

On Using Shadow Prices for the Asymptotic
Analysis of Portfolio Optimization under
Proportional Transaction Costs

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Zusammenfassung

Diese Dissertation enthält drei wissenschaftliche Arbeiten zum Problem der Nutzenmaximierung auf einem Finanzmarkt mit proportionalen Transaktionskosten. Eine der aufschlussreichsten Herangehensweisen hierzu ist der Übergang zu der asymptotischen Analysis der Auswirkung kleiner bzw. verschwindender Transaktionskosten, wobei der maximal erwartete Nutzen durch Taylor-Entwicklung lokal approximiert wird. Dieses Vorgehen findet Anwendung in den aktuellen Arbeiten von Kallsen und Muhle-Karbe (2012, 2013), in welchen mittels heuristischer Argumente ein Kandidat für die näherungsweise optimale Handelsstrategie, das asymptotische Handelsvolumen dazu sowie der Effekt der kleinen Transaktionskosten ermittelt werden. Ziel dieser Dissertation ist ein rigoroser Beweis zur Verifikation der Resultate von Kallsen und Muhle-Karbe mit präziser Formulierung hinreichender Regularitätsannahmen an das Marktmodell. Die Betrachtung erfolgt in drei Stufen jeweils mit einem der folgenden Schwerpunkte: exponentielle Nutzenfunktion, allgemeine Nutzenfunktionen sowie Handelsvolumen der approximativ optimalen Strategie. Durchgehend besteht die Kernidee in der Verwendung des so genannten Schattenpreis-Prozesses, welcher eine Transformation des ursprünglichen Optimierungsproblems mit Transaktionskosten in ein äquivalentes Problem ohne Transaktionskosten ermöglicht. Der Beweis der approximativen Optimalität der vorgeschlagenen Handelsstrategie basiert auf der Martingalmethode bzw. Dualitätstheorie, während sich die Verifikation der asymptotischen Formeln für das dazugehörige Handelsvolumen und den Kosteneffekt auf die Ergodeneigenschaft der reflektierten Brownschen Bewegung stützt. Die gesamte Methode funktioniert in einem allgemeinen eindimensionalen Itô-Rahmen (d.h. der Verlauf des Aktienkurses wird durch einen allgemeinen Itô-Prozess modelliert), was einen Zugang zu stochastischen Volatilitätsmodellen bietet.

Abstract

This thesis contains three scientific articles on the problem of utility maximization in a financial market with proportional transaction costs. One of the most revealing approaches to this problem consists in passing to the asymptotic analysis of the impact of small transaction costs, where the maximum expected utility is approximated locally by a Taylor series expansion. In recent works by Kallsen and Muhle-Karbe (2012, 2013) dealing with this issue, heuristic arguments are used for deriving a candidate trading strategy which is expected to be optimal to the leading order, along with asymptotic formulas for the associated trading volume and the effect of the small transaction costs. The subject of this thesis is a rigorous proof verifying the results of Kallsen and Muhle-Karbe with precise formulation of sufficient regularity assumptions on the market model. The study is carried out in three stages considering the exponential utility function, general utility functions and trading volume of the leading-order optimal strategy, respectively. The key idea throughout is the utilization of the so-called shadow price process which facilitates a transformation of the original problem with transaction costs into an equivalent frictionless one. The proof of the approximate optimality of the proposed strategy is based on the martingale method (duality theory), while the verification of the asymptotic formula for the respective trading volume and the impact of transaction costs relies on the ergodic property of reflected Brownian motion. The entire method works within a general one-dimensional Itô framework (i.e., the stock price is modelled by a general Itô process), which provides an access to stochastic volatility models.

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Chapter 1

Introduction

A classical problem of mathematical finance concerns an economic agent who invests in a financial market so as to maximize the expected utility of his terminal wealth and intermediate consumption. A possible approach to tackle such a problem is based on the dual characterization of admissible portfolios with the help of convex analysis. This has been studied mostly in frictionless environments, for instance in [21, 24, 34, 22]. In the context of markets with friction, Cvitanić and Karatzas [5] extend this approach to problems with proportional transaction costs. They introduce more or less explicitly the concept of consistent price systems and shadow price processes, which allow to translate the original problem into a more tractable frictionless one, cf. in particular Loewenstein [27], Kallsen and Muhle-Karbe [17] and Gerhold et al. [10] in this context.

The issue of utility optimization under proportional transaction costs is addressed for the first time by Magill and Constantinides [28]. Their heuristic derivation suggests that the optimal policy maintains the proportion of stock holdings within a wedge-shaped no-trade region containing the frictionless optimal risky fraction called Merton line. Davis and Norman [6] make this analysis rigorous and reveal that the optimal policy is linked to reflecting diffusions within the buying and selling boundaries of the no-trade region. Shreve and Soner [35] remove the restrictive conditions in [6] by merely assuming a finite value function which is shown to solve the Hamilton-Jacobi-Bellman equation in a viscosity sense. Shreve also proves that the sensitivity of the maximum expected utility to small transaction costs ε is of the order $O(\varepsilon^{2/3})$; and the width of the no-trade interval for the risky proportion is suggested to have order $O(\varepsilon^{1/3})$.

In recent studies, Kallsen and Muhle-Karbe [20, 19] investigate optimal portfolio choice under small proportional transaction costs ε for a stock price modelled by a general Itô process S over a finite time horizon T . In [20] dealing with optimization of exponential utility from terminal wealth, a leading-order optimal trading policy and the associated welfare impact are formally derived. Continuing the study of [20], [19] introduces a generalized risk-tolerance process R and provides parallel results for settings with general utility functions and allowing for intermediate consumption. Let us recall the

main statements therein. The leading-order optimal strategy φ^ε (in the sense of number of shares) is expected to stay within a no-trade region around the frictionless optimizer φ ; it increases and decreases only while hitting the lower and upper bound, respectively. In general, the midpoint of the no-trade region is shifted from φ by a corrector term accounting for the wealth effect caused by the past transaction costs. The half-width of the no-trade region reads as

$$\Delta\varphi^+ = \left(\frac{3}{2} R \frac{d[\varphi, \varphi]}{d[S, S]} S \varepsilon \right)^{1/3}. \quad (1.1)$$

The welfare impact of transaction costs is expressed in terms of the so-called certainty equivalent loss which refers to the initial capital necessary to balance the effect of transaction costs. To the leading order, this amounts to

$$\frac{1}{2} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \frac{(\Delta\varphi_t^+)^2}{R_t} d[S, S]_t \right], \quad (1.2)$$

where \mathbf{Q} denotes the equivalent martingale measure minimizing the frictionless dual problem. Moreover, the trading volume of strategy φ^ε quantified by its total variation can be approximated by a finite rate:

$$\int_0^T |d\varphi_t^\varepsilon| = \int_0^T (\varepsilon S_t)^{-1/3} \left(\frac{1}{12R_t} \right)^{1/3} \left(\frac{d[\varphi, \varphi]_t}{d[S, S]_t} \right)^{2/3} d[S, S]_t + o(\varepsilon^{-1/3}). \quad (1.3)$$

The purpose of the present study is to rigorously prove the main statements of [20, 19] under well-defined regularity conditions. Our approach resembles that of Henderson [12] and Kramkov and Sîrbu [26] in the sense that an explicitly known dual control provides us with an upper bound for the optimization problem. Since this bound coincides to the leading order with the utility of a candidate strategy to the primal problem, the latter must be approximately optimal. The idea of our proof can be outlined as follows:

- [20, 19] each derive a possibly suboptimal candidate strategy φ^ε along with a *shadow price* S^ε . This term here refers to a frictionless price process moving within the bid-ask bounds $[(1 - \varepsilon)S, (1 + \varepsilon)S]$ and such that strategy φ_t^ε only buys and sells stock if the shadow price S_t^ε coincides with the ask price $(1 + \varepsilon)S_t$ and bid price $(1 - \varepsilon)S_t$, respectively. Evidently, following φ^ε in the frictionless market S^ε yields the same wealth process and hence the same expected utility as in the original market with proportional transaction costs. This expected utility can be computed explicitly to the leading order because both φ^ε and S^ε are given in closed form.
- According to [21, 24, 34, 22] dealing with the issue of hedging duality in frictionless markets, the utility maximization problem for S^ε without transaction costs is related to a dual minimization problem on the set of equivalent local martingale measures.

Specifically, the value of the dual problem dominates the expected utility of any admissible trading strategy. In a second step we therefore construct a carefully chosen, explicitly known local martingale measure (identified with its Radon-Nikodym density Z^ε) for S^ε . Since trading in the frictionless market S^ε leads to higher profit than in the original market with transaction costs, the Lagrange dual function evaluated at Z^ε provides an upper bound to the maximum expected utility in the market with friction. This upper bound can be computed explicitly to the leading order because Z^ε is known in closed form.

- In a final step we observe that the suboptimal expected utility of φ^ε coincides to the leading order with the upper bound above, which proves the approximate optimality of the candidate strategy.

In the language of [5], $(Z^\varepsilon, Z^\varepsilon S^\varepsilon)$ is a *state-price density*, which, by duality to the set of self-financing portfolios in the market with friction, provides an upper bound to the expected utility under transaction costs.

In contrast to our method, the classical approach to the leading-order optimization is based on stochastic control theory in a Markovian framework. Roughly speaking, the main purpose therein is to develop an asymptotic series expansion of the value function for small transaction costs. Based on the Hamilton-Jacobi-Bellman equation, appropriate differential equations for the leading-order terms can be derived heuristically in a first step and then shown to possess viscosity solutions of the latter. For an account of the literature, the reader is referred to [20, 19]. Here we point out again some essential works in the related areas. Focusing on geometric Brownian motion, Whalley and Wilmott [40] deal with exponential utility and heuristically develop power series expansions of the value function with and without contingent claim liability, which provides an approximate utility indifference option price; Bichuch [3] proves the results in [40] rigorously and provides a “nearly optimal” strategy. For power type utility functions, Janeček and Shreve [16] derive asymptotic expansions of the value function and the no-trade boundary over an infinite time horizon; Bichuch [2] carries out parallel analysis for a finite time horizon. Soner and Touzi [38, 31] (Possamaï to be added to the latter) treat infinite horizon consumption problems for utility functions on the positive real line and general Markovian models in one- and multi-dimensional cases, respectively. They identify the first-order term in the asymptotic expansion of the value function with a singular ergodic control problem and make use of the deep theory of homogenization. In the one-dimensional case, the no-trade boundary is calculated explicitly to the leading order by solving the corrector equation for the potential function of the ergodic control. The cited papers carry a strong analytic flavor. By contrast and as noted above, our more probabilistic approach relies on dual considerations and our emphasis is put on an exact description of the leading-order optimal policy.

The thesis is organized as follows. In Chapter 2 we focus on exponential utility and verify the main results of [20] concerning formulas (1.1, 1.2) for a constant risk-tolerance

process R . In Chapter 3 we generalize the main theorem of Chapter 2 and provide a rigorous proof of formulas (1.1, 1.2) heuristically derived in [19] for general utility functions on \mathbb{R}_+ and allowing for intermediate consumption, provided that the related risk-tolerance process R exists. In Chapter 4, relying on ergodic properties of reflected Brownian motion, the asymptotic formula for the trading volume (1.3) proposed in [19] is proved in the sense of convergence in probability; using the same approach, the incurred utility loss while following the leading-order optimal trading strategy φ^ε can be approximated by an expression that does not depend on φ^ε itself, which completes the proof of formula (1.2). Due to different references and for the sake of better exposition, each chapter is written as a self-contained article with separate notation. Appendix A contains some essential inequalities which are used for the estimations throughout the thesis. In order to provide examples for the main theorem of Chapter 2 in concrete models, in Appendix B we derive the frictionless optimal trading strategy φ maximizing exponential utility in stochastic volatility models with conditionally independent increments.

We generally use the notation of [15] and [32]. In particular, $H \bullet Y := \int_0^\cdot H_t dY_t$ stands for the stochastic integral of H with respect to Y .

Chapter 2

Exponential Utility

At this first stage of our research, we consider an investor with constant absolute risk aversion who trades a risky asset with price following general Itô dynamics, in the presence of small proportional transaction costs. Kallsen and Muhle-Karbe [20] formally derive a leading-order optimal trading policy and the associated welfare impact of transaction costs. In the present chapter we carry out a convex duality approach facilitated by the concept of shadow price processes in order to verify the main results of [20] under well-defined regularity conditions. Furthermore, due to the nature of exponential utility, our theorem can be directly applied to the issue of utility indifference pricing for deriving an asymptotic price formula and the related hedging strategy. In particular, the results of Whalley and Wilmott [40] and Bichuch [3] can be verified here as a byproduct. It turns out that our regularity conditions imposed on the frictionless optimization problem in the presence of contingent claim liability are consistent with the regularity assumptions on the payoff function required in [3] for the rigorous proof.

The remainder of this chapter is organized as follows. The market model is introduced in Section 2.1. Subsequently, we present the main results concerning the leading-order optimal investment and their application in utility indifference pricing. In Section 2.3, we provide examples in concrete models, where the frictionless optimizer is known explicitly; we also verify the asymptotic utility indifference price formula proposed in [40, 3] by applying our main theorem. For comparison with existing literature, in Section 2.4 we state the Markovian counterparts of the assumptions required for our main theorem and the auxiliary shadow price process as our key object for the proof. The proofs of the main results are provided in Section 2.5.

2.1 The market model

We consider the same setup as in [20]: fixing a finite time horizon $T \in (0, \infty)$, the financial market consists of a riskless asset (bond) with price normalized to 1 and a risky asset (stock) traded with proportional transaction costs. The stock price S is modelled by a

general positive Itô process

$$dS_t = b_t^S dt + \sigma_t^S dW_t$$

defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$. Here, W is a one-dimensional, standard Brownian motion and b, σ are predictable processes satisfying

$$\int_0^T |b_t^S| + (\sigma_t^S)^2 dt < \infty \quad \text{a.s.}$$

Let $\varepsilon \in (0, 1)$ denote the relative bid-ask spread, i.e., an investor has to pay the higher ask price $(1 + \varepsilon)S$ but only receives the lower bid-price $(1 - \varepsilon)S$ for buying and selling the stock, respectively.

Definition 2.1.1. 1. A *trading strategy* is an \mathbb{R}^2 -valued predictable process (ψ^0, ψ) of finite variation, where ψ_t^0 and ψ_t denote the number of shares held in the bank account and in stock at time t , respectively. The *liquidation wealth process* of a trading strategy (ψ^0, ψ) is defined as

$$X^{\psi, \varepsilon} := \psi^0 + \psi \mathbf{1}_{\{\psi \geq 0\}}(1 - \varepsilon)S + \psi \mathbf{1}_{\{\psi < 0\}}(1 + \varepsilon)S.$$

2. Writing $\psi = \psi^\uparrow - \psi^\downarrow$ with increasing predictable processes $\psi^\uparrow, \psi^\downarrow$ which do not increase at the same time, a trading strategy (ψ^0, ψ) is called *self financing* if

$$d\psi_t^0 = (1 - \varepsilon)S_t d\psi_t^\downarrow - (1 + \varepsilon)S_t d\psi_t^\uparrow,$$

cf. [17]. For given initial value ψ_0^0 , a self-financing trading strategy (ψ^0, ψ) will be identified with its second component ψ in the sequel.

3. Given *initial wealth* $(x^B, x^S) \in \mathbb{R}^2$ in the bank account and the stock, respectively, a self-financing trading strategy (ψ^0, ψ) is said to be *admissible* for (x^B, x^S) and written as

$$\psi \in \mathcal{A}^\varepsilon(x^B, x^S)$$

if $x^B = \psi_0^0, x^S = \psi_0 S_0$, and if the related liquidation wealth is bounded from below, i.e.,

$$X^{\psi, \varepsilon} \geq -K \text{ for some } K \in \mathbb{R}_+.$$

Remark 2.1.2. The liquidation wealth of a self-financing strategy (ψ^0, ψ) with $x^B = \psi_0^0, x^S = \psi_0 S_0$ can be written as

$$\begin{aligned} X_t^{\psi, \varepsilon} &= \psi_t^0 + \psi_t \mathbf{1}_{\{\psi_t \geq 0\}}(1 - \varepsilon)S_t + \psi_t \mathbf{1}_{\{\psi_t < 0\}}(1 + \varepsilon)S_t \\ &= \psi_0^0 + (1 - \varepsilon)S \bullet \psi_t^\downarrow - (1 + \varepsilon)S \bullet \psi_t^\uparrow + \psi_t \mathbf{1}_{\{\psi_t \geq 0\}}(1 - \varepsilon)S_t + \psi_t \mathbf{1}_{\{\psi_t < 0\}}(1 + \varepsilon)S_t. \end{aligned} \quad (2.1)$$

If $(1 - \varepsilon)S \cdot \psi^\downarrow = \tilde{S} \cdot \psi^\downarrow$ and $(1 + \varepsilon)S \cdot \psi^\uparrow = \tilde{S} \cdot \psi^\uparrow$ for some Itô process \tilde{S} with values in $[(1 - \varepsilon)S, (1 + \varepsilon)S]$, then (2.1) and integration by parts yield

$$\begin{aligned} X_t^{\psi, \varepsilon} &= \psi_0^0 - \tilde{S} \cdot \psi_t + \psi_t \tilde{S}_t - \psi_t \mathbf{1}_{\{\psi_t \geq 0\}} (\tilde{S}_t - (1 - \varepsilon)S_t) - \psi_t \mathbf{1}_{\{\psi_t < 0\}} (\tilde{S}_t - (1 + \varepsilon)S_t) \\ &= \psi_0^0 + \psi_0 \tilde{S}_0 + \psi \cdot \tilde{S}_t - \psi_t \mathbf{1}_{\{\psi_t \geq 0\}} (\tilde{S}_t - (1 - \varepsilon)S_t) - \psi_t \mathbf{1}_{\{\psi_t < 0\}} (\tilde{S}_t - (1 + \varepsilon)S_t) \end{aligned}$$

and hence

$$|X_t^{\psi, \varepsilon} - (x + \psi \cdot \tilde{S}_t)| \leq \varepsilon x^S + 2\varepsilon |\psi_t S_t|.$$

In this setting, we focus on the exponential utility function with constant absolute risk aversion $p > 0$:

$$U(x) := -e^{-px}.$$

Our optimization problem consists in maximizing the expected utility or, equivalently, the certainty equivalent $\mathbf{CE}(X_T^{\psi, \varepsilon})$ of terminal wealth over all admissible trading strategies ψ with given initial wealth (x^B, x^S) . As usual, the *certainty equivalent* of a random payoff X refers to the deterministic amount with the same utility, i.e.,

$$\mathbf{CE}(X) := -\frac{1}{p} \ln \mathbf{E}[e^{-pX}].$$

2.2 Main results

In this section we present the main theorem of this chapter concerning the leading-order optimal investment under small transaction costs. To this end, we require some regularity conditions on the corresponding frictionless market.

Assumption 2.2.1. We suppose that the frictionless price process S allows for an equivalent local martingale measure with finite relative entropy.

Denote the initial wealth before liquidation by $x := x^B + x^S$. According to [8, Theorems 2.1, 2.2], Assumption 2.2.1 implies that the *minimal entropy (local) martingale measure (MEMM)* \mathbf{Q} for S exists. By [34, Theorem 2.2 (iv)], there is a predictable, S -integrable process φ such that $\varphi \cdot S$ is a \mathbf{Q} -martingale and

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \frac{U'(x + \varphi \cdot S_T)}{y}$$

with $y := \mathbf{E}[U'(x + \varphi \cdot S_T)]$. Interpreted as number of shares, strategy φ is the optimal solution to the frictionless counterpart of the above utility maximization problem.

For any Itô process X , we denote by b^X and $c^{X, X}$ its local \mathbf{Q} -drift and quadratic variation, respectively, i.e.,

$$dX_t = b_t^X dt + dM_t^{X, \mathbf{Q}}, \quad c_t^{X, X} := \frac{d[X, X]_t}{dt},$$

where $M^{X,\mathbf{Q}}$ is a continuous \mathbf{Q} -local martingale starting in 0. Similarly, for Itô processes X and Y , their local covariation is denoted by

$$c_t^{X,Y} := \frac{d[X,Y]_t}{dt}.$$

The local drift rate of Itô process X relative to \mathbf{P} rather than \mathbf{Q} is written as $b^{X,\mathbf{P}}$.

Assumption 2.2.2. We suppose that the frictionless optimizer φ and the *activity rate*

$$\rho := \frac{c^{\varphi,\varphi}}{c^{S,S}}$$

are well-defined Itô processes such that $\rho > 0$.

The processes S , φ , ρ and their dynamics depend on the current level of the stock price. In concrete models, the following related normalized processes are easier to work with:

- the *stock return process* $Y := \ln S$,
- the *stock holdings* $\pi := \varphi S$,
- the *normalized activity rate* $\eta := \rho S^4$.

Assumption 2.2.3. We assume that

$$\mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0,T]} |X_t|^n \right] < \infty \quad \text{for any } n \in \mathbb{N} \text{ and any } X \in \mathcal{H}, \quad (2.2)$$

where

$$\mathcal{H} := \{\pi, \eta, \eta^{-1}, b^\pi, b^\eta, c^{Y,Y}, (c^{Y,Y})^{-1}, c^{\pi,\pi}, c^{\eta,\eta}\}.$$

Moreover, we suppose

$$\mathbf{E}_{\mathbf{Q}}[\exp(|9p\varphi \bullet S_T|)] < \infty. \quad (2.3)$$

Finally, $c^{\varphi,\varphi}$ is assumed to be a continuous process.

2.2.1 Optimal investment

The following theorem provides the main results of this chapter.

Theorem 2.2.4. *Suppose that Assumptions 2.2.1, 2.2.2, 2.2.3 hold. Let*

$$\Delta\varphi^\pm := \pm \left(\frac{3\rho}{2p} S\varepsilon \right)^{1/3} = \pm \left(\frac{3\eta}{2p} \varepsilon \right)^{1/3} S^{-1}. \quad (2.4)$$

1. There exists a continuous adapted process

$$\varphi^\varepsilon = \varphi + \Delta\varphi = \varphi^{\varepsilon\uparrow} - \varphi^{\varepsilon\downarrow},$$

where $\Delta\varphi$ has values in $[\Delta\varphi^-, \Delta\varphi^+]$,

$$\varphi_0^\varepsilon = \begin{cases} \varphi_0 + \Delta\varphi_0^+ & \text{if } x^S > (\varphi_0 + \Delta\varphi_0^+)S_0, \\ \varphi_0 + \Delta\varphi_0^- & \text{if } x^S < (\varphi_0 + \Delta\varphi_0^-)S_0, \\ x^S/S_0 & \text{otherwise,} \end{cases}$$

and $\varphi^{\varepsilon\uparrow}, \varphi^{\varepsilon\downarrow}$ are increasing process such that

$$\varphi^{\varepsilon\uparrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^-\} \subseteq \Omega \times [0, T],$$

$$\varphi^{\varepsilon\downarrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^+\} \subseteq \Omega \times [0, T].$$

2. By slight abuse of notation, we identify φ^ε with the unique self-financing strategy (ψ^0, ψ) that satisfies $\psi_0^0 = x^B$, $\psi_0 S_0 = x^S$, $\psi_t = \varphi_t^\varepsilon$ for $t \in (0, T]$. Define

$$\tau^\varepsilon := \inf \left\{ t \in [0, T] : |X_t^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_t)| > 1 \text{ or } |X_t^{\varphi^\varepsilon, \varepsilon}| > \varepsilon^{-4/3} \right\} \wedge T. \quad (2.5)$$

Then $\mathbf{P}(\tau^\varepsilon = T) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Moreover, $\varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}$ is a utility-maximizing strategy to the leading order $O(\varepsilon^{2/3})$, i.e.,

$$\sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} [U(X_T^{\psi, \varepsilon})] = \mathbf{E} [U(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon})] + o(\varepsilon^{2/3}).$$

(As above, $\varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}$ here refers to the strategy $\psi \in \mathcal{A}^\varepsilon(x^B, x^S)$ with $\psi_t = \varphi_t^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}(t)$ for $t \in (0, T]$.)

3. The optimal certainty equivalent amounts to

$$\begin{aligned} \sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{CE}(X_T^{\psi, \varepsilon}) &= \mathbf{CE}(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon}) + o(\varepsilon^{2/3}) \\ &= \mathbf{CE}(x + \varphi \cdot S_T) - \frac{p}{2} \mathbf{E}_{\mathbf{Q}} [(\Delta\varphi^+)^2 \cdot [S, S]_T] + o(\varepsilon^{2/3}). \end{aligned}$$

Proof. The proof is split up into several steps given in Section 2.5. The existence of φ^ε is linked to the Skorohod problem with time-dependent reflecting barriers (cf. Lemma 2.5.1). With the help of the shadow price process S^ε derived heuristically in [20] (cf. Corollary 2.5.2 for rigorous arguments), the utility generated by φ^ε stopped at τ^ε is computed in Lemma 2.5.6. The optimality of φ^ε is proved by means of some dual considerations (cf. Lemma 2.5.9) in conjunction with the conjugate relation (cf. Lemma 2.5.10). Finally, the proof of the explicit expression for the certainty equivalent loss relies on a random time change and the ergodic property of reflected Brownian motion presented in Chapter 4 (cf. Corollary 2.5.11). \square

From our theorem, the leading-order optimal strategy under transaction costs φ^ε stays within the random no-trade region $[\varphi + \Delta\varphi^-, \varphi + \Delta\varphi^+]$ around the frictionless optimizer φ ; it increases and decreases only while hitting the lower and upper bound, respectively. In this sense, $\varphi + \Delta\varphi^+$ and $\varphi + \Delta\varphi^-$ correspond to the *selling* and *buying boundary*, respectively. At the random time τ^ε , the portfolio is liquidated primarily in order to bound losses.

Roughly speaking, the assumptions in Theorem 2.2.4 concern sufficient integrability of the solution to the frictionless utility maximization problem in order to guarantee that the maximum expected utility is twice differentiable as a function of $\varepsilon^{1/3}$. In Section 2.3 these assumptions are verified in a general stochastic volatility setup. For comparison with the classical approach of stochastic control, their Markovian counterparts are stated in Section 2.4.

2.2.2 Utility indifference pricing

As a byproduct, due to the nature of exponential utility, Theorem 2.2.4 yields an asymptotic utility indifference price and the corresponding hedging strategy. Let H be an \mathcal{F}_T -measurable random variable modelling a contingent claim maturing at time T . For simplicity, we suppose that H is bounded. The *utility indifference price* of H denoted by $\pi^\varepsilon(H)$ refers to the initial premium compensating the certainty equivalent with and without the contingent claim liability $-H$, i.e.,

$$\pi^\varepsilon(H) := \sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{CE}(X_T^{\psi, \varepsilon}) - \sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{CE}(X_T^{\psi, \varepsilon} - H).$$

Notice that letting \mathbf{P}^H be the equivalent probability measure with density

$$\frac{d\mathbf{P}^H}{d\mathbf{P}} := \frac{e^{pH}}{\mathbf{E}[e^{pH}]},$$

for exponential utility, we have

$$\sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E}[U(X_T^{\psi, \varepsilon} - H)] = \mathbf{E}[e^{pH}] \sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E}_{\mathbf{P}^H}[U(X_T^{\psi, \varepsilon})]. \quad (2.6)$$

Hence the utility maximization problem in the presence of random endowment $-H$ is equivalent to the pure optimal investment problem treated in Theorem 2.2.4 under measure \mathbf{P}^H . From Girsanov's Theorem, change of measure from \mathbf{P} to \mathbf{P}^H results in change of the local drift of stock price S , which in turn leads to a different frictionless optimizer φ^H rather than φ and a new MEMM \mathbf{Q}^H such that $\frac{d\mathbf{Q}^H}{d\mathbf{P}^H}$ minimizes the relative entropy with respect to \mathbf{P}^H . (Note that from [7], the set of local martingale measures with finite relative entropy with respect to \mathbf{P}^H equals that with respect to \mathbf{P} .) Theorem 2.2.4 applied to problem (2.6) on the underlying probability space (Ω, \mathbf{P}^H) then yields the asymptotic utility indifference price of H . We assert this as a direct consequence of Theorem 2.2.4 in the following corollary.

Corollary 2.2.5. *Let H be a bounded \mathcal{F}_T -measurable random variable representing a contingent claim with maturity T . Suppose that Assumption 2.2.1 holds and denote by \mathbf{Q}^H, φ^H the MEMM and the frictionless optimal trading strategy in the presence of contingent claim liability H , i.e., \mathbf{Q}^H, φ^H are related by*

$$\frac{d\mathbf{Q}^H}{d\mathbf{P}} = \frac{U'(x + \varphi^H \cdot S_T - H)}{\mathbf{E}[U'(x + \varphi^H \cdot S_T - H)]}.$$

Further, suppose that the counterparts of Assumptions 2.2.2, 2.2.3 in case of frictionless optimization with random endowment $-H$ hold, i.e., all objects initially related to \mathbf{Q}, φ are now replaced with those related to \mathbf{Q}^H, φ^H defined in the same manner. Then the utility indifference price of H amounts to

$$\pi^\varepsilon(H) = \pi^0(H) + \frac{P}{2} (\mathbf{E}_{\mathbf{Q}^H} [(\Delta\varphi^{H,+})^2 \cdot [S, S]_T] - \mathbf{E}_{\mathbf{Q}} [(\Delta\varphi^+)^2 \cdot [S, S]_T]) + o(\varepsilon^{2/3}),$$

where $\pi^0(H)$ is the frictionless utility indifference price, $\Delta\varphi^+$ is given by (2.4) and

$$\Delta\varphi^{H,\pm} := \pm \left(\frac{3}{2p} \frac{c^{\varphi^H, \varphi^H}}{c^{S, S}} S\varepsilon \right)^{1/3}.$$

Moreover, relating $\varphi^{H,\varepsilon}, \tau^{H,\varepsilon}$ to $\varphi^H, \Delta\varphi^{H,+}$ in the same manner as $\varphi^\varepsilon, \tau^\varepsilon$ in Theorem 2.2.4 related to $\varphi, \Delta\varphi^+$, respectively, $\varphi^{H,\varepsilon} \mathbf{1}_{[0, \tau^{H,\varepsilon}]}$ is the hedging strategy corresponding to the indifference price $\pi^\varepsilon(H)$ to the leading order $O(\varepsilon^{2/3})$.

Remark 2.2.6. If the market is complete, $\mathbf{Q}^H = \mathbf{Q}$ is the unique equivalent martingale measure and the frictionless utility optimization in the presence of contingent claim liability is split up into the pure optimal investment problem and hedging away the random endowment $-H$, i.e., $\pi^0(H) = \mathbf{E}_{\mathbf{Q}}[H]$ and $\varphi^H = \varphi + \Delta$ with Δ the replicating strategy of claim H . Writing the \mathbf{Q} -local martingale parts of φ, Δ as $dM_t^{\varphi, \mathbf{Q}} = \Gamma_t^\varphi dS_t$, $dM_t^{\Delta, \mathbf{Q}} = \Gamma_t^\Delta dS_t$, the no-trade region and asymptotic utility indifference price provided in Corollary 2.2.5 can be expressed by

$$\Delta\varphi^{H,\pm} S = \pm \left(\frac{3}{2p} (\Gamma^\varphi S^2 + \Gamma S^2)^2 \varepsilon \right)^{1/3}, \quad (2.7)$$

$$\pi^\varepsilon(H) = \mathbf{E}_{\mathbf{Q}}[H] + \left(\frac{9p\varepsilon^2}{32} \right)^{1/3} \mathbf{E}_{\mathbf{Q}} \left[\frac{(\Gamma^\varphi S^2 + \Gamma S^2)^{4/3} - (\Gamma^\varphi S^2)^{4/3}}{S^2} \cdot [S, S]_T \right] + o(\varepsilon^{2/3}). \quad (2.8)$$

In the Black-Scholes model, (2.7, 2.8) coincide with the results of Whalley and Wilmott [40] which are proved under well-defined conditions by Bichuch [3]. In Section 2.3.2 we show that our regularity conditions on the frictionless optimizer φ^H in form of Assumptions 2.2.2, 2.2.3 are indeed consistent with the conditions imposed on the payoff function g with $H = g(S_T)$ in [3, Assumptions 3.1, 3.2].

2.3 Examples

We now verify the assumptions for Theorem 2.2.4 in concrete models, where the frictionless optimizer φ is explicitly known. In general, it is difficult to solve the frictionless optimization problem in closed form. Here we discuss two tractable cases: the Black-Scholes model with and without contingent claim liability, and stochastic volatility models with conditionally independent increments.

2.3.1 Black-Scholes model

Let us consider the so-called *Black-Scholes model*:

$$dS_t = S_t(bdt + \sigma dW_t) \quad (2.9)$$

with $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_+ \setminus \{0\}$. We show that Assumptions 2.2.1, 2.2.2, 2.2.3 hold if $b \neq 0$.

From Theorem B.1 in the appendix, the frictionless optimal strategy φ satisfies

$$\pi_t = \varphi_t S_t = \frac{b}{p\sigma^2} \quad \text{for all } t \in [0, T]. \quad (2.10)$$

By Itô's formula, we have

$$d\varphi_t = -\frac{b}{p\sigma^2 S_t^2} dS_t + \frac{b}{p\sigma^2 S_t^3} d[S, S]_t$$

and hence

$$c_t^{\varphi, \varphi} = \frac{b^2}{p^2 \sigma^2 S_t^2}, \quad \rho_t = \frac{b^2}{p^2 \sigma^4 S_t^4}, \quad \eta_t = \frac{b^2}{p^2 \sigma^4}$$

for all $t \in [0, T]$. Therefore, all processes in set \mathcal{H} of (2.2) are constant, which in particular yields Condition (2.2). The frictionless optimal terminal gains are of the form

$$\varphi \bullet S_T = \frac{b}{p\sigma} W_T^{\mathbf{Q}},$$

where $W_t^{\mathbf{Q}} = W_t + (b/\sigma)t$, $t \in [0, T]$ is a standard Brownian motion under measure \mathbf{Q} . Thus, Condition (2.3) is satisfied. Then from Theorem 2.2.4, the no-trade bounds are determined by

$$\Delta\varphi^\pm S = \pm \left(\frac{3b^2 \varepsilon}{2p^3 \sigma^4} \right)^{1/3} \quad (2.11)$$

and the certainty equivalent loss due to transaction costs refers to

$$\sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{CE}(X_T^{\psi, \varepsilon}) - \mathbf{CE}(x + \varphi \bullet S_T) = - \left(\frac{9b^4 \varepsilon^2}{32\sigma^2} \right)^{1/3} \frac{T}{p} + o(\varepsilon^{2/3}). \quad (2.12)$$

(2.11) coincides with the formulas in [40, p.319] and [3, (3.4)]. The expression in (2.12), on the other hand, is obtained from [3, (3.7, 3.8)] if one uses the equation for V_2 in [40, p.317]. Note, however, that [3] considers a slightly more involved notion of admissibility.

2.3.2 Verification of Bichuch

As an application of Corollary 2.2.5, we verify the asymptotic utility indifference price formula proposed by Whalley and Wilmott [40] and proved rigorously by Bichuch [3].

Proposition 2.3.1. *For the Black-Scholes model (2.9) with $b > 0$, let $H = g(S_T)$ be a European type contingent claim with bounded payoff function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying [3, Assumptions 3.1, 3.2], that is,*

(C1) *g is four times continuously differentiable such that $g(s) - g'(s)s$, $s^2 g''(s)$, $s^3 g'''(s)$ are bounded for all $s \in \mathbb{R}_+$*

(C2) $\frac{b}{p\sigma^2} - \sup_{s \in \mathbb{R}_+} |s^2 g''(s)| \geq K$ for some $K \in \mathbb{R}_+ \setminus \{0\}$.

Let $V_0(S_t, t)$, $\Gamma(S_t, t)$ be the Black-Scholes price of $g(S_T)$ and the related option's gamma, respectively, i.e.,

$$\Gamma = \partial_s^2 V_0$$

and the function V_0 satisfies the Black-Scholes PDE:

$$\begin{aligned} \partial_t V_0 + \frac{\sigma^2 s^2}{2} \partial_s^2 V_0 &= 0, \\ V_0(s, T) &= g(s). \end{aligned}$$

Then the utility indifference price of H amounts to

$$\begin{aligned} \pi^\varepsilon(H) &= V_0(S_0, 0) \\ &+ \left(\frac{9p\varepsilon^2}{32} \right)^{1/3} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \sigma^2 \left((\Gamma(S_t, t) S_t^2 - \frac{b}{p\sigma^2})^{4/3} - \left(\frac{b}{p\sigma^2} \right)^{4/3} \right) dt \right] \\ &+ o(\varepsilon^{2/3}). \end{aligned}$$

Moreover, the no-trade region for hedging strategy $\varphi^{H, \varepsilon}$ is given by

$$\Delta \varphi_t^{H, \pm} S_t = \pm \left(\frac{3}{2p} \left(\Gamma(S_t, t) S_t^2 - \frac{b}{p\sigma^2} \right)^2 \varepsilon \right)^{1/3}.$$

(Cf. [3, Corollary 3.8, Lemma 5.5] for equivalent results.)

Proof. In order to apply Corollary 2.2.5, we verify the counterparts of Assumptions 2.2.2, 2.2.3 required on the parameters related to φ^H . Recalling Remark 2.2.6, we have

$$\varphi_t^H = \varphi_t + \partial_s V_0(S_t, t)$$

with φ given by (2.10). Then the stock holdings following strategy φ^H are

$$\pi_t^H = \varphi_t^H S_t = \pi_t + \partial_s V_0(S_t, t) S_t \quad (2.13)$$

with π as in (2.10). Due to the four-times differentiability imposed on g , it holds that $V_0 \in C^{4,1}$ (cf. [3, Lemma 4.2]). In view of (2.10) and using Itô's formula, we obtain

$$\begin{aligned} d\pi_t^H &= (\partial_s V_0(S_t, t) + S_t \partial_s^2 V_0(S_t, t)) dS_t + S_t \partial_t \partial_s V_0(S_t, t) dt \\ &\quad + (\partial_s^2 V_0(S_t, t) + S_t \partial_s^3 V_0(S_t, t)) d[S, S]_t, \end{aligned} \quad (2.14)$$

and hence

$$b_t^{\pi^H} = S_t \partial_t \partial_s V_0(S_t, t) + \sigma^2 (S_t^2 \partial_s^2 V_0(S_t, t) + S_t^3 \partial_s^3 V_0(S_t, t)), \quad (2.15)$$

$$c_t^{\pi^H, \pi^H} = \sigma^2 (S_t \partial_s V_0(S_t, t) + S_t^2 \partial_s^2 V_0(S_t, t))^2. \quad (2.16)$$

Itô's formula applied to $\varphi^H = \pi^H / S$ yields

$$d\varphi_t^H = -\frac{\pi_t^H}{S_t^2} dS_t + \frac{\pi_t^H}{S_t^3} d[S, S]_t + \frac{1}{S_t} d\pi_t^H + d\left[\frac{1}{S}, \pi^H\right]_t;$$

in conjunction with (2.13, 2.14), this implies

$$\eta_t^H = \frac{c_t^{\varphi^H, \varphi^H}}{c_t^{S, S}} S_t^4 = (\pi_t - S_t^2 \partial_s^2 V_0(S_t, t))^2. \quad (2.17)$$

Using again Itô's formula, we obtain

$$b_t^{\eta^H} = -2\sqrt{\eta_t^H} [S_t^2 \partial_t \partial_s^2 V_0(S_t, t) + \sigma^2 (S_t^4 \partial_s^4 V_0(S_t, t) + 4S_t^3 \partial_s^3 V_0(S_t, t) + 2S_t^2 \partial_s^2 V_0(S_t, t))], \quad (2.18)$$

$$c_t^{\eta^H, \eta^H} = 4\eta_t^H \sigma^2 (S_t^3 \partial_s^3 V_0(S_t, t) + 2S_t^2 \partial_s^2 V_0(S_t, t))^2. \quad (2.19)$$

Now, by [3, Lemma 4.2], Condition (C1) on g (cf. [3, Assumption 3.1]) implies that $V_0 - s\partial_s V_0$, $s^2 \partial_s^2 V_0$, $s^3 \partial_s^3 V_0$, $s^4 \partial_s^4 V_0$ as well as $s\partial_t \partial_s V_0$, $s^2 \partial_t \partial_s^2 V_0$ are globally bounded on $\mathbb{R}_+ \times [0, T]$; combined with the boundedness of g and thus that of V_0 , $s\partial_s V_0$ is also globally bounded. In view of (2.13, 2.15, 2.16, 2.17, 2.18, 2.19), we then deduce that Condition (C1) on g implies the boundedness of π^H , b^{π^H} , c^{π^H, π^H} , η^H , b^{η^H} , c^{η^H, η^H} . Moreover, since $s^2 \partial_s^2 V_0$ is bounded by $\sup_{s \in \mathbb{R}_+} |s^2 g''(s)|$ according to [3, Lemma 4.2], in view of (2.17, 2.10), Condition (C2) on g (cf. [3, Assumption 3.2]) implies the boundedness of $(\eta^H)^{-1}$. Therefore, Assumptions 2.2.2, 2.2.3 imposed on φ^H instead of φ are fulfilled. Then, applying Corollary 2.2.5 and recalling (2.7, 2.8, 2.11, 2.12), the assertion follows. \square

2.3.3 Stochastic volatility model

Let us turn to the following stochastic volatility model:

$$dS_t = S_t(b(\tilde{Y}_t)dt + \sigma(\tilde{Y}_t)dW_t) \quad (2.20)$$

with continuous functions $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ and a *non-necessarily Markovian* Itô process \tilde{Y} which is independent of the Brownian motion W . The filtration is assumed to be generated by W and \tilde{Y} .

Proposition 2.3.2. *Suppose that the stochastic volatility model (2.20) is such that*

- *the coefficients $b(\tilde{Y}), \sigma(\tilde{Y})$ are bounded processes that are bounded away from 0,*
- *the processes*

$$\pi := \frac{b(\tilde{Y})}{p\sigma(\tilde{Y})^2}, \quad \eta := \pi^2 + \frac{c^{\pi,\pi}}{\sigma(\tilde{Y})^2}, \quad (2.21)$$

are Itô processes with bounded coefficients $b^{\pi,\mathbf{P}}, c^{\pi,\pi}, b^{\eta,\mathbf{P}}, c^{\eta,\eta}$ and continuous $c^{\pi,\pi}$,

- *for*

$$\tilde{Z}_t := \mathbf{E} \left[\exp \left(-\frac{1}{2} \int_0^T \left(\frac{b(\tilde{Y}_t)}{\sigma(\tilde{Y}_t)} \right)^2 dt \right) \middle| \mathcal{F}_t \right], \quad (2.22)$$

the process $c^{\tilde{Z},\tilde{Z}}/\tilde{Z}^2$ is bounded.

Then Assumptions 2.2.1, 2.2.2, 2.2.3 are satisfied. Moreover, π and η are the corresponding stock holdings and normalized activity rate. The no-trade boundary is given by

$$\Delta\phi^\pm S = \pm \left(\frac{3\eta\epsilon}{2p} \right)^{1/3}$$

and the certainty equivalent loss amounts to

$$\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left(\frac{9p\eta_t^2}{32} \right)^{1/3} \sigma(\tilde{Y}_t)^2 dt \right] \epsilon^{2/3} + o(\epsilon^{2/3}).$$

Proof. Step 1. We show that $[f(S), X] = 0$ for any C^2 -function f and any Itô process X which is $\sigma(\tilde{Y})$ -measurable. Indeed, by Itô's formula it suffices to prove that $[W, X] = 0$. Using [32, Theorem II.4], it is easy to show that the martingale part of X is $\sigma(\tilde{Y})$ -measurable. Hence without loss of generality, X is a local martingale. By localization it suffices to consider the case where X is a square-integrable martingale. Let \mathbf{G} be the filtration defined in (B.2) in the appendix. Then X_t is \mathcal{G}_0 -measurable for any $t \in [0, T]$ and W is a Brownian motion relative to both \mathbf{F} and \mathbf{G} . We obtain

$$\mathbf{E}[W_t X_t | \mathcal{F}_s] = \mathbf{E}[\mathbf{E}[W_t X_t | \mathcal{G}_s] | \mathcal{F}_s] = \mathbf{E}[\mathbf{E}[W_t | \mathcal{G}_s] X_t | \mathcal{F}_s] = W_s \mathbf{E}[X_t | \mathcal{F}_s] = W_s X_s$$

for any $s < t$. Hence WX is a martingale, which implies $[W, X] = 0$ as desired.

Step 2. We show that π, η in (2.21) coincide with the stock holdings and the normalized activity rate. By Theorem B.1, the frictionless optimizer φ satisfies $\varphi_t S_t = \pi_t$ for any $t \in [0, T]$. From Itô's formula we get

$$d\varphi_t = -\frac{\pi_t}{S_t^2} dS_t + \frac{\pi_t}{S_t^3} d[S, S]_t + \frac{1}{S_t} d\pi_t + d\left[\frac{1}{S}, \pi\right]_t.$$

Step 1 yields $[S, \pi] = 0$, which implies

$$c_t^{\varphi, \varphi} = \frac{\pi_t^2 \sigma(\tilde{Y}_t)^2 + c_t^{\pi, \pi}}{S_t^2}, \quad \frac{c_t^{\varphi, \varphi}}{c_t^{S, S}} S_t^4 = \pi_t^2 + \frac{c_t^{\pi, \pi}}{\sigma(\tilde{Y}_t)^2} = \eta_t. \quad (2.23)$$

Step 3. Let \bar{Z} be defined as in (B.3). By Theorem B.1, $\tilde{Z}\bar{Z}/\tilde{Z}_0$ is the density process of the MEMM \mathbf{Q} . For any Itô process X , Girsanov's theorem implies

$$b_t^X = b_t^{X, \mathbf{P}} + \frac{\tilde{Z}\bar{Z}, X}{\tilde{Z}_t \bar{Z}_t}.$$

Integration by parts yields

$$\left| \frac{c^{\tilde{Z}\bar{Z}, X}}{\tilde{Z}\bar{Z}} \right| = \left| \frac{c^{\tilde{Z}, X}}{\tilde{Z}} + \frac{c^{\bar{Z}, X}}{\bar{Z}} \right| \leq \sqrt{\frac{c^{\tilde{Z}, \tilde{Z}}}{\tilde{Z}^2}} \sqrt{c^{X, X}} + \sqrt{\frac{c^{\bar{Z}, \bar{Z}}}{\bar{Z}^2}} \sqrt{c^{X, X}} = \left(\sqrt{\frac{c^{\tilde{Z}, \tilde{Z}}}{\tilde{Z}^2}} + \left| \frac{b(\tilde{Y})}{\sigma(\tilde{Y})} \right| \right) \sqrt{c^{X, X}}.$$

In view of our boundedness assumptions, we conclude that b^π and b^η are bounded. Consequently, all processes in set \mathcal{H} are bounded, which implies Condition (2.2).

Moreover, the frictionless optimal terminal gains are of the form

$$\varphi \bullet S_T = \int_0^T \frac{b(\tilde{Y}_t)}{p\sigma(\tilde{Y}_t)} dW_t^{\mathbf{Q}},$$

where $W^{\mathbf{Q}} = W + \int_0^\cdot (b(\tilde{Y}_t)/\sigma(\tilde{Y}_t)) dt$ is a standard Brownian motion under measure \mathbf{Q} . If an integrand H is bounded by $m \in \mathbb{R}_+$, we have

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \left[\exp(H \bullet W_T^{\mathbf{Q}}) \right] &\leq \mathbf{E}_{\mathbf{Q}} \left[\exp\left(H \bullet W_T^{\mathbf{Q}} - \frac{1}{2} \int_0^T H_t^2 dt \right) \right] \exp\left(\frac{1}{2} m^2 T \right) \\ &\leq \exp\left(\frac{1}{2} m^2 T \right) < \infty. \end{aligned}$$

Together, we conclude that Condition (2.3) holds. Finally, in view of (2.23), requiring $c^{\pi, \pi}$ to have continuous paths ensures that the process $c^{\varphi, \varphi}$ is continuous, which completes the proof of Assumption 2.2.3. \square

2.4 Comparison and Interpretation

In this section we comment on the regularity conditions in Theorem 2.2.4 and state the connection of our approach to the classical methods of stochastic control in a Markovian framework.

2.4.1 Markovian setting

For the sake of better exposition, let us focus on the following setup (cf. e.g. [41, 39]) with stock price S evolving as:

$$dS_t = S_t(\mu(\Upsilon_t)dt + \sigma(\Upsilon_t)dW_t) \quad (2.24)$$

for a one-dimensional state variable Υ with dynamics

$$d\Upsilon_t = b(\Upsilon_t)dt + a(\Upsilon_t)d(\zeta W_t + \sqrt{1 - \zeta^2}W_t^\perp),$$

where W, W^\perp are one-dimensional standard Brownian motions orthogonal to each other, $\zeta \in [-1, 1]$, and μ, σ, b, a are Lipschitz-continuous functions from \mathbb{R} to \mathbb{R} ; furthermore, σ is assumed to be bounded away from zero. Note that in case of $\zeta = 1$, (2.24) corresponds to the model considered in the recent study by Soner and Touzi [38].

2.4.2 Parameters in the frictionless market

Since φ is supposed to be the frictionless optimal trading strategy, letting $\hat{X} := \hat{X}^{v,x,\varphi}$ denote the wealth process generated by φ and starting in x at time t , i.e., $d\hat{X}_r = \varphi_r dS_r$ with $\hat{X}_t = x$, the value function v in the frictionless case can be written as

$$v(t, v, x) = \mathbf{E} [U(\hat{X}_T) | (\Upsilon_t, \hat{X}_t) = (v, x)].$$

Note that for exponential utility $U(\cdot) = -e^{-p(\cdot)}$, the value function is of the form

$$v(t, v, x) = e^{-px}u(t, v). \quad (2.25)$$

Now, let us link our regularity conditions on the frictionless parameters to requirements on the value function v and the market coefficients:

i) On Assumption 2.2.1. From (2.25), we deduce $\partial_x^n v = (-p)^n v$ for all $n \in \mathbb{N}$. In particular, for a finite value function v , the process $\{\partial_x v(t, \Upsilon_t, \hat{X}_t)\}_{t \in [0, T]}$ is a martingale. Duality theory (cf. [34, Theorem 2.2]) yields that $\{\partial_x v(t, \Upsilon_t, \hat{X}_t) / \partial_x v(0, v, x)\}_{t \in [0, T]}$ is the density process of the MEMM \mathbf{Q} , which leads to Assumption 2.2.1.

ii) On Assumption 2.2.2. From [39, Theorem 4.1], for smooth value function v with $\partial_x^2 v < 0$, the frictionless optimal stock holdings satisfy

$$\pi_t = \pi(t, \Upsilon_t) := \frac{1}{p} \left(\frac{\mu}{\sigma^2} + \zeta \frac{a}{\sigma} \frac{\partial_v u}{u} \right) (t, \Upsilon_t). \quad (2.26)$$

Define the differential operator $\mathcal{L} := \partial_t + b\partial_v + \frac{1}{2}a^2\partial_v^2$. In case of $\pi \in C^{1,2}$, Itô's formula yields the local characteristics of the process π :

$$b_t^{\pi, \mathbf{P}} = \mathcal{L}\pi(t, \Upsilon_t), \quad c_t^{\pi, \pi} = a(\Upsilon_t)^2(\partial_v\pi(t, \Upsilon_t))^2. \quad (2.27)$$

Integration by parts applied to $\varphi = \pi/S$ combined with (2.27) then yields the normalized activity rate η :

$$\eta_t = \frac{c_t^{\varphi, \varphi}}{S_t^4} S_t^4 = \eta(t, \Upsilon_t) := \pi(t, \Upsilon_t)^2 + \frac{a(\Upsilon_t)^2}{\sigma(\Upsilon_t)^2} (\partial_v\pi(t, \Upsilon_t))^2 - 2\zeta \frac{a(\Upsilon_t)}{\sigma(\Upsilon_t)} \pi(t, \Upsilon_t) \partial_v\pi(t, \Upsilon_t). \quad (2.28)$$

If the function $\eta \in C^{1,2}$, Itô's formula yields the local characteristics of the process η :

$$b_t^{\eta, \mathbf{P}} = \mathcal{L}\eta(t, \Upsilon_t), \quad c_t^{\eta, \eta} = a(\Upsilon_t)^2(\partial_v\eta(t, \Upsilon_t))^2. \quad (2.29)$$

Together, Assumption 2.2.2 refers to the two times continuous differentiability of functions π, η in (2.26, 2.28), respectively, as well as the requirement of $\eta > 0$ for the process η in (2.28). We then find out that in case of $\zeta = 1$, these regularity conditions are intimately connected to [38, Assumption 5.2 (smoothness)]. Indeed, the Markovian counterparts of our frictionless parameters π, η expressed in terms of (2.26, 2.28) correspond to $\mathbf{y}, (\alpha/\sigma)^2$ in [38], respectively, with α given by [38, Equation (3.8)].

iii) On Assumption 2.2.3. Note that a sufficient condition for Assumption 2.2.3 is the boundedness of all processes in the set \mathcal{H} in Condition (2.2). Aiming to verify this stronger but more illustrative condition, we suppose that the functions $\mu, \sigma, b, a, \sigma^{-1}$ are bounded and smooth with bounded derivatives at all orders. Due to [29, Section 8] and in view of (2.26, 2.27, 2.28, 2.29), the above regularity conditions imposed on the coefficients ensure the boundedness of $\pi, b^{\pi, \mathbf{P}}, c^{\pi, \pi}, \eta, b^{\eta, \mathbf{P}}, c^{\eta, \eta}$. Moreover, let Z denote the density process of the MEMM \mathbf{Q} . For any Itô process X , Girsanov's theorem yields

$$b^X = b^{X, \mathbf{P}} + \frac{c^{X, Z}}{Z} \quad \text{with} \quad \left| \frac{c^{X, Z}}{Z} \right|^2 \leq c^{X, X} \frac{c^{Z, Z}}{Z^2}.$$

Due to [39, Lemma 3.1, Equation (3.12)], the above regularity conditions on the coefficients also guarantee the boundedness of $c^{Z, Z}/Z^2$. Therefore, we deduce the boundedness of all processes in the set \mathcal{H} in Condition (2.2) except for η^{-1} . In order to obtain the boundedness of the process η^{-1} , we have to require additionally that the function η in (2.28) satisfies

$$\eta \geq \eta^* \quad \text{for some} \quad \eta^* \in \mathbb{R}_+ \setminus \{0\}. \quad (2.30)$$

As mentioned above, in case of $\zeta = 1$, η in our context corresponds to $(\alpha/\sigma)^2$ in [38]. Hence (2.30) is linked to [38, Equation (5.3)] implied by [38, Assumption 5.2, (5.2)]. (Note that since [38] deals with utility functions defined on the positive real line, the different risk tolerances result in the slight difference between (2.30) and [38, Equation (5.3)].) As [38], we do not attempt to verify (2.30) in general but simply point out that as a direct consequence of (2.28), the requirement (2.30) can be verified e.g.

- for $\zeta = 1$ and constant coefficients μ, σ , cf. [38, Section 6] or Section 2.3.1;
- for $\zeta = 0$ which is a special case of the example considered in Section 2.3.3.

In the context of utility indifference pricing, the counterparts of Assumptions 2.2.2, 2.2.3 required on the frictionless optimizer φ^H in the presence of random endowment $-H$ are additionally linked to the regularity of the payoff function g with $H = g(S_T)$. The considerations in Section 2.3.2 (cf. Proposition 2.3.1) reveal that for bounded claim H , the boundedness of all processes in the set \mathcal{H} of Condition (2.2) relative to φ^H, \mathbf{Q}^H – except for $(\eta^H)^{-1}$ – is connected to [3, Assumption 3.1]; the counterpart of requirement (2.30) on $(\eta^H)^{-1}$ corresponds to [3, Assumption 3.2].

2.4.3 Shadow price

As in [20] for the heuristic derivation, the concept of an auxiliary shadow price is again essential for our rigorous proof of the results concerning the optimal investment under small transaction costs. The shadow price process corresponds to the dual optimizer in a market with proportional transaction costs, which appears for the first time in [5], cf. the process $\hat{R}P = \hat{Z}_1/\hat{Z}_0$ in [5, Theorem 6.1]. It is a process taking values within the bid-ask spread $[(1 - \varepsilon)S, (1 + \varepsilon)S]$. [5, Theorem 6.1] reveals a general relationship between the primal and dual optimizers in frictional markets, namely the optimal trading strategy (number of shares) increases and decreases only when the shadow price process hits its upper and lower bound, respectively. This indicates that the utility maximization problem under transaction costs is equivalent to the frictionless optimization at shadow price. In particular, at any instant, any transaction between the optimal holdings in the bank and the stock would result in utility loss, even if this could be conducted at the potentially more favorable shadow price rather than the real bid or ask price. In the Markovian case, this can be interpreted as follows. In the model stated in Section 2.4.1, letting $(\hat{X}^B, \hat{X}^S) := (\hat{X}^B, \hat{X}^S)_{v, x^B, x^S}$ denote the optimal portfolio holdings in bond and stock starting in (x^B, x^S) at time t , the value function v^ε in the market with transaction costs can be written as

$$v^\varepsilon(t, v, x^B, x^S) = \mathbf{E} \left[U(\hat{X}_T^B + \hat{X}_T^S - \varepsilon|\hat{X}_T^S|) \mid (\Upsilon_t, \hat{X}_t^B, \hat{X}_t^S) = (v, x^B, x^S) \right].$$

Let $q_t := q(t, \Upsilon_t, \hat{X}_t^B, \hat{X}_t^S)$ denote the shadow price at time t divided by S_t . Then the above consideration suggests that the mapping $\mathbb{R} \ni \delta \mapsto v^\varepsilon(t, v, x^B - \delta q(t, v, x^B, x^S), x^S + \delta)$ attains its maximum at $\delta = 0$. In particular, its derivative should be equal to zero at $\delta = 0$, which leads to

$$q(t, v, x^B, x^S) = \frac{\partial_{x^S} v^\varepsilon}{\partial_{x^B} v^\varepsilon}(t, v, x^B, x^S) \in [1 - \varepsilon, 1 + \varepsilon]. \quad (2.31)$$

Note that in the buying and selling region where $q_t = 1 + \varepsilon$ and $q_t = 1 - \varepsilon$, respectively, (2.31) coincides with the Hamilton-Jacobi-Bellman equation e.g. in [6, Equation (3.3)], [35, Equation (4.6)].

Passing to the asymptotic analysis for small transaction costs, the approximate shadow price process S^ε along with the candidate strategy φ^ε proposed in [20] reads as

$$S^\varepsilon = S + \frac{pc^{S,S}}{3c^{\varphi,\varphi}}(\varphi^\varepsilon - \varphi)^3 - \left(\frac{9pc^{S,S}}{4c^{\varphi,\varphi}}\right)^{1/3} (S\varepsilon)^{2/3}(\varphi^\varepsilon - \varphi)$$

(cf. Section 2.5.1 below for existence and detailed analysis). On the other hand, previous literature dealing with Markovian settings (cf. [35, 40, 16, 3, 38]) suggests a Taylor series expansion of the value function v^ε in form of

$$v^\varepsilon(t, v, x^B, x^S) = v(t, v, x^B + x^S) + u(t, v, x^B + x^S)\varepsilon^{2/3} + w(t, v, x^B + x^S, \xi)\varepsilon^{4/3} + o(\varepsilon^{2/3}), \quad (2.32)$$

where $\xi := (x^S - \pi(t, v))/\varepsilon^{1/3}$ stands for the so-called “fast variable” (cf. [38]). In the classical asymptotic analysis carried out in the literature [35, 40, 16, 3, 38], partial differential equations for the leading-order terms u, w are derived by combining (2.32) and the Hamilton-Jacobi-Bellman equation for v^ε ; in [38] these are called the “corrector equations”. In [40, 3], the partial differential equations for u, w can be solved explicitly, cf. [40, 3, functions H_2, H_4]. The detailed analysis in Section 2.3.1 shows that the counterparts of our $\Delta\varphi^\pm, -S(\varphi^\varepsilon - \varphi)/\varepsilon^{1/3}$ in [40, 3] are Y^\pm, H_{4y} , respectively. In the recent study [38], the function w is identified with the solution of an appropriate corrector equation related to some ergodic control problem, cf. [38, Equation (3.11)]. As stated in Section 2.4.2, our normalized activity rate η is connected to $(\alpha/\sigma)^2$ in [38]; in conjunction with (2.4) and [38, Equation (4.4)] for $\lambda^{1,0} = \lambda^{0,1} = 1$, we find out that $p\Delta\varphi^+S/\varepsilon^{1/3}, p(\varphi^\varepsilon - \varphi)S/\varepsilon^{1/3}$ and $(S^\varepsilon - S)/(S\varepsilon)$ in our context correspond to $\rho_0, \rho = -(\partial_x^2 v/\partial_x v)\xi$ and $-\bar{w}_\rho = -\partial_\xi w/\partial_x v$ in [38, Section 4.1], respectively.

Note that according to Section 3.4, considering the risk-tolerance process R introduced in [19] as the risk tolerance of the value function evaluated along the optimal wealth process, the interpretation in this section also holds for the setting of Chapter 3 for general utility functions by replacing p and η with R and $\tilde{\eta}$, respectively.

2.5 Proof of the main results

Now, we prove Theorem 2.2.4 and assume throughout this section that Assumptions 2.2.1, 2.2.2 hold.

2.5.1 Analysis of shadow price process S^ε

Let us start with the investigation of the shadow price process S^ε formally derived in [20] along with the candidate strategy φ^ε .

Existence

First, we prove the existence of the candidate strategy φ^ε in Theorem 2.2.4-1.

Lemma 2.5.1. *For*

$$\Delta\varphi_0 := \begin{cases} \Delta\varphi_0^+ & \text{if } x^S > (\varphi_0 + \Delta\varphi_0^+)S_0, \\ \Delta\varphi_0^- & \text{if } x^S < (\varphi_0 + \Delta\varphi_0^-)S_0, \\ \frac{x^S}{S_0} - \varphi_0 & \text{otherwise,} \end{cases}$$

there exists a solution $\Delta\varphi$ to the Skorohod stochastic differential equation (SDE)

$$d\Delta\varphi_t = -d\varphi_t \quad (2.33)$$

with reflection at $\Delta\varphi^-$, $\Delta\varphi^+$, i.e., there exist a continuous adapted $[\Delta\varphi^-, \Delta\varphi^+]$ -valued process $\Delta\varphi$ and adapted increasing processes $\varphi^{\varepsilon\uparrow}$, $\varphi^{\varepsilon\downarrow}$ such that

$$\varphi^{\varepsilon\uparrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^-\} \subseteq \Omega \times [0, T], \quad (2.34)$$

$$\varphi^{\varepsilon\downarrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^+\} \subseteq \Omega \times [0, T], \quad (2.35)$$

and

$$\Delta\varphi = -\varphi + \varphi^{\varepsilon\uparrow} - \varphi^{\varepsilon\downarrow}. \quad (2.36)$$

Proof. Notice that (2.33) is an SDE related to the semimartingale φ with constant coefficients which are obviously Lipschitz continuous with respect to $\Delta\varphi$: the time-dependent reflecting barriers $\Delta\varphi^\pm$ are Lipschitz operators in the sense of [37, Definition 3.1] evaluated at process $\Delta\varphi$. Hence the assertion follows from [37, Theorem 3.3]. \square

Let us now pass to the auxiliary shadow price process S^ε proposed in [20].

Corollary 2.5.2. *Let $\Delta\varphi$, $\varphi^{\varepsilon\uparrow}$, $\varphi^{\varepsilon\downarrow}$ be as in Lemma 2.5.1 (or equivalently in Theorem 2.2.4). Define*

$$\alpha := \frac{pc^{S,S}}{3c^{\varphi,\varphi}}, \quad \beta := \left(\frac{S}{\alpha}\right)^{1/3} = \Delta\varphi^+ \left(\frac{2}{\varepsilon}\right)^{1/3}, \quad \gamma := 3\alpha\beta^2 \left(\frac{\varepsilon}{2}\right)^{2/3} \quad (2.37)$$

and

$$\Delta S := \alpha\Delta\varphi^3 - \gamma\Delta\varphi. \quad (2.38)$$

Then ΔS is an Itô process with values in $[-\varepsilon S, \varepsilon S]$ such that

$$\varphi^{\varepsilon\uparrow} \text{ increases only on the set } \{\Delta S = \varepsilon S\} \subseteq \Omega \times [0, T], \quad (2.39)$$

$$\varphi^{\varepsilon\downarrow} \text{ increases only on the set } \{\Delta S = -\varepsilon S\} \subseteq \Omega \times [0, T]. \quad (2.40)$$

Proof. Consider the function $f(x, a, g) := ax^3 - gx$. By Assumption 2.2.2, α, γ, φ are Itô processes. From (2.34, 2.35, 2.36) we deduce that

$$d\Delta\varphi_t = -d\varphi_t \text{ on } \{\Delta\varphi \neq \Delta\varphi^\pm\}. \quad (2.41)$$

Since

$$\frac{\partial f}{\partial x}(\Delta\varphi^\pm, \alpha, \gamma) = 3\alpha(\Delta\varphi^\pm)^2 - \gamma = 0, \quad (2.42)$$

Itô's formula yields

$$\begin{aligned} d\Delta S_t &= df(\Delta\varphi_t, \alpha_t, \gamma_t) \\ &= -(3\alpha_t\Delta\varphi_t^2 - \gamma_t)d\varphi_t + 3\alpha_t\Delta\varphi_t d[\varphi, \varphi]_t \\ &\quad - 3\Delta\varphi_t^2 d[\alpha, \varphi]_t + \Delta\varphi_t^3 d\alpha_t - \Delta\varphi_t d\gamma_t + d[\gamma, \varphi]_t. \end{aligned} \quad (2.43)$$

In particular, ΔS is an Itô process. Moreover, (2.34, 2.35, 2.38) and $f(\Delta\varphi^\pm, \alpha, \gamma) = \mp\varepsilon S$ imply (2.39, 2.40). \square

Remark 2.5.3. For $\Delta\varphi$ and ΔS as in Lemma 2.5.1 and Corollary 2.5.2, define

$$\varphi^\varepsilon := \varphi + \Delta\varphi, \quad S^\varepsilon := S + \Delta S. \quad (2.44)$$

S^ε serves as a proxy to the so-called *shadow price process* which corresponds to the dual optimizer in the market with transaction costs, cf. the process $\hat{R}P = \hat{Z}_1/\hat{Z}_0$ in [5, Theorem 6.1]. Due to (2.39, 2.40), trading with φ^ε at price S^ε without friction or at the bid/ask prices $S(1 \pm \varepsilon)$ are equivalent. This will be exploited in the next sections for computing the expected utility generated by φ^ε and constructing a suitable dual upper bound.

Estimation of local parameters

In order to prepare for the calculations in coming sections, as a next step we look into the local dynamics of the shadow price process. Keeping the notations of α, β, γ in (2.37), we define the following sets of processes:

$$\begin{aligned} \mathcal{H}^{b\Delta S} &:= \left\{ \beta c^{S,S}, \alpha\beta^2 b^\varphi, \beta^2 c^{\alpha,\varphi}, \beta^3 b^\alpha, \beta b^{\alpha\beta^2}, c^{\alpha\beta^2,\varphi} \right\}, \\ \mathcal{H}^{c\Delta S,S} &:= \left\{ \alpha\beta^2 c^{\varphi,S}, \beta^3 c^{\alpha,S}, \beta c^{\alpha\beta^2,S} \right\}, \\ \mathcal{H}^{c\Delta S,\Delta S} &:= \left\{ \alpha^2\beta^4 c^{\varphi,\varphi}, \beta^6 c^{\alpha,\alpha}, \beta^2 c^{\alpha\beta^2,\alpha\beta^2} \right\}, \\ \mathcal{H}^{c\Delta S,\varphi} &:= \left\{ \alpha\beta^2 c^{\varphi,\varphi}, \beta^3 c^{\alpha,\varphi}, \beta c^{\alpha\beta^2,\varphi} \right\}, \\ \mathcal{G}_1 &:= \{Sb^\varphi\} \cup \mathcal{H}^{c\Delta S,\varphi} \cup \left\{ \beta b^\Delta : b^\Delta \in \mathcal{H}^{b\Delta S} \right\}, \\ \mathcal{G}_2 &:= \left\{ \beta^2 c^{S,S}, S^2 c^{\varphi,\varphi} \right\} \cup \left\{ \beta^2 c^{\Delta,\Delta} : c^{\Delta,\Delta} \in \mathcal{H}^{c\Delta S,\Delta S} \right\}, \\ \mathcal{G} &:= \mathcal{G}_1 \cup \mathcal{G}_2. \end{aligned}$$

From (2.43), the coefficients related to ΔS can be estimated as follows:

$$b_t^{\Delta S} = \underbrace{p\Delta\varphi_t c_t^{S,S}}_{=:G_t^{(1)}} - \underbrace{(3\alpha_t\Delta\varphi_t^2 - \gamma_t)b_t^\varphi - 3\Delta\varphi_t^2 c_t^{\alpha,\varphi} + \Delta\varphi_t^3 b_t^\alpha - \Delta\varphi_t b_t^\gamma + c_t^{\gamma,\varphi}}_{=:G_t^{(2)}}, \quad (2.45)$$

$$|G_t^{(1)}| \leq \text{cst.} \beta_t c_t^{S,S} \varepsilon^{1/3}, \quad |G_t^{(2)}| \leq \text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b^{\Delta S}} \setminus \{\beta c^{S,S}\}} |b_t^\Delta| \varepsilon^{2/3}, \quad (2.46)$$

$$|c_t^{\Delta S,S}| \leq \text{cst.} \sum_{c^{\Delta,S} \in \mathcal{H}^{c^{\Delta S,S}}} |c_t^{\Delta,S}| \varepsilon^{2/3}, \quad (2.47)$$

$$|c_t^{\Delta S,\Delta S}| \leq \text{cst.} \sum_{c^{\Delta,\Delta} \in \mathcal{H}^{c^{\Delta S,\Delta S}}} |c_t^{\Delta,\Delta}| \varepsilon^{4/3} \quad (2.48)$$

and

$$c_t^{\Delta S,\varphi} = \underbrace{-p c_t^{S,S} (\Delta\varphi_t^2 - (\Delta\varphi_t^+)^2)}_{=:G_t^{(3)}} + \underbrace{\Delta\varphi_t^3 c_t^{\alpha,\varphi} - \Delta\varphi_t c_t^{\gamma,\varphi}}_{=:G_t^{(4)}}, \quad (2.49)$$

$$|G_t^{(3)}| \leq \text{cst.} |\alpha_t \beta_t^2 c_t^{\varphi,\varphi}| \varepsilon^{2/3}, \quad |G_t^{(4)}| \leq \text{cst.} \sum_{c^{\Delta,\varphi} \in \mathcal{H}^{c^{\Delta S,\varphi}} \setminus \{\alpha \beta^2 c^{\varphi,\varphi}\}} |c_t^{\Delta,\varphi}| \varepsilon. \quad (2.50)$$

Note that here and in the sequel, inequalities of the form $A \leq \text{cst.} B$ are to be interpreted in the sense that there exists a constant $k \in \mathbb{R}$ which does not depend on ε and such that $A \leq kB$.

Actually, for the proof of Theorem 2.2.4, Condition (2.2) can be slightly weakened. We formulate the weaker assumptions as a consequence of Condition (2.2) in the subsequent lemma.

Lemma 2.5.4. *Condition (2.2) implies*

$$\sum_{g \in \mathcal{G}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |g_t|^4 dt \right] + \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0,T]} |\varphi_t S_t|^4 + \sup_{t \in [0,T]} |\beta_t S_t|^4 \right] < \infty, \quad (2.51)$$

$$\sum_{c^{\Delta,S} \in \mathcal{H}^{c^{\Delta S,S}}} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0,T]} \left| \frac{c_t^{\Delta,S}}{c_t^{S,S}} \right|^{16} \right] + \sum_{b^\Delta \in \mathcal{H}^{b^{\Delta S}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{b_t^\Delta}{\sigma_t^S} \right|^{16} dt \right] < \infty. \quad (2.52)$$

Proof. This follows from Itô's formula, straightforward but tedious calculations and Lemma A.2. \square

2.5.2 Primal considerations

With the help of the shadow price process S^ε , in this section we approximate the expected utility generated by the candidate strategy φ^ε . Here we proceed similarly to [20], in that we develop a Taylor series expansion of the expected utility generated by φ^ε in $\varepsilon^{1/3}$. Aiming for this, we first use the estimations of the local dynamics of S^ε provided in the previous section and analyze the deviation of the liquidation wealth corresponding to φ^ε from the frictionless optimal wealth process $x + \varphi \cdot S$. This is done in the subsequent lemma. The approximation relies on the basic inequalities provided in Appendix A.

Lemma 2.5.5. *Assume Condition (2.2).*

1. *For any stopping time τ , we have*

$$\mathbf{E}_{\mathbf{Q}} \left[X_\tau^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_\tau) \right] = p \mathbf{E}_{\mathbf{Q}} \left[(2\Delta\varphi^2 - (\Delta\varphi^+)^2) \cdot [S, S]_\tau \right] + O(\varepsilon), \quad (2.53)$$

$$\mathbf{E}_{\mathbf{Q}} \left[(X_\tau^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_\tau))^2 \right] = \mathbf{E}_{\mathbf{Q}} \left[\Delta\varphi^2 \cdot [S, S]_\tau \right] + O(\varepsilon), \quad (2.54)$$

$$\left\| X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S) \right\|_{S^3(\mathbf{Q})}^3 = O(\varepsilon). \quad (2.55)$$

2. *Define stopping times*

$$\tau^{\varepsilon, 1} := \inf \left\{ t \in [0, T] : |X_t^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_t)| > 1 \right\}, \quad (2.56)$$

$$\tau^{\varepsilon, 2} := \inf \left\{ t \in [0, T] : |X_t^{\varphi^\varepsilon, \varepsilon}| > \varepsilon^{-4/3} \right\}, \quad (2.57)$$

$$\tau^\varepsilon := \tau^{\varepsilon, 1} \wedge \tau^{\varepsilon, 2} \wedge T. \quad (2.58)$$

Then

$$\mathbf{Q}(\tau^\varepsilon < T) = O(\varepsilon^{4/3}) \quad (2.59)$$

and $\lim_{\varepsilon \downarrow 0} \mathbf{P}(\tau^\varepsilon < T) = 0$.

Proof. 1. Let τ be a stopping time and q be a natural number less than or equal to 4. In view of (2.44), the deviation of the liquidation wealth $X^{\varphi^\varepsilon, \varepsilon}$ from $x + \varphi \cdot S$ can be separated into four parts:

$$X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S) = \Delta\varphi \cdot S + \varphi \cdot \Delta S + \Delta\varphi \cdot \Delta S + (X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi^\varepsilon \cdot S^\varepsilon)). \quad (2.60)$$

We look into the terms in (2.60) in the following four steps.

Step 1. From (A.2, A.7) and (2.51), we get

$$\|\Delta\varphi \cdot S\|_{S^q(\mathbf{Q})}^q \leq \text{cst.} \|\Delta\varphi \cdot S\|_{H^q(\mathbf{Q})}^q \leq \text{cst.} \underbrace{\sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 c_t^{S, S}|^q dt \right]}}_{< \infty} \varepsilon^{q/3}. \quad (2.61)$$

By letting $q = 2$ we deduce that $\Delta\varphi \cdot S$ is a square-integrable \mathbf{Q} -martingale. Thus

$$\mathbf{E}_{\mathbf{Q}}[\Delta\varphi \cdot S_\tau] = 0, \quad (2.62)$$

$$\mathbf{E}_{\mathbf{Q}}[(\Delta\varphi \cdot S_\tau)^2] = \mathbf{E}_{\mathbf{Q}}[\Delta\varphi^2 \cdot [S, S]_\tau]. \quad (2.63)$$

Step 2. Integration by parts yields

$$\varphi \cdot \Delta S = \varphi \Delta S - \Delta S \cdot \varphi - [\Delta S, \varphi].$$

By $|\Delta S| \leq \varepsilon S$ and (2.51), we have

$$\|\varphi \Delta S\|_{S^q(\mathbf{Q})}^q \leq \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |\varphi_t S_t|^q \right]}_{< \infty} \varepsilon^q. \quad (2.64)$$

From $|\Delta S| \leq \varepsilon S$, (A.7) and (2.51), we obtain

$$\|\Delta S \cdot \varphi\|_{H^q(\mathbf{Q})}^q \leq \underbrace{\text{cst.} \left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |S_t b_t^\varphi|^q dt \right] + \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |S_t^2 c_t^{\varphi, \varphi}|^q dt \right]} \right)}_{< \infty} \varepsilon^q. \quad (2.65)$$

In view of (2.49, 2.50, A.7) and (2.51), it holds that

$$\mathbf{E}_{\mathbf{Q}}[[\Delta S, \varphi]_\tau] = -p \mathbf{E}_{\mathbf{Q}}[(\Delta\varphi^2 - (\Delta\varphi^+)^2) \cdot [S, S]_\tau] + O(\varepsilon), \quad (2.66)$$

$$\|[\Delta S, \varphi]\|_{H^q(\mathbf{Q})}^q \leq \underbrace{\text{cst.} \sum_{c^{\Delta, \varphi} \in \mathcal{H}^{c^{\Delta S, \varphi}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |c_t^{\Delta, \varphi}|^q dt \right]}_{< \infty} \varepsilon^{2q/3}. \quad (2.67)$$

Together, (2.64–2.67) then yield

$$\mathbf{E}_{\mathbf{Q}}[\varphi \cdot \Delta S_\tau] = p \mathbf{E}_{\mathbf{Q}}[(\Delta\varphi^2 - (\Delta\varphi^+)^2) \cdot [S, S]_\tau] + O(\varepsilon), \quad (2.68)$$

$$\|\varphi \cdot \Delta S\|_{S^q(\mathbf{Q})}^q = O(\varepsilon^{2q/3}). \quad (2.69)$$

Step 3. From (2.45, 2.46, 2.48, A.7) and (2.51), we have

$$\mathbf{E}_{\mathbf{Q}}[\Delta\varphi \cdot \Delta S_\tau] = p \mathbf{E}_{\mathbf{Q}}[\Delta\varphi^2 \cdot [S, S]_\tau] + O(\varepsilon), \quad (2.70)$$

$$\begin{aligned} \|\Delta\varphi \cdot \Delta S\|_{H^q(\mathbf{Q})}^q &\leq \underbrace{\text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b^{\Delta S}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t b_t^\Delta|^q dt \right]}_{< \infty} \varepsilon^{2q/3} \\ &\quad + \underbrace{\text{cst.} \sqrt{\sum_{c^{\Delta, \Delta} \in \mathcal{H}^{c^{\Delta S, \Delta}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 c_t^{\Delta, \Delta}|^q dt \right]}}_{< \infty} \varepsilon^{2q/3}. \end{aligned} \quad (2.71)$$

Step 4. Due to (2.39, 2.40) and Remark 2.1.2, taking (A.4, 2.51) into account, we deduce

$$\begin{aligned} \left\| X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi^\varepsilon \cdot S^\varepsilon) \right\|_{S^q(\mathbf{Q})}^q &\leq 2\mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |\varphi_t^\varepsilon S_t| \right] \varepsilon + x^S \varepsilon \\ &\leq \underbrace{\text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |\varphi_t S_t|^q + \sup_{t \in [0, T]} |\beta_t S_t|^q + x^S \right]}_{< \infty} \varepsilon^q. \end{aligned} \quad (2.72)$$

Together, in view of (A.4), relations (2.60) and (2.62, 2.68, 2.70, 2.72) for $q = 1$ imply (2.53). (2.60) and (2.63, 2.69, 2.71, 2.72) for $q = 2$ yield (2.54). (2.60) and (2.61, 2.69, 2.71, 2.72) for $q = 3$ lead to (2.55).

2. (2.56), Markov's inequality, and (A.4, 2.60, 2.61, 2.69, 2.71, 2.72) yield

$$\begin{aligned} \mathbf{Q}(\tau^{\varepsilon, 1} < T) &\leq \mathbf{Q} \left(\sup_{t \in [0, T]} |X_t^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_t)| > 1 \right) \\ &\leq \left\| X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S) \right\|_{S^4(\mathbf{Q})}^4 \\ &= O(\varepsilon^{4/3}). \end{aligned}$$

Condition (2.3) implies that the \mathbf{Q} -martingale $\varphi \cdot S$ is in fact square-integrable and hence $\mathbf{E}_{\mathbf{Q}}[\varphi^2 \cdot [S, S]_T] < \infty$. Moreover, from (2.57), Markov's inequality, (A.2, 2.60) and (A.7, 2.61, 2.69, 2.71, 2.72) for $q = 1$, we get

$$\begin{aligned} \mathbf{Q}(\tau^{\varepsilon, 2} < T) &\leq \mathbf{Q} \left(\sup_{t \in [0, T]} |X_t^{\varphi^\varepsilon, \varepsilon}| > \varepsilon^{-4/3} \right) \\ &\leq \|X^{\varphi^\varepsilon, \varepsilon}\|_{S^1(\mathbf{Q})} \varepsilon^{4/3} \\ &\leq \left(\|x + \varphi \cdot S\|_{S^1(\mathbf{Q})} + \|X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S)\|_{S^1(\mathbf{Q})} \right) \varepsilon^{4/3} \\ &\leq \underbrace{\text{cst.} \left(1 + \sqrt{\mathbf{E}_{\mathbf{Q}}[\varphi^2 \cdot [S, S]_T]} + \|X^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S)\|_{S^1(\mathbf{Q})} \right)}_{< \infty} \varepsilon^{4/3}. \end{aligned}$$

Therefore,

$$\mathbf{Q}(\tau^\varepsilon < T) \leq \mathbf{Q}(\tau^{\varepsilon, 1} < T) + \mathbf{Q}(\tau^{\varepsilon, 2} < T) = O(\varepsilon^{4/3}).$$

Since \mathbf{P} and \mathbf{Q} are equivalent, $\mathbf{Q}(\tau^\varepsilon < T) \rightarrow 0$ implies $\mathbf{P}(\tau^\varepsilon < T) \rightarrow 0$. \square

Using the considerations above, we now compute the expected utility when following strategy $\varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}$ to the leading order.

Lemma 2.5.6. *Suppose that Conditions (2.2, 2.3) hold. For τ^ε as in (2.5) or equivalently in (2.58), we have*

$$\mathbf{E} \left[U(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon}) \right] = \mathbf{E}[U(x + \varphi \cdot S_T)] - y p \mathbf{E}_{\mathbf{Q}} \left[\left((\Delta\varphi^+)^2 - \frac{3}{2} \Delta\varphi^2 \right) \cdot [S, S]_T \right] + O(\varepsilon). \quad (2.73)$$

Proof. Let Z denote the density process of \mathbf{Q} . Taylor expansion of $U(x) = -e^{-px}$ yields

$$\begin{aligned} U(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon}) &= U(x + \varphi \cdot S_T) + \underbrace{y Z_T (X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T))}_{=: G^{(5)}} - \underbrace{\frac{p}{2} y Z_T (X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T))^2}_{=: G^{(6)}} \\ &\quad + \underbrace{\frac{p^2}{6} y Z_T \exp\left(-p\theta (X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T))\right) (X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T))^3}_{=: G^{(7)}} \end{aligned}$$

for some random $\theta \in (0, 1)$. In view of (2.53, 2.54, 2.55), the terms $G^{(5)}, G^{(6)}, G^{(7)}$ are expected to have expectation of order $O(\varepsilon^{2/3}), O(\varepsilon^{2/3}), O(\varepsilon)$, respectively. We prove this rigorously in the following steps.

Step 1: Expectation of terms $G^{(5)}, G^{(6)}$. Notice that

$$X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T) = (X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_{\tau^\varepsilon})) + (\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T) \quad (2.74)$$

with

$$\left| X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_{\tau^\varepsilon}) \right| \stackrel{(2.56)}{\leq} 1 \quad (2.75)$$

and

$$\mathbf{E}_{\mathbf{Q}} [|\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T|^{2n}] \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} [|\varphi \cdot S_T|^{2n}] \stackrel{(2.3)}{<} \infty \quad (2.76)$$

for all $n \in \mathbb{N}$, where the first inequality in (2.76) is due to the fact that $|\varphi \cdot S|^{2n}$ is a \mathbf{Q} -submartingale. In view of (2.74), Hölder's inequality and (2.59, 2.76, 2.75), we have

$$\begin{aligned} \left| \mathbf{E}[G^{(5)}] - y \mathbf{E}_{\mathbf{Q}} \left[X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_{\tau^\varepsilon}) \right] \right| &\leq y \mathbf{E}_{\mathbf{Q}} [\mathbf{1}_{\{\tau^\varepsilon < T\}} |\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T|] \\ &\leq y \mathbf{Q}(\tau^\varepsilon < T)^{\frac{3}{4}} \sqrt[4]{\mathbf{E}_{\mathbf{Q}} [|\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T|^4]} \\ &= O(\varepsilon) \end{aligned} \quad (2.77)$$

and

$$\begin{aligned} &\left| \mathbf{E}[G^{(6)}] - \frac{p}{2} y \mathbf{E}_{\mathbf{Q}} \left[(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_{\tau^\varepsilon}))^2 \right] \right| \\ &\leq \text{cst.} (\mathbf{E}_{\mathbf{Q}} [\mathbf{1}_{\{\tau^\varepsilon < T\}} |\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T|^2] + \mathbf{E}_{\mathbf{Q}} [\mathbf{1}_{\{\tau^\varepsilon < T\}} |\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T|]) \\ &\leq \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{3}{4}} \sqrt[4]{\mathbf{E}_{\mathbf{Q}} [|\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T|^8]} + O(\varepsilon) \\ &= O(\varepsilon). \end{aligned} \quad (2.78)$$

Step 2: Expectation of the remainder term $G^{(7)}$. From (2.75) and (2.55), we obtain

$$\begin{aligned} |\mathbf{E}[G^{(7)}]| &\leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\exp(-p\theta(\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T)) |X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T)|^3 \right] \\ &\leq \text{cst.} \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{\tau^\varepsilon = T\}} |X_T^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T)|^3 \right]}_{=O(\varepsilon)} \\ &\quad + \text{cst.} \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{\tau^\varepsilon < T\}} \exp(-p\theta(\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T)) |X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T)|^3 \right]}_{=:G^{(8)}}. \end{aligned}$$

By Hölder's inequality and (2.59) we have

$$G^{(8)} \leq \underbrace{\mathbf{Q}(\tau^\varepsilon < T)^{\frac{3}{4}}}_{=O(\varepsilon)} \underbrace{\sqrt[4]{\mathbf{E}_{\mathbf{Q}} \left[\exp(-4p\theta(\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T)) |X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T)|^{12} \right]}}_{=:G^{(9)}}.$$

Using again Hölder's inequality, we get

$$G^{(9)} \leq \underbrace{\sqrt[9/2]{\mathbf{E}_{\mathbf{Q}} \left[\exp\left(-\frac{9}{2}p\theta(\varphi \cdot S_{\tau^\varepsilon} - \varphi \cdot S_T)\right) \right]}}_{=:G^{(10)}} \underbrace{\sqrt[36]{\mathbf{E}_{\mathbf{Q}} \left[|X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon} - (x + \varphi \cdot S_T)|^{108} \right]}}_{\stackrel{(2.76)}{< \infty}}$$

with

$$G^{(10)} \leq \sqrt[9]{\mathbf{E}_{\mathbf{Q}} [\exp(-9p\theta\varphi \cdot S_{\tau^\varepsilon})]} \underbrace{\sqrt[9]{\mathbf{E}_{\mathbf{Q}} [\exp(9p\theta\varphi \cdot S_T)]}}_{\stackrel{(2.3)}{< \infty}}.$$

Since $\varphi \cdot S$ is a \mathbf{Q} -martingale, $\exp(-9p\theta\varphi \cdot S)$ is a \mathbf{Q} -submartingale by Jensen's inequality. Hence,

$$\mathbf{E}_{\mathbf{Q}}[\exp(-9p\theta\varphi \cdot S_{\tau^\varepsilon})] \leq \mathbf{E}_{\mathbf{Q}}[\exp(-9p\theta\varphi \cdot S_T)] \stackrel{(2.3)}{< \infty}.$$

Therefore,

$$\mathbf{E}[G^{(7)}] = O(\varepsilon). \quad (2.79)$$

Step 3: Further estimations. In a final step, we sum up the expectations of $G^{(5)}$, $G^{(6)}$, $G^{(7)}$ estimated as above. Combining (2.77, 2.78, 2.79) with (2.53, 2.54), we obtain

$$\mathbf{E} \left[U(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon}) \right] = \mathbf{E} [U(x + \varphi \cdot S_T)] - yp \mathbf{E}_{\mathbf{Q}} \left[\left((\Delta\varphi^+)^2 - \frac{3}{2}\Delta\varphi^2 \right) \cdot [S, S]_{\tau^\varepsilon} \right] + O(\varepsilon).$$

Moreover, Hölder's inequality and (2.59) yield

$$\left| \mathbf{E}_{\mathbf{Q}} \left[\int_{\tau^\varepsilon}^T \left((\Delta\varphi_t^+)^2 - \frac{3}{2}\Delta\varphi_t^2 \right) d[S, S]_t \right] \right| \leq \text{cst.} \underbrace{\mathbf{Q}(\tau^\varepsilon < T)^{\frac{3}{4}}}_{=O(\varepsilon)} \underbrace{\sqrt[4]{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 c_t^{S, S}|^4 dt \right]}}_{\stackrel{(2.51)}{< \infty}} \varepsilon^{2/3}. \quad (2.80)$$

This completes the proof. \square

2.5.3 Dual considerations

In order to obtain an appropriate upper bound for the maximum expected utility, we construct a dual variable based on Girsanov's theorem. More specifically, we consider the minimal martingale measure for the appropriately stopped process S^ε relative to \mathbf{Q} . This martingale measure turns out to be optimal to the leading order. Throughout this section, we keep the notations of $\varphi^\varepsilon, \Delta\varphi, S^\varepsilon, \Delta S$ in (2.44), τ^ε in (2.58) as well as all sets of processes defined in Section 2.5.1.

Let

$$\rho^{\varepsilon,1} := \inf \left\{ t \in [0, T] : \left| \frac{c_t^{\Delta S, S}}{c_t^{S, S}} \right| > \frac{1}{2} \right\} \quad (2.81)$$

and define

$$Z^{\varepsilon, \mathbf{Q}} := \exp(N^\varepsilon), \quad N^\varepsilon := - \int_0^\cdot \theta_t^\varepsilon dS_t - \frac{1}{2} \int_0^\cdot (\theta_t^\varepsilon)^2 d[S, S]_t, \quad (2.82)$$

where

$$\theta^\varepsilon := \frac{b^{S^\varepsilon}}{c^{S^\varepsilon, S}} \mathbf{1}_{[[0, \rho^{\varepsilon,1}]]} = \frac{b^{\Delta S}}{c^{S, S} + c^{\Delta S, S}} \mathbf{1}_{[[0, \rho^{\varepsilon,1}]]}. \quad (2.83)$$

Furthermore, let

$$\rho^{\varepsilon,2} := \inf \left\{ t \in [0, T] : |Z_t^{\varepsilon, \mathbf{Q}} - 1| > \frac{1}{2} \right\}, \quad (2.84)$$

$$\rho^\varepsilon := \rho^{\varepsilon,1} \wedge \rho^{\varepsilon,2} \wedge T. \quad (2.85)$$

and define the ‘‘stopped’’ processes

$$\begin{aligned} \bar{\varphi}^\varepsilon &:= \varphi^\varepsilon \mathbf{1}_{[[0, \tau^\varepsilon \wedge \rho^\varepsilon]]}, & \Delta \bar{\varphi} &:= \Delta \varphi \mathbf{1}_{[[0, \tau^\varepsilon \wedge \rho^\varepsilon]]}, \\ \bar{S}^\varepsilon &:= S \left(1 + \frac{\Delta S^{\tau^\varepsilon \wedge \rho^\varepsilon}}{S^{\tau^\varepsilon \wedge \rho^\varepsilon}} \right), & \bar{N}^\varepsilon &:= (N^\varepsilon)^{\tau^\varepsilon \wedge \rho^\varepsilon}, & \bar{Z}^\varepsilon &:= (Z^{\varepsilon, \mathbf{Q}})^{\tau^\varepsilon \wedge \rho^\varepsilon}. \end{aligned} \quad (2.86)$$

Remark 2.5.7. By construction, \bar{Z}^ε is a bounded \mathbf{Q} -local martingale and hence a \mathbf{Q} -martingale. If Z denotes the density process of \mathbf{Q} , the process $Z^\varepsilon := Z \bar{Z}^\varepsilon$ is a \mathbf{P} -martingale. Integration by parts yields that $\bar{Z}^\varepsilon \bar{S}^\varepsilon$ is a \mathbf{Q} -local martingale and hence $Z^\varepsilon \bar{S}^\varepsilon = Z \bar{Z}^\varepsilon \bar{S}^\varepsilon$ is a \mathbf{P} -local martingale. Consequently, \bar{S}^ε is a local martingale under the probability measure with density process Z^ε . Therefore, Z^ε corresponds to an equivalent (local) martingale measure for \bar{S}^ε . It serves as a dual variable in the frictionless market at shadow price \bar{S}^ε .

Passing to convex duality theory, we denote by \tilde{U} the conjugate function of U , i.e.,

$$\tilde{U}(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \quad y \geq 0, \quad (2.87)$$

which satisfies $-\tilde{U}' = (U')^{-1}$. Since $U(x) = -e^{-px}$, we obtain

$$\tilde{U}'(y) = \frac{1}{p} \ln \frac{y}{p}, \quad \tilde{U}''(y) = \frac{1}{py}, \quad \tilde{U}'''(y) = -\frac{1}{py^2}. \quad (2.88)$$

Our purpose is to calculate the expected convex conjugate \tilde{U} evaluated at the dual variable $yZ^\varepsilon = yZ\bar{Z}^\varepsilon$ to the leading order. Parallel to the procedure in the previous section, we develop a Taylor series expansion of $\mathbf{E}[\tilde{U}(yZ^\varepsilon)]$ in $\varepsilon^{1/3}$. Since Z^ε is expected to approach the frictionless dual optimizer Z for ε near 0, this will be accomplished by looking into the deviation of Z^ε from Z and applying the frictionless duality theory linking yZ to φ . This is done in the subsequent two lemmas.

Lemma 2.5.8. *Assume Condition (2.2).*

1. For \bar{Z}^ε defined by (2.82, 2.86), we have

$$\mathbf{E}_Q \left[(\bar{Z}_T^\varepsilon - 1)^2 \right] = p^2 \mathbf{E}_Q [\Delta\bar{\varphi}^2 \cdot [S, S]_T] + O(\varepsilon), \quad (2.89)$$

$$\mathbf{E}_Q \left[|\bar{Z}_T^\varepsilon - 1|^3 \right] = O(\varepsilon). \quad (2.90)$$

2. For ρ^ε defined in (2.85), it holds that

$$\mathbf{Q}(\rho^\varepsilon < T) = O(\varepsilon^{4/3}). \quad (2.91)$$

Proof. Note that on $\{\rho^\varepsilon \geq t\}$, (2.45, 2.46, 2.47, 2.81, 2.83) yield

$$\theta_t^\varepsilon \sigma_t^S = p\Delta\varphi_t \sigma_t^S + \underbrace{\frac{b_t^{\Delta S} \sigma_t^S}{c_t^{S,S} + c_t^{\Delta S,S}} - p\Delta\varphi_t \sigma_t^S}_{=: G_t^{(12)}} \quad (2.92)$$

with

$$|\theta_t^\varepsilon \sigma_t^S| \leq \text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b^\Delta S}} \left| \frac{b_t^\Delta}{\sigma_t^S} \right| \varepsilon^{1/3}, \quad |p\Delta\varphi \sigma^S| \leq \text{cst.} \beta \sigma^S \varepsilon^{1/3}, \quad (2.93)$$

$$|G_t^{(12)}| \leq \text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b^\Delta S}} \left| \frac{b_t^\Delta}{\sigma_t^S} \right| \left(1 + \sum_{c^{\Delta,S} \in \mathcal{H}^{c^{\Delta,S}}} \left| \frac{c_t^{\Delta,S}}{c^{S,S}} \right| \right) \varepsilon^{2/3}. \quad (2.94)$$

From (2.84) and Taylor expansion of $x \mapsto e^x$, we obtain

$$\bar{Z}^\varepsilon - 1 = -p\Delta\bar{\varphi} \cdot S + (\bar{N}^\varepsilon + p\Delta\bar{\varphi} \cdot S) + G^{(13)} \quad \text{with} \quad |G^{(13)}| \leq \frac{3}{4} |\bar{N}^\varepsilon|^2. \quad (2.95)$$

1. From (2.95), (A.4) for $q = 2$, (A.2) for $q = 2, 4$, (2.92, 2.93, 2.94, A.7), the Cauchy-Schwarz inequality and (2.52), we deduce that

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1 + p\Delta\bar{\varphi} \cdot S_T)^2 \right] &\leq \text{cst.} \|\bar{N}^\varepsilon + p\Delta\bar{\varphi} \cdot S\|_{H^2(\mathbf{Q})}^2 + \text{cst.} \|\bar{N}^\varepsilon\|_{H^4(\mathbf{Q})}^4 \\
&\leq \text{cst.} \sum_{m=1,2} \sum_{n=1,2} \underbrace{\left(\sum_{b^\Delta \in \mathcal{H}^{b^\Delta S}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{b_t^\Delta}{\sigma_t^S} \right|^{8m} dt \right] \right)}_{< \infty}^{\frac{1}{2n}} \\
&\quad \times \underbrace{\left(1 + \sum_{c^{\Delta, S} \in \mathcal{H}^{c^{\Delta, S, S}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{c_t^{\Delta, S}}{c_t^{S, S}} \right|^{8m} dt \right] \right)}_{< \infty}^{\frac{1}{2n}} \varepsilon^{4/3}. \quad (2.96)
\end{aligned}$$

Combined with (2.63) and the Cauchy-Schwarz inequality, this yields (2.89). Analogously, from (2.95), (A.4) for $q = 3$, (A.2) for $q = 3, 6$, (2.92, 2.93, A.7) and (2.52), we get

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}} \left[|\bar{Z}_T^\varepsilon - 1|^3 \right] &\leq \text{cst.} \|\bar{N}^\varepsilon\|_{H^3(\mathbf{Q})}^3 + \text{cst.} \|\bar{N}^\varepsilon\|_{H^6(\mathbf{Q})}^6 \\
&\leq \text{cst.} \sum_{m=1,2} \sum_{n=1,2} \underbrace{\left(\sum_{b^\Delta \in \mathcal{H}^{b^\Delta S}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{b_t^\Delta}{\sigma_t^S} \right|^{6m} dt \right] \right)}_{< \infty}^{\frac{1}{2n}} \varepsilon.
\end{aligned}$$

2. By (2.81), Markov's inequality, (2.47), (A.4) for $q = 2$ and (2.52), it holds that

$$\mathbf{Q}(\rho^{\varepsilon,1} < T) \leq 4 \left\| \frac{c_t^{\Delta, S, S}}{c_t^{S, S}} \right\|_{S^2(\mathbf{Q})}^2 \leq \text{cst.} \underbrace{\sum_{\sigma^\Delta \in \mathcal{H}^{c^{\Delta, S, S}}} \left\| \frac{c_t^{\Delta, S}}{c_t^{S, S}} \right\|_{S^2(\mathbf{Q})}^2}_{< \infty} \varepsilon^{4/3}. \quad (2.97)$$

Further, notice that

$$\mathbf{Q}(\rho^{\varepsilon,2} < T) = \mathbf{Q}(\{\rho^{\varepsilon,2} < T\} \cap \{\rho^{\varepsilon,2} < \rho^{\varepsilon,1}\}) + \mathbf{Q}(\rho^{\varepsilon,1} \leq \rho^{\varepsilon,2} < T),$$

where

$$\mathbf{Q}(\rho^{\varepsilon,1} \leq \rho^{\varepsilon,2} < T) \leq \mathbf{Q}(\rho^{\varepsilon,1} < T) \stackrel{(2.97)}{=} O(\varepsilon^{4/3}).$$

From (2.84), Markov's inequality, (A.2) for $q = 6$, (2.92, 2.93, A.7) and (2.52), we deduce

that

$$\begin{aligned}
\mathbf{Q}(\{\rho^{\varepsilon,2} < T\} \cap \{\rho^{\varepsilon,2} < \rho^{\varepsilon,1}\}) &\leq \mathbf{Q}\left(\exists t \leq T : |Z_t^{\varepsilon, \mathbf{Q}} - 1| > \frac{1}{2}\right) \\
&\leq \mathbf{Q}\left(\exists t \leq T : |N_t^\varepsilon| > \ln \frac{3}{2}\right) \\
&\leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |N_t^\varepsilon|^6 \right] \\
&\leq \text{cst.} \|N^\varepsilon\|_{H^6(\mathbf{Q})}^6 \\
&\leq \text{cst.} \sum_{n=1,2} \left(\sum_{b^\Delta \in \mathcal{H}^{b^\Delta S}} \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{b_t^\Delta}{\sigma_t^S} \right|^{12} dt \right]}_{< \infty} \right)^{\frac{1}{2n}} \varepsilon^2.
\end{aligned}$$

Together, (2.91) follows. \square

Now, let us apply the conjugate relation connecting the frictionless primal and dual optimizers φ and yZ , respectively, and gather the above considerations to draw the following conclusion.

Lemma 2.5.9. *Under Condition (2.2), we have*

$$\mathbf{E} \left[\tilde{U}(yZ_T \bar{Z}_T^\varepsilon) \right] + xy = \mathbf{E}[U(x + \varphi \cdot S_T)] - yp \mathbf{E}_{\mathbf{Q}} \left[\left((\Delta \varphi^+)^2 - \frac{3}{2} \Delta \varphi^2 \right) \cdot [S, S]_T \right] + O(\varepsilon). \quad (2.98)$$

Proof. By (2.88), Taylor expansion of \tilde{U} yields

$$\begin{aligned}
\tilde{U}(yZ_T \bar{Z}_T^\varepsilon) &= \tilde{U}(yZ_T) + \underbrace{\tilde{U}'(yZ_T) yZ_T (\bar{Z}_T^\varepsilon - 1)}_{=: G^{(14)}} + \underbrace{\frac{1}{2} \tilde{U}''(yZ_T) (yZ_T)^2 (\bar{Z}_T^\varepsilon - 1)^2}_{=: G^{(15)}} \\
&\quad - \underbrace{\frac{1}{6p} \left(1 + \theta (\bar{Z}_T^\varepsilon - 1) \right)^{-1} yZ_T (\bar{Z}_T^\varepsilon - 1)^3}_{=: G^{(16)}}
\end{aligned}$$

for some random $\theta \in (0, 1)$. Due to the optimality of φ and yZ_T as well as by the conjugate relation (cf. [34, Theorem 2.2]), we conclude

$$\mathbf{E}[\tilde{U}(yZ_T)] = \mathbf{E}[U(x + \varphi \cdot S_T)] - xy, \quad (2.99)$$

$$-\tilde{U}'(yZ_T) = x + \varphi \cdot S_T, \quad (2.100)$$

$$\tilde{U}''(yZ_T) (yZ_T)^2 = \frac{1}{p} yZ_T, \quad (2.101)$$

and $\varphi \cdot S$ is a \mathbf{Q} -martingale, i.e.,

$$\mathbf{E}_{\mathbf{Q}}[x + \varphi \cdot S_T] = x. \quad (2.102)$$

Step 1. By means of the above conjugate relations, we now calculate the expectation of the first two terms $G^{(14)}, G^{(15)}$ in the Taylor expansion of \tilde{U} . (2.100, 2.102) yield

$$\mathbf{E} \left[G^{(14)} \right] = \underbrace{y \mathbf{E}_{\mathbf{Q}}[x + \varphi \cdot S_T] - xy}_{=0} - y \mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\varphi \cdot S_T) \right]$$

and

$$\begin{aligned} -\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\varphi \cdot S_T) \right] &= -\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon) \right] + \mathbf{E}_{\mathbf{Q}} \left[\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T \right] \\ &\quad + \mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1) (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T) \right]. \end{aligned}$$

Note that $\bar{Z}^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon)$ is a \mathbf{Q} -local martingale, cf. Remark 2.5.7. (2.84, 2.72, 2.57) and (2.51) yield

$$\left\| \bar{Z}^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon) \right\|_{S^1(\mathbf{Q})} \leq \text{cst.} \left(\varepsilon^{-4/3} + \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |\varphi_t S_t| + \sup_{t \in [0, T]} |\beta_t S_t| \right] \right) < \infty,$$

which implies that $\bar{Z}^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon)$ is a uniformly integrable \mathbf{Q} -martingale and hence

$$\mathbf{E}_{\mathbf{Q}}[\bar{Z}_T^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon)] = 0.$$

(2.62, 2.68, 2.70, 2.77) in conjunction with (2.59, 2.91) and the argument in (2.77) yield

$$\mathbf{E}_{\mathbf{Q}} \left[\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T \right] = p \mathbf{E}_{\mathbf{Q}} \left[(2\Delta\varphi^2 - (\Delta\varphi^+)^2) \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon).$$

Further, we have

$$\begin{aligned} &\mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1) (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T) \right] \\ &= \underbrace{\mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1) (\Delta\bar{\varphi} \cdot S_T) \right]}_{=: G^{(17)}} + \underbrace{\mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1) (\bar{\varphi}^\varepsilon \cdot \Delta\bar{S}_T + \varphi \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} - \varphi \cdot S_T) \right]}_{=: G^{(18)}}. \end{aligned}$$

From (2.63), the Cauchy-Schwarz inequality and (2.96), we conclude

$$\begin{aligned} G^{(17)} &= -p \mathbf{E}_{\mathbf{Q}} \left[(\Delta\bar{\varphi} \cdot S_T)^2 \right] + \mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1 + p\Delta\bar{\varphi} \cdot S_T) (\Delta\bar{\varphi} \cdot S_T) \right] \\ &= -p \mathbf{E}_{\mathbf{Q}} \left[\Delta\bar{\varphi}^2 \cdot [S, S]_T \right] + O(\varepsilon). \end{aligned}$$

By the Cauchy-Schwarz inequality, (2.84) and (2.89), using (2.69, 2.71, A.4) for $q = 2$ and combining the argument in (2.77) with (2.91), it follows that

$$\begin{aligned} |G^{(18)}| &\leq \text{cst.} \sqrt{\mathbf{E}_{\mathbf{Q}} [(\bar{Z}_T^\varepsilon - 1)^2]} \sqrt{\mathbf{E}_{\mathbf{Q}} [(\bar{\varphi}^\varepsilon \cdot \Delta \bar{S}_T)^2]} \\ &\quad + \text{cst.} \mathbf{E}_{\mathbf{Q}} [\mathbf{1}_{\{\tau^\varepsilon \wedge \rho^\varepsilon < T\}} |\varphi \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} - \varphi \cdot S_T|] \\ &= O(\varepsilon). \end{aligned}$$

Together, we obtain

$$\mathbf{E} [G^{(14)}] = yp \mathbf{E}_{\mathbf{Q}} [(\Delta \varphi^2 - (\Delta \varphi^+)^2) \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon}] + O(\varepsilon). \quad (2.103)$$

Moreover, (2.101) and (2.89) yield

$$\mathbf{E} [G^{(15)}] = \frac{y}{2p} \mathbf{E}_{\mathbf{Q}} [(\bar{Z}_T^\varepsilon - 1)^2] = \frac{yp}{2} \mathbf{E}_{\mathbf{Q}} [\Delta \bar{\varphi}^2 \cdot [S, S]_T] + O(\varepsilon). \quad (2.104)$$

Step 2. let us pass to the remainder term $G^{(16)}$. By (2.84) and (2.90), we have

$$\left| \mathbf{E} [G^{(16)}] \right| \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} [|\bar{Z}_T^\varepsilon - 1|^3] = O(\varepsilon). \quad (2.105)$$

Step 3. Finally, from (2.99, 2.103, 2.104, 2.105) we obtain

$$\begin{aligned} &\mathbf{E} [\tilde{U}(yZ_T \bar{Z}_T^\varepsilon)] + xy \\ &= \mathbf{E} [U(x + \varphi \cdot S_T)] - yp \mathbf{E}_{\mathbf{Q}} \left[\left((\Delta \varphi^+)^2 - \frac{3}{2} \Delta \varphi^2 \right) \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon). \end{aligned}$$

Combining this with (2.91) and the argument in (2.80), the assertion follows. \square

2.5.4 Optimality

Having calculated both the primal and dual value of the optimization problem related to the approximate shadow price, recalling the link and equivalence stated in Remark 2.5.3, we are now able to prove the leading-order optimality of the candidate strategy φ^ε .

Lemma 2.5.10. *Under Assumption 2.2.3, we have*

$$\sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} [U(X_T^{\psi, \varepsilon})] = \mathbf{E} [U(X_{\tau^\varepsilon}^{\varphi^\varepsilon, \varepsilon})] + O(\varepsilon).$$

Proof. Take an arbitrary admissible trading strategy $\psi \in \mathcal{A}^\varepsilon(x^B, x^S)$ and let $\bar{S}^\varepsilon, \bar{Z}^\varepsilon$ be as in (2.86). Since \bar{S}^ε has values in $[(1 - \varepsilon)S, (1 + \varepsilon)S]$, we have

$$x + \psi \cdot \bar{S}^\varepsilon \geq X^{\psi, \varepsilon} \geq -K \quad \text{a.s.} \quad (2.106)$$

for some $K \in \mathbb{R}_+$. Recalling Remark 2.5.7, $\bar{Z}^\varepsilon(x + \psi \cdot \bar{S}^\varepsilon)$ is a \mathbf{Q} -local martingale and hence a \mathbf{Q} -supermartingale by (2.106). Therefore,

$$\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon(x + \psi \cdot \bar{S}_T^\varepsilon) \right] \leq x. \quad (2.107)$$

Together, we conclude

$$\begin{aligned} \mathbf{E} [U(X_T^{\psi, \varepsilon})] &\stackrel{(2.106)}{\leq} \mathbf{E} \left[U(x + \psi \cdot \bar{S}_T^\varepsilon) \right] \\ &\stackrel{(2.87)}{\leq} \mathbf{E} \left[\tilde{U}(y Z_T \bar{Z}_T^\varepsilon) \right] + y \mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon(x + \psi \cdot \bar{S}_T^\varepsilon) \right] \\ &\stackrel{(2.107)}{\leq} \mathbf{E} \left[\tilde{U}(y Z_T \bar{Z}_T^\varepsilon) \right] + xy \\ &\stackrel{\text{Lemma 2.5.9}}{=} \mathbf{E} [U(x + \varphi \cdot S_T)] \\ &\quad - yp \mathbf{E}_{\mathbf{Q}} \left[\left((\Delta \varphi^+)^2 - \frac{3}{2} \Delta \varphi^2 \right) \cdot [S, S]_T \right] + O(\varepsilon) \\ &\stackrel{\text{Lemma 2.5.6}}{=} \mathbf{E} \left[U(X_{T^\varepsilon}^{\varphi^\varepsilon, \varepsilon}) \right] + O(\varepsilon). \end{aligned}$$

Since $\varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]} \in \mathcal{A}^\varepsilon(x^B, x^S)$ by definition, this proves the assertion. \square

Finally, we apply the ergodic argument in Section 4.3 in order to express the minimum utility loss caused by transaction costs in terms of $\Delta \varphi^+$ rather than both $\Delta \varphi^+$ and $\Delta \varphi$, cf. (2.73, 2.98).

Corollary 2.5.11. *Under Assumption 2.2.3, we have*

$$\sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} [U(X_T^{\psi, \varepsilon})] = \mathbf{E} [U(x + \varphi \cdot S_T)] - \frac{yp}{2} \mathbf{E}_{\mathbf{Q}} [(\Delta \varphi^+)^2 \cdot [S, S]_T] + o(\varepsilon^{2/3})$$

and hence

$$\sup_{\psi \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{CE}(X_T^{\psi, \varepsilon}) = \mathbf{CE}(x + \varphi \cdot S_T) - \frac{p}{2} \mathbf{E}_{\mathbf{Q}} [(\Delta \varphi^+)^2 \cdot [S, S]_T] + o(\varepsilon^{2/3}).$$

Proof. The proof relies on the ergodic property of reflected Brownian motion stated in Section 4.3. More specifically, we are aiming to apply Lemma 4.3.4 to our context with $(\Delta \varphi, K, L, \Delta Y)$ on (Ω, \mathbf{P}) in Section 4.3 corresponding to $(\Delta \varphi, (\Delta \varphi^+ / \varepsilon^{1/3})^2 c^{S, S}, 0, 0)$ on (Ω, \mathbf{Q}) in the present setting. We therefore verify Assumption 4.2.1 and Condition (4.17). Note that our strategy φ^ε characterized by (2.33, 2.34, 2.35) corresponds to the process φ^δ following the dynamics of (4.1, 4.2, 4.3) for $\delta \triangleq (\varepsilon/2)^{1/3}$, $\Delta \Phi \triangleq 0$, $\beta \triangleq \beta$. Hence it suffices to show that Condition (2.2) implies

$$\left\| \frac{c^{\beta, \beta}}{c^{\varphi, \varphi}} \right\|_{S^1(\mathbf{Q})} + \left\| \frac{\beta b^\varphi}{c^{\varphi, \varphi}} \right\|_{S^1(\mathbf{Q})} + \left\| \frac{\beta b^\beta}{c^{\varphi, \varphi}} \right\|_{S^1(\mathbf{Q})} + \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \beta_t^2 c_t^{S, S} dt \right] < \infty,$$

which follows from Itô's formula applied to $\varphi = \pi/S$ and $\beta = 3c^{\varphi, \varphi}/(pc^{S, S})S$ and Lemma A.2. Therefore, Lemma 4.3.4 can be applied, which yields

$$\mathbf{E}_{\mathbf{Q}} [\Delta\varphi^2 \bullet [S, S]_T] = \frac{1}{3} \mathbf{E}_{\mathbf{Q}} [(\Delta\varphi^+)^2 \bullet [S, S]_T] + o(\varepsilon^{2/3}). \quad (2.108)$$

The assertion in turn is a consequence of Lemma 2.5.10 in conjunction with (2.73, 2.108) and Taylor expansion of $y \mapsto -\frac{1}{p} \ln(-y)$ at $\mathbf{E}[U(x + \varphi \bullet S_T)]$. \square

Chapter 3

General Utility

In order to continue the study on the utility maximization problem addressed in the introduction, this chapter provides a rigorous proof of the main results of Kallsen and Muhle-Karbe [19] concerning an explicit leading-order optimal trading-consumption policy and the associated welfare under small proportional transaction costs. Here, a market with one risky asset modelled by a general Itô process and utility functions on \mathbb{R}_+ with bounded relative risk aversion are considered. Assuming the existence of the generalized risk-tolerance process introduced in [19], we apply the duality theory of [22, 5] in order to obtain a parallel theorem to Theorem 2.2.4.

The procedure here resembles that in Chapter 2. Constructing a suitable dual variable to the frictionless market at an auxiliary shadow price enables us to find an upper bound to the original optimization problem matching the expected utility generated by the proposed candidate strategy in [19] to the leading order, which obviously yields the desired approximate optimality. In contrast to Chapter 2, however, the concerned dual variable is in general not necessarily the minimal martingale measure for the shadow price. Instead, it depends on the generalized risk-tolerance process introduced in [19]. The existence of such a generalized risk-tolerance process is not proved here. Since we are dealing with general utility functions, our rigorous proof also relies on some convex analytical considerations by Kramkov and Sîrbu [25] assuming a bounded relative risk aversion. The literature related to the subject of this chapter is provided in Chapter 1.

In the subsequent section, we recall the setup and address the utility maximization problem allowing for intermediate consumption. The main results are stated in Section 3.2. In Section 3.3 we verify the regularity conditions required in the main theorem for logarithmic utility in a model with general Itô process as well as power utility in a stochastic volatility model with conditionally independent increments. In Section 3.4 we compare our assumptions on the frictionless parameters to the requirements of the recent work by Soner and Touzi [38] dealing with the same issue in a Markovian framework. The proof of the main theorem is split up in several steps provided in Section 3.5.

Throughout this chapter $\mathcal{E}(H) := \exp\left(H - \frac{1}{2}[H, H]\right)$ denotes the stochastic exponen-

tial with respect to an Itô process H ; and $I_t := t$ stands for the identity process.

3.1 The market model

Let us fix a finite time horizon $T \in (0, \infty)$. Our financial market consists of one riskless asset (bond) with price normalized to 1 and one risky asset (stock) with price modelled by a general positive Itô process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$:

$$dS_t = b_t^S dt + \sigma_t^S dW_t,$$

where W is a one-dimensional standard Brownian motion and b^S, σ^S are predictable processes with

$$\int_0^T |b_t^S| + (\sigma_t^S)^2 dt < \infty \quad \text{a.s.}$$

For $\varepsilon \in (0, 1)$, the stock is supposed to be traded with proportional transaction costs εS . The price processes $(1 - \varepsilon)S$ and $(1 + \varepsilon)S$ for buying and selling one share are called the bid- and ask-price, respectively.

Definition 3.1.1. 1. A *trading strategy* (ψ^0, ψ) is considered to be a two-dimensional predictable process of finite variation with ψ_t^0 and ψ_t representing the number of shares held in the bank account and in stock at time t , respectively. The *consumption rate* k is a non-negative adapted process satisfying

$$\int_0^T k_t dt < \infty \quad \text{a.s.}$$

2. A trading-consumption policy (ψ^0, ψ, k) is called *self financing* if

$$d\psi_t^0 = (1 - \varepsilon)S_t d\psi_t^\downarrow - (1 + \varepsilon)S_t d\psi_t^\uparrow - k_t dt.$$

Here, ψ_t^\uparrow and ψ_t^\downarrow stand for the number of shares bought and sold by time t , respectively. Mathematically, $\psi^\uparrow, \psi^\downarrow$ are increasing processes which do not increase at the same time and such that $\psi = \psi^\uparrow - \psi^\downarrow$. In the sequel, a self-financing trading-consumption policy (ψ^0, ψ, k) for given initial value ψ_0^0 is identified with (ψ, k) for simplification of the notation.

3. In order to value a self-financing portfolio/consumption pair (ψ, k) , we consider the *liquidation wealth process* defined as

$$X^\varepsilon(\psi, k) := \psi^0 + \psi S - |\psi| S \varepsilon.$$

4. We call a self-financing portfolio/consumption pair (ψ^0, ψ) *admissible* for initial wealth $(x^B, x^S) \in \mathbb{R}^2$ in the bank account and the stock, respectively, and write

$$(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)$$

if $x^B = \psi_0^0$, $x^S = \psi_0 S_0$, and if the related liquidation wealth is positive, i.e.,

$$X^\varepsilon(\psi, k) > 0.$$

Let $U \in \{U_1(t, \cdot) \text{ for } t \in [0, T], U_2\}$ be our utility function which is defined on the positive real line, strictly concave, strictly increasing and continuously differentiable. Our purpose is to maximize the expected utility from consumption and terminal wealth, i.e., to find

$$\arg \sup_{(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(X_T^\varepsilon(\psi, k)) \right],$$

where¹ $\iota \in \{0, 1\}$ (the mapping $t \mapsto U_1(t, \cdot)$ is considered to be measurable). We also consider the *minimum certainty equivalent loss* $\Delta \text{CE}^\varepsilon$ which refers to the initial capital necessary to balance the effect of transaction costs. Mathematically, $\Delta \text{CE}^\varepsilon$ is defined implicitly by

$$\begin{aligned} & \sup_{(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(X_T^\varepsilon(\psi, k)) \right] \\ &= \sup_{(\psi, k) \in \mathcal{A}^0(x^B + x^S - \Delta \text{CE}^\varepsilon)} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(x^B + x^S - \Delta \text{CE}^\varepsilon + \psi \cdot S_T - k \cdot I_T) \right], \end{aligned} \quad (3.1)$$

where $\mathcal{A}^0(z)$ denotes the set of admissible investment-consumption strategies (ψ, k) with initial capital z in the frictionless market in the sense of Karatzas and Žitković [22], that is, ψ is a predictable, S -integrable process and k is non-negative adapted such that $z + \psi \cdot S - k \cdot I \geq 0$.

3.2 Main results

This section contains the main theorem of this chapter which states the approximate optimality of the candidate trading-consumption policy proposed in [19]. In order to provide a rigorous proof, we impose the following regularity conditions on the frictionless market.

¹Throughout this chapter, we use the convention that $\iota \Theta = 0$ for $\iota = 0$ if the object Θ is not well defined in case of $\iota = 0$.

Assumption 3.2.1. Let $U \in \{U_1(t, \cdot) \text{ for } t \in [0, T], U_2\}$. As in [25, Assumption 1], the relative risk aversion is assumed to be bounded, that is,

$$-\frac{U''(x)x}{U'(x)} \in (a_1, a_2) \quad \text{for some } a_1, a_2 > 0 \text{ and all } x > 0. \quad (3.2)$$

Let $x := x^B + x^S$ denote the initial wealth before liquidation. Suppose that the frictionless price process S allows for an equivalent local martingale measure (*EMM*) and the value function of the frictionless utility maximization problem has finite value in x . According to Kramkov and Sîrbu [25, Lemma 3], utility functions with bounded relative risk aversion fulfill both the Inada conditions and the requirement that the asymptotic elasticity of U is strictly less than one. Then by Karatzas and Žitković [22, Theorem 3.10], there exist a predictable S -integrable process φ , a non-negative adapted process κ and a finitely additive measure \mathbf{Q} with regular part \mathbf{Q}^r (cf. [22, Proposition 2.1] for accurate definition) such that

$${}_t \frac{d(\mathbf{Q}|_{\mathcal{F}_t})^r}{d\mathbf{P}|_{\mathcal{F}_t}} = {}_t \frac{U_1'(t, \kappa_t)}{y}, \quad t \in [0, T), \quad \frac{d\mathbf{Q}^r}{d\mathbf{P}} = \frac{U_2'(x + \varphi \cdot S_T - \kappa \cdot I_T)}{y}$$

with

$${}_t y = {}_t \mathbf{E}[U_1'(t, \kappa_t)], \quad y = \mathbf{E}[U_2'(x + \varphi \cdot S_T - \kappa \cdot I_T)],$$

where (φ, κ) interpreted as number of shares and consumption rate, respectively, is the optimal solution to the frictionless counterpart of the above utility maximization problem and \mathbf{Q} minimizes the related dual problem.

Assumption 3.2.2. We suppose that the dual optimizer \mathbf{Q} to the frictionless utility maximization of consumption and terminal wealth is an equivalent local martingale measure (*EMM*) and thus $\mathbf{Q}^r = \mathbf{Q}$.

In the sequel, the density process of \mathbf{Q} is denoted by Z . For any Itô process X , we write b^X and $c^{X,X}$ for its local \mathbf{Q} -drift and quadratic variation, respectively, i.e.,

$$dX_t = b_t^X dt + dM_t^{X,\mathbf{Q}}, \quad c_t^{X,X} := \frac{d[X, X]_t}{dt},$$

where $M^{X,\mathbf{Q}}$ is a continuous \mathbf{Q} -local martingale starting in 0. For Itô processes X and Y , their local covariation is denoted by

$$c_t^{X,Y} := \frac{d[X, Y]_t}{dt}.$$

The local drift rate of Itô process X relative to \mathbf{P} rather than \mathbf{Q} is written as $b^{X,\mathbf{P}}$.

Assumption 3.2.3. We assume the existence of the *risk-tolerance process* R in [19], that is, a positive Itô process R satisfying

$$b^R = \frac{1}{R} \left(c^{R,R} - \frac{(c^{R,S})^2}{c^{S,S}} \right) - r, \quad R_T = - \frac{U_2'(x + \varphi \cdot S_T - \kappa \cdot I_T)}{U_2''(x + \varphi \cdot S_T - \kappa \cdot I_T)} \quad (3.3)$$

(cf. [19, Equation (B.15)]), where r stands for the risk tolerance with respect to the current consumption:

$$r_t := -1 \frac{U_1'(t, \kappa_t)}{U_1''(t, \kappa_t)}$$

for $t \in [0, T)$. In line with [19], we also define the sensitivity parameters

$$\varphi' := \frac{c^{R,S}}{R c^{S,S}}, \quad \kappa' := \frac{r}{R}$$

and suppose that φ' is an Itô process.

In case of $t = 0$ and $b^R = 0$, R coincides with the risk-tolerance wealth process in [26, Definition 2] which exists either if the market is complete or if the utility function is of power or exponential type, cf. [26, Remark 4]. Applying the Galtchouk-Kunita-Watanabe decomposition to the \mathbf{Q} -local martingale part of R , we obtain

$$dR_t = b_t^R dt + \frac{c_t^{R,S}}{c_t^{S,S}} dS_t + dR_t^\perp,$$

where R^\perp is a \mathbf{Q} -local martingale with $[R^\perp, S] = 0$. In particular, it holds that

$$c^{R,R^\perp} = c^{R^\perp,R^\perp} = R(b^R + r).$$

According to [19], the parameters φ' and κ' model the sensitivity of the frictionless optimal trading and consumption rate to the current wealth, respectively.

Assumption 3.2.4. The frictionless optimizer φ and the *activity rate*

$$\rho := \frac{c^{\varphi,\varphi}}{c^{S,S}}$$

are assumed to be well-defined Itô processes. Moreover, we suppose that $c^{\varphi,\varphi} > 0$.

Due to the nature of the utility functions which we are dealing with and for the sake of interpretation in concrete models, we introduce the following normalized processes:

- the *stock return process* Y with $dS_t = S_t dY_t$,
- the *frictionless optimal wealth process* $X(\varphi, \kappa) := x + \varphi \cdot S - \kappa \cdot I$,

- the risky fraction $\pi := \frac{\varphi^S}{R}$,
- the normalized activity rate

$$\eta := \frac{3\rho S^4}{2R^2} = \frac{3}{2} \left(\pi^2 + \frac{\pi^2 c^{R,R}}{R^2 c^{Y,Y}} + \frac{c^{\pi,\pi}}{c^{Y,Y}} - \frac{2\pi^2 c^{Y,R}}{R c^{Y,Y}} - \frac{2\pi c^{Y,\pi}}{c^{Y,Y}} + \frac{2\pi c^{R,\pi}}{R c^{Y,Y}} \right).$$

Assumption 3.2.5. We assume that

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |H_t|^n \right] + \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} U_2'(X_t(\varphi, \kappa))^8 \right] + \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} Z_t^{\pm 8} \right] \\ & + \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |\iota U_1(t, \kappa_t)|^4 \right] + \mathbf{E}_{\mathbf{Q}} \left[\sup_{t, s \in [0, T]} \left| \iota U_1 \left(t, \frac{X_s(\varphi, \kappa)}{4T} \right) \right|^4 \right] < \infty \end{aligned} \quad (3.4)$$

for any $n \in \mathbb{N}$ and any $H \in \mathcal{H}_1 \cup \mathcal{H}_2$, where

$$\begin{aligned} \mathcal{H}_1 & := \left\{ \pi, \eta^{\pm 1}, b^\pi, b^\eta, (c^{Y,Y})^{\pm 1}, c^{\pi,\pi}, c^{\eta,\eta}, R^{\pm 1}, b^R, c^{R,R}, S b^{\varphi'}, S^2 c^{\varphi', \varphi'}, \iota r^{\pm 1} \right\}, \\ \mathcal{H}_2 & := \left\{ X(\varphi, \kappa)^{\pm 1}, \left[\mathcal{E} \left(\varphi' \cdot S - \kappa' \cdot I - \frac{1}{R} \cdot R + \frac{1}{R^2} \cdot [R, R] \right) \right]^{\pm 1}, \left[\mathcal{E} \left(-\frac{1}{R} \cdot R^\perp \right) \right]^{\pm 1} \right\}. \end{aligned}$$

Further, the process $c^{\varphi, \varphi}$ is assumed to have continuous paths.

The subsequent theorem presents the main results of this chapter concerning the general structure of leading-order utility maximization under small proportional transaction costs. It parallels Theorem 2.2.4 of the previous chapter for exponential utility.

Theorem 3.2.6. Suppose that Assumptions 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5 hold. Let

$$\Delta\varphi^\pm := \pm \left(\frac{3}{2} R \rho S \varepsilon \right)^{1/3} = \pm \frac{(\eta \varepsilon)^{1/3} R}{S}.$$

1. There exist continuous adapted processes $\varphi^\varepsilon, \kappa^\varepsilon$ with

$$\begin{aligned} \varphi^\varepsilon & = \varphi + \varphi'((\varphi^\varepsilon - \varphi) \cdot S - (\kappa^\varepsilon - \kappa) \cdot I) + \Delta\varphi = \varphi^{\varepsilon\uparrow} - \varphi^{\varepsilon\downarrow}, \\ \kappa^\varepsilon & = \kappa + \kappa'((\varphi^\varepsilon - \varphi) \cdot S - (\kappa^\varepsilon - \kappa) \cdot I), \end{aligned}$$

where

$$\varphi_0^\varepsilon = \begin{cases} \varphi_0 + \Delta\varphi_0^+ & \text{if } x^S > (\varphi_0 + \Delta\varphi_0^+) S_0, \\ \varphi_0 + \Delta\varphi_0^- & \text{if } x^S < (\varphi_0 + \Delta\varphi_0^-) S_0, \\ x^S / S_0 & \text{otherwise,} \end{cases}$$

$\Delta\varphi$ takes values in $[\Delta\varphi^-, \Delta\varphi^+]$, and $\varphi^{\varepsilon\uparrow}, \varphi^{\varepsilon\downarrow}$ are increasing process such that

$$\varphi^{\varepsilon\uparrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^-\} \subseteq \Omega \times [0, T],$$

$$\varphi^{\varepsilon\downarrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^+\} \subseteq \Omega \times [0, T].$$

2. In order to ease the notation, let $(\varphi^\varepsilon, \kappa^\varepsilon)$ represent the unique self-financing trading-consumption policy (ψ^0, ψ, k) with $\psi_0^0 = x^B$, $\psi_0 S_0 = x^S$, $(\psi_t, k_t) = (\varphi^\varepsilon, \kappa^\varepsilon)$ for $t \in (0, T]$. Define

$$\tau^\varepsilon := \inf \left\{ t \in [0, T] : \left| \frac{X_t^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_t(\varphi, \kappa)}{X_t(\varphi, \kappa)} \right| > \frac{1}{2} \text{ or } \left| \frac{\kappa_t^\varepsilon - \kappa_t}{\kappa_t} \right| > \frac{1}{2} \right\} \wedge T. \quad (3.5)$$

Then $\mathbf{P}(\tau^\varepsilon = T) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Moreover, setting

$$\tilde{\varphi}^\varepsilon := \varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}, \quad \tilde{\kappa}^\varepsilon := \kappa^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]} + \frac{X_{\tau^\varepsilon}(\varphi, \kappa)}{4T} \mathbf{1}_{[\tau^\varepsilon, T]}, \quad (3.6)$$

$(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ is an admissible utility-maximizing portfolio/consumption pair to the leading order $O(\varepsilon^{2/3})$, i.e.,

$$\begin{aligned} & \sup_{(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(X_T^\varepsilon(\psi, k)) \right] \\ &= \mathbf{E} \left[\int_0^T \iota U_1(t, \tilde{\kappa}_t^\varepsilon) dt + U_2(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)) \right] + o(\varepsilon^{2/3}). \end{aligned}$$

(Analogously, $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ stands for the portfolio/consumption pair $(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)$ with $(\psi_t, k_t) = (\tilde{\varphi}_t^\varepsilon, \tilde{\kappa}_t^\varepsilon)$ for $t \in (0, T]$.)

3. The minimum certainty equivalent loss amounts to

$$\Delta \mathbf{CE}^\varepsilon = \frac{1}{2} \mathbf{E}_Q \left[\frac{(\Delta \varphi^+)^2}{R} \cdot [S, S]_T \right] + o(\varepsilon^{2/3}).$$

Proof. As in Chapter 2, the proof is done in several steps in Section 3.5. The existence of our candidate portfolio/consumption pair $(\varphi^\varepsilon, \kappa^\varepsilon)$ is related to a multi-dimensional Skorohod problem (cf. Lemma 3.5.2). The utility generated by $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ is approximated by means of an auxiliary shadow price process S^ε , exploiting the equivalence of following strategy φ^ε in the market with transaction costs and in the frictionless market at price S^ε (cf. Corollary 3.5.3 and Section 3.5.2). The optimality of $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ relies on some dual considerations (cf. Section 3.5.3) as well as the ergodic property of reflected Brownian motion stated in Chapter 4 (cf. Section 3.5.4). \square

Note that the stopping argument in (3.5, 3.6) is essential for our approach. Focusing on utility functions defined on \mathbb{R}_+ , we follow the trading and consumption policy $(\varphi^\varepsilon, \kappa^\varepsilon)$ only by time τ^ε in order to prevent negative wealth. In addition, up to τ^ε , the losses are bounded so that some convex analytical considerations of Kramkov and Sîrbu [25] can be applied to our context (cf. Lemma 3.5.7 and Lemma 3.5.10 for more details).

According to Theorem 3.2.6, it is to the leading order optimal to keep the number of shares φ^ε within a random corridor called no-trade region at a half-width of $\Delta \varphi^+$.

In general, the midpoint of the corridor refers to the frictionless optimizer φ with an offset accounting for the wealth effect of the past transaction costs expressed in terms of $\varphi'((\varphi^\varepsilon - \varphi) \cdot S - (\kappa^\varepsilon - \kappa) \cdot I)$. Note that from the self-financing restriction, we have

$$X^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X(\varphi, \kappa) = (\varphi^\varepsilon - \varphi) \cdot S - (\kappa^\varepsilon - \kappa) \cdot I - \varepsilon S \cdot (\varphi^{\varepsilon\uparrow} + \varphi^{\varepsilon\downarrow}) - \varepsilon |\varphi^\varepsilon| S;$$

thus to the leading order, $(\varphi^\varepsilon - \varphi) \cdot S - (\kappa^\varepsilon - \kappa) \cdot I$ can be calculated by the loss in liquidation wealth $X^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X(\varphi, \kappa)$ plus the incurred transaction costs $\varepsilon S \cdot (\varphi^{\varepsilon\uparrow} + \varphi^{\varepsilon\downarrow})$. For a general asymptotic analysis of the trading volume and transaction costs related to “aisle-type” strategies like φ^ε , the reader is referred to Chapter 4.

3.3 Examples

In order to present applications of Theorem 3.2.6, in this section we consider some concrete models and specific utility functions such that the frictionless optimizer (φ, κ) is known explicitly.

3.3.1 Logarithmic utility

Let $U_1(t, \cdot) := e^{-\delta t} \ln(\cdot)$ for all $t \in [0, T]$, $U_2 := \ln$ for some $\delta \in \mathbb{R}_+$. Consider the model with stock price S determined by

$$dS_t = S_t(b_t dt + \sigma_t dW_t),$$

where W is a standard Brownian motion and the processes b, σ are continuous, bounded and bounded away from 0. We assume additionally that the local \mathbf{P} -drift and quadratic variation of σ, b are bounded. Further, setting $\bar{\pi} := b/\sigma^2$, the processes $c^{\bar{\pi}, \bar{\pi}}, c^{\bar{\pi}, W}$ are assumed to be continuous.

Standard martingale method (cf. e.g. [11, Theorem 3.1, Example 4.2]) yields that the *Merton proportion* for the frictionless problem equals

$$\frac{\varphi_t S_t}{X_t(\varphi, \kappa)} = \frac{b_t}{\sigma_t^2} \quad \text{for all } t \in [0, T]$$

and the corresponding dual optimizer is the minimal *EMM* with density process

$$Z = \mathcal{E} \left(-\frac{b}{\sigma} \cdot W \right).$$

Therefore, the frictionless optimal consumption rate and wealth process are of the form

$$\kappa_t = \frac{t e^{-\delta t}}{t(1 - \frac{1}{\delta} e^{-\delta T}) + 1} x_{\mathcal{E}} \left(\frac{b}{\sigma} \cdot W^{\mathbf{Q}} \right)_t,$$

$$X_t(\varphi, \kappa) = \frac{\iota \left(\frac{1}{\delta} e^{-\delta t} - \frac{1}{\delta} e^{-\delta T} \right) + 1}{\iota \left(1 - \frac{1}{\delta} e^{-\delta T} \right) + 1} x \mathcal{E}^{\mathbf{Q}} \left(\frac{b}{\sigma} \cdot W^{\mathbf{Q}} \right)_t,$$

respectively, where $W^{\mathbf{Q}} := W + \int_0^\cdot (b_t/\sigma_t) dt$ is a standard Brownian motion under the measure \mathbf{Q} . Since $-U''(x)x/U'(x) = 1$ for $U \in \{U_1(t, \cdot) \text{ for } t \in [0, T], U_2\}$ and all $x > 0$, the risk-tolerance process R and the risk tolerance with respect to the consumption rate read as

$$R = X(\varphi, \kappa), \quad r = \iota \kappa,$$

respectively. Hence by definition, we have

$$\begin{aligned} \pi &= \frac{b}{\sigma^2}, & \eta &= \frac{3}{2} \left(\pi^2 (1 - \pi)^2 - 2\pi(1 - \pi) \frac{c^{\pi, Y}}{c^{Y, Y}} + \frac{c^{\pi, \pi}}{c^{Y, Y}} \right), \\ \kappa' &= \frac{\kappa}{X(\varphi, \kappa)}, & \varphi' &= \frac{\varphi}{X(\varphi, \kappa)} = \frac{b}{\sigma^2 S}, & b^R &= -r, & R^\perp &= 0, \end{aligned}$$

and

$$\varphi' \cdot S - \kappa' \cdot I - \frac{1}{R} \cdot R + \frac{1}{R^2} \cdot [R, R] = \pi \cdot Y - \pi \cdot Y + (\pi \varphi' S) \cdot [Y, Y] = \frac{b^2}{\sigma^2} \cdot I.$$

Together, using Itô's formula and due to the boundedness assumptions on b, σ , it can be shown that all processes in Condition (3.4) are either bounded or in form of

$$\bar{H} \exp(H \cdot W^{\mathbf{Q}}) \quad \text{with bounded } \bar{H}, H.$$

Note that if the processes \bar{H}, H are bounded by $m \in \mathbb{R}_+$, it holds for any $n \in \mathbb{N}$ that

$$\mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} \left| \bar{H}_t \exp(H \cdot W_t^{\mathbf{Q}}) \right|^n \right] \leq m^n \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} \underbrace{\mathcal{E}^{\mathbf{Q}}(nH \cdot W^{\mathbf{Q}})_t}_{=: M_t} \right] \exp\left(\frac{nm^2 T}{2}\right). \quad (3.7)$$

By boundedness of H , the process M in (3.7) is an L^2 -martingale under measure \mathbf{Q} ; Doob's quadratic inequality yields that it is a uniformly integrable \mathbf{Q} -martingale. Therefore, Assumptions 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5 are fulfilled. Then from Theorem 3.2.6, the no-trade bounds are given by

$$\frac{\Delta \varphi_t^\pm S_t}{X_t(\varphi, \kappa)} = \pm (\eta_t \varepsilon)^{1/3} \quad \text{for all } t \in [0, T]$$

and the certainty equivalent loss is equal to

$$\frac{1}{2} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T X_t(\varphi, \kappa) \eta_t^2 \sigma_t^2 dt \right] \varepsilon^{2/3} + o(\varepsilon^{2/3}).$$

3.3.2 Power utility

We now focus on the problem without consumption with utility function of power type, i.e., $\iota = 0$, $U_2(x) = x^{1-p}/(1-p)$ for some $p \in \mathbb{R}_+ \setminus \{0, 1\}$. Consider the following stochastic volatility model:

$$dS_t = S_t \left(b(\tilde{Y}_t) dt + \sigma(\tilde{Y}_t) dW_t \right), \quad (3.8)$$

where the functions $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, \tilde{Y} stands for an Itô process which is independent of the Brownian motion W . Here, we suppose that the filtration is generated by W and \tilde{Y} .

Proposition 3.3.1. *Let us assume that the stochastic volatility model (3.8) fulfills the following requirements:*

- the processes $b(\tilde{Y})$, $\sigma(\tilde{Y})$ are bounded and bounded away from 0,
- letting

$$\pi := \frac{b(\tilde{Y})}{\sigma(\tilde{Y})^2}, \quad \eta := \frac{3}{2} \left(\pi^2 (1 - \pi)^2 + \frac{c^{\pi, \pi}}{c^{Y, Y}} \right), \quad (3.9)$$

the processes π and η are Itô processes with bounded coefficients $b^{\pi, \mathbf{P}}, c^{\pi, \pi}, b^{\eta, \mathbf{P}}, c^{\eta, \eta}$ and continuous $c^{\pi, \pi}$,

- defining

$$\tilde{Z}_t := \mathbf{E} \left[\exp \left(-\frac{1}{2} \int_0^T \left(\frac{b(\tilde{Y}_t)}{\sigma(\tilde{Y}_t)} \right)^2 dt \right) \middle| \mathcal{F}_t \right], \quad (3.10)$$

the process $c^{\tilde{Z}, \tilde{Z}} / \tilde{Z}^2$ is bounded.

Then Assumptions 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5 are fulfilled. Moreover, the processes π and η coincide with the risky fraction and normalized activity rate, respectively. Setting $X := x + \varphi \bullet S$, the no-trade boundary reads as

$$\frac{\Delta \varphi^\pm S}{X} = \pm \frac{1}{p} (\eta \varepsilon)^{1/3}$$

and the minimum certainty equivalent loss equals

$$\frac{1}{2p} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T X_t \eta_t^2 \sigma(\tilde{Y}_t)^2 dt \right] \varepsilon^{2/3} + o(\varepsilon^{2/3}).$$

Proof. Step 1. Due to [18, Theorem 3.1], the frictionless optimal stock holdings satisfy

$$\frac{\varphi_t S_t}{X_t} = \frac{b(\tilde{Y}_t)}{p \sigma(\tilde{Y}_t)^2} \quad \text{for all } t \in [0, T].$$

Thus, the frictionless optimal wealth process is

$$X = x \mathcal{E} \left(\frac{b(\tilde{Y})}{p\sigma(\tilde{Y})} \bullet W + \int_0^\cdot \frac{b(\tilde{Y}_t)^2}{p\sigma(\tilde{Y}_t)^2} dt \right).$$

From the conjugate relation, the related dual optimizer satisfies $yZ_T = X_T^{-p}$. Since $-U_2''(x)x/U_2'(x) = p$ for all $x > 0$, the risk-tolerance wealth process is $R = X/p$. Further, as shown in Proposition 2.3.2, it holds that $c^{\pi,Y} = c^{\pi,R} = 0$. Hence by definition, π and η defined in (3.9) are the corresponding risky fraction and activity rate and we have

$$\varphi' = \frac{\varphi}{X} = \frac{b(\tilde{Y})}{p\sigma(\tilde{Y})^2 S}.$$

Step 2. Define filtration $\mathbf{G} = (\mathcal{G}_t)_{t \in [0, T]}$ by

$$\mathcal{G}_t := \bigcap_{s>t} \sigma \left(\mathcal{F}_s \cup \sigma((\tilde{Y}_r)_{r \in [0, T]}) \right), \quad t \in [0, T],$$

and set

$$\bar{Z} := \mathcal{E} \left(-\frac{b(\tilde{Y})}{\sigma(\tilde{Y})} \bullet W \right).$$

From the definition of filtration \mathbf{G} , the random variable \tilde{Z}_T defined by (3.10) is \mathcal{G}_0 -measurable. Due to [1, Theorem 15.5], since the processes \tilde{Y} and W are independent, W is also a standard Brownian motion relative to \mathbf{G} . Since the process $b(\tilde{Y})/\sigma(\tilde{Y})$ is assumed to be bounded, the local martingale \bar{Z} satisfies Novikov's condition, and thus it is a martingale with respect to both \mathbf{F} and \mathbf{G} . Hence, it holds for any $t \in [0, T]$ that

$$\begin{aligned} Z_t^{\mathbf{G}} &:= \mathbf{E} \left[X_T^{-p} \middle| \mathcal{G}_t \right] \\ &= \mathbf{E} \left[x^{-p} \exp \left(-\int_0^T \frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} dW_s - \int_0^T \left(\frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} \right)^2 ds \right) \middle| \mathcal{G}_t \right] \\ &= \mathbf{E} \left[x^{-p} \tilde{Z}_T \bar{Z}_T \middle| \mathcal{G}_t \right] \\ &= \underbrace{x^{-p} \tilde{Z}_T}_{\mathcal{G}_0\text{-measurable}} \bar{Z}_t. \end{aligned}$$

In particular, $Z_T^{\mathbf{G}}$ is integrable. From Step 1, the normalized \mathbf{G} -martingale $Z^{\mathbf{G}}$ is the \mathbf{G} -density process of the probability measure \mathbf{Q} . Taking the conditional expectation of $Z^{\mathbf{G}}$ with respect to the filtration \mathbf{F} , we deduce that $\bar{Z}\tilde{Z} = Z$.

Step 3. For any Itô process H , using Girsanov's theorem twice, we deduce

$$b^H = b^{H, \mathbf{P}} + \frac{c^{\tilde{Z}, H}}{\tilde{Z}} + \frac{c^{\bar{Z}, H}}{\bar{Z}}$$

with

$$\left| \frac{c^{\tilde{Z},H}}{\tilde{Z}} + \frac{c^{\bar{Z},H}}{\bar{Z}} \right| \leq \sqrt{\frac{c^{\tilde{Z},\tilde{Z}}}{\tilde{Z}^2}} \sqrt{c^{H,\bar{H}}} + \sqrt{\frac{c^{\bar{Z},\bar{Z}}}{\bar{Z}^2}} \sqrt{c^{H,\bar{H}}} = \left(\sqrt{\frac{c^{\tilde{Z},\tilde{Z}}}{\tilde{Z}^2}} + \left| \frac{b(\tilde{Y})}{\sigma(\tilde{Y})} \right| \right) \sqrt{c^{H,\bar{H}}}.$$

Thus, our boundedness assumptions ensure that b^π and b^η are bounded. Therefore, all processes in Condition (3.4) are either bounded or of the form

$$\bar{H} \exp(H \cdot W^{\mathbf{Q}}) \quad \text{with bounded } \bar{H}, H,$$

and hence, as shown in the previous example (cf. (3.7)), the assertion follows. \square

3.4 Comments on the regularity conditions

In order to compare Theorem 3.2.6 with existing literature based on stochastic control theory, in this section we focus on the Markovian setting in Section 2.4.1 and provide the Markovian counterparts of Assumptions 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5.

Let us denote by $\hat{X} := \hat{X}^{v,x,\varphi,\kappa}$ the frictionless optimal wealth process starting with x at time t following the portfolio/consumption pair (φ, κ) , i.e., $\hat{X}_r = x + \int_t^r \varphi_s dS_s - \int_t^r \kappa_s ds$. Then the value function (also called the indirect utility in [19]) representing the maximum expected utility achieved on the time interval $[t, T]$ from initial wealth x can be written as

$$v(t, v, x) = \mathbf{E} \left[\int_t^T U_1(s, \kappa_s) ds + U_2(\hat{X}_T) \mid (\Upsilon_t, \hat{X}_t) = (v, x) \right].$$

We now comment on the assumptions in Theorem 3.2.6 by formally relating them to some regularity conditions on the value function v which are commonly required in classical approaches of stochastic control:

On Assumption 3.2.1. Dealing with utility functions on the positive real line, the special case of utility with constant relative risk aversion is of particular interest. In a Markovian setting, Condition (3.2) requiring a bounded risk aversion of the utility function reminds us of the analogous assumption imposed on the value function in [38, Equation (7.1)], provided that the value function is two-times differentiable.

On Assumption 3.2.2. By the formal recursion in [19, Section B.1], up to normalization, the density process Z of the dual optimizer \mathbf{Q} refers to the marginal indirect utility evaluated along the optimal wealth process, i.e., $Z_t = \partial_x v(t, \Upsilon_t, \hat{X}_t) / \partial_x v(0, v, x)$.

On Assumption 3.2.3. In [19], the process R following the dynamics in (3.3) is expected to be the risk tolerance of the indirect utility evaluated along the optimal wealth process, i.e., $R_t = -\partial_x v(t, \Upsilon_t, \hat{X}_t) / \partial_x^2 v(t, \Upsilon_t, \hat{X}_t)$, which generalizes the risk-tolerance wealth process in [26]. Suppose that this can be rigorously proved. Itô's formula then yields that requiring R, φ' to be Itô processes refers to a sufficient regularity condition on the value function, e.g. $v \in C^{1,3,5}$.

On Assumption 3.2.4. Based on the dynamic programming principle, the Hamilton-Jacobi-Bellman equation which the value function v is expected to satisfy (cf. e.g. [30]) reads as:

$$\begin{aligned} & \partial_t v + \left(b \partial_v v + \frac{1}{2} a^2 \partial_v^2 v \right) \\ & + \sup_{k>0} (U_1(\cdot, k) - k \partial_x v) + \sup_{\theta \in \mathbb{R}} \left(\frac{1}{2} \theta^2 \sigma^2 \partial_x^2 v + \theta (\mu \partial_x v + \zeta a \sigma \partial_v \partial_x v) \right) = 0, \\ & v(T, v, x) = U_2(x). \end{aligned}$$

Suppose that the value function is a solution of this partial differential equation with terminal condition. Then the optimal consumption rate κ and the optimal stock holdings ϑ are obtained by solving the maximization problems in the Hamilton-Jacobi-Bellman equation. From convex analysis, κ is given by

$$\kappa_t = \kappa(t, \Upsilon_t, \hat{X}_t) := (U_1(t, \cdot)')^{-1}(\partial_x v(t, \Upsilon_t, \hat{X}_t)).$$

Assuming strict concavity of v in x , the optimal stock holdings ϑ refer to

$$\vartheta_t := \varphi_t S_t = \vartheta(t, \Upsilon_t, \hat{X}_t) := - \left(\frac{\mu}{\sigma^2} \frac{\partial_x v}{\partial_x^2 v} + \zeta \frac{a}{\sigma} \frac{\partial_v \partial_x v}{\partial_x^2 v} \right) (t, \Upsilon_t, \hat{X}_t). \quad (3.11)$$

Let $\mathcal{L} := \partial_t + b \partial_v + \frac{1}{2} a^2 \partial_v^2 + \vartheta^2 \sigma^2 \partial_x + \vartheta (\mu \partial_x + \zeta a \sigma \partial_v \partial_x)$. If the function $\vartheta \in C^{1,2,2}$, then from Itô's formula, the coefficients of the process ϑ in (3.11) are given by

$$\begin{aligned} b_t^{\vartheta, \mathbf{P}} &= \mathcal{L} \vartheta(t, \Upsilon_t, \hat{X}_t), \\ c_t^{\vartheta, \vartheta} &= \left(a^2 (\partial_v \vartheta)^2 + \vartheta^2 \sigma^2 (\partial_x \vartheta)^2 + 2 \zeta a \sigma \vartheta \partial_v \vartheta \partial_x \vartheta \right) (t, \Upsilon_t, \hat{X}_t). \end{aligned}$$

Moreover, since $\varphi = \vartheta/S$, integration by parts yields

$$\begin{aligned} \frac{3}{2} \eta_t R_t^2 &= \frac{c_t^{\varphi, \varphi}}{c_t^{S, S}} S_t^4 \\ &= \vartheta^2(t, \Upsilon_t, \hat{X}_t) - 2 \left(\frac{\vartheta}{\sigma} (\zeta a \partial_v \vartheta + \sigma \vartheta \partial_x \vartheta) \right) (t, \Upsilon_t, \hat{X}_t) \\ &\quad + \left(\frac{1}{\sigma^2} \left(a^2 (\partial_v \vartheta)^2 + \vartheta^2 \sigma^2 (\partial_x \vartheta)^2 + 2 \zeta a \sigma \vartheta \partial_v \vartheta \partial_x \vartheta \right) \right) (t, \Upsilon_t, \hat{X}_t) \\ &=: \tilde{\eta}(t, \Upsilon_t, \hat{X}_t) =: \tilde{\eta}_t. \end{aligned} \quad (3.12)$$

Again, assuming $\tilde{\eta} \in C^{1,2,2}$ and applying Itô's formula lead to

$$b_t^{\tilde{\eta}, \mathbf{P}} = \mathcal{L} \tilde{\eta}(t, \Upsilon_t, \hat{X}_t),$$

$$c_t^{\tilde{\eta}, \tilde{\eta}} = \left(a^2 (\partial_v \tilde{\eta})^2 + \vartheta^2 \sigma^2 (\partial_x \tilde{\eta})^2 + 2\zeta a \sigma \vartheta \partial_v \tilde{\eta} \partial_x \tilde{\eta} \right) (t, \Upsilon_t, \hat{X}_t).$$

Therefore, Assumption 3.2.4 is linked to the two-times differentiability of the functions ϑ and $\tilde{\eta}$ above and the requirement that $\tilde{\eta} > 0$. For $\zeta = 1$, the parameters $\vartheta = \pi R$ and $2/3\eta R^2$ in our context refer to \mathbf{y} and $(\alpha/\sigma)^2$ in [38, Equation (3.8)], respectively. If the relative risk aversion X/R is bounded, Assumption 3.2.4 is related to [38, Assumption 5.2 (Smoothness)].

On Assumption 3.2.5. For better exposition, here we suppose that there is no intermediate consumption and focus on the utility functions with constant relative risk aversion, i.e., $\iota = 0$ and $-xU_2''(x)/U_2'(x) = p \in \mathbb{R}_+ \setminus \{0\}$. In this case, the risk-tolerance process R is given by $X(\varphi, \kappa)/p$ and we have $R^\perp = 0$ (see [19, Section B.2] or Section 3.3 for more details). From [41, Theorems 3.2, 3.3], the value function and the optimal stock holdings are of the form

$$v(t, v, x) = \frac{x^{1-p}}{1-p} u(v, t)^\delta, \quad (3.13)$$

$$\vartheta(t, v, x) = x \left(\frac{1}{p} \frac{\mu}{\sigma^2} + \frac{\zeta}{p + \zeta^2(1-p)} \frac{a \partial_v u}{\sigma u} \right) (v, t).$$

Hence, the optimal risky fraction is

$$\pi_t = \pi(t, \Upsilon_t, \hat{X}_t) := \frac{\vartheta(t, \Upsilon_t, \hat{X}_t)}{R_t} = \left(\frac{\mu}{\sigma^2} + p \frac{\zeta}{p + \zeta^2(1-p)} \frac{a \partial_v u}{\sigma u} \right) (t, \Upsilon_t), \quad (3.14)$$

and by definition, the sensitivity parameter φ' is given by

$$\varphi'_t = \frac{1}{S_t} \left(\frac{1}{p} \frac{\mu}{\sigma^2} + \frac{\zeta}{p + \zeta^2(1-p)} \frac{a \partial_v u}{\sigma u} \right) (t, \Upsilon_t). \quad (3.15)$$

Now, let us suppose the boundedness and smoothness of the coefficient functions μ , σ , b , a , σ^{-1} and their derivatives at all orders. From (3.14, 3.12, 3.15) and applying Itô's formula as well as the same arguments as in [29, Section 8, footnote 25], the processes π , η , $b^{\pi, \mathbf{P}}$, $b^{\eta, \mathbf{P}}$, $(c^{Y, Y})^{\pm 1}$, $c^{\pi, \pi}$, $c^{\eta, \eta}$, $Sb^{\varphi', \mathbf{P}}$, $S^2 c^{\varphi', \varphi'}$ are then expected to be bounded. Moreover, as in [19, Section B.1] by formal recursion and combining the duality relation in [24], i.e., $\partial_x v(T, \Upsilon_T, \hat{X}_T) / \partial_x v(0, v, x) = Z_T$, the density process Z of the dual optimizer \mathbf{Q} is up to normalization expected to be $\partial_x v(\cdot, \Upsilon, \hat{X})$. Itô's formula and (3.13) yield

$$\frac{c_t^{Z, Z}}{Z_t^2} \leq (1-p)^2 \frac{c_t^{\hat{X}, \hat{X}}}{\hat{X}_t^2} + \delta^2 \frac{c_t^{\Upsilon, \Upsilon}}{u(\Upsilon_t, t)},$$

and so the process $c^{Z, Z}/Z^2$ is bounded due to the above arguments. Then from Girsanov's theorem, for any Itô process X , it holds that

$$b^X = b^{X, \mathbf{P}} + \frac{c^{X, Z}}{Z} \quad \text{with} \quad \left| \frac{c^{X, Z}}{Z} \right|^2 \leq c^{X, X} \frac{c^{Z, Z}}{Z^2},$$

which implies that the processes b^π , b^η , $Sb^{\phi'}$ are bounded as well and all processes in (3.4) except for η^{-1} are either bounded or of the form

$$\bar{H} \exp\left(H \cdot W^{\mathbf{Q}}\right) \quad \text{with bounded } H, \bar{H}$$

possessing bounded $S^n(\mathbf{Q})$ -norm for all $n \in \mathbb{N}$ (cf. (3.7) for more details). Finally, we consider the process η^{-1} . Since for bounded indirect relative risk tolerance $-x\partial_x^2 v / \partial_x v$, $2/3\eta R^2 = \tilde{\eta}$ corresponds to $(\alpha/\sigma)^2$ in [38, Equation (3.8)], requiring $\eta^{-1} \geq \eta^*$ for some $\eta^* \in \mathbb{R}_+ \setminus \{0\}$ in our context refers to [38, Equation (5.3)] guaranteed by [38, Assumption 5.2, Equation (5.2)]. In case of logarithmic utility, this requirement is fulfilled by imposing boundedness and regularity assumptions of the market coefficients (cf. Section 3.3.1 for more details). In general, $\zeta = 0$ provides a sufficient condition which is considered in Section 3.3.2.

3.5 Proof of the main results

In this section we prove Theorem 3.2.6. We generally proceed as in Chapter 2 and suppose throughout that Assumptions 3.2.1, 3.2.2, 3.2.3, 3.2.4 hold.

First, we recall some properties of the utility function fulfilling Assumption 3.2.1 which are stated in Kramkov and Sîrbu [25].

Remark 3.5.1. Consider $U \in \{U_1(t, \cdot) \text{ for } t \in [0, T], U_2\}$.

1. Let \tilde{U} denote the conjugate function of U , that is,

$$\tilde{U}(y) := \sup_{x>0} (U(x) - xy), \quad y > 0. \quad (3.16)$$

From the conjugate relation $-\tilde{U}' = (U')^{-1}$, Condition (3.2) is equivalent to

$$-\frac{\tilde{U}''(y)y}{\tilde{U}'(y)} \in \left(\frac{1}{a_2}, \frac{1}{a_1}\right) \quad \text{for all } y > 0. \quad (3.17)$$

2. Due to [25, Lemma 3], for any $a > 1$, there exist $b_1, b_2 \in (0, 1)$ such that

$$\frac{U'(ax)}{U'(x)} \in (b_1, b_2) \quad \text{for all } x > 0. \quad (3.18)$$

Analogously, it can be shown that for any $a > 1$, there exist $\tilde{b}_1, \tilde{b}_2 \in (0, 1)$ such that

$$\frac{\tilde{U}'(ay)}{\tilde{U}'(y)} \in (\tilde{b}_1, \tilde{b}_2) \quad \text{for all } y > 0. \quad (3.19)$$

3. The functions $x \mapsto \frac{U'(x)}{x}$ and $y \mapsto \frac{\tilde{U}'(y)}{y}$ are strictly decreasing and increasing, respectively.

3.5.1 Analysis of shadow price process S^ε

In this section we carry out a detailed analysis of the shadow price process S^ε proposed in [19] which serves as the key object of our proof.

Existence

Let us start with the existence of the candidate trading-consumption policy $(\varphi^\varepsilon, \kappa^\varepsilon)$ in Theorem 3.2.6-1 derived along with the shadow price process S^ε in [19].

Lemma 3.5.2. For $\Delta\tilde{X}_0 = 0$,

$$\Delta\varphi_0 := \begin{cases} \Delta\varphi_0^+ & \text{if } x^S > (\varphi_0 + \Delta\varphi_0^+)S_0, \\ \Delta\varphi_0^- & \text{if } x^S < (\varphi_0 + \Delta\varphi_0^-)S_0, \\ \frac{x^S}{S_0} - \varphi_0 & \text{otherwise,} \end{cases}$$

there exist continuous adapted processes $\Delta\tilde{X}, \Delta\varphi \in [\Delta\varphi^-, \Delta\varphi^+]$, φ^ε and adapted increasing processes $\varphi^{\varepsilon\uparrow}, \varphi^{\varepsilon\downarrow}$ such that

$$\varphi^{\varepsilon\uparrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^-\} \subseteq \Omega \times [0, T], \quad (3.20)$$

$$\varphi^{\varepsilon\downarrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^+\} \subseteq \Omega \times [0, T], \quad (3.21)$$

and

$$\varphi^\varepsilon = \varphi + \varphi' \Delta\tilde{X} + \Delta\varphi = \varphi^{\varepsilon\uparrow} - \varphi^{\varepsilon\downarrow}, \quad d\Delta\tilde{X}_t = \Delta\tilde{X}_t d(\varphi' \cdot S_t - \kappa' \cdot I_t) + \Delta\varphi_t dS_t. \quad (3.22)$$

Proof. Consider the \mathbb{R}^4 -valued semimartingales

$$Z := (Z^1, Z^2, Z^3, Z^4), \quad H := (H^1, H^2, H^3, H^4)$$

with

$$\begin{aligned} Z^1 &:= \Delta\varphi^+ \cdot \frac{1}{\Delta\varphi^+} - \varphi' \cdot S - [\varphi', S] + (\Delta\varphi^+ \varphi') \cdot \left[\frac{1}{\Delta\varphi^+}, S \right], \\ Z^2 &:= -\frac{1}{\Delta\varphi^+} \cdot \varphi' - \frac{(\varphi')^2}{\Delta\varphi^+} \cdot S + \frac{\varphi' \kappa'}{\Delta\varphi^+} \cdot I - \frac{\varphi'}{\Delta\varphi^+} \cdot [\varphi', S] - (\varphi')^2 \cdot \left[S, \frac{1}{\Delta\varphi^+} \right] - \left[\varphi', \frac{1}{\Delta\varphi^+} \right], \\ Z^3 &:= \varphi' \cdot S - \kappa' \cdot I, \quad Z^4 := \Delta\varphi^+ \cdot S, \\ H^1 &:= -\frac{1}{\Delta\varphi^+} \cdot \varphi - \left[\varphi, \frac{1}{\Delta\varphi^+} \right], \quad H^2 = H^3 = H^4 := 0, \end{aligned}$$

and the function $f: [-1, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}^{4 \times 4}$, $x = (x_1, x_2, x_3, x_4) \mapsto f(x) := \{f_{ij}(x)\}_{i,j=1,\dots,4}$ with

$$f_{11}(x) = f_{24}(x) := x_1, \quad f_{12}(x) = f_{23}(x) := x_2,$$

$$f_{ij}(x) := 0 \quad \text{for } (i, j) \notin \{(1, 1), (1, 2), (2, 3), (2, 4)\}.$$

For each $m, n \in \mathbb{N}$, define

$$\tau_m := \inf \left\{ t \in [0, T] : \sum_{i=1}^4 \left[|H_t^i| + |b_t^{Z^i}| + \sqrt{c_t^{Z^i, Z^i}} \right] > m \right\},$$

$$f^{(n)} = \left\{ f_{ij}^{(n)} \right\}_{i,j=1,\dots,4} \quad \text{with } f_{ij}^{(n)} := (f_{ij} \wedge n) \vee (-n).$$

Since $f^{(n)}$ is Lipschitz-continuous and bounded on $[-1, 1] \times \mathbb{R}^3$ with linear growth rate, from [36, Theorem 5], we deduce that there exists a pair of \mathbb{R}^4 -valued continuous adapted processes $(X^{(n)}, K^{(n)})$ solving the Skorohod SDE,

$$dX_t^{(m,n)} = dH_t^{\tau_m} + f^{(n)}(X_t^{(m,n)}) dZ_t^{\tau_m} + dK_t^{(m,n)},$$

where $X^{(m,n)1} \in [-1, 1]$, $X^{(m,n)2} \in \mathbb{R}$, $X^{(m,n)3} = X^{(m,n)4} = 0$; $K^{(m,n)1} = K^{(m,n)\uparrow} - K^{(m,n)\downarrow}$ with increasing processes $K^{(m,n)\uparrow}$ and $K^{(m,n)\downarrow}$ which increase only on the set $\{X^{(m,n)1} = -1\}$ and $\{X^{(m,n)1} = 1\}$, respectively, and $K^{(m,n)2} = K^{(m,n)3} = K^{(m,n)4} = 0$. Since f fulfills the linear growth condition:

$$\|f(x)\|^2 \leq C \|x\|^2 \quad \text{for some } C \in \mathbb{R}_+ \text{ and all } x \in [-1, 1] \times \mathbb{R}^3,$$

by (A.2, A.4, A.7) and Fubini's theorem, we get

$$\begin{aligned} \mathbf{E} \left[\sup_{t \in [0, T]} \left\| X_t^{(m,n)} \right\|^2 \right] &\leq 1 + \mathbf{E} \left[\sup_{t \in [0, T]} \left\| H_t^{\tau_m} \right\|^2 \right] + \mathbf{E} \left[\sup_{t \in [0, T]} \left\| f(X_t^{(m,n)}) \cdot Z_t^{\tau_m} \right\|^2 \right] \\ &\leq 1 + m^2 + \text{cst.} \mathbf{E} \left[\sum_{i,j=1}^4 \int_0^{\tau_m} \left| f_{ij}^{(n)}(X_t^{(m,n)}) \right|^2 c_t^{Z^i, Z^j} dt \right] \\ &\quad + \text{cst.} \mathbf{E} \left[\sum_{i,j=1}^4 \int_0^{\tau_m} \left| f_{ij}^{(n)}(X_t^{(m,n)}) b_t^{Z^j} \right|^2 dt \right] \\ &\leq 1 + m^2 + \text{cst.} m^2 C \int_0^T \sup_{s \in [0, t]} \left\| X_s^{(m,n)} \right\|^2 dt. \end{aligned}$$

Monotone convergence and Gronwall's inequality (cf. e.g. [32, Theorem V.68]) yield

$$\mathbf{E} \left[\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left\| X_t^{(m,n)} \right\|^2 \right] = \lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \in [0, T]} \left\| X_t^{(m,n)} \right\|^2 \right] \leq (m^2 + 1) e^{\text{cst.} m^2 C T}.$$

Since $\tau_m \rightarrow \infty$ a.s. when $m \rightarrow \infty$, putting together the solutions on $[[0, \tau_m]]$,

$$X := \sum_{m,n \in \mathbb{N}} X^{(m,n)} \mathbf{1}_{[[0, \tau_m]]} \quad \text{with} \quad K := \sum_{m,n \in \mathbb{N}} K^{(m,n)} \mathbf{1}_{[[0, \tau_m]]}$$

provides the unique solution to the Skorohod SDE on $[0, T]$:

$$dX_t = dH_t + f(X_t) dZ_t + dK_t.$$

Finally, setting

$$\begin{aligned} \Delta\varphi &:= X^1 \Delta\varphi^+, \quad \Delta\tilde{X} := X^2, \quad \varphi^\varepsilon := \varphi + \varphi' X^2 + \Delta\varphi, \\ \varphi^{\varepsilon\uparrow} &:= \Delta\varphi^+ \cdot K^\uparrow, \quad \varphi^{\varepsilon\downarrow} := \Delta\varphi^+ \cdot K^\downarrow, \end{aligned}$$

Itô's formula yields that $\Delta\tilde{X}$, $\Delta\varphi$, φ^ε , $\varphi^{\varepsilon\uparrow}$, $\varphi^{\varepsilon\downarrow}$ are the desired processes fulfilling (3.20, 3.21, 3.22). \square

As a consequence of the previous lemma, the following corollary provides the existence of the shadow price process S^ε in [19].

Corollary 3.5.3. *Let $\Delta\varphi$, $\varphi^{\varepsilon\uparrow}$, $\varphi^{\varepsilon\downarrow}$ be as in Lemma 3.5.2. Set*

$$\begin{aligned} \Delta\pi &:= \frac{\Delta\varphi S}{R}, \quad \Delta\pi^\pm := \frac{\Delta\varphi^\pm S}{R} = \pm (\eta\varepsilon)^{1/3}, \\ \beta &:= \Delta\pi^+ \left(\frac{2}{\varepsilon}\right)^{1/3}, \quad \alpha := \frac{S}{\beta^3}, \quad \gamma := 3\alpha\beta^2 \left(\frac{\varepsilon}{2}\right)^{2/3}, \end{aligned}$$

and define

$$\Delta S := \alpha\Delta\pi^3 - \gamma\Delta\pi. \quad (3.23)$$

Then ΔS is an Itô process valued in $[-\varepsilon S, \varepsilon S]$ with

$$\varphi^{\varepsilon\uparrow} \text{ increases only on the set } \{\Delta S = \varepsilon S\} \subseteq \Omega \times [0, T], \quad (3.24)$$

$$\varphi^{\varepsilon\downarrow} \text{ increases only on the set } \{\Delta S = -\varepsilon S\} \subseteq \Omega \times [0, T]. \quad (3.25)$$

Proof. Corresponding to (3.23), let us consider the function $f(x, a, b) := ax^3 - bx$. Then by definition, it holds that

$$\frac{\partial f}{\partial x}(\pm\Delta\pi, \alpha, \gamma) = 3\alpha(\pm\Delta\pi)^2 - \gamma = 0. \quad (3.26)$$

Note that (3.20, 3.21, 3.22) yield

$$d\Delta\pi_t = d\frac{\Delta\varphi_t S_t}{R_t} = -d\frac{(\varphi_t + \varphi'_t \Delta\tilde{X}_t) S_t}{R_t} \text{ on } \{\Delta\varphi \neq \Delta\varphi^\pm\}, \quad (3.27)$$

and φ , φ' , R , α , γ are assumed to be Itô processes. Therefore, applying Itô's formula, we have

$$\begin{aligned} d\Delta S_t &= df(\Delta\pi_t, \alpha_t, \gamma_t) \\ &= (3\alpha_t \Delta\pi_t^2 - \gamma_t) d\Delta\pi_t + 3\alpha_t \Delta\pi_t d[\Delta\pi, \Delta\pi]_t \\ &\quad + 3\Delta\pi_t^2 d[\alpha, \Delta\pi]_t + \Delta\pi_t^3 d\alpha_t - \Delta\pi_t d\gamma_t - d[\gamma, \Delta\pi]_t. \end{aligned} \quad (3.28)$$

In particular, ΔS is an Itô process. Furthermore, in view of (3.20, 3.21, 3.23) and from $f(\pm\Delta\pi, \alpha, \gamma) = \mp\varepsilon S$, we conclude (3.24, 3.25). \square

Remark 3.5.4. 1. For φ^ε , $\Delta\tilde{X}$, $\Delta\pi$ and ΔS in Lemma 3.5.2 and Corollary 3.5.3, set

$$S^\varepsilon := S + \Delta S, \quad \Delta\kappa := \kappa' \Delta\tilde{X}, \quad \kappa^\varepsilon := \kappa + \Delta\kappa. \quad (3.29)$$

Observe that (3.22) yields

$$\Delta\tilde{X} = (\varphi^\varepsilon - \varphi) \cdot S - \Delta\kappa \cdot I.$$

2. As indicated above, the process S^ε models the so-called shadow price in the sense that it is an Itô process moving within the bid-ask spread $[(1 - \varepsilon)S, (1 + \varepsilon)S]$ and it coincides with the ask and bid price only if the strategy φ^ε hits the buying and selling boundary of the no-trade region, respectively. Obviously, following strategy φ^ε , trading in a frictionless market at price S^ε generates the same value as in the original market with transaction costs. This is the key idea which our proof of the approximate optimality of $(\varphi^\varepsilon, \kappa^\varepsilon)$ is based on.

Estimation of local parameters

As preparation for the primal and dual considerations related to the frictionless market with shadow price process S^ε , we next investigate some local coefficients related to the process ΔS . Let us throughout keep the notations of the processes $\Delta\varphi$, $\Delta\tilde{X}$, $\Delta\pi$, $\Delta\pi^+$, ΔS as well as the parameters α , β , γ in Lemma 3.5.2 and Corollary 3.5.3.

First, using Itô's formula and (3.27), we have

$$d\varphi_t = -\frac{\pi_t R_t}{S_t} dY_t + \frac{\pi_t R_t}{S_t} d[Y, Y]_t + \frac{\pi_t}{S_t} dR_t + \frac{R_t}{S_t} d\pi_t + \frac{1}{S_t} d[\pi, R]_t - \frac{\pi_t}{S_t} d[Y, R]_t - \frac{R_t}{S_t} d[\pi, Y]_t, \quad (3.30)$$

$$\begin{aligned} d\Delta\pi_t &= -d\pi_t - \frac{\pi_t}{R_t} dR_t + \frac{\pi_t}{R_t^2} d[R, R]_t + \pi_t dY_t - \frac{\pi_t}{R_t} d[R, Y]_t - \frac{\Delta\pi_t}{R_t} dR_t + \frac{\Delta\pi_t}{R_t^2} d[R, R]_t \\ &\quad + \Delta\pi_t (1 - \varphi'_t S_t) dY_t - \frac{\Delta\pi_t (1 - \varphi'_t S_t)}{R_t} d[R, Y]_t - \Delta\pi_t d[S, \varphi'_t] - \Delta\pi_t \varphi'_t S_t d[Y, Y]_t \\ &\quad - \frac{\Delta\tilde{X}_t}{R_t} (\varphi'_t S_t)^2 dY_t - \frac{\Delta\tilde{X}_t}{R_t} S_t d\varphi'_t + \frac{\Delta\tilde{X}_t}{R_t} \kappa'_t \varphi'_t S_t dt - \frac{\Delta\tilde{X}_t}{R_t} (1 + \varphi'_t S_t) d[S, \varphi'_t]_t \\ &\quad + \frac{\Delta\tilde{X}_t}{R_t^2} (\varphi'_t S_t)^2 d[R, Y]_t + \frac{\Delta\tilde{X}_t}{R_t^2} S_t d[\varphi', R]_t - \frac{\Delta\tilde{X}_t}{R_t} (\varphi'_t S_t)^2 d[Y, Y]_t. \end{aligned} \quad (3.31)$$

Now, set

$$X(\Delta) := \frac{\Delta\tilde{X}}{R} \varepsilon^{-1/3}, \quad b(\Delta\pi) := -b^\pi - \pi \frac{b^R}{R} + \pi \frac{c^{R,R}}{R^2} - \pi \frac{c^{R,Y}}{R}$$

and define the following sets of processes:

$$\begin{aligned}
\mathcal{R}^{c^{\Delta\pi,\Delta\pi}} &:= \left\{ \beta^2 \frac{c^{R,R}}{R^2}, \beta^2 (1 - \varphi'S)^2 c^{Y,Y}, X(\Delta)^2 (\varphi'S)^4 c^{Y,Y}, X(\Delta)^2 S^2 c^{\varphi',\varphi'} \right\}, \\
\mathcal{R}^{b^{\Delta\pi}} &:= \left\{ \beta \frac{b^R}{R}, \beta \frac{c^{R,R}}{R^2}, \beta (1 - \varphi'S) \frac{c^{R,Y}}{R}, \beta c^{S,\varphi'}, \beta \varphi'S c^{Y,Y}, X(\Delta) S b^{\varphi'}, X(\Delta) \kappa' \varphi'S, \right. \\
&\quad \left. X(\Delta) (1 + \varphi'S) c^{S,\varphi'}, X(\Delta) (\varphi'S)^2 \frac{c^{R,Y}}{R}, X(\Delta) \frac{S c^{\varphi',R}}{R}, X(\Delta) (\varphi'S)^2 c^{Y,Y} \right\}, \\
\mathcal{R}^{c^{\Delta\pi,\varphi}} &:= \left\{ \beta \frac{c^{R,\varphi}}{R}, \beta (1 - \varphi'S) c^{Y,\varphi}, X(\Delta) (\varphi'S)^2 c^{Y,\varphi}, X(\Delta) S c^{\varphi',\varphi} \right\}, \\
\mathcal{H}^{b^{\Delta S}} &:= \left\{ \alpha \beta^4 c^{Y,Y}, \alpha \beta^2 b(\Delta\pi), \beta^5 c^{Y,Y}, \beta^2 c^{\alpha,\alpha}, \beta^3 b^\alpha, \beta b^{\alpha\beta^2}, \beta^3 c^{Y,Y}, c^{\alpha\beta^2,\alpha\beta^2} \right\}, \\
\mathcal{R}^{b^{\Delta S}} &:= \left\{ \alpha \beta r^{\Delta,\Delta}, \alpha \beta^2 r^\Delta, \beta^2 r^{\Delta,\Delta}, r^{\Delta,\Delta} : r^{\Delta,\Delta} \in \mathcal{R}^{c^{\Delta\pi,\Delta\pi}}, r^\Delta \in \mathcal{R}^{b^{\Delta\pi}} \right\}, \\
\mathcal{H}^{c^{\Delta S,R^\perp}} &:= \left\{ \alpha \beta^5 c^{Y,Y}, \alpha \beta^2 c^{R^\perp,R^\perp}, \beta^3 c^{\alpha,R^\perp}, \beta c^{\alpha\beta^2,R^\perp}, \alpha \beta^2 r^{\Delta,\Delta} : r^{\Delta,\Delta} \in \mathcal{R}^{c^{\Delta\pi,\Delta\pi}} \right\}, \\
\mathcal{H}^{c^{\Delta S,S}} &:= \left\{ \alpha \beta^5 c^{Y,Y}, \alpha \beta^2 c^{S,S}, \beta^3 c^{\alpha,S}, \beta c^{\alpha\beta^2,S}, \alpha \beta^2 r^{\Delta,\Delta} : r^{\Delta,\Delta} \in \mathcal{R}^{c^{\Delta\pi,\Delta\pi}} \right\}, \\
\mathcal{H}^{c^{\Delta S,\Delta S}} &:= \left\{ \alpha^2 \beta^7 c^{Y,Y}, \beta^6 c^{\alpha,\alpha}, \beta^2 c^{\alpha\beta^2,\alpha\beta^2}, \alpha^2 \beta^4 r^{\Delta,\Delta} : r^{\Delta,\Delta} \in \mathcal{R}^{c^{\Delta\pi,\Delta\pi}} \right\}, \\
\mathcal{H}^{c^{\Delta S,\varphi}} &:= \left\{ \beta^2 R c^{Y,Y}, \alpha \beta^2 c^{\varphi,\varphi}, \beta^3 c^{\alpha,\varphi}, \beta c^{\alpha\beta^2,\varphi}, \alpha \beta^2 r^{\Delta,\varphi} : r^{\Delta,\varphi} \in \mathcal{R}^{c^{\Delta\pi,\Delta\pi}} \right\}.
\end{aligned}$$

Note that from (3.31, 3.30), we obtain

$$\begin{aligned}
\left| c^{\Delta\pi,\Delta\pi} - \frac{1}{3} \beta^3 c^{Y,Y} \right| &\leq \text{cst.} \sum_{r^{\Delta,\Delta} \in \mathcal{R}^{c^{\Delta\pi,\Delta\pi}}} |r^{\Delta,\Delta}| \varepsilon^{2/3}, \\
|b^{\Delta\pi} - b(\Delta\pi)| &\leq \text{cst.} \sum_{r^\Delta \in \mathcal{R}^{b^{\Delta\pi}}} |r^\Delta| \varepsilon^{1/3}, \\
\left| c^{\Delta\pi,\varphi} + \frac{\beta^3 R}{3S} c^{Y,Y} \right| &\leq \text{cst.} \sum_{r^{\Delta,\varphi} \in \mathcal{R}^{c^{\Delta\pi,\varphi}}} |r^{\Delta,\varphi}| \varepsilon^{1/3}.
\end{aligned}$$

Then by (3.28), the coefficients related to ΔS can be estimated in the following manner:

$$b_t^{\Delta S} = \underbrace{\Delta\pi_t S_t c_t^{Y,Y}}_{=: G_t^{(1)}} + G_t^{(2)} \quad (3.32)$$

with

$$|G^{(1)}| \leq \text{cst.} \beta S c^{Y,Y} \varepsilon^{1/3}, \quad |G^{(2)}| \leq \text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b^{\Delta S}} \cup \mathcal{R}^{b^{\Delta S}}} |b^\Delta| \varepsilon^{2/3}, \quad (3.33)$$

$$\left| c_t^{\Delta S, R^\perp} \right| \leq \text{cst.} \sum_{c^{\Delta, R^\perp} \in \mathcal{H}^{c^{\Delta S, R^\perp}}} \left| c_t^{\Delta, R^\perp} \right| \varepsilon^{2/3}, \quad (3.34)$$

$$\left| c_t^{\Delta S, S} \right| \leq \text{cst.} \sum_{c^{\Delta, S} \in \mathcal{H}^{c^{\Delta S, S}}} \left| c_t^{\Delta, S} \right| \varepsilon^{2/3}, \quad (3.35)$$

$$\left| c_t^{\Delta S, \Delta S} \right| \leq \text{cst.} \sum_{c^{\Delta, \Delta} \in \mathcal{H}^{c^{\Delta S, \Delta S}}} \left| c_t^{\Delta, \Delta} \right| \varepsilon^{4/3}, \quad (3.36)$$

and

$$c_t^{\Delta S, \varphi} = - \underbrace{\left(\Delta \pi_t^2 - (\Delta \pi_t^+)^2 \right)}_{=: G_t^{(3)}} R_t c_t^{Y, Y} + G_t^{(4)} \quad (3.37)$$

with

$$\left| G^{(3)} \right| \leq \text{cst.} \left| \beta^2 R c^{Y, Y} \right| \varepsilon^{2/3}, \quad \left| G^{(4)} \right| \leq \text{cst.} \sum_{c^{\Delta, \varphi} \in \mathcal{H}^{c^{\Delta S, \varphi}}} \left| c^{\Delta, \varphi} \right| \varepsilon. \quad (3.38)$$

The following lemma transforms Condition (3.4) into a more technical version which will be directly worked with in coming sections.

Lemma 3.5.5. *Assume Condition (3.4).*

1. For any $q \in \mathbb{N}$, we have

$$\left\| \frac{\Delta \tilde{X}}{R} \right\|_{S^q(\mathbf{Q})} \leq \text{cst.} \varepsilon^{1/3}. \quad (3.39)$$

2. Set

$$\mathcal{G}_1 := \{ S b^\varphi \} \cup \mathcal{H}^{c^{\Delta S, \varphi}} \cup \left\{ \frac{\beta R}{S} b^{\Delta}, \varphi' \frac{\Delta \tilde{X}}{\varepsilon^{1/3}} b^{\Delta} : b^{\Delta} \in \mathcal{H}^{b^{\Delta S}} \cup \mathcal{R}^{b^{\Delta S}} \right\},$$

$$\mathcal{G}_2 := \{ S^2 c^{\varphi, \varphi} \} \cup \left\{ \frac{\beta^2 R^2}{S^2} c^{\Delta, \Delta}, \left(\varphi' \frac{\Delta \tilde{X}}{\varepsilon^{1/3}} \right)^2 c^{\Delta, \Delta} : c^{\Delta, \Delta} \in \mathcal{H}^{c^{\Delta S, \Delta S}} \right\},$$

$$\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2.$$

Then for any $n \in \mathbb{N}$, we have

$$\sum_{g \in \mathcal{G}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |g_t|^n dt \right] < \text{cst.}, \quad (3.40)$$

$$\sum_{h \in \mathcal{H}^{b^{\Delta S}} \cup \mathcal{R}^{b^{\Delta S}} \cup \mathcal{H}^{c^{\Delta S, R^\perp}}} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} \left| \frac{h_t}{\sigma_t^S} \right|^n \right] + \sum_{c^{\Delta, S} \in \mathcal{H}^{c^{\Delta S, S}}} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} \left| \frac{c_t^{\Delta, S}}{c_t^{S, S}} \right|^n \right] < \text{cst.} \quad (3.41)$$

Proof. 1. Note that from (3.22) and Itô's formula, $\Delta\tilde{X}/R$ solves the affine linear SDE

$$d\frac{\Delta\tilde{X}_t}{R_t} = \frac{\Delta\tilde{X}_t}{R_t} d\left(\varphi' \cdot S_t - \kappa' \cdot I_t - \frac{1}{R} \cdot R_t + \frac{1}{R^2} \cdot [R, R]_t\right) + d\frac{\Delta\varphi}{R} \cdot S_t,$$

which implies

$$\frac{\Delta\tilde{X}}{R} = \mathcal{E}(H) \frac{\Delta\varphi}{R\mathcal{E}(H)} \cdot S \quad \text{with} \quad H := \varphi' \cdot S - \kappa' \cdot I - \frac{1}{R} \cdot R + \frac{1}{R^2} \cdot [R, R]. \quad (3.42)$$

By the Cauchy-Schwarz inequality, (A.2, A.7), Condition (3.4) and Lemma A.2, we obtain for any $q \in \mathbb{N}$ that

$$\left\| \frac{\Delta\tilde{X}}{R} \right\|_{S^q(\mathbf{Q})}^q \leq \underbrace{\text{cst.} \|\mathcal{E}(H)\|_{S^{2q}(\mathbf{Q})}^q}_{<\text{cst.}} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{\beta_t^2 c_t^{Y,Y}}{[\mathcal{E}(H)_t]^2} \right|^q dt \right]} \varepsilon^{q/3}, \quad (3.43)$$

which implies (3.39).

2. Analogously to Lemma 2.5.4 in Chapter 2, (3.40, 3.41) follow from (3.30), Itô's formula applied to $\varphi', \alpha, \alpha\beta^2$, Condition (3.4) combined with (3.39) and Lemma A.2. \square

3.5.2 Primal considerations

Having analyzed the local dynamics of the shadow price process S^ε , we are now able to compute the wealth and utility generated by the candidate trading-consumption policy $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$.

As a first step, we consider the deviation of the wealth process generated by $(\varphi^\varepsilon, \kappa^\varepsilon)$ under transaction costs from the frictionless optimal wealth process (φ, κ) .

Lemma 3.5.6. *Assume Condition (3.4).*

1. *For any stopping time τ , we have*

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} [X_\tau^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_\tau(\varphi, \kappa)] &= \mathbf{E}_{\mathbf{Q}} \left[\frac{2\Delta\varphi^2 - (\Delta\varphi^+)^2}{R} \cdot [S, S]_\tau \right] - \mathbf{E}_{\mathbf{Q}} [\Delta\kappa \cdot I_\tau] \\ &\quad + \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta\tilde{X} \Delta\varphi}{R} \cdot [S, S]_\tau \right] + O(\varepsilon). \end{aligned} \quad (3.44)$$

Furthermore,

$$\|X^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X(\varphi, \kappa)\|_{S^q(\mathbf{Q})} = O(\varepsilon^{1/3}) \quad (3.45)$$

holds for any $q \in \mathbb{N}$.

2. Define stopping times

$$\tau^{\varepsilon,1} := \inf \left\{ t \in [0, T] : \left| \frac{X_t^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_t(\varphi, \kappa)}{X_t(\varphi, \kappa)} \right| > \frac{1}{2} \right\}, \quad (3.46)$$

$$\tau^{\varepsilon,2} := \inf \left\{ t \in [0, T] : t \left| \frac{\Delta \kappa_t}{\kappa_t} \right| > \frac{1}{2} \right\}, \quad (3.47)$$

$$\tau^\varepsilon := \tau^{\varepsilon,1} \wedge \tau^{\varepsilon,2} \wedge T. \quad (3.48)$$

Then

$$\mathbf{Q}(\tau^\varepsilon < T) = O(\varepsilon^2) \quad (3.49)$$

and in particular, $\lim_{\varepsilon \downarrow 0} \mathbf{P}(\tau^\varepsilon < T) = 0$.

3. It holds that

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{(X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_{\tau^\varepsilon}(\varphi, \kappa))^2}{R_{\tau^\varepsilon}} \right] = \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2}{R} \cdot [S, S]_{\tau^\varepsilon} \right] - \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \kappa^2}{r} \cdot I_{\tau^\varepsilon} \right] + O(\varepsilon). \quad (3.50)$$

Proof. 1. Let $q \in \mathbb{N}$ and τ be a stopping time. The loss due to transaction costs can be split up into five parts:

$$\begin{aligned} & X^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X(\varphi, \kappa) \\ &= (\varphi^\varepsilon - \varphi) \cdot S + \varphi \cdot \Delta S + (\varphi^\varepsilon - \varphi) \cdot \Delta S + (X^\varepsilon(\varphi^\varepsilon, 0) - (x + \varphi^\varepsilon \cdot S^\varepsilon)) - \Delta \kappa \cdot I. \end{aligned}$$

We will estimate each term separately.

Step 1. (A.2, A.7), Condition (3.4), (3.39) and Lemma A.2 yield

$$\begin{aligned} \|(\varphi^\varepsilon - \varphi) \cdot S\|_{S^q(\mathbf{Q})}^q &= \left\| \left(\varphi' \Delta \tilde{X} + \Delta \varphi \right) \cdot S \right\|_{S^q(\mathbf{Q})}^q \\ &\leq \underbrace{\text{cst.} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \left((\varphi'_t S_t)^2 \left(\frac{\Delta \tilde{X}}{\varepsilon^{1/3}} \right)^2 + \beta_t^2 \right) c_t^{Y,Y} \right|^{2q} dt \right]}}_{< \text{cst.}} \varepsilon^{q/3}, \end{aligned}$$

and thus

$$\|(\varphi^\varepsilon - \varphi) \cdot S\|_{S^q(\mathbf{Q})} = O(\varepsilon^{1/3}). \quad (3.51)$$

In particular, the \mathbf{Q} -local martingale $(\varphi^\varepsilon - \varphi) \cdot S$ is a square-integrable \mathbf{Q} -martingale, which implies

$$\mathbf{E}_{\mathbf{Q}}[(\varphi^\varepsilon - \varphi) \cdot S]_\tau = 0. \quad (3.52)$$

Step 2. Integration by parts leads to

$$\varphi \cdot \Delta S = \varphi \Delta S - \Delta S \cdot \varphi - [\Delta S, \varphi].$$

Since $|\Delta S| \leq \varepsilon S$, Condition (3.4) and Lemma A.2 yield

$$\|\varphi \Delta S\|_{S^q(\mathbf{Q})}^q \leq \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |\varphi_t S_t|^q \right]}_{< \infty} \varepsilon^q; \quad (3.53)$$

and by (A.7, 3.40), we have

$$\|\Delta S \cdot \varphi\|_{H^q(\mathbf{Q})}^q \leq \underbrace{\text{cst.} \left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |S_t b_t^\varphi|^q dt \right] + \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |S_t^2 c_t^{\varphi, \varphi}|^q dt \right]} \right)}_{< \infty} \varepsilon^q. \quad (3.54)$$

Further, from (3.37, 3.38) and using (A.7, 3.40), we deduce

$$\left\| \frac{\Delta \varphi^2 - (\Delta \varphi^+)^2}{R} \cdot [S, S] \right\|_{S^q(\mathbf{Q})}^q \leq \underbrace{\text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 R_t c_t^{Y, Y}|^q dt \right]}_{< \infty} \varepsilon^{2q/3},$$

$$\left\| [\Delta S, \varphi] + \frac{\Delta \varphi^2 - (\Delta \varphi^+)^2}{R} \cdot [S, S] \right\|_{S^q(\mathbf{Q})}^q \leq \underbrace{\text{cst.} \sum_{c^{\Delta, \varphi} \in \mathcal{H}^{c^{\Delta S, \varphi}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |c_t^{\Delta, \varphi}|^q dt \right]}_{< \text{cst.}} \varepsilon^q.$$

In conjunction with (A.4), this implies

$$\mathbf{E}_{\mathbf{Q}} [[\Delta S, \varphi]_\tau] = -\mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2 - (\Delta \varphi^+)^2}{R} \cdot [S, S]_\tau \right] + O(\varepsilon), \quad (3.55)$$

$$\|[\Delta S, \varphi]\|_{S^q(\mathbf{Q})} = O(\varepsilon^{2/3}). \quad (3.56)$$

(3.53–3.56) combined with (A.2) then yield

$$\mathbf{E}_{\mathbf{Q}} [\varphi \cdot \Delta S_\tau] = \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2 - (\Delta \varphi^+)^2}{R} \cdot [S, S]_\tau \right] + O(\varepsilon), \quad (3.57)$$

$$\|\varphi \cdot \Delta S\|_{S^q(\mathbf{Q})} = O(\varepsilon^{2/3}). \quad (3.58)$$

Step 3. (3.22) yields

$$(\varphi^\varepsilon - \varphi) \cdot \Delta S = \Delta \varphi \cdot \Delta S + (\varphi' \Delta \tilde{X}) \cdot \Delta S.$$

In view of (3.32, 3.33, 3.36) and using (A.7, 3.40), we obtain

$$\begin{aligned} \left\| \frac{\Delta\varphi^2}{R} \cdot [S, S] \right\|_{H^q(\mathbf{Q})}^q &\leq \underbrace{\text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 R_t c_t^{Y,Y}|^q dt \right]}_{< \infty} \varepsilon^{2q/3}, \\ \left\| \Delta\varphi \cdot \Delta S - \frac{\Delta\varphi^2}{R} \cdot [S, S] \right\|_{H^q(\mathbf{Q})}^q &\leq \underbrace{\text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b\Delta S} \cup \mathcal{R}^{b\Delta S}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{\beta_t R_t}{S_t} b_t^\Delta \right|^{2q} dt \right]}_{< \text{cst.}} \varepsilon^q \\ &\quad + \underbrace{\text{cst.} \sqrt{\sum_{c^{\Delta, \Delta} \in \mathcal{H}^{c^{\Delta S, \Delta S}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{\beta_t^2 R_t^2}{S_t^2} c_t^{\Delta, \Delta} \right|^{2q} dt \right]}}_{< \text{cst.}} \varepsilon^q. \end{aligned}$$

Combined with (A.4), this yields

$$\mathbf{E}_{\mathbf{Q}} [\Delta\varphi \cdot \Delta S_\tau] = \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta\varphi^2}{R} \cdot [S, S]_\tau \right] + O(\varepsilon), \quad (3.59)$$

$$\|\Delta\varphi \cdot \Delta S\|_{H^q(\mathbf{Q})} = O(\varepsilon^{2/3}). \quad (3.60)$$

Again, from (3.32, 3.33, 3.36) and by (A.7, 3.40), we have

$$\begin{aligned} \left\| \frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S] \right\|_{H^q(\mathbf{Q})}^q &\leq \underbrace{\text{cst.} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \varphi'_t \frac{\Delta \tilde{X}_t}{\varepsilon^{1/3}} \beta_t S_t c_t^{Y,Y} \right|^{2q} dt \right]}}_{< \text{cst.}} \varepsilon^{q/3}, \\ \left\| \left(\varphi' \Delta \tilde{X} \right) \cdot \Delta S - \frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S] \right\|_{H^q(\mathbf{Q})}^q &\leq \underbrace{\text{cst.} \sum_{b^\Delta \in \mathcal{H}^{b\Delta S} \cup \mathcal{R}^{b\Delta S}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \varphi'_t \frac{\Delta \tilde{X}_t}{\varepsilon^{1/3}} b_t^\Delta \right|^{2q} dt \right]}_{< \text{cst.}} \varepsilon^q \\ &\quad + \underbrace{\text{cst.} \sqrt{\sum_{c^{\Delta, \Delta} \in \mathcal{H}^{c^{\Delta S, \Delta S}}} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \left(\varphi'_t \frac{\Delta \tilde{X}_t}{\varepsilon^{1/3}} \right)^2 c_t^{\Delta, \Delta} \right|^{2q} dt \right]}}_{< \text{cst.}} \varepsilon^{4q/3}, \end{aligned}$$

and hence applying (A.4), we obtain

$$\mathbf{E}_{\mathbf{Q}} \left[\left(\varphi' \Delta \tilde{X} \right) \cdot \Delta S_{\tau} \right] = \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S]_{\tau} \right] + O(\varepsilon), \quad (3.61)$$

$$\left\| \left(\varphi' \Delta \tilde{X} \right) \cdot \Delta S \right\|_{H^q(\mathbf{Q})} = O(\varepsilon^{2/3}). \quad (3.62)$$

Combining (3.59–3.62) with (A.2) then yields

$$\mathbf{E}_{\mathbf{Q}} [(\varphi^{\varepsilon} - \varphi) \cdot \Delta S_{\tau}] = \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2}{R} \cdot [S, S]_{\tau} \right] + \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S]_{\tau} \right] + O(\varepsilon), \quad (3.63)$$

$$\|(\varphi^{\varepsilon} - \varphi) \cdot \Delta S\|_{S^q(\mathbf{Q})} = O(\varepsilon^{2/3}). \quad (3.64)$$

Step 4. In view of (3.24, 3.25) and recalling Remark 2.1.2 regarding the self-financing constraint, by (3.39), Condition (3.4) and Lemma A.2, it holds that

$$\begin{aligned} & \|X^{\varepsilon}(\varphi^{\varepsilon}, 0) - (x + \varphi^{\varepsilon} \cdot S^{\varepsilon})\|_{S^q(\mathbf{Q})} \\ & \leq \|\varphi^{\varepsilon} S\|_{S^q(\mathbf{Q})} \varepsilon + 2x^B \varepsilon \\ & \leq \underbrace{\text{cst.} \left(\|\varphi S\|_{S^q(\mathbf{Q})} + \left\| \varphi' S \Delta \tilde{X} \right\|_{S^q(\mathbf{Q})} + \|\beta R\|_{S^q(\mathbf{Q})} + x^S \right)}_{< \text{cst.}} \varepsilon. \end{aligned} \quad (3.65)$$

Step 5. From (3.29, 3.39), Condition (3.4) and Lemma A.2, we deduce

$$\|\Delta \kappa\|_{S^q(\mathbf{Q})} \leq \|r\|_{S^{2q}(\mathbf{Q})} \left\| \frac{\Delta \tilde{X}}{R} \right\|_{S^{2q}(\mathbf{Q})} = O(\varepsilon^{1/3}). \quad (3.66)$$

Now, applying (A.4), relations (3.52, 3.57, 3.63) and (3.65) for $q = 1$ imply (3.44); (3.39, 3.58, 3.64, 3.65, 3.66) yield (3.45).

2. From (3.46), Markov's inequality, the Cauchy-Schwarz inequality, (3.45) and Condition (3.4), we have

$$\begin{aligned} \mathbf{Q}(\tau^{\varepsilon, 1} < T) & \leq \mathbf{Q} \left(\sup_{t \in [0, T]} \left| \frac{X_t^{\varepsilon}(\varphi^{\varepsilon}, \kappa^{\varepsilon}) - X_t(\varphi, \kappa)}{X_t(\varphi, \kappa)} \right| > \frac{1}{2} \right) \\ & \leq \text{cst.} \left\| \frac{X^{\varepsilon}(\varphi^{\varepsilon}, \kappa^{\varepsilon}) - X(\varphi, \kappa)}{X(\varphi, \kappa)} \right\|_{S^6(\mathbf{Q})}^6 \\ & \leq \text{cst.} \left\| X(\varphi, \kappa)^{-1} \right\|_{S^{12}(\mathbf{Q})}^6 \|X^{\varepsilon}(\varphi^{\varepsilon}, \kappa^{\varepsilon}) - X(\varphi, \kappa)\|_{S^{12}(\mathbf{Q})}^6 \\ & = O(\varepsilon^2). \end{aligned} \quad (3.67)$$

(3.47), Markov's inequality, the Cauchy-Schwarz inequality, (3.66) and Conditions (3.2, 3.4) yield

$$\begin{aligned}
\mathbf{Q}(\tau^{\varepsilon,2} < T) &\leq \mathbf{Q}\left(\sup_{t \in [0, T]} t \left| \frac{\Delta \kappa_t}{\kappa_t} \right| > \frac{1}{2}\right) \\
&\leq \text{cst.} \left\| t \frac{\Delta \kappa}{\kappa} \right\|_{S^6(\mathbf{Q})}^6 \\
&\leq \text{cst.} \underbrace{\|\Delta \kappa\|_{S^{12}(\mathbf{Q})}^6}_{=O(\varepsilon^2)} \underbrace{\|t \kappa^{-1}\|_{S^{12}(\mathbf{Q})}^6}_{< \infty}. \tag{3.68}
\end{aligned}$$

Hence, we deduce

$$\mathbf{Q}(\tau^\varepsilon < T) = \mathbf{Q}(\tau^{\varepsilon,1} < T) + \mathbf{Q}(\tau^{\varepsilon,2} < T) = O(\varepsilon^2).$$

Since \mathbf{P} and \mathbf{Q} are equivalent, $\mathbf{Q}(\tau^\varepsilon < T) \rightarrow 0$ implies $\mathbf{P}(\tau^\varepsilon < T) \rightarrow 0$.

3. Integration by parts and Itô's formula yield

$$\begin{aligned}
\frac{\Delta \tilde{X}^2}{R} &= -\frac{\Delta \tilde{X}^2}{R^2} \cdot R + \frac{\Delta \tilde{X}^2}{R^3} \cdot [R, R] + 2 \frac{\Delta \tilde{X} (\varphi^\varepsilon - \varphi)}{R} \cdot S \\
&\quad - 2 \frac{\Delta \tilde{X} \Delta \kappa}{R} \cdot I + \frac{(\varphi^\varepsilon - \varphi)^2}{R} \cdot [S, S] - 2 \frac{\Delta \tilde{X} (\varphi^\varepsilon - \varphi)}{R^2} \cdot [R, S].
\end{aligned}$$

By (3.22) and definition of φ' , κ' , the \mathbf{Q} -drift of $\Delta \tilde{X}^2/R$ is

$$\int_0^\cdot -\frac{\Delta \tilde{X}^2}{R^2} \underbrace{\left(b_t^R - \frac{1}{R_t} \left(c_t^{R,R} - \frac{(c_t^{R,S})^2}{c_t^{S,S}} \right) + 2r_t \right)}_{\stackrel{(3.3)}{=} r_t} dt + \frac{\Delta \varphi^2}{R} \cdot [S, S] = -\frac{\Delta \kappa^2}{r} \cdot I + \frac{\Delta \varphi^2}{R} \cdot [S, S].$$

Moreover, from (3.39, 3.66), Condition (3.4) and Lemma A.2, we get

$$\begin{aligned}
\left\| \frac{\Delta \tilde{X}^2}{R^2} \cdot R \right\|_{H^2(\mathbf{Q})}^2 &= \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left(\frac{\Delta \tilde{X}_t}{R_t} \right)^4 c_t^{R,R} dt \right] < \infty, \\
\left\| \frac{\Delta \tilde{X} (\varphi^\varepsilon - \varphi)}{R} \cdot S \right\|_{H^2(\mathbf{Q})}^2 &\leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left(\frac{\Delta \tilde{X}_t (\varphi'_t \Delta \tilde{X}_t + R_t \beta_t)}{R_t} \right)^2 c_t^{Y,Y} dt \right] < \infty.
\end{aligned}$$

Therefore, the \mathbf{Q} -local martingale part of $\Delta \tilde{X}^2/R$ is a square-integrable \mathbf{Q} -martingale. Together, we have

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \tilde{X}_{\tau^\varepsilon}^2}{R_{\tau^\varepsilon}} \right] = \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2}{R} \cdot [S, S]_{\tau^\varepsilon} \right] - \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \kappa^2}{r} \cdot I_{\tau^\varepsilon} \right] + O(\varepsilon). \tag{3.69}$$

Moreover, using the Cauchy-Schwarz inequality and (A.4), Condition (3.4) and (3.58, 3.64, 3.65) yield

$$\begin{aligned} & \left| \mathbf{E}_{\mathbf{Q}} \left[\frac{(X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_{\tau^\varepsilon}(\varphi, \kappa) - \Delta \tilde{X}_{\tau^\varepsilon})^2}{R_{\tau^\varepsilon}} \right] \right| \\ & \leq \underbrace{\|R^{-1}\|_{S^2(\mathbf{Q})}}_{< \infty} \underbrace{\|X^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X(\varphi, \kappa) - \Delta \tilde{X}\|_{S^4(\mathbf{Q})}^2}_{=O(\varepsilon^{4/3})}; \end{aligned}$$

this in conjunction with (3.69) and the Cauchy-Schwarz inequality implies (3.50). \square

Now, we approximate the expected utility generated by $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ by developing a Taylor series expansion around the frictionless optimizer (φ, κ) up to the leading order $O(\varepsilon^{2/3})$.

Lemma 3.5.7. *Assume Condition (3.4). Let τ^ε be the stopping time in (3.5) or equivalently in (3.48) and $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ be defined by (3.29) and (3.6). Then we have*

$$(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) \in \mathcal{A}^\varepsilon(x^B, x^S)$$

and

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \iota U_1(t, \tilde{\kappa}_t^\varepsilon) dt + U_2(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)) \right] \\ & = \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \iota U_1(t, \kappa_t) dt + U_2(X_T(\varphi, \kappa)) \right] - y \mathbf{E}_{\mathbf{Q}} \left[\frac{(\Delta \varphi^+)^2 - \frac{3}{2} \Delta \varphi^2}{R} \cdot [S, S]_T \right] \\ & \quad + y \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S]_T \right] + o(\varepsilon^{2/3}). \end{aligned} \quad (3.70)$$

Proof. First, we show that $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ is an admissible portfolio/consumption pair. From (3.6, 3.46), we deduce that

$$X_t^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) = X_t^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) \in \left[\frac{1}{2}, \frac{3}{2} \right] X_t(\varphi, \kappa) \quad \text{for } t \leq \tau^\varepsilon, \quad (3.71)$$

$$X_t^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) = X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - \underbrace{\int_{\tau^\varepsilon}^t \frac{X_{\tau^\varepsilon}(\varphi, \kappa)}{4T} ds}_{\in [0, \frac{1}{4} X_{\tau^\varepsilon}(\varphi, \kappa)]} \quad \text{for } t > \tau^\varepsilon,$$

and thus

$$\frac{1}{4} X_{\tau^\varepsilon}(\varphi, \kappa) \leq X_t^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) \leq \frac{3}{2} X_{\tau^\varepsilon}(\varphi, \kappa) \quad \text{for } t > \tau^\varepsilon. \quad (3.72)$$

In particular, we have $X^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) \geq 0$. Taking into account that $\tilde{\kappa}^\varepsilon \geq 0$ (cf. (3.6, 3.47)), we then obtain the admissibility of $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$.

Next, we prove (3.70). Taylor expansion of $U_1(t, \cdot)$ for $t \in [0, T]$, U_2 yields

$$\begin{aligned}
& \int_0^T \imath U_1(t, \tilde{\kappa}_t^\varepsilon) dt + U_2(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)) \\
&= \int_0^T \imath U_1(t, \kappa_t) dt + U_2(X_T(\varphi, \kappa)) \\
&+ \underbrace{\imath \int_{\tau^\varepsilon}^T U_1(t, \tilde{\kappa}_t^\varepsilon) - U_1(t, \kappa_t) dt}_{=:G^{(5)}} + \underbrace{\int_0^{\tau^\varepsilon} \imath y Z_t \Delta \kappa_t dt}_{=:G^{(6)}} - \underbrace{\frac{1}{2} \int_0^{\tau^\varepsilon} \imath y Z_t \frac{\Delta \kappa_t^2}{r_t} dt}_{=:G^{(7)}} \\
&+ \underbrace{y Z_T (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))}_{=:G^{(8)}} - \underbrace{\frac{1}{2} y Z_T \frac{(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))^2}{R_T}}_{=:G^{(9)}} \\
&+ \frac{1}{2} \underbrace{\int_0^{\tau^\varepsilon} \imath (U_1''(t, \kappa_t + \vartheta_t \Delta \kappa_t) - U_1''(t, \kappa_t)) \Delta \kappa_t^2 dt}_{=:G^{(10)}} \\
&+ \frac{1}{2} \underbrace{(U_2''(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon, \vartheta_T)) - U_2''(X_T(\varphi, \kappa))) (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))^2}_{=:G^{(11)}}
\end{aligned}$$

for some random $(\vartheta_t)_{t \in [0, T]} \subseteq (0, 1)$, where

$$X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon, \vartheta_T) := X_T(\varphi, \kappa) + \vartheta_T (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa)).$$

We consider the expectation of the terms $G^{(5)}$ to $G^{(11)}$ separately in the following steps.

Step 1: Expectation of term $G^{(5)}$. From (3.6), Fubini's theorem, the Cauchy-Schwarz inequality, (3.49) and Condition (3.4), we deduce

$$\begin{aligned}
|\mathbf{E}[G^{(5)}]| &\leq \mathbf{E} \left[\mathbf{1}_{\{\tau^\varepsilon < T\}} \int_0^T |\imath U_1(t, \kappa_t)| + \left| \imath U_1 \left(t, \frac{X_{\tau^\varepsilon}(\varphi, \kappa)}{4T} \right) \right| dt \right] \\
&= \mathbf{E}_{\mathbf{Q}} \left[\frac{1}{Z_T} \mathbf{1}_{\{\tau^\varepsilon < T\}} \int_0^T |\imath U_1(t, \kappa_t)| + \left| \imath U_1 \left(t, \frac{X_{\tau^\varepsilon}(\varphi, \kappa)}{4T} \right) \right| dt \right] \\
&\leq \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}} \|Z^{-1}\|_{S^4(\mathbf{Q})} \|\imath U_1(\cdot, \kappa)\|_{S^4(\mathbf{Q})} \\
&\quad + \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}} \|Z^{-1}\|_{S^4(\mathbf{Q})} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t, s \in [0, T]} \left| \imath U_1 \left(t, \frac{X_s(\varphi, \kappa)}{4T} \right) \right|^4 \right] \\
&= O(\varepsilon).
\end{aligned}$$

Step 2: Expectation of terms $G^{(6)}$ to $G^{(9)}$. Fubini's theorem yields

$$\mathbf{E}[G^{(6)}] = y \mathbf{E}_{\mathbf{Q}}[\Delta \kappa \bullet I_{\tau^\varepsilon}],$$

$$\mathbf{E} \left[G^{(7)} \right] = \frac{y}{2} \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \kappa^2}{r} \cdot I_{\tau^\varepsilon} \right].$$

Notice that by (3.72) and condition (3.4),

$$\mathbf{E}_{\mathbf{Q}} \left[|X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa)|^q \right] \leq \text{cst.} \|X(\varphi, \kappa)\|_{S^q(\mathbf{Q})}^q < \infty \quad (3.73)$$

holds for any $q \in \mathbb{N}$. Using the Cauchy-Schwarz inequality, (3.73, 3.71), (3.49) and Condition (3.4) lead to

$$\begin{aligned} & \left| \mathbf{E} \left[G^{(8)} \right] - y \mathbf{E}_{\mathbf{Q}} \left[X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_{\tau^\varepsilon}(\varphi, \kappa) \right] \right| \\ & \leq y \mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{\tau^\varepsilon < T\}} \left| (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa)) - (X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_{\tau^\varepsilon}(\varphi, \kappa)) \right| \right] \\ & \leq \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}} \|X(\varphi, \kappa)\|_{S^2(\mathbf{Q})} \\ & = O(\varepsilon) \end{aligned} \quad (3.74)$$

and

$$\begin{aligned} & \left| \mathbf{E} \left[G^{(9)} \right] - \frac{y}{2} \mathbf{E}_{\mathbf{Q}} \left[\frac{(X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_{\tau^\varepsilon}(\varphi, \kappa))^2}{R_{\tau^\varepsilon}} \right] \right| \\ & \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{\tau^\varepsilon < T\}} \left| \frac{1}{R_{\tau^\varepsilon}} \left| (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))^2 - (X_{\tau^\varepsilon}^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_{\tau^\varepsilon}(\varphi, \kappa))^2 \right| \right] \right. \\ & \quad \left. + \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{\tau^\varepsilon < T\}} \left| \frac{1}{R_T} - \frac{1}{R_{\tau^\varepsilon}} \right| (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))^2 \right] \right] \\ & \leq \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}} \|R^{-1}\|_{S^4(\mathbf{Q})} \|X(\varphi, \kappa)\|_{S^4(\mathbf{Q})}^2 \\ & = O(\varepsilon). \end{aligned} \quad (3.75)$$

Now, (3.74, 3.44) yield

$$\mathbf{E} \left[G^{(6)} + G^{(8)} \right] = y \mathbf{E}_{\mathbf{Q}} \left[\frac{2\Delta\varphi^2 - (\Delta\varphi^+)^2}{R} \cdot [S, S]_{\tau^\varepsilon} \right] + y \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta X \Delta \varphi}{R} \cdot [S, S]_{\tau^\varepsilon} \right] + O(\varepsilon);$$

(3.75, 3.50) yield

$$\mathbf{E} \left[G^{(7)} + G^{(9)} \right] = \frac{y}{2} \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta\varphi^2}{R} \cdot [S, S]_{\tau^\varepsilon} \right] + O(\varepsilon).$$

Step 3: Expectation of term $G^{(10)}$. Consider the case where $\iota = 1$. From Condition (3.2), (3.47), Remark 3.5.1-3 and (3.18), we obtain for $t \leq \tau^\varepsilon$ that

$$\left| U_1''(t, \kappa_t + \vartheta_t \Delta \kappa_t) \right| \leq a_2 \frac{U_1'(t, \kappa_t + \vartheta_t \Delta \kappa_t)}{\kappa_t + \vartheta_t \Delta \kappa_t} \leq a_2 \frac{U_1'(t, \frac{1}{2} \kappa_t)}{\frac{1}{2} \kappa_t} \leq \text{cst.} \frac{U_1'(t, \kappa_t)}{\kappa_t}.$$

In conjunction with the Cauchy-Schwarz inequality, Conditions (3.2, 3.4) and (3.66), this leads to

$$\begin{aligned} \sup_{\varepsilon \in (0,1)} \mathbf{E} \left[\left| \frac{G^{(10)}}{\varepsilon^{2/3}} \right|^2 \right] &\leq \text{cst.} \sup_{\varepsilon \in (0,1)} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in (0,T)} Z_t \left| \frac{\kappa_t^{-1} \Delta \kappa_t^2}{\varepsilon^{2/3}} \right|^2 \right] \\ &\leq \text{cst.} \|Z\|_{S^2(\mathbf{Q})} \|\iota \kappa^{-1}\|_{S^8(\mathbf{Q})}^2 \sup_{\varepsilon \in (0,1)} \left\| \frac{\Delta \kappa}{\varepsilon^{1/3}} \right\|_{S^8(\mathbf{Q})}^2 < \infty. \end{aligned}$$

Therefore, by [14, Theorem 27.2], $\{G^{(10)}/\varepsilon^{2/3}\}_{\varepsilon \in (0,1)}$ is uniformly integrable. Since $G^{(10)}/\varepsilon^{2/3} \rightarrow 0$ in probability for $\varepsilon \downarrow 0$, this implies

$$\mathbf{E}[G^{(10)}] = o(\varepsilon^{2/3}).$$

Step 4: Expectation of term $G^{(11)}$. Note that

$$\begin{aligned} &\frac{G^{(11)}}{\varepsilon^{2/3}} \\ &= \underbrace{\mathbf{1}_{\{\tau^\varepsilon = T\}} \left((U_2''(X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon, \vartheta_T)) - U_2''(X_T(\varphi, \kappa))) \frac{(X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_T(\varphi, \kappa))^2}{\varepsilon^{2/3}} \right)}_{=: G^{(12)}} \\ &\quad + \underbrace{\frac{\mathbf{1}_{\{\tau^\varepsilon < T\}}}{\varepsilon^{2/3}} \left((U_2''(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon, \vartheta_T)) - U_2''(X_T(\varphi, \kappa))) (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))^2 \right)}_{=: G^{(13)}}. \end{aligned}$$

From Condition (3.2), (3.71), Remark 3.5.1-3 and (3.18), we deduce

$$|U_2''(X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon, \vartheta_T))| \leq \frac{U_2'(X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon, \vartheta_T))}{X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon, \vartheta_T)} \leq \frac{U_2'(\frac{1}{2}X_T(\varphi, \kappa))}{\frac{1}{2}X_T(\varphi, \kappa)} \leq \text{cst.} \frac{U_2'(X_T(\varphi, \kappa))}{X_T(\varphi, \kappa)};$$

combined with the Cauchy-Schwarz inequality, Condition (3.4) and (3.45), this yields

$$\begin{aligned} \sup_{\varepsilon \in (0,1)} \mathbf{E} \left[\left| G^{(12)} \right|^{\frac{3}{2}} \right] &\leq \text{cst.} \sup_{\varepsilon \in (0,1)} \mathbf{E}_{\mathbf{Q}} \left[\left| Z_T^{\frac{1}{2}} \left| \frac{1}{X_T(\varphi, \kappa)} \frac{(X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_T(\varphi, \kappa))^2}{\varepsilon^{2/3}} \right|^2 \right|^{\frac{3}{2}} \right] \\ &\leq \text{cst.} \|Z\|_{S^1(\mathbf{Q})}^{\frac{1}{2}} \|X(\varphi, \kappa)^{-1}\|_{S^{12}(\mathbf{Q})}^{\frac{3}{2}} \\ &\quad \times \sup_{\varepsilon \in (0,1)} \left\| \frac{X^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X(\varphi, \kappa)}{\varepsilon^{1/3}} \right\|_{S^{12}(\mathbf{Q})}^3 \\ &< \infty. \end{aligned}$$

Due to Condition (3.2), (3.72), Remark 3.5.1-3 and (3.18), it holds that

$$\begin{aligned}
& |U_2''(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon, \vartheta_T))| \\
& \leq a_2 \frac{U_2'(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon, \vartheta_T))}{X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon, \vartheta_T)} \\
& \leq a_2 \frac{U_2'\left((1-\vartheta_T)X_T(\varphi, \kappa) + \frac{\vartheta_T}{4}X_{\tau^\varepsilon}(\varphi, \kappa)\right)}{(1-\vartheta_T)X_T(\varphi, \kappa) + \frac{\vartheta_T}{4}X_{\tau^\varepsilon}(\varphi, \kappa)} \\
& \leq a_2 \mathbf{1}_{\{X_T \leq \frac{1}{4}X_{\tau^\varepsilon}\}} \frac{U_2'(X_T(\varphi, \kappa))}{X_T(\varphi, \kappa)} + a_2 \mathbf{1}_{\{X_T > \frac{1}{4}X_{\tau^\varepsilon}\}} \frac{U_2'(\frac{1}{4}X_{\tau^\varepsilon}(\varphi, \kappa))}{\frac{1}{4}X_{\tau^\varepsilon}(\varphi, \kappa)} \\
& \leq \text{cst.} \sup_{t \in [0, T]} \frac{U_2'(X_t(\varphi, \kappa))}{X_t(\varphi, \kappa)}.
\end{aligned}$$

Thus, Hölder's inequality, (3.49), (3.73) and Condition (3.4) yield

$$\begin{aligned}
& \sup_{\varepsilon \in (0, 1)} \mathbf{E} \left[\left| G^{(13)} \right|^{\frac{3}{2}} \right] \\
& \leq \mathbf{E}_{\mathbf{Q}} \left[\frac{1}{Z_T} \left| \frac{\mathbf{1}_{\{\tau^\varepsilon < T\}}}{\varepsilon^{2/3}} \sup_{t \in [0, T]} \frac{U_2'(X_t(\varphi, \kappa))}{X_t(\varphi, \kappa)} (X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) - X_T(\varphi, \kappa))^2 \right|^{\frac{3}{2}} \right] \\
& \leq \text{cst.} \sup_{\varepsilon \in (0, 1)} \frac{\mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}}}{\varepsilon} \|Z^{-1}\|_{S^8(\mathbf{Q})} \|U_2'(X(\varphi, \kappa))\|_{S^8(\mathbf{Q})}^{\frac{3}{2}} \|X(\varphi, \kappa)^{-1}\|_{S^{24}(\mathbf{Q})}^{\frac{3}{2}} \\
& \quad \times \|X(\varphi, \kappa)\|_{S^{24}(\mathbf{Q})}^3 \\
& < \infty.
\end{aligned}$$

Combined with (A.4) and [14, Theorem 27.2], we deduce that $\{G^{(11)}/\varepsilon^{2/3}\}_{\varepsilon \in (0, 1)}$ is uniformly integrable. Since $G^{(11)}/\varepsilon^{2/3} \rightarrow 0$ in probability when $\varepsilon \downarrow 0$, this yields

$$\mathbf{E}[G^{(11)}] = o(\varepsilon^{2/3}).$$

Step 5: Further estimations. Finally, from the Cauchy-Schwarz inequality, (3.49), Hölder's inequality, Condition (3.4) and Lemma A.2 as well as (3.39), we have

$$\begin{aligned}
\left| \mathbf{E}_{\mathbf{Q}} \left[\int_{\tau^\varepsilon}^T \frac{(\Delta\varphi_t^+)^2 - \frac{3}{2}\Delta\varphi_t^2}{R_t} d[S, S]_t \right] \right| & \leq \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 R_t c_t^{Y, Y}|^2 dt \right]} \\
& = O(\varepsilon), \tag{3.76}
\end{aligned}$$

$$\begin{aligned}
\left| \mathbf{E}_{\mathbf{Q}} \left[\int_{\tau^\varepsilon}^T \frac{\varphi_t' \Delta \tilde{X}_t \Delta \varphi_t}{R_t} d[S, S]_t \right] \right| & \leq \text{cst.} \mathbf{Q}(\tau^\varepsilon < T)^{\frac{1}{2}} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\varphi_t' S_t \Delta \tilde{X}_t \beta_t c_t^{Y, Y}|^2 dt \right]} \\
& = O(\varepsilon), \tag{3.77}
\end{aligned}$$

which completes the proof. \square

3.5.3 Dual considerations

Aiming to prove the optimality of the candidate strategy $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$, we intend to find an upper bound to the utility optimization problem which matches the utility generated by $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ to the leading order. In the previous section, we have transformed the initial optimization problem under transaction costs into a frictionless one at price S^ε . Standard duality theory (cf. e.g. [24, 22]) yields that the dual problem of minimizing the expected convex conjugate \tilde{U} over all *EMM* for S^ε dominates the frictionless maximum expected utility for S^ε . We therefore apply Girsanov's theorem and construct the density process of an appropriate *EMM* for S^ε at which the dual object function related to the frictionless market at price S^ε equals the expected utility attained by $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ to the leading order.

Define the stopping time

$$\rho^{\varepsilon,1} := \inf \left\{ t \in [0, T] : \left| \frac{c_t^{\Delta S, S}}{c_t^{S, S}} \right| > \frac{1}{2} \right\} \quad (3.78)$$

and the process $N^\varepsilon := \ln Z^{\varepsilon, \mathbf{Q}}$ with

$$Z^{\varepsilon, \mathbf{Q}} := \mathcal{E} \left(-H^{\varepsilon, S} \cdot S - H^{\varepsilon, R} \cdot R^\perp \right), \quad (3.79)$$

where

$$H^{\varepsilon, S} := \frac{b^{\Delta S} - H^{\varepsilon, R} c^{\Delta S, R^\perp}}{c^{S, S} + c^{\Delta S, S}} \mathbf{1}_{[0, \rho^{\varepsilon,1}]}, \quad H^{\varepsilon, R} := -\frac{\mathcal{E} \left(-\frac{1}{R} \cdot R^\perp \right)}{R} \left(\frac{\Delta \varphi}{R \mathcal{E} \left(-\frac{1}{R} \cdot R^\perp \right)} \cdot S \right). \quad (3.80)$$

Further, define the stopping times

$$\rho^{\varepsilon,2} := \inf \left\{ t \in [0, T] : |Z_t^{\varepsilon, \mathbf{Q}} - 1| > \frac{1}{2} \right\}, \quad (3.81)$$

$$\rho^\varepsilon := \rho^{\varepsilon,1} \wedge \rho^{\varepsilon,2} \wedge T \quad (3.82)$$

and the following ‘‘stopped’’ processes:

$$\begin{aligned} \bar{\varphi}^\varepsilon &:= \varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon \wedge \rho^\varepsilon]}, & \Delta \bar{\varphi} &:= \Delta \varphi \mathbf{1}_{[0, \tau^\varepsilon \wedge \rho^\varepsilon]}, & \Delta \bar{\pi} &:= \Delta \pi \mathbf{1}_{[0, \tau^\varepsilon \wedge \rho^\varepsilon]}, \\ \bar{S}^\varepsilon &:= S \left(1 + \frac{\Delta S^{\tau^\varepsilon \wedge \rho^\varepsilon}}{S^{\tau^\varepsilon \wedge \rho^\varepsilon}} \right), & \bar{H}^{\varepsilon, R} &:= (H^{\varepsilon, R})^{\tau^\varepsilon \wedge \rho^\varepsilon}, & \bar{N}^\varepsilon &:= (N^\varepsilon)^{\tau^\varepsilon \wedge \rho^\varepsilon}, & \bar{Z}^\varepsilon &:= (Z^{\varepsilon, \mathbf{Q}})^{\tau^\varepsilon \wedge \rho^\varepsilon}. \end{aligned} \quad (3.83)$$

Remark 3.5.8. Define $Z^\varepsilon := Z \bar{Z}^\varepsilon$. From (3.79, 3.81), \bar{Z}^ε is a bounded \mathbf{Q} -local martingale and thus a \mathbf{Q} -martingale. Therefore, Z^ε is a \mathbf{P} -martingale. (3.80) and integration by parts yield that $\bar{Z}^\varepsilon \bar{S}^\varepsilon$ is a \mathbf{Q} -local martingale and hence $Z^\varepsilon \bar{S}^\varepsilon = Z \bar{Z}^\varepsilon \bar{S}^\varepsilon$ is a \mathbf{P} -local martingale. Consequently, Z^ε is the density process of an equivalent (local) martingale measure for \bar{S}^ε .

Note that in contrast to Chapter 2, the dual variable Z^ε constructed in the present general setting is not necessarily the minimal martingale measure for \bar{S}^ε . According to [5], the pair $(Z^\varepsilon, Z^\varepsilon \bar{S}^\varepsilon)$ is a state price density which is dual to the set of self-financing portfolios in the market with friction and therefore provides an upper bound to our original utility maximization problem under transaction costs.

In order to approximate the expectation of the convex conjugate \tilde{U} at the dual variable yZ^ε for small ε , we first estimate the deviation of Z^ε from Z , utilizing the investigation of the local parameters related to the shadow price process \bar{S}^ε provided in Section 3.5.1.

Lemma 3.5.9. *Under Condition (3.4), we have*

$$\left\| \bar{Z}^\varepsilon - 1 \right\|_{S^q(\mathbf{Q})} = O(\varepsilon^{1/3}) \quad \text{for any } q \in \mathbb{N}, \quad (3.84)$$

$$\mathbf{Q}(\rho^\varepsilon < T) = O(\varepsilon^2) \quad (3.85)$$

and

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}} \left[\int_0^T r_t \left(\bar{Z}_t^\varepsilon - 1 \right)^2 dt + R_T \left(\bar{Z}_T^\varepsilon - 1 \right)^2 \right] \\ &= \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2}{R} \cdot [S, S]_T \right] - 2\mathbf{E}_{\mathbf{Q}} \left[(\Delta \varphi H^{\varepsilon, R}) \cdot [R, S]_T \right] + o(\varepsilon^{2/3}). \end{aligned} \quad (3.86)$$

Proof. Let $q \in \mathbb{N}$. First, note that from (3.80), $H^{\varepsilon, R}R$ satisfies the affine linear SDE

$$d \left(H_t^{\varepsilon, R} R_t \right) = H_t^{\varepsilon, R} R_t d \left(-\frac{1}{R} \cdot R_t^\perp \right) - d\Delta\pi \cdot Y_t$$

and hence

$$H^{\varepsilon, R}R = -H^{\varepsilon, R} \cdot R^\perp - \Delta\pi \cdot Y. \quad (3.87)$$

By the Cauchy-Schwarz inequality, (A.2, A.7), Condition (3.4) and Lemma A.2, we have

$$\|H^{\varepsilon, R}\|_{S^q(\mathbf{Q})}^q \leq \text{cst.} \left\| \frac{\mathcal{E} \left(-\frac{1}{R} \cdot R^\perp \right)}{R} \right\|_{S^{2q}(\mathbf{Q})}^q \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{\beta_t^2 c_t^{Y, Y}}{[\mathcal{E} \left(-\frac{1}{R} \cdot R^\perp \right)]_t^2} \right|^q dt \right]} \varepsilon^{q/3}. \quad (3.88)$$

In view of (3.78, 3.80), from (3.32, 3.33, 3.34) and using Taylor expansion of $x \mapsto (1+x)^{-1}$ and (3.35), we deduce that on $\{\rho^\varepsilon \geq t\}$,

$$H_t^{\varepsilon, S} \sigma_t^S = \frac{\Delta \varphi_t}{R_t} \sigma_t^S + \underbrace{\frac{b_t^{\Delta S} \sigma_t^S - H_t^{\varepsilon, R} c_t^{\Delta S, R^\perp} \sigma_t^S}{c_t^{S, S} + c_t^{\Delta S, S}} - \frac{\Delta \varphi_t}{R_t} \sigma_t^S}_{=: G_t^{(14)}} \quad (3.89)$$

with

$$\left| \frac{\Delta \varphi_t}{R_t} \sigma_t^S \right| \leq \text{cst.} \beta_t S_t c_t^{Y,Y} \varepsilon^{1/3},$$

$$\left| H_t^{\varepsilon,S} \sigma_t^S \right| \leq \text{cst.} \underbrace{\left(\sum_{b^\Delta \in \mathcal{H}^{b^\Delta S} \cup \mathcal{R}^{b^\Delta S}} \left| \frac{b_t^\Delta}{\sigma_t^S} \right| + \sum_{c^{\Delta,R^\perp} \in \mathcal{H}^{c^{\Delta S,R^\perp}}} \left| \frac{c_t^{\Delta,R^\perp}}{\sigma_t^S} \right| \left| H_t^{\varepsilon,R} \right| \right)}_{=: G_t^{(15)}} \varepsilon^{1/3}, \quad (3.90)$$

$$\left| G_t^{(14)} \right| \leq \text{cst.} G_t^{(15)} \underbrace{\left(1 + \sum_{c^{\Delta,S} \in \mathcal{H}^{c^{\Delta S,S}}} \left| \frac{c_t^{\Delta,S}}{c_t^{S,S}} \right| \right)}_{=: G_t^{(16)}} \varepsilon^{2/3}. \quad (3.91)$$

From (3.81) and Taylor expansion of $x \mapsto e^x$, we obtain

$$\bar{Z}^\varepsilon - 1 = -\Delta \bar{\pi} \cdot Y - \bar{H}^{\varepsilon,R} \cdot R^\perp + (\bar{N}^\varepsilon + \Delta \bar{\pi} \cdot Y + H^{\varepsilon,R} \cdot R^\perp) + G^{(17)} \quad \text{with} \quad \left| G^{(17)} \right| \leq \frac{3}{4} |\bar{N}^\varepsilon|^2. \quad (3.92)$$

1. From (A.4, A.2, A.7), in view of (3.92, 3.89, 3.90, 3.91), and by (3.41, 3.88), Condition (3.4) and Lemma A.2, we deduce

$$\begin{aligned} & \left\| \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon,R} \cdot R^\perp \right\|_{S^q(\mathbf{Q})}^q \\ & \leq \text{cst.} \left\| \Delta \bar{\pi} \cdot Y \right\|_{H^q(\mathbf{Q})}^q + \text{cst.} \left\| \bar{H}^{\varepsilon,R} \cdot R^\perp \right\|_{H^q(\mathbf{Q})}^q \\ & \leq \text{cst.} \underbrace{\sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \beta_t^2 c_t^{Y,Y} \right|^q dt \right]}}_{< \text{cst.}} \varepsilon^{q/3} + \text{cst.} \underbrace{\sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \left(\frac{H_t^{\varepsilon,R}}{\varepsilon^{1/3}} \right)^2 c_t^{R^\perp, R^\perp} \right|^q dt \right]}}_{< \text{cst.}} \varepsilon^{q/3} \end{aligned}$$

and thus

$$\left\| \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon,R} \cdot R^\perp \right\|_{S^q(\mathbf{Q})} = O(\varepsilon^{1/3}); \quad (3.93)$$

$$\begin{aligned}
& \left\| \bar{Z}^\varepsilon - 1 + \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon, R} \cdot R^\perp \right\|_{S^q(\mathbf{Q})}^q \\
& \leq \text{cst.} \left\| \bar{N}^\varepsilon + \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon, R} \cdot R^\perp \right\|_{H^q(\mathbf{Q})}^q + \text{cst.} \left\| \bar{N}^\varepsilon \right\|_{H^{2q}(\mathbf{Q})}^{2q} \\
& \leq \text{cst.} \sum_{m=1,2} \sum_{n=1,2} \underbrace{\left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| G_t^{(15)} + G_t^{(15)} G_t^{(16)} \right|^{2qm} dt \right] \right)^{\frac{1}{mn}}}_{< \text{cst.}} \varepsilon^{2q/3} \\
& \quad + \text{cst.} \sum_{m=1,2} \sum_{n=1,2} \underbrace{\left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \left(\frac{H_t^{\varepsilon, R}}{\varepsilon^{1/3}} \right)^2 c_t^{R^\perp, R^\perp} \right|^{2qm} dt \right] \right)^{\frac{1}{mn}}}_{< \text{cst.}} \varepsilon^{2q/3}
\end{aligned}$$

and hence

$$\left\| \bar{Z}^\varepsilon - 1 + \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon, R} \cdot R^\perp \right\|_{S^q(\mathbf{Q})}^q = O(\varepsilon^{2q/3}). \quad (3.94)$$

Combined with (A.4), this implies (3.84).

2. (3.78), Markov's inequality, (3.35, A.4) and (3.41) yield

$$\mathbf{Q}(\rho^{\varepsilon, 1} < T) \leq 8 \left\| \frac{c^{\Delta S, S}}{c^{S, S}} \right\|_{S^3(\mathbf{Q})}^3 \leq \text{cst.} \underbrace{\sum_{c^{\Delta S, S} \in \mathcal{H} c^{\Delta S, S}} \left\| \frac{c^{\Delta S, S}}{c^{S, S}} \right\|_{S^3(\mathbf{Q})}^3}_{< \infty} \varepsilon^2. \quad (3.95)$$

Further, from (3.81), Markov's inequality, (3.95), (A.2), (3.89, 3.90) and (A.7), by (3.41, 3.88), Condition(3.4) and Lemma A.2, we deduce that

$$\begin{aligned}
\mathbf{Q}(\rho^{\varepsilon, 2} < T) &= \mathbf{Q}(\{\rho^{\varepsilon, 2} < T\} \cap \{\rho^{\varepsilon, 2} < \rho^{\varepsilon, 1}\}) + \mathbf{Q}(\rho^{\varepsilon, 1} \leq \rho^{\varepsilon, 2} < T) \\
&\leq \mathbf{Q}\left(\exists t \in [0, T] : |N_t^\varepsilon| > \ln \frac{3}{2}\right) + \mathbf{Q}(\rho^{\varepsilon, 1} < T) \\
&\leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\sup_{t \in [0, T]} |N_t^\varepsilon|^6 \right] + O(\varepsilon^2) \\
&\leq \text{cst.} \|N^\varepsilon\|_{H^6(\mathbf{Q})}^6 + O(\varepsilon^2) \\
&\leq \text{cst.} \sum_{n=1,2} \underbrace{\left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |G_t^{(15)}|^{12} dt \right] \right)^{\frac{1}{n}}}_{< \text{cst.}} \varepsilon^2 \\
& \quad + \text{cst.} \sum_{n=1,2} \underbrace{\left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \left(\frac{H_t^{\varepsilon, R}}{\varepsilon^{1/3}} \right)^2 c_t^{R^\perp, R^\perp} \right|^{12} dt \right] \right)^{\frac{1}{n}}}_{< \text{cst.}} \varepsilon^2 + O(\varepsilon^2).
\end{aligned}$$

Together, (3.85) follows.

3. By the Cauchy-Schwarz inequality, (3.84, 3.94) and Condition (3.4), we get

$$\begin{aligned}
& \left| \mathbf{E}_{\mathbf{Q}} \left[\int_0^T r_t \left(\bar{Z}_t^\varepsilon - 1 \right)^2 - r_t \left(\Delta \bar{\pi} \cdot Y_t + \bar{H}^{\varepsilon, R} \cdot R_t^\perp \right)^2 dt \right] \right| \\
& \quad + \left| \mathbf{E}_{\mathbf{Q}} \left[R_T \left(\bar{Z}_T^\varepsilon - 1 \right)^2 - R_T \left(\Delta \bar{\pi} \cdot Y_T + \bar{H}^{\varepsilon, R} \cdot R_T^\perp \right)^2 \right] \right| \\
& \leq \text{cst.} \underbrace{\left(\|r\|_{S^2(\mathbf{Q})} + \|R\|_{S^2(\mathbf{Q})} \right)}_{< \infty} \underbrace{\left\| \bar{Z}^\varepsilon - 1 + \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon, R} \cdot R^\perp \right\|_{S^4(\mathbf{Q})}^2}_{=O(\varepsilon^{4/3})} \\
& \quad + \text{cst.} \underbrace{\left(\|r\|_{S^2(\mathbf{Q})} + \|R\|_{S^2(\mathbf{Q})} \right)}_{< \infty} \underbrace{\left\| \bar{Z}^\varepsilon - 1 + \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon, R} \cdot R^\perp \right\|_{S^4(\mathbf{Q})}}_{=O(\varepsilon^{2/3})} \underbrace{\left\| \bar{Z}^\varepsilon - 1 \right\|_{S^4(\mathbf{Q})}}_{=O(\varepsilon^{1/3})} \\
& = O(\varepsilon).
\end{aligned}$$

Moreover, using the Cauchy-Schwarz inequality, from (3.49, 3.85), Condition (3.4) and (3.93), we obtain

$$\begin{aligned}
& \left| \mathbf{E}_{\mathbf{Q}} \left[\int_{\tau^\varepsilon \wedge \rho^\varepsilon}^T r_t \left(\Delta \bar{\pi} \cdot Y_t + \bar{H}^{\varepsilon, R} \cdot R_t^\perp \right)^2 dt \right] \right| \\
& \quad + \left| \mathbf{E}_{\mathbf{Q}} \left[(R_T - R_{\tau^\varepsilon \wedge \rho^\varepsilon}) \left(\Delta \bar{\pi} \cdot Y_T + \bar{H}^{\varepsilon, R} \cdot R_T^\perp \right)^2 \right] \right| \\
& \leq \text{cst.} \mathbf{Q} \left(\tau^\varepsilon \wedge \rho^\varepsilon < T \right)^{\frac{1}{2}} \underbrace{\left(\|r\|_{S^4(\mathbf{Q})} + \|R\|_{S^4(\mathbf{Q})} \right) \left\| \Delta \bar{\pi} \cdot Y + \bar{H}^{\varepsilon, R} \cdot R^\perp \right\|_{S^8(\mathbf{Q})}^2}_{< \infty} \\
& = O(\varepsilon).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \mathbf{E}_{\mathbf{Q}} \left[\int_0^T r_t \left(\bar{Z}_t^\varepsilon - 1 \right)^2 dt + R_T \left(\bar{Z}_T^\varepsilon - 1 \right)^2 \right] \\
& = \mathbf{E}_{\mathbf{Q}} \left[\int_0^{\tau^\varepsilon \wedge \rho^\varepsilon} r_t \left(\Delta \bar{\pi} \cdot Y_t + \bar{H}^{\varepsilon, R} \cdot R_t^\perp \right)^2 dt + R_{\tau^\varepsilon \wedge \rho^\varepsilon} \left(\Delta \bar{\pi} \cdot Y_{\tau^\varepsilon \wedge \rho^\varepsilon} + \bar{H}^{\varepsilon, R} \cdot R_{\tau^\varepsilon \wedge \rho^\varepsilon}^\perp \right)^2 \right] \\
& \quad + O(\varepsilon).
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
& R \left(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp \right)^2 \\
&= \underbrace{2 \left(R\Delta\pi(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp) \right) \cdot Y}_{=:G^{(18)}} + \underbrace{2 \left(H^{\varepsilon,R}R(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp) \right) \cdot R^\perp}_{=:G^{(19)}} \\
&\quad + \underbrace{\left(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp \right)^2 \cdot R}_{=:G^{(20)}} + \underbrace{2 \left(H^{\varepsilon,R}(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp) \right) \cdot [R, R^\perp]}_{=:G^{(21)}} \\
&\quad + \underbrace{\left(R(H^{\varepsilon,R})^2 \right) \cdot [R^\perp, R^\perp]}_{=:G^{(22)}} + \underbrace{\left(R\Delta\pi^2 \right) \cdot [Y, Y] + 2 \left(\Delta\pi(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp) \right) \cdot [Y, R]}_{=:G^{(23)}}.
\end{aligned}$$

From $c^{R,R^\perp} = c^{R^\perp,R^\perp} = R(b^R + r)$ and in view of (3.87), we deduce that the \mathbf{Q} -drift of $\int_0^\cdot r_t (\Delta\pi \cdot Y_t + H^{\varepsilon,R} \cdot R_t^\perp)^2 dt + R(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp)^2$ equals

$$G^{(23)} + \int_0^\cdot \underbrace{\left(G_t^{(20)} + 2G_t^{(21)}R_t + G_t^{(22)}R_t \right)}_{\stackrel{(3.87)}{=} 0} (b_t^R + r_t) dt.$$

Furthermore, from (3.87, 3.88), Condition (3.4) and Lemma A.2, we have

$$\begin{aligned}
\|G^{(18)}\|_{H^2(\mathbf{Q})}^2 &= 2 \left\| (R\Delta\pi H^{\varepsilon,R}R) \cdot Y \right\|_{H^2(\mathbf{Q})}^2 \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left(R_t^2 \beta_t H_t^{\varepsilon,R} \right)^2 c_t^{Y,Y} dt \right] < \infty, \\
\|G^{(19)}\|_{H^2(\mathbf{Q})}^2 &= 2 \left\| (H^{\varepsilon,R}R)^2 \cdot R^\perp \right\|_{H^2(\mathbf{Q})}^2 \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left(H_t^{\varepsilon,R} R_t \right)^4 c_t^{R^\perp, R^\perp} dt \right] < \infty, \\
\|G^{(20)} \cdot R\|_{H^2(\mathbf{Q})}^2 &= \left\| (H^{\varepsilon,R}R) \cdot R \right\|_{H^2(\mathbf{Q})}^2 \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left(H_t^{\varepsilon,R} R_t \right)^2 c_t^{R,R} dt \right] < \infty.
\end{aligned}$$

Therefore, the \mathbf{Q} -local martingale part of $R(\Delta\pi \cdot Y + H^{\varepsilon,R} \cdot R^\perp)^2$ is a square-integrable \mathbf{Q} -martingale. Together, we obtain

$$\begin{aligned}
& \mathbf{E}_{\mathbf{Q}} \left[\int_0^T r_t \left(\bar{Z}_t^\varepsilon - 1 \right)^2 dt + R_T \left(\bar{Z}_T^\varepsilon - 1 \right)^2 \right] \\
&= \mathbf{E}_{\mathbf{Q}} \left[\left(R\Delta\pi^2 \right) \cdot [Y, Y]_{\tau^\varepsilon \wedge \rho^\varepsilon} - 2 \left(\Delta\pi H^{\varepsilon,R}R \right) \cdot [Y, R]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon).
\end{aligned}$$

Finally, the Cauchy-Schwarz inequality, Hölder's inequality, (3.49, 3.85), Condition (3.4) and Lemma A.2 as well as (3.88) yield

$$\begin{aligned}
\left| \mathbf{E}_{\mathbf{Q}} \left[\int_{\tau^\varepsilon \wedge \rho^\varepsilon}^T R_t \Delta\pi^2 d[Y, Y]_t \right] \right| &\leq \text{cst.} \mathbf{Q}(\tau^\varepsilon \wedge \rho^\varepsilon < T)^{\frac{1}{2}} \underbrace{\sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |R_t \beta_t^2 c_t^{Y,Y}|^2 dt \right]}}_{< \infty} \\
&= O(\varepsilon),
\end{aligned}$$

$$\begin{aligned} \left| \mathbf{E}_{\mathbf{Q}} \left[\int_{\tau^\varepsilon \wedge \rho^\varepsilon}^T \Delta \pi_t (\Delta \pi \bullet Y_t) d[Y, R]_t \right] \right| &\leq \text{cst.} \mathbf{Q} (\tau^\varepsilon \wedge \rho^\varepsilon < T)^{\frac{1}{2}} \underbrace{\sqrt{\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t H_t^{\varepsilon, R} c_t^{Y, R}|^2 dt \right]}}_{< \text{cst.}} \\ &= O(\varepsilon). \end{aligned}$$

(3.86) then follows. □

By means of the estimations in the previous lemma and applying the duality theory in [24, 22] linking yZ to (φ, κ) , we are now able to calculate the desired dual upper bound provided by Z^ε .

Lemma 3.5.10. *Assume Condition (3.4). It holds that*

$$\begin{aligned} &\mathbf{E} \left[\int_0^T \mathfrak{U}_1 \left(t, yZ_t \bar{Z}_t^\varepsilon \right) dt + \tilde{U}_2 \left(yZ_T \bar{Z}_T^\varepsilon \right) \right] + xy \\ &= \mathbf{E} \left[\int_0^T \mathfrak{U}_1 \left(t, \kappa_t \right) dt + U_2 \left(X_T \left(\varphi, \kappa \right) \right) \right] - y \mathbf{E}_{\mathbf{Q}} \left[\frac{(\Delta \varphi^+)^2 - \frac{3}{2} \Delta \varphi^2}{R} \bullet [S, S]_T \right] \\ &\quad - y \mathbf{E}_{\mathbf{Q}} \left[(\Delta \varphi H^{\varepsilon, R}) \bullet [R, S]_T \right] + o(\varepsilon^{2/3}). \end{aligned} \tag{3.96}$$

Proof. Taylor expansion of $\tilde{U}_1(t, \cdot)$ for $t \in [0, T]$ and \tilde{U}_2 yields

$$\begin{aligned} &\int_0^T \mathfrak{U}_1 \left(t, yZ_t \bar{Z}_t^\varepsilon \right) dt + \tilde{U}_2 \left(yZ_T \bar{Z}_T^\varepsilon \right) \\ &= \int_0^T \mathfrak{U}_1 \left(t, yZ_t \right) dt + \tilde{U}_2 \left(yZ_T \right) \\ &\quad + \underbrace{\int_0^T \mathfrak{U}'_1 \left(t, yZ_t \right) yZ_t \left(\bar{Z}_t^\varepsilon - 1 \right) dt + \tilde{U}'_2 \left(yZ_T \right) yZ_T \left(\bar{Z}_T^\varepsilon - 1 \right)}_{=: G^{(24)}} \\ &\quad + \underbrace{\frac{1}{2} \int_0^T \mathfrak{U}''_1 \left(t, yZ_t \right) \left(yZ_t \right)^2 \left(\bar{Z}_t^\varepsilon - 1 \right)^2 dt + \frac{1}{2} \tilde{U}''_2 \left(yZ_T \right) \left(yZ_T \right)^2 \left(\bar{Z}_T^\varepsilon - 1 \right)^2}_{=: G^{(25)}} \\ &\quad + \underbrace{\frac{1}{2} \int_0^T \left(\tilde{U}''_1 \left(t, yZ_t \left(1 + \vartheta_t \left(\bar{Z}_t^\varepsilon - 1 \right) \right) \right) - \tilde{U}''_1 \left(t, yZ_t \right) \right) \left(yZ_t \right)^2 \left(\bar{Z}_t^\varepsilon - 1 \right)^2 dt}_{=: G^{(26)}} \\ &\quad + \underbrace{\frac{1}{2} \left(\tilde{U}''_2 \left(yZ_T \left(1 + \vartheta_T \left(\bar{Z}_T^\varepsilon - 1 \right) \right) \right) - \tilde{U}''_2 \left(yZ_T \right) \right) \left(yZ_T \right)^2 \left(\bar{Z}_T^\varepsilon - 1 \right)^2}_{=: G^{(27)}} \end{aligned}$$

for some random $(\vartheta_t)_{t \in [0, T]} \subseteq (0, 1)$. Since $X(\varphi, \kappa)$ and yZ are considered to be the frictionless optimizer for the primal and dual problem, respectively, by the conjugate

relation (cf. [22, Theorem 3.10]), it holds that

$$\mathbf{E} \left[\int_0^T \imath \tilde{U}_1(t, yZ_t) dt + \tilde{U}_2(yZ_T) \right] = \mathbf{E} \left[\int_0^T \imath U_1(t, \kappa_t) dt + U_2(X_T(\varphi, \kappa)) \right] - xy, \quad (3.97)$$

$$-\imath \tilde{U}_1'(t, yZ_t) = \kappa_t, \quad -\tilde{U}_2'(yZ_T) = X_T(\varphi, \kappa), \quad (3.98)$$

$$\imath \tilde{U}_1''(t, yZ_t)(yZ_t)^2 = \imath r_t y Z_t, \quad \tilde{U}_2''(yZ_T)(yZ_T)^2 = R_T y Z_T, \quad (3.99)$$

and

$$x = \mathbf{E} \left[\int_0^T Z_t \kappa_t dt + Z_T X_T(\varphi, \kappa) \right] = \mathbf{E}_{\mathbf{Q}} [x + \varphi \cdot S_T], \quad (3.100)$$

where the second equality in (3.100) follows from Fubini's theorem.

Step 1. From (3.98), (3.100), Remark 3.5.8 and Fubini's theorem, we have

$$\begin{aligned} \mathbf{E} \left[G^{(24)} \right] &= y \mathbf{E} \left[\int_0^T Z_t \kappa_t dt + Z_T X_T(\varphi, \kappa) \right] - y \mathbf{E} \left[\int_0^T Z_t \bar{Z}_t^\varepsilon \kappa_t dt + Z_T \bar{Z}_T^\varepsilon X_T(\varphi, \kappa) \right] \\ &= \underbrace{y \mathbf{E}_{\mathbf{Q}} [x + \varphi \cdot S_T]}_{=0} - xy - y \mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\varphi \cdot S_T) \right], \end{aligned}$$

and

$$\begin{aligned} -\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\varphi \cdot S_T) \right] &= -\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon) \right] + \mathbf{E}_{\mathbf{Q}} \left[\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T \right] \\ &\quad + \mathbf{E}_{\mathbf{Q}} \left[(\bar{Z}_T^\varepsilon - 1) (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T) \right]. \end{aligned}$$

Note that from Remark 3.5.8, $\bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon$ is a local martingale under the measure with density process $Z^\varepsilon = \bar{Z}^\varepsilon$. (3.81, 3.65) and Conditions (3.2, 3.4) yield

$$\begin{aligned} &\left\| \bar{Z}^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon) \right\|_{S^1(\mathbf{Q})} \\ &\leq \text{cst.} \left(\|X(\varphi, \kappa)\|_{S^1(\mathbf{Q})} + \|\kappa\|_{S^1(\mathbf{Q})} + \|X^\varepsilon(\varphi^\varepsilon, 0) - (x + \bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon)\|_{S^1(\mathbf{Q})} \right) \\ &< \infty, \end{aligned}$$

and hence the \mathbf{Q} -local martingale $\bar{Z}^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}^\varepsilon)$ is a uniformly integrable \mathbf{Q} -martingale and

$$\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon) \right] = 0.$$

From (3.52, 3.57, 3.63), in view of (3.49, 3.85) and by the same argument as in (3.74), we obtain

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \left[\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T \right] &= \mathbf{E}_{\mathbf{Q}} \left[\frac{2\Delta\varphi^2 - (\Delta\varphi^+)^2}{R} \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] \\ &\quad + \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon). \end{aligned}$$

Observe that

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}} \left[\left(\bar{Z}_T^\varepsilon - 1 \right) \left(\bar{\varphi}^\varepsilon \cdot \bar{S}_T^\varepsilon - \varphi \cdot S_T \right) \right] \\ &= \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\left(\bar{Z}_T^\varepsilon - 1 \right) \left(\varphi^\varepsilon - \varphi \right) \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} \right]}_{=: G^{(28)}} + \underbrace{\mathbf{E}_{\mathbf{Q}} \left[\left(\bar{Z}_T^\varepsilon - 1 \right) \left(\bar{\varphi}^\varepsilon \cdot \Delta \bar{S}_T + \varphi \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} - \varphi \cdot S_T \right) \right]}_{=: G^{(29)}}. \end{aligned}$$

The Cauchy-Schwarz inequality, (3.42) and (3.94) yield

$$\begin{aligned} |G^{(28)}| &= -\mathbf{E}_{\mathbf{Q}} \left[\left(\frac{\Delta \bar{\varphi}}{R} \cdot S_T + \bar{H}^{\varepsilon, R} \cdot R_T^\perp \right) \left(\varphi^\varepsilon - \varphi \right) \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] \\ &\quad + \mathbf{E}_{\mathbf{Q}} \left[\left(\bar{Z}_T^\varepsilon - 1 + \frac{\Delta \bar{\varphi}}{R} \cdot S_T + \bar{H}^{\varepsilon, R} \cdot R_T^\perp \right) \left(\varphi^\varepsilon - \varphi \right) \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] \\ &= -\mathbf{E}_{\mathbf{Q}} \left[\left(\frac{\Delta \bar{\varphi}}{R} \cdot S_T + \bar{H}^{\varepsilon, R} \cdot R_T^\perp \right) \left(\varphi^\varepsilon - \varphi \right) \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon) \\ &= -\mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2}{R} \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] - \mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon). \end{aligned}$$

Using the Cauchy-Schwarz inequality and (A.4), in view of (3.81, 3.84, 3.58, 3.64) and combining the argument in (3.74) with (3.85), we conclude

$$\begin{aligned} |G^{(29)}| &\leq \text{cst.} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\left(\bar{Z}_T^\varepsilon - 1 \right)^2 \right]} \sqrt{\mathbf{E}_{\mathbf{Q}} \left[\left(\bar{\varphi}^\varepsilon \cdot \Delta \bar{S}_T \right)^2 \right]} \\ &\quad + \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{\tau^\varepsilon \wedge \rho^\varepsilon < T\}} \left| \varphi \cdot S_{\tau^\varepsilon \wedge \rho^\varepsilon} - \varphi \cdot S_T \right| \right] \\ &= O(\varepsilon). \end{aligned}$$

Together, it follows that

$$\mathbf{E} \left[G^{(24)} \right] = y \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2 - (\Delta \varphi^+)^2}{R} \cdot [S, S]_{\tau^\varepsilon \wedge \rho^\varepsilon} \right] + O(\varepsilon). \quad (3.101)$$

Step 2. (3.99), Fubini's theorem and (3.86) imply

$$\mathbf{E} \left[G^{(25)} \right] = \frac{y}{2} \mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta \varphi^2}{R} \cdot [S, S]_T \right] - y \mathbf{E}_{\mathbf{Q}} \left[(\Delta \varphi H^{\varepsilon, R}) \cdot [R, S]_T \right] + o(\varepsilon^{2/3}). \quad (3.102)$$

Step 3. We show that the family $\{(G^{(26)} + G^{(27)})/\varepsilon^{2/3}\}_{\varepsilon \in (0,1)}$ is uniformly integrable.

By Condition (3.17), (3.81), Remark 3.5.1-3 and (3.19), we deduce

$$\begin{aligned} \left| \tilde{U}_1'' \left(t, yZ_t \left(1 + \vartheta_t \left(\bar{Z}_t^\varepsilon - 1 \right) \right) \right) \right| &\leq \frac{1}{a_1} \left| \frac{\tilde{U}_1'(t, yZ_t(1 + \vartheta_t(\bar{Z}_t^\varepsilon - 1)))}{yZ_t(1 + \vartheta_t(\bar{Z}_t^\varepsilon - 1))} \right| \\ &\leq \frac{1}{a_1} \left| \frac{\tilde{U}_1'(t, \frac{3}{2}yZ_t)}{\frac{3}{2}yZ_t} \right| \\ &\leq \text{cst.} \left| \frac{\tilde{U}_1'(t, yZ_t)}{yZ_t} \right| \end{aligned}$$

and

$$\begin{aligned} \left| \tilde{U}_2'' \left(yZ_T \left(1 + \vartheta_T \left(\bar{Z}_T^\varepsilon - 1 \right) \right) \right) \right| &\leq \frac{1}{a_1} \left| \frac{\tilde{U}_2'(yZ_T(1 + \vartheta_T(\bar{Z}_T^\varepsilon - 1)))}{yZ_T(1 + \vartheta_T(\bar{Z}_T^\varepsilon - 1))} \right| \\ &\leq \frac{1}{a_1} \left| \frac{\tilde{U}_2'(\frac{3}{2}yZ_T)}{\frac{3}{2}yZ_T} \right| \\ &\leq \text{cst.} \left| \frac{\tilde{U}_2'(yZ_T)}{yZ_T} \right|; \end{aligned}$$

using again (3.17) and combining this with (3.98), Fubini's theorem, the Cauchy-Schwarz inequality, (3.84) and Conditions (3.2, 3.4), we obtain

$$\begin{aligned} &\sup_{\varepsilon \in (0,1)} \mathbf{E} \left[\left| \frac{G^{(26)} + G^{(27)}}{\varepsilon^{2/3}} \right|^2 \right] \\ &\leq \text{cst.} \sup_{\varepsilon \in (0,1)} \mathbf{E}_{\mathbf{Q}} \left[\left| \int_0^T \kappa_t \frac{(\bar{Z}_t^\varepsilon - 1)^2}{\varepsilon^{2/3}} dt \right|^2 \right] + \text{cst.} \sup_{\varepsilon \in (0,1)} \mathbf{E}_{\mathbf{Q}} \left[\left| X_T(\varphi, \kappa) \frac{(\bar{Z}_T^\varepsilon - 1)^2}{\varepsilon^{2/3}} \right|^2 \right] \\ &\leq \text{cst.} \|\kappa\|_{S^4(\mathbf{Q})}^2 \sup_{\varepsilon \in (0,1)} \left\| \frac{\bar{Z}^\varepsilon - 1}{\varepsilon^{1/3}} \right\|_{S^8(\mathbf{Q})}^4 + \text{cst.} \|X(\varphi, \kappa)\|_{S^4(\mathbf{Q})}^2 \sup_{\varepsilon \in (0,1)} \left\| \frac{\bar{Z}^\varepsilon - 1}{\varepsilon^{1/3}} \right\|_{S^8(\mathbf{Q})}^4 \\ &< \infty. \end{aligned}$$

Thus by [14, Theorem 27.2], the family $\{(G^{(26)} + G^{(27)})/\varepsilon^{2/3}\}_{\varepsilon \in (0,1)}$ is uniformly integrable. Since $(G^{(26)} + G^{(27)})/\varepsilon^{2/3} \rightarrow 0$ in probability when $\varepsilon \downarrow 0$, this implies

$$\mathbf{E} \left[G^{(26)} + G^{(27)} \right] = o(\varepsilon^{2/3}). \quad (3.103)$$

Finally, the assertion follows from (3.97, 3.101, 3.102, 3.103) in conjunction with (3.85) and the argument in (3.76). \square

3.5.4 Optimality

Based on the primal and dual considerations above, in this section we prove the leading-order optimality of policy $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$.

First, we apply the ergodic argument stated in Section 4.3 to remove the dependence of the expressions (3.70, 3.96) on the reflecting process $\Delta\varphi$.

Lemma 3.5.11. *Under Assumptions 3.2.5, we have*

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{\Delta\varphi^2}{R} \cdot [S, S]_T \right] = \frac{1}{3} \mathbf{E}_{\mathbf{Q}} \left[\frac{(\Delta\varphi^+)^2}{R} \cdot [S, S]_T \right] + o(\varepsilon^{2/3}), \quad (3.104)$$

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{\varphi' \Delta \tilde{X} \Delta \varphi}{R} \cdot [S, S]_T \right] = o(\varepsilon^{2/3}), \quad (3.105)$$

$$\mathbf{E}_{\mathbf{Q}} \left[(\Delta\varphi H^{\varepsilon, R}) \cdot [R, S]_T \right] = o(\varepsilon^{2/3}). \quad (3.106)$$

Proof. We verify the assumptions for Lemma 4.3.4 in the context of this chapter. Note that the ‘‘aisle type’’ strategy φ^δ characterized by (4.1, 4.2, 4.3) refers to our candidate strategy φ^ε satisfying (3.20, 3.21, 3.22) with $\mathbf{P} \triangleq \mathbf{Q}$, $\delta \triangleq (\varepsilon/2)^{1/3}$, $\varphi \triangleq \varphi$, $\Delta\Phi \triangleq \varphi' \Delta \tilde{X}$ and $\beta \triangleq \beta$. In view of (3.22, 3.30) and applying Itô’s formula to $\Delta\varphi^+/\varepsilon^{1/3}$, $\varphi' \Delta \tilde{X}/\varepsilon^{1/3}$, we deduce that Condition (3.4) combined with (3.39) and Lemma A.2 implies Condition (4.4). Moreover, according to Assumption 3.2.5, $c^{\varphi, \varphi}$ is considered to be continuous and strictly positive. Therefore, Assumption 4.2.1 is fulfilled. Further, corresponding to (3.42, 3.80), let us define the following processes:

$$\Delta Y^{(1)} := \frac{\Delta\varphi}{R \mathcal{E}(H)} \cdot S, \quad \Delta Y^{(2)} := \frac{\Delta\varphi}{R \mathcal{E}\left(-\frac{1}{R} \cdot R^\perp\right)} \cdot S,$$

$$K := \left(\frac{\Delta\varphi^+}{\varepsilon^{1/3}} \right)^2 \frac{c^{S, S}}{R}, \quad L^{(1)} := \mathcal{E}(H) \varphi' c^{S, S} \frac{\Delta\varphi^+}{\varepsilon^{1/3}}, \quad L^{(2)} := \mathcal{E}\left(-\frac{1}{R} \cdot R^\perp\right) \frac{\Delta\varphi^+}{\varepsilon^{1/3}} \frac{c^{R, S}}{R},$$

where

$$H := \varphi' \cdot S + \kappa' \cdot I - \frac{1}{R} \cdot R + \frac{1}{R} \cdot [R, R].$$

From Condition (3.4) and Lemma A.2, we have

$$\begin{aligned}
& \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |K_t| dt \right] + \sum_{i=1,2} \left(\mathbf{E}_{\mathbf{Q}} \left[\int_0^T |c_t^{\Delta Y^{(i)}, \Delta Y^{(i)}}|^2 ds \right] + \|L^{(i)}\|_{S^4} \right) \\
& \leq \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T |\beta_t^2 R_t c_t^{Y,Y}| dt \right] + \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{\beta_t^2 c_t^{Y,Y}}{[\mathcal{E}(H)_t]^2} \right|^2 dt \right] \\
& \quad + \text{cst.} \mathbf{E}_{\mathbf{Q}} \left[\int_0^T \left| \frac{\beta_t^2 c_t^{Y,Y}}{[\mathcal{E}(-\frac{1}{R} \cdot R^\perp)_t]^2} \right|^2 dt \right] \\
& \quad + \text{cst.} \left\| \mathcal{E}(H) \phi' S c^{Y,Y} \beta R \right\|_{S^4(\mathbf{Q})} + \text{cst.} \left\| \mathcal{E} \left(-\frac{1}{R} \cdot R^\perp \right) \beta c^{R,Y} \right\|_{S^4(\mathbf{Q})} \\
& < \text{cst.}
\end{aligned}$$

(cf. Condition (4.17)). Applying Lemma 4.3.4 to K , $L \triangleq L^{(i)}$, $\Delta Y \triangleq \Delta Y^{(i)}$ for $i = 1$ and $i = 2$, we then obtain (3.104), (3.105) and (3.106), respectively. \square

Lemma 3.5.12. *Under Assumption 3.2.5, we have*

$$\begin{aligned}
& \sup_{(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(X_T^\varepsilon(\psi, k)) \right] \\
& = \mathbf{E} \left[\int_0^T \iota U_1(t, \tilde{\kappa}_t^\varepsilon) dt + U_2(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)) \right] + o(\varepsilon^{2/3}).
\end{aligned}$$

Further, the minimum certainty equivalent loss is

$$\Delta \text{CE}^\varepsilon = \frac{1}{2} \mathbf{E}_{\mathbf{Q}} \left[\frac{(\Delta \varphi^+)^2}{R} \cdot [S, S]_T \right] + o(\varepsilon^{2/3}).$$

Proof. 1. We prove the approximate optimality of $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ via the conjugate relation. Let $(\psi, k) \in \mathcal{A}^\varepsilon(x^B, x^S)$ be an arbitrary admissible trading strategy. In view of (3.83), we have $\bar{S}^\varepsilon \in [(1 - \varepsilon)S, (1 + \varepsilon)S]$, which implies

$$x + \psi \cdot \bar{S}^\varepsilon - k \cdot I \geq X^\varepsilon(\psi, k) > 0. \tag{3.107}$$

Since by Remark 3.5.8, $x + \psi \cdot \bar{S}^\varepsilon$ is a local martingale under the measure with density process $Z^\varepsilon = \bar{Z}^\varepsilon$, in conjunction with (3.107), we obtain

$$\mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (x + \psi \cdot \bar{S}_T^\varepsilon) \right] \leq x. \tag{3.108}$$

Combining the above considerations with Fubini's theorem, we deduce

$$\begin{aligned}
& \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(X_T^\varepsilon(\psi, k)) \right] \\
& \stackrel{(3.107)}{\leq} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(x + \psi \cdot \bar{S}_T^\varepsilon - k \cdot I_T) \right] \\
& \stackrel{(3.16)}{\leq} \mathbf{E} \left[\int_0^T \iota \tilde{U}_1(t, y Z_t \bar{Z}_t^\varepsilon) dt + \tilde{U}_2(y Z_T \bar{Z}_T^\varepsilon) \right] + y \mathbf{E}_{\mathbf{Q}} \left[\bar{Z}_T^\varepsilon (x + \psi \cdot \bar{S}_T^\varepsilon) \right] \\
& \stackrel{(3.108)}{\leq} \mathbf{E} \left[\int_0^T \iota \tilde{U}_1(t, y Z_t \bar{Z}_t^\varepsilon) dt + \tilde{U}_2(y Z_T \bar{Z}_T^\varepsilon) \right] + xy \\
& \stackrel{(3.96, 3.104, 3.106)}{=} \mathbf{E} \left[\int_0^T \iota U_1(t, \kappa_t) dt + U_2(X_T(\varphi, \kappa)) \right] \\
& \quad - \frac{y}{2} \mathbf{E}_{\mathbf{Q}} \left[\frac{(\Delta \varphi^+)^2}{R} \cdot [S, S]_T \right] + o(\varepsilon^{2/3}) \\
& \stackrel{(3.70, 3.104, 3.105)}{=} \mathbf{E} \left[\int_0^T \iota U_1(t, \tilde{\kappa}_t^\varepsilon) dt + U_2(X_T^\varepsilon(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)) \right] + o(\varepsilon^{2/3}). \tag{3.109}
\end{aligned}$$

In addition, from Lemma 3.5.7, we have $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon) \in \mathcal{A}^\varepsilon(x^B, x^S)$. Then, the optimality of $(\tilde{\varphi}^\varepsilon, \tilde{\kappa}^\varepsilon)$ follows.

2. Consider the value function of the frictionless optimization problem, i.e., the mapping

$$z \mapsto v(z) := \sup_{(\psi, k) \in \mathcal{A}^0(z)} \mathbf{E} \left[\int_0^T \iota U_1(t, k_t) dt + U_2(z + \psi \cdot S_T - k \cdot I_T) \right].$$

From [22, Theorem 3.10 (v)], we have $v'(x) = y$. Taylor expansion of v around x in conjunction with (3.1, 3.109) then yields the minimum certainty equivalent loss. \square

Chapter 4

Trading Volume

In this chapter we investigate the class of “aisle-type” trading strategies staying constant while between some small, time-dependent barriers given by general Itô processes over a finite time horizon, i.e., trading does not occur inside some random corridor but happens at an infinite rate at the boundary in order to remain within the corridor. By means of a random time change and looking into the local dynamics of such a strategy, we prove the asymptotic formula for its total variation (trading volume) proposed in [19] providing a finite rate with respect to time.

In the context of utility maximization under proportional transaction costs with one risky asset, the behavior of the optimal trading strategy can be characterized by the aisle-type dynamics considered in this chapter, cf. [28, 6, 35]. For small transaction costs and the related asymptotics, in the previous two chapters we have rigorously proved the leading-order optimality of the candidate strategy φ^ε keeping the number of shares within an explicit random corridor (no-trade region) proposed in [20, 19] for related settings; applying the main theorem of this chapter, the trading volume and incurred transaction costs while following strategy φ^ε can be calculated to the leading order. Moreover, utilizing the method in the last section of this chapter, the minimum utility loss can be simplified to an expression that does not depend on the aisle-type strategy φ^ε itself, which completes the proofs in Chapters 2 and 3.

Concerning asymptotic results for the trading volume in portfolio optimization under proportional transaction costs, Janeček and Shreve [16] provide heuristic arguments in the context of maximizing power utility of consumption in the Black-Scholes model over an infinite time horizon, rescaling time and applying the ergodic theorem on the local time of reflected Brownian motion. For the corresponding long-run optimal investment problem in the same setting, Gerhold et al. [9] make the idea in [16] precise to derive rigorously the long-term average of the share- and wealth-turnover. In case of power utility and Black-Scholes model, the asymptotic turnover proposed in [19] and proved in this chapter coincide with the asymptotic results in [9]. By contrast, our approach inherits the spirit of the previous two chapters in respect of using an auxiliary shadow price

process for calculating the displacement loss and transaction costs caused by trading with the aisle-type strategy φ^ε , and relies on the uniform stationary distribution of standard Brownian motion reflected to remain within a fixed interval.

The remainder of this chapter is organized as follows. The subsequent section provides a precise definition of the aisle-type strategies considered in the study of this chapter. The main theorem concerning the asymptotic total variation of those strategies and applications are stated in Section 4.2. In Section 4.3 we carry out a random time change and apply ergodic arguments to reflected Brownian motion in order to remove the dependence of the asymptotic formulas on the aisle-type strategy itself, which completes the proof in Section 4.3 as well as the proofs of the minimum utility loss in Chapters 2 and 3.

4.1 Setup

First, let us recall some of the notations in [32]. For a process A of finite variation, $\int_0^\cdot |dA_t|$ denotes the total variation of A . Fixing a compact time interval $[0, T]$, we say that a sequence of processes $\{H^n\}_{n \in \mathbb{N}}$ converges to a process H *uniformly in probability* and write

$$H^n \xrightarrow[u.p.]{} H \quad \text{as } n \rightarrow \infty$$

if

$$\sup_{t \in [0, T]} |H_t^n - H_t| \longrightarrow 0 \text{ in probability}$$

for $n \rightarrow \infty$.

Now, fix a finite time horizon T and a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$. For any Itô process X , b^X and $c^{X,X}$ stand for its local drift and quadratic variation, respectively, i.e.,

$$dX_t = b_t^X dt + dM_t^X, \quad c_t^{X,X} := \frac{d[X, X]_t}{dt},$$

where M^X is a continuous local martingale starting in 0. Analogously, the local covariation of Itô processes X and Y is denoted by

$$c_t^{X,Y} := \frac{d[X, Y]_t}{dt}.$$

In this chapter we consider the following quantities to be known. We assume that δ is a real number in $(0, 1)$ and φ, β, Ψ are Itô processes which do not depend on δ , where β is strictly positive. Further, we suppose that $\Delta\Phi$ is an Itô process starting in 0.

Lemma 4.1.1. *Set $\Delta\varphi^\pm := \pm\beta\delta$. There exists a continuous adapted process φ^δ with*

$$\varphi^\delta = \varphi + \Delta\Phi + \Delta\varphi = \varphi^{\delta\uparrow} - \varphi^{\delta\downarrow}, \quad (4.1)$$

where $\Delta\varphi$ has values in $[\Delta\varphi^-, \Delta\varphi^+]$ and $\varphi^{\delta\uparrow}, \varphi^{\delta\downarrow}$ are increasing process such that

$$\varphi^{\delta\uparrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^-\} \subseteq \Omega \times [0, T], \quad (4.2)$$

$$\varphi^{\delta\downarrow} \text{ increases only on the set } \{\Delta\varphi = \Delta\varphi^+\} \subseteq \Omega \times [0, T]. \quad (4.3)$$

Proof. The assertion follows from [37, Theorem 3.3]. Indeed, (4.1) is related to a Skorohod SDE driven by the semimartingale $\varphi + \Delta\Phi$ with constant coefficients which are in particular Lipschitz continuous with respect to $\Delta\varphi$. The time-dependent reflecting barriers $\Delta\varphi^\pm$ are Lipschitz operators in the sense of [37, Definition 3.1] evaluated at process $\Delta\varphi$. \square

In Chapters 2 and 3 dealing with the problem of utility maximization under small transaction costs, we have proved the leading-order optimality of the candidate strategy φ^ε proposed in [20, 19] behaving like the process φ^δ in Lemma 4.1.1 for suitable $\varphi + \Delta\Phi$ and $\Delta\varphi^\pm = \pm\beta\delta$ representing the midpoint and the spread of the approximately optimal no-trade region, respectively. Generally, we call a strategy following the dynamics of (4.1, 4.2, 4.3) an ‘‘aisle-type’’ strategy. The aim of this chapter consists in verifying the asymptotic formula for the total variation of the aisle-type strategy φ^δ in Lemma 4.1.1 for small δ proposed in [19] under well-defined regularity conditions.

4.2 Main results

In this section we rigorously prove the asymptotic turnover of the aisle-type strategies heuristically derived in [19]. To this end, let us first impose some regularity conditions on the processes $\varphi, \beta, \Psi, \Delta\Phi$ which we are starting with.

Assumption 4.2.1. Set

$$\mathcal{H} := \left\{ \varphi, \beta, \frac{\Delta\Phi}{\delta} \right\}.$$

We suppose that

$$\sum_{X \in \mathcal{H}} \left\| \frac{c^{X,X}}{c^{\varphi,\varphi}} \right\|_{S^1} + \sum_{X \in \mathcal{H}} \left\| \frac{\beta b^X}{c^{\varphi,\varphi}} \right\|_{S^1} < \text{cst.}, \quad (4.4)$$

where cst. is used to denote a constant independent of δ and $\|\cdot\|_{S^q}$ stands for the S^q -norm defined in (A.1). Further, $c^{\varphi,\varphi}$ is assumed to be a positive process with continuous paths.

Assumption 4.2.2. Set

$$\mathcal{G} := \left\{ \Psi Y, \frac{Y}{\beta} \Psi b^X, Y \Psi b^\Psi, \frac{Y}{\beta^2} \Psi c^{X,X}, \frac{Y}{\beta} c^{X,X}, \frac{Y}{\beta} c^{\Psi,\Psi}, \frac{Y^2}{\beta^2} \Psi^2 c^{X,X}, Y^2 c^{\Psi,\Psi} : \begin{array}{l} X, Y \in \mathcal{H} \\ Y \neq \varphi \end{array} \right\}.$$

We assume that

$$\sum_{g \in \mathcal{G}} \mathbf{E} \left[\int_0^T |g_t| dt \right] + \mathbf{E} \left[\int_0^T \left| b_t^{\frac{\Delta\Phi}{\delta}} \right|^4 + \left(c_t^{\frac{\Delta\Phi}{\delta}, \frac{\Delta\Phi}{\delta}} \right)^2 dt \right] + \left\| \Psi \frac{c^{\varphi,\varphi}}{\beta^2} \right\|_{S^4} < \text{cst.} \quad (4.5)$$

The following theorem states the main result of this chapter.

Theorem 4.2.3. *Suppose that Assumptions 4.2.1, 4.2.2 hold. Let φ^δ be the process satisfying (4.1, 4.2, 4.3). Then*

$$\delta \int_0^\cdot \Psi_t \left| d\varphi_t^\delta \right| \xrightarrow[u.p.]{} \frac{1}{2} \int_0^\cdot \Psi_t \frac{c_t^{\varphi, \varphi}}{\beta_t} dt \quad \text{as } \delta \downarrow 0. \quad (4.6)$$

Proof. Set

$$\alpha := \frac{1}{2} \Psi \beta^{-3}, \quad \gamma := 3\alpha \beta^2 \delta^2$$

and define

$$\Delta \xi := \alpha \Delta \varphi^3 - \gamma \Delta \varphi.$$

By (4.1, 4.2, 4.3), we have

$$d\Delta \varphi_t = -d\varphi_t - d\Delta \Phi_t \quad \text{on } \{\Delta \varphi \neq \Delta \varphi^\pm\}. \quad (4.7)$$

For the function $f(x, a, b) := ax^3 - bx$, it holds that

$$\frac{\partial f}{\partial x}(\Delta \varphi^\pm, \alpha, \gamma) = 0;$$

since α, γ are Itô processes by assumption, in view of (4.7) and applying Itô's formula, we deduce that

$$\begin{aligned} d\Delta \xi_t &= df(\Delta \varphi_t, \alpha_t, \gamma_t) \\ &= (3\alpha_t \Delta \varphi_t^2 - \gamma_t) d\Delta \varphi_t + 3\alpha_t \Delta \varphi_t d[\Delta \varphi, \Delta \varphi]_t \\ &\quad + 3\Delta \varphi_t^2 d[\alpha, \Delta \varphi]_t + \Delta \varphi_t^3 d\alpha_t - \Delta \varphi_t d\gamma_t - d[\gamma, \Delta \varphi]_t \end{aligned}$$

and $\Delta \xi$ is an Itô process taking values in $[-|\Psi| \delta^3, |\Psi| \delta^3]$. By applying Itô's formula to α, γ , we obtain

$$\left| c^{\Delta \xi, \varphi} + 3\alpha \left(\Delta \varphi^2 - (\Delta \varphi^+)^2 \right) c^{\varphi, \varphi} \right| \leq \text{cst.} \sum_{g \in \mathcal{G}} |g| \delta^3, \quad (4.8)$$

$$\left| \left(b^{\Delta \xi} - 3\alpha \Delta \varphi c^{\varphi, \varphi} \right) (\Delta \Phi + \Delta \varphi) \right| \leq \text{cst.} \sum_{g \in \mathcal{G}} |g| \delta^3, \quad (4.9)$$

$$c^{\Delta \xi, \Delta \xi} (\Delta \Phi + \Delta \varphi)^2 \leq \text{cst.} \sum_{g \in \mathcal{G}} |g| \delta^6. \quad (4.10)$$

Using integration by parts twice, we get

$$\Delta \xi \cdot \varphi^\delta = \Delta \xi (\varphi^\delta - \varphi) + \Delta \xi \cdot \varphi + [\Delta \xi, \varphi] - (\Delta \Phi + \Delta \varphi) \cdot \Delta \xi - \underbrace{[\varphi^\delta, \Delta \xi]}_{=0}.$$

From $|\Delta\xi| \leq |\Psi| \delta^3$, (A.2, A.7) and Assumption 4.2.2, it holds that

$$\begin{aligned} \left\| \Delta\xi(\varphi^\delta - \varphi) \right\|_{S^1} &\leq \underbrace{\left(\|\Psi\beta\|_{S^1} + \left\| \Psi \frac{\Delta\Phi}{\delta} \right\|_{S^1} \right)}_{< \text{cst.}} \delta^3, \\ \|\Delta\xi \cdot \varphi\|_{S^1} &\leq \underbrace{\text{cst.} \left(\mathbf{E} \left[\int_0^T |\Psi_t b_t^\varphi| dt \right] + \mathbf{E} \left[\sqrt{\int_0^T |\Psi_t^2 c_t^{\varphi, \varphi}| dt} \right] \right)}_{< \text{cst.}} \delta^3. \end{aligned}$$

In view of (4.8, 4.9, 4.10), by (A.2, A.7) and Assumption 4.2.2, we have

$$\begin{aligned} &\left\| [\Delta\xi, \varphi] + \int_0^\cdot 3\alpha_t \left(\Delta\varphi_t^2 - (\Delta\varphi_t^+)^2 \right) c_t^{\varphi, \varphi} dt \right\|_{S^1} \\ &\leq \mathbf{E} \left[\int_0^T \left| c_t^{\Delta\xi, \varphi} + 3\alpha_t \left(\Delta\varphi_t^2 - (\Delta\varphi_t^+)^2 \right) c_t^{\varphi, \varphi} \right| dt \right] \\ &\leq \underbrace{\text{cst.} \mathbf{E} \left[\int_0^T \sum_{g \in \mathcal{G}} |g_t| dt \right]}_{< \text{cst.}} \delta^3 \end{aligned}$$

and

$$\begin{aligned} &\left\| (\Delta\Phi + \Delta\varphi) \cdot \Delta\xi - \int_0^\cdot 3\alpha_t (\Delta\Phi_t + \Delta\varphi_t) \Delta\varphi_t c_t^{\varphi, \varphi} dt \right\|_{S^1} \\ &\leq \text{cst.} \mathbf{E} \left[\int_0^T \left| (\Delta\Phi_t + \Delta\varphi_t) \left(b_t^{\Delta\xi} - 3\alpha_t \Delta\varphi_t c_t^{\varphi, \varphi} \right) \right| dt \right] \\ &\quad + \text{cst.} \mathbf{E} \left[\sqrt{\int_0^T \left| (\Delta\Phi_t + \Delta\varphi_t)^2 c_t^{\Delta\xi, \Delta\xi} \right| dt} \right] \\ &\leq \underbrace{\text{cst.} \left(\mathbf{E} \left[\int_0^T \sum_{g \in \mathcal{G}} |g_t| dt \right] + \mathbf{E} \left[\sqrt{\sum_{g \in \mathcal{G}} \int_0^T |g_t| dt} \right] \right)}_{< \text{cst.}} \delta^3. \end{aligned}$$

Together, we obtain

$$\left\| \Delta\xi \cdot \varphi^\delta - \int_0^\cdot 3\alpha_t \left((\Delta\varphi_t^+)^2 - 2\Delta\varphi_t^2 - \Delta\Phi_t \Delta\varphi_t \right) c_t^{\varphi, \varphi} dt \right\|_{S^1} = O(\delta^3). \quad (4.11)$$

Since (4.2, 4.3) imply that $\varphi^{\delta\uparrow}$ and $\varphi^{\delta\downarrow}$ increase only on $\{\Delta\xi = \Psi\delta^3\}$ and $\{\Delta\xi = -\Psi\delta^3\}$, respectively, we deduce

$$\Delta\xi \cdot \varphi^\delta = \int_0^\cdot \Psi_t \delta^3 d\varphi_t^{\delta\uparrow} + \int_0^\cdot \Psi_t \delta^3 d\varphi_t^{\delta\downarrow} = \delta^3 \int_0^\cdot \Psi_t |d\varphi_t^\delta|.$$

Combining this with (4.11), we conclude

$$\left\| \delta \int_0^\cdot \Psi_t |d\varphi_t^\delta| - \int_0^\cdot 3\alpha_t \left(\frac{(\Delta\varphi_t^+)^2 - 2\Delta\varphi_t^2 - \Delta\Phi_t \Delta\varphi_t}{\delta^2} \right) c_t^{\varphi, \varphi} dt \right\|_{S^1} = O(\delta). \quad (4.12)$$

In conjunction with Lemma 4.3.4, setting

$$K := \alpha\beta^2 c^{\varphi, \varphi}, \quad L := \alpha\beta c^{\varphi, \varphi}, \quad \Delta Y := \frac{\Delta\Phi}{\delta},$$

the assertion follows. \square

Remark 4.2.4. 1. In view of the proof of Theorem 4.2.3, the asymptotic formula in (4.6) is still valid for convergence in S^1 -norm, cf. (4.12) and Lemma 4.3.4.

2. In the setting of Chapter 2 dealing with the issue of portfolio optimization under small proportional transaction costs ε for exponential utility, the leading-order optimal strategy $\varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}$ follows the dynamics of (4.1, 4.2, 4.3) at scale $\delta := \varepsilon^{1/3}$, for midpoint φ representing the frictionless target, $\Delta\Phi := 0$ and the half-width of the no-trade region given by

$$\Delta\varphi^\pm = \pm\beta\delta := \pm \left(\frac{3}{2p} \frac{c^{\varphi, \varphi}}{c^{S, S}} S\varepsilon \right)^{1/3},$$

up to a stopping time τ^ε which converges to T in the sense of (2.59) (cf. Theorem 2.2.4). Using the same argument as in Lemma 2.5.4, it can be shown that for $\Psi = 1$ or $\Psi = S$, Assumption 2.2.3 and the additional condition $\|S^{\pm 1}\|_{S^n(\mathbf{Q})} < \infty$ for all $n \in \mathbb{N}$ ensure the counterparts of Assumptions 4.2.1, 4.2.2 in the setting of Chapter 2. Theorem 4.2.3 applied to $\Psi := 1$ then yields the asymptotic formula¹ for the trading volume of strategy φ^ε :

$$\int_0^\cdot |d\varphi_t^\varepsilon| = \int_0^\cdot (\varepsilon S_t)^{-1/3} \left(\frac{p}{12} \right)^{1/3} \left(\frac{c_t^{\varphi, \varphi}}{c_t^{S, S}} \right)^{2/3} d[S, S]_t + o_{u.p.}(\varepsilon^{-1/3}); \quad (4.13)$$

and the incurred transaction costs obtained by applying Theorem 4.2.3 for $\Psi := S$ amount to

$$\int_0^\cdot \varepsilon S_t |d\varphi_t^\varepsilon| = \varepsilon^{2/3} \int_0^\cdot \left(\frac{p}{12} \right)^{1/3} \left(\frac{c_t^{\varphi, \varphi}}{c_t^{S, S}} S_t \right)^{2/3} d[S, S]_t + o_{u.p.}(\varepsilon^{2/3}). \quad (4.14)$$

In conjunction with Remark 4.2.4-1, (2.59) and by the same argument as in (2.80), it can be shown that (4.13) and (4.14) are still valid when replacing φ^ε with the leading-order optimal strategy $\varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}$. In particular, all concrete models verified in Section 2.3 fulfill Assumptions 4.2.1, 4.2.2 and hence can be considered as examples of Theorem 4.2.3 in this chapter.

¹Here and in the sequel, the expression $H(\zeta) = o_{u.p.}(\zeta)$ refers to $\left| \frac{H(\zeta)}{\zeta} \right| \xrightarrow[u.p.]{} 0$ for $\zeta \downarrow 0$.

3. Parallel to Remark 4.2.4-2, let us now pass to the setting of Chapter 3 concerning the general structure of utility maximization under small proportional transaction costs ε . Theorem 3.2.6 states that the leading-order optimal investment strategy $\varphi^\varepsilon \mathbf{1}_{\llbracket 0, \tau^\varepsilon \rrbracket}$ behaves like the process φ^δ following (4.1, 4.2, 4.3) at scale $\delta := \varepsilon^{1/3}$, for midpoint φ referring to the frictionless optimizer with offset

$$\Delta\Phi := \varphi'((\varphi^\varepsilon - \varphi) \cdot S - (\kappa^\varepsilon - \kappa) \cdot I)$$

and the spread of the no-trade region determined by

$$\Delta\varphi^\pm = \pm\beta\delta := \pm \left(\frac{3}{2} R \frac{c^{\varphi, \varphi}}{c^{S, S}} S \varepsilon \right)^{1/3},$$

up to a stopping time τ^ε converging to T in the sense of (3.49). For $\Psi = 1$ or $\Psi = S$, by the argument in Lemma 3.5.5, it can be shown that Assumptions 4.2.1, 4.2.2 with respect to the setting of Chapter 3 follow from Assumption 3.2.5 in conjunction with the regularity condition $\|S^{\pm 1}\|_{S^n(\mathbf{Q})} < \infty$ for all $n \in \mathbb{N}$. Then from Theorem 4.2.3 for $\Psi = 1$, the asymptotic trading volume of strategy φ^ε reads as

$$\int_0^\cdot |d\varphi_t^\varepsilon| = \int_0^\cdot (\varepsilon S_t)^{-1/3} \left(\frac{1}{12R_t} \right)^{1/3} \left(\frac{c_t^{\varphi, \varphi}}{c_t^{S, S}} \right)^{2/3} d[S, S]_t + o_{u.p.}(\varepsilon^{-1/3}); \quad (4.15)$$

and Theorem 4.2.3 for $\Psi = S$ yields the incurred transaction costs:

$$\int_0^\cdot \varepsilon S_t |d\varphi_t^\varepsilon| = \varepsilon^{2/3} \int_0^\cdot \left(\frac{1}{12R_t} \right)^{1/3} \left(\frac{c_t^{\varphi, \varphi}}{c_t^{S, S}} S_t \right)^{2/3} d[S, S]_t + o_{u.p.}(\varepsilon^{2/3}). \quad (4.16)$$

Taking Remark 4.2.4-1, (3.49) into account and by the same argument as in (3.76), the results in (4.15) and (4.16) also hold for the approximately optimal strategy $\varphi^\varepsilon \mathbf{1}_{\llbracket 0, \tau^\varepsilon \rrbracket}$. In particular, Theorem 4.2.3 proves the asymptotic formulas proposed in [19, (3.5), (3.6)] and all examples provided in Section 3.3 are valid in the context of this chapter. Note that in case of power utility and Black-Scholes model, (4.15) confirms the asymptotic rates of the share- and wealth-turnover in [9, Theorem 2.2 (vi)].

4.3 Further asymptotics

In this section we apply some ergodic property of reflected Brownian motion in order to simplify the expression for the asymptotic total variation of the aisle-type strategy φ^δ following (4.1, 4.2, 4.3), so that formula (4.12) does not depend on the reflecting process $\Delta\varphi$ and the offset of the corridor $\Delta\Phi$. Moreover, the results of this section (cf. Lemma 4.3.4) can be used in Chapters 2 and 3 for approximating the minimum utility loss due to small proportional transaction costs, cf. Corollary 2.5.11, Lemma 3.5.11.

Throughout this section, let $K, L, \Delta Y$ be Itô processes, where K, L do not depend on δ and ΔY starts in 0. Suppose that

$$\mathbf{E} \left[\int_0^T |K_t| dt \right] + \mathbf{E} \left[\int_0^T |b_t^{\Delta Y}|^4 + (c_t^{\Delta Y, \Delta Y})^2 dt \right] + \|L\|_{S^4} < \text{cst.} \quad (4.17)$$

First, let us focus on a constant corridor $[-1, 1]$ and set

$$q := \frac{\Delta \varphi}{\Delta \varphi^+}.$$

Then q is a semimartingale reflected to stay between ± 1 . Aiming to apply the ergodic property of standard Brownian motion reflected to remain within $[-1, 1]$, we shall carry out changes of time and measure. To prepare for this, we modify the process q in the following manner. Using Itô's formula, the dynamics of q can be characterized by

$$dq_t = b_t^q dt + dM_t^q + dA_t^{\uparrow} - dA_t^{\downarrow},$$

where M^q is a continuous local martingale starting in 0, A^{\uparrow} and A^{\downarrow} are increasing processes which grow only on $\{q = -1\}$ and $\{q = 1\}$, respectively, and

$$b_t^q = \frac{b_t^{\Delta \varphi}}{\beta_t \delta} - \frac{c_t^{\Delta \varphi, \beta}}{\beta_t^2 \delta} - \frac{q_t b_t^{\beta}}{\beta_t} + \frac{q_t c_t^{\beta, \beta}}{\beta_t^2}, \quad (4.18)$$

$$c_t^{q, q} = \frac{c_t^{\Delta \varphi, \Delta \varphi}}{\beta_t^2 \delta^2} - 2 \frac{q_t c_t^{\beta, \Delta \varphi}}{\beta_t^2 \delta} + \frac{q_t^2 c_t^{\beta, \beta}}{\beta_t^2}. \quad (4.19)$$

Define stopping times

$$\sigma^{\delta, 1} := \inf \left\{ t \in [0, T] : \left| c_t^{q, q} - \frac{c_t^{\varphi, \varphi}}{\beta_t^2 \delta^2} \right| > \frac{c_t^{\varphi, \varphi}}{2\beta_t^2 \delta^{3/2}} \right\}, \quad (4.20)$$

$$\sigma^{\delta, 2} := \inf \left\{ t \in [0, T] : \left| \frac{b_t^q}{c_t^{q, q}} \right| > 1 \right\}, \quad (4.21)$$

$$\sigma^{\delta} := \sigma^{\delta, 1} \wedge \sigma^{\delta, 2} \wedge T.$$

Since $c^{\varphi, \varphi} > 0$ by assumption, we have

$$d\varphi_t = b_t^{\varphi} dt + \sqrt{c_t^{\varphi, \varphi}} dW_t$$

for a standard Brownian motion W on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbf{P})$. Set

$$b^{\bar{q}} := (b^q)^{\sigma^{\delta}}, \quad c^{\bar{q}, \bar{q}} := (c^{q, q})^{\sigma^{\delta}} \quad (4.22)$$

and define

$$M^{\bar{q}} := M^q \mathbf{1}_{[[0, \sigma^\delta]]} + \left(M_{\sigma^\delta}^q + \int_{\sigma^\delta}^{\cdot} \sqrt{c_t^{\bar{q}, \bar{q}}} dW_{t-\sigma^\delta} \right) \mathbf{1}_{[[\sigma^\delta, T]]}.$$

Then $M^{\bar{q}}$ is a continuous local martingale for which the local quadratic variation equals $c^{\bar{q}, \bar{q}}$. From [37, Theorem 3.3], there exists a unique solution \bar{q} to the Skorohod SDE:

$$d\bar{q}_t = b_t^{\bar{q}} dt + dM_t^{\bar{q}} + d\bar{A}_t^\uparrow - d\bar{A}_t^\downarrow, \quad \bar{q}_0 = q_0,$$

where \bar{A}^\uparrow and \bar{A}^\downarrow are increasing and increase only on $\{\bar{q} = -1\}$ and $\{\bar{q} = 1\}$, respectively. In particular, $q = \bar{q}$ on $[[0, \sigma^\delta]]$.

4.3.1 Time change

Fix $t \in [0, T)$. Consider the time change $(t(\vartheta))_{\vartheta \in \mathbb{R}_+}$ defined by

$$t(\vartheta) := \inf \{s \in [t, T] : [\bar{q}, \bar{q}]_s - [\bar{q}, \bar{q}]_t > \vartheta\} \wedge T.$$

Set

$$\bar{\vartheta} := [\bar{q}, \bar{q}]_T - [\bar{q}, \bar{q}]_t = \int_t^T c_s^{\bar{q}, \bar{q}} ds$$

and

$$\tilde{q}_\vartheta := \bar{q}_{t(\vartheta)} \quad \text{for } \vartheta \in \mathbb{R}_+.$$

Note that setting

$$\bar{c}^{\varphi, \varphi} := (c^{\varphi, \varphi})^{\sigma^\delta}, \quad \bar{\beta} := \beta^{\sigma^\delta},$$

(4.20, 4.22) yield

$$c^{\bar{q}, \bar{q}} \in \frac{\bar{c}^{\varphi, \varphi}}{\bar{\beta}^2 \delta^2} + \left[-\frac{\bar{c}^{\varphi, \varphi}}{2\bar{\beta}^2 \delta^2}, \frac{\bar{c}^{\varphi, \varphi}}{2\bar{\beta}^2 \delta^2} \right] \delta^{1/2} \subseteq \left[\frac{\bar{c}^{\varphi, \varphi}}{2\bar{\beta}^2 \delta^2}, \frac{3\bar{c}^{\varphi, \varphi}}{2\bar{\beta}^2 \delta^2} \right]. \quad (4.23)$$

Since $c^{\varphi, \varphi} > 0$, $\beta > 0$ by assumption, we have $c^{\bar{q}, \bar{q}} > 0$. Therefore, the mapping $\vartheta \mapsto t(\vartheta)$ is continuously differentiable on the interval $(0, \bar{\vartheta})$ with derivative $(c_t^{\bar{q}, \bar{q}})^{-1}$.

Lemma 4.3.1. *Suppose that Assumption 4.2.1 holds. Set*

$$\vartheta^\delta := \int_t^{(t+\delta) \wedge T} c_s^{\bar{q}, \bar{q}} ds.$$

Then we have

$$\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} q_s^2 K_s ds - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^2 d\vartheta K_t \right| \xrightarrow{\delta \downarrow 0} 0 \quad \text{in probability} \quad (4.24)$$

and

$$\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} q_s \Delta Y_s L_s ds - \frac{1}{\vartheta \delta} \int_0^{\vartheta \delta} \tilde{q}_{\vartheta} d\vartheta \Delta Y_t L_t \right| \xrightarrow{\delta \downarrow 0} 0 \quad \text{in probability.} \quad (4.25)$$

Moreover, for any $\omega \in \Omega$, there exist some $\delta_0(\omega)$, $\underline{K}(\omega)$, $\overline{K}(\omega) > 0$ such that

$$\underline{K}(\omega) \delta^{-1} \leq \vartheta^\delta(\omega) \leq \overline{K}(\omega) \delta^{-1} \quad (4.26)$$

holds for all $\delta \leq \delta_0(\omega)$.

Proof. Step 1. From (4.20), Markov's inequality, (4.19, 4.1) and Condition (4.4), we have

$$\begin{aligned} \mathbf{P}(\sigma^{\delta,1} < T) &\leq \mathbf{P}\left(\sup_{s \in [0, T]} \left| \frac{c_s^{q,q} - \frac{c_s^{\varphi,\varphi}}{\beta_s^2 \delta^2}}{\frac{c_s^{\varphi,\varphi}}{2\beta_s^2 \delta^{3/2}}} \right| > 1\right) \\ &\leq 2\delta^{1/2} \underbrace{\left\| \frac{c^{\frac{\Delta\Phi}{\delta}, \frac{\Delta\Phi}{\delta}} + 2|c^{\frac{\Delta\Phi}{\delta}, \varphi}| + 2|c^{\beta, \varphi}| + 2|c^{\beta, \Delta\Phi}| + c^{\beta, \beta}}{c^{\varphi, \varphi}} \right\|_{S^1}}_{< \text{cst.}}. \end{aligned} \quad (4.27)$$

Note that on $\{\sigma^{\delta,1} \geq s\}$, it holds that

$$c_s^{q,q} \in \left[\frac{c_s^{\varphi,\varphi}}{2\beta_s^2 \delta^2}, \frac{3c_s^{\varphi,\varphi}}{2\beta_s^2 \delta^2} \right].$$

Then by (4.21, 4.18, 4.27), Markov's inequality and Condition (4.4), we get

$$\begin{aligned} \mathbf{P}(\sigma^{\delta,2} < T) &\leq \mathbf{P}(\sigma^{\delta,2} < \sigma^{\delta,1} \leq T) + \mathbf{P}(\sigma^{\delta,1} \leq \sigma^{\delta,2} < T) \\ &\leq \mathbf{P}\left(\sup_{s \in [0, T]} \left| \frac{2b_s^q \beta_s^2 \delta^2}{c_s^{\varphi,\varphi}} \right| > 1\right) + \mathbf{P}(\sigma^{\delta,1} < T) \\ &\leq \delta \underbrace{\left\| \frac{\beta(|b^\varphi| + |b^{\Delta\Phi}| + |b^\beta|) + |c^{\varphi, \beta}| + |c^{\Delta\Phi, \beta}| + c^{\beta, \beta}}{c^{\varphi, \varphi}} \right\|_{S^1}}_{< \text{cst.}} + O(\delta^{1/2}). \end{aligned} \quad (4.28)$$

Together, we obtain

$$\lim_{\delta \downarrow 0} \mathbf{P}(\sigma^\delta < T) \leq \lim_{\delta \downarrow 0} \mathbf{P}(\sigma^{\delta,1} < T) + \lim_{\delta \downarrow 0} \mathbf{P}(\sigma^{\delta,2} < T) = 0.$$

Therefore, for any $\zeta > 0$, we have

$$\lim_{\delta \downarrow 0} \mathbf{P} \left(\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} (q_s^2 - \bar{q}_s^2) K_s ds \right| > \zeta \right) \leq \lim_{\delta \downarrow 0} \mathbf{P} (\sigma^\delta < T) = 0, \quad (4.29)$$

$$\lim_{\delta \downarrow 0} \mathbf{P} \left(\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} (q_s^2 - \bar{q}_s^2) \Delta Y_s L_s ds \right| > \zeta \right) \leq \lim_{\delta \downarrow 0} \mathbf{P} (\sigma^\delta < T) = 0. \quad (4.30)$$

Step 2. By continuity of the process K and using the mean value theorem, we deduce

$$\lim_{\delta \downarrow 0} \left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s^2 (K_s - K_t) ds \right| (\omega) = 0 \quad \text{for all } \omega \in \Omega. \quad (4.31)$$

Further, (A.2, A.7), the Cauchy-Schwarz inequality, Condition (4.17), continuity of L and dominated convergence yield

$$\begin{aligned} & \lim_{\delta \downarrow 0} \mathbf{E} \left[\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s^2 (\Delta Y_s L_s - \Delta Y_t L_t) ds \right| \right] \\ & \leq \lim_{\delta \downarrow 0} \mathbf{E} \left[\sup_{s \in [t, t+\delta]} |(\Delta Y_s L_s - \Delta Y_t L_t)| \right] \\ & \leq \lim_{\delta \downarrow 0} \mathbf{E} \left[\sup_{s \in [t, t+\delta]} |(\Delta Y_s - \Delta Y_t) L_s| \right] + \lim_{\delta \downarrow 0} \mathbf{E} \left[\sup_{s \in [t, t+\delta]} |\Delta Y_t (L_s - L_t)| \right] \\ & \leq \text{cst.} \lim_{\delta \downarrow 0} \left\| \int_0^T \mathbf{1}_{[t, t+\delta]}(s) |b_s^{\Delta Y}| ds + \sqrt{\int_0^T \mathbf{1}_{[t, t+\delta]}(s) c_s^{\Delta Y, \Delta Y} ds} \right\|_{L^2(\mathbf{P})} \underbrace{\|L\|_{S^2}}_{< \text{cst.}} \\ & \quad + \underbrace{\lim_{\delta \downarrow 0} \left[\sup_{s \in [t, t+\delta]} |L_s - L_t|^2 \right]}_{=0} \underbrace{\|\Delta Y\|_{S^2}}_{< \text{cst.}} \\ & \leq \underbrace{\text{cst.} \lim_{\delta \downarrow 0} \sqrt{\int_0^T \mathbf{1}_{[t, t+\delta]}(s) ds}}_{=0} \underbrace{\sqrt{\mathbf{E} \left[\int_0^T |b_s^{\Delta Y}|^2 ds \right]}}_{< \text{cst.}} \\ & \quad + \underbrace{\text{cst.} \lim_{\delta \downarrow 0} \sqrt[4]{\int_0^T \mathbf{1}_{[t, t+\delta]}(s) ds}}_{=0} \underbrace{\sqrt[4]{\mathbf{E} \left[\int_0^T (c_s^{\Delta Y, \Delta Y})^2 ds \right]}}_{< \text{cst.}} \end{aligned}$$

which implies

$$\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s^2 (\Delta Y_s L_s - \Delta Y_t L_t) ds \right| \longrightarrow 0 \quad \text{in probability as } \delta \downarrow 0. \quad (4.32)$$

Step 3. Fix $\omega \in \Omega$. Applying the mean value theorem to the mapping $t \mapsto t(\vartheta)$, we get

$$\begin{aligned} \delta &= \left(t(\vartheta^\delta) - t(0) \right) (\omega) \\ &= \left((c_{t(\xi)}^{\bar{q}, \bar{q}})^{-1} \vartheta^\delta \right) (\omega) \text{ for some } \xi \in [0, \vartheta^\delta(\omega)]. \end{aligned} \quad (4.33)$$

From (4.23) as well as the continuity of the mapping $s \mapsto \frac{c_s^{\varphi, \varphi}}{\beta_s^2}(\omega)$, we deduce

$$\lim_{\delta \downarrow 0} \left| \frac{1}{\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^n \left((c_{t(\vartheta)}^{\bar{q}, \bar{q}})^{-1} - (c_{t(\xi)}^{\bar{q}, \bar{q}})^{-1} \right) d\vartheta \right| (\omega) = 0, \quad n = 1, 2.$$

Change of variables yields

$$\lim_{\delta \downarrow 0} \left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s^n ds - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^n d\vartheta \right| (\omega) = 0, \quad n = 1, 2. \quad (4.34)$$

Therefore, (4.34) for $n = 2$ leads to

$$\lim_{\delta \downarrow 0} \left| \left(\frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s^2 ds - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^2 d\vartheta \right) K_t \right| (\omega) = 0; \quad (4.35)$$

the Cauchy-Schwarz inequality, (4.34) for $n = 1$, dominated convergence, (A.2, A.7) and Condition (4.17) yield

$$\begin{aligned} & \lim_{\delta \downarrow 0} \mathbf{E} \left[\left| \left(\frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s ds - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \right) \Delta Y_t L_t \right| \right] \\ & \leq \underbrace{\lim_{\delta \downarrow 0} \mathbf{E} \left[\left| \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} \bar{q}_s ds - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \right|^2 \right]}_{=0} \underbrace{\|\Delta Y\|_{S^4} \|L\|_{S^4}}_{< \text{cst.}}. \end{aligned} \quad (4.36)$$

Finally, (4.29, 4.31, 4.35) and (4.30, 4.32, 4.36) imply (4.24) and (4.25), respectively, and (4.26) follows from (4.33, 4.23). \square

4.3.2 Change of measure

We use the same notation as in Section 4.3.1. From the Dambis-Dubins-Schwarz theorem (cf. [33, Theorems V.1.6, V.1.7]), there exists an enlargement $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_\vartheta)_{\vartheta \in \mathbb{R}_+}, \tilde{\mathbf{P}})$ of the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_{t(\vartheta)})_{\vartheta \in \mathbb{R}_+}, \mathbf{P})$ and a standard Brownian motion \tilde{W} on that space such that $\tilde{W}_\vartheta = M_{t(\vartheta)}^{\bar{q}} - M_{t(0)}^{\bar{q}}$ for $\vartheta < \bar{\vartheta}$.

Since the process $(\tilde{b}_\vartheta)_{\vartheta \in \mathbb{R}_+}$ defined by

$$\tilde{b}_\vartheta := \frac{b_{t(\vartheta)}^{\bar{q}}}{c_{t(\vartheta)}^{\bar{q}, \bar{q}}} \mathbf{1}_{[[0, \vartheta]]}(\vartheta)$$

is bounded,

$$\frac{d\mathbf{P}^\delta}{d\tilde{\mathbf{P}}} = \exp \left(- \int_0^{\vartheta^\delta} \tilde{b}_\vartheta d\tilde{W}_\vartheta - \frac{1}{2} \int_0^{\vartheta^\delta} \tilde{b}_\vartheta^2 d\vartheta \right)$$

defines a probability measure on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_\vartheta)_{\vartheta \in \mathbb{R}_+})$ whose Hellinger process $h(\frac{1}{2}, \mathbf{P}^\delta, \tilde{\mathbf{P}})$ is given by

$$h(\frac{1}{2}, \mathbf{P}^\delta, \tilde{\mathbf{P}})_\vartheta = \int_0^{\vartheta \wedge \vartheta^\delta} \tilde{b}_\zeta^2 d\zeta, \quad \vartheta \in \mathbb{R}_+$$

(cf. [15, Theorem IV.1.33]) and such that

$$\tilde{W}^\delta := \tilde{W} + \int_0^\cdot \mathbf{1}_{[[0, \vartheta^\delta]]}(\vartheta) \tilde{b}_\vartheta d\vartheta$$

is a \mathbf{P}^δ -standard Brownian motion. In view of (4.18, 4.19, 4.26), we have

$$\lim_{\delta \downarrow 0} \int_0^\infty \mathbf{1}_{[[0, \vartheta^\delta]]}(\vartheta) \tilde{b}_\vartheta^2 d\vartheta = 0 \quad \text{a.s.}$$

By [15, Theorem V.4.31 and Lemma V.4.3], this implies

$$\lim_{\delta \downarrow 0} \sup_{A \in \tilde{\mathcal{F}}_\infty} |\mathbf{P}^\delta(A) - \tilde{\mathbf{P}}(A)| = 0. \quad (4.37)$$

Lemma 4.3.2. *On the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we have*

$$\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^2 d\vartheta - \frac{1}{3} \right| \xrightarrow{\delta \downarrow 0} 0 \quad \text{in probability,} \quad (4.38)$$

$$\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \right| \xrightarrow{\delta \downarrow 0} 0 \quad \text{in probability.} \quad (4.39)$$

Proof. Let process Y starting at \tilde{q}_0 be the unique solution to the Skorohod SDE

$$dY_\vartheta = d\tilde{W}_\vartheta^\delta$$

with reflection at ± 1 (cf. e.g. [37, Theorem 3.3] for existence and uniqueness). Observe that Y coincides with \tilde{q} on $[[0, \vartheta^\delta]]$. Indeed, according to [13, 10.18], we have

$$d\tilde{q}_\vartheta = \mathbf{1}_{[[0, \vartheta]]}(\vartheta) d\tilde{W}_\vartheta + \tilde{b}_\vartheta d\vartheta + d\bar{A}_{t(\vartheta)}^\uparrow - d\bar{A}_{t(\vartheta)}^\downarrow,$$

i.e., \tilde{q} solves the Skorohod SDE on $[[0, \vartheta^\delta]]$, which yields $Y = \tilde{q}$ on $[[0, \vartheta^\delta]]$ by uniqueness of the solution to the stopped Skorohod SDE. Note that standard Brownian motion reflected at ± 1 is a Markov process with uniform stationary distribution, cf. e.g. [4, Appendix 1.5].

Let $\xi > 0$ and f be an arbitrary bounded measurable function defined on $[-1, 1]$. Due to [23, Theorem] and in view of (4.26), there exist two constants $C < \infty$ and $\zeta < 1$ such that for all $\delta_0 \in (0, 1)$, $\delta \leq \delta_0$, $\vartheta \in [0, 1]$ we have

$$\mathbf{P}^\delta \left(A_{\vartheta}^\delta \right) < C \zeta^{\frac{1}{\delta_0}}$$

with

$$A_{\vartheta}^\delta := \left\{ \left| \frac{1}{[\vartheta^\delta]} \sum_{i=0}^{[\vartheta^\delta]-1} f(\tilde{q}_{\vartheta+i}) - \frac{1}{2} \int_{-1}^1 f(x) dx \right| > \xi \right\}.$$

In combination with (4.37) and interpreting A_{ϑ}^δ naturally as a subset of Ω , we obtain

$$\lim_{\delta \downarrow 0} \mathbf{P} \left(A_{\vartheta}^\delta \right) = \lim_{\delta \downarrow 0} \tilde{\mathbf{P}} \left(A_{\vartheta}^\delta \right) = 0.$$

Fubini's theorem and dominated convergence yield

$$\lim_{\delta \downarrow 0} \mathbf{E} \left[\int_0^1 \mathbf{1}_{A_{\vartheta}^\delta} d\vartheta \right] = \lim_{\delta \downarrow 0} \int_0^1 \mathbf{P}(A_{\vartheta}^\delta) d\vartheta = \int_0^1 \lim_{\delta \downarrow 0} \mathbf{P}(A_{\vartheta}^\delta) d\vartheta = 0.$$

Again by dominated convergence, we obtain

$$\begin{aligned} & \lim_{\delta \downarrow 0} \mathbf{E} \left[\left| \int_0^1 \left(\frac{1}{[\vartheta^\delta]} \sum_{i=0}^{[\vartheta^\delta]-1} f(\tilde{q}_{\vartheta+i}) \right) d\vartheta - \frac{1}{2} \int_{-1}^1 f(x) dx \right| \right] \\ & \leq \lim_{\delta \downarrow 0} \mathbf{E} \left[\int_0^1 \left| \frac{1}{[\vartheta^\delta]} \sum_{i=0}^{[\vartheta^\delta]-1} f(\tilde{q}_{\vartheta+i}) - \frac{1}{2} \int_{-1}^1 f(x) dx \right| d\vartheta \right] = 0, \end{aligned}$$

and hence

$$\left| \int_0^1 \left(\frac{1}{[\vartheta^\delta]} \sum_{i=0}^{[\vartheta^\delta]-1} f(\tilde{q}_{\vartheta+i}) \right) d\vartheta - \frac{1}{2} \int_{-1}^1 f(x) dx \right| \longrightarrow 0 \text{ in probability}$$

for $\delta \downarrow 0$. Since

$$\frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} f(\tilde{q}_\vartheta) d\vartheta - \int_0^1 \left(\frac{1}{[\vartheta^\delta]} \sum_{i=0}^{[\vartheta^\delta]-1} f(\tilde{q}_{\vartheta+i}) \right) d\vartheta \longrightarrow 0 \quad \text{a.s.}$$

as $\delta \downarrow 0$, (4.38) and (4.39) follow by setting $f(x) := x^2$ and $f(x) := x$, respectively. \square

Corollary 4.3.3. *On the space $(\Omega, \mathcal{F}, \mathbf{P})$, we have*

$$\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \right| |\Delta Y_t L_t| \xrightarrow{\delta \downarrow 0} 0 \quad \text{in probability.} \quad (4.40)$$

Proof. Using the Cauchy-Schwarz inequality, (4.39) and dominated convergence, (A.2, A.7) and Condition (4.17), we have

$$\lim_{\delta \downarrow 0} \mathbf{E} \left[\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \right| |\Delta Y_t L_t| \right] \leq \underbrace{\lim_{\delta \downarrow 0} \sqrt{\mathbf{E} \left[\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \right|^2 \right]}}_{=0} \underbrace{\|\Delta Y\|_{S^4}^2 \|L\|_{S^4}^2}_{< \text{cst.}}$$

The assertion follows. \square

4.3.3 Asymptotics

Now, we are able to draw the following conclusion which completes the proofs of the asymptotic formulas in Theorem 4.2.3, Corollary 2.5.11 and Lemma 3.5.11 in the related contexts.

Lemma 4.3.4. *Suppose that Assumption 4.2.1 holds. Let $\Delta\varphi$ be the reflecting process related to the “aisle-type” strategy φ^δ satisfying (4.1, 4.2, 4.3). Let $K, L, \Delta Y$ be Itô processes satisfying Condition (4.17), where K, L do not depend on δ and ΔY starts in 0. Then*

$$\left\| \int_0^\cdot \left(\frac{\Delta\varphi_t}{\Delta\varphi_t^+} \right)^2 K_t dt - \frac{1}{3} \int_0^\cdot K_t dt \right\|_{S^1} \longrightarrow 0 \quad \text{as } \delta \downarrow 0, \quad (4.41)$$

$$\left\| \int_0^\cdot \frac{\Delta\varphi_t}{\Delta\varphi_t^+} \Delta Y_t L_t dt \right\|_{S^1} \longrightarrow 0 \quad \text{as } \delta \downarrow 0. \quad (4.42)$$

Proof. Step 1. Let $\varepsilon > 0$ be arbitrary. Fix $t \in [0, T)$ and define $\vartheta^\delta, \tilde{q}$ as in Section 4.3.1. Let

$$\Delta X_t^\delta := \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} q_s^2 K_s ds, \quad \Delta Y_t^\delta := \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} q_s \Delta Y_s L_s ds$$

and set

$$A_t^\delta := \left\{ \left| \Delta X_t^\delta - \frac{1}{3} K_t \right| > \varepsilon \right\}, \quad B_t^\delta := \left\{ \left| \Delta Y_t^\delta \right| > \varepsilon \right\}.$$

By (4.24, 4.38) and (4.25, 4.40), we have

$$\begin{aligned} 0 &\leq \lim_{\delta \downarrow 0} \mathbf{P} \left(A_t^\delta \right) \\ &\leq \lim_{\delta \downarrow 0} \mathbf{P} \left(\left| \Delta X_t^\delta - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^2 d\vartheta K_t \right| > \frac{\varepsilon}{2} \right) + \lim_{\delta \downarrow 0} \mathbf{P} \left(\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^2 d\vartheta - \frac{1}{3} \right| |K_t| > \frac{\varepsilon}{2} \right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
0 &\leq \lim_{\delta \downarrow 0} \mathbf{P} \left(B_t^\delta \right) \\
&\leq \lim_{\delta \downarrow 0} \mathbf{P} \left(\left| \Delta Y_t^\delta - \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta d\vartheta \Delta Y_t L_t \right| > \frac{\varepsilon}{2} \right) + \lim_{\delta \downarrow 0} \mathbf{P} \left(\left| \frac{1}{\vartheta^\delta} \int_0^{\vartheta^\delta} \tilde{q}_\vartheta^2 d\vartheta \right| |\Delta Y_t L_t| > \frac{\varepsilon}{2} \right) \\
&= 0,
\end{aligned}$$

respectively.

Step 2. Fubini's theorem, dominated convergence and Step 1 yield

$$\lim_{\delta \downarrow 0} \mathbf{E} \left[\int_0^T \mathbf{1}_{A_t^\delta} dt \right] = \lim_{\delta \downarrow 0} \int_0^T \mathbf{P} \left(A_t^\delta \right) dt = \int_0^T \lim_{\delta \downarrow 0} \mathbf{P} \left(A_t^\delta \right) dt = 0, \quad (4.43)$$

$$\lim_{\delta \downarrow 0} \mathbf{E} \left[\int_0^T \mathbf{1}_{B_t^\delta} dt \right] = \lim_{\delta \downarrow 0} \int_0^T \mathbf{P} \left(B_t^\delta \right) dt = \int_0^T \lim_{\delta \downarrow 0} \mathbf{P} \left(B_t^\delta \right) dt = 0. \quad (4.44)$$

Observe that for any $t \in [0, T]$,

$$\left| \Delta X_t^\delta \right| \leq \frac{1}{\delta} \int_t^{(t+\delta) \wedge T} |K_s| ds =: \Delta X_t^{\delta+}.$$

Using Fubini's theorem, we conclude

$$\sup_{t \in [0, T]} \left| \int_0^t q_s^2 K_s ds - \int_0^t \Delta X_s^\delta ds \right| \leq \int_0^\delta |q_s^2 K_s| ds \leq \int_0^T \mathbf{1}_{[0, \delta]}(s) |K_s| ds \quad (4.45)$$

and

$$\int_0^T \Delta X_t^{\delta+} dt \leq \int_0^T |K_s| ds. \quad (4.46)$$

So by Condition (4.17), it holds that

$$\mathbf{E} \left[\int_0^T \sup_{\delta \in (0, 1)} \left| \Delta X_t^\delta \right| dt \right] \leq \mathbf{E} \left[\int_0^T |K_s| ds \right] < \infty.$$

Then, (4.41) follows from (4.43, 4.45, 4.46) and dominated convergence. Similarly, Fubini's

$$\sup_{t \in [0, T]} \left| \int_0^t q_s \Delta Y_s L_s ds - \int_0^t \Delta Y_s^\delta ds \right| \leq \int_0^\delta |q_s \Delta Y_s L_s| ds \leq \delta \sup_{s \in [0, T]} |q_s \Delta Y_s L_s|, \quad (4.47)$$

$$\sup_{\delta \in (0, 1)} \mathbf{E} \left[\sup_{s \in [0, T]} |q_s \Delta Y_s L_s|^2 \right] \leq \text{cst.} \|\Delta Y\|_{S^4} \|L\|_{S^4} < \infty. \quad (4.48)$$

(4.42) then follows from (4.44, 4.47, 4.48) and [14, Theorem 27.2]. \square

Appendix A

This appendix contains some fundamental tools which are used throughout the thesis.

Let us fix a probability measure $\tilde{\mathbf{P}}$ and denote for any Itô process X its local $\tilde{\mathbf{P}}$ -drift and quadratic variation by $b^{X,\tilde{\mathbf{P}}}$ and $c^{X,X}$, respectively, i.e.,

$$dX_t = b_t^{X,\tilde{\mathbf{P}}} dt + dM_t^{X,\tilde{\mathbf{P}}}, \quad c^{X,X} := \frac{d[X,X]_t}{dt},$$

where $M^{X,\tilde{\mathbf{P}}}$ is a continuous $\tilde{\mathbf{P}}$ -martingale starting in 0. From [32, Section V.2], we recall the S^q - and H^q -norms, $q \in [1, \infty)$, for an Itô process X :

$$\|X\|_{S^q(\tilde{\mathbf{P}})} := \left\| \sup_{t \in [0, T]} |X_t| \right\|_{L^q(\tilde{\mathbf{P}})}, \quad (\text{A.1})$$

$$\|X\|_{H^q(\tilde{\mathbf{P}})} := \left\| \int_0^T |b_t^{X,\tilde{\mathbf{P}}}| dt + \sqrt{\int_0^T c_t^{X,X} dt} \right\|_{L^q(\tilde{\mathbf{P}})}.$$

To be more precise, [32] requires $X_0 = 0$ in the definition of the H^q -norm.

Remark A.1. 1. Due to [32, Theorem V.2],

$$\|X\|_{S^q(\tilde{\mathbf{P}})} \leq \text{cst.} \|X\|_{H^q(\tilde{\mathbf{P}})} \quad (\text{A.2})$$

holds if $X_0 = 0$.

2. By convexity of the mapping $x \mapsto |x|^q$ and Jensen's inequality, we have

$$\left| \sum_{n=1}^N Y_n \right|^q = N^q \left| \sum_{n=1}^N \frac{Y_n}{N} \right|^q \leq N^{q-1} \sum_{n=1}^N |Y_n|^q \quad (\text{A.3})$$

for any $N \in \mathbb{N}$ and any random variables Y_1, \dots, Y_N . In particular, for any $N \in \mathbb{N}$ and Itô processes $X^{(1)}, \dots, X^{(N)}$, it holds that

$$\left\| \sum_{n=1}^N X^{(n)} \right\|_q^q \leq \text{cst.} \sum_{n=1}^N \left\| X^{(n)} \right\|_q^q, \quad (\text{A.4})$$

where $\|\cdot\|_q \in \{\|\cdot\|_{S^q(\tilde{\mathbf{P}})}, \|\cdot\|_{H^q(\tilde{\mathbf{P}})}\}$.

3. For any $g \in L^q([0, T])$, Hölder's inequality yields

$$\left(\int_0^T |g(t)| dt \right)^q \leq \|g\|_{L^q([0, T])}^q \|1\|_{L^r([0, T])}^q \leq T^{q-1} \int_0^T |g(t)|^q dt \quad (\text{A.5})$$

with $\frac{1}{q} + \frac{1}{r} = 1$. Moreover,

$$\left\| \sqrt{Y} \right\|_{L^q(\tilde{\mathbf{P}})}^q = \|Y\|_{L^{q/2}(\tilde{\mathbf{P}})}^{\frac{q}{2}} \leq \|Y\|_{L^q(\tilde{\mathbf{P}})}^{\frac{q}{2}} \quad (\text{A.6})$$

for any random variable $Y \in L^q(\tilde{\mathbf{P}})$.

4. Gathering the above inequalities, we obtain

$$\begin{aligned} & \left\| \int_0^T \left(\sum_{m=1}^M X_t^{(m)} \right) dt + \sqrt{\int_0^T \left(\sum_{n=1}^N Y_t^{(n)} \right) dt} \right\|_{L^q(\tilde{\mathbf{P}})}^q \\ & \stackrel{(\text{A.3})}{\leq} \text{cst.} \left(\left\| \int_0^T \left(\sum_{m=1}^M X_t^{(m)} \right) dt \right\|_{L^q(\tilde{\mathbf{P}})}^q + \left\| \sqrt{\int_0^T \left(\sum_{n=1}^N Y_t^{(n)} \right) dt} \right\|_{L^q(\tilde{\mathbf{P}})}^q \right) \\ & \stackrel{(\text{A.6})}{\leq} \text{cst.} \left(\left\| \int_0^T \left(\sum_{m=1}^M X_t^{(m)} \right) dt \right\|_{L^q(\tilde{\mathbf{P}})}^q + \left\| \int_0^T \left(\sum_{n=1}^N Y_t^{(n)} \right) dt \right\|_{L^q(\tilde{\mathbf{P}})}^{\frac{q}{2}} \right) \\ & \stackrel{(\text{A.5})}{\leq} \text{cst.} \left(\mathbf{E}_{\tilde{\mathbf{P}}} \left[\int_0^T \left| \sum_{m=1}^M X_t^{(m)} \right|^q dt \right] + \sqrt{\mathbf{E}_{\tilde{\mathbf{P}}} \left[\int_0^T \left| \sum_{n=1}^N Y_t^{(n)} \right|^q dt \right]} \right) \\ & \stackrel{(\text{A.3})}{\leq} \text{cst.} \left(\sum_{m=1}^M \mathbf{E}_{\tilde{\mathbf{P}}} \left[\int_0^T |X_t^{(m)}|^q dt \right] + \sqrt{\sum_{n=1}^N \mathbf{E}_{\tilde{\mathbf{P}}} \left[\int_0^T |Y_t^{(n)}|^q dt \right]} \right) \quad (\text{A.7}) \end{aligned}$$

for any $M, N \in \mathbb{N}$ and processes $X^{(m)}$, $m = 1, \dots, M$ and $Y^{(n)}$, $n = 1, \dots, N$.

The following lemma contains some basic observations.

Lemma A.2. *Define*

$$\mathcal{X} := \left\{ (X_t)_{t \in [0, T]} : \mathbf{E}_{\tilde{\mathbf{P}}} \left[\sup_{t \in [0, T]} |X_t|^n \right] < \infty \text{ for any } n \in \mathbb{N} \right\}.$$

Then

1. For $X, Y \in \mathcal{X}$, $c \in \mathbb{R}$, it holds that $X + Y, XY, cX \in \mathcal{X}$.

2. If $X \in \mathcal{X}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ with $|f(x)| \leq 1 + |x|$ for any $x \in \mathbb{R}$, then $f(X) \in \mathcal{X}$.
3. If $X \in \mathcal{X}$, then $\mathbf{E}_{\tilde{\mathbf{P}}}\left[\int_0^T |X_t|^n dt\right] < \infty$ for any $n \in \mathbb{N}$.

Proof. This is straightforward. □

Appendix B

As an auxiliary result, we determine the explicit solution to the frictionless optimization problem related to the stochastic volatility model in Section 2.3.3. We proceed analogously to [18, Theorem 3.1] dealing with power utility. Here, let us focus on the setup of Chapter 2 considering the exponential utility function.

Theorem B.1. *For the stochastic volatility model characterized by (2.20) with bounded $b(\tilde{Y})/\sigma(\tilde{Y})$, the frictionless optimizer φ satisfies*

$$\varphi_t S_t = \frac{b(\tilde{Y}_t)}{p\sigma(\tilde{Y}_t)^2} \quad \text{for all } t \in [0, T]. \quad (\text{B.1})$$

The MEMM \mathbf{Q} has density process

$$\mathbf{E} \left[\frac{d\mathbf{Q}}{d\mathbf{P}} \middle| \mathcal{F}_t \right] = \frac{\tilde{Z}_t}{\tilde{Z}_0} \exp \left(- \int_0^t \frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} dW_s - \frac{1}{2} \int_0^t \left(\frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} \right)^2 ds \right), \quad t \in [0, T],$$

where the process \tilde{Z} is defined as in (2.22).

Proof. Step 1. Define filtration $\mathbf{G} = (\mathcal{G}_t)_{t \in [0, T]}$ by

$$\mathcal{G}_t := \bigcap_{s>t} \sigma \left(\mathcal{F}_s \cup \sigma((\tilde{Y}_r)_{r \in [0, T]}) \right), \quad t \in [0, T] \quad (\text{B.2})$$

and let

$$\varphi := \frac{b(\tilde{Y})}{p\sigma(\tilde{Y})^2 S}$$

in line with (B.1). Further, define a \mathbf{G} -martingale $Z^{\mathbf{G}}$ by

$$Z_t^{\mathbf{G}} := \mathbf{E} [U'(x + \varphi \bullet S_T) | \mathcal{G}_t], \quad t \in [0, T]$$

and a probability measure \mathbf{Q} by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \frac{Z_T^{\mathbf{G}}}{\mathbf{E} [Z_T^{\mathbf{G}}]}.$$

Note that the integrability of $Z_T^{\mathbf{G}}$ follows from the boundedness of $b(\tilde{Y})/\sigma(\tilde{Y})$. Finally, set

$$\bar{Z}_t := \exp\left(-\int_0^t \frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} dW_s - \frac{1}{2} \int_0^t \left(\frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)}\right)^2 ds\right), \quad t \in [0, T]. \quad (\text{B.3})$$

By definition of \mathbf{G} , the random variable \tilde{Z}_T is \mathcal{G}_0 -measurable. Since \tilde{Y} is independent of W , it follows from [1, Theorem 15.5] that W is a standard Brownian motion with respect to \mathbf{G} as well. Due to the boundedness of $b(\tilde{Y})/\sigma(\tilde{Y})$, the local martingale \bar{Z} satisfies Novikov's condition, whence it is a martingale relative to both \mathbf{F} and \mathbf{G} . Therefore, we deduce that for any $t \in [0, T]$,

$$\begin{aligned} Z_t^{\mathbf{G}} &= \mathbf{E}\left[p \exp\left(-px - \int_0^T \frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} dW_s - \int_0^T \left(\frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)}\right)^2 ds\right) \middle| \mathcal{G}_t\right] \\ &= \mathbf{E}\left[pe^{-px} \tilde{Z}_T \bar{Z}_T \middle| \mathcal{G}_t\right] \\ &= \underbrace{pe^{-px} \tilde{Z}_T}_{\mathcal{G}_0\text{-measurable}} \bar{Z}_t. \end{aligned} \quad (\text{B.4})$$

Step 2. Let $\bar{\mathbf{Q}}$ be the probability measure with density process \bar{Z} . By Girsanov's theorem,

$$W^{\bar{\mathbf{Q}}} := W + \int_0^\cdot \frac{b(\tilde{Y}_t)}{\sigma(\tilde{Y}_t)} dt$$

is a standard Brownian motion under measure $\bar{\mathbf{Q}}$ relative to both \mathbf{F} and \mathbf{G} . Since $b(\tilde{Y})/\sigma(\tilde{Y})$ is bounded,

$$\varphi \cdot S = \frac{1}{p} \int_0^\cdot \frac{b(\tilde{Y}_t)}{\sigma(\tilde{Y}_t)} dW_t^{\bar{\mathbf{Q}}}$$

is a $\bar{\mathbf{Q}}$ -martingale with respect to \mathbf{G} . Moreover, S is a $\bar{\mathbf{Q}}$ -local martingale relative to both \mathbf{F} and \mathbf{G} because $dS_t = S_t \sigma(\tilde{Y}_t) dW_t^{\bar{\mathbf{Q}}}$, cf. [32, Theorem IV.33].

Let ψ be an admissible strategy in the sense of [34, Definition 1.2], i.e., ψ is an S -integrable process such that the related wealth process is uniformly bounded from below. Note that $\psi \cdot S$ is a $\bar{\mathbf{Q}}$ -local martingale which is bounded from below and hence a $\bar{\mathbf{Q}}$ -supermartingale. By the generalized Bayes' formula and in view of (B.4), $\varphi \cdot S$ is a $\bar{\mathbf{Q}}$ -martingale and $\psi \cdot S$ is a $\bar{\mathbf{Q}}$ -supermartingale, both with respect to filtration \mathbf{G} . Hence, by concavity of U , we have

$$\begin{aligned} \mathbf{E}[U(x + \psi \cdot S_T)] &\leq \mathbf{E}[U(x + \varphi \cdot S_T)] + \mathbf{E}[U'(x + \varphi \cdot S_T)(\psi \cdot S_T - \varphi \cdot S_T)] \\ &= \mathbf{E}[U(x + \varphi \cdot S_T)] + \underbrace{\mathbf{E}[U'(x + \varphi \cdot S_T)] \mathbf{E}_{\bar{\mathbf{Q}}}[\psi \cdot S_T - \varphi \cdot S_T]}_{\leq 0}. \end{aligned}$$

Step 3. We show that $U(x + \varphi \cdot S_T)$ lies in the L^1 -closure of the set

$$\{U(x + \psi \cdot S_T) : \psi \text{ is admissible}\}.$$

Indeed, letting

$$\tau_n := \inf\{t \in [0, T] : x + \varphi \bullet S_t < -n\},$$

we can approximate φ by the sequence $(\varphi^{(n)})_{n \in \mathbb{N}}$ defined as $\varphi^{(n)} := \varphi \mathbf{1}_{[0, \tau_n]}$, which fulfills the admissibility requirement in [34, Definition 1.2]. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \mathbf{E} \left[\sup_{n \in \mathbb{N}} \left| U(x + \varphi^{(n)} \bullet S_T) \right| \right] \\ &= \mathbf{E} \left[\sup_{n \in \mathbb{N}} \exp(-p(x + \varphi^{(n)} \bullet S_T)) \right] \\ &\leq \mathbf{E} \left[\sup_{t \in [0, T]} \exp(-p(x + \varphi \bullet S_t)) \right] \\ &= e^{-px} \mathbf{E} \left[\sup_{t \in [0, T]} \exp \left(- \int_0^t \frac{b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} dW_s - \frac{1}{4} \int_0^t \left(\frac{2b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} \right)^2 ds \right) \right] \\ &\leq e^{-px} \sqrt{\mathbf{E} \left[\sup_{t \in [0, T]} \underbrace{\exp \left(- \int_0^t \frac{2b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} dW_s - \frac{1}{2} \int_0^t \left(\frac{2b(\tilde{Y}_s)}{\sigma(\tilde{Y}_s)} \right)^2 ds \right)}_{=: M_t} \right]}. \end{aligned}$$

By boundedness of $(2b(\tilde{Y}_t)/\sigma(\tilde{Y}_t))_{t \in [0, T]}$, the process $(M_t)_{t \in [0, T]}$ is an L^2 -martingale, which implies

$$\sup_{n \in \mathbb{N}} U(x + \varphi^{(n)} \bullet S_T) \in L^1(\mathbf{P})$$

by Doob's quadratic inequality. Dominated convergence yields

$$\left\| U(x + \varphi^{(n)} \bullet S_T) - U(x + \varphi \bullet S_T) \right\|_{L^1(\mathbf{P})} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Step 4. By Steps 2 and 3, the payoff $x + \varphi \bullet S_T$ is optimal in the sense of [34, Theorem 2.2 (iii)], which implies that \mathbf{Q} is the dual optimizer, cf. [34, Equation (42)]. Moreover, we have shown in Step 2 that $\varphi \bullet S$ is a \mathbf{Q} -martingale with respect to filtration \mathbf{G} and hence \mathbf{F} as well, which yields that φ is the optimal strategy in the sense of [34, Theorem 2.2 (iv)]. \square

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Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation eigenständig angefertigt habe. Davon ausgenommen ist der fachliche Rat durch Prof. Dr. Jan Kallsen, der diese Arbeit betreut hat. Darüber hinaus basiert ein Teil des Inhalts von Kapitel 2 und Abschnitt 4.3 sowie der Anhang auf der Preprint-Fassung eines gemeinsam mit meinem Betreuer bei der Zeitschrift *The Annals of Applied Probability* eingereichten Artikels:

- J. Kallsen und S. Li. Portfolio Optimization under Small Transaction Costs: a Convex Duality Approach. Preprint, 2013.

Bei der Erstellung der Dissertation habe ich die Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft eingehalten. Zudem versichere ich, dass diese Arbeit weder ganz noch teilweise einem anderen Prüfungsverfahren zu Grunde lag. Des Weiteren erkläre ich, dass ich noch keinen Promotionsversuch unternommen habe.

Kiel, den 10. Februar 2015