& = \sum_{l' \geq 0} 2^{\frac{l'}{p}(\frac{1}{r} \frac{1}{m_1 - 2} - \frac{m_2}{m_1 + m_2})}.
\end{aligned} \tag{3.6.14}$$

The sum is finite if

$$\boxed{\frac{1}{r} < \frac{m_2(m_1 - 2)}{m_1 + m_2}}, \tag{3.6.15}$$

which gives another condition for our collection.

In the second case $l' \leq 0$, we have

$$\begin{aligned}
& \sum_{l' < 0} 2^{\frac{l'}{p}(p_0 - \frac{m_1 + 2m_2}{m_1 + m_2})} \left[\sum_{k \in I} k_1^{(m_1 - 2)(1 - p_0)r} k_2^{(m_2 - 2)(p_0 - 2)r} \log^{\gamma p_0 r} \left(1 + \frac{k_1^{m_1 - 2}}{k_2^{m_2 - 2}} 2^{-l'} \right) \right]^{\frac{1}{rp}} \\
& \lesssim \sum_{l' < 0} 2^{\frac{l'}{p}(p_0 - \frac{m_1 + 2m_2}{m_1 + m_2})} |l'|^{\frac{\gamma p_0}{p}} 2^{|l'|(\frac{1}{pr} \frac{1}{m_2 - 2} + \frac{p_0 - 2}{p})_+} \\
& = \sum_{l' < 0} |l'|^{\frac{\gamma p_0}{p}} 2^{\frac{l'}{p}(p_0 - \frac{m_1 + 2m_2}{m_1 + m_2} - (\frac{1}{r} \frac{1}{m_2 - 2} + p_0 - 2)_+)}.
\end{aligned} \tag{3.6.16}$$

Notice that $p_0 > p_c = 2\frac{\bar{m} + 3}{\bar{m} + 4} = 1 + \frac{\bar{m} + 2}{\bar{m} + 4} \geq 1 + \frac{\bar{m}}{\bar{m} + 2} \geq 1 + \frac{m_2}{m_1 + m_2} = \frac{m_1 + 2m_2}{m_1 + m_2}$ yields

$$p_0 - \frac{m_1 + 2m_2}{m_1 + m_2} > 0, \tag{3.6.17}$$

therefore the sum converges in the case $\frac{b}{r} = \frac{1}{r} \frac{1}{m_2 - 2} + p_0 - 2 \leq 0$. However, we will not take this as condition, but instead discuss the case $b > 0$, where we need

$$\begin{aligned}
0 < p_0 - \frac{m_1 + 2m_2}{m_1 + m_2} - \frac{1}{r} \frac{1}{m_2 - 2} - p_0 + 2 \\
= \frac{m_1}{m_1 + m_2} - \frac{1}{r} \frac{1}{m_2 - 2}
\end{aligned}$$

which is equivalent to

$$\boxed{\frac{1}{r} < \frac{m_1(m_2 - 2)}{m_1 + m_2}}, \tag{3.6.18}$$

the symmetric condition to (3.6.15).

Summation in l

Recall that we want to show

$$(3.6.8) \quad \sum_{l=0}^{\infty} 2^{l(\frac{h+1}{p}-1)} \min\{2^{-l}, |\Omega|\}^{1-\frac{1}{p^*}+\frac{1}{pr}} \lesssim |\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}+\frac{1}{pr}}.$$

We claim that it is sufficient to show for $\mu > 0$, $\nu - \mu > 0$

$$\int_0^{\infty} e^{x\mu} \min\{e^{-x}, A\}^{\nu} dx \lesssim A^{\nu-\mu}. \quad (3.6.19)$$

Provided that (3.6.19) is valid, we apply it to $A = |\Omega|$, $\mu = \frac{h+1}{p} - 1$ and $\nu = 1 - \frac{1}{p^*} + \frac{1}{pr}$. It is already known due to [IKM] that $R^* : L^2(\Gamma) \rightarrow L^{2p}(\mathbb{R}^3)$ is bounded for all $p \geq h+1$. Hence, we are interested in results for $p < h+1$, and this means we may assume $\mu = \frac{h+1}{p} - 1 > 0$. Moreover we want $0 < \nu - \mu = 2 - \frac{1}{p^*} + \frac{1}{pr} - \frac{h+1}{p} = \frac{1}{p} \left(2p - h - 1 - \frac{p}{p^*} + \frac{1}{r} \right)$. Notice that if $p \leq 2$, then $\frac{p}{p^*} = 1$, but if $p > 2$, then $\frac{p}{p^*} = p \left(1 - \frac{1}{p} \right) = p - 1$. Thus $\frac{p}{p^*} = 1 + (p-2)_+$ for all $1 \leq p \leq \infty$, i.e. the condition which is required is

$$\boxed{\frac{1}{r} > h + 2 - 2p + (p-2)_+}. \quad (3.6.20)$$

We still need to check (3.6.19). Observe

$$\begin{aligned} \int_0^{\infty} e^{x\mu} \min\{e^{-x}, A\}^{\nu} dx &= \int_{\ln A}^{\infty} e^{y\mu} A^{-\mu} \min\{e^{-y} A, A\}^{\nu} dy \\ &= A^{\nu-\mu} \int_{\ln A}^{\infty} e^{y\mu} \min\{e^{-y}, 1\}^{\nu} dy. \end{aligned}$$

The integral can be estimated by

$$\int_{\ln A}^{\infty} e^{y\mu} \min\{e^{-y}, 1\}^{\nu} dy \leq \int_{-\infty}^0 e^{y\mu} dy + \int_0^{\infty} e^{-y(\nu-\mu)} dy$$

and converges since $\mu > 0$ and $\nu - \mu > 0$.

It still remains to check whether there exist $1 \leq r < \infty$ (for $m > 2$) satisfying the conditions (3.6.10), (3.6.12), (3.6.15), (3.6.18) and (3.6.20). This task will be accomplished in Lemma 3.6.1. First, we discuss the situation where $m = 2$.

3.6.3 The case $m = 2$

We will just give some hints how to modify the previous proof. In this case, $r = \infty$ turns out to be an appropriate choice. Rewriting the integral in (3.6.9) in terms of the L^r -norm yields

$$\left\| \left(k_1^{(m_1-2)(1-p_0)} k_2^{(m_2-2)(p_0-2)} \log^{\gamma p_0} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-l'} \right) \right)_{k \in I} \right\|_r \lesssim \begin{cases} 2^{|l'|(\frac{1}{r} \frac{1}{m_1-2} + 1 - p_0)} & l' \geq 0 \\ |l'| \gamma p_0 2^{|l'|(\frac{1}{r} \frac{1}{m_2-2} + p_0 - 2)} & l' < 0 \end{cases}$$

provided the conditions

$$(3.6.10) \quad \frac{1}{m_1 - 2} + (1 - p_0)r < 0,$$

$$(3.6.12) \quad \frac{1}{m_1 - 2} + \frac{1}{m_2 - 2} - r < 0$$

are satisfied. This already gives rise to the conjecture

$$\begin{aligned} \sup_{k \in I} k_1^{(m_1-2)(1-p_0)} k_2^{(m_2-2)(p_0-2)} \log^{\gamma p_0} \left(1 + \frac{k_1^{m_1-2}}{k_2^{m_2-2}} 2^{-l'}\right) \\ \leq \sup_{s \geq t 2^{l'}} s^{1-p_0} t^{p_0-2} \log^{\gamma p_0} \left(1 + \frac{s}{t} 2^{-l'}\right) \lesssim \begin{cases} 2^{|l'|(1-p_0)} & l' \geq 0 \\ |l'|^{\gamma p_0} 2^{|l'|(p_0-2)_+} & l' < 0 \end{cases}. \end{aligned} \quad (3.6.21)$$

Observe that the conditions (3.6.10) and (3.6.12) are automatically fulfilled for $r = \infty$. It is easy to check that (3.6.21) indeed holds true, even in the case $m = 2$.

Summation in l'

The summation in l' becomes simpler. We split again into $l' \geq 0$ and $l' < 0$, and obtain for the first half of the sum (cf. (3.6.14))

$$\sum_{l' \geq 0} 2^{\frac{l'}{p}(p_0 - \frac{m_1+2m_2}{m_1+m_2})} 2^{l' \frac{1-p_0}{p}} = \sum_{l' \geq 0} 2^{-\frac{l'}{p} \frac{m_2}{m_1+m_2}} < \infty.$$

The second part of the sum becomes (cf. (3.6.16))

$$\sum_{l' < 0} |l'|^{\gamma \frac{p_0}{p}} 2^{\frac{l'}{p}(p_0 - \frac{m_1+2m_2}{m_1+m_2} - (p_0-2)_+)}.$$

We already know from (3.6.17) that $p_0 - \frac{m_1+2m_2}{m_1+m_2} > 0$. Thus the sum converges if $p_0 \leq 2$. For $p_0 > 2$, notice

$$p_0 - \frac{m_1 + 2m_2}{m_1 + m_2} - (p_0 - 2)_+ = \frac{m_1}{m_1 + m_2} > 0,$$

and thus the sum is finite.

Summation in l

It remains to show

$$\sum_{l=0}^{\infty} 2^{l(\frac{h+1}{p}-1)} \min\{|\Omega|, 2^{-l}\}^{1-\frac{1}{p^*}} \lesssim |\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}},$$

which is the special case $r = \infty$ of (3.6.8). We saw that this holds true provided (3.6.20): $\frac{1}{r} > h + 2 - 2p + (p - 2)_+$.

However,

$$2p > 2p_c = 4 \frac{\bar{m} + 3}{\bar{m} + 4} = \frac{2\bar{m} + 4}{\bar{m} + 4} + 2 > \frac{2\bar{m}}{\bar{m} + 2} + 2 \stackrel{m=2}{=} h + 2.$$

Thus for the case $p \leq 2$ we have $h + 2 - 2p + (p - 2)_+ = h + 2 - 2p < 0$. For the case $p > 2$ notice that

$$h + 2 - 2p + (p - 2)_+ = h - p = \frac{2\bar{m}}{\bar{m} + 2} - p < 2 - p < 0.$$

□

3.6.4 Final considerations

It remains to check whether we could satisfy all our conditions in the case $m > 2$.

LEMMA 3.6.1

Let $m > 2$, $p, p_0 \geq p_c = 2\frac{\bar{m}+3}{\bar{m}+4}$ and $h + 1 < 2p$. Define

$$J =]0, 1 + (p - 2)_+[\cap]h + 2 - 2p + (p - 2)_+, \frac{\bar{m}(m - 2)}{\bar{m} + m}[\quad (3.6.22)$$

Then $J \neq \emptyset$ and for every $\frac{1}{r} \in J$ we have $r^* = \frac{rp}{p^*} \geq 1$ and

$$(3.6.10) \quad \frac{1}{r} < (m_1 - 2)(p_0 - 1)$$

$$(3.6.12) \quad \frac{1}{r} < \frac{(m_1 - 2)(m_2 - 2)}{m_1 + m_2 - 4}$$

$$(3.6.15) \quad \frac{1}{r} < \frac{m_2(m_1 - 2)}{m_1 + m_2}$$

$$(3.6.18) \quad \frac{1}{r} < \frac{m_1(m_2 - 2)}{m_1 + m_2}$$

$$(3.6.20) \quad \frac{1}{r} > h + 2 - 2p + (p - 2)_+$$

Proof: First of all, we will show $h + 2 - 2p + (p - 2)_+ < \frac{\bar{m}(m-2)}{\bar{m}+m} = h - \frac{2\bar{m}}{\bar{m}+m}$. We begin with the case $p \leq 2$. Since $p > p_c$, it suffices to prove

$$\begin{aligned} h + 2 - 2p_c &< h - \frac{2\bar{m}}{\bar{m} + m} \\ \Leftrightarrow \frac{\bar{m}}{\bar{m} + m} &< p_c - 1 = \frac{\bar{m} + 2}{\bar{m} + 4} \\ \Leftrightarrow 2\bar{m} &< m\bar{m} + 2m, \end{aligned} \quad (3.6.23)$$

which is true since $m \geq 2$.

If $p > 2$, observe

$$h + 2 - 2p + (p - 2)_+ = h - p < h - 2 \leq h - \frac{2\bar{m}}{\bar{m} + m} = \frac{\bar{m}(m - 2)}{\bar{m} + m}.$$

Thus both intervals used for the definition of J are not empty, but we still have to check that their intersection is not trivial. Since we assumed $h + 1 < 2p$, we have

$$h + 2 - 2p + (p - 2)_+ < 1 + (p - 2)_+;$$

since $m > 2$, we have $0 < \frac{\bar{m}(m-2)}{\bar{m}+m}$, hence $J \neq \emptyset$.

For $\frac{1}{r} \in J$, especially $\frac{1}{r} \leq 1 + (p-2)_+ = \frac{p}{p^*}$, thus $r^* = r \frac{p}{p^*} \geq 1$. Considering the conditions, (3.6.20) is obvious by the definition of J . Furthermore do we have

$$\frac{1}{r} < \frac{\bar{m}(m-2)}{\bar{m}+m} = \frac{m_1 m_2 - 2\bar{m}}{m_1 + m_2} \leq \frac{m_1 m_2 - 2m_i}{m_1 + m_2}$$

for both $i = 1, 2$, which gives (3.6.15) and (3.6.18). To obtain (3.6.10), we estimate

$$\frac{1}{r} < \frac{\bar{m}(m-2)}{\bar{m}+m} \leq \frac{\bar{m}(m_1-2)}{\bar{m}+m} \stackrel{(3.6.23)}{<} (p_c - 1)(m_1 - 2) \leq (p_0 - 1)(m_1 - 2).$$

Finally, observe that the following statements are equivalent:

$$\begin{aligned} & \frac{\bar{m}(m-2)}{\bar{m}+m} \leq \frac{(m_1-2)(m_2-2)}{m_1+m_2-4} \\ \Leftrightarrow & \frac{\bar{m}}{\bar{m}+m} \leq \frac{\bar{m}-2}{\bar{m}+m-4} \\ \Leftrightarrow & \bar{m}(\bar{m}+m) - 4\bar{m} \leq \bar{m}(\bar{m}+m) - 2(\bar{m}+m) \\ \Leftrightarrow & m \leq \bar{m}. \end{aligned}$$

Hence (3.6.12) holds true as well. □

3.7 On the sharpness of the bilinear estimate

3.7.1 General approach

In this section, we briefly discuss the sharpness of the bilinear estimates that we obtained, namely (3.4.4) at the endpoint. Let $0 \in V \subset \mathbb{R}^2$, $\phi, \tilde{\phi} : V \rightarrow \mathbb{R}$ be smooth on a neighbourhood of V with $\phi(0) = \tilde{\phi}(0)$, denote by $\sigma, \tilde{\sigma}$ the surface measures of $S = \text{Graph}(\phi)$ and $\tilde{S} = \text{Graph}(\tilde{\phi})$. Let the curve Γ be a parametrisation of the intersection of the two surfaces, i.e. of the zero set of $\psi(x) = \phi(x) - \tilde{\phi}(x)$. If we define $\nu = \frac{\nabla\psi(0)}{|\nabla\psi(0)|}$ as the normal direction of Γ and τ as its tangential direction, Γ is of the form $\Gamma(t) = t\tau + (at^k + \mathcal{O}(t^{k+1}))\nu$ for some $k \geq 2$, $|t| \lesssim 1$. The order of the non-linear part $\phi_{\text{nonlin}}(\xi) = \phi(\xi) - \phi(0) - \nabla\phi(0)\xi$ of ϕ on a box containing Γ shall be l , i.e. $\phi_{\text{nonlin}}(\varepsilon t\tau + a\varepsilon^k s\nu) = b(\varepsilon t)^l + \mathcal{O}(\varepsilon^{l+1})$ for $|s|, |t| \lesssim 1$, $\varepsilon \ll 1$ (here, the Landau symbol is a function of t and s). Correspondingly we also assume $\tilde{\phi}_{\text{nonlin}}(\varepsilon t\tau + a\varepsilon^k s\nu) = \tilde{b}(\varepsilon t)^{\tilde{l}} + \mathcal{O}(\varepsilon^{\tilde{l}+1})$ for $|s|, |t| \lesssim 1$, $\varepsilon \ll 1$. Without loss of generality, we may assume $\tilde{l} \leq l$. In the case $\tilde{l} = l$ we may even assume $b \leq \tilde{b}$.

LEMMA 3.7.1

Assume $k = \tilde{l}$ and $a|\nabla\psi(0)| \approx \tilde{b}$. Then the bilinear restriction estimate

$$\|\widehat{f d\sigma} \widehat{g d\tilde{\sigma}}\|_p \leq C_p \|f\|_2 \|g\|_2 \text{ for all } f \in L^2(S), g \in L^2(\tilde{S}) \quad (3.7.1)$$

implies

$$p \geq p_c := \frac{2k + 2l + 2}{k + l + 2} \quad (3.7.2)$$

and

$$C_{p_c} \gtrsim a^{1-\frac{1}{p}} \frac{b^{\frac{1}{2}-\frac{1}{p}}}{\tilde{b}^{\frac{1}{2}}}. \quad (3.7.3)$$

Proof: Observe that $\partial_\tau \psi(0) = \langle \tau, \nabla\psi(0) \rangle = \langle \tau, \nu \rangle |\nabla\psi(0)| = 0$.

By a rotation of coordinates, we may assume $\tau = (1, 0)$, $\nu = (0, 1)$. In these coordinates, $\partial_1 \psi(0) = 0$.

Let f, g_0 be smooth versions of the characteristic function of the set

$$\{\xi \in \mathbb{R}^2 | 0 \leq \xi_1 \leq \varepsilon, 0 \leq \xi_2 \leq a\varepsilon^k\}$$

well adapted to the curve Γ , i.e. let $\chi \in C_0^\infty(B(0, \frac{1}{10}))$, $0 \leq \chi \leq 1$, $\chi(0) = 1$ and define $f(\xi) = g_0(\xi) = \chi(\frac{\xi_1}{\varepsilon}, \frac{\xi_2}{a\varepsilon^k})$. Using the notation $x' = (x_1, x_2)$, we conclude

$$\begin{aligned} |\widehat{f d\sigma}(x)| &= \left| \int \int e^{-i[x' \cdot \xi + x_3 \phi(\xi)]} f(\xi) d\xi \right| \\ &= \left| \int \int e^{-i[(x' + \nabla\phi(0)x_3) \cdot \xi + x_3 \phi_{\text{nonlin}}(\xi)]} f(\xi) d\xi \right| \\ &= a\varepsilon^{k+1} \left| \int \int e^{-i[\varepsilon(x_1 + \partial_1\phi(0)x_3)t + a\varepsilon^k(x_2 + \partial_2\phi(0)x_3)s + x_3 \phi_{\text{nonlin}}(\varepsilon t, a\varepsilon^k s)]} \chi(t, s) dt ds \right| \\ &\gtrsim a\varepsilon^{k+1} \chi_{Q(\varepsilon)}(x), \end{aligned}$$

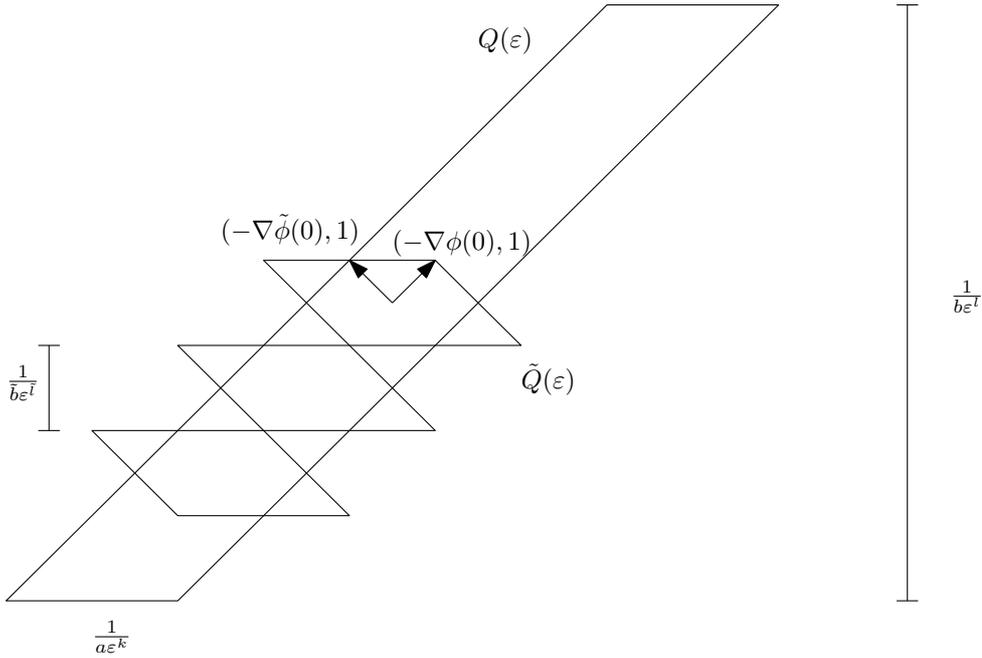


Figure 3.8: Shifting of the boxes

where $Q(\varepsilon) = \{x \in \mathbb{R}^3 : |x_1 + \partial_1 \phi(0)x_3| \ll \frac{1}{\varepsilon}, |x_2 + \partial_2 \phi(0)x_3| \ll \frac{1}{a\varepsilon^k}, |x_3| \ll \frac{1}{b\varepsilon^l}\}$. In a similar manner, we conclude

$$|\widehat{g_0 d\tilde{\sigma}}(x)| \gtrsim a\varepsilon^{k+1} \chi_{\tilde{Q}(\varepsilon)}(x), \quad (3.7.4)$$

with $\tilde{Q}(\varepsilon) = \{x \in \mathbb{R}^3 : |x_1 + \partial_1 \tilde{\phi}(0)x_3| \ll \frac{1}{\varepsilon}, |x_2 + \partial_2 \tilde{\phi}(0)x_3| \ll \frac{1}{a\varepsilon^k}, |x_3| \ll \frac{1}{b\varepsilon^l}\}$.

Notice that $\frac{1}{b\varepsilon^l} \leq \frac{1}{b\varepsilon^l}$ for ε small enough since $\tilde{l} \leq l$ (or since $b \leq \tilde{b}$ if $\tilde{l} = l$). We aim at covering the big box $Q(\varepsilon)$ or at least a large part of it, by translations of the smaller box $\tilde{Q}(\varepsilon)$. This is possible if

$$|Q(\varepsilon) \cap \tilde{Q}(\varepsilon)| \approx |\tilde{Q}(\varepsilon)|, \quad (3.7.5)$$

i.e. if the gradients $\nabla \phi(0)$ and $\nabla \tilde{\phi}(0)$ do not differ too much. The quantitative statement is that we need $\frac{1}{b\varepsilon^l} |\partial_1(\phi - \tilde{\phi})(0)| \lesssim \frac{1}{\varepsilon}$ and $\frac{1}{b\varepsilon^l} |\partial_2(\phi - \tilde{\phi})(0)| \lesssim \frac{1}{a\varepsilon^k}$. The first condition is trivial, since $\partial_1(\phi - \tilde{\phi})(0) = \partial_1 \psi(0) = 0$. The second one is equivalent to

$$a|\partial_\nu(\phi - \tilde{\phi})(0)|\varepsilon^{k-\tilde{l}} = a|\nabla \psi(0)| \lesssim \tilde{b}, \quad (3.7.6)$$

This holds true due to our assumption $a|\nabla \psi(0)| \approx \tilde{b}$.

Introduce the modulations of g_0 by $g_j(\xi) = e^{-ij\tilde{b}^{-1}\varepsilon^{-\tilde{l}}[\tilde{\phi}(\xi) - \nabla \phi(0) \cdot \xi]} g_0(\xi)$, $|j| \leq \frac{\tilde{b}\varepsilon^{\tilde{l}}}{b\varepsilon^l}$ such that $\widehat{g_j d\tilde{\sigma}}(x) = \widehat{g_0 d\tilde{\sigma}}(x + j\tilde{b}^{-1}\varepsilon^{-\tilde{l}}(-\nabla \phi(0), 1))$ are translations in the direction of $(-\nabla \phi(0), 1)$ (cf. Figure 3.8). We claim that the functions $\widehat{g_j d\tilde{\sigma}}$ have essentially disjoint supports on

the big box $Q(\varepsilon)$. The first observation to verify this is that for $|x_1 + \partial_1\phi(0)x_3| \leq \frac{1}{\varepsilon}$ and $|x_2 + \partial_2\phi(0)x_3| \leq \frac{1}{a\varepsilon^k}$ we have

$$\begin{aligned} |\widehat{g_0 d\tilde{\sigma}}(x)| &\approx a\varepsilon^{k+1} \left| \int \int e^{-i[\varepsilon(x_1 + \partial_1\tilde{\phi}(0)x_3)t + a\varepsilon^k(x_2 + \partial_2\tilde{\phi}(0)x_3)s + x_3\tilde{\phi}_{nonlin}(\varepsilon t, a\varepsilon^k s)]} \chi(s, t) dt ds \right| \\ &\leq C_N a\varepsilon^{k+1} \left(1 + \tilde{b}\varepsilon^{\tilde{l}}|x_3|\right)^{-N} \end{aligned} \quad (3.7.7)$$

for any $N \geq 1$. For the proof we may assume $\tilde{b}\varepsilon^{\tilde{l}}|x_3| \gg 1$. For the phase function

$$\begin{aligned} \varphi(t, s) &= \varepsilon(x_1 + \partial_1\tilde{\phi}(0)x_3)t + a\varepsilon^k(x_2 + \partial_2\tilde{\phi}(0)x_3)s + x_3\tilde{\phi}_{nonlin}(\varepsilon t, a\varepsilon^k s) \\ &= \varepsilon(x_1 + \partial_1\tilde{\phi}(0)x_3)t + a\varepsilon^k(x_2 + \partial_2\tilde{\phi}(0)x_3)s + x_3[\tilde{b}(\varepsilon t)^{\tilde{l}} + \mathcal{O}(\varepsilon^{\tilde{l}+1})] \end{aligned}$$

we have

$$\begin{aligned} \partial_s \varphi(t, s) &\approx a\varepsilon^k(x_2 + \partial_2\tilde{\phi}(0)x_3) + x_3\mathcal{O}(\varepsilon^{\tilde{l}+1}) \\ &= a\varepsilon^k(x_2 + \partial_2\phi(0)x_3) - a\varepsilon^k\partial_2\psi(0)x_3 + x_3\mathcal{O}(\varepsilon^{\tilde{l}+1}). \end{aligned}$$

Since $\partial_2\psi(0) = \partial_\nu\psi(0) = |\nabla\psi(0)|$, we have $a\varepsilon^k\partial_2\psi(0) \approx \tilde{b}\varepsilon^{\tilde{l}}$ and thus

$$\partial_s \varphi(t, s) \approx a\varepsilon^k(x_2 + \partial_2\phi(0)x_3) - x_3(\tilde{b}\varepsilon^{\tilde{l}} + \mathcal{O}(\varepsilon^{\tilde{l}+1})).$$

As we assumed $a\varepsilon^k|x_2 + \partial_2\phi(0)x_3| \leq 1$, integration by parts in s gives the estimate (3.7.7). Consequently, we obtain

$$\begin{aligned} |\widehat{g_j d\tilde{\sigma}}(x)| &= |\widehat{g_0 d\tilde{\sigma}}(x + j\tilde{b}^{-1}\varepsilon^{-\tilde{l}}(-\nabla\phi(0), 1))| \\ &\lesssim a\varepsilon^{k+1}(1 + |\tilde{b}\varepsilon^{\tilde{l}}x_3 + j|)^{-N}, \end{aligned} \quad (3.7.8)$$

which implies for $g = \sum_j g_j$ and $x \in Q(\varepsilon)$

$$|\widehat{g d\tilde{\sigma}}(x)| \approx \sum_j |\widehat{g_j d\tilde{\sigma}}(x)| \gtrsim a\varepsilon^{k+1} \chi_{\bigcup_j (\tilde{Q}(\varepsilon) + j\tilde{b}^{-1}\varepsilon^{-\tilde{l}}(-\nabla\phi(0), 1))}. \quad (3.7.9)$$

Notice that by construction (see (3.7.5)), we have

$$\begin{aligned} &\left| Q(\varepsilon) \cap \bigcup_j \left(\tilde{Q}(\varepsilon) + j\tilde{b}^{-1}\varepsilon^{-\tilde{l}}(-\nabla\phi(0), 1) \right) \right| \\ &= \sum_j \left| Q(\varepsilon) \cap \left(\tilde{Q}(\varepsilon) + j\tilde{b}^{-1}\varepsilon^{-\tilde{l}}(-\nabla\phi(0), 1) \right) \right| \\ &\approx \sum_j |\tilde{Q}(\varepsilon)| \\ &\approx \frac{\tilde{b}\varepsilon^{\tilde{l}}}{b\varepsilon^{\tilde{l}}} (\varepsilon \cdot a\varepsilon^k \cdot \tilde{b}\varepsilon^{\tilde{l}})^{-1} \\ &\approx |Q(\varepsilon)|. \end{aligned}$$

Observe further that the g_j are almost orthogonal in the following sense:

$$\begin{aligned} \int g_j \bar{g}_{j'} d\xi &= \int e^{i(j'-j)\tilde{b}^{-1}\varepsilon^{-\tilde{l}}[\tilde{\phi}(\xi)-\nabla\phi(0)\xi]} |g_0(\xi)|^2 d\xi \\ &= e^{i(j'-j)\tilde{b}^{-1}\varepsilon^{-\tilde{l}}\tilde{\phi}(0)} \int e^{i(j'-j)\tilde{b}^{-1}\varepsilon^{-\tilde{l}}[\tilde{\phi}_{nonlin}(\xi)-\nabla\psi(0)\xi]} |g_0(\xi)| d\xi \\ &= e^{i(j'-j)\tilde{b}^{-1}\varepsilon^{-\tilde{l}}\tilde{\phi}(0)} a\varepsilon^{k+1} \int \int e^{i(j'-j)\tilde{b}^{-1}\varepsilon^{-\tilde{l}}[\tilde{\phi}_{nonlin}(\varepsilon t, a\varepsilon^k s)-\nabla\psi(0)\cdot(\varepsilon t, a\varepsilon^k s)]} \chi(t, s) ds dt. \end{aligned}$$

Since $\partial_s \tilde{\phi}_{nonlin}(\varepsilon t, a\varepsilon^k s) \leq \mathcal{O}(\varepsilon^{\tilde{l}+1}) \ll \tilde{b}\varepsilon^{\tilde{l}}$ and $\partial_1 \psi(0) = 0$, i.e.

$$|\partial_s(\nabla\psi(0) \cdot (\varepsilon t, a\varepsilon^k s))| = |\partial_2 \psi(0)| a\varepsilon^k \approx \tilde{b}\varepsilon^{\tilde{l}},$$

integration by parts in s gives

$$|\langle g_j, g_{j'} \rangle| \lesssim a\varepsilon^{k+1} |j - j'|^{-N} \approx \|g_0\|_2^2 |j - j'|^{-N} \quad (3.7.10)$$

for every $N \in \mathbb{N}$. We conclude

$$\|g\|_2^2 = \sum_{j, j'} \langle g_j, g_{j'} \rangle \approx \sum_j \|g_0\|_2^2 \approx \frac{\tilde{b}\varepsilon^{\tilde{l}}}{b\varepsilon^l} \|g_0\|_2^2. \quad (3.7.11)$$

Therefore we obtain

$$\|f\|_2 \|g\|_2 \approx a\varepsilon^{k+1} \sqrt{\frac{\tilde{b}\varepsilon^{\tilde{l}}}{b\varepsilon^l}} \quad (3.7.12)$$

and

$$\begin{aligned} \|\widehat{f d\sigma g d\tilde{\sigma}}\|_p &\gtrsim a^2 \varepsilon^{2k+2} |Q(\varepsilon)|^{\frac{1}{p}} \\ &\approx a^2 \varepsilon^{2k+2} (ab)^{-\frac{1}{p}} \varepsilon^{-\frac{1}{p}(k+l+1)} \\ &= a^{2-\frac{1}{p}} b^{-\frac{1}{p}} \varepsilon^{(k+1)(2-\frac{1}{p})-l\frac{1}{p}}. \end{aligned} \quad (3.7.13)$$

Thus

$$a^{2-\frac{1}{p}} b^{-\frac{1}{p}} \varepsilon^{(k+1)(2-\frac{1}{p})-l\frac{1}{p}} \lesssim C_p a\varepsilon^{k+1} \sqrt{\frac{\tilde{b}\varepsilon^{\tilde{l}}}{b\varepsilon^l}},$$

i.e.

$$C_p \gtrsim \varepsilon^{(k+1)(1-\frac{1}{p})-l\frac{1}{p}+\frac{1}{2}(l-\tilde{l})} a^{1-\frac{1}{p}} \frac{b^{\frac{1}{2}-\frac{1}{p}}}{\tilde{b}^{\frac{1}{2}}}. \quad (3.7.14)$$

Since ε can be chosen arbitrarily small, we must have $(k+1)(1-\frac{1}{p})-l\frac{1}{p}+\frac{1}{2}(l-\tilde{l}) \geq 0$, which gives

$$p \geq p_c = \frac{2k+2l+2}{k+l+2}. \quad (3.7.15)$$

Moreover

$$C_p \gtrsim a^{1-\frac{1}{p}} \frac{b^{\frac{1}{2}-\frac{1}{p}}}{\tilde{b}^{\frac{1}{2}}} \quad (3.7.16)$$

for $p = p_c$. □

3.7.2 A lower bound for the constant in the bilinear estimate

Recall (3.4.1)

$$M(a, b) = M_{m_1, m_2}(a, b) = \begin{cases} a & \text{if } m_1 > m_2 \\ b & \text{if } m_1 < m_2 \\ a \vee b & \text{if } m_1 = m_2, \end{cases}$$

and ϕ shall be again our function of normalised finite type (m_1, m_2) (cf. page 52).

LEMMA 3.7.2

Let $V = [0, \rho_1] \times [0, \rho_2]$, $\tilde{V} = \tilde{r} + V$, $2\rho_i \leq \tilde{r}_i \lesssim \rho_i$, $S = \text{Graph}(\phi|_V)$, $\tilde{S} = \text{Graph}(\phi|_{\tilde{V}})$ and let $\varkappa_i = \rho_i^{m_i-2}$. For $p = 2\frac{m+3}{m+4}$, $m = m_1 \wedge m_2$, we have

$$\|R_{S, \tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S, \tilde{S}}(R))} \gtrsim [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p} - \frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2} - \frac{5}{2p}} \left(\frac{M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{4}{p} - \frac{5}{2}}.$$

Comparison with Theorem 3.4.1 yields that the lemma states (essentially) the sharpness of (3.4.4).

Proof:

Let $\tilde{\phi}(\xi) = \phi(\tilde{r} + \xi) - \phi(\tilde{r})$, and $\psi(\xi) = \phi(\xi) - \tilde{\phi}(\xi) = \phi(\xi) - \phi(\tilde{r} + \xi) + \phi(\tilde{r})$. Then $\nabla\psi(0) = \nabla\phi(0) - \nabla\phi(\tilde{r}) = -\nabla\phi(\tilde{r})$. Introduce a orthonormal system $\tau, \nu = \nabla\psi(0)$ like described above, i.e. $\nu_i \approx -\frac{\varkappa_i \rho_i}{\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2}$, $\tau_1 \approx \frac{\varkappa_2 \rho_2}{\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2}$ and $\tau_2 \approx -\frac{\varkappa_1 \rho_1}{\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2}$. We parametrise the intersection curve of S and \tilde{S} , more precisely, the zero set of ψ , by $\Gamma(t) = t\tau + \gamma(t)\nu$. Since $0 = \psi(t\tau + \gamma(t)\nu)$, by differentiation we obtain

$$0 = (\nabla\phi)(\Gamma(t)) \cdot (\tau + \dot{\gamma}(t)\nu) \quad (3.7.17)$$

and for $t = 0$

$$0 = (\nabla\psi)(0) \cdot (\tau + \dot{\gamma}(0)\nu) = \dot{\gamma}(0)|\nabla\psi(0)|, \quad (3.7.18)$$

thus $\dot{\gamma}(0) = 0$. Differentiating (3.7.17) once more in $t = 0$ yields

$$\begin{aligned} 0 &= \tau^t \text{Hess}(\psi)(0)\tau + |\nabla\psi(0)|\ddot{\gamma}(0) \\ \Rightarrow \ddot{\gamma}(0) &= -\frac{1}{|\nabla\psi(0)|} \tau^t \text{Hess}(\psi)(0)\tau \\ &\approx \frac{1}{\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2} \frac{\varkappa_1 \varkappa_2^2 \rho_2^2 + \varkappa_2 \varkappa_1^2 \rho_1^2}{(\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2)^2} \\ &= \varkappa_1 \varkappa_2 \frac{\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2}{(\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2)^3} \end{aligned}$$

Therefore $\Gamma(t) = t\tau + \nu(at^k + \mathcal{O}(t^{k+1}))$, $\gamma(t) = at^k + \mathcal{O}(t^{k+1})$ with $\boxed{k=2}$ and

$$\boxed{a \approx \varkappa_1 \varkappa_2 \frac{\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2}{(\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2)^3}}. \quad (3.7.19)$$

Moreover we have

$$\varepsilon t\tau + a\varepsilon^2 s\nu \approx \frac{(\varepsilon t\kappa_2\rho_2 - a\varepsilon^2 s\kappa_1\rho_1, -\varepsilon t\kappa_1\rho_1 - a\varepsilon^2 s\kappa_2\rho_2)}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \quad (3.7.20)$$

$$= \varepsilon t \frac{(\kappa_2\rho_2, -\kappa_1\rho_1)}{\kappa_1\rho_1 \vee \kappa_2\rho_2} + \mathcal{O}(\varepsilon^2), \quad (3.7.21)$$

such that

$$\begin{aligned} |\phi_{nonlin}(\varepsilon t\tau + a\varepsilon^2 s\nu)| &= |\phi(\varepsilon t\tau + a\varepsilon^2 s\nu)| \\ &\approx \left| \frac{\varepsilon t\kappa_2\rho_2}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right|^{m_1} + \left| \frac{\varepsilon t\kappa_1\rho_1}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right|^{m_2} + \mathcal{O}(\varepsilon^{m+1}) \\ &\approx \begin{cases} (\varepsilon t)^{m_2} \left(\frac{\kappa_1\rho_1}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right)^{m_2} + \mathcal{O}(\varepsilon^{m_2+1}) & \text{if } m_1 > m_2 \\ (\varepsilon t)^{m_1} \left(\frac{\kappa_2\rho_2}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right)^{m_1} + \mathcal{O}(\varepsilon^{m_1+1}) & \text{if } m_1 < m_2 \\ (\varepsilon t)^m + \mathcal{O}(\varepsilon^{m+1}) & \text{if } m_1 = m_2. \end{cases} \\ &= (\varepsilon t)^m \left(\frac{M(\kappa_1\rho_1, \kappa_2\rho_2)}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right)^m + \mathcal{O}(\varepsilon^{m+1}) \\ &= b(\varepsilon t)^l + \mathcal{O}(\varepsilon^{l+1}), \end{aligned}$$

with $\boxed{l = m}$ and

$$\boxed{b \approx \left(\frac{M(\kappa_1\rho_1, \kappa_2\rho_2)}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right)^m}. \quad (3.7.22)$$

In a similar manner, we conclude

$$\begin{aligned} |\tilde{\phi}_{nonlin}(\varepsilon t\tau + a\varepsilon^2 s\nu)| &\approx \kappa_1 \left| \frac{\varepsilon t\kappa_2\rho_2}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right|^2 + \kappa_2 \left| \frac{\varepsilon t\kappa_1\rho_1}{\kappa_1\rho_1 \vee \kappa_2\rho_2} \right|^2 + \mathcal{O}(\varepsilon^3) \\ &= (\varepsilon t)^2 \kappa_1 \kappa_2 \frac{\kappa_1\rho_1^2 + \kappa_2\rho_2^2}{(\kappa_1\rho_1 \vee \kappa_2\rho_2)^2} + \mathcal{O}(\varepsilon^3), \end{aligned}$$

hence $\boxed{\tilde{l} = 2 \leq l}$ and

$$\boxed{\tilde{b} \approx \kappa_1 \kappa_2 \frac{\kappa_1\rho_1^2 + \kappa_2\rho_2^2}{(\kappa_1\rho_1 \vee \kappa_2\rho_2)^2} = a(\kappa_1\rho_1 \vee \kappa_2\rho_2)}, \quad (3.7.23)$$

so the conditions $k = \tilde{l}$ and $a|\nabla\psi(0)| \approx \tilde{b}$ hold true. Applying Lemma 3.7.1, we obtain

$$p_c = \frac{2k + 2l + 2}{k + l + 2} = \frac{2m + 6}{m + 4} = p, \quad (3.7.24)$$

which we will use in the form

$$m \left(\frac{1}{2} - \frac{1}{p} \right) = \frac{3}{p} - 2 \quad (3.7.25)$$

$$\text{and } (m - 2) \left(\frac{3}{2p} - 1 \right) = m \left(\frac{5}{2p} - \frac{3}{2} \right). \quad (3.7.26)$$

By collecting (3.7.19), (3.7.22) and (3.7.23), we have

$$\begin{aligned}
\|R_{S,\tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S,\tilde{S}}(R))} &\gtrsim a^{1-\frac{1}{p}} \frac{b^{\frac{1}{2}-\frac{1}{p}}}{\tilde{b}^{\frac{1}{2}}} \\
&\approx b^{\frac{1}{2}-\frac{1}{p}} (\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2)^{-\frac{1}{2}} a^{\frac{1}{2}-\frac{1}{p}} \\
&\approx \left(\frac{M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)}{\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2} \right)^{m(\frac{1}{2}-\frac{1}{p})} (\varkappa_1 \rho_1 \vee \varkappa_2 \rho_2)^{\frac{3}{p}-2} \\
&\quad \cdot [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2)]^{\frac{1}{2}-\frac{1}{p}} \\
&\stackrel{(3.7.25)}{=} M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^{\frac{3}{p}-2} (\varkappa_1 \varkappa_2)^{\frac{1}{2}-\frac{1}{p}} (\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2)^{\frac{1}{2}-\frac{1}{p}} \\
&= \left(\frac{M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^2}{\varkappa_1 \varkappa_2} \right)^{\frac{3}{2p}-1} (\varkappa_1 \varkappa_2)^{\frac{1}{2p}-\frac{1}{2}} (\varkappa_1 \rho_1^2 + \varkappa_2 \rho_2^2)^{\frac{1}{2}-\frac{1}{p}}.
\end{aligned}$$

We claim that

$$\left(\frac{M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^2}{\varkappa_1 \varkappa_2} \right)^{\frac{3}{2p}-1} = (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}}. \quad (3.7.27)$$

Taking this for granted for a moment, we see that

$$\begin{aligned}
&\|R_{S,\tilde{S}}^*\|_{L^2 \times L^2 \rightarrow L^p(Q_{S,\tilde{S}}(R))} \\
&\gtrsim M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}} (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} (\varkappa_1 \varkappa_2)^{\frac{1}{2p}-\frac{1}{2}} (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{1}{2}-\frac{1}{p}} \\
&= \left(\frac{M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{4}{p}-\frac{5}{2}} [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p}-\frac{1}{2}} \left(\frac{\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{3}{2}-\frac{5}{2p}} \\
&= \left(\frac{M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)}{\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2} \right)^{\frac{4}{p}-\frac{5}{2}} [\varkappa_1 \varkappa_2 (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)]^{\frac{1}{2p}-\frac{1}{2}} (\varkappa_1 \rho_1^2 \wedge \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}}.
\end{aligned}$$

To prove (3.7.27), we distinguish between three cases:

Case 1: $m_1 < m_2$

Since $m = m_1$, we have

$$\begin{aligned}
\varkappa_1^{-\left(\frac{3}{2p}-1\right)} &= \rho_1^{-(m-2)\left(\frac{3}{2p}-1\right)} \\
&\stackrel{(3.7.26)}{=} \rho_1^{m\left(\frac{3}{2}-\frac{5}{2p}\right)} \\
&= (\varkappa_1 \rho_1^2)^{\frac{3}{2}-\frac{5}{2p}}.
\end{aligned} \quad (3.7.28)$$

Because $M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^2 = \varkappa_2^2 \rho_2^2$, we obtain

$$\begin{aligned}
\left(\frac{M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^2}{\varkappa_1 \varkappa_2} \right)^{\frac{3}{2p}-1} &= \left(\frac{\varkappa_2 \rho_2^2}{\varkappa_1} \right)^{\frac{3}{2p}-1} \\
&= (\varkappa_1 \rho_1^2)^{\frac{3}{2}-\frac{5}{2p}} (\varkappa_2 \rho_2^2)^{\frac{3}{2p}-1} \\
&= (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} (\varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}} \\
&= (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}}.
\end{aligned} \quad (3.7.29)$$

Case 2: $m_1 > m_2$

A symmetric argument shows

$$\varkappa_2^{-(\frac{3}{2p}-1)} = (\varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}}. \quad (3.7.30)$$

and

$$\left(\frac{M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^2}{\varkappa_1 \varkappa_2} \right)^{\frac{3}{2p}-1} = (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} (\varkappa_1 \rho_1^2)^{\frac{4}{p}-\frac{5}{2}} \quad (3.7.31)$$

$$(\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}} M(\varkappa_1 \rho_1^2, \varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}}. \quad (3.7.32)$$

Case 3: $m_1 = m_2$

Observe that in this case $m_1 = m = m_2$, thus both (3.7.28) and (3.7.30) apply, i.e.

$$(\varkappa_1 \varkappa_2)^{-(\frac{3}{2p}-1)} = (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}}. \quad (3.7.33)$$

We may assume without loss of generality $\rho_1 \leq \rho_2$. Then $\varkappa_1 \rho_1 = \rho_1^{m-1} \leq \rho_2^{m-1} = \varkappa_2 \rho_2$. By definition of M , we have

$$\begin{aligned} M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^{2(\frac{3}{2p}-1)} &= \rho_2^{(2m-2)(\frac{3}{2p}-1)} \\ &\stackrel{(3.7.26)}{=} \rho_2^{m(\frac{3}{2p}-1)+m(\frac{5}{2p}-\frac{3}{2})} \\ &= (\varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}} \\ &= (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}}, \end{aligned} \quad (3.7.34)$$

since also $\varkappa_1 \rho_1^2 = \rho_1^m \leq \rho_2^m = \varkappa_2 \rho_2^2$. Thus

$$\left(\frac{M(\varkappa_1 \rho_1, \varkappa_2 \rho_2)^2}{\varkappa_1 \varkappa_2} \right)^{\frac{3}{2p}-1} = (\varkappa_1 \rho_1^2 \vee \varkappa_2 \rho_2^2)^{\frac{4}{p}-\frac{5}{2}} (\varkappa_1 \rho_1^2 \varkappa_2 \rho_2^2)^{\frac{3}{2}-\frac{5}{2p}}. \quad (3.7.35)$$

□

3.8 Appendix

LEMMA 3.3.1

Let $S_j = \{(\xi, \phi(\xi)) | \xi \in \tau_j\}$, $\tau_j \subset \mathbb{R}^{d-1}$, $j = 1, 2$. Let $A \in GL(d-1)$, $a > 0$, and put $\phi^s(\eta) = \frac{1}{a}\phi(A\eta)$. By σ , we denote the surface measure on $\text{Graph}(\phi)$, by σ_s the surface measure on $\text{Graph}(\phi^s)$, and let $S_j^s = \{(\eta, \phi(\eta)) | \eta \in \tau_j^s\}$, $\tau_j^s = A^{-1}(\tau_j)$, $j = 1, 2$. For any $Q^s \subset \mathbb{R}^{d+1}$, let $Q = \{(A^{-1}x', a^{-1}x_{d+1}) | (x', x_{d+1}) \in Q^s\}$. Assume that

$$\|\widehat{g_1 d\sigma^s g_2 d\sigma^s}\|_{L_p(Q^s)} \leq C_s \|g_1\|_2 \|g_2\|_2 \text{ for all } g_j \in L_2(S_j^s, \sigma^s). \quad (3.8.1)$$

Then

$$\|\widehat{f_1 d\sigma f_2 d\sigma}\|_{L_p(Q)} \leq C_s |\det A|^{\frac{1}{p}} a^{-\frac{1}{p}} \|f_1\|_2 \|f_2\|_2 \text{ for all } f_j \in L_2(S_j, \sigma). \quad (3.8.2)$$

Proof: Since

$$\begin{aligned} \widehat{f_j d\sigma}(x) &= \int_{\tau_j} f_j(\xi) e^{i(x' \cdot \xi + x_d \phi(\xi))} d\xi \\ &= |\det A| \int_{\tau_j^s} f_j(A\xi) e^{i(A^t x' \cdot \eta + a x_d \phi^s(\eta))} d\eta \\ &= |\det A| \mathfrak{F}(f_j \circ A d\sigma)(Ax', ax_d), \end{aligned}$$

we have

$$\begin{aligned} \|\widehat{f_1 d\sigma f_2 d\sigma}\|_{L_p(Q)} &= |\det A|^2 (|\det A| a)^{-\frac{1}{p}} \|\mathfrak{F}(f_1 \circ A d\sigma) \mathfrak{F}(f_2 \circ A d\sigma)\|_{L_p(Q^s)} \\ &\leq C_s |\det A|^{2-\frac{1}{p}} a^{-\frac{1}{p}} \|f_1 \circ A\|_2 \|f_2 \circ A\|_2 \\ &= C_s |\det A|^{1-\frac{1}{p}} a^{-\frac{1}{p}} \|f_1\|_2 \|f_2\|_2. \end{aligned}$$

REMARK 3.8.1

Let $U \subset \mathbb{R}^n$ open, $\phi : U \rightarrow \mathbb{R}$ such that $\nabla\phi$ is bounded, and let $N(x) = (\nabla\phi(x), -1)$ and $N_0(x) = \frac{N(x)}{|N(x)|}$, $x \in U$. Then for all $x_1, x_2 \in U$ we have

$$|N_0(x_1) - N_0(x_2)| \approx |N(x_1) - N(x_2)| = |\nabla\phi(x_1) - \nabla\phi(x_2)|, \quad (3.8.3)$$

where the constants in the inequality only depend on $\|\nabla\phi\|_\infty$ and the dimension n .

Proof: Write $v_i = \nabla\phi(x_i)$, $c_i = |N(x_i)|$, $i = 1, 2$. Then $|v_i| \lesssim 1$ and

$$1 \leq c_i = |N(x_i)| = \sqrt{1 + |v_i|^2} \lesssim 1$$

for $i = 1, 2$. Observe further that

$$|c_1 - c_2| = \left| |N(x_1)| - |N(x_2)| \right| \leq |N(x_1) - N(x_2)| = |v_1 - v_2|. \quad (3.8.4)$$

Hence we have

$$\begin{aligned} |N_0(x_1) - N_0(x_2)| &= \frac{|c_2(v_1, -1) - c_1(v_2, -1)|}{c_1 c_2} \\ &\approx |c_2(v_1 - v_2, 0) - (c_1 - c_2)(v_2, -1)| \\ &\approx |c_2(v_1 - v_2) - (c_1 - c_2)(v_2)| + |c_1 - c_2|. \end{aligned}$$

Thus on the one hand,

$$\begin{aligned} |N_0(x_1) - N_0(x_2)| &\lesssim c_2|v_1 - v_2| + |c_1 - c_2|(|v_2| + 1) \\ &\lesssim |v_1 - v_2|. \end{aligned}$$

On the other hand

$$|N_0(x_1) - N_0(x_2)| \gtrsim |c_2|v_1 - v_2| - |c_1 - c_2||v_2| + |c_1 - c_2|.$$

In case we have $c_2|v_1 - v_2| \geq 2|c_1 - c_2||v_2|$, we can conclude

$$|N_0(x_1) - N_0(x_2)| \gtrsim \frac{1}{2}c_2|v_1 - v_2| + |c_1 - c_2| \gtrsim |v_1 - v_2|.$$

If the converse $c_2|v_1 - v_2| < 2|c_1 - c_2||v_2|$ holds true, then we estimate

$$|N_0(x_1) - N_0(x_2)| \gtrsim |c_1 - c_2| \gtrsim 2|c_1 - c_2||v_2| \gtrsim |v_1 - v_2|.$$

□

REMARK 3.8.2

Let $1 \leq p_0, p_1 \leq \infty$ and $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$, $t \in (0, 1)$. Then for all $F \in L_{p_0} \cap L_{p_1}$ we have $F \in L_{p_t}$ and

$$\|F\|_{p_t} \leq \|F\|_{p_0}^{1-t} \|F\|_{p_1}^t. \quad (3.8.5)$$

Proof: Application of Hölder's inequality yields

$$\|F\|_{p_t} = \| |F|^{1-t} |F|^t \|_{p_t} \leq \| |F|^{1-t} \|_{\frac{p_0}{1-t}} \| |F|^t \|_{\frac{p_1}{t}} = \|F\|_{p_0}^{1-t} \|F\|_{p_1}^t. \quad (3.8.6)$$

DEFINITION

For $\alpha \in \mathbb{N}^n$ let $M_\alpha = \{\beta \in \mathbb{N}^n : \beta_i \leq \alpha_i\}$ and let

$P_\alpha = \{p : M_\alpha \rightarrow \mathbb{N} : \sum_{\beta \in M_\alpha} |\beta| p_\beta = |\alpha|, p_0 = 0\}$. For every $p \in P_\alpha$, $m \in \mathbb{N}$ we define the

set $A_m(p) = \{(k^\beta)_{\beta \in M_\alpha} | k^\beta : \{1, \dots, p_\beta\} \rightarrow \{1, \dots, m\}\}$. We further introduce the symbol $p_k(x) = \prod_{\beta \in M_\alpha} \prod_{i=1}^{p_\beta} x_{k_i^\beta}$, $k \in A_m(p)$, $x \in \mathbb{R}^m$.

LEMMA 3.8.3 (FORMULA OF FAA DI BRUNO)

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ and $g \in C^\infty(U, V)$, $f \in C^\infty(V, \mathbb{R}^l)$. Then for every $\alpha \in \mathbb{N}^n$ there exists a sequence $\{c_p\}_{p \in P_\alpha} \in \mathbb{R}^{P_\alpha}$ such that

$$\partial^\alpha (f \circ g) = \sum_{p \in P_\alpha} c_p ([D^{\sum p_\beta} f] \circ g) (\bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta}). \quad (3.8.7)$$

REMARK

For $K = \sum_{\beta \in M_\alpha} p_\beta$, the application of the K -tensor $[D^{\sum p_\beta} f] \circ g$ to the K arguments

$\bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta}$ is defined by

$$([D^{\sum p_\beta} f] \circ g) (\bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta})(x) = \sum_{k \in A_m(p)} [(D^{\sum p_\beta} f)(g(x))]_{(k_1^\beta, \dots, k_{p_\beta}^\beta)_{\beta \in M_\alpha}} \prod_{\beta \in M_\alpha} \prod_{i=1}^{p_\beta} [\partial^\beta g(x)]_{k_i^\beta}.$$

Notice that the order of appearance of each term in the argument is unimportant due to the symmetry of the tensor.

Proof: The proof is by induction. The start at $\alpha = 0$ is obvious. Let $j = 1, \dots, n$. We have to prove that if the statement holds true for a certain α , then it holds true for $\alpha + e_j$ as well. We compute

$$\begin{aligned} \partial^{\alpha+e_j}(f \circ g) &= \partial_j \sum_{p \in P_\alpha} c_p([D^{\sum_{\beta \in M_\alpha} p_\beta} f] \circ g) \left(\bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta} \right) \\ &= \sum_{p \in P_\alpha} c_p([D^{\sum_{\beta \in M_\alpha} p_\beta+1} f] \circ g) (\partial_j g, \bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta}) \\ &\quad + \sum_{p \in P_\alpha} \sum_{\gamma \in M_\alpha, p_\gamma \neq 0} c_p([D^{\sum_{\beta \in M_\alpha} p_\beta} f] \circ g) \left(\bigotimes_{\beta \in M_\alpha, \beta \neq \gamma} [\partial^\beta g]^{p_\beta}, p_\gamma [\partial^\gamma g]^{p_\gamma-1}, \partial_j \partial^\gamma g \right). \end{aligned}$$

Extend p to $M_{\alpha+e_j}$: For $\beta \in M_{\alpha+e_j}$, define $\tilde{p}_\beta = p_\beta$ if $\beta \in M_\alpha$ and $\tilde{p}_\beta = 0$ if not. This is just an auxiliary construction; we do not claim that \tilde{p} is contained in $P_{\alpha+e_j}$. To handle the first part of the sum, define $\bar{p}_\beta = \tilde{p}_\beta$, if $\beta \neq e_j$, and $\bar{p}_\beta = \tilde{p}_\beta + 1$ if $\beta = e_j$. Then

$$\begin{aligned} \sum_{\beta \in M_{\alpha+e_j}} |\beta| \bar{p}_\beta &= |e_j| + \sum_{\beta \in M_{\alpha+e_j}} |\beta| \tilde{p}_\beta \\ &= 1 + \sum_{\beta \in M_\alpha} |\beta| p_\beta \\ &= |\alpha| + 1 = |\alpha + e_j|, \end{aligned}$$

hence $\bar{p} \in P_{\alpha+e_j}$. Notice further that $\sum_{\beta \in M_\alpha} p_\beta + 1 = \sum_{\beta \in M_{\alpha+e_j}} \bar{p}_\beta$ and

$$\begin{aligned} (\partial_j g, \bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta}) &= (\partial_j g, [\partial_j g]^{\tilde{p}_{e_j}}, \bigotimes_{\beta \in M_{\alpha+e_j}, \beta \neq e_j} [\partial^\beta g]^{\tilde{p}_\beta}) \\ &= ([\partial_j g]^{\tilde{p}_{e_j}}, \bigotimes_{\beta \in M_{\alpha+e_j}, \beta \neq e_j} [\partial^\beta g]^{\tilde{p}_\beta}) \\ &= \left(\bigotimes_{\beta \in M_{\alpha+e_j}} [\partial^\beta g]^{\bar{p}_\beta} \right). \end{aligned}$$

Thus

$$([D^{\sum_{\beta \in M_\alpha} p_\beta+1} f] \circ g) (\partial_j g, \bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta}) = ([D^{\sum_{\beta \in M_{\alpha+e_j}} \bar{p}_\beta} f] \circ g) \left(\bigotimes_{\beta \in M_{\alpha+e_j}} [\partial^\beta g]^{\bar{p}_\beta} \right),$$

which is of the desired form.

To deal with the second part, for a fixed $\gamma \in M_\alpha$ (i.e. $\gamma + e_j \in M_{\alpha+e_j}$) we define p^γ by

$$p_\beta^\gamma = \begin{cases} \tilde{p}_\gamma - 1, & \text{if } \beta = \gamma \\ \tilde{p}_{\gamma+e_j} + 1, & \text{if } \beta = \gamma + e_j \\ \tilde{p}_\beta, & \text{else.} \end{cases}$$

Then

$$\begin{aligned}
\sum_{\beta \in M_{\alpha+e_j}} |\beta| p_\beta^\gamma &= |\gamma| (\tilde{p}_\gamma - 1) + |\gamma + e_j| (\tilde{p}_{\gamma+e_j} + 1) + \sum_{\beta \in M_{\alpha+e_j}, \beta \neq \gamma, \gamma+e_j} |\beta| \tilde{p}_\beta \\
&= -|\gamma| + |\gamma + e_j| + \sum_{\beta \in M_{\alpha+e_j}} |\beta| \tilde{p}_\beta \\
&= 1 + \sum_{\beta \in M_\alpha} |\beta| p_\beta \\
&= |\alpha| + 1 = |\alpha + e_j|,
\end{aligned}$$

hence $p^\gamma \in M_{\alpha+e_j}$ and furthermore

$$[\partial^\gamma g]^{p_{\gamma-1}} = [\partial^\gamma g]^{p_\gamma^\gamma}$$

and $([\partial^{\gamma+e_j} g]^{\tilde{p}_{\gamma+e_j}}, \partial_j \partial^\gamma g) = [\partial^{\gamma+e_j} g]^{\tilde{p}_{\gamma+e_j}+1} = [\partial^{\gamma+e_j} g]^{p_{\gamma+e_j}^\gamma}$. Thus

$$([D^{\sum_{\beta \in M_\alpha} p_{\beta+1}} f] \circ g) \left(\bigotimes_{\beta \in M_\alpha, \beta \neq \gamma} [\partial^\beta g]^{p_\beta}, p_\gamma [\partial^\gamma g]^{p_{\gamma-1}}, \partial_j g \right) = ([D^{\sum_{\beta \in M_{\alpha+e_j}} p_{\beta+1}} f] \circ g) \left(\bigotimes_{\beta \in M_{\alpha+e_j}} [\partial^\beta g]^{p_\beta^\gamma} \right).$$

To summarise, we can choose $\bar{c}_p, p \in P_{\alpha+e_j}$ appropriate such that

$$\partial_j \partial^\alpha (f \circ g) = \sum_{p \in P_{\alpha+e_j}} \bar{c}_p ([D^{\sum_{\beta \in M_{\alpha+e_j}} p_\beta} f] \circ g) \left(\bigotimes_{\beta \in M_{\alpha+e_j}} [\partial^\beta g]^{p_\beta} \right).$$

□

DEFINITION

Let S be a surface in \mathbb{R}^d . A unit vector $n \in \mathbb{R}^d$ is called c -non-degenerated if $|\langle N(x), n \rangle| > c$ for every $x \in S$, where N is the unit normal field of S .

If H is a hypersurface in \mathbb{R}^d with normal vector n , $\iota : \mathbb{R}^{d-1} \rightarrow H$ is an ℓ_2 -isometry and $S = \{\iota(u) + \phi(u)n : u \in U\}$ for some $U \subset \mathbb{R}^{d-1}$, we say that S is the graph of $\phi : U \rightarrow \mathbb{R}$ over the hypersurface H .

LEMMA 3.8.4

Let $d \in \mathbb{N}, c, c_1 > 0$. Then there exists a sequence $C : \mathbb{N}^d \rightarrow \mathbb{R}_+, \alpha \rightarrow C_\alpha$ with the following property:

Let $\phi_0 \in C^\infty(U_0, \mathbb{R})$, $U_0 \subset \mathbb{R}^{d-1}$ with $|\nabla \phi_0| \leq c_1$, $S = \{(u, \phi_0(u)) | u \in U_0\}$, $A, r > 0$ such that $Ar \leq 1$, n a c -non-degenerated unit vector, and assume that S is the graph of ϕ over $H = n^\perp$. If

$$|\partial^\alpha \phi_0| \leq Ar^{|\alpha|} \tag{3.8.8}$$

for every $|\alpha| \geq 2$, then

$$|\partial^\alpha \phi| \leq C_\alpha Ar^{|\alpha|} \tag{3.8.9}$$

for every $|\alpha| \geq 2$.

Proof: S is the set of zeros of the function $f(u, t) = t - \phi_0(u)$. Obviously f satisfies (3.8.8) as well. (3.8.10)

Moreover we have an isometry $\iota(u) = \iota(\sum_{i=1}^{d-1} u_i e_i) = \sum_{i=1}^{d-1} u_i \iota(e_i)$ with $\partial_j \iota = \iota(e_j)$. Let $g(u) = \iota(u) + \phi(u)n$, then

$$\partial_j g = \iota(e_j) + \partial_j \phi n, \quad (3.8.11)$$

for all $j = 1, \dots, d$,

$$\partial^\alpha g = \partial^\alpha \phi n \quad (3.8.12)$$

for all $\alpha \in \mathbb{N}^d$, $|\alpha| \geq 2$ and

$$f(g(u)) = 0 \quad (3.8.13)$$

for all $u \in U$. The implicit function theorem then tells us

$$\begin{aligned} |\partial_j \phi(u)| &= |[\partial_n f]^{-1} \partial_{\iota(e_j)} f|(\iota(u) + \phi(u)n) \\ &\leq \frac{|\nabla f(\iota(u) + \phi(u)n)|}{|\langle \nabla f(\iota(u) + \phi(u)n), n \rangle|} \\ &\leq \frac{c_1}{c}. \end{aligned} \quad (3.8.14)$$

Now let $\alpha \in \mathbb{N}^d$, $|\alpha| > 0$ and assume that for every $\beta \in \mathbb{N}^d$ with $2 \leq |\beta| < |\alpha|$ inequality (3.8.9) holds true⁷. We apply the formula of Faa di Bruno (Lemma 3.8.3). Notice that there is just a single $\bar{p} \in P_\alpha$ with $\bar{p}_\alpha \neq 0$: Since $\sum_{\beta \in M_\alpha} |\beta| \bar{p}_\beta = |\alpha|$, we must have $\bar{p}_\alpha = 1$ and $\bar{p}_\beta = 0$ for any $\beta \neq \alpha$. Therefore

$$\begin{aligned} 0 &\stackrel{(3.8.13)}{=} \partial^\alpha (f \circ g) \\ &= \sum_{p \in P_\alpha} c_p ([D^{\sum p_\beta} f] \circ g) \left(\bigotimes_{\beta \in M_\alpha} [\partial^\beta g]^{p_\beta} \right) \\ &= \sum_{p \in P_\alpha, p_\alpha=0} c_p ([D^{\sum p_\beta} f] \circ g) \left(\bigotimes_{\beta \in M_\alpha, \beta \neq \alpha} [\partial^\beta g]^{p_\beta} \right) + c_{\bar{p}} ([\nabla f] \circ g) \cdot (\partial^\alpha g). \end{aligned}$$

Hence

$$\begin{aligned} c |\partial^\alpha \phi| &\leq |\langle (\nabla f) \circ g, n \rangle \partial^\alpha \phi| \\ &= |\langle (\nabla f) \circ g, \partial^\alpha g \rangle| \\ &\leq \frac{1}{|c_{\bar{p}}|} \sum_{p \in P_\alpha, p_\alpha=0} |c_p| | [D^{\sum p_\beta} f] \circ g | \prod_{\beta \in M_\alpha, \beta \neq \alpha} |\partial^\beta g|^{p_\beta}. \end{aligned} \quad (3.8.15)$$

Notice that $\beta \in M_\alpha, \beta \neq \alpha$ implies $|\beta| < |\alpha|$. Thus for $|\beta| \geq 2$ we have

$$|\partial^\beta g| \stackrel{(3.8.12)}{=} |\partial^\beta \phi| \leq C_\beta A r^{|\beta|}.$$

⁷The beginning of the induction on $|\alpha|$ is indeed included, since for $|\alpha| = 2$, the range of β 's is empty.

For $|\beta| = 1$ however, we do not have such an estimate. But we may apply (3.8.14) to see that $|\partial^\beta \phi| \lesssim 1$. Observe further that $p_0 = 0$ by definition, so we do not need to worry about $\beta = 0$. Finally, according to (3.8.10), we can estimate the derivatives of f by $|D^{\sum \beta} f| \leq Ar^{\sum \beta}$. We obtain for $p \neq \bar{p}$

$$\begin{aligned}
|[D^{\sum \beta} f] \circ g| \prod_{\beta \in M_\alpha, \beta \neq \alpha} |\partial^\beta g|^{p_\beta} &\lesssim Ar^{\sum \beta} \prod_{\beta \in M_\alpha, \beta \neq \alpha, |\beta| \geq 2} (Ar^{|\beta|})^{p_\beta} \\
&= A^{1 + \sum_{|\beta| \geq 2} p_\beta} r^{\sum_\beta p_\beta + \sum_{|\beta| \geq 2} |\beta| p_\beta} \\
&= A(Ar)^{\sum_{|\beta| \geq 2} p_\beta} r^{\sum_\beta |\beta| p_\beta} \\
&\leq Ar^{|\alpha|}.
\end{aligned}$$

Putting this bound into (3.8.15) gives the correct estimate. \square

Summary

My thesis deals with problems in Fourier restriction theory. The restriction problem associated to a hypersurface $S \subset \mathbb{R}^n$ is the question for which values p, q the adjoint restriction operator $R_S^*(f) = \widehat{f d\sigma_S}$ is bounded from $L^q(S)$ to $L^p(\mathbb{R}^n)$, where σ_S is the surface measure of S .

The introduction in Chapter 1 outlines the background of the problem and gives a brief survey of previous work by other authors and the development in this field, mainly for surfaces with nonvanishing Gaussian curvature. The surfaces discussed in this thesis are so-called surfaces of finite type in \mathbb{R}^3 , where the gaussian curvature is allowed to vanish at certain points, but the surface still has finite order of contact.

Part 2 discusses the case of a conical surface of finite type, which can be obtained by taking a cone and replacing the underlying curve, the circle, by a curve of finite type. This situation allows to adapt techniques applied in restriction estimates for curves. We describe the previous partial progress on this problem and give the complete solution of the question.

We first establish an approximation theorem, which allows to approximate the adjoint restriction operator by certain discrete operators. The discussion of the discrete operators eventually reduces to very concrete geometric problems. For some technical reasons, we even need to study more general problems. This leads to restriction inequalities with certain weighted measures, which turn out to include a version of the so-called affine arclength measure. It is known that making this detour with the affine arclength measure allows to pass to the desired estimates.

Part 3 is joint work with my advisor Prof. Dr. Detlef Müller and with Prof. Dr. Ana Vargas and discusses a different surface which contains a curve of finite type in every direction. It requires totally different techniques, namely the bilinear method, which means studying the bilinear operators $R_{S_1, S_2}^*(f_1, f_2) = \widehat{f_1 d\sigma_{S_1}} \widehat{f_2 d\sigma_{S_2}}$ for a whole family of certain pairs of subsurfaces (S_1, S_2) . Classically, when dealing with the paraboloid, for instance, the family can be reduced to a single pair of subsurfaces by rescaling and affine transformations. However, our surface lacks enough homogeneity for such transformations. Therefore we modify and generalise recently developed bilinear techniques to obtain more quantitative versions to meet our requirements. We also briefly discuss the sharpness of the bilinear results. Finally we pass from the bilinear estimates to a new result for the original linear operator.

Zusammenfassung

Meine Arbeit behandelt Probleme aus der Theorie der Fourierrestriktionssätze. Dabei geht es um die Frage, für welche Werte p, q der adjungierte Restriktionsoperator $R_S^*(f) = \widehat{f d\sigma_S}$ einer Hyperfläche $S \subset \mathbb{R}^n$ beschränkt ist von $L^q(S)$ nach $L^p(\mathbb{R}^n)$, wobei σ_S das Oberflächenmaß von S ist.

Die Einführung in Kapitel I erläutert den Hintergrund des Problems und gibt einen kurzen Abriss der Arbeit anderer Autoren und der Entwicklung in diesem Gebiet, hauptsächlich für Flächen mit nichtverschwindender Gaußkrümmung. Die Flächen, die in dieser Arbeit behandelt werden, sind sogenannte Flächen von endlichem Typ im \mathbb{R}^3 , bei denen die Gaußkrümmung zwar in gewissen Punkten verschwinden darf, aber die Fläche immer noch endliche Kontaktordnung besitzt.

Teil I behandelt den Fall einer kegelartigen Fläche von endlichem Typ, die man erhält, indem man bei einem Kegel die zugrundeliegende Kurve, den Kreis, durch eine Kurve von endlichem Typ ersetzt. Diese Situation erlaubt es einem, Techniken, wie sie auch für Restriktionsabschätzungen für Kurven benutzt werden, zu adaptieren. Wir beschreiben den Stand der Forschung für dieses Problem und liefern die vollständige Lösung der Frage. Zunächst entwickeln wir einen Approximationssatz, welcher uns erlaubt, den adjungierten Restriktionsoperator durch gewisse diskrete Operatoren anzunähern. Die Behandlung dieser diskreten Operatoren kann letztendlich auf sehr anschauliche geometrische Fragestellungen zurückgeführt werden. Aufgrund gewisser technischer Hindernisse müssen dabei sogar allgemeinere Probleme studiert werden. Dies führt zu Restriktionsabschätzungen mit gewissen gewichteten Maßen. Eines dieser Maße stellt sich als eine Variation des sogenannten affinen Bogenmaßes heraus. Es ist bekannt, dass der Umweg über dieses affine Bogenmaß auch wieder zu der gewünschten Abschätzung führt.

Teil II ist eine gemeinsame Arbeit mit meinem Betreuer Prof. Dr. Detlef Müller und mit Prof. Dr. Ana Vargas und behandelt eine weitere Fläche, die in jeder Richtung eine Kurve von endlichem Typ enthält. Dafür werden komplett andere Techniken benötigt, wie etwa die bilineare Methode. Dabei studiert man die bilinearen Operatoren $R_{S_1, S_2}^*(f_1, f_2) = \widehat{f_1 d\sigma_{S_1}} \widehat{f_2 d\sigma_{S_2}}$ für eine ganze Familie von gewissen Paaren von Teilflächen (S_1, S_2) . In der klassischen Theorie, etwa für den Paraboloiden, kann diese Familie durch Skalierung und affine Transformationen auf ein einzelnes Paar von Teilflächen zurückgeführt werden. Unserer Fläche fehlt es jedoch an der nötigen Homogenität für diese Transformationen. Daher modifizieren und verallgemeinern wir bisherige bilineare Techniken, um quantitative Versionen zu erhalten, die unseren Anforderungen genügen. Wir geben ferner eine kurze Erläuterung zur Schärfe der bilinearen Abschätzung. Schlussendlich folgern wir aus den bilinearen Abschätzungen ein neues Resultat für den ursprünglichen linearen Operator.

Bibliography

- [ACK] G. I. Arkhipov, V. N. Chubarikov, A. A. Karatsuba , Trigonometric integrals, Mathematics of the USSR Izvestiya 15 No2 (1980), 211-239
- [Ba1] B. Barcelo Taberner, On the restriction of the Fourier transform to a conical surface, Transactions of the American Mathematical Society, 292 No1 (1985)
- [Ba2] B. Barcelo, The restriction of the Fourier transform to some curves and surfaces, Studia Mathematica 84 No1 (1986), 39–69
- [Bo1] J. Bourgain, Besicovitch-type maximal operators and applications to Fourier analysis, Geometric and Functional Analysis 22 (1991), 147-187+
- [Bo2] J. Bourgain, Estimates for cone multipliers, Operator Theory: Advances and Applications 77 (1995), 1-16
- [BG] J. Bourgain, L. Guth, Bounds on oscillatory integral operators based on multilinear estimates, Geometric and Functional Analysis, Vol.21 (2011) 1239-1295
- [C] A. Córdoba, The Kakeya maximal function and the spherical summation multipliers, American Journal of Mathematics 99 (1977), 1-22
- [DM] S. W. Drury, B. P. Marshall, Fourier restriction theorems for curves with affine and Euclidean arclengths, Mathematical Proceedings of the Cambridge Philosophical Society 97 (1985), 111-125
- [F] C. Fefferman, The multiplier problem for the ball, Annals of Mathematics 94 (1971) 330-336
- [FU] E. Ferreyra, M. Urciuolo, Fourier restriction estimates to mixed homogeneous surfaces, Journal of inequalities in pure and applied mathematics, Vol. 10 (2009) Issue 2, Article 35, pp. 11
- [Gra] L. Grafakos, Classical Fourier Analysis, Graduate Texts in Mathematics, Springer, 2008
- [Gre] A. Greenleaf, Principal Curvature and Harmonic Analysis, Indiana University Mathematics Journal, Vol. 30, No. 4 (1981)
- [GW] P. M. Gruber, J. M. Wills, Handbook of convex geometry, North-Holland (1993)

- [IKM] I. Ikromov, M. Kempe, D. Müller, Estimates for maximal functions associated with hypersurfaces in \mathbb{R}^3 and related problems in harmonic analysis, *Acta Mathematica* 204 (2010), 151-271
- [L1] S. Lee, Bilinear restriction estimates for surfaces with curvatures of different signs, *Transactions of the American Mathematical Society*, Vol. 358, No. 8, 3511-2533, 2005
- [L2] S. Lee, Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators, *Duke Mathematical Journal*, Volume 122, Number 1 (2004), 1-232
- [LV] S. Lee, A. Vargas, Restriction estimates for some surfaces with vanishing curvatures, *Journal of Functional Analysis* 258 (2010), 2884-2909
- [S] C. D. Sogge, A sharp restriction theorem for degenerate curves in \mathbb{R}^2 , *American Journal of Mathematics*, Vol. 109, No. 2 (1987), pp. 223-228
- [St] E. Stein, *Harmonic Analysis*, Princeton University Press (1993)
- [T1] T. Tao, A Sharp bilinear restriction estimate for paraboloids, *Geometric and Functional Analysis* 13, 1359-1384, 2003
- [T2] T. Tao, Recent progress on the Restriction conjecture, arXiv:math/0311181v1 [math.CA] 12 Nov 2003
- [T3] T. Tao, From Rotating Needles to stability of waves: emerging connections between combinatorics, analysis, and PDE, *Notices of the American Mathematical Society*, 48 (2001)
- [TV1] T. Tao, A. Vargas, A bilinear approach to cone multipliers I. Restriction estimates, *Geometric and Functional Analysis* 10, 185-215, 2000
- [TV2] T. Tao, A. Vargas, A bilinear approach to cone multipliers II. Applications, *Geometric and Functional Analysis* 10, 216-258, 2000
- [V] A. Vargas, Restriction theorems for a surface with negative curvature, *Mathematische Zeitschrift* 249, 97-111 (2005)
- [W] T. Wolff, A Sharp Bilinear Cone Restriction Estimate, *The Annals of Mathematics*, Second Series, Vol. 153, No. 3, 661-698, 2001
- [Z] A. Zygmund, On Fourier coefficients and transforms of functions of two variables, *Studia Mathematica* 50 (1974), 189-201

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Abhandlung - abgesehen von der Beratung durch meinen Betreuer und den Beiträgen meiner Mitautoren im zweiten Teil meiner Arbeit (gemäß der Erklärung meines Betreuers) - inhaltlich und der Form nach unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht.

Kiel, den 22. April 2014

(Stefan Buschenhenke)