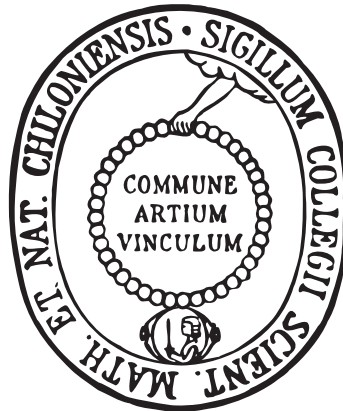


On Functions in the Speiser Class with One Tract



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Abstract

Let f be a transcendental entire map. A point $w \in \mathbb{C}$ is called *critical value* of f if there exists a point $z \in \mathbb{C}$ such that $f'(z) = 0$ and $f(z) = w$. A point $a \in \mathbb{C}$ is called an *asymptotic value* of f if there exists a curve γ with $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$ but $f(\gamma(t)) \rightarrow a$ as $t \rightarrow \infty$. The *singular set* of f is the set consisting of all critical and asymptotic values of f . The set \mathcal{B} of all transcendental entire functions with a bounded singular set is called *Eremenko-Lyubich class*. The *Speiser class* \mathcal{S} consists of all functions in class \mathcal{B} where the singular set is not only bounded but finite.

These classes have been thoroughly studied, notably, in complex dynamics, which deals with the behaviour of an entire or rational map f under *iteration*. Of particular interest is the construction of functions in classes \mathcal{B} and \mathcal{S} with prescribed behaviour. One method to obtain maps in class \mathcal{B} is using so-called *Cauchy integrals*. Gwyneth Stallard used this method to prove that for any $d \in (1, 2)$ there exists a function in class \mathcal{B} whose Julia set has *Hausdorff dimension* equal to d . It is the shape of the *tracts* of her maps which yields the desired Hausdorff dimension. Here, a tract is a connected component of the set where the modulus of the function is large.

The question arises whether Stallard's result also holds for maps in class \mathcal{S} . While we are not able to answer this question, we show that there exist functions in class \mathcal{S} whose tracts are in some sense similar to the tracts used by Stallard. The method of Cauchy integrals does generally not generate maps in class \mathcal{S} . An alternative construction method is *quasiconformal folding*, which was recently introduced by Christopher Bishop.

We use Bishop's method to construct *quasiregular maps* which only grow in one parabola shaped tract which is symmetric to the real axis and are bounded otherwise. Furthermore, we prove that for each constructed quasiregular map g there exists an entire function f in class \mathcal{S} such that $g = f \circ \phi$ for some quasiconformal homeomorphism ϕ . Thus, the tract of f , which is still symmetric to the real axis, is the quasiconformal image of the tract of g . Moreover, the quasiconformal map involved is *asymptotically conformal* at infinity. We use this to prove that the *maximum modulus* $M(r, f)$ of f on the circle with radius r is bounded below by a function which depends on the shape of the tract. In particular, we prove that there exists an entire map f in the class \mathcal{S} with only one tract, which is symmetric to the real axis, such that $\log \log M(r, f)$ is bounded below by $d \cdot \sqrt{r}$ for some $d > 0$.

Zusammenfassung

Sei f eine ganz transzendente Funktion. Ein Punkt $w \in \mathbb{C}$ heißt *kritischer Wert* der Funktion f , falls es einen Punkt $z \in \mathbb{C}$ mit $f'(z) = 0$ und $f(z) = w$ gibt. Ein Punkt $a \in \mathbb{C}$ heißt *asymptotischer Wert* von f , falls es eine Kurve γ mit $\gamma(t) \rightarrow \infty$ für $t \rightarrow \infty$ gibt, so dass $f(\gamma(t)) \rightarrow a$ gilt. Die Menge $\text{sing}(f^{-1})$ ist die *Menge der Singularitäten der Umkehrfunktion* von f und besteht aus allen kritischen und allen asymptotischen Werten von f . Die Menge \mathcal{B} aller ganz transzendenten Funktionen derart, dass $\text{sing}(f^{-1})$ beschränkt ist, heißt *Eremenko-Lyubich-Klasse*. Die *Speiser-Klasse* \mathcal{S} besteht aus allen Funktionen der Klasse \mathcal{B} , deren Menge der Singularitäten der Umkehrfunktion sogar endlich ist.

Insbesondere in der komplexen Dynamik, die sich mit dem Verhalten einer ganzen oder rationalen Funktion unter *Iteration* befasst, wurden diese Funktionenklassen ausgiebig untersucht. Die Konstruktion von Funktionen in den Klassen \mathcal{B} und \mathcal{S} mit vorgeschriebenem Verhalten ist von besonderem Interesse. Eine Möglichkeit, Funktionen der Klasse \mathcal{B} zu konstruieren, sind sogenannte *Cauchyintegrale*. Gwyneth Stallard nutzte diese Methode, um zu beweisen, dass es für jedes $d \in (1, 2)$ eine Funktion in der Klasse \mathcal{B} gibt, deren Julia-Menge *Hausdorff-Dimension* d hat. Die Form der *Trakte* ihrer Funktionen bestimmt dabei die Hausdorff-Dimension. Dabei ist ein Trakt eine Zusammenhangskomponente der Menge, auf welcher der Absolutbetrag der Funktion groß ist.

Es ergibt sich die Frage, ob Stallards Resultat auch für Funktionen der Klasse \mathcal{S} gilt. Auch wenn wir diese Frage nicht beantworten können, so zeigen wir, dass es Funktionen in der Klasse \mathcal{S} gibt, deren Trakte in gewissem Sinne den von Stallard genutzten Trakten ähneln. Im Allgemeinen sind Funktionen, die mithilfe von Cauchyintegralen konstruiert wurden, nicht in der Klasse \mathcal{S} . Eine alternative Konstruktionsmethode ist die *quasikonforme Faltung*, die kürzlich von Christopher Bishop vorgestellt wurde.

Wir nutzen Bishops Methode, um quasireguläre Funktionen zu konstruieren, die nur in einem parabelförmigen, zur reellen Achse symmetrischen Trakt wachsen und ansonsten beschränkt sind. Des Weiteren beweisen wir, dass es zu jeder so konstruierten quasiregulären Funktion g eine ganze Funktion f in der Klasse \mathcal{S} gibt, so dass $g = f \circ \phi$ für eine quasikonforme Abbildung ϕ gilt. Somit ist der Trakt von f , welcher ebenfalls symmetrisch zur reellen Achse ist, ein quasikonformes Bild des Traktes von g . Ferner ist die hierbei genutzte quasikonforme Abbildung *asymptotisch konform* bei unendlich. Wir nutzen die-

ses, um zu zeigen, dass der *Maximalbetrag* $M(r, f)$ von f auf dem Kreis mit Radius r von unten durch eine Funktion beschränkt ist, die von der Form des Traktes abhängt. Insbesondere beweisen wir, dass es eine ganze Funktion f in der Klasse \mathcal{S} gibt, die nur einen Trakt hat, der ferner symmetrisch zur reellen Achse ist, so dass $\log \log M(r, f)$ von unten durch $d \cdot \sqrt{r}$ für ein $d > 0$ beschränkt ist.

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Contents

1	Introduction	1
2	Preliminaries	7
2.1	Notations and miscellaneous	7
2.2	Complex analysis	11
2.3	Conformal mappings of infinite strips	14
2.4	Quasiconformal maps	18
2.4.1	Definitions and results	18
2.4.2	Asymptotic conformality	23
2.5	Quasiconformal folding	24
2.5.1	Idea and setup	25
2.5.2	Examples	34
3	Quasiregular Maps with One Parabola Shaped Tract	43
3.1	Notations	43
3.2	Asymptotic behaviour of h and τ	46
3.3	Shape of Ω	52
3.4	Asymptotic behaviour of ν	59
3.5	Comparison of the lengths of the edges	66
3.6	The ratio of diameter and distance	74
3.7	The graph (V, E) has bounded geometry	78
3.8	Applying Bishop's construction	79
3.9	Approximating Ω from inside	82
4	Growth of Finite-Type Maps with One Tract	89
	Bibliography	97
	Nomenclature	101

1 Introduction

In many function theoretic issues, the number of singularities of the inverse of a function plays an important role. Let f be a transcendental entire map, that is an entire map which is not a polynomial. The *singular set* $\text{sing}(f^{-1})$ of f consists of all those points $w \in \mathbb{C}$ where there exists a branch of f^{-1} which cannot be continued into w . See Section 2.2 for the precise definitions of this and the following concepts. Elements of $\text{sing}(f^{-1})$ are called *singular values* of f . For a transcendental entire map f , the singular values are the critical and asymptotic values of f . A point $w \in \mathbb{C}$ is called *critical value* of f if there exists a point $z \in \mathbb{C}$ such that $f(z) = w$ and $f'(z) = 0$. An *asymptotic value* $a \in \mathbb{C}$ of f is a point such that there exists a curve γ with $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$ but $f(\gamma(t)) \rightarrow a$ as $t \rightarrow \infty$. For example, one can check that 0 is the only singular value of the exponential map. Also, this point is an asymptotic value. This is exactly the point to which no branch of the logarithm can be extended.

The set \mathcal{B} of all transcendental entire maps which have a bounded singular set is called *Eremenko-Lyubich class* and the set \mathcal{S} of all those transcendental entire maps where the singular set is not only bounded but finite is called *Speiser class*. A function $f \in \mathcal{B}$ is also said to be of *bounded-type* and a function $f \in \mathcal{S}$ is called of *finite-type*. As mentioned above, the exponential map \exp has only one singular value, thus $\exp \in \mathcal{S}$. In contrast, the map si given by $\text{si}(z) = \sin(z)/z$ has infinitely many singular values, but those are all contained in $\{z \in \mathbb{C} : |z| \leq 1\}$. By definition of the Eremenko-Lyubich class, we see that $\text{si} \in \mathcal{B}$.

The classes \mathcal{S} and \mathcal{B} have been thoroughly studied, in particular, in connection with complex dynamics. This theory deals with the behaviour of an entire or rational map f under *iteration*, which is the repeated application of the map. In the late 1910s, Gaston Julia [Jul18] and Pierre Fatou [Fat19] laid the cornerstone to complex dynamics. During their studies, they observed that the complex plane can be divided into two sets – one set where the iterates of the map act *normal* and one set where these iterates have a chaotic behaviour. This means that two points which are close together and lie in the first set show the same behaviour under iteration, whilst two such points in the latter set might behave completely different under iteration. Later on, these sets were named *Fatou set* and *Julia set*, respectively. An elementary description of these sets is only possible in very few cases. For example, the Julia set of the quadratic polynomial $q(z) = z^2$

is just $\mathcal{J}(q) = \{z \in \mathbb{C} : |z| = 1\}$ and the Fatou set $\mathcal{F}(q) = \mathbb{C} \setminus \mathcal{J}(q)$. Investigating the exponential map yields $\mathcal{F}(\exp) = \emptyset$ and $\mathcal{J}(\exp) = \mathbb{C}$, see [Mis81]. In most other cases, neither the Fatou set nor the Julia set can be described in such a nice manner. For instance, the Julia set consists of uncountably many connected components if it is not connected. A lot of work has been done in order to get a better understanding of these sets. For more information see for example the books by John Milnor [Mil06], Alan Beardon [Bea91] and Norbert Steinmetz [Ste93].

An example for the thorough study of the classes \mathcal{B} and \mathcal{S} is that Sullivan's No Wandering Domain Theorem [Sul85, Theorem 1] – rational maps do not have wandering domains – also holds for finite-type maps [EL92, Theorem 3], [GK86, Theorem 4.2], but in general bounded-type maps may have wandering domains [Bis15, Theorem 17.1]. Particularly, the Eremenko-Lyubich class turned out to be important in complex dynamics. The so-called *logarithmic change of variables* is the main tool to study this class (see [EL92] for more information). Due to the importance of the Eremenko-Lyubich class, the construction of bounded-type maps with given properties is of great interest. One method to obtain such maps is using so-called *Cauchy integrals*.

As an example for this method, let $g(z) = \exp(e^z)$ and let T be the half-strip given by $T = \{x + iy : x > 0, |y| < \pi\}$. Let γ be a curve which parametrises ∂T in clockwise direction. The integral

$$h(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta$$

converges for every $z \in \mathbb{C} \setminus \overline{T}$. The function h can be extended analytically to an entire map f . Note that this entire map is bounded on $\mathbb{C} \setminus \overline{T}$. Thus, its *tracts* – the components of the set where $|f(z)|$ is large – are contained in \overline{T} . For an introduction to Cauchy integrals and recent results see [RG14, Section 2]. The method of Cauchy integrals allows to construct entire functions in class \mathcal{B} with only one tract. This tract can be arbitrarily narrow, corresponding to the arbitrarily fast growth of the function. We consider functions in class \mathcal{S} with one tract and study how narrow the tract can be.

The *Hausdorff dimension* is a generalisation of the intuitive concept of dimension. For a definition see [Fal86, Section 1.2]. There is a lot of literature on the Hausdorff dimension of Julia sets of transcendental entire and rational maps. One of these is a work by Gwyneth Stallard [Sta00]. She proved that for any real number $d \in (1, 2)$ there exists a transcendental entire map $f \in \mathcal{B}$ such that the Hausdorff dimension of $\mathcal{J}(f)$ is d . Since the dimension of $\mathcal{J}(\exp)$ is 2 and for any $f \in \mathcal{B}$ the Hausdorff dimension of $\mathcal{J}(f)$ is strictly larger than 1 (see [Sta96, Theorem 2]), her result answers the question which dimensions of Julia sets in class \mathcal{B} are possible. Namely, every Hausdorff dimension in $(1, 2]$ can be achieved.

Stallard constructs the maps using Cauchy integrals. Her functions essentially only grow in a certain domain and are bounded otherwise. These domains are given by

$$G_p = \left\{ x + iy : x > 3, |y| < \frac{\pi x}{(1+p)(\log x)^p} \right\}$$

where p is a positive real number. These sets are shaped like parabolas opened to the right and the width of G_p is determined by p . It is the shape of the sets used by Stallard which later on yields the desired Hausdorff dimension. The Julia set of the map which is constructed with respect to the parameter p has Hausdorff dimension $1 + \frac{1}{1+p}$.

The question arises whether Stallard's result still holds if we only consider maps with finitely many singular values, hence maps in class \mathcal{S} . While we are not able to answer this question, we show that there exist functions in class \mathcal{S} whose tracts are close to the domains G_p in some sense. Since the method of Cauchy integrals does generally not generate maps with finite singular sets, it is not possible to use this technique to obtain maps in class \mathcal{S} . An alternative construction method is *quasiconformal folding*. This technique was introduced by Christopher Bishop in [Bis15] and will be explained in Section 2.5.

Quasiconformal maps have turned out to be very important in complex analysis. For example, they are used in the proofs of the No Wandering Domain Theorems mentioned above. A quasiconformal map is an orientation preserving homeomorphism which maps infinitesimal small circles onto infinitesimal small ellipses where the ratio of the major and minor axis is uniformly bounded. If we do not require the map to be injective, it is called *quasiregular*. Simple examples are $x + iy \mapsto 2x + iy$ for a quasiconformal map and $x + iy \mapsto (2x + iy)^2$ for a quasiregular map which is not quasiconformal. For precise definitions, see Section 2.4.

The idea of quasiconformal folding uses that the preimage of $[-1, 1]$ under a transcendental entire map $f \in \mathcal{S}$ with the only critical values 1 and -1 is a locally finite, unbounded plane tree T where the vertices are the preimages of 1 and -1 and the edges are the preimages of $(-1, 1)$. A plane tree is given by a set of vertices $V \subset \mathbb{C}$ and a set of edges E which are all Jordan curves that are pairwise disjoint with the possible exception of one common endpoint in V , see Section 2.1. Every connected component Ω_j of $\mathbb{C} \setminus T = \mathbb{C} \setminus f^{-1}([-1, 1])$ is simply connected and one can find conformal maps τ_j mapping Ω_j onto the right half-plane such that $f = \cosh \circ \tau_j$ on Ω_j (see Section 2.5.1). Bishop reverses this procedure and starts with an unbounded, locally finite plane tree T and conformal maps τ_j from each component Ω_j of $\mathbb{C} \setminus T$ onto the right half-plane. In general, the map g defined by $g = \cosh \circ \tau_j$ on Ω_j cannot be continued to an entire map. Bishop shows that under certain geometric conditions on T – the *bounded geometry conditions*, see Definition 2.65 – the map g can be modified into a global quasiregular

map coinciding with g on \mathbb{C} with the exception of a small neighbourhood of T . Applying the Measurable Riemann Mapping Theorem (Theorem 2.51) yields a finite-type map f which essentially behaves like g .

For $r_0 > 0$ and a map $w : (r_0, \infty) \rightarrow (0, \infty)$ we define

$$G(r_0, w) := \{re^{i\varphi} : r > r_0, |\varphi| < w(r)\}.$$

By using *quasiconformal folding*, we will construct quasiregular maps whose only tract is parabola shaped.

Theorem 1. *Let $\delta \in [0, \frac{1}{2}]$. For $\delta = 0$ let $\alpha < -1$, for $\delta = \frac{1}{2}$ let $\alpha \geq 0$ and let $\alpha \in \mathbb{R}$ otherwise. Let $w : (1, \infty) \rightarrow (0, \infty)$, $r \mapsto \frac{(\log r)^\alpha}{r^\delta}$ and let $k_0 > 0$. Then there exist constants $k \in (0, k_0)$, $\rho > 1$ and $R \geq 1$, and a quasiregular map g such that*

$$G(\rho, kw) \subset \{z \in \mathbb{C} : |g(z)| \geq R\} \subset G(1, k_0w).$$

Furthermore, there exist a quasiconformal map ϕ and an entire map $f \in \mathcal{S}$ such that $g = f \circ \phi$. The only critical values of f are 1 and -1 and the only asymptotic value of f is 0. Moreover, g has only one tract. This tract is symmetric to the real axis.

For $\delta = 0$, the tracts of the quasiregular maps constructed in Theorem 1 are similar to those used by Stallard in her construction of functions in class \mathcal{B} with preassigned Hausdorff dimension of their corresponding Julia sets.

For g , ϕ and f with $g = f \circ \phi$ constructed by Theorem 1, we prove that the involved quasiconformal map ϕ is *asymptotically conformal* at infinity, which means that the map behaves like the multiplication with a non-zero complex constant near infinity, see Section 2.4. We will use this to prove a lower bound of the *maximum modulus* of f . The maximum modulus of f on the circle with radius $r > 0$ is defined by $M(r, f) = \max_{|z|=r} |f(z)|$.

Theorem 2. *There exists a function $f \in \mathcal{S}$ with only one tract, which is symmetric to the real axis, and a constant $d > 0$ such that*

$$\log \log M(r, f) \geq d \cdot \sqrt{r}$$

for r sufficiently large.

Chapter 2 contains notations and preliminaries on the topics needed. In particular, we discuss conformal mappings of infinite strips, quasiconformal and quasiregular maps, and quasiconformal folding. The section on quasiconformal folding contains two examples on how to apply the technique. The first can be found in Bishop's article [Bis15, page 10]. The second is also mentioned there [Bis15, page 3] but we write it out in more

detail. In Chapter 3 we prove Theorem 1. The first section deals with the construction of the bounded geometry tree and the definitions of several functions and constants. The next sections deal with the behaviour of these functions. Afterwards, we prove that the constructed tree has in fact bounded geometry and use Bishop's construction to obtain the quasiregular map g . In Chapter 4 we rephrase Theorem 2 and prove this result using Theorem 1.

2 Preliminaries

2.1 Notations and miscellaneous

In this section, we introduce several notations used throughout this thesis. In addition, we will state definitions and results which do not fit into the context of the following sections. At the end of this thesis, we provide a nomenclature which contains the following symbols and notations as well as those defined in the other chapters.

Notations

We denote the one-point compactification of the complex plane \mathbb{C} by $\widehat{\mathbb{C}}$. This set is also called the *Riemann sphere*. Furthermore, we denote the left half-plane by \mathbb{H}_l . Thus, $\mathbb{H}_l = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. The right half-plane is denoted by \mathbb{H}_r and the upper half-plane by \mathbb{H}_u . The open unit disk in \mathbb{C} is denoted by \mathbb{D} .

Let $A, B \subset \mathbb{C}$ be non-empty. The *diameter* of A is given by

$$\operatorname{diam}(A) = \sup\{|z - w| : z, w \in A\}$$

and the *distance* between A and B by

$$\operatorname{dist}(A, B) = \inf\{|z - w| : z \in A, w \in B\}.$$

The open disk in \mathbb{C} with radius $r > 0$ and centre $z \in \mathbb{C}$ is denoted by $D(z, r)$.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function and $R > 0$. A connected component of

$$\{z \in \mathbb{C} : |f(z)| > R\}$$

is called *tract* of f .

Graph theory

Recall that a graph $G = (V, E)$ is given by a non-empty set V of *vertices* and a usually non-empty set $E \subset \{\{u, v\} : u, v \in V, u \neq v\}$ of *edges*. For an introduction to graph theory and the definitions in this subsection, see [Die05].

Let $G = (V, E)$ be a graph. The *degree* $d(v)$ of a vertex $v \in V$ is the number of edges adjacent to v , that is $d(v) = |\{u \in V : \{v, u\} \in E\}|$. If the degree of v is finite for all $v \in V$, the graph is called *locally finite*. A *path* in G is a tuple (v_1, \dots, v_n) of vertices such that $\{v_j, v_{j+1}\} \in E$ for all $1 \leq j < n$ and v_1, \dots, v_n are pairwise disjoint. A *cycle* in G is a tuple (v_1, \dots, v_n) of vertices such that $n \geq 3$, $v_1 = v_n$, $\{v_{n-1}, v_n\} \in E$ and (v_1, \dots, v_{n-1}) is a path. A graph is *connected* if for all $x, y \in V$ there exists a path (v_1, \dots, v_n) in G such that $v_1 = x$ and $v_n = y$. A connected graph without cycles is called *tree*.

A graph is called *planar* if there exists an embedding of G into the plane \mathbb{R}^2 (or \mathbb{C}) in the following way: The vertices are identified with pairwise disjoint points in \mathbb{R}^2 (or \mathbb{C}) and the edges are represented by Jordan arcs with starting points and endpoints in V which at most intersect in one common endpoint or starting point.

Let $V \subset \mathbb{C}$ be a set consisting of pairwise disjoint points and let E be a set of Jordan arcs each of which connects two elements of V . If the elements of E are disjoint with the possible exception of one common endpoint or starting point, we call $G = (V, E)$ a *plane graph*. Note that every embedding of a planar graph is a plane graph. If we speak about graphs in this thesis, we always refer to plane graphs.

Lipschitz continuity and absolute continuity

In Section 2.3, we utilise that Lipschitz continuous functions are differentiable almost everywhere and can be written as the integral over their derivative. We first state the definitions of Lipschitz continuity and absolute continuity and then give the result needed.

Definition 2.1 (Lipschitz continuity). Let $I \subset \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be a function. We say that f is *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

holds for all $x, y \in I$.

Definition 2.2 (Absolute continuity). Let $I \subset \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be a function. We say that f is *absolutely continuous* if for all $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for any finite sequence $(a_i, b_i)_{i=1}^k$ of pairwise disjoint open intervals in I with $\sum_{i=1}^k |b_i - a_i| < \delta$ we have $\sum_{i=1}^k |f(b_i) - f(a_i)| < \varepsilon$.

We immediately see that every Lipschitz continuous function is absolutely continuous. The following theorem can be found in [Wal90, Satz 9.23].

Theorem 2.3. *If f is absolutely continuous on $I = [a, b]$, then f is differentiable almost everywhere, f' is (Lebesgue-) integrable and*

$$f(b) - f(a) = \int_a^b f'(t) dt.$$

Riemann-Stieltjes integral

In Section 2.3, we need the so-called *Riemann-Stieltjes integral*. Before we can give the definition, we need to know what functions of *bounded variation* are. The following definitions can be found in [Wal90, Section 5.20] and [Wal90, Chapter 6].

Definition 2.4 (Bounded variation). Let $[a, b] \subset \mathbb{R}$ and let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Let $Z = \{t_0, \dots, t_n\}$ with $t_0 = a < t_1 < \dots < t_n = b$ be a partition of $[a, b]$. The *variation* of f with respect to Z is defined by

$$\text{var}(Z) = \text{var}(Z; f) := \sum_{k=1}^n |f(t_k) - f(t_{k-1})|.$$

The *total variation* is given by

$$V_a^b(f) := \sup_Z \text{var}(Z; f),$$

where the supremum is taken over all partitions Z of $[a, b]$. The function f is called of *bounded variation* if $V_a^b(f) < \infty$. The set of all functions of bounded variation on $[a, b]$ is denoted by $BV([a, b])$.

Remark 2.5. Any monotone and any piecewise continuously differentiable function is of bounded variation.

Definition 2.6 (Riemann-Stieltjes integral). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be functions. For any partition $Z = \{t_0, \dots, t_n\}$ with $a = t_0 < \dots < t_n = b$ and any set $X = \{x_1, \dots, x_n\}$ of points such that $x_k \in (t_{k-1}, t_k)$ we define the *Riemann-Stieltjes sum* by

$$\sigma(Z, x; f dg) = \sigma(Z, x) := \sum_{k=1}^n f(x_k)(g(t_k) - g(t_{k-1})).$$

If there exists $J \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions Z and intermediate points X with $|t_{k+1} - t_k| < \delta$ for all $k \in \{0, \dots, n-1\}$ the inequality

$$|J - \sigma(Z, x)| < \varepsilon$$

holds, we call

$$\int_a^b f dg = \int_a^b f(t) dg(t) := \lim_Z \sigma(Z, x)$$

the *Riemann-Stieltjes integral* of f with respect to g .

The Riemann integral is the special case $g(t) = t$ of the Riemann-Stieltjes integral.

Remark 2.7. In particular, the Riemann-Stieltjes integral exists if f is continuous and $g \in BV([a, b])$. If g is continuously differentiable, the Riemann-Stieltjes integral fulfils the equality

$$\int_a^b f \, dg = \int_a^b f(t)g'(t) \, dt,$$

where the latter integral is a Riemann integral.

Coverings

In Section 2.5, we need the concept of coverings. The following definitions are extracted from [Jos97, Section 1.3], [NR11, Section 10.2] and [NR11, Section 5.1].

Definition 2.8. Let M , M' and M'' be manifolds.

- A map $\pi : M' \rightarrow M$ is said to be a *local homeomorphism* if each $x \in M'$ has a neighbourhood U such that $\pi(U)$ is open in M and $\pi|_U$ is a homeomorphism (onto $\pi(U)$).
- A local homeomorphism $\pi : M' \rightarrow M$ is called a *covering* if each $x \in M$ has a (connected) neighbourhood V such that every connected component of $\pi^{-1}(V)$ is mapped by π homeomorphically onto V .
- A covering $\pi : M' \rightarrow M$ is called *universal* if M' is simply connected.
- A covering between Riemann surfaces which is holomorphic is called *holomorphic covering*.
- Let $\pi : M' \rightarrow M$ and $f : M'' \rightarrow M$ be continuous. A continuous map $\tilde{f} : M'' \rightarrow M'$ such that $\pi \circ \tilde{f} = f$ is called *lift* (of f with respect to π).

Theorem 2.9. *Let $\pi : M' \rightarrow M$ be a covering, M'' a simply connected manifold, and $f : M'' \rightarrow M$ a continuous map. Then there exists a lift of f with respect to π .*

The last theorem can be found in [Jos97, Theorem 1.3.1] and the following theorems can be found in [Jos97, Corollary 1.3.2] and [NR11, Theorem 5.1.3], respectively.

Theorem 2.10. *If M is simply connected, then every covering $M' \rightarrow M$ is a homeomorphism.*

Theorem 2.11. *Let M , M' and M'' be Riemann surfaces and let $\pi_{M'} : M' \rightarrow M$ and $\pi_{M''} : M'' \rightarrow M$ be holomorphic coverings. If $f : M' \rightarrow M''$ is a lift of $\pi_{M'}$, then f is a holomorphic covering.*

Theorem 2.12. *Let M , M' and M'' be Riemann surfaces. Assume that M' and M'' are simply connected. Let $f : M' \rightarrow M$ and $g : M'' \rightarrow M$ be holomorphic coverings. Then there exists a biholomorphic map $\tau : M' \rightarrow M''$ such that $f = g \circ \tau$.*

Proof. By Theorem 2.9, there exists a lift $\tau : M' \rightarrow M''$ of f with respect to g . By Theorem 2.11, we get that τ is a holomorphic covering. Since M'' is simply connected, Theorem 2.10 yields that τ is a homeomorphism. Hence, τ is biholomorphic. \square

2.2 Complex analysis

This section deals with definitions and results in complex analysis. We assume that the reader is familiar with the basic notation of holomorphic maps and the main results in this field. For an introduction, see for example the books by Lars Ahlfors [Ahl66] or Reinhold Remmert [Rem84, Rem91].

We start with the notation of a conformal map followed by the notation of a transcendental entire map and the notation of the maximum modulus.

Notation 2.13 (Conformal map). Let $G \subset \mathbb{C}$ be a domain. A biholomorphic map $f : G \rightarrow f(G)$ is called *conformal*.

Definition 2.14 (Transcendental entire map). A map $f : \mathbb{C} \rightarrow \mathbb{C}$ which is holomorphic everywhere is called *entire*. If f is not a polynomial, we call f *transcendental*. In this case, ∞ is an essential singularity.

Notation 2.15 (Maximum modulus). Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be entire and let $r > 0$. The *maximum modulus* of f is defined by

$$M(r, f) := \max_{|z|=r} |f(z)|.$$

Although Schwarz's reflection principle is part of some introductory lectures on complex analysis, we restate the theorem here since it will be used in Chapter 3. Before we state the theorem, one definition is needed.

Definition 2.16 (Symmetric domain). Let $\Omega \subset \mathbb{C}$ be a domain. The *complex conjugate domain* Ω^* is defined by

$$\Omega^* = \{z \in \mathbb{C} : \bar{z} \in \Omega\}.$$

We say that Ω is *symmetric* if $\Omega = \Omega^*$.

For a proof of the next theorem, see [Ahl66, Section 6.5].

Theorem 2.17 (Schwarz's reflection principle). *Let $\Omega \subset \mathbb{C}$ be a symmetric domain. Let $f : \Omega \cap \overline{\mathbb{H}_u} \rightarrow \mathbb{C}$ be a continuous function which is holomorphic on $\Omega \cap \mathbb{H}_u$ and fulfils $f(\Omega \cap \mathbb{R}) \subset \mathbb{R}$. Then the reflected map*

$$\tilde{f} : \Omega \rightarrow \mathbb{C}, z \mapsto \begin{cases} f(z) & , z \in \Omega \cap \overline{\mathbb{H}_u} \\ \overline{f(\bar{z})} & , z \notin \Omega \cap \overline{\mathbb{H}_u} \end{cases}$$

is holomorphic.

Given two functions f and g defined on the same domain, we oftentimes want to know whether these two maps behave in the same way in neighbourhoods of specific points. One possibility to express this is the asymptotic equality.

Notation 2.18 (Asymptotic equality). Let $G \subset \mathbb{C}$ and let $f, g : G \rightarrow \mathbb{C}$. Let $z_0 \in \overline{G}$. If G is unbounded, $z_0 = \infty$ is also allowed. We say that f and g are *asymptotically equal* at z_0 if

$$f(z) = (1 + o(1)) \cdot g(z) \quad \text{as } z \rightarrow z_0.$$

In other words,

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = 1.$$

We write $f \sim g$ as $z \rightarrow z_0$ if f and g are asymptotically equal at z_0 .

We already mentioned the Eremenko-Lyubich class and the Speiser class in the introduction. The next goal in this section is to define these sets precisely. For an introduction and some results, see [EL92] and [MNTU00, Section 2.4].

Definition 2.19 (Singular value). Let f be a transcendental entire map. A point $w \in \mathbb{C}$ is called *critical value* of f if there exists a point $z \in \mathbb{C}$ such that $f'(z) = 0$ and $f(z) = w$. A point $a \in \mathbb{C}$ is called *asymptotic value* of f if there exists a curve $\gamma : (0, \infty) \rightarrow \mathbb{C}$ such that $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$ but $f(\gamma(t)) \rightarrow a$ as $t \rightarrow \infty$. A point $z \in \mathbb{C}$ is a *singular value* of f if it is a critical value or an asymptotic value. The set consisting of all singular values is called *singular set* of f and is denoted by $\text{sing}(f^{-1})$.

Remark 2.20. Let f be a transcendental entire map. A point $z \in \mathbb{C}$ is a singular value of f if and only if there exists a branch of f^{-1} which cannot be continued into z .

Definition 2.21. The *Eremenko-Lyubich class* is defined by

$$\mathcal{B} = \{f \text{ transcendental entire} : \text{sing}(f^{-1}) \text{ is bounded}\}$$

and the *Speiser class* by

$$\mathcal{S} = \{f \text{ transcendental entire} : \text{sing}(f^{-1}) \text{ is finite}\}.$$

Remark 2.22. Functions in class \mathcal{B} are called *bounded-type* entire maps and functions in class \mathcal{S} are called *finite-type* entire maps.

We will need Carathéodory's Theorem during the introduction of quasiconformal maps in Section 2.4. The theorem can be found in [Pom92, Theorem 2.1].

Theorem 2.23 (Carathéodory). *Let $\psi : \mathbb{D} \rightarrow D \subset \widehat{\mathbb{C}}$ be biholomorphic. The following statements are equivalent:*

- (i) ψ can be extended continuously onto $\partial\mathbb{D}$.
- (ii) ∂D is locally connected.
- (iii) $\widehat{\mathbb{C}} \setminus D$ is locally connected.

The following theorem deals with univalent functions defined on the unit disk.

Theorem 2.24 (Koebe Distortion Theorem). *Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be univalent with $f(0) = 0$ and $f'(0) = 1$. If $z \in \mathbb{D}$, then the inequalities*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$$

hold. Furthermore,

$$D\left(0, \frac{1}{4}\right) \subset f(\mathbb{D}).$$

For a proof of the last theorem, see [Pom75, Theorem 1.6, Corollary 1.4]. The result can be extended to univalent functions defined on some disk in the complex plane.

Corollary 2.25. *Let $r > 0$, $z_0 \in \mathbb{C}$, $\varrho \in (0, 1)$ and let $f : D(z_0, r) \rightarrow \mathbb{C}$ be univalent. For all $z \in D(z_0, \varrho r)$ the inequality*

$$\frac{1-\varrho}{(1+\varrho)^3} \leq \frac{|f'(z)|}{|f'(z_0)|} \leq \frac{1+\varrho}{(1-\varrho)^3}$$

holds. If $z \in \partial D(z_0, \varrho r)$, then also

$$\frac{\varrho}{(1+\varrho)^2} \leq \frac{|f(z) - f(z_0)|}{r|f'(z_0)|} \leq \frac{\varrho}{(1-\varrho)^2}$$

holds. Furthermore,

$$D\left(f(z_0), \frac{1}{4}|f'(z_0)|r\right) \subset f(D(z_0, r)).$$

Proof. Apply Theorem 2.24 to

$$g : \mathbb{D} \rightarrow \mathbb{C}, \quad z \mapsto \frac{f(z_0 + rz) - f(z_0)}{rf'(z_0)}.$$

□

We will need the following lemma in Chapter 3. A proof of this result can be found in [Pom92, Proposition 1.10].

Lemma 2.26. *Let f be analytic in the convex domain H . If*

$$\operatorname{Re} f'(z) > 0 \quad \text{for all } z \in H,$$

then f is univalent in H .

2.3 Conformal mappings of infinite strips

In the proof of Theorem 1, we need to know more about the behaviour of a conformal function mapping an unbounded domain, symmetric to the real axis, onto a parallel strip. In his thesis, Lars Ahlfors proved the following inequality, see [Ahl30, Erste Hauptungleichung] or [Nev70, Chapter IV, Section 4]. Here, we restrict the theorem to symmetric strips.

Theorem 2.27. *Let $\Omega = \{x + iy : |y| < \psi(x)\}$ be a horizontal strip where $\psi : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is a positive continuous function. Let σ map Ω conformally onto $\{x + iy : |y| < \frac{\pi}{2}\}$ such that $\operatorname{Re} \sigma(z) \rightarrow \pm\infty$ as $\operatorname{Re} z \rightarrow \pm\infty$. Let*

$$\beta(x) = \inf_{|y| < \psi(x)} \operatorname{Re} \sigma(x + iy) \quad \text{and} \quad \alpha(x) = \sup_{|y| < \psi(x)} \operatorname{Re} \sigma(x + iy).$$

If $\int_{x_1}^{x_2} \frac{dt}{\psi(t)} > 4$, then

$$\beta(x_2) - \alpha(x_1) \geq \frac{\pi}{2} \int_{x_1}^{x_2} \frac{dt}{\psi(t)} - 4\pi.$$

This inequality states that a conformal map from a symmetric strip onto a parallel strip expands at a rate which depends on the width of the initial strip. In his thesis, Ahlfors also proved that a similar inequality holds from above under slightly more restrictive conditions.

Stefan Warschawski investigated conformal mappings of infinite strips in [War42]. Under some conditions on the strip, he sharpens Ahlfors' results and does not only give an asymptotic expression of such a map but also results on the inverse of the map. Before we state Warschawski's theorem, we will give some definitions needed. The following definitions and results can all be found in [War42]. For the following definition, see [War42, page 282].

Definition 2.28 (Simple Jordan strip). Let $x_0 \in \mathbb{R}$ and let $\phi_+, \phi_- : [x_0, \infty) \rightarrow \mathbb{R}$ be continuous. Assume that $\phi_-(x) < \phi_+(x)$ for all $x \in [x_0, \infty)$. Let C_1 be a Jordan arc in $\{x + iy : x \leq x_0\}$ which connects $x_0 + i\phi_+(x_0)$ and $x_0 + i\phi_-(x_0)$. The curve consisting of C_1 and the graphs of ϕ_+ and ϕ_- is a Jordan arc which decomposes the complex plane in two regions. The region containing $\{x + iy : x \geq x_0, \phi_-(x) < y < \phi_+(x)\}$ is called a *simple Jordan strip*.

We define $\Theta : [x_0, \infty) \rightarrow (0, \infty)$, $x \mapsto \phi_+(x) - \phi_-(x)$ and for $x \in [x_0, \infty)$ we denote by Θ_x the segment $\{x + iy : \phi_-(x) \leq y \leq \phi_+(x)\}$. Thus, $\Theta(x)$ is the width of the strip at $\text{Re} = x$.

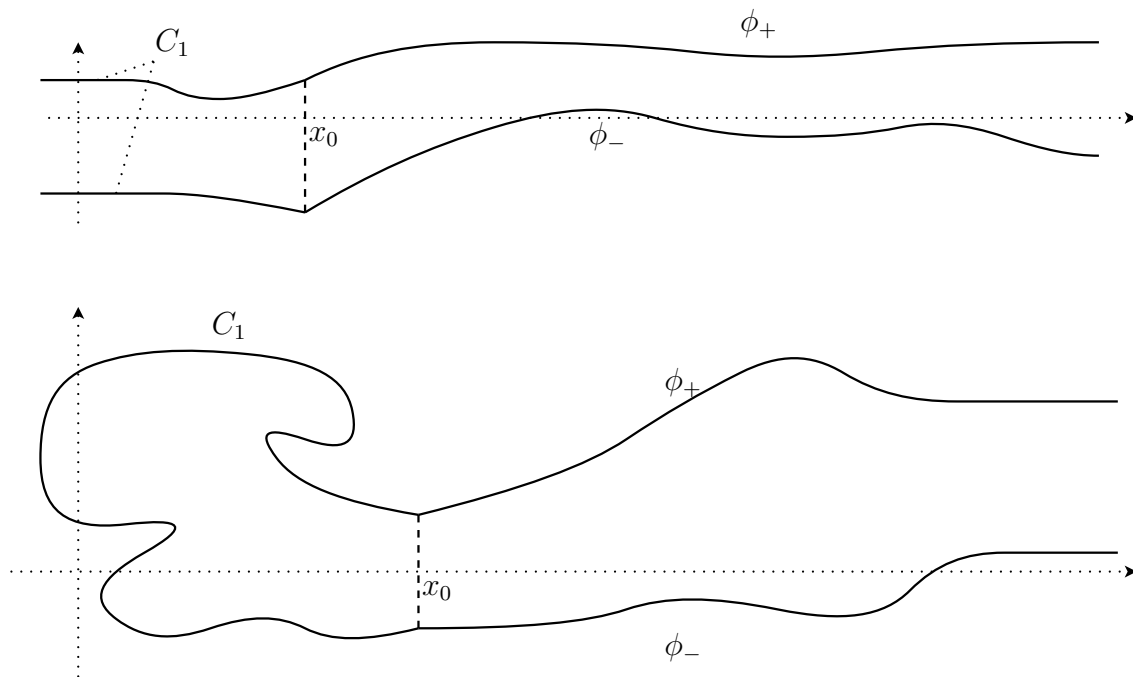


Figure 2.1: Examples of simple Jordan strips. The second one is an L-strip with boundary inclination 0 at $u = +\infty$, compare Definition 2.30.

Remark 2.29. The curve C_1 may pass through “ $u = -\infty$ ”. Of course, the curve consisting of C_1 and the graphs of ϕ_+ and ϕ_- does not decompose the complex plane in two regions but in three. But the region containing $\{x + iy : x \geq x_0, \phi_-(x) < y < \phi_+(x)\}$ is still one of these.

In particular, horizontal strips, horizontal half-strips, and domains symmetric to the real axis and bounded by the graph of some continuous map are simple Jordan strips. A vertical strip is no simple Jordan strip, but a simple rotation converts such a strip into a simple Jordan strip.

Definition 2.30 (L-strip). Let C be the curve $u \mapsto u + i\phi(u)$ where ϕ is a continuous, real function defined on $[u_0, \infty)$ for some $u_0 \in \mathbb{R}$. We say that C has an *L-tangent* with angle of inclination $\gamma \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ at $u = +\infty$ if for any $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that for all $u_2 > u_1 > R(\varepsilon)$ we have

$$|\arg((u_2 - u_1) + i(\phi(u_2) - \phi(u_1))) - \gamma| < \varepsilon.$$

If both L-tangents of the boundary curves of a simple Jordan strip S have the same angle of inclination γ at $u = +\infty$, we call S an *L-strip* with boundary inclination γ .

The definition of an L-strip can be found in [War42, page 282]. Notably, horizontal strips are L-strips with boundary inclination 0. Strips symmetric to the real axis and bounded by a differentiable map ϕ such that $\phi'(x) \rightarrow 0$ as $x \rightarrow \infty$ are L-strips with boundary inclination 0 as well. In contrast, a sector symmetric to the real axis with opening angle ϑ is not an L-strip since the boundary inclination of the upper boundary is $\frac{\vartheta}{2}$, the one of the lower boundary $-\frac{\vartheta}{2}$.

Let S be a simple Jordan strip where both boundary curves ϕ_+ and ϕ_- have an L-tangent at ∞ with boundary inclination $\gamma_+ \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\gamma_- \in (-\frac{\pi}{2}, \frac{\pi}{2})$, respectively. Thus, for $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that for all $x > y \geq R(\varepsilon)$ we have

$$\varepsilon > |\arg((x - y) + i(\phi_+(x) - \phi_+(y))) - \gamma_+| = \left| \arctan\left(\frac{\phi_+(x) - \phi_+(y)}{x - y}\right) - \gamma_+ \right|.$$

Choose $\varepsilon_0 > 0$ small enough such that $-\frac{\pi}{2} < \gamma_+ - \varepsilon_0 < \gamma_+ + \varepsilon_0 < \frac{\pi}{2}$. Since \tan is continuous, there exists a constant $C > 0$ such that

$$\left| \left(\frac{\phi_+(x) - \phi_+(y)}{x - y}\right) - \tan(\gamma_+) \right| < C$$

for all $x > y \geq R(\varepsilon_0)$. We get that

$$|\phi_+(x) - \phi_+(y)| \leq (|\tan(\gamma_+)| + C)|x - y|$$

for all $x > y \geq R(\varepsilon_0)$. Thus, $\phi_+|_{[R(\varepsilon_0), \infty)}$ is Lipschitz continuous. If ε_0 is small enough, we have $-\frac{\pi}{2} < \gamma_- - \varepsilon_0 < \gamma_- + \varepsilon_0 < \frac{\pi}{2}$ as well. The same reasoning as above shows that also $\phi_-|_{[R(\varepsilon_0), \infty)}$ is Lipschitz continuous. Thus, also $\Theta = \phi_+ - \phi_-$ is Lipschitz continuous. Since Lipschitz continuous maps are also absolutely continuous, Theorem 2.3 implies that Θ is differentiable almost everywhere and

$$\Theta(x) - \Theta(y) = \int_y^x \Theta'(t) dt$$

holds for all $x > y \geq R(\varepsilon_0)$. Furthermore, Θ' is bounded since Θ is Lipschitz continuous and thus Θ has bounded difference quotients.

Before we can finally state Warschawski's Theorem, we need the notation of *finite boundary turning*, which can be found in [War42, page 306].

Definition 2.31 (Finite boundary turning). Let S be an L-strip with boundary inclination γ , $|\gamma| < \frac{\pi}{2}$, at $u = +\infty$. Let ϕ_+ and ϕ_- represent the boundary curves of S . We say that S is a strip with *finite boundary turning* if ϕ'_+ and ϕ'_- are continuous for sufficiently large u and

$$\int_{u=u_1}^{\infty} |d\phi'_+(u)|, \int_{u=u_1}^{\infty} |d\phi'_-(u)|$$

converge.

Note that the latter integrals converge if ϕ'_+ and ϕ'_- , respectively, are non-negative, continuously differentiable, and both $\phi'_+(x) \rightarrow 0$ and $\phi'_-(x) \rightarrow 0$ as $x \rightarrow \infty$. We can now state Warschawski's Theorem. The version given below combines results from [War42, Theorem IX] and [War42, Theorem X].

Theorem 2.32 (Warschawski). *Let S be an L-strip with boundary inclination $\gamma = 0$ and finite boundary turning at $u = +\infty$. Furthermore, let*

$$\int^{\infty} \frac{\Theta'^2(t)}{\Theta(t)} dt \tag{2.1}$$

be convergent. Let Z map S conformally onto the strip $|y| < \frac{\pi}{2}$ such that $\operatorname{Re} Z(w) \rightarrow \infty$ as $\operatorname{Re} w \rightarrow \infty$. Then for $w = u + iv \in \bar{S}$ we have

$$Z(w) = \lambda_0 + \pi \int_{u_0}^u \frac{dt}{\Theta(t)} + i \frac{\pi v}{\Theta(u)} + o(1)$$

as $u \rightarrow \infty$, uniformly in v , for some real constants λ_0 and u_0 . Let W denote the inverse of Z . Uniformly in any fixed strip $|y| \leq \theta < \frac{\pi}{2}$ ($z = x + iy$),

$$\lim_{x \rightarrow \infty} \frac{|W'(z)|}{\Theta(\operatorname{Re} W(z))} = \frac{1}{\pi}.$$

Remark 2.33. In many cases, the condition (2.1) is not hard to verify. For example, if there exist constants $c, d > 0$ such that $d < \Theta(t) < c$ for large t , the condition is fulfilled. In this case, we have

$$\int_a^b \frac{\Theta'(x)^2}{\Theta(x)} dx = \Theta'(x) \log(\Theta(x)) \Big|_a^b - \int_a^b \log(\Theta(x)) d\Theta'(x).$$

Since $\log(\Theta(x))$ is bounded and Θ' is bounded due to the boundary inclination $\gamma = 0$ and $\int^{\infty} |d\Theta'(x)|$ exists due to the finite boundary turning assumptions, the assertion follows.

2.4 Quasiconformal maps

The theory of quasiconformal maps started with a problem of Herbert Grötzsch in 1928. He looked at functions from one rectangle onto another, mapping the vertices of the first rectangle onto those of the second preserving the orientation. In some cases, this is possible by using a conformal map. But in general such a conformal map does not exist because three points on the boundary define the conformal map uniquely. Grötzsch searched for maps with the above properties which are as close to a conformal map as possible. The result were *quasiconformal* maps.

Quasiconformal maps had a huge impact in function theory. One of the results obtained by the usage of quasiconformal maps is Sullivan's No Wandering Domain Theorem [Sul85, Theorem 1] and its extension to entire functions in class \mathcal{S} by Eremenko and Lyubich [EL92, Theorem 3] and Goldberg and Keen [GK86, Theorem 4.2]. The goal of this section is to introduce quasiconformal maps and give some results.

2.4.1 Definitions and results

There are two different definitions of quasiconformal maps: a geometric and an analytic one. Before we can define quasiconformal maps, we need some definitions. We roughly follow the book by Olli Lehto and Kaarlo Virtanen [LV73], which contains many more results and applications of quasiconformal maps. The book by Lars Ahlfors [Ahl06] deals with quasiconformal maps as well.

Definition 2.34 (Quadrilateral). Let $Q \subset \mathbb{C}$ be a Jordan domain and let z_1, z_2, z_3, z_4 be boundary points of Q . Assume that the numbering of these points is consistent with the positive orientation of ∂Q . We then call Q together with the sequence of boundary points a *quadrilateral* and denote it by $Q(z_1, z_2, z_3, z_4)$. The points z_1, z_2, z_3, z_4 are called *vertices* of Q and divide ∂Q into four Jordan arcs. The arcs $\overline{z_1 z_2}$ and $\overline{z_3 z_4}$ are called *a-sides* and the other two *b-sides* of Q .

Using the Riemann Mapping Theorem and Carathéodory's Theorem (Theorem 2.23), a given quadrilateral $Q(z_1, z_2, z_3, z_4)$ can be conformally mapped onto a quadrilateral $Q'(\frac{-1}{k}, -1, 1, \frac{1}{k})$ where $0 < k < 1$ and Q' is the upper half plane. The number k is uniquely defined by $Q(z_1, z_2, z_3, z_4)$. The map

$$\omega(z) = \int_0^z \frac{1}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} d\xi$$

maps the quadrilateral $Q'(\frac{-1}{k}, -1, 1, \frac{1}{k})$ conformally onto a rectangle such that the points $\frac{-1}{k}, -1, 1$ and $\frac{1}{k}$ are mapped onto the vertices of the rectangle. Hence, every quadrilateral

can be conformally mapped onto a rectangle. Such a conformal map is called *canonical* and the obtained rectangle also *canonical rectangle*. This corresponds to Grötzsch's question mentioned above.

Definition 2.35 (Conformal module). Let $Q(z_1, z_2, z_3, z_4)$ be a quadrilateral and let a be the length of the a -sides and b the length of the b -sides of a canonical rectangle of Q . The *conformal module* of Q is defined by

$$M(Q) = \frac{a}{b}.$$

Remark 2.36. The conformal module of a quadrilateral is independent of the choice of the canonical rectangle.

Rearranging the vertices of a quadrilateral changes the conformal module as follows:

$$M(Q(z_1, z_2, z_3, z_4)) = M(Q(z_3, z_4, z_1, z_2)) = \frac{1}{M(Q(z_2, z_3, z_4, z_1))} \quad (2.2)$$

Let $G \subset \mathbb{C}$ be a domain and w be an orientation preserving homeomorphism of G . A quadrilateral Q is then mapped onto a quadrilateral Q' by w .

Definition 2.37 (Dilatation of a quadrilateral). The *dilatation* of a quadrilateral Q under an orientation preserving homeomorphism w is given by

$$D(Q) = \frac{M(Q)}{M(Q')}.$$

The number

$$K(G) = \sup_{\overline{Q} \subset G} D(Q)$$

is called the *maximal dilatation* of w in the domain G .

Due to (2.2), the maximal dilatation is at least 1. We can now state the geometric definition of quasiconformal maps.

Definition 2.38 (Geometric definition of quasiconformal maps). Let $G \subset \widehat{\mathbb{C}}$ be a domain and let w be an orientation preserving homeomorphism of G . If there exists some $K \in \mathbb{R}$ such that

$$K(G) \leq K$$

holds, w is called *K -quasiconformal*.

Before we can state the analytic definition of a quasiconformal map, we need the concept of Wirtinger derivatives.

Definition 2.39 (Wirtinger derivatives). Let $G \subset \mathbb{C}$ be a domain and let $f : G \rightarrow \mathbb{C}$ be differentiable in the real sense. The *Wirtinger derivatives* are defined by

$$f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Remark 2.40. We observe the following.

1. Writing $f = u + iv$ the condition $f_{\bar{z}} \equiv 0$ matches the Cauchy-Riemann equations. Thus, a function f which is differentiable in the real sense is holomorphic if and only if $f_{\bar{z}} \equiv 0$.
2. If f is holomorphic, then $f' = f_z$.

Lemma 2.41. Let $G, H \subset \mathbb{C}$ be domains and let $f : G \rightarrow f(G)$ and $g : H \rightarrow \mathbb{C}$ be differentiable in the real sense. Furthermore, let $f(G) \subset H$. Then $g \circ f$ is differentiable in the real sense and

$$\begin{aligned} (g \circ f)_z &= (g_\zeta \circ f)f_z + (g_{\bar{\zeta}} \circ f)\overline{f_{\bar{z}}}, \\ (g \circ f)_{\bar{z}} &= (g_\zeta \circ f)f_{\bar{z}} + (g_{\bar{\zeta}} \circ f)\overline{f_z}. \end{aligned}$$

Remark 2.42. If f and g are both holomorphic, the latter equation yields that $g \circ f$ is holomorphic as well. In this case, the first equation is just the standard chain rule for complex differentiable functions.

We can now state the analytic definition of quasiconformal maps.

Definition 2.43 (Analytic definition of quasiconformal maps). Let $G \subset \mathbb{C}$ be a domain. A homeomorphism $w : G \rightarrow w(G)$ is called *K-quasiconformal* if

- (1) w has square-integrable weak derivatives and
- (2) $|w_{\bar{z}}| \leq \frac{K-1}{K+1}|w_z|$ holds almost everywhere.

The following theorem is a fundamental result in the theory of quasiconformal maps.

Theorem 2.44. *Definitions 2.38 and 2.43 are equivalent.*

A proof of the theorem can be found in [LV73, Chapter IV].

Theorem 2.45 (Properties of quasiconformal maps). Let $G, H \subset \mathbb{C}$ be domains and let $f : G \rightarrow f(G)$ be K -quasiconformal and $g : H \rightarrow g(H)$ be K' -quasiconformal. Moreover, let $f(G) \subset H$. Then the following holds:

- (1) The inverse $f^{-1} : f(G) \rightarrow G$ is K -quasiconformal.

(2) *The composition $g \circ f$ is KK' -quasiconformal.*

These properties follow from the geometric definition of quasiconformal maps. The following two theorems give rise to a better understanding of the geometric behaviour of quasiconformal maps. They can be found in [LV73, Chapter II, Section 4.2] and [LV73, Chapter I, Theorem 5.1], respectively.

Theorem 2.46. *A K -quasiconformal map $\phi : G \rightarrow \mathbb{C}$ is uniformly Hölder continuous with exponent $\frac{1}{K}$ in every compact subset of G .*

Theorem 2.47. *A 1-quasiconformal map is conformal.*

Before we can state the Measurable Riemann Mapping Theorem, we need to introduce two more notations.

Definition 2.48 (Beltrami coefficient). Let $G \subset \mathbb{C}$ be a domain. Let $\mu : G \rightarrow \mathbb{C}$ be measurable. If there exists k satisfying $0 \leq k < 1$ such that $|\mu(z)| \leq k$ for almost every $z \in G$, the map μ is called k -Beltrami coefficient or short *Beltrami coefficient*.

Definition 2.49 (Quasiregular map). Let U and V be open sets in \mathbb{C} . A map $\phi : U \rightarrow V$ is called K -quasiregular if ϕ has locally square-integrable weak derivatives and the map

$$\mu_\phi(z) := \frac{\phi_{\bar{z}}}{\phi_z}(z),$$

which is defined almost everywhere, is a $\frac{K-1}{K+1}$ -Beltrami coefficient. If the exact value of K is not important, we call such a map just *quasiregular*.

Thus, a quasiconformal map is nothing but a quasiregular homeomorphism. If a quasiregular map ϕ is given, the (almost everywhere defined) map $\mu_\phi = \frac{\phi_{\bar{z}}}{\phi_z}$ is a Beltrami coefficient. We may consider this as a partial differential equation.

Definition 2.50. Let $G \subset \mathbb{C}$ be a domain and let $\mu : G \rightarrow \mathbb{C}$ be a k -Beltrami coefficient. The partial differential equation

$$\phi_{\bar{z}} = \mu \cdot \phi_z$$

is called *Beltrami equation*.

The question arises whether a given Beltrami equation is solvable. It is answered positively by the Measurable Riemann Mapping Theorem. Since every Beltrami coefficient defined on a domain $G \subset \mathbb{C}$ can be extended to a Beltrami coefficient on \mathbb{C} , it suffices to consider the case $\mu : \mathbb{C} \rightarrow \mathbb{C}$.

Theorem 2.51 (Measurable Riemann Mapping Theorem). *Let μ be a Beltrami coefficient of \mathbb{C} . Then there exists a unique quasiconformal map $\phi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\phi(0) = 0$, $\phi(1) = 1$, $\phi(\infty) = \infty$ and $\mu_\phi = \mu$.*

The theorem was proved by Morrey, Bojarski, Ahlfors and Bers. A proof can be found in [Ahl06, Chapter V]. Later on, we will need the following corollary.

Corollary 2.52. *Let g be quasiregular. Then there exists a quasiconformal map ϕ such that $f = g \circ \phi^{-1}$ is entire.*

Proof. Let μ_g be the Beltrami coefficient of g . Apply Theorem 2.51 to obtain a quasiconformal map ϕ such that $\mu = \mu_\phi$. Define $f = g \circ \phi^{-1}$. This function has all the properties needed. \square

When we speak about applying the Measurable Riemann Mapping Theorem to obtain an entire map f from a given quasiregular map g , we will always mean the map constructed in the last corollary. The next lemma in this subsection treats the question, which curves are *removable* for quasiregular maps.

Definition 2.53. Let $\Omega \subset \mathbb{C}$ be a domain and let $\gamma \subset \Omega$ be a curve. We say that γ is *removable for quasiconformal (quasiregular/conformal) maps* if every continuous map of Ω which is quasiconformal (quasiregular/conformal) on $\Omega \setminus \gamma$ is quasiconformal (quasiregular/conformal) on all of Ω .

Lemma 2.54. *Let $\Omega \subset \mathbb{C}$ be a domain and let $\gamma \subset \Omega$ be a curve. The curve γ is removable for conformal maps if and only if γ is removable for quasiconformal maps.*

Proof. By Theorem 2.51, there is a correspondence between conformal and quasiconformal maps which immediately yields the result. \square

For this result and more information on removable sets for various families of functions, see [Bis94].

Let $\Omega \subset \mathbb{C}$ be a domain and let $\gamma \subset \Omega$ be removable for conformal maps. Let f be continuous on Ω and holomorphic on $\Omega \setminus \gamma$. Since f is locally conformal with the possible exception of a discrete set, the map f can be holomorphically extended onto γ with the possible exception of a discrete set. But f is continuous on Ω which implies that every point in this discrete set is a removable singularity. Thus, γ is removable for holomorphic maps as well. Hence, the following lemma holds.

Lemma 2.55. *Let $\Omega \subset \mathbb{C}$ be a domain and let $\gamma \subset \Omega$ a curve. The curve γ is removable for conformal maps if and only if γ is removable for holomorphic maps.*

By Theorem 2.51, every quasiregular map g can be written as the composition $g = f \circ \phi$ where f is holomorphic and ϕ is quasiconformal. Hence, a curve γ which is removable for quasiconformal maps is also removable for quasiregular maps because the image curve $\phi(\gamma)$ is removable for holomorphic maps as seen above.

Remark 2.56. Since straight line segments and circles are removable for conformal maps, quasiconformal images of these are removable for conformal maps as well. Rectifiable curves are removable for conformal maps as well.

Remark 2.57. The Maximum Principle also holds for quasiregular maps. The reason is that every quasiregular map can be written as the composition of a holomorphic map and a quasiconformal map by Theorem 2.51. Since quasiconformal maps are homeomorphisms, they are open maps. Holomorphic maps are open as well, thus also a quasiregular map is open.

2.4.2 Asymptotic conformality

A quasiconformal map is a function which maps infinitesimal small circles onto infinitesimal small ellipses with bounded distortion. This is by nature a local property. In some cases, one can say more about the global behaviour of such a map. For example, it is possible to show that the quasiconformal map is asymptotically conformal at infinity provided that the set of non-conformality is not too big. This result is given by the following theorem. A proof can be found in [LV73, Chapter V, Paragraph 6] and the theorem itself in [LV73, Chapter V, Theorem 6.1].

Theorem 2.58 (Teichmüller, Wittich, Belinski). *Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be quasiconformal and let μ be the Beltrami coefficient corresponding to ϕ . If*

$$\iint_{|z|>R} \frac{|\mu(z)|}{x^2 + y^2} dx dy < \infty$$

for some $R > 0$, then there exist constants $A > 0$ and $\alpha \in [0, 2\pi)$ and a function $\varepsilon : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that

$$\left| \frac{\phi(z)}{z} - Ae^{i\alpha} \right| < A\varepsilon(|z|) \quad \text{and} \quad \lim_{r \rightarrow \infty} \varepsilon(r) = 0,$$

in other words

$$\lim_{z \rightarrow \infty} \frac{\phi(z)}{z} = Ae^{i\alpha}.$$

Let E be the set where ϕ is not conformal. Then $\mu_{\bar{z}}(z) = 0$ for all $z \in \mathbb{C} \setminus E$ and $|\mu(z)| \leq k < 1$ for all $z \in E$. Thus, we get

$$\iint_{|z|>R} \frac{|\mu(z)|}{x^2 + y^2} dx dy \leq k \iint_E \frac{1}{x^2 + y^2} dx dy.$$

The value of the latter integral is known as the *logarithmic area* of E .

Definition 2.59 (Logarithmic area). The *logarithmic area* of a planar set E is given by

$$\log \text{area}(E) = \iint_E \frac{dy \, dx}{x^2 + y^2}.$$

If E is given in polar coordinates, the formula changes to

$$\log \text{area}(E) = \iint_E \frac{1}{r} \, d\varphi \, dr.$$

With the notation of the logarithmic area, we obtain the following special case of Theorem 2.58.

Theorem 2.60. *Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be quasiconformal. Let E be the set where ϕ is not conformal. If $\log \text{area}(E)$ is finite, then there exists a constant $c \in \mathbb{C} \setminus \{0\}$ and a function $\varepsilon : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that*

$$\left| \frac{\phi(z)}{z} - c \right| < |c| \cdot \varepsilon(|z|) \quad \text{and} \quad \lim_{r \rightarrow \infty} \varepsilon(r) = 0,$$

in other words

$$\lim_{z \rightarrow \infty} \frac{\phi(z)}{z} = c.$$

The theorem above yields an asymptotic expression for a quasiconformal map ϕ under conditions on the set of non-conformality E . If this set does not only have finite logarithmic area but the area decreases exponentially, that is

$$\log \text{area}(E \cap \{z \in \mathbb{C} : |z| \geq n\}) = \mathcal{O}(e^{-cn}) \text{ as } n \rightarrow \infty,$$

a theorem by Dyn'kin (see [Dyn97]) even yields the asymptotic expression

$$\phi(z) = \sum_{k=-m}^1 a_k z^k + \mathcal{O}(z^{-m-1})$$

in $\{z \in \mathbb{C} : |z| > 1\}$ for any $m \in \mathbb{N}$. Unfortunately, the maps in this thesis will never fulfil this stronger condition.

2.5 Quasiconformal folding

In 2011, Christopher Bishop published the first preprint on *quasiconformal folding* (now published in [Bis15]), which is a method to construct entire functions f in class \mathcal{S} with a good control of both the singular values of f and the geometric behaviour of f . The idea

itself is based on the fact that any finite plane tree T is associated to a polynomial p with only two critical values (± 1) such that $T_p = p^{-1}([-1, 1])$ is equivalent to T in a certain sense (see [SZ94, Theorem 1.1]). Polynomials with exactly two critical values are called *Shabat polynomials* or *generalised Chebyshev polynomials*. Trees arising as preimages of $[-1, 1]$ under a Shabat polynomial with the critical values ± 1 are called *true trees*. In [Bis14, Theorem 1.1], Bishop showed that the set of true trees is dense in the set of all continua with respect to the Hausdorff metric. This result is generalised in [Bis15] from finite trees and polynomials to infinite trees and entire maps. The role of the Shabat polynomials is played by the functions in the Speiser class \mathcal{S} .

2.5.1 Idea and setup

The goal of this subsection is to introduce quasiconformal folding without giving too many details. Although the idea of Bishop's construction can be described in a comparatively simple way, the proof itself is quite involved.

The idea

Let f be a transcendental entire function in class \mathcal{S} without finite asymptotic values and with exactly two critical values, which we assume to be 1 and -1 . Let $T := f^{-1}([-1, 1])$. Since f is a transcendental entire map, T is unbounded and connected, and every component of $\mathbb{C} \setminus T$ is simply connected. Thus, we can consider T as a tree. The vertices of T are precisely the preimages of 1 and -1 and the edges are the parts of T which connect two vertices without passing through another vertex. In other words, the edges are just the preimages of $(-1, 1)$ under f together with the corresponding endpoints.

Let Ω be a connected component of $\mathbb{C} \setminus T$.

Lemma 2.61. *The map f is a holomorphic covering from Ω onto $\mathbb{C} \setminus [-1, 1]$.*

Proof. For this proof, let $f = f|_{\Omega}$. No critical point of f is contained in Ω by construction. Thus, f is a local homeomorphism. Let φ_i , $i \in I$, denote the branches of f^{-1} . Let $z_0 \in \mathbb{C} \setminus [-1, 1]$ and let $\varepsilon > 0$ such that $D := D(z_0, \varepsilon) \subset \mathbb{C} \setminus [-1, 1]$. Since there are no singular values of f in $\mathbb{C} \setminus [-1, 1]$, every φ_i can be defined on D . Let $U := \bigcup_{i \in I} \varphi_i(D)$. We then have $U \subset f^{-1}(D)$. The other inclusion holds as well: Let $z \in f^{-1}(D)$. Then $w := f(z) \in D$. Since $f'(z) \neq 0$, there exist neighbourhoods $V(w)$ of w in D and $V(z)$ of z in Ω and a branch φ of f^{-1} such that $f \circ \varphi = \text{id}_{V(w)}$ and $\varphi \circ f = \text{id}_{V(z)}$. Since D contains no singular values of f , the branch φ can be extended onto D . Thus, there exists some $i \in I$ such that $\varphi = \varphi_i$. Hence, $z = \varphi(w) = \varphi_i(w) \in U$.

Assume that f is not a covering. Then there exists some point $z_0 \in \mathbb{C} \setminus [-1, 1]$ in which f is not a covering. If there would exist an $\varepsilon > 0$ such that $\varphi_i(D) \cap \varphi_j(D) = \emptyset$

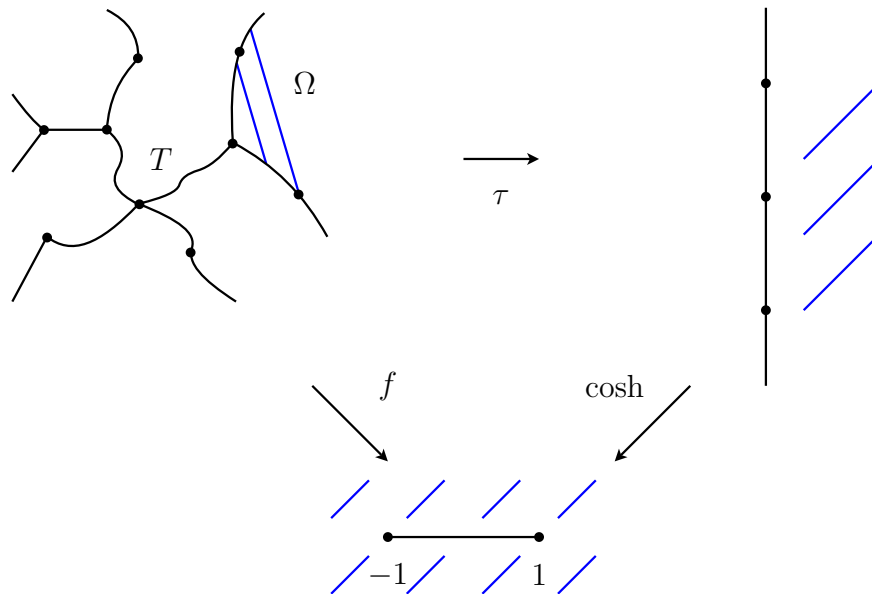


Figure 2.2: Illustration of the idea of quasiconformal folding. On every component Ω of $\mathbb{C} \setminus T$, the map f is decomposed into the composition of \cosh and a conformal map τ mapping Ω onto \mathbb{H}_r .

for all $i, j \in I$ with $i \neq j$, where $D = D(z_0, \varepsilon)$ as above, the map f would be a covering. Thus, for every $\varepsilon > 0$ there exist $i, j \in I$ with $i \neq j$ and $\varphi_i(D) \cap \varphi_j(D) \neq \emptyset$. Note that $\varphi_i(z_0) \neq \varphi_j(z_0)$ for $i \neq j$. Since φ_i and φ_j are holomorphic, the intersection $\varphi_i(D) \cap \varphi_j(D)$ is open. Let $w \in \varphi_i(D) \cap \varphi_j(D)$. We then have $\varphi_i(f(w)) = w = \varphi_j(f(w))$. Thus, $\varphi_i = \varphi_j$ on $f(\varphi_i(D) \cap \varphi_j(D))$ which is an open set as well. By the identity theorem, we get that $\varphi_i = \varphi_j$, a contradiction. \square

Since \cosh is a holomorphic covering from \mathbb{H}_r onto $\mathbb{C} \setminus [-1, 1]$ and both \mathbb{H}_r and Ω are simply connected, Theorem 2.12 yields the existence of a conformal map $\tau : \Omega \rightarrow \mathbb{H}_r$ such that $f = \cosh \circ \tau$. Hence, it is possible to decompose f into the composition of a conformal map and \cosh on each complementary component of T , see Figure 2.2.

Quasiconformal folding now reverses this procedure. Instead of starting with a transcendental entire map f without finite asymptotic values and with the only critical values 1 and -1 and then obtaining an infinite tree T and conformal maps τ_j from each component Ω_j of $\mathbb{C} \setminus T$ onto \mathbb{H}_r , we start with an infinite tree T such that each component Ω_j is simply connected and search for conformal maps τ_j such that the map g given by $\cosh \circ \tau_j$ on the corresponding component of $\mathbb{C} \setminus T$ can be extended to an entire map.

It is quite obvious that g cannot be extended in every case. In [Bis15, page 4], Bishop gives sufficient conditions which allow to alter g in a small neighbourhood of T such

that the newly obtained map can then be extended on all of \mathbb{C} . Of course, this map will in general not be entire, but at least it is possible to change g in such a way that the extension is quasiregular on \mathbb{C} . An application of the Measurable Riemann Mapping Theorem then yields the desired map f .

The setup

This subsection deals with the setup of Bishop's construction. For the remainder of this section, let T be an infinite, unbounded, connected and locally finite tree in the plane where the set of vertices V of T is an infinite, discrete set. Furthermore, we will denote by E the set of edges of T . Thus, an edge $e \in E$ is an injective curve which does not intersect any other edge with the possible exception of one common vertex.

Because T is a connected tree, every component of $\mathbb{C} \setminus T$ is simply connected and unbounded. Let $\Omega = \mathbb{C} \setminus T$. For every connected component Ω_j of Ω , we take a conformal map σ_j which maps \mathbb{H}_r onto Ω_j . This function has a continuous extension onto $\partial\mathbb{H}_r$. It may be normalised to map ∞ onto itself. By τ_j we will denote the inverse of σ_j and by τ the map $\tau : \Omega \rightarrow \mathbb{H}_r$ which coincides with τ_j on Ω_j . Moreover, let $g = \cosh \circ \tau$ and let $V_j := \{z \in \partial\mathbb{H}_r : \sigma_j(z) \in V\}$ and $I_j := \partial\mathbb{H}_r \setminus V_j$.

Remark 2.62. The connected components of I_j form a partition of $\partial\mathbb{H}_r \setminus V_j$.

Let $e \in E$. There are two possibilities: Either e without the endpoints is part of the boundaries of different components Ω_j and Ω_k or the edge e without the endpoints is part of the boundary of just one component Ω_j . In the first case, there is one interval a_j on $\partial\mathbb{H}_r$ and one interval a_k on $\partial\mathbb{H}_r$ such that $\sigma_j(a_j) = e = \sigma_k(a_k)$. In the second case, there are two different intervals a_j^1 and a_j^2 on $\partial\mathbb{H}_r$ such that $\sigma_j(a_j^1) = e = \sigma_j(a_j^2)$. In an informal way, we can say that the edge e has two sides which are each mapped by τ onto intervals on $i\mathbb{R}$, the so called τ -images. The τ -size of e is the minimum of the lengths of the two τ -images, see Figure 2.3.

Remark 2.63. If $T = f^{-1}([-1, 1])$ for some transcendental entire function f with the only critical values -1 and 1 and no finite asymptotic value, the associated partitions I_j of $i\mathbb{R}$ are given by $i\mathbb{R} \setminus \pi i\mathbb{Z}$, provided that the maps σ_j are chosen according to Theorem 2.12. Thus, the τ -size of every edge is exactly π .

Let now I be a connected component of I_j . Denote by Q_I the closed square in $\overline{\mathbb{H}_r}$ which has I as one side. Let V_{I_j} be the interior of the union of all those squares. Then V_{I_j} is an open subset of \mathbb{H}_r which has the imaginary axis as part of its boundary. This is the domain in which the folding for the component Ω_j takes place. For $r > 0$ define

$$T(r) = \bigcup_{e \in E} \{z \in \mathbb{C} : \text{dist}(z, e) < r \cdot \text{diam}(e)\}. \quad (2.3)$$

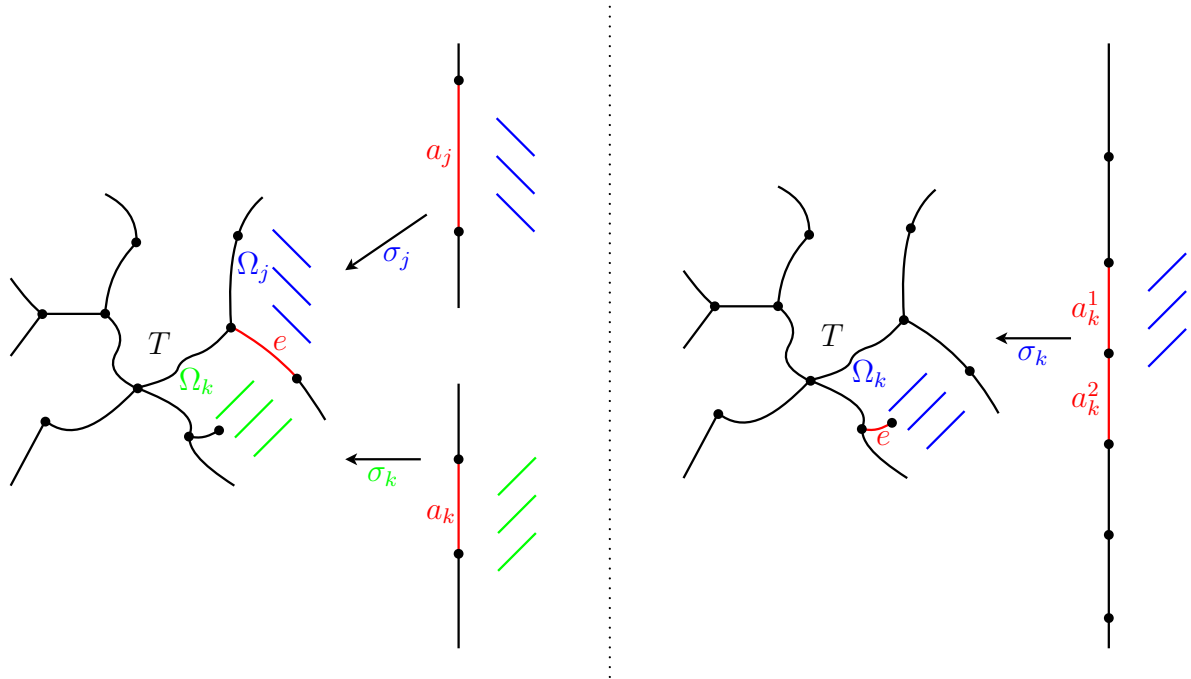


Figure 2.3: On the left, the τ -size of the edge e is the minimum of the lengths of a_j and a_k , and on the right, the τ -size of the edge e is the minimum of the lengths of a_k^1 and a_k^2 .

Bishop proves that there exists some $r_0 > 0$ such that $V_{I_j} \subset \tau_j(T(r_0) \cap \Omega_j)$ for any j . It is important that r_0 does not depend on j . See [Bis15, Lemma 2.1] for the following lemma.

Lemma 2.64. $\tau^{-1}(V_{I_j}) \subset T(r_0)$ for some $r_0 \leq 25.3$.

As stated earlier, it is obvious that not every map g obtained by the construction above can be extended analytically onto \mathbb{C} . In many cases, it is not even possible to just find a continuous extension of g . To approach this problem, Bishop makes the following two assumptions:

- (1) Adjacent elements of the induced partitions of $i\mathbb{R}$ have comparable lengths
- (2) There is a positive lower bound for the τ -lengths of edges.

These two assumptions provide all the material needed to make the modification of g into a global quasiregular map possible. Bishop shows that the first condition is basically local and follows from the so-called *bounded geometry conditions*. The second condition is more global and roughly states that complementary components of T are “smaller than half planes” near infinity. For more details, see Subsection 2.5.2.

With the notations and comments above, we can now introduce graphs with bounded geometry and state Bishop's first theorem. See [Bis15, page 4] for the following definition.

Definition 2.65 (Bounded Geometry). Let T be an unbounded graph in \mathbb{C} . We say that T has *bounded geometry*, if

- (1) The edges of T are twice differentiable with uniform bounds.
- (2) The angles between adjacent edges are uniformly bounded away from zero.
- (3) Adjacent edges have uniformly comparable lengths.
- (4) For non-adjacent edges e and f , $\frac{\text{diam}(e)}{\text{dist}(e,f)}$ is uniformly bounded.

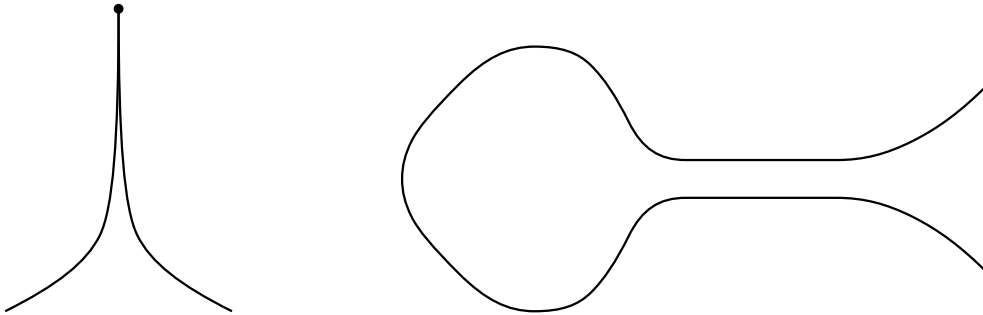


Figure 2.4: The second and fourth part of the bounded geometry condition prevent spikes and thin corridors.

It is worth noticing that these conditions are purely geometrical and often easily verified. For example, if the graph does only have straight lines as edges, the first assumption is fulfilled. Moreover, in many cases it is not hard to ensure that the second assumption is fulfilled as well. Note that the degree of every vertex is uniformly bounded above by some constant due to the second assumption. The third and fourth part can usually be achieved by adding some more vertices to the already existing edges. We will see this in a more precise way when we get to the examples in Subsection 2.5.2. The following theorem can be found in [Bis15, Theorem 1.1].

Theorem 2.66 (Bishop). *Suppose that T has bounded geometry and that every edge has τ -size $\geq \pi$. Then there is an entire function f and a K -quasiconformal map ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r_0)$. The constant K only depends on the bounded geometry constants of T and ϕ is conformal off $T(r_0)$. The only critical values of f are ± 1 and f has no finite asymptotic values.*

In the proof of this theorem, Bishop constructs a new tree $T' = (V', E')$ such that $T \subset T' \subset T(r_0)$ in the sense that $V \subset V'$, $E \subset E'$ and all vertices and edges of T' lie in $T(r_0)$. The edges of T' which are not already edges of T are images of straight line segments under a quasiconformal map. Furthermore, he exchanges τ by a map η which is quasiconformal from each component of $\mathbb{C} \setminus T'$ onto \mathbb{H}_r . The components of $\mathbb{C} \setminus T'$ are just the components of $\mathbb{C} \setminus T$ with trees removed which are rooted at the vertices of T on $\partial\Omega$. Bishop shows that the map η can be constructed such that $\eta(V) \subset \pi i\mathbb{Z}$, $\eta = \tau$ off $T(r_0)$ and $g = \cosh \circ \eta$ is continuous across T' . Since the edges of T are rectifiable and the other edges of T' are quasiconformal images of straight line segments, T' is removable for quasiregular maps. This means that a quasiregular map which is defined on $\mathbb{C} \setminus T'$ and continuous on \mathbb{C} can be extended to a quasiregular map on \mathbb{C} , compare Lemma 2.56. Thus, g is quasiregular on \mathbb{C} . The bounded geometry conditions of T ensure that the map η exists. By the Measurable Riemann Mapping Theorem (Theorem 2.51), there exists a quasiconformal map ϕ such that $f = g \circ \phi^{-1}$ is entire. The quasiregular map g is locally bijective except at the vertices of T' . Thus, the only critical values of f are ± 1 . Furthermore, Bishop shows that f has no finite asymptotic value. Hence, $f \in \mathcal{S}$. We refer to [Bis15] for a complete proof.

As already mentioned, a function constructed via Bishop's Theorem has only two critical values (± 1) and no finite asymptotic value. Furthermore, the degree of the critical points is uniformly bounded above by some constant. The reason for this is that the critical points correspond to the vertices of T and the degree of a critical point is the degree of the corresponding vertex. But the degree of the vertices is bounded due to condition (2) of the bounded geometry conditions on T .

In order to introduce higher order critical points and finite asymptotic values to the constructed map f , Bishop exchanges the tree T by an unbounded, locally finite, connected graph such that each of the components of $\mathbb{C} \setminus T$ is one of the following:

- (I) **D-components:** A D-component is a bounded component Ω whose boundary is a closed Jordan curve. This curve is the union of finitely many edges of T . Let d be this number. Furthermore, we have a quasiconformal map $\eta : \Omega \rightarrow \mathbb{D}$ which is length respecting on $\partial\Omega$ and maps the d vertices on $\partial\Omega$ to the d th roots of unity on $\partial\mathbb{D}$. Thereafter, \mathbb{D} is mapped onto itself by $\sigma : \mathbb{D} \rightarrow \mathbb{D}$, $z \mapsto z^d$. This yields a critical point of degree d with critical value 0. If a critical value $a \in \mathbb{D} \setminus \{0\}$ is desired, we post-compose σ with a quasiconformal map $\rho : \mathbb{D} \rightarrow \mathbb{D}$ such that $\rho(0) = a$ and ρ is the identity on $\partial\mathbb{D}$.
- (II) **L-components:** An L-component is an unbounded Jordan domain Ω . It comes together with a length respecting quasiconformal map $\eta : \Omega \rightarrow \mathbb{H}_l$. The map σ is the exponential map. This yields that 0 is a finite asymptotic value. If some other

asymptotic value $a \in \mathbb{D} \setminus \{0\}$ is requested, σ is post-composed with a quasiconformal map $\rho : \mathbb{D} \rightarrow \mathbb{D}$ such that $\rho(0) = a$ and ρ is the identity on $\partial\mathbb{D}$, just as in the case of a D-component.

(III) **R-component:** R-components are the components used in Theorem 2.66. Thus, Ω is an unbounded, simply connected domain which is mapped onto \mathbb{H}_r by a length respecting quasiconformal map η . Note that the boundary of Ω might be a tree instead of a Jordan curve. In Theorem 2.66, the map $\sigma : \mathbb{H}_r \rightarrow \mathbb{C}$ was just cosh. But in this more general case, we need to change σ slightly. There are two possibilities for every edge e of $\partial\Omega$. Either the edge is only part of the boundary of Ω or it is part of $\partial\Omega$ and $\partial\Omega_j$ for some different component Ω_j . This component Ω_j can be a D-, L- or R-component. We need to adjust σ such that it coincides with the definition of the σ of Ω_j .

Therefore, the intervals of the partition of $\partial\mathbb{H}_r$ induced by Ω and η are divided into two types. An interval I is of type 1 if the corresponding opposite site of $\eta^{-1}(I)$ belongs to $\partial\Omega$ or to the boundary of another R-component. The interval is of type 2 if the corresponding opposite site of $\eta^{-1}(I)$ is on the boundary of a D- or L-component. On type 1 intervals, $\sigma = \cosh$ and on type 2 intervals, $\sigma = \exp$. Furthermore, $\sigma = \cosh$ far away from $\partial\mathbb{H}_r$. In between, σ is quasiregular where the dilatation is independent of Ω and η . Denote the set of type 1 intervals by J_1 and the set of type 2 intervals by J_2 . The precise statement reads as follows, compare [Bis15, Lemma 7.1].

Lemma 2.67. *There exists a quasiregular map $\sigma : \mathbb{H}_r \rightarrow \mathbb{C} \setminus [-1, 1]$ so that*

$$\sigma(z) = \begin{cases} \cosh(z), & \text{if } z \in I \in J_1 \\ \exp(z), & \text{if } z \in I \in J_2 \\ \cosh(z), & \text{if } z \in \mathbb{H}_r + 1 = \{x + iy : x > 1\}. \end{cases}$$

The dilatation of σ is uniformly bounded, independent of all the choices.

With these changes to the construction, Bishop's Theorem takes the following form, see [Bis15, Theorem 7.2].

Theorem 2.68. *Suppose T is a bounded geometry graph and suppose τ is conformal from each complementary component of T to its standard version (i.e. left/right half-plane or unit disc). Assume that*

- *D- and L-components only share edges with R-components.*
- *τ on a D-component with n edges maps the vertices to n th roots of unity.*

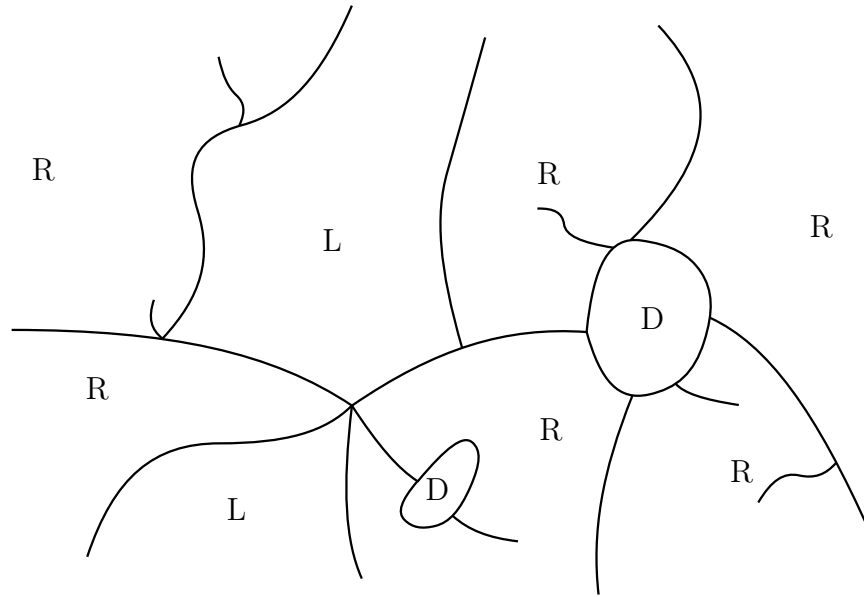


Figure 2.5: A sketch of a possible tree which can be used in Theorem 2.68. The components of $\mathbb{C} \setminus T$ are labelled as R-, L- or D-components.

- on L-components τ maps edges to intervals of length 2π on $\partial\mathbb{H}_l$ with endpoints in $2\pi i\mathbb{Z}$,
- on R-components the τ -sizes of all edges are $\geq 2\pi$.

Then there is an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = \sigma \circ \tau$ off $T(r_0)$ (a neighbourhood of T). The only singular values of f are ± 1 (critical values coming from the vertices of T) and the critical values and singular values assigned by the D- and L-components.

Remark 2.69. In the definition of the different components, we speak about length respecting quasiconformal maps but in the theorem above we start with conformal maps. Of course, not every conformal map is length respecting. But by using Lemma 2.74, it is possible to exchange the conformal map by a length respecting quasiconformal map.

Remark 2.70. Note that D- and L-components may share vertices of T with other D- and L-components. But the components must not touch along edges. The reason for this is Bishop's folding construction, which mainly takes place in R-components. In the other components, the conformal maps are only changed in a small neighbourhood of the boundary to render them length respecting on the boundary. The last assumption can be replaced by "there exists $d > 0$ such that the τ -sizes of all edges are $\geq d$ ". Exchanging τ by $\frac{2\pi}{d}\tau$ then ensures that the τ -sizes are $\geq 2\pi$.

Later in this thesis, we want to use Bishop's folding technique but only to obtain the quasiregular map on \mathbb{C} which is the modified version of g . Therefore, we will decompose his theorem in two – the folding construction itself and the application of the Measurable Riemann Mapping Theorem, separately.

Theorem 2.71 (Quasiregular version of Bishop's Theorem). *Suppose T is a bounded geometry graph and suppose τ is conformal from each complementary component of T to its standard version (i.e. left/right half-plane or unit disc). Assume that*

- *D - and L -components only share edges with R -components.*
- *τ on a D -component with n edges maps the vertices to n th roots of unity.*
- *on L -components τ maps edges to intervals of length 2π on $\partial\mathbb{H}_l$ with endpoints in $2\pi i\mathbb{Z}$,*
- *on R -components the τ -sizes of all edges are $\geq 2\pi$.*

Then there is a quasiregular map $g : \mathbb{C} \rightarrow \mathbb{C}$ which coincides with $\sigma \circ \tau$ off $T(r_0)$.

Proof. We apply Bishop's construction up to the point where he applies the Measurable Riemann Mapping Theorem. \square

Corollary 2.72. *There exists an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = g$ off $T(r_0)$. The only singular values of f are ± 1 (critical values coming from the vertices of T) and the critical values and singular values assigned by the D - and L -components.*

Proof. Apply the Measurable Riemann Mapping Theorem (Theorem 2.51) to $\mu = \mu_g$ to obtain a quasiconformal map ϕ such that $f = g \circ \phi^{-1}$ is entire. Thereafter, use the same reasoning as Bishop to obtain the desired result. \square

The following lemmas will be needed later on in order to obtain better estimates on the folding domain itself. They can be found in [Bis15, Lemma 4.1] and [Bis15, Lemma 4.3], respectively.

Lemma 2.73. *If T is a bounded geometry tree, then adjacent elements of I_j have comparable lengths.*

Lemma 2.74. *Let T be a bounded geometry tree, Ω_j be a component of $\mathbb{C} \setminus T$, and $\sigma_j : \mathbb{H}_r \rightarrow \Omega_j$ be the inverse to τ for this component. Suppose that the partition of $\partial\mathbb{H}_r$ induced by Ω_j has bounded geometry. Then there is a quasiconformal map $\beta : \mathbb{H}_r \rightarrow \mathbb{H}_r$ such that $\sigma_j \circ \beta$ is length-respecting on every element of I_j . The map β is the identity on $V_j \subset \partial\mathbb{H}_r$ and on $\mathbb{H}_r \setminus V_{I_j}$.*

2.5.2 Examples

The next two subsections deal with examples of applying quasiconformal folding. After these examples, the reader should be convinced that the application of quasiconformal folding is oftentimes way easier than the proof of Bishop's Theorem. But as we will see in Chapter 3, proving the conditions needed in Bishop's Theorem is usually more difficult than in these examples.

Functions in class \mathcal{S} of arbitrarily rapid growth

In this paragraph, we want to prove the following theorem using quasiconformal folding.

Theorem 2.75 (Merenkov). *There exist entire functions in class \mathcal{S} of arbitrarily rapid growth.*

This result is originally due to Sergei Merenkov. His proof and the statement can be found in [Mer08, Theorem 1]. In his paper [Bis15, page 10], Bishop already states this example. Since the tree used for proving Merenkov's Theorem using quasiconformal folding is quite simple, Merenkov's Theorem is a good example to illustrate quasiconformal folding.

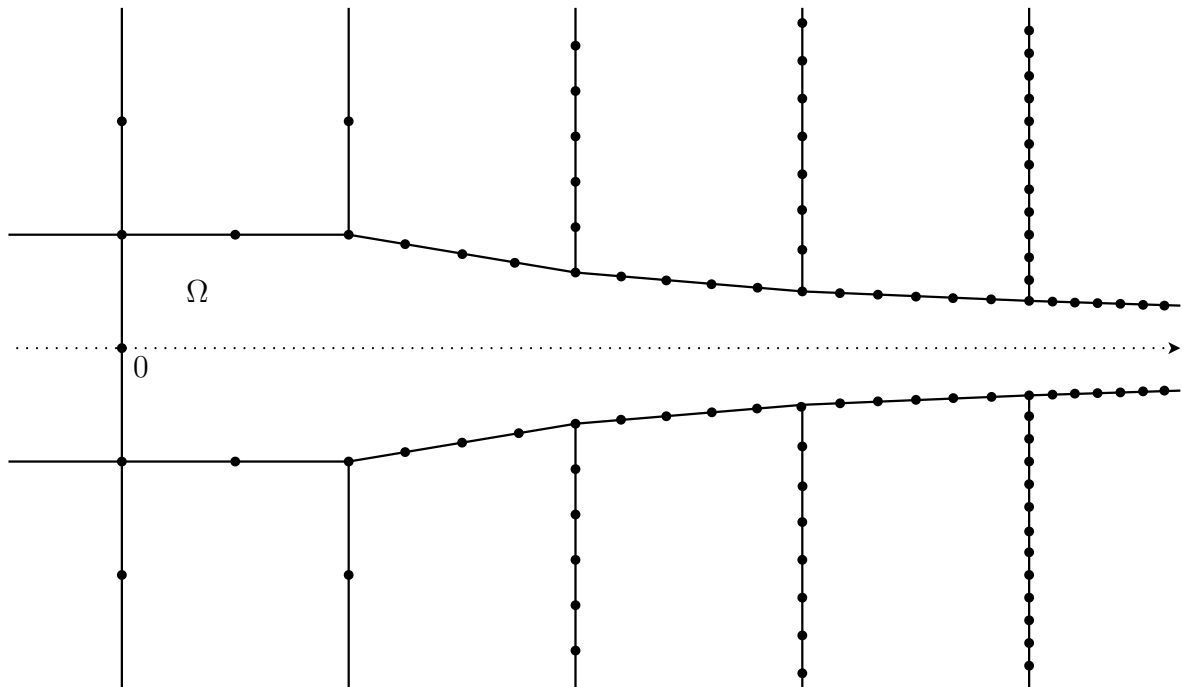


Figure 2.6: Illustration of the tree used for Merenkov's Theorem.

We consider the situation depicted in Figure 2.6. The complex plane is divided into several simply connected domains which are essentially half-strips with the only exception being the domain Ω containing the positive real axis. Along the edges of the strips, we add vertices. Each point in which two or more boundary parts of the strips meet is a vertex. Alongside $\partial\Omega$ we add vertices in such a way that the resulting edges have lengths which are uniformly comparable to the width of Ω . We add vertices along the boundaries of the half-strips in the left half-plane with roughly unit spacing. On the boundaries of the other half-strips, we add vertices such that the lengths of the corresponding edges are almost equal. The constructed tree has bounded geometry: Since the edges are only straight lines which meet at bounded angles ($\frac{\pi}{2}$ or π and close to $\frac{\pi}{2}$ or π in the remaining cases), the first two conditions in the definition of a bounded geometry graph are fulfilled. Due to the construction, adjacent edges have uniformly comparable lengths and the ratio of $\text{diam}(e_1)/\text{dist}(e_1, e_2)$ for non-adjacent edges e_1 and e_2 is uniformly bounded. Hence, also the third and fourth conditions are satisfied.

By adding more vertices along the boundaries in the fashion above, we can achieve that the edges get so small that $T(r_0)$ does not intersect $[1, \infty)$. Since each domain is essentially a half-strip, a conformal function mapping the half-strip onto the right half-plane is basically \sinh . Thus, the τ -sizes of the edges are bounded below by some positive constant.

Let τ map Ω conformally onto \mathbb{H}_r such that τ is symmetric with respect to the real axis. Using Ahlfors' Distortion Theorem (Theorem 2.27), we see that

$$\log(\tau(x)) \geq \inf_{|y| < \frac{1}{2}\Theta(x)} \log(|\tau(x + iy)|) \geq \sup_{|y| < \frac{1}{2}\Theta(\pi)} \log(|\tau(\pi + iy)|) + \pi \int_{\pi}^x \frac{dt}{\Theta(t)} - 4\pi$$

for $x \gg \pi$, where $\Theta(t)$ denotes the width of Ω at $\text{Re} = t$. Thus, we see that the narrower Ω is the more τ expands.

We can now apply Theorem 2.66 and obtain a quasiconformal map ϕ and an entire map $f \in \mathcal{S}$ such that $f \circ \phi = \cosh \circ \tau$ on $\Omega \setminus T(r_0)$. In particular, $f = \cosh \circ \tau \circ \phi^{-1}$ on $[1, \infty)$. Since ϕ is quasiconformal, thus Hölder-continuous, and τ is expanding as quickly as we wish, also f is expanding as quickly as we wish. This finishes the proof of Merenkov's Theorem.

Functions in class \mathcal{S} with tracts shaped like sectors

In his paper [Bis15, page 3], Bishop already mentions the examples given in this paragraph. We want to treat them in more detail, thereby providing a better understanding of quasiconformal folding in general.

A single tract

In this example, we want to construct functions in class \mathcal{S} where we only use sectors as R- and L-components. We start with a map which has only one tract. Therefore, let $0 < \beta \leq 1$ and define $R := \{re^{i\varphi} : r > 0, |\varphi| < \beta \cdot \frac{\pi}{2}\}$ and $L := \mathbb{C} \setminus \overline{R}$. Furthermore, define

$$\eta : \mathbb{H}_l \rightarrow L, z \mapsto -(-z)^{2-\beta} \text{ and } \tau : R \rightarrow \mathbb{H}_r, z \mapsto z^{\frac{1}{\beta}}.$$

Then both η and τ are conformal and can be extended onto $\partial\mathbb{H}_l$ and ∂R , respectively. We want R to be an R-component and L to be an L-component. The boundary of R just consists of the two rays with angles $\beta\frac{\pi}{2}$ and $-\beta\frac{\pi}{2}$ starting at 0 and going off to infinity. In order to fulfil the conditions required in Bishop's Theorem, we need to add vertices to ∂R such that every edge is mapped to an interval of length 2π with endpoints in $2\pi i\mathbb{Z}$ by a conformal map from L to \mathbb{H}_l . We will take η^{-1} to be this conformal map.

For $n \in \mathbb{Z}$, define $z_n := \eta(2\pi in)$ and $e_n := [z_n, z_{n+1}]$. Since every edge is just a straight line segment, the edges are twice differentiable with uniform bounds so that condition (1) in Definition 2.65 is satisfied. The angle between adjacent edges is just π , with the only exception being the angle between e_0 and e_{-1} which is given by $\beta\pi$. Thus, these angles are uniformly bounded away from 0. Hence, also condition (2) in Definition 2.65 is fulfilled.

To check whether adjacent edges have uniformly comparable lengths, we need to determine the vertices z_n . Let $z = re^{i\varphi} \in \overline{\mathbb{H}_l}$, $\frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2}$. Then $-z \in \overline{\mathbb{H}_r}$. Since we want $\arg(-z)$ to be in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we write $-z = e^{-i\pi}re^{i\varphi} = re^{i(\varphi-\pi)}$. Thus, $\eta(z)$ is given by

$$\eta(z) = e^{i\pi} (re^{i(\varphi-\pi)})^{2-\beta} = r^{2-\beta} e^{i((2-\beta)(\varphi-\pi)+\pi)}.$$

For $n > 0$, this yields

$$z_n = \eta(2\pi in) = \eta\left(2\pi n \cdot e^{i\frac{\pi}{2}}\right) = (2\pi n)^{2-\beta} e^{i((2-\beta)(-\frac{\pi}{2})+\pi)} = (2\pi n)^{2-\beta} e^{i\beta\frac{\pi}{2}}$$

and for $n < 0$

$$z_n = \eta(2\pi in) = \eta\left(2\pi |n| \cdot e^{i\frac{3\pi}{2}}\right) = (2\pi |n|)^{2-\beta} e^{i((2-\beta)\frac{\pi}{2}+\pi)} = (2\pi |n|)^{2-\beta} e^{-i\beta\frac{\pi}{2}}.$$

Furthermore,

$$z_0 = \eta(0) = 0.$$

We see that for $n \geq 0$ the length of e_n is given by

$$\ell(e_n) = (2\pi)^{2-\beta} \left((n+1)^{2-\beta} - n^{2-\beta} \right) = \ell(e_{-(n+1)}).$$

Thus, we can restrict to $n \geq 0$ for the remainder of this section. Let $n \geq 0$. We have

$$\frac{\ell(e_{n+1})}{\ell(e_n)} = \frac{(n+2)^{2-\beta} - (n+1)^{2-\beta}}{(n+1)^{2-\beta} - n^{2-\beta}} = \frac{\left(1 + \frac{2}{n}\right)^{2-\beta} - \left(1 + \frac{1}{n}\right)^{2-\beta}}{\left(1 + \frac{1}{n}\right)^{2-\beta} - 1}.$$

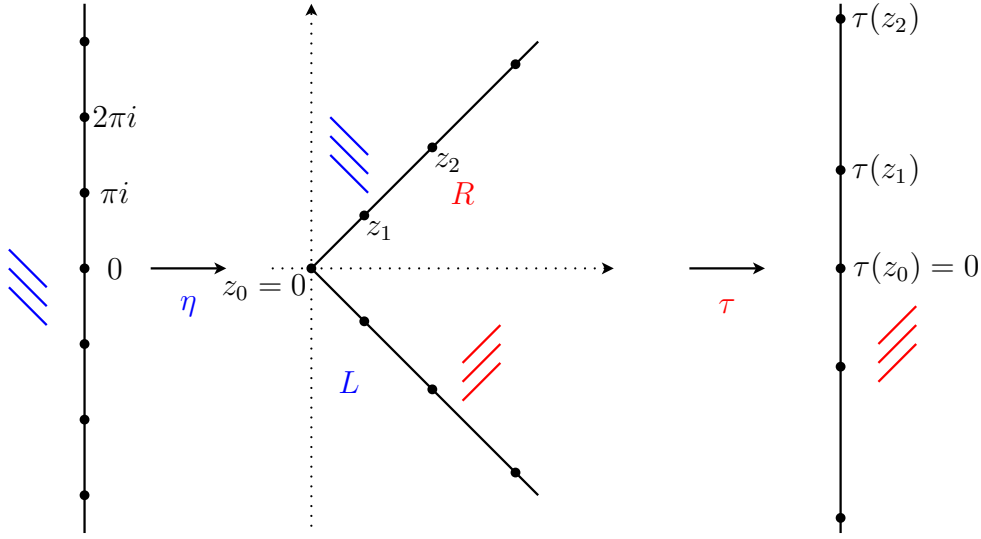


Figure 2.7: Illustration of the case with only one sector R .

Using l'Hôpital's rule, we obtain

$$\lim_{n \rightarrow \infty} \frac{\ell(e_{n+1})}{\ell(e_n)} = \lim_{n \rightarrow \infty} \frac{(2 - \beta) \left(1 + \frac{2}{n}\right)^{1-\beta} \frac{-2}{n^2} - (2 - \beta) \left(1 + \frac{1}{n}\right)^{1-\beta} \frac{-1}{n^2}}{(2 - \beta) \left(1 + \frac{1}{n}\right)^{1-\beta} \frac{-1}{n^2}} = \frac{2 - 1}{1} = 1.$$

Thus, there exists some constant $M > 0$ such that $\frac{1}{M} \leq \frac{\ell(e_{n+1})}{\ell(e_n)} \leq M$ for all $n \geq 0$. Due to the reasoning above, the same estimate holds for all $n \in \mathbb{Z}$. Therefore, adjacent edges have comparable lengths, which proves condition (3) in Definition 2.65.

It remains to show that the ratio $\frac{\text{diam}(e_n)}{\text{dist}(e_n, e_m)}$ is uniformly bounded for all non-adjacent edges e_n and e_m . Let $n, m \in \mathbb{Z}$ with $|n - m| > 1$. First, we consider the case that $n, m \geq 0$. Without loss of generality, we may assume that $n < m$. Then we have

$$\text{dist}(e_n, e_m) \geq \max\{\ell(e_{n+1}), \ell(e_{m-1})\},$$

which implies both

$$\frac{\text{diam}(e_n)}{\text{dist}(e_n, e_m)} \leq \frac{\ell(e_{n+1})}{\ell(e_n)} \leq M \quad \text{and} \quad \frac{\text{diam}(e_m)}{\text{dist}(e_n, e_m)} \leq \frac{\ell(e_m)}{\ell(e_{m-1})} \leq M.$$

The same estimates hold for $n, m < 0$. The only case remaining is that one of n and m is negative and one non-negative. We may assume that $n \geq 0$, $m < 0$ and $|n| \leq |m + 1|$. Then,

$$\text{dist}(e_n, e_m) \geq (2\pi)^{2-\beta} \sin\left(\frac{\beta\pi}{2}\right) \max\{n^{2-\beta}, |m + 1|^{2-\beta}\}$$

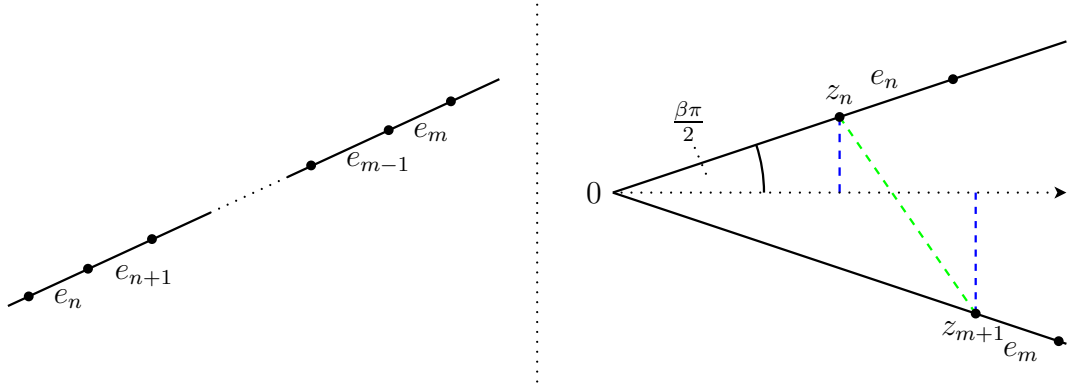


Figure 2.8: Illustration of the distance between non-adjacent edges e_n and e_m . On the left, $\text{dist}(e_n, e_m) = \sum_{k=n+1}^{m-1} \ell(e_k)$. On the right, the green dashed line is $\text{dist}(e_n, e_m)$, which is larger than the blue dashed lines.

which again yields both

$$\frac{\text{diam}(e_n)}{\text{dist}(e_n, e_m)} \leq \frac{\ell(e_n)}{(2\pi)^{2-\beta} \sin\left(\frac{\beta\pi}{2}\right) n^{2-\beta}} \leq C$$

and

$$\frac{\text{diam}(e_m)}{\text{dist}(e_n, e_m)} \leq \frac{\ell(e_m)}{(2\pi)^{2-\beta} \sin\left(\frac{\beta\pi}{2}\right) |m+1|^{2-\beta}} \leq C$$

for some constant $C > 0$. Hence, we see that also the fourth condition of a bounded geometry graph is fulfilled (compare Definition 2.65). Thus, (V, E) is a bounded geometry tree.

In order to apply Bishop's Theorem, we need to check that the τ -sizes of the edges are bounded below by some constant. In the case under consideration, we need to check that $\ell(\tau(e_n)) \geq d > 0$ for all $n \in \mathbb{Z}$, where d is some constant independent of n . Since our construction is symmetric, it suffices to consider the edges e_n with $n \geq 0$. Let e_n be such an edge. We then have

$$\begin{aligned} \ell(\tau(e_n)) &= |\tau(z_{n+1}) - \tau(z_n)| = \left| (2\pi)^{\frac{2-\beta}{\beta}} e^{i\frac{\pi}{2}} \left((n+1)^{\frac{2-\beta}{\beta}} - n^{\frac{2-\beta}{\beta}} \right) \right| \\ &= (2\pi)^{\frac{2-\beta}{\beta}} \left((n+1)^{\frac{2-\beta}{\beta}} - n^{\frac{2-\beta}{\beta}} \right) > \pi \end{aligned}$$

since $\beta \leq 1$ and thus $\frac{2-\beta}{\beta} \geq 1$.

Remark 2.76. If $\beta > 1$, then $\ell(\tau(e_n)) \rightarrow 0$ as $n \rightarrow \infty$. Thus, we cannot apply quasi-conformal folding in this case. The condition $\beta \leq 1$ means that the R-component is not allowed to be larger than a half-plane.

In conclusion, we can apply Bishop's Theorem and obtain a quasiconformal map ϕ and an entire map $f \in \mathcal{S}$ such that $f \circ \phi = \exp \circ \eta^{-1}$ on $L \setminus N$ and $f \circ \phi = \cosh \circ \tau$ on $R \setminus N$ where N is a small neighbourhood of ∂R in which Bishop's folding takes place.

Exchanging $\eta(z)$ by $\eta\left(\frac{1}{p} \cdot z\right)$ for a large number $p \in \mathbb{N}$ divides each edge e_n into p new, smaller edges. Thus, the folding domain N gets smaller, but only by a constant factor. Although this method increases the set where ϕ is conformal, the asymptotic behaviour of ϕ remains unchanged.

We will now prove that the logarithmic area of N is finite. The following lemma shows that the logarithmic area of domains in the right half-plane only changes by a constant if a power mapping $f(z) = z^\alpha$ is applied.

Lemma 2.77. *Let $\alpha \in (0, 2)$ and let $A \subset \mathbb{H}_r$ be a domain. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be given by $f(z) = z^\alpha$ and $B = f(A)$. Then,*

$$\log \text{area}(B) = \alpha^2 \cdot \log \text{area}(A).$$

Proof. Since $A \subset \mathbb{H}_r$, the map $f|_A$ is biholomorphic. With $f'(z) = \alpha z^{\alpha-1}$ we obtain by substitution

$$\begin{aligned} \log \text{area}(B) &= \iint_B \frac{1}{x^2 + y^2} dx dy = \int_B \frac{1}{|z|^2} d(x, y) = \int_A \frac{\det J_f(z)}{|f(z)|^2} d(x, y) \\ &= \int_A \frac{|f'(z)|^2}{|f(z)|^2} d(x, y) = \int_A \frac{\alpha^2}{|z|^2} d(x, y) = \alpha^2 \cdot \log \text{area}(A) \end{aligned}$$

where $J_f(z)$ denotes the Jacobian of f at z . □

By construction, the only changes to $\exp \circ \eta^{-1}$ in L are in a small domain near ∂L which is given as the image of $A := \{x + iy : -2\pi < x < 0\}$ under η . The map η is conjugated by $z \mapsto -z$ to the power mapping $z \mapsto z^{2-\beta}$. By Lemma 2.77, it suffices to prove that the logarithmic area of A is finite in order to get that the logarithmic area of $L \cap N$ is finite.

If the logarithmic area of $A' = \{x + iy : -2\pi < x < 0, y > 1\}$ is finite, the logarithmic area of A is finite as well. We have

$$\log \text{area}(A') = \int_1^\infty \int_{-2\pi}^0 \frac{1}{x^2 + y^2} dx dy \leq \int_1^\infty \frac{2\pi}{y^2} dy < \infty.$$

Hence, $\log \text{area}(A)$ is finite.

We now want to determine $R \cap N$. If $\beta = 1$, then both η and τ are just the identity and the domain N is the vertical strip $\{x + iy : -2\pi < x < 2\pi, y \in \mathbb{R}\}$. With the reasoning above, the logarithmic area of this set is finite. Let now $\beta < 1$. For convenience, we rotate the situation by 90° to the right. The imaginary axis is then replaced by the real

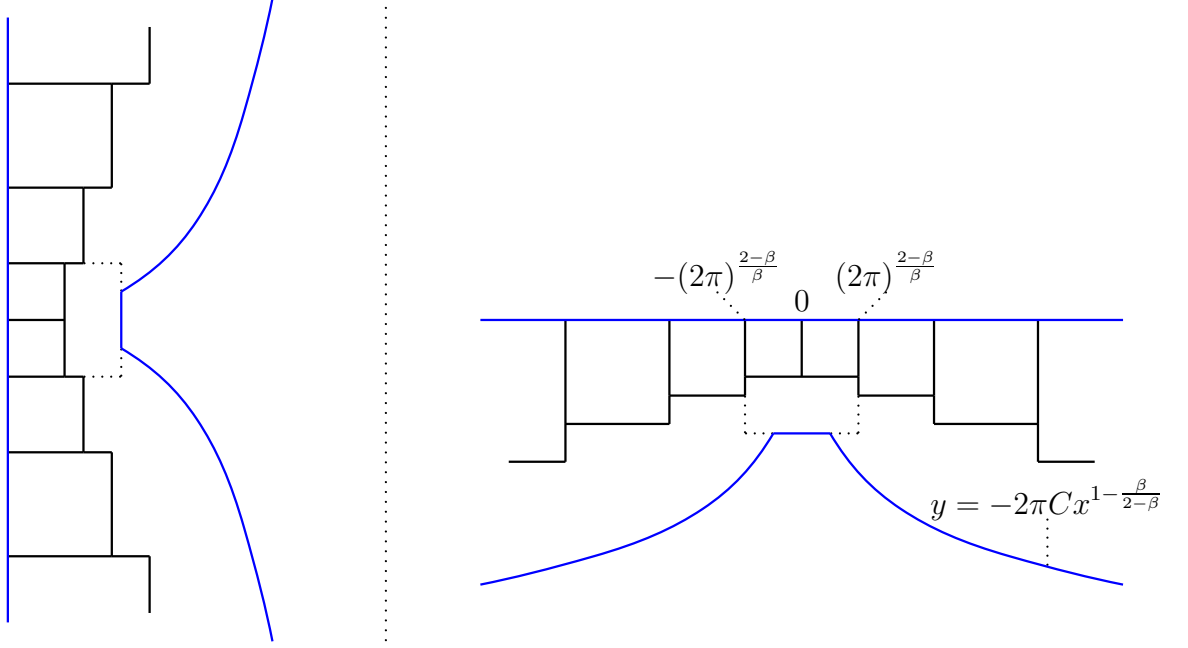


Figure 2.9: Illustration of the domain in \mathbb{H}_r in which the folding takes place. On the left, the domain in the right half-plane. On the right, the situation rotated by 90° to the right. The blue lines illustrate the boundary of the folding domain.

axis and the right half-plane by the lower half-plane. For any $n \in \mathbb{Z}$, the point $\tau(z_n)$ is then given by $\tau(z_n) = (2\pi)^{\frac{2-\beta}{\beta}} n^{\frac{2-\beta}{\beta}}$ for $n > 0$ and $\tau(z_n) = -(2\pi)^{\frac{2-\beta}{\beta}} |n|^{\frac{2-\beta}{\beta}}$ for $n < 0$. Recall that the domain in which the folding occurs in the right half-plane was given by the interior of the union of closed squares with one side being the image of an edge e_n under τ . Due to symmetry, we may again restrict to the case $n \geq 0$. Then, the height of this folding domain at $\tau(z_n)$ is given by $\ell(\tau(e_n))$. Let $x = \tau(z_n) = (2\pi)^{\frac{2-\beta}{\beta}} n^{\frac{2-\beta}{\beta}}$. Thus, $n = (2\pi)^{-1} x^{\frac{\beta}{2-\beta}}$ which yields

$$\ell(\tau(z_n)) = (2\pi)^{\frac{2-\beta}{\beta}} \left((n+1)^{\frac{2-\beta}{\beta}} - n^{\frac{2-\beta}{\beta}} \right) = x \left(\left(1 + \frac{1}{n} \right)^{\frac{2-\beta}{\beta}} - 1 \right).$$

Since

$$\left(1 + \frac{1}{n} \right)^{\frac{2-\beta}{\beta}} \leq 1 + \frac{C_1}{n}$$

for some $C_1 > 0$, we get

$$\ell(\tau(e_n)) \leq \frac{x \cdot C_1}{n} = 2\pi C_1 \cdot x^{1-\frac{\beta}{2-\beta}}.$$

We see that the folding domain in question is contained in the union of

$$\left\{ x + iy : x \in \mathbb{R}, -2\pi C_1 |x|^{1-\frac{\beta}{2-\beta}} < y < 0 \right\}$$

and

$$\left\{ x + iy : x \in \left(-(2\pi)^{\frac{2-\beta}{\beta}}, (2\pi)^{\frac{2-\beta}{\beta}} \right), -2(2\pi)^{\frac{2-\beta}{\beta}} < y < 0 \right\}.$$

Rotating back yields that the original folding domain is contained in the union of

$$\left\{ x + iy : y \in \mathbb{R}, 0 < x < 2\pi C_1 |y|^{1-\frac{\beta}{2-\beta}} \right\}$$

and

$$\left\{ x + iy : y \in \left(-(2\pi)^{\frac{2-\beta}{\beta}}, (2\pi)^{\frac{2-\beta}{\beta}} \right), 0 < x < 2(2\pi)^{\frac{2-\beta}{\beta}} \right\}.$$

Let $C_2 = 2\pi C_1$. To check whether the logarithmic area of the union above is finite, it suffices to prove that the logarithmic area of

$$B := \left\{ x + iy : y > 1, 0 < x < C_2 \cdot y^{1-\frac{\beta}{2-\beta}} \right\}$$

is finite. But since $\beta \in (0, 1)$,

$$\log \text{area}(B) = \int_1^\infty \int_0^{C_2 \cdot y^{1-\frac{\beta}{2-\beta}}} \frac{1}{x^2 + y^2} dx dy \leq C_2 \int_1^\infty y^{-(1+\frac{\beta}{2-\beta})} dy < \infty.$$

By Lemma 2.77, we obtain that the logarithmic area of $R \cap N$ is finite. Combined, we see that $\log \text{area}(N) < \infty$. By Theorem 2.58, we get that ϕ is asymptotically conformal at infinity.

Multiple tracts

The next step will be to consider maps which have not only one tract but multiple tracts shaped like sectors. Let $m \in \mathbb{N}$ and let $\alpha_k \in (0, 2)$, $1 \leq k \leq m$, such that $\sum_{k=1}^m \alpha_k = 2$. Let

$$\Omega_k := \left\{ r e^{i\varphi} : r > 0, \sum_{j=1}^{k-1} \alpha_j \pi < \varphi < \sum_{j=1}^k \alpha_j \pi \right\}$$

and let

$$\sigma_k : \Omega_k \rightarrow \mathbb{C}, z \mapsto z \cdot e^{-i\pi(\sum_{j=1}^{k-1} \alpha_j + \frac{\alpha_k}{2})}.$$

The maps σ_k rotate Ω_k such that the image is symmetric with respect to \mathbb{R} . We now label the sectors Ω_k as R- or L-components such that no two adjacent sectors are L-components and the opening angle of an L-component is larger than the opening angles of the adjacent R-components. This corresponds to the condition $\beta \leq 1$ in the case of a single tract.

We define for an R-component Ω_k

$$\tau_k : \Omega_k \rightarrow \mathbb{H}_r, z \mapsto (\sigma_k(z))^{\frac{2}{\alpha_k}}$$

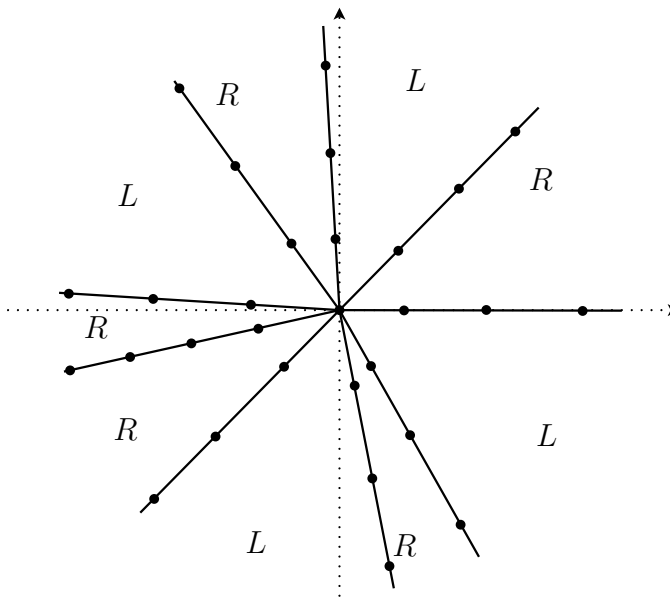


Figure 2.10: Illustration of the case with multiple sectors. The sectors are labelled by L and R for L-components and R-components, respectively.

and for an L-component Ω_k

$$\tau_k : \Omega_k \rightarrow \mathbb{H}_l, z \mapsto -(\sigma_k(z))^{\frac{2}{\alpha_k}}.$$

These functions map R-components conformally onto \mathbb{H}_r and L-components conformally on \mathbb{H}_l . We will now add vertices onto the rays separating the sectors Ω_j . For every L-component Ω_k , we add vertices onto $\partial\Omega_k$ by pulling back $2\pi i\mathbb{Z}$ under τ_k . With this procedure, we added vertices onto every ray separating one L- and one R-component. On a ray separating two R-components Ω_k and Ω_j with $\alpha_k\pi \leq \alpha_j\pi$ we define vertices via $\tau_k^{-1}(2\pi i\mathbb{Z}) \cap \partial\Omega_j$.

Our tree $T = (V, E)$ is now given by V being the set of vertices defined above and E being the set of edges where an edge is given as the part of some ray between two vertices. With the same reasoning as in the last subsection, we see that T has bounded geometry. Thus, Bishop's Theorem can be applied to obtain a map $f \in \mathcal{S}$ which essentially has the R-components as tracts.

As in the previous case of only one sector, exchanging the maps τ_k with $K \cdot \tau_k$ for some large constant $K \in \mathbb{N}$ divides every edge e_n into K smaller edges. Thus, the set N where all the folding takes place gets smaller. In fact, we can make this set as small as we want. With a similar argument as in the last subsection, we obtain that ϕ is asymptotically conformal at infinity in the case of multiple tracts as well.

3 Quasiregular Maps with One Parabola Shaped Tract

In this chapter, we want to prove Theorem 1. For the reader's convenience, we restate the theorem here. Beforehand, we need to give the following notation.

Notation 3.1. Let $r_0 > 0$ and let $w : (r_0, \infty) \rightarrow (0, \infty)$. We then define

$$G(r_0, w) := \{re^{i\phi} : r > r_0, |\phi| < w(r)\}.$$

Theorem 1. Let $\delta \in [0, \frac{1}{2}]$. For $\delta = 0$ let $\alpha < -1$, for $\delta = \frac{1}{2}$ let $\alpha \geq 0$ and let $\alpha \in \mathbb{R}$ otherwise. Let $w : (1, \infty) \rightarrow (0, \infty)$, $r \mapsto \frac{(\log r)^\alpha}{r^\delta}$ and let $k_0 > 0$. Then there exist constants $k \in (0, k_0)$, $\rho > 1$ and $R \geq 1$, and a quasiregular map g such that

$$G(\rho, kw) \subset \{z \in \mathbb{C} : |g(z)| \geq R\} \subset G(1, k_0w).$$

Furthermore, there exist a quasiconformal map ϕ and an entire map $f \in \mathcal{S}$ such that $g = f \circ \phi$. The only critical values of f are 1 and -1 and the only asymptotic value of f is 0. Moreover, g has only one tract. This tract is symmetric to the real axis.

The idea of the proof of Theorem 1 is to alter $G(1, k_0w)$ into a domain Ω such that the boundary of Ω is a bounded geometry tree. Applying Bishop's folding technique without the last step (Theorem 2.71) – the usage of the Measurable Riemann Mapping Theorem – will then yield the desired quasiregular map g .

The proof is structured as follows. We will first state some definitions and notations needed. Then we will analyse the newly defined maps and domains to obtain a better understanding of these. Thereafter, we will use Bishop's folding technique to construct our quasiregular maps.

Let us start with some notations used throughout this and the following chapter.

3.1 Notations

Throughout the next two chapters, we will use the following notations. If not stated otherwise, \log denotes the branch of the logarithm with argument in $(-\pi, \pi)$. Let δ, α, k_0 and w be as in Theorem 1.

Notation 3.2. Let

$$\beta := \begin{cases} -\alpha & , \delta \neq 0 \\ 1 - \alpha & , \delta = 0 \end{cases} \quad (3.1)$$

and $0 < \vartheta < \frac{\pi}{8}$. Furthermore, let $R_0 > 0$ be large such that

$$R_0 > \begin{cases} 2 & , \delta = 0 \\ \max \{2, 1 + e^{\frac{\alpha}{\delta}}\} & , \delta \neq 0 \end{cases}. \quad (3.2)$$

The value of R_0 will be determined later. Define

$$\mathcal{G} := \{R_0 + re^{i\phi} : r > 0, |\phi| < \pi - \vartheta\}$$

and

$$h : \mathcal{G} \rightarrow \mathbb{C}, h(z) = z^\delta (\log z)^\beta.$$

Let $K > \exp(c \cdot h(R_0 + 1))$ and let $0 < \varepsilon \ll \frac{\pi}{4}$. The value of ε will be determined later. Furthermore, define

$$c = c(\alpha, \delta, \varepsilon) := \begin{cases} \frac{\pi(1+\varepsilon)}{2k_0(1-\alpha)} & , \delta = 0 \\ \frac{\pi(1+\varepsilon)}{2k_0\delta} & , \delta \neq 0 \end{cases}.$$

Let Ω be the component of $\{z \in \mathbb{C} : \operatorname{Re} \exp(c \cdot h(z)) > K\}$ which contains $\mathbb{R}_{>\tilde{K}}$ for some $\tilde{K} > 0$. Define

$$\tau : \mathcal{G} \rightarrow \mathbb{C}, \tau(z) = \exp(c \cdot h(z)) - K.$$

As we will see in Section 3.2, this function maps Ω conformally onto \mathbb{H}_r . Let η map $\mathbb{C} \setminus \overline{\Omega}$ conformally onto \mathbb{H}_l . We may assume that η is symmetric with respect to \mathbb{R} . Define

$$V := \eta^{-1}(2\pi i\mathbb{Z})$$

and let $z_n \in V$ such that

$$\eta(z_n) = 2\pi in$$

for all $n \in \mathbb{Z}$. Furthermore, for $n \in \mathbb{Z}$, denote by e_n the part of $\partial\Omega$ which lies between z_n and z_{n+1} . Let

$$E := \{e_n : n \in \mathbb{Z}\}.$$

Denote by $2\varphi(r)$ the opening angle of Ω at $|z| = r$, that is

$$2\varphi(r) = \operatorname{meas} \{t \in [-\pi, \pi] : re^{it} \in \Omega\}.$$

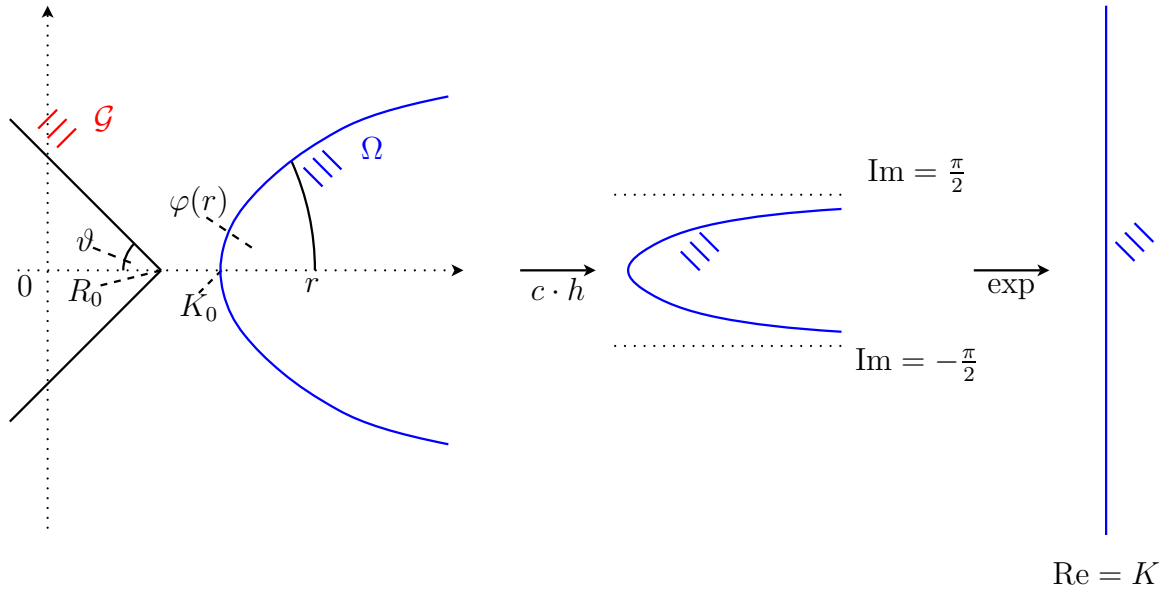


Figure 3.1: Illustration of the setup for the proof of Theorem 1.

Remark 3.3. The following statements will be used later.

- If $\delta = 0$, we have $\beta > 2$.
- If $\delta = \frac{1}{2}$, we have $\beta \leq 0$.
- The map η decomposes into

$$\eta(z) = -\exp(\mu(\log(z - K_0) - i\pi)) \quad (3.3)$$

where $K_0 = \inf\{x \in \mathbb{R} : x \in \Omega\} > R_0 + 1$ and μ maps the strip

$$S := \log(\mathbb{C} \setminus \overline{\Omega} - K_0) - i\pi$$

conformally onto $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{\pi}{2}\}$ such that $\operatorname{Re} \mu(z) \rightarrow \infty$ as $\operatorname{Re} z \rightarrow \infty$. Here, the logarithm with argument between 0 and 2π is used. Denote by ν the inverse of μ . Compare Figure 3.2.

- The map η^{-1} decomposes into

$$\eta^{-1}(z) = K_0 + \exp(\nu(\log(-z)) + i\pi). \quad (3.4)$$

For any $n \in \mathbb{N}$, the point $2\pi in$ is first mapped onto $-2\pi in$ and then by the logarithm with argument between $-\pi$ and π onto $\log(2\pi n) - i\frac{\pi}{2}$. This point is then mapped by ν onto some point w_n . We then have $z_n = \exp(w_n + i\pi) + K_0$.

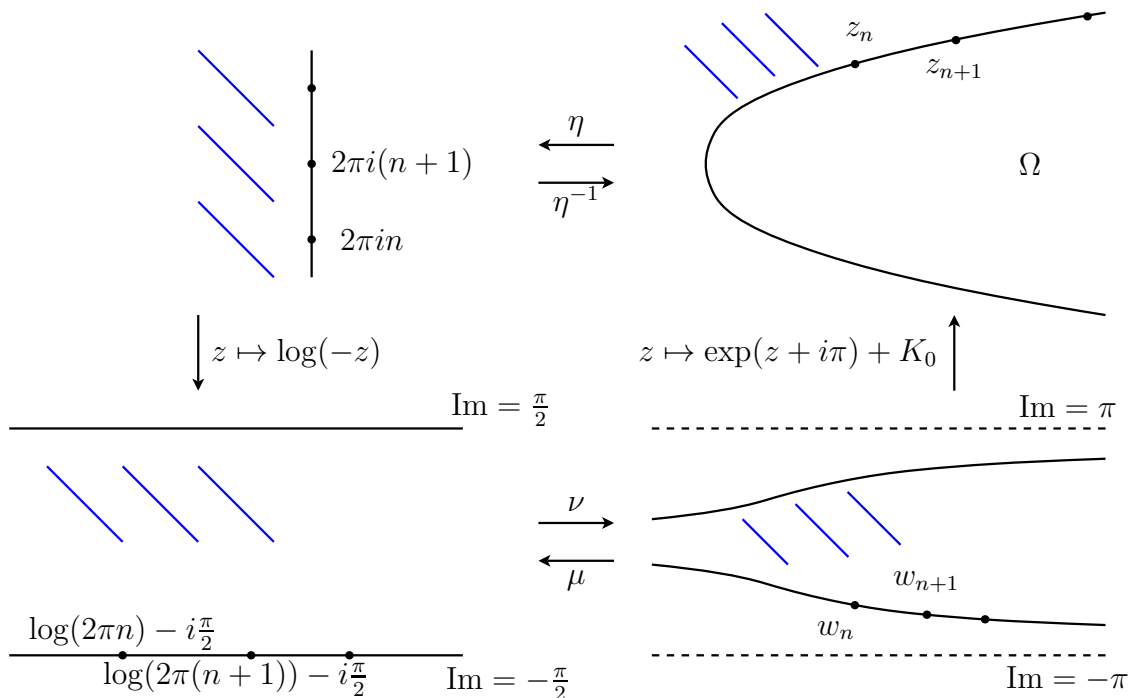


Figure 3.2: Illustration of the decomposition of η and η^{-1} , and the points z_n and w_n .

The proof of Theorem 1 falls into several separate steps:

- (1) Prove that (V, E) is a bounded geometry tree.
- (2) Prove that τ and η fulfil the inequalities needed for Bishop's construction, that is the τ -size of every edge is $\geq 2\pi$ and η maps edges to intervals of length 2π on $\partial\mathbb{H}_t$ with endpoints in $2\pi i\mathbb{Z}$. The latter condition is fulfilled by construction.
- (3) Prove that the constants k , ρ and R as in Theorem 1 exist for the quasiregular map g obtained.

Before we start proving (1), we will take a closer look at the asymptotic behaviour of h and τ . In particular, we will prove that Ω defined as above is roughly equal to $G(1, k_0w)$.

3.2 Asymptotic behaviour of h and τ

In this section, we investigate the asymptotic behaviour of h and τ as $|z| \rightarrow \infty$. Later on, this will help obtaining the needed estimates for Bishop's folding construction.

Lemma 3.4. *For $re^{i\phi} \in \mathcal{G}$, we have*

$$\operatorname{Re} h(re^{i\phi}) = h(r) \left((1 + a(r)) \cos(\delta\phi) - \left(\frac{\beta\phi}{\log r} + b(r) \right) \sin(\delta\phi) \right)$$

and

$$\operatorname{Im} h(re^{i\phi}) = h(r) \left((1 + a(r)) \sin(\delta\phi) + \left(\frac{\beta\phi}{\log r} + b(r) \right) \cos(\delta\phi) \right)$$

as $r \rightarrow \infty$ where $a, b : (1, \infty) \rightarrow \mathbb{R}$ are functions such that $a(r), b(r) = \mathcal{O}\left(\frac{\phi^2}{(\log r)^2}\right)$ as $r \rightarrow \infty$.

Proof. Let $z = re^{i\phi} \in \mathcal{G}$ with $r > 1$. Then,

$$\begin{aligned} h(z) &= z^\delta (\log z)^\beta = r^\delta e^{i\delta\phi(r)} (\log r + i\phi(r))^\beta = h(r) e^{i\delta\phi(r)} \left(1 + i \frac{\phi(r)}{\log r} \right)^\beta \\ &= h(r) e^{i\delta\phi(r)} \left(1 + i \frac{\beta\phi(r)}{\log r} + \mathcal{O}\left(\frac{\phi(r)^2}{(\log r)^2}\right) \right) \quad \text{as } r \rightarrow \infty. \end{aligned}$$

We write the \mathcal{O} term as $a(r) + ib(r)$ where $a, b : (1, \infty) \rightarrow \mathbb{R}$. In particular, we have $a(r), b(r) = \mathcal{O}\left(\frac{\phi^2}{(\log r)^2}\right)$ as $r \rightarrow \infty$. We obtain

$$\begin{aligned} h(z) &= h(r) (\cos(\delta\phi(r)) + i \sin(\delta\phi(r))) \left(1 + i \frac{\beta\phi(r)}{\log r} + (a(r) + ib(r)) \right) \\ &= h(r) (\cos(\delta\phi(r)) + i \sin(\delta\phi(r))) \left((1 + a(r)) + i \left(\frac{\beta\phi(r)}{\log r} + b(r) \right) \right) \end{aligned}$$

and hence, the conclusion follows. \square

Thus, we get a good control of both the real and imaginary part of h . If we only consider points $re^{i\phi(r)}$ such that $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$, we see that

$$\operatorname{Re} h(re^{i\phi}) \sim h(r) \quad \text{as } r \rightarrow \infty.$$

In this case, obviously $\operatorname{Re} h(z) \rightarrow \infty$ as $r \rightarrow \infty$. In fact, $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$ is not necessary.

Lemma 3.5. For $z \in \mathcal{G}$,

$$\operatorname{Re} h(z) \geq \frac{1}{2} \cos(\delta(\pi - \vartheta)) h(|z|).$$

In particular,

$$\operatorname{Re} h(z) \rightarrow \infty \quad \text{as } |z| \rightarrow \infty.$$

Proof. Let $z = R_0 + re^{i\phi} \in \mathcal{G}$. We have (compare Figure 3.3)

$$|\arg z| = |\arg(R_0 + re^{i\phi})| \leq |\phi| < \pi - \vartheta.$$

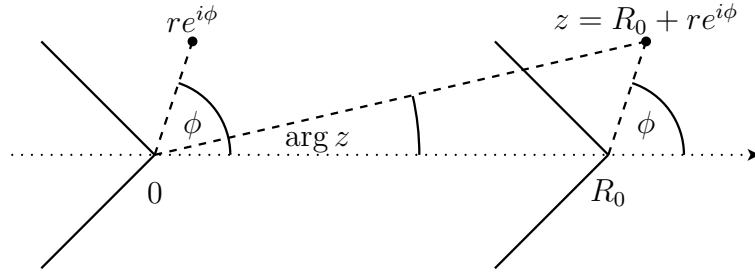


Figure 3.3: Illustration of the estimate on the argument of $R_0 + re^{i\phi}$.

Furthermore, $|\delta \arg z| \leq \delta(\pi - \vartheta) \leq \frac{1}{2}(\pi - \vartheta) < \frac{\pi}{2}$. Hence,

$$|\cos(\delta \arg z)| = \cos(\delta \arg z) \geq \cos(\delta(\pi - \vartheta)) > 0.$$

As shown in Lemma 3.4, we have

$$\operatorname{Re} h(z) = h(|z|) \left((1 + a(|z|)) \cos(\delta \arg z) - \left(\frac{\beta \arg z}{\log |z|} + b(|z|) \right) \sin(\delta \arg z) \right).$$

If $|z|$ is large enough such that $1 - |a(|z|)| \geq \frac{3}{4}$ and

$$\left| \left(\frac{\beta \arg z}{\log |z|} + b(|z|) \right) \sin(\delta \arg z) \right| < \frac{1}{4} \cos(\delta(\pi - \vartheta)),$$

recall that $a(r), b(r) = \mathcal{O}((\log r)^{-2})$, we get

$$\operatorname{Re} h(z) \geq \frac{1}{2} \cos(\delta(\pi - \vartheta)) h(|z|).$$

Since by definition $h(|z|) \rightarrow \infty$ as $|z| \rightarrow \infty$, the assertion follows. \square

Now we understand the asymptotic behaviour of h . The next task is to take a closer look at τ or, to be more precise, at $\log \tau$.

Lemma 3.6. *For $z \in \mathcal{G}$,*

$$\log(\tau(z)) \sim c \cdot h(z)$$

as well as

$$\operatorname{Re} \log(\tau(z)) \sim c \cdot \operatorname{Re} h(z) \quad \text{and} \quad \operatorname{Im} \log(\tau(z)) \sim c \cdot \operatorname{Im} h(z)$$

hold as $\operatorname{Re} z \rightarrow \infty$.

Proof. We have

$$\begin{aligned}
\log(\tau(z)) &= \log(\exp(c \cdot h(z)) - K) \\
&= \log\left(\exp(c \cdot h(z)) \left(1 - \frac{K}{\exp(c \cdot h(z))}\right)\right) \\
&= c \cdot h(z) + \log\left(1 - \frac{K}{\exp(c \cdot h(z))}\right) \\
&= c \cdot h(z) - \frac{K}{\exp(c \cdot h(z))} + \mathcal{O}(\exp(-2c \cdot h(z))) \\
&\sim c \cdot h(z) \quad \text{as } \operatorname{Re} h(z) \rightarrow \infty.
\end{aligned}$$

By Lemma 3.5, $\operatorname{Re} h(z) \rightarrow \infty$ as $\operatorname{Re} z \rightarrow \infty$ which yields

$$\begin{aligned}
\operatorname{Re} \log(\tau(z)) &= c \cdot \operatorname{Re} h(z) - K e^{-c \cdot \operatorname{Re} h(z)} \cos(-c \cdot \operatorname{Im} h(z)) \\
&\quad + \mathcal{O}(e^{-2c \cdot \operatorname{Re} h(z)} \cos(-c \cdot \operatorname{Im} h(z))) \\
&\sim c \cdot \operatorname{Re} h(z) \quad \text{as } \operatorname{Re} z \rightarrow \infty
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Im} \log(\tau(z)) &= c \cdot \operatorname{Im} h(z) - K e^{-c \cdot \operatorname{Re} h(z)} \sin(-c \cdot \operatorname{Im} h(z)) \\
&\quad + \mathcal{O}(e^{-2c \cdot \operatorname{Re} h(z)} \sin(-c \cdot \operatorname{Im} h(z))) \\
&\sim c \cdot \operatorname{Im} h(z) \quad \text{as } \operatorname{Re} z \rightarrow \infty.
\end{aligned}$$

□

After investigating the asymptotic behaviour of h and τ , we will now prove that h is injective. Beforehand, we will prove the following lemma on the argument of h' .

Lemma 3.7. *For $\delta = 0$,*

$$\arg(h'(z)) = -\arg(z) + (\beta - 1) \arctan\left(\frac{\arg z}{\log |z|}\right)$$

and for $\delta \neq 0$

$$\arg(h'(z)) = \arg(z) (\delta - 1 + o(1))$$

hold for $z \in \mathcal{G}$ as $|z| \rightarrow \infty$.

Proof. The derivative of h is given by

$$h'(z) = z^{\delta-1} (\log z)^\beta \left(\delta + \frac{\beta}{\log z} \right).$$

Let $\delta = 0$. Then,

$$\arg(h'(z)) = \arg(z^{-1}(\log z)^{\beta-1}) = \operatorname{Im} \log(z^{-1}(\log z)^{\beta-1}).$$

But

$$\log(z^{-1}(\log z)^{\beta-1}) = -\log z + (\beta - 1) \log(\log z)$$

which yields

$$\arg(h'(z)) = \operatorname{Im}(-\log z + (\beta - 1) \log(\log z)) = -\arg(z) + (\beta - 1) \arctan\left(\frac{\arg z}{\log |z|}\right)$$

for $|z| > 1$. Let now $\delta \neq 0$. Then,

$$\arg(h'(z)) = \operatorname{Im} \log\left(z^{\delta-1}(\log z)^{\beta} \left(\delta + \frac{\beta}{\log z}\right)\right).$$

Thus,

$$\begin{aligned} \arg(h'(z)) &= \operatorname{Im} \left((\delta - 1) \log z + \beta \log(\log z) + \log \left(\delta + \frac{\beta}{\log z} \right) \right) \\ &= (\delta - 1) \arg(z) + \beta \arctan\left(\frac{\arg(z)}{\log |z|}\right) + \operatorname{Im} \log \left(\delta + \beta \frac{\log |z| - i \arg(z)}{|\log z|^2} \right) \\ &= (\delta - 1) \arg(z) + \beta \arctan\left(\frac{\arg(z)}{\log |z|}\right) + \arctan\left(\frac{-\beta \arg(z)}{\delta |\log z|^2 + \beta \log |z|}\right) \\ &= \arg(z) (\delta - 1 + o(1)) \end{aligned}$$

as $|z| \rightarrow \infty$. □

Lemma 3.8. *The function h is injective if R_0 is large enough.*

Proof. Let $H_1 := \{R_0 + re^{i\phi} : r > 0, -\vartheta < \phi < \pi - \vartheta\}$. Let $h_1 : H_1 \rightarrow \mathbb{C}$ be defined by $h_1(z) = e^{i(\frac{\pi}{2}-\vartheta)} h(z)$. Let $\delta = 0$. For $z \in H_1$, we have $-\vartheta < \arg(z) < \pi - \vartheta$, compare Figure 3.3. Since by Lemma 3.7

$$\arg(h'(z)) = -\arg(z) + (\beta - 1) \arctan\left(\frac{\arg z}{\log |z|}\right),$$

this yields

$$-\pi + \vartheta < \arg(h'(z)) < \vartheta$$

for $|z|$ sufficiently large. Thus,

$$-\frac{\pi}{2} < \arg(h'_1(z)) < \frac{\pi}{2} \tag{3.5}$$

which implies

$$\operatorname{Re} h'_1(z) > 0$$

for $|z|$ sufficiently large. By increasing R_0 , we get that (3.5) holds for all $z \in H_1$. By Lemma 2.26 we obtain that h_1 is injective. Thus, h is injective on H_1 .

Let now $\delta \neq 0$. Since $|\delta - 1| < 1$ and $-\vartheta < \arg(z) < \pi - \vartheta$ for all $z \in H_1$, Lemma 3.7 yields also in this case

$$-\pi + \vartheta < \arg(h'(z)) < \vartheta$$

for $|z|$ sufficiently large. Again,

$$-\frac{\pi}{2} < \arg(h'_1(z)) < \frac{\pi}{2}$$

for $|z|$ sufficiently large. By increasing R_0 , we get that h_1 is injective in this case as well. Hence, h is injective on H_1 .

Let $H_2 := \{R_0 + re^{i\phi} : r > 0, -\pi + \vartheta < \phi < \vartheta\}$. With the same arguments as above, we see that h is injective on H_2 if R_0 is large enough. Since $h((R_0, \infty)) \subset \mathbb{R}$, we obtain that h is injective if R_0 is large. \square

Let from now on R_0 be sufficiently large so that h is injective.

Lemma 3.9. *The restriction of τ onto Ω is conformal.*

Proof. By definition, $\tau(z) = e^{c \cdot h(z)} - K$. Since the principle branch of the logarithm can be defined on \mathbb{H}_r and h is injective by Lemma 3.8, the branch ϕ of τ^{-1} given by $\phi(z) = h^{-1}\left(\frac{1}{c} \log(z + K)\right)$ can be defined on \mathbb{H}_r and maps this set onto Ω . This yields the desired result. \square

The last part of this section will deal with the growth of $|h''|$.

Lemma 3.10. *For $z \in \mathcal{G}$ with $|z|$ sufficiently large, we have*

$$|h''(z)| \leq |h'(z)|.$$

Proof. We have

$$h'(z) = \delta z^{\delta-1} (\log z)^\beta + z^\delta \beta (\log z)^{\beta-1} \frac{1}{z} = z^{\delta-1} (\log z)^\beta \left(\delta + \frac{\beta}{\log z} \right)$$

and

$$\begin{aligned} h''(z) &= ((\delta - 1)z^{\delta-2} (\log z)^\beta + \beta z^{\delta-2} (\log z)^{\beta-1}) \left(\delta + \frac{\beta}{\log z} \right) + z^{\delta-1} (\log z)^\beta \frac{-\beta \frac{1}{z}}{(\log z)^2} \\ &= z^{\delta-2} (\log z)^\beta \left(\left(\delta - 1 + \frac{\beta}{\log z} \right) \left(\delta + \frac{\beta}{\log z} \right) - \frac{\beta}{(\log z)^2} \right) \\ &= z^{\delta-2} (\log z)^\beta \left((\delta - 1)\delta + \frac{\beta\delta + (\delta - 1)\beta}{\log z} + \frac{\beta(\beta - 1)}{(\log z)^2} \right). \end{aligned}$$

Since, for some constant $A > 0$,

$$\left| \frac{(\delta - 1)\delta}{z} + \frac{\beta\delta + (\delta - 1)\beta}{z \log z} + \frac{\beta(\beta - 1)}{z(\log z)^2} \right| \leq \frac{A}{|z|} < \left| \delta + \frac{\beta}{\log z} \right| \quad \text{as } |z| \rightarrow \infty,$$

the assertion follows. \square

After obtaining a good picture of h and τ , the next step is to look at Ω .

3.3 Shape of Ω

This section deals with the shape of Ω . In particular, we will see that Ω is roughly equal to $G(1, k_0 w)$ in the sense that $\Omega \cap \{z \in \mathbb{C} : \operatorname{Re} z > r_1\} \subset G(1, k_0 w)$ for r_1 sufficiently large and the difference between the opening angle of Ω and $G(1, k_0 w)$ tends to 0. Thus, the consideration of Ω instead of $G(1, k_0 w)$ is justified.

Lemma 3.11.

$$\varphi(r) \sim \frac{k_0}{1 + \varepsilon} \cdot \frac{(\log r)^\alpha}{r^\delta} \quad \text{as } r \rightarrow \infty.$$

Proof. Let $z = r e^{i\varphi(r)} \in \partial\Omega$ with $r > 1$ and $\varphi(r) > 0$. Then Lemma 3.6 yields

$$\frac{\pi}{2} = \operatorname{Im} \log(\tau(z)) \sim c \cdot \operatorname{Im} h(z).$$

But by Lemma 3.4,

$$\operatorname{Im} h(r e^{i\varphi(r)}) = h(r) \left((1 + a(r)) \sin(\delta\varphi(r)) + \left(\frac{\beta\varphi(r)}{\log r} + b(r) \right) \cos(\delta\varphi(r)) \right).$$

Since $a(r), b(r) = \mathcal{O}\left(\frac{\varphi(r)^2}{(\log r)^2}\right)$ and $\frac{h(r)}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$, we must have $\varphi(r) \rightarrow 0$ as $r \rightarrow \infty$.

If $\delta = 0$, this yields

$$\operatorname{Im} h(z) = h(r) \left(\frac{\beta\varphi(r)}{\log r} + b(r) \right) \sim \frac{h(r)\beta\varphi(r)}{\log r} \quad \text{as } r \rightarrow \infty.$$

Therefore,

$$\varphi(r) \sim \frac{\pi}{2\beta c} \frac{\log r}{h(r)} = \frac{k_0}{1 + \varepsilon} \cdot \frac{(\log r)^\alpha}{r^\delta} \quad \text{as } r \rightarrow \infty.$$

If $\delta \neq 0$, the factor in front of $\cos(\delta\varphi(r))$ is way smaller than the first summand. Hence,

$$\operatorname{Im} h(z) \sim h(r) ((1 + a(r)) \sin(\delta\varphi(r))) \sim h(r)\delta\varphi(r) \quad \text{as } r \rightarrow \infty.$$

Thus,

$$\varphi(r) \sim \frac{\pi}{2\delta c} \frac{1}{h(r)} = \frac{k_0}{1 + \varepsilon} \cdot \frac{(\log r)^\alpha}{r^\delta} \quad \text{as } r \rightarrow \infty.$$

\square

Remark 3.12. For $\delta = 0$, we have

$$\varphi(r) \sim \frac{k_0}{1 + \varepsilon} \cdot \frac{\log r}{h(r)}$$

and for $\delta \neq 0$

$$\varphi(r) \sim \frac{k_0}{1 + \varepsilon} \cdot \frac{1}{h(r)}$$

both as $r \rightarrow \infty$.

Thus, we see that there exists some constant $r_1 > 0$ such that

$$\Omega \cap \{z \in \mathbb{C} : \operatorname{Re} z > r_1\} \subset G(1, k_0 w).$$

Now we will investigate the boundary of Ω and prove that it is an analytic curve with bounded curvature. Furthermore, we will see that it can be represented as the graph of a differentiable function – at least if $\operatorname{Re} z$ is large enough.

Lemma 3.13. *The boundary of Ω is an analytic curve.*

Proof. The principle branch of the logarithm can be defined on \mathbb{H}_r . The branch ϕ of τ^{-1} which is given by $\phi(z) = h^{-1}\left(\frac{1}{c} \log(z + K)\right)$ maps the positive real axis into the positive real axis. This branch can be analytically extended across $i\mathbb{R}$. Thus, we get that $\partial\Omega$ is the image of $\Gamma : \mathbb{R} \rightarrow \mathbb{C}, t \mapsto it$ under ϕ . Therefore, $\partial\Omega$ is the image of an analytic curve under an analytic map. Hence, $\partial\Omega$ itself is an analytic curve. \square

We define $\lambda : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\lambda(t) = \tau^{-1} \circ \Gamma(t) = \tau^{-1}(it) \tag{3.6}$$

as seen in the previous lemma. In the following lemmas, we describe the asymptotic behaviour of $\lambda(t)$ as $t \rightarrow \infty$ and investigate the behaviour of $h'(\lambda(t))$ for t large.

Lemma 3.14.

$$|\lambda(t)| \rightarrow \infty \quad \text{as} \quad |t| \rightarrow \infty.$$

Proof. By definition,

$$\lambda(t) = \tau^{-1} \circ \Gamma(t).$$

Thus,

$$\tau(\lambda(t)) = \Gamma(t).$$

Since τ has no poles and $|\Gamma(t)| \rightarrow \infty$ as $|t| \rightarrow \infty$, the assertion follows. \square

Lemma 3.15. *There exists $t_0 > 0$ and $C > 0$ such that*

$$|h'(\lambda(t))| \leq C \cdot |t|$$

for all $t \in \mathbb{R}$ with $|t| \geq t_0$.

Proof. Since $\tau(z) = e^{c \cdot h(z)} - K$, we have $c \cdot h(z) = \log(\tau(z) + K)$. Since $\lambda(t) = \tau^{-1}(it)$, we have $c \cdot h(\lambda(t)) = \log(K + it)$. With $h'(z) = h(z) \left(\frac{\delta}{z} + \frac{\beta}{z \log z} \right)$ this yields

$$c \cdot h'(\lambda(t)) = \log(K + it) \left(\frac{\delta}{\lambda(t)} + \frac{\beta}{\lambda(t) \log \lambda(t)} \right).$$

Furthermore, $|\log(K + it)| \leq \tilde{C}|t|$ for some constant $\tilde{C} > 0$ and $t \in \mathbb{R}$ such that $|t|$ is sufficiently large (of course, this estimate is weak, but it suffices for our purpose). Therefore, using Lemma 3.14,

$$|h'(\lambda(t))| \leq C|t|$$

for some constant $C > 0$ and $t \in \mathbb{R}$ such that $|t|$ is sufficiently large. \square

Since we now know the behaviour of $|h'|$ along the boundary of Ω , we will take a look at the curvature of $\partial\Omega$. First, we will prove a general result on the curvature of the image of an analytic curve under a holomorphic map provided that the curvature of the curve to start with is already known. Thereafter, we will see that the curvature of $\partial\Omega$ is bounded.

Lemma 3.16. *Let $D \subset \mathbb{C}$ be a domain and let $f : D \rightarrow \mathbb{C}$ be holomorphic. Let $\gamma : \mathbb{R} \rightarrow D$ be analytic and let $\sigma := f \circ \gamma$. Denote by κ_σ and κ_γ the curvature of σ and γ , respectively. Then both*

$$|\kappa_\sigma(t)| \leq \frac{|f''(\gamma(t))|}{|f'(\gamma(t))|^2} + \frac{1}{|f'(\gamma(t))|} |\kappa_\gamma(t)| \quad \text{and} \quad |\kappa_\gamma(t)| \leq |f'(\gamma(t))| \cdot |\kappa_\sigma(t)| + \frac{|f''(\gamma(t))|}{|f'(\gamma(t))|}$$

hold for all $t \in \mathbb{R}$.

Proof. For any analytic curve $\Gamma : \mathbb{R} \rightarrow \mathbb{C}$, the curvature of $\Gamma = x + iy$ can be computed by

$$\kappa_\Gamma(t) = \text{Im} \left(\frac{\overline{\Gamma'(t)} \cdot \Gamma''(t)}{|\Gamma'(t)|^3} \right)$$

because

$$\begin{aligned} \overline{\Gamma'(t)} \cdot \Gamma''(t) &= (x'(t) - iy'(t)) (x''(t) + iy''(t)) \\ &= x'(t)x''(t) + y'(t)y''(t) + i(x'(t)y''(t) - y'(t)x''(t)). \end{aligned}$$

Furthermore,

$$\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t) \quad \text{and} \quad \sigma''(t) = f''(\gamma(t)) \cdot \gamma'(t)^2 + f'(\gamma(t)) \cdot \gamma''(t).$$

Thus, the curvature of σ is

$$\begin{aligned} \kappa_\sigma(t) &= \operatorname{Im} \left(\frac{\overline{f'(\gamma(t))} \cdot f''(\gamma(t)) \cdot |\gamma'(t)|^2 \cdot \gamma'(t)}{|f'(\gamma(t))|^3 \cdot |\gamma'(t)|^3} + \frac{\overline{\gamma'(t)} \cdot \gamma''(t)}{|f'(\gamma(t))| \cdot |\gamma'(t)|^3} \right) \\ &= \operatorname{Im} \left(\frac{f''(\gamma(t)) \cdot \gamma'(t)}{f'(\gamma(t)) \cdot |f'(\gamma(t))| \cdot |\gamma'(t)|} \right) + \frac{1}{|f'(\gamma(t))|} \cdot \kappa_\gamma(t). \end{aligned}$$

Hence,

$$|\kappa_\sigma(t)| \leq \frac{|f''(\gamma(t))|}{|f'(\gamma(t))|^2} + \frac{1}{|f'(\gamma(t))|} \cdot |\kappa_\gamma(t)|$$

which proves the first assertion. Moreover,

$$\kappa_\gamma(t) = |f'(\gamma(t))| \cdot \kappa_\sigma(t) - \operatorname{Im} \left(\frac{f''(\gamma(t)) \cdot \gamma'(t)}{f'(\gamma(t)) \cdot |\gamma'(t)|} \right)$$

which yields the second assertion

$$|\kappa_\gamma(t)| \leq |f'(\gamma(t))| \cdot |\kappa_\sigma(t)| + \frac{|f''(\gamma(t))|}{|f'(\gamma(t))|}.$$

□

Lemma 3.17. *The curvature of $\partial\Omega$ is bounded.*

Proof. By construction, $\partial\Omega$ is represented by the curve $\lambda(t) = \tau^{-1}(it)$. Rewritten,

$$c \cdot h(\lambda(t)) = \log(K + it).$$

Using Lemma 3.16,

$$|\kappa_\lambda(t)| \leq c \cdot |h'(\lambda(t))| \cdot |\kappa_{\log(K+it)}| + \frac{|h''(\lambda(t))|}{|h'(\lambda(t))|}.$$

By Lemma 3.14, we have $|\lambda(t)| \rightarrow \infty$ as $|t| \rightarrow \infty$. Thus, for $|t|$ large, Lemma 3.15 and Lemma 3.10 imply

$$|h'(\lambda(t))| \leq c_1 \cdot |t| \quad \text{and} \quad |h''(\lambda(t))| \leq |h'(\lambda(t))|.$$

Hence,

$$|\kappa_\lambda(t)| \leq c \cdot c_1 |t| \cdot |\kappa_{\log(K+it)}| + 1. \tag{3.7}$$

It remains to show that $t \cdot \kappa_{\log(K+it)}$ is bounded. Note that a second application of Lemma 3.16 yields that $|\kappa_{\log(K+it)}| \leq 1$. Since this estimate is not sufficient, we compute $\kappa_{\log(K+it)}$ directly. We have

$$\log(K + it) = \frac{1}{2} \log(t^2 + K^2) + i \arctan\left(\frac{t}{K}\right).$$

Writing $\log(K + it) = x(t) + iy(t)$, we obtain

$$x'(t) = \frac{t}{K^2 + t^2} \quad \text{and} \quad x''(t) = \frac{K^2 - t^2}{(K^2 + t^2)^2}.$$

Furthermore,

$$y'(t) = \frac{K}{K^2 + t^2} \quad \text{and} \quad y''(t) = \frac{-2tK}{(K^2 + t^2)^2}.$$

Hence,

$$x'(t)y''(t) - y'(t)x''(t) = \frac{-2t^2K - K^3 + t^2K}{(K^2 + t^2)^3} = \frac{-K(t^2 + K^2)}{(K^2 + t^2)^3} = \frac{-K}{(K^2 + t^2)^2}$$

and

$$x'(t)^2 + y'(t)^2 = \frac{t^2 + K^2}{(K^2 + t^2)^2} = \frac{1}{K^2 + t^2}.$$

Thus, the curvature $\kappa_{\log(K+it)}$ is

$$\kappa_{\log(K+it)} = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}} = \frac{-K}{\sqrt{K^2 + t^2}}.$$

This implies that $|t| \cdot |\kappa_{\log(K+it)}|$ is bounded by some constant c_2 . Therefore, (3.7) yields the boundedness of $\kappa_\lambda(t)$ for $|t|$ sufficiently large. Since λ is analytic, κ_λ is continuous. Thus, κ_λ is bounded. \square

Hence, $\partial\Omega$ is an analytic curve with bounded curvature. The question at hand is whether we can describe $\partial\Omega$ in a better way than done before.

Lemma 3.18. *There exists a constant $x_0 > 0$ such that $\partial\Omega \cap \{x + iy : x > x_0, y > 0\}$ is the graph of an increasing function ψ . Furthermore, ψ is differentiable with $\psi'(x) \rightarrow 0$ and $\frac{\psi(x)}{x} \rightarrow 0$ as $x \rightarrow \infty$. In addition, there exists a constant $d > 0$ such that the inequality $\psi(x) \geq d \cdot \sqrt{x}$ is fulfilled for sufficiently large x .*

Proof. We will first prove that the boundary is indeed the graph of an increasing, differentiable function ψ from some point on. Thereafter, we are going to confirm the properties of ψ .

To prove our claim that the boundary is the graph of an increasing function from some point on, it suffices to show

$$0 < \arg(\lambda'(t)) < \frac{\pi}{2}$$

for large $t > 0$. Recall that by (3.6) we have $\lambda(t) = \tau^{-1}(\Gamma(t))$, where $\Gamma(t) = it$. Thus,

$$\lambda'(t) = (\tau^{-1})'(\Gamma(t)) \cdot \Gamma'(t) = \frac{\Gamma'(t)}{\tau'(\tau^{-1} \circ \Gamma(t))}.$$

Furthermore, $\Gamma' \equiv i$ and $\tau'(z) = c \cdot h'(z)e^{c \cdot h(z)} = c \cdot (\tau(z) + K) h'(z)$ which yields

$$c \cdot \lambda'(t) = \frac{i}{(K + \Gamma(t)) h'(\tau^{-1} \circ \Gamma(t))} = \frac{i}{(K + it) h'(\lambda(t))}.$$

We will verify later in this proof that

$$\arg(\lambda'(t)) = \frac{\pi}{2} - \arg(K + it) - \arg(h'(\lambda(t))) \quad (3.8)$$

holds where the principle value of the argument is used. Let us first consider the case $\delta = 0$. We then have $\beta > 2$ (compare (3.1) and Theorem 1). By Lemma 3.7, we have

$$\arg(h'(z)) = -\arg(z) + (\beta - 1) \arctan\left(\frac{\arg z}{\log |z|}\right)$$

for $z \in \mathcal{G}$ with $|z| > 1$. Since $\arctan(x) \sim x$ as $x \rightarrow 0$, we obtain

$$\arg(h'(z)) = \arg(z) \left(-1 + \frac{(\beta - 1)(1 + o(1))}{\log |z|} \right)$$

as $|z| \rightarrow \infty$. For $t > 0$, we have $\arg(\lambda(t)) > 0$. Since by Lemma 3.14 $|\lambda(t)| \rightarrow \infty$ as $t \rightarrow \infty$, we get

$$\arg(h'(\lambda(t))) = \arg(\lambda(t)) \left(-1 + \frac{(\beta - 1)(1 + o(1))}{\log |\lambda(t)|} \right) < 0$$

for sufficiently large $t > 0$. Since $\lambda(t) = r(t)e^{i\varphi(r(t))} \in \partial\Omega$ and $\varphi(r) \rightarrow 0$ as $r \rightarrow \infty$ by Lemma 3.11, we even get

$$\arg(h'(\lambda(t))) \rightarrow 0 \quad (3.9)$$

as $t \rightarrow \infty$, justifying equation (3.8) in the case $\delta = 0$. Let us now consider the case $\delta \neq 0$. By Lemma 3.7, we have

$$\arg(h'(z)) = \arg(z)(\delta - 1 + o(1))$$

for $z \in \mathcal{G}$ as $|z| \rightarrow \infty$. Since $\delta - 1 < 0$, $\arg(\lambda(t)) > 0$ for $t > 0$, and $|\lambda(t)| \rightarrow \infty$ as $t \rightarrow \infty$, we get both

$$\arg(h'(\lambda(t))) < 0 \quad \text{and} \quad \arg(h'(\lambda(t))) \rightarrow 0 \quad (3.10)$$

as $t \rightarrow \infty$, justifying equation 3.8 in this case as well. With this in mind, we get

$$0 < \arg \lambda'(t) < \frac{\pi}{2}$$

for sufficiently large t . Thus, there exists a constant $x_0 > 0$ such that

$$\partial\Omega \cap \{x + iy : x > x_0, y > 0\}$$

is the graph of an increasing function ψ . In particular, (3.8), (3.9) and (3.10) imply

$$\arg(\lambda'(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.11)$$

It remains to show that ψ is differentiable and both $\psi'(x) \rightarrow 0$ and $\frac{\psi(x)}{x} \rightarrow 0$ as $x \rightarrow \infty$. By Lemma 3.13, the boundary of Ω is an analytic curve. Thus, ψ is arbitrarily differentiable. With $\lambda(t) = x(t) + iy(t)$ we have $\psi(x(t)) = y(t)$. Hence,

$$\psi'(x(t)) = \frac{y'(t)}{x'(t)} = \tan(\arg(\lambda'(t))) \rightarrow 0$$

as $t \rightarrow \infty$ by (3.11), which yields

$$\psi'(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

But this implies

$$\psi(x) = \psi(x_0) + \int_{x_0}^x \psi'(t) dt = o(x)$$

and hence,

$$\frac{\psi(x)}{x} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Lemma 3.11 also implies for $\delta = 0$

$$\psi(x) = r(x) \sin(\varphi(r(x))) \sim r(x) \varphi(r(x)) \sim \frac{k}{1 + \varepsilon} \frac{r(x)}{(\log r(x))^{\beta-1}}.$$

Since in this case $\beta > 2$, there exists some constant $d_1 > 0$ such that for x and consequently also $r(x)$ large we have

$$\psi(x) \geq d_1 \cdot \sqrt{r(x)} \geq d_1 \cdot \sqrt{x}.$$

If in contrast $\delta \neq 0$, we obtain using Lemma 3.11

$$\psi(x) = r(x) \sin(\varphi(r(x))) \sim r(x) \varphi(r(x)) \sim \frac{k}{1 + \varepsilon} \frac{r(x)^{1-\delta}}{(\log r(x))^\beta}.$$

By the definition of β , again we see that there exists a constant $d_2 > 0$ such that for sufficiently large x

$$\psi(x) \geq d_2 \cdot \sqrt{r(x)} \geq d_2 \cdot \sqrt{x}.$$

Choosing $d := \max\{d_1, d_2\}$ and x_0 large enough, the assertion follows. \square

This section gave us a nice and clear description of $\partial\Omega$ and showed that $\partial\Omega$ is in fact given by the graph of an increasing differentiable function – starting at some point.

3.4 Asymptotic behaviour of ν

Recall that the map η , which maps $\mathbb{C} \setminus \overline{\Omega}$ conformally onto \mathbb{H}_t , can be decomposed into

$$\eta(z) = -\exp(\mu(\log(z - K_0) - i\pi)),$$

compare (3.3) and Figure 3.2. The function μ maps $S = \log(\mathbb{C} \setminus \overline{\Omega} - K_0) - i\pi$ conformally onto $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{\pi}{2}\}$. Here, the branch of \log with argument in $(0, 2\pi)$ is chosen. The goal of this section is to analyse the map μ and the inverse ν of μ . Our first step is to prove that S is an L-Strip with boundary inclination 0 at ∞ . See Section 2.3 for details.

Lemma 3.19. *The strip $S = \log(\mathbb{C} \setminus \overline{\Omega} - K_0) - i\pi$ where the argument is chosen to be in $(0, 2\pi)$ is an L-Strip with boundary inclination 0 in the sense of Warschawski. Furthermore, there exists a constant t_0 such that the upper boundary of the intersection $S \cap \{x + iy : x > t_0, y > 0\}$ is given by the graph of a differentiable function γ .*

Proof. Let $\Lambda : \mathbb{C} \setminus [K_0, \infty) \rightarrow \mathbb{C}$ be given by $\Lambda(z) = \log(z - K_0)$ where the argument is chosen to be in $(0, 2\pi)$. We need to prove that the boundary curves of S have an L-tangent at infinity with angle of inclination 0. Due to the symmetry of our construction, again it suffices to prove our claim for the boundary curve which lies in the upper half-plane. Since we consider asymptotic behaviour at infinity, we can restrict ourselves to $S \cap \{z \in \mathbb{C} : \operatorname{Re} z > t_0\}$ for any t_0 . First, we will prove that the boundary is given by the graph of a differentiable function. Thereafter, we prove that S fulfils the L-strip properties (see Section 2.3).

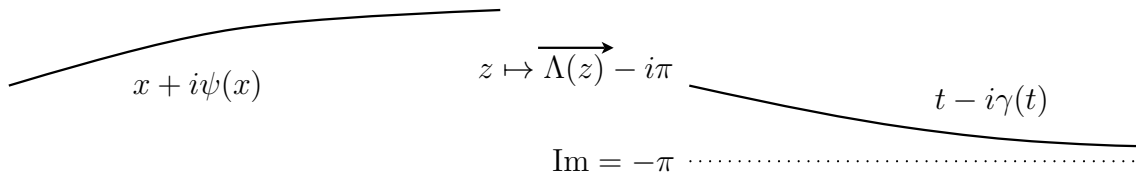


Figure 3.4: Illustration of the situation in Lemma 3.19.

By Lemma 3.18, $\partial\Omega$ is the graph of an increasing, differentiable function ψ from some point $x_0 > 0$ on.

Let $x > x_0$. We have

$$\Lambda(x + i\psi(x)) = \log \sqrt{(x - K_0)^2 + \psi(x)^2} + i \arg(x - K_0 + i\psi(x))$$

where $0 < \arg(z) < 2\pi$. Since $x - K_0 > 0$ and $\psi(x) > 0$, we get

$$\arg(x - K_0 + i\psi(x)) = \arctan \left(\frac{\psi(x)}{x - K_0} \right)$$

and therefore, by Lemma 3.18,

$$\arg(x - K_0 + i\psi(x)) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Define $g(x) := \log \sqrt{(x - K_0)^2 + \psi(x)^2}$. Since ψ is increasing, the map g is strictly increasing, hence invertible. Therefore, the set $\{\Lambda(x + i\psi(x)) : x > x_0\}$ (where we take the branch of the logarithm with argument between 0 and 2π) can be represented as the graph of a function $\tilde{\gamma} : (t_0, \infty) \rightarrow (0, 2\pi)$ with $t_0 = g(x_0)$. We may assume that x_0 is sufficiently large such that we even get $\tilde{\gamma}(t) < \pi$ for all $t \geq t_0$. Moreover,

$$\begin{aligned} \Lambda(x - i\psi(x)) &= \log \sqrt{(x - K_0)^2 + \psi(x)^2} + \arg(x - K_0 - i\psi(x)) \\ &= \log \sqrt{(x - K_0)^2 + \psi(x)^2} + 2\pi - \arg(x - K_0 + i\psi(x)), \end{aligned}$$

which implies that $\{\Lambda(x - i\psi(x)) : x > x_0\}$ can be represented as the graph of the function $t \mapsto 2\pi - \tilde{\gamma}(t)$.

Define $\gamma : \mathbb{R} \rightarrow (0, \pi)$, $t \mapsto \pi - \tilde{\gamma}(t)$ and $\tilde{S} := \{x + iy : x > t_0, |y| < \gamma(x)\}$. Since $\psi'(x) \rightarrow 0$ as $x \rightarrow \infty$, there exists for every $\kappa > 0$ an $R_\kappa > 0$ such that for all $x_2 > x_1 > R_\kappa$ we have

$$|\arg((x_2 - K_0 + i\psi(x_2)) - (x_1 - K_0 + i\psi(x_1)))| = \left| \arctan \left(\frac{\psi(x_2) - \psi(x_1)}{x_2 - x_1} \right) \right| < \kappa.$$

Furthermore, we have

$$t - i\gamma(t) = g(x) + i \arctan \left(\frac{\psi(x)}{x - K_0} \right) - i\pi$$

for $x = g^{-1}(t)$. Therefore,

$$\gamma(t) = \pi - \arctan \left(\frac{\psi \circ g^{-1}(t)}{g^{-1}(t) - K_0} \right),$$

which yields

$$\gamma'(t) = \frac{-1}{1 + \left(\frac{\psi(x)}{x - K_0} \right)^2} \cdot \frac{\psi'(x)(x - K_0) - \psi(x)}{(x - K_0)^2} \cdot \frac{1}{g'(x)}.$$

With

$$g'(x) = \frac{1}{2} \cdot \frac{2(x - K_0) + 2\psi(x)\psi'(x)}{(x - K_0)^2 + \psi(x)^2} = \frac{x - K_0 + \psi(x)\psi'(x)}{(x - K_0)^2 + \psi(x)^2}$$

we obtain

$$\begin{aligned}\gamma'(t) &= \frac{-1}{1 + \left(\frac{\psi(x)}{x-K_0}\right)^2} \cdot \frac{1}{x-K_0} \left(\psi'(x) - \frac{\psi(x)}{x-K_0}\right) \cdot \frac{(x-K_0)^2 + \psi(x)^2}{x-K_0 + \psi(x)\psi'(x)} \\ &= \frac{-1}{1 + \left(\frac{\psi(x)}{x-K_0}\right)^2} \cdot \left(\psi'(x) - \frac{\psi(x)}{x-K_0}\right) \cdot \frac{(x-K_0)^2 + \psi(x)^2}{(x-K_0)^2 + (x-K_0)\psi(x)\psi'(x)} \\ &= \frac{-1}{1 + \left(\frac{\psi(x)}{x-K_0}\right)^2} \cdot \left(\psi'(x) - \frac{\psi(x)}{x-K_0}\right) \cdot \frac{1 + \frac{\psi(x)^2}{(x-K_0)^2}}{1 + \frac{\psi(x)}{x-K_0}\psi'(x)}.\end{aligned}$$

Since for $x \rightarrow \infty$ also $t = g^{-1}(x) \rightarrow \infty$ and since $\psi'(x) \rightarrow 0$ and $\frac{\psi(x)}{x} \rightarrow 0$ as $x \rightarrow \infty$, we have

$$\gamma'(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

This implies that for any $\kappa > 0$ there exists $R_\kappa > 0$ such that for all $x_2 > x_1 > R_\kappa$ the inequality

$$|\arg((x_2 + i\gamma(x_2)) - (x_1 + i\gamma(x_1)))| < \kappa$$

holds. By symmetry,

$$|\arg((x_2 - i\gamma(x_2)) - (x_1 - i\gamma(x_1)))| < \kappa$$

holds as well. Therefore, both boundary curves of \tilde{S} have an L-tangent at $u = +\infty$ with angle of inclination 0 which yields that \tilde{S} is an L-strip with boundary inclination 0 at $u = +\infty$. Since S is just \tilde{S} extended to the left, the same is true for S . \square

Remark 3.20. Since S is an L-Strip with boundary inclination 0 at ∞ , Theorem 2.32 shows that μ has the asymptotic expression

$$\mu(u + iv) = \lambda_0 + \pi \int_{u_0}^u \frac{dt}{\Theta(t)} + i \frac{\pi v}{\Theta(u)} + o(1) \quad \text{as } u \rightarrow \infty.$$

Here, the convergence is uniform in v and $\Theta(t) = 2\gamma(t)$ with γ as in the previous proof denotes the width of S at $\text{Re} = t$. The numbers u_0 and λ_0 are real constants. Note that $\gamma(t) \rightarrow \pi$ as $t \rightarrow \infty$.

Remark 3.21. Recall that $z_n = e^{w_n + i\pi} + K_0$ for $n \in \mathbb{Z}$ (see Notation 3.2). By construction, the points w_n with $n > 0$ lie on the graph of $-\gamma$ and the points w_n with $n < 0$ on the graph of γ . Hence, $\text{Im } w_n = -\gamma(\text{Re } w_n)$ for $n > 0$ and $\text{Im } w_n = \gamma(\text{Re } w_n)$ for $n < 0$. Compare Figure 3.2 on page 46.

Theorem 2.32 also yields an asymptotic expression for $\nu'(z)$ provided that we only consider points in $\{x + iy : x \in \mathbb{R}, |y| < \frac{\pi}{2} - \theta\}$ for some $\theta \in (0, \frac{\pi}{2})$. Recall that $\nu = \mu^{-1}$. We will now show that ν can be extended onto a strictly larger domain than $\{x + iy : x > x_0, |y| < \frac{\pi}{2}\}$ for some $x_0 > 0$. Thereafter, we will use this and Warschawski's Theorem to obtain estimates on ν' on the lines $\{t \mp i\frac{\pi}{2} : t \in \mathbb{R}\}$.

Lemma 3.22. *The map ν can be extended analytically onto the half-strip*

$$\{z \in \mathbb{C} : \operatorname{Re} z > A_0, |\operatorname{Im} z| < \pi - \vartheta\}$$

for some $A_0 > 0$ and ϑ as defined in Notation 3.2.

Proof. The idea of this proof is to construct a conformal function which maps a half-strip into the left half-plane such that a horizontal line is mapped onto a vertical one. This map is essentially given by the composition of (a part of) η^{-1} and τ . In particular, the map ν will be part of this composition. We will use Schwarz's reflection principle to extend this constructed map onto a larger half-strip. After that, we will use that the function has no singular values to show that ν itself can be extended onto the half-strip in question.

Choose $A > 0$ such that $A > (2 + \sqrt{2})K_0$ and

$$\frac{1}{2} \cos(\delta(\pi - \vartheta)) h\left(\frac{A}{\sqrt{2}}\right) > \frac{\log(2K)}{c}.$$

Define

$$\begin{aligned} \tilde{\Omega} &:= \{z \in \mathbb{C} \setminus \bar{\Omega} : |z - K_0| > A, 0 < \arg(z - K_0) < \pi - \vartheta\} \\ &= \{K_0 + re^{i\phi} : r > A, \varphi(K_0 + r) < \phi < \pi - \vartheta\}. \end{aligned}$$

Let $z = K_0 + re^{i\phi} \in \tilde{\Omega}$. Then,

$$|z|^2 = (K_0)^2 + r^2 + 2K_0r \cos(\phi) \geq (K_0)^2 + r^2 - 2K_0r = (r - K_0)^2$$

and

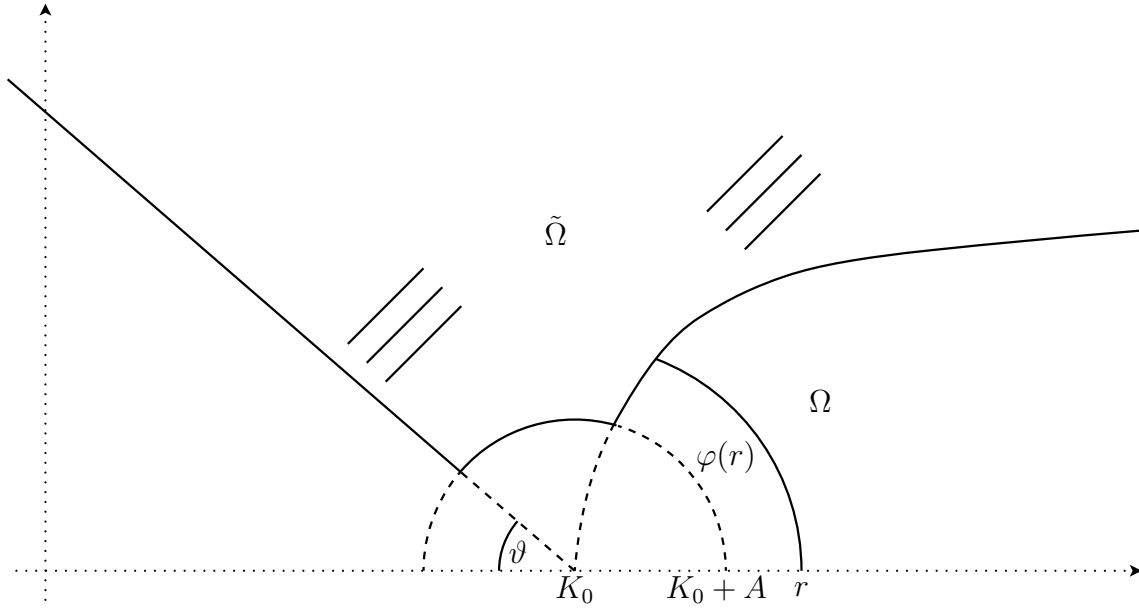
$$(r - K_0)^2 = \frac{r^2}{2} + \frac{r^2}{2} - 2rK_0 + K_0^2 = \frac{r^2}{2} + \left(\frac{r}{\sqrt{2}} - \sqrt{2}K_0\right)^2 - K_0^2.$$

Since $r > A > (2 + \sqrt{2})K_0$, we have

$$\frac{r}{\sqrt{2}} - \sqrt{2}K_0 \geq (\sqrt{2} + 1)K_0 - \sqrt{2}K_0 = K_0 > R_0 + 1 > 1$$

with R_0 from (3.2) in Notation 3.2. Therefore,

$$|z| \geq \frac{r}{\sqrt{2}} \geq \frac{A}{\sqrt{2}}.$$

Figure 3.5: Illustration of the domain $\tilde{\Omega}$.

Hence, as seen in Lemma 3.5,

$$\operatorname{Re} h(z) \geq \frac{1}{2} \cos(\delta(\pi - \vartheta)) h(|z|) \geq \frac{1}{2} \cos(\delta(\pi - \vartheta)) h\left(\frac{A}{\sqrt{2}}\right) > \frac{\log(2K)}{c}.$$

This implies

$$|\exp(c \cdot h(z))| = \exp(c \cdot \operatorname{Re} h(z)) \geq 2K. \quad (3.12)$$

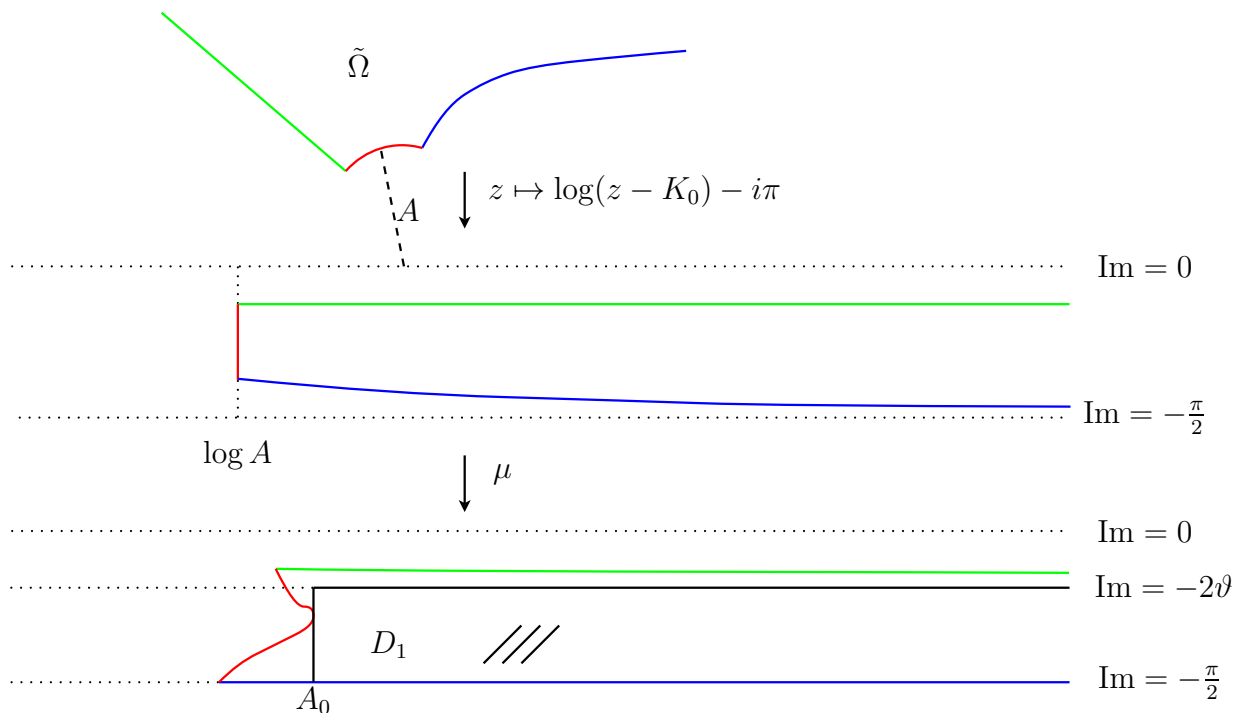
The map $z \mapsto \log(z - K_0) - i\pi$, where the branch of the logarithm with argument between 0 and 2π is taken, maps the domain $\tilde{\Omega}$ onto a domain D in the parallel strip $\{z \in \mathbb{C} : -\pi < \operatorname{Im} z < 0\}$. Here, the left boundary of D is part of the line $\operatorname{Re}(z) = \log(A)$ and the upper boundary is the line $\{t - i\vartheta : t > \log A\}$. After applying μ to D , we obtain another strip-like domain, this time in $\{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Im} z < 0\}$. By Remark 3.20, μ has the asymptotic expression

$$\mu(u + iv) = \lambda_0 + \pi \int_{u_0}^u \frac{dt}{\Theta(t)} + i \frac{\pi v}{\Theta(u)} + o(1) \quad \text{as } u \rightarrow \infty.$$

Here, λ_0 and u_0 are real constants and $\Theta(t)$ denotes the width of the strip S at $\operatorname{Re} z = t$. By construction, $\Theta(t) \rightarrow 2\pi$ as $t \rightarrow \infty$. Therefore, $\pi \leq \Theta(t) \leq 2\pi$ for all $t > \log(A)$ if A is chosen large enough. This yields for $t > \log(A)$

$$\operatorname{Im}(\mu(t - i\vartheta)) = \frac{-\pi\vartheta}{\Theta(t)} + o(1) \geq -\vartheta + o(1) \geq -2\vartheta \geq -\frac{\pi}{4}$$

as long as A is large enough.

Figure 3.6: Illustration of D_1 which is the domain of s .

Let $A_0 := \sup\{\operatorname{Re} \mu(\log(A) + iy) : \varphi(K_0 + A) - \pi < y < -\vartheta\}$. Define

$$D_1 := \left\{ z \in \mathbb{C} : \operatorname{Re} z > A_0, -\frac{\pi}{2} < \operatorname{Im} z < -2\vartheta \right\}$$

and let $s : D_1 \rightarrow \mathbb{C}$ be defined by

$$s(z) = \exp(c \cdot h(\exp(\nu(z) + i\pi) + K_0)).$$

This function maps $\{t - i\frac{\pi}{2} : t > A_0\}$ onto $\{K + iy : y > y_0\}$ for some constant y_0 . By Schwarz's reflection principle, s can be extended onto

$$D_2 := \{z \in \mathbb{C} : \operatorname{Re} z > A_0, -\pi + 2\vartheta < \operatorname{Im} z < -2\vartheta\}.$$

Since $|s(z)| = \exp(c \cdot \operatorname{Re}(h(w)))$ for some $w \in \tilde{\Omega}$, we have $|s(z)| > 2K$ for all $z \in D_1$ as seen above. Hence by (3.12), s does neither take the value 0 nor $2K$. Therefore, the extension of s has no zero. Since D_2 is simply connected, we can define a branch of $\log(s)$ on D_2 . This implies, that not only s but also $z \mapsto c \cdot h(\exp(\nu(z) + i\pi) + K_0)$ can be extended onto D_2 . Since h is injective, we get that $h^{-1}(h(\exp(\nu(z) + i\pi) + K_0)) = \exp(\nu(z) + i\pi) + K_0$ for all $z \in D_2$. Hence, also $z \mapsto \exp(\nu(z) + i\pi) + K_0$ can be extended onto D_2 . This yields the extensibility of ν onto D_2 . \square

Since we now know that ν can be extended onto a larger domain, we can establish estimates on ν' on the lines $\{t \mp i\frac{\pi}{2} : t \in \mathbb{R}\}$ using Koebe's Theorem, see Theorem 2.24.

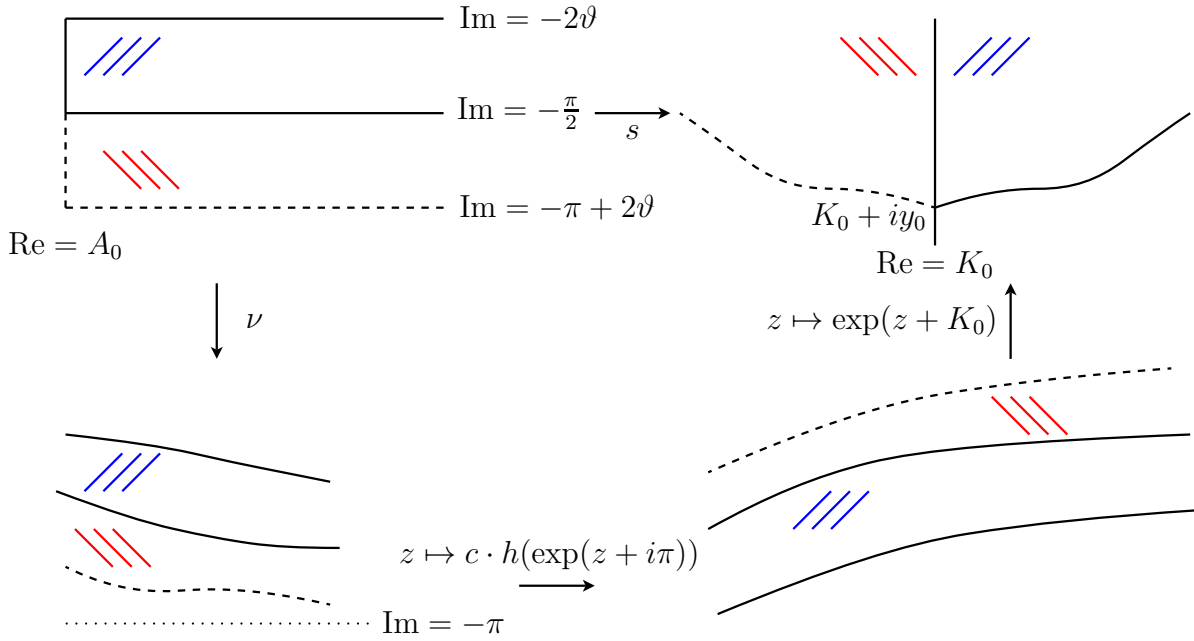


Figure 3.7: Illustration of the proof of Lemma 3.22.

Lemma 3.23.

$$1 \leq \left| \nu' \left(x - i \frac{\pi}{2} \right) \right| \leq 4$$

for sufficiently large x .

Proof. Let $\Theta(t)$ denote the width of S at $\operatorname{Re} z = t$. By Theorem 2.32, we have with $|y| \leq \frac{\pi}{2} - \varepsilon$

$$\lim_{x \rightarrow \infty} \frac{|\nu'(x + iy)|}{\Theta(\operatorname{Re} \nu(x + iy))} = \frac{1}{\pi}$$

uniformly in y . For $x \rightarrow \infty$, also $\operatorname{Re} \nu(x + iy) \rightarrow \infty$ and $\Theta(t) \rightarrow 2\pi$ as $t \rightarrow \infty$. Therefore, there exists some x_1 such that for all $x > x_1$ and $|y| \leq \frac{\pi}{2} - \varepsilon$ we obtain

$$\frac{1}{\pi} - \frac{\varepsilon}{2\pi} < \frac{|\nu'(x + iy)|}{\Theta(\operatorname{Re} \nu(x + iy))} < \frac{1}{\pi} + \frac{\varepsilon}{2\pi}.$$

Furthermore,

$$2\pi - \varepsilon < \Theta(\operatorname{Re} \nu(x + iy)) < 2\pi$$

which yields both

$$|\nu'(x + iy)| < \left(\frac{1}{\pi} + \frac{\varepsilon}{2\pi} \right) \Theta(\operatorname{Re} \nu(x + iy)) \leq (2 + \varepsilon) \leq 3 \tag{3.13}$$

and

$$\begin{aligned}
|\nu'(x+iy)| &> \Theta(\operatorname{Re} \nu(x+iy)) \left(\frac{1}{\pi} - \frac{\varepsilon}{2\pi} \right) \\
&\geq \frac{\Theta(\operatorname{Re} \nu(x+iy))}{\pi} - \varepsilon \\
&\geq \frac{2\pi - \varepsilon}{\pi} - \varepsilon \\
&= 2 - \varepsilon \left(1 + \frac{1}{\pi} \right) \\
&\geq \frac{3}{2}
\end{aligned} \tag{3.14}$$

if ε from Notation 3.2 is chosen small enough. Let $x > x_1$. The extension of ν is univalent in $B := D(x - i\frac{\pi}{2}, \frac{\pi}{2} - 2\varepsilon)$. Consider the point $z := x - i(\frac{\pi}{2} - \varepsilon) \in B$. Let $z_0 := x - i\frac{\pi}{2}$ and let $\varsigma = \frac{4\varepsilon}{\pi}$. Since $0 < \varepsilon < \frac{\pi}{4}$ (compare Notation 3.2), both $\varsigma < 1$ and $\varsigma\frac{\pi}{2} > \varepsilon$. By Corollary 2.25 (corollary of Koebe's Distortion Theorem), we have

$$\frac{1 - \varsigma}{(1 + \varsigma)^3} \leq \frac{|\nu'(z)|}{|\nu'(z_0)|} \leq \frac{1 + \varsigma}{(1 - \varsigma)^3}$$

and therefore,

$$\frac{(1 - \varsigma)^3}{1 + \varsigma} |\nu'(z)| \leq |\nu'(z_0)| \leq \frac{(1 + \varsigma)^3}{1 - \varsigma} |\nu'(z)|.$$

If ε is sufficiently small, we even get

$$\frac{2}{3} |\nu'(z)| \leq |\nu'(z_0)| \leq \frac{4}{3} |\nu'(z)|.$$

Combining this with (3.13) and (3.14), we obtain the desired estimates. \square

In conclusion, we see that we have a good control of ν' on the boundary of the strip $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{\pi}{2}\}$. Thus, we get estimates on how strongly the map η^{-1} distorts lengths of curves on $\partial\Omega$.

3.5 Comparison of the lengths of the edges

In this section, we prove that the lengths of adjacent edges of (V, E) are uniformly comparable. This means that we prove the existence of a constant $M > 0$ such that

$$\frac{1}{M} \leq \frac{\ell(e_{n+1})}{\ell(e_n)} \leq M$$

is true for all $n \in \mathbb{Z}$.

But at first we need the following estimate for the exponential map.

Lemma 3.24.

$$\exp(|z|) - 1 \leq 3|\exp(z) - 1|$$

for $z = x + iy$ with $x > 0$ and $0 < y < \varrho$ for sufficiently small ϱ .

Proof. Let $z = x + iy$ with $x > 0$ and $0 < y \leq \varrho \leq \frac{\pi}{2}$. Since

$$\exp(\sqrt{x^2 + y^2}) - \exp(x) \leq \exp(x+y) - \exp(x) = \exp(x)(\exp(y) - 1) = \exp(x)|\exp(y) - 1|,$$

we have

$$\begin{aligned} \exp(|z|) - 1 &= \exp(x) \cos(y) - 1 + \exp(x) - \exp(x) \cos(y) + \exp(\sqrt{x^2 + y^2}) - \exp(x) \\ &\leq |\exp(x) \cos(y) - 1| + \exp(x)|1 - \cos(y)| + \exp(x)|\exp(y) - 1|. \end{aligned}$$

Because of

$$|1 - \cos(y)| = 1 - \cos(y) \leq \frac{1}{2}y^2,$$

we obtain

$$\exp(x)|1 - \cos(y)| \leq \exp(x)\frac{1}{2}y^2.$$

Furthermore,

$$0 \leq \exp(y) - 1 = \sum_{n=1}^{\infty} \frac{y^n}{n!} = y \sum_{n=0}^{\infty} \frac{y^n}{(n+1)!} \leq y \exp(y) \leq y \exp(\varrho)$$

and

$$y \leq \frac{\pi}{2} \sin(y),$$

which yields

$$\exp(x)|1 - \cos(y)| + \exp(x)|\exp(y) - 1| \leq \frac{\pi}{2} \exp(x) \sin(y) \left(\frac{\varrho}{2} + \exp(\varrho) \right).$$

Since

$$\begin{aligned} |\exp(x) \cos(y) - 1| + |\exp(x) \sin(y)| &\leq \sqrt{2} \sqrt{|\exp(x) \cos(y) - 1|^2 + |\exp(x) \sin(y)|^2} \\ &= \sqrt{2} |\exp(z) - 1| \end{aligned}$$

and

$$\frac{\varrho}{2} + \exp(\varrho) \geq 1,$$

we obtain

$$\exp(|z|) - 1 \leq \left(\frac{\varrho}{2} + \exp(\varrho) \right) \frac{\pi}{2} \sqrt{2} |\exp(z) - 1| \leq 3 |\exp(z) - 1|$$

for sufficiently small ϱ . □

With this in mind, we can now establish upper and lower estimates for the lengths of the edges. We will start with an upper bound.

Lemma 3.25. *There exists $n_0 \in \mathbb{N}$ and there exists a constant $C > 0$ such that*

$$\ell(e_n) \leq \frac{C}{n} e^{\operatorname{Re} w_n}$$

for all $n \in \mathbb{N}_{\geq n_0}$.

Proof. Let $n \in \mathbb{N}$. Recall that $\eta(z_n) = 2\pi in$. Consider the points $2\pi in$, $2\pi i(n+1)$ and $2\pi i(n+2)$. As seen in (3.4), η^{-1} is given by

$$\eta^{-1}(z) = K_0 + \exp(\nu(\log(-z)) + i\pi).$$

The points above are first mapped onto $-2\pi in$, $-2\pi i(n+1)$ and $-2\pi i(n+2)$ and then by the logarithm with argument between $-\pi$ and π onto $\log(2\pi n) - i\frac{\pi}{2}$, $\log(2\pi(n+1)) - i\frac{\pi}{2}$ and $\log(2\pi(n+2)) - i\frac{\pi}{2}$, respectively. Those points are then mapped onto w_n , w_{n+1} and w_{n+2} under ν . We then have $z_n = \exp(w_n + i\pi) + K_0$, see Figure 3.2 on page 46.

Choose x_0 big enough such that for all $x > x_0$ the inequality $|\psi'(x)| < \varepsilon$ is fulfilled. For sufficiently large n , we have $\operatorname{Re} z_n > x_0$. This yields

$$\ell(e_n) = \int_{\operatorname{Re} z_n}^{\operatorname{Re} z_{n+1}} |1 + i\psi'(t)| dt \leq \sqrt{1 + \varepsilon^2} (\operatorname{Re} z_{n+1} - \operatorname{Re} z_n). \quad (3.15)$$

Therefore,

$$\ell(e_n) \leq \sqrt{1 + \varepsilon^2} \operatorname{Re}(z_{n+1} - z_n) \leq 2|z_{n+1} - z_n|.$$

Since $z_n = \exp(w_n + i\pi) + K_0$, we have both

$$|z_n - K_0| = \exp(\operatorname{Re}(w_n + i\pi)) = \exp(\operatorname{Re} w_n)$$

and

$$\frac{z_{n+1} - K_0}{z_n - K_0} = \exp(w_{n+1} - w_n).$$

Hence,

$$\begin{aligned} |z_{n+1} - z_n| &= |z_n - K_0| \left| \frac{z_{n+1} - K_0}{z_n - K_0} - 1 \right| \\ &= e^{\operatorname{Re} w_n} |\exp(w_{n+1} - w_n) - 1| \\ &\leq e^{\operatorname{Re} w_n} (\exp(|w_{n+1} - w_n|) - 1). \end{aligned} \quad (3.16)$$

Furthermore, we get using Lemma 3.23

$$|w_{n+1} - w_n| \leq \int_{\log(2\pi n)}^{\log(2\pi(n+1))} \left| \nu' \left(t - i\frac{\pi}{2} \right) \right| dt \leq 4 \log \left(1 + \frac{1}{n} \right).$$

Thus,

$$|w_{n+1} - w_n| \leq \log \left(\left(1 + \frac{1}{n}\right)^4 \right). \quad (3.17)$$

Therefore,

$$\exp(|w_{n+1} - w_n|) - 1 \leq \left(1 + \frac{1}{n}\right)^4 - 1 = \frac{4}{n} + \frac{6}{n^2} + \frac{4}{n^3} + \frac{1}{n^4} \leq \frac{15}{n}.$$

Combining the equations above, we obtain

$$\ell(e_n) \leq \frac{30}{n} e^{\operatorname{Re} w_n}$$

for large n . □

The next step is to prove that roughly the same estimate also holds from below.

Lemma 3.26. *There exists $n_0 \in \mathbb{N}$ and there exists a constant $D > 0$ such that*

$$\ell(e_n) \geq \frac{D}{n} e^{\operatorname{Re} w_n}$$

for all $n \in \mathbb{N}_{\geq n_0}$.

Proof. The proof follows the same steps as the proof of Lemma 3.25. Let w_n be as in Notation 3.2 and x_0 be defined as in Lemma 3.25. For sufficiently large n ,

$$\ell(e_n) = \int_{\operatorname{Re} z_n}^{\operatorname{Re} z_{n+1}} |1 + i\psi'(t)| dt \geq \operatorname{Re} z_{n+1} - \operatorname{Re} z_n.$$

If n is large enough such $\operatorname{Re} z_n > x_0$, we have

$$\begin{aligned} |z_{n+1} - z_n| &= \sqrt{(\operatorname{Re} z_{n+1} - \operatorname{Re} z_n)^2 + (\psi(\operatorname{Re} z_{n+1}) - \psi(\operatorname{Re} z_n))^2} \\ &\leq \sqrt{1 + \varepsilon^2} |\operatorname{Re} z_{n+1} - \operatorname{Re} z_n| \\ &\leq \sqrt{2} |\operatorname{Re} z_{n+1} - \operatorname{Re} z_n|. \end{aligned}$$

Hence,

$$\ell(e_n) \geq \frac{1}{\sqrt{2}} |z_{n+1} - z_n|,$$

which corresponds to (3.15). Denote by γ_n the part of the boundary of S between w_n and w_{n+1} . As in (3.16) in the proof of Lemma 3.25, we have

$$|z_{n+1} - z_n| = |z_n - K_0| \left| \frac{z_{n+1} - K_0}{z_n - K_0} - 1 \right| = e^{\operatorname{Re} w_n} |\exp(w_{n+1} - w_n) - 1|.$$

If n is chosen such that $\text{Im}(w_{n+1} - w_n) < \varepsilon$, we get by Lemma 3.24 (passing to a smaller value of ε if needed)

$$|\exp(w_{n+1} - w_n) - 1| \geq \frac{1}{3}(\exp(|w_{n+1} - w_n|) - 1).$$

Hence,

$$\ell(e_n) \geq \frac{1}{3\sqrt{2}}e^{\text{Re } w_n}(\exp(|w_{n+1} - w_n|) - 1).$$

Using Lemma 3.19, we get for sufficiently large n

$$\begin{aligned} \int_{\log(2\pi n)}^{\log(2\pi(n+1))} \left| \nu' \left(t - i\frac{\pi}{2} \right) \right| dt &= \ell(\gamma_n) = \int_{\text{Re } w_n}^{\text{Re } w_{n+1}} \sqrt{1 + \gamma'(t)^2} dt \\ &\leq \sqrt{2}(\text{Re } w_{n+1} - \text{Re } w_n) \leq \sqrt{2}|w_{n+1} - w_n|. \end{aligned}$$

Therefore,

$$\ell(e_n) \geq \frac{1}{3\sqrt{2}}e^{\text{Re } w_n} \left(\exp \left(\frac{1}{\sqrt{2}} \int_{\log(2\pi n)}^{\log(2\pi(n+1))} \left| \nu' \left(t - i\frac{\pi}{2} \right) \right| dt \right) - 1 \right).$$

By Lemma 3.23,

$$\int_{\log(2\pi n)}^{\log(2\pi(n+1))} \left| \nu' \left(t - i\frac{\pi}{2} \right) \right| dt \geq \log \left(1 + \frac{1}{n} \right) \geq \frac{1}{\sqrt{2}} \log \left(1 + \frac{1}{n} \right),$$

which corresponds to (3.17). Since

$$\left(1 + \frac{1}{n} \right)^{\frac{1}{2}} - 1 = \frac{\frac{1}{n}}{\left(1 + \frac{1}{n} \right)^{\frac{1}{2}} + 1} \geq \frac{1}{3n},$$

we get

$$\ell(e_n) \geq \frac{1}{9\sqrt{2}}e^{\text{Re } w_n} \frac{1}{n}.$$

□

Thus, we see that the lengths of the edges can be controlled if we know more about the behaviour of $\text{Re } w_n$ as $n \rightarrow \infty$. In fact, that is possible as we will see in the next lemma.

Lemma 3.27.

$$\text{Re } w_n = 2 \log |n| + \mathcal{O}(1) \quad \text{as } |n| \rightarrow \infty.$$

Proof. Let S and μ be as before (see Notation 3.2). By Remark 3.20, μ has the asymptotic expression

$$\mu(u + iv) = \lambda_0 + \pi \int_{u_0}^u \frac{dt}{\Theta(t)} + i\pi \frac{v}{\Theta(u)} + o(1) \quad \text{as } u \rightarrow \infty$$

uniformly in v , where λ_0 and u_0 are real constants and $\Theta(t)$ denotes the width of the strip S at $\operatorname{Re} z = t$. By construction, we have $\Theta(t) = 2\pi - 2\varphi(e^t)$. Hence, we can write

$$\mu(u + iv) = \lambda_0 + \frac{1}{2} \int_{u_0}^u \frac{2\pi - 2\varphi(e^t) + 2\varphi(e^t)}{2\pi - 2\varphi(e^t)} dt + i \frac{v}{2} \frac{2\pi - 2\varphi(e^u) + 2\varphi(e^u)}{2\pi - 2\varphi(e^u)} + o(1)$$

as $u \rightarrow \infty$. Thus,

$$\mu(u + iv) = \lambda_0 + \frac{1}{2} \int_{u_0}^u dt + \int_{u_0}^u \frac{\varphi(e^t)}{2\pi - 2\varphi(e^t)} dt + i \frac{v}{2} + i \frac{v\varphi(e^u)}{2\pi - 2\varphi(e^u)} + o(1) \quad \text{as } u \rightarrow \infty.$$

Since $\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, we obtain

$$\mu(u + iv) = \frac{1}{2}(u + iv) + \lambda_0 - \frac{u_0}{2} + o(1) + \int_{u_0}^u \frac{\varphi(e^t)}{2\pi - 2\varphi(e^t)} dt \quad \text{as } u \rightarrow \infty.$$

Let us first consider the case $\delta = 0$. By Remark 3.12, we have

$$\varphi(r) \sim \frac{k_0}{1 + \varepsilon} \cdot \frac{\log r}{h(r)} = \frac{k_0}{1 + \varepsilon} (\log r)^{1-\beta} \quad \text{as } r \rightarrow \infty.$$

Since $\delta = 0$, we have $\beta > 2$. For sufficiently large t this yields

$$|\varphi(e^t)| \leq k_0 \cdot t^{1-\beta} \leq \frac{\pi}{2}.$$

Hence,

$$|2\pi - 2\varphi(e^t)| \geq 2\pi - 2|\varphi(e^t)| \geq \pi.$$

Adjusting λ_0 and u_0 if necessary, we may assume that $u_0 > 0$. With this in mind, we get

$$\left| \int_{u_0}^u \frac{\varphi(e^t)}{2\pi - 2\varphi(e^t)} dt \right| \leq \frac{k_0}{\pi} \int_{u_0}^u |t|^{1-\beta} dt \leq \frac{k_0}{\pi} \int_{u_0}^{\infty} |t|^{1-\beta} dt < \infty$$

since $\beta > 2$. Let us now consider the case $\delta \neq 0$. By Remark 3.12, we have

$$\varphi(r) \sim \frac{k_0}{1 + \varepsilon} \cdot \frac{1}{h(r)} = \frac{k_0}{1 + \varepsilon} \cdot \frac{1}{r^\delta (\log r)^\beta} \quad \text{as } r \rightarrow \infty.$$

If t is large, this implies

$$|\varphi(e^t)| \leq k_0 \cdot e^{-\delta t} t^{-\beta} \leq \frac{\pi}{2}.$$

Again

$$|2\pi - 2\varphi(e^t)| \geq 2\pi - 2|\varphi(e^t)| \geq \pi$$

and

$$\left| \int_{u_0}^u \frac{\varphi(e^t)}{2\pi - 2\varphi(e^t)} dt \right| \leq \frac{k_0}{\pi} \int_{u_0}^{\infty} e^{-\delta t} t^{-\beta} dt < \infty.$$

Therefore,

$$\operatorname{Re} \mu(u + iv) = \frac{1}{2}u + \mathcal{O}(1) \quad \text{as } u \rightarrow \infty.$$

Since by construction (for $n \neq 0$)

$$\mu(w_n) = \log |n| \mp i\frac{\pi}{2},$$

we get

$$\frac{1}{2} \operatorname{Re} w_n + \mathcal{O}(1) = \operatorname{Re} \mu(w_n) = \log |n|$$

as $|n| \rightarrow \infty$, which implies the assertion. \square

Since we now know the behaviour of $\operatorname{Re} w_n$ as $|n| \rightarrow \infty$, we can take a closer look at $\operatorname{Re} z_n$ and $|z_n|$.

Lemma 3.28. *We have*

$$\operatorname{Re} z_n \sim |z_n|$$

as $|n| \rightarrow \infty$. Furthermore, there exists a constant $M > 0$ such that

$$\frac{n^2}{M} \leq |z_n - K_0| \leq Mn^2$$

for all $n \in \mathbb{Z}$.

Proof. By construction, we have $z_n = K_0 + e^{w_n + i\pi}$ for $n \in \mathbb{Z}$ (see Notation 3.2). Thus,

$$\operatorname{Re} z_n = e^{\operatorname{Re} w_n} \cos(\pi + \operatorname{Im} w_n) + K_0.$$

By Lemma 3.27, we have $\operatorname{Re} w_n \rightarrow \infty$ as $|n| \rightarrow \infty$. Hence, Remark 3.20 and Remark 3.21 imply that $\pi + \operatorname{Im} w_n \rightarrow 0$ as $n \rightarrow \infty$ and $\pi + \operatorname{Im} w_n \rightarrow 2\pi$ as $n \rightarrow -\infty$. In particular, $\cos(\pi + \operatorname{Im} w_n) \rightarrow 1$ as $|n| \rightarrow \infty$. Thus, we obtain

$$\operatorname{Re} z_n \sim e^{\operatorname{Re} w_n}$$

as $|n| \rightarrow \infty$. Furthermore,

$$\begin{aligned} |z_n|^2 &= (e^{\operatorname{Re} w_n} \cos(\pi + \operatorname{Im} w_n) + K_0)^2 + e^{2\operatorname{Re} w_n} (\sin(\pi + \operatorname{Im} w_n))^2 \\ &= e^{2\operatorname{Re} w_n} + K_0^2 + 2K_0 e^{\operatorname{Re} w_n} \cos(\pi + \operatorname{Im} w_n) \end{aligned}$$

which yields

$$|z_n| \sim e^{\operatorname{Re} w_n}$$

as $|n| \rightarrow \infty$. The first assertion follows. By Lemma 3.27, there exist $n_0 \in \mathbb{N}$ and constants $C, D > 0$ such that

$$2 \log |n| - D \leq \operatorname{Re} w_n \leq 2 \log |n| + C$$

for all $n \in \mathbb{Z}$ with $|n| \geq n_0$. Since

$$|z_n - K_0| = e^{\operatorname{Re} w_n},$$

we get

$$e^{-D}n^2 \leq |z_n - K_0| \leq e^C n^2$$

for $n \geq n_0$. Since $z_0 = K_0$, choosing $M > 0$ large enough implies the second assertion. \square

The last step in this section is to combine the results above to obtain the desired comparability of the edges.

Lemma 3.29. *There exists $n_0 \in \mathbb{N}$ and a constant $M > 0$ such that*

$$\frac{1}{M} \leq \frac{\ell(e_n)}{n} \leq M$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

Proof. By Lemma 3.25 and Lemma 3.26, there exist $n_0 \in \mathbb{N}$ and constants $C, D > 0$ such that for all $n \in \mathbb{N}_{\geq n_0}$ we have

$$\frac{D}{n} e^{\operatorname{Re} w_n} \leq \ell(e_n) \leq \frac{C}{n} e^{\operatorname{Re} w_n}.$$

By Lemma 3.27, there exist $C', D' > 0$ such that

$$2 \log n - D' \leq \operatorname{Re} w_n \leq 2 \log n + C'$$

for $n \geq n_0$. Thus,

$$D e^{-D'} \leq \frac{\ell(e_n)}{n} \leq C e^{C'}$$

which implies the assertion. \square

Lemma 3.30. *There exists a constant $M > 0$ such that*

$$\frac{1}{M} \leq \frac{\ell(e_{n+1})}{\ell(e_n)} \leq M$$

for all $n \in \mathbb{Z}$.

Proof. By Lemma 3.29, there exists $n_0 \in \mathbb{N}$ and $M' > 0$ such that

$$\frac{1}{M'} \leq \frac{\ell(e_n)}{n} \leq M'$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. Hence,

$$\frac{1}{(M')^2} \frac{n+1}{n} \leq \frac{\ell(e_{n+1})}{\ell(e_n)} \leq (M')^2 \frac{n+1}{n}$$

which yields the existence of a constant $M > 0$ such that

$$\frac{1}{M} \leq \frac{\ell(e_{n+1})}{\ell(e_n)} \leq M$$

for $n \geq n_0$. By symmetry, the same estimates hold for $n \leq -n_0 - 1$. If we choose M large enough, the inequality holds for all $n \in \mathbb{Z}$. \square

3.6 The ratio of diameter and distance

One of the things needed for the graph (V, E) to have bounded geometry is that the ratio

$$\frac{\text{diam}(e_n)}{\text{dist}(e_n, e_m)}$$

for non adjacent edges e_n and e_m is uniformly bounded from above. This section deals with this bound. It is here that the hypothesis $\delta \in [0, \frac{1}{2}]$ from Theorem 1 is needed.

By Lemma 3.29, we know that the length of the edge e_n is comparable to n . In order to get an upper bound of $\text{diam}(e_n)/\text{dist}(e_n, e_m)$ for edges e_m which are not adjacent to e_n , we need that $\text{dist}(e_n, e_m)$ is at least $d \cdot n$ for some constant $d > 0$. In particular, we need that the distance between e_n and the edge $e_{-(n+1)}$, which is e_n reflected at the real line, is at least $d \cdot n$. Since $\text{dist}(e_n, e_{-(n+1)}) = 2 \text{Im } z_n$, we need that $\text{Im } z_n$ is at least $d \cdot n$. By Lemma 3.28, $|z_n| \sim \text{Re } z_n$ and $|z_n|$ is comparable to n^2 . Since by Lemma 3.18 the point z_n lies on the graph of a function ψ , we have $\text{Im } z_n = \psi(\text{Re } z_n)$. Thus, we need that $\psi(x) \geq d \cdot \sqrt{x}$, which implies that the smallest domain with which the construction works is given by $\delta = \frac{1}{2}$ and $\alpha = 0$.

The proof is divided into several parts, each part treating a different case. At last, these cases will be combined to obtain the desired bound.

We start with the case that both n and m are non-negative. In other words, both edges e_n and e_m lie in the upper half-plane.

Lemma 3.31. *There exists a constant $D > 0$ such that*

$$\text{dist}(e_n, e_m) \geq D \cdot \max\{\ell(e_n), \ell(e_m)\}$$

for all $n, m \in \mathbb{N}_0$ with $|n - m| > 1$.

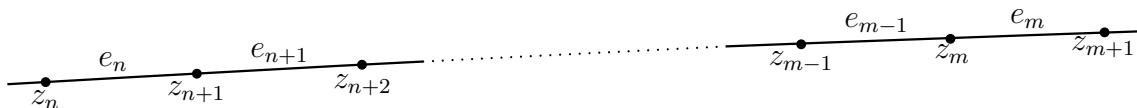


Figure 3.8: Illustration of the situation in Lemma 3.31.

Proof. Let $n, m \in \mathbb{N}_0$ with $|n - m| > 1$. This means that the edges e_n and e_m do not share a common endpoint and are non-adjacent. Without loss of generality, we may assume that $n < m$. Since $|n - m| > 1$, we then have $n + 1 < m$. Let x_0 be as in Lemma 3.18 and choose $n_0 \in \mathbb{N}$ such that $\text{Re } z_l > x_0$ for all $l \geq n_0$ and $|\psi'(x)| < 1$ for all $x \geq \text{Re } z_{n_0}$.

First, let us consider the case that $n_0 \leq n < n+1 < m$. We have

$$\ell(e_{n+1}) = \int_{\operatorname{Re} z_{n+1}}^{\operatorname{Re} z_{n+2}} \sqrt{1 + \psi'(t)^2} dt \leq \int_{\operatorname{Re} z_{n+1}}^{\operatorname{Re} z_m} \sqrt{1 + \psi'(t)^2} dt$$

and because of $\psi'(x)^2 \leq 1$, we get

$$l(e_{n+1}) \leq 2(\operatorname{Re} z_m - \operatorname{Re} z_{n+1}).$$

Since

$$\operatorname{dist}(e_n, e_m) = |z_m - z_{n+1}| \geq \operatorname{Re} z_m - \operatorname{Re} z_{n+1},$$

we obtain using Lemma 3.30

$$\ell(e_n) \leq M \cdot \ell(e_{n+1}) \leq 2M \cdot \operatorname{dist}(e_n, e_m).$$

Hence,

$$\operatorname{dist}(e_n, e_m) \geq \frac{1}{2M} \cdot \ell(e_n).$$

With a similar argument, we obtain

$$\ell(e_m) \leq M \cdot \ell(e_{m-1}) \leq 2M \cdot \operatorname{dist}(e_n, e_m)$$

which yields

$$\operatorname{dist}(e_n, e_m) \geq \frac{1}{2M} \cdot \ell(e_m).$$

Second, let $0 \leq n < n_0 < m$. By Lemma 3.30, we get

$$\ell(e_n) \leq M^{n_0} \cdot \ell(e_{n_0}).$$

Using the first case, we know that

$$\operatorname{dist}(e_{n_0}, e_m) \geq \frac{1}{2M} \cdot \ell(e_{n_0}).$$

Therefore,

$$\operatorname{dist}(e_n, e_m) \geq \operatorname{dist}(e_{n_0}, e_m) \geq \frac{1}{2M} \cdot \ell(e_{n_0}) \geq \frac{1}{2M^{n_0+1}} \cdot \ell(e_n).$$

Likewise,

$$\operatorname{dist}(e_n, e_m) \geq \operatorname{dist}(e_{n_0}, e_m) \geq \frac{1}{2M} \cdot \ell(e_m).$$

Third, let $0 \leq n < m \leq n_0$. In this case, there are only finitely many possibilities for the pair (n, m) . Hence, we can find a constant $\tilde{D} > 0$ such that in this case

$$\operatorname{dist}(e_n, e_m) \geq \tilde{D} \cdot \max\{\ell(e_n), \ell(e_m)\}.$$

Define $D := \min\left\{\tilde{D}, \frac{1}{2M}, \frac{1}{2M^{n_0+1}}\right\}$. Then,

$$\operatorname{dist}(e_n, e_m) \geq D \cdot \max\{\ell(e_n), \ell(e_m)\}$$

for all $n, m \in \mathbb{N}_0$ with $|n - m| > 1$. □

Due to the symmetry of our construction, the same estimates hold if both edges lie in the lower half-plane. We will now consider the case that one edge lies in the upper, the other in the lower half-plane. Without loss of generality, we may assume that $n + 1 \leq m$ and consider the edges e_n and e_{-m} .

Lemma 3.32. *There exists a constant $D > 0$ such that*

$$\text{dist}(e_n, e_{-m}) \geq D \cdot \max \{ \ell(e_n), \ell(e_{-m}) \}$$

for all $n, m \in \mathbb{N}$ with $n + 1 \leq m$.

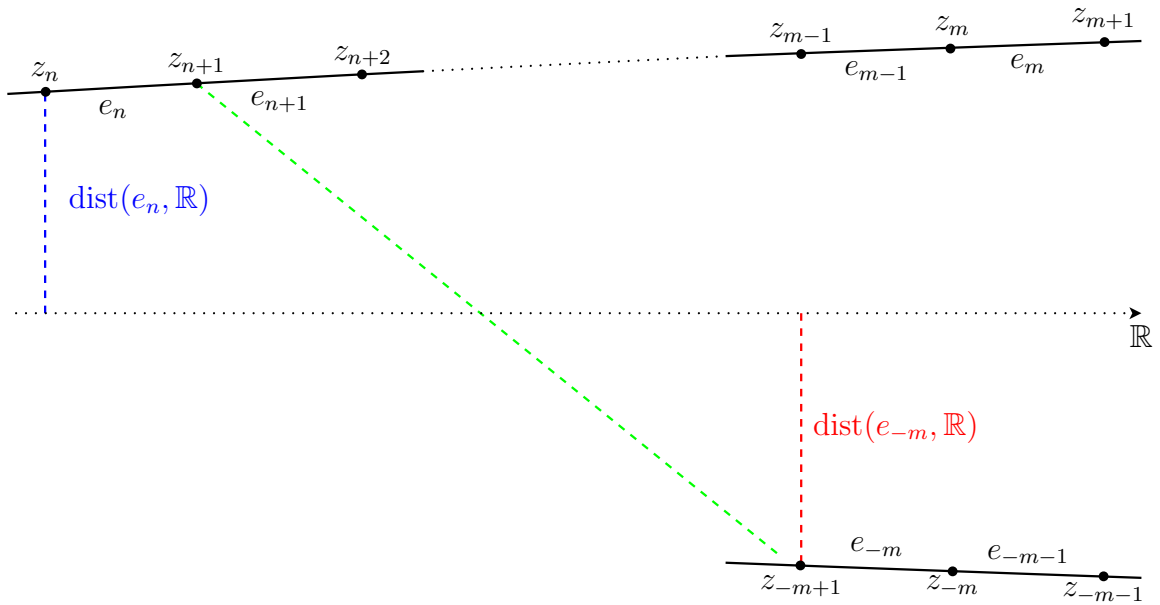


Figure 3.9: Illustration of the situation in Lemma 3.32. The dashed green line is roughly the distance between e_n and e_{-m} .

Proof. Let $n, m \in \mathbb{N}$ with $n + 1 \leq m$. Let x_0 and d be as in Lemma 3.18. Let $n_0 \in \mathbb{N}$ such that $\text{Re } z_l > x_0$ for all $l \geq n_0$.

First, let $n_0 \leq n$. We have both

$$\text{dist}(e_n, e_{-m}) \geq \text{dist}(e_n, \mathbb{R}) = \text{Im } z_n$$

and

$$\text{dist}(e_n, e_{-m}) \geq \text{dist}(e_{-m}, \mathbb{R}) = |\text{Im } z_{-m+1}| = \text{Im } z_{m-1}.$$

Let $l \in \{n, m - 1\}$. Then $l \geq n_0$ and

$$\text{Im } z_l = \psi(\text{Re } z_l) \geq d \cdot \sqrt{\text{Re } z_l}.$$

By Lemma 3.28 there exists a constant $D_1 > 0$ such that $\operatorname{Re} z_q \geq D_1 q^2$ for all $q \geq n_0$ and by Lemma 3.29 there exists a constant $M > 0$ such that $q > \frac{1}{M} \ell(e_q)$ for all $q \geq n_0$. Thus, we get

$$\operatorname{Im} z_l = \psi(\operatorname{Re} z_l) \geq D_2 \cdot \ell(e_l)$$

for some constant $D_2 > 0$. Thus,

$$\operatorname{dist}(e_n, e_{-m}) \geq D_2 \cdot \ell(e_n) \quad \text{and} \quad \operatorname{dist}(e_n, e_{-m}) \geq D_2 \cdot \ell(e_{m-1}) = D_2 \cdot \ell(e_{-m}).$$

If $n, m \leq 2n_0$, we find a constant $D_3 > 0$ which is independent of n and m such that

$$\operatorname{dist}(e_n, e_{-m}) \geq D_3 \cdot \max \{ \ell(e_n), \ell(e_{-m}) \}.$$

Let now $n \leq n_0$ and $m \geq 2n_0$. Since $m \geq n_0$, we immediately get

$$\operatorname{dist}(e_n, e_{-m}) \geq D_2 \cdot \ell(e_{-m})$$

as in the first case. But also

$$\operatorname{dist}(e_n, e_{-m}) \geq \operatorname{dist}(e_n, e_{-2n_0}) \geq D_3 \cdot \ell(e_n)$$

as in the second case. If $n \geq 2n_0$ and $m \leq n_0$, the same arguments hold. Thus, the assertion is proved. \square

The last lemma did not yet deal with the case $n = 0$. This will be done in the following lemma.

Lemma 3.33. *There exists a constant $D > 0$ such that for all $n \in \mathbb{N}_{>1}$ we have*

$$\operatorname{dist}(e_0, e_{-n}) \geq D \cdot \max \{ \ell(e_0), \ell(e_n) \}.$$

Proof. Let x_0 be as in Lemma 3.18. Let $n_0 \in \mathbb{N}$ such that $\operatorname{Re} z_l > x_0$ for all $l \geq n_0$. First, let $n \geq n_0$. With the same reasoning as in Lemma 3.32, we obtain

$$\operatorname{dist}(e_0, e_{-n}) \geq \operatorname{dist}(e_{-n}, \mathbb{R}) = |\operatorname{Im} z_{-n+1}| \geq D_1 \cdot \ell(e_{-n}),$$

which yields one of the desired estimates. Furthermore,

$$\operatorname{dist}(e_0, e_{-n}) \geq \operatorname{dist}(e_0, e_{-n_0}) \geq D_1 \cdot \ell(e_{-n_0}).$$

By Lemma 3.30,

$$\ell(e_0) \leq M^{n_0} \cdot \ell(e_{n_0}).$$

Therefore,

$$\operatorname{dist}(e_0, e_{-n}) \geq \frac{D_1}{M^{n_0}} \cdot \ell(e_0).$$

Since there are only finitely many non-considered cases, the proof is finished. \square

Although the previous lemmas did not deal with all the possible cases, the symmetry in the construction yields the boundedness in the missing cases.

Lemma 3.34. *There exists a constant $D > 0$ such that*

- (i) $\text{dist}(e_{-n}, e_{-m}) \geq D \cdot \max\{\ell(e_{-n}), \ell(e_{-m})\}$ for all $n, m \in \mathbb{N}_0$;
- (ii) $\text{dist}(e_{-n}, e_m) \geq D \cdot \max\{\ell(e_{-n}), \ell(e_m)\}$ for all $n, m \in \mathbb{N}$ with $1 - n \leq m$;
- (iii) $\text{dist}(e_{-1}, e_n) \geq D \cdot \max\{\ell(e_{-1}), \ell(e_n)\}$ for all $n \in \mathbb{N}$.

Proof. Since by construction

$$\text{Re } z_n = \text{Re } z_{-n} \text{ and } \text{Im } z_n = -\text{Im } z_{-n}$$

and

$$\ell(e_n) = \ell(e_{-(n+1)})$$

for all $n \in \mathbb{N}_0$, we can just reflect everything at the real axis and apply Lemma 3.31, Lemma 3.32 and Lemma 3.33. After reflecting back, we obtain the desired estimates. \square

So far, we established a lower bound for the distance of two non-adjacent edges. It remains to show that the ratio in question is bounded.

Lemma 3.35. *There exists a constant $N > 0$ such that*

$$\frac{\text{diam}(e_n)}{\text{dist}(e_n, e_m)} \leq N$$

for all $n, m \in \mathbb{Z}$ with $|n - m| > 1$.

Proof. Since $\text{diam}(e_n) \leq \ell(e_n)$ for all $n \in \mathbb{Z}$, the assertion follows immediately from Lemma 3.31, Lemma 3.32, Lemma 3.33 and Lemma 3.34. \square

3.7 The graph (V, E) has bounded geometry

In order to apply Bishop's folding construction, we need that (V, E) is a bounded geometry graph. In the last two sections, we saw that the lengths of adjacent edges are uniformly comparable and that the width of Ω is comparable to the lengths of the adjoining edges.

It is needed in Bishop's construction that the angles between adjacent edges are uniformly bounded away from zero (see (2) in Definition 2.65). But that is quite obvious:

Lemma 3.36. *For all $n \in \mathbb{Z}$, the angle between e_n and e_{n+1} is uniformly bounded.*

Proof. By Lemma 3.13, $\partial\Omega$ is an analytic curve. Since all edges are parts of this curve, the angle between adjacent edges is always π . \square

Combining the results above, we can finally prove that the graph given by (V, E) has bounded geometry.

Lemma 3.37. *The graph given by (V, E) has bounded geometry.*

Proof. By Definition 2.65 there are four things to prove:

- 1) Every edge $e_n \in E$ is twice continuously differentiable with uniform bounds.
- 2) The angles between adjacent edges are uniformly bounded away from 0.
- 3) Adjacent edges have uniformly comparable lengths.
- 4) For non-adjacent edges e_n and e_m , the ratio $\frac{\text{diam}(e_n)}{\text{dist}(e_n, e_m)}$ is uniformly bounded.

By Lemma 3.13 and Lemma 3.17, the first part follows. The second point follows immediately from Lemma 3.36. Lemma 3.30 implies the third statement. Lemma 3.35 proves the last statement. Thus, the graph given by (V, E) has bounded geometry. \square

3.8 Applying Bishop's construction

This section finally uses the ideas of Bishop's quasiconformal folding to obtain the desired quasiregular map g . But before we can do this, one last thing is to be proven: The lengths of the images of the edges under τ need to have a uniform positive lower bound.

Lemma 3.38. *There exists a constant $D > 0$ such that $\ell(\tau(e_n)) \geq D$ for all $n \in \mathbb{Z}$.*

Remark 3.39. Exchanging τ by $\tilde{\tau} := \frac{2\pi}{D} \cdot \tau$ then yields $\ell(\tilde{\tau}(e_n)) \geq 2\pi$ for all $n \in \mathbb{Z}$.

Proof. Let $\hat{\tau}(z) := \tau(z) + K = e^{c \cdot h(z)}$. We then have

$$\ell(\tau(e_n)) = |\tau(z_{n+1}) - \tau(z_n)| = \ell(\hat{\tau}(e_n))$$

for all $n \in \mathbb{N}$. Furthermore,

$$\hat{\tau}'(z) = \tau'(z) = e^{c \cdot h(z)} \cdot c \cdot h'(z).$$

By Lemma 3.5, there exists a constant $d_1 > 0$ such that $\text{Re } h(z) \geq d_1 \cdot h(|z|)$ for all $z \in \mathcal{G} = \{R_0 + re^{i\phi} : r > 0, |\phi| < \pi - \vartheta\}$ with $|z|$ sufficiently large. Moreover,

$$h'(z) = h(z) \left(\frac{\delta}{z} + \frac{\beta}{z \log(z)} \right)$$

and

$$|h(z)| = |z|^\delta |\log(z)|^\beta.$$

Since for $|z|$ large both

$$|\log z|^2 = (\log |z|)^2 + (\arg z)^2 \leq 4(\log |z|)^2$$

and

$$|\log z|^2 = (\log |z|)^2 + (\arg z)^2 \geq (\log |z|)^2,$$

there exists a constant $d_2 > 0$ such that

$$|h(z)| \geq d_2 \cdot h(|z|)$$

for $|z|$ sufficiently large.

Let $\delta = 0$. By definition (see Notation 3.2), we have $\beta > 2$. Hence,

$$\left| \frac{\delta}{z} + \frac{\beta}{z \log z} \right| = \left| \frac{\beta}{z \log z} \right|$$

and

$$c \cdot d_1 \cdot h(|z|) = c \cdot d_1 \cdot (\log |z|)^\beta \geq \log(|z|)$$

for $|z|$ large. Therefore,

$$\begin{aligned} |\hat{\tau}'(z)| &\geq c \cdot e^{c \cdot \operatorname{Re} h(z)} \cdot |h(z)| \cdot \left| \frac{\beta}{z \log z} \right| \\ &\geq d_3 \cdot e^{c \cdot d_1 \cdot h(|z|)} \cdot h(|z|) \cdot \frac{1}{|z| \log |z|} \\ &\geq d_3 \cdot (\log |z|)^{\beta-1}, \end{aligned}$$

which yields $|\hat{\tau}'(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$.

If $\delta \neq 0$, then β is some real number. We have

$$\left| \frac{\delta}{z} + \frac{\beta}{z \log z} \right| \geq \frac{\delta}{|z|} - \frac{|\beta|}{|z \log z|} \geq \frac{\delta}{2|z|}$$

for $|z|$ large. Moreover,

$$h(|z|) \geq d_4 \cdot |z|^{\frac{\delta}{2}}$$

for $|z|$ large. Therefore,

$$\begin{aligned} |\hat{\tau}'(z)| &\geq c \cdot e^{c \cdot \operatorname{Re} h(z)} \cdot |h(z)| \cdot \left| \frac{\delta}{z} + \frac{\beta}{z \log z} \right| \\ &\geq d_5 \cdot e^{c \cdot d_1 \cdot d_4 \cdot |z|^{\frac{\delta}{2}}} \cdot |z|^{\delta-1} \cdot (\log |z|)^\beta \\ &\geq d_6 \cdot |z|^\delta \cdot (\log |z|)^\beta, \end{aligned}$$

for $|z|$ sufficiently large. Again, $|\hat{\tau}'(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$.

By Lemma 3.28, we have $|z_n| \rightarrow \infty$ as $|n| \rightarrow \infty$. By Lemma 3.18, the boundary of Ω is the graph of an increasing function ψ from some point on. Thus, we have $|z| \geq |z_n|$ for all $z \in e_n$ as long as n is large enough. This yields

$$|\hat{\tau}'(z)| \geq 1$$

for all $z \in e_n$. Hence,

$$\ell(\tau(e_n)) = \ell(\hat{\tau}(e_n)) = \int_{e_n} |\hat{\tau}'(z)| |dz| \geq \ell(e_n).$$

By Lemma 3.29, we have

$$\ell(e_n) \geq \frac{1}{M} \cdot n$$

for some constant $M > 0$. Thus,

$$\ell(\tau(e_n)) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Due to the symmetry of our construction, we get

$$\ell(\tau(e_{-n})) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

as well. In particular, there exists a positive lower bound for $\ell(\tau(e_n))$. \square

As seen in Section 3.7, the boundary of Ω , described as the graph (V, E) , has bounded geometry. The complementary components of this graph are just Ω and $\mathbb{C} \setminus \bar{\Omega}$. Using the maps τ and η as above and taking Ω to be an R-component and $\mathbb{C} \setminus \bar{\Omega}$ to be an L-component, we have that the edge e_n is mapped under η onto $i[2\pi n, 2\pi(n+1)]$ for every $n \in \mathbb{Z}$. By Lemma 3.38, the lengths of the τ -images of every edge $e \in E$ are uniformly bounded below by some positive constant. Thus, Bishop's folding technique can be applied (Theorem 2.71). This yields a quasiregular map g which coincides with $\cosh \circ \tau$ on $\Omega \setminus T(r_0)$, where $T(r_0)$ is the neighbourhood of T in which the folding takes place. The domain $T(r_0)$ was defined in Bishop's construction. Furthermore, g coincides with $\exp \circ \eta$ on $\mathbb{C} \setminus (\bar{\Omega} \cup T(r_0))$. In particular, we have a good control on the behaviour of g as long as we stay away from $T(r_0)$. By construction, we still have $g(\mathbb{C} \setminus \bar{\Omega}) \subseteq \mathbb{D}$. Thus,

$$\{z \in \mathbb{C} : |g(z)| > r\} \subseteq \{z \in \mathbb{C} : |g(z)| > 1\} \subset \Omega$$

for all $r \geq 1$. As seen in Section 3.3, there exists a constant $r_1 > 0$ such that

$$\Omega \cap \{z \in \mathbb{C} : \operatorname{Re} z > r_1\} \subset G(1, k_0 w).$$

Let $\Delta = \Omega \cap \{z \in \mathbb{C} : \operatorname{Re}(z) < r_1\}$. Since g is a non-constant quasiregular map on \mathbb{C} , the maximum modulus principle provides the existence of a constant $C > 0$ such that $|g(z)| \leq C$ for all $z \in \Delta$, compare Remark 2.57. Choosing $R > C$, we obtain

$$\{z \in \mathbb{C} : |g(z)| \geq R\} \subset \Omega \cap \{z \in \mathbb{C} : \operatorname{Re} z > r_1\}.$$

Thus,

$$\{z \in \mathbb{C} : |g(z)| \geq R\} \subset G(1, k_0 w).$$

This yields the constant R in Theorem 1. It remains to prove, that there exist constants $\rho > 1$ and $k > 0$ such that

$$G(\rho, kw) = \left\{ r e^{i\phi} : r > \rho, |\phi| < k \cdot \frac{(\log r)^\alpha}{r^\delta} \right\} \subset \{z \in \mathbb{C} : |g(z)| \geq R\}.$$

The following section will focus on this problem.

3.9 Approximating Ω from inside

As seen before, the map g being the composition of \cosh and τ or the exponential and η on Ω or $\mathbb{C} \setminus \overline{\Omega}$, respectively, is altered in Bishop's construction in order to get a quasiregular map on \mathbb{C} . In this section, we want to investigate the region where Bishop's folding technique takes place. This region is a subset of $T(r_0)$ and it is needed for the question where g is exactly given by the composition above.

First, we are going to take a look at the domains in the right and left half-planes where the folding occurs.

Lemma 3.40. *There exist constants $C_1, C_2 > 0$ such that the folding in the right half-plane takes place in*

$$\{x + iy : 0 < x < C_1 \cdot y\} \cup \{x + iy : 0 < x < C_2, |y| < C_2\}.$$

The folding in the left half-plane takes place in $\{x + iy : -2\pi < x < 0\}$.

Proof. As seen in Section 2.5, the folding in the right half-plane takes place in the domain V_I which has the imaginary axis as one part of the boundary and consists of squares Q_n whose one side is the edge $\tau(e_n)$. By Lemma 3.37, the graph $T = (V, E)$ has bounded geometry. Thus, Lemma 2.73 implies that there exists a constant $N > 0$ such that

$$\frac{1}{N} \leq \frac{\ell(\tau(e_{n+1}))}{\ell(\tau(e_n))} \leq N$$

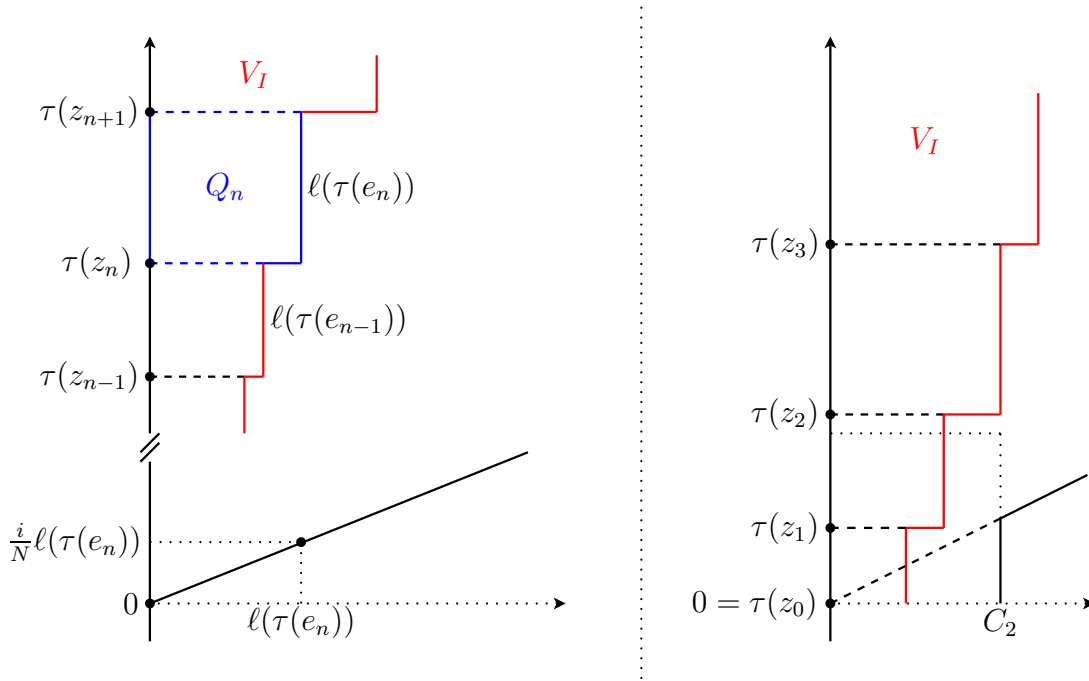


Figure 3.10: Illustration of the folding domain in the right half-plane. On the left, we show the case for $n > 1$, and on the right, we show the case $n = 1$.

for all $n \in \mathbb{Z}$. Hence,

$$|\tau(z_n) - 0| = |\tau(z_n)| \geq \ell(\tau(e_{n-1})) \geq \frac{1}{N} \ell(\tau(e_n))$$

for $n \in \mathbb{N}$. This implies

$$Q_n \subset \left\{ x + iy : x > 0, y > x \cdot \frac{1}{N} \right\} = \{x + iy : 0 < x < C_1 \cdot y\}$$

where $C_1 := N$. Choosing $C_2 > \ell(\tau(e_0))$ together with the symmetry of the construction yields the assertion for the right half-plane.

The only modification for the map η is that it needs to be length respecting along the edges of T . This can be done in the same way as in Lemma 2.74. The changes only occur in $\{x + iy : -2\pi < x < 0\}$, which implies the assertion for the folding in the left half-plane. \square

We will need the next lemma in the proof of Lemma 3.42.

Lemma 3.41. *For all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq \log 2$ we have*

$$|\cosh(z)| \geq \frac{1}{4} e^{\operatorname{Re} z}.$$

Proof. Let $z = x + iy \in \mathbb{C}$ with $x \geq \log 2$. Then

$$|\cosh(z)| = \frac{1}{2} |e^z + e^{-z}| \geq \frac{1}{2} (|e^z| - |e^{-z}|) = \frac{1}{2} (e^x - e^{-x}) \geq \frac{1}{2} (e^x - 1).$$

Since $x \geq \log 2$, we have

$$\frac{1}{4}e^x \geq \frac{1}{2}$$

which yields

$$|\cosh(z)| \geq \frac{1}{2}e^x - \frac{1}{2} \geq \frac{1}{4}e^x.$$

□

After knowing where Bishop's folding takes place, we can start searching for the domain where $g = \cosh \circ \tau$. Essentially, this is Ω up to a multiplicative constant at the opening angle. Using the last lemma, we can find such a set as a subdomain of the set of those points where $|g|$ is large.

Lemma 3.42. *There exist constants $\rho > 1$ and $0 < k < k_0$ such that*

$$G(\rho, kw) \subset \{z \in \mathbb{C} : |g(z)| \geq R\}.$$

Proof. By Lemma 3.40, the folding in the right half-plane takes place in

$$D := \{x + iy : 0 < x < C_1 \cdot y\} \cup \{x + iy : 0 < x < C_2, |y| < C_2\}.$$

The idea of this proof is to find constants ρ and k such that $\tau(z) \in \mathbb{H}_r \setminus D$ for all $z \in G(\rho, kw)$. Thus, there is no folding in $G(\rho, kw)$. This fact and Lemma 3.41 will then yield the result.

Let $k := \frac{2k_0 \arctan\left(\frac{1}{C_1}\right)}{(1+\varepsilon)^2\pi} < k_0$. Let $z \in \mathbb{C}$ such that $|\phi| = |\arg(z)| < k \cdot \frac{(\log|z|)^\alpha}{|z|^\delta}$ and $r = |z| > 1$. Let us first consider the case that $\delta = 0$. As seen in Lemma 3.4, we can write the imaginary part of h as

$$\operatorname{Im} h(z) \sim \frac{h(r)\beta\phi}{\log r} \leq \frac{\beta 2k_0 \arctan\left(\frac{1}{C_1}\right)}{(1+\varepsilon)^2\pi} \cdot \frac{h(r)(\log r)^\alpha}{r^\delta \log r}.$$

Since in this case $\beta = 1 - \alpha$ and $h(r) = r^\delta (\log r)^\beta = (\log r)^{1-\alpha}$ (see Notation 3.2), we see that

$$\frac{h(r)(\log r)^\alpha}{r^\delta \log r} = 1.$$

Furthermore, $c = \frac{(1+\varepsilon)\pi}{2(1-\alpha)k_0}$ (compare Notation 3.2). Thus, $c \cdot \frac{\beta 2k_0 \arctan\left(\frac{1}{C_1}\right)}{(1+\varepsilon)^2\pi} = \frac{\arctan\left(\frac{1}{C_1}\right)}{1+\varepsilon}$. In conclusion, we get

$$c \operatorname{Im} h(z) \sim \frac{h(r)\beta\phi c}{\log r} \leq \frac{\arctan\left(\frac{1}{C_1}\right)}{1+\varepsilon}.$$

By construction and Lemma 3.6,

$$\arg \tau(z) = \operatorname{Im} \log \tau(z) \sim c \cdot \operatorname{Im} h(z).$$

If we choose $\rho > C_2 > 0$ large enough, we have for all z as above with $|z| > \rho$

$$|\arg \tau(z)| \leq (1 + \varepsilon) \cdot c \cdot |\operatorname{Im} h(z)| < \arctan \left(\frac{1}{C_1} \right).$$

If in contrast $\delta \neq 0$, the imaginary part of h fulfils

$$\operatorname{Im} h(z) \sim h(r) \delta \phi \leq \frac{\delta 2k_0 \arctan \left(\frac{1}{C_1} \right)}{(1 + \varepsilon)^2 \pi} \cdot \frac{h(r) (\log r)^\alpha}{r^\delta}$$

as seen in Lemma 3.4. Again, $\frac{h(r) (\log r)^\alpha}{r^\delta} = 1$ and $\frac{c \delta 2k_0 \arctan \left(\frac{1}{C_1} \right)}{(1 + \varepsilon)^2 \pi} = \frac{\arctan \left(\frac{1}{C_1} \right)}{1 + \varepsilon}$ because in this case $c = \frac{(1 + \varepsilon) \pi}{2 \delta k_0}$ and $\beta = -\alpha$. Similarly to the first case,

$$|\arg \tau(z)| \leq (1 + \varepsilon) \cdot c \cdot |\operatorname{Im} h(z)| < \arctan \left(\frac{1}{C_1} \right)$$

for all z as above with $|z| > \rho$.

The real part of τ fulfils

$$\operatorname{Re} \tau(z) = \operatorname{Re} (e^{c \cdot h(z)} - K) = e^{c \cdot \operatorname{Re} h(z)} \cos (c \cdot \operatorname{Im} h(z)) - K.$$

As seen above, $|c \cdot \operatorname{Im} h(z)| < \arctan \left(\frac{1}{C_1} \right) < \frac{\pi}{2}$ for all the points z in consideration. Thus, with $d := \cos \left(\arctan \left(\frac{1}{C_1} \right) \right)$, we get $|\cos (c \cdot \operatorname{Im} h(z))| > d > 0$. Therefore,

$$\operatorname{Re} \tau(z) \geq d \cdot e^{c \cdot \operatorname{Re} h(z)} - K$$

which tends to ∞ as $|z| \rightarrow \infty$ by Lemma 3.5. Possibly by increasing ρ , we get

$$\operatorname{Re} \tau(z) \geq C_2 \quad \text{and} \quad \frac{1}{4} \cdot e^{\operatorname{Re} \tau(z)} \geq R$$

for all $z \in \mathbb{C}$ with $|z| > \rho$ and $|\arg z| < k \cdot \frac{(\log |z|)^\alpha}{|z|^\delta}$. In particular, $\tau(z) \in \mathbb{H}_r \setminus D$ for those values of z .

We now consider the set

$$G(\rho, kw) = \left\{ z \in \mathbb{C} : |z| > \rho, |\arg z| < k \cdot \frac{(\log |z|)^\alpha}{|z|^\delta} \right\}.$$

As seen above, $\tau(z) \in \mathbb{H}_r \setminus D$ for all $z \in G(\rho, kw)$. Hence, there is no folding at all in $G(\rho, kw)$ and

$$g = \cosh \circ \tau \quad \text{on} \quad G(\rho, kw).$$

Thus, for $z \in G(\rho, kw)$ we have

$$|g(z)| \geq R \quad \Leftrightarrow \quad |\cosh \circ \tau(z)| \geq R.$$

Using Lemma 3.41, we get $|g(z)| \geq R$ for all $z \in G(\rho, kw)$ which implies

$$G(\rho, kw) \subset \{z \in \mathbb{C} : |g(z)| \geq R\},$$

finishing the proof of this lemma. □

Combining the results from Section 3.8 and Lemma 3.42 proves that there exist constants $\rho > 0$, $k \in (0, k_0)$ and $R > 1$ and a quasiregular map g such that

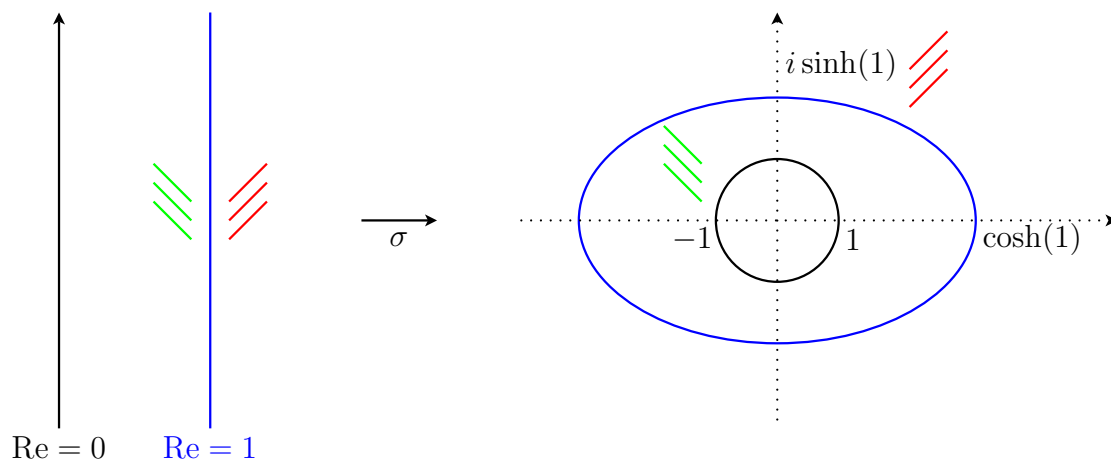
$$G(\rho, kw) \subset \{z \in \mathbb{C} : |g(z)| \geq R\} \subset G(1, k_0w). \quad (3.18)$$

We now show that g has only one tract which is parabola shaped and symmetric to the real axis.

As we have seen while proving that the constants ρ , k and R exist, g is bounded on $\mathbb{C} \setminus \overline{\Omega}$. Thus, we need to investigate the behaviour of g on Ω . Recall that in Bishop's construction the tree T was exchanged by a tree T' such that $T \subset T'$. Again, $\mathbb{C} \setminus T'$ has two connected components. Since the folding technique is only applied in the R-components, one of the connected components is still $\mathbb{C} \setminus \overline{\Omega}$ (see Section 2.5). The other one is a domain Ω' which is Ω with some small trees rooted at the vertices z_n removed. The domain Ω' is then mapped onto \mathbb{H}_r by a quasiconformal map $\hat{\tau}$ which coincides with τ off $T(r_0)$. The right half-plane is then mapped onto $\mathbb{C} \setminus \overline{\mathbb{D}}$ by a quasiregular map σ which coincides with \exp on $\partial\mathbb{H}_r$ and with \cosh on $\{x+iy : x \geq 1\}$, see Lemma 2.67. The function σ maps $i\mathbb{R}$ onto $\partial\mathbb{D}$ and $1+i\mathbb{R}$ onto an ellipse with major axis $\cosh(1)$ and minor axis $\sinh(1)$, see Figure 3.11. Since σ is quasiregular, we see that σ is bounded on $\{x+iy : 0 < x < 1\}$. We may assume that $R > 1$. But then $\sigma^{-1}(\{z \in \mathbb{C} : |z| = R\}) = \cosh^{-1}(\{z \in \mathbb{C} : |z| = R\})$ which is roughly a vertical line. Thus, $\sigma^{-1}(\{z \in \mathbb{C} : |z| > R\})$ is roughly a right half-plane H . In particular, this set is connected. Since $\hat{\tau}$ is quasiconformal, also $\hat{\tau}^{-1}(H)$ is connected. Thus, g has only one tract. Due to (3.18), this tract is shaped like a parabola. Since our construction is symmetric to the real axis, this tract is also symmetric to the real axis.

It remains to prove that there exists a transcendental entire function $f \in \mathcal{S}$ and a quasiconformal map ϕ such that $g = f \circ \phi$. The only critical values of this map f are 1 and -1 and the only asymptotic value of f is 0.

The application of the Measurable Riemann Mapping Theorem (Theorem 2.51) yields the existence of a transcendental entire map f and a quasiconformal map ϕ such that $f \circ \phi = g$. As Bishop has proven (see [Bis15, page 4, page 32]), the only critical values

Figure 3.11: Illustration of the mapping properties of σ .

of f are ± 1 , since g is locally 1-to-1 except at the vertices of T' , the tree constructed in Bishop's proof (see Section 2.5). Furthermore, the only finite asymptotic value of f is 0. Thus, $f \in \mathcal{S}$. This finishes the proof of Theorem 1.

4 Growth of Finite-Type Maps with One Tract

In this chapter, we prove the following theorem which includes Theorem 2.

Theorem 4.1. *Let δ , α , g , ϕ and f be as in Theorem 1. Let h be as in Notation 3.2. Then f has only one tract, which is symmetric to the real axis. Furthermore, there exists a constant $d > 0$ such that*

$$\log \log M(r, f) \geq d \cdot h(r)$$

holds for r sufficiently large.

Remark 4.2. If we choose $\delta = \frac{1}{2}$, $\alpha = 0$ and $k_0 = 1$, we get $h(r) = r^\delta = \sqrt{r}$. Thus, we obtain Theorem 2.

The idea of the proof is to obtain estimates on the growth of g . Thereafter, we show that the quasiconformal map ϕ is asymptotically conformal at infinity. This yields the desired growth of f and proves the theorem.

By construction (see Section 2.4), the Beltrami coefficients μ_g and μ_ϕ belonging to g and ϕ , respectively, are identical. Therefore, ϕ is holomorphic (thus conformal) where g is holomorphic. Since g is holomorphic off $T(r_0)$, the set N where ϕ is not conformal is contained in $T(r_0)$. We will now show that the logarithmic area of N is finite. The Teichmüller-Wittich-Belinski Theorem (Theorem 2.58) will then yield that ϕ is asymptotically conformal at infinity. Since $N \subset T(r_0)$, it suffices to prove that the logarithmic area of $T(r_0)$ is finite.

Lemma 4.3. *The logarithmic area of $T(r_0)$ is finite.*

Proof. Since the exact form of $T(r_0)$ can hardly be computed, we will construct a set \mathcal{M} such that $T(r_0) \subset \mathcal{M}$ and $\log \text{area}(\mathcal{M}) < \infty$. This set will have the form

$$\mathcal{M} = \{x + iy : x > x_0, \psi(x) - \varrho(x) \leq y \leq \psi(x) + \varrho(x)\}$$

where ϱ is a suitable step function.

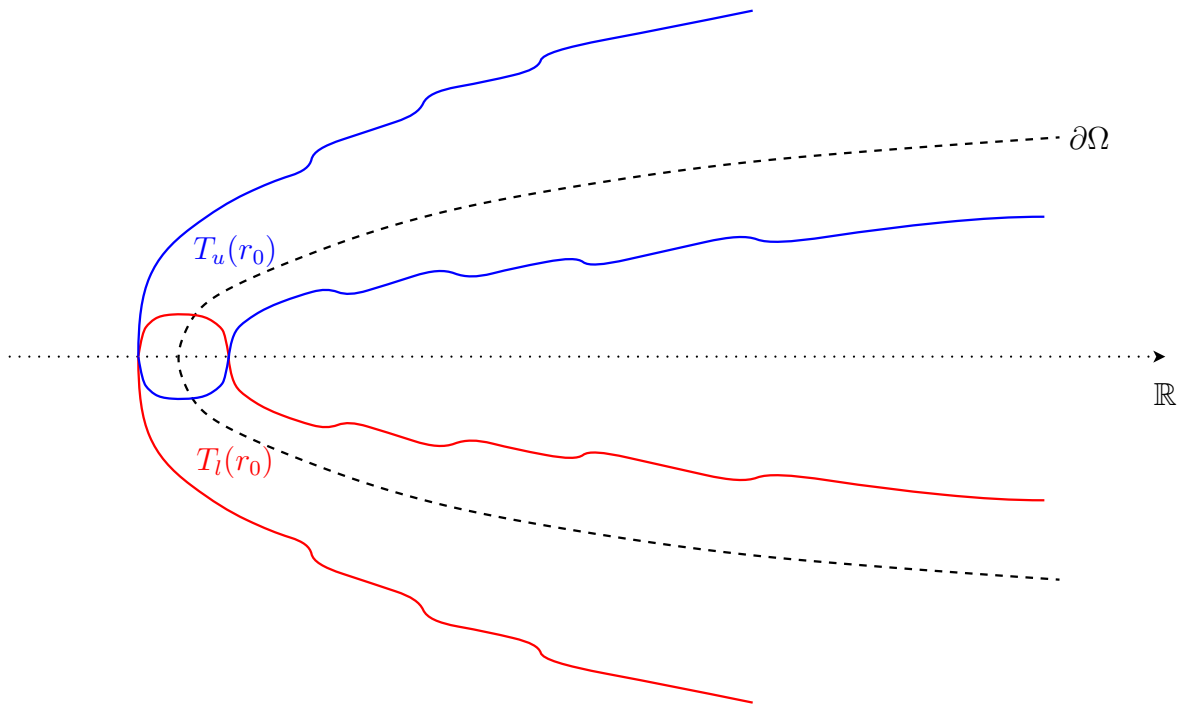


Figure 4.1: Illustration of the domain $T(r_0) = T_u(r_0) \cup T_l(r_0)$.

Recall that the set $T(r_0)$ (compare (2.3)) is a neighbourhood of T and is defined by

$$T(r_0) = \bigcup_{e \in E} \{z \in \mathbb{C} : \text{dist}(z, e) < r_0 \cdot \text{diam}(e)\}.$$

Due to the symmetry of T , it suffices to find bounds in the upper half-plane. Let

$$T_u(r_0) = \bigcup_{n \in \mathbb{N}_0} \{z \in \mathbb{C} : \text{dist}(z, e_n) < r_0 \cdot \text{diam}(e_n)\}$$

and

$$T_l(r_0) = \bigcup_{n \in \mathbb{N}} \{z \in \mathbb{C} : \text{dist}(z, e_{-n}) < r_0 \cdot \text{diam}(e_{-n})\}.$$

Then $T_u(r_0)$ is the complex conjugate of $T_l(r_0)$ and $T(r_0) \subset T_u(r_0) \cup T_l(r_0)$. In the following paragraphs, we are going to prove that the logarithmic area of $T_u(r_0)$ (thus also of $T_l(r_0)$) is finite.

By Lemma 3.18, the boundary of Ω is the graph of an increasing function ψ from some point x_0 on. Furthermore, $\psi'(x) \rightarrow 0$ as $x \rightarrow \infty$. Let $n_0 \in \mathbb{N}$ be large enough such that $\text{Re } z_{n_0} \geq x_0$ and $\psi'(x) < \varepsilon$ for all $x \geq \text{Re } z_{n_0}$. Let $n \in \mathbb{N}$ such that $n \gg n_0$. The exact lower bound will be determined later. Furthermore, let $w = x + iy \in \partial T_u(r_0)$ such that $\text{Re } z_n \leq x < \text{Re } z_{n+1}$, $y > \psi(x)$ and let $z = x + i\psi(x)$. Then, $z \in e_n$. Since

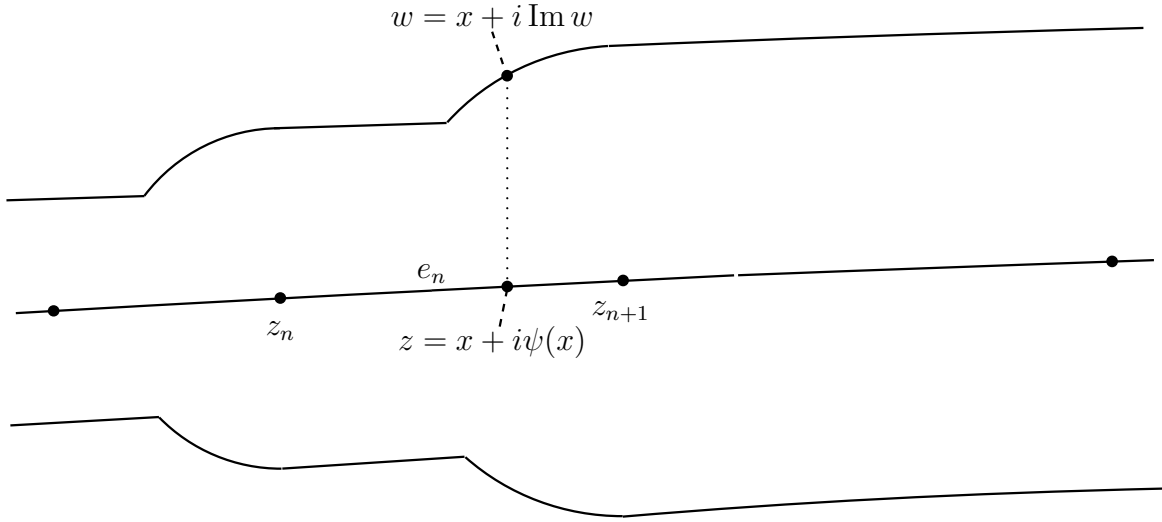


Figure 4.2: Illustration of the situation in the proof of Lemma 4.3. Note that the distance between $\partial\Omega$ and $\partial T_u(r_0)$ is a lot longer than depicted above as r_0 is large.

$w \in \partial T_u(r_0)$ and x is large, we have $w \in \partial T(r_0)$. Thus, there exists $m \in \mathbb{N}$ such that $\operatorname{dist}(w, e_m) = r_0 \cdot \operatorname{diam}(e_m)$. We will now prove that there exists a constant $n_1 > 0$ such that

$$|m - n| < n_1. \quad (4.1)$$

We have

$$\ell(e_j) = \int_{\operatorname{Re} z_j}^{\operatorname{Re} z_{j+1}} \sqrt{1 + |\psi'(t)|} dt \leq 2(\operatorname{Re} z_{j+1} - \operatorname{Re} z_j)$$

for all $j > n_0$. Let us first assume that $n < m$. Hence,

$$\operatorname{Re} z_m - \operatorname{Re} z_{n+1} = \sum_{j=n+1}^{m-1} (\operatorname{Re} z_{j+1} - \operatorname{Re} z_j) \geq \frac{1}{2} \sum_{j=n+1}^{m-1} \ell(e_j).$$

By Lemma 3.29, we have

$$\ell(e_j) \geq \frac{1}{M} \cdot j$$

for some constant $M > 1$, provided that j is large enough. Thus,

$$r_0 \cdot \operatorname{diam}(e_m) = \operatorname{dist}(w, e_m) \geq \operatorname{Re} z_m - \operatorname{Re} z_{n+1} \geq \frac{1}{2} \sum_{j=n+1}^{m-1} \ell(e_j) \geq \frac{1}{2M} \sum_{j=n+1}^{m-1} j.$$

Since by Lemma 3.29 also

$$\operatorname{diam}(e_m) \leq \ell(e_m) \leq M \cdot m,$$

we obtain

$$\begin{aligned}
Mr_0m &\geq r_0 \cdot \text{diam}(e_m) \geq \frac{1}{2M} \sum_{j=0}^{m-n-2} (j+n+1) = \frac{1}{2M} \left(\sum_{j=0}^{m-n-2} (n+1) + \sum_{j=0}^{m-n-2} j \right) \\
&= \frac{1}{2M} \left((m-n-1)(n+1) + \frac{(m-n-2)(m-n-1)}{2} \right) \\
&= \frac{1}{4M} (m-n-1)(n+m) \\
&\geq \frac{1}{4M} (m-n-1)m.
\end{aligned}$$

Thus,

$$m - n \leq 4M^2r_0 + 1.$$

If $m < n$, we have

$$r_0 \cdot \text{diam}(e_m) \geq \text{Re } z_n - \text{Re } z_{m+1}$$

and the same computation shows that

$$n - m \leq 4M^2r_0 + 1.$$

In particular, (4.1) holds with $n_1 = 4M^2r_0 + 1$.

The next step is to prove that the distance between $\text{Im } w$ and $\text{Im } z = \psi(x)$ is bounded above by $c_1 \cdot \ell(e_n)$ for some constant $c_1 > 0$ which is independent of n .

By (4.1), $|m - n| < n_1$ for some constant n_1 . By Lemma 3.30, there exists a constant $M_1 > 1$ such that $\frac{1}{M_1} \leq \frac{\ell(e_{j+1})}{\ell(e_j)} \leq M_1$ for all $j \in \mathbb{N}$. If $m < n$, we get

$$|\text{Im } w - \text{Im } z| \leq |\text{Im } w - \text{Im } z_{m+1}| \leq \text{dist}(w, e_m).$$

Since $\text{dist}(w, e_m) = r_0 \cdot \text{diam}(e_m)$,

$$|\text{Im } w - \text{Im } z| \leq r_0 \cdot \text{diam } e_m \leq r_0 \cdot \ell(e_m) \leq r_0 \cdot M_1^{n-m} \ell(e_n) \leq c_3 \cdot \ell(e_n),$$

compare Figure 4.3. If $m > n$, we have

$$|\text{Im } w - \text{Im } z| = \text{dist}(w, z) \leq \text{dist}(w, z_n)$$

since $\text{Im } z_n \leq \text{Im } z$ and $\text{Re } z_n \leq \text{Re } z$. Thus,

$$|\text{Im } w - \text{Im } z| \leq \text{dist}(w, e_m) + \text{dist}(z_n, e_m).$$

We have $\text{dist}(e_m, w) = r_0 \cdot \text{diam}(e_m)$ and

$$\text{dist}(z_n, e_m) \leq \sum_{j=n}^m \ell(e_j).$$

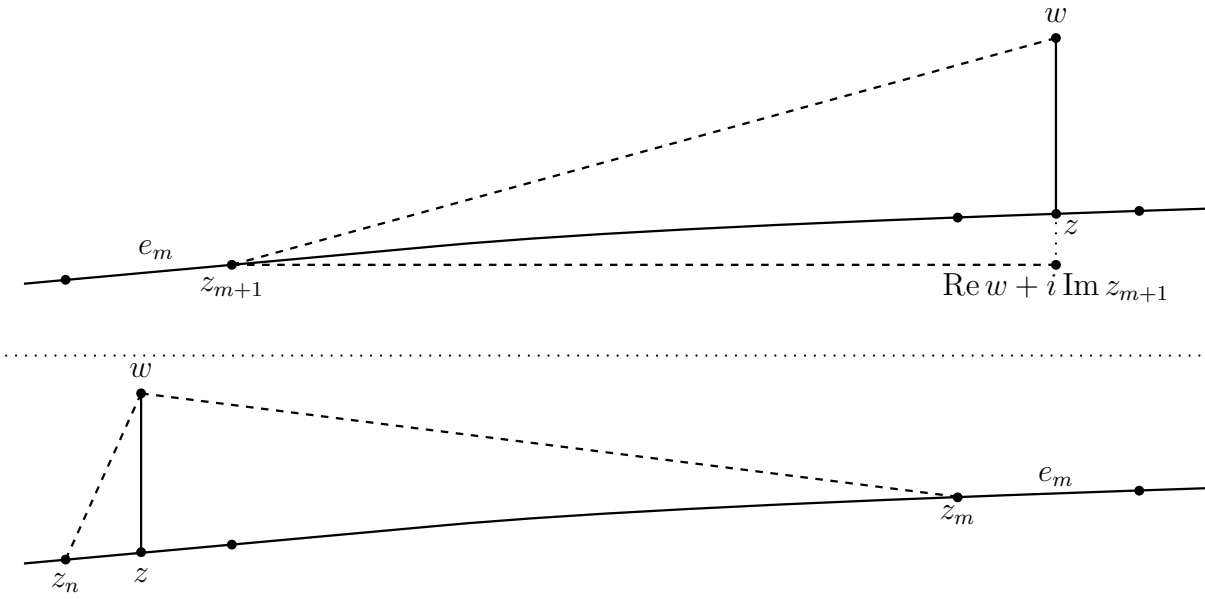


Figure 4.3: Illustration of the distance of $\text{Im } w$ and $\text{Im } z$. On top the case $m < n$ and on bottom the case $m > n$.

Hence,

$$|\text{Im } w - \text{Im } z| \leq r_0 \cdot \text{diam } e_m + \sum_{j=n}^m \ell(e_j).$$

Since

$$\sum_{j=n}^m \ell(e_j) \leq (m - n + 1) M_1^{m-n} \ell(e_n),$$

a similar computation shows that also in this case

$$|\text{Im } w - \text{Im } z| \leq c_4 \cdot \ell(e_n)$$

with a suitable constant $c_4 > 0$. Compare Figure 4.3.

Let now $w = x + iy \in \partial T_u(r_0)$ such that $\text{Re } z_n \leq x < \text{Re } z_{n+1}$ and $y < \psi(x)$. Again, let $z = x + i\psi(x)$. With very similar calculations to those above, also in this case we get

$$|\text{Im } w - \text{Im } z| \leq c_5 \cdot \ell(e_n)$$

for a suitable constant c_5 . By Lemma 3.29,

$$\ell(e_n) \leq M \cdot n$$

for all $n \geq n_0$. Thus, we get

$$|\text{Im } w - \text{Im } z| \leq c_6 \cdot n.$$

Define a step function ϱ by $\varrho(x) = c_7 \cdot n$ for $\operatorname{Re} z_n \leq x < \operatorname{Re} z_{n+1}$ where the constant c_7 is chosen such that for all $w \in \partial T_u(r_0)$ with $\operatorname{Re} z_n \leq \operatorname{Re} w = x < \operatorname{Re} z_{n+1}$ we have $|\operatorname{Im} w - \psi(x)| < c_7 \cdot n$. We then have

$$|\operatorname{Im} w - \psi(\operatorname{Re} w)| \leq \varrho(\operatorname{Re} w)$$

for all $w \in \partial T_u(r_0)$ with $\operatorname{Re} w \geq \operatorname{Re} z_{n_0}$. Our next step is to get an estimation of $\varrho(x)$ from above. Therefore, we investigate the relationship of n and x . By Lemma 3.29, we have

$$\operatorname{diam}(e_n) \leq \ell(e_n) \leq M \cdot n.$$

Furthermore,

$$\operatorname{Re} z_n \leq x = \operatorname{Re} z \leq \operatorname{Re} z_{n+1}.$$

By Lemma 3.28, there exists a constant $M_2 > 0$ such that

$$\frac{n^2}{M_2} \leq |z_n - K_0| \leq M_2 n^2.$$

Thus, there exists a constant $M_3 > 0$ such that

$$\frac{n^2}{M_3} \leq |z_n| \leq M_3 n^2$$

for n sufficiently large. Since by Lemma 3.29 we have $\operatorname{Re} z_n \sim |z_n|$ as $|n| \rightarrow \infty$, the same estimates hold for $\operatorname{Re} z_n$. Thus,

$$\frac{d_1}{\sqrt{x}} \leq \frac{1}{n} \leq \frac{c_8}{\sqrt{x}}$$

for some constants $d_1, c_8 > 0$. Hence,

$$\varrho(x) = c_7 \cdot n \leq c_9 \cdot \sqrt{x}. \quad (4.2)$$

Let $\mathcal{M} = \{x + iy : x > x_0, \psi(x) - \varrho(x) \leq y \leq \psi(x) + \varrho(x)\}$. By the computations above, we have $T_u(r_0) \cap \{x + iy : x > x_0\} \subset \mathcal{M}$. The logarithmic area of this set is given by

$$\int_{x_0}^{\infty} \int_{\psi(x)-\varrho(x)}^{\psi(x)+\varrho(x)} \frac{1}{x^2 + y^2} dy dx \leq \int_{x_0}^{\infty} \frac{2\varrho(x)}{x^2} dx.$$

Equation (4.2) yields

$$\int_{x_0}^{\infty} \frac{2\varrho(x)}{x^2} dx \leq 2 \int_{x_0}^{\infty} \frac{c_9}{x^{\frac{3}{2}}} dx.$$

Since $\int_{x_0}^{\infty} x^{-\frac{3}{2}} dx$ converges, the logarithmic area of \mathcal{M} is finite. In conclusion, we get that the logarithmic area of $T(r_0)$ is finite. \square

Thus, ϕ is asymptotically conformal at ∞ . The next step in the proof is to show that the map g has a minimal growth which depends on h , hence on δ and α .

Lemma 4.4. *There exists a constant $d > 0$ such that*

$$\log \log \cosh(\tau(r)) \geq d \cdot h(r)$$

holds for r sufficiently large. In particular,

$$\log \log M(r, g) \geq d \cdot h(r)$$

holds for r sufficiently large.

Proof. Let $r > 0$ be large enough such that $\tau(r)$ is defined. Then,

$$\log \cosh(\tau(r)) = \log \left(\frac{1}{2} (e^{\tau(r)} + e^{-\tau(r)}) \right) = \tau(r) + \log \left(\frac{1 + e^{-2\tau(r)}}{2} \right).$$

Since $\tau(r) = e^{c \cdot h(r)} - K = e^{c \cdot \operatorname{Re} h(r)} - K \rightarrow \infty$ as $r \rightarrow \infty$ by Lemma 3.5, we have

$$\log \cosh(\tau(r)) \geq \tau(r) - \log 2 + o(1)$$

as $r \rightarrow \infty$. Furthermore,

$$\log \tau(r) = \log (e^{c \cdot h(r)} - K) = c \cdot h(r) + \log (1 - K \cdot e^{-c \cdot h(r)}) = c \cdot h(r) + o(1)$$

as $r \rightarrow \infty$ since $h(r) \rightarrow \infty$ as $r \rightarrow \infty$. This implies

$$\begin{aligned} \log \log \cosh(\tau(r)) &\geq \log(\tau(r) - \log 2 + o(1)) = \log \tau(r) + \log \left(1 - \frac{\log 2}{\tau(r)} + \frac{o(1)}{\tau(r)} \right) \\ &\geq c \cdot h(r) + o(1) \end{aligned}$$

as $r \rightarrow \infty$. Thus, there exists some constant $d > 0$ such that

$$\log \log \cosh(\tau(r)) \geq d \cdot h(r) \tag{4.3}$$

for r sufficiently large. This proves the first part of the lemma.

By Lemma 3.40, the folding in the right half-plane takes place in

$$D := \{x + iy : 0 < x < C_1 \cdot y\} \cup \{x + iy : 0 < x < C_2, |y| < C_2\}.$$

Since $D \cap \mathbb{R}_{>C_2} = \emptyset$, the map g coincides with the composition $\sigma \circ \tau$ on $\mathbb{R}_{>C_3}$ for some constant $C_3 > 0$. Here, σ is the map which maps \mathbb{H}_r to $\mathbb{C} \setminus [-1, 1]$ which has been constructed by Lemma 2.67. Since $\sigma = \cosh$ on $\{x + iy : x > 1\}$, there exists a constant $C_4 > 0$ such that $g(s) = \sigma(\tau(s)) = \cosh(\tau(s))$ for all $s > C_4$. Hence,

$$M(r, g) \geq |g(r)| = g(r) = \cosh(\tau(r))$$

for $r > C_4$. The second assertion follows from (4.3) and the monotonicity of \log on $\mathbb{R}_{>0}$. \square

Lemma 4.5. *There exists a constant $d > 0$ such that*

$$\log \log M(r, f) \geq d \cdot h(r)$$

for r sufficiently large.

Proof. Since ϕ is asymptotically conformal at infinity, there exists a constant $a \in \mathbb{C} \setminus \{0\}$ such that $\lim_{z \rightarrow \infty} \frac{\phi(z)}{z} = a$. By construction, ϕ can be chosen to be symmetric to the real axis. Thus, we have $a \in \mathbb{R}_{>0}$. This yields $\phi^{-1}(r) \geq \frac{r}{2a}$ for r sufficiently large. Hence,

$$h(\phi^{-1}(r)) \geq h\left(\frac{r}{2a}\right) = \left(\frac{r}{2a}\right)^\delta \left(\log\left(\frac{r}{2a}\right)\right)^\beta = (2a)^{-\delta} h(r) \left(1 - \frac{\log(2a)}{\log r}\right)^\beta$$

which yields

$$h(\phi^{-1}(r)) \geq \frac{(2a)^{-\delta}}{2} h(r) \tag{4.4}$$

for r sufficiently large. Since $g(x) = \cosh(\tau(x))$ for x sufficiently large, we have

$$M(r, f) \geq |f(r)| = |g(\phi^{-1}(r))| = \cosh(\tau(\phi^{-1}(r))). \tag{4.5}$$

By Lemma 4.4, we have

$$\log \log \cosh(\tau(r)) \geq \tilde{d} \cdot h(r)$$

for some $\tilde{d} > 0$ and r sufficiently large. Thus, equations (4.4) and (4.5) yield

$$\log \log M(r, f) \geq \log \log \cosh(\tau(r)) \geq \tilde{d} \cdot h(\phi^{-1}(r)) \geq d \cdot h(r)$$

for a suitable constant $d > 0$ and r sufficiently large. □

The map g has only one tract which is symmetric to the real axis. Thus, also f has only one tract which is the image of the tract of g under the quasiconformal map ϕ . Since ϕ is symmetric to the real axis, the tract of f is symmetric as well. This finishes the proof of Theorem 4.1.

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Nomenclature

α	Real number depending on δ , see Theorem 1, page 43
β	Real constant depending on α , $\beta := \begin{cases} -\alpha & , \delta \neq 0 \\ 1 - \alpha & , \delta = 0 \end{cases}$, page 44
c	Constant depending on α , δ , k_0 and ε , $c = \begin{cases} \frac{\pi(1+\varepsilon)}{2k_0(1-\alpha)} & , \delta = 0 \\ \frac{\pi(1+\varepsilon)}{2k_0\delta} & , \delta \neq 0 \end{cases}$, page 44
\mathbb{C}	Field of complex numbers, page 7
$\widehat{\mathbb{C}}$	Riemann sphere, page 7
\mathbb{D}	Open unit disk in \mathbb{C} , page 7
δ	Real number in $[0, \frac{1}{2}]$, see Theorem 1, page 43
$D(z, r)$	Open disk in \mathbb{C} with radius r and centre $z \in \mathbb{C}$, page 7
$\text{diam}(A)$	Diameter of $A \subset \mathbb{C}$, page 7
$\text{dist}(A, B)$	Distance between $A, B \subset \mathbb{C}$, page 7
η	Conformal map symmetric to the real axis from $\mathbb{C} \setminus \overline{\Omega}$ onto \mathbb{H}_l , page 44
ε	Small positive real number, page 44
E	$\{e_n : n \in \mathbb{Z}\}$, page 44
e_n	Part of $\partial\Omega$ between z_n and z_{n+1} for $n \in \mathbb{Z}$. The n th edge, page 44
\mathcal{G}	$\{R_0 + re^{i\phi} : r > 0, \phi < \pi - \vartheta\}$, page 44
$G(r_0, w)$	$\{re^{i\phi} : r > r_0, \phi < w(r)\}$, page 43
h	$h : \mathcal{G} \rightarrow \mathbb{C}$, $h(z) = z^\delta(\log z)^\beta$, page 44
\mathbb{H}_l	Left half-plane, page 7

\mathbb{H}_r	Right half-plane, page 7
\mathbb{H}_u	Upper half-plane, page 7
k_0	Positive real constant, see Theorem 1, page 43
K	Real constant larger than $\exp(c \cdot h(R_0 + 1))$, page 44
K_0	$\inf\{x \in \mathbb{R} : x \in \Omega\}$, page 45
$\ell(\gamma)$	Length of the curve γ , page 36
λ	Representation of $\partial\Omega$ as an analytic curve, page 53
μ	Conformal map from S onto $\{z \in \mathbb{C} : \operatorname{Im}(z) < \frac{\pi}{2}\}$, symmetric to the real axis, page 45
ν	Inverse of μ , page 45
Ω	Component of $\{z \in \mathbb{C} : \operatorname{Re} \exp(c \cdot h(z)) > K\}$ which contains $\mathbb{R}_{>\tilde{K}}$ for some $\tilde{K} > 0$, page 44
ψ	Increasing, differentiable function representing $\partial\Omega \cap \{x + iy : x > x_0, y > 0\}$ for some x_0 . We have $\frac{\psi(x)}{x} \rightarrow 0$ as $x \rightarrow \infty$ and $\psi'(x) \rightarrow 0$ as $x \rightarrow \infty$, page 56
φ	Angular function such that $2\varphi(r)$ is the opening angle of Ω at $ z = r$, page 44
R_0	Real constant sufficiently large. At least $R_0 > \begin{cases} 2 & , 1 + \delta = 0 \\ \max\{1, e^{\frac{\pi}{\delta}}\} & , \delta \neq 0 \end{cases}$, page 44
S	$S = \log(\mathbb{C} \setminus \overline{\Omega} - K_0) - i\pi$, page 45
τ	$\tau : \mathcal{G} \rightarrow \mathbb{C}$, $\tau(z) = \exp(c \cdot h(z)) - K$, page 44
ϑ	Positive real constant less than $\frac{\pi}{8}$, page 44
V	$V := \eta^{-1}(2\pi i\mathbb{Z})$, page 44
w	$w : (1, \infty) \rightarrow (0, \infty)$, $r \mapsto \frac{(\log r)^\alpha}{r^\delta}$, see Theorem 1, page 43
w_n	Complex number such that $z_n = \exp(w_n + i\pi) + K_0$ for $n \in \mathbb{Z}$, page 45
z_n	Point on $\partial\Omega$ such that $\eta(z_n) = 2\pi in$ for $n \in \mathbb{Z}$, page 44

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit – abgesehen von der Beratung durch den Betreuer meiner Promotion – in Inhalt und Form selbstständig angefertigt habe und dabei die Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft eingehalten habe.

Diese Arbeit hat weder ganz noch in Teilen einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen und wurde weder veröffentlicht noch zur Veröffentlichung eingereicht.

Kiel, den

Simon Albrecht