

**MAXIMAL FUNCTIONS ASSOCIATED WITH
NONISOTROPIC DILATIONS OF
HYPERSURFACES IN \mathbb{R}^3**

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Abstract

The goal of this thesis is to establish L^p -estimates for maximal functions associated with nonisotropic dilations of hypersurfaces in \mathbb{R}^3 . More precisely, let S be a smooth hypersurface in \mathbb{R}^3 with a surface measure $d\mu$ and $\eta \in C_0^\infty(\mathbb{R}^3)$ be a smooth non-negative function with compact support. Suppose that $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, t^{a_3}x_3)$ is a family of dilations with $a_j > 0$. Then the associated maximal operator is given by

$$\mathcal{M}f(x) := \sup_{t>0} \left| \int_S f(x - \delta_t(y)) \eta(y) d\mu(y) \right|, \quad f \in \mathcal{S}(\mathbb{R}^3).$$

The problem of characterizing the range of all p for which \mathcal{M} is bounded on $L^p(\mathbb{R}^3)$ has been extensively considered. The best understood class are maximal functions associated with isotropic standard dilations $\delta_t(x) = tx$, see the articles by Greenleaf, Iosevich-Sawyer, Ikromov-Kempe-Müller and Zimmermann. Some results for maximal functions associated with nonisotropic dilations have already been obtained by Greenleaf, Iosevich-Sawyer and Zimmermann, but for some situations such as the hypersurface parameterized as the graph of a smooth function $\Phi(x_1, x_2) = x_2^d(1 + \mathcal{O}(x_2^m))$ near the origin, where $d \geq 2$, $m \geq 1$, and associated dilations $\delta_t(x) = (t^a x_1, tx_2, t^d x_3)$ for an arbitrary real number $a > 0$, the question was open until recently. In fact, such problems do arise already in lower dimensions. For instance, we consider the curve $\gamma(x) = (x, x^2(1 + \phi(x)))$, where $\phi(x) = \mathcal{O}(x^m)$ and $m \geq 1$, and associated dilations $(x_1, x_2) \rightarrow (tx_1, t^2x_2)$. If $\phi \equiv 0$, then the corresponding maximal function is the maximal function along parabolas in the plane, which plays an important role in the study of singular Randon transforms, and which is very well understood due to the work by Nagel-Riviere-Wainger and others. If $\phi \neq 0$, the problem was open until recently, however, the corresponding maximal function shows features related to the Bourgain circular maximal function, which required deep ideas and L^4 -techniques. An alternative approach was later given by Mockenhaupt, Seeger and Sogge, who established local smoothing estimates for Fourier integral operators satisfying the so-called "cinematic curvature" condition. However, we observe that in the study of \mathcal{M} related

to the mentioned curve $\gamma(x)$ and associated dilations, in our situation, we will consider a family of corresponding Fourier integral operators which fail to satisfy the "cinematic curvature condition" uniformly, which means that classical local smoothing estimates could not be directly applied to our problem.

In this thesis, we develop new ideas in order to overcome the above difficulty and finally establish sharp L^p -estimates for the maximal function related to the curve $\gamma(x)$ with associated dilations in the plane. Later, we generalize the result to curves of finite type d ($d \geq 2$) and associated dilations $(x_1, x_2) \rightarrow (tx_1, t^d x_2)$. Furthermore, we also obtain L^p -estimates for the maximal function related to the mentioned hypersurface $\Phi(x_1, x_2)$ in \mathbb{R}^3 with associated dilations. Moreover, by an alternative approach, we also get L^p -estimates for some classes of maximal functions in \mathbb{R}^3 established earlier by Greenleaf, Iosevich-Sawyer, Ikromov-Kempe-Müller and Zimmermann.

Zusammenfassung

Das Ziel dieser Arbeit ist es, L^p -Abschätzungen für Maximalfunktionen zu nichtisotropen Dilatation von Hyperflächen im \mathbb{R}^3 zu untersuchen. Genauer gesagt, sei S eine glatte Hyperfläche im \mathbb{R}^3 mit Oberflächenmaß $d\mu$ und sei $\eta \in C_0^\infty(\mathbb{R}^3)$ eine glatte nichtnegative Funktion mit kompaktem Träger. Angenommen, $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, t^{a_3}x_3)$ ist eine Familie von Dilatation. Dann ist der zugehörige Maximaloperator gegeben durch

$$\mathcal{M}f(x) := \sup_{t>0} \left| \int_S f(x - \delta_t(y)) \eta(y) d\mu(y) \right|, \quad f \in \mathcal{S}(\mathbb{R}^3).$$

Die Frage nach dem Bereich aller p , für die \mathcal{M} auf $L^p(\mathbb{R}^3)$ beschränkt ist, wurde intensiv studiert. Die am besten verstandene Klasse sind Maximalfunktionen zu den isotropen Standard-Dilatation $\delta_t(x) = tx$, siehe die Artikel von Greenleaf, Iosevich-Sawyer, Ikromov-Kempe-Müller und Zimmermann. Für Maximalfunktionen zu nichtisotropen Dilatation gibt es einige Resultate von Greenleaf, Iosevich, Sawyer und Zimmermann, aber für einige Situationen, wie zum Beispiel die Hyperfläche $\Phi(x_1, x_2) = x_2^d(1 + \mathcal{O}(x_2^m))$, wobei $d \geq 2$, $m \geq 1$, und zugehörigen Dilatation $\delta_t(x) = (t^a x_1, t x_2, t^d x_3)$ für eine beliebige reelle Zahl $a > 0$, war die Frage bislang offen. In der Tat treten analoge Probleme schon in niedriger Dimension auf. Betrachten wir zum Beispiel die Kurve $\gamma(x) = (x, x^2(1 + \phi(x)))$, wobei $\phi(x) = \mathcal{O}(x^m)$ und $m \geq 1$, und die damit verbundenen Dilatation $(x_1, x_2) \rightarrow (tx_1, t^2x_2)$. Für $\phi \equiv 0$ ist die entsprechende Maximalfunktion gerade die Maximalfunktion entlang Parabeln in der Ebene, die im Studium von singulären Randon-Transformationen eine wichtige Rolle spielt, und die aufgrund von Arbeiten von Stein, Nagel-Riviere-Wainger und anderen sehr gut verstanden sind. Für $\phi \neq 0$ zeigt die entsprechende Maximalfunktion Charakteristiken des Bourgain'schen Maximaloperators für Kreise, diese Behandlung tiefliegende Ideen und L^4 -Techniken erforderten. Ein alternativer Ansatz von Mockenhaupt, Seeger und Sogge beruht auf sogenannten "local smoothing"-Abschätzungen für Fourier-Integraloperatoren, die gewissen "kinematischen" Krümmungsbedingungen genügen. Wir werden sehen, dass beim Studium des Maximaloperators zu der erwähnten Kurve $\gamma(x)$ und zugehörige Dilata-

tion die Familie der entsprechenden Fourier-Integraloperatoren nicht gleichmäßig der "kinematischen" Krümmungsbedingung genügt, was bedeutet, dass die klassische sogenannte "local smoothing"-Abschätzung nicht direkt auf unser Problem angewendet werden kann.

In dieser Arbeit entwickeln wir neue Ideen, um die obigen Schwierigkeiten zu überwinden und schließlich die scharfe L^p -Abschätzung für die Maximalfunktion zur Kurve $\gamma(x)$ und die verbundenen Dilatationen in der Ebene zu beweisen. Später verallgemeinern wir das Resultat für Kurven vom endlichen Typ d ($d \geq 2$) und zugehörige Dilatation $(x_1, x_2) \rightarrow (tx_1, t^d x_2)$. Wir erhalten die L^p -Abschätzung auch für die Maximalfunktion der genannten Hyperfläche $\Gamma(x_1, x_2)$ im \mathbb{R}^3 mit die verbundenen Dilatation. Darüber hinaus geben wir einen alternativen Beweis für L^p -Abschätzungen einiger Klassen von Maximalfunktionen im \mathbb{R}^3 , die von Greenleaf, Iosevich-Sawyer, Ikromov-Kempe-Müller und Zimmermann bereits mit anderen Methoden gezeigt wurden.

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Chapter 1

Introduction

1.1 Background on maximal functions

If $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a locally integrable function, the classical Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \rightarrow \mathbb{C}$ is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad (1.1)$$

where $B(x,r)$ is the ball of radius r centered at x , and $|E|$ denotes the measure of a measurable set E . It arises very naturally in analysis, for proving theorems about the existence almost everywhere of limits, for controlling pointwise important objects such as the Poisson integrals or for controlling, not pointwise but at least in average, other basic operators such as singular integral operators. For example, by a standard density argument, the weak (1,1) boundedness of M implies in particular that Lebesgue differentiation theorem, according to which

$$\lim_{r \rightarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy = f(x)$$

for all $f \in L^1(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$.

In the past 40 years, the interest in taking averages over low dimensional sets has been increasing. E.g. in (1.1) one replaces the balls of radius r by a suitable family of lower-dimensional set. The study of such averages is of interest not only for its own sake, but because it has connection with other problems, such as the behavior of the classical Radon transform (see Nagel, Riviere and Wainger [27], Marstrand [22], Falconer [7], Strichartz [45], and Oberlin and Stein [29]), and the behavior of solutions of the wave equation, and more generally other hyperbolic equations (see Stein [39], Greenleaf [10], Ruiz [31], and Sogge [34]).

The starting point for intensive studies associated with averages over low dimensional sets is based on an earlier bound of Stein [39] from 1976 on the spherical maximal function

$$\mathcal{M}f(x) := \sup_{t>0} |M_t f(x)|,$$

where M_t are the spherical averaging operators

$$\mathcal{M}_t f(x) = \int_{|y|=1} f(x - ty) d\sigma(y),$$

and $d\sigma$ is normalized surface measure on the sphere S^{n-1} . Then Stein's fundamental result shows that for $n \geq 3$, the corresponding spherical maximal operator is bounded on $L^p(\mathbb{R}^n)$ for every $p > n/(n-1)$. One can use this bound to show the pointwise

covergence $\lim_{t \rightarrow 0} M_t f(x) = f(x)$ a.e.. The analogous result in dimension two was later proved by J. Bougain [2].

Then one turned to deal with generalizations of \mathcal{M}_t defined as before, i.e. the sphere is replaced by a more general smooth hypersurface $S \in \mathbb{R}^n$. Let $\rho \in C_0^\infty(S)$ be a smooth non-negative function with compact support. Then the associated maximal operator is defined as

$$\mathcal{M}f(x) := \sup_{t>0} \left| \int_S f(x - ty)\rho(y)d\mu(y) \right|, \quad x \in \mathbb{R}^n, \quad (1.2)$$

where $d\mu$ denotes the surface measure on S . Greenleaf [10] proved that \mathcal{M} is bounded on $L^p(\mathbb{R}^n)$ if $n \geq 3$ and $p > (k + 1)/k$, provided S has at least $k \geq 2$ non-vanishing principal curvatures and S is starshaped with respect to the origin.

A fundamental open problem is to characterize the L^p boundedness properties of the maximal operator associated to hypersurface where the Gaussian curvature at some points is allowed to vanish, with the exception of the 2 dimensional case, i.e. the case of finite type curves in \mathbb{R}^2 studied by A. Iosevich in [12].

Many authors put a lot of effort on the development of this subject and obtained partial results in high dimension. Sogge and Stein [37] showed that if the Gaussian curvature of S does not vanish of infinite order at any point of S , then there exists a $p_0(S) < \infty$ so that the maximal function is bounded on L^p , $p > p_0(S)$. However, the exponent $p_0(S)$ given in that paper is in general far from being optimal. In order to find the smallest value of $p_0(S)$, one will put more restriction on the surfaces. It is worth mentioning that in 1992, Sogge [36] employed the local smoothing estimate to get the L^p -boundedness $p > 2$ for \mathcal{M} , where the surface has at least one non-vanishing principal curvature everywhere in \mathbb{R}^n ($n \geq 2$). The perhaps best understood class in higher dimension is the class of convex hypersurface of finite line type, and the related work in this settings included Cowling and Mauceri [4] [5], Nagel, Seeger and Wainger[28], Iosevich and Sawyer [13] [14], and Iosevich, Sawyer and Seeger [15]. In particular, Iosevich and Sawyer proved in [14] sharp L^p -estimates for maximal functions related to the surface which are given by smooth convex functions of finite line type for $p > 2$. In 2010, Ikoromov, Kempe and Müller [11] discovered a connection between the L^p -boundedness ($p > \max\{h(S), 2\}$) of \mathcal{M} and the height $h(S)$ of a smooth, compact hypersurface S of finite type in \mathbb{R}^3 satisfying the transversality assumption (i.e. for every $x \in S$, the affine tangent plane $x + T_x S$ to S through x does not pass through the origin in \mathbb{R}^3) on S . Recently, Zimmermann in his thesis [47] proved that maximal averages over analytic hypersurfaces passing through the origin in general behave more regularly than the maximal averages over hypersurfaces satisfying the transversality condition.

Maximal operators defined by averages over curves or surfaces with non-isotropic dilations have also been extensively considered, i.e. the dilation ty appearing in (1.2) is replaced by $\delta_t(y) = (t^{a_1}y_1, t^{a_2}y_2, \dots, t^{a_n}y_n)$, where $a_j > 0$, $j = 1, 2, \dots, n$. In 1970, in the study of a problem related to Poisson integrals for symmetric spaces, Stein raised the question as to when the operator \mathcal{M}_γ defined by

$$\mathcal{M}_\gamma f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(x - \gamma(t))| dt,$$

where $\gamma(t) = (A_1 t^{a_1}, A_2 t^{a_2}, \dots, A_n t^{a_n})$ and A_1, A_2, \dots, A_n are real, $a_i > 0$, is bounded on $L^p(\mathbb{R}^n)$. Nagel, Riviere and Wainger [27] showed that the L^p -boundedness of \mathcal{M} holds for $p > 1$ for the special case $\gamma(t) = (t, t^2)$ in \mathbb{R}^2 and Stein [40] for homogeneous curves in \mathbb{R}^n . More general maximal operators involving two-parameter dilations related to homogeneous curves have studied by Marletta and Ricci [21]. For maximal functions \mathcal{M} associated with nonisotropic dilations in higher dimensions, one can see the work by Greenleaf [10], Sogge and Stein [37], Iosevich and Sawyer [14], Ikoromov, Kempe and Müller [11], Zimmermann [47].

1.2 Local smoothing estimates

Let $S \subset \mathbb{R}^2$ be a smooth curve whose curvature does not vanish anywhere, and let $d\sigma$ be the induced Lebesgue measure. The averaging operators are defined by

$$A_t f(x) := \int_S f(x - ty) d\sigma(y), \quad f \in C_0^\infty(\mathbb{R}^2),$$

the associated maximal operator is given by

$$\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|, \quad x \in \mathbb{R}^n.$$

Then $\|\mathcal{M}f\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)}$, for $2 < p < \infty$. This is a theorem of Bourgain [2]. An alternative approach to the result for circular maximal operators has been devised by Mockenhaupt, Seeger and Sogge [26]. Since the averaging operators A_t can be expressed as Fourier integral operators, the authors of [25] develop a more sophisticated theory of local smoothing estimates of Fourier integral operators satisfying the so-called cinematic curvature condition so that they can improve many estimates for maximal operators. In particular, they are able to prove the variable coefficient version of the circular maximal theorem, which includes estimates for averages over geodesic circles. These general local smoothing estimates for Fourier integral operators have also other

applications, one of which is to deal with a special class which contains the solution operators for the Cauchy problem associated to variable coefficient wave equations.

The phrase "local smoothing" was first used for certain types of estimates for dispersive equations which go back to Kato [16], Sjölin [33], and Vega [46]. Here we only show one of the deep results obtained by Mockenhaupt, Seeger and Sogge in order to easily understand the phrase "local smoothing" from estimates for the following Cauchy problem for the wave equation

$$\begin{aligned} ((\partial/\partial t)^2 - \Delta)u &= 0, \\ u|_{t=0} &= f, \quad (\partial/\partial t)u|_{t=0} = g, \quad f, g \in C_0^\infty(\mathbb{R}^2), \end{aligned}$$

where the space variable x is in \mathbb{R}^2 and Δ is the Laplace operator. Let $\alpha(p) = 1/p$, if $4 \leq p < \infty$, and $\alpha(p) = 1/2 - 1/p$, if $2 < p < 4$. Assume $2 < p < \infty$, then for any compact time interval I ,

$$\left(\int \int_{\mathbb{R}^2 \times I} |u(x, t)|^p dx dt \right)^{1/p} \leq C(\|f\|_{L_\gamma^p} + \|g\|_{L_{\gamma-1}^p}), \quad \text{if } \gamma > 1/2 - 1/p - \alpha(p)/2, \quad (1.3)$$

where C only depends on I , p , and L_γ^p denotes the usual Sobolev space of order γ .

For fixed $t \neq 0$ the analogue of (1.3) holds if and only if $\gamma > 1/2 - 1/p$ (see Miyachi [24], Peral [30] and Seeger [32]). Thus this result says that there is "local smoothing" of order $\alpha(p)/2$.

Furthermore, we show some main ideas of the proof of the L^p -boundedness of the circular maximal operator in [26]. However, in \mathbb{R}^2 one cannot use the argument from [10], since the square functions that control the maximal operator are not bounded on any L^p space. To avoid this phenomenon, Mockenhaupt, Seeger and Sogge in [26] make use of the Little-Paley theory and the Sobolev embedding theorem in order to show that it is sufficient to prove certain estimates involving L^p -averages of derivatives of A_t with respect to the t variable. It is not difficult to see that there can never be $L^p \rightarrow L^p$ local smoothing when $p \leq 2$ or $p = \infty$. From the above arguments, local smoothing estimates for the corresponding Fourier integral operator gives L^4 -estimates with an extra bit of regularity, together with the easy endpoint estimates on L^2 and L^∞ , finally the M. Riesz interpolation theorem implies the L^p -boundedness ($p > 2$) of corresponding maximal operators. Now the challenge is how to get the extra bit of regularity in the L^4 -estimate for the corresponding Fourier integral operator. Their proof relies greatly on the fact that a natural cone attached to the Fourier integral operator associated to the wave equation is the light cone $\{(\xi, |\xi|)\}$, whose level curves $|\xi| = \text{const}$ have non-vanishing curvature. Note that this is equivalent to saying that the cinematic curvature of the

Fourier integral operator (i.e. the curvature of the curve $\{\xi : |\xi| = 1\}$) is equal to 1. Mockenhaupt, Seeger and Sogge employ dyadic decompositions and angular decompositions to give orthogonality arguments and reduce the problem to Kakeya type maximal functions.

A very natural question is to ask how the L^4 -estimate for a Fourier integral operator depends on the cinematic curvature. In this thesis, we show that a class of Fourier integral operators which do not satisfy the cinematic curvature condition uniformly, still satisfy a local smoothing estimate. Just when we finished our work and searched for background materials on "cinematic curvature" online, it came to our attention that in 2000, Kung already obtained some results for a related problem in his thesis [17] which has not been published until now and only show the results in [18] which we had not been aware of. We are thankful to Kung who provided us with his thesis soon after we wrote an Email to him. Through reading his thesis we know that the basic structure of both Kung's and our approach might appear similar, since both strategies rely on papers [26] and [25] by Mockenhaupt, Seeger and Sogge. Nevertheless, our approach differs from [17], since we made use of a different angular decomposition. In this way, we obtain stronger L^4 estimates for a class of Fourier integral operators than in [17], which can be applied on establishing the L^p -boundedness of the maximal operator associated with dilations $(x_1, x_2) \rightarrow (tx_1, t^d x_2)$ of the curve $x \rightarrow (x, x^d(1 + \phi(x)))$, where $\phi(x) = \mathcal{O}(x^m)$, $m \geq 1$ and $d \geq 2$. In a model case, the associated cone for the corresponding Fourier integral operator $\mathcal{F}_\lambda^\delta$ which is localized to frequencies $|\xi| \approx \lambda$ is of the form $\{(\xi, \delta q(\xi))\}$, where $\delta > 0$ is very small and $q(\xi)$ is homogeneous of degree one, smooth on the support of the symbol of $\mathcal{F}_\lambda^\delta$. Comparing the light cone, we observe that the level curves of the cone $\{(\xi, \delta q(\xi))\}$ become flatter and the cinematic curvature of $\mathcal{F}_\lambda^\delta$ is only greater than or equal to δ . In [18], Kung got that $\|\mathcal{F}_\lambda^\delta\|_{L^4 \rightarrow L^4} \leq \lambda^{1/8+\epsilon_1} \delta^{-1/2}$. However, in this thesis we obtain the better estimate that $\|\mathcal{F}_\lambda^\delta\|_{L^4 \rightarrow L^4} \leq \lambda^{1/8+\epsilon_1} \delta^{-(3/8+\epsilon_2-\epsilon_1)}$ (see Chapter 3 of this thesis).

Moreover, in [18], Kung extends his estimate for the operator $\mathcal{F}_\lambda^\delta$ to more general Fourier integral operators which correspond the cones $\{(\xi, q(x, t, \xi))\}$, where for all x, t , the curvature of the curve $\{\xi : q(x, t, \xi) = 1\}$ is greater than or equal to δ . In view of an application to our problems, we extend our estimate for the operator $\mathcal{F}_\lambda^\delta$ to a class of Fourier integral operators which correspond the cones $\{(\xi, \delta q(t, \xi, \delta))\}$, where $q(t, \xi, \delta)$ is homogeneous of degree one in ξ , smooth on the support of the symbol of the corresponding Fourier integral operator (see Chapter 4 of this thesis). However, it is still open which exponent of δ is optimal.

1.3 Organization of the thesis

In our settings, the averaging operator is always written as a Fourier integral operator, then we turn to prove L^p -boundedness for the Fourier integral operator. In Chapter 2, first, we give an overview of the theory of Fourier integral operators and a concrete description of the so-called cinematic curvature condition. Furthermore, we establish L^p -boundedness of the maximal operator related to the curve $x \rightarrow (x, x^2(1 + \phi(x)))$, where $\phi(x) = \mathcal{O}(x^m)$ and $m \geq 1$, with associated dilations $(x_1, x_2) \rightarrow (tx_1, t^2x_2)$. After a scaling argument, application of the method of stationary phase and Littlewood-Paley theory, we observe that the phase function of the corresponding Fourier integral operator will not satisfy the cinematic curvature condition uniformly, which causes major difficulties to apply the local smoothing estimate directly to our Fourier integral operator. We separate the problem into two parts, depending on a dyadic decomposition for the frequency variables of the corresponding Fourier integral operator, i.e. low frequency and high frequency. For low frequency, we mainly use the better endpoint regularity estimates, the M. Riesz interpolation theorem to get the desired result. For high frequency, we develop a new method based on the main idea of the local smoothing estimate from [25] to get the key L^4 -boundedness of the corresponding Fourier integral operator $\tilde{\mathcal{F}}_\lambda^\delta$. Later, we generalize the result to curves of finite type d ($d \geq 2$) and associated dilations $(x_1, x_2) \rightarrow (tx_1, t^d x_2)$. These results answer the question which was unsolved until recently as to how to characterize the range of all p for which maximal functions associated with non-isotropic dilations of non-homogeneous curves of finite type in the plane is bounded.

Chapter 3 gives a detailed proof of the L^4 -estimate of the model operator, which is translation-invariant and can approximate the Fourier integral operator $\tilde{\mathcal{F}}_\lambda^\delta$.

The proof of the L^4 -estimate of the Fourier integral operator $\tilde{\mathcal{F}}_\lambda^\delta$ will be given in Chapter 4.

In Chapter 5, employing some arguments used and some results obtained in the plane, we also establish L^p -estimates for the maximal function related to the hypersurface $(x_1, x_2) \rightarrow (x_1, x_2, x_2^d(1 + \mathcal{O}(x_2^m)))$ with associated dilations $\delta_t(x) = (t^a x_1, tx_2, t^d x_3)$ for an arbitrary real number $a > 0$. These results could not be covered by the theorems about maximal functions associated with nonisotropic dilations of hypersurfaces in \mathbb{R}^3 from Greenleaf [10], Iosevich-Sawyer [14], Ikromov-Kempe-Müller [11] and Zimmermann [47]. Moreover, by an alternative approach, we also get L^p -estimates for some classes of maximal functions in \mathbb{R}^3 proved in [10], [14], [11] and [47].

Conventions: Throughout this thesis, we shall use the well known notation $A \ll B$,

which means if there is a sufficiently large constant G , which does not depend on the relevant parameters arising in the context in which the quantities A and B appear, such that $GA \leq B$. We write $A \approx B$, and mean that A and B are comparable. We write $A \lesssim B$ if $A \ll B$ or $A \approx B$. $A \wedge B$ means if $A \leq B$, then $A \wedge B = A$; if $A \geq B$, then $A \wedge B = B$.

Chapter 2

Maximal functions associated with
nonisotropic dilations
of curves in the plane

2.1 Statement of the main results

In this chapter, we will show some L^p -estimates for maximal functions associated with nonisotropic dilations of curves in the plane.

The maximal operator along curves in \mathbb{R}^2 is defined by

$$\mathcal{M}f(x) = \sup_{t>0} |f * \mu_t(x)|,$$

where μ is a arc length measure supported on the curve and μ_t is the same measure dilated by $t > 0$ and appropriately normalized.

Two different situations will arise. (1) A maximal operator of the first type: the curve $x_2 = \gamma(x_1)$ is homogeneous under the given dilations (such as $\gamma(x_1) = x_1^d, d > 0, d \neq 1$). Basically one obtains the same operator by restricting the supremum to $t = 2^j$. Under appropriate assumptions on the curve, one can prove that \mathcal{M} is bounded on L^p for $p > 1$. The exact results and proofs can be found in page 472-492 of [41]. (2) A maximal operator of the second type: the curve $x_2 = \gamma(x_1)$ is not homogeneous under the given dilations. Then the various μ_t are supported on different curves and the problem becomes much more complicated. We have the following subcases:

(2a) if the dilations are isotropic and the Gaussian curvature of the curve does not vanish, then \mathcal{M} is bounded on L^p for $p > 2$, see [2] and [25]. The range of p must be further restricted if the curvature allows to vanish at some point [12] unless one introduces a damping factor [19];

(2b) if the dilations are non-isotropic and the curve with non-vanishing Gaussian curvature away from the origin is homogeneous, Marletta and Ricci [20] combine Bourgain's theorem with a Littlewood-Paley decomposition to show that \mathcal{M} is bounded on L^p if and only if $p > 2$;

A natural question to ask is whether there exists some positive constant $p_0 > 1$ such that \mathcal{M} is bounded on L^p for $p > p_0$ in the setting (2c) if the dilations are non-isotropic and the curve $x_2 = \gamma(x_1)$ is non-homogeneous and of finite type (such as $\gamma(x_1) = x_1^d(1 + x_1^m + \mathcal{O}(x_1^{m+1}))$, $d \geq 2$ and $m \geq 1$)? This chapter will give the answers. Now we list our main results.

Let $\phi \in C^\infty(I, \mathbb{R})$, where I is a bounded interval containing the origin, and

$$\phi(0) \neq 0; \phi^{(j)}(0) = 0, j = 1, 2, \dots, m-1; \phi^{(m)}(0) \neq 0 (m \geq 1). \quad (2.1)$$

Theorem 2.1.1. *Define the maximal operator*

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}} f(y_1 - tx, y_2 - t^2x^2\phi(x))\eta(x)dx \right|, \quad (2.2)$$

where $\eta(x)$ is supported in a sufficiently small neighborhood of the origin. If ϕ satisfies (2.1), then for $p > 2$, there exists a constant C_p such that the following inequality holds true:

$$\|\mathcal{M}f\|_{L^p} \leq C_p \|f\|_{L^p}, \quad f \in C_0^\infty(\mathbb{R}^2). \quad (2.3)$$

Remark 2.1.1. If $m \rightarrow \infty$, then we obtain the maximal operator of the first type described above which is bounded on L^p for all $p > 1$.

In fact, the above theorem is sharp if $1 \leq m < \infty$, which follow from the following proposition.

Proposition 2.1.2. *If $1 \leq m < \infty$, then $p > 2$ is a necessary condition for the maximal inequality (2.3).*

Proof. Let $f(x) = \frac{\chi_{[-1/2, 1/2]^2}(x)}{|x| \log \frac{1}{|x|}}$. It's clear that $f \in L^p(\mathbb{R}^2)$ if $p \leq 2$.

Now we take $\epsilon < \frac{1}{100}$ such that $[\epsilon/2, 4\epsilon] \subset \text{supp } \eta$, then fix it. We may assume that $\phi(0) > 0$, $\phi^{(j)}(0) = 0$, $j = 1, 2, \dots, m-1$, and $\phi^{(m)}(0) > 0$. Let $y_1 \in [1/8, 1/4]$ and $y_1^2 \phi(\epsilon) \leq y_2 \leq y_1^2 \phi(2\epsilon)$. By the inverse function theorem, we can set $a := a(y) = \phi^{-1}(y_2/y_1^2)$, then $\epsilon \leq a \leq 2\epsilon$ (otherwise, if we assume $\phi(0) > 0$ and $\phi^{(m)}(0) < 0$, since $\phi'(x) = \frac{\phi^{(m)}(0)}{(m-1)!} x^{m-1} + \mathcal{O}(x^m)$, then $\phi(x)$ is strictly decreasing in the interval $[\epsilon/2, 4\epsilon]$. In this case, we take $y_1^2 \phi(2\epsilon) \leq y_2 \leq y_1^2 \phi(\epsilon)$). We choose $\epsilon_1 > 0$ such that $\epsilon_1 \ll \epsilon$, then $\{x : |x - a| < \epsilon_1\} \subset \text{supp } \eta$. Now we have

$$\begin{aligned} Mf(y) &= \sup_{t>0} \left| \int f(y_1 - tx, y_2 - t^2 x^2 \phi(x)) \eta(x) dx \right| \\ &= \sup_{t>0} \int \frac{\chi_{[-1/2, 1/2]^2}(y_1 - tx, y_2 - t^2 x^2 \phi(x))}{|(y_1 - tx, y_2 - t^2 x^2 \phi(x))| \log \frac{1}{|(y_1 - tx, y_2 - t^2 x^2 \phi(x))|}} \eta(x) dx \\ &\geq \int_{\{x: |x-a| < \epsilon_1\}} \frac{\chi_{[-1/2, 1/2]^2}(y_1 - t_0 x, y_2 - t_0^2 x^2 \phi(x))}{|(y_1 - t_0 x, y_2 - t_0^2 x^2 \phi(x))| \log \frac{1}{|(y_1 - t_0 x, y_2 - t_0^2 x^2 \phi(x))|}} dx, \end{aligned}$$

where $t_0 = y_1/a$. We also observe that $\frac{1}{16\epsilon} \leq t_0 \leq \frac{1}{4\epsilon}$.

If $|x - a| < \epsilon_1$, then x can be written as $x = a + z = \frac{y_1}{t_0} + z$, $|z| < \epsilon_1$, which gives $|y_1 - t_0 x| = t_0 |x - \frac{y_1}{t_0}| = t_0 |z| \leq \frac{\epsilon_1}{4\epsilon} < 1/4$, and $|y_2 - t_0^2 x^2 \phi(x)|$ is equal to

$$\begin{aligned} &|y_2 - \frac{y_1^2}{a^2} (a+z)^2 (\phi(a) + \phi'(a)z + \mathcal{O}(z^2))| \\ &= |y_2 - \frac{y_1^2}{a^2} a^2 \phi(a) - \frac{y_1^2}{a^2} (2az + z^2) \phi(a+z) - \frac{y_1^2}{a^2} a^2 (\phi'(a)z + \mathcal{O}(z^2))| \\ &= |\frac{y_1^2}{a^2} (2az + z^2) \phi(a+z) + y_1^2 (\phi'(a)z + \mathcal{O}(z^2))| \\ &= t_0 |z| |y_1 (2 + \frac{z}{a}) \phi(a+z) + ay_1 (\phi'(a) + \mathcal{O}(z))| \end{aligned}$$

$$\leq \frac{\epsilon_1}{4\epsilon} \left[\frac{1}{4} \left(2 + \frac{\epsilon_1}{\epsilon} \right) \phi(4\epsilon) + \frac{\epsilon}{2} (\phi(a) + \mathcal{O}(\epsilon_1)) \right] t_0 |z| < 1/4.$$

Set $g(z) = y_1(2 + \frac{z}{a})\phi(a+z) + ay_1(\phi'(a) + \mathcal{O}(z))$ which is bounded from above by, say \tilde{C} . From the above argument, we have $(y_1 - t_0x, y_2 - t_0^2x^2\phi(x)) \in [-1/2, 1/2]$. Then making a change of variables, we have

$$\begin{aligned} Mf(y) &\geq \int_{\{x: |x-a| < \epsilon_1\}} \frac{dx}{|(y_1 - t_0x, y_2 - t_0^2x^2\phi(x))| \log \frac{1}{|(y_1 - t_0x, y_2 - t_0^2x^2\phi(x))|}} \\ &\geq \int_0^{\epsilon_1} \frac{dz}{t_0z |(1, g(z))| \log \frac{1}{t_0z |(1, g(z))|}}. \end{aligned}$$

Let $C = \sqrt{1 + \tilde{C}^2}$. It is a fact that $\frac{1}{x \log \frac{1}{x}}$ decrease monotonically in the neighborhood of the origin. Together with the above estimates, we will have

$$\frac{1}{t_0z |(1, g(z))| \log \frac{1}{t_0z |(1, g(z))|}} \geq \frac{1}{Ct_0z \log \frac{1}{Ct_0z}}.$$

Finally, for the fixed ϵ and $\epsilon_1 \ll \epsilon$, we have

$$Mf(y) \geq \int_0^{\epsilon_1} \frac{dz}{Ct_0z \log \frac{1}{Ct_0z}} \geq \frac{4\epsilon}{C} \int_0^{C\epsilon_1/(16\epsilon)} \frac{dz}{z \log \frac{1}{z}} = \infty.$$

□

Next we will generalize Theorem 2.1.1 to the curve of finite type $d (\geq 2)$ associated with the function ϕ defined by (2.1).

Theorem 2.1.3. *Define the maximal operator*

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}} f(y_1 - tx, y_2 - t^d x^d \phi(x)) \eta(x) dx \right|, \quad (2.4)$$

where $\eta(x)$ is supported in a sufficiently small neighborhood of the origin. If ϕ satisfies (2.1) and $d \geq 2$, then for $p > 2$, there exists a constant C_p such that

$$\|\mathcal{M}f\|_{L^p} \leq C_p \|f\|_{L^p}, \quad f \in C_0^\infty(\mathbb{R}^2). \quad (2.5)$$

Remark 2.1.2. If $m \rightarrow \infty$, then \mathcal{M} coincides with the maximal operator of the second type described above.

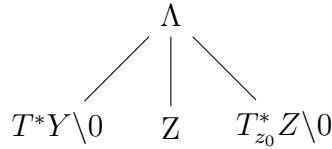
2.2 Background on Fourier integral operators and auxiliary results

2.2.1 Local smoothing of Fourier integral operators

In our settings, the averaging operator is always expressed as a Fourier integral operator, then we turn to prove the L^p boundedness for these operators. So here we will make a brief introduction to local smoothing of Fourier integral operators in [25].

We consider a class of Fourier integral operators $I^\nu(\mathbb{R}^{n+1}, \mathbb{R}^n; \Lambda)$, which is determined by the properties of its canonical relation Λ , which is a conic Lagrangian in $T^*Y \setminus 0$ to $T^*Z \setminus 0$ with respect to the symplectic form $d\zeta \wedge dz - d\eta \wedge dy$, and closed in $T^*Z \setminus 0 \times T^*Y \setminus 0$. In fact, these assumptions imply that $\Lambda \subset T^*Z \setminus 0 \times T^*Y \setminus 0$ is a conic (immersed) submanifold of dimension $2n + 1$.

To guarantee local regularity properties of operators $\mathcal{F} \in I^\nu(\mathbb{R}^{n+1}, \mathbb{R}^n; \Lambda)$, we shall impose conditions on Λ which are based on the properties of the following three projections



We assume that Π_X is the projection of Λ onto X , $X = T^*_{z_0} Z \setminus 0$, Z , or $T^*Y \setminus 0$. The condition has two parts:

(1) non-degeneracy condition:

$$\text{rank } d\Pi_{T^*Y} \equiv 2n, \tag{2.6}$$

$$\text{rank } d\Pi_Z \equiv n + 1; \tag{2.7}$$

analogue of the Carleson-Sjölin condition for nonhomogeneous phase (for every $z_0 \in Z$, $\Gamma_{z_0} := \Pi_{T^*_{z_0} Z}(\Lambda)$ has n non-vanishing principal curvature);

(2) cone condition: for every $z_0 \in Z$, Γ_{z_0} is a smooth conic n -dimensional hypersurface, $n - 1$ principal curvatures do not vanish. A Fourier integral operator which satisfies (2.6), (2.7) and the cone condition is said to satisfy the "cinematic curvature" condition.

The exact description of this condition can be found in [35]. Here we like to see how the condition can be reformulated if we use local coordinates. The non-degeneracy

condition implies that near a given point $(z_0, \zeta_0, y_0, \eta_0) \in \Lambda$, local coordinates can be chosen so that $\Lambda \ni (z, \zeta, y, \eta) \rightarrow (z, \eta)$ has bijective differential and there must be a phase function $\varphi(z, \eta)$ so that Λ takes the form

$$\{(z, \varphi'_z(z, \eta), \varphi'_\eta(z, \eta), \eta) : \eta \in \mathbb{R}^n \setminus 0 \text{ in a conic neighborhood of } \eta_0\}. \quad (2.8)$$

In this case, the condition (2.6) becomes

$$\text{rank } \varphi''_{z, \eta} \equiv n, \quad (2.9)$$

which means if we fix z_0 , then,

$$\Gamma_{z_0} = \{\varphi'_z(z_0, \eta) : \eta \in \mathbb{R}^n \setminus 0 \text{ in a conic neighborhood of } \eta_0\} \subset T_{z_0}^* Z \setminus 0$$

must be a smooth conic submanifold of dimension n . Then if $\Gamma_{z_0} \ni \zeta = \varphi'_z(z_0, \eta)$ and $\theta \in S^n$ is normal to Γ_{z_0} at ζ , it follows that $\pm\theta$ are the unique directions for which $\nabla_\eta \langle \varphi'_z(z_0, \eta), \theta \rangle = 0$. The cone condition (2) is just that

$$\text{rank} \left(\frac{\partial^2}{\partial \eta_j \partial \eta_k} \right) \langle \varphi'_z(z_0, \eta), \theta \rangle = n - 1 \text{ if } \eta, \theta \text{ are obtained from the above.} \quad (2.10)$$

Suppose $\mathcal{F} \in I^{\mu-1/4}(Z, Y; \Lambda)$, where Λ satisfies (2.6), (2.7) and (2.8), $\mathcal{F}f$ can be written as a finite sum of the form

$$\int_{\mathbb{R}^n} e^{i\varphi(z, \eta)} a(z, \eta) \widehat{f}(\eta) d\eta, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (2.11)$$

where the phase function φ satisfies (2.9) and (2.10) and the symbol a of order μ has small conic support in $\mathbb{R}^{n+1} \times \mathbb{R}^n$, which means that a vanishes for all z outside a small compact set and for all $\eta = (\eta_1, \eta')$ outside a narrow cone $\{\eta : |\eta'| \leq \varepsilon \eta_1\}$.

In this thesis, we always consider the case $n = 2$ and $z = (x, t) \in \mathbb{R}^2 \times \mathbb{R}$. We fix $\beta \in C_0^\infty(\mathbb{R})$ supported in $[1/2, 2]$ and set $a_\lambda(x, t, \eta) = \lambda^{-\mu} \beta(|\eta|/\lambda) a(x, t, \eta)$ for fixed $\lambda > 1$. Then a_λ is a symbol of order zero and satisfies the usual symbol estimates uniformly in λ . Mockenhaupt, Seeger and Sogge show that the dyadic estimate of the Fourier integral operator \mathcal{F} is as following.

Theorem 2.2.1. [26]

$$\left(\int_{1/2}^4 \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{i\varphi(x, t, \eta)} a_\lambda(x, t, \eta) \widehat{f}(\eta) d\eta \right|^p dx dt \right)^{1/p} \leq C_p \lambda^{1/2-1/p-\epsilon(p)},$$

where $\epsilon(p) = \frac{1}{2p}$, if $4 \leq p < \infty$; $\epsilon(p) = \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$, if $2 < p \leq 4$.

2.2.2 Auxiliary results

We will often use the following method of stationary phase.

Lemma 2.2.2. (Theorem 1.2.1 in [35]) *Let S be a smooth hypersurface in \mathbb{R}^n with non-vanishing Gaussian curvature and $d\mu$ be the Lebesgue measure on S . Then,*

$$|\widehat{d\mu}(\xi)| \leq C(1 + |\xi|)^{-\frac{n-1}{2}}. \quad (2.12)$$

Moreover, suppose that $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ is the cone consisting of all ξ which are normal to S at some point $x \in S$ belonging to a fixed relatively compact neighborhood \mathcal{N} of $\text{supp } d\mu$. Then

$$\begin{aligned} \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \widehat{d\mu}(\xi) \right| &= \mathcal{O}(1 + |\xi|)^{-N} \text{ for all } N \in \mathbb{N}, \text{ if } \xi \notin \Gamma, \\ \widehat{d\mu}(\xi) &= \sum e^{-i\langle x_j, \xi \rangle} a_j(\xi) \text{ if } \xi \in \Gamma, \end{aligned} \quad (2.13)$$

where the finite sum is taken over all $x_j \in \mathcal{N}$ having ξ as the normal and

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha a_j(\xi) \right| \leq C_\alpha (1 + |\xi|)^{-\frac{n-1}{2} - |\alpha|}. \quad (2.14)$$

We also use the following well-known estimate.

Lemma 2.2.3. (Theorem 2.4.2 in [35]) *Suppose that F is $C^1(\mathbb{R})$. Then if $p > 1$ and $1/p + 1/p' = 1$,*

$$\sup_\lambda |F(\lambda)|^p \leq |F(0)|^p + p \left(\int |F(\lambda)|^p d\lambda \right)^{1/p'} \left(\int |F'(\lambda)|^p d\lambda \right)^{1/p}.$$

Next, we will introduce the definition of the nonisotropic Littlewood-Paley operator associated with a parameter and show a related lemma.

Let $\beta \in C_0^\infty(\mathbb{R})$ be non-negative and $\text{supp } \beta \subset [1/\sqrt{c_0}, \sqrt{c_0}]$, where $c_0 > 1$ is a real number. For a fixed integer $j > 0$, real numbers $a_1, a_2 > 0$ and $\ell \in \mathbb{Z}$, we define the non-isotropic Littlewood-Paley operator with parameter j as

$$\widehat{\Delta_\ell^j f}(\xi) = \beta(2^{-j}|\delta_{2^\ell} \xi|) \hat{f}(\xi), \quad (2.15)$$

where $\delta_{2^\ell} \xi = (2^{\ell a_1} \xi_1, 2^{\ell a_2} \xi_2)$.

Lemma 2.2.4. *Let*

$$g_j(f) = \left(\sum_{\ell \in \mathbb{Z}} |\Delta_\ell^j f|^2 \right)^{1/2},$$

then the following holds true:

$$\|g_j(f)\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}. \quad (2.16)$$

if $p \in (1, \infty)$, and C_p only depends on p .

Proof. The idea of proof is very similar to the classic one in [38]. For completeness, we give the proof here. For $a = (a_1, a_2)$ and $\alpha = (\alpha_1, \alpha_2)$, we define $(a, \alpha) = a_1\alpha_1 + a_2\alpha_2$, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}$. Let $|\cdot|_\delta$ be a homogeneous norm, i.e. $|\delta_r\xi|_\delta = r|\xi|_\delta$ for $r > 0$. Hörmander's theorem in nonisotropic case is usually stated in the following way.

Lemma 2.2.5. [6] *Let $m(x) \in L^\infty(\mathbb{R}^2)$, and assume $m(x)$ is N times continuously differentiable where $N > |a|/2$; moreover, assume that*

$$\int_{R/2 \leq |\xi|_\delta \leq 2R} \left| R^{(a,\alpha)} \left(\frac{\partial}{\partial \xi} \right)^\alpha m(\xi) \right|^2 \frac{d\xi}{R^{|a|}} \leq C_\alpha, \quad |m(\xi)| \leq C \quad \text{a.e.},$$

where C is independent of R , say $C \geq 1$. Then there exists a constant A_p such that

$$\|T_m f\|_{L^p} = \|\mathcal{F}^{-1}(m\hat{f})\|_{L^p} \leq A_p \|f\|_{L^p}, \quad 1 < p < \infty, \quad f \in C_0^\infty(\mathbb{R}^2), \quad (2.17)$$

where A_p depends only on a and p . In particular, if m satisfies that

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha m(\xi) \right| \leq C |\xi|_\delta^{-(a,\alpha)}, \quad (2.18)$$

for all $\xi \neq 0$, then T_m is bounded on $L^p(\mathbb{R}^2)$.

Set

$$R_\ell := \begin{cases} r_{2\ell}, & \text{if } \ell \in \mathbb{Z}, \ell > 0; \\ r_{|2\ell|+1}, & \text{if } \ell \in \mathbb{Z}, \ell \leq 0, \end{cases}$$

where $\{r_k\}_{k \in \mathbb{N}}$ is the Rademacher function system on $[0, 1]$. Then

$$g_j(f)(x)^p \approx \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} \Delta_\ell^j f(x) R_\ell(t) \right|^p dt. \quad (2.19)$$

Since $|\cdot|_\delta \approx |\cdot|$ for $|\xi| \approx 1$, then it is easy to see that $\sum_{\ell \in \mathbb{Z}} \beta(2^{-j}|\delta_{2^\ell}\xi|) \leq C$. So we can define

$$T_{m_j,t} f := \left(\sum_{\ell \in \mathbb{Z}} R_\ell(t) \Delta_\ell^j \right) f, \quad (2.20)$$

then $T_{m_j,t}$ is a Fourier multiplier operator with multiplier

$$m_{j,t}(\xi) := \sum_{\ell \in \mathbb{Z}} R_\ell(t) \beta(2^{-j}|\delta_{2^\ell}\xi|).$$

However, $|R_\ell(t)| \leq 1$ for all $\ell \in \mathbb{Z}$ and $\|T_{m_j,t}\|_{L^p \rightarrow L^p} = \|T_{m_0,t}\|_{L^p \rightarrow L^p}$. In fact, if $\tau_j f(x) = 2^{2j/p} f(2^j x)$, then

$$\begin{aligned} T_{m_j,t} f &= T_{2^{2j/p} \tau_j m_{0,t}} f \\ &= \mathcal{F}^{-1}(2^{2j/p} \tau_j m_{0,t} \hat{f}) \\ &= 2^{2j/p} \mathcal{F}^{-1}(\tau_j(m_{0,t} \hat{f}(2^j \cdot))) \\ &= 2^{2j} 2^{-2j/p} \tau_j \mathcal{F}^{-1}(m_{0,t} \hat{f}(2^j \cdot)) \\ &= \tau_j \mathcal{F}^{-1}(m_{0,t} 2^{-2j/p} 2^{2j} \hat{f}(2^j \cdot)) \\ &= \tau_j \mathcal{F}^{-1}(m_{0,t} \widehat{\tau_j f}) = \tau_j T_{m_0,t}(\tau_j f), \end{aligned}$$

and $\|\tau_j T_{m_0,t} \tau_j\|_{L^p \rightarrow L^p} = \|T_{m_0,t}\|_{L^p \rightarrow L^p}$ implies the desired estimate.

It thus suffices to show that $m_{0,t}$ satisfies (2.18). Since for some $\ell \in \mathbb{Z}$, $\beta(|\delta_{2^\ell} \xi|) \neq 0 \Rightarrow |\xi|_\delta \approx 2^{-\ell}$, then

$$\sum_{\ell \in \mathbb{Z}} \left(\frac{\partial}{\partial \xi} \right)^\alpha \beta(|\delta_{2^\ell} \xi|) \leq C_\alpha C |\xi|_\delta^{-(a,\alpha)},$$

for arbitrary $\xi \neq 0$, which implies that $m_{0,t}$ satisfies (2.18). Then there exists a constant A_p such that

$$\|T_{m_j,t} f\|_{L^p} \leq A_p \|f\|_{L^p}, \quad \text{for all } f \in L^p(\mathbb{R}^2).$$

Furthermore, (2.19) and Fubini's theorem gives

$$\begin{aligned} \|g_j f\|_{L^p}^p &\approx \int_{\mathbb{R}^2} \left(\int_0^1 |T_{m_j,t} f|^p dt \right) dx \\ &= \int_0^1 \|T_{m_j,t} f\|_{L^p}^p dt \\ &\leq \int_0^1 A_p^p \|f\|_{L^p}^p dt = A_p^p \|f\|_{L^p}^p. \end{aligned}$$

□

2.3 The proof for curves with non-vanishing Gaussian curvature

In this section, we will give the proof of Theorem 2.1.1.

2.3.1 The case when $m = 1$

In this section, $\phi \in C^\infty(\mathbb{R})$ satisfies the following condition

$$\phi(0) \neq 0, \quad \phi'(0) \neq 0. \quad (2.21)$$

We choose $B > 0$ very small and $\tilde{\rho} \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \tilde{\rho} \subset \{x : B/2 \leq |x| \leq 2B\}$ and $\sum_k \tilde{\rho}(2^k x) = 1$ for $x \in \mathbb{R}$.

Put

$$\begin{aligned} A_t f(y) &:= \int f(y_1 - tx, y_2 - t^2 x^2 \phi(x)) \eta(x) dx \\ &= \sum_k \int f(y_1 - tx, y_2 - t^2 x^2 \phi(x)) \tilde{\rho}(2^k x) \eta(x) dx = \sum_k A_t^k f(y), \end{aligned}$$

where

$$A_t^k f(y) := \int f(y_1 - tx, y_2 - t^2 x^2 \phi(x)) \tilde{\rho}(2^k x) \eta(x) dx. \quad (2.22)$$

Since η is supported in a sufficiently small neighborhood of the origin, then we only need to consider $k > 0$ sufficiently large.

Considering the isometric operator on $L^p(\mathbb{R}^2)$ defined by $T_k f(x_1, x_2) = 2^{3k/p} f(2^k x_1, 2^{2k} x_2)$, one can compute that

$$T_k^{-1} A_t^k T_k f(y) = 2^{-k} \int f(y_1 - tx, y_2 - t^2 x^2 \phi(\frac{x}{2^k})) \tilde{\rho}(x) \eta(2^{-k} x) dx. \quad (2.23)$$

Then it suffices to prove the following estimate

$$\sum_k 2^{-k} \left\| \sup_{t>0} |\widetilde{A}_t^k| \right\|_{L^p \rightarrow L^p} \leq C_p, \quad \text{for all } p > 2, \quad (2.24)$$

where

$$\widetilde{A}_t^k f(y) := \int f(y_1 - tx, y_2 - t^2 x^2 \phi(\frac{x}{2^k})) \tilde{\rho}(x) \eta(2^{-k} x) dx. \quad (2.25)$$

By means of the Fourier inversion formula, we have

$$\begin{aligned} \widetilde{A}_t^k f(y) &= \int_{\mathbb{R}^2} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-i(t\xi_1 x + t^2 \xi_2 x^2 \phi(\frac{x}{2^k}))} \tilde{\rho}(x) \eta(2^{-k} x) dx \hat{f}(\xi) d\xi \\ &= \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_k}(\delta_t \xi) \hat{f}(\xi) d\xi, \end{aligned}$$

where

$$\widehat{d\mu_k}(\xi) := \int_{\mathbb{R}} e^{-i(\xi_1 x + \xi_2 x^2 \phi(\frac{x}{2^k}))} \tilde{\rho}(x) \eta(2^{-k} x) dx. \quad (2.26)$$

We choose a non-negative function $\beta \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \beta \subset [1/2, 2]$ and $\sum_{j \in \mathbb{Z}} \beta(2^{-j}r) = 1$ for $r > 0$. Define the dyadic operators

$$\widetilde{A}_{t,j}^k f(y) = \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_k}(\delta_t \xi) \beta(2^{-j}|\delta_t \xi|) \widehat{f}(\xi) d\xi, \quad (2.27)$$

and denote by $\widetilde{\mathcal{M}}_j^k$ the corresponding maximal operator. Now we have that

$$\sup_{t>0} |\widetilde{A}_t^k f(y)| \leq \widetilde{\mathcal{M}}^{k,0} f(y) + \sum_{j \geq 1} \widetilde{\mathcal{M}}_j^k f(y), \quad \text{for } y \in \mathbb{R}^2.$$

where

$$\widetilde{\mathcal{M}}^{k,0} f(y) := \sup_{t>0} \left| \sum_{j \leq 0} \widetilde{A}_{t,j}^k f(y) \right|. \quad (2.28)$$

We observe that $\widetilde{\mathcal{M}}^{k,0} f(y) = \sup_{t>0} |f * K_{\delta_{t^{-1}}}(y)|$, where $K_{\delta_{t^{-1}}}(x) = t^{-3} K(\frac{x_1}{t}, \frac{x_2}{t^2})$ and

$$K(y) := \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_k}(\xi) \rho(|\xi|) d\xi, \quad (2.29)$$

where $\rho \in C_0^\infty(\mathbb{R})$ is supported in $[0, 1]$. Since ϕ satisfies (2.21), $\text{supp } \tilde{\rho} \subset \{x : B/2 \leq |x| \leq 2B\}$, then Lemma 2.2.2 implies that for a multi-index α ,

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \widehat{d\mu_k}(\xi) \right| \leq C_{B,\alpha} (1 + |\xi|)^{-1/2}. \quad (2.30)$$

By integration by parts, we obtain that

$$|K(y)| \leq C_N (1 + |y|)^{-N}. \quad (2.31)$$

Then $\widetilde{\mathcal{M}}^{k,0} f(y) \leq C'_N Mf(y)$, where M is the Hardy-Littlewood maximal operator defined by

$$Mf(y) := \sup_{t>0} \frac{1}{|B(y,t)|} \int_{B(y,t)} f(x) dx, \quad (2.32)$$

where $B(y,t) = \{x : |y - x|_\delta < t\}$ and $|x|_\delta = \max\{|x_1|, |x_2|^{1/2}\}$. For the exact description, one can see page 8-13 in [41]. So it suffices to prove that

$$\sum_k 2^{-k} \sum_{j \geq 1} \|\widetilde{\mathcal{M}}_j^k\|_{L^p \rightarrow L^p} \leq C_p. \quad (2.33)$$

Since $\widetilde{A}_{t,j}^k$ is localized to frequency $|\delta_t \xi| \approx 2^j$, we will show that

$$\|\widetilde{\mathcal{M}}_j^k\|_{L^p \rightarrow L^p} \lesssim \|\mathcal{M}_{j,loc}^k\|_{L^p \rightarrow L^p},$$

where $\mathcal{M}_{j,loc}^k f(y) = \sup_{t \in [1,2]} |\widehat{A_{t,j}^k} f(y)|$. In fact, for fixed $j \geq 1$ and all $\ell \in \mathbb{Z}$, let Δ_ℓ^j be the non-isotropic Littlewood-Paley operator in \mathbb{R}^2 defined by $\widehat{\Delta_\ell^j f}(\xi) = \tilde{\beta}(2^{-j}|\delta_{2^\ell \xi}|)\hat{f}(\xi)$, here $\tilde{\beta} \in C_0^\infty(\mathbb{R})$ is nonnegative and satisfies $\beta(|\delta_t \xi|) = \beta(|\delta_t \xi|)\tilde{\beta}(|\xi|)$, for any $t \in [1, 2]$. Then

$$\begin{aligned} \widehat{M_j^k} f(y) &= \sup_{\ell \in \mathbb{Z}} \sup_{t \in [1,2]} \left| \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_k}(\delta_{2^\ell t \xi}) \beta(2^{-j}|\delta_{2^\ell t \xi}|) \tilde{\beta}(2^{-j}|\delta_{2^\ell t \xi}|) \hat{f}(\xi) d\xi \right| \\ &\leq \left(\sum_{\ell \in \mathbb{Z}} \sup_{t \in [1,2]} \left| \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_k}(\delta_{2^\ell t \xi}) \beta(2^{-j}|\delta_{2^\ell t \xi}|) \widehat{\Delta_\ell^j f}(\xi) d\xi \right|^p \right)^{1/p} \\ &= \left(\sum_{\ell \in \mathbb{Z}} \sup_{t \in [1,2]} \left| 2^{-3\ell} \int_{\mathbb{R}^2} e^{i\xi \cdot \delta_{2^{-\ell}} y} \widehat{d\mu_k}(\delta_t \xi) \beta(2^{-j}|\delta_t \xi|) \widehat{\Delta_\ell^j f}(\delta_{2^{-\ell}} \xi) d\xi \right|^p \right)^{1/p} \\ &= \left(\sum_{\ell \in \mathbb{Z}} \left| \mathcal{M}_{j,loc}^k(\Delta_\ell^j f \circ \delta_{2^\ell})(\delta_{2^{-\ell}} y) \right|^p \right)^{1/p}. \end{aligned}$$

Since $p > 2$, Lemma 2.2.4 implies that

$$\begin{aligned} \|\widehat{M_j^k} f\|_{L^p}^p &\leq \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^2} \left| \mathcal{M}_{j,loc}^k(\Delta_\ell^j f \circ \delta_{2^\ell})(\delta_{2^{-\ell}} y) \right|^p dy \\ &= \sum_{\ell \in \mathbb{Z}} 2^{3\ell} \left\| \mathcal{M}_{j,loc}^k(\Delta_\ell^j f \circ \delta_{2^\ell}) \right\|_{L^p}^p \\ &\leq \|\mathcal{M}_{j,loc}^k\|_{L^p \rightarrow L^p}^p \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^2} |\Delta_\ell^j f(y)|^p dy \\ &\leq \|\mathcal{M}_{j,loc}^k\|_{L^p \rightarrow L^p}^p \left\| \left(\sum_{\ell \in \mathbb{Z}} |\Delta_\ell^j f(y)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)}^p \\ &\lesssim \|\mathcal{M}_{j,loc}^k\|_{L^p \rightarrow L^p}^p \|f\|_{L^p(\mathbb{R}^2)}^p. \end{aligned}$$

Based on the above argument, next we will only consider

$$\sum_k 2^{-k} \sum_{j \geq 1} \|\mathcal{M}_{j,loc}^k\|_{L^p \rightarrow L^p} \leq C_p. \quad (2.34)$$

In order to get (2.34), first we will estimate

$$\widehat{d\mu_k}(\delta_t \xi) = \int_{\mathbb{R}} e^{-it^2 \xi_2(-sx + x^2 \phi(\delta x))} \tilde{\rho}(x) \eta(\delta x) dx, \quad (2.35)$$

where $2^{-k} = \delta$ and

$$s := s(\xi, t) = -\frac{\xi_1}{t\xi_2}, \quad \text{for } \xi_2 \neq 0. \quad (2.36)$$

If $\xi_2 = 0$, then

$$|\widehat{d\mu_k}(\delta_t\xi)| = |(\eta(\delta\cdot)\tilde{\rho})^\wedge(t\xi_1)| \leq \frac{C'_N}{(1 + |t\xi_1|)^N} = \frac{C'_N}{(1 + |\delta_t\xi|)^N},$$

and for multi-index α ,

$$|D_\xi^\alpha \widehat{d\mu_k}(\delta_t\xi)| = |D_\xi^\alpha (\eta(\delta\cdot)\tilde{\rho})^\wedge(t\xi_1)| \leq \frac{C_{\alpha,N}}{(1 + |\delta_t\xi|)^N}.$$

Since $t \approx 1$, then we can deduce the case $\xi_2 = 0$ into B_k of the following (2.39).

Put

$$\Phi(s, x, \delta) = -sx + x^2\phi(\delta x), \quad (2.37)$$

then we have

$$\partial_x \Phi(s, x, \delta) = -s + 2x\phi(\delta x) + x^2\delta\phi'(\delta x)$$

and

$$\partial_x^2 \Phi(s, x, \delta) = 2\phi(\delta x) + 4x\delta\phi'(\delta x) + x^2\delta^2\phi''(\delta x).$$

Since k is sufficiently large and $\phi(0) \neq 0$, then the implicit function theorem implies that there exists a smooth solution $x_c = \tilde{q}(s, \delta)$ of the equation $\partial_x \Phi(s, x, \delta) = 0$. For the sake of simplicity, we may assume $\phi(0) = 1/2$.

Meanwhile, we observe when k tends to infinity, $\tilde{q}(s, \delta)$ smoothly converges to the solution $\tilde{q}(s, 0) = s$ of the equation $\partial_x \Phi(s, x, 0) = 0$.

Let $\tilde{\Phi}(s, \delta) := \Phi(s, \tilde{q}(s, \delta), \delta)$. From the above arguments and Taylor expansion of smooth functions $\tilde{q}(s, \delta)$ and $\phi(x)$, the phase function

$$-t^2\xi_2\tilde{\Phi}(s, \delta) = \frac{\xi_1^2}{2\xi_2} + \delta\frac{\xi_1^3}{t\xi_2^2}\phi'(0) + \delta^2R(t, \xi, \delta), \quad (2.38)$$

where $R(t, \xi, \delta)$ is homogeneous of degree one in ξ . $-t^2\xi_2\tilde{\Phi}(s, \delta)$ can be considered as a small perturbation of $\frac{\xi_1^2}{2\xi_2} + \delta\frac{\xi_1^3}{t\xi_2^2}\phi'(0)$.

By applying the method of stationary phase, we have

$$\widehat{d\mu_k}(\delta_t\xi) = e^{-it^2\xi_2\tilde{\Phi}(s,\delta)} \chi_k\left(\frac{\xi_1}{t\xi_2}\right) \frac{A_k(\delta_t\xi)}{(1 + |\delta_t\xi|)^{1/2}} + B_k(\delta_t\xi), \quad (2.39)$$

where χ_k is a smooth function supported in the interval $[c_k, \tilde{c}_k]$, for certain non-zero positive constants $c_1 \leq c_k, \tilde{c}_k \leq c_2$ depending only on k . $\{A_k(\delta_t\xi)\}_k$ is contained in a bounded subset of symbols of order zero. More precisely, for arbitrary $t \in [1, 2]$,

$$|D_\xi^\alpha A_k(\delta_t\xi)| \leq C_\alpha(1 + |\xi|)^{-\alpha}, \quad (2.40)$$

where C_α do not depend on k and t . Furthermore, B_k is a remainder term and satisfies for arbitrary $t \in [1, 2]$,

$$|D_\xi^\alpha B_k(\delta_t \xi)| \leq C_{\alpha, N}(1 + |\xi|)^{-N}, \quad (2.41)$$

where $C_{\alpha, N}$ are admissible constants and again do not depend on k and t .

First, let us consider the remainder part of (2.34). Set

$$M_j^{k,0} f(y) := \sup_{t \in [1,2]} \left| \int_{\mathbb{R}^2} e^{i\xi \cdot y} B_k(\delta_t \xi) \beta(2^{-j} |\delta_t \xi|) \hat{f}(\xi) d\xi \right|. \quad (2.42)$$

By (2.41) and integration by parts, it is easy to get $|(B_k \beta(2^{-j} \cdot))^\vee(x)| \leq C_N 2^{-jN} (1 + |x|)^{-N}$. So we have that

$$\begin{aligned} \|M_j^{k,0}\|_{L^p \rightarrow L^p} &= \sup\{\|M_j^{k,0} f\|_{L^p} : \|f\|_{L^p} = 1\} \\ &= \sup\{\| \sup_{t \in [1,2]} |t^{-3} (B_k \beta(2^{-j} \cdot))^\vee(\delta_{t^{-1}} \cdot) * f \|_{L^p} : \|f\|_{L^p} = 1\} \\ &\leq \sup\{C_N 2^{-jN} \|Mf\|_{L^p} : \|f\|_{L^p} = 1\} \\ &\leq C_N 2^{-jN}, \end{aligned}$$

where M is the Hardy-Littlewood maximal operator defined by (2.32), and the L^p -boundedness ($1 < p < \infty$) of M implies (2.24) for remainder part of (2.34).

Put

$$A_{t,j}^k f(y) := \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} \chi_k\left(\frac{\xi_1}{t \xi_2}\right) \frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \beta(2^{-j} |\delta_t \xi|) \hat{f}(\xi) d\xi. \quad (2.43)$$

Denote by $M_j^{k,1}$ the corresponding maximal operator over $[1, 2]$. It remains to prove that

$$\sum_k 2^{-k} \sum_{j \geq 1} \|M_j^{k,1}\|_{L^p \rightarrow L^p} \leq C_{p, N}. \quad (2.44)$$

Since $\tilde{\Phi}(s, \delta)$ is homogeneous of degree zero in ξ and $\frac{\xi_1}{\xi_2} \approx 1$, then

$$\left| \nabla_\xi [\xi \cdot (y - x) - t^2 \xi_2 \tilde{\Phi}(s, \delta)] \right| \geq C|y - x|$$

provided $|y - x| \geq L$, where L is very large and determined by c_1, c_2 and $\|\phi\|_\infty(I)$. By integration by parts, we will see that the kernel of the operator $A_{t,j}^k$ is dominated by $2^{-jN} \mathcal{O}(|y - x|^{-N})$ if $|y - x| \geq L$. From now on, we will restrict our view on the situation

$$|y - x| \leq L. \quad (2.45)$$

Let $B_i(L)$ be a ball with center i and radius L . Furthermore, we will show that

$$\sup\{\|M_j^{k,1}f\|_{L^p} : \|f\|_{L^p} = 1, \text{supp } f \subset B_0(L)\} \leq C_p 2^{-j\tilde{\epsilon}_1(p)} 2^{k\tilde{\epsilon}_2(p)} \quad (2.46)$$

implies that

$$\|M_j^{k,1}\|_{L^p \rightarrow L^p} \leq C_p 2^{-j\tilde{\epsilon}_1(p)} 2^{k\tilde{\epsilon}_2(p)}, \quad (2.47)$$

where C_p depends on p , c_1 , c_2 and $\|\phi\|_\infty(I)$, and $\tilde{\epsilon}_1(p)$, $\tilde{\epsilon}_2(p) > 0$. Then in order to prove inequality (2.34), it suffices to prove inequality (2.46).

In fact, we can decompose $f = \sum_{i \in LZ^2} f_i$, where $\text{supp } f_i \subset B_i(L)$. If we have proved (2.46), then

$$\begin{aligned} \|M_j^{k,1}f\|_{L^p} &\leq \left\| \sum_{i \in LZ^2} M_j^{k,1}f_i \right\|_{L^p} \\ &\leq \left(\sum_{i \in LZ^2} \|M_j^{k,1}(f_i(\cdot + i))\|_{L^p}^p \right)^{1/p} \\ &\leq C_p 2^{-j\tilde{\epsilon}_1(p)} 2^{k\tilde{\epsilon}_2(p)} \left(\sum_{i \in LZ^2} \|f_i(\cdot + i)\|_{L^p}^p \right)^{1/p} \\ &\leq C_p 2^{-j\tilde{\epsilon}_1(p)} 2^{k\tilde{\epsilon}_2(p)} \|f\|_{L^p}, \end{aligned}$$

which implies (2.47).

Now we observe inequality (2.46), together with the assumption (2.45), we can choose $\rho_1 \in C_0^\infty(\mathbb{R}^2 \times [\frac{1}{2}, 4])$ such that (2.46) will follow from that

$$\|\widetilde{M_j^{k,1}}\|_{L^p \rightarrow L^p} \leq C_p 2^{-j\tilde{\epsilon}_1(p)} 2^{k\tilde{\epsilon}_2(p)}, \quad (2.48)$$

where

$$\widetilde{M_j^{k,1}}f(y) := \sup_{t \in [1,2]} |\rho_1(y, t) A_{t,j}^k f(y)|. \quad (2.49)$$

Lemma 2.3.1.

$$\|\widetilde{M_j^{k,1}}f\|_{L^p} \leq C_p 2^{-(j \wedge k)/p} \|f\|_{L^p}, \quad 2 \leq p \leq \infty.$$

Proof. To prove the above lemma, we employ the M. Riesz interpolation theorem between the L^2 and the L^∞ -estimate. By Lemma 2.2.3, we have

$$\begin{aligned} &\|\widetilde{M_j^{k,1}}f\|_{L^2}^2 \\ &\leq \left(\int_{1/2}^4 \int_{\mathbb{R}^2} |\rho_1(y, t) A_{t,j}^k f(y)|^2 dy dt \right)^{1/2} \left(\int_{1/2}^4 \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial t} (\rho_1(y, t) A_{t,j}^k f(y)) \right|^2 dy dt \right)^{1/2} \\ &= \left\| \left\| \rho_1(y, t) A_{t,j}^k f(y) \right\|_{(L^2, dy)} \right\|_{(L^2, dt)} \left\| \left\| \frac{\partial}{\partial t} (\rho_1(y, t) A_{t,j}^k f(y)) \right\|_{(L^2, dy)} \right\|_{(L^2, dt)}. \end{aligned}$$

Moreover,

$$\frac{\partial}{\partial t}(\rho_1(y, t)A_{t,j}^k f(y)) = \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} h_k(y, t, \xi, j) \hat{f}(\xi) d\xi, \quad (2.50)$$

where

$$\begin{aligned} h_k(y, t, \xi, j) &= \left(\frac{\partial}{\partial t} \rho_1(y, t) + i \frac{\partial}{\partial t} (-t^2 \xi_2 \tilde{\Phi}(s, \delta)) \rho_1(y, t) \right) \chi_k \left(\frac{\xi_1}{t \xi_2} \right) \frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \beta(2^{-j} |\delta_t \xi|) \\ &\quad + \rho_1(y, t) \frac{\partial}{\partial t} \left(\chi_k \left(\frac{\xi_1}{t \xi_2} \right) \right) \frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \beta(2^{-j} |\delta_t \xi|) \\ &\quad + \rho_1(y, t) \chi_k \left(\frac{\xi_1}{t \xi_2} \right) \frac{\partial}{\partial t} \left(\frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \right) \beta(2^{-j} |\delta_t \xi|) \\ &\quad + \rho_1(y, t) \chi_k \left(\frac{\xi_1}{t \xi_2} \right) \frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \frac{\partial}{\partial t} (\beta(2^{-j} |\delta_t \xi|)). \end{aligned}$$

From (2.38) and (2.40), we get $|\frac{\partial}{\partial t}(-t^2 \xi_2 \tilde{\Phi}(s, \delta))| = |-\delta \frac{\xi_1^3}{t^2 \xi_2^2} \phi'(0) + \delta^2 \frac{\partial}{\partial t} R(t, \xi, \delta)| \lesssim \delta 2^j$, $|\frac{\partial}{\partial t}(\chi_k(\frac{\xi_1}{t \xi_2}))| \approx 1$, $|\frac{\partial}{\partial t}(\frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}})| \lesssim 2^{-j/2}$ and $|\frac{\partial}{\partial t}(\beta(2^{-j} |\delta_t \xi|))| \lesssim 1$. Therefore $|h_k(y, t, \xi, j)| \lesssim 2^{j/2} \delta + 2^{-j/2}$. Then $(2^{j/2} \delta + 2^{-j/2})^{-1} \frac{\partial}{\partial t}(\rho_1(y, t)A_{t,j}^k f(y))$ behaves like $2^{j/2} A_{t,j}^k$ which is a symbol of order zero. So we only consider the L^2 -boundedness of the operator $2^{j/2} \rho_1 A_{t,j}^k$, for fixed $t \in [1/2, 4]$, however this is easily obtained by the Plancherel theorem, i.e.

$$\begin{aligned} \|2^{j/2} \rho_1(y, t) A_{t,j}^k f(y)\|_{(L^2, dy)} &\leq \|e^{-it^2 \xi_2 \tilde{\Phi}(s, \delta)} 2^{j/2} \chi_k \left(\frac{\xi_1}{t \xi_2} \right) \frac{A_k(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \beta(2^{-j} |\delta_t \xi|) \hat{f}(\xi)\|_{(L^2, d\xi)} \\ &\leq C \|f\|_{L^2}. \end{aligned}$$

From the above arguments, we obtain

$$\begin{aligned} \|\widetilde{M_j^{k,1}} f\|_{L^2}^2 &\leq 2^{-j/2} (2^{j/2} \delta + 2^{-j/2}) \|\|2^{j/2} \rho_1(y, t) A_{t,j}^k f(y)\|_{(L^2, dy)}\|_{(L^2, dt)} \\ &\quad \times \left\| \left\| \frac{\partial}{\partial t} ((2^{j/2} \delta + 2^{-j/2})^{-1} \rho_1(y, t) A_{t,j}^k f(y)) \right\|_{(L^2, dy)} \right\|_{(L^2, dt)} \\ &\leq C 2^{-j/2} (2^{j/2} \delta + 2^{-j/2}) \|f\|_{L^2}^2 \\ &\leq C(\delta + 2^{-j}) \|f\|_{L^2}^2, \end{aligned}$$

then

$$\|\widetilde{M_j^{k,1}} f\|_{L^2} \leq C(\delta + 2^{-j})^{1/2} \|f\|_{L^2}. \quad (2.51)$$

Next, let us turn to prove

$$\|\widetilde{M_j^{k,1}} f\|_{L^\infty} \leq C \|f\|_{L^\infty}. \quad (2.52)$$

Inequality (2.52) follows from the fact that the kernels of $A_{t,j}^k$ are uniformly bounded in $L^1(\mathbb{R}^2)$. The idea of the proof can be found in page 406-408 in [41]. For completeness, in fact, this idea also will be used later, so we give a brief overview here.

First, we introduce the angular decomposition as in [41] of the ξ -space in the plane. For each positive integer j , we consider a roughly equally spaced set of points with grid length $2^{-j/2}$ on the unit circle S^1 ; that is, we fix a collection $\{\xi_j^\nu\}_\nu$ of unit vectors, that satisfy:

- (a) $|\xi_j^\nu - \xi_j^{\nu'}| \geq 2^{-j/2}$, if $\nu \neq \nu'$;
- (b) if $\xi \in S^1$, then there exists a ξ_j^ν so that $|\xi - \xi_j^\nu| < 2^{-j/2}$.

Let Γ_j^ν denote the corresponding cone in the ξ -space whose central direction is ξ_j^ν , i.e.

$$\Gamma_j^\nu = \{\xi : |\xi/|\xi| - \xi_j^\nu| \leq 2 \cdot 2^{-j/2}\}.$$

We can construct an associated partition of unity:

Lemma 2.3.2. [41] χ_j^ν is homogeneous of degree zero in ξ and supported in Γ_j^ν , with

$$\sum_{\nu} \chi_j^\nu(\xi) = 1 \quad \text{for all } \xi \neq 0 \text{ and all } j, \quad (2.53)$$

and

$$|\partial_{\xi}^{\alpha} \chi_j^{\nu}(\xi)| \leq A_{\alpha} 2^{|\alpha|j/2} |\xi|^{-|\alpha|}. \quad (2.54)$$

Hence, in order to establish (2.52), it is sufficient to prove

$$\int_{\mathbb{R}^2} |K_t^{\nu}(y)| dy \leq C 2^{-j/2}, \quad (2.55)$$

where C does not depend on t, j, k and ν , and

$$K_t^{\nu}(y) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} \widetilde{A}_k(\xi, t) \beta(2^{-j} |\delta_t \xi|) \chi_j^{\nu}(\xi) d\xi, \quad (2.56)$$

and $\widetilde{A}_k(\xi, t) = \chi_k\left(\frac{\xi_1}{t\xi_2}\right) \frac{A_k(\delta_t \xi)}{(1+|\delta_t \xi|)^{1/2}}$.

Then we have

$$\begin{aligned} \|\widetilde{M}_j^{k,1} f\|_{L^{\infty}} &:= \left\| \sup_{t>0} |K_t * f| \right\|_{L^{\infty}} \\ &\leq \|f\|_{L^{\infty}} \sup_{t>0} \sum_{\nu \lesssim 2^{j/2}} \|K_t^{\nu}\|_{L^1} \\ &\lesssim \|f\|_{L^{\infty}}, \end{aligned}$$

and get inequality (2.52).

For fixed ν , the inner integral (2.56) is supported in the truncated cone Γ_j^ν . Let T be the transpose operator, i.e. $T(x_1, x_2) = (x_1, x_2)^t$. We can find a rotation ρ_j^ν such that $T^{-1}\rho_j^\nu T(1, 0) = \xi_j^\nu$. Write $\tilde{T} = T^{-1}\rho_j^\nu T$ and \tilde{T}_i the i -th variable after the action of \tilde{T} , this leads to prove the following estimate

$$\int_{\mathbb{R}^2} \left| \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\tilde{T}\xi \cdot y - t^2 \tilde{T}_2 \xi \tilde{\Phi}(\tilde{s}, \delta))} \tilde{A}_k(\tilde{T}\xi, t) \beta(2^{-j}|\delta_t \tilde{T}\xi|) \chi_j^\nu(\tilde{T}\xi) d\xi \right| dy \leq C 2^{-j/2}, \quad (2.57)$$

where $\tilde{s} = -\frac{\tilde{T}_1 \xi}{t \tilde{T}_2 \xi}$. From the ξ -support of the inner integral of (2.57), we know that ξ belongs to the following region, see Figure 1.

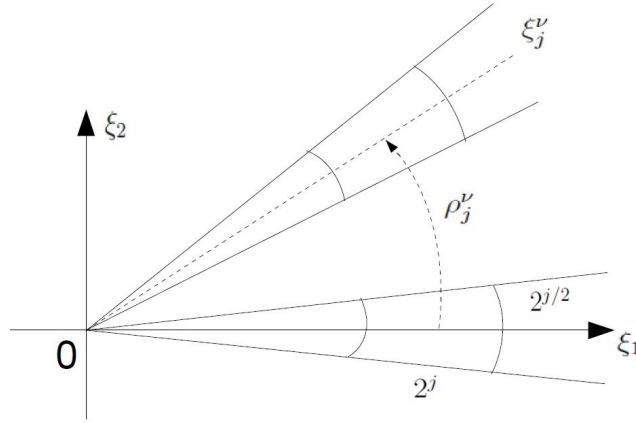


Figure 1

Put $\tilde{\Phi}(y, \xi) = \tilde{T}\xi \cdot y - t^2 \tilde{T}_2 \xi \tilde{\Phi}(\tilde{s}, \delta)$. Clearly $\tilde{\Phi}$ is homogeneous of degree one in ξ , hence $\tilde{\Phi}(y, \xi) = \xi \cdot \nabla_\xi \tilde{\Phi}(y, \xi)$. Let $h(\xi) = \tilde{\Phi}(y, \xi) - \xi \cdot \nabla_\xi \tilde{\Phi}(y, \bar{\xi})$, where $\bar{\xi} = (1, 0)$. We claim that

$$\left| \left(\frac{\partial}{\partial \xi_1} \right)^N h(\xi) \right| \leq A_N \cdot 2^{-jN}, \quad (2.58)$$

$$\left| \left(\frac{\partial}{\partial \xi_2} \right)^N h(\xi) \right| \leq A_N \cdot 2^{-jN/2}. \quad (2.59)$$

Since $h(\xi)$ is also homogeneous of degree one, then $h(\bar{\xi}) = 0$ and $\nabla_\xi h(\bar{\xi}) = 0$. As a result

$$\left(\frac{\partial}{\partial \xi_1} \right)^N h(\bar{\xi}) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial}{\partial \xi_1} \right)^N h(\bar{\xi}) = 0.$$

Thus $\left(\frac{\partial}{\partial \xi_1}\right)^N h(\xi) = \mathcal{O}\left(\frac{|\xi_2|^2}{|\xi|^{N+1}}\right)$. Since $|\xi_2| \lesssim 2^{j/2}$ and $|\xi| \approx 2^j$, we have that $\frac{|\xi_2|^2}{|\xi|^{N+1}} \leq \mathcal{O}(2^{-jN})$, we get inequality (2.58).

Again, $\frac{\partial}{\partial \xi_2} h(\bar{\xi}) = 0$, so $\frac{\partial}{\partial \xi_2} h(\xi) = \mathcal{O}\left(\frac{\xi_2}{|\xi|}\right) \approx 2^{-j/2}$. For $N > 1$, we have that $\left|\left(\frac{\partial}{\partial \xi_2}\right)^N h(\xi)\right| \leq A_N |\xi|^{1-N}$, by homogeneity, and $2^{j(1-N)} \leq 2^{-Nj/2}$, so we finish the proof of inequality (2.59).

We observe that $\chi_j^\nu \circ \tilde{T}$ satisfies the same conditions (2.53) and (2.54) as χ_j^ν . Meanwhile, $\text{supp}(\chi_j^\nu \circ \tilde{T}) \subset \{\xi : \left|\frac{\xi}{|\xi|} - (1, 0)\right| \leq 2 \cdot 2^{-j/2}\}$. If we set $\epsilon = 2^{-j/2}$ in (iii) of Lemma 14 in [23], then it follows that (2.54) has an improvement for $\chi_j^\nu \circ \tilde{T}$, i.e.

$$\left|\left(\frac{\partial}{\partial \xi_1}\right)^N (\chi_j^\nu \circ \tilde{T})(\xi)\right| \leq C_N |\xi|^{-N} \approx 2^{-jN}. \quad (2.60)$$

We now rewrite

$$K_t^\nu(y) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i\xi \cdot \nabla_\xi \tilde{\Phi}(y, \bar{\xi})} e^{ih(\xi)} \widetilde{A}_k(\tilde{T}\xi, t) \beta(2^{-j} |\delta_t \tilde{T}\xi|) (\chi_j^\nu \circ \tilde{T})(\xi) d\xi.$$

Next, we introduce the operator \mathcal{L} defined by $\mathcal{L} = I - 2^{2j} \frac{\partial^2}{\partial \xi_1^2} - 2^j \frac{\partial^2}{\partial \xi_2^2}$. Because of (2.54), (2.60) and the fact that \widetilde{A}_k is a symbol of order $-1/2$, we get that

$$|\mathcal{L}^N (e^{ih(\xi)} \widetilde{A}_k(\tilde{T}\xi, t) \beta(2^{-j} |\delta_t \tilde{T}\xi|) (\chi_j^\nu \circ \tilde{T})(\xi))| \leq A_N 2^{-j/2}.$$

However,

$$\mathcal{L}^N e^{i\xi \cdot \nabla_\xi \tilde{\Phi}(y, \bar{\xi})} = \left(1 + 2^{2j} \left|\frac{\partial}{\partial \xi_1} \tilde{\Phi}(y, \bar{\xi})\right|^2 + 2^j \left|\frac{\partial}{\partial \xi_2} \tilde{\Phi}(y, \bar{\xi})\right|^2\right)^N \cdot e^{i\xi \cdot \nabla_\xi \tilde{\Phi}(y, \bar{\xi})}.$$

From Figure 1, we note that the support of the integrand has volume at most $\mathcal{O}(2^j \cdot 2^{j/2}) \approx 2^{3j/2}$, thus we obtain by integration by parts that

$$|K_t^\nu(y)| \leq A_N 2^{3j/2} 2^{-j/2} (1 + 2^{2j} \left|\frac{\partial}{\partial \xi_1} \tilde{\Phi}(y, \bar{\xi})\right|^2 + 2^j \left|\frac{\partial}{\partial \xi_2} \tilde{\Phi}(y, \bar{\xi})\right|^2)^{-N}.$$

In order to prove (2.55), we make the change of variables $y \mapsto \left(\frac{\partial}{\partial \xi_1} \tilde{\Phi}(y, \bar{\xi}), \frac{\partial}{\partial \xi_2} \tilde{\Phi}(y, \bar{\xi})\right)$. Since

$$\det \left(\frac{\partial^2 \tilde{\Phi}(y, \bar{\xi})}{\partial \xi_i \partial y_j} \right) \neq 0,$$

and the Jacobian is bounded from below, then we have

$$\int_{\mathbb{R}^2} |K_t^\nu(y)| dy \leq A_N 2^j \int_{\mathbb{R}^2} \frac{dy}{(1 + |2^j y_1| + |2^{j/2} y_2|)^{2N}} \leq C 2^{-j/2}.$$

□

Now we split the set of j into two parts $j > 2k$ and $j \leq 2k$. By Lemma 2.3.1, we get

$$\sum_k 2^{-k} \sum_{j \leq 2k} \|\widetilde{M}_j^{k,1}\|_{L^p \rightarrow L^p} \leq C_p. \quad (2.61)$$

For $j > 2k$, we introduce some new notations. Let $j = k + j'$, then $j' > k$. Set $\lambda = 2^{j'} > 2^k = \delta^{-1}$, then $2^j = \frac{\lambda}{\delta}$. By Lemma 2.2.3,

$$\begin{aligned} & \|\widetilde{M}_j^{k,1} f\|_{L^4}^4 \\ & \leq C \frac{\delta^2}{\lambda} \left(\int_{\mathbb{R}^2} \int_{1/2}^4 \left| \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} (\lambda/\delta)^{1/2} \widetilde{A}_k(\xi, t) \beta\left(\frac{\delta}{\lambda} |\delta_t \xi|\right) \hat{f}(\xi) d\xi \right|^4 dy dt \right)^{3/4} \\ & \quad \times \left(\int_{\mathbb{R}^2} \int_{1/2}^4 \left| \frac{\partial}{\partial t} (\rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} (\delta \lambda)^{-1/2} \widetilde{A}_k(\xi, t) \beta\left(\frac{\delta}{\lambda} |\delta_t \xi|\right) \hat{f}(\xi) d\xi \right|^4 dy dt \right)^{1/4}. \end{aligned} \quad (2.62)$$

In order to simplify the notations, we choose $\tilde{\chi} \in C_0^\infty([c_1, c_2])$ so that $\tilde{\chi}(\frac{\xi_1}{\xi_2}) \chi(\frac{\xi_1}{t\xi_2}) = \chi(\frac{\xi_1}{t\xi_2})$ for arbitrary $t \in [1/2, 4]$ and k sufficiently large. In a similar way we choose $\rho_0 \in C_0^\infty((-10, 10))$ such that $\rho_0(|\xi|) \beta(|\delta_t \xi|) = \beta(|\delta_t \xi|)$ for arbitrary $t \in [1/2, 4]$. Furthermore, since A_k satisfies (2.40), if $a(\xi, t) := (\lambda/\delta)^{1/2} \widetilde{A}_k(\xi, t) \beta(\frac{\delta}{\lambda} |\delta_t \xi|)$ for k sufficiently large, then $a(\xi, t)$ is a symbol of order zero, i.e. for any $t \in [1/2, 4]$, $\alpha \in \mathbb{N}^2$,

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha a(\xi, t) \right| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}. \quad (2.63)$$

Assume that we have obtained the following theorem:

Theorem 2.3.3. *For all $\lambda > 1/\delta$,*

$$\left(\int_{\mathbb{R}^2} \int_{1/2}^4 |\tilde{F}_\lambda^\delta f(y, t)|^4 dy dt \right)^{1/4} \leq C \lambda^{1/8 + \epsilon_1} \delta^{-(1/2 + \epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)}, \text{ some } \epsilon_1, \epsilon_2 > 0, \quad (2.64)$$

where

$$\tilde{F}_\lambda^\delta f(y, t) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} a(\xi, t) \rho_0\left(\frac{\delta}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi. \quad (2.65)$$

Remark 2.3.1. If we replace the localized frequency $|\xi| \approx \lambda$ of Theorem 3.3 in [18] by $|\xi| \approx \frac{\lambda}{\delta}$, then we obtain the factor $\lambda^{1/8 + \epsilon} 2^{5k/8 + \epsilon}$ which is larger than the factor $\lambda^{1/8 + \epsilon_1} 2^{k/2 + \epsilon_2}$ appeared in the above Theorem 2.3.3. This means that we get a better L^4 -estimate for Fourier integral operators not satisfying the cinematic curvature uniformly.

By (2.62), we obtain

$$\begin{aligned}\|\widetilde{M}_j^{k,1} f\|_{L^4(\mathbb{R}^2)} &\leq C \left(\frac{\delta^2}{\lambda}\right)^{1/4} \lambda^{1/8+\epsilon_1} \delta^{-(1/2+\epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)} \\ &= C 2^{j'(-1/8+\epsilon_1)} 2^{k\epsilon_2} \|f\|_{L^4(\mathbb{R}^2)} \\ &= C 2^{j(-1/8+\epsilon_1)} 2^{k(1/8-\epsilon_1+\epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)}.\end{aligned}$$

Employing the M. Riesz interpolation theorem and Lemma 2.3.1, we have for $2 < p \leq 4$,

$$\|\widetilde{M}_j^{k,1} f\|_{L^p(\mathbb{R}^2)} \leq C 2^{j(-1/8+\epsilon_1)(2-4/p)} 2^{k(3/4-5/(2p)+(2-4/p)(\epsilon_2-\epsilon_1))} \|f\|_{L^p(\mathbb{R}^2)},$$

and

$$\sum_k 2^{-k} \sum_{j>2k} \|\widetilde{M}_j^{k,1}\|_{L^p \rightarrow L^p} \leq C \sum_k 2^{-k(1/4+5/(2p)-(2-4/p)(\epsilon_2-\epsilon_1))} \sum_{j>2k} 2^{j(-1/8+\epsilon_1)(2-4/p)} \leq C_p.$$

For $4 \leq p < \infty$, the M. Riesz interpolation theorem and Lemma 2.3.1 imply that

$$\|\widetilde{M}_j^{k,1} f\|_{L^p(\mathbb{R}^2)} \leq C 2^{j(-1/2+4\epsilon_1)/p} 2^{4k(1/8-\epsilon_1+\epsilon_2)/p} \|f\|_{L^p(\mathbb{R}^2)},$$

and

$$\sum_k 2^{-k} \sum_{j>2k} \|\widetilde{M}_j^{k,1}\|_{L^p \rightarrow L^p} \leq C \sum_k 2^{k(1/(2p)-1+4(\epsilon_2-\epsilon_1)/p)} \sum_{j>2k} 2^{j(-1/(2p)+4\epsilon_1/p)} \leq C_p.$$

We have finished the proof of Theorem 2.1.1.

It remains to prove Theorem 2.3.3. First we introduce a model operator which is translation-invariant. We define it by

$$\mathcal{F}_\lambda^\delta f(y, t) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\xi \cdot y + E(\xi) + tq_\delta(\xi))} a(\xi, t) \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi, \quad (2.66)$$

where

$$E(\xi) = \frac{\xi_1^2}{2\xi_2} \quad \text{and} \quad q_\delta(\xi) = \delta \cdot \frac{\xi_1^3}{\xi_2^2} \phi'(0). \quad (2.67)$$

Notice that compared to (2.38), we have replaced t by $1/t$. This is no problem since $t \approx 1$. Then the model operator $\mathcal{F}_\lambda^\delta$ approximates the operator $\tilde{\mathcal{F}}_\lambda^\delta$ whose phase function has of the form (2.38).

Theorem 2.3.4. *For all $\lambda > 1/\delta$, we have*

$$\|\mathcal{F}_\lambda^\delta f(y, t)\|_{L^4(\mathbb{R}^3)} \leq C \lambda^{1/8+\epsilon_1} \delta^{-(1/2+\epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)}. \quad (2.68)$$

We give the proof of Theorem 2.3.4 in Chapter 3. In Chapter 4, the proof of Theorem 2.3.3 will be shown for the phase function $-t^2 \xi_2 \tilde{\Phi}(s, \delta)$ in (2.38).

2.3.2 The case when $m > 1$

Here we only show the differences with the case $m = 1$ and continue to follow the notation used in the previous section. We will mainly estimate

$$\widehat{d\mu_{k,m}}(\delta_t\xi) = \int_{\mathbb{R}} e^{-i(t\xi_1x + t^2\xi_2x^2\phi(\delta x))} \tilde{\rho}(x)\eta(\delta x)dx,$$

where $\delta = 2^{-k}$.

Set

$$s := s(\xi, t) = -\frac{\xi_1}{t\xi_2}, \quad \text{for } \xi_2 \neq 0, \quad (2.69)$$

and

$$\Phi(s, x, \delta) = -sx + x^2\phi\left(\frac{x}{2^k}\right). \quad (2.70)$$

The implicit function theorem implies that for enough large k , there exists a smooth solution $\tilde{q}(s, \delta)$ for the equation $\partial_2\Phi(s, x, \delta) = 0$. Meanwhile, if we choose k sufficiently large and assume $\phi(0) = 1/2$, then $\tilde{q}(s, \delta)$ smoothly converges to the solution $\tilde{q}(s, 0) = s$ of the equation $\partial_2\Phi(s, x, 0) = 0$.

Let $\tilde{\Phi}(s, \delta) := \Phi(s, q(s, \delta), \delta)$. The phase function can be written as

$$-t^2\xi_2\tilde{\Phi}(s, \delta) = \frac{\xi_1^2}{2\xi_2} + (-1)^{m+1}\frac{\phi^{(m)}(0)}{m!}\delta^m\frac{\xi_1^{m+2}}{t^m\xi_2^{m+1}} + R(t, \xi, \delta),$$

where $R(t, \xi, \delta)$ is homogeneous of degree one in ξ and has at least $m+1$ power of δ .

Using the similar argument as in the last section, it suffices to prove that

$$\sum_k 2^{-k} \sum_{j \geq 1} \left\| \sup_{t \in [1, 2]} |\widetilde{A_{t,j}^k}| \right\|_{L^p \rightarrow L^p} \leq C_p, \quad (2.71)$$

where

$$\widetilde{A_{t,j}^k} f(y) := \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_{k,m}}(\delta_t\xi) \beta(2^{-j}|\delta_t\xi|) \hat{f}(\xi) d\xi.$$

A standard application of the method of stationary phase yields that

$$\widehat{d\mu_{k,m}}(\delta_t\xi) = e^{-it^2\xi_2\tilde{\Phi}(s,\delta)} \chi_{k,m}\left(\frac{\xi_1}{t\xi_2}\right) \frac{A_{k,m}(\delta_t\xi)}{(1 + |\delta_t\xi|)^{1/2}} + B_{k,m}(\delta_t\xi), \quad (2.72)$$

where $\chi_{k,m}$ is a smooth function supported in $[c_{k,m}, \widetilde{c_{k,m}}]$, for certain non-zero constant $c_{k,m}$ and $\widetilde{c_{k,m}}$ dependent only on k and m . $A_{k,m}$ is a symbol of order zero in ξ and $\{A_{k,m}(\delta_t\xi)\}_{k,m}$ is contained in a bounded subset of symbol of order zero. More precisely, for arbitrary $t \in [1, 2]$,

$$|D_\xi^\alpha A_{k,m}(\delta_t\xi)| \leq C_\alpha (1 + |\xi|)^{-\alpha}, \quad (2.73)$$

where C_α do not depend on k and m . Furthermore, $B_{k,m}$ is a remainder term and satisfies for arbitrary $t \in [1, 2]$,

$$|D_\xi^\alpha B_{k,m}(\delta_t \xi)| \leq C_{\alpha,N} (1 + |\xi|)^{-N}, \quad (2.74)$$

where $C_{\alpha,N}$ are admissible constants and again do not depend on k and m .

Put

$$A_{t,j}^k f(y) := \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} \chi_{k,m} \left(\frac{\xi_1}{t \xi_2} \right) \frac{A_{k,m}(\delta_t \xi)}{(1 + |\delta_t \xi|)^{1/2}} \beta(2^{-j} |\delta_t \xi|) \hat{f}(\xi) d\xi. \quad (2.75)$$

A similar discussion as before allows us to choose $\rho_1(y, t) \in C_0^\infty(\mathbb{R}^2 \times [1/2, 4])$ and to show that it is sufficient to prove that

$$\sum_k 2^{-k} \sum_{j \geq 1} \|\widetilde{M_j^{k,1}}\|_{L^p \rightarrow L^p} \leq C_p, \quad (2.76)$$

where

$$\widetilde{M_j^{k,1}} f(y) := \sup_{t \in [1, 2]} |\rho_1(y, t) A_{t,j}^k f(y)|,$$

(compare (2.46), (2.47)).

Moreover,

$$\frac{\partial}{\partial t} (\rho_1(y, t) A_{t,j}^k f(y)) = \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} h_{k,m}(y, t, \xi, j) \hat{f}(\xi) d\xi, \quad (2.77)$$

where $|h_{k,m}(y, t, \xi, j)| \lesssim 2^{j/2} \delta^m + 2^{-j/2}$, which follows from the similar argument with h_k in (2.50), together with the facts that $|\frac{\partial}{\partial t}(-t^2 \xi_2 \tilde{\Phi}(s, \delta))| = |(-1)^m \frac{\phi^{(m)}(0)}{m!} \delta^m \frac{\xi_1^{m+2}}{t^{m+1} \xi_2^{m+1}} + \frac{\partial}{\partial t} R(t, \xi, \delta)| \lesssim \delta^m 2^j$.

So we still have the following regularity estimates.

Lemma 2.3.5.

$$\|\widetilde{M_j^{k,1}} f\|_{L^p(\mathbb{R}^3)} \leq C_p 2^{-(j \wedge k)/p} \|f\|_{L^p(\mathbb{R}^2)}, \quad 2 \leq p \leq \infty. \quad (2.78)$$

The proof is similar to the one of Lemma 2.3.1. Hence, we have that

$$\begin{aligned} \sum_k 2^{-k} \sum_{0 < j \leq 9km} \|\widetilde{M_j^{k,1}}\|_{L^p \rightarrow L^p} &\leq C_p + \sum_k 2^{-k} \sum_{km < j \leq 9km} 2^{-km/p} \\ &\leq C_p + 8m \sum_k k 2^{-k(1+m/p)} \lesssim_p 1, \end{aligned}$$

where $2 < p < \infty$.

Based on these results, let us now assume that $j > 9km$. We introduce some notations. Let $j = km + j'$ and $\delta' = \delta^m$, then $\lambda = 2^{j'} > \delta'^8$ and $2^j = \frac{\lambda}{\delta'}$. After the same simplification as in the previous section, we will prove the following theorem:

Theorem 2.3.6. For $\lambda > \delta'^{-8}$, the following inequality

$$\left(\int_{\mathbb{R}^2} \int_{1/2}^4 |\tilde{F}_\lambda^{\delta'} f(y, t)|^4 dy dt \right)^{1/4} \leq C \lambda^{1/8+\epsilon_1} \delta'^{-(1/2+\epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)} \quad (2.79)$$

holds true for some $\epsilon_1, \epsilon_2 > 0$, where

$$\tilde{F}_\lambda^{\delta'} f(y, t) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^2 \xi_2 \tilde{\Phi}(s, \delta))} a(\xi, t) \rho_0\left(\frac{\delta'}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi.$$

In order to use the proof of Chapter 4 to get (2.79), we make the change of variable $t \rightarrow \tilde{t}^{-1/m}$, then it suffices to prove that

$$\left(\int_{\mathbb{R}^2} \int_{2^{-2m}}^{2^m} |\tilde{F}_\lambda^{\delta'} f(y, \tilde{t}^{-1/m})|^4 dy d\tilde{t} \right)^{1/4} \leq C \lambda^{1/8+\epsilon_1} \delta'^{-(1/2+\epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)}, \quad (2.80)$$

where

$$\tilde{F}_\lambda^{\delta'} f(y, \tilde{t}^{-1/m}) = \rho_1(y, \tilde{t}^{-1/m}) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - \tilde{t}^{-2/m} \xi_2 \tilde{\Phi}(\tilde{s}, \delta))} a(\xi, \tilde{t}^{-1/m}) \rho_0\left(\frac{\delta'}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi,$$

and

$$\tilde{s} = -\tilde{t}^{1/m} \frac{\xi_1}{\xi_2}, \quad -\tilde{t}^{-2/m} \xi_2 \tilde{\Phi}(\tilde{s}, \delta) = \frac{\xi_1^2}{2\xi_2} + \tilde{t} \delta' (-1)^{m+1} \frac{\phi^{(m)}(0)}{m!} \cdot \frac{\xi_1^{m+2}}{\xi_2^{m+1}} + R(\tilde{t}^{-1/m}, \xi, \delta). \quad (2.81)$$

Hence, we employ the similar idea of Chapter 4 to obtain (2.80), then Theorem 2.3.6 is proved.

Based on the above theorem, we get that

$$\begin{aligned} \|\widetilde{M}_j^{k,1} f\|_{L^4} &\leq C \left(\frac{\delta'^2}{\lambda}\right)^{1/4} \lambda^{1/8+\epsilon_1} \delta'^{-(1/2+\epsilon_2)} \|f\|_{L^4} \\ &= C 2^{j(-1/8+\epsilon_1)} 2^{km\epsilon_2} \|f\|_{L^4} \\ &= C 2^{j(-1/8+\epsilon_1)} 2^{km(1/8+\epsilon_2-\epsilon_1)} \|f\|_{L^4}. \end{aligned} \quad (2.82)$$

Employing the M. Riesz interpolation theorem between (2.78) for $p = 2$ and (2.82), we have for $2 < p \leq 4$,

$$\|\widetilde{M}_j^{k,1} f\|_{L^p(\mathbb{R}^3)} \leq C 2^{-j(1/8-\epsilon_1)(2-4/p)} 2^{km[3/4-5/(2p)+(2-4/p)(\epsilon_2-\epsilon_1)]} \|f\|_{L^p(\mathbb{R}^2)}.$$

If $3/4 - 5/(2p) + (2 - 4/p)(\epsilon_2 - \epsilon_1) \leq 0$, then (2.76) will follow directly for $j > 9km$.

If $3/4 - 5/(2p) + (2 - 4/p)(\epsilon_2 - \epsilon_1) > 0$, then $km < j/9$ yields that

$$\begin{aligned} \|\widetilde{M}_j^{k,1} f\|_{L^p(\mathbb{R}^3)} &\leq C 2^{-j(1/8-\epsilon_1)(2-4/p)} 2^{j[3/4-5/(2p)+(2-4/p)(\epsilon_2-\epsilon_1)]/9} \|f\|_{L^p(\mathbb{R}^2)} \\ &= C 2^{j[-1/6+2/(9p)+(2-4/p)(8\epsilon_1+\epsilon_2)/9]} \|f\|_{L^p(\mathbb{R}^2)}. \end{aligned}$$

Employing the M. Riesz interpolation theorem (2.78) for $p = \infty$ and (2.82), we have for $4 \leq p < \infty$,

$$\|\widetilde{M}_j^{k,1} f\|_{L^p(\mathbb{R}^3)} \leq C 2^{4j(-1/8+\epsilon_1)/p} 2^{4km(1/8+(\epsilon_2-\epsilon_1))/p} \|f\|_{L^p(\mathbb{R}^2)}.$$

Since $1/8 + (\epsilon_2 - \epsilon_1) > 0$, then $km < j/9$ implies that

$$\begin{aligned} \|\widetilde{M}_j^{k,1} f\|_{L^p(\mathbb{R}^3)} &\leq C 2^{4j(-1/8+\epsilon_1)/p} 2^{4j(1/8+(\epsilon_2-\epsilon_1))/(9p)} \|f\|_{L^p(\mathbb{R}^2)} \\ &= C 2^{4j(-1+8\epsilon_1+\epsilon_2)/(9p)} \|f\|_{L^p(\mathbb{R}^2)}, \end{aligned}$$

which finishes the proof of (2.76), hence of Theorem 2.1.1 when $m > 1$.

2.4 The proof for curves of finite type

In this section, we will prove Theorem 2.1.3.

The proof of Theorem 2.1.1 inspires us to prove Theorem 2.1.3 directly. Because at the beginning of the proof, we employ a dyadic decomposition to restrict on an interval far away from the origin, and this means whether $d = 2$, the curve has still non-vanishing Gaussian curvature in this constant interval and we can still apply the method of stationary phase. We proceed as in the proof of Theorem 2.1.1.

We choose $\tilde{\rho} \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \tilde{\rho} \subset \{x : B/2 \leq |x| \leq 2B\}$ and $\sum_k \tilde{\rho}(2^k x) = 1$. Since the support of η is sufficiently small, then we can choose k sufficiently large.

Put

$$A_t f(y) := \int_{\mathbb{R}} f(y_1 - tx, y_2 - t^d x^d \phi(x)) \eta(x) dx = \sum_k \widetilde{A}_t^k f(y),$$

where $\widetilde{A}_t^k f(y) := \int_{\mathbb{R}} f(y_1 - tx, y_2 - t^d x^d \phi(x)) \eta(x) \tilde{\rho}(2^k x) dx$.

Consider the isometric operator on $L^p(\mathbb{R}^2)$ defined by

$$T f(x_1, x_2) = 2^{(d+1)k/p} f(2^k x_1, 2^{dk} x_2).$$

As the argument in the last section, it suffices to prove the following estimate

$$\sum_k 2^{-k} \|\sup_{t>0} |A_t^k| \|_{L^p} \leq C_p, \text{ for } p > 2,$$

where $A_t^k f(y) := \int_{\mathbb{R}} f(y_1 - tx, y_2 - t^d x^d \phi(\frac{x}{2^k})) \tilde{\rho}(x) \eta(2^{-k} x) dx$.

By means of the Fourier inversion formula, we can write

$$A_t^k f(y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_{k,d,m}}(\delta_t \xi) \hat{f}(\xi) d\xi,$$

where

$$\widehat{d\mu_{k,d,m}}(\delta_t\xi) = \int_{\mathbb{R}} e^{-i(t\xi_1x + t^d\xi_2x^d\phi(\frac{x}{2^k}))} \tilde{\rho}(x)\eta(2^{-k}x)dx.$$

Choosing a non-negative function $\beta \in C_0^\infty(\mathbb{R})$ as before, set

$$A_{t,j}^k f(y) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot y} \widehat{d\mu_{k,d,m}}(\delta_t\xi) \beta(2^{-j}|\delta_t\xi|) \hat{f}(\xi) d\xi$$

and denote by M_j^k the corresponding maximal operator.

Since $A_{t,j}^k$ is localized to frequencies $|\delta_t\xi| \approx 2^j$, we can still use Lemma 2.2.4 to prove that

$$\|M_j^k\|_{L^p \rightarrow L^p} \lesssim \|M_{j,loc}^k\|_{L^p \rightarrow L^p},$$

where $M_{j,loc}^k f(y) := \sup_{t \in [1,2]} |A_{t,j}^k f(y)|$.

Set

$$\delta := 2^{-k}, \quad s := s(\xi, t) = -\frac{\xi_1}{t^{d-1}\xi_2}, \quad \text{for } \xi_2 \neq 0, \quad (2.83)$$

and

$$\Phi(s, x, \delta) = -sx + x^d\phi(\delta x). \quad (2.84)$$

From the proof of Theorem 2.1.1, for fixed $t \in [1, 2]$, we would mainly estimate

$$A_{t,j}^k f(y) := \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^d\xi_2\tilde{\Phi}(s,\delta))} \chi_{k,d,m}\left(\frac{\xi_1}{t^{d-1}\xi_2}\right) \frac{A_{k,d,m}(\delta_t\xi)}{(1+|\delta_t\xi|)^{1/2}} \beta(2^{-j}|\delta_t\xi|) \hat{f}(\xi) d\xi,$$

where $\chi_{k,d,m}$ is a smooth function supported in the conical region $[c_{k,d,m}, \widetilde{c_{k,d,m}}]$, for certain non-zero constant $c_{k,d,m}$ and $\widetilde{c_{k,d,m}}$ dependent only on k, m and d . $A_{k,d,m}$ is a symbol of order zero in ξ and $\{A_{k,d,m}(\delta_t\xi)\}_k$ is contained in a bounded subset of symbol of order zero. Let $\phi(0) = 1/d$ and $\tilde{\Phi}(s, \delta) := \Phi(s, \tilde{q}(s, \delta), \delta)$, then the phase function can be written as

$$-t^d\xi_2\tilde{\Phi}(s, \delta) = \left(\frac{1}{d^{\frac{d}{d-1}}} - \frac{1}{d^{\frac{1}{d-1}}}\right) \left(-\frac{d\xi_1^d}{\xi_2}\right)^{\frac{1}{d-1}} - \frac{\delta^m\phi^{(m)}(0)}{t^m m!} \left(-\frac{\xi_1}{\xi_2^{\frac{m+1}{m+d}}}\right)^{\frac{d+m}{d-1}} + R(t, \xi, \delta, d),$$

where $R(t, \xi, \delta, d)$ is homogeneous of degree one in ξ and has at least $m+1$ power of δ .

The similar argument with (2.46) allows us to choose $\rho_1(y, t) \in C_0^\infty(\mathbb{R}^2 \times [1/2, 4])$ and it is sufficient to prove that

$$\sum_k 2^{-k} \sum_{j \geq 1} \|\widetilde{M_j^{k,1}}\|_{L^p \rightarrow L^p} \leq C_p, \quad (2.85)$$

where

$$\widetilde{M_j^{k,1}} f(y) := \sup_{t \in [1,2]} |\rho_1(y, t) A_{t,j}^k f(y)|.$$

Moreover,

$$\frac{\partial}{\partial t}(\rho_1(y, t)A_{t,j}^k f(y)) = \int_{\mathbb{R}^2} e^{i(\xi \cdot y - t^d \xi_2 \tilde{\Phi}(s, \delta))} h_{k,m,d}(y, t, \xi, j) \hat{f}(\xi) d\xi, \quad (2.86)$$

where $|h_{k,m}(y, t, \xi, j)| \lesssim 2^{j/2} \delta^m + 2^{-j/2}$, which follows from the similar argument with h_k in (2.50), together with the facts that

$$\left| \frac{\partial}{\partial t}(-t^2 \xi_2 \tilde{\Phi}(s, \delta)) \right| = \left| m \frac{\delta^m \phi^{(m)}(0)}{t^{m+1} m!} \left(-\frac{\xi_1}{\xi_2^{\frac{m+1}{m+d}}} \right)^{\frac{d+m}{d-1}} + \frac{\partial}{\partial t} R(t, \xi, \delta, d) \right| \lesssim \delta^m 2^j.$$

So we still have the same regularity estimates as (2.3.5) and

$$\sum_k 2^{-k} \sum_{0 < j \leq 9km} \|\widetilde{M_j^{k,1}}\|_{L^p \rightarrow L^p} \leq C_p, \text{ for } 2 < p < \infty.$$

Based on these arguments, for $j > 9km$, let $j = km + j'$ and $\delta' = \delta^m$, then $j' > 8km$ and $\lambda = 2^{j'} > \delta'^{-8}$. We would be done if we could prove the following theorem.

Theorem 2.4.1.

$$\left(\int_{\mathbb{R}^2} \int_{2^{-2m}}^{2^m} |\tilde{F}_\lambda^{\delta'} f(y, \tilde{t}^{-1/m})|^4 dy d\tilde{t} \right)^{1/4} \leq C \lambda^{1/8 + \epsilon_1} \delta'^{-(1/2 + \epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)}, \quad (2.87)$$

where

$$\tilde{F}_\lambda^{\delta'} f(y, \tilde{t}^{-1/m}) = \rho_1(y, \tilde{t}^{-1/m}) \int_{\mathbb{R}^2} e^{i(\xi \cdot y - \tilde{t}^{-d/m} \xi_2 \tilde{\Phi}(\tilde{s}, \delta))} a(\xi, \tilde{t}^{-1/m}) \rho_0\left(\frac{\delta'}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi,$$

and

$$\begin{aligned} \tilde{s} &= -\tilde{t}^{(d-1)/m} \frac{\xi_1}{\xi_2}, \\ -\tilde{t}^{-d/m} \xi_2 \tilde{\Phi}(\tilde{s}, \delta) &= \left(\frac{1}{d^{\frac{d}{d-1}}} - \frac{1}{d^{\frac{1}{d-1}}} \right) \left(-\frac{d \xi_1^d}{\xi_2} \right)^{\frac{1}{d-1}} - \tilde{t} \delta' \frac{\phi^{(m)}(0)}{m!} \left(-\frac{\xi_1}{\xi_2^{\frac{m+1}{m+d}}} \right)^{\frac{d+m}{d-1}} + R(\tilde{t}^{-1/m}, \xi, \delta, d). \end{aligned} \quad (2.88)$$

The proof is similar to the one in Chapter 4.

Chapter 3

A model operator

In this chapter, we will prove Theorem 2.3.4, which will be obtained in a similar way as the proof of L^4 -boundedness of the following operator:

$$\mathcal{F}_\lambda^\delta f(y, t) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(y \cdot \xi + E(\xi) + tq_\delta(\xi))} a(\xi, t) \rho_0\left(\frac{\delta}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi, \quad (3.1)$$

where $E(\xi) = \frac{\xi_1^2}{\xi_2}$, $q_\delta(\xi) = \delta \cdot \frac{\xi_1^3}{\xi_2^2}$, $\tilde{\chi} \in C_0^\infty([c_1, c_2])$ (c_1, c_2 are very small positive constants), $\rho_0 \in C_0^\infty((-10, 10))$ and a is a symbol of order zero. So we need to prove that for all $\lambda > 1/\delta$,

$$\|\mathcal{F}_\lambda^\delta f(y, t)\|_{L^4(\mathbb{R}^3)} \leq C \lambda^{1/8 + \epsilon_1} \delta^{-(1/2 + \epsilon_2)} \|f\|_{L^4(\mathbb{R}^2)}. \quad (3.2)$$

The following approach follows the proof of Proposition 3 in [25], but we need various modifications.

We may assume that the ξ -support of the symbol a is in the first quadrant. We rewrite

$$\begin{aligned} \mathcal{F}_\lambda^\delta f(y, t) &= \int_{\mathbb{R}} e^{it\tau} \mathcal{F}_\lambda^\delta f(y, \hat{\tau}) d\tau \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{i(y \cdot \xi + t\tau)} e^{iE(\xi)} (\rho_1(y, \cdot) a(\xi, \cdot))^\wedge(\tau - q_\delta(\xi)) \rho_0\left(\frac{\delta}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \hat{f}(\xi) d\xi d\tau. \end{aligned}$$

We can reduce our proof to the situation $|\tau - q_\delta(\xi)| \leq C_0 \lambda$ for an appropriate constant C_0 which is small and satisfies $q_\delta(\xi) - 2C_0 \lambda > 0$. In fact, if $|\tau - q_\delta(\xi)| \geq C_0 \lambda$, then

(i) if $\tau \gg q_\delta(\xi) \approx \lambda > 1$, by $|\xi| \approx \lambda/\delta$, then

$$|\tau - q_\delta(\xi)| \geq C\tau \geq C(\lambda + |(\delta\xi, \tau)|);$$

(ii) if $\tau \lesssim q_\delta(\xi) \approx \lambda > 1$, by assumption, then

$$|\tau - q_\delta(\xi)| \geq C_0 \lambda \geq C(\lambda + |(\delta\xi, \tau)|).$$

Define the operator $\mathcal{L} = I - \Delta_{\xi, \tau}$, where $\Delta_{\xi, \tau} = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \tau^2}$, then the above arguments of (i) and (ii), $\rho_1 \in C_0^\infty(\mathbb{R}^2 \times [1/2, 4])$, together with the fact that a is a symbol of order zero, imply that for sufficiently large N' ,

$$|\mathcal{L}^2(\rho_1(y, \cdot)a(\cdot, \xi))^{\wedge}(\tau - q_\delta(\xi))| \leq C_{N'}(\lambda + |(\delta\xi, \tau)|)^{-N'} \quad (3.3)$$

and

$$\begin{aligned} & \left| \mathcal{L}^2 \left(e^{iE(\xi)}(\rho_1(y, \cdot)a(\cdot, \xi))^{\wedge}(\tau - q_\delta(\xi))\rho_0\left(\frac{\delta}{\lambda}|\xi|\right)\tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \right) \right| \\ & \leq C_{N'}(\lambda + |(\delta\xi, \tau)|)^{-N'} \\ & \leq C_{N'}\lambda^{-N'/2}(1 + |(\delta\xi, \tau)|)^{-N'/2}. \end{aligned}$$

Since $\mathcal{L}^2(e^{i((y-x)\cdot\xi+t\tau)}) = (1 + |(y-x, t)|^2)^2$, by integration by parts in (ξ, τ) and the assumption $|\tau - q_\delta(\xi)| \geq C_0\lambda$, the kernel $K(y, t; x)$ of the operator $\mathcal{F}_\lambda^\delta$ can be controlled by

$$\begin{aligned} |K(y, t; x)| & \leq C_{N'} \frac{1}{(1 + |(y-x, t)|^2)^2} \lambda^{-N'/2} \int_{\mathbb{R}^3} (1 + |(\delta\xi, \tau)|)^{-N'/2} d\xi d\tau \\ & \leq C_{N'} \frac{1}{(1 + |(y-x, t)|^2)^2} \lambda^{-N'/2} \delta^{-2}. \end{aligned}$$

Since $\delta\lambda > 1$, then for sufficiently large $N < N'$,

$$|K(y, t; x)| \leq C_N \lambda^{-N} \frac{1}{(1 + |(y-x, t)|^2)^2}. \quad (3.4)$$

In the next steps, we will always assume that $|\tau - q_\delta(\xi)| \leq C_0\lambda$ for a small appropriate constant C_0 , which shows that

$$\lambda \approx q_\delta(\xi) - C_0\lambda \leq \tau \leq q_\delta(\xi) + C_0\lambda \approx \lambda. \quad (3.5)$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ satisfy $\text{supp } \varphi \subset [-1, 1]$ and $\sum_{n \in \mathbb{Z}} \varphi^2(\cdot - n) = 1$. In order to give a decomposition for τ , the dual to t , we define the operator P^n on functions in \mathbb{R}^3 by

$$(P^n g)^{\wedge}(\eta, \tau) = \varphi(\lambda^{-1/2}\tau - n)\widehat{g}(\eta, \tau). \quad (3.6)$$

Similarly, we define $\widehat{f}_n(\xi) = \varphi((\lambda^{-1/2}q_\delta(\xi) - n)/10)\widehat{f}(\xi)$.

Lemma 3.0.2. For $N > 0$ and $1 \leq p \leq \infty$,

$$\left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta(f - f_n) \right\|_{L^p(\mathbb{R}^3)} \leq C_N \lambda^{-N} \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.7)$$

Proof. For fixed n , we have

$$(P^n)^2 \mathcal{F}_\lambda^\delta(f - f_n)(y, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{i(y \cdot \xi + t\tau + E(\xi))} \varphi^2(\lambda^{-1/2}\tau - n) (\rho_1(y, \cdot) a(\xi, \cdot))^\wedge(\tau - q_\delta(\xi)) \\ \times \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) [1 - \varphi((\lambda^{-1/2}q_\delta(\xi) - n)/10)] \hat{f}(\xi) d\xi d\tau.$$

From the support of φ , we have $\lambda^{1/2}(n-1) \leq \tau \leq \lambda^{1/2}(n+1)$, together with (3.5), we get $n \approx \lambda^{1/2}$. We observe that $q_\delta(\xi) \notin [\lambda^{1/2}(n-10), \lambda^{1/2}(n+10)]$, then we get $|\tau - q_\delta(\xi)| \gtrsim (\lambda + |(\delta\xi, \tau)|)^{1/2}$. Finally, as in the estimate (3.4), the kernel of $(P^n)^2 \mathcal{F}_\lambda^\delta$ defined by

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{i[(y-x) \cdot \xi + t\tau + E(\xi)]} \varphi^2(\lambda^{-1/2}\tau - n) (\rho_1(y, \cdot) a(\xi, \cdot))^\wedge(\tau - q_\delta(\xi)) \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \\ \times [1 - \varphi((\lambda^{-1/2}q_\delta(\xi) - n)/10)] d\xi d\tau$$

can be dominated by $\mathcal{O}(\lambda^{-N}(1 + |(y-x, t)|^2)^{-2})$ and

$$\left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta(f - f_n) \right\|_{L^p(\mathbb{R}^3)} \leq C_{N'} \sum_{n \approx \lambda^{1/2}} \lambda^{-N'} \|f\|_{L^p(\mathbb{R}^2)} \leq C_N \lambda^{-N} \|f\|_{L^p(\mathbb{R}^2)}.$$

□

Lemma 3.0.3. *For $2 \leq p \leq \infty$, we have*

$$\left\| \sum_{n \approx \lambda^{1/2}} (P^n)^2 \mathcal{F}_\lambda^\delta f_n \right\|_{L^p(\mathbb{R}^3)} \leq C \lambda^{(1/4 - 1/(2p))} \left\| \left(\sum_{n \approx \lambda^{1/2}} |P^n \mathcal{F}_\lambda^\delta f_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)}.$$

The idea can be found in [25].

So far we have used a radial decomposition of the Fourier integral operator $\mathcal{F}_\lambda^\delta$ with respect to frequency variables. Now we make a further angular decomposition. As we have introduced in Lemma 2.3.2, here we redefine the homogeneous partition of unity of $\mathbb{R}^2 \setminus \{0\}$ that depends on the scale λ/δ . Specially, for fixed $j > 0$, we choose functions $\chi_\nu, \nu = 0, 1, 2, \dots$, and the size of every angle is $(\delta/\lambda)^{1/2}$. Meanwhile χ_ν satisfies the following conditions:

$$\sum_\nu \chi_\nu(\xi) = 1 \text{ for all } \xi \neq 0, \quad (3.8)$$

$$|\partial_\xi^\alpha \chi_\nu| \leq A_\alpha (\lambda/\delta)^{|\alpha|/2} |\xi|^{-|\alpha|} \text{ for all } \xi \neq 0. \quad (3.9)$$

Define Q_ν by

$$(Q_\nu g)^\wedge(\eta, \tau) = \Psi_\nu^\delta(\eta, \tau) \hat{g}(\eta, \tau) = \psi\left(\frac{\tau - q_\delta(\eta)}{|\tau/\delta|^\epsilon}\right) \chi_\nu(\eta') \hat{g}(\eta, \tau),$$

where $\eta' = \frac{\eta}{|\eta|}$ and $\psi \in C_0^\infty(\mathbb{R})$ is supported in $[-2, 2]$ and equal to 1 in $[-1, 1]$. In fact, $(Q_\nu g)^\wedge(\eta, \tau)$ is supported in a thin sector intersected with a thin neighborhood of the cone $\{(\eta, q_\delta(\eta))\}$, see Figure 2.

Put

$$\mathcal{F}_{\lambda, \nu}^\delta f(y, t) = \rho_1(y, t) \int_{\mathbb{R}^2} e^{i(y \cdot \xi + E(\xi) + tq_\delta(\xi))} a(\xi, t) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \rho_0\left(\frac{\delta}{\lambda} |\xi|\right) \chi_\nu(\xi) \widehat{f}(\xi) d\xi.$$

Lemma 3.0.4. *Given a fixed $\epsilon > 0$ and $n \approx \lambda^{1/2}$, we define*

$$R_{\lambda, \nu}^\delta f = P^n(\mathcal{F}_{\lambda, \nu}^\delta f) - P^n Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f). \quad (3.10)$$

For any $N > 0$, there is a uniform constant C_N so that

$$\|R_{\lambda, \nu}^\delta f_n\|_{L^p(\mathbb{R}^3)} \leq C_N \left(\frac{\lambda}{\delta}\right)^{-N} \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.11)$$

Proof.

$$\begin{aligned} R_{\lambda, \nu}^\delta f_n(y, t) &= \int_{\mathbb{R}^2} e^{iE(\xi)} \left[\int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i(y \cdot \eta + t\tau)} (1 - \Psi_\nu^\delta(\eta, \tau)) \varphi(\lambda^{-1/2} \tau - n) \right. \\ &\quad \left. \times (\rho_1 \cdot a(\xi, \cdot))^\wedge(\eta - \xi, \tau - q_\delta(\xi)) d\eta d\tau \right] \rho_0\left(\frac{\delta}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \chi_\nu(\xi) \widehat{f}_n(\xi) d\xi. \end{aligned}$$

In order to estimate $|(\eta - \xi, \tau - q_\delta(\xi))|$, we split our consideration into different cases:

1. When $\eta' \notin \text{supp } \chi_\nu$, but $\xi \in \text{supp } \chi_\nu$, then $|\eta - \xi| \geq C\left(\frac{\lambda}{\delta}\right)^{1/2}$ and the following inequalities will hold true:

(1). if $|\eta| \gg |\xi| \approx \frac{\lambda}{\delta}$,

$$|\eta - \xi| \geq C|\eta| \geq C\left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right),$$

(2). if $|\eta| \lesssim |\xi| \approx \frac{\lambda}{\delta}$,

$$|\eta - \xi| \geq C\left(\frac{\lambda}{\delta}\right)^{1/2} \geq C\left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right)^{1/2}.$$

2. When $\eta' \in \text{supp } \chi_\nu$, but $\frac{|\tau - q_\delta(\eta)|}{|\tau/\delta|^\epsilon} \geq 2$ and $\xi \in \text{supp } \chi_\nu$, then the following inequalities will hold true:

(1). if $|\eta| \gg |\xi| \approx \frac{\lambda}{\delta}$,

$$|\eta - \xi| \geq C|\eta| \geq C\left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right),$$

(2). if $|\eta| \lesssim |\xi| \approx \frac{\lambda}{\delta}$, for $\mu \ll \epsilon$, we consider two subcases:

a. when $|\eta - \xi| \geq C|\tau/\delta|^\mu$, we have

$$|\eta - \xi| \geq C\left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right)^\mu,$$

b. when $|\eta - \xi| \leq C|\tau/\delta|^\mu$, since

$$|\tau - q_\delta(\xi)| \geq |\tau - q_\delta(\eta)| - |q_\delta(\xi) - q_\delta(\eta)|$$

and

$$|q_\delta(\xi) - q_\delta(\eta)| \approx \delta|\xi - \eta| \leq C\delta\left|\frac{\tau}{\delta}\right|^\mu,$$

then we have

$$|\tau - q_\delta(\xi)| \geq 2\left|\frac{\tau}{\delta}\right|^\epsilon - C\delta\left|\frac{\tau}{\delta}\right|^\mu \approx \left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right)^\epsilon.$$

Combing all the cases, if we set $\mathcal{L} = I - \Delta_{\eta, \tau}$, for any $2 < M \ll \mu N'$, we obtain,

$$|\mathcal{L}^M(\widehat{\rho}_1(\eta - \xi, \tau - q_\delta(\xi)))| \leq C_{N'}\left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right)^{-\mu N'}. \quad (3.12)$$

The estimate (3.9) implies that $|\chi_\nu(\eta')| \leq (\lambda/\delta)^{1/2}$, in addition to (3.12), we get

$$\mathcal{L}^M\left(\left(1 - \Psi_\nu^\delta(\eta, \tau)\right)\varphi(\lambda^{-1/2}\tau - n)\widehat{\rho}_1(\eta - \xi, \tau - q_\delta(\xi))\right) \leq C_{M, N'}\left(\frac{\lambda}{\delta}\right)^{M/2}\left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right)^{-\mu N'}. \quad (3.13)$$

By integration by parts for (η, τ) , we have

$$\begin{aligned} & |R_{\lambda, \nu}^\delta f_n(y, t)| \\ & \leq \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i(y \cdot \eta + t\tau)} (1 - \Psi_\nu^\delta(\eta, \tau)) \varphi(\lambda^{-1/2}\tau - n) (\rho_1 \cdot a(\xi, \cdot))^{\wedge}(\eta - \xi, \tau - q_\delta(\xi)) d\eta d\tau \right| \\ & \quad \times \left| \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \chi_\nu(\xi) \widehat{f}_n(\xi) \right| d\xi \\ & \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| \mathcal{L}^M\left(\left(1 - \Psi_\nu^\delta(\eta, \tau)\right)\varphi(\lambda^{-1/2}\tau - n) (\rho_1 \cdot a(\xi, \cdot))^{\wedge}(\eta - \xi, \tau - q_\delta(\xi))\right) \right| d\eta d\tau \\ & \quad \times \frac{1}{(1 + |(y, t)|^2)^M} \left| \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \chi_\nu(\xi) \widehat{f}_n(\xi) \right| d\xi \\ & \leq \frac{C_{M, N'}}{(1 + |(y, t)|^2)^M} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left(\frac{\lambda}{\delta}\right)^M \left(\frac{\lambda}{\delta} + \left|\left(\eta, \frac{\tau}{\delta}\right)\right|\right)^{-\mu N'} d\eta d\tau \left| \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \chi_\nu(\xi) \widehat{f}_n(\xi) \right| d\xi \\ & \leq C_{M, N'} \left(\frac{\lambda}{\delta}\right)^{M - \mu N'/2} \frac{\delta}{(1 + |(y, t)|^2)^M} \int_{\mathbb{R}^2} \left| \rho_0\left(\frac{\delta}{\lambda}|\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) \chi_\nu(\xi) \varphi\left(\frac{\lambda^{-1/2}q_\delta(\xi) - n}{10}\right) \right| d\xi \|\widehat{f}\|_{L^\infty} \\ & \leq C_{M, N'} \delta \cdot \frac{\lambda}{\delta^{3/2}} \left(\frac{\lambda}{\delta}\right)^{M - \mu N'/2} \frac{1}{(1 + |(y, t)|^2)^M} \|\widehat{f}\|_{L^\infty}. \end{aligned}$$

The last inequality follows from the volume of set $\{\xi : \rho_0(\frac{\delta}{\lambda}|\xi|)\tilde{\chi}(\frac{\xi_1}{\xi_2})\chi_\nu(\xi) \neq 0\}$.

Since we have supposed that f is supported in a fixed compact set, then there exists a sufficiently large $C_{M,N',p}$ so that the following inequality holds true,

$$|R_{\lambda,\nu}^\delta f_n(y,t)| \leq C_{M,N',p} \left(\frac{\lambda}{\delta}\right)^{-N} \frac{1}{(1+|(y,t)|^2)^M} \|f\|_{L^p}.$$

We have finished the proof of Lemma 3.0.4. \square

Until now, since $\sum_n (P^n)^2$ is the identity, applying Lemma 3.0.2, then we get

$$\begin{aligned} \|\mathcal{F}_\lambda^\delta f\|_{L^4} &= \left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta f \right\|_{L^4} \\ &\leq \left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta f_n \right\|_{L^4} + \left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta (f - f_n) \right\|_{L^4} \\ &\leq \left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta f_n \right\|_{L^4} + C_N \lambda^{-N} \|f\|_{L^4}. \end{aligned}$$

It is more convenient to consider a related square function than to estimate the sum of the main term in the right-hand side directly, so by Lemma 3.0.3 and Lemma 3.0.4, we obtain

$$\begin{aligned} \left\| \sum_n (P^n)^2 \mathcal{F}_\lambda^\delta f_n \right\|_{L^4} &\leq C \lambda^{1/8} \left\| \left(\sum_n |P^n \mathcal{F}_\lambda^\delta f_n|^2 \right)^{1/2} \right\|_{L^4} \\ &\leq C \lambda^{1/8} \left\| \left(\sum_n |P^n \sum_\nu Q_\nu(\mathcal{F}_{\lambda,\nu}^\delta f_n)|^2 \right)^{1/2} \right\|_{L^4} + C_N \lambda^{-N} \|f\|_{L^4} \end{aligned}$$

To estimate the main term in the last expression we notice that, for each fixed $n \approx \lambda^{1/2}$ and ν , the Fourier transform of $P^n Q_\nu g$ is supported in the following region (see Figure 2),

$$\mathcal{U}_\nu^n = \{(\eta, \tau) \in \mathbb{R}^3 : (\eta, \tau) \in \text{supp } \Psi_\nu^\delta(\eta, \tau), |\lambda^{-1/2}\tau - n| \leq 1\}.$$

We need an estimate of the number of overlaps of algebraic sums of the sets $\mathcal{U}_\nu^n, \mathcal{U}_{\nu'}^{n'}$, for fixed n, n' corresponding to indices ν for which ξ_ν lie in the first quadrant.

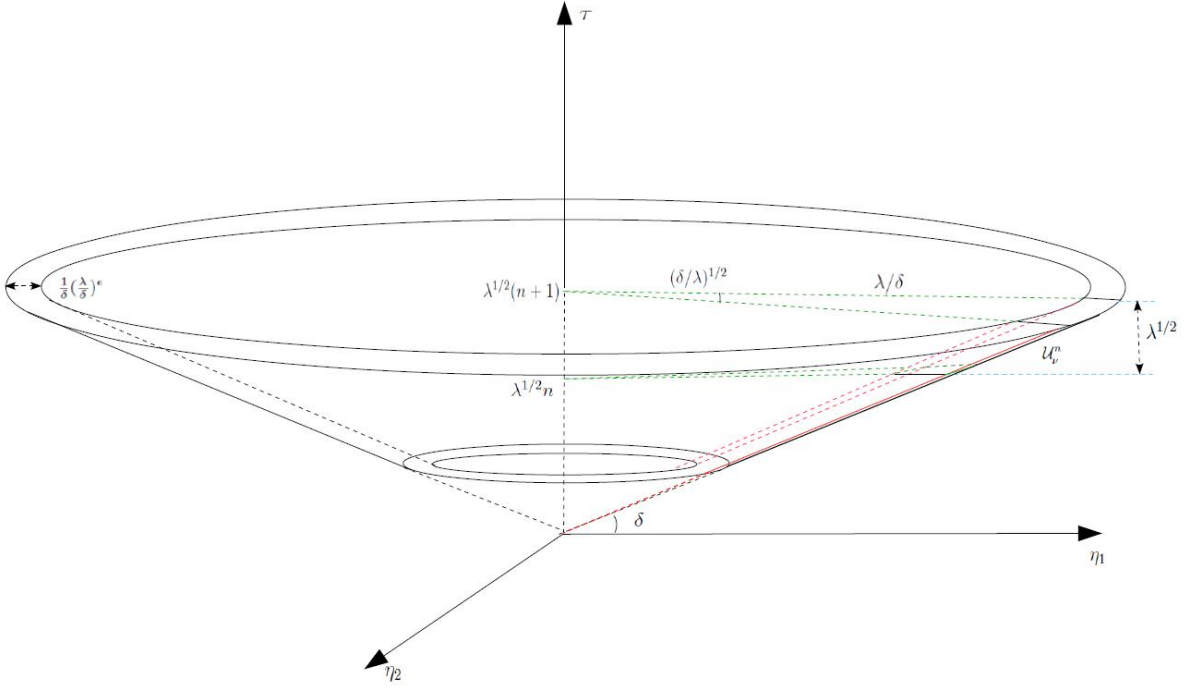


Figure 2: A thin neighborhood of the cone $\tau = \delta|\eta|$ and \mathcal{U}_ν^n
 (The figure of the cone $\tau = q_\delta(\eta)$ is similar to a sector of the cone $\tau = \delta|\eta|$.)

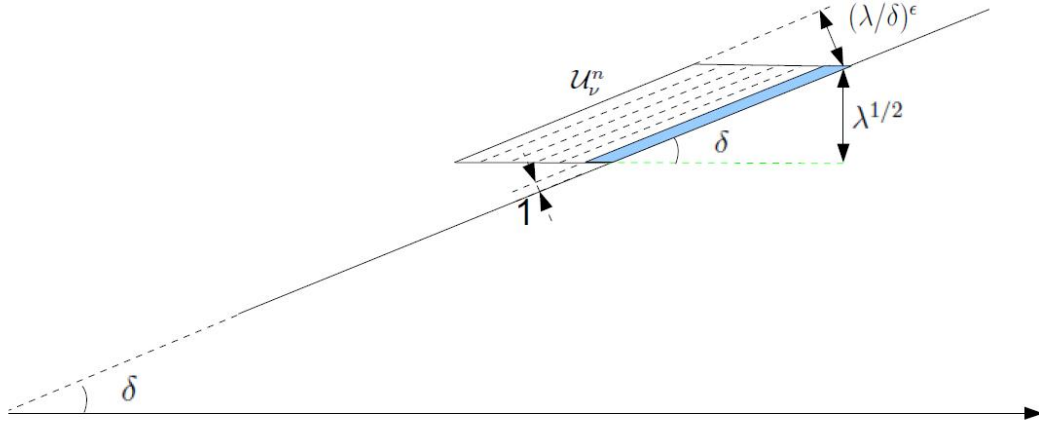
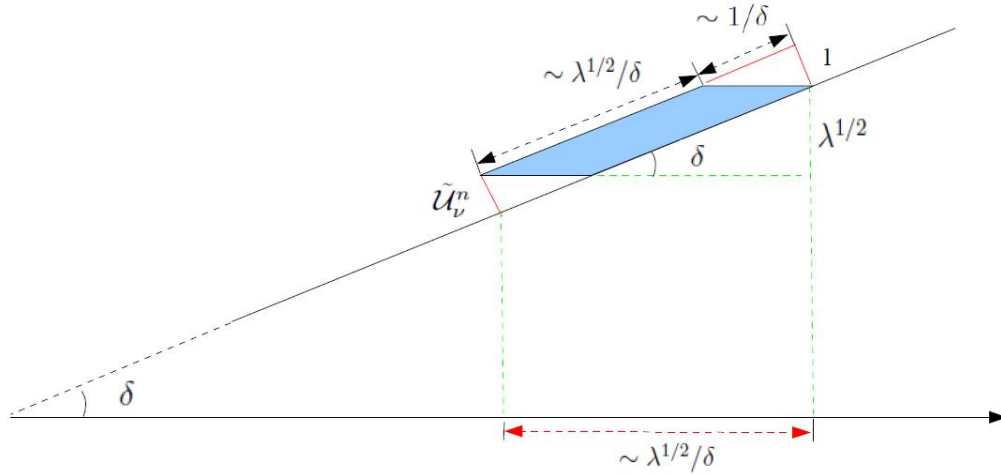
Lemma 3.0.5. *For fixed $n, n' \in \mathbb{Z}$, there is a constant C , independent of n, n', λ and δ , such that*

$$\sum_{\nu, \nu'} \chi_{\mathcal{U}_\nu^n + \mathcal{U}_{\nu'}^{n'}}(\eta, \tau) \leq C \left(\frac{\lambda}{\delta}\right)^{2\epsilon} \log_2 \left(\frac{\lambda}{\delta}\right)^{1/2}. \quad (3.14)$$

Proof. First, we split every \mathcal{U}_ν^n into $(\frac{\lambda}{\delta})^\epsilon$ pieces along the normal direction to the cone $\tau = q_\delta(\eta)$ (vertical cross-section of \mathcal{U}_ν^n will be shown in Figure 3 (1)), then it suffices to prove that

$$\sum_{\nu, \nu'} \chi_{\tilde{\mathcal{U}}_\nu^n + \tilde{\mathcal{U}}_{\nu'}^{n'}}(\eta, \tau) \leq C \log_2 \left(\frac{\lambda}{\delta}\right)^{1/2}, \quad (3.15)$$

where $\tilde{\mathcal{U}}_\nu^n$ is comparable to a rectangle tangential to the cone of dimension $1 \times \frac{\lambda^{1/2}}{\delta} \times (\frac{\lambda}{\delta})^{1/2}$ (vertical cross-section of $\tilde{\mathcal{U}}_\nu^n$ and the comparable rectangle will be shown in Figure 3 (2) and the left-hand one of Figure 4).

Figure 3 (1): Vertical cross-section of \mathcal{U}_ν^n Figure 3 (2): Vertical cross-section of $\tilde{\mathcal{U}}_\nu^n$ and the comparable rectangle

Furthermore, here we use similar arguments as in [26] to prove a scaled version by $1/\lambda$ of inequality (3.15).

To this end we define $\Gamma_\nu = \{\xi \in \mathbb{R}^2 \setminus 0 : \sqrt{\frac{\delta}{\lambda}}\nu \leq |\frac{\xi_2}{\xi_1}| \leq \sqrt{\frac{\delta}{\lambda}}(\nu + 1) \text{ for } \nu = (\frac{\pi}{2} - c_2)\sqrt{\frac{\lambda}{\delta}}, \dots, (\frac{\pi}{2} - c_1)\sqrt{\frac{\lambda}{\delta}} \text{ and here non-zero positive numbers } c_1, c_2 \text{ with } c_1 < c_2 \text{ depend on the support of } \tilde{\chi}\}$. Next, for $n = 0, 1, \dots, \lambda^{1/2}$, let Λ_ν^n be the set of all (η, τ) such that $\text{dist}((\eta, \tau), (\xi, q_\delta(\xi))) \leq \lambda^{-1}$ for some $\xi \in \Gamma_\nu$ with $q_\delta(\xi) \in [(\frac{\pi}{2} - c_1)^{-2} + \frac{n}{\sqrt{\lambda}}, (\frac{\pi}{2} - c_1)^{-2} + \frac{(n+1)}{\sqrt{\lambda}}]$. Thus Λ_ν^n is basically a $\lambda^{-1} \times \frac{1}{\delta\sqrt{\lambda}} \times \frac{1}{\sqrt{\delta\lambda}}$ rectangle (see the right one of

Figure 4) lying on $\{(\xi, q_\delta(\xi)) : q_\delta(\xi) \in [(\frac{\pi}{2} - c_1)^{-2}, (\frac{\pi}{2} - c_2)^{-2}]\}$. Its shortest side points in the normal direction to the cone.

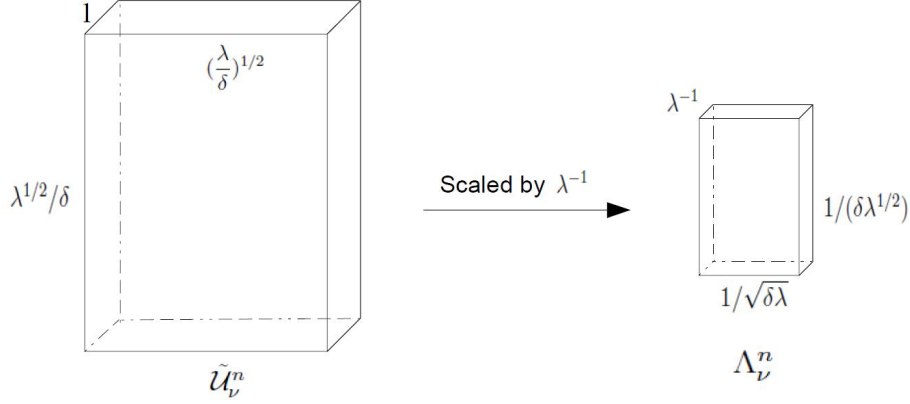


Figure 4

The scaled version of the overlap lemma is that there is a constant C independent of n , n' , λ and δ such that

$$\sum_{\nu, \nu'} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau) \leq C \log_2\left(\frac{\lambda}{\delta}\right)^{1/2}. \quad (3.16)$$

However, inequality (3.16) follows from

$$\sum_{|\nu - \nu'| \approx 2^\ell} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau) \leq C, \quad 0 \leq \ell \leq \log_2\left(\frac{\lambda}{\delta}\right)^{1/2}. \quad (3.17)$$

To deduce inequality (3.17), we give a simple argument to show that for $\bar{\xi} \in \Gamma_\nu$ and $\tilde{\xi} \in \Gamma_{\nu'}$, the angle between the normals to the cone at $(\bar{\xi}, q_\delta(\bar{\xi}))$ and $(\tilde{\xi}, q_\delta(\tilde{\xi}))$ is $\approx 2^\ell (\frac{\delta}{\lambda})^{1/2}$ if $|\nu - \nu'| \approx 2^\ell$. It will follow from the claim that $\text{dist}(P, A) \approx \text{dist}(P, D) \approx 1/\delta$ (See Figure 5 for $q_\delta(\xi) = 1$). Next, we will first prove this claim.

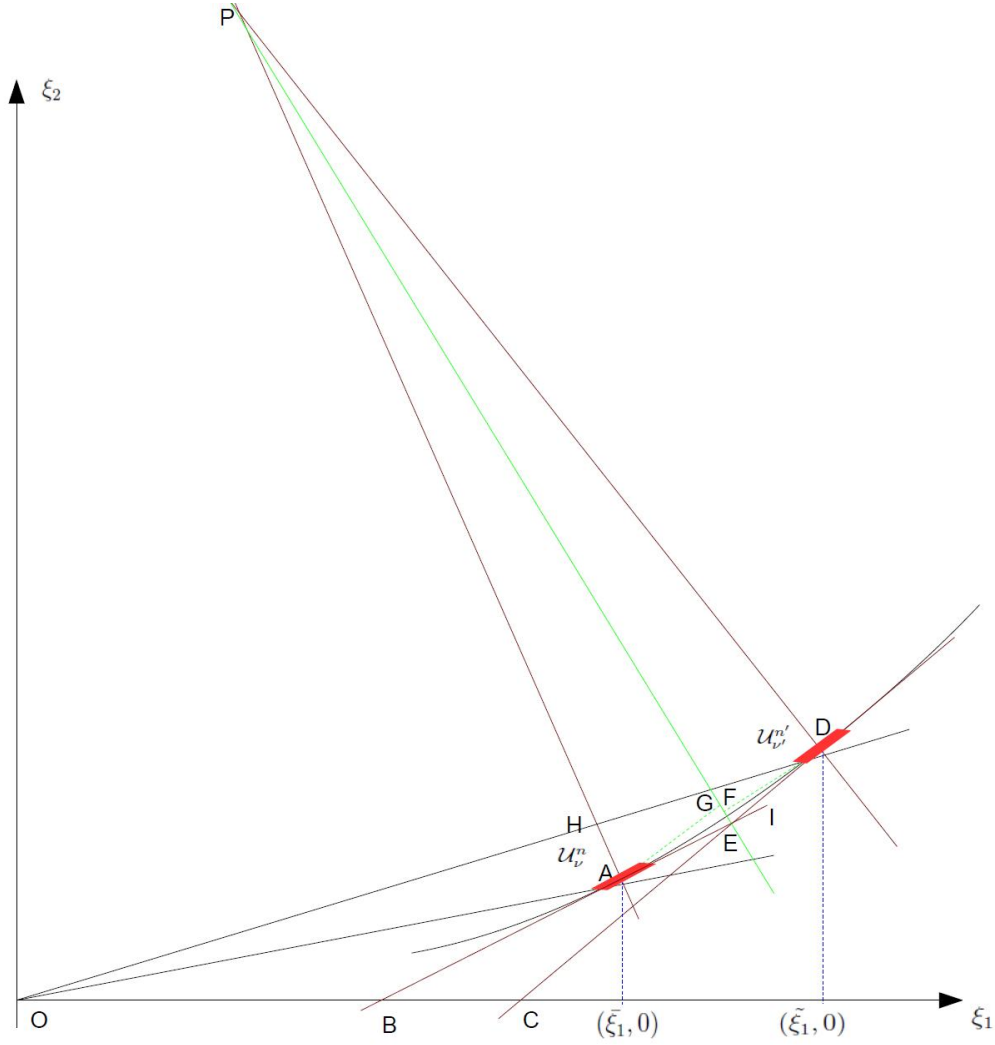
In Figure 5, AE and DE are the tangent line of the curve $q_\delta(\xi) = 1$ at point $(\bar{\xi}_1, 0)$, $(\tilde{\xi}_1, 0)$, where $\bar{\xi}_1, \tilde{\xi}_1 \approx 1/\delta$. PA and PD are perpendicular respectively AE and DE .

From the expression of the curve $q_\delta(\xi) = 1$, it is easy to know that $\frac{\pi}{2} - c_2 \leq \tan \angle AOB \approx \tan \angle B$, $\tan \angle C \approx \tan \angle DOB \leq \frac{\pi}{2} - c_1$.

Since $\tan(\angle DOB - \angle AOB) \approx 2^\ell (\frac{\delta}{\lambda})^{1/2}$, then

$$\tan \angle C - \tan \angle B \approx \delta^{1/2} (\tilde{\xi}_1^{1/2} - \bar{\xi}_1^{1/2}) \approx \tan \angle DOB - \tan \angle AOB \approx 2^\ell \left(\frac{\delta}{\lambda}\right)^{1/2},$$

which implies $\tan(\angle C - \angle B) \approx 2^\ell (\frac{\delta}{\lambda})^{1/2}$. As a result of $\angle APD + \angle AED = \angle DEI + \angle AED = 180^\circ$, then $\angle APD = \angle DEI = \angle C - \angle B \approx 2^\ell (\frac{\delta}{\lambda})^{1/2}$.

Figure 5: Curve $q_\delta(\xi) = 1$

Since $q_\delta(\xi) = 1$, then $\delta^{-1/2} \frac{\xi_2}{\xi_1} = \xi_1^{1/2}$, and the slope of the line PA (the line PD) is equal to $\frac{3}{2} \delta^{1/2} \bar{\xi}_1^{-1/2}$ ($\frac{3}{2} \delta^{1/2} \tilde{\xi}_1^{1/2}$). So $\tilde{\xi}_1^{1/2} - \bar{\xi}_1^{1/2} = \delta^{-1/2} (\tan \angle DOB - \tan \angle AOB) \approx \delta^{-1/2} 2^\ell (\frac{\delta}{\lambda})^{1/2} = 2^\ell / \lambda^{1/2}$. Then the line PA can be written as

$$\xi_2 = -\frac{\xi_1}{\frac{3}{2} \delta^{1/2} \bar{\xi}_1^{1/2}} + \delta^{1/2} \bar{\xi}_1^{3/2} + \frac{2}{3} \delta^{-1/2} \bar{\xi}_1^{1/2}$$

and the line PD is

$$\xi_2 = -\frac{\xi_1}{\frac{3}{2} \delta^{1/2} \tilde{\xi}_1^{1/2}} + \delta^{1/2} \tilde{\xi}_1^{3/2} + \frac{2}{3} \delta^{-1/2} \tilde{\xi}_1^{1/2}.$$

It's easy to compute the point $P \approx (-\frac{1}{\delta}, \frac{1}{\delta})$. The claim was proved.

Since $AG \perp PG$, $DF \perp PF$, $PA \perp AE$ and $PD \perp DE$, then $\angle GAE = \angle APG \approx 2^\ell (\frac{\delta}{\lambda})^{1/2}$ and $\angle EDF = \angle DPF \approx 2^\ell (\frac{\delta}{\lambda})^{1/2}$.

The projection on PG of the width of \mathcal{U}_ν^n is equal to $\sin \angle GAE \times \frac{1}{\sqrt{\delta\lambda}} \approx \frac{2^\ell}{\lambda} \geq \lambda^{-1}$. For ν, ν' with $|\nu - \nu'| \approx 2^\ell$, $\Lambda_\nu^n + \Lambda_{\nu'}^{n'}$ is comparable to a rectangle of size $2^\ell \lambda^{-1} \times \frac{1}{\sqrt{\delta\lambda}} \times 1/(\delta\lambda^{1/2})$. In other words, $vol(\Lambda_\nu^n + \Lambda_{\nu'}^{n'}) \approx 2^\ell \delta^{-1/2} \lambda^{-2}$.

Next we define

$$\Omega = \{(\eta, \tau) : \sum_{|\nu - \nu'| \approx 2^\ell} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau) \geq \frac{1}{2} \max_{(\eta, \tau)} \sum_{|\nu - \nu'| \approx 2^\ell} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau)\}.$$

Since the sum is equally distributed on Ω , one can check that

$$vol(\Omega) \approx \underbrace{(2^\ell (\frac{\delta}{\lambda})^{1/2} \times \frac{2^\ell}{\sqrt{\delta\lambda}})}_d \times \lambda^{-1/2} \times \delta^{-1} = \frac{2^{2\ell}}{\delta^2 \lambda^{3/2}}.$$

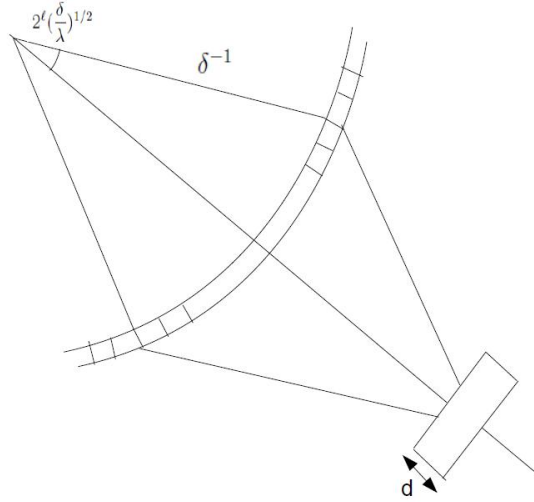


Figure 6 : $\Lambda_\nu^n + \Lambda_{\nu'}^{n'}$ for $|\nu - \nu'| \approx 2^\ell$

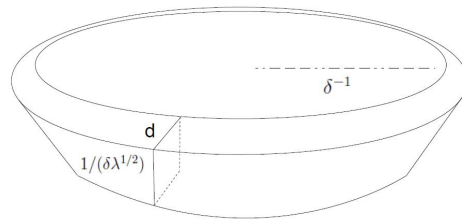


Figure 7 : Ω

Set $\bar{C} = \max_{(\eta, \tau)} \sum_{|\nu - \nu'| \approx 2^\ell} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau)$, then

$$\begin{aligned} \bar{C} \text{vol}(\Omega) &= \int_{\Omega} \bar{C} d\eta d\tau \leq 2 \int_{\mathbb{R}^3} \sum_{|\nu - \nu'| \approx 2^\ell} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau) d\eta d\tau \\ &= 2 \sum_{|\nu - \nu'| \approx 2^\ell} \text{vol}(\Lambda_\nu^n + \Lambda_{\nu'}^{n'}) \\ &\lesssim 2^\ell \left(\frac{\lambda}{\delta}\right)^{1/2} 2^\ell \delta^{-3/2} \lambda^{-2} \lesssim 2^{2\ell} \frac{1}{\delta^2 \lambda^{3/2}}, \end{aligned}$$

together with the estimate of $\text{vol}(\Omega)$, we know $\bar{C} \leq C$, i.e. $\sum_{|\nu - \nu'| \approx 2^\ell} \chi_{\Lambda_\nu^n + \Lambda_{\nu'}^{n'}}(\eta, \tau) \leq C$. \square

Let us continue our estimate for $\left\| \left(\sum_n |P^n \sum_\nu Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n)|^2 \right)^{1/2} \right\|_{L^4}$. By Plancherel's theorem and the above overlap lemma, we have

$$\begin{aligned} &\left\| \left(\sum_n |P^n \sum_\nu Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n)|^2 \right)^{1/2} \right\|_{L^4}^4 \\ &= \int_{\mathbb{R}^3} \left(\sum_n |P^n \sum_\nu Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n)|^2 \right)^2 dy dt \\ &= \int_{\mathbb{R}^3} \sum_n \left| P^n \sum_\nu Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n) \right|^2 \sum_{n'} \left| P^{n'} \sum_{\nu'} Q_{\nu'}(\mathcal{F}_{\lambda, \nu'}^\delta f_{n'}) \right|^2 dy dt \\ &= \int_{\mathbb{R}^3} \sum_{n, n'} \left| \sum_{\nu, \nu'} P^n Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n) P^{n'} Q_{\nu'}(\mathcal{F}_{\lambda, \nu'}^\delta f_{n'}) \right|^2 dy dt \\ &= \int_{\mathbb{R}^3} \sum_{n, n'} \left| \sum_{\nu, \nu'} \chi_{U_\nu^n + U_{\nu'}^{n'}}(\eta, \tau) (P^n Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n))^\wedge * (P^{n'} Q_{\nu'}(\mathcal{F}_{\lambda, \nu'}^\delta f_{n'}))^\wedge(\eta, \tau) \right|^2 d\eta d\tau \\ &\leq \int_{\mathbb{R}^3} \sum_{n, n'} \sum_{\nu, \nu'} \chi_{U_\nu^n + U_{\nu'}^{n'}}^2(\eta, \tau) \sum_{\nu, \nu'} \left| (P^n Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n))^\wedge * (P^{n'} Q_{\nu'}(\mathcal{F}_{\lambda, \nu'}^\delta f_{n'}))^\wedge(\eta, \tau) \right|^2 d\eta d\tau \\ &\leq C \left(\frac{\lambda}{\delta}\right)^{2\epsilon} \log_2 \left(\frac{\lambda}{\delta}\right)^{1/2} \int_{\mathbb{R}^3} \sum_{n, n'} \sum_{\nu, \nu'} \left| P^n Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n) P^{n'} Q_{\nu'}(\mathcal{F}_{\lambda, \nu'}^\delta f_{n'}) \right|^2 dy dt \\ &= C \left(\frac{\lambda}{\delta}\right)^{2\epsilon} \log_2 \left(\frac{\lambda}{\delta}\right)^{1/2} \left\| \left(\sum_{n, \nu} |P^n Q_\nu(\mathcal{F}_{\lambda, \nu}^\delta f_n)|^2 \right)^{1/2} \right\|_{L^4}^4. \end{aligned}$$

Hence by Lemma 3.0.4, it suffices to estimate $\left\| \left(\sum_{n, \nu} |P^n \mathcal{F}_{\lambda, \nu}^\delta f_n|^2 \right)^{1/2} \right\|_{L^4}$.

Furthermore,

$$\begin{aligned}
& \left\| \left(\sum_{n,\nu} |P^n(\mathcal{F}_{\lambda,\nu}^\delta f_n)|^2 \right)^{1/2} \right\|_{L^4}^4 \\
&= \left\| \left(\sum_{n,\nu} |\lambda^{1/2} e^{-i\lambda^{1/2}nt} \check{\varphi}(\lambda^{1/2}\cdot) *_t \mathcal{F}_{\lambda,\nu}^\delta f_n(x,t)|^2 \right)^{1/2} \right\|_{L^4}^4 \\
&\leq C'_N \left\| \left(\sum_{n,\nu} |\lambda^{1/2} \int_{\mathbb{R}} \frac{|\mathcal{F}_{\lambda,\nu}^\delta f_n(y,t-s)|}{(1+\lambda^{1/2}|s|)^N} ds|^2 \right)^{1/2} \right\|_{L^4}^4 \\
&\leq C'_N \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left[\sum_{n,\nu} \lambda^{1/2} \int_{\mathbb{R}} \frac{1}{(1+\lambda^{1/2}|s|)^N} ds \lambda^{1/2} \int_{\mathbb{R}} \frac{|\mathcal{F}_{\lambda,\nu}^\delta f_n(y,t-s)|^2}{(1+\lambda^{1/2}|s|)^N} ds \right]^2 dy dt \\
&\leq C''_N \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left(\sum_{n,\nu} \lambda^{1/2} \int_{\mathbb{R}} \frac{|\mathcal{F}_{\lambda,\nu}^\delta f_n(y,t-s)|^2}{(1+\lambda^{1/2}|s|)^N} ds \right)^2 dy dt \\
&\leq C''_N \left\| \sum_{n,\nu} \lambda^{1/2} \int_{\mathbb{R}} \frac{|\mathcal{F}_{\lambda,\nu}^\delta f_n(y,t-s)|^2}{(1+\lambda^{1/2}|s|)^N} ds \right\|_{L^2}^2 \\
&\leq C''_N \left(\int_{\mathbb{R}} \lambda^{1/2} (1+\lambda^{1/2}|s|)^{-N} \left\| \sum_{n,\nu} |\mathcal{F}_{\lambda,\nu}^\delta f_n(y,t-s)|^2 \right\|_{L^2} ds \right)^2 \\
&\leq C'''_N \left\| \left(\sum_{n,\nu} |\mathcal{F}_{\lambda,\nu}^\delta f_n(y,t)|^2 \right)^{1/2} \right\|_{L^4}^4,
\end{aligned}$$

where $*_t$ means that convolution acts only on the variable t .

At present we have

$$\|\mathcal{F}_\lambda^\delta f\|_{L^4} \leq C \lambda^{1/8} \left(\frac{\lambda}{\delta}\right)^{\epsilon/2} [\log_2(\frac{\lambda}{\delta})^{1/2}]^{1/4} \left\| \left(\sum_{n,\nu} |\mathcal{F}_{\lambda,\nu}^\delta f_n(x,t)|^2 \right)^{1/2} \right\|_{L^4} + C_N \lambda^{-N} \|f\|_{L^4}.$$

It remains to estimate $\left\| \left(\sum_{n,\nu} |\mathcal{F}_{\lambda,\nu}^\delta f_n(x,t)|^2 \right)^{1/2} \right\|_{L^4}$. For $m = (m_1, m_2) \in \mathbb{Z}^2$, we define the operator P_m^δ acting on the functions in \mathbb{R}^2 by

$$(P_m^\delta f)^\wedge(\xi) = \varphi\left(\frac{\delta}{\lambda^{1/2}}\xi_1 - m_1\right) \varphi\left(\frac{\delta}{\lambda^{1/2}}\xi_2 - m_2\right) \widehat{f}(\xi), \quad (3.18)$$

then $f = \sum_{m \in \mathbb{Z}^2} P_m^\delta f$.

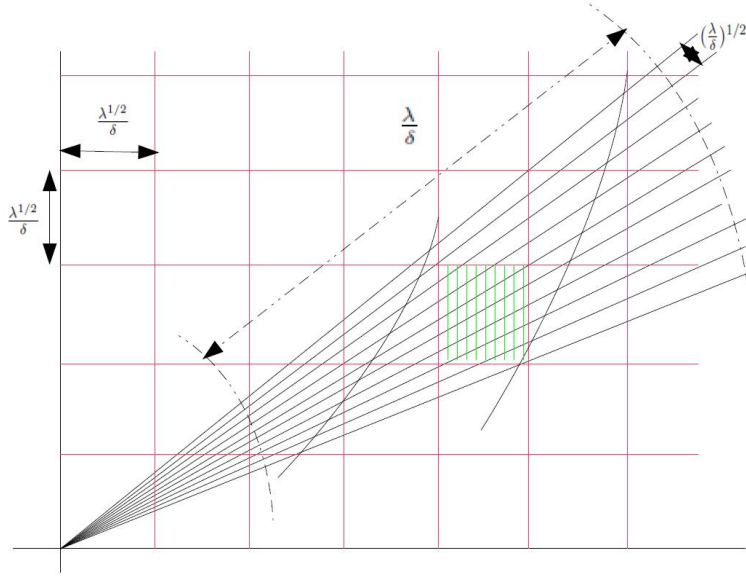


Figure 8 : $\text{Card} \{(n, \nu) : m \in J_{n, \nu}\} \lesssim \delta^{-1/2}$

The definition of f_n and the support properties of χ_ν imply

$$\mathcal{F}_{\lambda, \nu}^\delta f_n = \mathcal{F}_{\lambda, \nu}^\delta f_{n, \nu},$$

where $f_{n, \nu} = \sum_{m \in J_{n, \nu}} P_m^\delta f$ and the index set $J_{n, \nu}$ is contained in the set of all $m \in \mathbb{Z}^2$ such that $\chi_\nu \widehat{P_m^\delta f}$ is not identically zero. It follows from the above discussion that

$$\text{card}\{J_{n, \nu}\} \leq C. \quad (3.19)$$

Meanwhile, we also get

$$\text{card}\{(n, \nu) : m \in J_{n, \nu}\} \lesssim \delta^{-1/2}. \quad (3.20)$$

Let the kernel $K_{\lambda, \nu, n}^\delta(y, t; x)$ of $\mathcal{F}_{\lambda, \nu}^\delta f_{n, \nu}$ be

$$\rho_1(y, t) \int_{\mathbb{R}^2} e^{i((y-x) \cdot \xi + E(\xi) + tq_\delta(\xi))} \rho_0\left(\frac{\delta}{\lambda} |\xi|\right) \tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right) a(\xi, t) \chi_\nu(\xi) \varphi((\lambda^{-1/2} q_\delta(\xi) - n)/10) d\xi. \quad (3.21)$$

We need the following estimates. Notice that (3.23) is an estimate of a Keakeya-type maximal operator.

Lemma 3.0.6. *For fixed n, ν and $g \in L^2(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} |K_{\lambda, \nu, n}^\delta(y, t; x)| dx \leq C; \quad (3.22)$$

$$\left(\int_{\mathbb{R}^2} \sup_{\nu, n} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}} |K_{\lambda, \nu, n}^\delta(y, t; x)| |g(y, t)| dy dt \right\}^2 dx \right)^{1/2} \leq C \delta^{-1/2} (\log_2 \frac{\lambda^{1/2}}{\delta})^2 \|g\|_{L^2}. \quad (3.23)$$

Lemma 3.0.7. [26] For $2 \leq p \leq \infty$,

$$\left\| \left(\sum_{m \in \mathbb{Z}^2} |P_m^\delta f(y)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}.$$

Supposing that Lemma 3.0.6 holds true, together with (3.19), (3.20) and Lemma 3.0.7, a duality argument and Hölder's inequality give

$$\begin{aligned} & \left\| \left(\sum_{n, \nu} |\mathcal{F}_{\lambda, \nu}^\delta f_n(y, t)|^2 \right)^{1/2} \right\|_{L^4}^2 \\ &= \sup_{\|g\|_{L^2}=1} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}} \sum_{n, \nu} |\mathcal{F}_{\lambda, \nu}^\delta f_n(y, t)|^2 g(y, t) dy dt \right| \\ &= \sup_{\|g\|_{L^2}=1} \sum_{n, \nu} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} (K_{\lambda, \nu, n}^\delta(y, t; x))^{1/2} [(K_{\lambda, \nu, n}^\delta(y, t; x))^{1/2} f_{n, \nu}(x)] dx \right|^2 |g(y, t)| dy dt \\ &\leq \sup_{\|g\|_{L^2}=1} \sum_{n, \nu} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |K_{\lambda, \nu, n}^\delta(y, t; x)| dx \int_{\mathbb{R}^2} |K_{\lambda, \nu, n}^\delta(y, t; x)| |f_{n, \nu}(x)|^2 dx |g(y, t)| dy dt \\ &\leq C \sup_{\|g\|_{L^2}=1} \sum_{n, \nu} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |K_{\lambda, \nu, n}^\delta(y, t; x)| |f_{n, \nu}(x)|^2 dx |g(y, t)| dy dt \\ &\leq C \sup_{\|g\|_{L^2}=1} \sum_{n, \nu} \int_{\mathbb{R}^2} |f_{n, \nu}(x)|^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}} |K_{\lambda, \nu, n}^\delta(y, t; x)| |g(y, t)| dy dt dx \\ &\leq C' \sup_{\|g\|_{L^2}=1} \int_{\mathbb{R}^2} \sum_{n, \nu} \sum_{m \in J_{n, \nu}} |P_m^\delta f(x)|^2 \sup_{n, \nu} \int_{\mathbb{R}^2} \int_{\mathbb{R}} |K_{\lambda, \nu, n}^\delta(y, t; x)| |g(y, t)| dy dt dx \\ &\leq C' \sup_{\|g\|_{L^2}=1} \int_{\mathbb{R}^2} \sum_{m \in \mathbb{Z}^2} \sum_{\{(n, \nu): m \in J_{n, \nu}\}} |P_m^\delta f(x)|^2 \sup_{n, \nu} \int_{\mathbb{R}^2} \int_{\mathbb{R}} |K_{\lambda, \nu, n}^\delta(y, t; x)| |g(y, t)| dy dt dx \\ &\leq C'' \delta^{-1/2} \sup_{\|g\|_{L^2}=1} \left\| \left(\sum_{m \in \mathbb{Z}^2} |P_m^\delta f(x)|^2 \right)^{1/2} \right\|_{L^4}^2 \\ &\quad \times \left(\int_{\mathbb{R}^2} \sup_{\nu, n} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}} |K_{\lambda, \nu, n}^\delta(y, t; x)| |g(y, t)| dy dt \right\}^2 dx \right)^{1/2} \\ &\leq C''' \delta^{-1} (\log_2 \frac{\lambda^{1/2}}{\delta})^2 \|f\|_{L^4}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{F}_\lambda^\delta f\|_{L^4} &\leq C\delta^{-1/2}\lambda^{1/8}\left(\frac{\lambda}{\delta}\right)^{\epsilon/2}[\log_2\left(\frac{\lambda}{\delta}\right)^{1/2}]^{1/4}\log_2\frac{\lambda^{1/2}}{\delta}\|f\|_{L^4} + C_N\lambda^{-N}\|f\|_{L^4} \\ &\leq C\lambda^{1/8+\epsilon_1}\delta^{-(1/2+\epsilon_2)}\|f\|_{L^4}, \end{aligned}$$

where ϵ_1 and ϵ_2 are very small.

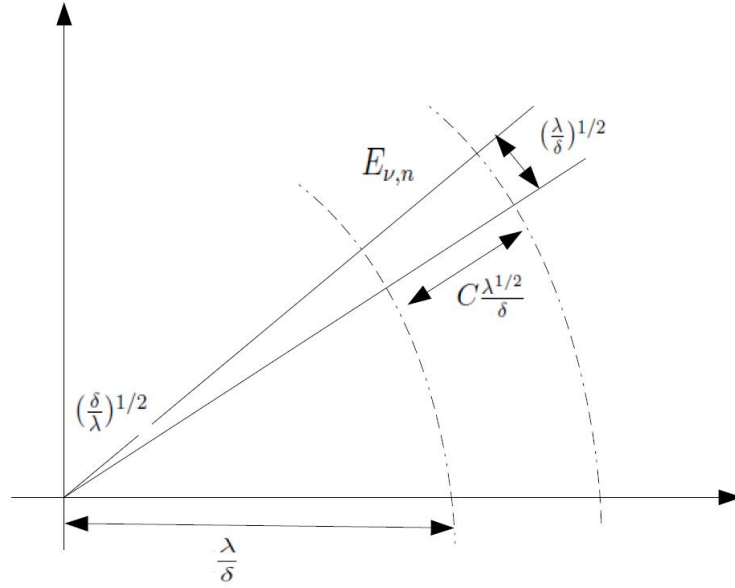


Figure 9 : $E_{\nu, n}$

We turn to prove Lemma 3.0.6. By (3.21), we put

$$E_{\nu, n} := \left\{ \xi : \rho_0\left(\frac{\delta}{\lambda}|\xi|\right)\tilde{\chi}\left(\frac{\xi_1}{\xi_2}\right)a(\xi, t)\chi_\nu(\xi)\varphi\left(\left(\lambda^{-1/2}q_\delta(\xi) - n\right)/10\right) \neq 0 \right\} \quad (3.24)$$

and consider the size of $E_{\nu, n}$ (see Figure 9), first we can adopt some ideas from the proof of inequality (2.55) to prove inequality (3.22), here we still use similar notations.

After a rotation by the transformation \tilde{T} , we put $\tilde{\Psi}(y, \xi) = \tilde{T}\xi \cdot y + E(\tilde{T}\xi) + tq_\delta(\tilde{T}\xi)$ and $h(\xi) = \tilde{\Psi}(y, \xi) - \tilde{\Psi}_\xi(y, \bar{\xi}) \cdot \xi$, where $\bar{\xi} = (1, 0)$. Since $\tilde{\Psi}$ is homogeneous of degree one, $|\xi| \approx \frac{\lambda}{\delta}$ and $|\xi_2| \leq C\left(\frac{\lambda}{\delta}\right)^{1/2}$, then we can get some similar results with (2.58), (2.59), (2.60),

$$\left| \left(\frac{\partial}{\partial \xi_1} \right)^N h(\xi) \right| \lesssim \left(\frac{\lambda}{\delta} \right)^{-N}, \quad \left| \left(\frac{\partial}{\partial \xi_2} \right)^N h(\xi) \right| \lesssim \left(\frac{\lambda}{\delta} \right)^{-N/2}, \quad (3.25)$$

$$\left| \left(\frac{\partial}{\partial \xi_1} \right)^N (\chi_\nu \circ \tilde{T})(\xi) \right| \lesssim \left(\frac{\lambda}{\delta} \right)^{-N}, \quad \left| \left(\frac{\partial}{\partial \xi_2} \right)^N (\chi_\nu \circ \tilde{T})(\xi) \right| \lesssim \left(\frac{\lambda}{\delta} \right)^{-N/2}, \quad N \geq 1. \quad (3.26)$$

Furthermore, set $\mathcal{L} := I - \frac{\lambda}{\delta^2} \frac{\partial^2}{\partial \xi_1^2} - \frac{\lambda}{\delta} \frac{\partial^2}{\partial \xi_2^2}$ and $D_t(z) := (t^2 z_1, t z_2)$. Because of (2.63), (3.25), (3.26) and $\mathcal{L}^N (\varphi((\lambda^{-1/2} q_\delta(T\xi) - n)/10)) \leq C$, then

$$\mathcal{L}^N \left(e^{ih(\xi)} \rho_0 \left(\frac{\delta}{\lambda} |\xi| \right) \tilde{\chi} \left(\frac{\xi_1}{\xi_2} \right) a(\xi, t) \chi_\nu(\xi) \varphi((\lambda^{-1/2} q_\delta(\xi) - n)/10) \right) \leq C.$$

Noting that the area of the region $E_{\nu,n}$ is at most $C \frac{\lambda}{\delta^{3/2}}$, so by integration by parts in ξ , we obtain

$$\begin{aligned} |K_{\lambda,\nu,n}^\delta(y, t; x)| &\leq \frac{C'_N \lambda / \delta^{3/2}}{\left(1 + \frac{\lambda}{\delta^2} |(\tilde{\Psi}_\xi(y-x, \bar{\xi}))_1|^2 + \frac{\lambda}{\delta} |(\tilde{\Psi}_\xi(y-x, \bar{\xi}))_2|^2 \right)^N} \\ &\leq \frac{C_N \lambda / \delta^{3/2}}{\left(1 + |D_{\delta^{-1/2}} \lambda^{1/2} (\tilde{\Psi}_\xi(y-x, \bar{\xi}))| \right)^{2N}}, \end{aligned} \quad (3.27)$$

where

$$|(\tilde{\Psi}_\xi(y-x, \bar{\xi}))_1| = |\langle y-x + \nabla E(\xi_\nu) + t \nabla q_\delta(\xi_\nu), \xi_\nu \rangle|$$

and

$$|(\tilde{\Psi}_\xi(y-x, \bar{\xi}))_2| = |y-x + \nabla E(\xi_\nu) + t \nabla q_\delta(\xi_\nu) - \langle y-x + \nabla E(\xi_\nu) + t \nabla q_\delta(\xi_\nu), \xi_\nu \rangle \xi_\nu|.$$

Inequality (3.27) implies inequality (3.22).

We introduce some notations. Given a direction $\xi_\nu = (\cos \theta_\nu, \sin \theta_\nu) \in S^1$, let $\gamma_{\theta_\nu} \subset \mathbb{R}^3$ be the ray defined by

$$\gamma_{\theta_\nu} = \{(y, t) : y + \nabla E(\cos \theta_\nu, \sin \theta_\nu) + t \nabla q_\delta(\cos \theta_\nu, \sin \theta_\nu) = 0\}. \quad (3.28)$$

From inequality (3.27), we note that for fixed x , the kernels $K_{\lambda,\nu,n}^\delta(y, t; x)$ are essentially supported in a rectangle of size $\frac{\delta}{\lambda^{1/2}} \times (\frac{\delta}{\lambda})^{1/2} \times 1$ around $\gamma_{\theta_\nu} + (x, 0)$, see Figure 10. This sheds some light on the proof of inequality (3.23).

We may assume $g \geq 0$ and choose non-negative functions $\varrho \in C_0^\infty([\pi/2 - c_2, \pi/2 - c_1])$ and $\beta_0 \in C_0^\infty([0, 2])$ which satisfies $\beta_0(r) + \sum_{\ell=1}^\infty \beta(2^{-\ell} r) = 1$, for $r > 0$. Let $|\cdot|_D$ be a homogeneous norm under $D_t(z) = (t^2 z_1, t z_2)$, i.e. $|D_t(z)|_D = t|z|_D$. Let $h_0(\xi) = \beta_0(|\xi|_D) \in C_0^\infty(\mathbb{R}^2)$ and $h(\xi) = \beta(|\xi|_D) \in C_0^\infty(\mathbb{R}^2)$, then $h_0(\xi) + \sum_{\ell=1}^\infty h(D_{2^{-\ell}} \xi) = 1$. It is clear that for $\ell > 1$, $\xi \neq 0$, $h(D_{2^{-\ell}} \xi) \neq 0 \Rightarrow 2^{\ell-1} \leq |\xi|_D \leq 2^{\ell+1} \Rightarrow |\xi| \gtrsim 2^\ell$.

Since the right side of inequality (3.27) does not depend on n , then the left side of

inequality (3.23) can be controlled by

$$\begin{aligned}
& \left(\int_{\mathbb{R}^2} \sup_{\nu, n} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}} |K_{\lambda, \nu, n}^\delta(y, t; x)| |g(y, t)| dy dt \right\}^2 dx \right)^{1/2} \\
& \leq C'_N \left(\left\| \sup_{\nu} \frac{1}{\delta^{3/2}/\lambda} \int_{\mathbb{R}^2 \times [1/2, 4]} h_0(D_{\delta^{-1/2}} \lambda^{1/2} \tilde{\Psi}_\xi(y, \bar{\xi})) g(x+y, t) dy dt \right\|_{L^2} \right. \\
& \quad \left. + \left\| \sup_{\nu} \sum_{\ell=1}^{\infty} \int_{\mathbb{R}^2 \times [1/2, 4]} h(D_{2^{-\ell}}(D_{\delta^{-1/2}} \lambda^{1/2} \tilde{\Psi}_\xi(y, \bar{\xi}))) \frac{\lambda/\delta^{3/2} g(x+y, t)}{(1 + |D_{\delta^{-1/2}} \lambda^{1/2} \tilde{\Psi}_\xi(y, \bar{\xi})|)^{2N}} dy dt \right\|_{L^2} \right) \\
& \leq C''_N \sum_{\ell=0}^{\infty} 2^{-2(N-3/2)\ell} \left\| \sup_{\nu} \frac{1}{2^{3(\ell+1)} \delta^{3/2}/\lambda} \int_{\{(y, t) \in \mathbb{R}^2 \times [0, 1] : |\lambda^{1/2}(y + \nabla E(\xi_\nu) + t \nabla q_\delta(\xi_\nu))|_D \leq 2^{\ell+1} \delta^{1/2}\}} \right. \\
& \quad \left. \times g(x+y, t) dy dt \right\|_{L^2} \\
& \leq C''_N \sum_{\ell=0}^{\infty} 2^{-2(N-3/2)\ell} \left\| \sup_{\nu} \frac{1}{|R_\nu^\ell|} \int_{R_\nu^\ell} g(x+y, t) dy dt \right\|_{L^2},
\end{aligned}$$

where

$$R_\nu^\ell = \{(y, t) \in \mathbb{R}^2 \times [1/2, 4] : |\lambda^{1/2}(y + \nabla E(\xi_\nu) + t \nabla q_\delta(\xi_\nu))|_D \leq 2^{\ell+1} \delta^{1/2}\}.$$

From the above argument, in order to prove inequality (3.23), we need to prove an $L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^2)$ maximal estimate involving averages over cuboids R_ν^0 of dimensions $\frac{\delta}{\lambda^{1/2}} \times (\frac{\delta}{\lambda})^{1/2} \times 1$, which is basically tangential to the cone $\{(y, t) \in \gamma_\theta : \gamma_\theta \text{ is defined as (3.28) and } \theta \in \text{supp } \rho\}$ (see Figure 10), because in a similar way one can obtain $L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^2)$ maximal estimates involving averages over $\{R_\nu^\ell\}_\nu$ which are $\frac{2^{2\ell}\delta}{\lambda^{1/2}} \times 2^\ell (\frac{\delta}{\lambda})^{1/2} \times 1$ cuboids for $\ell = 1, 2, \dots$. However, we can split every R_ν^0 along its longer side of size $(\frac{\delta}{\lambda})^{1/2}$ into $\delta^{-1/2}$ pieces, i.e. $\{R_{\nu, i}^0\}_{i=1}^{\delta^{-1/2}}$, where $R_{\nu, i}^0$ is a $\frac{\delta}{\lambda^{1/2}} \times \frac{\delta}{\lambda^{1/2}} \times 1$ tube around $\gamma_{\theta_{\nu, i}} + (x, 0)$ (see Figure 10). Then next we will prove a stronger maximal estimate involving averages over tubes R_θ defined by

$$R_\theta := \{(y, t) \in \mathbb{R}^2 \times [0, 1] : \text{dist}\{(y, t), \gamma_\theta\} < \frac{\delta}{\lambda^{1/2}}\}. \quad (3.29)$$

That is because

$$\begin{aligned}
\left\| \sup_{\nu} \frac{1}{|R_\nu^0|} \int_{R_\nu^0} g(x+y, t) dy dt \right\|_{L^2} & \leq \delta^{1/2} \sum_{i=1}^{\delta^{-1/2}} \left\| \sup_{\nu} \frac{1}{|R_{\nu, i}^0|} \int_{R_{\nu, i}^0} g(x+y, t) dy dt \right\|_{L^2} \\
& \leq \delta^{1/2} \sum_{i=1}^{\delta^{-1/2}} \left\| \sup_{\theta} \frac{1}{|R_\theta|} \int_{R_\theta} \varrho(\theta) g(x+y, t) dy dt \right\|_{L^2}.
\end{aligned}$$

Hence, we would be done if we can obtain the following Kakeya type estimate.

Lemma 3.0.8.

$$\left\| \sup_{\theta} \frac{1}{|R_{\theta}|} \int_{R_{\theta}} \varrho(\theta) g(x - y, t) dy dt \right\|_{L^2} \leq C \delta^{-1/2} (\log_2 \frac{\lambda^{1/2}}{\delta})^2 \|g\|_{L^2}. \quad (3.30)$$

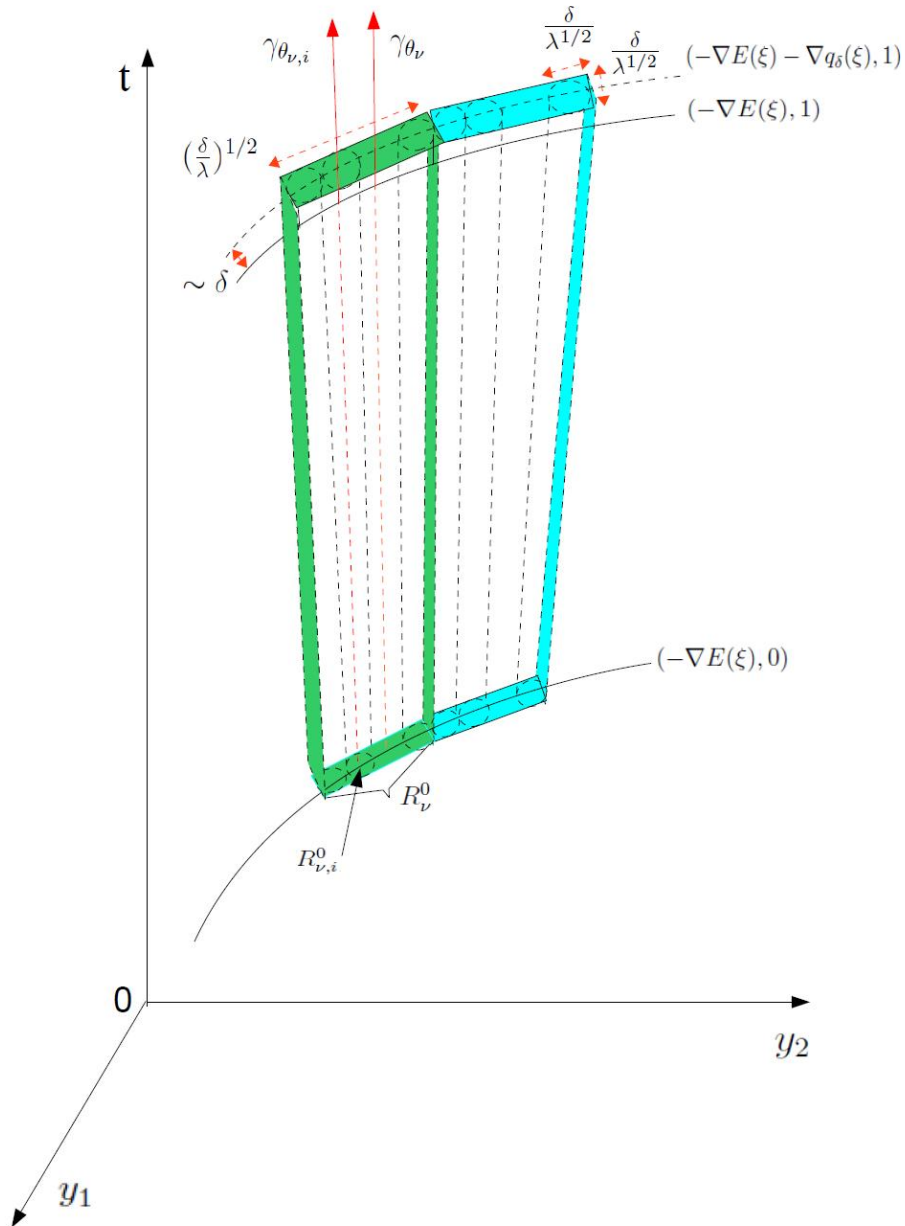


Figure 10 : cuboid R_{ν}^0 and tube $R_{\nu,i}^0$

Remark 3.0.1. The constant $\delta^{-1/2}$ in inequality (3.30) is sharp if we take $E(\xi) = |\xi|$

and $q_\delta(\xi) = \delta|\xi|$. Let

$$M_{\delta,\lambda}g(x) = \sup_{\theta} \frac{1}{|R_\theta|} \int_{R_\theta} \varrho(\theta)g(x-y, t)dydt.$$

In particular, we take $g(z, t) = \chi_{P_\delta}(z, t)$, where P_δ is a tubular neighborhood of width δ around the t -axis with height 1. Assume that $x \in A_\delta := \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 1 + \delta, c_1 \leq |x_1/x_2| \leq c_2\}$. Now taking θ so that $(\cos \theta, \sin \theta) = \frac{x}{|x|}$, then we note that

$$g * \chi_{R_\theta}(x) = |R_\theta \cap (P_\delta - (x, 0))| \approx |R_\theta|,$$

which implies $M_{\delta,\lambda}g(x) \gtrsim 1$ on A_δ . Let $C_\delta := \|M_{\delta,\lambda}\|_{L^2 \rightarrow L^2}$. Since $|A_\delta| \approx \delta$, then

$$\delta^{1/2} \lesssim \|M_{\delta,\lambda}g\|_{L^2} \leq C_\delta \delta \Rightarrow C_\delta \gtrsim \delta^{-1/2}.$$

In fact, if we take $E(\xi) = \frac{\xi_1^2}{\xi_2}$ and $q_\delta(\xi) = \delta \frac{\xi_1^3}{\xi_2^2}$, in the similar way as above, we still get the same conclusion, regarding complicated computations, we omit here.

Proof. We choose a suitable $\sigma \in C_0^\infty(\mathbb{R}^2)$ satisfying $\check{\sigma} \geq 0$. Let $\bar{\delta} = \frac{\delta}{\lambda^{1/2}}$ and $\sigma_{\bar{\delta}}(\theta, t, \xi) = \varrho(\theta)\sigma(\bar{\delta}\xi)\chi_{[0,1]}(t)$. Then We have

$$\begin{aligned} & \int_{\{(y,t) \in \mathbb{R}^2 \times [0,1] : \text{dist}\{(y,t), \gamma_\theta\} < \bar{\delta}\}} g(x-y, t)dydt \\ & \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}} \varrho(\theta)\check{\sigma} \left(\frac{y + t\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)}{\bar{\delta}} \right) g(x-y, t)\chi_{[0,1]}(t)dydt \\ & = \bar{\delta}^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}} \varrho(\theta)[\sigma(\bar{\delta}\cdot)]^\vee (y + t\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta))g(x-y, t) \\ & \quad \times \chi_{[0,1]}(t)dydt \\ & = \bar{\delta}^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{i\xi \cdot [x-y+t\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi)d\xi g(y, t)dydt \\ & = \bar{\delta}^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i\xi \cdot [x+t\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi)g(\hat{\xi}, t)d\xi dt \\ & =: \bar{\delta}^2 A_\theta g(x), \end{aligned}$$

where $g(\hat{\xi}, t)$ denotes the partial Fourier transform of g with respect to the ξ -variables. Therefore, we may further reduce to prove that

$$\| \sup_{\theta} |A_\theta g| \|_{L^2} \leq C\delta^{-1/2} |\log_2 \bar{\delta}|^2 \|g\|_{L^2}. \quad (3.31)$$

In order to prove the above inequality, we need to break up the operator A_θ . Just as before, we take $\beta \in C_0^\infty(\mathbb{R})$ and define the dyadic operators A_θ^τ by

$$A_\theta^\tau g(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i\xi \cdot [x+t\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi)\beta\left(\frac{|\xi|}{\tau}\right)g(\hat{\xi}, t)d\xi dt,$$

so it suffices to prove that

$$\left\| \sup_{\theta} |A_{\theta}^{\tau} g| \right\|_{L^2} \leq C \delta^{-1/2} \log_2 \tau^{1/2} \|g\|_{L^2}, \quad \tau > 2. \quad (3.32)$$

This is because $A_{\theta} = \sum_{1 < \kappa < |\log_2 \bar{\delta}| + C} A_{\theta}^{2^{\kappa}} + \tilde{R}_{\theta}$, where C is a fixed constant and the kernel of \tilde{R}_{θ} defined by

$$\int_{\mathbb{R}^2} e^{i\xi \cdot [(x-y) + t \nabla q_{\delta}(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi) \beta_0(|\xi|) d\xi$$

is controlled by $\mathcal{O}((1 + |y - x|)^{-N})$ for any N with bounds independent of θ .

Let $\theta_{\xi} = \arg \xi$. We need to make one final reduction based on the following observation:

$$\begin{aligned} & \frac{\partial}{\partial \theta} \langle \xi, \nabla q_{\delta}(\cos \theta, \sin \theta) \rangle \\ &= \left\langle \xi, (-\sin \theta, \cos \theta) \begin{pmatrix} \partial_{11} q_{\delta}(\cos \theta, \sin \theta) & \partial_{12} q_{\delta}(\cos \theta, \sin \theta) \\ \partial_{21} q_{\delta}(\cos \theta, \sin \theta) & \partial_{22} q_{\delta}(\cos \theta, \sin \theta) \end{pmatrix} \right\rangle \\ &= 6\delta |\xi| \frac{\cos \theta}{\sin^4 \theta} \left\langle \frac{\xi}{|\xi|}, (-\sin \theta, \cos \theta) \right\rangle = 6\delta |\xi| \frac{\cos \theta}{\sin^4 \theta} \sin(\theta_{\xi} - \theta) = 0 \Leftrightarrow \theta = \theta_{\xi}. \end{aligned} \quad (3.33)$$

For $\ell > 0$, put

$$A_{\theta}^{\tau, \ell} g(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i\xi \cdot [x + t \nabla q_{\delta}(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi) \beta\left(\frac{|\xi|}{\tau}\right) \beta_{\ell, \tau}(\xi, \theta) g(\hat{\xi}, t) d\xi dt,$$

where $\beta_{\ell, \tau}(\xi, \theta) = \beta\left(2^{-\ell} \tau^{1/2} \left| \langle (-\sin \theta, \cos \theta), \frac{\xi}{|\xi|} \rangle \right| \right)$. In fact, $\left| \langle (-\sin \theta, \cos \theta), \frac{\xi}{|\xi|} \rangle \right| \leq C$, C is very small, which implies that $\ell \leq C \log_2 \tau^{1/2}$. We define $A_{\theta}^{\tau, 0} = A_{\theta}^{\tau} - \sum_{0 < \ell \leq C \log_2 \tau^{1/2}} A_{\theta}^{\tau, \ell}$.

Again we use Lemma 2.2.3, for $\ell \geq 0$,

$$\left\| \sup_{\theta} |A_{\theta}^{\tau, \ell} g| \right\|_{L^2}^2 \leq C \left(\int \int |A_{\theta}^{\tau, \ell} g(x)|^2 d\theta dx \right)^{1/2} \left(\int \int \left| \frac{\partial}{\partial \theta} A_{\theta}^{\tau, \ell} g(x) \right|^2 d\theta dx \right)^{1/2}.$$

Inequality (3.32) will be obtained if we can show that

$$\left(\int \int |A_{\theta}^{\tau, \ell} g(x)|^2 d\theta dx \right)^{1/2} \leq C \delta^{-1/2} 2^{-\ell/2} \tau^{-1/4} \|g\|_{L^2}, \quad (3.34)$$

$$\left(\int \int \left| \frac{\partial}{\partial \theta} A_{\theta}^{\tau, \ell} g(x) \right|^2 d\theta dx \right)^{1/2} \leq C \delta^{-1/2} 2^{\ell/2} \tau^{1/4} \|g\|_{L^2}. \quad (3.35)$$

From $\beta\left(2^{-\ell} \tau^{1/2} \left| \langle (-\sin \theta, \cos \theta), \frac{\xi}{|\xi|} \rangle \right| \right) \neq 0$ and $\beta\left(\frac{|\xi|}{\tau}\right) \neq 0$, we know that on the support of the symbol of the operator $A_{\theta}^{\tau, \ell}$, $\left| \langle (-\sin \theta, \cos \theta), \frac{\xi}{|\xi|} \rangle \right| \approx 2^{\ell} \tau^{-1/2}$ and $|\xi| \approx \tau$.

Since

$$\begin{aligned}
\frac{\partial}{\partial \theta} \xi \cdot \nabla E(\cos \theta, \sin \theta) &= \langle \xi, 2(-\sin \theta, \cos \theta) \rangle \begin{pmatrix} \partial_{11} E(\cos \theta, \sin \theta) & \partial_{12} E(\cos \theta, \sin \theta) \\ \partial_{21} E(\cos \theta, \sin \theta) & \partial_{22} E(\cos \theta, \sin \theta) \end{pmatrix} \\
&= \langle \xi, 2(-\sin \theta, \cos \theta) \rangle \begin{pmatrix} \frac{1}{\sin \theta} & -\frac{\cos \theta}{\sin^2 \theta} \\ -\frac{\cos \theta}{\sin^2 \theta} & \frac{\cos^2 \theta}{\sin^3 \theta} \end{pmatrix} \\
&= \frac{2}{\sin \theta} \langle \xi, (-\sin \theta, \cos \theta) \rangle \begin{pmatrix} 1 & -\frac{\cos \theta}{\sin \theta} \\ -\frac{\cos \theta}{\sin \theta} & \frac{\cos^2 \theta}{\sin^2 \theta} \end{pmatrix} \\
&= |\xi| \frac{2}{\sin^3 \theta} \langle \frac{\xi}{|\xi|}, (-\sin \theta, \cos \theta) \rangle,
\end{aligned}$$

then

$$\left| \frac{\partial}{\partial \theta} \xi \cdot \left[t \nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta) \right] \right| \leq C 2^\ell \tau^{1/2}.$$

It is easy to check that

$$\left| \frac{\partial}{\partial \theta} \beta \left(2^{-\ell} \tau^{1/2} |\langle (-\sin \theta, \cos \theta), \frac{\xi}{|\xi|} \rangle| \right) \right| \leq C 2^{-\ell} \tau^{1/2},$$

one can see that $\frac{\partial}{\partial \theta} A_\theta^{\tau, \ell}$ behaves like $2^\ell \tau^{1/2} A_\theta^{\tau, \ell}$, then we only prove inequality (3.34).

Employing Plancherel's theorem, We have that

$$\begin{aligned}
&\int \int |A_\theta^{\tau, \ell} g(x)|^2 d\theta dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}} A_\theta^{\tau, \ell} g(x) \overline{A_\theta^{\tau, \ell} g(x)} d\theta dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i\xi \cdot [x+t\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi) \beta\left(\frac{|\xi|}{\tau}\right) \beta_{\ell, \tau}(\xi, \theta) g(\hat{\xi}, t) d\xi dt \right) \\
&\quad \times \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{-i\xi' \cdot [x+t'\nabla q_\delta(\cos \theta, \sin \theta) + \nabla E(\cos \theta, \sin \theta)]} \overline{\sigma_{\bar{\delta}}(\theta, t', \xi')} \overline{\beta\left(\frac{|\xi'|}{\tau}\right)} \overline{\beta_{\ell, \tau}(\xi', \theta)} \overline{g(\hat{\xi}', t')} d\xi' dt' \right) \\
&\quad \times d\theta dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{i[(t\xi - t'\xi') \cdot \nabla q_\delta(\cos \theta, \sin \theta) + (\xi - \xi') \cdot \nabla E(\cos \theta, \sin \theta)]} \sigma_{\bar{\delta}}(\theta, t, \xi) \overline{\sigma_{\bar{\delta}}(\theta, t', \xi')} \right. \\
&\quad \left. \times \beta_{\ell, \tau}(\xi, \theta) \overline{\beta_{\ell, \tau}(\xi', \theta)} d\theta \right) \delta_0(\xi' - \xi) \beta\left(\frac{|\xi|}{\tau}\right) \overline{\beta\left(\frac{|\xi'|}{\tau}\right)} g(\hat{\xi}, t) \overline{g(\hat{\xi}', t')} d\xi dt d\xi' dt' \\
&= \int_{\mathbb{R}^2} \int_0^1 \int_0^1 \left(\int_{\mathbb{R}} e^{i[(t-t')\xi \cdot \nabla q_\delta(\cos \theta, \sin \theta)]} |\rho(\theta) \beta_{\ell, \tau}(\xi, \theta)|^2 d\theta \right) |\sigma(\bar{\delta}\xi) \beta\left(\frac{|\xi|}{\tau}\right)|^2 g(\hat{\xi}, t) \\
&\quad \times \overline{g(\hat{\xi}, t')} d\xi dt dt',
\end{aligned}$$

where δ_0 denotes the two-dimensional Dirac delta function.

Hence

$$\int \int |A_\theta^{\tau, \ell} g(x)|^2 d\theta dx = \int_{\mathbb{R}^2} \left\{ \int_0^1 \int_0^1 H^{\ell, \tau}(t, t', \xi) \left| \beta\left(\frac{|\xi|}{\tau}\right) \sigma(\bar{\delta}\xi) \right|^2 g(\hat{\xi}, t) \overline{g(\hat{\xi}, t')} dt dt' \right\} d\xi,$$

where

$$H^{\ell, \tau}(t, t', \xi) = \int_{\mathbb{R}} e^{i(t-t') \langle \xi, \nabla_{q_\delta}(\cos \theta, \sin \theta) \rangle} |\varrho(\theta) \beta_{\ell, \tau}(\xi, \theta)|^2 d\theta.$$

First we claim that for $\ell > 0$,

$$|H^{\ell, \tau}(t, t', \xi)| \leq C 2^\ell \tau^{-1/2} (1 + \delta 2^{2\ell} |t - t'|)^{-N}, \quad |\xi| \approx \tau. \quad (3.36)$$

Since $\langle (-\sin \theta, \cos \theta), \frac{\xi}{|\xi|} \rangle = \sin(\theta_\xi - \theta)$ and $|\theta_\xi - \theta| \approx 2^\ell \tau^{-1/2}$ on $\text{supp } \beta_{\ell, \tau}$, we make the change of variable $\theta = \theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)$, then

$$\begin{aligned} H^{\ell, \tau}(t, t', \xi) &= \int_{\mathbb{R}} e^{i(t-t') \langle \xi, \nabla_{q_\delta}(\cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)), \sin(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)) \rangle} \\ &\quad \times |\varrho(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)) \beta(|\omega|)|^2 \frac{2^\ell \tau^{-1/2}}{\sqrt{1 - 2^{2\ell} \tau^{-1} \omega^2}} d\omega. \end{aligned}$$

In a similar way as (3.33) we have that

$$\begin{aligned} &\frac{\partial}{\partial \omega} \langle \xi, \nabla_{q_\delta}(\cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)), \sin(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))) \rangle \\ &= \frac{62^\ell \tau^{-1/2}}{\sqrt{1 - 2^{2\ell} \tau^{-1} \omega^2}} \delta |\xi| \frac{\cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))}{\sin^4(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))} \\ &\quad \times \left\langle \left(-\sin(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)), \cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)) \right), \frac{\xi}{|\xi|} \right\rangle \\ &= \frac{62^\ell \tau^{-1/2} \delta |\xi| \cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))}{\sqrt{1 - 2^{2\ell} \tau^{-1} \omega^2} \sin^4(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))} \langle \sin(\theta_\xi - (\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))) \rangle \\ &= \frac{-62^{2\ell} \tau^{-1} \delta |\xi| \omega \cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))}{\sqrt{1 - 2^{2\ell} \tau^{-1} \omega^2} \sin^4(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))}. \end{aligned}$$

Since the support of ρ and β gives that $\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega) \approx 1$ and $|\omega| \approx 1$, in addition to $|\xi| \approx \tau$ and $\ell \leq C \log_2 \tau^{1/2}$, then

$$\frac{\partial}{\partial \omega} \langle \xi, \nabla_{q_\delta}(\cos(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega)), \sin(\theta_\xi + \arcsin(2^\ell \tau^{-1/2} \omega))) \rangle \approx 2^{2\ell} \delta.$$

The claim (3.36) will follow from integration by parts. Based on the above claim,

Hölder's inequality and Plancherel's theorem, we have

$$\begin{aligned}
& \int \int |A_\theta^{\tau, \ell} g(y)|^2 d\theta dy \\
& \leq C_N 2^\ell \tau^{-1/2} \int_{\mathbb{R}^2} \left\{ \int_0^1 \int_0^1 \frac{\overline{|g(\hat{\xi}, t')|}}{(1 + \delta 2^{2\ell} |t - t'|)^N} dt' |g(\hat{\xi}, t)| dt \right\} |\beta(\frac{|\xi|}{\tau}) \sigma(\bar{\delta}\xi)|^2 d\xi \\
& \leq C_N 2^\ell \tau^{-1/2} \int_{\mathbb{R}^2} \left\{ \int_0^1 \overline{|g(\hat{\xi}, \cdot)|} * (1 + \delta 2^{2\ell} |\cdot|)^{-N}(t) |g(\hat{\xi}, t)| dt \right\} |\beta(\frac{|\xi|}{\tau}) \sigma(\bar{\delta}\xi)|^2 d\xi \\
& \leq C_N 2^\ell \tau^{-1/2} \int_{\mathbb{R}^2} \left\| \overline{|g(\hat{\xi}, \cdot)|} * (1 + \delta 2^{2\ell} |\cdot|)^{-N} \right\|_{(L^2, dt)} \|g(\hat{\xi}, t)\|_{(L^2, dt)} |\beta(\frac{|\xi|}{\tau}) \sigma(\bar{\delta}\xi)|^2 d\xi \\
& \leq C_N 2^{-\ell} \tau^{-1/2} \delta^{-1} \|g\|_{L^2}^2,
\end{aligned}$$

and inequality (3.34) has been proved for $\ell > 0$.

For $\ell = 0$, the proof is simpler. We still make the change of variable $\theta \rightarrow \theta_\xi + \arcsin(\tau^{-1/2}\omega)$, then the support of the symbol of the operator $H^{0, \tau}$ gives that 0 is the non-degenerate critical point, since

$$\begin{aligned}
& \frac{\partial^2}{\partial \omega^2} \langle \xi, \nabla q_\delta(\cos(\theta_\xi + \arcsin(\tau^{-1/2}\omega)), \sin(\theta_\xi + \arcsin(\tau^{-1/2}\omega))) \rangle \Big|_{\omega=0} \\
& = -6\delta |\xi| \tau^{-1} \frac{\cos \theta_\xi}{\sin^4 \theta_\xi} \approx \delta.
\end{aligned}$$

Finally, integration by parts implies that

$$|H^{0, \tau}(t, t', \xi)| \leq C \tau^{-1/2} (1 + \delta |t - t'|)^{-1/2}. \quad (3.37)$$

We still use Hölder's inequality and Plancherel's theorem to get inequality (3.34) for $\ell = 0$. \square

Chapter 4

More general Fourier integral operators not satisfying the cinematic curvature condition uniformly

The purpose of this chapter is to prove Theorem 2.3.3. The main idea follows from the proof of Theorem 6.1 given in [26]. In fact, the proof has a similar structure with Chapter 3. Instead of repeating the proof here, we give a brief overview, and only a detailed argument partly. In this chapter, we still assume $\delta\lambda > 1$ and the ξ -support of the symbol a is in the first quadrant.

Since $t \approx 1$, we can replace t by $1/t$ in (2.65). By (2.38), we write

$$-t^{-2}\xi_2\Phi(s, \tilde{q}(s, \delta), \delta) := E(\xi) + q_\delta(\xi, t), \quad s = s(\xi, t^{-1}),$$

where

$$E(\xi) = \frac{\xi_1^2}{2\xi_2}, \quad q_\delta(\xi, t) = t\delta\phi'(0)\frac{\xi_1^3}{\xi_2^2} + \delta^2R(t, \delta, \xi), \quad (4.1)$$

by abuse of notation, we have written $R(t, \delta, \xi)$ in place of $R(t^{-1}, \delta, \xi)$ in (4.1).

Then

$$\tilde{F}_\lambda f(y, t) := \tilde{F}_\lambda^\delta f(y, t) = \int_{\mathbb{R}^2} e^{i(\xi \cdot y + E(\xi) + q_\delta(\xi, t))} \tilde{a}_\lambda(y, t, \xi) \hat{f}(\xi) d\xi, \quad (4.2)$$

where $\tilde{a}_\lambda(y, t, \xi) = \rho_1(y, t)a(\xi, t)\rho_0(\frac{\delta}{\lambda}|\xi|)\tilde{\chi}(\frac{\xi_1}{\xi_2})$, which is a symbol of order zero in ξ .

The support of $\tilde{\chi}$ implies that we are working in a fixed, small conic region $c_1 \leq |\xi_1/\xi_2| \leq c_2$. Using the angular decomposition appeared in Chapter 3, we can write

$\tilde{F}_\lambda = \sum_\nu F_\lambda^\nu$, where the sum runs only over $(\frac{\pi}{2} - c_2)\sqrt{\frac{\lambda}{\delta}} \lesssim \nu \lesssim (\frac{\pi}{2} - c_1)\sqrt{\frac{\lambda}{\delta}}$ and

$$\tilde{F}_\lambda^\nu f(y, t) := \int_{\mathbb{R}^2} e^{i(\xi \cdot y + E(\xi) + q_\delta(\xi, t))} \tilde{a}_\lambda(y, t, \xi) \chi_\nu(\xi) \hat{f}(\xi) d\xi. \quad (4.3)$$

By Minkowski's inequality, we have

$$\begin{aligned} \|\tilde{F}_\lambda f\|_{L^4}^2 &= \left\| \sum_{\nu, \mu} \tilde{F}_\lambda^\nu f \tilde{F}_\lambda^\mu f \right\|_{L^2} \\ &= \left\| \sum_\ell \sum_{|\nu - \mu| \approx 2^\ell} \tilde{F}_\lambda^\nu f \tilde{F}_\lambda^\mu f \right\|_{L^2} \\ &\leq C \sum_\ell \left\| \sum_{|\nu - \mu| \approx 2^\ell, \nu \leq \mu} \tilde{F}_\lambda^\nu f \tilde{F}_\lambda^\mu f \right\|_{L^2}, \end{aligned}$$

where $2^\ell \leq (c_2 - c_1)\sqrt{\frac{\lambda}{\delta}}$.

Now we make a further decomposition so that the symbol becomes

$$\tilde{a}_{\lambda, \ell}^{\nu, j}(y, t, \xi) = \tilde{a}_\lambda(y, t, \xi) \chi_\nu(\xi) \varphi(2^\ell \lambda^{-1} q'_\delta(\xi, t) - j), \quad (4.4)$$

where φ is defined as in Chapter 3 and $q'_\delta(\xi, t) = \partial_t q_\delta(\xi, t)$. Since $|\xi| \approx \frac{\lambda}{\delta}$, then we have $\tilde{F}_\lambda^\nu = \sum_{j \approx 2^\ell} \tilde{F}_{\lambda, \ell}^{\nu, j}$, where

$$\tilde{F}_{\lambda, \ell}^{\nu, j} f(y, t) = \int_{\mathbb{R}^2} e^{i(\xi \cdot y + E(\xi) + q_\delta(\xi, t))} \tilde{a}_{\lambda, \ell}^{\nu, j}(y, t, \xi) \hat{f}(\xi) d\xi.$$

Since $c_1 \leq |\xi_1/\xi_2| \leq c_2$ and $q'_\delta(\xi, t) = \delta \phi'(0) \frac{\xi_1^3}{\xi_2^2} + \delta^2 \partial_t R(t, \delta, \xi)$, then $2^{-\ell} \lambda(j-1) \lesssim \delta |\xi| \lesssim 2^{-\ell} \lambda(j+1)$. The support of χ_ν implies that the ξ -support of the symbol is comparable to a $2^{-\ell} \frac{\lambda}{\delta} \times (\frac{\lambda}{\delta})^{1/2}$ rectangle. Meanwhile, by (3.9) and (4.1), we notice that

$$|\partial_\xi^\alpha \partial_{y, t}^\beta \tilde{a}_{\lambda, \ell}^{\nu, j}(y, t, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha|/2} (\delta \lambda)^{|\beta|/2}. \quad (4.5)$$

Now we use two almost orthogonal lemmas. One of them is as follows

Lemma 4.0.9. *Suppose that $|j + k - j' - k'| \geq \lambda^\varepsilon$. Then for any $N > 0$,*

$$\left| \int_{\mathbb{R}^3} \tilde{F}_{\lambda, \ell}^{\nu, j} f \tilde{F}_{\lambda, \ell}^{\mu, k} f \overline{\tilde{F}_{\lambda, \ell}^{\nu', j'} f \tilde{F}_{\lambda, \ell}^{\mu', k'}} f dy dt \right| \leq C_{\varepsilon, N} \lambda^{-N} \|f\|_{L^4}^4. \quad (4.6)$$

Proof.

$$\int_{\mathbb{R}^3} \tilde{F}_{\lambda, \ell}^{\nu, j} f \tilde{F}_{\lambda, \ell}^{\mu, k} f \overline{\tilde{F}_{\lambda, \ell}^{\nu', j'} f \tilde{F}_{\lambda, \ell}^{\mu', k'}} f dy dt = \int H_{j, k, j', k'}^{\nu, \mu, \nu', \mu'}(\eta, \xi, \eta', \xi') \hat{f}(\eta) \hat{f}(\xi) \overline{\hat{f}(\eta') \hat{f}(\xi')} d\eta d\xi d\eta' d\xi', \quad (4.7)$$

where

$$H_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(\eta, \xi, \eta', \xi') = \int_{\mathbb{R}^3} e^{i\Psi(y,t,\eta,\xi,\eta',\xi')} b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(y, t, \eta, \xi, \eta', \xi') dy dt, \quad (4.8)$$

$$\begin{aligned} \Psi(y, t, \eta, \xi, \eta', \xi') &= y \cdot (\eta + \xi - \eta' - \xi') + \frac{1}{2} \left(\frac{\eta_1^2}{\eta_2} + \frac{\xi_1^2}{\xi_2} - \frac{\eta_1^2}{\eta_2'} - \frac{\xi_1^2}{\xi_2'} \right) + F(t, \eta, \xi, \eta', \xi'), \\ F(t, \eta, \xi, \eta', \xi') &= q_\delta(\eta, t) + q_\delta(\xi, t) - q_\delta(\eta', t) - q_\delta(\xi', t) \text{ and} \end{aligned}$$

$$b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(y, t, \eta, \xi, \eta', \xi') = \tilde{a}_{\lambda,\ell}^{\nu,j}(y, t, \eta) \tilde{a}_{\lambda,\ell}^{\mu,k}(y, t, \xi) \overline{\tilde{a}_{\lambda,\ell}^{\nu',j'}(y, t, \eta')} \tilde{a}_{\lambda,\ell}^{\mu',k'}(y, t, \xi'). \quad (4.9)$$

Let $\mathcal{L}g = \frac{\partial}{\partial t}(\frac{g}{\partial_t \Psi})$. By integration by parts, we have

$$|H_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(\eta, \xi, \eta', \xi')| \leq \int_{\mathbb{R}^3} \left| \mathcal{L}^N b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(y, t, \eta, \xi, \eta', \xi') \right| dy dt. \quad (4.10)$$

Since $\partial_t \Psi = \partial_t F$, $\mathcal{L}g = \frac{\partial}{\partial t}(\frac{g}{\partial_t F})$, then the inner integral can be written as the sum of expressions of the form

$$\frac{\prod_{i=0}^{N_2} (\Psi_i(t, \eta) + \Psi_i(t, \xi) - \Psi_i(t, \eta') - \Psi_i(t, \xi'))}{(\partial_t F)^{N+N_1}} \partial_t^\beta b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(y, t, \eta, \xi, \eta', \xi'), \quad (4.11)$$

where $0 \leq N_1 \leq N$, $0 \leq N_2 \leq N_1$, $\beta \leq N - N_2$, and Ψ_i are partial derivative of $\partial_t F$ with respect to t -variable. Note that Ψ_i are still homogeneous of degree one in ξ .

Now from the estimate (4.5) and the support of φ , it is easy to see that

$$|\partial_t F(t, \eta, \xi, \eta', \xi')| \geq C 2^{-\ell} \lambda |j + k - j' - k'| \geq C(\delta \lambda)^{1/2} |j + k - j' - k'|.$$

Since $|\Psi_i(t, \cdot)| \leq |\partial_t^{2+\alpha} q_\delta(\cdot, t)| \leq \delta^2 \frac{\lambda}{\delta} = \delta \lambda$ for all $\alpha \geq 0$, then

$$\left| \prod_{i=0}^{N_2} (\Psi_i(t, \eta) + \Psi_i(t, \xi) - \Psi_i(t, \eta') - \Psi_i(t, \xi')) \right| \leq C(\delta \lambda)^{N_2+1},$$

in addition to the fact that

$$|\partial_t b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}| \leq 2^\ell \lambda^{-1} \delta^2 \frac{\lambda}{\delta} = 2^\ell \delta \leq \left(\frac{\lambda}{\delta}\right)^{1/2} \delta = (\delta \lambda)^{1/2}, \quad (4.12)$$

we have

$$\left| H_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(y, \eta, \xi, \eta', \xi') \right| \leq C \frac{(\delta \lambda)^{N_2+1} (\delta \lambda)^{(N-N_2)/2}}{((\delta \lambda)^{1/2} |j + k - j' - k'|)^{N+N_1}} \leq C \lambda^{-\varepsilon N} (\delta \lambda),$$

provided that $|j + k - j' - k'| \geq \lambda^\varepsilon$. Since $b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}$ is supported on a set of measure $2^{-4\ell} (\frac{\lambda}{\delta})^6$ and $\delta^{-1} < \lambda$, then the integral is bounded by $C_{\varepsilon, N} \lambda^{-N} \|f\|_{L^1}^4$, in addition to the compact support of f , we get the desired result. \square

The above lemma implies that

$$\begin{aligned}
& \int \left| \sum_{|\nu-\mu|\approx 2^\ell} \tilde{F}_\lambda^\nu f \tilde{F}_\lambda^\mu f \right|^2 dy dt \\
& \leq \int \left| \sum_{|\nu-\mu|\approx 2^\ell} \sum_{j,k\leq 2^\ell} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 dy dt \\
& = \int \left| \sum_{s\leq 2^{\ell+1}} \sum_{j+k=s} \sum_{|\nu-\mu|\approx 2^\ell} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 dy dt \\
& = \int \left| \sum_{\substack{s\leq 2^{\ell+1} \\ s'\leq 2^{\ell+1}}} \sum_{\substack{j+k=s \\ j'+k'=s'}} \sum_{\substack{|\nu-\mu|\approx 2^\ell \\ |\nu'-\mu'|\approx 2^\ell}} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j'} f \tilde{F}_{\lambda,\ell}^{\mu',k'} f}} \right| dy dt \\
& \leq \int \left| \sum_{s,s',|s-s'|\leq \lambda^\varepsilon} \sum_{\substack{j+k=s \\ j'+k'=s'}} \sum_{\substack{|\nu-\mu|\approx 2^\ell \\ |\nu'-\mu'|\approx 2^\ell}} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j'} f \tilde{F}_{\lambda,\ell}^{\mu',k'} f}} \right| dy dt + C\lambda^{-N} \|f\|_{L^4}^4.
\end{aligned} \tag{4.13}$$

By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
& \int \left| \sum_{s,s',|s-s'|\leq \lambda^\varepsilon} \sum_{\substack{j+k=s \\ j'+k'=s'}} \sum_{\substack{|\nu-\mu|\approx 2^\ell \\ |\nu'-\mu'|\approx 2^\ell}} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j'} f \tilde{F}_{\lambda,\ell}^{\mu',k'} f}} \right| dy dt \\
& \leq \lambda^\varepsilon \sum_{s\leq 2^{\ell+1}} \int \left| \sum_{\substack{j+k=s \\ j'+k'=s}} \sum_{\substack{|\nu-\mu|\approx 2^\ell \\ |\nu'-\mu'|\approx 2^\ell}} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j'} f \tilde{F}_{\lambda,\ell}^{\mu',k'} f}} \right| dy dt \\
& = \lambda^\varepsilon \sum_{s\leq 2^{\ell+1}} \int \left| \sum_{j+k=s} \sum_{|\nu-\mu|\approx 2^\ell} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 dy dt \\
& \leq C\lambda^{\varepsilon 2^{\ell+1}} \sum_{s\leq 2^{\ell+1}} \int \sum_{j+k=s} \left| \sum_{|\nu-\mu|\approx 2^\ell} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 dy dt \\
& \leq C\lambda^{\varepsilon 2^{\ell+1}} \left\| \left(\sum_{j,k} \left| \sum_{|\nu-\mu|\approx 2^\ell} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 \right)^{1/2} \right\|_{L^2}^2.
\end{aligned} \tag{4.14}$$

Now we use another almost orthogonality lemma to bring the sum in μ and ν outside of the square function.

Lemma 4.0.10. *Suppose that $|\nu - \mu| \approx 2^\ell$, $|\nu' - \mu'| \approx 2^\ell$, and $|\nu - \nu'| + |\mu - \mu'| \geq \lambda^\varepsilon$. Then for any $N > 0$,*

$$\left| \int_{\mathbb{R}^3} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j'} f \tilde{F}_{\lambda,\ell}^{\mu',k'} f}} dy dt \right| \leq C_{\varepsilon,N} \lambda^{-N} \|f\|_{L^4}^4. \tag{4.15}$$

Proof. First we will introduce Lemma 6.8 in [26] as follows.

Lemma 4.0.11. [26] *Let $\tilde{\Psi} \in C^4(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree one. Let $\varsigma < \pi/4$, $a_0 < 1/4$, $A_0 \geq 1$. Let \mathcal{S}_λ be the intersection of a sector which subtends an angle of size ς with the annulus $\{\eta : (1-a_0)\lambda \leq |\eta| \leq (1+a_0)\lambda\}$. Let $h \in C^1(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree one such that $b_0|\eta| \leq h(\eta) \leq b_1|\eta|$, $|\nabla_\eta h(\eta)| \leq b_2$ for some positive constants b_0, b_1, b_2 .*

Suppose that $a_0^{-1} \leq 2^n$, $n \leq \ell$, $2^\ell \leq \varsigma\lambda^{1/2}$, and that $\eta, \xi, \eta', \xi' \in \mathcal{S}_\lambda$ are chosen such that for given integers ν, μ, ν', μ'

- (1) $|\arg(\eta) - \nu\lambda^{-1/2}| \leq \lambda^{-1/2}$; $|\arg(\eta') - \nu'\lambda^{-1/2}| \leq \lambda^{-1/2}$;
- (2) $|\arg(\eta) - \nu\lambda^{-1/2}| \leq \lambda^{-1/2}$; $|\arg(\eta') - \nu'\lambda^{-1/2}| \leq \lambda^{-1/2}$;
- (3) $2^{\ell-1}\lambda^{-1/2} \leq \max\{|\arg(\eta) - \arg(\xi)|, |\arg(\eta') - \arg(\xi')|\} \leq 2^{\ell+1}\lambda^{-1/2}$;
- (4) $|h(\eta) - h(\eta')| \leq 2^{-n}\lambda$;
- (5) $|h(\xi) - h(\xi')| \leq 2^{-n}\lambda$.

Then one can choose ς, a_0 sufficiently small, and A_0 sufficiently large (only depending on $\tilde{\Psi}, b_0, b_1, b_2$) such that for all ν, μ, ν', μ' with $A_0 2^{\ell-n} \leq |\nu - \nu'| + |\mu - \mu'| \leq \varsigma\lambda^{1/2}$

$$\begin{aligned} & |\tilde{\Psi}(\eta) + \tilde{\Psi}(\xi) - \tilde{\Psi}(\eta') - \tilde{\Psi}(\xi')| \\ & \leq C[(2^\ell|\nu - \nu'| + |\nu - \nu'|^2) + (2^\ell|\mu - \mu'| + |\mu - \mu'|^2) + |\eta + \xi - \eta' - \xi'|]. \end{aligned} \quad (4.16)$$

Suppose now that $\tilde{\Psi}$ satisfies the additional assumption $\text{rank } \tilde{\Psi}''_{\eta\eta} = 1$. Then if either $\mu \leq \nu$ and $\mu' \leq \nu'$ or $\nu \leq \mu$ and $\nu' \leq \mu'$ and if $A_0 2^{\ell-n} \leq |\nu - \nu'| + |\mu - \mu'| \leq \varsigma\lambda^{1/2}$ we have also with suitable positive constants c_0, C_0

$$\begin{aligned} & |\tilde{\Psi}(\eta) + \tilde{\Psi}(\xi) - \tilde{\Psi}(\eta') - \tilde{\Psi}(\xi')| \\ & \geq c_0[(2^\ell|\nu - \nu'| + |\nu - \nu'|^2) + (2^\ell|\mu - \mu'| + |\mu - \mu'|^2)] - C_0|\eta + \xi - \eta' - \xi'|. \end{aligned} \quad (4.17)$$

Now in order to apply the above lemma to evaluate (4.11), we rewrite (4.11) as $\frac{\delta^{2(N_2+1)}}{\delta^{N+N_1}}$ times

$$\frac{\prod_{i=0}^{N_2} (\Psi_i(t, \eta)/\delta^2 + \Psi_i(t, \xi)/\delta^2 - \Psi_i(t, \eta')/\delta^2 - \Psi_i(t, \xi')/\delta^2)}{(\partial_t F/\delta)^{N+N_1}} \partial_t^\beta b_{j,k,j',k'}^{\nu,\mu,\nu',\mu'}(y, t, \eta, \xi, \eta', \xi'). \quad (4.18)$$

The worst case to bound the numerator of (4.18) from above is that we apply inequality (4.16) in the above lemma with $\tilde{\Psi} = \frac{1}{\delta^2} \partial_t^2 q_\delta(\cdot, t)$, at the same time, we will replace ς by $c_2 - c_1$, $h(\eta)$ by $\frac{1}{\delta^2} \partial_t^2 q_\delta(\eta, t)$ and λ by λ/δ in the above lemma. In order to bound the denominator of (4.18) from below, we apply inequality (4.17) in the above lemma with $\tilde{\Psi} = \frac{1}{\delta} q'_\delta$, in this case, we will replace ς by $c_2 - c_1$, $h(\eta)$ by $\frac{1}{\delta} q'_\delta(\eta, t)$ and λ by λ/δ in

the above lemma. Note that $\partial_y \Psi(y, t, \eta, \xi, \eta', \xi') = \eta + \xi - \eta' - \xi'$, by (4.12), $\frac{\delta^{2(N_2+1)}}{\delta^{N+N_1}} \times$ (4.18) can be controlled by

$$\frac{C\delta^{N_2+1}(\delta\lambda)^{(N-N_2)/2} [\delta ((2^\ell|\nu - \nu'| + |\nu - \nu'|^2) + (2^\ell|\mu - \mu'| + |\mu - \mu'|^2) + |\partial_y \Psi|)]^{N_2+1}}{\left(\delta \max \{|\partial_y \Psi|, (2^\ell|\nu - \nu'| + |\nu - \nu'|^2) + (2^\ell|\mu - \mu'| + |\mu - \mu'|^2) - |\partial_y \Psi|\} \right)^{N+N_1}} \quad (4.19)$$

If $|\nu - \nu'| + |\mu - \mu'| \geq \lambda^\epsilon$, together with the fact that $(\frac{\pi}{2} - c_2)\sqrt{\frac{\lambda}{\delta}} \lesssim \nu \lesssim (\frac{\pi}{2} - c_1)\sqrt{\frac{\lambda}{\delta}}$, then

$$\begin{aligned} & \frac{1}{\delta ((2^\ell|\nu - \nu'| + |\nu - \nu'|^2) + (2^\ell|\mu - \mu'| + |\mu - \mu'|^2))} \\ & \leq C\delta^{-1} \left(2^\ell \lambda^\epsilon + \sqrt{\frac{\lambda}{\delta}} \lambda^\epsilon \right)^{-1} \\ & \leq C(\delta\lambda)^{-1/2} \lambda^{-\epsilon}. \end{aligned}$$

Since $\delta\lambda > 1$, (4.19) can be controlled by $\mathcal{O}\left(\frac{\delta^{N_2+1}(\delta\lambda)^{(N-N_2)/2}}{(\delta\lambda)^{(N-1)/2}\lambda^\epsilon(N-1)}\right) = \mathcal{O}\left(\frac{1}{\lambda^\epsilon(N-1)-1/2}\right)$. Now if we choose N sufficiently large, the similar argument with Lemma 4.0.9 gives the desired result. \square

Let us continue the proof of our theorem. Based on Lemma 4.0.10 and the Cauchy-Schwarz inequality, then we have that

$$\begin{aligned} & \left\| \left(\sum_{j,k} \left| \sum_{|\nu-\mu| \approx 2^\ell} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 \right)^{1/2} \right\|_{L^2}^2 \\ & = \int \sum_{j,k} \left| \sum_{\substack{|\nu-\mu| \approx 2^\ell \\ |\nu'-\mu'| \approx 2^\ell}} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j} f \tilde{F}_{\lambda,\ell}^{\mu',k} f} \right| dy dt \\ & \leq \int \sum_{j,k} \left| \sum_{\substack{|\nu-\mu|, |\nu'-\mu'| \approx 2^\ell \\ |\nu-\nu'| + |\mu-\mu'| \leq \lambda^\epsilon}} \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \overline{\tilde{F}_{\lambda,\ell}^{\nu',j} f \tilde{F}_{\lambda,\ell}^{\mu',k} f} \right| dy dt + C\lambda^{-N} \|f\|_{L^4}^4 \\ & \leq \lambda^{2\epsilon} \int \sum_{j,k} \sum_{|\nu-\mu| \approx 2^\ell} \left| \tilde{F}_{\lambda,\ell}^{\nu,j} f \tilde{F}_{\lambda,\ell}^{\mu,k} f \right|^2 dy dt + C\lambda^{-N} \|f\|_{L^4}^4 \\ & \leq \lambda^{2\epsilon} \left\| \left(\sum_{j,\nu} |\tilde{F}_{\lambda,\ell}^{\nu,j} f|^2 \right)^{1/2} \right\|_{L^4}^4 + C\lambda^{-N} \|f\|_{L^4}^4. \end{aligned}$$

At present, we put all estimates together and get

$$\|\tilde{F}_\lambda f\|_{L^4}^2 \leq C\lambda^\epsilon \sum_\ell 2^{\ell/2} \left\| \left(\sum_{j,\nu} |\tilde{F}_{\lambda,\ell}^{\nu,j} f|^2 \right)^{1/2} \right\|_{L^4}^2 + C\lambda^{-N} \|f\|_{L^4}^2.$$

Furthermore, we introduce a finer decomposition of the operators $\tilde{F}_{\lambda,\ell}^{\nu,j}$, namely, we set

$$\tilde{F}_{\lambda,\ell}^{\nu,j,n} f(y, t) := \int_{\mathbb{R}^2} e^{i(\xi \cdot y + E(\xi) + q_\delta(\xi, t))} \tilde{a}_{\lambda,\ell}^{\nu,j,n}(y, t, \xi) \hat{f}(\xi) d\xi, \quad (4.20)$$

where

$$\tilde{a}_{\lambda,\ell}^{\nu,j,n}(y, t, \xi) = \varphi(\lambda^{-1/2} q'_\delta(\xi, t) - n) \tilde{a}_{\lambda,\ell}^{\nu,j}(y, t, \xi). \quad (4.21)$$

For fixed y, t , $\text{supp } \xi \rightarrow \tilde{a}_{\lambda,\ell}^{\nu,j,n}$ is comparable to a $\frac{\lambda^{1/2}}{\delta} \times (\frac{\lambda}{\delta})^{1/2}$ rectangle and $\tilde{F}_{\lambda,\ell}^{\nu,j} = \sum_n \tilde{F}_{\lambda,\ell}^{\nu,j,n}$, where the sum involves less than $2^{-\ell} \lambda^{1/2}$. Since on the support of the symbols $\tilde{a}_{\lambda,\ell}^{\nu,j,n}$, $\lambda^{1/2}(n-1) \leq q'_\delta(\xi, t) \leq \lambda^{1/2}(n+1)$ and on the support of the symbols $\tilde{a}_{\lambda,\ell}^{\nu,j}$, $2^{-\ell} \lambda(j-1) \leq q'_\delta(\xi, t) \leq 2^{-\ell} \lambda(j+1)$, for fixed j , then which implies that if n satisfies $|n - n(j)| \leq 2^{-\ell} \lambda^{1/2}$ with $n(j) = [2^{-\ell} \lambda^{1/2} j]$, then $\tilde{F}_{\lambda,\ell}^{\nu,j,n} \neq 0$, where $[\cdot]$ denotes the nearest integer function.

One can follow the proof of Lemma 4.0.9 and get an almost orthogonality lemma similar to Lemma 4.0.9:

Lemma 4.0.12. *Suppose that $|m + n - m' - n'| \geq \lambda^\epsilon$. Then for any $N > 0$,*

$$\left| \int_{\mathbb{R}^3} \tilde{F}_{\lambda,\ell}^{\nu,j,m} f \tilde{F}_{\lambda,\ell}^{\mu,k,n} f \overline{\tilde{F}_{\lambda,\ell}^{\nu,j,m'} f \tilde{F}_{\lambda,\ell}^{\mu,k,n'} f} dy dt \right| \leq C_{\epsilon,N} \lambda^{-N} \|f\|_{L^4}^4. \quad (4.22)$$

Using this lemma and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \left\| \left(\sum_{j,\nu} |\tilde{F}_{\lambda,\ell}^{\nu,j} f|^2 \right)^{1/2} \right\|_{L^4}^4 \\ &= \sum_{\substack{\nu,\mu \\ j,k}} \int \left| \sum_{m,n} \tilde{F}_{\lambda,\ell}^{\nu,j,m} f \tilde{F}_{\lambda,\ell}^{\mu,k,n} f \right|^2 dy dt \\ &= \sum_{\substack{\nu,\mu \\ j,k}} \int \left| \sum_{\substack{s \leq 2^{-\ell} \lambda^{1/2} \\ s' \leq 2^{-\ell} \lambda^{1/2}}} \sum_{\substack{m-m'=s \\ n'-n=s'}} \tilde{F}_{\lambda,\ell}^{\nu,j,m} f \tilde{F}_{\lambda,\ell}^{\mu,k,n} f \overline{\tilde{F}_{\lambda,\ell}^{\nu,j,m'} f \tilde{F}_{\lambda,\ell}^{\mu,k,n'} f} \right| dy dt \\ &\leq \sum_{\substack{\nu,\mu \\ j,k}} \int \left| \sum_{\substack{s \leq 2^{-\ell} \lambda^{1/2} \\ s' \leq 2^{-\ell} \lambda^{1/2} \\ |s-s'| \leq \lambda^\epsilon}} \sum_{\substack{m-m'=s \\ n'-n=s'}} \tilde{F}_{\lambda,\ell}^{\nu,j,m} f \tilde{F}_{\lambda,\ell}^{\mu,k,n} f \overline{\tilde{F}_{\lambda,\ell}^{\nu,j,m'} f \tilde{F}_{\lambda,\ell}^{\mu,k,n'} f} \right| dy dt \\ &+ C_N \lambda^{-N} \|f\|_{L^4}^4 \end{aligned}$$

$$\begin{aligned}
&\leq \lambda^\epsilon \sum_{\substack{\nu, \mu \\ j, k}} \sum_{s \leq 2^{-\ell} \lambda^{1/2}} \int \left| \sum_{\substack{m-m'=s \\ n'-n=s}} \tilde{F}_{\lambda, \ell}^{\nu, j, m} f \tilde{F}_{\lambda, \ell}^{\mu, k, n} f \overline{\tilde{F}_{\lambda, \ell}^{\nu, j, m'} f \tilde{F}_{\lambda, \ell}^{\mu, k, n'} f} \right| dy dt \\
&= \lambda^\epsilon \sum_{\substack{\nu, \mu \\ j, k}} \sum_{s \leq 2^{-\ell} \lambda^{1/2}} \int \left| \sum_{m-m'=s} \tilde{F}_{\lambda, \ell}^{\nu, j, m} f \overline{\tilde{F}_{\lambda, \ell}^{\nu, j, m'} f} \sum_{n-n'=s} \overline{\tilde{F}_{\lambda, \ell}^{\mu, k, n} f \tilde{F}_{\lambda, \ell}^{\mu, k, n'} f} \right| dy dt \\
&\leq \lambda^\epsilon \sum_{\substack{\nu, \mu \\ j, k}} \int \left(\sum_{s \leq 2^{-\ell} \lambda^{1/2}} \left| \sum_{m-m'=s} \tilde{F}_{\lambda, \ell}^{\nu, j, m} f \overline{\tilde{F}_{\lambda, \ell}^{\nu, j, m'} f} \right|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{s \leq 2^{-\ell} \lambda^{1/2}} \left| \sum_{n-n'=s} \tilde{F}_{\lambda, \ell}^{\mu, k, n} f \overline{\tilde{F}_{\lambda, \ell}^{\mu, k, n'} f} \right|^2 \right)^{1/2} dy dt \\
&= \lambda^\epsilon \int \left(\sum_{j, \nu} \left(\sum_{s \leq 2^{-\ell} \lambda^{1/2}} \left| \sum_{n-n'=s} \tilde{F}_{\lambda, \ell}^{\nu, j, n} f \overline{\tilde{F}_{\lambda, \ell}^{\nu, j, n'} f} \right|^2 \right)^{1/2} \right)^2 dy dt \\
&\leq C \lambda^\epsilon 2^{-\ell} \lambda^{1/2} \int \left(\sum_{j, \nu} \left(\sum_{s \leq 2^{-\ell} \lambda^{1/2}} \sum_{n-n'=s} \left| \tilde{F}_{\lambda, \ell}^{j, \nu, n} f \overline{\tilde{F}_{\lambda, \ell}^{\nu, j, n'} f} \right|^2 \right)^{1/2} \right)^2 dy dt \\
&\leq C \lambda^\epsilon 2^{-\ell} \lambda^{1/2} \int \left(\sum_{j, \nu} \left(\sum_n |\tilde{F}_{\lambda, \ell}^{\nu, j, n} f|^4 \right)^{1/2} \right)^2 dy dt \\
&\leq C \lambda^\epsilon 2^{-\ell} \lambda^{1/2} \left\| \left(\sum_{j, \nu, n} |\tilde{F}_{\lambda, \ell}^{\nu, j, n} f|^2 \right)^{1/2} \right\|_{L^4}^4.
\end{aligned}$$

We would be done if we could prove that

$$\left\| \left(\sum_{j, \nu, n} |\tilde{F}_{\lambda, \ell}^{\nu, j, n} f|^2 \right)^{1/2} \right\|_{L^4} \leq C \delta^{-(1/2+\epsilon_2)} \lambda^{\epsilon_1} \|f\|_{L^4}. \quad (4.23)$$

Now comparing expressions of $\tilde{F}_{\lambda, \ell}^{\nu, j, n} f$ and $\mathcal{F}_{\lambda, \nu}^\delta f_n$ which appeared in Chapter 3, we see that they have a similar support in ξ , see Figure 8. Since all estimates after (3.0.6) in Chapter 3 are valid under small smooth perturbations q'_δ , then from now on, we can combine the idea from Chapter 3 and the proof of Theorem 6.1 in [26], the L^1 boundness on the kernel of the remaining operator, Carleson's square function estimate, and a Kakeya-type maximal estimate yield the factor $\delta^{-(1/2+\epsilon_2)} \lambda^{\epsilon_1}$. We thus finish the proof for the more general phase function.

Chapter 5

Maximal functions associated with
nonisotropic dilations of
some classes of hypersurfaces in \mathbb{R}^3

5.1 Statement of the main results

The purpose of this chapter is to study the L^p -boundedness of certain maximal functions associated with nonisotropic dilations of surfaces in \mathbb{R}^3 . We first recall some results proved by Greenleaf in [10]. Let S be a smooth hypersurface in \mathbb{R}^n with a smooth measure $d\mu$ compactly supported away from the boundary. Given an n -tuple (a_1, a_2, \dots, a_n) of nonnegative real numbers, consider the maximal operator \mathcal{M} defined by

$$\mathcal{M}f(x) = \sup_{t>0} \left| \int_S f(x - \delta_t(y)) d\mu(y) \right|,$$

where δ_t denotes the nonisotropic dilation given by $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n)$. Supposing that at each point of $\text{supp } d\mu$, S has everywhere non-vanishing Gaussian curvature and in addition S is starshaped with respect to the origin, Greenleaf [10] showed that \mathcal{M} is bounded on $L^p(\mathbb{R}^n)$ if $n \geq 3$ and $p > n/(n-1)$. Iosevich and Sawyer [14] observed that \mathcal{M} often behaves much better than the maximal function with standard dilations due to a "rotational curvature" in the time parameter t . Let \mathcal{H} be a hyperplane and $d(x, \mathcal{H})$ denote the distance from x on S to \mathcal{H} . They showed that in the case $a_1 = a_2 = \dots = a_{n-1} \neq a_n$, the condition $d(x, \mathcal{H})^{-1} \in L_{loc}^{1/p}(S)$ over all horizontal hyperplanes \mathcal{H} is sufficient for the L^p -boundedness of \mathcal{M} when the surface S is given as the graph of a mixed homogeneous function with finite-type level sets. In 2010, Ikoromov, Kempe and Müller [11] discovered a connection between the L^p -boundedness of \mathcal{M} and the height of a smooth, compact hypersurface of finite type in \mathbb{R}^3 satisfying a transversality assumption on S . Recently, Zimmermann [47] proved that the maximal averages over analytic hypersurfaces located at the origin in \mathbb{R}^3 generally behave more regularly than the maximal averages over hypersurfaces satisfying the transversality condition. It is worth to mention that in [47], he could not handle the case when the Newton polyhedron of the analytic function is given by $\{(t_1, t_2) : t_1 \geq n, t_2 \geq 0\}$ or $\{(t_1, t_2) : t_1 \geq 0, t_2 \geq m\}$ and does not have any line with slope which equals to minus one after any linear change of coordinates.

In this chapter, we establish L^p -estimates for maximal functions related to hypersurfaces $(x_1, x_2) \rightarrow (x_1, x_2, x_2^d(1 + \mathcal{O}(x_2^m)))$ with associated dilations $\delta_t(x) = (t^a x_1, t x_2, t^d x_3)$ for an arbitrary real number $a > 0$. These results are stated in Theorem 5.1.2 and Theorem 5.1.4, and could not be covered by the theorems respectively developed in the above references. Moreover, by an alternative approach, we also get L^p -estimates for a large class of maximal functions in \mathbb{R}^3 proved in [10], [14], [11] and [47], see Theorem 5.1.1, Theorem 5.1.3 and Theorem 5.1.5.

Let Ω be an open neighborhood of the origin. Suppose Γ is a hypersurface in \mathbb{R}^3

which is parametrized as the graph of a smooth function $\Phi : \Omega \rightarrow \mathbb{R}$ at the origin, i.e. $\Gamma = \{(x, \Phi(x)), x \in \Omega \subset \mathbb{R}^2\}$.

In this chapter, our proofs always follow the following idea. First we "freeze" the first variable x_1 and apply the method of stationary phase in Lemma 2.2.2 to curves in (x_2, x_3) - plane, then by Lemma 2.2.3, we can reduce to apply L^p -estimates for certain Fourier integral operators which appeared in Theorem 2.2.1, Theorem 2.3.3, Theorem 2.3.6 and Theorem 2.4.1.

Let $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, t^{a_3}x_3)$, $a_j > 0$ be a family of dilations.

First, we show L^p estimates for maximal functions related to hypersurfaces with at least one non-vanishing principal curvature.

Theorem 5.1.1. *Assume that $\Phi(x_1, x_2) \in C^\infty(\Omega)$ satisfies*

$$\partial_2\Phi(0, 0) = 0, \quad \partial_2^2\Phi(0, 0) \neq 0, \quad (5.1)$$

and $2a_2 \neq a_3$. Then there exists a sufficiently small neighborhood of the origin $U \subset \Omega$ such that for every positive smooth function $\eta \in C_0^\infty(U)$, the associated maximal function

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, \Phi(x_1, x_2)))\eta(x)dx \right|, \quad (5.2)$$

initially defined on $C_0^\infty(\mathbb{R}^3)$, is bounded on $L^p(\mathbb{R}^3)$ for $p > 2$.

Theorem 5.1.2. *Let $\phi \in C^\infty(I)$, where I is a bounded interval containing the origin. Define the maximal function by*

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, x_2^2\phi(x_2)))\eta(x)dx \right|, \quad (5.3)$$

where η is supported in a sufficiently small neighborhood U of the origin. Assume that ϕ satisfies (2.1), i.e.

$$\phi(0) \neq 0; \quad \phi^{(j)}(0) = 0, \quad j = 1, 2, \dots, m-1; \quad \phi^{(m)}(0) \neq 0 \quad (m \geq 1),$$

and $2a_2 = a_3$. Then for $p > 2$, there exists a constant C_p such that the maximal operator satisfies the following estimate:

$$\|\mathcal{M}f\|_{L^p} \leq C_p \|f\|_{L^p}, \quad f \in C_0^\infty(\mathbb{R}^3).$$

Then we extend the above theorems to hypersurfaces of finite type.

Theorem 5.1.3. *Assume that $\Phi(x_1, x_2) \in C^\infty(\Omega)$ satisfies $\Phi(0, 0) \neq 0$ and $da_2 \neq a_3$, $d \geq 2$. Then there exists a sufficiently small neighborhood of the origin $U \subset \Omega$ such that for every positive smooth function $\eta \in C_0^\infty(U)$, the associated maximal function*

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, x_2^d \Phi(x_1, x_2))) \eta(x) dx \right|, \quad (5.4)$$

initially defined on $C_0^\infty(\mathbb{R}^3)$, is bounded on $L^p(\mathbb{R}^3)$ for $p > 2$.

The case when $da_2 = a_3$ in Lemma 5.1.2 turns out to be much harder. For this case, we have the following result.

Theorem 5.1.4. *Let $\phi \in C^\infty(I)$, where I is a bounded interval containing the origin. Define the maximal function by*

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, x_2^d \phi(x_2))) \eta(x) dx \right|, \quad (5.5)$$

where η is supported in a sufficiently small neighborhood U of the origin. Assume that ϕ satisfies (2.1), $d \geq 2$ and $da_2 = a_3$. Then for $p > 2$, there exists a constant C_p such that the maximal operator satisfies the following estimate:

$$\|\mathcal{M}f\|_{L^p} \leq C_p \|f\|_{L^p}, \quad f \in C_0^\infty(\mathbb{R}^3).$$

Finally, we notice that the surfaces in Lemma 5.1.3 are required to go through the origin. Now we consider the other case.

Theorem 5.1.5. *Assume that $\Phi(x_1, x_2) \in C^\infty(\Omega)$ satisfies $\Phi(0, 0) \neq 0$ and $da_2 \neq a_3$, $d \geq 2$. Then there exists a sufficiently small neighborhood of the origin $U \subset \Omega$ such that for every positive smooth function $\eta \in C_0^\infty(U)$, the associated maximal function*

$$\mathcal{M}f(y) := \sup_{t>0} \left| \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, 1 + x_2^d \Phi(x_1, x_2))) \eta(x) dx \right|, \quad (5.6)$$

initially defined on $C_0^\infty(\mathbb{R}^3)$, is bounded on $L^p(\mathbb{R}^3)$ for $p > d$.

5.2 Proofs for surfaces with one non-vanishing principal curvature

5.2.1 Maximal theorem with $2a_2 \neq a_3$

In this section we give the proof for Theorem 5.1.1.

We can always choose non-negative functions $\eta_1, \eta_2 \in C_0^\infty(\mathbb{R})$ so that $\eta(x) \leq \eta_1(x_1)\eta_2(x_2)$.

Since

$$\left| \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, \Phi(x_1, x_2)))\eta(x)dx \right| \leq \int_{\mathbb{R}^2} |f|(y - \delta_t(x_1, x_2, \Phi(x_1, x_2)))\eta_1(x_1)\eta_2(x_2)dx,$$

then we may assume $\eta(x) = \eta_1(x_1)\eta_2(x_2)$ and $f \geq 0$, $a_1 = 1$. Set $(y_2, y_3) = y'$ and $(\xi_2, \xi_3) = \xi'$. Denote $1 + a_2 + a_3$ by Q and $(t^{a_2}\xi_2, t^{a_3}\xi_3)$ by $\delta'_t\xi'$.

Put

$$A_t f(y) := \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, \Phi(x_1, x_2)))\eta_1(x_1)\eta_2(x_2)dx.$$

By means of the Fourier inversion formula, we can write

$$A_t f(y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_t \xi') dx_1 \hat{f}(\xi) d\xi,$$

where

$$\widehat{d\mu_{x_1}}(\xi') = \int_{\mathbb{R}} e^{-i(\xi_2 x_2 + \xi_3 \Phi(x_1, x_2))} \eta_2(x_2) dx_2. \quad (5.7)$$

Choose a non-negative function $\beta \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp } \beta \subset [1/2, 2] \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \beta(2^{-j}r) = 1 \quad \text{for } r > 0.$$

Put

$$A_{t,j} f(y) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_t \xi') dx_1 \beta(2^{-j}|\delta'_t \xi'|) \hat{f}(\xi) d\xi,$$

and M_j is the corresponding maximal operator.

$$\begin{aligned} A_t^0 f(y) &:= A_t f(y) - \sum_{j=1}^{\infty} A_{t,j} f(y) \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_t \xi') dx_1 \rho(|\delta'_t \xi'|) \hat{f}(\xi) d\xi, \end{aligned}$$

where ρ is supported in a neighborhood of the origin. Since

$$\begin{aligned} &\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_t \xi') dx_1 \rho(|\delta'_t \xi'|) d\xi \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \eta_1(x_1) \int_{\mathbb{R}^2} e^{i\xi' \cdot y'} \widehat{d\mu_{x_1}}(\delta'_t \xi') \rho(|\delta'_t \xi'|) d\xi' \delta_0(y_1 - tx_1) dx_1 \\ &= \frac{1}{(2\pi)^2} \frac{1}{t} \eta_1\left(\frac{y_1}{t}\right) \int_{\mathbb{R}^2} e^{i\xi' \cdot y'} \widehat{d\mu_{\frac{y_1}{t}}}(\delta'_t \xi') \rho(|\delta'_t \xi'|) d\xi' \\ &= \frac{1}{(2\pi)^2} t^{-Q} \eta_1\left(\frac{y_1}{t}\right) \int_{\mathbb{R}^2} e^{i\delta'_{t^{-1}\xi'} \cdot y'} \widehat{d\mu_{\frac{y_1}{t}}}(\xi') \rho(|\xi'|) d\xi', \end{aligned}$$

then $A_t^0 f(y) = t^{-Q} f * K(t^{-1}y_1, t^{-a_2}y_2, t^{-a_3}y_3) = f * K_{\delta_{t^{-1}}}(y)$, where

$$K(y) = \frac{1}{(2\pi)^2} \eta_1(y_1) \int_{\mathbb{R}^2} e^{i\xi' \cdot y'} \widehat{d\mu_{y_1}}(\xi') \rho(|\xi'|) d\xi'.$$

Since Φ satisfies (5.1) and $y_1 \in \text{supp } \eta_1$, for $N \in \mathbb{N}$ and multi-index α with $|\alpha| = N$, then Theorem 2.2.2 implies that

$$|D_{\xi'}^\alpha \widehat{d\mu_{y_1}}(\xi')| \leq C'_N (1 + |\xi'|)^{-1/2}. \quad (5.8)$$

By integration by parts in ξ' , we get

$$\begin{aligned} |K(y)| &\leq C''_N \frac{1}{(1 + |y_1|)^N} \frac{1}{(1 + |y'|)^N} \int_{\mathbb{R}^2} \left| D_{\xi'}^\alpha \left(\widehat{d\mu_{y_1}}(\xi') \rho(|\xi'|) \right) \right| d\xi' \\ &\leq C_N \prod_{i=1}^3 \frac{1}{(1 + |y_i|)^N}. \end{aligned}$$

Now we choose N sufficiently large, then

$$\sup_{t>0} |A_t^0 f(y)| \leq C_N M f(y), \quad (5.9)$$

where M is the non-isotropic Hardy-Littlewood maximal operator defined by

$$M f(y) := \sup_{t>0} \frac{1}{|B(y, t)|} \int_{B(y, t)} f(x) dx, \quad (5.10)$$

where $B(y, t) = \{x : |y - x|_\delta < t\}$ and $|x|_\delta = \max_{i=1}^3 |x_i|^{1/a_i}$.

So it suffices to prove that

$$\|M_j f\|_{L^p} \leq C_p 2^{-j\epsilon(p)} \|f\|_{L^p}, \quad j \geq 1, \quad 2 < p < \infty, \quad \text{some } \epsilon(p) > 0. \quad (5.11)$$

Since $A_{t,j}$ is localized to frequencies $|\delta_t \xi| \approx 2^j$, we can employ Lemma 2.2.4 in a similar way as in Section 2.3 to prove that

$$\|M_j\|_{L^p \rightarrow L^p} \lesssim \|M_{j,loc}\|_{L^p \rightarrow L^p},$$

where $M_{j,loc} f(y) := \sup_{t \in [1, 2]} |A_{t,j} f(y)|$.

Indeed, for fixed $j \geq 1$ and all $\ell \in \mathbb{Z}$, let Δ_ℓ^j be Littlewood-Paley operator in \mathbb{R}^2 defined by $\widehat{\Delta_\ell^j f}(\xi) = \tilde{\beta}(2^{-j} |\delta'_{2^\ell} \xi'|) \hat{f}(\xi)$, where $\tilde{\beta} \in C_0^\infty(\mathbb{R})$ is nonnegative and satisfies $\beta(|\delta'_t \xi'|) = \beta(|\delta'_t \xi'|) \tilde{\beta}(|\xi'|)$. $M_j f(y)$ is equal to

$$\begin{aligned}
& \sup_{\ell \in \mathbb{Z}} \sup_{t \in [1,2]} \left| \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-i2^\ell t \xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_{2^\ell t} \xi') dx_1 \beta(2^{-j} |\delta'_{2^\ell t} \xi'|) \tilde{\beta}(2^{-j} |\delta'_{2^\ell t} \xi'|) \hat{f}(\xi) d\xi \right| \\
&= \left(\sum_{\ell \in \mathbb{Z}} \sup_{t \in [1,2]} \left| \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-i2^\ell t \xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_{2^\ell t} \xi') dx_1 \beta(2^{-j} |\delta'_{2^\ell t} \xi'|) \widehat{\Delta_\ell^j f}(\xi) d\xi \right|^p \right)^{1/p} \\
&= \left(\sum_{\ell \in \mathbb{Z}} \sup_{t \in [1,2]} \left| 2^{-\ell Q} \int_{\mathbb{R}^3} e^{i\xi \cdot \delta_{2^{-\ell}} y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{x_1}}(\delta'_t \xi') dx_1 \beta(2^{-j} |\delta'_t \xi'|) \right. \right. \\
&\quad \left. \left. \times \widehat{\Delta_\ell^j f}(\delta_{2^{-\ell}} \xi) d\xi \right|^p \right)^{1/p} \\
&= \left(\sum_{\ell \in \mathbb{Z}} \left| \mathcal{M}_{j,loc}(\Delta_\ell^j f \circ \delta_{2^\ell})(\delta_{2^{-\ell}} y) \right|^p \right)^{1/p}.
\end{aligned}$$

Since $p > 2$, then Lemma 2.2.4 implies that

$$\begin{aligned}
\|M_j f(y)\|_{L^p}^p &= \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^3} \left| \mathcal{M}_{j,loc}(\Delta_\ell^j f \circ \delta_{2^\ell})(\delta_{2^{-\ell}} y) \right|^p dy \\
&= \sum_{\ell \in \mathbb{Z}} 2^{\ell Q} \left\| \mathcal{M}_{j,loc}(\Delta_\ell^j f \circ \delta_{2^\ell}) \right\|_{L^p}^p \\
&\leq \|\mathcal{M}_{j,loc}\|_{L^p \rightarrow L^p}^p \sum_{\ell \in \mathbb{Z}} 2^{\ell Q} \int_{\mathbb{R}^3} |\Delta_\ell^j f(\delta_{2^\ell} y)|^p dy \\
&= \|\mathcal{M}_{j,loc}\|_{L^p \rightarrow L^p}^p \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^3} |\Delta_\ell^j f(y)|^p dy \\
&\leq \|\mathcal{M}_{j,loc}\|_{L^p \rightarrow L^p}^p \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left(\sum_{\ell \in \mathbb{Z}} |\Delta_\ell^j f(y)|^2 \right)^{p/2} dy \\
&= \|\mathcal{M}_{j,loc}\|_{L^p \rightarrow L^p}^p \left\| \left\| \left(\sum_{\ell \in \mathbb{Z}} |\Delta_\ell^j f(y)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)} \right\|_{L^p(\mathbb{R})}^p \\
&\leq C_p \|\mathcal{M}_{j,loc}\|_{L^p \rightarrow L^p}^p \|f\|_{L^p(\mathbb{R}^3)}.
\end{aligned}$$

For fixed $t \in [1, 2]$, let us estimate $\widehat{d\mu_{x_1}}(\delta'_t \xi')$.

Set

$$s := s(\xi', t) = -\frac{t^{a_2} \xi_2}{t^{a_3} \xi_3}, \quad \text{for } \xi_3 \neq 0, \quad (5.12)$$

and

$$\Psi(x_1, x_2, s) := -s x_2 + \Phi(x_1, x_2). \quad (5.13)$$

We observe that

$$\partial_2 \Psi(0, 0, 0) = 0 \quad \text{and} \quad \partial_2^2 \Psi(0, 0, 0) \neq 0, \quad (5.14)$$

since Φ satisfies the condition (5.1). The implicit function theorem implies that there must be a smooth solution $x_2 = \psi(x_1, s)$ to the equation

$$\partial_2 \Phi(x_1, x_2) = s, \quad (5.15)$$

where x_1 and s are enough small. Here if $t \in [\frac{1}{2}, 4]$, we can choose x_1 and s sufficiently small such that

$$\partial_2 \Phi\left(\frac{x_1}{t}, \psi\left(\frac{x_1}{t}, s\right)\right) = s. \quad (5.16)$$

For above sufficiently small x_1 , a standard application of the method of stationary phase in x_2 yields that

$$\widehat{d\mu_{x_1}}(\delta'_t \xi') = e^{-it^{a_3} \xi_3 \tilde{\Psi}(x_1, s)} \frac{\chi_{x_1}(t^{a_2} \xi_2 / t^{a_3} \xi_3)}{(1 + |\delta'_t \xi'|)^{1/2}} A_{x_1}(\delta'_t \xi') + B_{x_1}(\delta'_t \xi'),$$

where $\tilde{\Psi}(x_1, s) := \Psi(x_1, \psi(x_1, s), s)$ and χ_{x_1} is a smooth function supported on the set $\{z : |z| < \epsilon_{x_1}\}$, where ϵ_{x_1} can be controlled by a small positive constant independent on x_1 . Moreover, $t^{a_3} \xi_3 \tilde{\Psi}(x_1, s)$ is a smooth function which is homogeneous of degree one in ξ' . Meanwhile, A_{x_1} is a symbol of order zero such that

$$|D_{\xi'}^\alpha A_{x_1}(\xi')| \leq C_\alpha (1 + |\xi'|)^{-|\alpha|}, \quad (5.17)$$

where α is a multi-index and C_α are admissible constants. B_{x_1} is a smooth function and satisfies

$$|D_{\xi'}^\alpha B_{x_1}(\xi')| \leq C_{\alpha, N} (1 + |\xi'|)^{-N}, \quad (5.18)$$

again with admissible constants $C_{\alpha, N}$ and $N \in \mathbb{N}$.

For fixed $t \in [1, 2]$,

$$\begin{aligned} A_{t, j}^0 f(y) &:= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) B_{x_1}(\delta'_t \xi') dx_1 \beta(2^{-j} |\delta'_t \xi'|) \hat{f}(\xi) d\xi \\ &= \mathcal{F}^{-1} \left\{ \int_{\mathbb{R}^3} e^{i\xi \cdot y} \left(\int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) B_{x_1}(\delta'_t \xi') dx_1 \right) \beta(2^{-j} |\delta'_t \xi'|) \right\} * f(y), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{F}^{-1} \left\{ \int_{\mathbb{R}^3} e^{i\xi \cdot y} \left(\int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) B_{x_1}(\delta'_t \xi') dx_1 \right) \beta(2^{-j} |\delta'_t \xi'|) \right\} (y) \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi' \cdot y'} \int_{\mathbb{R}} \eta_1(x_1) B_{x_1}(\delta'_t \xi') \delta_0(y_1 - tx_1) dx_1 \beta(2^{-j} |\delta'_t \xi'|) d\xi' \\ &= \frac{1}{(2\pi)^2} t^{-Q} \eta_1\left(\frac{y_1}{t}\right) \int_{\mathbb{R}^2} e^{i\xi' \cdot \delta'_{t^{-1}} y'} B_{y_1/t}(\xi') \beta(2^{-j} |\xi'|) d\xi' \\ &= \tilde{K}_{\delta_{t^{-1}}}(y), \end{aligned}$$

where $\tilde{K}(y) = \frac{1}{(2\pi)^2} \eta_1(y_1) \int_{\mathbb{R}^2} e^{i\xi' \cdot y'} B_{y_1}(\xi') \beta(2^{-j}|\xi'|) d\xi'$. By (5.18) and the support of β , it is easy to get

$$|\tilde{K}(y)| \leq C_N 2^{-jN} \prod_{i=1}^3 \frac{1}{(1 + |y_i|)^N},$$

and

$$\sup_{t \in [1,2]} |A_{t,j}^0 f(y)| \leq C_N 2^{-jN} Mf(y), \quad (5.19)$$

where M denotes the Hardy-Littlewood maximal operator defined by (5.10).

Put

$$A_{t,j}^1 f(y) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) e^{-it^{a_3} \xi_3 \tilde{\Psi}(x_1, s)} E_{x_1}(\delta'_t \xi') dx_1 \beta(2^{-j}|\delta'_t \xi'|) \hat{f}(\xi) d\xi$$

where

$$E_{x_1}(\delta'_t \xi') := \frac{\chi_{x_1}(t^{a_2} \xi_2 / t^{a_3} \xi_3)}{(1 + |\delta'_t \xi'|)^{1/2}} A_{x_1}(\delta'_t \xi'),$$

and denote by $\mathcal{M}_{j,loc}^1$ the corresponding maximal operator. It remains to prove that

$$\|\mathcal{M}_{j,loc}^1 f\|_{L^p} \leq C_p 2^{-j\epsilon(p)} \|f\|_{L^p}, \quad j \geq 1, \quad 2 < p < \infty, \quad \text{some } \epsilon(p) > 0. \quad (5.20)$$

$$\begin{aligned} \|\mathcal{M}_{j,loc}^1 f\|_{L^p} &= \left\| \sup_{t \in [1,2]} \frac{1}{(2\pi)^2} \left| \int_{\mathbb{R}} \eta_1(x_1) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^{a_3} \xi_3 \tilde{\Psi}(x_1, s))} E_{x_1}(\delta'_t \xi') \right. \right. \\ &\quad \times \left. \left. \beta(2^{-j}|\delta'_t \xi'|) f(y_1 - tx_1, \hat{\xi}') d\xi' dx_1 \right\|_{L^p(dy)} \\ &\leq C \left\| \sup_{t \in [1,2]} \left| \int_{\mathbb{R}} \eta_1\left(\frac{x_1}{t}\right) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^{a_3} \xi_3 \tilde{\Psi}\left(\frac{x_1}{t}, s\right))} E_{x_1/t}(\delta'_t \xi') \right. \right. \\ &\quad \times \left. \left. \beta(2^{-j}|\delta'_t \xi'|) f(y_1 - x_1, \hat{\xi}') d\xi' dx_1 \right\|_{L^p(dy)} \\ &= C \left\| \sup_{t \in [1,2]} |\widetilde{A}_{t,j}^1 f| \right\|_{L^p(dy)} = C \|\widetilde{\mathcal{M}_{j,loc}^1} f\|_{L^p}, \end{aligned}$$

where

$$\widetilde{A}_{t,j}^1 f(y) := \int_{\mathbb{R}} \eta_1\left(\frac{x_1}{t}\right) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^{a_3} \xi_3 \tilde{\Psi}\left(\frac{x_1}{t}, s\right))} E_{x_1/t}(\delta'_t \xi') \beta(2^{-j}|\delta'_t \xi'|) f(y_1 - x_1, \hat{\xi}') d\xi' dx_1, \quad (5.21)$$

and

$$\widetilde{\mathcal{M}_{j,loc}^1} f(y) := \sup_{t \in [1,2]} |\widetilde{A}_{t,j}^1 f(y)|. \quad (5.22)$$

Set

$$Q_{x_1}(y', t, \xi') = \xi' \cdot y' - t^{a_3} \xi_3 \tilde{\Psi}\left(\frac{x_1}{t}, s\right). \quad (5.23)$$

Choose a bump function $\tilde{\rho} \in C_0^\infty(\mathbb{R})$ supported in $[1/2, 4]$ such that $\tilde{\rho}(t) = 1$ if $1 \leq t \leq 2$. By Lemma 2.2.3, we have

$$\begin{aligned} \|\widetilde{\mathcal{M}}_{j,loc}^1 f(y)\|_{L^p}^p &\leq C_p \left(\int_{\mathbb{R}^3} \int_{1/2}^4 \left| \tilde{\rho}(t) \widetilde{A}_{t,j}^1 f(y) \right|^p dt dy \right)^{1/p'} \\ &\quad \times \left(\int_{\mathbb{R}^3} \int_{1/2}^4 \left| \frac{\partial}{\partial t} \left(\tilde{\rho}(t) \widetilde{A}_{t,j}^1 f(y) \right) \right|^p dt dy \right)^{1/p}. \end{aligned} \quad (5.24)$$

Moreover, we can choose $\tilde{\eta}_1 \in C_0^\infty(\mathbb{R})$ such that $\tilde{\eta}_1 = 1$ on the support of η_1 , then we have

$$\frac{\partial}{\partial t} \widetilde{A}_{t,j}^1 f(y) = \int_{\mathbb{R}} \tilde{\eta}_1\left(\frac{x_1}{t}\right) \int_{\mathbb{R}^2} e^{iQ_{x_1}(y', t, \xi')} h(t, j, x_1, \xi') f(y_1 - x_1, \widehat{\xi}') d\xi' dx_1,$$

where

$$\begin{aligned} h(t, j, x_1, \xi') &= \left(-t^{-2} x_1 \eta_1'\left(\frac{x_1}{t}\right) + \frac{\partial}{\partial t} Q_{x_1}(y', t, \xi') \right) E_{x_1/t}(\delta'_t \xi') \beta(2^{-j} |\delta'_t \xi'|) \\ &\quad + \eta_1\left(\frac{x_1}{t}\right) \frac{\partial}{\partial t} E_{x_1/t}(\delta'_t \xi') \beta(2^{-j} |\delta'_t \xi'|) + \eta_1\left(\frac{x_1}{t}\right) E_{x_1/t}(\delta'_t \xi') \frac{\partial}{\partial t} \beta(2^{-j} |\delta'_t \xi'|), \end{aligned}$$

and

$$\left| \frac{\partial}{\partial t} Q_{x_1}(y', t, \xi') \right| = \left| \xi_3 \frac{\partial}{\partial t} \left(t^{a_3} \Psi\left(\frac{x_1}{t}, \psi\left(\frac{x_1}{t}, s\right), s\right) \right) \right| \leq C |\xi_3|,$$

since $t \approx 1$, x_1 and the support of χ_{x_1} are sufficiently small. Note that A_{x_1} satisfies (5.17), then it is easy to see that $\frac{\partial}{\partial t} (\tilde{\rho}(t) \widetilde{A}_{t,j}^1)$ behaves like $2^j \widetilde{A}_{t,j}^1$. Clearly we only need to show the L^p -boundedness of the operator $\widetilde{A}_{t,j}^1$.

Furthermore, choose a function $\eta_0 \in C_0^\infty(\mathbb{R})$, non-negative so that for arbitrary $t \in [1/2, 4]$, $\eta_1(\frac{x_1}{t}) \leq \eta_0(x_1)$, then

$$\begin{aligned} &\left(\int_{\mathbb{R}^3} \int_{1/2}^4 \left| \tilde{\rho}(t) \widetilde{A}_{t,j}^1 f(y) \right|^p dt dy \right)^{1/p} \\ &\lesssim \left\| \int_{\mathbb{R}} \eta_0(x_1) \left| \tilde{\rho}(t) \int_{\mathbb{R}^2} e^{iQ_{x_1}(y', t, \xi')} E_{x_1/t}(\delta'_t \xi') \beta(2^{-j} |\delta'_t \xi'|) \right. \right. \\ &\quad \left. \left. \times f(y_1 - x_1, \widehat{\xi}') d\xi' \right| dx_1 \right\|_{L^p([1/2, 4] \times \mathbb{R}^3, dt dy' dy_1)} \\ &\lesssim \left\| \int_{\mathbb{R}} \eta_0(x_1) \left| \tilde{\rho}(t) \int_{\mathbb{R}^2} e^{iQ_{x_1}(y', t, \xi')} E_{x_1/t}(\delta'_t \xi') \beta(2^{-j} |\delta'_t \xi'|) \right. \right. \\ &\quad \left. \left. \times f(y_1 - x_1, \widehat{\xi}') d\xi' \right| dx_1 \right\|_{L^p([1/2, 4] \times \mathbb{R}^2, dt dy')} \left\| dx_1 \right\|_{L^p(\mathbb{R}, dy_1)}. \end{aligned}$$

Now for the inner norm we would apply the local smoothing estimate from Theorem 2.2.1, so we should verify that $Q_{x_1}(y', t, \xi')$ satisfies the non-degeneracy condition (2.9) and the cone condition (2.10). Apparently, the non-degeneracy condition will follow from

$$\text{rank } \partial_{(y', t), \xi'}^2 Q_{x_1}(y', t, \xi') = \text{rank} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ * & * \end{pmatrix} = 2.$$

Since $\partial_2^2 \Phi(0, 0) \neq 0$, we can choose U sufficiently small such that

$$C_1 |\partial_2^2 \Phi(0, 0)| \geq |\partial_2^2 \Phi(x_1, x_2)| \geq C_2 |\partial_2^2 \Phi(0, 0)|. \quad (5.25)$$

Also since

$$\partial_2 \Phi\left(\frac{x_1}{t}, \psi\left(\frac{x_1}{t}, s\right)\right) = s, \quad (5.26)$$

by applying $\frac{\partial}{\partial s}$ on both sides, then we have

$$\partial_2^2 \Phi\left(\frac{x_1}{t}, \psi\left(\frac{x_1}{t}, s\right)\right) \partial_2 \psi\left(\frac{x_1}{t}, s\right) = 1,$$

which implies that

$$\partial_2 \psi\left(\frac{x_1}{t}, s\right) = \frac{1}{\partial_2^2 \Phi\left(\frac{x_1}{t}, \psi\left(\frac{x_1}{t}, s\right)\right)}. \quad (5.27)$$

Next, we will use the fact that $|\xi_2| \ll |\xi_3| \approx |\xi'| \approx 2^j$ ($j \geq 1$).

By the above arguments, we get

$$\begin{aligned} \partial_t Q_{x_1}(y'_0, t_0, \xi') &= \xi_3 \left[a_2 t_0^{a_2-1} \frac{\xi_2}{\xi_3} \psi\left(\frac{x_1}{t_0}, s\right) + t_0^{a_2} \frac{\xi_2}{\xi_3} \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) (a_2 - a_3) t_0^{a_2-a_3-1} \frac{\xi_2}{\xi_3} \right. \\ &\quad \left. + a_3 t_0^{a_3-1} \Phi\left(\frac{x_1}{t_0}, \psi\left(\frac{x_1}{t_0}, s\right)\right) + t_0^{a_3} \partial_2 \Phi\left(\frac{x_1}{t_0}, \psi\left(\frac{x_1}{t_0}, s\right)\right) \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) \right. \\ &\quad \left. \times (a_2 - a_3) t_0^{a_2-a_3-1} \frac{\xi_2}{\xi_3} + x_1 R^1(x_1, t_0, s) \right] \\ &= \xi_3 \left[a_2 t_0^{a_2-1} \frac{\xi_2}{\xi_3} \psi\left(\frac{x_1}{t_0}, s\right) + a_3 t_0^{a_3-1} \Phi\left(\frac{x_1}{t_0}, \psi\left(\frac{x_1}{t_0}, s\right)\right) + x_1 R^1(x_1, t_0, s) \right]. \end{aligned}$$

Furthermore, we will prove that

$$|\partial_{\xi_2}^2 (\partial_t Q_{x_1}(y'_0, t_0, \xi'))| \approx 1. \quad (5.28)$$

Note that $\partial_t Q_{x_1}(y'_0, t_0, \xi')$ is homogeneous of degree one in ξ' , in addition to (5.28), then we obtain that

$$\text{rank} \left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} \right) \langle \partial_{(y', t)} Q_{x_1}(y'_0, t_0, \xi'), \theta \rangle = 1, \quad i, j = 2, 3, \quad (5.29)$$

where $\pm\theta$ are the unique directions for which $\nabla_{\xi'} \langle \partial_{(y',t)} Q_{x_1}(y'_0, t_0, \xi'), \theta \rangle = 0$. This implies the cone condition (2.10).

Now, let us turn to prove (5.28). Since x_1 is sufficiently small, $\left| \left(\frac{\partial}{\partial \xi_2} \right)^\alpha R^1(x_1, t_0, s) \right|$, $\alpha \leq 2$ is a remainder term. So we only consider

$$\left| \frac{\partial^2}{\partial \xi_2^2} \left(a_2 t_0^{a_2-1} \frac{\xi_2}{\xi_3} \psi\left(\frac{x_1}{t_0}, s\right) + a_3 t_0^{a_3-1} \Phi\left(\frac{x_1}{t_0}, \psi\left(\frac{x_1}{t_0}, s\right)\right) \right) \right| \approx 1. \quad (5.30)$$

$$\begin{aligned} & \frac{\partial}{\partial \xi_2} \left(a_2 t_0^{a_2-1} \frac{\xi_2}{\xi_3} \psi\left(\frac{x_1}{t_0}, s\right) + a_3 t_0^{a_3-1} \Phi\left(\frac{x_1}{t_0}, \psi\left(\frac{x_1}{t_0}, s\right)\right) \right) \\ &= a_2 t_0^{a_2-1} \frac{1}{\xi_3} \psi\left(\frac{x_1}{t_0}, s\right) - a_2 t_0^{a_2-1} \frac{\xi_2}{\xi_3} \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) t_0^{a_2-a_3} \frac{1}{\xi_3} \\ & \quad - a_3 t_0^{a_3-1} \partial_2 \Phi\left(\frac{x_1}{t_0}, \psi\left(\frac{x_1}{t_0}, s\right)\right) \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) t_0^{a_2-a_3} \frac{1}{\xi_3} + x_1 R^2(x_1, t_0, s) \\ &= t_0^{a_2-1} \frac{1}{\xi_3} \left(a_2 \psi\left(\frac{x_1}{t_0}, s\right) + (a_3 - a_2) t_0^{a_2-a_3} \frac{\xi_2}{\xi_3} \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) + x_1 R^2(x_1, t_0, s) \right). \end{aligned}$$

By the same reason as before, we only estimate

$$\begin{aligned} & \frac{\partial^2}{\partial \xi_2^2} \left(a_2 \psi\left(\frac{x_1}{t_0}, s\right) + (a_3 - a_2) t_0^{a_2-a_3} \frac{\xi_2}{\xi_3} \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) \right) \\ &= -a_2 \partial_2^2 \psi\left(\frac{x_1}{t_0}, s\right) t_0^{a_2-a_3} \frac{1}{\xi_3} + (a_3 - a_2) t_0^{a_2-a_3} \frac{1}{\xi_3} \partial_2 \psi\left(\frac{x_1}{t_0}, s\right) - (a_3 - a_2) t_0^{a_2-a_3} \frac{\xi_2}{\xi_3} \\ & \quad \times \partial_2^2 \psi\left(\frac{x_1}{t_0}, s\right) t_0^{a_2-a_3} \frac{1}{\xi_3} \\ &= t_0^{a_2-a_3} \frac{1}{\xi_3} \left((a_3 - 2a_2) \partial_2^2 \psi\left(\frac{x_1}{t_0}, s\right) + (a_2 - a_3) t_0^{a_2-a_3} \frac{\xi_2}{\xi_3} \partial_2^2 \psi\left(\frac{x_1}{t_0}, s\right) \right). \end{aligned}$$

(5.27) and (5.25) imply that $\partial_2 \psi\left(\frac{x_1}{t_0}, s\right) \geq \frac{1}{C_1 |\Phi^2(0,0)|} \neq 0$. Moreover, since $2a_2 \neq a_3$ and $|\xi_2| \ll |\xi_3|$, then $|(a_3 - 2a_2) \partial_2^2 \psi\left(\frac{x_1}{t_0}, s\right)| \approx 1$ and $|(a_2 - a_3) t_0^{a_2-a_3} \frac{\xi_2}{\xi_3} \partial_2^2 \psi\left(\frac{x_1}{t_0}, s\right)| \ll 1$. We finish the proof of (5.30) and (5.28).

Hence we can apply Theorem 2.2.1 for $\mu = -1/2$, and obtain

$$\begin{aligned} & \left(\int_{\mathbb{R}^3} \int_{1/2}^4 |\tilde{\rho}(t) \widetilde{A_{t,j}^1} f(y)|^p dt dy \right)^{1/p} \\ & \leq C_p 2^{-j(1/p+\epsilon(p))} \left\| \int_{\mathbb{R}} \eta_0(x_1) \|f(y_1 - x_1, y')\|_{L^p(\mathbb{R}^2, dy')} dx_1 \right\|_{L^p(\mathbb{R}, dy_1)} \\ & \leq C_p 2^{-j(1/p+\epsilon(p))} \|f\|_{L^p} \|\eta_0\|_{L^1}, \end{aligned}$$

and thus by (5.24), we get

$$\begin{aligned} & \|\widetilde{\mathcal{M}}_{j,loc}^1 f(y)\|_{L^p}^p \\ & \leq C_p \left(2^{-j(1/p+\epsilon(p))} \|f\|_{L^p} \|\eta_0\|_{L^1} \right)^{p-1} \left(2^{-j(-1+1/p+\epsilon(p))} \|f\|_{L^p} \|\eta_0\|_{L^1} \right) \\ & = C_p 2^{-jp\epsilon(p)} \|f\|_{L^p}^p \|\eta_0\|_{L^1}^p. \end{aligned}$$

Now we have finished the proof of Theorem 5.1.1.

5.2.2 Maximal theorem with $2a_2 = a_3$

In this section, we will prove Theorem 5.1.2.

First we may assume $\eta(x) = \eta_1(x_1)\eta_2(x_2)$ with non-negative functions $\eta_1, \eta_2 \in C_0^\infty(\mathbb{R})$, and $f \geq 0$, $a_2 = 1$, $a_3 = 2$. Let $(y_2, y_3) = y'$ and $(\xi_2, \xi_3) = \xi'$. Denote $a_1 + 3$ by Q and $(t\xi_2, t^2\xi_3)$ by $\tilde{\delta}_t\xi'$. We choose $B > 0$ very small and $\tilde{\rho} \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \tilde{\rho} \subset \{x \in \mathbb{R} : B/2 \leq |x| \leq 2B\}$ and $\sum_k \tilde{\rho}(2^k x) = 1$.

Put

$$A_t f(y) := \int_{\mathbb{R}^2} f(y_1 - t^{a_1} x_1, y_2 - t x_2, y_3 - t^2 x_2^2 \phi(x_2)) \eta(x) dx = \sum_k \widetilde{A}_t^k f(y),$$

where $\widetilde{A}_t^k f(y) = \int_{\mathbb{R}^2} f(y_1 - t^{a_1} x_1, y_2 - t x_2, y_3 - t^2 x_2^2 \phi(x_2)) \eta_1(x_1) \eta_2(x_2) \tilde{\rho}(2^k x_2) dx$.

Define the isometric operator T on $L^p(\mathbb{R}^3)$ by

$$Tf(x_1, x_2, x_3) = 2^{3k/p} f(x_1, 2^k x_2, 2^{2k} x_3).$$

One can easily compute that

$$T^{-1} \widetilde{A}_t^k T f(y) = 2^{-k} \int_{\mathbb{R}^2} f(y_1 - t^{a_1} x_1, y_2 - t x_2, y_3 - t^2 x_2^2 \phi(\frac{x_2}{2^k})) \eta_1(x_1) \eta_2(\frac{x_2}{2^k}) \rho(x_2) dx.$$

It suffices to prove the following estimate

$$\sum_k 2^{-k} \left\| \sup_{t>0} |A_t^k| \right\|_{L^p \rightarrow L^p} \leq C_p, \quad p > 2,$$

where

$$A_t^k f(y) := \int_{\mathbb{R}^2} f(y_1 - t^{a_1} x_1, y_2 - t x_2, y_3 - t^2 x_2^2 \phi(\frac{x_2}{2^k})) \eta_1(x_1) \eta_2(\frac{x_2}{2^k}) \rho(x_2) dx. \quad (5.31)$$

By means of the Fourier inversion formula, we can write

$$A_t^k f(y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \widehat{\eta}_1(t^{a_1} \xi_1) \widehat{d\mu_{k,m}}(\tilde{\delta}_t \xi') \hat{f}(\xi) d\xi,$$

where

$$\widehat{d\mu_{k,m}}(\tilde{\delta}_t \xi') = \int_{\mathbb{R}} e^{-i(t\xi_2 x_2 + t^2 \xi_3 x_2^2 \phi(\frac{x_2}{2^k}))} \tilde{\rho}(x_2) \eta_2(2^{-k} x_2) dx_2.$$

Choose a non-negative function $\beta \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp } \beta \subset [1/2, 2] \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \beta(2^{-j} r) = 1 \quad \text{for } r > 0.$$

and set

$$A_{t,j}^k f(y) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \widehat{\eta}_1(t^{a_1} \xi_1) \widehat{d\mu_{k,m}}(\tilde{\delta}_t \xi') \beta(2^{-j} |\tilde{\delta}_t \xi'|) \hat{f}(\xi) d\xi, \quad (5.32)$$

and denote by M_j^k the corresponding maximal operator.

From the proof of inequality (5.9), it is easy to see that the supremum of the absolute value of the difference between $A_t^k f(y)$ and $\sum_{j=1}^\infty A_{t,j}^k f(y)$ is dominated by the Hardy-Littlewood maximal function $Mf(y)$ defined by (5.10). It remains to consider the L^p -boundedness ($2 < p < \infty$) of the maximal operator M_j^k for $j \geq 1$.

Since $A_{t,j}^k$ is localized to frequencies $|\tilde{\delta}_t \xi'| \approx 2^j$, combining the method of Section 5.2.1 and Lemma 2.2.4, then we will have

$$\|M_j^k\|_{L^p \rightarrow L^p} \lesssim \|M_{j,loc}^k\|_{L^p \rightarrow L^p},$$

where $M_{j,loc}^k f(y) := \sup_{t \in [1,2]} |A_{t,j}^k f(y)|$.

For fixed $t \in [1, 2]$, let us estimate $\widehat{d\mu_{k,m}}(\tilde{\delta}_t \xi')$.

Set

$$\delta := 2^{-k}, \quad s := s(\xi', t) = -\frac{\xi_2}{t\xi_3}, \quad \text{for } \xi_3 \neq 0, \quad (5.33)$$

$$\Phi(s, x_2, \delta) := -sx_2 + x_2^2 \phi(\delta x_2). \quad (5.34)$$

A similar argument as in Section 5.2.1 and Section 2.3.2 shows that we can reduce to considering for $j \geq 1$, the L^p -boundedness of the operator $\widetilde{A_{t,j}^k}$ given by

$$\begin{aligned} \widetilde{A_{t,j}^k} f(y) &:= \int_{\mathbb{R}} \eta_1\left(\frac{x_1}{t^{a_1}}\right) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \Phi(s, \delta))} \chi_{k,m}\left(\frac{\xi_2}{t\xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) \\ &\quad \times f(y_1 - x_1, \widehat{\xi}') d\xi' dx_1, \end{aligned}$$

where $\tilde{\Phi}(s, \delta) := \Phi(s, \tilde{q}(s, \delta), \delta)$ and $x_2 = \tilde{q}(s, \delta)$ is the solution of the equation $\partial_2 \Phi(s, x_2, \delta) = 0$ and smoothly converges to the solution $\tilde{q}(s, 0) = s$ of the equation $\partial_2 \Phi(s, x, 0) = 0$ if we assume $\phi(0) = 1/2$. The phase function can be written as

$$-t^2 \xi_3 \tilde{\Phi}(s, \delta) := \frac{\xi_2^2}{2\xi_3} + (-1)^{m+1} \delta^m \frac{\phi^{(m)}(0)}{m!} \frac{\xi_2^{m+2}}{t^m \xi_3^{m+1}} + R(t, \xi', \delta), \quad (5.35)$$

which is homogeneous of degree one and can be considered as a small perturbation of $\frac{\xi_2^2}{2\xi_3} + (-1)^{m+1} \delta^m \frac{\phi^{(m)}(0)}{m!} \frac{\xi_2^{m+2}}{t^m \xi_3^{m+1}}$. $\chi_{k,m}$ is a smooth function supported in the interval $[c_{k,m}, \tilde{c}_{k,m}]$, for certain non-zero constants $c_{k,m}$ and $\tilde{c}_{k,m}$ dependent only on k and m . $A_{k,m}$ is a symbol of order zero and $\{A_{k,m}(\tilde{\delta}_t \xi')\}_k$ is contained in a bounded subset of symbols of order zero. Denote by $\widetilde{\mathcal{M}}_{j,loc}^k$ the corresponding maximal operator.

Following the argument of (2.46), we can choose a bump function $\rho_1 \in C_0^\infty(\mathbb{R}^2 \times [1/2, 4])$, by Lemma 2.2.3, we have

$$\begin{aligned} \|\widetilde{\mathcal{M}}_{j,loc}^k f\|_{L^p} &\leq C_p \left(\int_{\mathbb{R}^3} \int_{1/2}^4 |\rho_1(y', t) \widetilde{A}_{t,j}^k f(y)|^p dt dy \right)^{1/p'} \\ &\quad \times \left(\int_{\mathbb{R}^3} \int_{1/2}^4 \left| \frac{\partial}{\partial t} (\rho_1(y', t) \widetilde{A}_{t,j}^k f(y)) \right|^p dt dy \right)^{1/p}. \end{aligned} \quad (5.36)$$

Moreover, we choose a non-negative function $\tilde{\eta}_1 \in C_0^\infty(\mathbb{R})$ such that $\eta_1 = 1$ on the support of η_1 , then

$$\frac{\partial}{\partial t} \left(\widetilde{A}_{t,j}^k f(y) \right) = \int_{\mathbb{R}} \tilde{\eta}_1\left(\frac{x_1}{t}\right) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \tilde{\Phi}(s, \delta))} h_{k,m}(y, t, j, \xi') f(y_1 - x_1, \hat{\xi}') d\xi' dx_1,$$

where

$$\begin{aligned} h_{k,m}(y, t, j, \xi') &= \left(\frac{\partial}{\partial t} \rho_1(y', t) \eta_1\left(\frac{x_1}{t}\right) - \rho_1(y', t) \frac{x_1}{t^2} \eta_1'\left(\frac{x_1}{t}\right) - \rho_1(y', t) \eta_1\left(\frac{x_1}{t}\right) \frac{\partial}{\partial t} (t^2 \xi_3 \tilde{\Phi}(s, \delta)) \right) \\ &\quad \times \chi_{k,m}\left(\frac{\xi_2}{t\xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) \\ &\quad - \rho_1(y', t) \eta_1\left(\frac{x_1}{t}\right) \frac{\xi_2}{t^2 \xi_3} \chi'_{k,m}\left(\frac{\xi_2}{t\xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) \\ &\quad + \rho_1(y', t) \eta_1\left(\frac{x_1}{t}\right) \chi_{k,m}\left(\frac{\xi_2}{t\xi_3}\right) \frac{\partial}{\partial t} \left(\frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \right) \beta(2^{-j} |\tilde{\delta}_t \xi'|) \\ &\quad + \rho_1(y', t) \eta_1\left(\frac{x_1}{t}\right) \chi_{k,m}\left(\frac{\xi_2}{t\xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \frac{\partial}{\partial t} \left(\beta(2^{-j} |\tilde{\delta}_t \xi'|) \right). \end{aligned}$$

Since $t \approx 1$, x_1 and support of $\chi_{k,m}$ are sufficiently small, by (5.35), then $|t^2 \xi_3 \tilde{\Phi}(s, \delta)| \approx 2^j \delta^m$. Moreover, $\{A_{k,m}(\tilde{\delta}_t \xi')\}_k$ is contained in a bounded subset of symbols of order

zero, so we obtain

$$|h_{k,m}(y, t, j, \xi')| \lesssim 2^{j/2} \delta^m + 2^{-j/2}. \quad (5.37)$$

Now it is easy to see that $(2^{j/2} \delta^m + 2^{-j/2})^{-1} \frac{\partial}{\partial t} (\rho_1(y', t) \widetilde{A_{t,j}^k} f)$ behaves like $2^{j/2} \widetilde{A_{t,j}^k} f$. It is sufficient to estimate $\|\rho_1(y', t) \widetilde{A_{t,j}^k} f(y)\|_{L^p(\mathbb{R}^3 \times [1/2, 4], dt dy)}$.

Furthermore, choosing a function $\eta_0 \in C_0^\infty(\mathbb{R})$, non-negative, such that for arbitrary $t \in [1, 2]$, $a_1 > 0$, $\eta_1(\frac{x_1}{t a_1}) \leq \eta_0(x_1)$, we get

$$\begin{aligned} & \left| \int_{\mathbb{R}} \eta_1\left(\frac{x_1}{t a_1}\right) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \tilde{\Phi}(s, \delta))} \chi_{k,m}\left(\frac{\xi_2}{t \xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) f(y_1 - x_1, \widehat{\xi}') d\xi' dx_1 \right| \\ & \leq \int_{\mathbb{R}} \eta_0(x_1) \left| \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \tilde{\Phi}(s, \delta))} \chi_{k,m}\left(\frac{\xi_2}{t \xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) f(y_1 - x_1, \widehat{\xi}') d\xi' \right| dx_1. \end{aligned}$$

In order to apply the regularity estimate Lemma 2.3.5 for $j \leq 9km$ and the local smoothing estimate Theorem 2.3.6 for $j > 9km$ of the Fourier integral operators not satisfying the "cinematic curvature condition" uniformly, we freeze x_1 , in fact, by Minkowski's and Young's inequalities, we have

$$\begin{aligned} & \|\rho_1(y', t) \widetilde{A_{t,j}^k} f(y)\|_{L^p(\mathbb{R}^3 \times [1/2, 4], dt dy)} \\ & \leq C \left(\int_{\mathbb{R}^3} \int \left(\rho_1(y', t) \int_{\mathbb{R}} \eta_0(x_1) \left| \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \tilde{\Phi}(s, \delta))} \chi_{k,m}\left(\frac{\xi_2}{t \xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) \right. \right. \right. \\ & \quad \left. \left. \left. \times f(y_1 - x_1, \widehat{\xi}') d\xi' \right| dx_1 \right)^p dt dy \right)^{1/p} \\ & \leq C \left\| \int_{\mathbb{R}} \eta_0(x_1) \left\| \rho_1(y', t) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \tilde{\Phi}(s, \delta))} \chi_{k,m}\left(\frac{\xi_2}{t \xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) \right. \right. \\ & \quad \left. \left. \times f(y_1 - x_1, \widehat{\xi}') d\xi' \right\|_{L^p(\mathbb{R}^2 \times [\frac{1}{2}, 4], dt dy')} \right\|_{L^p(\mathbb{R}, dy_1)} \\ & \leq C \|\eta_0\|_{L^1} \left\| \left\| \rho_1(y', t) \int_{\mathbb{R}^2} e^{i(\xi' \cdot y' - t^2 \xi_3 \tilde{\Phi}(s, \delta))} \chi_{k,m}\left(\frac{\xi_2}{t \xi_3}\right) \frac{A_{k,m}(\tilde{\delta}_t \xi')}{(1 + |\tilde{\delta}_t \xi'|)^{1/2}} \beta(2^{-j} |\tilde{\delta}_t \xi'|) \right. \right. \\ & \quad \left. \left. \times f(y_1, \widehat{\xi}') d\xi' \right\|_{L^p(\mathbb{R}^2 \times [\frac{1}{2}, 4], dt dy')} \right\|_{L^p(\mathbb{R}, dy_1)}. \end{aligned}$$

Finally, together with the arguments from Section 2.3.2, we finish the proof.

5.3 Proofs for surfaces of finite type

5.3.1 Maximal function theorem with $da_2 \neq a_3$

Theorem 5.1.3 will be proved in this section.

First we assume $\eta(x) = \eta_1(x_1)\eta_2(x_2)$ with non-negative functions $\eta_1, \eta_2 \in C_0^\infty(\mathbb{R})$ and $f \geq 0$, $a_1 = 1$. Let $(y_2, y_3) = y'$ and $(\xi_2, \xi_3) = \xi'$. Denote $(t^{a_2}\xi_2, t^{a_3}\xi_3)$ by $\delta'_t\xi'$. We choose $B > 0$ very small and $\tilde{\rho} \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \tilde{\rho} \subset \{x \in \mathbb{R} : B/2 \leq |x| \leq 2B\}$ and $\sum_k \tilde{\rho}(2^k x) = 1$.

Put

$$A_t f(y) := \int_{\mathbb{R}^2} f(y_1 - tx_1, y_2 - t^{a_2}x_2, y_3 - t^{a_3}x_2^d \Phi(x)) \eta(x) dx := \sum_k \widetilde{A}_t^k f(y),$$

where $\widetilde{A}_t^k f(y) := \int_{\mathbb{R}^2} f(y_1 - tx_1, y_2 - t^{a_2}x_2, y_3 - t^{a_3}x_2^d \Phi(x)) \eta(x) \tilde{\rho}(2^k x_2) dx$.

Define the isometric operator on $L^p(\mathbb{R}^3)$ by

$$Tf(x_1, x_2, x_3) = 2^{k(d+1)/p} f(x_1, 2^k x_2, 2^{dk} x_3).$$

By the arguments in Section 5.2.1, it suffices to prove the following inequality

$$\sum_k 2^{-k} \left\| \sup_{t>0} |A_t^k| \right\|_{L^p \rightarrow L^p} \leq C_p,$$

where

$$A_t^k f(y) = \int_{\mathbb{R}^2} f(y_1 - tx_1, y_2 - t^{a_2}x_2, y_3 - t^{a_3}x_2^d \Phi(x_1, \frac{x_2}{2^k})) \eta_1(x_1) \eta_2(2^{-k}x_2) \tilde{\rho}(x_2) dx. \quad (5.38)$$

By means of the Fourier inversion formula, we can write

$$A_t^k f(y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{k,x_1,d}(\delta'_t\xi')} dx_1 \hat{f}(\xi) d\xi,$$

where

$$\widehat{d\mu_{k,x_1,d}(\delta'_t\xi')} = \int_{\mathbb{R}} e^{-i(t^{a_2}\xi_2 x_2 + t^{a_3}\xi_3 x_2^d \Phi(x_1, \frac{x_2}{2^k}))} \tilde{\rho}(x_2) \eta_2(2^{-k}x_2) dx_2.$$

Choose a non-negative function $\beta \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp } \beta \subset [1/2, 2] \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \beta(2^{-j}r) = 1 \quad \text{for } r > 0.$$

Define

$$A_{t,j}^k f(y) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot y} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{k,x_1,d}(\delta'_t \xi')} dx_1 \beta(2^{-j} |\delta'_t \xi'|) \hat{f}(\xi) d\xi,$$

and denote by M_j^k the corresponding maximal operator.

From the arguments of Section 5.2.1, we see that $\sup_{t>0} \left| \sum_{j \leq 0} A_{t,j}^k f(y) \right|$ can be dominated by the Hardy-Littlewood maximal function $Mf(y)$ defined by (5.10), then it suffices to prove that

$$\|M_j^k\|_{L^p \rightarrow L^p} \leq C_p 2^{-j\epsilon(p)}, \quad j \geq 1, \quad 2 < p < \infty, \quad \text{some } \epsilon(p) > 0. \quad (5.39)$$

Since $A_{t,j}^k$ is localized to frequencies $|\delta'_t \xi'| \approx 2^j$, we can still use Lemma 2.2.4 to prove that

$$\|M_j^k\|_{L^p \rightarrow L^p} \lesssim \|M_{j,loc}^k\|_{L^p \rightarrow L^p},$$

where $M_{j,loc}^k f(y) := \sup_{t \in [1,2]} |A_{t,j}^k f(y)|$.

For fixed $t \in [1, 2]$, let us estimate $\widehat{d\mu_{k,x_1,d}(\delta'_t \xi')}$.

Set

$$\delta := 2^{-k}, \quad s := s(\xi', t) = -\frac{t^{a_2} \xi_2}{t^{a_3} \xi_3}, \quad \text{for } \xi_3 \neq 0, \quad (5.40)$$

and

$$\Psi(x_1, x_2, s, \delta) := -sx_2 + \Phi_k(x_1, x_2), \quad (5.41)$$

where $\Phi_k(x_1, x_2) := x_2^d \Phi(x_1, \delta x_2)$.

Since $x_2 \approx 1$ here, in addition to $\Phi(0, 0) \neq 0$, then we can reduce our proof to the case $d = 2$. So from now on we can proceed similarly as in Section 5.2.1.

This finishes the proof of Theorem 5.1.3.

5.3.2 Maximal theorem with $da_2 = a_3$

Combining the proofs of Section 5.2.2 and Section 2.4, it is easy to get Theorem 5.1.4. We omit the details here.

5.4 Proofs for surfaces not passing through the origin

We will show the proof of Theorem 5.1.5, where $da_2 \neq a_3$.

Following the proof of Section 5.3.1, we only modify some places.

Define

$$\begin{aligned} A_{t,j}^k f(y) &:= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\langle \xi, (y_1, y_2, y_3 - t^{\alpha_3} 2^{dk}) \rangle} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{k,x_1,d}}(\delta'_t \xi') dx_1 \\ &\quad \times \beta(2^{-j} |\delta'_t \xi'|) \widehat{f}(\xi) d\xi. \end{aligned} \quad (5.42)$$

Set $A_t^{k,0} f(y) := \sum_{j \leq 0} A_{t,j}^k f(y)$ and $M^{k,0} f(y) := \sup_{t>0} |A_t^{k,0} f(y)|$. It is easy to see that

$$A_t^{k,0} f(y) = f * K_{\delta_{t^{-1}}}^\sigma(y), \quad (5.43)$$

where $\sigma = (0, 0, 2^{dk})$ and $K_{\delta_{t^{-1}}}^\sigma(y) = t^{-Q} K^\sigma(t^{-1}y_1, t^{-\alpha_2}y_2, t^{-\alpha_3}y_3)$ and K^σ is the translate

$$K^\sigma(y) = K(y - \sigma). \quad (5.44)$$

Indeed, since

$$\begin{aligned} &\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\langle \xi, (y_1, y_2, y_3 - t^{\alpha_3} 2^{dk}) \rangle} \int_{\mathbb{R}} e^{-it\xi_1 x_1} \eta_1(x_1) \widehat{d\mu_{k,x_1,d}}(\delta'_t \xi') dx_1 \rho(|\delta'_t \xi'|) d\xi \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \eta_1(x_1) \int_{\mathbb{R}^2} e^{i\langle \xi', (y_2, y_3 - t^{\alpha_3} 2^{dk}) \rangle} \widehat{d\mu_{k,x_1,d}}(\delta'_t \xi') \rho(|\delta'_t \xi'|) d\xi' \delta_0(y_1 - tx_1) dx_1 \\ &= \frac{1}{(2\pi)^2} \frac{1}{t} \eta_1\left(\frac{y_1}{t}\right) \int_{\mathbb{R}^2} e^{i\langle \xi', (y_2, y_3 - t^{\alpha_3} 2^{dk}) \rangle} \widehat{d\mu_{k, \frac{y_1}{t}, d}}(\delta'_t \xi') \rho(|\delta'_t \xi'|) d\xi' \\ &= \frac{1}{(2\pi)^2} t^{-Q} \eta_1\left(\frac{y_1}{t}\right) \int_{\mathbb{R}^2} e^{i\langle \xi', \delta'_{t^{-1}} y' - (0, 2^{dk}) \rangle} \widehat{d\mu_{k, \frac{y_1}{t}, d}}(\xi') \rho(|\xi'|) d\xi' \\ &= K_{\delta_{t^{-1}}}^\sigma(y), \end{aligned}$$

then

$$K(y) := \frac{1}{(2\pi)^2} \eta_1(y_1) \int_{\mathbb{R}^2} e^{i\xi' \cdot y'} \widehat{d\mu_{k,y_1,d}}(\xi') \rho(|\xi'|) d\xi',$$

and by integration by parts,

$$|K(y)| \leq C_N (1 + |y|)^{-N}, \quad N \in \mathbb{N}. \quad (5.45)$$

Now, we choose N large enough and (5.44) and (5.45) show that

$$\|M^{k,0}\|_{L^\infty \rightarrow L^\infty} \leq C, \quad (5.46)$$

with a constant C which does not depend on σ . Moreover, scaling by the factor 2^{-kd} in the direction of the vector σ , we see that

$$\|M^{k,0}\|_{L^1 \rightarrow L^1, \infty} \leq C 2^{kd}, \quad (5.47)$$

since we can compare with $2^{kd}M$, where M is the Hardy-Littlewood maximal operator defined by (5.10).

Finally, by interpolation between (5.46) and (5.47), we obtain that

$$\|M^{k,0}\|_{L^p \rightarrow L^p} \leq C2^{kd/p}, \quad p > 1. \quad (5.48)$$

Then $p > d$ implies that $\sum_k 2^{-k} \|M^{k,0}\|_{L^p \rightarrow L^p} \leq C \sum_k 2^{k(d/p-1)} \lesssim 1$.

Hence, it suffices to prove that

$$\|M_j^k\|_{L^p \rightarrow L^p} \leq C_p 2^{-j\epsilon(p)} 2^{kd/p}, \quad j \geq 1, \quad d < p < \infty, \quad \text{some } \epsilon(p) > 0,$$

where $M_j^k f(y) := \sup_{t>0} |A_{t,j}^k f(y)|$.

Since $A_{t,j}^k$ is localized to frequencies $|\delta'_t \xi'| \approx 2^j$, we still use Lemma 2.2.4 to prove that

$$\|M_j^k\|_{L^p \rightarrow L^p} \lesssim \|M_{j,loc}^k\|_{L^p \rightarrow L^p},$$

where $M_{j,loc}^k f(y) := \sup_{t \in [1,2]} |A_{t,j}^k f(y)|$.

Furthermore, a standard application of the method of stationary phase requires us to show the L^p -boundedness of the operator $\widetilde{\mathcal{M}}_{j,loc}^{k,1}$ given by

$$\begin{aligned} \widetilde{\mathcal{M}}_{j,loc}^{k,1} f(y) &:= \sup_{t \in [1,2]} |\widetilde{A}_{t,j}^{k,1} f(y)| \\ &= \sup_{t \in [1,2]} \left| \int_{\mathbb{R}} \eta_1\left(\frac{x_1}{t}\right) \int_{\mathbb{R}^2} e^{iQ_{k,x_1,d}(y',t,\xi')} E_{k,x_1/t,d}(\delta'_t \xi') dx_1 \right. \\ &\quad \left. \times \beta(2^{-j} |\delta'_t \xi'|) f(y_1 - x_1, \widehat{\xi}') d\xi' dx_1, \right. \end{aligned} \quad (5.49)$$

where

$$Q_{k,x_1,d}(y',t,\xi') := \langle \xi', (y_2, y_3 - t^{a_3} 2^{dk}) \rangle - t^{a_3} \xi_3 \tilde{\Psi}\left(\frac{x_1}{t}, s, \delta\right), \quad (5.50)$$

$$\tilde{\Psi}\left(\frac{x_1}{t}, s, \delta\right) := \Psi\left(\frac{x_1}{t}, \psi\left(\frac{x_1}{t}, s, \delta\right), s, \delta\right), \quad (5.51)$$

and

$$\Psi(x_1, x_2, s, \delta) := -sx_2 + x_2^d \Phi(x_1, \delta x_2). \quad (5.52)$$

By Lemma 2.2.3, we have

$$\|\widetilde{\mathcal{M}}_{j,loc}^{k,1} f\|_{L^p}^p \leq C_p \left(\int_{\mathbb{R}^3} |\rho(t) \widetilde{A}_{t,j}^{k,1} f(y)|^p \right)^{1/p'} \left(\int_{\mathbb{R}^3} \left| \frac{\partial}{\partial t} (\rho(t) \widetilde{A}_{t,j}^{k,1} f(y)) \right|^p \right)^{1/p}.$$

Since

$$\begin{aligned} \left| \frac{\partial}{\partial t} Q_{k,x_1,d}(y',t,\xi') \right| &= \left| -a_3 t^{a_3-1} \xi_3 2^{kd} + \xi_3 \frac{\partial}{\partial t} (t^{a_3} \tilde{\Psi}\left(\frac{x_1}{t}, s, \delta\right)) \right| \leq C 2^{kd} |\xi_3| \\ &\approx 2^{dk} 2^j, \end{aligned}$$

then $\frac{\partial}{\partial t}(\rho(t)\widetilde{A_{t,j}^{k,1}})$ behaves like $2^{kd}2^j\widetilde{A_{t,j}^{k,1}}$. Clearly we only consider the L^p -estimate for the operator $\widetilde{A_{t,j}^{k,1}}$.

The same estimates will follow from the proof of Section 5.3.1, by the assumption $p > d$, finally we obtain

$$\begin{aligned} & \sum_k 2^{-k} \sum_j \|\widetilde{\mathcal{M}_{j,loc}^{k,1}}\|_{L^p \rightarrow L^p} \\ & \leq C_p \sum_k 2^{-k} \sum_j \left(2^{-j(1/p+\epsilon(p))} \|\eta_0\|_{L^1} \right)^{(p-1)/p} \left(2^{kd} 2^{-j(-1+1/p+\epsilon(p))} \|\eta_0\|_{L^1} \right)^{1/p} \\ & = C_p \sum_k 2^{-k(1-d/p)} \sum_j 2^{-jp\epsilon(p)} \|\eta_0\|_{L^1} \leq C_p. \end{aligned}$$

We have finished the proof of Theorem 5.1.5.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht.

Kiel, den 21. 05. 2015

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