

On the antichain structure of compact tree forcings

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Zusammenfassung

Wir untersuchen die Strukturen der klassischen kompakten Baum-Forcings von Sacks und Silver, sowie von dem Spaltungsbaum Forcing. Besonderes Augenmerk liegt dabei auf den Strukturen sowie den kardinalen Koeffizienten der zugehörigen Forcing Ideale und maximalen Antiketten.

Hierbei wurden folgende Resultate bewiesen

- Es wurde der Nachweis der Unabhängigkeit der Überdeckungszahlen des Sacks Forcing Ideals und des Spaltungsbaum Forcing Ideals geführt.
- Die $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$ - nirgends Distributivität des Spaltungsbaum Forcings wurde in ZFC bewiesen.
- Es findet sich in der Arbeit ein Nachweis der relativen Konsistenz von ZFC zu $(ZFC + \neg CH + \text{es existieren kurze Antiketten des Spaltungsbaum Forcings})$.
- Die dominating number \mathfrak{d} wird als untere Schranke für die Antikettenzahl des Silver Forcings nachgewiesen.

Abstract

We investigate the structures of the classical compact tree forcings of Sacks and Silver, as well as of the splitting tree forcing. We focus on the study of the structures and the cardinal coefficients of their respective forcing ideals and maximal antichains.

The following results are proven.

- A proof is given for the independence of the covering numbers of the Sacks forcing ideal and the splitting tree forcing ideal.
- The $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$ –nowhere distributivity of the splitting tree forcings is shown in *ZFC*.
- This thesis contains the proof for the relative consistency of *ZFC* and $(ZFC + \neg CH + \text{there exist short maximal antichains of the splitting tree forcing})$.
- We show that the dominating number is a lower bound for the antichain number of the Silver forcing.

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1 Introduction and basic definitions

In the first subsection of this chapter a short introduction to the topic of this PhD thesis is given. It also contains the motivation behind the results of this thesis and how they are connected to the current state of research. The following subsections introduce some basic concepts together with their respective definitions and some notations that are used in this thesis. We assume that the reader is familiar with the basic concepts behind set theory and forcing.

Introduction

Forcing is one of the main methods to produce independence results in modern set theory. A very common type of forcings that occur frequently is tree forcing. A classical tree forcing consists of a certain class of trees that are subsets of $2^{<\omega}$ or $\omega^{<\omega}$. The closure of those trees is a closed set in the Cantor space or in the Baire space respectively, thus a certain type of tree forcing often characterizes a certain type of closed sets with special analytical or combinatorial properties. Probably, the most known example is the Sacks forcing, whose tree closures are exactly the perfect sets of the Cantor space.

Another example is the splitting tree forcing. O. Spinas found that it characterizes the analytic countably splitting sets of the Cantor Space (see [Spi04]).

He also found the following canonization result for Borel functions $f : (2^\omega)^n \rightarrow 2^\omega$. Whenever we have some splitting block $\bar{p} = \prod_{i < n} p_i$, with p_i is a splitting tree, then there exists a splitting tree $q_i \subseteq p_i$ for each $i < n$ and some $E \subseteq n$ such that the following holds. For all $x, y \in \prod_{i < n} [q_i]$ we have $f(x) = f(y) \Leftrightarrow x_i = y_i$ for all $i \in E$.

In other words, for every Borel function and every such splitting block we find a splitting subblock, such that the function is either one to one or constant with respect to one side of the subblock ([Spi07]).

A lot of the fusion techniques for splitting trees that are used in this thesis were developed in his paper. The class of splitting trees is a subclass of Sacks trees and seems to be quite similar. Unfortunately, the fusion techniques are much more complicated than those for Sacks trees.

The starting point of my research was to analyze how similar those types of trees were with respect to forcing. One aspect, which is important to understand the structure of a tree forcing, is the structure of its respective forcing ideal. Hence, the first thing that is shown is that the splitting tree forcing ideal is neither a subset of the Sacks forcing ideal nor the Silver forcing ideal and vice versa. This is similar to the analysis that J. Brendle did for several other tree forcings ([Bre95]).

The next question that comes up is, whether the ideals are different in

terms of cardinal coefficients. Take for example the covering number, which is the smallest number of ideal sets that are needed to cover the whole Cantor space and is denoted by $cov(I)$ with I being the respective ideal. H. Judah A. Miller and S. Shelah have shown that by starting with a $ZFC + CH$ model and doing a countable support iteration of Sacks forcing of length ω_2 , one obtains a model that satisfies $\mathfrak{c} = cov(I(\mathbb{S})) = \aleph_2$.

In chapter 3 I use this result along with a result by J.E. Baumgartner and R. Laver [BL79] and the techniques mentioned above by O. Spinas. As a result, I can show that the covering numbers of the ideals of the splitting tree forcing and the Sacks forcing respectively are independent from each other. That means that there exists no ZFC provable connection between these numbers. The relation of other coefficients of these ideals is still not known.

P. Simon has shown that the Sacks forcing is $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$ -nowhere distributive, which implies that it collapses \mathfrak{c} to \mathfrak{b} ([Sim93]). Around the same time A. Roslanowski and S. Shelah have shown that Sacks forcing collapses \mathfrak{c} to $\mathfrak{b}^{+\epsilon}$, which is between \mathfrak{b} and \mathfrak{d} ([RS96]). In chapter 4 I show a similar result for splitting tree forcing. Actually, splitting tree forcing is $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$ -nowhere distributive.

Other important aspects of forcings are the structures and possible cardinalities of their maximal antichains. It was possible to use similar ideas as in [RS96] to obtain a model, where the continuum hypothesis fails but short uncountable maximal antichains of the splitting tree forcing exist.

For Silver forcing this proof can not be reproduced in the same manner. Inspired by a question of G. Laguzzi ([Lag13]) and others, whether $add(I(S_i)) \leq \mathfrak{b}$ holds true, O. Spinas and I analyzed the structure of Silver antichains. We conjectured that the smallest size of an uncountable Silver antichain is \mathfrak{c} . We were able to prove this to some extent. O. Spinas found that maximal antichains that consist solely of Silver functions with infinite domain have to be of size \mathfrak{c} . He could also use this theorem to answer the question of G. Laguzzi positively. Furthermore, note the following. Given two incompatible Silver functions with infinite domains, the reason for their incompatibility can either be that they have different values for a natural that is in both of their domains, or they are identical on the common part of their domains but they are incompatible due to their codomains being almost disjoint. We were able to show that an antichain of size smaller than \mathfrak{c} (with finite and infinite conditions) that has an uncountable subset of conditions that are incompatible only due to the second reason, is not maximal. In conclusion, I could show that uncountable maximal antichains of Sil-

ver forcing have to be at least of size \mathfrak{d} . These results for the Silver forcing are presented in the last chapter of the PhD thesis and will also be published in an article that is about to be released ([SW]).

Tree forcings

A common type of forcings are *tree forcings*, whose members are usually subtrees T of 2^ω or $\omega^{<\omega}$ that feature a certain property that defines the respective forcing and which is ordered by inclusion. By $[T]$ we denote the set of all infinite branches of T . $[T]$ is a closed set in the Cantor space or in the Baire space respectively and often certain types of forcings are used to characterize closed sets in the Cantor space or Baire space with special analytical and/or combinatorial properties.

We define $T(n) := \{\sigma \in T : |\sigma| = n + 1\}$ and $T \upharpoonright n := \{\sigma \in T : |\sigma| \leq n\} = \bigcup_{k < n} T(k)$. We denote by $T[n]$ the values that the nodes of T have on the n -th level of T , i.e. $T[n] := \{\sigma(n) : \sigma \in T(n)\}$. For any $\sigma \in T$ let $T \upharpoonright \sigma T := \{\tau \in T : \tau \subseteq \sigma \vee \tau \supseteq \sigma\}$ be the subtree of T consisting of all nodes that are initial segments or extensions of σ (sometimes we use the notation p_σ). For a given tree T let

$sp(T) := \{\sigma \in T : \exists i, j (i \neq j \wedge \sigma \hat{\ } i, \sigma \hat{\ } j \in T)\}$ be the set of all splitting nodes of T .

Furthermore, if we have a subset X of $2^{<\omega}$ or $\omega^{<\omega}$, by $dwcl(X)$ we denote the downward closure of X , which is the tree $\{\sigma \in 2^{<\omega} : \exists \tau \in X \sigma \subseteq \tau\}$ or $\{\sigma \in \omega^{<\omega} : \exists \tau \in X \sigma \subseteq \tau\}$ respectively.

If we have some uncountable $X \subseteq 2^\omega$ or ω^ω then

$cond(X) := \{\sigma \in 2^{<\omega} : |\{x \in X : \sigma \subseteq x\}| \geq \aleph_1\}$ or

$\{\sigma \in \omega^{<\omega} : |\{x \in X : \sigma \subseteq x\}| \geq \aleph_1\}$ respectively is the condensation tree of X . It is easy to see that then $X \setminus [cond(X)]$ is countable and $cond(X)$ is a perfect tree, i.e. $\forall \sigma \in T \exists \tau \in T (\tau \supseteq \sigma \wedge \tau \in sp(T))$.

The tree forcings that are the main object of this thesis are the Sacks forcing \mathbb{S} , the splitting tree forcing S and the Silver forcing Si . A short introduction to each of these forcings will be given.

Definition 1.1. The *Sacks forcing* (\mathbb{S}, \leq) is defined by:

$$p \in \mathbb{S} :\Leftrightarrow p \subseteq 2^{<\omega} \text{ is a tree } \wedge p \text{ is perfect,} \\ \text{i.e. } \forall \sigma \in T \exists \tau \in p (\tau \supseteq \sigma \wedge \tau \in sp(p)).$$

For $p, q \in \mathbb{S}$ we define $p \leq q :\Leftrightarrow p \subseteq q$.

Remark. It is easy to see that the perfect sets of the Cantor space are exactly the closures of members of the Sacks forcing.

\mathbb{S} is a quite natural tree forcing and several other tree forcings, in particular the ones that will be of interest in this thesis, are subforcings of \mathbb{S} .

Definition 1.2. The *splitting tree forcing* S is defined by:

$$p \in S :\Leftrightarrow p \subseteq 2^{<\omega} \text{ is a tree} \\ \wedge \forall \sigma \in p \exists K \in \omega \forall n \geq K \exists \tau_0, \tau_1 \in p (\tau_0, \tau_1 \supseteq \sigma \wedge \tau_0(n) = 0 \wedge \tau_1(n) = 1).$$

For $p, q \in S$ we define $p \leq q :\Leftrightarrow p \subseteq q$.

Remark. Note that the defining property of splitting trees implies that every splitting tree is a perfect tree.

If we have given a splitting tree p and a node $\sigma \in p$, we denote by $K_p(\sigma)$ the smallest possible witness for the splitting property of p with respect to σ .

As found by O. Spinas (see [Spi04]) splitting trees can be used to characterize the *analytic countably splitting subsets* of $[\omega]^\omega$.

Definition 1.3. A subset $X \subseteq [\omega]^\omega$ is called *countably splitting* iff for every countable $A \subseteq [\omega]^\omega$ there exists $b \in X$ that *splits* all $a \in A$, i.e. $\forall a \in A (|a \cap b| = \omega \wedge |a \cap b^c| = \omega)$.

By identifying members of $[\omega]^\omega$ with their respective characteristic functions ($\in 2^\omega$) one can easily see that the closures of splitting trees are closed countably splitting sets and in addition we get the following theorem.

Theorem 1.4 (O. Spinas 2004). *An analytic subset X of the Cantor space is countably splitting, if and only if it contains the closure of a splitting tree.*

If we have $a, b \in [\omega]^\omega$ and a is not splitting b , i.e. $b \subseteq^* a$ or $b \subseteq^* a^c$ we say b *refines* a .

The third relevant tree forcing in this thesis is the Silver forcing.

Definition 1.5. The *Silver forcing* S_i is defined by:

$$p \in S_i :\Leftrightarrow p \subseteq 2^{<\omega} \text{ is perfect and for all nodes } \sigma, \sigma' \in p \text{ of the same length and all } i \in 2 \text{ we have } \sigma \frown i \in p \Leftrightarrow \sigma' \frown i \in p, \text{ i.e. on each level}$$

either all nodes go left, all nodes go right or all nodes split at that level.

For $p, q \in Si$ we define $p \leq q :\Leftrightarrow p \subseteq q$.

Remarks. We will identify a Silver tree with a function $f : dom(f) \rightarrow 2$ with $dom(f) \subseteq \omega$ and infinite codomain. Such f represents the Silver tree consisting of all nodes $\sigma \in 2^{<\omega}$ with $\sigma(n) = f(n)$ for $n \in dom(f)$. The respective ordering is reverse inclusion. Sometimes it is convenient to identify such f with $f' \in (2 \cup \{*\})^\omega$ with $f'^{-1}[\{*\}]$ being the codomain of f .

Note the following important fact. All the above mentioned tree forcings satisfy axiom A. The fusion order that witnesses axiom A is defined for \mathbb{S} and Si by $p \leq_n q :\Leftrightarrow (p \leq q \wedge \forall \sigma n\text{-th splitting node of } p (\sigma \in q))$.

For a fusion order witnessing axiom A for S we point to chapter 3. This implies the properness of all three forcings, thus these forcings all preserve the cardinality of \aleph_1 . Moreover, all these forcings have the size of the continuum \mathfrak{c} . Hence, in a ground model that satisfies the continuum hypothesis, the forcings satisfy the \aleph_2 -chain condition and therefore also preserve all cardinals. Because properness is preserved by countable support iterations this is also true if we take a model of $ZFC + CH$ and do a countable support iteration of length \aleph_2 . (See [She98] for preservation results for proper forcings.)

Forcing ideals and cardinal coefficients

Given a tree forcing $P \subseteq \mathfrak{P}(2^{<\omega})$ we can define the ideal of “small” subsets of the Cantor space with respect to this forcing.

Definition 1.6. Let P be a forcing, whose members are subtrees of $2^{<\omega}$, ordered by inclusion. Then we define the forcing ideal $I(P)$ by

$$I(P) := \{X \subseteq 2^\omega : \forall p \in P \exists q \leq p [q] \cap X = \emptyset\}.$$

As with other ideals on the Cantor space we can analyze the cardinal coefficients of this forcing ideals. In this thesis we will only need the covering number, which is defined in the following way.

Definition 1.7. Let Y be a set and $I \subseteq \mathfrak{P}(Y)$ be an ideal, then the *covering number* of the ideal $cov(I)$ is defined by

$$cov(I) := \min\{|\mathfrak{F}| : \mathfrak{F} \subseteq I \wedge \bigcup \mathfrak{F} = Y\},$$

i.e. the smallest cardinality of a family \mathfrak{F} of ideal sets, whose union covers the whole space Y .

To understand the structure of forcings and forcing ideals it is also helpful to study their antichains. Recall that two members p and q of a forcing P are called compatible ($p \parallel q$), if there exists $r \in P$ with $r \leq p$ and $r \leq q$. Else we say that p and q are incompatible ($p \perp q$). An antichain is a subset of the forcing, such that any two of its members are incompatible. An antichain is maximal if every member of the forcing is compatible to at least one element of the antichain.

For every $n \in \omega$ let $f_n := \{(i, 0) : i < n\} \cup \{(n, 1)\} \in Si$. Then it is easy to see that an arbitrary Silver function f is compatible to f_n with n being the minimal natural number, such that either $f(n) = 1$ or $n \notin \text{dom}(f)$. Hence, $\{f_n : n \in \omega\}$ forms a countable maximal antichain. To get a maximal Sacks antichain define $p_n := 2^{<\omega} \upharpoonright f_n$ for every $n \in \omega$. Given $p \in \mathbb{S}$, we have $p \parallel p_n$ with n minimal, such that $1 \in p[n]$. In addition, we have $p_n \perp p_m$ for $m \neq n$. By the same reasons $\{p_n : n \in \omega\}$ is also a maximal antichain with respect to the splitting tree forcing.

Hence, for these forcings we define the *antichain number* in the following way.

Definition 1.8. Let P be either Sacks, Silver or splitting tree forcing. We define the *antichain number* of P by

$$\mathfrak{a}(P) := \min\{|A| : A \text{ is a maximal antichain} \wedge |A| \geq \aleph_1\}.$$

One of the main results of chapter 5 will be that $\mathfrak{a}(Si)$ is at least of the same size as the *dominating number* \mathfrak{d} .

Definition 1.9. The *dominating number* \mathfrak{d} is the smallest cardinality of a dominating family of functions of ω^ω , i.e

$$\mathfrak{d} := \min\{|\mathfrak{F}| : \mathfrak{F} \subseteq \omega^\omega \wedge \forall g \in \omega^\omega \exists f \in \mathfrak{F} \forall^\infty n \in \omega g(n) < f(n)\}.$$

2 Set theoretic inclusions between the splitting tree forcing ideal and other forcing ideals

One of the most fundamental questions one may ask with respect to the interplay of two different types of tree forcings is if one of the forcing ideals is a subset of the other. Jörg Brendle has already shown in ZFC that the Sacks forcing ideal is not a subset of the Silver forcing ideal [Bre95]. Hence, before we investigate the cardinal invariants of the splitting tree forcing and its respective ideal, it is shown in this chapter that in every model of ZFC the splitting forcing ideal is not a subset of the afore mentioned two ideals and vice versa. Brendle also mentions without giving a proof that there exists a member of the Silver forcing ideal that is not an element of the Sacks ideal. For completeness sake a small proof of this fact is included in this chapter as well.

We start with the following well known fact:

Proposition 2.1. $I(Si) \not\subseteq I(\mathbb{S})$

Proof. Let $T \in \mathbb{S}$ be some skew tree. It suffices to show that $[T] \in I(Si) \setminus I(\mathbb{S})$.

Because T is a skew tree it is pretty obvious that T does not contain a Silver tree as a subset. Hence, for every $p \in Si$ we have $\sigma \in p$ such that $\sigma \notin T$, which implies that $[p \upharpoonright \sigma] \cap [T] = \emptyset$.

We can conclude that $[T] \in I(Si)$.

On the other hand $T \in \mathbb{S}$ witnesses that $[T]$ is not a member of $I(\mathbb{S})$, thus the claim is proven. \square

Now, let us compare the Silver ideal and the splitting ideal.

Proposition 2.2. $I(S) \not\subseteq I(Si)$

Proof. Let f be a Silver function with $|\text{dom}(f)| = \omega$ and let $T \in Si$ be the respective tree.

We have $\forall p \subseteq T (p \in Si \rightarrow [p] \cap [T] \neq \emptyset)$.

That means $[T] \notin I(Si)$.

Now let $q \in S$ be an arbitrary splitting tree. Pick some $n \in \text{dom}(f)$ with $n \geq K_q(\emptyset)$. Then there exists, by definition of $K_q(\emptyset)$, $\sigma \in q$ with $|\sigma| \geq n + 1$ and $\sigma(n) = 1 - f(n)$. In particular we have $\sigma \notin T$, which implies $[q \upharpoonright \sigma] \cap [T] = \emptyset$.

We can conclude that $[T] \in I(S)$. \square

Proposition 2.3. $I(Si) \not\subseteq I(S)$

Proof. Let $\langle p_\zeta : \zeta < 2^{\aleph_0} \rangle$ be an enumeration of all Silver trees (with $\langle f_\zeta : \zeta < 2^{\aleph_0} \rangle$ being the respective Silver functions) and let $\langle q_\xi : \xi < 2^{\aleph_0} \rangle$ be an enumeration of all splitting trees.

For a Silver tree p with a Silver function f we say p is of *Type(Even)*, if all $n \in \omega \setminus \text{dom}(f)$ are even, and p is of *Type(Odd)*, if all $n \in \omega \setminus \text{dom}(f)$ are odd.

Because the set of Silver trees that are either of *Type(Even)* or of *Type(Odd)* is dense in *Si* we can assume without restriction that for all $\zeta < 2^{\aleph_0}$ we have that p_ζ is of one of these types.

Let $\xi \in 2^{\aleph_0}$ be arbitrary.

We construct a perfect subtree $\overline{q_\xi} \subseteq q_\xi$ in the following way:

For $n \in \omega$ and $(s, t) \in 2^n \times 2^n$ we pick $\sigma_{s,t} \in q_\xi$ by recursion.

$n = 0$:

We define $\sigma_{\emptyset, \emptyset} := \emptyset$.

$n \rightsquigarrow n + 1$:

Let $\sigma_{s,t}$ be constructed for any $(s, t) \in 2^k \times 2^k$ with $k \leq n$.

Let $k_n \geq \max\{N \in \omega : \exists (s, t) \in 2^n \times 2^n (N = K_{q_\xi}(\sigma_{s,t}))\}$ with k_n even.

For every $(s, t) \in 2^n \times 2^n$ and every $i, j \in 2$ pick $\sigma'_{s \smallfrown i, t \smallfrown j} \in q_\xi$ with $\sigma'_{s \smallfrown i, t \smallfrown j} \supseteq \sigma_{s,t}$, $|\sigma'_{s \smallfrown i, t \smallfrown j}| = k_n + 1$ and $\sigma'_{s \smallfrown i, t \smallfrown j}(k_n) = i$.

Let $l_n \geq \max\{N \in \omega : \exists (s, t) \in 2^n \times 2^n \exists i, j \in 2 (N = K_{q_\xi}(\sigma'_{s \smallfrown i, t \smallfrown j}))\}$ with l_n odd.

For every $(s, t) \in 2^n \times 2^n$ and every $i, j \in 2$ pick $\sigma_{s \smallfrown i, t \smallfrown j} \in q_\xi$ with $\sigma_{s \smallfrown i, t \smallfrown j} \supseteq \sigma'_{s \smallfrown i, t \smallfrown j}$, $|\sigma_{s \smallfrown i, t \smallfrown j}| = l_n + 1$ and $\sigma_{s \smallfrown i, t \smallfrown j}(l_n) = j$.

Let $\overline{q_\xi} := \text{dwcl}\{\sigma_{s,t} : \exists n \in \omega (s, t) \in 2^n \times 2^n\}$.

Now, for any $f \in 2^\omega$ we denote by $\overline{q_\xi}^f$ the downward closure of the set of all $\sigma_{s,t}$ with $s \subseteq f$.

Note that for all $\sigma \in \overline{q_\xi}^f$ and $n \in \omega$ with $k_n < |\sigma|$ we have $\sigma(k_n) = f(n)$.

Hence, we can conclude that for all $p \in \text{Si}$ of *Type(Odd)* we have $|\{f \in 2^\omega : [p] \cap [\overline{q_\xi}^f] \neq \emptyset\}| \leq 1$.

Thus, we can pick some $f \in 2^\omega$ with

$\forall \zeta < \xi$ (p_ζ is of Type(*Odd*) $\rightarrow [p_\zeta] \cap [\overline{q_\xi^f}] = \emptyset$).

By construction of $\overline{q_\xi^f}$ we have, for any $\sigma \in \text{split}(\overline{q_\xi^f})$, that $|\sigma|$ is even. Therefore $\forall p \in Si$ (p is of Type(*Even*) $\rightarrow |[p] \cap [\overline{q_\xi^f}]| \leq 1$).

Because of this we can pick some x_ξ with $\forall \zeta < \xi$ ($x_\xi \notin [p_\zeta]$).

Define $X := \{x_\xi : \xi < 2^{\aleph_0}\}$.

For any $\zeta < 2^{\aleph_0}$ we have $X_\xi \notin [p_\zeta]$ for all ξ with $\zeta < \xi < 2^{\aleph_0}$. It follows $|X \cap [p_\zeta]| < 2^{\aleph_0}$. Obviously, every Silver tree can be decomposed into continuum many Silver subtrees with pairwise disjoint closure. Hence, we can find a Silver subtree $\overline{p_\xi} \subseteq p_\xi$ such that $[\overline{p_\xi}] \cap X = \emptyset$. We conclude $X \in I(Si)$.

On the other hand, for any $q \in S$, we have $X \cap [q] \neq \emptyset$ by construction. Thus, $X \notin I(S)$. □

Proposition 2.4. $I(S) \not\subseteq I(\mathbb{S})$

Proof. Pick some $p \in \mathbb{S}$ that does not contain a splitting tree as a subset (for example pick a perfect tree such that every node has a fixed value 0 on even levels or alternatively take a perfect Type(i_0, i_1) tree for some $i_0, i_1 \in 2$ (see the following definition/remark and Proposition 2.8)).

By a result of O.Spinas an analytic subset of 2^ω is countably splitting, if and only if it contains the closure of a splitting tree as a subset (see [Spi04]).

Thus, we know that $[p]$ is not countably splitting. We have a countable set $A \subseteq [\omega]^\omega$ such that there is no $x \in [p]$ that splits all members of A .

By the same result, for any $q \in S$ we have that $[q]$ is countably splitting. By definition for any $q \in S$ we have that

$Q_q := [q] \cap \{x \in 2^\omega : x \text{ splits } A\}$ is countably splitting as well and therefore contains the closure of some splitting subtree $\overline{q} \subseteq q$. Because $[\overline{q}] \subseteq Q$ we have $[\overline{q}] \cap [p] = \emptyset$.

We can conclude that $[p] \in I(S)$.

On the other hand $p \in \mathbb{S}$, hence $[p] \notin I(\mathbb{S})$. □

Before we can answer the question, whether a member of the Sacks

ideal exists that is not an element of the splitting ideal, we need to make some preparations.

Definition 2.5. Let $p \in \mathbb{S}$ be a perfect tree and let $i_0, i_1 \in 2$. We say p is of Type (i_0, i_1) , if the following holds true:

For all $\sigma \in \text{split}(p)$ and for all $x_0, x_1 \in [p]$ with $x_0 \supseteq \sigma \hat{\ } 0$ and $x_1 \supseteq \sigma \hat{\ } 1$ we have:

$$\forall j \in 2 \forall \tau \in \text{split}(p) (\sigma \hat{\ } j \subseteq \tau \subseteq x_j \rightarrow x_{1-j}(|\tau|) = i_j).$$

Definition 2.6. Let $p \in \mathbb{S}$ be a perfect tree. Let $\langle \sigma_s : s \in 2^{<\omega} \rangle$ be a canonical enumeration of the splitting nodes of p , i.e. $\sigma_\emptyset = \text{stem}(p)$, $\sigma_s \subseteq \sigma_t$ for $s \subseteq t$ and $\sigma_{s \hat{\ } j} \supseteq \sigma_s \hat{\ } j$ for $j \in 2$.

We call p an *alternating tree*, if the following holds true:

For all $s \in 2^{<\omega}$, $\sigma \in \text{split}(p)$, $x \in [p]$ with $x, \sigma \supseteq \sigma_s$, $\sigma = \sigma_{s \hat{\ } t}$, and such that σ_s is the common initial segment of x and σ we have

$$x(|\sigma|) = \begin{cases} 0, & \text{if } |t| \text{ is odd} \\ 1, & \text{if } |t| \text{ is even.} \end{cases}$$

Remark. By definition all perfect trees that have a certain type or are alternating trees are skew trees. Also, a perfect tree can not be of two different types. If a perfect tree is of Type (i_0, i_1) , every perfect subtree is of Type (i_0, i_1) as well.

Proposition 2.7. Let $p \in \mathbb{S}$ be arbitrary. Then there exists $q \in \mathbb{S}$ with $q \subseteq p$ and q is an alternating tree.

Proof. We build a fusion sequence of splitting trees $\langle p_n : n \in \omega \rangle$ with $p_0 = p$, $p_{n+1} \leq_{n+1} p_n$ for all $n \in \omega$ and such that we have the following property (*):

For all natural $n \geq 1$ and for all $\sigma, \tau \in p$ with common initial segment μ and σ being an $(n+1)$ -th splitting node and $\text{length}(\sigma) < \text{length}(\tau)$ we have:

$$\tau(|\sigma|) = \begin{cases} 0, & \text{if } |\text{split}(p_n) \cap \{\rho : \mu \hat{\ } j \subseteq \rho \subseteq \sigma\}| \text{ is odd} \\ 1, & \text{otherwise} \end{cases}$$

We construct the p_n by recursion:

$n = 1$:

Choose $\sigma_0 \supseteq \text{stem}(p \hat{\ } 0)$ such that $\sigma_0 \in \text{split}(p)$ and $|\sigma_0| \geq K(\text{stem}(p \hat{\ } 1))$.

Then there exists $\tau \supseteq \text{stem}(p \frown 1)$ with $\text{lgth}(\tau) > |\sigma_0|$ such that $\tau(|\sigma_0|) = 0$.

Now let $K := \max\{K(\sigma_0 \frown 0), K(\sigma_0 \frown 1)\}$ and choose $\tau_0 \in \text{split}(p)$ with $\tau_0 \supseteq \tau$ and $|\tau_0| \geq K$.

By definition of K , there exist $\sigma_{00} \supseteq \sigma_0 \frown 0$ and $\sigma_{01} \supseteq \sigma_0 \frown 1$ with $\text{lgth}(\sigma_{00}) = \text{lgth}(\sigma_{01}) > |\tau_0|$ and $\sigma_{00}(|\tau_0|) = \sigma_{01}(|\tau_0|) = 0$

Define $p_1 := \bigcup_{\rho \in \{\tau_0, \sigma_{00}, \sigma_{01}\}} p \upharpoonright \rho$.

$n \rightsquigarrow n + 1$:

Let p_n be already constructed with $p_n = \bigcup_{j \in 2^{n+1}} p_n \upharpoonright \sigma_j$ with σ_j extensions of the 2^n ($n + 1$)-th splitting nodes.

We construct extensions of these nodes by recursion:

First, choose $\sigma_0^0 \in \text{split}(p_n)$ with $\sigma_0^0 \supseteq \sigma_0$ and $|\sigma_0^0| \geq \max\{K(\sigma_j) : 1 \leq j < 2^{n+1}\}$. Also, for all j with $1 \leq j < 2^{n+1}$, pick $\sigma_j^0 \in p_n$ with $\sigma_j^0 \supseteq \sigma_j$, such that

$$\sigma_j^0(|\sigma_0^0|) = \begin{cases} 0, & \text{if } |\text{split}(p_n) \cap \{\rho : \sigma_0 \cap \sigma_j \subseteq \rho \subseteq \sigma_0^0\}| + 1 \text{ is odd} \\ 1, & \text{otherwise} \end{cases},$$

thus respecting (*).

Define $\sigma_{0,0}^0 := \sigma_0^0 \frown 0$ and $\sigma_{0,1}^0 := \sigma_0^0 \frown 1$

Now let $i < 2^{n+1} - 1$ and suppose that $\sigma_{j,k}^i$ are constructed for all $j \leq i$ and $k \in 2$ and that σ_j^i are constructed for $i < j < 2^n$. Choose $\sigma_{i+1}^{i+1} \in \text{split}(p_n)$ with $\sigma_{i+1}^{i+1} \supseteq \sigma_{i+1}^i$ and $|\sigma_{i+1}^{i+1}| \geq \max(\{K(\sigma_j^i) : j > i + 1\} \cup \{K(\sigma_{j,k}^i) : j \leq i; k \in 2\})$.

For all $j \leq i, k \in 2$ pick $\sigma_{j,k}^{i+1} \in p_n$ with $\sigma_{j,k}^{i+1} \supseteq \sigma_{j,k}^i$ and

$$\sigma_{j,k}^{i+1}(|\sigma_{i+1}^{i+1}|) = \begin{cases} 0, & \text{if } |\text{split}(p_n) \cap \{\rho : \sigma_{i+1}^i \cap \sigma_j \subseteq \rho \subseteq \sigma_{i+1}^{i+1}\}| + 1 \text{ is odd} \\ 1, & \text{otherwise} \end{cases}.$$

For all j with $i + 1 < j < 2^n$ pick $\sigma_j^{i+1} \in p_n$ with $\sigma_j^{i+1} \supseteq \sigma_j^i$ and

$$\sigma_j^{i+1}(|\sigma_{i+1}^{i+1}|) = \begin{cases} 0, & \text{if } |\text{split}(p_n) \cap \{\rho : \sigma_{i+1}^i \cap \sigma_j \subseteq \rho \subseteq \sigma_{i+1}^{i+1}\}| + 1 \text{ is odd} \\ 1, & \text{otherwise} \end{cases}.$$

Define $\sigma_{i+1,0}^{i+1} := \sigma_{i+1}^{i+1} \frown 0$ and $\sigma_{i+1,1}^{i+1} := \sigma_{i+1}^{i+1} \frown 1$.

After we have recursively constructed all $\sigma_{i,k}^{2^{n+1}-1}$ for $i < 2^{n+1}, k \in 2$ we define $p_{n+1} := \bigcup_{i \in 2^{n+1}; k \in 2} p_n \upharpoonright \sigma_{i,k}^{2^{n+1}-1}$.

After we have obtained the fusion sequence of perfect trees let $q := \bigcap_{n \in \omega} p_n \in \mathbb{S}$.

Obviously by construction q is an alternating subtree of p . □

Proposition 2.8. *Let $p \in \mathbb{S}$ and $i_0, i_1 \in 2$ be arbitrary. Then there exists $q \in \mathbb{S}$ with $q \subseteq p$ and q is of Type (i_0, i_1) .*

Proof. Similar to the proof of Proposition 2.7 we build a fusion sequence of splitting trees $\langle p_n : n \in \omega \rangle$ with $p_0 = p, p_{n+1} \leq_{n+1} p_n$ for all $n \in \omega$ and such that we have the following property (*):

For all natural $n \geq 1$ and for all $\sigma, \tau \in p$ with common initial segment μ and σ being an $(n+1)$ -th splitting node and $length(\sigma) < length(\tau)$ we have:

$$\tau(|\sigma|) = \begin{cases} i_0, & \text{if } \tau \text{ is right of } \sigma \text{ by lexicographic order} \\ i_1, & \text{otherwise} \end{cases}$$

The sequence can be constructed analogously as in the proof of Proposition 2.7 with values chosen for $\sigma_j^i(|\sigma_i^i|)$ and $\sigma_{j,k}^i(|\sigma_i^i|)$ with respect to the altered (*) property.

It is easy to see that $q := \bigcup_{n \in \omega} p_n \in \mathbb{S}$ is the desired subtree of p . □

Lemma 2.9. *Let p be an alternating tree. Then there exists a perfect subtree $q \subseteq p$ with q is of Type $(1, 1)$.*

Proof. Let $f : Even \rightarrow 2$ be a function and let $\langle \sigma_s : s \in 2^{<\omega} \rangle$ be a canonical enumeration of the splitting nodes (an order isomorphism w.r.t. inclusion order). Let $q := dwcl(\{\sigma_s : s \in 2^{<\omega} \wedge s \upharpoonright Even \subseteq f\})$. Then it can be easily seen that q is a Type(1, 1) subtree of p . □

Corollary 2.10. *Let $\lambda < 2^{\aleph_0}$ be an ordinal and let $\langle p_\xi : \xi < \lambda \rangle$ be a sequence of perfect trees such that $\forall \xi < \lambda \forall \bar{p} \in \mathbb{S} (\bar{p} \subseteq p \rightarrow \bar{p} \text{ is not of Type } (1, 1))$. And let q be an*

alternating perfect tree.

Then there exists a perfect tree $\bar{q} \subseteq q$ with $\forall \xi < \lambda$ ($[p_\xi] \cap [\bar{q}] = \emptyset$).

Proof. By Lemma 2.9 we can find some perfect $q' \subseteq q$ of Type (1, 1). Assume by contradiction that for any $\xi < \lambda$ we would have that $[q'] \cap [p_\xi]$ is uncountable, then we would have some perfect subtree of $q' \cap p_\xi$ of Type (1, 1), which is not possible because of our premise that p_ξ does not contain a Type (1, 1,) tree.

Thus, the assumption is wrong and we can conclude that

$$|[q'] \cap (\bigcup_{\xi < \lambda} [p_\xi])| < 2^{\aleph_0}.$$

Because every perfect tree can be partitioned into 2^{\aleph_0} many perfect subtrees with disjoint closures, we can find a perfect subtree $\bar{q} \subseteq q'$ with $\forall \xi < \lambda$ ($[\bar{q}] \cap [p_\xi] = \emptyset$).

□

The next Lemma is a specific case of Lemma 2 on page 11 from [Bre95].

Lemma 2.11. *Let $p \in \mathbb{S}$. Then there exists $(i_0, i_1) \in 2^2$ and a perfect subtree $\bar{p} \subseteq p$ with \bar{p} is of Type (i_0, i_1) .*

The following two lemmata will be core elements in answering the question, weather the Sacks ideal is a subset of the splitting ideal or not.

Lemma 2.12. *For every $p \in \mathbb{S}$ exists a $q \in \mathbb{S}$ with $q \subseteq p$ and $\{\sigma_{s,t} : (s, t) \in 2^{<\omega} \times 2^{<\omega} \wedge |s| = |t|\} \subseteq \text{split}(q)$ such that*

$\forall n, m \in \omega \forall s, t \in 2^n \forall s', t' \in 2^m (\sigma_{s,t} \subseteq \sigma_{s',t'} \leftrightarrow (s \subseteq s' \wedge t \subseteq t'))$ and the following two conditions hold true:

i): For all $f \in 2^\omega$ we have that

$q_f := \text{dwcl}(\{\sigma_{s,t} : s, t \in 2^{<\omega} \wedge |s| = |t| \wedge s \subseteq f\})$ is alternating.

ii):

For $i \in 2$ let $f_i, g_i \in 2^\omega$ and $s \in 2^{<\omega}$ such that for all $i \in 2$ we have $s \subseteq f_i$ and $f_i(|s|) = i$.

Then for $\phi_i := \bigcup_{n \in \omega} \sigma_{f_i|n, g_i|n}$ the following holds true:

$\forall i \in 2 \forall \rho \in \text{split}(q) ((\sigma_{s \frown i, g_i|s+1|} \subseteq \rho \subseteq \phi_i) \rightarrow \phi_{1-i}(|\rho|) = 0)$.

Proof. We construct recursively a sequence $\langle p_n : n \in \omega \rangle$ as well as $\{\sigma_{s,t} : (s, t) \in 2^{<\omega} \times 2^{<\omega} \wedge |s| = |t|\} \subseteq \text{split}(p)$ and $\{\tau_{s,t,k} : (s, t) \in 2^{<\omega} \times 2^{<\omega} \wedge |s| = |t| \wedge k \in 2\} \subseteq p$, such that

$$p_0 = p$$

and for all $n \in \omega$ we have $p_{n+1} \leq p_n$

and for all $n \in \omega$ and for every σ which is at most a $2n$ -th splitting node of p_n and for all $\tau \in p_n$ with $|\tau| > |\sigma|$ the following two properties hold true:

$$(\star) : (\sigma = \sigma_{s,t} \wedge \tau \subseteq \sigma_{s',t'} \wedge s \parallel s' \wedge t \perp t')$$

$$\rightarrow \tau(|\sigma|) = \begin{cases} 0 & , \text{if } |\{\rho \in p_n : \tau \cap \sigma \subsetneq \rho \subseteq \sigma\} \cap \text{split}(p_n)|/2 \text{ is even} \\ 1 & , \text{if } |\{\rho \in p_n : \tau \cap \sigma \subsetneq \rho \subseteq \sigma\} \cap \text{split}(p_n)|/2 \text{ is odd} \end{cases}$$

$$\text{and } (\star\star) : (\sigma_{s,t} \subseteq \sigma \wedge \sigma_{s',t'} \subseteq \tau \wedge s \perp s') \rightarrow \tau(|\sigma|) = 0.$$

$$\underline{n = 0} :$$

Define $p_0 := p$, $\sigma_{\emptyset, \emptyset} := \text{stem}(p)$ and for $j \in 2$ define $\tau_{\emptyset, \emptyset, j} := \text{stem}(p) \hat{\ } j$.

$$\underline{n \rightsquigarrow n + 1} :$$

Fix some $n \in \omega$. Let p_n be constructed as well as $\sigma_{s,t} \in \text{split}(p_n)$ and $\tau_{s,t,j} \in p_n$ with $\tau_{s,t,j} \supseteq \sigma_{s,t} \hat{\ } j$ for all $s, t \in 2^n, j \in 2$.

First, we will recursively construct nodes $\overline{\sigma_{s,t} \hat{\ } j} \in \text{split}(p_n)$ with $\overline{\sigma_{s,t} \hat{\ } j} \supseteq \tau_{s,t,j}$ that shall be the successor splitting nodes of the $\sigma_{s,t}$ in p_{n+1} , and also extensions $\overline{\tau_{s,t} \hat{\ } j, i} \in p_n$ with $\overline{\tau_{s,t} \hat{\ } j, i} \supseteq \overline{\sigma_{s,t} \hat{\ } j} \hat{\ } i$. While doing this we have to make sure that (\star) will hold in p_{n+1} .

Let $\pi : 2^{2n+1} \rightarrow 2^n \times 2^n \times 2$ be a bijection.

For $\pi(0) =: (s_0, t_0, j_0)$ choose $\overline{\sigma_{s_0, t_0, j_0}^{(0)}} \in \text{split}(p_n)$ with $\overline{\sigma_{s_0, t_0, j_0}^{(0)}} \supseteq \tau_{s_0, t_0, j_0}$ such that $k_0 := |\overline{\sigma_{s_0, t_0, j_0}^{(0)}}| \geq \max\{K_{p_n}(\tau_{s,t,j}) : (s, t, j) \in 2^n \times 2^n \times 2\}$.

Now, for all $(s, t, j) \in 2^n \times 2^n \times 2$ with $\pi(0) \neq (s, t, j)$ pick $\overline{\sigma_{s,t} \hat{\ } j} \in p_n$ with $\overline{\sigma_{s,t} \hat{\ } j} \supseteq \tau_{s,t,j}$ and $|\overline{\sigma_{s,t} \hat{\ } j}| \geq k_0$ such that $\overline{\sigma_{s,t} \hat{\ } j}(k_0) = 0$.

For $i \in 2$ define $\overline{\tau_{s_0, t_0, j_0, i}^{(0)}} := \overline{\sigma_{s_0, t_0, j_0}^{(0)}} \hat{\ } i$.

Now let $m \in 2^{2n+1}$, $m \geq 1$ and assume that for all $(s, t, j) \in 2^n \times 2^n \times 2$ with $\pi^{-1}(s, t, j) < m$ and $i \in 2$ the nodes $\overline{\tau_{s,t \frown j,i}^{(m-1)}}$ have been constructed and assume that for all $(s, t, j) \in 2^n \times 2^n \times 2$ with $\pi^{-1}(s, t, j) \geq m$ the nodes $\overline{\sigma_{s,t \frown j}^{(m-1)}}$ have been constructed.

For $\pi(m) =: (s_m, t_m, j_m)$ choose $\overline{\sigma_{s_m, t_m \frown j_m}^{(m)}} \in \text{split}(p_n)$ with $\overline{\sigma_{s_m, t_m \frown j_m}^{(m)}} \supseteq \overline{\sigma_{s_m, t_m \frown j_m}^{(m-1)}}$, such that

$$k_m := |\overline{\sigma_{s_m, t_m \frown j_m}^{(m)}}| \geq \max \left(\left\{ K_{p_n} \left(\overline{\tau_{s,t \frown j,i}^{(m-1)}} \right) : \pi^{-1}(s, t, j) < m \right\} \cup \left\{ K_{p_n} \left(\overline{\sigma_{s,t \frown j}^{(m-1)}} \right) : \pi^{-1}(s, t, j) > m \right\} \right).$$

For all $s, t \in 2^n$ and $i, j \in 2$ with $\pi^{-1}(s, t, j) < m$ choose $\overline{\tau_{s,t \frown j,i}^{(m)}} \in p_n$ with $\overline{\tau_{s,t \frown j,i}^{(m)}} \supseteq \overline{\tau_{s,t \frown j,i}^{(m-1)}}$ and $\overline{\tau_{s,t \frown j,i}^{(m)}}(k_m) = 0$.

Also, for all $s, t \in 2^n$ and $j \in 2$ with $\pi^{-1}(s, t, j) > m$ choose $\overline{\sigma_{s,t \frown j}^{(m)}} \in p_n$ with $\overline{\sigma_{s,t \frown j}^{(m)}} \supseteq \overline{\sigma_{s,t \frown j}^{(m-1)}}$ and $\overline{\sigma_{s,t \frown j}^{(m)}}(k_m) = 0$.

For $i \in 2$ define $\overline{\tau_{s_m, t_m \frown j_m, i}^{(m)}} := \overline{\sigma_{s_m, t_m \frown j_m}^{(m)}} \frown i$.

After we have constructed all $\overline{\sigma_{s,t \frown j}^{(m)}}$ for $s, t \in 2^n$, $j \in 2$ and $m \in 2^{2n+1}$ with $\pi(s, t, j) \leq m$ and all $\overline{\tau_{s,t \frown j,i}^{(m)}}$ for $s, t \in 2^n$, $i, j \in 2$ and $m \in 2^{2n+1}$ with $\pi(s, t, j) \geq m$

we define, for all $s, t \in 2^n$ and $j \in 2$,

$$\overline{\sigma_{s,t \frown j}} := \overline{\sigma_{s,t \frown j}^{(m_{s,t,j})}} \text{ with } m_{s,t,j} := \pi(s, t, j)$$

and we define, for all $s, t \in 2^n$ and $i, j \in 2$,

$$\overline{\tau_{s,t \frown j,i}} := \overline{\tau_{s,t \frown j,i}^{(2^{2n+1}-1)}}.$$

After having defined the $(2n+1)$ -th splitting nodes of p_{n+1} , namely the nodes $\overline{\sigma_{s,t \frown j}}$ and their extensions $\overline{\tau_{s,t \frown j,i}}$, we will now recursively construct nodes $\sigma_{s \frown i, t \frown j} \in \text{split}(p_n)$ with $\sigma_{s \frown i, t \frown j} \supseteq \overline{\tau_{s,t \frown j,i}}$, that will be the successor splitting nodes of the nodes $\overline{\sigma_{s,t \frown j}}$ in p_{n+1} , as well as extensions $\tau_{s \frown i, t \frown j, k} \in p_n$ with $\tau_{s \frown i, t \frown j, k} \in p_n$ and $\tau_{s \frown i, t \frown j, k} \supseteq \sigma_{s \frown i, t \frown j} \frown k$.

Let $\pi : 2^{2n+2} \rightarrow 2^n \times 2^n \times 2 \times 2$ be a bijection.

For $\pi(0) =: (s_0, t_0, i_0, j_0)$ choose $\sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)} \in \text{split}(p_n)$ with $\sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)} \supseteq \overline{\tau_{s_0, t_0, 0, i_0}}$ such that $l_0 := |\sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)}| \geq \max\{K_{p_n}(\overline{\tau_{s, t, i, j}}) : (s, t, i, j) \in 2^n \times 2^n \times 2 \times 2\}$.

For all $(s, t, i, j) \in 2^n \times 2^n \times 2 \times 2$ with $\pi(0) \neq (s, t, i, j)$ pick $\sigma_{s \widehat{i}, t \widehat{j}}^{(0)} \in p_n$ with $\sigma_{s \widehat{i}, t \widehat{j}}^{(0)} \supseteq \overline{\tau_{s, t, i, j}}$, such that:

if $s \perp s_0$, then $\sigma_{s \widehat{i}, t \widehat{j}}^{(0)}(l_0) = 0$

and if $s \parallel s_0$ and $t \widehat{j} \perp t_0 \widehat{j}_0$, then

$$\sigma_{s \widehat{i}, t \widehat{j}}^{(0)}(l_0) = \begin{cases} 0 & \text{,if } |\{\rho \in \text{split}(p_n) : \sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)} \cap \overline{\tau_{s, t, i, j}} \subsetneq \rho \subseteq \sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)}\}|/2 \\ & \text{is even} \\ 1 & \text{,if } |\{\rho \in \text{split}(p_n) : \sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)} \cap \overline{\tau_{s, t, i, j}} \subsetneq \rho \subseteq \sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)}\}|/2 \\ & \text{is odd} \end{cases}.$$

For $k \in 2$ define $\tau_{s_0 \widehat{i}_0, t_0 \widehat{j}_0, k}^{(0)} := \sigma_{s_0 \widehat{i}_0, t_0 \widehat{j}_0}^{(0)} \widehat{k}$.

Now let $m \in 2^{2n+2}$, $m \geq 1$ and assume that for all $(s, t, i, j) \in 2^n \times 2^n \times 2 \times 2$ with $\pi^{-1}(s, t, i, j) < m$ and $k \in 2$ the nodes $\tau_{s \widehat{i}, t \widehat{j}, k}^{(m-1)}$ have been constructed and that for all $(s, t, i, j) \in 2^n \times 2^n \times 2 \times 2$ with $\pi^{-1}(s, t, i, j) \geq m$ the nodes $\sigma_{s \widehat{i}, t \widehat{j}}^{(m-1)}$ have been constructed.

For $\pi(m) =: (s_m, t_m, i_m, j_m)$ choose $\sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)} \in \text{split}(p_n)$ with $\sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)} \in \text{split}(p_n) \supseteq \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m-1)} \in \text{split}(p_n)$, such that

$$l_m := |\sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)}| \geq \max \left(\left\{ K_{p_n} \left(\tau_{s \widehat{i}, t \widehat{j}, k}^{(m-1)} \right) : \pi^{-1}(s, t, i, j) < m \right\} \cup \left\{ K_{p_n} \left(\sigma_{s \widehat{i}, t \widehat{j}}^{(m-1)} \right) : \pi^{-1}(s, t, i, j) > m \right\} \right).$$

For all $(s, t, i, j) \in 2^n \times 2^n \times 2 \times 2$ with $\pi^{-1}(s, t, i, j) < m$ and all $k \in 2$ pick $\tau_{s \widehat{i}, t \widehat{j}, k}^{(m)} \in p_n$ with $\tau_{s \widehat{i}, t \widehat{j}, k}^{(m)} \supseteq \tau_{s \widehat{i}, t \widehat{j}, k}^{(m-1)}$, such that:

if $s_0 \widehat{i}_0 \perp s \widehat{i}$, then $\tau_{s \widehat{i}, t \widehat{j}, k}^{(m)}(l_m) = 0$

and if $s_0 \widehat{i}_0 \parallel s \widehat{i}$ and $t_0 \widehat{j}_0 \perp t \widehat{j}$, then

$$\tau_{s \widehat{i}, t \widehat{j}, k}^{(m)}(l_m) = \begin{cases} 0 & \text{,if } |\{\rho \in \text{split}(p_n) : \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)} \cap \tau_{s \widehat{i}, t \widehat{j}, k}^{(m-1)} \subsetneq \rho \subseteq \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)}\}|/2 \\ & \text{is even} \\ 1 & \text{,if } |\{\rho \in \text{split}(p_n) : \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)} \cap \tau_{s \widehat{i}, t \widehat{j}, k}^{(m-1)} \subsetneq \rho \subseteq \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)}\}|/2 \\ & \text{is odd} \end{cases} .$$

For all $(s, t, i, j) \in 2^n \times 2^n \times 2 \times 2$ with $\pi^{-1}(s, t, i, j) > m$ pick $\sigma_{s \widehat{i}, t \widehat{j}}^{(m)} \in p_n$ with $\sigma_{s \widehat{i}, t \widehat{j}}^{(m)} \supseteq \sigma_{s \widehat{i}, t \widehat{j}}^{(m-1)}$, such that:

if $s_m \perp s$, then $\sigma_{s \widehat{i}, t \widehat{j}}^{(m)}(l_m) = 0$

and if $s_m \parallel s$ and $t_m \widehat{j}_m \perp t \widehat{j}$, then

$$\sigma_{s \widehat{i}, t \widehat{j}}^{(m)}(l_m) = \begin{cases} 0 & \text{,if } |\{\rho \in \text{split}(p_n) : \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)} \cap \sigma_{s \widehat{i}, t \widehat{j}}^{(m-1)} \subsetneq \rho \subseteq \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)}\}|/2 \\ & \text{is even} \\ 1 & \text{,if } |\{\rho \in \text{split}(p_n) : \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)} \cap \sigma_{s \widehat{i}, t \widehat{j}}^{(m-1)} \subsetneq \rho \subseteq \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)}\}|/2 \\ & \text{is odd} \end{cases} .$$

For $k \in 2$ define $\tau_{s_m \widehat{i}_m, t_m \widehat{j}_m, k}^{(m)} := \sigma_{s_m \widehat{i}_m, t_m \widehat{j}_m}^{(m)}, \widehat{k}$

If we have constructed all $\sigma_{s \widehat{i}, t \widehat{j}}^{(m)}$ for $s, t \in 2^n$, $i, j \in 2$ and $m \in 2^{2n+2}$ with $\pi(s, t, i, j) \leq m$

and if we have constructed all $\tau_{s \widehat{i}, t \widehat{j}, k}^{(m)}$ for $s, t \in 2^n$, $i, j, k \in 2$ and $m \in 2^{2n+2}$ with $\pi(s, t, i, j) \geq m$

we define, for all $s, t \in 2^n$ and $i, j \in 2$,

$$\sigma_{s \widehat{i}, t \widehat{j}} := \sigma_{s \widehat{i}, t \widehat{j}}^{(m_{s,t,i,j})} \text{ with } m_{s,t,i,j} := \pi(s, t, i, j)$$

and we define, for all $s, t \in 2^n$ and $i, j, k \in 2$,

$$\tau_{s \widehat{i}, t \widehat{j}, k} := \tau_{s \widehat{i}, t \widehat{j}, k}^{(2^{2n+2}-1)} .$$

We finish the construction of p_{n+1} by defining

$$p_{n+1} := \bigcup_{\substack{s,t \in 2^n \\ i,j,k \in 2}} p_n \upharpoonright \tau_{s \widehat{i}, t \widehat{j}, k} .$$

It is now easy to see that by construction the sequence $\langle p_n : n \in \omega \rangle$

is as desired and that by defining

$q := \bigcap_{n \in \omega} p_n = \text{dwcl}(\{\sigma_{s,t} : s, t \in 2^{<\omega}\}) \in \mathbb{S}$, q is a witness for the statement of the lemma. □

Lemma 2.13. *Let q and $\{\sigma_{s,t} : (s, t) \in 2^{<\omega} \times 2^{<\omega} \wedge |s| = |t|\} \subseteq \text{split}(q)$ be as in Lemma 2.12. For every $f \in 2^\omega$ let $q_f := \text{dwcl}(\{\sigma_{s,t} : (s, t) \in 2^{<\omega} \times 2^{<\omega} \wedge |s| = |t| \wedge s \subseteq f\})$. Then for every $p \in \mathbb{S}$ such that p is of Type $(i_0, i_1) \neq (0, 0)$ the set $\{f \in 2^\omega : [q_f] \cap [p] \neq \emptyset\}$ is countable.*

Proof. Fix some perfect p that is without restriction of Type (i_0, i_1) with $i_0 = 0$. Assume for contradiction that $A := \{f \in 2^\omega : [q_f] \cap [p] \neq \emptyset\}$ is uncountable. Hence, $T := \text{cond}(A)$ is a perfect tree and therefore we can pick some $s \in \text{split}(T)$. Now, by definition of A and T , there exist uncountably many functions $f \in 2^\omega$, such that $s \frown 0 \subseteq f$ and there exists $g \in 2^\omega$ with $\bigcup_{n \in \omega} \sigma_{f \upharpoonright n, g \upharpoonright n} \in [q_f] \cap [p]$.

By a simple pigeonhole argument we find $f_0, f_1, g_0, g_1 \in 2^\omega$ with $s \frown 0 \subseteq f_0, f_1, g_0 \upharpoonright |s| + 1 = g_1 \upharpoonright |s| + 1$ and

$\Phi_i := \bigcup_{n \in \omega} \sigma_{f_i \upharpoonright n, g_i \upharpoonright n} \in [q_{f_i}] \cap [p]$ for every $i \in 2$.

Analogously we find $f_2, f_3, g_2, g_3 \in 2^\omega$ with

$s \frown 1 \subseteq f_2, f_3, g_2 \upharpoonright |s| + 1 = g_3 \upharpoonright |s| + 1$ and

$\Phi_i := \bigcup_{n \in \omega} \sigma_{f_i \upharpoonright n, g_i \upharpoonright n} \in [q_{f_i}] \cap [p]$ for every $i \in \{2, 3\}$.

For $\sigma = \Phi_0 \cap \Phi_1 \in \text{split}(p)$ we have that $\sigma_{s \frown 0, g_0 \upharpoonright |s| + 1} \subseteq \sigma \subseteq \Phi_0$.

So, by definition of q , for $\Phi_2 \in [p]$ we have $\Phi_2(|\sigma|) = 0$. Analogous we got $\Phi_0(|\Phi_2 \cap \Phi_3|) = 0$. This is obviously a contradiction because with respect to the type of p at least one of the values has to be 1.

We can conclude that $\{f \in 2^\omega : [q_f] \cap [p] \neq \emptyset\}$ is countable. □

Lemma 2.13 will enable us to prove the following theorem.

Theorem 2.14. $I(\mathbb{S}) \not\subseteq I(S)$

Proof. Let $\langle p_\zeta : \zeta < 2^{\aleph_0} \rangle$ enumerate all perfect trees of a certain type and let

$\langle p_\xi : \xi < 2^{\aleph_0} \rangle$ enumerate all splitting trees.

We construct a set $X = \{x_\lambda : \lambda < 2^{\aleph_0}\} \in \mathbb{S} \setminus I(S)$ in the following way.

For $\lambda < 2^{\aleph_0}$ we first pick some $q \subseteq q_\lambda$ as in Lemma 2.12. By Lemma 2.13 we can conclude that

$A := \{f \in 2^\omega : \exists \zeta < \lambda ([q_f] \cap [p_\zeta] \neq \emptyset \wedge p_\zeta \text{ is of Type } \neq (0,0))\}$ has a cardinality equal or less than $|\lambda| \cdot |\aleph_0|$. Thus, we can pick some $f \in 2^\omega \setminus A$.

q_f is an alternating perfect tree and by Lemma 2.9 we can choose a perfect subtree $\bar{q}_\lambda \subseteq q_f$ of type $(1, 1)$.

We can conclude that $\forall \zeta < \lambda [\bar{q}_\lambda] \cap [p_\zeta] = \emptyset$.

We obtain x_λ by just picking some arbitrary member of $[\bar{q}_\lambda]$.

To see that X is as desired, fix an arbitrary $p \in \mathbb{S}$. By Lemma 2.11 there exists a $\zeta < 2^{\aleph_0}$ with $p_\zeta \subseteq p$. For all $\lambda > \zeta$ we have $x_\lambda \notin [p_\zeta]$, thus we can conclude that $|[p_\zeta] \cap X| < 2^{\aleph_0}$ and we get that there exists $\bar{p}_\zeta \in \mathbb{S}$ with $\bar{p}_\zeta \subseteq p_\zeta$ and $[\bar{p}_\zeta] \cap X = \emptyset$.

Because $p \in \mathbb{S}$ was chosen arbitrary we have $X \in I(\mathbb{S})$.

On the other hand for every $\lambda < 2^{\aleph_0}$ we have that $[q_\lambda]$ meets X in x_λ . This implies that $X \notin I(S)$.

□

3 The independence of the covering numbers of $I(S)$ and $I(\mathbb{S})$

In the last chapter we have seen that, in every ZFC model, the splitting tree forcing ideal is not included in the Sacks forcing ideal and vice versa. This raises the question of whether their cardinal invariants are related to each other. In this chapter we will focus on the covering number of the two ideals. H.Judah, A.Miller and S.Shelah have shown that starting with a ground model that satisfies CH and

then iterating the Sacks forcing ω_2 times with countable supports leads to a model that satisfies that $\mathfrak{c} = cov(I(\mathbb{S})) = \omega_2$ (see [JMS92]). We will use a result of

Baumgartner and Laver [BL79] to show that this model also satisfies $cov(I(S)) = \omega_1$. To show that a model of $ZFC + CH$ has a forcing extension that satisfies $cov(I(\mathbb{S})) < cov(I(S))$ we have to delve deeper into the countable support iteration of splitting tree forcing. We will use techniques similar to the ones that

O. Spinas used, when he analyzed finite products of splitting tree forcing in [Spi07], to get a continuous reading of names that is one to one for the iterated forcing. This will lead to the desired forcing extension. As the main result of this chapter we will get that the covering numbers of the two ideals are independent.

At first we want to look at some basic properties of the splitting tree forcing. If we have some $G \subseteq S$ that is a generic filter over some ground model V , as with other tree forcings, one can look at $g := \bigcup \bigcap G$ which is a member of 2^ω . We say that g is a *generic splitting real (over V)*. Then g has the following property.

Proposition 3.1. *Let V be some ground model and g splitting generic over V . Then for every $x \in V \cap 2^\omega$ we have that g splits x .*

Proof. The proof is an easy density argument. □

In contrast we have a theorem that has been proven by Baumgartner and Laver [BL79].

Theorem 3.2. *Let V be a ground model, let $\alpha \geq 1$ be an ordinal, let \mathbb{S}_α be the countable support iteration of \mathbb{S} of length α and let G be a \mathbb{S}_α generic filter over V . Then every $x \in V[G]$ is refined by some $y \in V$, i.e. x does not split y .*

Proof. For reference see the Theorem 4.3 in [BL79] □

One can use Laver generic and Sacks generic reals to reconstruct the respective Laver generic or Sacks generic filters. It is well known that in the same fashion one can use the splitting generic real to reconstruct the splitting generic filter as shown in the following proof.

Proposition 3.3. *Let V be some ground model and let g be a splitting generic real over V . Then $G := \{p \in V \cap S : g \in [p]\}$ defines a S generic filter over V .*

Proof. If G' is a generic filter with $g = \bigcup \bigcap G'$, then by definition of G we have $G' \subseteq G$. Thus, for every dense set $D \subset S$ of the ground model we have $D \cap G \neq \emptyset$.

We still have to prove that two splitting trees that are members of G contain a splitting tree as a subset.

So let $p, q \in G$. We have $g \in [p \cap q]$. We will use a Cantor-Bendixon style procedure in V to construct a subset of $p \cap q$:

For every tree $A \subseteq 2^{<\omega}$ let

$$\bar{A} := \{\sigma \in A : \exists X \in [\omega]^\omega \exists f : X \rightarrow 2 \forall \tau \in (A \upharpoonright \sigma) \forall k \in \text{dom}(\tau) \cap X \\ (\tau(k)) = f(x)\},$$

which is, in other words, the set of nodes such that the tree restricted to one of these nodes has fixed values on a ground model set.

Now define recursively (in V) $r_0 := p \cap q$, $r_{\alpha+1} := r_\alpha \setminus \bar{r}_\alpha$ and $r_\lambda := \bigcap_{\xi < \lambda} r_\xi$ for $\lambda \in \text{Lim}$.

This sequence becomes constant at some $\theta < \omega_1$, i.e. $r_\theta = r_{\theta+1}$.

Assume that $g \notin [r_\theta]$, thus we have $\alpha < \theta$ minimal with $g \in [r_\alpha]$ and $g \notin [r_{\alpha+1}]$, hence we have $\sigma \subseteq g$ with $\sigma \in r_\alpha$ and $\sigma \notin r_{\alpha+1}$. We can assume without loss of generality that (in V) we have some $X \in [\omega]^\omega$ such that all $x \in [r_\alpha \upharpoonright \sigma]$ have constant value 0 on X . Because of the Shoenfield absoluteness theorem, this also holds true in the generic extension, hence $g \upharpoonright X \equiv 0$. But then g would not split the ground model set X , which is obviously a contradiction.

We can conclude that $g \in [r_\theta]$. This means that r_θ is not empty. But then, by definition of r_θ , we have $r_\theta \in S$. Furthermore p and q are compatible, which is witnessed by r_θ .

□

As a consequence of Proposition 3.1 we have that forcing with the splitting tree forcing adds new reals. By a result of O. Spinas ([Spi07]) we know, if G is a S generic filter over V and $x \in V[G]$ is a new real, that $V[x] = V[G]$. With respect to the theorem of Baumgartner and Laver we can conclude that for a S generic filter G and a \mathbb{S} generic filter H the new reals of $V[G]$ and $V[H]$ are disjoint.

Another important fact is the following.

Lemma 3.4. *S is proper.*

Proof. S satisfies Axiom A (see [Spi07] or below).

□

Thus, we get:

Proposition 3.5. *Let V be a ground model that satisfies CH and let P be either the splitting tree forcing or the Sacks forcing. Let $V_{\omega_2}^P$ be a generic extension obtained by a countable support iteration of P of length ω_2 . Then the following two facts hold.*

(i): $V \models \kappa$ is a cardinal $\Leftrightarrow V_{\omega_2}^P \models \kappa$ is a cardinal

(ii): $V_{\omega_2}^P \models \mathfrak{c} = \omega_2$

Proof. (i): It is well known fact that, if we start with a model of CH and we do a countable support iteration of length $\leq \omega_2$ of proper forcings of size \aleph_1 , the iteration satisfies the \aleph_2 chain condition. Also, properness is preserved under the iteration (see [She98] for these and other more general preservation theorems).

That Sacks forcing adds new reals is a well known fact. The same holds true for splitting tree forcing as we know by Proposition 3.1. Hence, $V_{\omega_2}^P \models \mathfrak{c} \geq \omega_2$. The fact that $V_{\omega_2}^P \models \mathfrak{c} \leq \omega_2$ is proven by a standard nice name argument. \square

Now that we have stated the basic facts, let us examine the situation of the covering number of the forcing ideals in $V_{\omega_2}^{\mathbb{S}}$ and $V_{\omega_2}^{\mathbb{S}}$. We will start with the former one.

We cite a theorem of [JMS92].

Theorem 3.6. *Let V be a ground model that satisfies CH and let $V_{\omega_2}^{\mathbb{S}}$ be a generic extension obtained by countable support iteration of Sacks forcing of length ω_2 . Then the following holds true.*

$$V_{\omega_2}^{\mathbb{S}} \models \text{cov}(I(\mathbb{S})) = \omega_2$$

Proof. See [JMS92]. \square

On the other hand the covering number of the splitting ideal is small in $V_{\omega_2}^{\mathbb{S}}$ as stated by the following theorem.

Theorem 3.7. *Let V be a ground model that satisfies CH and let $V_{\omega_2}^{\mathbb{S}}$ be a generic extension obtained by countable support iteration of Sacks forcing of length ω_2 . Then the following holds true.*

$$V_{\omega_2}^{\mathbb{S}} \models \text{cov}(I(S)) = \omega_1$$

Proof. We work in $V_{\omega_2}^{\mathbb{S}}$. Let D be a dense subset of S . Let $\langle p_\xi : \xi < \omega_2 \rangle$ enumerate all members of D .

We will use this enumeration to construct a matrix with ω_1 many rows such that every row is a dense subset of S and such that for every real $x \in 2^\omega \cap V_{\omega_2}^{\mathbb{S}}$ there exists a row such that x is not an element of any of the closures of the trees in that row.

Let $\langle a_\zeta : \zeta < \omega_1 \rangle$ enumerate all ground model reals. Now for every $\zeta < \omega_1$ and every $\xi < \omega_2$ we can find a splitting tree $q_{\zeta,\xi} \subseteq p_\xi$ with the property that every $y \in [q_{\zeta,\xi}]$ splits a_ζ (see the proof of Theorem 2.4).

Now because by Theorem 3.2 for every real $x \in V_{\omega_2}^{\mathbb{S}} \cap 2^\omega$ there exists $\zeta < \omega_1$, such that x does not split a_ζ , thus $\forall x \in V_{\omega_2}^{\mathbb{S}} \cap 2^\omega \exists \zeta < \omega_1 \forall \xi < \omega_2 (x \notin [q_{\zeta,\xi}])$ and the matrix $\langle q_{\zeta,\xi} : \zeta < \omega_1, \xi < \omega_2 \rangle$ is as desired.

For $\zeta < \omega_1$ define $X_\zeta := 2^\omega \setminus \bigcup_{\xi < \omega_2} [q_{\zeta,\xi}]$.

Because every row of the matrix is a dense subset of S , we can conclude that $X_\zeta \in I(S)$ for all $\zeta < \omega_1$. And because for every real in the forcing extension there is a row of the matrix, such that no closure of the trees in that row contains that real, we can conclude that $2^\omega \subseteq \bigcup_{\zeta < \omega_1} X_\zeta$ (in $V_{\omega_2}^{\mathbb{S}}$). □

We have the following corollary.

Corollary 3.8. *Let V be a ground model that satisfies CH and let $V_{\omega_2}^{\mathbb{S}}$ be a generic extension obtained by countable support iteration of Sacks forcing of length ω_2 . Then the following holds true.*

$$V_{\omega_2}^{\mathbb{S}} \models \text{cov}(I(S)) < \text{cov}(I(\mathbb{S}))$$

To show that $\text{cov}(I(S))$ and $\text{cov}(I(\mathbb{S}))$ are independent of each other, we will take a closer look at the model $V_{\omega_2}^S$, which is a generic extension of a $ZFC + CH$ model we get by iterating splitting tree forcing ω_2 times with countable support.

With Proposition 3.5 in mind we get the following theorem.

Theorem 3.9. *Let V be a ground model that satisfies CH and let $V_{\omega_2}^S$ be a generic extension obtained by countable support iteration of splitting tree forcing of length ω_2 . Then the following holds true.*

$$V_{\omega_2}^S \models \text{cov}(I(S)) = \omega_2$$

Proof. Because of Proposition 3.5 we know that in $V_{\omega_2}^S$ the cardinals of the ground model are preserved and that the continuum has size ω_2 . The rest of the proof is done analogously to the proof of Theorem 1.2 in [JMS92]. \square

To we analyze $\text{cov}(I(\mathbb{S}))$ in $V_{\omega_2}^S$ a little closer, we will go the same route that J.E. Baumgartner and R.Laver went in [BL79] combined with the fusion techniques for splitting trees of [Spi07]. We show that there is a continuous one to one reading of names for the iterated splitting tree forcing. For this we need the following definitions.

Definition 3.10. Let $p \in S$ be a splitting tree. We call $F \subset p$ a *front* of p , if and only if $|F| < \omega \wedge p = \bigcup_{\sigma \in F} p \upharpoonright \sigma \wedge \forall \sigma, \sigma' \in F (\sigma \neq \sigma' \rightarrow \sigma \perp \sigma')$. In other words, a front of p is a finite maximal antichain of nodes of the tree with respect to inverse inclusion order.

Definition 3.11. Let $\alpha \in OR$ and S_α denote the countable support iteration of the splitting tree forcing of length α . Let $H \in [\alpha]^{<\omega}$ be arbitrary.

For a $p \in S_\alpha$ we say the set \dot{F} is a *H-front* of p if and only if the following holds true:

- (i): \dot{F} is a function with $\text{dom}(\dot{F}) = H$ and $\dot{F}(\beta)$ is a S_β -name for every $\beta \in H$.
- (ii): If $0 \in H$ then $\dot{F}(0)$ is a front of $p(0)$.
- (iii): For all $\beta \in H$ with $0 < \beta$ we have that $p \upharpoonright \beta \Vdash \dot{F}(\beta)$ is a front of $p(\beta)$.

Remark: For $\tilde{H} \in [\alpha]^{<\omega}$ with $\tilde{H} \supseteq H$ we can assume that \dot{F} is also a \tilde{H} -front of p by defining $\dot{F}(\beta)$ as a canonical name for the stem of $p(\beta)$ for all $\beta \in \tilde{H} \setminus H$.

Definition 3.12. We say that $\langle (p_i, F_i) : i \in \omega \rangle$, with $p_i \in S$ and $F_i \in [p_i]^{<\omega}$ for all $i \in \omega$, is a *front-fusion sequence*, iff for all $i \in \omega$ we have that $(p_{i+1}, F_{i+1}) \leq (p_i, F_i)$ and F_{i+1} strictly refines F_i .

With $(q, G) \leq (p, F) :\Leftrightarrow$

(i): $q \leq p$

(ii): F is a front of p and G is a front of q .

(iii): $\forall \sigma \in F; j \in \omega : p(\sigma)[j] = q(\sigma)[j]$

Remark: In this case we have that $p := \bigcap_{i \in \omega} p_i \in S$ and p is generated by $\bigcup_{i \in \omega} F_i$.

Definition 3.13. Let $\alpha \in OR$ and let $p_i \in S_\alpha$, $H_i \in [\alpha]^{<\omega}$ and \dot{F}_i be a function with domain H_i for all $i \in \omega$. We say that $\langle (p_i, \dot{F}_i, H_i) : i \in \omega \rangle$ is a *front-fusion sequence* for S_α , iff the following holds true:

(i): $H_i \subseteq H_{i+1}$ for all $i \in \omega$.

(ii): $\bigcup_{i \in \omega} H_i = \bigcup_{i \in \omega} \text{supp}(p_i)$

(iii): \dot{F}_i is a H_i -front of p_i for all $i \in \omega$.

(iv): $(p_{i+1}, \dot{F}_{i+1}) \leq_{H_{i+1}} (p_i, \dot{F}_i)$ for all $i \in \omega$.

Given $p, q \in S_\alpha$ with \dot{F} is a H -front of p and \dot{G} is a \tilde{H} -front of q , for some $H, \tilde{H} \in [\alpha]^{<\omega}$ with $\tilde{H} \supseteq H$, we define $(q, \dot{G}) \leq_{\tilde{H}} (p, \dot{F}) :\Leftrightarrow$

(a): If $\emptyset \in H$ then $(q(\emptyset), \dot{G}(\emptyset)) \leq (p(\emptyset), \dot{F}(\emptyset))$.

(b): For all $\beta \in H$ with $\emptyset < \beta$ we have
 $q \upharpoonright \beta \Vdash (q(\beta), \dot{G}(\beta)) \leq (p(\beta), \dot{F}(\beta))$.

Remark: A simple induction shows that $p := \bigcap_{i \in \omega} p_i \in S_\alpha$.

Definition 3.14. Let $\alpha \in OR, p \in S_\alpha, H \in [\text{supp}(p)]^{<\omega}$ and \dot{F} be a H -front of p .

We say that p is (H, \dot{F}) -*deciding* iff:

For all $\bar{\sigma} \in (2^{<\omega})^H$

either: $\forall \beta \in H [(p \upharpoonright \beta) \upharpoonright (\bar{\sigma} \upharpoonright \beta) \Vdash \bar{\sigma}(\beta) \in \dot{F}(\beta)]$

or: $\exists \gamma \in H \forall \beta \in H [\beta < \gamma \rightarrow (p \upharpoonright \beta) \upharpoonright (\bar{\sigma} \upharpoonright \beta) \Vdash \bar{\sigma}(\beta) \in \dot{F}(\beta)$
 $\wedge (p \upharpoonright \gamma) \upharpoonright (\bar{\sigma} \upharpoonright \gamma) \Vdash \bar{\sigma}(\gamma) \notin \dot{F}(\gamma)]$

With $(p \upharpoonright \beta) \upharpoonright (\bar{\sigma} \upharpoonright \beta)$ meaning the tree that is restricted in each coordinate $\zeta \in H \cap \beta$ to the nodes that are compatible to $\bar{\sigma}(\zeta)$.

Remark: Let p be (H, \dot{F}) -deciding and $\beta_0 < .. < \beta_n$ an increasing enumeration of H . Then $p \upharpoonright \beta_0$ decides, whether a node is a member of the front $\dot{F}(\beta_0)$ of $p(\beta_0)$ or not. If we restrict p in the β_0 -th coordinate to some σ_0 that is forced to be in the front of $p(\beta_0)$ the restriction decides, what $\dot{F}(\beta_1)$ looks like. More generally, we can gradually restrict p coordinatewise to nodes that are in the respective fronts to decide what the front of the next coordinate is. Because of this, for $\bar{\sigma} \in (2^{<\omega})^H$ we say sometimes “ $\bar{\sigma} \in \dot{F}$ ”, if $\bar{\sigma}$ satisfies the first condition of the definition above and in this case we denote by $p_{\bar{\sigma}}$ the restriction of $p(\beta)$ by $\bar{\sigma}(\beta)$ for all $\beta \in H$.

Before we can go on and prove the existence of an injective continuous reading of names of countable support iterations of the splitting tree forcing, we need to prove a somewhat technical lemma.

Again, V will be a model for $ZFC + CH$ and $V_{\omega_2}^S$ will be a model that is obtained by countable support iteration of the splitting tree forcing of length ω_2 . With respect to this, we will denote, for $\delta < \omega_2$, by V_δ the intermediate model that the iteration produces at the δ -th step of the iteration. In the following context we will identify (by a dense embedding) the forcing $S_{OT([\delta, \alpha])}$ in V_δ , with the forcing $S_{\delta\alpha} := \{p \in S_\alpha : \text{supp}(p) \subseteq [\delta, \alpha]\}$ ordered by $p \leq q \Leftrightarrow \exists f \in G_\delta (f \frown p \leq f \frown q)$.

Lemma 3.15. *Let $\delta < \alpha \in OR$, $H = \{\beta_0, \dots, \beta_n\} \in [\delta, \alpha]^{<\omega}$ Furthermore let $p \in S_{\delta\alpha}$ and \dot{F} a H -front of p .*

Then the following holds true (in V_δ):

(i):

For any \dot{f} that is a $S_{\delta\alpha}$ -name for a function $f : \omega \rightarrow V_\delta$ and for any $i \in \omega$ there exists a condition $q \in S_{\delta\alpha}$ and a H -front \dot{G} of q such that $(q, \dot{G}) \leq_H (p, \dot{F})$, q is (H, \dot{G}) -deciding and for all $\bar{\sigma} \in \dot{G}$ we have that $q_{\bar{\sigma}}$ decides the value of $\dot{f}(i)$.

(ii):

For any \dot{f} that is a $S_{\delta\alpha}$ -name for a function $f : \omega \rightarrow V_\delta$ there exists a condition $q \in S_{\delta\alpha}$ and a sequence $\langle (G_i, H_i) : i \in \omega \rangle$, $H_i \in [\delta, \alpha]^{<\omega}$ and the \dot{G}_i are H_i -fronts of q such that $(q, \dot{G}_i) \leq_{H_i} (p, \dot{F})$, q is (H_i, \dot{G}_i) -

deciding and for all $i \in \omega$ and $\bar{\sigma} \in \dot{G}_i$ we have that $q_{\bar{\sigma}}$ decides the value of $\dot{f}(i)$.

(iii):

For all $\gamma > \alpha$ and \dot{f} that are $S_{\delta\alpha}$ -names for elements of $S_{\alpha\gamma}$ there exists $(q, \dot{G}) \leq_H (p, \dot{F})$ and $f \in S_{\alpha\gamma}$ ($f \in V_\delta$) with q is (H, \dot{G}) -deciding and $q \Vdash \dot{f} = f$.

Proof. For each δ, α (i) \Rightarrow (ii):

By iteration we get a sequence $\langle (q_i, H_i, \dot{G}_i) : i \in \omega \rangle$ with $H_{i+1} \supseteq H_i$, $H_0 \supseteq H$ and $(q_{i+1}, \dot{G}_{i+1}) \leq_{H_{i+1}} (q_i, \dot{G}_i)$, $(q_0, \dot{H}_0) \leq_{H_i} (p, \dot{F})$ and $\bigcup_{i \in \omega} H_i = \bigcup_{i \in \omega} \text{supp}(q_i)$ as in (i).

We get the desired q by defining $q := \bigcap_{i \in \omega} q_i$ (see the remark in Definition 3.13).

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Also for each δ, α (ii) \Rightarrow (iii):

This is pretty much the same as in the proof of Lemma 2.3 (iv) in [BL79].

By (ii) we have $q \leq p$ that forces that $\text{supp}(\dot{f}) \subseteq A = \{a_i : i \in \omega\}$ for countable $A \in V$ and we have a sequence $\langle \dot{G}_i : i \in \omega \rangle$, such that for every $\bar{\sigma} \in \dot{G}_i$ we have a set $f_{\bar{\sigma}}$ with $q \upharpoonright \bar{\sigma} \Vdash \dot{f}(a_i) = f_{\bar{\sigma}}$.

Define (in V_δ)

$f(a_i) := \{(\tau, \bar{r}) : \exists r \exists \bar{\sigma} \in G_i ((\tau, r) \in f_{\bar{\sigma}} \wedge \bar{r} \leq r \wedge \bar{r} \leq (q \upharpoonright \bar{\sigma}) \frown \mathbb{1}_{[\alpha, a_i]})\}$
as well as $f \upharpoonright [\alpha, \gamma] \setminus A \equiv \mathbb{1}$.

An easy calculation shows that $q \Vdash \dot{f} = f$

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It remains to show that (i) holds true for all $\delta, \alpha \in OR$. This will be done by induction on α (i.e. for $\alpha = 1$ and $\alpha > 1$) using (i)-(iii) for $\xi < \alpha$ in the induction hypothesis.

$\alpha = 1$:

This is similar to the proof of Lemma 3 in [Spi07] (here it will be easier because we do not need to force incompatible decisions here). Let $p \in S$ be arbitrary with a front F (and $H = \{\emptyset\}$).

Let $F = \{\rho_0, \dots, \rho_r\}$ enumerate the front.

Fix some $v \leq r$. We want to construct our new subtree q_{ρ_v} of p_{ρ_v} in

the following way.

Step 1:

Let $s := \text{stem}(p_{\rho_v})$, define $K := K_{p_{\rho_v}}(s \smallfrown 0)$.

Enumerate $2^K \cap p_{s \smallfrown 1}$ by $\{\sigma_j : j < l\}$.

Next, for any $j < l$, choose a subtree q_{σ_j} of p_{σ_j} such that $q_{\sigma_j} \Vdash \dot{f}(i) = x_{\sigma_j}$ for some $x_{\sigma_j} \in V$.

Step 2:

Let $K^* := \min\{K_{q_{\sigma_j}}(\sigma_j) : j < l\}$.

Enumerate $2^{K^*} \cap p_{s \smallfrown 0}$ by $\{\mu_j : j < t\}$.

Choose subtrees $q_{\mu_j} \leq p_{\mu_j}$ with $q_{\mu_j} \Vdash \dot{f}(i) = x_{\mu_j}$ for some $x_{\mu_j} \in V$.

After these two steps we define $q^{(v)} := \bigcup_{j < l} q_{\sigma_j} \cup \bigcup_{j < t} q_{\mu_j}$ with a front $G^{(v)} := \{\sigma_j : j < l\} \cup \{\mu_j : j < t\}$.

After doing this construction for each $v \leq r$ define $q := \bigcup_{v \leq r} q^{(v)}$ together with a front $G := \bigcup_{v \leq r} G^{(v)}$.

It is easy to see that $p_\sigma[n] = q_\sigma[n]$ for all $\sigma \in F$ and $n \in \omega$ and the other statements of (i) should be pretty clear by construction.

⊣

$\alpha > 1$:

We show the lemma for $\delta = \emptyset$ (For $\delta > \emptyset$ the proof is completely analogous).

We first work in V_{β_n} and prove (i) for $H_n := \{\beta_n\}$ instead of H .

If $\alpha = \beta_n + 1$ the proof is the same as in the case $\alpha = 1$. So we can assume that $\alpha > \beta_n + 1$.

Let $F(\beta_n) = \{\rho_0, \dots, \rho_r\}$ and fix some $v \leq r$.

Step 1:

We define $s := \text{stem}(p(\beta_n))_{\rho_v}$, $K := K_{p(\beta_n)}(s \smallfrown 0)$

and enumerate $2^K \cap p(\beta_n)_{s \smallfrown 1}$ by $\{\sigma_j : j < l\}$.

For each σ_j we find $q(\beta_n)_{\sigma_j} \in S$ and $g_{\sigma_j} \in S_{\beta_n+1}^\alpha$ such that $q(\beta_n)_{\sigma_j} \smallfrown g_{\sigma_j} \leq p(\beta_n)_{\sigma_j} \smallfrown p \upharpoonright [\beta_n+1, \alpha)$ and $q(\beta_n)_{\sigma_j} \smallfrown g_{\sigma_j} \Vdash \dot{f}(i) = x_{\sigma_j}$ for some $x_{\sigma_j} \in V$.

Step 2:

Let $K^* := \min\{K_{q(\beta_n)_{\sigma_j}}(\sigma_j) : j < l\}$.

Enumerate $2^{K^*} \cap p(\beta_n)_{s \setminus 0}$ by $\{\mu_j : j < t\}$.

Choose as above $q(\beta_n)_{\mu_j} \dot{\wedge} g_{\mu_j} \leq p(\beta_n)_{\mu_j} \dot{\wedge} p \upharpoonright [\beta_n + 1, \alpha)$ with $q(\beta_n)_{\mu_j} \dot{\wedge} g_{\mu_j} \Vdash \dot{f}(i) = x_{\mu_j}$ for some $x_{\mu_j} \in V$.

After doing these two steps we define $q^{(v)}(\beta_n) := \bigcup_{j < l} q(\beta_n)_{\sigma_j} \cup \bigcup_{j < t} q(\beta_n)_{\mu_j}$ and define a front $G^{(v)} := \{\sigma_j : j < l\} \cup \{\mu_j : j < t\}$.

After doing this construction for each $v \leq r$ define $q(\beta_n) := \bigcup_{v \leq r} q^{(v)}(\beta_n)$ with the front $G := \bigcup_{v \leq r} G^{(v)}$.

Let \dot{g} be a name for the union of $\{(g_\sigma, q(\beta_n)_\sigma) : \sigma \in G\}$. Thus, $q(\beta_n)_\sigma \Vdash (\dot{g} \Vdash^* \dot{f}(i) = x_\sigma)$.

Now, by induction hypothesis (i.e. (iii) for $\beta_n + 1 < \alpha$), we know that there exists $g \in S_{\beta_n+1} \alpha$ and $(q'(\beta_n), G') \leq_{\{\beta_n\}} (q(\beta_n), G)$ ($\Rightarrow (q'(\beta_n), G) \leq_{\{\beta_n\}} (p(\beta_n), F(\beta_n))$) such that $q'(\beta_n) \Vdash \dot{g} = g$.

So for each $\sigma \in G$ we have that $q'(\beta_n)_\sigma \dot{\wedge} g \Vdash \dot{f}(i) = x_\sigma$.

We can conclude that $q'(\beta_n) \dot{\wedge} g$ with the $\{\beta_n\}$ -front G witnesses (i) of the lemma for H_n in V_{β_n} .

If $\beta_{n-1} + 1 < \beta_n$ and we work in $V_{\beta_{n-1}+1}$ we can conclude that $p \upharpoonright [\beta_{n-1} + 1, \beta_n) \Vdash (\dot{\bar{q}}, \dot{G})$ witnesses the lemma for $\{\beta_n\}$ in V_{β_n} for some names $\dot{\bar{q}}$ and \dot{G} .

We can pick $q \upharpoonright [\beta_{n-1} + 1, \beta_n) \leq p \upharpoonright [\beta_{n-1} + 1, \beta_n)$ that forces $\dot{G} = G$, $\dot{\bar{q}} = \bar{q}$ for some G, \bar{q} and decides the respective x_σ . It follows that in this case $q := q \upharpoonright [\beta_{n-1} + 1, \beta_n) \dot{\wedge} \bar{q}$ and G witnesses the lemma in $V_{\beta_{n-1}+1}$ for H_n .

To finish the induction step for $\alpha > 1$ we have to show that, if the lemma holds true in $V_{\beta_{k+1}}$ for $H_{k+1} := \{\beta_{k+1}, \dots, \beta_n\}$, it holds true in $V_{\beta_{k-1}+1}$ (or V for $k = 0$) for $H_k := \{\beta_k, \beta_{k+1}, \dots, \beta_n\}$ (with $H_0 = H$).

We start to work in V_{β_k} . Let $p \in S_{\beta_k \alpha}$ and \dot{F} a H_k -front. Let $\dot{F}(\beta_k) = \{\rho_0, \dots, \rho_r\}$ and fix $v \leq r$.

Step 1 ' :

Define $s := \text{stem}(p(\beta_k)_{\rho_v})$ and $K := K(s \smallfrown 0)$.

Enumerate $2^K \cap p(\beta_k)_{s \smallfrown 1}$ by $\{\sigma_j : j < l\}$.

Now by the induction hypothesis we know that

$p(\beta_k)_{\sigma_j} \Vdash (\dot{g}^{(\sigma_j)}, \dot{G}_{\sigma_j}) \leq_{H_{k+1}} (p \upharpoonright [\beta_k + 1, \alpha], \dot{F} \upharpoonright H_{k+1})$, such that the lemma holds true.

We find $q(\beta_k)_{\sigma_j} \leq p(\beta_k)_{\sigma_j}$ that forces $\dot{g}^{(\sigma_j)} = g^{(\sigma_j)}$, decides for all $\bar{\sigma} \in {}^{H_{k+1}}(2^{<\omega})$, if $\bar{\sigma} \in \dot{G}_{\sigma_j}$ (w.r.t Definition 3.14) and forces $x_{\bar{\sigma}} = x_{\bar{\sigma}}$ for $\bar{\sigma} \in \dot{G}_{\sigma_j}$.

Hence, for $q(\beta_k)_{\sigma_j} \Vdash \bar{\sigma} \in \dot{G}_{\sigma_j}$ we have $q(\beta_k)_{\sigma_j} \smallfrown g_{\bar{\sigma}}^{(\sigma_j)} \Vdash \dot{f}(i) = x_{\sigma_j \bar{\sigma}}$.

Step 2 ' :

Define $K^* := \min\{K_{q(\beta_k)_{\sigma_j}}(\sigma_j) : j < l\}$.

Enumerate $2^{K^*} \cap p(\beta_k)_{s \smallfrown 0}$ by $\{\mu_j : j < t\}$.

Again we find $q(\beta_k)_{\mu_j} \leq p(\beta_k)_{\mu_j}$ and $g^{(\mu_j)}, \dot{G}_{\mu_j}, x_{\mu_j \bar{\sigma}}$ as in Step 1 ' .

After these two steps define $q(\beta_k)^{(v)} := \bigcup_{j < l} q(\beta_k)_{\sigma_j} \cup \bigcup_{j < t} q(\beta_k)_{\mu_j}$. Let $G^{(v)}(\beta_k) := \{\sigma_j : j < l\} \cup \{\mu_j : j < t\}$.

After doing this construction for all $v \leq r$ define $q(\beta_k) := \bigcup_{v \leq r} q(\beta_k)^{(v)}$ and $G(\beta_k) := \bigcup_{v \leq r} G^{(v)}(\beta_k)$.

Now let \dot{g} be a name for the union of $\{(g^{(\sigma)}, q(\beta_k)_{\sigma}) : \sigma \in G(\beta_k)\}$.

By induction hypothesis for $\beta_k + 1 < \alpha$ we get that there exists a \bar{g} in

$S_{\beta_k+1} \alpha$ and $(q'(\beta_k), \tilde{G}) \leq_{\{\beta_k\}} (q(\beta_k), G(\beta_k))$ such that $q'(\beta_k) \Vdash \dot{g} = \bar{g}$.

Define $q' := q'(\beta_k) \hat{\ } \bar{g}$.

Because, for $\sigma \in G(\beta_k)$, \dot{G}_σ is decided by $q'(\beta_k)_\sigma$ and forced to be as in the lemma for H_{k+1} we can define us canonically a H_k -front \dot{G} such that q' is (H_i, \dot{G}) -deciding $(q', \dot{G}) \leq_{H_i} (p, \dot{F})$ and $q'_\sigma \Vdash \dot{f}(i) = x_\sigma$ for $\sigma \in \dot{G}$.

Thus, the lemma holds true in V_{β_k} for H_k and as before we can show that, if $\beta_{k-1} + 1 < \beta_k$, the lemma also holds true in $V_{\beta_{k-1}+1}$ for H_k or, if $k = 0$, we can show that in V the lemma holds true for H . This proves the lemma for $\alpha > 1$. □

We get the following corollary.

Corollary 3.16. *Let $\alpha \in OR$, $p \in S$, F a front of p that is partitioned into two disjoint subsets, lets say $F = F_{good} \dot{\cup} F_{bad}$ and let $g \in S_{[1,\alpha]}$. Furthermore let \dot{f} be a S -Name for an element of $S_{[1,\alpha]}$, such that $\forall \sigma \in F_{bad} (p \upharpoonright \sigma \Vdash \dot{f} = g)$.*

Then there exists $q \in S$ and a front G of q with $(q, G) \leq (p, F)$ and $\forall \sigma \in F_{bad} q \upharpoonright \sigma = p \upharpoonright \sigma$ and there exists $f \in V_{[1,\alpha]}$ (in V), such that $q \Vdash \dot{f} = f$.

Proof. By Lemma 3.15 ii) we have $(q', F_{good}) \leq (\bigcup_{\sigma \in F_{good}} p \upharpoonright \sigma, F_{good})$, such that q forces that $supp(\dot{f}) \cup supp(g) \subseteq A = \{a_i : i \in \omega\}$ for countable $A \in V$ and we have a sequence $\langle G_i : i \in \omega \rangle$ of refining fronts of q' , such that for every $\sigma \in G_i$ we have a set f_σ with $q' \upharpoonright \sigma \Vdash \dot{f}(a_i) = f_\sigma$. Define (in V)

$$f(a_i) := \{(\tau, \bar{r}) : \exists \sigma \in G_i \exists r ((\tau, r) \in f_\sigma \wedge \bar{r} \leq r \wedge \bar{r} \leq (q' \upharpoonright \sigma) \hat{\ } \mathbb{1}_{[1,a_i]}) \vee \exists \sigma \in F_{bad} \exists r ((\tau, r) \in g(a_i) \wedge \bar{r} \leq r \wedge \bar{r} \leq (p \upharpoonright \sigma) \hat{\ } \mathbb{1}_{[1,a_i]})\}.$$

Define $q := q' \cup \bigcup_{\sigma \in F_{bad}} (p \upharpoonright \sigma)$. Obviously we have $q \upharpoonright \sigma = p \upharpoonright \sigma$ for all $\sigma \in F_{bad}$ and an easy calculation shows that $q \Vdash \dot{f} = f$. □

Now, we prove the existence of an injective continuous reading of names for the iterated splitting tree forcing.

Lemma 3.17. *Let $\alpha \leq \omega_2$, $b \in \omega$, $H = \{\beta_0, \dots, \beta_n\} \in [\alpha]^{<\omega}$, \dot{x} a S_α -Name.*

Furthermore let $p \in S_\alpha$ with
 $p \Vdash \dot{x} \in 2^\omega \wedge \dot{x} \notin V_\xi$ for all $\xi < \alpha$.

Let \dot{F} be an H -front such that p is (H, \dot{F}) -deciding.

Then there exists $q \leq p$ and a H -front \dot{G} of q , such that

(i): $(q, \dot{G}) \leq (p, \dot{F})$ with \dot{G} being a strict refinement of \dot{F} .

(ii): q is (H, \dot{G}) -deciding.

(iii): For all $\bar{\sigma} \in \dot{G}$ exists $\xi_{\bar{\sigma}} \in 2^{<\omega}$ with $q \upharpoonright \bar{\sigma} \Vdash \xi_{\bar{\sigma}} \subseteq \dot{x} \wedge \text{length}(\xi_{\bar{\sigma}}) \geq b$

and for $\bar{\sigma}, \bar{\sigma}'$ with $\bar{\sigma} \neq \bar{\sigma}'$ we have $\xi_{\bar{\sigma}} \perp \xi_{\bar{\sigma}'}$.

Proof. We start with a remark. The induction in the following proof has a similar structure as in Lemma 3.15. Coordinate-wise we are doing a similar construction as in [Spi07] to force incompatible initial segments of \dot{x} .

Consider the two dimensional case. We are able to construct a condition q that is stronger than p and a front in the first coordinate, such that by restricting q to a member of this front, the respective restriction forces that q has in the second component a front as desired. Namely, every member of the second front decides an initial segment of \dot{x} and all these initial segments are incompatible. The problem is, if we restrict q to two different members of the first front, we do not necessarily force incompatible initial segments any more. If we had Sacks forcing instead of splitting tree forcing, for each member of the first front, we could just make the respective front in the second coordinate big enough and then drop some nodes in each of the fronts of the second component. If we did the same with splitting tree forcing, in general, our trees would lose the splitting tree property. We can solve the problem by using the same idea as in the case of Sacks forcing and interweaving it more carefully into the induction.

We proof the Lemma by induction on α .

$\alpha = 1$:

Essentially this is the same construction as in Lemma 3 of [Spi07]. In addition we do a precise bookkeeping of all substeps that occur. (For easier reading we write F instead of $\dot{F}(0)$).

Let $F = \{\rho_0, \dots, \rho_v\}$. For $v \leq r$ define $s_v := \text{stem}(p \upharpoonright \rho_v)$, $s'_v := \text{stem}(p \upharpoonright s_v \frown 0)$ and $K := \max\{K_p(\rho_v) : v \leq r\}$.

Step 1:

Define

$$F^{(0)} := p \cap 2^K,$$

$$F_{\text{good}}^{(0)} := \bigcup_{v \leq r} [(p \upharpoonright s'_v \frown 0) \cup (p \upharpoonright s'_v \frown 1)] \cap 2^K,$$

$$F_{\text{bad}}^{(0)} := \bigcup_{v \leq r} (p \upharpoonright s_v \frown 1) \cap 2^K.$$

That means we partition the front $F^{(0)}$ into “good” and “bad” nodes. Above the good nodes we will be allowed to thin out the tree to decide initial segments of \dot{x} (or rather other statements). The subtrees that consist of all nodes that are compatible to a bad node stay fixed. This ensures that for the tree q that we will construct, we will have $(q \upharpoonright \rho_v)[k] = (p \upharpoonright \rho_v)[k]$ for all $v \leq r$ and $k \in \omega$.

Let $\{\sigma_i : i < l\}$ enumerate the members of $F_{\text{good}}^{(0)}$. For all $i < l$ exists a condition $q^{(0)} \upharpoonright \sigma_i \leq p \upharpoonright \sigma_i$ with $q^{(0)} \upharpoonright \sigma_i \Vdash \xi_{\sigma_i} \subseteq \dot{x}$, such that $\xi_{\sigma_i} \perp \xi_{\sigma_j}$ and $|\xi_{\sigma_i}| \geq b$ for all $i, j < l; i \neq j$ ($p \Vdash \dot{x} \notin V$).

$$\text{Define } q^{(0)} := \bigcup_{i < l} (q^{(0)} \upharpoonright \sigma_i) \cup \bigcup_{\tau \in F_{\text{bad}}^{(0)}} (p \upharpoonright \tau).$$

This finishes the first construction step.

For later use let $\Phi^{(0)}((p, F), (p, F), (q^{(0)}, F_{\text{good}}^{(0)}, F_{\text{bad}}^{(0)}))$ be the statement that $q^{(0)} \leq p$ is built as in the first construction step with respective good and bad nodes.

The first parameter is the pair (p, F) of the premise of the lemma. The nodes of the front of the first parameter are used to calculate the the good and bad nodes (i.e to define s_v , etc.. in this case). The second component lists the condition/front of the previous construction step. The number 0 indicates the construction step (in this case that the bad nodes are exactly the ones of length K that are compatible to some $s_v \frown 1$).

Note that the statement that incompatible initial segments of \dot{x} are forced by q^0 restricted to the respective good nodes is not included in $\Phi^{(0)}$. This will be handled separately.

Step 2:

$$\text{Let } K^* := \max : \{K_{q^{(0)}}(\tau) : \tau \in F^{(0)}\}.$$

Define

$$\begin{aligned}
F^{(1)} &:= q^{(0)} \cap 2^{K^*}, \\
F_{good}^{(1)} &:= \bigcup_{v \leq r} [(q^{(0)} \upharpoonright s'_v \hat{\ } 0) \cup (q^{(0)} \upharpoonright s_v \hat{\ } 1)] \cap 2^{K^*}, \\
F_{bad}^{(1)} &:= \bigcup_{v \leq r} (q^{(0)} \upharpoonright s'_v \hat{\ } 1) \cap 2^{K^*}.
\end{aligned}$$

Let $\{\mu_i : i < k\}$ enumerate $F_{good}^{(1)}$ and choose $q^{(1)} \upharpoonright \mu_i \leq q^{(0)} \upharpoonright \mu_i$ with $q^{(1)} \upharpoonright \mu_i \Vdash \xi_{\mu_i} \subseteq \dot{x} \wedge |\xi_{\mu_i}| \geq b$ and $\xi_{\mu_i} \perp \xi_{\mu_j}$ for $i, j < k; i \neq j$ and $\xi_{\mu_i} \supseteq \xi_\sigma$ for $\sigma \in F_{good}^{(0)}$ with $\mu_i \supseteq \sigma$.

$$\text{Define } q^{(1)} := \bigcup_{i < k} (q^{(1)} \upharpoonright \mu_i) \cup \bigcup_{\tau \in F_{bad}^{(1)}} (q^{(0)} \upharpoonright \tau).$$

As before, we let $\Phi^{(1)}((p, F), (q^{(0)}, F_{good}^{(0)} \cup F_{bad}^{(0)}), (q^{(1)}, F_{good}^{(1)}, F_{bad}^{(1)}))$ be the respective statement for the second construction step.

Step 3:

$$\text{Let } K^{**} := \max\{K_{q^{(1)}}(\tau) : \tau \in F^{(1)}\}.$$

Define

$$\begin{aligned}
F^{(2)} &:= q^{(1)} \cap 2^{K^{**}}, \\
F_{good}^{(2)} &:= \bigcup_{v \leq r} [(q^{(1)} \upharpoonright s'_v \hat{\ } 1) \cup (q^{(1)} \upharpoonright s_v \hat{\ } 1)] \cap 2^{K^{**}}, \\
F_{bad}^{(2)} &:= \bigcup_{v \leq r} (q^{(1)} \upharpoonright s'_v \hat{\ } 0) \cap 2^{K^{**}}.
\end{aligned}$$

Let $\{\nu_i : i < m\}$ enumerate $F_{good}^{(2)}$ and choose $q^{(2)} \upharpoonright \nu_i \leq q^{(1)} \upharpoonright \nu_i$ with $q^{(2)} \upharpoonright \nu_i \Vdash \xi_{\nu_i} \subseteq \dot{x} \wedge |\xi_{\nu_i}| \geq b$ and $\xi_{\nu_i} \perp \xi_{\nu_j}$ for $i, j < m; i \neq j$ and $\xi_{\nu_i} \supseteq \xi_\sigma$ for σ that is a good node of the previous steps with $\nu_i \supseteq \sigma$.

$$\text{Define } q^{(2)} := \bigcup_{i < m} (q^{(2)} \upharpoonright \nu_i) \cup \bigcup_{\tau \in F_{bad}^{(2)}} (q^{(1)} \upharpoonright \tau).$$

Also $\Phi^{(2)}((p, F), (q^{(1)}, F_{good}^{(1)} \cup F_{bad}^{(1)}), (q^{(2)}, F_{good}^{(2)}, F_{bad}^{(2)}))$ is the formula that describes the third construction step with respect to the good and bad nodes.

After we have done the three construction steps we define

$$G := F_{good}^{(2)} \cup \{\sigma \in F_{good}^{(1)} : \exists v \leq r \ s'_v \hat{\ } 0 \subseteq \sigma\}.$$

We have $(q^{(2)}, G) \leq_H (p, F)$ and one can easily see that $(q^{(2)}, G)$ witnesses the statement of this lemma. This proves the lemma for the case that $\alpha = 1$.

Remarks:

- Without loss of generality, in every step we can choose the ξ_σ of equal length.
- if $\Phi^{(i)}((p, F), (q^{(i-1)}, F^{(i-1)}), (q^{(i)}, F_{good}^{(i)}, F_{bad}^{(i)}))$ is true and we have $q'^{(i)} \leq q^{(i)}$, such that $F_{good}^{(i)} \cup F_{bad}^{(i)}$ is a front of $q'^{(i)}$ and with $q'^{(i)} \upharpoonright \sigma = q^{(i)} \upharpoonright \sigma$ for all $\sigma \in F_{bad}^{(i)}$, then we can conclude that $\Phi^{(i)}((p, F), (q^{(i-1)}, F^{(i-1)}), (q'^{(i)}, F_{good}^{(i)}, F_{bad}^{(i)}))$.
That means that above the good nodes we are allowed to thin out the trees of the respective steps as we wish.

$\alpha \geq 1$:

Before we start with the actual proof of the induction step, we observe the situation for the special case that $\alpha = 2$ and $H = \{0, 1\}$. This will make the description of the general case a little easier.

We have given $p = \langle p(0), \dot{p}(1) \rangle \in S_2$ as well as \dot{x} and \dot{F} as in the premise of the lemma. Also let $\{\rho_v : v \leq r\}$ enumerate the members of $\dot{F}(0)$.

We start to work in V and do the first construction step for $p(0)$, i.e

we define $s_v := stem(p(0) \upharpoonright \rho_v)$, $s'_v := stem(p(0) \upharpoonright s_v \frown 0)$,
 $K := max\{K_{p(0)}(s_v) : v \leq r\}$,

$$\begin{aligned} \dot{F}^{(0)}(0) &:= p(0) \cap 2^K \\ \dot{F}_{good}^{(0)}(0) &:= \bigcup_{v \leq r} [(p(0) \upharpoonright s'_v \frown 0) \cup (p(0) \upharpoonright s'_v \frown 1)] \cap 2^K, \\ \dot{F}_{bad}^{(0)}(0) &:= \bigcup_{v \leq r} (p(0) \upharpoonright s_v \frown 1) \cap 2^K. \end{aligned}$$

From the view of V_1 (the intermediate model that we get after forcing with S) we can prove the statement of the lemma with some modifications.

To be more precise, for every $\sigma \in \dot{F}_{good}^{(0)}(0)$ there exists $q^{(0)}(0) \upharpoonright \sigma \leq p(0) \upharpoonright \sigma$ and S -names $\dot{q}_\sigma^{(0)}(1)$, $\dot{F}_{\sigma, good}^{(0)}(1)$, $\dot{F}_{\sigma, bad}^{(0)}(1)$, such that
 $q^{(0)}(0) \upharpoonright \sigma \Vdash \Phi^{(0)}((\dot{p}(1), \dot{F}(1)), (\dot{p}(1), \dot{F}(1)), (\dot{q}_\sigma^{(0)}(1), \dot{F}_{\sigma, good}^{(0)}(1), \dot{F}_{\sigma, bad}^{(0)}(1)))$

and
 $q^{(0)}(0) \upharpoonright \sigma \Vdash$
 $[\forall \sigma(1) \in \dot{F}_{\sigma, \text{good}}^{(0)}(1) \exists \xi_{\sigma, \sigma(1)} : (\dot{q}_{\sigma}^{(0)}(1) \upharpoonright \sigma(1) \Vdash^* \xi_{\sigma, \sigma(1)} \subseteq \dot{x} \wedge |\xi_{\sigma, \sigma(1)}| \geq b)]$

and
 $q^{(0)}(0) \upharpoonright \sigma \Vdash \forall \sigma(1), \sigma'(1) \in \dot{F}_{\sigma, \text{good}}^{(0)}(1) : (\sigma(1) \neq \sigma'(1) \rightarrow \xi_{\sigma, \sigma(1)} \perp \xi_{\sigma, \sigma'(1)})$.

In addition we can assume that $q^{(0)}(0) \upharpoonright \sigma$ decides $\dot{F}_{\sigma, \text{good}}^{(0)}(1)$, $\dot{F}_{\sigma, \text{bad}}^{(0)}(1)$ and $\xi_{\sigma, \sigma(1)}$, and that there exists $L \in \omega$ with $|\xi_{\sigma, \sigma(1)}| = L$, for all $\sigma \in \dot{F}_{\text{good}}^{(0)}(0)$, $\sigma(1) \in \dot{F}_{\text{good}}^{(0)}(1)$.

We can make this assumption, because if this is not the case, we can thin out $q^{(0)}(0)$ and $\dot{q}_{\sigma}^{(0)}(1)$ above the good nodes, such that the statement holds true.

Now, let $m := |\dot{F}_{\text{good}}^{(0)}(0)|$. We can assume, by the same argument as above, that for all $\sigma(0) \in \dot{F}_{\text{good}}^{(0)}(0)$, $\sigma(1) \in \dot{F}_{\sigma(0), \text{good}}^{(0)}(1)$ and $i < m$ there exists $\tau_{\sigma(0), \sigma(1), i} \supseteq \sigma(1)$ and $\xi_{\sigma(0), \sigma(1), i} \supseteq \xi_{\sigma(0), \sigma(1)}$ with

$$q^{(0)}(0) \upharpoonright \sigma(0) \Vdash [\tau_{\sigma(0), \sigma(1), i} \in \dot{q}_{\sigma(0)}^{(0)}(1) \wedge \dot{q}_{\sigma(0)}^{(0)}(1) \upharpoonright \tau_{\sigma(0), \sigma(1), i} \Vdash^* \xi_{\sigma(0), \sigma(1), i} \subseteq \dot{x}],$$

such that $\xi_{\sigma(0), \sigma(1), i} \perp \xi_{\sigma(0), \sigma(1), j}$ for $i \neq j$.

For all $\sigma(0), \sigma'(0) \in \dot{F}_{\text{good}}^{(0)}(0)$ with $\sigma(0) \neq \sigma'(0)$ and $\sigma(1) \in \dot{F}_{\sigma(0), \text{good}}^{(0)}(1)$ exists at most one node $\sigma'(1)$ with $\xi_{\sigma(0), \sigma(1)} \parallel \xi_{\sigma'(0), \sigma'(1)}$. Therefore, we can choose $i_{\sigma(0), \sigma(1)} \in m$ for all $\sigma(0) \in \dot{F}_{\text{good}}^{(0)}(0)$, $\sigma(1) \in \dot{F}_{\sigma(0), \text{good}}^{(0)}(1)$, such that $\xi_{\sigma(0), \sigma(1), i_{\sigma(0), \sigma(1)}} \perp \xi_{\sigma'(0), \sigma'(1), i_{\sigma'(0), \sigma'(1)}}$ for $(\sigma(0), \sigma(1)) \neq (\sigma'(0), \sigma'(1))$.

That means that we can assume without loss of generality (by shrinking the subtrees that sit above the good nodes), that

$$q^{(0)}(0) \upharpoonright \sigma(0) \Vdash [\dot{q}_{\sigma(0)}^{(0)}(1) \upharpoonright \sigma(1) \Vdash^* \xi_{\sigma(0), \sigma(1)} \subseteq \dot{x}],$$

with $\xi_{\sigma(0), \sigma(1)} \perp \xi_{\sigma'(0), \sigma'(1)}$ for $(\sigma(0), \sigma(1)) \neq (\sigma'(0), \sigma'(1))$.

We define a S -name $\overline{\dot{q}^{(0)}(1)}$ for the union of $\{(\dot{q}_{\sigma}^{(0)}(1), q^{(0)}(0) \upharpoonright \sigma) : \sigma \in \dot{F}_{\text{good}}^{(0)}(0)\} \cup \{(\dot{p}(1), q^{(0)}(0)) : \sigma \in \dot{F}_{\text{bad}}^{(0)}(0)\}$.

By Corollary 3.16 we can assume without loss of generality (by shrinking $q^{(0)}(0)$ above the good nodes) that we have a name $\dot{q}^{(0,0)}(1)$

for a member of S , with $q^{(0)}(0) \Vdash \dot{q}^{(0,0)}(1) = \overline{\dot{q}^{(0)}(1)}$, in particular

$$q^{(0)}(0) \upharpoonright \sigma \Vdash \dot{q}^{(0,0)}(1) = \dot{q}_\sigma^{(0)}(1) \text{ for all } \sigma \in \dot{F}_{good}^{(0)}(0) \text{ and}$$

$$q^{(0)}(0) \upharpoonright \sigma \Vdash \dot{q}^{(0,0)}(1) = \dot{p}(1) \text{ for all } \sigma \in \dot{F}_{bad}^{(0)}(0).$$

Obviously there exist canonical names $\dot{F}^{(0,0)}(1), \dot{F}_{good}^{(0,0)}(1), \dot{F}_{bad}^{(0,0)}(1)$ with

$$q \Vdash \dot{F}^{(0,0)}(1) = \dot{F}_{good}^{(0,0)}(1) \cup \dot{F}_{bad}^{(0,0)}(1),$$

$$q^{(0)}(0) \upharpoonright \sigma \Vdash \dot{F}_{good}^{(0,0)}(1) = \dot{F}_{\sigma(0),good}^{(0)}(1) \wedge \dot{F}_{bad}^{(0,0)}(1) = \dot{F}_{\sigma(0),bad}^{(0)}(1) \text{ for all}$$

$$\sigma \in \dot{F}_{good}^{(0)}(0) \text{ and}$$

$$q^{(0)}(0) \upharpoonright \sigma \Vdash \dot{F}_{good}^{(0,0)}(1) = \emptyset \wedge \dot{F}_{bad}^{(0,0)}(1) = \dot{F}(1) \text{ for all } \sigma \in \dot{F}_{bad}^{(0)}(0).$$

Now we do the second construction step in V_1 analogously and decide the respective objects by shrinking $q^{(0)}(0)$ above the good nodes. This way we get $q^{(0,1)}(0) \leq q^{(0)}(0), \dot{F}^{(0,1)}(1), \dot{F}_{good}^{(0,1)}(1), \dot{F}_{bad}^{(0,1)}(1), \dot{q}^{(0,1)}(1)$ such that

$$q^{(0,1)}(0) \Vdash \dot{F}^{(0,1)}(1) = \dot{F}_{good}^{(0,1)}(1) \cup \dot{F}_{bad}^{(0,1)}(1),$$

$$\dot{q}^{(0,1)}(0) \upharpoonright \sigma \text{ decides the members of } \dot{F}^{(0,1)}(1), \dot{F}_{good}^{(0,1)}(1), \dot{F}_{bad}^{(0,1)}(1) \text{ for all}$$

$$\sigma \in \dot{F}^{(0)}(0) \text{ (as above) and}$$

$$q^{(0,1)}(0) \upharpoonright \sigma \Vdash$$

$$\Phi^{(1)}((p(0), \dot{F}(0)), (\dot{q}^{(0,0)}(1), \dot{F}^{(0,0)}(1), (\dot{q}^{(0,1)}, \dot{F}_{good}^{(0,1)}(1), \dot{F}_{bad}^{(0,1)}(1))))$$

$$\text{for all } \sigma \in \dot{F}_{good}^{(0)}(0),$$

$$\text{and such that } q^{(0,1)}(0) \upharpoonright \sigma = q^{(0)}(0) \upharpoonright \sigma \text{ for all } \sigma \in \dot{F}_{bad}^{(0)}(0)$$

In addition, for all $\sigma(0) \in \dot{F}_{good}^{(0)}(0)$ and all $\sigma(1)$ with $q^{(0,1)}(0) \upharpoonright \sigma(0) \Vdash \sigma(1) \in \dot{F}_{good}^{(0,1)}(1)$, we have $q^{(0,1)}(0) \upharpoonright \sigma(0) \Vdash [\dot{q}^{(0,1)}(1) \upharpoonright \sigma(1) \Vdash^* \xi_{\sigma(0),\sigma(1)} \subseteq \dot{x}]$.

and for all $\sigma(0), \sigma(1)$ as above and all $\tau(0), \tau(1)$ that are good nodes (of one of the good fronts constructed so far) with $\sigma(0) \supseteq \tau(0)$ and $\sigma(1) \supseteq \tau(1)$ we have $\xi_{\sigma(0),\sigma(1)} \supseteq \xi_{\tau(0),\tau(1)}$.

Again, we have $\xi_{\sigma(0),\sigma(1)} \perp \xi_{\sigma'(0),\sigma'(1)}$ for $(\sigma(0), \sigma(1)) \neq (\sigma'(0), \sigma'(1))$.

Analogously we do the third step in V_3 and decide the respective objects by shrinking $q^{(0,1)}(0)$ above the good nodes. This way we get $q^{(0,2)}(0) \leq q^{(0,1)}(0), \dot{F}^{(0,2)}(1), \dot{F}_{good}^{(0,2)}(1), \dot{F}_{bad}^{(0,2)}(1), \dot{q}^{(0,2)}(1)$ such that $q^{(0,2)}(0) \Vdash \dot{F}^{(0,2)}(1) = \dot{F}_{good}^{(0,2)}(1) \cup \dot{F}_{bad}^{(0,2)}(1)$, $\dot{q}^{(0,2)}(0) \upharpoonright \sigma$ decides the members of $\dot{F}^{(0,2)}(1), \dot{F}_{good}^{(0,2)}(1), \dot{F}_{bad}^{(0,2)}(1)$ for all $\sigma \in \dot{F}^{(0)}(0)$ (as above),

$q^{(0,2)}(0) \upharpoonright \sigma \Vdash$
 $\Phi^{(2)}((p(0), \dot{F}(0)), (\dot{q}^{(0,1)}(1), \dot{F}^{(0,1)}(1), (\dot{q}^{(0,2)}, \dot{F}_{good}^{(0,2)}(1), \dot{F}_{bad}^{(0,2)}(1))))$
for all $\sigma \in \dot{F}_{good}^{(0)}(0)$,
and such that $q^{(0,2)}(0) \upharpoonright \sigma = q^{(0,1)}(0) \upharpoonright \sigma$ for all $\sigma \in \dot{F}_{bad}^{(0)}(0)$.

In addition, for all $\sigma(0) \in \dot{F}_{good}^{(0)}(0)$ and all $\sigma(1)$ with $q^{(0,2)}(0) \upharpoonright \sigma(0) \Vdash \sigma(1) \in \dot{F}_{good}^{(0,2)}(1)$, we have $q^{(0,2)}(0) \upharpoonright \sigma(0) \Vdash [\dot{q}^{(0,2)}(1) \upharpoonright \sigma(1) \Vdash^* \xi_{\sigma(0), \sigma(1)} \subseteq \dot{x}]$

and for all $\sigma(0), \sigma(1)$ as above and all $\tau(0), \tau(1)$ that are good nodes with $\sigma(0) \supseteq \tau(0)$ and $\sigma(1) \supseteq \tau(1)$ we have $\xi_{\sigma(0), \sigma(1)} \supseteq \xi_{\tau(0), \tau(1)}$.

We have $\xi_{\sigma(0), \sigma(1)} \perp \xi_{\sigma'(0), \sigma'(1)}$ for $(\sigma(0), \sigma(1)) \neq (\sigma'(0), \sigma'(1))$.

Note the following. Let $\bar{\sigma} = \langle \sigma(0), \sigma(1) \rangle$ and $\bar{\sigma}' = \langle \sigma'(0), \sigma'(1) \rangle$ be sequences with $\bar{\sigma} \neq \bar{\sigma}'$,

$\sigma(0), \sigma'(0) \in \dot{F}_{good}^{(0)}(0)$ and

$q^{(0,2)} \upharpoonright \sigma(0) \Vdash$
 $\sigma(1) \in \dot{F}_{good}^{(0,2)}(1)$
 $\cup \{\sigma \in \dot{F}_{good}^{(0,1)}(1) : \exists \rho \in \dot{F}(1) (\sigma \supseteq \text{stem}(\dot{p}(1) \upharpoonright \text{stem}(\dot{p}(1) \upharpoonright \rho) \frown 0) \frown 0)\}$
and

$q^{(0,2)} \upharpoonright \sigma'(0) \Vdash$
 $\sigma'(1) \in \dot{F}_{good}^{(0,2)}(1)$
 $\cup \{\sigma \in \dot{F}_{good}^{(0,1)}(1) : \exists \rho \in \dot{F}(1) (\sigma \supseteq \text{stem}(\dot{p}(1) \upharpoonright \text{stem}(\dot{p}(1) \upharpoonright \rho) \frown 0) \frown 0)\}$.

Then there exist $\tau \subseteq \sigma(1), \tau' \subseteq \sigma'(1)$, which are both forced (by $q^{(0,2)} \upharpoonright \sigma(0)$ and $q^{(0,2)} \upharpoonright \sigma'(0)$ respectively) to be in $\dot{F}_{good}^{(0,i)}(1)$ for some $i \in 3$, such that $\langle \sigma(0), \tau \rangle \neq \langle \sigma'(0), \tau' \rangle$. We can conclude that $\xi_{\sigma(0),\tau} \perp \xi_{\sigma'(0),\tau'}$ and thus $\xi_{\bar{\sigma}} \perp \xi_{\bar{\sigma}'}$.

After we have done the three steps in the second coordinate, we go back to our first coordinate and do the second construction step.

We define $K^* := \max\{K_{q^{(0,2)}(0)}(\tau) : \tau \in \dot{F}^{(0)}(0)\}$,

$$\begin{aligned} \dot{F}^{(1)}(0) &:= q^{(0,2)}(0) \cap 2^{K^*} \\ \dot{F}_{good}^{(1)}(0) &:= \bigcup_{v \leq r} [(q^{(0,2)}(0) \upharpoonright s'_v \hat{\ } 0) \cup (q^{(0,2)}(0) \upharpoonright s_v \hat{\ } 1)] \cap 2^{K^*}, \\ \dot{F}_{bad}^{(1)}(0) &:= \bigcup_{v \leq r} (q^{(0,2)}(0) \upharpoonright s'_v \hat{\ } 1) \cap 2^{K^*}. \end{aligned}$$

We are then doing the three construction steps in V_1 in the same fashion as before, thus we get for all $i \in 3$:

$q^{(1,i)}, \dot{F}^{(1,i)}(1), \dot{F}_{good}^{(1,i)}(1), \dot{F}_{bad}^{(1,i)}(1)$ with:

$$q^{(0,2)} \leq q^{(1,0)} \leq q^{(1,1)} \leq q^{(1,2)},$$

$q^{(0,2)}(0) \upharpoonright \sigma = q^{(1,i)}(0) \upharpoonright \sigma$ and $q^{(1,i)}(0) \upharpoonright \sigma \Vdash q^{(0,2)}(1) = q^{(1,i)}(1)$ for all $\sigma \in \dot{F}_{bad}^{(1)}(0)$,

$q^{(1,i)}$ is $(H, \langle F^{(1)}(0), \dot{F}^{(1,i)}(1) \rangle)$ -deciding,

$$q^{(1,i)} \Vdash \dot{F}^{(1,i)}(1) = \dot{F}_{good}^{(1,i)}(1) \cup \dot{F}_{bad}^{(1,i)}(1),$$

$q^{(1,0)}(0) \upharpoonright \sigma \Vdash$
 $\Phi^{(0)}(\langle \dot{p}(1), \dot{F}(1) \rangle, \langle \dot{q}^{(0,2)}(1), \dot{F}^{(0,2)}(1) \rangle, \langle \dot{q}^{(1,0)}(1), \dot{F}_{good}^{(1,0)}(1), \dot{F}_{bad}^{(1,0)}(1) \rangle)$
for all $\sigma \in \dot{F}_{good}^{(1)}(0)$,

$q^{(1,i+1)}(0) \upharpoonright \sigma \Vdash$
 $\Phi^{(i+1)}(\langle \dot{p}(1), \dot{F}(1) \rangle, \langle \dot{q}^{(1,i)}(1), \dot{F}^{(1,i)}(1) \rangle, \langle \dot{q}^{(1,i)}(1), \dot{F}_{good}^{(1,i+1)}(1), \dot{F}_{bad}^{(1,i+1)}(1) \rangle)$
for all $\sigma \in \dot{F}_{good}^{(1)}(0)$.

In particular we can conclude that $(p, F) \leq_H (q^{(1,i)}, \langle F^{(1)}(0), \dot{F}^{(1,i)}(1) \rangle)$.

In addition we have for a fixed $i \in 3$,
that for all $\sigma(0) \in \dot{F}_{good}^{(1)}(0)$

and all $\sigma(1)$ with $q^{(1,i)}(0) \uparrow \sigma(0) \Vdash \sigma(1) \in \dot{F}_{good}^{(1,i)}(1)$,
that $q^{(1,i)} \uparrow \langle \sigma(0), \sigma(1) \rangle \Vdash \xi_{\sigma(0), \sigma(1)} \subseteq \dot{x}$, with $\xi_{\sigma(0), \sigma(1)} \perp \xi_{\sigma'(0), \sigma'(1)}$
for $(\sigma(0), \sigma(1)) \neq (\sigma'(0), \sigma'(1))$ and for all $\tau(0), \tau(1)$ that are good
nodes (of one of the good fronts constructed so far) with $\sigma(0) \supseteq \tau(0)$
and $\sigma(1) \supseteq \tau(1)$ we have $\xi_{\sigma(0), \sigma(1)} \supseteq \xi_{\tau(0), \tau(1)}$.

Next, we go back to our first coordinate and do the third construction step. Define $K^{**} := \max\{K_{q^{(1,2)}(0)}(\tau) : \tau \in \dot{F}^{(1)}(0)\}$,

$$\begin{aligned}\dot{F}^{(2)}(0) &:= q^{(0,2)}(0) \cap 2^{K^{**}} \\ \dot{F}_{good}^{(2)}(0) &:= \bigcup_{v \leq r} [(q^{(1,2)}(0) \uparrow s'_v \frown 1) \cup (q^{(1,2)}(0) \uparrow s_v \frown 1)] \cap 2^{K^{**}}, \\ \dot{F}_{bad}^{(2)}(0) &:= \bigcup_{v \leq r} (q^{(1,2)}(0) \uparrow s'_v \frown 0) \cap 2^{K^{**}}.\end{aligned}$$

We do the three construction steps from the view of V_1 and get for all $i \in 3$ $q^{(2,i)}$, $\dot{F}^{(2,i)}(1)$, $\dot{F}_{good}^{(2,i)}(1)$, $\dot{F}_{bad}^{(2,i)}(1)$ with the analog properties as above.

We can define a name \dot{G} for the H -front of $q^{(2,2)}$ that consists of all $\bar{\sigma} = \langle \sigma(0), \sigma(1) \rangle$ with either

$$\begin{aligned}\sigma(0) \in \dot{F}_{good}^{(2)}(0) \text{ and} \\ q^{(2,2)}(0) \uparrow \sigma(0) \Vdash \sigma(1) \in \dot{F}_{good}^{(2,2)}(1) \\ \cup \{\sigma \in \dot{F}_{good}^{2,1}(1) : \exists \rho \in \dot{F}(1) (\sigma \supseteq \text{stem}(\dot{p}(1) \uparrow \text{stem}(\dot{p}(1) \uparrow \rho) \frown 0) \frown 0)\}\end{aligned}$$

or

$$\begin{aligned}\sigma(0) \in \{\sigma \in \dot{F}_{good}^{(1)}(0) : \exists \rho \in \dot{F}(0) (\sigma \supseteq \text{stem}(p(0) \uparrow \text{stem}(p(0) \uparrow \rho) \frown 0) \frown 0)\} \\ \text{and} \\ q^{(2,2)}(0) \uparrow \sigma(0) \Vdash \sigma(1) \in \dot{F}_{good}^{(1,2)}(1) \\ \cup \{\sigma \in \dot{F}_{good}^{(1,1)}(1) : \exists \rho \in \dot{F}(1) (\sigma \supseteq \text{stem}(\dot{p}(1) \uparrow \text{stem}(\dot{p}(1) \uparrow \rho) \frown 0) \frown 0)\}.\end{aligned}$$

By construction $q^{(2,2)}$ is (H, \dot{G}) -deciding. Also for all $\bar{\sigma} \in \dot{G}$ we have defined $\xi_{\bar{\sigma}}$ with $q^{(2,2)} \uparrow \bar{\sigma} \Vdash \xi_{\bar{\sigma}} \subseteq \dot{x} \wedge |\xi_{\bar{\sigma}}| \geq b$.

Now let $\bar{\sigma} = \langle \sigma(0), \sigma(1) \rangle$, $\bar{\sigma}' = \langle \sigma'(0), \sigma'(1) \rangle \in \dot{G}$, with $\bar{\sigma} \neq \bar{\sigma}'$. It can be seen easily that there exist $\bar{\tau} = \langle \tau(0), \tau(1) \rangle$, $\bar{\tau}' = \langle \tau'(0), \tau'(1) \rangle$ and $i, j \in 3$ with $\tau(0), \tau'(0) \in \dot{F}_{good}^{(i)}(0)$ and $\tau(1), \tau'(1) \in \dot{F}_{good}^{(i,j)}(1)$, such that $\tau(0) \subseteq \sigma(0), \tau'(0) \subseteq \sigma'(0), \tau(1) \subseteq \sigma(1), \tau'(1) \subseteq \sigma'(1)$ and $\bar{\tau} \neq \bar{\tau}'$. Thus, $\xi_{\bar{\tau}} \perp \xi_{\bar{\tau}'}$, which implies $\xi_{\bar{\sigma}} \perp \xi_{\bar{\sigma}'}$.

We can conclude that $q^{(2,2)}$ and \dot{G} witness the lemma for the special case.

Now, we prove the lemma for $\alpha > 1$ for the general case. The construction that has to be done is basically the same as in the special case. The main difference is that depending on H we can have some arbitrary finite number of coordinates that we have to handle, which are not necessarily next to each other.

Let $\alpha > 1, H = \{\beta_0, \dots, \beta_n\} \in [\alpha]^{<\omega}$ and p, \dot{F} as in the premise of the lemma.

To capture the single substeps of the construction we will introduce a new relation in the following way:

Let $\phi : 3^{n+1} \rightarrow \{0, 1, 2\}^{n+1}$ the function that maps a natural number to its trinary representation.

Let $q, q' \in S_\alpha, \dot{G}, \dot{G}'$ H -fronts of q and q' with q is (H, \dot{G}) -deciding and q' is (H, \dot{G}') -deciding,

and let $\dot{G}_{good}, \dot{G}_{bad}, \dot{G}'_{good}, \dot{G}'_{bad} \in^H V$ with

$q \upharpoonright \beta_k \Vdash \dot{G}(\beta_k) = \dot{G}_{good}(\beta_k) \dot{\cup} \dot{G}_{bad}(\beta_k)$ for all $k \leq n$ and

$q' \upharpoonright \beta_k \Vdash \dot{G}'(\beta_k) = \dot{G}'_{good}(\beta_k) \dot{\cup} \dot{G}'_{bad}(\beta_k)$ for all $k \leq n$.

We say for $\bar{\sigma} \in (2^{<\omega})^H$ that $\bar{\sigma} \in \dot{G}_{good}$, if $\forall k \leq n ((q \upharpoonright \beta_k) \upharpoonright (\bar{\sigma} \upharpoonright \beta_k) \Vdash \bar{\sigma}(\beta_k) \in \dot{G}_{good}(\beta_k))$.

Also let c and c' natural numbers.

For $j \in 3^{n+1}$ we write

$((p, \dot{F}), (q, \dot{G}), (\dot{G}_{good}, \dot{G}_{bad}), c) \leq_{\phi(j)} ((p, \dot{F}'), (q', \dot{G}'), (\dot{G}'_{good}, \dot{G}'_{bad}), c')$, if and only if the following conditions hold true:

i): $q \leq q'$

ii): \dot{G} refines \dot{G}' , i.e. for $\bar{\sigma} \in \dot{G}$ and $\bar{\sigma}' \in \dot{G}'$ we have $\bar{\sigma}(\beta_k) \supseteq \bar{\sigma}'(\beta_k)$ for all $k \leq n$.

iii): For N the leftmost digit with $(\phi(j))_N \neq (\phi(j-1))_N$ (or the first digit in the case of $j = 0$) we have:

$\forall k < N \forall \bar{\sigma} \in \dot{G} ((q \upharpoonright \beta_k) \upharpoonright (\bar{\sigma} \upharpoonright \beta_k) \Vdash \dot{G}_{good}(\beta_k) = \dot{G}'_{good}(\beta_k))$

$$\wedge \dot{G}_{bad}(\beta_k) = \dot{G}'_{bad}(\beta_k)).$$

iv): With N as above we have:

$$\forall k \geq N \forall \bar{\sigma} \in \dot{G}_{good}((q \upharpoonright \beta_k) \upharpoonright (\bar{\sigma} \upharpoonright \beta_k) \Vdash \Phi^{(\phi^{(j)})_k}((\dot{p}(\beta_k), \dot{F}(\beta_k)), (\dot{q}'(\beta_k), \dot{G}'(\beta_k)), (\dot{q}(\beta_k), \dot{G}_{good}(\beta_k), \dot{G}_{bad}(\beta_k)))).$$

$$\text{v): } \forall \bar{\sigma} \in \dot{G} \forall l \leq n [(q \upharpoonright \beta_l) \upharpoonright (\bar{\sigma} \upharpoonright \beta_l) \Vdash \bar{\sigma}(\beta_l) \in \dot{G}_{bad} \\ \rightarrow \forall k \geq l ((q \upharpoonright \beta_k) \upharpoonright (\bar{\sigma} \upharpoonright \beta_k) \Vdash \dot{q}(\beta_k) \upharpoonright \bar{\sigma}(\beta_k) = \dot{q}'(\beta_k) \upharpoonright \bar{\sigma}(\beta_k))]$$

$$\text{vi): For all } \bar{\sigma} \in \dot{G}_{good} \text{ exists } \xi_{\bar{\sigma}} \in 2^{<\omega} \text{ with } |\xi_{\bar{\sigma}}| = c \geq c', \text{ such that} \\ \forall \bar{\sigma}, \bar{\sigma}'; \bar{\sigma} \neq \bar{\sigma}' (q \upharpoonright \bar{\sigma} \Vdash \xi_{\bar{\sigma}} \subseteq \dot{x} \wedge \xi_{\bar{\sigma}} \perp \xi_{\bar{\sigma}'}).$$

The properties i) to vi) describe the transition to the next construction step. iv) makes sure that we have $(q, \dot{G}) \leq_H (p, \dot{F})$. v) means that the trees stay fixed above the bad nodes (and in all further coordinates). vi) is the statement that, if we restrict our condition to the good nodes, it forces initial segments of \dot{x} , which are pairwise incompatible. The condition that $c \geq c'$ makes sure that in a descending chain of tuples we have that $\xi_{\bar{\sigma}} \supseteq \xi_{\bar{\sigma}'}$ for sequences of good nodes $\bar{\sigma}$ and $\bar{\sigma}'$ where the first one is a member of the front of a later tuple and an extension of the second one, which is a member of the front of a previous tuple of the chain.

Consider the situation that there exists a full sequence, i.e.

$$((p, \dot{F}), (p, \dot{F}), (\dot{F}, \emptyset^H), b) \geq_{00..00} \dots \geq_{\phi^{(j)}} \\ ((p, \dot{F}), (q^{(j)}, \dot{F}^{(j)}), (\dot{F}_{good}^{(j)}, \dot{F}_{bad}^{(j)}), b_j) \geq_{\phi^{(j+1)}} \dots \geq_{22..22} \\ ((p, \dot{F}), (q^{((3^{n+1}-1)}, \dot{F}^{((3^{n+1}-1)}, (\dot{F}_{good}^{((3^{n+1}-1)}, \dot{F}_{bad}^{((3^{n+1}-1)}, b_{3^{n+1}-1}).$$

We can then define a witness (q, \dot{G}) for the lemma in the following way (analogously to the special case):

$$\text{Define } q := q^{((3^{n+1}-1)} \text{ and let } \dot{G} \text{ be a canonical definition of the } H\text{-front, such that } q \text{ is } (H, \dot{G})\text{-deciding and we have } \bar{\sigma} \in \dot{G} \\ :\Leftrightarrow \exists j [\forall k \leq n ((\phi^{(j)})_k = 1 \vee (\phi^{(j)})_k = 2) \\ \wedge \bar{\sigma} \in \dot{F}_{good}^{(j)} \\ \wedge \forall k \leq n ((\phi^{(j)})_k = 1 \rightarrow (q \upharpoonright \beta_k) \upharpoonright (\bar{\sigma} \upharpoonright \beta_k) \Vdash \\ \exists \rho \in \dot{F}(\beta_k) (\text{stem}(\dot{p}(\beta_k) \upharpoonright (\text{stem}(\dot{p}(\beta_k) \upharpoonright \rho)) \wedge 0) \wedge 0) \subseteq \\ \bar{\sigma}(\beta_k)]].$$

Now, similarly to our special case, if we have given $\bar{\sigma}, \bar{\sigma}' \in \dot{G}$ with $\bar{\sigma} \neq \bar{\sigma}'$, there exist $j < 3^{n+1}$ and $\bar{\tau}, \bar{\tau}' \in \dot{F}_{good}^{(j)}$ with $\bar{\tau} \neq \bar{\tau}'$ and $\forall k \leq n (\bar{\tau}(\beta_k) \subseteq \bar{\sigma}(\beta_k) \wedge \bar{\tau}'(\beta_k) \subseteq \bar{\sigma}'(\beta_k))$. By property vi) for the

respective tuple of the sequence we can conclude that $\xi_{\bar{\tau}} \perp \xi_{\bar{\tau}'}$ and thus $\xi_{\bar{\sigma}} \perp \xi_{\bar{\sigma}'}$.

Thus, such a pair (q, \dot{G}) is actually a witness for the lemma.

Now we need to prove that, given the premise of the lemma, such a full sequence exists.

We show that for each $j \in 3^{n+1} - 1$ and every tuple $t_j = ((p, \dot{F}), (q^{(j)}, \dot{F}^{(j)}), (\dot{F}_{good}^{(j)}, \dot{F}_{bad}^{(j)}), b_j)$ there exists a successor tuple $t_{j+1} = ((p, \dot{F}), (q^{(j+1)}, \dot{F}^{(j+1)}), (\dot{F}_{good}^{(j+1)}, \dot{F}_{bad}^{(j+1)}), b_{j+1})$ with $t_{j+1} \leq_{\phi(j+1)} t_j$.

We show by backwards induction that, if in $V_{\beta_{k+1}}$ for each tuple as above exists a successor tuple with respect to $H_{k+1} := \{\beta_{k+1}, \dots, \beta_n\}$, this claim also holds true in $V_{\beta_{k-1}+1}$ (or V for $k = 0$) with respect to H_k .

We start the induction by showing that the claim holds true in V_{β_n} with respect to $H_n = \{\beta_n\}$. If $\alpha = \beta_n + 1$ the proof is completely analog to the case $\alpha = 1$. Thus, we can assume that $\alpha > \beta_n + 1$.

We work in V_{β_n} and have given $j \in 2$ and a tuple $((p \upharpoonright [\beta_n, \alpha], \dot{F} \upharpoonright [\beta_n, \alpha]), (q^{(j)} \upharpoonright [\beta_n, \alpha], \dot{F}^{(j)} \upharpoonright [\beta_n, \alpha]), (\dot{F}_{good}^{(j)} \upharpoonright [\beta_n, \alpha], \dot{F}_{bad}^{(j)} \upharpoonright [\beta_n, \alpha]), b_j)$.

In the same way as before we can find (in V_{β_n}) $q'^{(j+1)}(\beta_n), F^{(j+1)}(\beta_n), F_{good}^{(j+1)}(\beta_n)$ and $F_{bad}^{(j+1)}(\beta_n)$ with $V_{\beta_n} \models \Phi^{(j+1)}((p(\beta_n), F(\beta_n)), (q^{(j)}(\beta_n), F^{(j)}(\beta_n)), (q'^{(j+1)}(\beta_n), F_{good}^{(j+1)}(\beta_n), F_{bad}^{(j+1)}(\beta_n)))$.

For all $\sigma \in F_{good}^{(j+1)}(\beta_n)$ there exists $q''_{\sigma}{}^{(j+1)}(\beta_n) \leq q'^{(j+1)}(\beta_n) \upharpoonright \sigma$ and $f^{(\sigma)} \in S_{\beta_n+1, \alpha}$ with $q''_{\sigma}{}^{(j+1)}(\beta_n) \widehat{\ } f^{(\sigma)} \leq q'^{(j+1)}(\beta_n) \upharpoonright \sigma \widehat{\ } q^{(j+1)} \upharpoonright [\beta_n + 1, \alpha]$ and $q''_{\sigma}{}^{(j+1)}(\beta_n) \widehat{\ } f^{(\sigma)} \Vdash \xi_{\sigma} \subseteq \dot{x}$.

In addition we can pick all those objects such that there exists $b_{j+1} \geq b_j$ with $|\xi_{\sigma}| = b_{j+1}$ and $\xi_{\sigma} \perp \xi_{\sigma'}$ for all $\sigma, \sigma' \in F_{good}^{j+1}(\beta_n)$ with $\sigma \neq \sigma'$.

Now, let \dot{f} be a S_{β_n+1} -name for the union of $\{(f^{(\sigma)}, q''_{\sigma}{}^{(j+1)}) : \sigma \in F_{good}^{j+1}(\beta_n)\} \cup$

$\{(q^{(j)} \upharpoonright [\beta_n + 1, \alpha), q^{(j)}(\beta_n) \upharpoonright \sigma) : \sigma \in F_{bad}^{(j+1)}(\beta_n)\}$.

By Corollary 3.16 we get
 $q^{(j+1)}(\beta_n) \leq \bigcup_{\sigma \in F_{good}^{(j+1)}(\beta_n)} q''_{\sigma}{}^{(j+1)} \cup \bigcup_{\sigma \in F_{bad}^{(j+1)}(\beta_n)} q^{(j)} \upharpoonright \sigma$,
and we get $q^{(j+1)} \upharpoonright [\beta_n + 1, \alpha) \in S_{\beta_n+1, \alpha}$, such that

$\Phi^{(j+1)}((p(\beta_n), F(\beta_n)), (q^{(j)}(\beta_n), F^{(j)}(\beta_n)), (q^{(j+1)}(\beta_n), F_{good}^{(j+1)}(\beta_n), F_{bad}^{(j+1)}(\beta_n)))$

and $q^{(j+1)}(\beta_n) \Vdash \dot{f} = q^{(j+1)} \upharpoonright [\beta_n, \alpha)$ hold true.

In particular, we have for
 $q^{(j+1)} \upharpoonright [\beta_n, \alpha) := q^{(j+1)}(\beta_n) \hat{\wedge} q^{(j+1)} \upharpoonright [\beta_n + 1, \alpha)$ that
 $(q^{(j+1)} \upharpoonright [\beta_n, \alpha)) \upharpoonright \sigma \Vdash \xi_{\sigma} \subseteq \dot{x}$.

Thus, by $((p \upharpoonright [\beta_n, \alpha), \dot{F} \upharpoonright [\beta_n, \alpha)), (q^{(j+1)} \upharpoonright [\beta_n, \alpha), \dot{F}^{(j+1)} \upharpoonright [\beta_n, \alpha)),$
 $(\dot{F}_{good}^{(j+1)} \upharpoonright [\beta_n, \alpha), \dot{F}_{bad}^{(j+1)} \upharpoonright [\beta_n, \alpha)), b_{j+1})$ we have the desired successor
in V_{β_n} with respect to H_n .

Analogously to the proof of 3.15 (i) we can see that the claim also
holds true in $V_{\beta_{n-1}+1}$ with respect to H_n .

Next, let $k < n$ and assume that we get successor tuples in V_{β_k+1}
with respect to H_{k+1} . We show that under this hypothesis we can get
successors in $V_{\beta_{k-1}+1}$ with respect to H_k .

We work in V_{β_k} . Let $j < 3^{(n-k)+1} - 1$ and
 $((p \upharpoonright [\beta_k, \alpha), \dot{F} \upharpoonright [\beta_k, \alpha)), (q^{(j)} \upharpoonright [\beta_k, \alpha), \dot{F}^{(j)} \upharpoonright [\beta_k, \alpha)),$
 $(\dot{F}_{good}^{(j)} \upharpoonright [\beta_k, \alpha), \dot{F}_{bad}^{(j)} \upharpoonright [\beta_k, \alpha)), b_j)$ a respective tuple.

At first, consider the case that $(\phi(j))_0 = (\phi(j+1))_0$, i.e. that the
leftmost digits of the ternary representations are equal.

We define $\dot{F}_{good}^{(j+1)}(\beta_k) := \dot{F}_{good}^{(j)}(\beta_k)$ and $\dot{F}_{bad}^{(j+1)}(\beta_k) := \dot{F}_{bad}^{(j)}(\beta_k)$.

By our induction hypothesis we have for all $\sigma \in \dot{F}_{good}^{(j+1)}(\beta_k)$ a con-
dition $q'_{\sigma}{}^{(j+1)}(\beta_k) \leq q^{(j)}(\beta_k) \upharpoonright \sigma$ and a tuple
 $t_{\sigma} = ((p \upharpoonright [\beta_k + 1, \alpha), \dot{F} \upharpoonright [\beta_k + 1, \alpha)), (q^{(\sigma)}, \dot{F}^{(\sigma)}), (\dot{F}_{good}^{(\sigma)}, \dot{F}_{bad}^{(\sigma)}), b^{(\sigma)}),$
such that

$q'_\sigma^{(j+1)}(\beta_k) \Vdash t_\sigma \leq_{\langle(\phi(j+1)_1, \dots, \phi(j+1)_n)\rangle}$
 $((p \upharpoonright [\beta_k + 1, \alpha), \dot{F} \upharpoonright [\beta_k + 1, \alpha)), (q^{(j)} \upharpoonright [\beta_k + 1, \alpha), \dot{F}^{(j)} \upharpoonright [\beta_k + 1, \alpha)),$
 $(\dot{F}_{good}^{(j)} \upharpoonright [\beta_k + 1, \alpha), \dot{F}_{bad}^{(j)} \upharpoonright [\beta_k + 1, \alpha)), b_j).$
 (Without loss of generality we have that $q'_\sigma^{(j+1)}(\beta_k)$ decides $\dot{F}^{(\sigma)}$ etc..)

In particular, we have $q'_\sigma^{(j+1)}(\beta_k) \Vdash [q^{(\sigma)} \Vdash^* \xi_{\sigma \smallfrown \bar{\sigma}} \subseteq \dot{x}]$ for all $\bar{\sigma}$ with
 $q'_\sigma^{(j+1)}(\beta_k) \Vdash \bar{\sigma} \in \dot{F}_{good}^{(\sigma)}$.

Analogously to the special case, that we have discussed before, we can assume without loss of generality (by thinning out $q^{(\sigma)}$ above the good nodes), that $\xi_{\sigma \smallfrown \bar{\sigma}} \perp \xi_{\sigma' \smallfrown \bar{\sigma}'}$, if $(\sigma, \bar{\sigma}) \neq (\sigma', \bar{\sigma}')$ and that all those initial segments have common length b_{j+1} .

Let \dot{f} be a name for the union of $\{(q^{(\sigma)}, q'_\sigma^{(j+1)}(\beta_k)) : \sigma \in \dot{F}_{good}^{(j+1)}(\beta_k)\} \cup$
 $\{(q^{(j)} \upharpoonright [\beta_k + 1, \alpha), q^{(j)}(\beta_k) \upharpoonright \sigma) : \sigma \in \dot{F}_{bad}^{(j+1)}(\beta_k)\}$.

By corollary 3.16 we get $q^{(j+1)}(\beta_k) \leq q'_\sigma^{(j+1)}(\beta_k)$ with $\dot{F}^{(j+1)}(\beta_k)$
 is a front of $q^{(j+1)}(\beta_k)$ and $q^{(j+1)}(\beta_k) \upharpoonright \sigma = q'_\sigma^{(j+1)}(\beta_k) \upharpoonright \sigma$ for all
 $\sigma \in \dot{F}_{bad}^{(j+1)}(\beta_k)$ and we get $q^{(j+1)} \upharpoonright [\beta_k + 1, \alpha) \in S_{\beta_k + 1, \alpha}$ with
 $q^{(j+1)}(\beta_k) \Vdash q^{(j+1)} \upharpoonright [\beta_k + 1, \alpha) = \dot{f}$.

If we define $q^{(j+1)} \upharpoonright [\beta_k, \alpha) := q'_\sigma^{(j+1)}(\beta_k) \widehat{\ } q^{(j+1)} \upharpoonright [\beta_k + 1, \alpha)$, we
 have
 $q^{(j+1)} \upharpoonright [\beta_k, \alpha) \upharpoonright \sigma \smallfrown \bar{\sigma} \Vdash \xi_{\sigma \smallfrown \bar{\sigma}} \subseteq \dot{x}$ for all $\sigma \in \dot{F}_{good}^{(j+1)}(\beta_k)$ and $\bar{\sigma}$ with
 $q^{(j+1)}(\beta_k) \upharpoonright \sigma \Vdash \bar{\sigma} \in \dot{F}^{(\sigma)}$.

If we define $\dot{F}^{(j+1)} \upharpoonright [\beta_k, \alpha), \dot{F}_{good}^{(j+1)} \upharpoonright [\beta_k, \alpha)$ and $\dot{F}_{bad}^{(j+1)} \upharpoonright [\beta_k, \alpha)$ in a
 canonical way, we have with

$((p \upharpoonright [\beta_k, \alpha), \dot{F} \upharpoonright [\beta_k, \alpha)), (q^{(j+1)} \upharpoonright [\beta_k, \alpha), \dot{F}^{(j+1)} \upharpoonright [\beta_k, \alpha)),$
 $(\dot{F}_{good}^{(j+1)} \upharpoonright [\beta_k, \alpha), \dot{F}_{bad}^{(j+1)} \upharpoonright [\beta_k, \alpha)), b_{j+1})$ the desired $\phi(j+1)$ -successor.

We still need to consider the case that $(\phi(j))_0 \neq (\phi(j+1))_0$. In this
 case we start by doing the respective construction step in V_{β_k} , i.e. we
 get $\bar{q}^{(j+1)}(\beta_k), \dot{F}_{good}^{(j+1)}(\beta_k)$ and $\dot{F}_{bad}^{(j+1)}(\beta_k)$ with

$\Phi^{((\phi(j+1))_0)}((p(\beta_k), \dot{F}(\beta_k)), (q^{(j)}(\beta_k), \dot{F}(\beta_k)), (\bar{q}^{(j+1)}(\beta_k),$
 $\dot{F}_{good}^{(j+1)}(\beta_k), \dot{F}_{bad}^{(j+1)}(\beta_k))).$

The rest of the proof is completely analog with $\bar{q}^{(j+1)}(\beta_k)$ taking the

role of $q^{(j)}(\beta_k)$.

Thus, we have proven that in V_{β_k} we can get successor tuples with respect to H_k . Analogously to the proof of 3.15 (i) we get the claim for $V_{\beta_{k-1}+1}$ with respect to H_k . Furthermore, by induction we can conclude that the claim holds true in V with respect to H . Hence, we can construct a full sequence, which proves the lemma as we have stated before. □

We can immediately conclude the existence of the desired injective continuous reading of names for the iterated splitting tree forcing.

Corollary 3.18. *Let $\alpha \leq \omega_2$, $p \in S_\alpha$ and \dot{x} a S_α -name for a real such that for all $\xi < \alpha$ we have that $p \Vdash \dot{x} \notin V_\xi$.*

Then there exists a $q \leq p$ and a sequence $\langle (H_i, \dot{F}_i, k_i) : i \in \omega \rangle$ such that $H_i \in [\alpha]^{<\omega}$, \dot{F}_i is a H_i -front for q , $k_i \in \omega$, $(q, \dot{F}_{i+1}) \leq_{H_{i+1}} (q, \dot{F}_i)$, q is (H_i, \dot{F}_i) -deciding and $k_{i+1} > k_i$ for all $i \in \omega$ and there exists a family $\{\xi_{\bar{\sigma}} \in 2^{<\omega} : \bar{\sigma} \in \bigcup_{i \in \omega} \dot{F}_i\}$, such that for every $i \in \omega$ and $\bar{\sigma} \in \dot{F}_i$ we have $q_{\bar{\sigma}} \Vdash \xi_{\bar{\sigma}} \subseteq \dot{x}$ and $\text{length}(\xi_{\bar{\sigma}}) \geq k_i$ and for two indices $\bar{\sigma}, \bar{\sigma}'$ that are incompatible in at least one coordinate we have $\xi_{\bar{\sigma}} \perp \xi_{\bar{\sigma}'}$.

Proof. We just need to iterate Lemma 3.17. In this way, for any given $(p_i, \dot{F}_i, H_i, k_i)$, we get $(p_{i+1}, \dot{F}_{i+1}, H_{i+1}, k_{i+1})$ with $H_{i+1} \supseteq H_i$ as in the lemma. If we choose the sets H_i carefully such that $\bigcup_{i \in \omega} H_i = \bigcup_{i \in \omega} \text{support}(p_i)$, we have that $\langle (p_i, \dot{F}_i, H_i) : i \in \omega \rangle$ is an iterated fusion sequence. It is now easy to see that its infimum, $q \in S_\alpha$, witnesses the corollary. □

Corollary 3.18 gives us the required tool to prove the main result of this chapter.

Theorem 3.19. *Let V be a model for $ZFC + CH$ and let $V_{\omega_2}^S$ be the model that we get by countable support iteration of splitting tree forcing of length ω_2 . Then the following holds true.*

$$V_{\omega_2}^S \models \text{Cov}(I(\mathbb{S})) = \aleph_1$$

Proof. Let $\{q_\lambda : \lambda < \omega_2\}$ be an enumeration of Sacks forcing \mathbb{S} in $V_{\omega_2}^S$. We will construct a matrix of perfect trees $\{q_{\zeta\lambda} : \zeta < \omega_1, \lambda < \omega_2\}$ such

that each row $D_\zeta := \{q_{\zeta\lambda} : \lambda < \omega_2\}$ is a dense subset of \mathbb{S} (in $V_{\omega_2}^S$). Also we want to arrange that for every real that is not already in the groundmodel, there exists a row such that the real is not included in the closure of any tree of that row.

By defining $X_\zeta := 2^\omega \setminus \bigcup_{q \in D_\zeta} [q]$ we get a set that is in the Sacks ideal $I(\mathbb{S})$ and by construction of our matrix, every real which is not in the ground model is a member of $\bigcup_{\zeta < \omega_1} X_\zeta$. Hence, it remains to show that we can construct this matrix.

To each name for a new real $x \in V_{\omega_2}^S$ we assign a function f (of the ground model) in the following way. We choose p from the generic filter that is as in Corollary 3.18, i.e. there exists a sequence $\langle H_i, \dot{F}_i, k_i : i \in \omega \rangle$ and a family $P := \{\xi_{\bar{\sigma}} : \bar{\sigma} \in \bigcup_{i \in \omega} \dot{F}_i\}$ (which induces the perfect tree of possible initial segments of \dot{x}) as in the corollary (i.e. the \dot{F}_i are strictly refining H_i -fronts, p is (H_i, \dot{F}_i) -deciding; $p_{\bar{\sigma}}$ forces \dot{x} to include some initial segment $\xi_{\bar{\sigma}}$ and for any $i \in \omega$ and for different $\bar{\sigma}, \bar{\sigma}'$ the respective $\xi_{\bar{\sigma}}$ and $\xi_{\bar{\sigma}'}$ are incompatible and at least of length k_i).

We collapse the support of p to its order type with the canonical isomorphism i . This induces us an isomorphism $j : \{\bar{\sigma} : \bar{\sigma} \in \bigcup_{i \in \omega} \dot{F}_i\} \rightarrow \text{im}(j)$ by defining $j(\bar{\sigma}) := \{(i(\alpha), \sigma) : (\alpha, \sigma) \in \bar{\sigma}\}$. We define the function f by $f : j(\bar{\sigma}) \mapsto \xi_{\bar{\sigma}}$.

Note that, for some real $x \in V_{\omega_2}^S$ and for the function f defined as mentioned above (w.r.t x), the closure of $f^{-1}[\{x \upharpoonright k : k \in \omega\}]$ is a sequence of generic reals that are introduced at iteration steps indicated by the ordinals of the support of the condition p that was used to define f above.

Also note that by collapsing the support, on one hand we lose the information which coordinate of the function represents which coordinate of the support of p , but on the other hand there are at most ω_1 -many such functions.

For each such function f we build a row of the matrix in the following way:

Let $\langle \zeta_k : k \in \omega \rangle$ be an enumeration of the coordinates of the domain of f .

For $\lambda \in \omega_2$ we have that q_λ is an element of some intermediate model V_γ of the iteration.

We have the following cases.

Case 1: $\exists_{\sigma \in q_\lambda} : \sigma \notin P$

In this case just define $q_{f,\lambda} := q_\lambda \upharpoonright \sigma \leq q_\lambda$

It is clear that no real x that has the function f assigned to it is a member of $[q_{f,\lambda}]$.

Case 2: $q_\lambda \subseteq P$

First, we construct $q'_\lambda \leq q_\lambda$ as an intersection of a Sacks fusion sequence.

Assume that $q_\lambda \geq q_\lambda^{(0)} \geq \dots \geq q_\lambda^{(k-1)}$ are already constructed.

Choose $m \in \omega$ such that every node in $2^m \cap q_\lambda^{(k-1)}$ is the extension of at least k many splitting nodes from $q_\lambda^{(k-1)}$. Let $\{\sigma_i : i < l\}$ be an enumeration of $2^m \cap q_\lambda^{(k-1)}$.

For each $i < l$ let $A_i := \pi_{\zeta_k}(f^{-1}[q_\lambda^{(k-1)} \upharpoonright \sigma_i]) \in V_\gamma$ be the projection to the ζ_k -th coordinate.

(Note that because of the assumptions of Case 2 we have that $[A_i] \neq \emptyset$).

Now, we have either:

(i): A_i is a perfect tree.

Then there exists a perfect subtree $A'_i \subseteq A_i$ in V_γ that is constant on a set a of $V_\gamma \cap [\omega]^\omega$, i.e. there exists a constant value $c \in 2$ such that for all $\sigma \in A'_i$ and all $j \in \text{dom}(\sigma) \cap a$ we have $\sigma(j) = c$. In this case define $q_\lambda^{(k)} \upharpoonright \sigma_i := f[f^{-1}[q_\lambda^{(k-1)} \upharpoonright \sigma_i \cap \pi_{\zeta_k}^{-1}[A'_i]]]$, which is a perfect tree.

or (ii): There exists an isolated branch $y \in [A_i]$.

Then there exists $\sigma \in q_\lambda^{(k-1)}$, such that $[\pi_{\zeta_k}[f^{-1}[q_\lambda^{(k-1)} \upharpoonright \sigma_i]]] = y$ ($\in V_\gamma \cap 2^\omega$). In this case define $q_\lambda^{(k)} \upharpoonright \sigma_i := q_\lambda^{(k-1)} \upharpoonright \sigma$.

After we have defined all $q_\lambda^{(k)} \upharpoonright \sigma_i$ for all $i < l$ we define $q_\lambda^{(k)} := \bigcup_{i < l} q_\lambda^{(k)} \upharpoonright \sigma_i$.

If we have constructed the whole fusion sequence, we can define

$$q'_{f,\lambda} := \bigcap_{k \in \omega} q_\lambda^{(k)}.$$

If we have that $x \in [q'_{f,\lambda}]$ for some x with the function f assigned to it, then for each $k \in \omega$ we have that x is a member of some $[q_\lambda^{(k)} \upharpoonright \sigma_i]$ as above.

So the projection of the inverse of the reverse image under f to the ζ_k -th coordinate either constant on a real from V_γ (Case (i)) or it is a real from V_γ itself (Case (ii)).

This means that the generic real in the coordinate that was collapsed to ζ_k cannot be a real that splits all reals from V_γ , hence it has to be a real that is generic for some V_ζ with $\zeta < \gamma$. We can conclude that x itself has to be an element from V_γ .

This means we can choose a subtree $q_{f,\lambda} \leq q'_{f,\lambda}$ that has none of the ω_1 -many reals of V_γ in its closure, and thus we can conclude again that no x that has f assigned to it can be an element of $[q_{f,\lambda}]$.

In this way we can construct ω_1 many rows (one for each function f) of perfect trees that are dense (from the point of view of $V_{\omega_2}^S$), such that for each real $x \in V_{\omega_2}^S$ there is one row of trees that do not contain the real x in their closures. So we are able to construct the desired matrix.

□

4 On the antichain number of the splitting tree forcing

In this chapter we look at the collapsing number and the antichain number of the splitting tree forcing. By a result of P. Simon the Sacks forcing is $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$ -distributive (see [Sim93]). One can conclude that Sacks forcing collapses the cardinality of the continuum to the unbounding number of the ground model. While it seems not possible to modify this proof to show the same result for the splitting tree forcing, we can use the fusion techniques for splitting tree forcing that are introduced in [Spi07] to show that the splitting tree forcing is $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$ -distributive. Thus, it collapses \mathfrak{c} to \mathfrak{d} . In [RS96] A. Roslanowski and S. Shelah have shown the relative consistency of ZFC and $ZFC + \neg CH + \mathfrak{a}(\mathbb{S}) = \aleph_1$, where $\mathfrak{a}(\mathbb{S})$ is the smallest possible cardinality of an uncountable antichain of Sacks forcing. This latter result also holds true for splitting tree forcing, which is shown in this chapter by a modification of the original proof.

We recall the following definitions.

Definition 4.1. Let P be a forcing and let κ, μ, λ be cardinals. P is called (κ, μ, λ) -*distributive*, if and only if for each family $\langle P_\xi : \xi < \kappa \rangle$ consisting of maximal antichains of cardinality equal or less than μ there exists a maximal antichain Q , such that $\forall q \in Q \forall \xi < \kappa : (|\{p \in P_\xi : p \parallel q\}| < \lambda)$.

A stronger property than not being (κ, μ, λ) -distributive is the following.

Definition 4.2. Let P be a forcing and let κ, μ, λ be cardinals. P is called (κ, μ, λ) -*nowhere distributive*, if and only if there exists a family $\langle P_\xi : \xi < \kappa \rangle$ consisting of antichains of cardinality equal or less than μ , such that $\forall q \in P \exists \xi < \kappa : (|\{p \in P_\xi : p \parallel q\}| \geq \lambda)$.

The next two propositions are well known facts.

Proposition 4.3. Let P be a forcing and κ, μ be infinite cardinals with $\kappa < \mu$. If P is (κ, μ, μ) -nowhere distributive, then forcing with P changes the cofinality of μ to equal or smaller than κ .

Proof. Let $\langle P_\xi : \xi < \kappa \rangle$ as in Definition 4.2. And for each $\xi < \kappa$ let $\langle p_\nu^{(\xi)} : \nu < \gamma_\xi \leq \mu \rangle$ be an enumeration of P_ξ .

By $\dot{f} := \{((\check{\xi}, \check{\nu}), p_\nu^{(\xi)}) : \xi < \kappa, \nu < \gamma_\xi\}$ we define a P -name for a function from κ to μ .

Assume that any $p \in P$ would force that \dot{f} is bounded. Without loss of generality we can assume that there exists a $\lambda < \kappa$ with

$p \Vdash \forall \xi < \check{\lambda} : \dot{f}(\xi) \leq \check{\lambda}$. Then, by the choice of our maximal antichains, there exists $\xi < \kappa$ with $|\{\nu < \gamma_\xi : p \parallel p_\nu^{(\xi)}\}| \geq \mu$. In particular, there exists a $\nu > \lambda$ such that $p \parallel p_\nu^{(\xi)}$. But this is a contradiction because, by definition of \dot{f} and choice of p , p and $p_\nu^{(\xi)}$ have to be incompatible. \square

Proposition 4.4. *Let P be a forcing and κ, μ infinite cardinals with $\kappa < \mu$. If P is (κ, μ, μ) -nowhere distributive and P has a dense subset of cardinality $\delta \leq \mu$, then P collapses μ to κ .*

Proof. Let $\langle P_\xi : \xi < \kappa \rangle$ as in Definition 4.2. and let $\langle d_\zeta : \zeta < \delta \rangle$ be an enumeration of a dense subset of P .

By choice of the maximal antichains, we can pick for each $\zeta < \delta$ a $\xi_\zeta < \kappa$ with $|\{p \in P_{\xi_\zeta} : p \parallel d_\zeta\}| \geq \mu$.

Next, we recursively construct for each ζ a set $X_\zeta \subseteq \{p \in P_{\xi_\zeta} : p \parallel d_\zeta\}$, such that we have for all $\zeta, \zeta' < \delta$ with $\zeta \neq \zeta'$ that $|X_\zeta| = \mu$ and $(\xi_\zeta = \xi_{\zeta'}) \rightarrow X_\zeta \cap X_{\zeta'} = \emptyset$.

Now, for each $\zeta < \delta$ let $\langle p_\nu^{(\zeta)} : \nu < \mu \rangle$ enumerate X_ζ . Choose a P -name \dot{f} for a function from κ to μ with $\{((\check{\xi}_\zeta, \check{\nu}), p_\nu^{(\zeta)}) : \zeta < \delta, \nu < \mu\} \subseteq \dot{f}$. We will show that \dot{f} is a name for a surjection.

Assume by contradiction that there exists $p \in P$ and $\nu < \mu$ with $p \Vdash \check{\nu} \notin \text{ran}(\dot{f})$. By density we have some ζ with $d_\zeta \leq p$. We also know by choice of $p_\nu^{(\zeta)}$ that $p_\nu^{(\zeta)} \parallel d_\zeta$ (and thus, $p_\nu^{(\zeta)} \parallel p$), and by choice of \dot{f} we know that $p_\nu^{(\zeta)} \Vdash \dot{f}(\check{\xi}_\zeta) = \check{\nu}$, which is obviously a contradiction. \square

Theorem 4.5. *The splitting tree forcing S is $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$ -nowhere distributive.*

Proof. First, fix some dominating family $F \subseteq \omega^\omega$ with $|F| = \mathfrak{d}$. Without loss of generality we can assume that for all functions $g \in F$ we have that g is strictly increasing and $\forall n \in \omega : g(n) \geq n + 1$.

We will show that there exists a family $\{S_g : g \in F\}$ with the following properties:

- i) $S_g \subseteq S$, $S_g = \emptyset \vee |S_g| = \mathfrak{c}$ for all $g \in F$
- ii) $\forall g \in F \forall T, T' \in S_g : (T \neq T' \rightarrow T \perp T')$ (i.e. $(T \cap T') \notin S$)
- iii) $\forall T \in S \exists g \in F : |\{T' \in S_g \mid T' \subseteq T\}| = \mathfrak{c}$

This family is obviously a witness for the $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$ -nowhere distributivity of S .

For every splitting tree T we define $k_T : \omega \rightarrow \omega$ by $k_T(n) := \max\{K_T(\sigma) : \sigma \in T \upharpoonright n + 1\}$. We pick a function $g_T \in F$ such that g_T dominates k_T . Now, if we have $N > 0$ with $\forall n \geq N : g_T(n) \geq k_T(n)$, we can conclude by our choice of F that for the iterated function $g_T^{\sum_{k=0}^m 2^k \cdot 2}(0) \geq \sum_{k=0}^m 2^k \cdot 2 \geq N$.

Now, fix some splitting tree T with a function g_T as above and $N \in \omega$ with $\forall n \geq N : g_T(n) \geq k_T(n)$. We will recursively construct a fusion sequence $\langle T_m : m \in \omega \rangle$ of splitting trees with $T \supseteq T_0$ and $\forall m \in \omega : T_m \supseteq T_{m+1}$ and $p_T := \bigcap_{m \in \omega} T_m \in S$.

$m = 0$:

If $N > 0$, we choose $\sigma \in T$ with $|\sigma| = g_T^2(0) + 1$ and define $T_0 := T \upharpoonright \sigma$. Otherwise, if $N = 0$, we know that $g_T^2(0) \geq k_T(0)$ and choose $\sigma_0, \sigma_1 \in T$ with $|\sigma_i| = g_T^2(0) + 1$ and $\sigma_i(g_T^2(0)) = i$ for $i \in 2$ and we define $T_0 := T \upharpoonright \sigma_0 \cup T \upharpoonright \sigma_1$. In both cases we define $n_\emptyset := g_T^2(0)$.

$m \rightsquigarrow m + 1$:

First, define $l := g_T^{\sum_{k=0}^m 2^k \cdot 2}(0)$ and let $\alpha : 2^m \rightarrow 2^{<\omega}$ enumerate all members of $\{s \in 2^{<\omega} : |s| = m + 1\}$ in lexicographic order. Define $\bar{n}_{\alpha(i)} := g_T^{2^{i+1}}(l)$ and $n_{\alpha(i)} := g_T^{2^{i+2}}(l)$ for $i < 2^{m+1}$. Note that $n_{\alpha(i)} = g_T(\bar{n}_{\alpha(i)})$ and $\bar{n}_{\alpha(i+1)} = g_T(n_{\alpha(i)})$.

If $N > m + 1$, choose $\sigma \in T_m$ with $|\sigma| = l + 1$ and define $T_{m+1} := T_m \upharpoonright \sigma$.

Otherwise, if $N \leq m + 1$, we know that $l \geq N$ and recursively construct a sequence $\langle T_{m+1}^{(0)}, \dots, T_{m+1}^{(2^{m+1}-1)} \rangle$ in the following way.

First, for every $\sigma \in T_m$ recursively define $\bar{\sigma} \in 2^{<\omega}$ by:
 $\bar{\sigma}(\emptyset) := \sigma(n_\emptyset)$, if $n_\emptyset \in \text{dom}(\sigma)$
 $\bar{\sigma}(k + 1) := \sigma(n_s)$, if $n_s \in \text{dom}(\sigma)$ and $(\bar{\sigma} \upharpoonright k + 1) = s$,
otherwise ($n_s \notin \text{dom}(\sigma)$ or $\bar{\sigma}$ is defined for all $k \leq m$) the finite recursion stops.

Now, for every $\sigma \in T_m$ with $|\sigma| = \bar{n}_{\alpha(0)} + 1$ and $\bar{\sigma} = \alpha(0)$, we pick extensions $\sigma_0, \sigma_1 \in T_m$ with $\sigma_0, \sigma_1 \supseteq \sigma$, $|\sigma_0| = |\sigma_1| = n_{\alpha(0)} + 1$ and $\sigma_j(n_{\alpha(0)}) = j$.

Let A be the set of extensions that we have chosen and define

$$T_{m+1}^{(0)} := \bigcup \{ (T_m \upharpoonright \tau : \tau \in A) \} \cup \bigcup \{ T_m \upharpoonright \tau : \tau \in T_m \wedge |\tau| = \bar{n}_{\alpha(0)} + 1 \wedge \bar{\tau} \neq \alpha(0) \}.$$

Now assume that for some i with $0 \leq i < 2^{m+1} - 1$ we have already constructed $T_{m+1}^{(i)}$.

Then, for every $\sigma \in T_m^{(i)}$ with $|\sigma| = \bar{n}_{\alpha(i+1)} + 1$ and $\bar{\sigma} = \alpha(i+1)$, we pick extensions $\sigma_0, \sigma_1 \in T_m^{(i)}$ with $\sigma_0, \sigma_1 \supseteq \sigma$, $|\sigma_0| = |\sigma_1| = n_{\alpha(i+1)} + 1$ and $\sigma_j(n_{\alpha(i+1)}) = j$.

Let A be the set of extensions that we have chosen and define

$$T_{m+1}^{(i+1)} := \bigcup \{ T_m^{(i)} \upharpoonright \tau : \tau \in A \} \cup \bigcup \{ T_m^{(i)} \upharpoonright \tau : \tau \in T_m^{(i)} \wedge |\tau| = \bar{n}_{\alpha(i+1)} + 1 \wedge \bar{\tau} \neq \alpha(i+1) \}.$$

After the construction of T_{m+1}^j for all $0 \leq i < 2^{m+1}$ define $T_{m+1} := T_{m+1}^{(2^{m+1}-1)}$. This concludes the $m+1$ -th recursion step.

One can easily check, that $\langle T_m : m \in \omega \rangle$ is a fusion sequence of splitting trees. Hence, we can define $p_T := \bigcap_{m \in \omega} T_m \in S$.

Again, for every $\sigma \in p_T$ we can recursively define:

$$\begin{aligned} \bar{\sigma}(\emptyset) &:= \sigma(n_\emptyset), \text{ if } n_\emptyset \in \text{dom}(\sigma) \\ \bar{\sigma}(k+1) &:= \sigma(n_s), \text{ if } n_s \in \text{dom}(\sigma) \text{ and } (\bar{\sigma} \upharpoonright k+1) = s, \\ &\text{otherwise } (n_s \notin \text{dom}(\sigma)) \text{ the finite recursion stops.} \end{aligned}$$

For pruned non-empty tree (i.e. a nonempty tree that has only infinite branches) $Q \subseteq 2^{<\omega}$ with $\text{stem}(Q) \parallel \text{stem}(p_T)$ we define $T^Q := \{ \sigma \in p_T : \bar{\sigma} \in Q \}$.

Assume that Q has some isolated branch, e.g. x being the only member of $[Q \upharpoonright \tau]$ for some $\tau \in 2^{<\omega}$. We can conclude, for all $\sigma \in T^Q$ with $\bar{\sigma} \supseteq \tau$, that $\bar{\sigma} \subseteq x$, and thus we know that $\sigma(n_{\sigma \upharpoonright k}) = \bar{\sigma}(k) = x(k)$ for every $k \in |\bar{\sigma}| + 1$. Hence, if we fix some σ with $\bar{\sigma} \supseteq \tau$ we get that $(T^Q \upharpoonright \sigma)[n_s] = x(|s|)$ for every initial segment $s \subseteq x$. That means that T^Q is not a splitting tree.

On the other hand consider the case, where Q is perfect. Thus, for every $\sigma \in T^Q$ we can find $\sigma_0, \sigma_1 \in Q$ with $\tau_0, \tau_1 \supseteq \bar{\sigma}$ and $\tau_0 \perp \tau_1$. Fur-

thermore, we can find $\sigma_0, \sigma_1 \in T^Q$ with $\sigma_0, \sigma_1 \supseteq \sigma$ and $\overline{\sigma_0} = \tau_0$ and $\overline{\sigma_1} = \tau_1$. Now by construction of p_T we have for all $n \geq g_T(\max\{n_{\tau_0}, n_{\tau_1}\})$ that either $(T^Q \upharpoonright \sigma_0)[n] = \{0, 1\}$ or $(T^Q \upharpoonright \sigma_1)[n] = \{0, 1\}$, thus n witnesses the splitting property for σ in T^Q . Because σ was arbitrary we can conclude that T^Q is a splitting tree.

In conclusion, we get that T^Q is splitting, if and only if Q is a perfect tree. We get even more: If Q is not perfect then T^Q does not contain a splitting tree. (*)

If we now let $\langle T_\xi : \xi < \mathfrak{c} \rangle$ be an enumeration of all splitting trees, each with a respective function g_{T_ξ} and a subtree p_{T_ξ} , and in addition define $\tau_\xi := \overline{\text{stem}(p_{T_\xi})}$, we can easily find an antichain $\{Q_{\xi, \zeta} : \xi, \zeta < \mathfrak{c}\}$ of perfect trees with $\tau_\xi \subseteq \text{stem}(Q_{\xi, \zeta})$ for all $\xi, \zeta < \mathfrak{c}$. For example, let $\{f_{\xi, \zeta} : \xi, \zeta < \mathfrak{c}\}$ be a family of functions from ω into 2, with two different functions of the family being in two different equivalence classes modulo finite. If, for every $\xi, \zeta < \mathfrak{c}$, we define a Silver function $h_{\xi, \zeta}$ by

$$h_{\xi, \zeta}(m) := \begin{cases} \tau_\xi(n) & , \text{if } m = 2n \wedge n < |\tau_\xi| \text{ for some } n \in \omega \\ f_{\xi, \zeta}(n) & , \text{if } m = 2n \wedge n \geq |\tau_\xi| \text{ for some } n \in \omega \\ * & , \text{otherwise} \end{cases} ,$$

then, the respective family of Silver trees is our desired antichain of perfect trees.

For $(\xi, \zeta) \neq (\xi', \zeta')$ with $g_{T_\xi} = g_{T_{\xi'}}$ we can conclude that either $\overline{\text{stem}(T_\xi)}$ or $\overline{\text{stem}(T_{\xi'})}$ is compatible to both stems of $Q_{\xi, \zeta}$ and $Q_{\xi', \zeta'}$. So without loss of generality assume that the latter fact is true for T_ξ . Then we have $T_\xi^{Q_{\xi, \zeta}} \cap T_{\xi'}^{Q_{\xi', \zeta'}} \subseteq T_\xi^{Q_{\xi, \zeta} \cap Q_{\xi', \zeta'}}$ with $Q_{\xi, \zeta} \cap Q_{\xi', \zeta'} \notin \mathbb{S}$, and thus $T_\xi^{Q_{\xi, \zeta}}$ and $T_{\xi'}^{Q_{\xi', \zeta'}}$ are not compatible.

Finally, define $S_g := \{T_\xi^{Q_{\xi, \zeta}} : \xi, \zeta \in \mathfrak{c} \wedge g_{T_\xi} = g\}$ for every $g \in F$. Every nonempty S_g is an antichain, as we have just seen, and for every $T \in S$ we have some ξ , such that $T = T_\xi$ and $S_{g_{T_\xi}}$ contains with $\{T_\xi^{Q_{\xi, \zeta}} : \zeta < \mathfrak{c}\} \subseteq S_{g_{T_\xi}}$ \mathfrak{c} many subtrees of T . Thus, $\{S_g : g \in F\}$ is as desired. \square

As a consequence of Proposition 4.4 and Theorem 4.5 we get the following.

Corollary 4.6. *Let V be a ZFC model and G be S -generic over V . Then the following holds true.*

$$V[G] \models |\mathfrak{c}^V| = |\mathfrak{d}^V|$$

As mentioned before, we also want to examine the antichain number of the splitting tree forcing. In particular, we want to show that it is relative consistent to ZFC to have a ZFC model that fails CH and contains a short maximal antichain of splitting trees (i.e. a maximal antichain of size \aleph_1).

This is basically a modified version of the proof for the same fact with respect to Sacks forcing by A.Roslanowski and S.Shelah (see [RS96]).

At first we need to introduce a special type of splitting trees and prove some lemmata.

Definition 4.7. We say that a splitting tree p is *slim*, $p \in S_{slim}$, iff

$$\begin{aligned} p \in S \wedge \exists^\infty (n, n_0, n_1) \in \omega^3 \exists c_0, c_1 \in 2 \forall \sigma \in p(n) \exists k \in (n, n_0] \exists \sigma_0, \sigma_1 \in \\ p \upharpoonright \sigma \\ (n < n_0 < n_1 \wedge (p \upharpoonright \sigma)(k) = \{\sigma_0, \sigma_1\} \wedge \sigma_0 \perp \sigma_1 \\ \wedge (p \upharpoonright \sigma_0)[n_0] = \{c_0\} \wedge (p \upharpoonright \sigma_1)[n_1] = \{c_1\}). \end{aligned}$$

In other words a slim splitting tree is a splitting tree with infinite many levels such that every node on such a level splits into two extensions, let us say a first one and a second one. And all subtrees of nodes compatible to a first extension have a fixed value c_0 on the same level n_0 , and all subtrees of nodes compatible to a second node have a fixed value c_1 on the same level n_1 .

Lemma 4.8. *Let $p \in S$ and F a front of p , then there exists a $q \in S_{slim}$ with $q \leq p$ and a Front G of q , such that $(q, G) \leq (p, F)$ (see Definition 3.12). In particular, S_{slim} is dense in S .*

Proof. We construct a fusion sequence $\langle (p_i, F_i) : i \in \omega \rangle$ by recursion in the following way.

Let $(p_0, F_0) := (p, F)$. We can assume without restriction that there exists $a \in \omega$ with $F = p(a)$.

Now for $n \in \omega$ assume that (p_n, F_n) be constructed and that there exists $a \in \omega$, such that $F_n = p_n(a)$. Also choose $c_j \in 2$ for every $j \in 2$. Let $\{\sigma_j : j \in k\}$ be an enumeration of F_n .

For each $j \in k, l \in 2$ pick $\sigma_{j,l} \in p_n$, such that $\sigma_{j,l} \supseteq \sigma_j$; $\sigma_{j,0} \perp \sigma_{j,1}$ and such that there exists $m \in \omega$ with $p_n \upharpoonright \sigma_j(m) = \{\sigma_{j,0}, \sigma_{j,1}\}$.

Choose $m_0 \geq \max\{K_{p_n}(\sigma_{j,l}) : j \in k; l \in 2\}$ and let $\{\mu_j : j \in k_0\}$ an enumeration of $\bigcup_{j \in k} (p_n \upharpoonright \sigma_{j,0})(m_0)$ and $\{\tau_j : j \in k_1\}$ an enumeration of $\bigcup_{j \in k} (p_n \upharpoonright \sigma_{j,1})(m_0)$.

Pick $m_1 \geq \max\{K_{p_n}(\sigma) : \sigma \in p_n; |\sigma| = m_0\}$ and for every $j \in k_0$ choose $\mu'_j \in p_n(m_1 + 1)$ with $\mu'_j \supseteq \mu_j$, such that $\forall_{j \in k_0} : \mu'_j(m_1) = c_0$. Define $\overline{p_{n+1}} := \bigcup_{j \in k_0} (p_n \upharpoonright \mu'_j) \cup \bigcup_{j \in k_1} (p_n \upharpoonright \tau_j)$.

Define $\overline{p_{n+1}} := \bigcup_{j \in k_0} (p_n \upharpoonright \mu'_j) \cup \bigcup_{j \in k_1} (p_n \upharpoonright \tau_j)$ and choose $m_2 \geq \max\{K_{\overline{p_{n+1}}}(\sigma) : \sigma \in \overline{p_{n+1}}; |\sigma| = m_1 + 1\}$. Let $\{\mu''_j : j \in k_2\}$ an enumeration of $\bigcup_{j \in k_0} (\overline{p_{n+1}} \upharpoonright \mu'_j)(m_2)$ and $\{\tau'_j : j \in k_3\}$ an enumeration of $\bigcup_{j \in k_1} (\overline{p_{n+1}} \upharpoonright \tau_j)(m_2)$.

Now choose $m_3 \geq \max\{K_{\overline{p_{n+1}}}(\sigma) : \sigma \in \overline{p_{n+1}}; |\sigma| = m_2\}$ and for every $j \in k_3$ pick $\tau''_j \in \overline{p_{n+1}}(m_3 + 1)$ with $\tau''_j \supseteq \tau'_j$, such that $\forall_{j \in k_3} : \tau''_j(m_3) = c_1$.

Define $p_{n+1} := \bigcup_{j \in k_2} (\overline{p_{n+1}} \upharpoonright \mu''_j) \cup \bigcup_{j \in k_3} (\overline{p_{n+1}} \upharpoonright \tau''_j)$ and $F_{n+1} := p_{n+1}(m_3 + 1)$.

By construction we have for $r < m_0$ that $(p_n \upharpoonright \sigma_j)[r] = (p_{n+1} \upharpoonright \sigma_j)[r]$.

For $m_0 \leq r < m_2$ we have that $(p_n \upharpoonright \sigma_j)[r] = \{\emptyset, 1\} = (\overline{p_{n+1}} \upharpoonright \sigma_{j,1})[r] = (\overline{p_{n+1}} \upharpoonright \sigma_j)[r] = (p_{n+1} \upharpoonright \sigma_j)[r]$

and for $r \geq m_2$: $(p_n \upharpoonright \sigma_j)[r] = \{\emptyset, 1\} = (p_{n+1} \upharpoonright \sigma_{j,0})[r] = (p_{n+1} \upharpoonright \sigma_j)[r]$.

This implies $(p_{n+1}, F_{n+1}) \leq (p_n, F_n)$ and we have $F_{n+1} = p_{n+1}(m_3)$ and for all $\sigma_j \in F_n = p_n(a)$ we have $\sigma_{j,i}$ for $i \in 2$ and $m_1 < m_3$ with $(p_{n+1} \upharpoonright \sigma_{j,0})[m_1] = \{c_0\}$ and $(p_{n+1} \upharpoonright$

$$\sigma_{j,1}[m_3] = \{c_1\}.$$

This finishes the $(n + 1)$ -th recursion step.

Let $q := \bigcap_{n \in \omega} p_n = \text{dwcl}(\bigcup_{n \in \omega} F_n) \in S$, then obviously $(q, F_n) \leq (p, F)$ for any $n \in \omega$ and by construction $q \in S_{\text{slim}}$.

□

Lemma 4.9. *Let $\delta < \omega_1$ and $\bar{T} = \langle T_\xi : \xi < \delta \rangle$ a countable sequence of slim splitting trees. Then there exists $T \in S_{\text{slim}}$ with $\forall \xi < \delta : T \perp T_\xi$.*

In particular, every countable antichain of slim splitting trees is not maximal.

Proof. Assume without loss of generality that $\delta = \omega$.

We construct T by recursion. For some $k_0 \in \omega$ let $T \upharpoonright k_0 \subseteq 2^{<k_0}$ be an arbitrary tree.

Now assume that, for some increasing sequence $k_0 < \dots < k_j$ of natural numbers, $T \upharpoonright k_i$ has been constructed for all $i \leq j$.

Pick (n, n_0, n_1) with $k_j \leq n$, as in Definition 4.7, witnessing the slim property for T_j . Let $\{\sigma_l : l < m\}$ be an enumeration of $T_j(n)$. For $l < m$ let $\sigma_{l,0}$ and $\sigma_{l,1}$ be the respective extensions as in the definition of the slimness property.

Now expand T to height $h := \max\{|\sigma_{l,0}| : l < m\}$ by the full binary tree (i.e. $T \upharpoonright h := \{\sigma \in 2^{<h} : \exists \tau \in T(k_j) (\tau \parallel \sigma)\}$). Then let $M_0 := \{\sigma \in T(h) : \exists l < m (\sigma \parallel \sigma_{l,0})\}$. Pick for every $\sigma \in M_0$ an extension $\mu_\sigma \in 2^{<\omega}$ with $|\mu_\sigma| = n_0 + 1$ and $\mu_\sigma(n_0) = 1 - c_0$ (with c_0 as in the definition of the slim property of T_j with respect to (n, n_0, n_1)). We expand T up to height $n_0 + 1$ by defining $T \upharpoonright n_0 + 1 := \text{dwcl}(\{\mu_\sigma : \sigma \in M_0\}) \cup \{\sigma \in 2^{<n_0} : \exists \tau \in T(h) \setminus M_0\}$. In other words we are expanding the members of M_0 by extensions that are not members of T_j and expand all other nodes of $T(h)$ by the full binary tree.

Analogously define $M_1 := \{\sigma \in T(n_0 + 1) : \exists l < m (\sigma \parallel \sigma_{l,1})\}$ and pick for every $\sigma \in M_1$ an extension $\mu_\sigma \in 2^{<\omega}$ with $|\mu_\sigma| = n_1 + 1$ and $\mu_\sigma(n_1) = 1 - c_1$ (with c_1 as in the definition of the slim property of T_j). Define $T \upharpoonright n_1 + 1 := \text{dwcl}(\{\mu_\sigma : \sigma \in M_1\}) \cup \{\sigma \in 2^{<n_1} : \exists \tau \in T(n_0 + 1) \setminus M_1\}$.

We conclude the induction step by defining $k_j + 1 := n_1 + 1$.

Again it is easy to see that T is a splitting tree and obviously incompatible to all T_ξ for $\xi < \delta$. Because of Lemma 4.8 we can reduce T to a slim splitting subtree. \square

We will now define a forcing, which is not trivial by Lemma 4.9

Definition 4.10. Let $\delta < \omega_1$ and $\bar{T} = \langle T_\xi : \xi < \delta \rangle$ a countable sequence of slim splitting trees.

We define $(Q(\bar{T}), \leq)$ in the following way.

$$\begin{aligned}
p \in Q(\bar{T}) &:\Leftrightarrow p = (\bar{S}, \bar{F}), \text{ such that } \bar{S} = \langle S_0, \dots, S_{k-1} \rangle \text{ with} \\
&S_i \in \{T \in S_{Slim} : \forall \xi < \delta (T \perp T_\xi)\} \\
&\wedge \bar{F} = \langle F_0, \dots, F_{k-1} \rangle \text{ with} \\
&F_i \text{ is a front of } S_i \text{ for all } i \in k. \\
&\wedge \forall_{\substack{i, j \in k \\ i \neq j}} \forall \sigma \in F_i \forall \tau \in F_j (\sigma \perp \tau).
\end{aligned}$$

For $p, q \in Q(\bar{T})$ with

$$\begin{aligned}
p &= (\langle S_0^{(0)}, \dots, S_{k_0-1}^{(0)} \rangle, \langle F_0^{(0)}, \dots, F_{k_0-1}^{(0)} \rangle) \text{ and} \\
q &= (\langle S_0^{(1)}, \dots, S_{k_1-1}^{(1)} \rangle, \langle F_0^{(1)}, \dots, F_{k_1-1}^{(1)} \rangle)
\end{aligned}$$

$$\begin{aligned}
\text{define } q \leq p &:\Leftrightarrow \text{(i) : } \bigcup_{j < k_1} F_j^{(1)} \text{ refines } \bigcup_{i < k_0} F_i^{(0)} \\
&\text{(ii): } \forall i < k_0 \exists j < k_1 ((S_j^{(1)}, F_j^{(1)}) \leq (S_i^{(0)}, F_i^{(0)})).
\end{aligned}$$

In addition if G is a $Q(\bar{T})$ -generic filter over V , we define $T_G := \text{dwcl}(\bigcup \{ \bigcup_{i < k} F_i : (\bar{S}, \bar{F}) \in G; \bar{F} = \langle F_0, \dots, F_{k-1} \rangle \})$, which is the downward closure of all fronts that occur in the conditions of the generic filter.

Lemma 4.11. *Let $V \models ZFC$ and G be a $Q(\bar{T})$ -generic Filter over V , then $V[G] \models T_G$ is a slim splitting tree.*

Proof. Let $\sigma \in dwcl(F)$ for $p = (\overline{S}, \overline{F}) \in G$ with F is a front of the sequence \overline{F} .

An easy density argument shows the following.

There exists a sequence of conditions of the generic Filter $\langle p_n : n \in \omega \rangle$ with $p_0 = p$ and for all $n \in \omega$ there exist F_n, S_n in the respective sequences of the condition p_n , such that $F_0 = F$, $(S_{n+1}, F_{n+1}) \leq (S_n, F_n)$ and F_{n+1} is a strict refinement of F_n .

$$\Rightarrow \bigcap_{n \in \omega} S_n \in S \text{ and } \sigma \in \bigcap_{n \in \omega} S_n \subseteq T_G.$$

Thus to witness the splitting property of σ in T_G we can choose the levels and extensions that witness the splitting property for σ in $\bigcap_{n \in \omega} S_n \Rightarrow$ in T_G holds the splitting property for σ .
 $\Rightarrow T_G \in S$

Now let $p = (\overline{S}, \overline{F}) \in Q(\overline{T})$ be an arbitrary condition with $\overline{F} = \langle F_0, \dots, F_{k-1} \rangle$ and $\overline{S} = \langle S_0, \dots, S_{k-1} \rangle$. Without loss of generality we can assume that there exists an $a \in \omega$, such that $\forall j < k F_j = S_j(a)$. Also let $m \in \omega$ be arbitrary.

By doing the construction in the proof of Lemma 4.8 simultaneously for all pairs (S_j, F_j) we are obviously able to construct slim splitting trees $S'_j \subseteq S_j$ for all $j \in k$ and $(n, n_0, n_1) \in \omega^3$ with $n \geq m$ as well as c_0, c_1 , such that all S'_j satisfy the slim property for the same (n, n_0, n_1) with the same respective values c_0 and c_1 .

If we define $q := (\langle S'_0, \dots, S'_{k-1} \rangle, \langle S'_0(n_1 + 1), \dots, S'_{k-1}(n_1 + 1) \rangle)$ we have $q \leq p$ and we can conclude, by a density argument, that T_G satisfies the slimness property with respect to a triple of levels (n, n_0, n_1) with $n \geq m$. Because m was chosen arbitrarily we get that $T_G \in S_{slim}$. \square

Lemma 4.12. *Let $V \models ZFC$ and G be a $Q(\overline{T})$ -generic filter over V , then*

$$V[G] \models \forall \xi < \delta T_G \perp T_\xi.$$

Proof. Let $\xi < \delta$ and $p = (\langle S_0, \dots, S_{k-1} \rangle, \langle F_0, \dots, F_{k-1} \rangle)$. By definition we have for all $j < k$ that $S_j \perp T_\xi$. Thus, we can conclude $\forall j < k \forall \sigma \in S_j \exists \overline{\sigma} \in S_j (\overline{\sigma} \supseteq \sigma \wedge \overline{\sigma} \notin T_\xi)$.

Hence, for each pair (S_j, F_j) we are able to do the construction in the proof of Lemma 4.8 and in addition in the first induction step everytime we choose an extension μ_σ of a $\sigma \in S_j$ we pick $\mu_\sigma \notin T_\xi$. That way, we obtain $S'_j \in S_{slim}$ with $S'_j \subseteq S_j$ and $S'_j(m) \cap T_\xi = \emptyset$ for every

$j \in k$ and sufficiently large $m \in \omega$.

Thus, we have for $q := (\langle S'_0, \dots, S'_{k-1} \rangle, \langle S'_0(m), \dots, S'_{k-1}(m) \rangle) \in Q(\overline{T})$ that $q \leq p$ and $q \Vdash T_G \perp T_\xi$.

By density and because $\xi < \delta$ was chosen arbitrarily we can conclude that $\forall \xi < \delta T_G \perp T_\xi$. □

Lemma 4.13. *Let $p \in Q(\overline{T})$ with $p = (\overline{F}, \langle S_0, \dots, S_{k-1} \rangle)$, then $p \Vdash_{Q(\overline{T})} \forall j < k : T_G \parallel S_j$.*

Proof. Let $p \in G$. Then there exists, by a density argument analogous to the proof of Lemma 4.11, a sequence $\langle p_n : n \in \omega \rangle$ with $p_n \in G$, $p_0 = p$ and $p_n = (\overline{F}^{(n)}, \overline{S}^{(n)})$, such that for every $j < k$ there exist $F_{k_j}^{(n)}, S_{k_j}^{(n)}$ in the sequences $\overline{F}^{(n)}, \overline{S}^{(n)}$ and with the property that for all $j \in k$ we have that $S'_j := \bigcap_{n \in \omega} S_{k_j}^{(n)}$ is a slim splitting tree.

In addition, it holds true (in $V[G]$) that $S'_j \leq T_G$ as well as $S'_j \leq S_j$ for all $j \in k$, hence $p \Vdash \forall j \in k T_j \parallel T_G$. □

Lemma 4.14. *The forcing $Q(\overline{T})$ has the countable chain condition.*

Proof. Let $X = \{p_\xi : \xi < \omega_1\}$ be a set of ω_1 -many conditions of $Q(\overline{T})$ with $p_\xi = (\overline{F}^{(\xi)}, \overline{S}^{(\xi)})$.

By the pigeonhole principle, there exist at least two conditions $p_{\xi_0}, p_{\xi_1} \in X$ with $\overline{F}^{(\xi_0)} = \overline{F} = \overline{F}^{(\xi_1)} = \langle F_0, \dots, F_{m-1} \rangle$ and $\overline{S}_0^{(\xi_0)} = \langle S_0^{(\xi_0)}, \dots, S_{m-1}^{(\xi_0)} \rangle$ and $\overline{S}^{\xi_1} = \langle S_0^{(\xi_1)}, \dots, S_{m-1}^{(\xi_1)} \rangle$. In addition we can assume, for every $i \in m$, that if we do the first induction step in the proof of Lemma 4.8 for $(S_i^{(\xi_0)}, F_i)$ as well as for $(S_i^{(\xi_1)}, F_i)$ we can choose for every $\sigma_j \in F_i = \{\sigma_j : j < k\}$ the same extensions $\sigma_{j,0}, \sigma_{j,1} \in S_i^{(\xi_0)} \cap S_i^{(\xi_1)}$. We can continue the first induction step with the same levels n_0 and n_1 to obtain subtrees $\tilde{S}_i^{(\xi_l)} \subseteq S_i^{(\xi_l)}$ with the property that $(\tilde{S}_i^{(\xi_0)} \upharpoonright \sigma_{j,0})[n_0] = (\tilde{S}_i^{(\xi_0)} \upharpoonright \sigma_{j,1})[n_1] = \{0\}$ and $(\tilde{S}_i^{(\xi_1)} \upharpoonright \sigma_{j,0})[n_0] = (\tilde{S}_i^{(\xi_1)} \upharpoonright \sigma_{j,1})[n_1] = \{1\}$ for all $j < k$.

Thus, we obtain trees $\tilde{S}_i^{(\xi_0)}, \tilde{S}_i^{(\xi_1)} \in S_{Slim}$ and fronts $\tilde{F}_i^{(\xi_0)} = \tilde{S}_i^{(\xi_0)}(n_1 + 1)$ and $\tilde{F}_i^{(\xi_1)} = \tilde{S}_i^{(\xi_1)}(n_1 + 1)$, such that $\forall \sigma \in \tilde{F}_i^{(\xi_0)} \forall \sigma' \in \tilde{F}_i^{(\xi_1)} \sigma \perp \sigma'$ and such that $(\tilde{S}_i^{(\xi_l)}, \tilde{F}_i^{(\xi_l)}) \leq (S_i^{(\xi_l)}, F_i^{(\xi_l)})$ for $l \in 2$.

If we do this for every $i \in m$ and define

$q := (\langle \tilde{S}_0^{(\xi_0)}, \dots, \tilde{S}_{k-1}^{(\xi_0)}, \tilde{S}_0^{(\xi_1)}, \dots, \tilde{S}_{k-1}^{(\xi_1)} \rangle, \langle \tilde{F}_0^{(\xi_0)}, \dots, \tilde{F}_{k-1}^{(\xi_0)}, \tilde{F}_0^{(\xi_1)}, \dots, \tilde{F}_{k-1}^{(\xi_1)} \rangle) \in Q(\overline{T})$, we can conclude that $q \leq p_{\xi_0}, p_{\xi_1}$ and thus $p_{\xi_0} \parallel p_{\xi_1}$. □

By combining the previous lemmata we obtain the following theorem.

Theorem 4.15. *Let V be a model of $ZFC + \neg CH(+2^{\aleph_1} = \aleph_2)$*

Define a finite support iteration of forcings

$\langle P_\xi, Q_\zeta \mid 1 \leq \xi \leq \omega_1; 0 \leq \zeta < \omega_1 \rangle$ of length ω_1 as follows.

Let Q_ξ be a P_ξ -Name for the forcing $Q(\overline{T}_\xi)$,

whith $\overline{T}_\xi := \langle T_\zeta : \zeta < \xi \rangle$ being the sequence of the generic trees that were introduced in the previous iteration steps.

Then for any G that is P_{ω_1} -generic over V the following holds

$V[G] \models 2^{\aleph_0} \geq \aleph_2 (= \aleph_2)$ and $\langle T_\zeta : \zeta < \omega_1 \rangle$ is a maximal antichain of splitting trees.

Proof. First note that the forcing is a finite support iteration of *c.c.c.*-forcings and therefore satisfies the countable chain condition itself (see Lemma 4.14). Hence, $V[G] \models 2^{\aleph_0} \geq \aleph_2$. If we have the additional premise that $2^{\aleph_1} = \aleph_2$ we get by an easy nice name argument that $V[G] \models 2^{\aleph_0} = \aleph_2$.

Lemma 4.11 and Lemma 4.12 (and Lemma 4.9) imply that $\{T_\zeta : \zeta < \omega_1\}$ is an antichain in S consisting of slim splitting trees.

Now assume by contradiction that $\{T_\zeta : \zeta < \omega_1\}$ is not a maximal antichain in $V[G]$.

In this case pick some $T \in V[G] \cap S$ with $T \perp T_\zeta$ for all $\zeta < \omega_1$.

By Lemma 4.8 we can assume without loss of generality that $T \in S_{Slim}$.

By the *c.c.c.* of P_{ω_1} and a simple nice name argument we know that there exists a P_ξ -name for T with $\xi < \omega_1$.

For some δ with $\xi \leq \delta < \omega_1$ there exists a condition $p \in P_\delta$ with

$p \Vdash \forall_{\zeta < \omega_1} : T \perp T_\zeta$ (or more precisely a $p \in P_{\omega_1}$ with $supp(p) \subseteq \delta$).

But on the other hand we have for $q := p \cup (\delta, (\langle \emptyset \rangle, \langle T \rangle)) \in P_{\delta+1} \subseteq P_{\omega_1}$, that $q \leq p$ and $q \Vdash T \parallel T_\delta$, which is obviously a contradiction.

□

5 The antichain number of Silver forcing

In this chapter we investigate the structure of uncountable maximal antichains of Silver forcing. This forms part of a *JSL* article that is the joint work of O. Spinas and me ([SW]). As mentioned in the introduction, G. Laguzzi asked whether $\text{add}(I(Si)) \leq \mathfrak{b}$ ([Lag13]). In order to answer this question, Spinas proved that uncountable maximal antichains of Silver forcing that consist solely of Silver functions with infinite domains are of size \mathfrak{c} (see Theorem 5.2). This result eventually led to a positive answer of Laguzzi's question ([SW]). Theorem 5.4, a joint work of Spinas and me, exhibits another situation in which maximal Silver antichains have cardinality \mathfrak{c} (Spinas proved the special case that is described at the beginning of the proof and I used this basic idea to show the general case). In Theorem 5.5 I used this to conclude that the antichain number of the Silver forcing is at least \mathfrak{d} , which is the main result of this chapter. We conjecture the Silver antichain number to be \mathfrak{c} , which remains an open question.

Recall that for $f, g \in Si$ Silver functions we write $f \parallel g$, if f and g are compatible, which is equivalent to $f \cup g \in Si$. If f and g are incompatible, we write $f \perp g$. Note that there are two reasons for $f, g \in Si$ to be incompatible: There exists $n \in \text{dom}(f) \cap \text{dom}(g)$ with $f(n) \neq g(n)$, or $(\text{dom}(f))^c \cap (\text{dom}(g))^c$ is finite. or $f \in Si$ and $x \in (2 \cup \{*\})^\omega$ we write $f \upharpoonright x$ if f does not contradict x , i.e. $\forall n \in \text{dom}(f) \cap \text{dom}(x) f(n) = x(n)$. If the opposite is true, we write $f \uparrow x$.

An antichain $A \subseteq Si$ is a set of pairwise incompatible Silver conditions. It is the interplay of the two reasons for incompatibility just mentioned that makes it hard in general to understand the structure of maximal Silver antichains.

For any antichain A of Silver conditions we denote by

$A_{fin} := \{f \in A : |\text{dom}(f)| < \omega\}$ the set of finite conditions and by

$A_{inf} := \{f \in A : |\text{dom}(f)| = \omega\}$ the set of infinite conditions of the antichain.

In some situations we shall have an infinite subset $X \subseteq \omega$ and we need to consider Silver conditions relative to X , i.e. $f \in Si$ with $X \setminus \text{dom}(f)$ infinite. The set of all these is denoted by $Si(X)$. For $f, g \in Si(X)$ we write $f \parallel_X g$ if f, g are compatible with respect to $Si(X)$ or $f \perp_X g$ otherwise.

Note the following well known fact.

Lemma 5.1. *Let $\{X_\xi : \xi < \gamma < \mathfrak{c}\}$ be a family of sets $X_\xi \in [\omega]^\omega$. Then there exists a $X \in [\omega]^\omega$ with $|\omega \setminus X| = \omega$, such that $|X \cap X_\xi| = \omega$ for all $\xi < \gamma$.*

Proof. Let $\{X_\xi : \xi < \gamma\}$ be as above. Let $T \subseteq \omega^{<\omega}$ be a perfect tree such that each natural number occurs exactly once and thus uniquely

determines a node that has this number as its last value.

Pick some $x \in [T]$ that is not definable from $T \cup \{T\} \cup \{X_\xi : \xi < \gamma\}$. What is meant by this phrase is that we choose $\lambda \in OR$ sufficiently large (such that the statements that are relevant for the proof are absolute for H_λ and V), and then we pick some $x \in (H_\lambda \cap [T]) \setminus N$ for some $N \prec H_\lambda$ with $T \cup \{T\} \cup \dots \subseteq N$ and $|N| < \mathfrak{c}$. (Note, that from now on we will use this phrase to simplify the notation).

This implies that $\forall \xi < \gamma X_\xi \not\subseteq^* \text{ran}(x)$. Hence, if we define $X := \omega \setminus \text{ran}(x)$ we have $\forall \xi < \gamma |X \cap X_\xi| = \omega$ and $|\omega \setminus X| = \omega$. □

O.Spinas used this fact to conclude the following

Theorem 5.2. *Let $A \subseteq Si$ be a set of Silver conditions with $|A| < \mathfrak{c}$ and $\forall f \in A |dom f| = \omega$. Let $\langle e_n : n \in \omega \rangle$ be an enumeration of $[\omega]^{<\omega}$ and let $A_n := \{f \in A : f^{-1}[\{1\}] = e_n\}$. If every A_n is an antichain, there exists $h \in Si$ such that $h \not\uparrow f$ for every $f \in A$.*

Proof. Let $B_n := \{(dom f)^c : f \in A_n\}$. Obviously every B_n is an a.d.family. Now, because we have countably many a.d. families, we can easily choose some $b \in [\omega]^\omega$ with the property that for all $n \in \omega$ there is at most one $c \in B_n$ with $|c \cap b| = \omega$, and if c is like that then actually $b \subseteq^* c$.

For all $f \in A$ with $|f^{-1}[\{1\}]| < \omega$ we have $f \in A_n$ for some $n \in \omega$.

Hence, by choice of b we have either $b \subseteq^* (dom f)^c$ from which we can conclude

$$|dom f \cap (\omega \setminus b)| = \omega,$$

$$\text{or } |b \cap (dom f)^c| < \omega, \text{ hence } b \subseteq^* dom f.$$

This enables us to define a partition of A as follows: Letting

$$C_0 := \{f \in A : |f^{-1}[\{1\}]| < \omega \wedge b \subseteq^* dom(f)\},$$

$$C_1 := \{f \in A : |f^{-1}[\{1\}]| < \omega \wedge b \subseteq^* (dom(f))^c\},$$

$$D_0 := \{f \in A : |f^{-1}[\{1\}]| = \omega \wedge |b \cap f^{-1}[\{1\}]| = \omega\},$$

$$D_1 := \{f \in A : |f^{-1}[\{1\}]| = \omega \wedge |b \cap f^{-1}[\{1\}]| < \omega\},$$

we have $A = C_0 \dot{\cup} C_1 \dot{\cup} D_0 \dot{\cup} D_1$ and C_1 is countable.

By Lemma 5.1 we can pick some infinite $b_0 \subseteq b$ with $|b_0 \cap f^{-1}[\{1\}]| = \omega$ for all $f \in D_0$ as well as $|b \setminus b_0| = \omega$.

For $b_1 := b \setminus b_0$ we have $\forall f \in C_0 b_1 \subseteq^* dom(f)$.

Next, let $\langle f_n : n \in \omega \rangle$ enumerate the elements of C_1 and construct a perfect tree $T \subseteq (\omega \setminus b)^{<\omega}$ with $\forall \sigma \in T \setminus \{\emptyset\} (\sigma(|\sigma| - 1) \in (\omega \setminus b) \cap f_{|\sigma|}^{-1}[\{0\}])$

and $\forall \sigma, \sigma' \in T \setminus \{\emptyset\}$ ($\sigma \neq \sigma' \rightarrow (\sigma(|\sigma| - 1) \neq \sigma'(|\sigma'| - 1))$), such that each natural number occurs in the tree at most once.

By our assumption $|A| < \mathfrak{c}$ we then pick some $b_2 \in [T]$ that is not definable by $T \cup \{T\} \cup D_1 \cup \{D_1\}$.

Assume by contradiction, that there exists $f \in D_1$, such that $f^{-1}[\{1\}] \subseteq^* b_2$, hence $f^{-1}[\{1\}] \setminus N \subseteq b_2$ for some $N \in \omega$.

Then by construction of the tree b_2 would be determined by $b_2 = \bigcup \{\sigma \in T : \sigma(|\sigma| - 1) \in f^{-1}[\{1\}] \setminus N\}$, which is clearly not possible.

So in conclusion we get: $\forall f \in C_1$ $f^{-1}[\{0\}] \cap b_2 \neq \emptyset$ by construction of our tree, as well as $\forall f \in D_1$ ($|(\omega \setminus b) \setminus b_2 \cap f^{-1}[\{1\}]| = \omega$) as stated above.

Again by Lemma 2.1 we can split up $(\omega \setminus b) \setminus b_2$ into two disjoint infinite sets b_3 and b_4 with $\forall f \in D_1$ $|f^{-1}[\{1\}] \cap b_3| = \omega$

If we define $h \in Si$ by:

$$h(n) := \begin{cases} 0 & , \text{if } n \in b_0 \cup b_3 \\ 1 & , \text{if } n \in b_1 \cup b_2 \\ * & , \text{otherwise} \end{cases}$$

we can easily check that $h \not\uparrow A$.

□

In particular, we get the following corollary:

Corollary 5.3. *If $A \subseteq Si$ is an antichain of Silver conditions with $|A| < \mathfrak{c}$ and $\forall f \in A$ $|dom f| = \omega$, then A is not maximal.*

As a consequence of Theorem 5.2 we can show that the additivity of the Silver ideal is at most \mathfrak{b} (see [SW]).

The core of this chapter is the following result which analyzes maximal Silver antichains that contain uncountably many elements which are pairwise incompatible by the second reason mentioned in the beginning of this chapter.

Theorem 5.4. *Let $A \subseteq Si$ be an antichain with $\omega_1 \leq |A| < \mathfrak{c}$. If for some real $x \in 2^\omega$ the set $\{f \in A : f \upharpoonright x\}$ is uncountable, then A is not maximal.*

Proof. We assume without loss of generality that $\{f \in A : \forall n \in \text{dom} f \ f(n) = 0\}$ is uncountable. Let $\langle e_n : n \in \omega \rangle$ enumerate the set $\{f^{-1}[\{1\}] : f \in A_{\text{inf}} \wedge |f^{-1}[\{1\}]| < \omega\} \subseteq \omega^{<\omega}$ with $e_0 = \emptyset$ and as in the previous proof define $A_n := \{f \in A_{\text{inf}} : f^{-1}[\{1\}] = e_n\}$ and $B_n := \{(dom f)^c : f \in A_n\}$. Clearly each B_n is an almost disjoint family and B_0 is uncountable. Also for each $n \in \omega$ define $x_n \in 2^\omega$ by:

$$x_n(k) := \begin{cases} 1 & , \text{if } k \in e_n \\ 0 & , \text{otherwise} \end{cases}$$

If for some $n \in \omega$ we have $b \in B_n$ and if it is clear which B_n we are referring to, we will often write f_b for a Silver condition that has b as its codomain with $f_b^{-1}[\{1\}] = e_n$. We will also identify each $b \in B_n$ with its strictly increasing enumeration.

Let $T \subseteq \omega^{<\omega}$ be the tree on ω consisting of all $\sigma \in \omega^{<\omega}$ that are initial segments of uncountably many members of B_0 . Thus, $T := \text{cond}(\{\sigma \in \omega^{<\omega} : \exists b \in B_0 \ \sigma \subseteq b\})$. As B_0 is an uncountable a.d family, T is perfect.

Let us first sketch the basic idea of the proof in the special situation that $B_n = \emptyset$ for all $n \geq 1$. It is clear that for no $f \in A_{\text{fin}}$ can there be $\sigma \in T$ with $f^{-1}[\{1\}] \subseteq \text{ran}(\sigma)$. Hence every Silver condition that is constantly 0 outside some fixed branch of T is incompatible with every $f \in A_{\text{fin}}$. We distinguish two cases:

Case 1:

$$\begin{aligned} \exists \sigma \in T \ \exists m \in \omega \ \exists b_0 \neq b_1 \in B_0 \cap [T \upharpoonright \sigma] \ \forall f \in A_{\text{fin}} \ (f^{-1}[\{1\}] \subseteq \text{ran}(b_0) \cup \text{ran}(b_1) \\ \rightarrow f^{-1}[\{1\}] \cap m \neq \emptyset) \end{aligned}$$

In this case choose b_0, b_1 and m as above and define a function $h \in (2 \cup \{*\})^\omega$ by:

$$h_0(n) := \begin{cases} 0 & , \text{if } n \notin \text{ran}(b_0) \cup \text{ran}(b_1) \\ 0 & , \text{if } n \in (\text{ran}(b_0) \cup \text{ran}(b_1)) \cap m \\ * & , \text{otherwise} \end{cases}$$

Then obviously $h_0 \perp A_{fin}$ and because B_0 is an a.d. family we have for all $f, g \in A_{inf}$ that $|dom(f) \cap (b_0 \cup b_1)| = \omega$ and $f \perp_{(b_0 \cup b_1) \setminus m} g$. Hence, by the latter fact and by Theorem 5.2 we can pick $h_1 : (b_0 \cup b_1) \setminus m \rightarrow 2 \cup \{*\}$ with $h_1 \perp_{(b_0 \cup b_1) \setminus m} A_{inf}$. We can conclude that for $h := h_0 \cup h_1$ we have $h \perp A$.

Case 2: If the first case does not hold true, we can construct recursively

$\langle \sigma_s : s \in 2^{<\omega} \rangle$ and $\langle f_s : s \in 2^{<\omega} \rangle$ with $\sigma_s \in T$ and $f_s \in A_{fin}$ for all $s \in 2^{<\omega}$ in the following way:

For a given $\sigma_s \in T$ we pick branches $b_0, b_1 \in [T]$ with $b_0 \neq b_1$ and $b_0, b_1 \supseteq \sigma_s$. Let n be the length of the common initial segment. Because we are not in case 1 we can find $f_s \in A_{fin}$ with $f_s^{-1}[\{1\}] \subseteq ran(b_0) \cup ran(b_1)$ and $f_s^{-1}[\{1\}] \cap n = \emptyset$. Let $\sigma_{s \smallfrown 0}, \sigma_{s \smallfrown 1} \in T$ be sufficiently long initial segments of b_0 and b_1 with $f_s^{-1}[\{1\}] \subseteq ran(\sigma_{s \smallfrown 0}) \cup ran(\sigma_{s \smallfrown 1})$.

After we have got all σ_s, f_s let $T_0 := dwcl(\{\sigma_s : s \in 2^{<\omega}\})$. Choose a branch $x \in [T_0]$ that is not definable from $T \cup \{T\} \cup A \cup \{A\}$. Note that because x is a branch in T we have $f^{-1}\{1\} \cap (\omega \setminus x) \neq \emptyset$ for all $f \in A_{fin}$.

Thus, if we define $h_0 \in (2 \cup \{*\})^\omega$ by

$$h_0(n) := \begin{cases} 0, & \text{if } n \notin x \\ * & \text{if } n \in x \end{cases}$$

we have $h_0 \perp A_{fin}$.

Now, assume by contradiction that we have $b \in B_0$ with $b \supseteq^* ran(x)$. Then we know by choice of x that $\{y \in [T_0] : b \supseteq^* ran(y)\}$ is uncountable (otherwise x would be definable by b). Then there exists $N \in \omega$ with $\{y \in [T_0] : b \cup N \supseteq ran(y)\}$ uncountable. Let T' the perfect condensation tree of the latter set. We can pick some σ_s that is a member of T' of length greater or equal to N , such that $\sigma_{s \smallfrown 0}$ and $\sigma_{s \smallfrown 1}$ are members of T' as well. For branches b_0, b_1 of T' with $ran(b_0) \supseteq ran(\sigma_{s \smallfrown 0})$ and $ran(b_1) \supseteq ran(\sigma_{s \smallfrown 1})$ we have by definition of T_0 that $(ran(b_0) \cup ran(b_1)) \setminus N \supseteq f_s^{-1}[\{1\}]$. Hence $b \supseteq f_s^{-1}[\{1\}]$. This implies $f_s \parallel f_b$, which is a contradiction.

Therefore $\{f \upharpoonright x : f \in A_{inf}\}$ is an antichain solely consisting of infinite conditions relative to x .

By Theorem 5.2 we can choose $h_1 \in Si(x)$ with $h_1 \perp_x A_{inf}$. For $h := h_0 \cup h_1$ we get $h \perp A$.

For the general case we can repeatedly apply this basic idea in a quite delicate recursion (to take care of all B_n) together with some new

ideas, which in particular exploit the freedom we had to choose the two branches b_0, b_1 in case 2.

We construct a subtree $T_0 \subseteq T$ by recursion in the following way:

At the beginning of the recursion:

Choose a splitting node $\tau \in sp(T)$ and $\bar{\sigma}_0, \bar{\sigma}_1 \in T$, $\bar{\sigma}_0, \bar{\sigma}_1 \supseteq \tau$ with $\bar{\sigma}_0(|\tau|) \neq \bar{\sigma}_1(|\tau|)$. For $\bar{\sigma}_0$ fix $\bar{b} \in B_0 \cap [T \upharpoonright \bar{\sigma}_0]$, which we call the "reference branch". First, we check whether there exist $f \in A_{\bar{f}n}$ and $\sigma \in T$ with $\sigma \supseteq \bar{\sigma}_1$ and $f^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \bar{b}) \setminus |\tau|$.

If we find such σ, f , we define $\sigma_1 := \sigma$, $f_{0,\emptyset} := f$, $c_0 := \{\omega\} =: c_1$ and $\sigma_0 := \bar{b} \upharpoonright k$ with $k \in \omega$ sufficiently large that $f^{-1}[\{1\}] \subseteq ran(\sigma_0) \cup ran(\sigma_1)$.

In addition set $\bar{T}^{(0)} := T \upharpoonright \sigma_0$ and $\bar{T}^{(1)} := T \upharpoonright \sigma_1$

If we are not able to find any such σ, f we try to construct a sequence $\langle \mu^{(m)} \mid m \in \omega \rangle$ recursively with

$\bar{\sigma}_1 \subseteq \mu^{(0)} \subseteq \mu^{(1)} \subseteq \dots$

and obtain by $y := \bigcup_{m \in \omega} \mu^{(m)} \in \omega^\omega$ an "opponent" to \bar{b} as follows:

$m = 0$:

Case 1: For all $a \in B_1$ we have that $a \not\supseteq^* \bar{b}$:

In this case simply define $\mu^{(0)} := \bar{\sigma}_1$.

Case 2: There exists $a \in B_1$ with $a \supseteq^* \bar{b}$:

Case 2.1: $\forall \sigma \in T (\sigma \supseteq \bar{\sigma}_1 \rightarrow f_a^{-1}[\{1\}] \not\subseteq (ran(\sigma) \cup \bar{b}) \setminus |\tau|)$:

In this case define $\mu^{(0)} := \bar{\sigma}_1$.

Case 2.2: $\exists \sigma \in T (\sigma \supseteq \bar{\sigma}_1 \wedge f_a^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \bar{b}) \setminus |\tau|)$:

Then let $\bar{\mu}^{(0)} \in T$ be a witness for this statement.

Consider the following two subcases:

Case 2.2.1: The set $X := \{b \in B_0 : b \in [T \upharpoonright \bar{\mu}^{(0)}] \wedge |b \cap a| = \omega\}$ is uncountable:

In this case the construction of the $\mu^{(m)}$ stops.

We define $f_{0,\emptyset} := f_a$, $c_{\langle 0 \rangle} := \{\omega\}$, $c_{\langle 1 \rangle} := \{a\}$ and $\sigma_1 := \bar{\mu}^{(0)}$, as well as $\sigma_0 := \bar{b} \upharpoonright k$ with $k \in \omega$ sufficiently large, such that $f_a^{-1}[\{1\}] \subseteq ran(\sigma_0) \cup ran(\sigma_1)$.

We also set $\bar{T}^{(0)} := T \upharpoonright \sigma_0$ and $\bar{T}^{(1)} := cond(X)$

and go on with the construction of the tree T_0 (as described later).

Case 2.2.2: The X of the former case is countable:

Then choose some $\mu^{(0)} \in T$, $\mu^{(0)} \supseteq \bar{\mu}^{(0)}$, such that $|\bar{b}^c \cap ran(\mu^{(0)})| \geq 1$ and

$|a^c \cap ran(\mu^{(0)})| \geq 1$. This is clearly possible since uncountably many $b \in B_0$ have $\bar{\mu}^{(0)}$ as an initial segment and B_0 is an a.d. family.

In this case we go on with the construction of the $\mu^{(m)}$.

$m \rightsquigarrow m + 1$:

For $m \in \omega$ let $\mu^{(m)}$ be constructed:

Case 1: For all $a \in B_{m+2}$ we have $a \not\supseteq^* \bar{b}$:

In this case choose $\mu^{(m+1)} \in T$, $\mu^{(m+1)} \supseteq \mu^{(m)}$, such that $|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m + 2$ and such that for all $k \leq m$, where we had case 2.2.2 witnessed by $\tilde{a} \in B_{k+1}$, we have: $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.

Case 2: There exists $a \in B_{m+2}$ with $a \supseteq^* \bar{b}$:

Case 2.1: $\forall \sigma \in T$ ($\sigma \supseteq \mu^{(m)} \rightarrow f_a^{-1}[\{1\}] \not\subseteq (\text{ran}(\sigma) \cup \bar{b}) \setminus |\tau|$):

In this case also choose $\mu^{(m+1)} \in T$, $\mu^{(m+1)} \supseteq \mu^{(m)}$, such that $|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m + 2$ and such that for all $k \leq m$, where we had case 2.2.2 witnessed by $\tilde{a} \in B_{k+1}$, we have: $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.

Case 2.2: $\exists \sigma \in T$ ($\sigma \supseteq \mu^{(m)} \wedge f_a^{-1}[\{1\}] \subseteq (\text{ran}(\sigma) \cup \bar{b}) \setminus |\tau|$):

Then let $\bar{\mu}^{(m+1)}$ be a witness.

Again we have two subcases:

Case 2.2.1: The set $X := \{b \in B_0 : b \in [T \upharpoonright \bar{\mu}^{(m+1)}] \wedge |b \cap a| = \omega\}$ is uncountable:

Then the construction of the $\mu^{(m)}$ stops.

Define $f_{0,\emptyset} \equiv f_a$, $c_{(0)} := \{\omega\}$, $c_{(1)} := \{a\}$

and $\sigma_1 := \bar{\mu}^{(m+1)}$ and $\sigma_0 := \bar{b} \upharpoonright k$ with $k \in \omega$ sufficiently large, such that $f_a^{-1}[\{1\}] \subseteq \text{ran}(\sigma_0) \cup \text{ran}(\sigma_1)$.

Also define: $\bar{T}^{(0)} := T \upharpoonright \sigma_0$ and $\bar{T}^{(1)} := \text{cond}(X)$

Case 2.2.2: The set X defined above is countable:

Then choose $\mu^{(m+1)} \in T$, $\mu^{(m+1)} \supseteq \bar{\mu}^{(m+1)}$, such that:

$|\text{ran}(\mu^{(m+1)}) \cap a^c| \geq 1$, $|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m + 2$ and for all $k \leq m$, where we had case 2.2.2 witnessed by $\tilde{a} \in B_{k+1}$, we have $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.

Let us first consider the situation, where the construction of the $\mu^{(m)}$ does not stop (which means that case 2.2.1 does not occur):

Define by $y := \bigcup_{m \in \omega} \mu^{(m)}$ the opponent to \bar{b} . By construction the following propositions are true:

i) For all $f \in A_{f_m}$ we have:

$f^{-1}[\{1\}] \cap |\tau| \neq \emptyset$ or $f^{-1}[\{1\}] \cap (\omega \setminus (y \cap \bar{b})) \neq \emptyset$.

ii) For $f = f_d$ with $d \in B_{k+1}$ for some $k \in \omega$ we have:

$|b \cap \text{dom}(f_d)| = \omega$ (case 1 or case 2 with $a \supseteq^* \bar{b}$, $a \neq d$)

or $f_d^{-1}[\{1\}] \cap |\tau| \neq \emptyset$ or $f_d^{-1}[\{1\}] \cap (\omega \setminus \bar{b} \cup y) \neq \emptyset$ (case 2.1 with $d \supseteq^* \bar{b}$)

or $|y \cap \text{dom}(f_d)| = \omega$ (case 2.2.2 with $d \supseteq^* \bar{b}$).

iii) For $f = f_{\bar{b}}$ we have $|y \cap \text{dom}(f)| = \omega$.

iv) For $f = f_d$ with $d \in B_0 \setminus \{\bar{b}\}$ we have $|\text{dom}(f) \cap \bar{b}| = \omega$.

v) For $f \in A$ with $|f^{(-1)}[\{1\}]| = \omega$ the following holds: $f^{-1}[\{1\}] \cap (\omega \setminus (y \cup \bar{b})) \neq \emptyset$ or $|\text{dom}(f) \cap (y \cup \bar{b})| = \omega$.

Thus, if we define $h \in Si$ by $h \upharpoonright (\omega \setminus (\bar{b} \cup y)) \cup |\tau| := 0$, each condition of the antichain is either incompatible to h or is an infinite Silver condition relativized to $(\bar{b} \cup y) \setminus |\tau|$. This means, we can use Corollary 5.3 to obtain $h \in Si$ with $h \perp A$.
 $\Rightarrow A$ is not maximal.

If on the other hand we did not start to construct an opponent or the construction of the opponent stops (case 2.2.1), we go on with the construction of T_0 in the following way:

For $i \in 2$ we have either:

Case a: There exists $a \in B_1$ with $X := \{b \in B_0 \cap [\bar{T}^{(i)}] : |b \cap \bigcup c_i| = \omega \wedge |b \cap a| = \omega\}$ uncountable:

In this case define $z_{\langle i \rangle} := a$ and $T^{(\langle i \rangle)} := \text{cond}(X) (\subseteq \bar{T}^{(i)})$.

Or:

Case b:

$\forall a \in B_1 \{b \in B_0 \cap [\bar{T}^{(i)}] : |b \cap \bigcup c_i| = \omega \wedge |b \cap a| = \omega\}$ is countable:

In this case define $z_{\langle i \rangle} := *$ and $T^{(\langle i \rangle)} := \bar{T}^{(i)}$.

Remark: Because $Z := \{b \in B_0 : |b \cap \bigcup c_i| = \omega\}$ is uncountable and $\bar{T}^{(i)} = \text{cond}(Z)$ it follows in case b, that for all $a \in B_1$ the set $\{\sigma \in T^{(\langle i \rangle)} : \sigma(|\sigma| - 1) \in a^c\}$ is dense in $T^{(\langle i \rangle)}$.

In fact more is true:

For each subtree $T' \subseteq T^{(\langle i \rangle)}$ with the property

$\forall \sigma \in T' ([T' \upharpoonright \sigma] \cap Z \text{ is uncountable})$

we have $\forall a \in B_1 \{\sigma \in T' : \sigma(|\sigma| - 1) \in a^c\}$ is dense in T' .

(Note that in case b all subtrees $T^{(s)}$ with $s \in 2^{<\omega}$ and $s(0) = i$ in the following construction actually will have the above property.)

This finishes the first step in the construction of T_0 .

Suppose that for some $n \in \omega$ and for all $s \in 2^n$ all σ_s together with the c_t and z_t for all $t \in 2^{\leq n} \setminus \emptyset$ have been constructed
such that for each $s \in 2^n$ we have gotten $T^{(s)} = \text{cond}(X)$ with uncountable

$$X = \{b \in B_0 : b \supseteq \sigma_s \wedge \forall 1 \leq k \leq |s| \forall d \in c_{s|k} (|b \cap d| = \omega) \\ \wedge \forall 1 \leq k \leq |s| ((z_{s|k} \neq *) \rightarrow |b \cap z_{s|k}| = \omega)\}.$$

Now fix some $s \in 2^n$.

Choose a splitting node $\tau_s \in \text{sp}(T^{(s)})$, such that

$\text{ran}(\tau_s \upharpoonright [|\sigma_s|, |\tau_s|]) \cap d \neq \emptyset$ for all $\emptyset \neq t \subseteq s$, $d \in c_t$
 and that $\text{ran}(\tau_s \upharpoonright [|\sigma_s|, |\tau_s|]) \cap z_t \neq \emptyset$ for all $\emptyset \neq t \subseteq s$ with $z_t \neq *$.
 (This is possible by the construction of $T^{(s)}$.)
 Next, choose $\bar{\sigma}_{s \smallfrown 0}, \bar{\sigma}_{s \smallfrown 1} \in T^{(s)}$ with $\bar{\sigma}_{s \smallfrown 0}, \bar{\sigma}_{s \smallfrown 1} \supseteq \tau_s$
 and $\bar{\sigma}_{s \smallfrown 0}(|\tau_s|) \neq \bar{\sigma}_{s \smallfrown 1}(|\tau_s|)$.

Again we take a reference branch and try to construct an opponent (with respect to B_0):

First choose a reference branch $\bar{b} \in [T^{(s)} \upharpoonright \bar{\sigma}_{s \smallfrown 0}] \cap B_0$.

Check if there exists $f \in A_{\text{fin}}$ and $\sigma \in T^{(s)}$, $\sigma \supseteq \bar{\sigma}_{s \smallfrown 1}$, such that $f^{-1}[\{1\}] \subseteq (\bar{b} \cup \text{ran}(\sigma)) \setminus |\tau_s|$.

If we find such σ, f , define $\sigma_{s \smallfrown 1}^{(0)} := \sigma$, $f_{0,s} := f$ and $\sigma_{s \smallfrown 0}^{(0)} := \bar{b} \upharpoonright k$ for a sufficiently large $k \in \omega$.

In this case the construction of the opponent (w.r.t. B_0) stops.

Otherwise try to construct recursively the sequence $\langle \mu^{(m)} \mid m \in \omega \rangle$ as follows:

$m = 0$:

Case 1: For all $a \in B_1$ we have $a \not\supseteq^* \bar{b}$:

Define $\mu^{(0)} := \bar{\sigma}_{s \smallfrown 1}$

Case 2: There exists $a \in B_1$ with $a \supseteq^* \bar{b}$:

Case 2.1: $\forall \sigma \in T^{(s)}$ ($\sigma \supseteq \bar{\sigma}_{s \smallfrown 1} \rightarrow f_a^{-1}[\{1\}] \not\subseteq (\text{ran}(\sigma) \cup \bar{b}) \setminus |\tau_s|$):

Define $\mu^{(0)} := \bar{\sigma}_{s \smallfrown 1}$.

Case 2.2: $\exists \sigma \in T^{(s)}$ ($\sigma \supseteq \bar{\sigma}_{s \smallfrown 1} \wedge f_a^{-1}[\{1\}] \subseteq (\text{ran}(\sigma) \cup \bar{b}) \setminus |\tau_s|$):

Then let $\bar{\mu}^{(0)}$ be a witness.

case 2.2.1:

$$\begin{aligned}
 X := \{b \in B_0 \cap [T^{(s)} \upharpoonright \bar{\mu}^{(0)}] : & \forall \emptyset \neq t \subseteq s \forall d \in c_t (|b \cap d| = \omega) \\
 & \wedge \forall \emptyset \neq t \subseteq s (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\
 & \wedge |b \cap a| = \omega\}
 \end{aligned}$$

is uncountable:

In this case the construction of the $\mu^{(m)}$ stops.

Define $f_{0,s} := f_a$, $\sigma_{s \smallfrown 1}^{(0)} := \bar{\mu}^{(0)}$, $\sigma_{s \smallfrown 0}^{(0)} := \bar{b} \upharpoonright k$ with $k \in \omega$ sufficiently large.

Also define $\bar{T}_0^{(s \smallfrown 0)} := T^{(s)} \upharpoonright \sigma_{s \smallfrown 0}^{(0)}$ and $\bar{T}_0^{(s \smallfrown 1)} := \text{cond}(X)$.

Case 2.2.2: The set X of the previous case is countable: Then choose $\mu^{(0)} \in T^{(s)}$, $\mu^{(0)} \supseteq \bar{\mu}^{(0)}$, such that:

$$|\text{ran}(\mu^{(0)}) \cap \bar{b}^c| \geq 1 \text{ and } |\text{ran}(\mu^{(0)}) \cap a^c| \geq 1.$$

Remark: Because of the definition of $T^{(s)}$ we can conclude in this case, that

$\{\sigma \in T^{(s)} \upharpoonright \bar{\mu}^{(0)} : \sigma(|\sigma| - 1) \in a^c\}$ is dense in $T^{(s)} \upharpoonright \bar{\mu}^{(0)}$.

So we can arrange in the further construction of the $\mu^{(m)}$, that the opponent (if it exists) will have an infinite intersection with a^c .

$m \rightsquigarrow m + 1$:

Let $\mu^{(m)}$ be constructed for a $m \in \omega$:

Case 1: $\forall a \in B_{m+2} \ a \not\subseteq^* \bar{b}$:

In this case choose $\mu^{(m+1)} \in T^{(s)}$, $\mu^{(m+1)} \supseteq \mu^{(m)}$, such that

$$|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m + 2$$

and for all $k \leq m$, where we had case 2.2.2 witnessed by some $\tilde{a} \in B_{k+1}$, we have $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.

Case 2: $\exists a \in B_{m+2} \ a \supseteq^* \bar{b}$:

Case 2.1: $\forall \sigma \in T^{(s)} \ (\sigma \supseteq \mu^{(m)} \rightarrow f_a^{-1}[\{1\}] \not\subseteq (\text{ran}(\sigma) \cup \bar{b}) \setminus |\tau_s|)$:

In this case choose $\mu^{(m+1)}$ the same way as in case 1.

case 2.2: $\exists \sigma \in T^{(s)} \ (\sigma \supseteq \mu^{(m)} \wedge f_a^{-1}[\{1\}] \subseteq (\text{ran}(\sigma) \cup \bar{b}) \setminus |\tau_s|)$

In this case let $\bar{\mu}^{(m+1)}$ be a witness.

Case 2.2.1:

$$\begin{aligned} X := \{b \in B_0 \cap [T^{(s)} \upharpoonright \bar{\mu}^{(m+1)}] : & \forall \emptyset \neq t \subseteq s \forall d \in c_t \ (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s \ (z_t \neq * \rightarrow (|b \cap z_t| = \omega)) \\ & \wedge |b \cap a| = \omega \} \end{aligned}$$

is uncountable:

In this case the construction of the $\mu^{(m)}$ stops.

Define $f_{0,s} := f_a$, $\sigma_{s \smallfrown 1}^{(0)} := \bar{\mu}^{(m+1)}$, $\sigma_{s \smallfrown 0}^{(0)} := \bar{b} \upharpoonright k$ with $k \in \omega$ sufficiently large.

Also define $\bar{T}_0^{(s \smallfrown 0)} := T^{(s)} \upharpoonright \sigma_{s \smallfrown 0}^{(0)}$ and $\bar{T}_0^{(s \smallfrown 1)} := \text{cond}(X)$.

case 2.2.2: The set X of the previous case is countable: In this case choose $\mu^{(m+1)} \in T^{(s)}$, $\mu^{(m+1)} \supseteq \bar{\mu}^{(m+1)}$, such that

$$|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m + 2,$$

$$|\text{ran}(\mu^{(m+1)}) \cap a^c| \geq 1,$$

and for all $k \leq m$, where we had case 2.2.2 witnessed by some $\tilde{a} \in B_{k+1}$ we have $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.

Remark: In this case we also have that $\{\sigma \in T^{(s)} \upharpoonright \bar{\mu}^{(m+1)} : \sigma(|\sigma| - 1) \in a^c\}$ is dense in $T^{(s)} \upharpoonright \bar{\mu}^{(m+1)}$.

Now, if the construction of the $\mu^{(m)}$ is successful, we can define our opponent by $y := \bigcup_{m \in \omega} \mu^{(m)}$. Again it is true, that for all $f \in A$ we have $f^{-1}[\{1\}] \cap ((\omega \setminus (y \cup \bar{b})) \cup |\tau_s|) \neq \emptyset$ or $|\text{dom}(f) \cap (\bar{b} \cup y)| = \omega$. As before we can find $h \in Si$ with $h \perp A$. Hence A is not maximal.

If on the other hand the construction of the opponent stops, we try to construct opponents with respect to $B_{|t|}$ for all $t \subseteq s$ with $t \neq \emptyset$ and $z_t = *$ in the following sense:

Assume that for some $i < |s|$ and for all $j \leq i$ the $f_{j,s}$, $\sigma_{s \smallfrown 0}^{(j)}$ and $\sigma_{s \smallfrown 1}^{(j)}$ have been constructed together with $\bar{T}_j^{(s \smallfrown 0)} = T^{(s)} \upharpoonright \sigma_{s \smallfrown 0}^{(j)}$ and $\bar{T}_j^{(s \smallfrown 1)} = \text{cond}(X)$ for uncountable

$$\begin{aligned} X = \{b \in B_0 \cap [\bar{T}_{j-1}^{(s \smallfrown 1)} \upharpoonright \sigma_{s \smallfrown 1}^{(j)}] : & \forall \emptyset \neq t \subseteq s \forall d \in c_t (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ & \wedge \forall 1 \leq l \leq j (z_{s \upharpoonright l} = * \rightarrow |b \cap \text{dom}(f_{l,s})^c| = \omega)\} \end{aligned}$$

and assume that $z_{s \upharpoonright i+1} = *$. (Otherwise just set $\sigma_{s \smallfrown k}^{(i+1)} := \sigma_{s \smallfrown k}^{(i)}$ and define $\bar{T}_{i+1}^{(s \smallfrown k)}$ accordingly.)

We know because of the fact that $z_{s \upharpoonright i+1} = *$ and the respective remark that for all $e \in B_{i+1}$ the set $\{\sigma \in \bar{T}_i^{(s \smallfrown 1)} : \sigma(|\sigma| - 1) \in e^c\}$ is dense in $\bar{T}_i^{(s \smallfrown 1)}$.

Pick some reference branch $\bar{b} \in B_0 \cap [\bar{T}_i^{(s \smallfrown 0)}]$ and check if there exist $f \in A_{fin}$, $\sigma \in \bar{T}_i^{(s \smallfrown 1)}$ with $\{k \in \omega : f(k) \neq x_{i+1}(k)\} \subseteq (\bar{b} \cup \text{ran}(\sigma)) \setminus |\tau_s|$. If such f, σ exist define $f_{i+1,s} := f$, $\sigma_{s \smallfrown 1}^{(i+1)} := \sigma$ and $\sigma_{s \smallfrown 0}^{(i+1)} := \bar{b} \upharpoonright k$ for $k \in \omega$ sufficiently large, as well as $\bar{T}_{i+1}^{(s \smallfrown 0)} := \bar{T}_i^{(s \smallfrown 0)} \upharpoonright \sigma_{s \smallfrown 0}^{(i+1)}$ and $\bar{T}_{i+1}^{(s \smallfrown 1)} := \bar{T}_i^{(s \smallfrown 1)} \upharpoonright \sigma_{s \smallfrown 1}^{(i+1)}$.

If such f, σ do not exist we try again to construct an opponent sequence $\langle \mu^{(m)} \mid m \in \omega \rangle$ by recursion:

Before we start the recursion we consider two different cases:

- (i) $\forall e \in B_{i+1} e \not\supseteq^* \bar{b}$
- (ii) $\exists! e \in B_{i+1} e \supseteq^* \bar{b}$ (remember that B_{i+1} is an a.d.-family)

These two cases will be handled slightly differently in the following recursion.

Also pick some bijection $\phi : \omega \rightarrow \omega \setminus \{0, i+1\}$.

$m = 0$:

Case 1: $\forall a \in B_{\phi(0)} a \not\supseteq^* \bar{b}$:

In this case just set $\mu^{(0)} := \sigma_{s \smallfrown 1}^{(i)}$.

Case 2: $\exists a \in B_{\phi(0)} a \supseteq^* \bar{b}$:

Case 2.1:

$\forall \sigma \in \bar{T}_i^{(s \smallfrown 1)}, \sigma \supseteq \sigma_{s \smallfrown 1}^{(i)} (\{k \in \omega : f_a(k) \neq x_{i+1}(k)\} \not\subseteq (\bar{b} \cup \text{ran}(\sigma)) \setminus |\tau_s|)$:

In this case set $\mu^{(0)} := \sigma_{s \smallfrown 1}^{(i)}$.

Case 2.2:

Otherwise let $\bar{\mu}^{(0)} \in \bar{T}_i^{(s \smallfrown 1)}$, $\bar{\mu}^{(0)} \supseteq \sigma_{s \smallfrown 1}^{(i)}$ with $\{k \in \omega : f_a(k) \neq x_{i+1}(k)\} \subseteq (\bar{b} \cup \text{ran}(\bar{\mu}^{(0)})) \setminus |\tau_s|$.

Case 2.2.1:

$$\begin{aligned} X := \{b \in B_0 \cap [\bar{T}_i^{(s \smallfrown 1)} \upharpoonright \bar{\mu}^{(0)}] : & \forall \emptyset \neq t \subseteq s \forall d \in c_t (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ & \wedge \forall 1 \leq l \leq i (z_{s \upharpoonright l} = * \rightarrow (|b \cap (\text{dom}(f_{l,s}))^c| = \omega) \\ & \wedge |b \cap a| = \omega) \end{aligned}$$

is uncountable:

Then define $f_{i+1,s} := f_a$, $\sigma_{s \smallfrown 1}^{(i+1)} := \bar{\mu}^{(0)}$, $\sigma_{s \smallfrown 0}^{(i+1)} := \bar{b} \upharpoonright k$ for $k \in \omega$ sufficiently large, such that $\{k \in \omega : f_a(k) \neq x_{i+1}(k)\} \subseteq (\text{ran}(\sigma_{s \smallfrown 0}^{(i+1)}) \cup \text{ran}(\sigma_{s \smallfrown 1}^{(i+1)})) \setminus |\tau_s|$.

Also define $\bar{T}_{i+1}^{(s \smallfrown 0)} := \bar{T}_i^{(s \smallfrown 0)} \upharpoonright \sigma_{s \smallfrown 0}^{(i+1)}$ and $\bar{T}_{i+1}^{(s \smallfrown 1)} := \text{cond}(X)$.

The construction of the opponent stops in this case.

Case 2.2.2: The set X of the previous case is countable:

Then choose $\mu^{(0)} \in \bar{T}_i^{(s \smallfrown 1)}$, $\mu^{(0)} \supseteq \bar{\mu}^{(0)}$ with $|\text{ran}(\mu^{(0)}) \cap \bar{b}^c| \geq 1$, $|\text{ran}(\mu^{(0)}) \cap a^c| \geq 1$ (and $|\text{ran}(\mu^{(0)}) \cap e^c| \geq 1$ in case of (ii)).

We also obtain our usual density property.

$$\underline{m \rightsquigarrow m + 1} :$$

Let $\mu^{(m)}$ be constructed for some $m \in \omega$:

Case 1: $\forall a \in B_{\phi(m+1)}$ $a \not\supseteq^* \bar{b}$:

In this case choose $\mu^{(m+1)} \in \bar{T}_i^{s \smallfrown 1}$, $\mu^{(m+1)} \supseteq \mu^{(m)}$

with

$|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m + 2$ (and $|\text{ran}(\mu^{(m+1)}) \cap e^c| \geq m + 2$ in the case of (ii))

and for each $k \leq m$, where we had 2.2.2 witnessed by $\tilde{a} \in B_{\phi(k)}$, we have: $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.

Case 2: There exists $a \in B_{\phi(m+1)}$ with $a \supseteq^* \bar{b}$:

Case 2.1:

$\forall \sigma \in \bar{T}_i^{(s \smallfrown 1)}$ ($\sigma \supseteq \mu^{(m)} \rightarrow \{k \in \omega : f_a(k) \neq x_{i+1}(k)\} \not\subseteq (\bar{b} \cup \text{ran}(\sigma)) \setminus |\tau_s|$):

Pick $\mu^{(m+1)}$ as in case 1.

Case 2.2:

$\exists \sigma \in \bar{T}_i^{(s \smallfrown 1)}$ ($\sigma \supseteq \mu^{(m)} \wedge \{k \in \omega : f_a(k) \neq x_{i+1}(k)\} \subseteq (\bar{b} \cup \text{ran}(\sigma)) \setminus |\tau_s|$):
Then let $\bar{\mu}^{(m+1)}$ be a witness.

Case 2.2.1:

$$\begin{aligned}
X := \{b \in B_0 \cap [\overline{T}_i^{(s-1)} \upharpoonright \overline{\mu}^{(m+1)}] : & \forall \emptyset \neq t \subseteq s \forall d \in c_t (|b \cap d| = \omega) \\
& \wedge \forall \emptyset \neq t \subseteq s (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\
& \wedge \forall 1 \leq l \leq i (z_{s|l} = * \rightarrow |b \cap (\text{dom}(f_{l,s}))^c| = \omega) \\
& \wedge |b \cap a| = \omega \}
\end{aligned}$$

is uncountable:

In this case define $f_{i+1,s} := f_a$, $\sigma_{s \smallfrown 1}^{(i+1)} := \overline{\mu}^{(m+1)}$ and $\sigma_{s \smallfrown 0}^{(i+1)} := \bar{b} \upharpoonright k$ for $k \in \omega$ sufficiently large. Also set $\overline{T}_{i+1}^{(s-0)} := \overline{T}_i^{(s-0)} \upharpoonright \sigma_{s \smallfrown 0}^{(i+1)}$ and $\overline{T}_{i+1}^{(s-1)} := \text{cond}(X)$ and stop the construction of the opponent.

Case 2.2.2: The set X of the previous case is countable:

Then choose $\mu^{(m+1)} \in \overline{T}_i^{(s-1)}$, $\mu^{(m+1)} \supseteq \overline{\mu}^{(m+1)}$ with $|\text{ran}(\mu^{(m+1)}) \cap \bar{b}^c| \geq m+2$ (and $|\text{ran}(\mu^{(m+1)}) \cap e^c| \geq m+2$ in the case of (ii)) and for each $k \leq m$, where had case 2.2.2 witnessed by $\tilde{a} \in B_{\phi(k)}$, we have $|\text{ran}(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$, as well as $|\text{ran}(\mu^{(m+1)}) \cap a^c| \geq 1$. Remark: Again we get $\{\sigma \in \overline{T}_i^{(s-1)} \upharpoonright \overline{\mu}^{(m+1)} : \sigma(|\sigma| - 1) \in a^c\}$ is dense in $\overline{T}_i^{(s-1)} \upharpoonright \overline{\mu}^{(m+1)}$.

If the construction of the opponent $y := \bigcup_{m \in \omega} \mu^{(m)}$ is successful, we have the following situation:

$$|\bar{b}^c \cap y| = \omega \text{ and for } d \in B_0, d \neq \bar{b}, \text{ we have } |\bar{b} \cap d^c| = \omega.$$

Also we either have that

$$\text{for all } d \in B_{i+1} \text{ holds } |\bar{b} \cap d^c| = \omega \quad (\text{case (i)})$$

$$\text{or else } |y \cap e^c| = \omega \wedge \forall d \in B_{i+1} (d \neq e \rightarrow |\bar{b} \cap d^c| = \omega). \quad (\text{case (ii)}).$$

Analogously as with the previous opponents we have for all $j \notin \{0, i+1\}$ and $d \in B_j$ that either

$$\exists k \in (\omega \setminus (\bar{b} \cup y) \cup |\tau_s|) \cap \text{dom}(f_d) (f_d(k) \neq x_{i+1}(k))$$

$$\text{or } |\text{dom}(f_d) \cap ((\bar{b} \cup y) \setminus |\tau_s|)| = \omega.$$

If we define $h_0 \in Si$ by $\text{dom}(h_0) := \omega \setminus (\bar{b} \cup y) \cup |\tau_s|$ and

$h_0 \upharpoonright \text{dom}(h_0) \equiv x_{i+1} \upharpoonright \text{dom}(h_0)$ and, by Corollary 5.3, choose some

$h_1 \in Si((\bar{b} \cup y) \setminus |\tau_s|)$ with $h_1 \perp_{(\bar{b} \cup y) \setminus |\tau_s|} \{f \in A : f \parallel h_0\}$, we have $h \perp A$, where $h := h_0 \cup h_1 \in Si$. Hence A is not maximal

If on the other hand the construction of the $\mu^{(m)}$ stops, then go on with the construction of T_0 (or the construction of the next opponent). If for all $i < s$ with $z_{s|i+1} = *$ the constructions of the opponents (with respect to $B_{|i+1|}$) fail, we have constructed in particular $\sigma_{s \smallfrown 0}^{(|s|)}$, $\sigma_{s \smallfrown 1}^{(|s|)}$, $\overline{T}_{|s|}^{(s-0)}$ and $\overline{T}_{|s|}^{(s-1)}$. In this case set $\sigma_{s \smallfrown 0} := \sigma_{s \smallfrown 0}^{(|s|)}$, $\sigma_{s \smallfrown 1} :=$

$\sigma_{s \smallfrown 1}^{(|s|)}$ and $\bar{T}^{(s \smallfrown 0)} := \bar{T}_{|s|}^{(s \smallfrown 0)}$, $\bar{T}^{(s \smallfrown 1)} := \bar{T}_{|s|}^{(s \smallfrown 1)}$ and define $c_{s \smallfrown 0} := \{\omega\}$ and $c_{s \smallfrown 1} := \{(dom(f_{i,s}))^c : (1 \leq i \leq |s| \wedge z_{s \smallfrown i} = *) \vee i = 0\}$.

Also in this case for each $j \in 2$ we have to check two different cases:

Case a:

There is an $a \in B_{|s|+1}$ with:

$$\begin{aligned} X := \{b \in B_0 \cap [\bar{T}^{(s \smallfrown j)}] : & \forall \emptyset \neq t \subseteq s \smallfrown j \forall d \in c_t (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ & \wedge |b \cap a| = \omega\} \end{aligned}$$

is uncountable:

Then define $z_{s \smallfrown j} := a$ and $T^{(s \smallfrown j)} := cond(X)$.

Case b:

Otherwise just define $T^{(s \smallfrown j)} := \bar{T}^{(s \smallfrown j)}$ and $z_{s \smallfrown j} := *$.

This finishes the recursion step in the construction of the $\sigma_{s \smallfrown j}$, $T^{(s \smallfrown j)}$.

If for no $s \in 2^{<\omega}$ the construction of the opponents is successful, then define $T_0 := dwcl(\{\sigma_s : s \in 2^{<\omega}\})$, which is obviously a perfect subtree of T . Let $x \in [T_0]$ be a branch of the tree that is not definable from $A \cup \{A\} \cup T_0 \cup \{T_0\}$. Associated to x is a real $r \in 2^\omega$ by $r := \bigcup \{s : \sigma_s \subseteq x\}$.

Now let $f \in A$ be an arbitrary Silver function of the antichain. We analyze the different classes that f can be a member of:

i) For $f = f_d$ for a $d \in B_k$ with $k \in \omega \setminus \{\emptyset\}$ there are the following subcases:

Case 1: $z_{r \smallfrown k} = a \in B_k$:

Assume that $f_a^{-1}[\{1\}] \subseteq x$ would hold and choose some $n \in \omega$ with $x \smallfrown n \supseteq f_a^{-1}[\{1\}]$. We have $x \in [T^{(r \smallfrown k)}]$ and therefore $x \smallfrown n \in T^{(r \smallfrown k)}$, and hence by definition of the tree and the fact that $z_{r \smallfrown k} = a$ we know, that there exist uncountably many $b \in B_0$ with $b \supseteq x \smallfrown n \supseteq f_a^{-1}[\{1\}]$ and $|b \cap a| = \omega$. But then $f_a \parallel f_b$, which is clearly a contradiction. So we can conclude $f_a^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$.

On the other hand, we also know by the fact that $x \in [T_0 \smallfrown \sigma_{r \smallfrown k}]$ with $z_{r \smallfrown k} = a$ and the construction of T_0 that $|x \cap a| = \omega$. For any $d \in B_k$ with $d \neq a$ we have $|dom(f_d)^c \cap a| < \omega$, hence we get $|dom(f_d) \cap x| = \omega$. So in this case we have $|dom(f) \cap x| = \omega$ or $f^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$.

Case 2: $z_{r \smallfrown k} = *$:

Assume by contradiction, that $\exists a \in B_k$ $a \supseteq^* x$. Because x is not definable from $A \cup \{A\} \cup T_0 \cup \{T_0\}$, we know that there exists $N \in \omega$ with $\{z \in [T_0 \smallfrown \sigma_{r \smallfrown k}] : N \cup a \supseteq z\}$ being uncountable. In particular, we can

find a $\sigma_s \in T_0 \upharpoonright \sigma_{r \upharpoonright k}$ with $|\sigma_s| \geq N$ and $z, z' \in [T_0 \upharpoonright \sigma_{r \upharpoonright k}]$ with $N \cup a \supseteq z, z'$ and $z \supseteq \sigma_{s \frown 0}$ and $z' \supseteq \sigma_{s \frown 1}$. Because $z_{r \upharpoonright k} = *$ with $r \upharpoonright k \subseteq s$ we have defined $f_{k,s}$ and $\{n \in \omega : f_{k,s}(n) \neq x_k\} \subseteq (z \cup z') \setminus N \subseteq a$ is true. Also by construction we have $(\text{dom}(f_{k,s}))^c \in c_{s \frown 1}$ and hence $|z' \cap (\text{dom}(f_{k,s}))^c| = \omega$ and hence $|a \cap (\text{dom}(f_{k,s}))^c| = \omega$. So we can conclude that $f_a \parallel f_{k,s}$ for some $f_{k,s}$ with $(\text{dom}(f_{k,s}))^c \notin B_k$ by construction. So there are two different members of the antichain that are compatible, which is a contradiction.

We can conclude that in this case $\forall a \in B_k |a^c \cap x| = \omega$, so in particular $|\text{dom}(f) \cap x| = \omega$.

ii) $f = f_b$ for $b \in B_0$:

Assume by contradiction that $b \supseteq^* x$.

As in i) we find z, z' in $[T_0]$ with a common initial segment τ_s , $|\tau_s| \geq N$ and $z \cup z' \subseteq b \cup N$. Then we have $f_{0,s}^{-1}[\{1\}] \subseteq (z \cup z') \setminus N$ and $|z' \cap (\text{dom}(f_{0,1}))^c| = \omega$, hence $f_b \parallel f_{0,s}$, which is a contradiction.

We can conclude $|\text{dom}(f) \cap x| = \omega$.

iii) $f \in A_{fin}$:

Assume by contradiction $f^{-1}[\{1\}] \cap (\omega \setminus x) = \emptyset$. Then for a sufficiently large $n \in \omega$ we would have $f^{-1}[\{1\}] \subseteq x \upharpoonright n$ and in conclusion $f^{-1}[\{1\}] \subseteq b$ for some $b \in B_0$. This would imply $f \parallel f_b$, a contradiction.

So in this case we can conclude $f^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$.

iv) $|f^{-1}[\{1\}]| = \omega$:

Then $f^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$ or $|\text{dom}(f) \cap x| = \omega$.

So in each case for every $f \in A$ we have $\exists k \in \omega \setminus x f(k) = 1 \vee |\text{dom}(f) \cap x| = \omega$. This means that if we define $h_0 \in Si$ by $(\text{dom}(h_0))^c := x$ and $h_0 \upharpoonright (\omega \setminus x) \equiv 0$, then the functions of the antichain that are compatible with h_0 form an antichain exclusively consisting of infinite conditions relativized to x . By Corollary 5.3 we can pick some $h_1 \in Si(x)$ that is incompatible to these functions relativized to x . If we define $h := h_0 \cup h_1$, we get $h \perp A$.

Hence A is not a maximal antichain. \square

We can now use the above theorem to prove the following:

Theorem 5.5. $\mathfrak{d} \leq \mathfrak{a}(Si)$

Proof. Suppose that we have an antichain A of Silver conditions with $\omega_1 \leq |A| < \mathfrak{d}$. We want to show that A is not maximal. Because of theorem 2.4 we can assume without loss of generality:

$\forall x \in 2^\omega \ |\{f \in A : f|x\}| \leq \omega$.

First we define a tree $T \subseteq (2 \cup \{*\})^{<\omega}$ as follows:

$T := \{\sigma \in (2 \cup \{*\})^{<\omega} : |\{f \in A : \sigma \parallel f \wedge \sigma^{-1}[\{*\}] \subseteq (dom(f))^c\}| \geq \aleph_1\}$

We will show that there exists a branch in this tree that is a Silver condition that is incompatible with the antichain. Note that obviously $[T] \not\subseteq Si$. Hence, we define $Z^* := [T] \cap Si$. T has the following properties, that are easy to prove:

i): $\forall x \in [T] \forall f \in A_{fin} \exists k \in dom(f) \cap dom(x) \ x(k) \neq f(k)$

ii): The $*$ are dense in T ; i.e. $\forall \sigma \in T \exists \tau \in T \ (\tau \supseteq \sigma \wedge \tau^{-1}[\{*\}] \cap [|\sigma|, |\tau|) \neq \emptyset)$

iii) : $\forall \sigma \in T \forall k \in \omega \exists \tau \in T \ (\tau \supseteq \sigma \wedge |\tau| \geq k \wedge \tau^{-1}[\{*\}] \cap [|\sigma|, \tau) = \emptyset)$

iv*): If $\sigma \in T, f \in A_{inf}$, there are uncountably many conditions of the antichain that are compatible with σ , contain the $*$ of σ in their codomain and are also incompatible with f . This implies the following: $\forall f \in A_{inf} \forall \sigma \in T \exists \tau \in T \exists k \in \omega \ (\tau \supseteq \sigma \wedge k \in (dom(f) \cap (dom(\tau)) \wedge \tau(k) \neq f(k))) \vee (k \in dom(f) \wedge \tau(k) = *)$.

Because for any $\sigma \frown * \in T$ we also have $\sigma \frown 0 \in T$ and $\sigma \frown 1 \in T$, we can replace iv*) by the following:

iv): $\forall f \in A_{inf} \forall \sigma \in T \exists \tau \in T; \tau \supseteq \sigma \exists k \in dom(\tau) \cap dom(f) \ \tau(k) \neq f(k)$

Because of ii) and iii) we can pick a function $I : T \rightarrow Z^*; \sigma \mapsto x_\sigma$ such that for all σ, σ' with $\sigma \neq \sigma'$ we have that $x_\sigma^{-1}[\{*\}]$ and $x_{\sigma'}^{-1}[\{*\}]$ are almost disjoint. Together with the assumption from the beginning of the proof we get for any $\sigma, \sigma' \in T$ with $\sigma \neq \sigma'$ that the set $B_{\sigma, \sigma'} := \{f \in A_{inf} : f|x_\sigma \wedge f|x_{\sigma'}\}$ is countable. Let $B := \bigcup_{\sigma, \sigma' \in T, \sigma \neq \sigma'} B_{\sigma, \sigma'}$ be the countable set of infinite conditions of the antichain, that do not contradict at least two different $x_\sigma, x_{\sigma'}$. Let

$\langle b_n : n \in \omega \rangle$ be an enumeration of B .

In order to construct our desired branch of T , we need to introduce some auxilliary functions S_f, F_f (for $f \in A_{inf}$), h_n, H_n ($n \in \omega$). They are defined as follows:

$S_f : T \rightarrow \omega$

$$S_f(\sigma) := \begin{cases} \min\{k \in \omega : k \in dom(x_\sigma) \cap dom(f) \\ \wedge x_\sigma(k) \neq f(k)\} & , \text{if } \neg(f|x_\sigma) \wedge f \parallel \sigma \\ |\sigma| & , \text{if } f \perp \sigma \\ 0 & \text{otherwise} \end{cases}$$

$F_f : \omega \rightarrow \omega$

$F_f(n) := \max\{k \in \omega : \exists \sigma \in T (k = S_f(\sigma) \wedge |\sigma| \leq n)\}$
Note that the F_f are increasing.

$h_n : T \rightarrow \omega$

$$h_n(\sigma) := \begin{cases} \min\{k \in \omega : \exists \tau \in T (\tau \supseteq \sigma \wedge k \in \text{dom}(\tau) \cap \text{dom}(b_n) \\ \wedge |\tau| = k + 1 \wedge \tau(k) \neq b_n(k))\} & , \text{if } b_n \parallel \sigma \\ |\sigma| & , \text{if } b_n \perp \sigma \end{cases}$$

Note that because of property iv) of our tree h_n is well-defined for any $n \in \omega$.

$H_n : \omega \rightarrow \omega$

$H_n(m) := \max\{k \in \omega : \exists \sigma \in T (k = h_n(\sigma) \wedge |\sigma| \leq m)\}$.

For any $n \in \omega$ we have that H_n is increasing.

$F_f(n)$ gives us, for all x_σ with $|\sigma| \leq n$ that contradict f , an upper bound for the level at which this is witnessed.

$H_n(m)$ gives us, for all nodes of the tree of length at most m , an upper bound for the length of an extension in the tree that contradicts b_n .

We will use this information to construct for each $s \in \omega^{<\omega}$ and $f \in A_{inf}$ a sequence $R_{s,f} := \langle r_k^{(s,f)} : k \in \omega \rangle$ by recursion, that will help us to construct the desired branch of T .

For $k < |s|$ define $r_k^{(s,f)} := s(k)$
and $r_{|s|}^{(s,f)} := F_f \circ H_{|s|-1}(r_{|s|-1}^{(s,f)})$, if $s \neq \emptyset$
or $r_0^{(s,f)} := 0$, if $s = \emptyset$.

Furthermore for $k \in \omega$ define by recursion:

$$r_{|s|+k+1}^{(s,f)} := F_f \circ H_{|s|+k}(r_{|s|+k}^{(s,f)}).$$

We have $|\{R_{s,f} : s \in \omega^{<\omega}, f \in A_{inf}\}| \leq |A| < \mathfrak{d}$. So we can choose a strictly increasing sequence $R = \langle r_k : k \in \omega \rangle$ in ω with $\forall s \in \omega^{<\omega} \forall f \in A_{inf} \exists^\infty n \in \omega r_n > r_n^{(s,f)}$.

We use this sequence to construct $y \in Z^*$ by recursion:

$n = 0$:

Define $\sigma_0 := \emptyset$.

$n \rightsquigarrow n + 1$:

Let $\sigma_0, \dots, \sigma_n$ be constructed. Choose $\tilde{\sigma}_{n+1} \in T, \tilde{\sigma}_{n+1} \supseteq \sigma_n$ of minimal length with $\tilde{\sigma}_{n+1} \perp b_n$. Let $k_{n+1} \geq |\tilde{\sigma}_{n+1}|, r_{n+1}$ be sufficiently large such

that

$(x_{\tilde{\sigma}_{n+1}} \upharpoonright k_{n+1})^{-1}[\{*\}] \cap [|\sigma_n|, k_{n+1}) \neq \emptyset$. Then define $\sigma_{n+1} := x_{\tilde{\sigma}_{n+1}} \upharpoonright k_{n+1}$.

After ω many steps let $y := \bigcup_{n \in \omega} \sigma_n$. Obviously, by construction we have $y \in Z^* \subseteq Si$.

We claim that $y \perp A$. For all $f \in A_{fin}$ we get $y \perp f$ by property i) . For all $b \in B$ we have $y \perp B$ by construction of y . We still have to show that $y \perp A_{inf} \setminus B$:

Let $f \in A_{inf} \setminus B$. Because $f \notin B$ we can choose $n_0 \in \omega$ sufficiently large such that $\forall |\sigma| \geq |\sigma_{n_0}| \neg(f|x_\sigma)$.

Define $s \in \omega^{n_0+1}$ by:

$$s(k) := \begin{cases} 0 & , \text{if } k < n_0 \\ |\sigma_{n_0}| & , \text{if } k = n_0 \end{cases}$$

Hence, we have $r_{n_0}^{(s,f)} = |\sigma_{n_0}|$. Now let $k \geq 1$ be minimal with $|\sigma_{n_0+k}| > r_{n_0+k}^{(s,f)}$ (exists by construction of σ_n). Because H_n is increasing and by the fact that $|\sigma_{n_0+k-1}| \leq r_{n_0+k-1}^{(s,f)}$ we can conclude

$|\tilde{\sigma}_{n_0+k}| \leq H_{n_0+k-1}(|\sigma_{n_0+k-1}|) \leq H_{n_0+k-1}(r_{n_0+k-1}^{(s,f)})$, and hence by monotony of F_f we get

$|\sigma_{n_0+k}| > r_{n_0+k}^{(s,f)} = F_f \circ H_{n_0+k-1}(r_{n_0+k-1}^{(s,f)}) \geq F_f(|\tilde{\sigma}_{n_0+k}|)$.

By choice of n_0 and definition of F_f we have $\sigma_{n_0+k} = x_{\tilde{\sigma}_{n_0+k}} \upharpoonright |\sigma_{n_0+k}| \perp f$.
 $\Rightarrow y \perp f$

Thus the claim is proven and we can conclude that A is not a maximal antichain. \square

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Erklärung

Ich versichere hiermit, dass ich die vorliegende Dissertation abgesehen von der Beratung meines Betreuers, Herrn Prof. Dr. Otmar Spinas, selbständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Diese Arbeit fand unter Einhaltung der Regeln der guten wissenschaftlichen Praxis der Deutschen Forschungsgemeinschaft statt.

Die Ergebnisse des letzten Kapitels erscheinen in der folgenden Publikation:

Spinas, O. & Wyszkowski, M. *Silver Antichains*, Journal of Symbolic Logic, (angenommen/noch nicht publiziert)

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(Marek Wyszkowski)