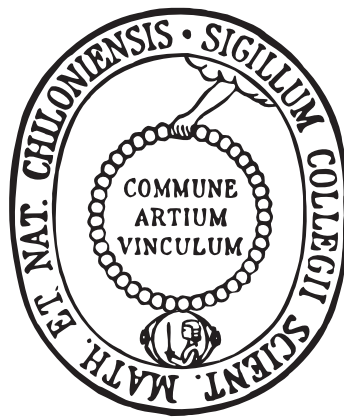


On the Measure of the Escaping Set of a Quasiregular Analogue of Sine

Dissertation

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Abstract

Quasiregular maps are a natural generalisation of holomorphic maps to higher dimensions. Recently, there is an increasing interest in studying the dynamics of quasiregular maps on the d -dimensional Euclidian space. Until now, not much is known about the dynamical behaviour of general quasiregular maps. However, quasiregular analogues of the complex exponential map as well as of trigonometric functions have been constructed and turned out to behave quite similarly to their counterparts in the plane.

In this thesis, we examine the dynamical behaviour of certain quasiregular maps on the d -dimensional Euclidian space which can be seen as quasiregular analogues of complex trigonometric functions. This class was introduced by Bergweiler and Eremenko in 2011. We will denote a member of this class by $\text{Sin}(x)$. We are interested in the iteration of these maps, in particular in the escaping set consisting of all points that tend to infinity under iteration.

We prove the following results: Theorem 1 is an analogue of a well known result by McMullen from 1987. We show that for $f(x) = \lambda \text{Sin}(x)$, where $\lambda > 0$, the escaping set $I(f)$ has positive Lebesgue measure. In fact, this is also true for the fast escaping set $A(f)$ consisting of all points that essentially escape as fast as possible.

Theorem 2 states that the measure of the non-escaping set of f in vertical square beams is finite. This corresponds to a result in the plane by Schubert from 2008 concerning the escaping set of $\sin(z)$ in vertical strips. Again, this is also true for the fast escaping set. We also use the map $\text{Sin}(x)$ to construct a quasiregular power mapping P_m which is similar to a quasiregular power mapping introduced by Mayer based on Zorich maps. We discuss some properties of the map P_m .

Theorem 3 then states that for $m \geq d + 1$, the composition of the power mapping P_m with a suitable quasiregular analogue of the trigonometric functions yields a quasiregular map with non-escaping set of finite measure. This generalises a result by Hemke from 2005. In the last section, we discuss the sharpness of Theorem 3.

Zusammenfassung

Quasireguläre Abbildungen sind eine natürliche Verallgemeinerung holomorpher Abbildungen auf höhere Dimensionen. In letzter Zeit gibt es ein zunehmendes Interesse an der Untersuchung des dynamischen Verhaltens von quasiregulären Abbildungen auf dem d -dimensionalen euklidischen Raum. Noch ist nicht viel über das dynamische Verhalten von allgemeinen quasiregulären Abbildungen bekannt. Allerdings wurden quasireguläre Analoga der komplexen Exponentialabbildung sowie von trigonometrischen Funktionen konstruiert, welche sich aus dynamischer Sicht sehr ähnlich wie ihre Gegenstücke in der Ebene verhalten.

In dieser Arbeit untersuchen wir das dynamische Verhalten von bestimmten quasiregulären Abbildungen auf dem d -dimensionalen euklidischen Raum, welche als quasireguläre Analoga von komplexen trigonometrischen Funktionen angesehen werden können. Diese Klasse wurde von Bergweiler und Eremenko im Jahr 2011 eingeführt. Wir bezeichnen eine Abbildung dieser Klasse mit $\text{Sin}(x)$. Wir interessieren uns für die Iteration dieser Abbildungen, insbesondere für die entkommende Menge bestehend aus all jenen Punkten, welche unter Iteration nach unendlich streben.

Wir beweisen die folgenden Ergebnisse: Theorem 1 ist ein Analogon von einem bekannten Resultat von McMullen aus dem Jahr 1987. Wir zeigen, dass die entkommende Menge von $f(x) = \lambda \text{Sin}(x)$ für alle $\lambda > 0$ positives Lebesguemaß hat. In der Tat stimmt dies auch für die schnell entkommende Menge $A(f)$, welche die Punkte enthält, die im Wesentlichen so schnell wie möglich entkommen.

Theorem 2 besagt, dass das Maß der nichtentkommenden Menge von f in vertikalen Balken endlich ist. Dies entspricht einem Resultat in der Ebene von Schubert aus dem Jahr 2008 über die entkommende Menge von $\sin(z)$ in vertikalen Streifen. Theorem 2 gilt ebenfalls für die schnell entkommende Menge.

Außerdem verwenden wir die Abbildung $\text{Sin}(x)$ um eine quasireguläre Potenzfunktion P_m zu konstruieren. Diese Abbildung ähnelt einer von Mayer vorgestellten quasiregulären Potenzfunktion, welche auf Zorich Abbildungen aufbaut. Wir diskutieren einige Eigenschaften der Abbildung P_m .

Theorem 3 besagt schließlich, dass die Komposition von P_m mit einem geeigneten quasiregulären Analogon der trigonometrischen Funktionen eine quasireguläre Abbildung mit nichtentkommender Menge von endlichem Maß ergibt, sofern $m \geq d + 1$. Dies verall-

gemeinert ein Ergebnis von Hemke aus dem Jahr 2005. Im letzten Abschnitt diskutieren wir die Schärfe von Theorem 3.

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1 Introduction

The iteration of holomorphic maps goes back to Julia [Jul18] and Fatou [Fat19] in the late 1910s. Independently of each other, they developed a theory which naturally partitions the complex plane into two sets, now called Fatou and Julia set. The *Fatou set* $\mathcal{F}(f)$ of an entire function f is defined as the set of all points in the complex plane where the iterates $\{f^n\}$ form a normal family. That is, every sequence in $\{f^n\}$ has a locally uniformly converging sub-sequence. Thus, the dynamical behaviour of f is stable in the Fatou set. The *Julia set* is defined as the complement of the Fatou set in the plane. The iterates behave chaotically in the Julia set. While some work was done in this field of mathematics in the following fifty years, it was only in the early 1980s that the awakened interest in this topic lead to extensive research. New mathematical techniques introduced primarily by Sullivan [Sul83, Sul85] and Douady and Hubbard [DH84, DH85] provided new perspectives. Furthermore, computer graphics of the Julia set and related sets lead to new ideas as well as created interest in the topic even among non-mathematicians.

Although Fatou [Fat26] already considered transcendental entire functions in 1926, that is entire maps with an essential singularity at infinity, the aforementioned research focused mainly on polynomials and rational functions. The first to thoroughly study the dynamics of arbitrary transcendental entire functions was Baker [Bak75] in 1975. In 1989, Eremenko [Erë89] introduced the *escaping set* $I(f)$ which consists of all points that tend to infinity under iteration. He first showed that the escaping set of a transcendental entire function is non-empty. Furthermore, he proved that the Julia set of every transcendental entire map is the boundary of the escaping set. Thus, there is a strong connection between the Julia set and the escaping set. This becomes even more evident in the *Eremenko-Lyubich class* consisting of all transcendental entire functions where the set of singularities of the inverse function is bounded. In this case, Eremenko and Lyubich [EL92] showed that the escaping set is contained in the Julia set and thus that the Julia set is the closure of the escaping set.

Recently, there is an increasing interest in the dynamics of quasiregular maps $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ on the d -dimensional Euclidian space. Quasiregular maps can be seen as higher dimensional generalisations of holomorphic maps in the plane. They were introduced by Lavrentiev [Lav38] in 1938 and studied systematically by Reshetnyak in a series of papers

starting with [Reš66]. Holomorphic maps are differentiable in the real sense and map infinitesimal small circles to infinitesimal small circles. To obtain an interesting class of functions in higher dimensions, we have to allow some non-differentiability as well as some distortion. Essentially, quasiregular maps are differentiable almost everywhere and map infinitesimal small spheres to infinitesimal small ellipsoids where the quotient of the largest and the smallest half-axes is uniformly bounded, see Section 2.4 for a formal definition. In this sense, quasiregular maps have bounded distortion. The maximal distortion $K(f)$ is called the *dilatation* of the map f .

Many results from function theory transfer to the setting of quasiregular maps. For example, quasiregular maps are open and discrete. Furthermore, Rickmann [Ric80] proved a quasiregular analogue of Picard's Theorem: a quasiregular map on \mathbb{R}^d that omits a finite number q of values is constant. The number q of omitted values depends only on the dimension and the dilatation of the map. Miniowitz [Min82] proved an analogue of Montel's Theorem: a family of K -quasiregular maps omitting q values is normal. Note that for this result the whole family needs to be K -quasiregular with the same K for every member of the family.

There are examples of quasiregular maps $f: \overline{\mathbb{R}^d} \rightarrow \overline{\mathbb{R}^d}$ such that the family $\{f^n\}$ is K -quasiregular with the same K for every iterate of f . These maps are called *uniformly quasiregular* and we can apply Miniowitz' Theorem. Therefore, many theorems of the Fatou-Julia theory for rational maps can be proved for uniformly quasiregular maps as well.

But uniformly quasiregular maps are rather special. While the composition of two quasiregular maps is again quasiregular, in general the dilatation grows. Thus, we cannot apply Miniowitz' analogue of Montel's Theorem for general quasiregular maps. Hence, we do not have an obvious definition of the Julia set. Bergweiler and Nicks [Ber13, BN14] suggest to define it by the so called "blowing up" property. They define the Julia set of a quasiregular map as set of all points $x \in \mathbb{R}^d$ such that the union of all forward iterates of every neighbourhood of x is the whole space minus a (in some sense) "small" set. In general, this "exceptional" set is not finite.

However, the escaping set $I(f)$ consisting of all points in \mathbb{R}^d that tend to infinity under iteration can be defined in the same way as before. Until now, only little is known about the dynamical behaviour of non-uniformly quasiregular maps. Just like for holomorphic maps, we distinguish between *polynomial type* and *transcendental type* quasiregular maps. A polynomial type quasiregular map extends to a quasiregular self-map of $\overline{\mathbb{R}^d}$, whereas a transcendental type quasiregular map has an essential singularity at infinity. All known examples of uniformly quasiregular maps are of polynomial type. It is not known whether there exist uniformly quasiregular maps of transcendental type.

For transcendental type quasiregular maps, Bergweiler, Fletcher, Langley and Meyer [BFLM09] showed that the escaping set is non-empty and has an unbounded component. The latter is an analogue of Rippon and Stallard's result in [RS05].

Furthermore, examples of quasiregular analogues of the exponential map and of trigonometric functions were studied under a dynamical point of view and turned out to behave quite similarly to their counterparts in the plane. The first examples are so called *Zorich maps* [Zor67]. These maps can be seen as higher dimensional analogues of the exponential map. The complex exponential map $\exp(x+iy)$ decomposes into an angular function $\cos y + i \sin y$ and a scaling function where the scaling is done by the real exponential map of x . Iwaniec and Martin [IM01] constructed Zorich maps in a similar fashion. The angular map is replaced by a bi-Lipschitz map from $[-1, 1]^{d-1} \times \{0\}$ onto the upper half-sphere of radius 1, the scaling is done by the real exponential map of the d th component. Bergweiler [Ber10b] showed that a certain kind of Zorich maps from \mathbb{R}^3 to \mathbb{R}^3 produces a dimension paradox like Karpińska's paradox for the complex exponential map. To state Karpińska's paradox, we first provide some historical background. In 1984, Devaney and Krych showed that for $0 < \lambda < 1/e$, the Julia set of $\lambda \exp(z)$ is an uncountable union of pairwise disjoint injective curves to infinity, called *hairs* (see Definition 2.17). Furthermore, $\lambda \exp(z)$ has an attracting fixed point ξ and the Julia set of $\lambda \exp(z)$ is equal to the set where the iterates do not tend to ξ . McMullen [McM87] showed that the Hausdorff dimension of the Julia set of $\lambda \exp(z)$ is 2 for all $\lambda \in \mathbb{C} \setminus \{0\}$. Karpińska [Kar99b] showed that for $0 < \lambda < 1/e$, the Hausdorff dimension collapses to 1 if we remove the finite endpoints of the hairs. Karpińska [Kar99a] also showed that the set of the finite endpoints of the hairs has Hausdorff dimension 2. Bergweiler's analogue of these results for Zorich maps is the following: with some extra conditions corresponding to the restriction to $0 < \lambda < 1/e$, each Zorich map has a unique fixed point ξ such that the set of points which do not tend to ξ under iteration consists of hairs. Like in the complex case, the endpoints of the hairs have Hausdorff dimension 3, but the union of the hairs without their endpoints has Hausdorff dimension 1.

Bergweiler and Eremenko [BE11] constructed quasiregular maps which can be seen as analogues of trigonometric functions (see Figure 1.1). They start with the upper half-cube with side length 2 and map it bi-Lipschitz onto the upper half-ball of radius 1 such that the upper face of the half-cube is mapped onto the upper unit-hemisphere. Scaling with the real exponential map of the d th component yields a homeomorphism from $T(r_0) := [-1, 1]^{d-1} \times [0, \infty[$ onto the upper half space. We call $T(r_0)$ the *initial tract*. Repeated reflections at hyperplanes yields a quasiregular map from \mathbb{R}^d to \mathbb{R}^d . These maps are 4-periodic in each coordinate except the last one. While the complex sine maps vertical half-strips of width π centred at multiples of π onto the upper or lower half-plane, the quasiregular counterpart maps half-square-beams $T(r)$, which are

obtained by $T(r_0)$ through reflection, homeomorphically onto the upper or lower half-space. The sets $T(r)$ are called *tracts*. While the choice of the initial bi-Lipschitz map may strongly affect the local dynamical behaviour, this is not the case for the following global results. Independently of the initial bi-Lipschitz map we hence denote an arbitrary map in this family by $\text{Sin}(x)$.

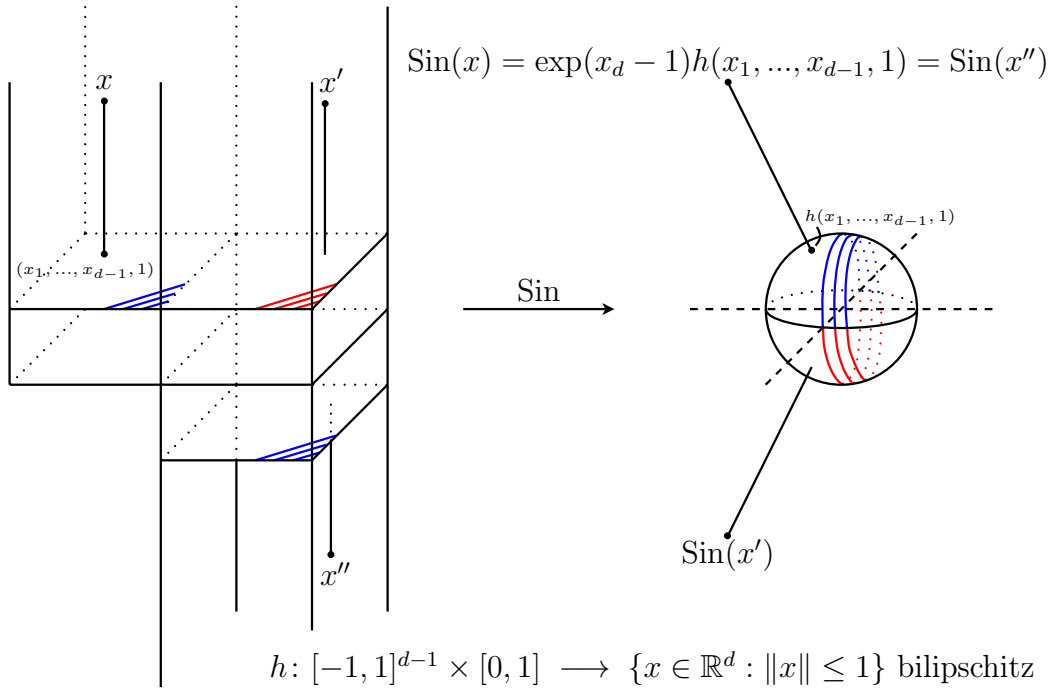


Figure 1.1: Construction of the quasiregular sine. The map h is the restriction of the initial bi-Lipschitz map to the upper face of the cube.

The motivation for Bergweiler and Eremenko to consider these maps came from a result by Schleicher [Sch07]. He considered maps in the sine family $\lambda \sin(z) + \mu$, where λ and μ are chosen such that the critical values are strictly pre-periodic. Schleicher showed that the Julia set consists of hairs, but this time they fill out the whole plane. Obviously, the Hausdorff dimension of the Julia set is 2, but if we remove the finite endpoints, the dimension collapses to 1, producing a particularly strong form of Karpińska's paradox. The easiest example in the class considered by Schleicher is $\pi \sin(z)$. Bergweiler and Eremenko [BE11] showed that this paradox can be reproduced by the quasiregular map $\lambda \text{Sin}(x)$, where $\lambda > 0$ is chosen so large that the map $\lambda \text{Sin}(x)$ becomes locally uniformly expanding. Fletcher and Nicks [FN13] showed that the periodic points of this map are dense in \mathbb{R}^d . Since the map is chosen to be locally uniformly expanding, they are all repelling. Furthermore, they showed that with the suggested definition of the Julia set by Bergweiler and Nicks, the Julia set of this map is the whole space. The results for

$\lambda \text{Sin}(x)$ in [BE11, FN13] require that $\lambda \text{Sin}(x)$ is a locally uniformly expanding map. This corresponds somehow to Schleicher's analogue result in the plane which only holds for a subset of the sine family.

We prove new results for $\lambda \text{Sin}(x)$ for arbitrary $\lambda > 0$. First, we prove an analogue of a result by McMullen from 1987 [McM87]. He showed that the measure of the escaping set of arbitrary functions in the sine family is positive. We obtain the following analogue of this result for the quasiregular map $\text{Sin}(x)$:

Theorem 1. *The measure of the escaping set of $\lambda \text{Sin}(x)$ is positive for every $\lambda > 0$.*

We actually prove that the measure of the set \mathcal{T} of points that escape exponentially fast in x_d -direction is already positive. Furthermore, we show that this set is contained in the *fast escaping set* introduced by Bergweiler and Hinkkanen in [BH99]. The fast escaping set essentially consists of all points which escape as fast as possible, see Definition 2.50. Thereby, Theorem 1 holds as well for the fast escaping set.

Theorem 1'. *The measure of the fast escaping set of $\lambda \text{Sin}(x)$ is positive for every $\lambda > 0$.*

Confirming a conjecture by Milnor, Schubert [Sch08] showed that the measure of the non-escaping set of the complex sine is finite in vertical strips of width 2π . Generalising Theorem 1, we show the following analogue of this statement:

Theorem 2. *The measure of the non-escaping set of $\lambda \text{Sin}(x)$ in a tract is finite for every $\lambda > 0$.*

We actually prove, that each tract without the set \mathcal{T} has finite measure and since \mathcal{T} is a subset of the escaping set, we thus obtain

Theorem 2'. *The measure of the non-fast-escaping set of $\lambda \text{Sin}(x)$ in a tract is finite for every $\lambda > 0$.*

Furthermore, we construct an analogue of the complex power mapping based on the map $\text{Sin}(x)$. The construction is similar to the construction of a quasiregular power mapping based on Zorich maps which was done by Mayer in [May97]. While complex powers are defined by $z^m = \exp(m \log z)$ with a suitable branch $\log(z)$ of the inverse of $\exp(z)$, he chose a Zorich map $Z(x)$ instead of the exponential map and some branch of the inverse of $Z(x)$ instead of $\log(z)$ to obtain a uniformly quasiregular map of polynomial type. We present a construction of this type by using a slightly modified version of the map $\text{Sin}(x)$ to obtain another quasiregular power mapping $P_m(x)$. The modification is analogous to the transformation $\sin(z + \frac{\pi}{2}) = \cos(z)$, hence we call the modified map $\text{Cos}(x)$. Like

Mayer's example, the power mapping P_m is uniformly quasiregular and of polynomial type with degree m^{d-1} . The map P_m can be considered as a higher dimensional analogue of the Chebyshev polynomials T_m satisfying $T_m(x) = \cos(m \arccos(x))$.

We obtain an analogue of a result by Hemke from 2005 [Hem05]. He showed that the measure of the whole non-escaping set of $\sinh(z^m)$ is finite if m is greater or equal to 3. The analogue of his result is

Theorem 3. *Let $m \geq d + 1$. Then the measure of the non-escaping set of $\text{Cos} \circ P_m$ is finite.*

This thesis is structured in the following way:

In Chapter 2, we give short surveys on complex dynamics and quasiregular maps. The precise statements of the results mentioned in the introduction can be found there. Furthermore, we present the formal constructions of Zorich maps as well as the quasiregular analogues of the trigonometric functions and the according preliminary results.

In Chapter 3, we prove Theorem 1, Theorem 1' and Theorem 2. Theorem 2' then follows directly from the proof of Theorem 2.

Chapter 4 is devoted to the construction of the quasiregular power mapping P_m and the proof of Theorem 3. Furthermore, we discuss certain properties of the map P_m . In the last section, we discuss the sharpness of Theorem 3.

2 Motivation and Preliminary Results

In this chapter, we present the theorems mentioned in the introduction. Furthermore, we state the tools that will be used later in the proofs.

2.1 Notations

The following basic notations will be used in the rest of the thesis. We will introduce further notations when needed. Furthermore, we provide a nomenclature containing the introduced notations and symbols at the end of the thesis.

We denote by $d \in \mathbb{N}$ with $d \geq 3$ the dimension of the Euclidian space \mathbb{R}^d . Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ (or $f: \mathbb{C} \rightarrow \mathbb{C}$). We denote by f^n the n th iterate of f , that is $f^n(z) = f(f^{n-1}(z))$ for $n \in \mathbb{N}$ and $f^0(z) = z$. We denote by $B(x, r)$ the closed ball in \mathbb{R}^d with centre $x \in \mathbb{R}^d$ and radius $r > 0$. We denote by $\mathbb{S} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ the $(d - 1)$ -dimensional unit-sphere. For a Lebesgue measurable set $A \subset \mathbb{R}^d$, we denote by $\text{meas}(A)$ the d -dimensional Lebesgue measure. We denote by ν the measure of the d -dimensional unit ball and by ω the surface area of the unit sphere \mathbb{S} . We denote the Riemann sphere by $\overline{\mathbb{C}}$ and by $\overline{\mathbb{R}^d}$ the one point compactification of \mathbb{R}^d endowed with the chordal metric. We denote by $\text{card}(A)$ the cardinality of a set A . We will need the notion of the density of two measurable sets.

Definition 2.1. Let $A, B \subset \mathbb{R}^d$ be measurable sets, $\text{meas}(B) \neq 0$. Then we call

$$\text{dens}(A, B) := \frac{\text{meas}(A \cap B)}{\text{meas}(B)}$$

the *density* of A in B .

2.2 Hausdorff measure and dimension

To give precise statements of the theorems stated in the introduction, we need the notion of the Hausdorff dimension. To define the Hausdorff dimension, we first introduce the Hausdorff measure. In this section, we follow [Fal86] and [Mat95].

Definition 2.2. For $A \subset \mathbb{R}^d$ and $s, \delta > 0$, put

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_j \text{diam}(E_j)^s : A \subset \bigcup_j E_j, \text{diam}(E_j) \leq \delta \right\},$$

where $E_j \subset \mathbb{R}^d$. We call

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A)$$

the s -dimensional Hausdorff measure of A and

$$\dim A := \inf \{s : \mathcal{H}^s(A) = 0\}$$

the Hausdorff dimension of A .

The following two remarks are from [Mat95, page 56]:

Remark 2.3. For $s = d \in \mathbb{N}$, the d -dimensional Hausdorff measure and the d -dimensional Lebesgue measure differ only by a multiplicative constant.

Remark 2.4. Let $1 \leq m \leq d$. If M is an m -dimensional C^1 -submanifold of \mathbb{R}^d , then the restriction of \mathcal{H}^m to M gives a constant multiple of the surface measure on M .

We will only apply this to $(d-1)$ -spheres and $m = d-1$.

We denote by

$$\text{dens}_{\mathcal{H}^s}(A, B)$$

the density of A in B with respect to the s -dimensional Hausdorff measure. Regarding Lipschitz maps, we have an upper bound for the distortion of the Hausdorff measure, see [Fal86, Lemma 1.8]:

Theorem 2.5. Let $E, F \subset \mathbb{R}^d$, L be a positive constant and $\Psi: E \rightarrow F$ be a surjective map such that

$$\|\Psi(x) - \Psi(y)\| \leq L\|x - y\|$$

for all $x, y \in E$. Then

$$\mathcal{H}^s(F) \leq L^s \mathcal{H}^s(E).$$

The image of a Lipschitz map of a Lebesgue measurable set is again Lebesgue measurable, see [AE01, Theorem IX.5.12]:

Theorem 2.6. Let f be a Lipschitz continuous map. Then the image $f(A)$ is Lebesgue measurable for every Lebesgue measurable set A .

Remark 2.7. For $s = d$ and $\Psi: E \rightarrow F$ as in Theorem 2.5 such that E is measurable, we thus obtain

$$\text{meas}(F) \leq L^d \text{meas}(E).$$

Theorem 2.5 holds in the other direction as well:

Lemma 2.8. *Let $E, F \subset \mathbb{R}^d$, L be a positive constant and $\Psi: E \rightarrow F$ be a bijective map such that*

$$L\|x - y\| \leq \|\Psi(x) - \Psi(y)\|$$

for all $x, y \in E$. Then

$$\mathcal{H}^s(F) \geq L^s \mathcal{H}^s(E).$$

Proof. Denote the inverse map of Ψ by Φ . Put $\tilde{x} := \Psi(x)$ and $\tilde{y} := \Psi(y)$. Then

$$\|\Phi(\tilde{x}) - \Phi(\tilde{y})\| \leq \frac{1}{L} \|\tilde{x} - \tilde{y}\|.$$

Theorem 2.5 yields

$$\mathcal{H}^s(E) \leq \frac{1}{L^s} \mathcal{H}^s(F)$$

and thus

$$\mathcal{H}^s(F) \geq L^s \mathcal{H}^s(E). \quad \square$$

2.3 Survey on complex dynamics

We want to give a short survey on the iteration of holomorphic maps by giving the basic definitions and stating those results we touched upon in the introduction. See [Ber93] for an introduction to transcendental dynamics.

Definition 2.9. Let $U \subseteq \overline{\mathbb{C}}$ be open and \mathfrak{F} be a family of functions from U to $\overline{\mathbb{C}}$. We call \mathfrak{F} *normal* if for each sequence in \mathfrak{F} there exists a locally uniformly converging subsequence with respect to the chordal metric. We call \mathfrak{F} *normal in z_0* for some $z_0 \in U$ if there exists a neighbourhood V of z_0 such that the family $\{f|_V : f \in \mathfrak{F}\}$ is normal.

Regarding meromorphic functions, Montel's Theorem is a powerful criterion for normality:

Theorem 2.10. *Let $G \subset \overline{\mathbb{C}}$ be open, $a_1, a_2, a_3 \in \overline{\mathbb{C}}$ distinct and \mathfrak{F} a family of functions meromorphic in G . If $f(z) \neq a_j$ holds for all $z \in G$, all functions $f \in \mathfrak{F}$ and $j \in \{1, 2, 3\}$, then \mathfrak{F} is normal.*

In the context of the iteration of holomorphic maps we obtain a partition of the plane into two sets.

Definition 2.11. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic map. We call

$$\mathcal{F}(f) := \{z \in \mathbb{C} : \{f^n : n \in \mathbb{N}\} \text{ is normal in } z\}$$

the *Fatou set of f* and its complement

$$\mathcal{J}(f) := \mathbb{C} \setminus \mathcal{F}(f)$$

the *Julia set of f* .

The Fatou set $\mathcal{F}(f)$ is the set where the iterates of f have stable behaviour, and the Julia set $\mathcal{J}(f)$ is the set where the iterates behave chaotically. Another important set in transcendental dynamics is the escaping set:

Definition 2.12. Let f be a transcendental entire map. We call the set

$$I(f) := \{z \in \mathbb{C} : |f^n(z)| \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

the *escaping set of f* .

The escaping set was introduced by Eremenko. He proved the following result [Er89, Theorem 1 and page 340]:

Theorem 2.13. *Let f be a transcendental entire function. Then $I(f) \neq \emptyset$ and*

$$J(f) = \partial I(f).$$

Thus, there is a close connection between the escaping set and the Julia set. This becomes even more apparent in an important subset of transcendental entire functions.

Definition 2.14. We call the class

$$\mathcal{B} := \{f : f \text{ is transcendental entire such that } \text{sing}(f^{-1}) \text{ is bounded}\}$$

Eremenko-Lyubich class, where $\text{sing}(f^{-1})$ denotes the set of the singularities of the inverse function.

For functions in the class \mathcal{B} , we have the following result by Eremenko and Lyubich [EL92, Theorem 1 and page 995]:

Theorem 2.15. *Let $f \in \mathcal{B}$. Then $I(f) \subset \mathcal{J}(f)$ and thus $J(f) = \overline{I(f)}$.*

For arbitrary transcendental entire functions, Rippon and Stallard [RS05, Theorem 1] proved the following result:

Theorem 2.16. *Let f be a transcendental entire function. Then the escaping set $I(f)$ has at least one unbounded component.*

Next, we state some results concerning the exponential and the trigonometric family. Most of these were already mentioned in the introduction and we will present some quasiregular analogues in the section about the dynamics of quasiregular maps and when we consider examples. We start with some results about the exponential family. We put

$$E_\lambda(z) = \lambda \exp(z),$$

where $\lambda \in \mathbb{C}$.

Definition 2.17. We call a subset H of \mathbb{C} or \mathbb{R}^d a *hair* if there exists a homeomorphism $\gamma: [0, \infty[\rightarrow H$ such that $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$. We call $\gamma(0)$ the *endpoint* of the hair H .

For $0 < \lambda < 1/e$, the function E_λ has an attracting fixed point ξ . Using this terminology, we can state the following result by Devaney and Krych [DK84, page 50]:

Theorem 2.18. *Let $0 < \lambda < 1/e$. Then*

$$\mathcal{J}(E_\lambda) = \{z \in \mathbb{C} : E_\lambda^n \rightarrow \xi \text{ as } n \rightarrow \infty\}$$

and $\mathcal{J}(E_\lambda)$ is an uncountable union of pairwise disjoint hairs.

McMullen proved the following result [McM87, Theorem 1.2]:

Theorem 2.19. *Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\dim \mathcal{J}(E_\lambda) = 2$.*

Theorem 2.19 directly implies that the union of the hairs in Theorem 2.18 has Hausdorff dimension 2. Karpińska showed that the dimension collapses if we remove the endpoints of the hairs. This phenomenon is known as “Karpińska’s paradox”. Formally she proved [Kar99b, Theorem 1.1]:

Theorem 2.20. *Let $0 < \lambda < 1/e$, and let C_λ be the set of endpoints of the hairs that form $\mathcal{J}(E_\lambda)$. Then $\dim(\mathcal{J}(E_\lambda) \setminus C_\lambda) = 1$.*

Karpińska [Kar99a, Theorem 1] also proved that C_λ has Hausdorff dimension 2.

Next, we state some results about the trigonometric family, i.e. functions of the form

$$g(z) = \lambda \sin(z) + \mu, \quad \lambda \neq 0.$$

Devaney and Tangerman [DT86] showed that for certain parameters the Julia set consists of hairs. In [McM87, Theorem 1.1], McMullen proved the following result:

Theorem 2.21. *For $g(z) = \lambda \sin(z) + \mu$, $\lambda \neq 0$, the Julia set $\mathcal{J}(g)$ has positive area.*

Remark 2.22. McMullen actually proved that $I(g)$ has positive area.

Remark 2.23. In [AB12], Aspenberg and Bergweiler proved a generalisation of Theorem 2.21. Essentially, for $g \in \mathcal{B}$ with a certain growth restriction, the escaping set $I(g)$ and the Julia set $\mathcal{J}(g)$ have positive area.

Schleicher [Sch07, Theorem 1] proved the following particularly strong form of Karpińska's paradox:

Theorem 2.24. *There exists a representation of \mathbb{C} as a union of hairs with the following properties:*

- *the intersection of two hairs is either empty or consists of the common endpoint;*
- *the union of the hairs without their endpoints has Hausdorff dimension 1.*

This representation is defined by the dynamics of the maps $g(z) = \lambda \sin(z) + \mu$, $\lambda \neq 0$, where λ and μ are chosen such that the critical values of g are strictly pre-periodic. For these maps, the Julia set $\mathcal{J}(g)$ consists of hairs. Moreover, $\mathcal{J}(g) = \mathbb{C}$. The easiest example in the trigonometric family with strictly pre-periodic critical values is given by $\pi \sin(z)$.

Confirming a conjecture by Milnor, Schubert proved the following result [Sch08, Theorem 1.2]:

Theorem 2.25. *The area of the non-escaping set of $\sin(z)$ in a vertical strip of width 2π is finite.*

Hemke [Hem05, Theorem 5.1] considered the following family:

Theorem 2.26. *Let $g(z) := P(z) \exp(Q(z)) + \tilde{P}(z) \exp(\tilde{Q}(z))$ for polynomials P, \tilde{P} and Q, \tilde{Q} such that $n := \deg(\tilde{Q}) = \deg(Q) \geq 0$ and the arguments of their n th coefficients q, \tilde{q} differ by some odd multiple of π/n . Then*

$$\text{meas}(I(g)) > 0.$$

If furthermore $\tilde{Q} = -Q$ and $n \geq 3$, then

$$\text{meas}(\mathbb{C} \setminus I(g)) < \infty.$$

As an example, he considered the map

$$g(z) = \exp(z^3) - \exp(-z^3) = 2 \sinh(z^3).$$

Its Fatou set is non-empty, since it contains a super attractive basin around zero. Thus,

$$0 < \text{meas}(\mathbb{C} \setminus I(g)) < \infty.$$

2.4 Survey on quasiregular maps

Quasiregular maps are natural generalisations of analytic maps in the plane to higher dimensions. We follow the survey of Väisälä [Väi80] for the historical background. Most of the results we are going to state can also be found in the book by Rickman [Ric93]. Higher dimensional quasiregular maps were introduced by Lavrentiev [Lav38] in 1938. They were first studied systematically by Reshetnyak in a series of papers starting with [Reš66].

To motivate the definition of quasiregular maps, we follow [Väi80] and recall the definition of analytic maps. Let $G \subset \mathbb{R}^2$ be open. A function $f : G \rightarrow \mathbb{R}^2$ is (complex) analytic if and only if

- f is C^1 in the real sense;
- $\|Df(x)\|^2 = J_f(x)$ for all $x \in G$,

where $Df(x)$ denotes the derivative,

$$\|Df(x)\| := \sup_{\|h\|=1} \|Df(x)(h)\|$$

denotes its norm and $J_f(x)$ denotes the Jacobian determinant of f at x . The second condition holds by the Cauchy-Riemann differential equations. Hence, holomorphic functions map infinitesimal small circles onto infinitesimal small circles. We can easily generalise this definition to higher dimensions. But if we take $G \subset \mathbb{R}^d$ open and $f : G \rightarrow \mathbb{R}^d$ satisfying

- f is C^1 in the real sense;
- $\|Df(x)\|^d = J_f(x)$ for all $x \in G$,

then f is either constant or a sense preserving Möbius transformation. In order to get a more interesting class of functions, we need to allow some distortion as well as some non-differentiability.

Definition 2.27. Let $U \subset \mathbb{R}^d$ be a domain. Then

$$W_{d,loc}^1(U)$$

denotes the Sobolev space consisting of all functions $f = (f_1, \dots, f_d) : U \rightarrow \mathbb{R}^d$ for which all first-order weak partial derivatives $\partial_k f_i$ exist and are locally in L^d .

Now we are able to give a precise definition of quasiregular maps.

Definition 2.28. Let $U \subset \mathbb{R}^d$ be a domain. A continuous function $f : U \rightarrow \mathbb{R}^d$ is called *quasiregular* if

- $f \in W_{d,loc}^1(U)$;
- there exists a constant $K_O \geq 1$ such that

$$\|Df(x)\|^d \leq K_O J_f(x) \quad (2.1)$$

for almost every $x \in U$.

Definition 2.29. Let $I \subset \mathbb{R}$ be an interval. We call a function $f : I \rightarrow \mathbb{R}$ *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^l |f(b_k) - f(a_k)| < \varepsilon$$

for every finite number of non-overlapping open intervals $]a_k, b_k[$ with $[a_k, b_k] \subset I$ for all $1 \leq k \leq l$ and

$$\sum_{k=1}^l (b_k - a_k) < \delta.$$

Remark 2.30. Quasiregular maps are often defined using the class ACL^d . These are functions from a domain $U \subset \mathbb{R}^d$ to \mathbb{R}^d where the coordinate functions are absolutely continuous on almost every line parallel to the coordinate axes and additionally almost every partial derivative is locally L^d -integrable.

For a complete definition and the following statement, see [Ric93, Proposition I.1.2].

Theorem 2.31. *A map $f : U \rightarrow \mathbb{R}^d$ is ACL^d if and only if it is continuous and belongs to $W_{d,loc}^1(U)$.*

Remark 2.32. If inequality (2.1) holds for some K_O , then

$$J_f(x) \leq K_I \ell(Df(x))^d \quad \text{a.e.} \quad (2.2)$$

holds for some $K_I \geq 1$, where

$$\ell(Df(x)) := \inf_{\|h\|=1} \|Df(x)(h)\|,$$

see [Ric93, page 11].

Remark 2.33. Geometrically, $Df(x)$ maps the unit ball onto an ellipsoid around zero. The length of the major half-axis is given by $\|Df(x)\|$, the length of the smallest half-axis by $\ell(Df(x))$ and the volume by $\nu \cdot J_f(x)$, which is equal to the product of all half-axes times ν . Hence, f maps infinitesimal small spheres to infinitesimal small ellipsoids with bounded eccentricity.

Definition 2.34. The smallest constants $K_O = K_O(f)$ and $K_I = K_I(f)$ for which (2.1) and (2.2) hold are called *outer* and *inner dilatation* of f . Moreover,

$$K(f) := \max\{K_I(f), K_O(f)\}$$

is called (maximal) *dilatation* of f . We say that f is K -*quasiregular* if $K(f) \leq K$.

Remark 2.35. Like for the operator norm, we have the following estimates for ℓ . Let $A, B: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be linear. Then

$$\ell(A) = \inf_{\|h\|=1} \|Ah\| \leq \frac{\|Aw\|}{\|w\|}$$

for every $w \in \mathbb{R}^d \setminus \{0\}$. Thus,

$$\ell(A \cdot B) = \inf_{\|h\|=1} \|A(Bh)\| \geq \inf_{\|h\|=1} \ell(A)\|Bh\| = \ell(A) \cdot \ell(B). \quad (2.3)$$

Quasiregularity can be defined on $\overline{\mathbb{R}^d}$ as well, see for example [Vuo88, Definition 10.6]:

Definition 2.36. Let $G \subset \overline{\mathbb{R}^d}$ be a domain. A continuous map $f: G \rightarrow \overline{\mathbb{R}^d}$ is quasiregular if either $f(G) = \{\infty\}$ or $f^{-1}(\infty)$ is discrete and $f|_{G \setminus (f^{-1}(\infty) \cup \{\infty\})}$ is quasiregular.

A proof of the following result can be found in [Ric93, Theorem I.2.4].

Theorem 2.37. *Quasiregular maps are differentiable almost everywhere.*

Theorem 2.38. *Non-constant quasiregular maps are open and discrete.*

A proof can be found in [Ric93, Theorem I.4.1].

The composition of two quasiregular maps is again quasiregular, but in general the dilatation grows, see [Ric93, Theorem II.6.8].

Theorem 2.39. *Let f and g be quasiregular such that the domain of f contains the range of g . Then*

$$K(f \circ g) \leq K(f)K(g).$$

Rickman [Ric80, Theorem 1.1] proved the following analogue of Picard's Theorem:

Theorem 2.40. *Let $d \in \mathbb{N}$, $d \geq 2$ and $K \geq 1$. Then there exists $q = q(d, K)$ with the following properties: if $a_1, \dots, a_q \in \mathbb{R}^d$ are distinct and if $f: \mathbb{R}^d \rightarrow \mathbb{R}^d \setminus \{a_1, \dots, a_q\}$ is K -quasiregular, then f is constant.*

Equivalently, a non-constant K -quasiregular map $f: \mathbb{R}^d \rightarrow \overline{\mathbb{R}^d}$ omits at most q values.

Remark 2.41. For $d = 2$ and $K = 1$ we have Picard's Theorem.

Miniowitz [Min82, Theorem 4] proved an analogue of Montel's Theorem:

Theorem 2.42. *Let $d \in \mathbb{N}$, $d \geq 2$, and $K \geq 1$. Let $a_1, \dots, a_q \in \mathbb{R}^d$ be distinct, where $q = q(d, K)$ is as in Rickman's Theorem. Let $\Omega \subset \mathbb{R}^d$ be a domain and \mathfrak{F} be a family of K -quasiregular maps $f: \Omega \rightarrow \mathbb{R}^d \setminus \{a_1, \dots, a_q\}$. Then \mathfrak{F} is normal.*

Note that the family needs to be K -quasiregular with the same K for all maps.

2.5 Dynamics of quasiregular maps

If we want to apply Miniowitz' Theorem to the iterates of quasiregular maps, we need that all iterates are K -quasiregular with the same constant K .

Definition 2.43. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be quasiregular. We call f *uniformly quasiregular* if there exists $K \geq 1$ such that f^n is K -quasiregular for all $n \in \mathbb{N}$.

Remark 2.44. The first example of a uniformly quasiregular self-map of \mathbb{R}^d where $d \geq 3$ was constructed by Iwaniec and Martin, see [IM01, page 496]. Another example was given by Mayer [May97, Section 5]. Both examples are presented in [Ber10a, Examples 4.1 and 4.2] as well. The quasiregular power mapping which is constructed in Chapter 4 is uniformly quasiregular as well.

While there is no obvious definition for the Julia set of a general quasiregular map, we can still define the escaping set in the same way we did for entire transcendental maps.

Definition 2.45. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a quasiregular map. We call the set

$$I(f) := \{x \in \mathbb{R}^d : \|f^n(x)\| \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

the *escaping set* of f .

There are only few results about the dynamics of non-uniformly quasiregular maps. Just like in the case of entire functions, we have polynomial type and transcendental type quasiregular maps:

Definition 2.46. We say that a quasiregular map $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is of *polynomial type* if $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ is satisfied. If f has an essential singularity at ∞ , we say that f is of *transcendental type*.

Remark 2.47. By putting $f(\infty) = \infty$, a quasiregular map $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ of polynomial type extends to a quasiregular self-map of $\overline{\mathbb{R}^d}$.

The dynamics of polynomial type quasiregular maps are studied for example in [FN11] and [Ber13].

Remark 2.48. Every known example of a uniformly quasiregular map $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ extends to a quasiregular self-map of $\overline{\mathbb{R}^d}$. It is not known whether there exists a uniformly quasiregular map $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ with an essential singularity at ∞ .

For transcendental type quasiregular maps, Bergweiler, Fletcher, Langley and Meyer [BFLM09, page 643] proved the following analogue of Theorem 2.13 and Theorem 2.16:

Theorem 2.49. *Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a transcendental type quasiregular map. Then $I(f) \neq \emptyset$. Moreover, $I(f)$ has an unbounded component.*

Further results on the dynamics of transcendental type quasiregular maps can be found in [BN14].

An important subset of the escaping set of transcendental entire maps was introduced by Bergweiler and Hinkkanen in [BH99]. Since the definition is essentially the same, we restrict ourselves to the quasiregular case, see [BDF14, page 2]:

Definition 2.50. For a quasiregular map $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ we denote by

$$M(r, f) := \max_{\|x\|=r} \|f(x)\|$$

the *maximum modulus* of f and by $M^n(r, f)$ the n th iterate of $M(r, f)$ with respect to the first variable (for example, $M^2(r, f) = M(M(r, f), f)$). Then the *fast escaping set* of f is defined by

$$A(f) := \{x \in \mathbb{R}^d : \exists N \in \mathbb{N} \forall n \in \mathbb{N} : \|f^{n+N}(x)\| \geq M^n(R, f)\},$$

with R so large that $M^n(R, f) \rightarrow \infty$ as $n \rightarrow \infty$.

2.6 Branch points of quasiregular maps

Definition 2.51. Let $G \subset \mathbb{R}^d$ be open and $f: G \subset \mathbb{R}^d$ be quasiregular. We call the set

$$B_f := \{x \in G : f \text{ is not locally homeomorphic at } x\}$$

the *branch set* of f .

Remark 2.52. Let $d \geq 3$. If $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is quasiregular and not injective, then $B_f \neq \emptyset$. In fact, the $(d - 2)$ -dimensional Hausdorff measure of the image $f(B_f)$ is positive, see [Ric93, Proposition III.3].

Similar to the concept of the multiplicity of a point, we follow Rickman and Srebro [RS86, page 247] to define the local index:

Definition 2.53. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be quasiregular. Then we call

$$i(f, x) := \inf_{U(x)} \sup_{y \in \mathbb{R}^d} \text{card} (f^{-1}(y) \cap U(x))$$

the *local index of f at x* , where $U(x)$ is a neighbourhood of x .

Remark 2.54. $x \in B_f$ if and only if $i(f, x) \geq 2$, see [MS75, Lemma 3.2].

Using the notion of the local index, we can assign a degree to non-constant quasiregular self-maps on \mathbb{R}^d which do not have an essential singularity at infinity, see [MS75, Theorem 4.1]:

Theorem 2.55. Let $f: \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$ be a non-constant quasiregular map. Then there exists a constant $\mu(f) \in \mathbb{N}$ such that

$$\mu(f) = \sum_{x \in f^{-1}(y)} i(x, f)$$

for all $y \in \overline{\mathbb{R}^n}$.

Definition 2.56. Let f and $\mu(f)$ be as in Theorem 2.55. Then $\mu(f)$ is called the *degree of f* .

2.7 Examples

Winding map

An easy example of a quasiregular map is given by the so called *winding map*, see for example [Ric93, Example I.3.1]: Let k be a positive integer and $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the map $(r, \varphi, x_3) \mapsto (r, k\varphi, x_3)$ in cylindrical coordinates. Then f is quasiregular with $K_O(f) = k^2$ and $K_I(f) = k$. The branch set B_f is the x_3 -axis.

Before we discuss examples of quasiregular analogues of the exponential map and of trigonometric functions, we want to give a brief reminder how the complex exponential map can be constructed using the real exponential map. This is just to show the

similarities in the construction, see Figure 2.1. We first map the vertical line segment $i[-\pi/2, \pi/2]$ onto the half-circle of radius 1 in the right half-plane. Then we scale with the real exponential of the real part. Repeated reflections at the sides of the horizontal strip and the imaginary axis yield the well known exponential map.

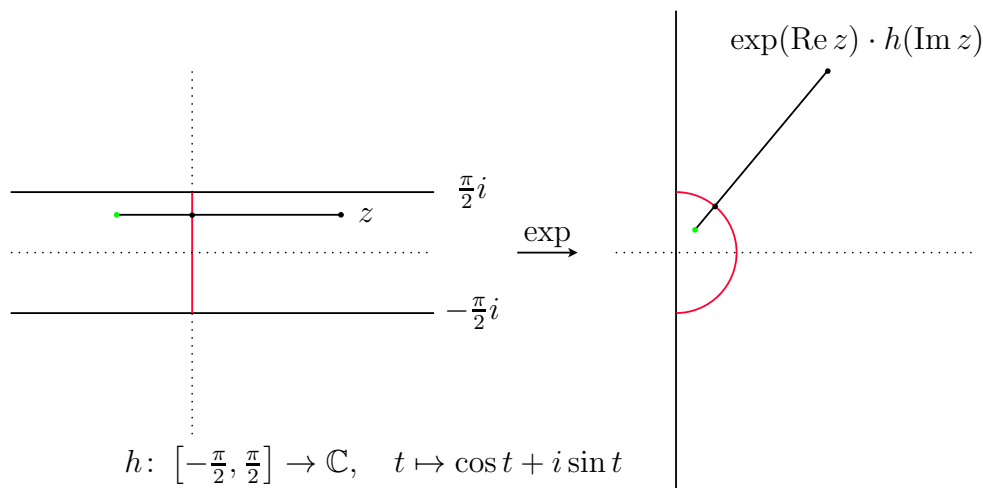


Figure 2.1: Construction of the complex exponential map.

Zorich maps

An important example of quasiregular maps was given by Zorich (according to [IM01], see [Zor67]). This map can be considered as a d -dimensional analogue of the exponential map. Following Iwaniec and Martin [IM01, Example 6.5.4], see also Bergweiler [Ber10b, page 601], we put

$$U := \{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} : |x_j| \leq 1\}.$$

Now let h_Z be a bi-Lipschitz map from U to the upper hemisphere

$$\mathbb{S}^+ := \{x \in \mathbb{R}^d : \|x\| = 1, x_d \geq 0\}$$

and define

$$Z: U \times \mathbb{R} \rightarrow \mathbb{R}^d, \quad Z(x) = \exp(x_d)h_Z(x_1, \dots, x_{d-1}).$$

Then Z maps the infinite square beam $U \times \mathbb{R}$ onto the upper half-space

$$\mathbb{H}^+ := \{x \in \mathbb{R}^d : x_d \geq 0\}.$$

Repeated reflections at the sides of square beams and the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ yields a quasiregular map $Z: \mathbb{R}^d \rightarrow \mathbb{R}^d$.

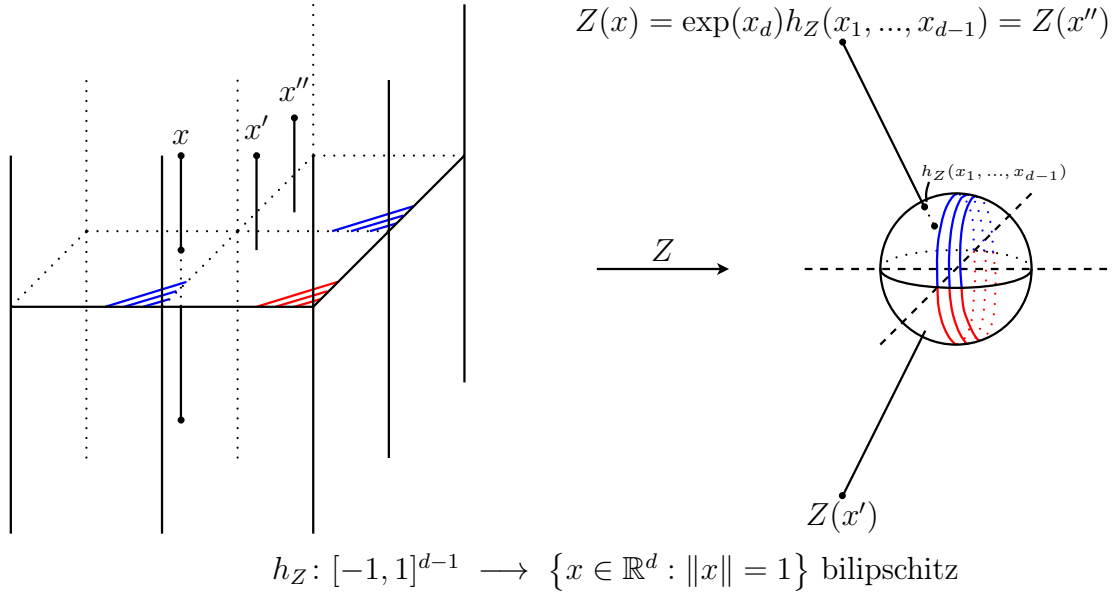


Figure 2.2: Construction of Zorich maps.

Definition 2.57. We call a function defined like this a *Zorich map*.

Remark 2.58. The map h_Z takes the role of the angular map in the construction of the complex exponential map.

For $d = 3$, Bergweiler [Ber10b, page 602] showed that there exist $\alpha, m, M \in \mathbb{R}$ with $0 < \alpha < 1$ and $m < M$, such that $\|DZ(x)\| \leq \alpha$ a.e. for $x_3 \leq m$, while $\ell(DZ(x)) \geq \frac{1}{\alpha}$ a.e. for $x_3 \geq M$. Now choose $a \geq \exp(M) - m$ and consider the map

$$Z_a: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad Z_a(x) := Z(x) - (0, 0, a).$$

This leads to the following result [Ber10b, Theorem 1]:

Theorem 2.59. *Let Z_a be as above. Then there exists a unique fixed point $\xi = (\xi_1, \xi_2, \xi_3)$ satisfying $\xi_3 \leq m$ and the set*

$$J := \{x \in \mathbb{R}^3 : Z_a^k(x) \rightarrow \xi \text{ as } k \rightarrow \infty\}$$

consists of uncountably many pairwise disjoint hairs. The set C of endpoints of these hairs has Hausdorff dimension 3, while $J \setminus C$ has Hausdorff dimension 1.

This can be seen as a 3-dimensional analogue of Karpińska's paradox in the plane [Kar99b].

Remark 2.60. The free choice of the bi-Lipschitz map h_Z generates a whole class of maps. Locally, the dynamical behaviour of these maps can be quite different, see [Ber10b, Section 7] for some examples.

Power mapping

Mayer [May97, Section 5] constructed quasiregular analogues of power mappings based on Zorich maps. For $z \in \mathbb{C}$ we have

$$z^m = \exp(m \log z),$$

with the branch $\log: \mathbb{C} \setminus [0, \infty[\rightarrow \{z \in \mathbb{C} : 0 < \text{Im } z < 2\pi\}$ of the exponential map. Mayer uses this exact approach. He replaced the exponential map by a Zorich map Z and the logarithm by a branch of the inverse of Z . The resulting map is in fact a uniformly quasiregular map with degree m^{d-1} .

Quasiregular sine

We follow Bergweiler and Eremenko [BE11] to construct a quasiregular analogue of sine. Let

$$\text{Sin} : [-1, 1]^{d-1} \times [0, 1] \rightarrow \{x \in \mathbb{R}^d : \|x\|_2 \leq 1, x_d \geq 0\}$$

be a bi-Lipschitz map such that the upper face $[-1, 1]^{d-1} \times \{1\}$ is mapped onto the upper hemisphere

$$\mathbb{S}^+ = \{x \in \mathbb{R}^d : \|x\| = 1, x_d \geq 0\}.$$

We then define $\text{Sin} : [-1, 1]^{d-1} \times [1, \infty[\rightarrow \mathbb{R}^d$ by

$$\text{Sin}(x) = \exp(x_d - 1) \text{Sin}(x_1, \dots, x_{d-1}, 1).$$

Then $\text{Sin}(x)$ maps $[-1, 1]^{d-1} \times [1, \infty[$ bijectively onto the upper half-space

$$\mathbb{H}^+ = \{x \in \mathbb{R}^d : x_d \geq 0\}.$$

Using repeated reflections at faces of square-beams and the hyperplane $\mathbb{R}^{d-1} \times \{0\}$, we can extend $\text{Sin}(x)$ to a map $\text{Sin} : \mathbb{R}^d \rightarrow \mathbb{R}^d$. We denote the restriction of the initial bi-Lipschitz map to the upper face of the half-cube by

$$h := \text{Sin}|_{[-1, 1]^{d-1} \times \{1\}}.$$

Definition 2.61. We will refer to h as *angular map* of $\text{Sin}(x)$.

The map h is bi-Lipschitz. Thus, there exists a constant $L_h > 0$ such that

$$\frac{1}{L_h} \|x - y\| \leq \|h(x) - h(y)\| \leq L_h \|x - y\| \quad (2.4)$$

for all $x, y \in [-1, 1]^{d-1} \times \{1\}$.

Remark 2.62. Again, this construction produces a whole class of maps. An explicit example of a bi-Lipschitz map from the upper half-cube onto the upper half-ball, where the upper face is mapped onto the upper hemisphere can be found in [BE11, page 173].

Remark 2.63. In [BE11, Section 5 and page 167], the authors show that $\text{Sin}(x)$ is indeed quasiregular and that

$$\beta := \text{ess inf}_{x \in \mathbb{R}^d} \ell(Df(x)) > 0.$$

Now we want to label the tracts which are generated by $[-1, 1]^{d-1} \times [0, \infty[$ through reflections. Again, we follow [BE11].

Definition 2.64. Put $R := \mathbb{Z}^{d-1} \times \{-1, 1\}$. For $r = (r_1, \dots, r_d) \in R$ we call

$$T(r) := \{x \in \mathbb{R}^d : |x_j - 2r_j| \leq 1 \text{ für } 1 \leq j \leq d-1, r_d x_d \geq 0\}$$

a *tract* of $\text{Sin}(x)$ with label r .

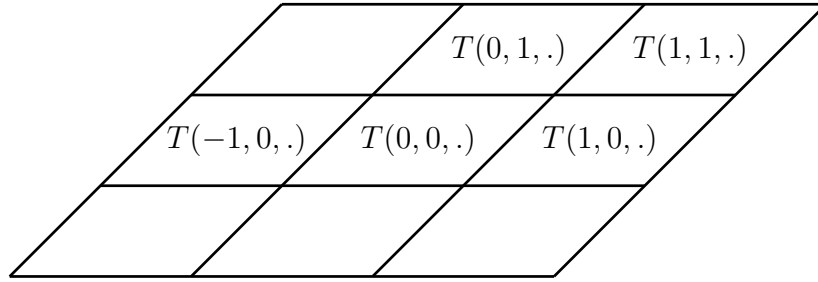


Figure 2.3: Labels of the tracts.

We put $r_0 := (0, \dots, 0, 1) \in R$ and call

$$T(r_0) = T(0, \dots, 0, 1)$$

the initial tract in the construction of $\text{Sin}(x)$. Since all tracts are obtained by reflections of $T(r_0)$ at hyperplanes, we find that if

$$\sigma(r) := \sum_{j=1}^{d-1} r_j + \frac{1}{2}(r_d - 1)$$

is even, then $\text{Sin}(x)$ maps $T(r)$ bijectively onto \mathbb{H}^+ . If $\sigma(r)$ is odd, $\text{Sin}(x)$ maps $T(r)$ bijectively onto

$$\mathbb{H}^- := \{x \in \mathbb{R}^d : x_d \leq 0\}.$$

So the map σ determines whether we need an even or an odd number of reflections to obtain the tract $T(r)$ by reflecting the initial tract at hyperplanes. For $r \in R$ we denote by Λ^r the inverse function of $f|_{T(r)}$. Thus,

$$\Lambda^r : \mathbb{H}^+ \rightarrow T(r)$$

if $\sigma(r)$ is even and

$$\Lambda^r : \mathbb{H}^- \rightarrow T(r)$$

if $\sigma(r)$ is odd.

Remark 2.65. The branch points of $\text{Sin}(x)$ are the edges of the tracts $T(r)$, where $r \in R$.

Definition 2.66. For $x \in \mathbb{R}^d$, we call the sequence $\underline{s} = (s_k)_{k \geq 0}$ in R such that

$$\text{Sin}^n(x) \in T(s_n)$$

an *external address* of x .

Remark 2.67. The external address is not unique if $\text{Sin}^n(x) \in \partial T(r)$ for some $n \geq 0$ and $r \in R$. But this will not matter for our purposes.

Remark 2.68. The complex sine and its quasiregular counterpart have similar geometric properties. While $\sin(x + iy)$ is 2π -periodic in x -direction, its quasiregular counterpart $\text{Sin}(x_1, \dots, x_d)$ is 4-periodic in each component except the last one. Under the complex sine, vertical half-strips of the form

$$\left\{ (x + iy) \in \mathbb{C} : |x - k\pi| \leq \frac{\pi}{2}, y \geq 0 \right\}$$

are mapped onto the upper or lower half-plane, corresponding to the tracts $T(r)$ for $r_d = 1$, which are mapped onto the upper or lower half-space. Likewise for $y \leq 0$ and $r_d = -1$.

2.8 Dynamical behaviour of $\lambda \text{Sin}(x)$

For the rest of this chapter, we investigate the dynamical behaviour of the map

$$f : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad f(x) := \lambda \text{Sin}(x),$$

where $\lambda > 0$. Apart from sharing geometric properties with the complex sine, there are even more similarities if we investigate the dynamical behaviour of f compared to sine. The next two results are analogues of Theorem 2.24. Bergweiler and Eremenko [BE11, Theorem 1] proved the following theorem:

Theorem 2.69. *There exists a representation of \mathbb{R}^d as a union of hairs with the following properties:*

- *the intersection of two hairs is either empty or consists of the common endpoint;*
- *the union of the hairs without their endpoints has Hausdorff dimension 1.*

This representation is defined by the dynamics of the (locally uniformly expanding) map $f(x) = \lambda \text{Sin}(x)$ with λ sufficiently large. In fact, λ can be chosen so large that

$$1 < \lambda\beta = \text{ess inf}_{x \in \mathbb{R}^d} \ell(Df(x)).$$

Fletcher and Nicks [FN13, Theorem 1.1] showed that a slightly modified version \tilde{f} of f behaves chaotically in the whole space:

Theorem 2.70. *Assume that $\text{Sin}(x)$ fixes the origin. Then for λ sufficiently large, the periodic points of $\tilde{f}(x) = \lambda \text{Sin}(x)$ are dense in \mathbb{R}^d (and all repelling). Furthermore, \tilde{f} has the blowing-up property everywhere in \mathbb{R}^d , that is*

$$\bigcup_{k=0}^{\infty} \tilde{f}^k(U) = \mathbb{R}^d$$

for any non-empty open set $U \subset \mathbb{R}^d$.

Remark 2.71. With the definition of the Julia set of non-uniformly quasiregular maps suggested by Bergweiler and Nicks [BN14, Definition 1.1], this means that the Julia set of \tilde{f} is the whole space.

2.9 Estimates for $\lambda \text{Sin}(x)$

We are going to need the following estimates for the derivatives and the Jacobian of the inverse branches, see [BE11, page 168]:

$$\|f(x)\| = \lambda \exp(|x_d| - 1), \quad x \in \mathbb{R}^d, |x_d| \geq 1. \quad (2.5)$$

For all $r \in \mathbb{R}$, we have

$$\|D\Lambda^r(y)\| \leq \frac{1}{\beta \|y\|}, \quad y \in \mathbb{R}^d, \|y\| \geq \lambda \quad \text{a.e.}, \quad (2.6)$$

where β is from Remark 2.63.

Furthermore, there exist constants $c_1, c_2, c_3 > 0$ such that

$$\ell(D(\Lambda^r(y))) \geq \frac{c_1}{\|y\|}, \quad y \in \mathbb{R}^d, \|y\| \geq \lambda \quad \text{a.e.} \quad (2.7)$$

and thus

$$\frac{c_2}{\|y\|^d} \leq J_{\Lambda^r(y)} \leq \frac{c_3}{\|y\|^d}, \quad y \in \mathbb{R}^d, \|y\| \geq \lambda \quad \text{a.e.} \quad (2.8)$$

2.10 Miscellaneous notations and results

We are going to need the following version of the Besicovitch Covering Lemma, see [Guz81, Theorem 3.2.1]:

Theorem 2.72. *Let $M \subset \mathbb{R}^d$ be bounded and $r : M \rightarrow]0, \infty[$ be a positive function. Then there exists an at most countable subset A of M satisfying*

$$M \subset \bigcup_{x \in A} B(x, r(x))$$

such that no point in \mathbb{R}^d is contained in more than 4^{2d} of the balls $B(x, r(x))$ with $x \in A$.

Remark 2.73. The upper bound of balls covering any point $x \in \mathbb{R}^d$ does neither depend on x nor the radius function r .

Infinite products

We will need the notion of infinite products, see [FL80, Chapter VII.2].

Definition 2.74. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{C} . We call

$$\prod_{n=1}^{\infty} a_n$$

convergent if there exists $N \in \mathbb{N}$ such that the limit $\lim_{n \rightarrow \infty} \prod_{k=N}^n a_k$ exists and is non-zero.

Remark 2.75. Let $\prod_{n=1}^{\infty} a_n$ be convergent. Then there exists $N \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq N$. In fact, $\lim_{n \rightarrow \infty} a_n = 1$.

The following result reduces the convergence of an infinite product to the convergence of an infinite sum, see [FL80, pages 173 and 174]:

Theorem 2.76. *Let $(b_n)_{n \in \mathbb{N}}$ be a real sequence. Then*

$$\prod_{n=1}^{\infty} (1 + |b_n|) \text{ converges} \iff \sum_{n=1}^{\infty} |b_n| \text{ converges.}$$

In particular, if

$$\sum_{n=1}^{\infty} |b_n|$$

converges, then

$$\prod_{n=1}^{\infty} (1 + b_n)$$

converges as well.

Spherical coordinates

We are going to use the following spherical coordinates, see [AE01, page 206]

$$\begin{aligned}x_1 &= r \cos \theta_1 \sin \theta_2 \dots \sin \theta_{d-1} \\x_2 &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{d-1} \\x_3 &= r \cos \theta_2 \sin \theta_3 \dots \sin \theta_{d-1} \\&\vdots \\x_{d-1} &= r \cos \theta_{d-2} \sin \theta_{d-1} \\x_d &= r \cos \theta_{d-1},\end{aligned}$$

where $\theta_j \in]0, \pi[$ for $2 \leq j \leq d-2$ and $0 < \theta_1 < 2\pi$. The volume element is given by

$$dV = r^{n-1} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{d-2} \theta_{d-1}.$$

3 The Escaping Set of $\lambda \text{Sin}(x)$

The results about $\lambda \text{Sin}(x)$ in the previous chapter rely on the fact that λ is chosen large enough to make the map $f(x) = \lambda \text{Sin}(x)$ locally expanding. In the first part of this chapter, we prove an analogue of McMullen's result [McM87] for the sine family (see Theorem 2.21) without these restrictions on λ .

Theorem 1. *Let $\lambda > 0$. Then the measure of the escaping set of $\lambda \text{Sin}(x)$ is positive, i.e.*

$$\text{meas}(I(\lambda \text{Sin})) > 0.$$

Note that $\lambda \text{Sin}(x)$ does not need to be locally expanding. Theorem 1 holds for the fast escaping set as well:

Theorem 1'. *Let $\lambda > 0$. Then the measure of the fast escaping set of $\lambda \text{Sin}(x)$ is positive.*

In the second part of this chapter, we generalise Theorem 1 and obtain an analogue of Schubert's result [Sch08], see Theorem 2.25.

Theorem 2. *Let $r \in \mathbb{R}$, $\lambda > 0$. Then the measure of the non-escaping set of $\lambda \text{Sin}(x)$ is finite in the tract $T(r)$, that is*

$$\text{meas}(I^c(\lambda \text{Sin}) \cap T(r)) < \infty,$$

where

$$I^c(g) := \mathbb{R}^d \setminus I(g)$$

denotes the non-escaping set of a quasiregular map g .

This holds for the fast escaping set as well:

Theorem 2'. *Let $r \in \mathbb{R}$, $\lambda > 0$. Then*

$$\text{meas}(A^c(\lambda \text{Sin}) \cap T(r)) < \infty.$$

We will prove Theorem 1 by adapting some of the techniques used by Aspenberg and Bergweiler in [AB12]. Generalising McMullen's Theorem, in this paper the authors show that the escaping set of a function g in the Eremenko-Lyubich class \mathcal{B} has positive measure if g satisfies certain growth restrictions. At some point in the proof they perform

a logarithmic change of variables and show that the measure of the escaping set of the logarithmic transform has positive measure. By definition, our map f shares some properties with a logarithmic transform, allowing us to adapt the main strategy at that point of their proof.

3.1 Notations

For this chapter, we put again $f(x) = \lambda \text{Sin}(x)$, $\lambda > 0$.

Consider the set

$$L := \left\{ x \in \mathbb{R}^d : |f_d(x)| \geq \exp\left(\frac{1}{2}|x_d|\right) \right\}$$

of all points in \mathbb{R}^d , where the modulus of the d th component of f is “large” and put

$$\mathcal{T} := \{x \in L : f^n(x) \in L \text{ for all } n \in \mathbb{N}\}.$$

The set \mathcal{T} consists of all points which stay in L under iteration. Hence, for $x \in \mathcal{T}$, $n \in \mathbb{N}$ we have

$$|f_d^n(x)| \geq E_{\frac{1}{2}}^n(|x_d|), \quad (3.1)$$

where $E_{\frac{1}{2}}$ is the map

$$E_{\frac{1}{2}} : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto \exp\left(\frac{1}{2}t\right),$$

$E_{\frac{1}{2}}^n$ denotes the n th iterate of $E_{\frac{1}{2}}$ and

$$f_d^n(x) = (f^n(x))_d$$

denotes the d th component of the n th iterate of f at x . This implies $\mathcal{T} \subset I(f)$. For $n \geq 0$ we denote by

$$\mathcal{T}_n := \{x \in L : f^k(x) \in L \text{ for } 0 \leq k \leq n\}$$

the set of points which stay in L for n iterations. Then $L = \mathcal{T}_0$ and we put

$$S := \mathbb{R}^d \setminus L$$

for the complement of L in \mathbb{R}^d . So S is the set where $|f_d|$ is “small”.

For $x \in \mathbb{R}^d$ we finally denote by

$$Q(x) := \left\{ y \in \mathbb{R}^d : |y_j - x_j| \leq \frac{|x_d|}{2} \right\}$$

the axis-aligned cube with centre x and side length $|x_d|$ and by

$$p: \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}, \quad (x_1, \dots, x_d) \mapsto (x_1, \dots, x_{d-1})$$

the projection from \mathbb{R}^d onto \mathbb{R}^{d-1} .

3.2 Proof of Theorem 1

To give a better understanding of what we are going to prove in detail, we provide a short idea of the main strategy used in the proof. Consider a cube $Q(x)$ with x_d large. We want to estimate the percentage of points in $Q(x)$ which stay under iteration $n - 1$ times in L and get mapped into S in the n th step. If the percentage of lost points decreases fast enough as n gets larger, we will end up with enough points which stay in L forever and thus are in the escaping set of f .

Formally, take $w \in \mathbb{R}^d$ with w_d large. We are going to construct a (rapidly) decreasing sequence $(\Delta_n(w_d))_n$ satisfying

$$\text{dens}(\mathcal{T}_{n-1} \setminus \mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)) \leq \Delta_n(w_d) \quad (3.2)$$

for all $n \in \mathbb{N}$ as well as

$$\text{dens}(\mathcal{T}, L \cap Q(w)) \geq \prod_{k=1}^{\infty} (1 - \Delta_k(w_d)) > 0. \quad (3.3)$$

Then $\text{meas}(\mathcal{T}) > 0$ and thus $\text{meas} I(f) > 0$.

Following the strategy outlined above, we need an estimate of $\text{dens}(S, Q(x))$. To achieve this, we estimate the density of S in the initial tract at a certain level $a \geq 1$. This estimate will depend on the Lipschitz constant of the angular map h . First we prove the following lemma about the density of a spherical cap in the upper hemisphere.

Lemma 3.1. *Let $b \in [0, 1]$. Then*

$$\text{dens}_{\mathcal{H}^{d-1}}(\mathbb{S}_b^c, \mathbb{S}_0) \leq C_1 \cdot b,$$

where C_1 is a positive constant,

$$\mathbb{S}_b := \{x \in \mathbb{R}^d : \|x\| = 1, x_d \geq b\}$$

and

$$\mathbb{S}_b^c := \mathbb{S}_0 \setminus \mathbb{S}_b.$$

Proof. To compute the density, we use the spherical coordinates stated in Section 2.10. Then

$$\begin{aligned} \text{area}(\mathbb{S}_0) &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} \dots \int_0^{\frac{\pi}{2}} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{d-2} \theta_{d-1} d\theta_{d-1} \dots d\theta_1 \\ &= 2\pi \int_0^{\frac{\pi}{2}} \sin^{d-2} \theta_{d-1} d\theta_{d-1} \int_0^{\pi} \dots \int_0^{\pi} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{d-3} \theta_{d-2} d\theta_{d-2} \dots d\theta_2 \end{aligned}$$

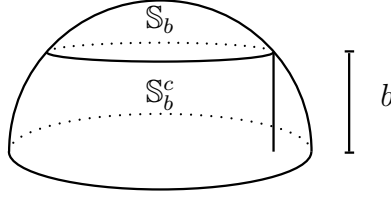


Figure 3.1: Spherical caps.

and

$$\text{area}(\mathbb{S}_b^c) = 2\pi \int_{\arccos b}^{\frac{\pi}{2}} \sin^{d-2} \theta_{d-1} \, d\theta_{d-1} \int_0^{\pi} \dots \int_0^{\pi} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{d-3} \theta_{d-2} \, d\theta_{d-2} \dots \, d\theta_2.$$

By putting

$$\frac{2}{C_1} := \int_0^{\frac{\pi}{2}} \sin^{d-2} \theta_{d-1} \, d\theta_{d-1},$$

we thus obtain

$$\begin{aligned} \frac{\text{area}(\mathbb{S}_b^c)}{\text{area}(\mathbb{S}_0)} &= \frac{\int_{\arccos b}^{\frac{\pi}{2}} \sin^{d-2} \theta_{d-1} \, d\theta_{d-1}}{\int_0^{\frac{\pi}{2}} \sin^{d-2} \theta_{d-1} \, d\theta_{d-1}} \\ &\leq \frac{C_1}{2} \cdot \int_{\arccos b}^{\frac{\pi}{2}} 1 \, d\theta_{d-1} \\ &= \frac{C_1}{2} \cdot \left(\frac{\pi}{2} - \arccos b \right) \\ &= \frac{C_1}{2} \cdot \arcsin b \\ &\leq \frac{C_1}{2} \cdot 2b. \end{aligned}$$

Remark 2.4 then yields

$$\text{dens}_{\mathcal{H}^{d-1}}(\mathbb{S}_b^c, \mathbb{S}_0) \leq C_1 \cdot b. \quad \square$$

Remark 3.2. For $d = 3$ we actually have

$$\text{dens}_{\mathcal{H}^{d-1}}(\mathbb{S}_b^c, \mathbb{S}_0) = b.$$

Lemma 3.3. *Let $a \geq 1$. Then*

$$\text{dens}_{\mathcal{H}^{d-1}}(S, [-1, 1]^{d-1} \times \{a\}) \leq C_1 \omega \frac{L_h^{d-1}}{2^d} \tilde{\delta}(a),$$

where

$$\tilde{\delta}(t) := \frac{1}{\lambda} \exp\left(-\frac{t}{2} + 1\right)$$

and $\omega = \text{area}(\mathbb{S})$.

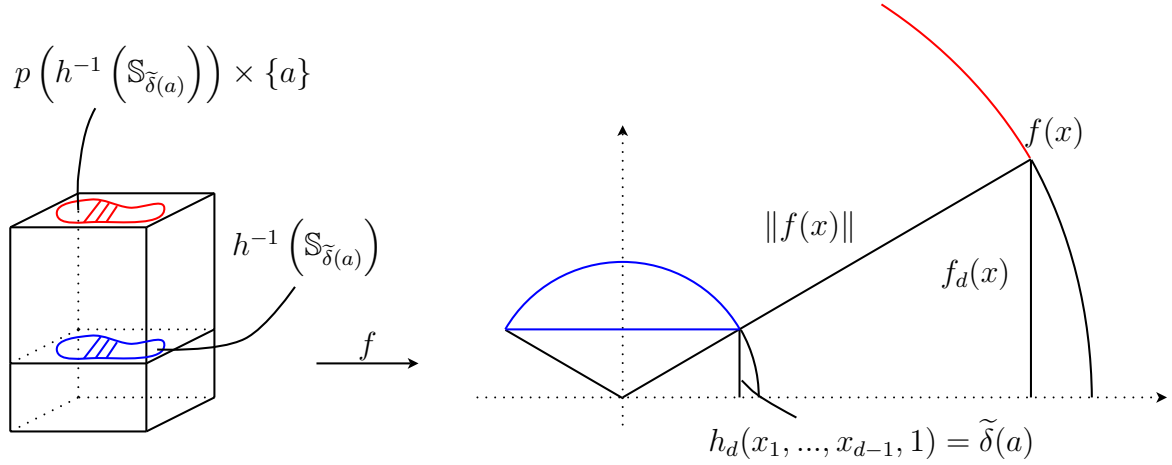


Figure 3.2: Illustrating the proof of Lemma 3.3.

Proof. We first decompose f into an angular map and a scaling map: for $a \geq 1$ and $x = (x_1, \dots, x_{d-1}, a) \in [-1, 1]^{d-1} \times \{a\}$ we have $\|f(x)\| = \lambda \exp(a - 1)$. Thus, we obtain

$$\begin{aligned} f_d(x) &= \|f(x)\| \cdot h_d(x_1, \dots, x_{d-1}, 1) \\ &= \lambda \exp(a - 1) \cdot h_d(x_1, \dots, x_{d-1}, 1). \end{aligned}$$

Points in S have a small d th component after applying f . In particular,

$$\begin{aligned} x \in S &\iff f_d(x) = \lambda \exp(a - 1) \cdot h_d(x_1, \dots, x_{d-1}, 1) < \exp\left(\frac{1}{2}a\right) \\ &\iff h_d(x_1, \dots, x_{d-1}, 1) < \frac{1}{\lambda} \exp\left(-\frac{a}{2} + 1\right) = \tilde{\delta}(a). \end{aligned}$$

This means

$$x \in S \iff (x_1, \dots, x_{d-1}, 1) \in h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right)$$

and therefore,

$$\begin{aligned} x \in L &\iff (x_1, \dots, x_{d-1}, 1) \in h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}\right) \\ &\iff h_d(x_1, \dots, x_{d-1}, 1) \geq \tilde{\delta}(a). \end{aligned}$$

This implies that

$$T(r_0) \cap S \cap (\mathbb{R}^{d-1} \times \{a\}) = p\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right)\right) \times \{a\}$$

and

$$T(r_0) \cap L \cap (\mathbb{R}^{d-1} \times \{a\}) = p\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}\right)\right) \times \{a\}.$$

Thus,

$$\begin{aligned} \text{dens}_{\mathcal{H}^{d-1}}(S, [-1, 1]^{d-1} \times \{a\}) &= \text{dens}_{\mathcal{H}^{d-1}}\left(p\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right)\right) \times \{a\}, [-1, 1]^{d-1} \times \{a\}\right) \\ &= \text{dens}_{\mathcal{H}^{d-1}}\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right), [-1, 1]^{d-1} \times \{1\}\right). \end{aligned}$$

Since h is bi-Lipschitz with Lipschitz constant L_h , Theorem 2.5 yields

$$\begin{aligned} \text{dens}_{\mathcal{H}^{d-1}}\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right), [-1, 1]^{d-1} \times \{1\}\right) &= \frac{\mathcal{H}^{d-1}\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right) \cap [-1, 1]^{d-1} \times \{1\}\right)}{\mathcal{H}^{d-1}\left([-1, 1]^{d-1} \times \{1\}\right)} \\ &= \mathcal{H}^{d-1}\left(h^{-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right)\right) \cdot \frac{1}{\mathcal{H}^{d-1}\left([-1, 1]^{d-1} \times \{1\}\right)} \\ &\leq L_h^{d-1} \mathcal{H}^{d-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right) \cdot \frac{1}{\mathcal{H}^{d-1}\left([-1, 1]^{d-1} \times \{1\}\right)}. \end{aligned}$$

Using Remark 2.4 and Lemma 3.1, we obtain

$$\begin{aligned} \frac{L_h^{d-1}}{\mathcal{H}^{d-1}\left([-1, 1]^{d-1} \times \{1\}\right)} \mathcal{H}^{d-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c\right) &= L_h^{d-1} \frac{\mathcal{H}^{d-1}\left(\mathbb{S}_0\right)}{\mathcal{H}^{d-1}\left([-1, 1]^{d-1} \times \{1\}\right)} \frac{\mathcal{H}^{d-1}\left(\mathbb{S}_{\tilde{\delta}(a)}^c \cap \mathbb{S}_0\right)}{\mathcal{H}^{d-1}\left(\mathbb{S}_0\right)} \\ &= L_h^{d-1} \frac{\text{area}\left(\mathbb{S}_0\right)}{2^{d-1}} \text{dens}_{\mathcal{H}^{d-1}}\left(\mathbb{S}_{\tilde{\delta}(a)}^c, \mathbb{S}_0\right) \\ &\leq L_h^{d-1} \frac{\text{area}\left(\mathbb{S}_0\right)}{2^{d-1}} \cdot C_1 \tilde{\delta}(a) \\ &= L_h^{d-1} \frac{\omega}{2^d} \cdot C_1 \tilde{\delta}(a), \end{aligned}$$

finishing the proof. □

Remark 3.4. Since $\tilde{\delta}$ is decreasing, we obtain

$$p\left(L \cap \left([-1, 1]^{d-1} \times \{a\}\right)\right) \times [a, \infty[\subset L.$$

Thus,

$$S \cap \left([-1, 1]^{d-1} \times [a, b]\right) \subset p\left(S \cap \left([-1, 1]^{d-1} \times \{a\}\right)\right) \times [a, \infty[.$$

Next, we estimate the density of S in a cube $Q(x)$:

Lemma 3.5. For x_0 large,

$$\text{dens}(S, Q(x)) \leq \delta(|x_d|)$$

holds for all $x \in \mathbb{R}^d$ with $|x_d| \geq x_0$, where

$$\delta(t) := \frac{3}{2^{d+1}\lambda} L_h^{d-1} C_1 \omega \exp\left(-\frac{t}{4} + 1\right).$$

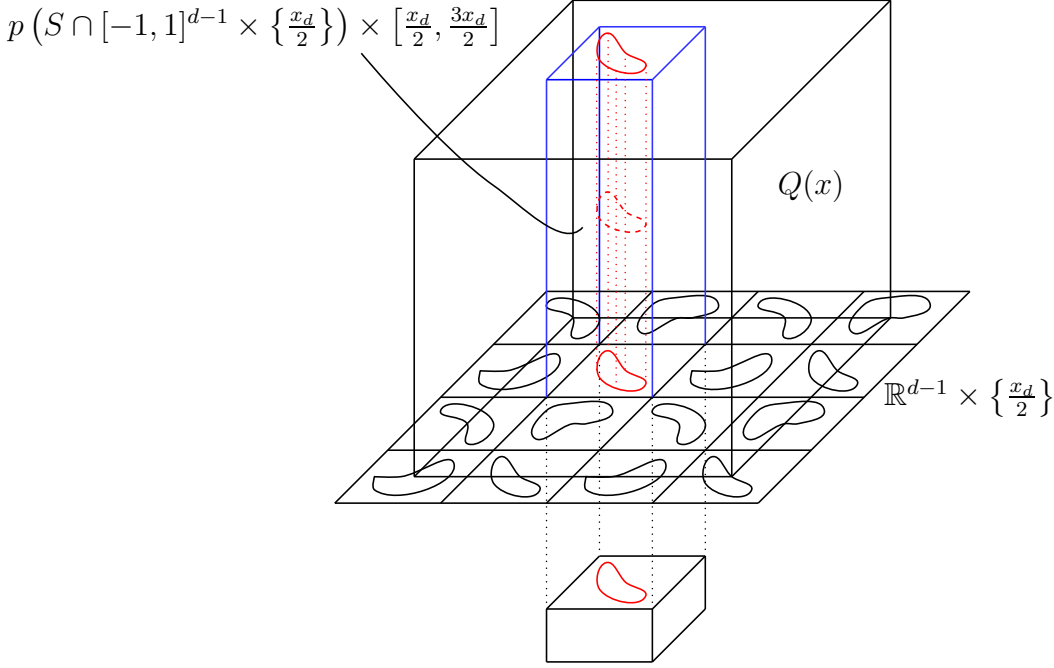


Figure 3.3: Illustrating the proof of Lemma 3.5. The red “pillar” is contained in L .

Proof. Let $r \in R$, $a \in \mathbb{R}$ with $|a| \geq 1$. Using the symmetry properties of f , we obtain

$$\text{dens}_{\mathcal{H}^{d-1}}(S, T(r) \cap (\mathbb{R}^{d-1} \times \{a\})) = \text{dens}_{\mathcal{H}^{d-1}}(S, [-1, 1]^{d-1} \times \{|a|\}).$$

Thus, without loss of generality, assume that $x \in T(r_0)$ with $x_d \geq 2$. To estimate the density of S in the cube $Q(x)$, we first estimate the density of S in the lower face of $Q(x)$. By periodicity, we can choose x_d large enough such that

$$\text{dens}_{\mathcal{H}^{d-1}}(S, p(Q(x)) \times \left\{\frac{x_d}{2}\right\}) \leq \frac{3}{2} \text{dens}_{\mathcal{H}^{d-1}}(S, T(r_0) \cap (\mathbb{R}^{d-1} \times \left\{\frac{x_d}{2}\right\})).$$

By Lemma 3.3, we obtain

$$\begin{aligned} \text{dens}_{\mathcal{H}^{d-1}}(S, p(Q(x)) \times \left\{\frac{x_d}{2}\right\}) &\leq \frac{3}{2} \text{dens}_{\mathcal{H}^{d-1}}(S, T(r_0) \cap (\mathbb{R}^{d-1} \times \left\{\frac{x_d}{2}\right\})) \\ &= \frac{3}{2} \text{dens}_{\mathcal{H}^{d-1}}(S, [-1, 1]^{d-1} \times \left\{\frac{x_d}{2}\right\}) \\ &\leq \frac{3}{2^{d+1}} L_h^{d-1} C_1 \omega \tilde{\delta}\left(\frac{x_d}{2}\right). \end{aligned}$$

By Remark 3.4,

$$S \cap \left([-1, 1]^{d-1} \times \left[\frac{x_d}{2}, \frac{3x_d}{2}\right]\right) \subset p\left(S \cap [-1, 1]^{d-1} \times \left\{\frac{x_d}{2}\right\}\right) \times \left[\frac{x_d}{2}, \frac{3x_d}{2}\right]$$

and thus by symmetry

$$S \cap Q(x) \subset p\left(S \cap \left(p(Q(x)) \times \left\{\frac{x_d}{2}\right\}\right)\right) \times \left[\frac{x_d}{2}, \frac{3x_d}{2}\right].$$

We finally obtain

$$\begin{aligned}
\text{dens}(S, Q(x)) &\leq \text{dens} \left(p \left(S \cap \left(p(Q(x)) \times \left\{ \frac{x_d}{2} \right\} \right) \right) \times \left[\frac{x_d}{2}, \frac{3x_d}{2} \right], Q(x) \right) \\
&= \text{dens}_{\mathcal{H}^{d-1}} \left(S, p(Q(x)) \times \left\{ \frac{x_d}{2} \right\} \right) \\
&\leq \frac{3}{2^{d+1}} L_h^{d-1} C_1 \omega \tilde{\delta} \left(\frac{x_d}{2} \right) \\
&= \delta(x_d).
\end{aligned}$$

□

We devote the remainder of this section to prove the following lemma:

Lemma 3.6. *For x_0 large and $w \in \mathbb{R}^d$ with $w_d = 2x_0$,*

$$\text{dens}(\mathcal{T}_n, Q(w)) \geq \frac{1}{2}$$

holds for all $n \in \mathbb{N}$.

Proof. We prove this lemma by induction. First note that by Lemma 3.5

$$\begin{aligned}
\text{dens}(S, Q(w)) &\leq \delta(w_d) \\
&\leq \frac{1}{3}
\end{aligned}$$

if x_0 and hence w_d is large enough. (The choice of $1/3$ will play a role in the end of the proof.) Since $\mathcal{T}_0 = L = \mathbb{R}^d \setminus S$, we obtain

$$\begin{aligned}
\text{dens}(\mathcal{T}_0, Q(w)) &= \text{dens}(L, Q(w)) \\
&= 1 - \text{dens}(S, Q(w)) \\
&\geq \frac{2}{3} \\
&\geq \frac{1}{2}.
\end{aligned}$$

For the remainder of the section, we now suppose that

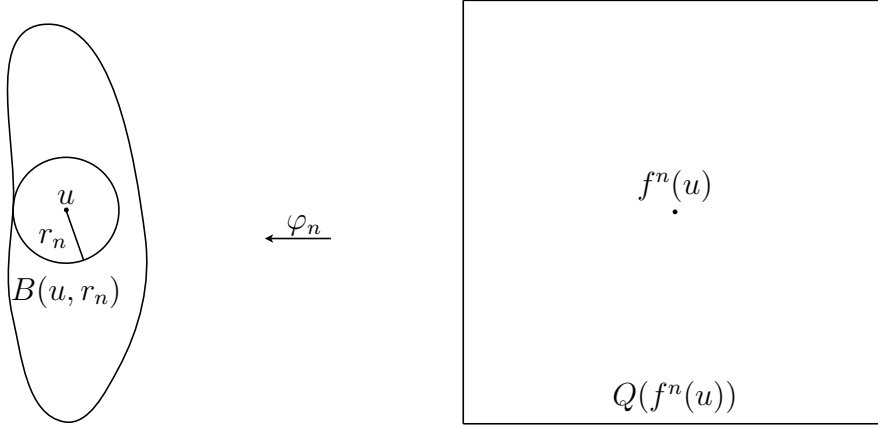
$$\text{dens}(\mathcal{T}_k, Q(w)) \geq \frac{1}{2} \tag{3.4}$$

holds for all $k \leq n-1$. Let $u \in \mathcal{T}_{n-1}$ with $|u_d| \geq x_0$. Put

$$v := f^n(u)$$

and denote by \underline{s} the external address of u . Note that the first n entries of \underline{s} are unique since the boundary points of tracts get mapped into $\mathbb{R}^{d-1} \times \{0\}$ and then stay there forever under iteration. Hence, these points are not in the set L . Put

$$\varphi_n := \Lambda^{s_0} \circ \dots \circ \Lambda^{s_{n-1}}.$$

Figure 3.4: Definition of $B(u, r_n)$.

We denote by

$$B(u, r_n(u)) = B(u, r_n)$$

the largest ball in $\varphi_n(Q(v))$ with centre u . Since φ_n is a homeomorphism, we have $r_n > 0$. We want to estimate the distortion of φ_n by estimating the distortion of the inverse branches which define φ_n .

Lemma 3.7. *Let x_0 be large, $u \in \mathcal{T}_{n-1}$ with $|u_d| \geq x_0$ and $0 \leq j \leq n-1$. Then*

$$\Lambda^{s_{n-j-1}}(Q(f^{n-j}(u))) \subset Q(f^{n-j-1}(u)).$$

Furthermore,

$$\operatorname{ess\,sup}_{y \in Q(v)} J_{\varphi_n}(y) \leq \prod_{j=0}^{n-1} \operatorname{ess\,sup}_{y \in Q(f^{n-j}(u))} J_{\Lambda^{s_{n-j-1}}}(y),$$

and

$$\operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \geq \prod_{j=0}^{n-1} \operatorname{ess\,inf}_{y \in Q(f^{n-j}(u))} \ell(D\Lambda^{s_{n-j-1}}(y)).$$

Proof. By the mean value inequality, we obtain

$$\begin{aligned} \operatorname{diam} \Lambda^{s_{n-1}}(Q(f^n(u))) &= \sup_{x_1, x_2 \in Q(f^n(u))} \|\Lambda^{s_{n-1}}(x_1) - \Lambda^{s_{n-1}}(x_2)\| \\ &\leq \sup_{x_1, x_2 \in Q(f^n(u))} \operatorname{ess\,sup}_{y \in [x_1, x_2]} \|D\Lambda^{s_{n-1}}(y)\| \|x_1 - x_2\| \\ &\leq \operatorname{ess\,sup}_{y \in Q(f^n(u))} \|D\Lambda^{s_{n-1}}(y)\| \operatorname{diam}(Q(f^n(u))), \end{aligned}$$

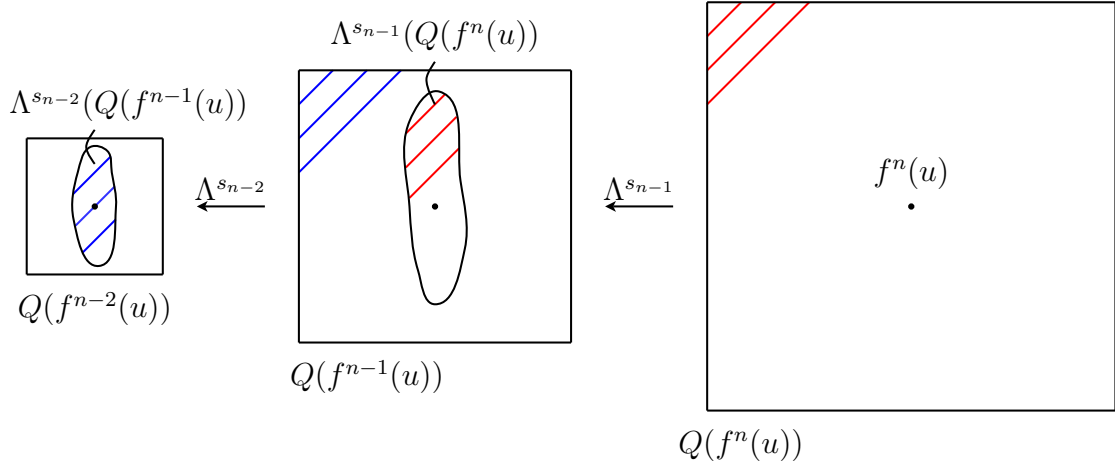


Figure 3.5: Illustrating Lemma 3.7.

where $[x_1, x_2]$ denotes the straight line segment connecting x_1 and x_2 . By using inequality (2.6), we obtain

$$\begin{aligned} \operatorname{ess\,sup}_{y \in Q(f^n(u))} \|D\Lambda^{s_{n-1}}(y)\| &\leq \operatorname{ess\,sup}_{y \in Q(f^n(u))} \frac{1}{\beta \|y\|} \\ &= \frac{1}{\beta \min_{y \in Q(f^n(u))} \|y\|}. \end{aligned}$$

Since $\|y\| \geq \frac{|v_d|}{2}$ for all $y \in Q(v) = Q(f^n(u))$, we end up with

$$\begin{aligned} \operatorname{diam} \Lambda^{s_{n-1}}(Q(f^n(u))) &\leq \operatorname{ess\,sup}_{y \in Q(f^n(u))} \|D\Lambda^{s_{n-1}}(y)\| \operatorname{diam}(Q(f^n(u))) \\ &\leq \frac{1}{\beta \min_{y \in Q(f^n(u))} \|y\|} \sqrt{d} |v_d| \\ &\leq \frac{1}{\beta} \frac{2}{|v_d|} \sqrt{d} |v_d| \\ &= \frac{2\sqrt{d}}{\beta}. \end{aligned}$$

This estimate does not depend on n , hence

$$\operatorname{diam} \Lambda^{s_{n-j-1}}(Q(f^{n-j}(u))) \leq \frac{2\sqrt{d}}{\beta} \quad (3.5)$$

for $0 \leq j \leq n-1$.

For $x_0 \geq \frac{8\sqrt{d}}{\beta}$, we obtain

$$\begin{aligned} |f_d^{n-j-1}(u)| &\geq E_{\frac{1}{2}}^{n-j-1}(x_0) \\ &\geq x_0 \\ &\geq \frac{8\sqrt{d}}{\beta} \\ &\geq 2 \operatorname{diam} \Lambda^{s_{n-j-1}}(Q(f^{n-j}(u))) \end{aligned}$$

and thus

$$\Lambda^{s_{n-j-1}}(Q(f^{n-j}(u))) \subset Q(f^{n-j-1}(u)) \quad (3.6)$$

for $0 \leq j \leq n-1$. Next, we estimate the norm of the derivative of φ_n . By the chain rule, we have

$$\begin{aligned} D\varphi_n(y) &= D(\Lambda^{s_0} \circ \dots \circ \Lambda^{s_{n-1}})(y) \\ &= D\Lambda^{s_0}(\Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(y)) \cdot \dots \cdot D\Lambda^{s_{n-1}}(y) \end{aligned}$$

almost everywhere. To estimate $\ell(D\varphi_n(y))$, we use estimate (2.3). Then (3.5) and (3.6) yield

$$\begin{aligned} \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) &= \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\Lambda^{s_0}(\Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(y)) \cdot \dots \cdot D\Lambda^{s_{n-1}}(y)) \\ &\geq \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\Lambda^{s_0}(\Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(y))) \cdot \dots \cdot \ell(D\Lambda^{s_{n-1}}(y)) \\ &\geq \operatorname{ess\,inf}_{y \in \Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(Q(v))} \ell(D\Lambda^{s_0}(y)) \cdot \dots \cdot \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\Lambda^{s_{n-1}}(y)) \\ &\geq \operatorname{ess\,inf}_{y \in Q(f(u))} \ell(D\Lambda^{s_0}(y)) \cdot \dots \cdot \operatorname{ess\,inf}_{y \in Q(f^n(u))} \ell(D\Lambda^{s_{n-1}}(y)) \\ &= \prod_{j=0}^{n-1} \operatorname{ess\,inf}_{y \in Q(f^{n-j}(u))} \ell(D\Lambda^{s_{n-j-1}}(y)). \end{aligned}$$

We can estimate the Jacobian in a similar way:

$$J_{\varphi_n}(y) = J_{\Lambda^{s_0}}(\Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(y)) \cdot \dots \cdot J_{\Lambda^{s_{n-1}}}(y)$$

and by applying (3.5) and (3.6) we have

$$\begin{aligned} \operatorname{ess\,sup}_{y \in Q(v)} J_{\varphi_n}(y) &= \operatorname{ess\,sup}_{y \in Q(v)} J_{\Lambda^{s_0}}(\Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(y)) \cdot \dots \cdot J_{\Lambda^{s_{n-1}}}(y) \\ &\leq \operatorname{ess\,sup}_{y \in \Lambda^{s_1} \circ \dots \circ \Lambda^{s_{n-1}}(Q(v))} J_{\Lambda^{s_0}}(y) \cdot \dots \cdot \operatorname{ess\,sup}_{y \in Q(v)} J_{\Lambda^{s_{n-1}}}(y) \\ &\leq \operatorname{ess\,sup}_{y \in Q(f(u))} J_{\Lambda^{s_0}}(y) \cdot \dots \cdot \operatorname{ess\,sup}_{y \in Q(f^n(u))} J_{\Lambda^{s_{n-1}}}(y) \\ &= \prod_{j=0}^{n-1} \operatorname{ess\,sup}_{y \in Q(f^{n-j}(u))} J_{\Lambda^{s_{n-j-1}}}(y). \end{aligned}$$

□

Now we pull the set S back to obtain an estimate of the percentage of points in $B(u, r_n)$ which get mapped into S in the n th step of iteration.

Lemma 3.8. *Let x_0 be large, $u \in \mathcal{T}_{n-1}$ with $|u_d| \geq x_0$. Then*

$$\text{dens}(f^{-n}(S), B(u, r_n)) \leq \tilde{\eta} \delta(x_{0,n}) K^n,$$

where

$$x_{0,n} := E_{\frac{1}{2}}^n(x_0),$$

$$K := \left(\frac{c_3}{c_1^d} (1 + 2\sqrt{d}) \right)^d,$$

and

$$\tilde{\eta} := \frac{2^d}{\nu}.$$

Again, $\nu = \text{meas}(B(0, 1))$ and c_1, c_2 are from inequalities (2.6) and (2.7), respectively.

Remark 3.9. This estimate does not depend on u .

Proof. By definition of the density, we obtain

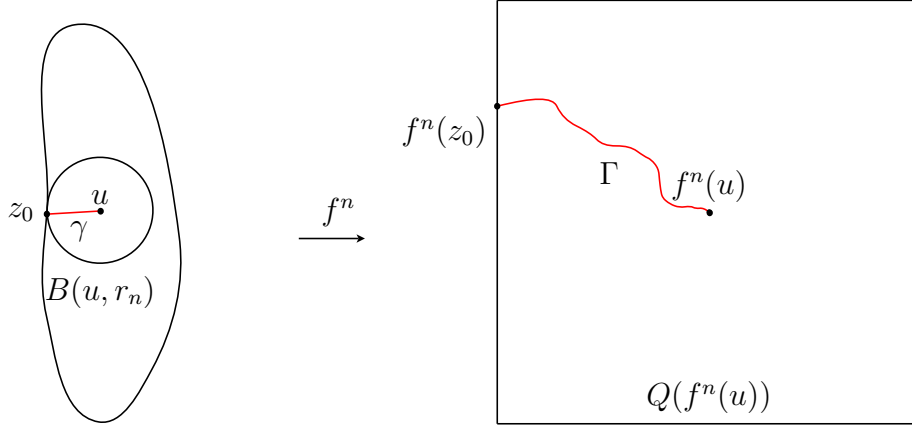
$$\begin{aligned} \text{dens}(\varphi_n(Q(v) \cap S), B(u, r_n)) &= \frac{\text{meas}(\varphi_n(Q(v) \cap S) \cap B(u, r_n))}{\text{meas}(B(u, r_n))} \\ &\leq \frac{\text{meas}(\varphi_n(Q(v) \cap S))}{\text{meas}(B(u, r_n))} \\ &= \frac{\text{meas}(\varphi_n(Q(v) \cap S))}{\nu \cdot r_n^d}. \end{aligned} \tag{3.7}$$

The Jacobian determines the distortion, i.e.

$$\begin{aligned} \text{meas}(\varphi_n(Q(v) \cap S)) &= \int_{Q(v) \cap S} |\det D\varphi_n(x)| dx \\ &\leq \text{ess sup}_{y \in Q(v) \cap S} J_{\varphi_n}(y) \text{meas}(Q(v) \cap S). \end{aligned} \tag{3.8}$$

Next, we want to find a lower bound for the radius r_n . Let $\gamma: [0, 1] \rightarrow \mathbb{R}^d$ be the straight line segment connecting u to a point $z_0 \in \partial B(u, r_n) \cap \partial \varphi_n(Q(v))$. Then $\Gamma := f^n \circ \gamma$ is an absolutely continuous Jordan arc in $Q(v)$ connecting v and $f^n(z_0) \in \partial Q(v)$. Denoting by $\mathcal{L}(\gamma)$ the length of γ , this yields

$$\begin{aligned} r_n &= \mathcal{L}(\gamma) \\ &= \mathcal{L}(\varphi_n \circ \Gamma) \\ &= \int_0^1 \|(\varphi_n \circ \Gamma)'(t)\| dt \\ &= \int_0^1 \|D\varphi_n(\Gamma(t))\Gamma'(t)\| dt \end{aligned}$$

Figure 3.6: Estimate for r_n .

and by standard estimates

$$\begin{aligned}
\int_0^1 \|D\varphi_n(\Gamma(t))\Gamma'(t)\| &\geq \int_0^1 \ell(D\varphi_n(\Gamma(t)))\|\Gamma'(t)\| dt \\
&\geq \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \int_0^1 \|\Gamma'(t)\| dt \\
&= \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \mathcal{L}(\Gamma) \\
&\geq \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \frac{|v_d|}{2}.
\end{aligned}$$

Thus,

$$r_n \geq \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \frac{|v_d|}{2}. \quad (3.9)$$

Using inequalities (3.8) and (3.9), we obtain

$$\begin{aligned}
\frac{\operatorname{meas}(\varphi_n(Q(v) \cap S))}{\nu \cdot r_n^d} &\leq \frac{\operatorname{meas}(Q(v) \cap S) \cdot \operatorname{ess\,sup}_{y \in Q(v)} J_{\varphi_n}(y)}{\nu \cdot r_n^d} \\
&\leq \frac{\operatorname{meas}(Q(v) \cap S) \cdot \operatorname{ess\,sup}_{y \in Q(v)} J_{\varphi_n}(y)}{\nu \left(\frac{|v_d|}{2} \operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \right)^d} \\
&= \frac{\operatorname{meas}(Q(v) \cap S) \cdot \operatorname{ess\,sup}_{y \in Q(v)} J_{\varphi_n}(y)}{2^{-d} \nu \operatorname{meas}(Q(v)) \left(\operatorname{ess\,inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \right)^d}.
\end{aligned}$$

By the definition of the density and by Lemma 3.7, we thus get

$$\begin{aligned} \frac{\text{meas}(\varphi_n(Q(v) \cap S))}{\nu \cdot r_n^d} &\leq \frac{2^d}{\nu} \text{dens}(S, Q(v)) \frac{\text{ess sup}_{y \in Q(v)} J_{\varphi_n}(y)}{\left(\text{ess inf}_{y \in Q(v)} \ell(D\varphi_n(y)) \right)^d} \\ &\leq \frac{2^d}{\nu} \text{dens}(S, Q(v)) \prod_{k=0}^{n-1} \frac{\text{ess sup}_{y \in Q(f^{n-j}(u))} J_{\Lambda^{s_{n-j-1}}}(y)}{\left(\text{ess inf}_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) \right)^d}. \end{aligned} \quad (3.10)$$

Furthermore, inequality (2.8) yields for $0 \leq j \leq n-1$ that

$$\begin{aligned} \text{ess sup}_{y \in Q(f^{n-j}(u))} J_{\Lambda^{s_{n-j-1}}}(y) &\leq \text{ess sup}_{y \in Q(f^{n-j}(u))} \frac{c_3}{\|y\|^d} \\ &= \frac{c_3}{\left(\min_{y \in Q(f^{n-j}(u))} \|y\| \right)^d}. \end{aligned}$$

By (2.7), we have

$$\begin{aligned} \text{ess inf}_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) &\geq \text{ess inf}_{y \in Q(f^{n-j}(u))} \frac{c_1}{\|y\|} \\ &= \frac{c_1}{\max_{y \in Q(f^{n-j}(u))} \|y\|}. \end{aligned}$$

Since

$$\begin{aligned} \max_{y \in Q(f^{n-j}(u))} \|y\| &\leq \min_{y \in Q(f^{n-j}(u))} \|y\| + \text{diam}(Q(f^{n-j}(u))) \\ &= \min_{y \in Q(f^{n-j}(u))} \|y\| + \sqrt{d} |f_d^{n-j}(u)| \end{aligned}$$

and $\min_{y \in Q(f^{n-j}(u))} \|y\| \geq \frac{|f_d^{n-j}(u)|}{2}$, we obtain

$$\begin{aligned} \frac{\text{ess sup}_{y \in Q(f^{n-j}(u))} J_{\Lambda^{s_{n-j-1}}}(y)}{\left(\text{ess inf}_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) \right)^d} &\leq \frac{c_3}{c_1^d} \left(\frac{\max_{y \in Q(f^{n-j}(u))} \|y\|}{\min_{y \in Q(f^{n-j}(u))} \|y\|} \right)^d \\ &\leq \frac{c_3}{c_1^d} \left(\frac{\min_{y \in Q(f^{n-j}(u))} \|y\| + \sqrt{d} |f_d^{n-j}(u)|}{\min_{y \in Q(f^{n-j}(u))} \|y\|} \right)^d \\ &\leq \frac{c_3}{c_1^d} \left(1 + \frac{2\sqrt{d} |f_d^{n-j}(u)|}{|f_d^{n-j}(u)|} \right)^d \\ &= \frac{c_3}{c_1^d} (1 + 2\sqrt{d})^d. \end{aligned}$$

Inequalities (3.7) and (3.10) then yield

$$\begin{aligned} \text{dens}(\varphi_n(Q(v) \cap S), B(u, r_n)) &\leq \frac{\text{meas}(\varphi_n(Q(v) \cap S))}{\nu \cdot r_n^d} \\ &\leq \frac{2^d}{\nu} \text{dens}(S, Q(v)) \prod_{k=0}^{n-1} \frac{\text{ess sup}_{y \in Q(f^{n-j}(u))} J_{\Lambda^{s_{n-j-1}}}(y)}{\left(\text{ess inf}_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) \right)^d} \\ &\leq \frac{2^d}{\nu} \text{dens}(S, Q(v)) \left(\frac{c_3}{c_1^d} (1 + 2\sqrt{d})^d \right)^n. \end{aligned}$$

Applying Lemma 3.5, we thus obtain

$$\begin{aligned} \text{dens}(\varphi_n(Q(v) \cap S), B(u, r_n)) &\leq \frac{2^d}{\nu} \text{dens}(S, Q(v)) \left(\frac{c_3}{c_1^d} (1 + 2\sqrt{d})^d \right)^n \\ &\leq \tilde{\eta} \delta(|v_d|) K^n \\ &\leq \tilde{\eta} \delta(x_{0,n}) K^n. \end{aligned}$$

Since $\varphi_n(Q(v))$ is the component of $f^{-n}(Q(v))$ that contains u , we have

$$f^{-n}(S) \cap B(u, r_n) = \varphi_n(Q(v) \cap S) \cap B(u, r_n)$$

and thus finally obtain

$$\text{dens}(f^{-n}(S), B(u, r_n)) \leq \tilde{\eta} \delta(x_{0,n}) K^n,$$

finishing the proof. □

Following the underlying heuristic principle, we prove the following lemma which gives an upper bound of the percentage of the points in $\mathcal{T}_{n-1} \cap Q(w)$ that get lost when we apply f once more.

Lemma 3.10. *Let $w \in \mathbb{R}^d$ with $w_d = 2x_0$ and suppose that $\text{dens}(\mathcal{T}_{n-1}, Q(w)) \geq \frac{1}{2}$. Then*

$$\text{dens}(\mathcal{T}_{n-1} \setminus \mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)) \leq \eta \delta(x_{0,n}) K^n,$$

where $\eta := 4^{2d+1} \tilde{\eta}$.

Remark 3.11. Denoting $\Delta_n(x_0) := \eta \delta(x_{0,n}) K^n$, this is essentially inequality (3.2) in the idea of the proof.

Proof of Lemma 3.10. Let $u \in \mathcal{T}_{n-1} \cap Q(w)$. Then $u_d \geq x_0$ and thus Lemma 3.8 holds. Using the Besicovitch Covering Lemma (Theorem 2.72), we can find an at most countable subset $A \subset \mathcal{T}_{n-1} \cap Q(w)$, such that the balls $B(u, r_n(u))$, $u \in A$, cover the set $\mathcal{T}_{n-1} \cap Q(w)$

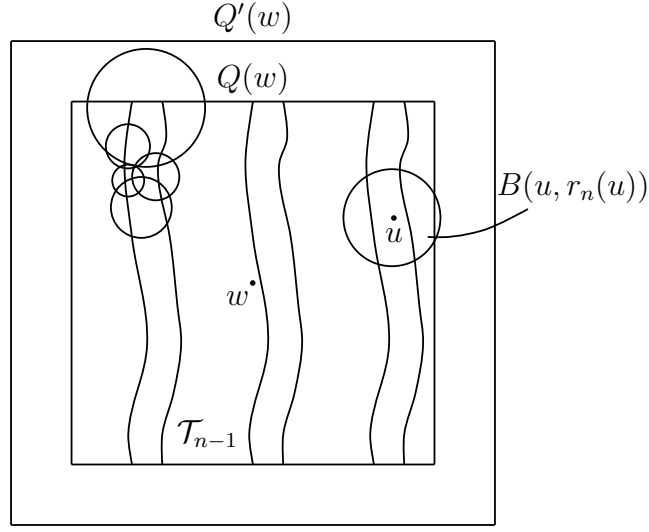


Figure 3.7: Applying the Besicovitch Covering Lemma.

with no point being covered by more than 4^{2d} of these balls. Using this covering property and Lemma 3.8, we obtain

$$\begin{aligned} \text{meas}(f^{-n}(S) \cap \mathcal{T}_{n-1} \cap Q(w)) &\leq \text{meas} \left(f^{-n}(S) \cap \bigcup_{u \in A} B(u, r_n(u)) \right) \\ &\leq \sum_{u \in A} \text{meas}(f^{-n}(S) \cap B(u, r_n(u))) \\ &\leq \tilde{\eta} \delta(x_{0,n}) K^n \sum_{u \in A} \text{meas} B(u, r_n(u)). \end{aligned}$$

Since all tracts $T(r)$ have width 2 and the balls $B(u, r_n)$ are obtained by pulling back into respective tracts, the radii r_n are bounded from above, in fact $r_n(u) \leq 1$. By putting

$$Q'(w) := \left\{ x \in \mathbb{R}^d : |x_j - w_j| \leq \frac{1}{2}|w_d| + 1 \right\},$$

we hence obtain

$$B(u, r_n(u)) \subset Q'(w).$$

For x_0 large we find that $\text{meas}(Q'(w)) \leq 2 \text{meas}(Q(w))$. By the choice of the set A , we know that each point in $Q'(w)$ is covered at most 4^{2d} times. Hence,

$$\begin{aligned} \sum_{u \in A} \text{meas} B(u, r_n(u)) &\leq 4^{2d} \text{meas}(Q'(w)) \\ &\leq 2 \cdot 4^{2d} \text{meas}(Q(w)). \end{aligned}$$

Assumption (3.4) implies that $\text{meas}(Q(w)) \leq 2 \text{meas}(\mathcal{T}_{n-1} \cap Q(w))$ and thus

$$\begin{aligned} \text{meas}(f^{-n}(S) \cap \mathcal{T}_{n-1} \cap Q(w)) &\leq \tilde{\eta} \delta(x_{0,n}) K^n 4^{2d+1} \text{meas}(\mathcal{T}_{n-1} \cap Q(w)) \\ &= \eta \delta(x_{0,n}) K^n \text{meas}(\mathcal{T}_{n-1} \cap Q(w)). \end{aligned}$$

The definition of the density then yields

$$\text{dens}(f^{-n}(S), \mathcal{T}_{n-1} \cap Q(w)) \leq \eta \delta(x_{0,n}) K^n. \quad (3.11)$$

The set $\mathcal{T}_{n-1} \setminus \mathcal{T}_n$ contains all points which stay $n-1$ times in L under iteration and are mapped into S in the n th step. Thus, we find that

$$f^{-n}(S) \cap \mathcal{T}_{n-1} = \mathcal{T}_{n-1} \setminus \mathcal{T}_n.$$

Hence,

$$\begin{aligned} \text{dens}(f^{-n}(S), \mathcal{T}_{n-1} \cap Q(w)) &= \frac{\text{meas}(f^{-n}(S) \cap \mathcal{T}_{n-1} \cap Q(w))}{\text{meas}(\mathcal{T}_{n-1} \cap Q(w))} \\ &= \frac{\text{meas}(\mathcal{T}_{n-1} \setminus \mathcal{T}_n \cap \mathcal{T}_{n-1} \cap Q(w))}{\text{meas}(\mathcal{T}_{n-1} \cap Q(w))} \\ &= \text{dens}(\mathcal{T}_{n-1} \setminus \mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)). \end{aligned}$$

In conclusion, we obtain

$$\text{dens}(\mathcal{T}_{n-1} \setminus \mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)) \leq \eta \delta(x_{0,n}) K^n. \quad \square$$

Remark 3.12. We shall apply Lemma 3.7, Lemma 3.8 and Lemma 3.10 for $l \in \mathbb{N}$ with $l \leq n$ as well.

Lemma 3.13. *Let x_0 be large and $w \in \mathbb{R}^d$ with $w_d = 2x_0$. Furthermore, suppose that $\text{dens}(\mathcal{T}_k, Q(w)) \geq \frac{1}{2}$ for $0 \leq k \leq n-1$. Then*

$$\text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) \geq \prod_{k=1}^n (1 - \eta \delta(x_{0,n}) K^k)$$

holds for x_0 large.

Proof. Since $\mathcal{T}_n \subset \mathcal{T}_{n-1}$ and by using Lemma 3.10, we obtain

$$\begin{aligned} \text{dens}(\mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)) &= 1 - \text{dens}(\mathcal{T}_{n-1} \setminus \mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)) \\ &\geq 1 - \eta \delta(x_{0,n}) K^n. \end{aligned} \quad (3.12)$$

We find that

$$\begin{aligned} \text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) &= \frac{\text{meas}(\mathcal{T}_n \cap \mathcal{T}_0 \cap Q(w))}{\text{meas}(\mathcal{T}_0 \cap Q(w))} \\ &= \frac{\text{meas}(\mathcal{T}_n \cap \mathcal{T}_{n-1} \cap Q(w))}{\text{meas}(\mathcal{T}_{n-1} \cap Q(w))} \frac{\text{meas}(\mathcal{T}_{n-1} \cap Q(w))}{\text{meas}(\mathcal{T}_0 \cap Q(w))} \\ &= \text{dens}(\mathcal{T}_n, \mathcal{T}_{n-1} \cap Q(w)) \text{dens}(\mathcal{T}_{n-1}, \mathcal{T}_0 \cap Q(w)). \end{aligned}$$

Lemma 3.10 and Remark 3.12 then yield

$$\begin{aligned} \text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) &= \prod_{k=1}^n \text{dens}(\mathcal{T}_k, \mathcal{T}_{k-1} \cap Q(w)) \\ &\geq \prod_{k=1}^n (1 - \eta \delta(x_{0,n}) K^k). \end{aligned}$$

□

Having the heuristic idea in mind, we still need to show that the sequence $\Delta_n(x_0)$ decreases fast enough if x_0 is chosen large enough. Lemma 3.13 holds for all $n \in \mathbb{N}$. On the right hand side, this leads to an infinite product:

Lemma 3.14. *For large x_0 , the infinite product*

$$\prod_{k=1}^{\infty} (1 - \eta \delta(x_{0,k}) K^k)$$

converges,

$$\prod_{k=1}^{\infty} (1 - \eta \delta(x_{0,k}) K^k) \geq \frac{3}{4}$$

and

$$\eta \delta(x_{0,k}) K^k < 1$$

for all $k \in \mathbb{N}$.

Proof. By Theorem 2.76, the infinite product converges if the sum

$$\sum_{k=1}^{\infty} \delta(x_{0,k}) K^k = \sum_{k=1}^{\infty} \frac{3}{2^{d+1} \lambda} L_h^{d-1} C_1 \omega \exp\left(-\frac{1}{4} E_{\frac{1}{2}}^k(x_0) + 1\right) K^k$$

converges. Using $E_{\frac{1}{2}}(t) \geq 2t$ for sufficiently large t and by choosing x_0 accordingly, we

obtain

$$\begin{aligned} \frac{\exp\left(-\frac{1}{4}E_{\frac{1}{2}}^{n+1}(x_0) + 1\right) K^{n+1}}{\exp\left(-\frac{1}{4}E_{\frac{1}{2}}^n(x_0) + 1\right) K^n} &= \frac{\exp\left(-\frac{1}{4}E_{\frac{1}{2}}\left(E_{\frac{1}{2}}^n(x_0)\right) + 1\right) K}{\exp\left(-\frac{1}{4}E_{\frac{1}{2}}^n(x_0) + 1\right)} \\ &\leq \frac{\exp\left(-\frac{1}{2}E_{\frac{1}{2}}^n(x_0) + 1\right) K}{\exp\left(-\frac{1}{4}E_{\frac{1}{2}}^n(x_0) + 1\right)} \\ &= \exp\left(-\frac{1}{4}E_{\frac{1}{2}}^n(x_0)\right) K. \end{aligned}$$

Since $\exp\left(-\frac{1}{4}E_{\frac{1}{2}}^n(x_0)\right) K \rightarrow 0$ as $n \rightarrow \infty$, the sum $\sum_{k=1}^{\infty} \delta(x_{0,k}) K^k$ converges and thus the product $\prod_{k=1}^{\infty} (1 - \eta\delta(x_{0,k}) K^k)$ converges. This implies that there exists $k_0 \in \mathbb{N}$ such that

$$0 < 1 - \eta\delta(x_{0,k}) K^k \tag{3.13}$$

holds for all $k \geq k_0$. By choosing x_0 large, we obtain that (3.13) holds for all $k \in \mathbb{N}$. Again by choosing x_0 large, we obtain

$$\prod_{k=1}^{\infty} (1 - \eta\delta(x_{0,k}) K^k) \geq \frac{3}{4}. \tag{3.14}$$

□

Now we are finally able to finish the proof of Lemma 3.6. The induction hypothesis (3.4), i.e. $\text{dens}(\mathcal{T}_k, Q(w)) \geq \frac{1}{2}$ for all $k \leq n - 1$ still holds, thus the premises of Lemmas 3.13 and 3.14 are satisfied yielding

$$\begin{aligned} \text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) &\geq \prod_{k=1}^n (1 - \eta\delta(x_{0,k}) K^k) \\ &\geq \prod_{k=1}^{\infty} (1 - \eta\delta(x_{0,k}) K^k) \\ &\geq \frac{3}{4}. \end{aligned}$$

Since $\mathcal{T}_n \subset \mathcal{T}_0$, we obtain

$$\begin{aligned} \text{dens}(\mathcal{T}_{n-1}, Q(w)) &= \frac{\text{meas}(\mathcal{T}_n \cap Q(w))}{\text{meas} Q(w)} \\ &= \frac{\text{meas}(\mathcal{T}_n \cap \mathcal{T}_0 \cap Q(w))}{\text{meas}(\mathcal{T}_0 \cap Q(w))} \frac{\text{meas}(\mathcal{T}_0 \cap Q(w))}{\text{meas}(Q(w))} \\ &= \text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) \text{dens}(\mathcal{T}_0, Q(w)). \end{aligned}$$

In the beginning of the proof of Lemma 3.6, we established that $\text{dens}(\mathcal{T}_0, Q(w)) \geq \frac{2}{3}$ and thus

$$\text{dens}(\mathcal{T}_n, Q(w)) \geq \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

finishes the proof of Lemma 3.6. \square

To finish the proof of Theorem 1, we note that Lemma 3.6 yields $\text{dens}(\mathcal{T}_n, Q(w)) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$ and hence $\text{dens}(\mathcal{T}, Q(w)) \geq \frac{1}{2}$. This already implies that $\text{meas}(\mathcal{T}) > 0$. Furthermore,

$$\text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) \geq \prod_{k=1}^{\infty} (1 - \eta \delta(x_{0,k}) K^k) \geq \frac{3}{4} \quad (3.15)$$

holds for all $n \in \mathbb{N}$. Thus, we obtain

$$\text{dens}(\mathcal{T}, \mathcal{T}_0 \cap Q(w)) \geq \frac{3}{4}.$$

Again, this implies that $\text{meas}(\mathcal{T}) > 0$ and since $\mathcal{T} \subset I(f)$ we conclude that the escaping set of f has positive measure.

3.3 Proof of Theorem 1'

The set \mathcal{T} is not only contained in the escaping set of f , but also in the fast escaping set. We prove the following lemma first:

Lemma 3.15. *Let $t > 0$. Then there exists $s > t$ such that*

$$E_{\frac{1}{2}}^n(s) \geq \exp^n(t)$$

for all $n \in \mathbb{N}$.

Proof. Choose $C_t > 2$ such that

$$\frac{1}{2} C_t \cdot t \geq t + \log C_t. \quad (3.16)$$

Let $s \geq C_t \cdot t$. Then

$$\begin{aligned} E_{\frac{1}{2}}(s) &\geq E_{\frac{1}{2}}(C_t \cdot t) \\ &= \exp\left(\frac{1}{2} C_t \cdot t\right) \\ &\geq \exp(t + \log C_t) \\ &= C_t \exp(t). \end{aligned}$$

Since $C_t > 2$, inequality (3.16) holds for all $r \geq t$, that is

$$\frac{1}{2}C_t \cdot r \geq r + \log C_t.$$

Since $\exp(t) > t$, we thus obtain

$$\begin{aligned} E_{\frac{1}{2}}^2(s) &\geq E_{\frac{1}{2}}(C_t \exp(t)) \\ &= \exp\left(\frac{1}{2}C_t \exp(t)\right) \\ &\geq \exp(\exp(t) + \log(C_t)) \\ &= C_t \exp^2(t). \end{aligned}$$

Inductively, we obtain

$$E_{\frac{1}{2}}^n(s) \geq C_t \exp^n(t) \geq \exp^n(t). \quad \square$$

Now we are able to prove Theorem 1'.

Proof of Theorem 1'. For simplicity, we only consider the case $\lambda = 1$. The general case follows analogously. Put $\lambda = 1$ and thus consider $f(x) = \sin(x)$. To estimate the maximum modulus of f , we use that $\|f(x)\| = \exp(|x_d| - 1)$ for $x \in \mathbb{R}^d$ with $|x_d| \geq 1$. Hence,

$$\begin{aligned} M^n(R, f) &\leq \exp(\dots \exp(\exp(R - 1) - 1) \dots - 1) \\ &\leq \exp^n(R), \end{aligned}$$

where $\exp^n(t)$ denotes the n th iterate of $\exp(t)$.

For R large, we have

$$\begin{aligned} M(R, f) &\geq \exp\left(\frac{R}{2} - 1\right) \\ &\geq 2R. \end{aligned}$$

Thus,

$$\begin{aligned} M^2(R, f) &= M(M(R, f), f) \\ &\geq 2M(R, f) \\ &\geq 4R \end{aligned}$$

and inductively

$$M^n(R, f) \geq 2^n R \rightarrow \infty \text{ as } n \rightarrow \infty.$$

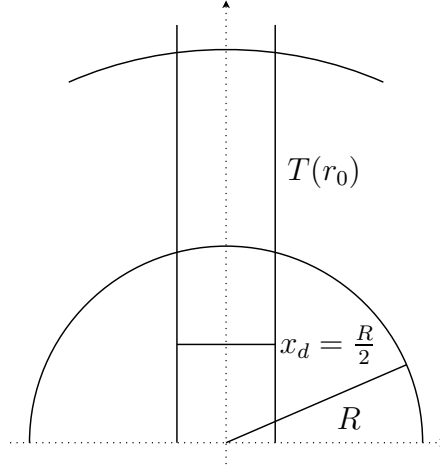


Figure 3.8: Illustrating the proof of Theorem 1'.

Let $x \in \mathcal{T}$. Without loss of generality, let $x_d \geq R$. By inequality (3.1), we have

$$|f_d^n(x_d)| \geq E_{\frac{1}{2}}^n(x_d).$$

Thus,

$$\begin{aligned} \|f^n(x)\| &= \|f(f^{n-1}(x))\| \\ &= \exp(|f_d^{n-1}(x)| - 1) \\ &\geq \exp\left(E_{\frac{1}{2}}^{n-1}(x_d) - 1\right). \end{aligned}$$

For $l \in \mathbb{N}$, we hence obtain

$$\begin{aligned} \|f^{n+l+1}(x)\| &\geq \exp\left(E_{\frac{1}{2}}^{n+l}(x_d) - 1\right) \\ &\geq \exp\left(E_{\frac{1}{2}}^{n+l}(R) - 1\right) \\ &= \exp\left(E_{\frac{1}{2}}^n\left(E_{\frac{1}{2}}^l(R)\right) - 1\right). \end{aligned}$$

Now choose l large enough such that

$$E_{\frac{1}{2}}^l(R) \geq s,$$

where s is chosen according to Lemma 3.15. Then Lemma 3.15 yields

$$\begin{aligned} \exp\left(E_{\frac{1}{2}}^n\left(E_{\frac{1}{2}}^l(R)\right) - 1\right) &\geq \exp\left(E_{\frac{1}{2}}^n(s) - 1\right) \\ &\geq \exp(\exp^n(R) - 1) \\ &\geq \exp^n(R). \end{aligned}$$

With $N = l + 1$ we obtain

$$\|f^{n+N}(x)\| \geq M^n(R, f)$$

for all $n \in \mathbb{N}$ and hence $x \in A(f)$. □

3.4 Proof of Theorem 2

We want to show that the measure of the non-escaping set of f is finite in tracts. By symmetry, it suffices to prove that this is true for the initial tract $T(r_0)$. In fact, we show that already $T(r_0) \setminus \mathcal{T}$ has finite measure. Since $\mathcal{T} \subset I(f)$ and $\mathcal{T} \subset L$, we obtain

$$\begin{aligned} \text{meas}(T(r_0) \setminus I(f)) &\leq \text{meas}(T(r_0) \setminus \mathcal{T}) \\ &= \text{meas}(S \cap T(r_0) \setminus \mathcal{T}) + \text{meas}(L \cap T(r_0) \setminus \mathcal{T}) \\ &= \text{meas}(S \cap T(r_0)) + \text{meas}(L \setminus \mathcal{T} \cap T(r_0)). \end{aligned}$$

Put $w = (0, \dots, 0, w_d) \in \mathbb{R}^d$. In the cube $Q(w)$, we get the following estimate:

Lemma 3.16. *If w_d is chosen large enough, then*

$$\text{meas}(T(r_0) \setminus I(f) \cap Q(w)) \leq 2^d w_d \delta(w_d) + 2^d w_d \left(1 - \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(E_{\frac{1}{2}}^k \left(\frac{w_d}{2} \right) \right) K^k \right) \right).$$

Proof. Put $w_d = 2x_0$, where x_0 is from Lemma 3.14. Then equation (3.15) yields

$$\text{dens}(\mathcal{T}_n, \mathcal{T}_0 \cap Q(w)) \geq \prod_{k=1}^{\infty} (1 - \eta \delta(x_{0,k}) K^k)$$

for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} \text{dens}(\mathcal{T}, \mathcal{T}_0 \cap Q(w)) &\geq \prod_{k=1}^{\infty} (1 - \eta \delta(x_{0,k}) K^k) \\ &= \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(E_{\frac{1}{2}}^k(x_0) \right) K^k \right) \\ &= \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(E_{\frac{1}{2}}^k \left(\frac{w_d}{2} \right) \right) K^k \right). \end{aligned} \tag{3.17}$$

In the cube $Q(w)$, the density of S in $T(r_0)$ is comparable to the density of S in $Q(w)$. Thus, by choosing w_d large, we can achieve that

$$\frac{1}{2} \text{dens}(S, T(r_0) \cap Q(w)) \leq \text{dens}(S, Q(w)) \leq \frac{3}{2} \text{dens}(S, T(r_0) \cap Q(w)),$$

and thus

$$\text{meas}(S \cap T(r_0) \cap Q(w)) \leq 2 \text{meas}(T(r_0) \cap Q(w)) \text{dens}(S, Q(w)).$$

With

$$\text{meas}(T(r_0) \cap Q(w)) = 2^{d-1} w_d,$$

Lemma 3.5 yields

$$\text{meas}(S \cap T(r_0) \cap Q(w)) \leq 2^d w_d \delta(w_d).$$

Similarly, we obtain

$$\text{dens}(L \setminus \mathcal{T}, T(r_0) \cap Q(w) \cap L) \leq 2 \text{dens}(L \setminus \mathcal{T}, Q(w) \cap L)$$

and hence

$$\begin{aligned} \text{meas}(L \setminus \mathcal{T} \cap T(r_0) \cap Q(w) \cap L) &\leq 2 \text{meas}(T(r_0) \cap Q(w) \cap L) \text{dens}(L \setminus \mathcal{T}, Q(w) \cap L) \\ &\leq 2 \text{meas}(T(r_0) \cap Q(w)) \text{dens}(L \setminus \mathcal{T}, Q(w) \cap L) \\ &= 2^d w_d (1 - \text{dens}(\mathcal{T}, Q(w) \cap L)). \end{aligned}$$

Using (3.17) we find that

$$\begin{aligned} \text{meas}(L \setminus \mathcal{T} \cap T(r_0) \cap Q(w)) &= \text{meas}(L \setminus \mathcal{T} \cap T(r_0) \cap Q(w) \cap L) \\ &\leq 2^d w_d \left(1 - \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(\mathbb{E}_{\frac{1}{2}}^k \left(\frac{w_d}{2} \right) \right) K^k \right) \right). \end{aligned}$$

We thus obtain

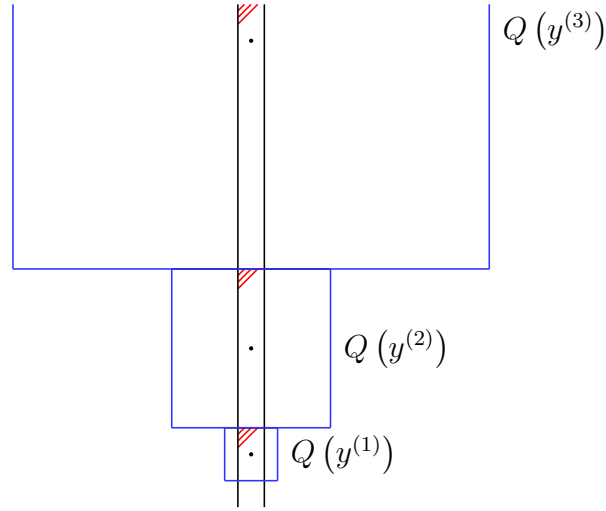
$$\text{meas}(T(r_0) \setminus I(f) \cap Q(w)) \leq 2^d w_d \delta(w_d) + 2^d w_d \left(1 - \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(\mathbb{E}_{\frac{1}{2}}^k \left(\frac{w_d}{2} \right) \right) K^k \right) \right),$$

finishing the proof. \square

Next, we choose a sequence of cubes which cover the tract $T(r_0)$. Let $y_1 = w_d$ from Lemma 3.16 and put $y_j := 3^{j-1} y_1$ and $y^{(j)} := (0, \dots, 0, y_j)$. Since $\frac{y_{j+1}}{2} = \frac{3}{2} y_j$ holds for all $j \in \mathbb{N}$, we find that the cubes $Q(y^{(j)})$ cover the set $T(r_0) \setminus [-1, 1]^{d-1} \times [0, \frac{y_1}{2}]$ such that the intersection of two cubes is a set of measure zero.

We can apply Lemma 3.16 to each of these cubes, particularly with w_d replaced by y_j . We thus find that

$$\begin{aligned} \text{meas}(T(r_0) \setminus I(f)) &\leq \frac{y_1}{2} 2^{d-1} + \text{meas} \left(\bigcup_{j=1}^{\infty} (T(r_0) \cap Q(y^{(j)}) \setminus I(f)) \right) \\ &= y_1 2^{d-2} + \sum_{j=1}^{\infty} \text{meas}(T(r_0) \setminus I(f) \cap Q(y^{(j)})) \\ &\leq y_1 2^{d-2} + 2^d \sum_{j=1}^{\infty} y_j \left(\delta(y_j) + \left(1 - \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(\mathbb{E}_{\frac{1}{2}}^k \left(\frac{y_j}{2} \right) \right) K^k \right) \right) \right). \end{aligned}$$

Figure 3.9: Sequence of cubes covering the tract $T(r_0)$.

It remains to prove that the infinite sum converges. For the first part, we note that with $y_{j+1} = 3y_j$, we obtain

$$\begin{aligned} \frac{y_{j+1}\delta(y_{j+1})}{y_j\delta(y_j)} &= \frac{y_{j+1} \left(\exp\left(-\frac{1}{4}y_{j+1} + 1\right) \right)}{y_j \left(\exp\left(-\frac{1}{4}y_j + 1\right) \right)} \\ &= 3 \exp\left(-\frac{1}{2}y_j\right) \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Hence,

$$\sum_{j=1}^{\infty} y_j \delta(y_j) = \sum_{j=1}^{\infty} y_j \frac{3}{2^{d+1}\lambda} L_h^{d-1} C_1 \omega \exp\left(-\frac{y_j}{4} + 1\right)$$

converges. The second part of the infinite sum converges as well:

Lemma 3.17. *For y_1 large, we obtain*

$$\sum_{j=1}^{\infty} y_j \left(1 - \prod_{k=1}^{\infty} \left(1 - \eta \delta \left(E_{\frac{1}{2}}^k \left(\frac{y_j}{2} \right) \right) K^k \right) \right) < \infty.$$

Proof. We first observe that

$$\log(1-t) \geq -2t \quad \text{holds for } 0 \leq t \leq \frac{1}{2} \quad (3.18)$$

and

$$\exp(-t) \geq 1 - 2t \quad \text{holds for } t \geq 0. \quad (3.19)$$

Put

$$a_{k,j} := \eta \delta \left(E_{\frac{1}{2}}^k \left(\frac{y_j}{2} \right) \right) K^k.$$

Then by Lemma 3.14, the infinite product $\prod_{k=1}^{\infty} (1 - a_{k,j})$ converges for all $j \in \mathbb{N}$. Hence, there exists $k_0 \in \mathbb{N}$ such that $a_{k,j} \leq \frac{1}{2}$ for all $k \geq k_0$. By choosing y_1 large enough, we obtain $a_{k,j} \leq \frac{1}{2}$ for all $k, j \in \mathbb{N}$. Using (3.18) and (3.19), we find that

$$\begin{aligned}
1 - \prod_{k=0}^{\infty} (1 - a_{k,j}) &= 1 - \exp\left(\sum_{k=1}^{\infty} \log(1 - a_{k,j})\right) \\
&\leq 1 - \exp\left(-2 \sum_{k=1}^{\infty} a_{k,j}\right) \\
&\leq 1 - \left(1 - 4 \sum_{k=1}^{\infty} a_{k,j}\right) \\
&= 4 \sum_{k=1}^{\infty} a_{k,j}.
\end{aligned} \tag{3.20}$$

Now,

$$\begin{aligned}
a_{k+1,j} &= \eta \frac{3}{2^{d+1}\lambda} L_h^{d-1} C_1 \omega \exp\left(-\frac{1}{4} \mathbb{E}_{\frac{1}{2}}^{k+1}\left(\frac{y_j}{2}\right) + 1\right) K^{k+1} \\
&=: \hat{\eta} \exp\left(-\frac{1}{4} \mathbb{E}_{\frac{1}{2}}^{k+1}\left(\frac{y_j}{2}\right) + 1\right) K^{k+1}.
\end{aligned}$$

Choose y_1 large enough such that

$$\mathbb{E}_{\frac{1}{2}}^{k+1}\left(\frac{y_j}{2}\right) \geq \mathbb{E}_{\frac{1}{2}}^k\left(\frac{y_j}{2}\right) + 8 + 4 \log K.$$

Then

$$\begin{aligned}
a_{k+1,j} &\leq \hat{\eta} \exp\left(-\frac{1}{4} \mathbb{E}_{\frac{1}{2}}^k\left(\frac{y_j}{2}\right) + 1 - \frac{1}{4}(8 + 4 \log K)\right) K^{k+1} \\
&= \hat{\eta} \exp\left(-\frac{1}{4} \mathbb{E}_{\frac{1}{2}}^k\left(\frac{y_j}{2}\right) + 1\right) \exp(-2) K^{-1} K^{k+1} \\
&\leq \frac{1}{2} \hat{\eta} \exp\left(-\frac{1}{4} \mathbb{E}_{\frac{1}{2}}^k\left(\frac{y_j}{2}\right) + 1\right) K^k \\
&= \frac{1}{2} a_{k,j}.
\end{aligned}$$

By induction, we therefore obtain

$$a_{k,j} \leq 2^{-k+1} a_{1,j}$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{k,j} &\leq \sum_{k=1}^{\infty} 2^{-k+1} a_{1,j} \\
&= 2a_{1,j} \\
&= 2\hat{\eta} \exp\left(-\frac{1}{4} E_{\frac{1}{2}}\left(\frac{y_j}{2}\right) + 1\right) \\
&\leq 2\hat{\eta} \exp\left(-\frac{1}{4}y_j + 1\right),
\end{aligned} \tag{3.21}$$

since $E_{\frac{1}{2}}(t) > 2t$ for t large enough. Using (3.20) and (3.21), we finally conclude

$$\begin{aligned}
\sum_{j=1}^{\infty} y_j \left(1 - \prod_{k=1}^{\infty} (1 - a_{k,j})\right) &\leq \sum_{j=1}^{\infty} y_j \left(4 \sum_{k=1}^{\infty} a_{k,j}\right) \\
&\leq 8\hat{\eta} \sum_{j=1}^{\infty} y_j \exp\left(-\frac{1}{4}y_j + 1\right) \\
&< \infty.
\end{aligned}$$

□

Thus, $\text{meas}(T(r_0) \setminus I(f)) < \infty$, finishing the proof of Theorem 2.

Proof of Theorem 2'. Since $\mathcal{T} \subset A(f)$ and we actually showed that the measure of $T(r_0) \setminus \mathcal{T}$ is finite, we obtain $\text{meas}(T(r_0) \setminus A(f)) < \infty$ and thus $\text{meas}(T(r) \setminus A(f)) < \infty$ for all $r \in R$. □

4 A Quasiregular Map with Non-Escaping Set of Finite Measure

In the previous chapter, we showed that the non-escaping set of the quasiregular sine is finite in each tract. By periodicity, the whole non-escaping set of sine has either measure zero or ∞ . In this chapter, we construct a quasiregular map with non-escaping set of positive, finite measure. In the plane, we have the following result by Hemke [Hem05] (see the example after Theorem 2.3): Let $g(z) = \sinh(z^3)$. Then the non-escaping set of g has finite measure.

To generalise this result for quasiregular maps, we first need to think about a suitable quasiregular power mapping. Mayer's [May97] quasiregular analogue of power mappings is based on Zorich maps, see Section 2.7. We construct another quasiregular power mapping by using the quasiregular sine instead of a Zorich map. We will see later in the proof that this enables us to apply Theorem 2. Furthermore, it is more convenient to slightly move the initial tract in the construction of $\text{Sin}(x)$. Thereby, we achieve that $m \cdot T(r_0)$ decomposes nicely into a union of tracts.

Definition 4.1. We call the map

$$\text{Cos}: \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \text{Cos}(x) = \text{Sin}(x - (1, \dots, 1, 0))$$

quasiregular cosine.

Keeping the construction of the quasiregular sine in mind, we simply start with the upper half-cube and shift it in the first $d - 1$ coordinates by one. The rest of the construction is the same.

Remark 4.2. The bi-Lipschitz angular map for $\text{Cos}(x)$ is given by

$$h: [0, 2]^{d-1} \times \{1\} \rightarrow \mathbb{S}_0.$$

Remark 4.3. Let $\lambda > 0$. Then Theorem 1 and 2 hold for $\lambda \text{Cos}(x)$ as well, with the same proofs.

4.1 A quasiregular power mapping using $\text{Cos}(x)$

To construct the quasiregular power mapping, we start with the upper half-space \mathbb{H}^+ . We denote by

$$\text{Cos}_0^{-1}: \mathbb{H}^+ \rightarrow [0, 2]^{d-1} \times [0, \infty[$$

the inverse branch of $\text{Cos}(x)$ which maps \mathbb{H}^+ onto the appropriate initial tract, which we denote again by $T(r_0)$. For $m \in \mathbb{N}$ we put

$$\mathcal{M}_m: [0, 2]^{d-1} \times [0, \infty[\rightarrow [0, 2m]^{d-1} \times [0, \infty[, \quad \mathcal{M}_m(x) = mx.$$

Finally, we map $[0, 2m]^{d-1} \times [0, \infty[$ onto \mathbb{R}^d using $\text{Cos}(x)$. Reflection at $\mathbb{R}^{d-1} \times \{0\}$ then yields the following map:

Definition 4.4. We call

$$P_m: \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad P_m(x) = \begin{cases} \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(x) & \text{for } x \in \mathbb{H}^+ \\ \overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\bar{x})} & \text{for } x \in \mathbb{H}^- \end{cases}$$

quasiregular power mapping, where

$$\bar{x} = \overline{(x_1, \dots, x_d)} = (x_1, \dots, x_{d-1}, -x_d)$$

denotes the reflection at the hyperplane $\mathbb{R}^{d-1} \times \{0\}$.

Remark 4.5. Since P_m is the composition of quasiregular maps and two reflections, this power mapping is indeed quasiregular. Later in this chapter, we will see that P_m is in fact uniformly quasiregular.

Remark 4.6. The map P_m can be considered as a higher dimensional analogue of the Chebyshev polynomials T_m satisfying

$$T_m(x) = \cos(m \arccos(x)).$$

Properties of P_m

Outside of the ball of radius 1, we can write P_m as a product of an angular function and a scaling function. In particular, P_m is of the following form:

Lemma 4.7. *Let $\|x\| \geq 1$ and $\lambda \geq 1$. Then*

$$P_m(\lambda x) = \lambda^m \|x\|^m P_m|_{\mathbb{S}} \left(\frac{x}{\|x\|} \right).$$

Proof. Without loss of generality, let $x_d \geq 0$ (the statement directly follows for $x_d \leq 0$ by reflection). Then $P_m(\lambda x) = \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\lambda x)$. By the definition of $\text{Cos}(x)$, we obtain

$$\begin{aligned} \text{Cos}_0^{-1}(\lambda x) &= \left(h^{-1} \left(\frac{\lambda x}{\|\lambda x\|} \right) + (0, \dots, 0, \log(\lambda \|x\|)) \right) \\ &= \left(p \left(h^{-1} \left(\frac{x}{\|x\|} \right) \right), \log(\lambda \|x\|) + 1 \right), \end{aligned}$$

where again $p: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $p(x_1, \dots, x_d) = (x_1, \dots, x_{d-1})$ denotes the projection. Thus,

$$\begin{aligned} \mathcal{M}_m \circ \text{Cos}_0^{-1}(\lambda x) &= m \text{Cos}_0^{-1}(\lambda x) \\ &= \left(p \left(m \cdot h^{-1} \left(\frac{x}{\|x\|} \right) \right), m \log(\lambda \|x\|) + m \right). \end{aligned}$$

With

$$\begin{aligned} P_m|_{\mathbb{S}} \left(\frac{x}{\|x\|} \right) &= \text{Cos} \left(m \cdot h^{-1} \left(\frac{x}{\|x\|} \right) \right) \\ &= \text{Cos} \left(p \left(m \cdot h^{-1} \left(\frac{x}{\|x\|} \right) \right), m \right) \\ &= \text{Cos} \left(p \left(m \cdot h^{-1} \left(\frac{x}{\|x\|} \right) \right), 1 \right) e^{m-1}, \end{aligned}$$

we get

$$\begin{aligned} P_m(\lambda x) &= \text{Cos} \left(\left(p \left(m \cdot h^{-1} \left(\frac{x}{\|x\|} \right) \right), m \log(\lambda \|x\|) + m \right) \right) \\ &= \text{Cos} \left(p \left(m \cdot h^{-1} \left(\frac{x}{\|x\|} \right) \right), 1 \right) e^{m-1} \exp(m \log(\lambda \|x\|)) \\ &= \lambda^m \|x\|^m P_m|_{\mathbb{S}} \left(\frac{x}{\|x\|} \right). \end{aligned}$$

□

Remark 4.8. P_m maps spheres with radius greater or equal to 1 to spheres. In particular, let $r \geq 1$. Then

$$P_m(r \cdot \mathbb{S}) = e^{m-1} r^m \cdot \mathbb{S}$$

Proof. Without loss of generality, let $x \in \mathbb{H}^+$ with $\|x\| = r \geq 1$. Lemma 4.7 yields

$$\|P_m(x)\| = r^m \left\| P_m|_{\mathbb{S}} \left(\frac{x}{\|x\|} \right) \right\|.$$

Put $y := \frac{x}{\|x\|}$ and $z := \text{Cos}_0^{-1}(y)$. Then

$$\begin{aligned} \|P_m(y)\| &= \|\text{Cos} \circ \mathcal{M} \circ \text{Cos}_0^{-1}(y)\| \\ &= \|\text{Cos}(mz_1, \dots, mz_{d-1}, m)\| \\ &= e^{m-1}. \end{aligned}$$

□

Remark 4.9. The branch points of $\text{Cos}(x)$ lie on the edges of tracts. Denote the set $B_{\text{Cos}} \cap [0, 2m]^{d-1} \times [0, \infty[$ without the vertical edges of $[0, 2m]^{d-1} \times [0, \infty[$ by E . Then the branch set of P_m is contained in the union of

$$\text{Cos} \circ \mathcal{M}_m^{-1}(E)$$

and

$$\overline{\text{Cos} \circ \mathcal{M}_m^{-1}(E)}.$$

Lemma 4.10. P_m has degree m^{d-1} .

Proof. By definition, $\text{Cos}_0^{-1}: \mathbb{H}^+ \rightarrow T(r_0)$ and $\mathcal{M}_m: T(r_0) \rightarrow [0, 2m]^{d-1} \times [0, \infty[$ are bijective maps.

For m even, we can write $[0, 2m]^{d-1} \times [0, \infty[$ as a union of exactly m^{d-1} tracts. Depending on their label, half of those tracts are mapped onto \mathbb{H}^+ , the other half onto \mathbb{H}^- . Thus, a point $y \in \mathbb{H}^+$ with $\|y\| = e^{m-1}$ and $y_d > 0$ has exactly $\frac{m^{d-1}}{2}$ preimages under the map

$$\text{Cos}: [0, 2m]^{d-1} \times [0, \infty[\rightarrow \mathbb{R}^d.$$

None of these preimages is contained in the boundary of some tract. Hence, there are exactly $\frac{m^{d-1}}{2}$ preimages of y under $P_m|_{\mathbb{H}^+}$ and for every $x \in (P_m|_{\mathbb{H}^+})^{-1}(y)$ we have $x_d > 0$ and $i(P_m, x) = 1$. The same holds for \bar{y} and the preimages of y and \bar{y} are distinct. By definition of P_m , we have $\overline{(P_m|_{\mathbb{H}^+})^{-1}(y)} = (P_m|_{\mathbb{H}^-})^{-1}(y)$ and thus m^{d-1} preimages of y under P_m .

For m odd, $[0, 2m]^{d-1} \times [0, \infty[$ contains exactly $\frac{m^{d-1}+1}{2}$ tracts which are mapped onto \mathbb{H}^+ under Cos and exactly $\frac{m^{d-1}-1}{2}$ tracts which are mapped onto \mathbb{H}^- under Cos . Hence, the set $(P_m|_{\mathbb{H}^+})^{-1}(y)$ contains exactly $\frac{m^{d-1}+1}{2}$ elements and $\overline{(P_m|_{\mathbb{H}^+})^{-1}(y)}$ contains exactly $\frac{m^{d-1}-1}{2}$ elements. Again, both sets are disjoint. With $\overline{(P_m|_{\mathbb{H}^+})^{-1}(y)} = (P_m|_{\mathbb{H}^-})^{-1}(y)$, we obtain that y has m^{d-1} preimages. Again, every preimage has index 1.

The definition of the degree now yields that the degree of P_m is in fact m^{d-1} . \square

We want to illustrate the concept of the local index for the case $d = 3$. Since $i(P_m, x) = 1$ for all $x \in \mathbb{R}^3 \setminus B_{P_m}$, we consider a point $x_0 \in B_{P_m}$ with $\|x\| \geq 1$ and its image $y = P_m(x_0)$. Then x_0 is mapped onto an edge of some tract $T(r_1)$ by $\mathcal{M}_m \circ \text{Cos}_0^{-1}$. Thus, preimages of y under $\text{Cos}|_{[0, 2m]^{d-1} \times [0, \infty[}$ lie on vertical edges of the tracts $T(r)$, where $0 \leq r_j \leq m-1$ for $1 \leq j \leq d-1$ and $r_d = 1$.

Now let m be odd. Then there are exactly m preimages of y in $\mathbb{R}^{d-1} \times \{0\}$ under P_m . Exactly one of those has index 1. The other $m-1$ have index 2. Finally, there are $\left(\frac{m-1}{2}\right)^2$ preimages in $\mathbb{H}^+ \setminus \mathbb{H}^-$ and $\left(\frac{m-1}{2}\right)^2$ preimages in $\mathbb{H}^- \setminus \mathbb{H}^+$, each with index 2.

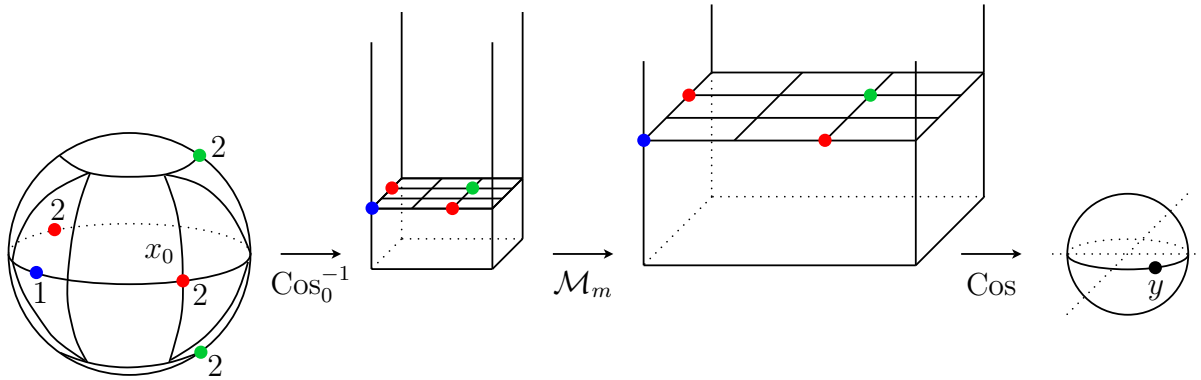


Figure 4.1: The case $d = 3$, $m = 3$. We consider $x_0 \in B_{P_3}$ and $y = P_3(x_0)$. In the case that $x \in P_3^{-1}(y)$ is a branch point of P_3 (red and green), the local index of P_3 at x is 2.

Counting the preimages with multiplicity (i.e. according to the local index) then yields

$$\sum_{x \in P_m^{-1}(y)} i(P_m, x) = 1 + 2(m - 1) + 4 \left(\frac{m - 1}{2} \right)^2 = m^2.$$

The case m even is similar, but slightly more complicated, see Figure 4.2.

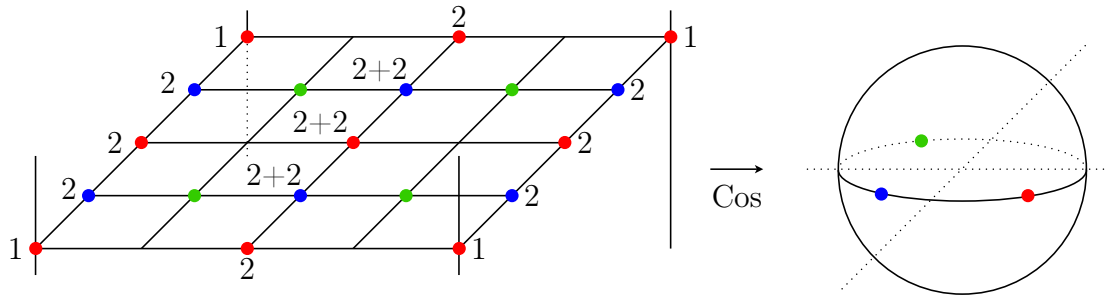


Figure 4.2: The case $d = 3$ and $m = 4$. This time, the three colors visualize three different types of points in $P_4(B_{P_4})$ and their preimages under $\text{Cos} |_{[-1,1]^{d-1} \times [0,\infty[}$. Inner points have counterparts in \mathbb{H}^- , indicated by “ $2 + 2$ ”.

Sectors

We label the tracts of $\text{Cos}(x)$ in the same manner as we did for $\text{Sin}(x)$. Since we are only considering the quasiregular cosine in this chapter, we use the same notation as before. In particular, we put again $R = \mathbb{Z}^{d-1} \times \{-1, 1\}$. For $r = (r_1, \dots, r_d) \in R$ we put

$$T(r) := \{x \in \mathbb{R}^d : |x_j - 2r_j - 1| \leq 1 \text{ for } 1 \leq j \leq d - 1, r_d x_d \geq 0\}.$$

Again, if

$$\sigma(r) = \sum_{j=1}^{d-1} r_j + \frac{1}{2}(r_d - 1)$$

is even, then $\text{Cos}(x)$ maps $T(r)$ bijectively onto \mathbb{H}^+ . If $\sigma(r)$ is odd, then $\text{Cos}(x)$ maps $T(r)$ bijectively onto \mathbb{H}^- .

Definition 4.11. Let $r \in R$ with $0 \leq r_j \leq m-1$ for $1 \leq j \leq d-1$. For $r_d = 1$ put

$$\begin{aligned} \Omega(r) &:= \text{Cos} \circ \mathcal{M}_m^{-1}(T(r)) \\ &= \text{Cos} \left(\prod_{j=1}^{d-1} \left[\frac{2r_j}{m}, \frac{2r_j+2}{m} \right] \times [0, \infty[\right), \end{aligned}$$

and for $r_d = -1$

$$\Omega(r) = \overline{\Omega(r_1, \dots, r_{d-1}, 1)}.$$

We call $\Omega(r)$ a *sector* of P_m .

Except for sets of measure zero, this defines a partition of \mathbb{R}^d into $2m^{d-1}$ sectors. If $\sigma(r)$ is even, then P_m maps $\Omega(r)$ bijectively onto \mathbb{H}^+ . If $\sigma(r)$ is odd, then P_m maps $\Omega(r)$ bijectively onto \mathbb{H}^- . By putting

$$\tilde{\Omega}(r) := \text{Cos} \left(\prod_{j=1}^{d-1} \left[\frac{2r_j}{m}, \frac{2r_j+2}{m} \right] \times \{1\} \right)$$

and again by reflection, we get sectors on the sphere which are mapped under P_m onto the upper or lower half-sphere of radius e^{m-1} , depending on r .

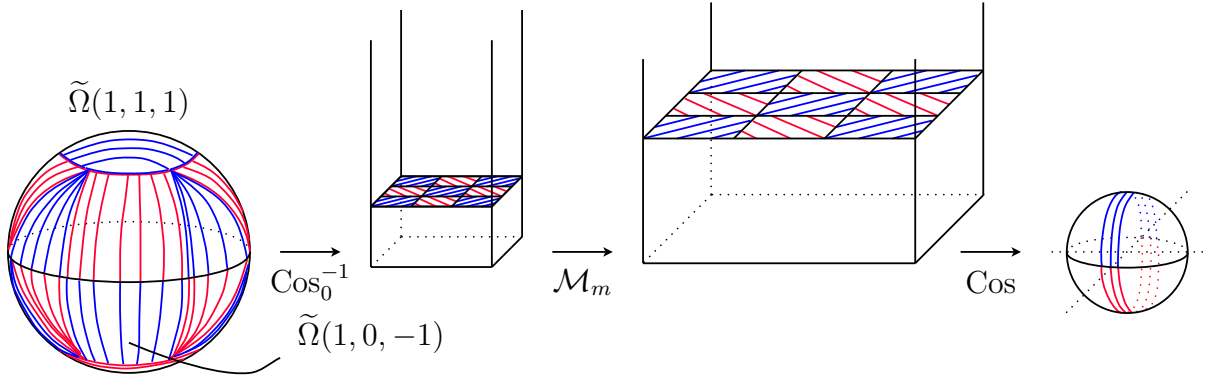


Figure 4.3: Construction of P_m : on the left we see the sectors $\tilde{\Omega}_r$, which get mapped onto the upper or lower half-sphere of radius e^{m-1} .

Inverse branches

We want to characterise those branches of the inverse of P_m which map the upper or lower half-space onto some sector $\Omega(r)$. Since P_m maps $\tilde{\Omega}(r)$ bijectively onto $e^{m-1}\mathbb{S}^\pm$, we

denote by

$$\tilde{\Psi}_r: e^{m-1}\mathbb{S}^\pm \rightarrow \tilde{\Omega}(r)$$

the corresponding inverse branch. Here \mathbb{S}^\pm denotes the upper or lower half-sphere, depending on the label of $\tilde{\Omega}$. This provides us with an angular map, the scaling is done by taking the appropriate root:

Lemma 4.12. *For $r \in R$, $0 \leq r_j \leq m-1$ and $1 \leq j \leq d-1$ the map*

$$\Psi_r: \mathbb{H}^\pm \setminus B(0, e^{m-1}) \rightarrow \Omega(r) \setminus B(0, 1), \quad y \mapsto e^{-\frac{m-1}{m}} \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right)$$

is the inverse map of $P_m|_{\Omega(r) \setminus B(0,1)}$.

Proof. Let $y \in \mathbb{R}^d$ with $\|y\| \geq e^{m-1}$. Since $\|\tilde{\Psi}_r(x)\| = 1$ for all $x \in e^{m-1}\mathbb{S}^\pm$ and by Lemma 4.7, we obtain

$$\begin{aligned} P_m(\Psi_r(y)) &= P_m \left(e^{-\frac{m-1}{m}} \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) \right) \\ &= \left\| e^{-\frac{m-1}{m}} \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) \right\|^m \cdot P_m|_{\tilde{\Omega}(r)} \left(\frac{e^{-\frac{m-1}{m}} \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right)}{\left\| e^{-\frac{m-1}{m}} \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) \right\|} \right) \\ &= \frac{\|y\|}{e^{m-1}} \cdot P_m|_{\tilde{\Omega}(r)} \left(\tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) \right) \\ &= y. \end{aligned} \quad \square$$

Remark 4.13. The branches

$$\tilde{\Psi}_r: e^{m-1}\mathbb{S}^\pm \rightarrow \tilde{\Omega}(r)$$

are bi-Lipschitz for all $r \in R$ such that $0 \leq r_j \leq m-1$ for $1 \leq j \leq d-1$, with a uniform Lipschitz constant $L_{\tilde{\Psi}}$.

Proof. For $x \in \mathbb{S}^+$, we have

$$P_m(x) = \text{Cos} \left(p \left(m \cdot h^{-1}(x) \right), 1 \right) e^{m-1}.$$

The map h is bi-Lipschitz by definition as well as Cos restricted to $T(r) \cap (\mathbb{R}^{d-1} \times \{1\})$ for $0 \leq r_j \leq m-1$, where the Lipschitz constant can be chosen independently of r . In fact, it is the same Lipschitz constant as for h . Put

$$\kappa: \mathbb{R}^{d-1} \times \{m\} \rightarrow \mathbb{R}^{d-1} \times \{1\}, \quad (x_1, \dots, x_{d-1}, m) \mapsto (x_1, \dots, x_{d-1}, 1).$$

Since \mathcal{M}_m is bi-Lipschitz and $P_m(x) = \text{Cos} \circ \kappa \circ \mathcal{M}_m \circ h^{-1}(x) e^{m-1}$, we hence conclude that for $\tilde{\Omega}(r) \subset \mathbb{H}^+$, the map $P_m|_{\tilde{\Omega}(r)}$ is bi-Lipschitz as composition of bi-Lipschitz maps. By reflection, the same holds for $P_m|_{\tilde{\Omega}(r)}$ if $\tilde{\Omega}(r) \subset \mathbb{H}^-$ and hence for the corresponding branches of the inverse function. \square

4.2 Theorem 3 and outline of the proof

We are now able to state the main result of this chapter and to give an idea, how we are going to prove it.

Theorem 3. *Let $m \geq d+1$. Then the measure of the non-escaping set of the quasiregular cosine composed by P_m is finite, i.e.*

$$\text{meas}(I^c(\text{Cos} \circ P_m)) < \infty.$$

To outline the proof of Theorem 3, consider the following diagram. We will show that the diagram commutes.

$$\begin{array}{ccc}
 \mathbb{R}^d & \xrightarrow{\text{Cos} \circ P_m} & \mathbb{R}^d \\
 P_m \downarrow & & \downarrow P_m \\
 \mathbb{R}^d & \xrightarrow{\text{Cos} \circ \mathcal{M}_m} & \mathbb{R}^d \\
 \mathcal{M}_m \downarrow & & \downarrow \mathcal{M}_m \\
 \mathbb{R}^d & \xrightarrow{\mathcal{M}_m \circ \text{Cos}} & \mathbb{R}^d
 \end{array}$$

Definition 4.14. Let $g, h: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be continuous maps. We say that g is *semi-conjugated* to h via Φ , if there exists a continuous and surjective map $\Phi: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\Phi \circ g = h \circ \Phi$. In the case that Φ is injective, we say that g is *conjugated* to h via Φ .

We immediately get that $\text{Cos} \circ \mathcal{M}_m$ is conjugated to $\mathcal{M}_m \circ \text{Cos}$ via \mathcal{M}_m .

By Remark 4.3, the measure of the non-escaping set of $m \text{Cos}(x)$ is finite in tracts, thus

$$\text{meas}(I^c(m \text{Cos}) \cap T(r)) < \infty$$

holds for all $r \in R$. By conjugacy, we have

$$I^c(\text{Cos} \circ \mathcal{M}_m) = \frac{1}{m} I^c(m \text{Cos}).$$

Therefore,

$$\text{meas} \left(I^c(\text{Cos} \circ \mathcal{M}_m) \cap \frac{1}{m} T(r) \right) < \infty,$$

where $\frac{1}{m} T(r)$ is a periodic tract of $\text{Cos} \circ \mathcal{M}_m$.

We will show that $\text{Cos} \circ P_m$ is semi-conjugated to $\text{Cos} \circ \mathcal{M}_m$ via P_m . Since P_m is of polynomial type, it will turn out that

$$I^c(\text{Cos} \circ P_m) = P_m^{-1}(I^c(\text{Cos} \circ \mathcal{M}_m)).$$

To prove Theorem 3, we are hence going to prove the semi-conjugacy and then show that

$$\text{meas}(P_m^{-1}(I^c(\text{Cos} \circ \mathcal{M}_m))) < \infty.$$

4.3 Proof of Theorem 3

Lemma 4.15. *The map $\text{Cos} \circ P_m$ is semi-conjugated to $\text{Cos} \circ \mathcal{M}_m$ via P_m , i.e.*

$$P_m \circ \text{Cos} \circ P_m = \text{Cos} \circ \mathcal{M}_m \circ P_m.$$

Proof. We show that

$$P_m \circ \text{Cos}(x) = \text{Cos} \circ \mathcal{M}_m(x)$$

for all $x \in \mathbb{R}^d$. First, let $\text{Cos}(x) \in \mathbb{H}^+$. Then

$$P_m \circ \text{Cos}(x) = \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \text{Cos}(x).$$

If additionally $x \in \mathbb{H}^+$, then there exists $r \in R$ with $r_d = 1$ and $\sigma(r)$ even such that $x \in T(r)$. To construct Cos , we used repeated reflections. Thus, there exists exactly one $y \in T(r_0) = T(0, \dots, 0, 1)$ and an even number of reflections S_1, \dots, S_{2k} at faces of tracts such that

$$x = S_1 \circ \dots \circ S_{2k}(y).$$

We get

$$\begin{aligned} \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \text{Cos}(x) &= \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \text{Cos}(S_1 \circ \dots \circ S_{2k}(y)) \\ &= \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \text{Cos}(y) \\ &= \text{Cos} \circ \mathcal{M}_m(y) \\ &= \text{Cos}(my). \end{aligned}$$

Furthermore, there exist $k_1, \dots, k_{d-1} \in \mathbb{Z}$ such that $x = (4k_1 \pm y_1, \dots, 4k_{d-1} \pm y_{d-1}, y_d)$ with an even number of minus signs. Since Cos is 4-periodic in each coordinate except the last one, we obtain

$$\begin{aligned} \text{Cos} \circ \mathcal{M}_m(x) &= \text{Cos}(4mk_1 \pm my_1, \dots, 4mk_{d-1} \pm my_{d-1}, my_d) \\ &= \text{Cos}(\pm my_1, \dots, \pm my_{d-1}, my_d). \end{aligned}$$

Since it requires an odd number of reflections to flip the negative sign in a single coordinate, we obtain

$$\text{Cos}(my_1, \dots, -my_j, \dots, my_d) = \overline{\text{Cos}((my_1, \dots, my_j, \dots, my_d))}.$$

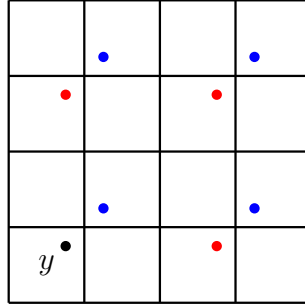


Figure 4.4: The case $r_d = 1$ and $\sigma(r)$ even. The blue and red points are obtained through y by an even number of reflections at faces of tracts.

Applying this an even number of times yields

$$\text{Cos} \circ \mathcal{M}_m(x) = \text{Cos}(my).$$

Next, let $x \in T(r)$ with $r_d = -1$ and $\sigma(r)$ even. Then there exists exactly one $y \in T(r_0)$ and an odd number of reflections S_1, \dots, S_{2k+1} at faces of tracts such that

$$x = S_1 \circ \dots \circ S_{2k+1}(\bar{y}).$$

Thus,

$$\begin{aligned} \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \text{Cos}(x) &= \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \text{Cos}(S_1 \circ \dots \circ S_{2k+1}(\bar{y})) \\ &= \text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1} \circ \overline{\text{Cos}(\bar{y})} \\ &= \text{Cos} \circ \mathcal{M}_m(y) \\ &= \text{Cos}(my). \end{aligned}$$

Since $r_d = -1$, we have $x_d = -y_d$. Again, there exist $k_1, \dots, k_{d-1} \in \mathbb{Z}$ such that $x = (4k_1 \pm y_1, \dots, 4k_{d-1} \pm y_{d-1}, -y_d)$, this time with an odd number of minus signs,

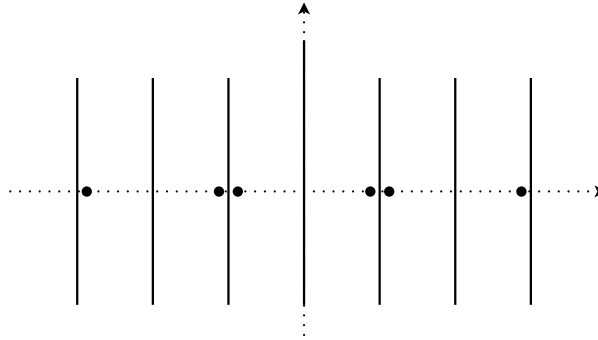


Figure 4.5: Flipping the sign in a single coordinate takes an odd number of reflections.

not counting the sign of $-y_d$. Thus, we obtain

$$\begin{aligned}\text{Cos} \circ \mathcal{M}_m(x) &= \text{Cos}(4mk_1 \pm my_1, \dots, 4mk_{d-1} \pm my_{d-1}, -my_d) \\ &= \text{Cos}(\pm my_1, \dots, \pm my_{d-1}, -my_d) \\ &= \overline{\text{Cos}(my_1, \dots, my_{d-1}, -my_d)} \\ &= \text{Cos}(my).\end{aligned}$$

Now for $x \in T(r)$ with $\sigma(r)$ odd, which implies that $\text{Cos}(x) \in \mathbb{H}^-$, we have

$$\begin{aligned}P_m \circ \text{Cos}(x) &= \overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(x))}} \\ &= \overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(\bar{x}))}.\end{aligned}$$

For $r_d = -1$ and thus $x_d \leq 0$ there exists a unique $y \in T(r_0)$ and an even number of reflections S_1, \dots, S_{2k} such that $x = S_1 \circ \dots \circ S_{2k}(\bar{y})$. This implies that

$$\begin{aligned}\overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(\bar{x}))}} &= \overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(\overline{S_1 \circ \dots \circ S_{2k}(\bar{y})}))}} \\ &= \overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(y))}} \\ &= \text{Cos}(m\bar{y}).\end{aligned}$$

Again, we find $k_1, \dots, k_{d-1} \in \mathbb{Z}$ such that $x = (4k_1 \pm y_1, \dots, 4k_{d-1} \pm y_{d-1}, -y_d)$ with an even number of minus signs, not counting the sign of $-y_d$. Thus,

$$\begin{aligned}\text{Cos} \circ \mathcal{M}_m(x) &= \text{Cos}(4mk_1 \pm my_1, \dots, 4mk_{d-1} \pm my_{d-1}, -my_d) \\ &= \text{Cos}(\pm my_1, \dots, \pm my_{d-1}, -my_d) \\ &= \text{Cos}(m\bar{y}).\end{aligned}$$

Finally, if $r_d = 1$, we find that $x = S_1 \circ \dots \circ S_{2k+1}(y)$ and thus

$$\begin{aligned}\overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(\bar{x}))}} &= \overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(\overline{S_1 \circ \dots \circ S_{2k+1}(y))}))}} \\ &= \overline{\overline{\text{Cos} \circ \mathcal{M}_m \circ \text{Cos}_0^{-1}(\text{Cos}(y))}} \\ &= \text{Cos}(m\bar{y}).\end{aligned}$$

For $x = (4k_1 \pm y_1, \dots, 4k_{d-1} \pm y_{d-1}, y_d)$ with an odd number of minus signs we finally conclude that

$$\begin{aligned}\text{Cos} \circ \mathcal{M}_m(x) &= \text{Cos}(4mk_1 \pm my_1, \dots, 4mk_{d-1} \pm my_{d-1}, my_d) \\ &= \text{Cos}(\pm my_1, \dots, \pm my_{d-1}, my_d) \\ &= \text{Cos}(m\bar{y}).\end{aligned}$$

□

Since P_m is not injective, we only get a semi-conjugacy. Nevertheless, we have the following relation between the escaping set of $\text{Cos} \circ P_m$ and $\text{Cos} \circ \mathcal{M}_m$:

Lemma 4.16.

$$I(\text{Cos} \circ P_m) = P_m^{-1}(I(\text{Cos} \circ \mathcal{M}_m)).$$

Proof. For this proof, we put $f := \text{Cos} \circ \mathcal{M}_m$ and $g := \text{Cos} \circ P_m$. Lemma 4.15 states that

$$f \circ P_m = P_m \circ g.$$

Then

$$f^n \circ P_m = f^{n-1} \circ f \circ P_m = f^{n-1} \circ P_m \circ g.$$

Inductively,

$$f^n \circ P_m = P_m \circ g^n$$

follows for all $n \in \mathbb{N}$. Let $x \in I(g)$. Then $\|g^n(x)\| \rightarrow \infty$ as $n \rightarrow \infty$. For n large we thus obtain

$$\begin{aligned} \|f^n \circ P_m(x)\| &= \|P_m(g^n(x))\| \\ &= \left\| \|g^n(x)\|^m P_m|_{\mathbb{S}} \left(\frac{g^n(x)}{\|g^n(x)\|} \right) \right\| \\ &= e^{m-1} \|g^n(x)\|^m \rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, $P_m(x) \in I(f)$ and thus $x \in P_m^{-1}(I(f))$.

Now, let $x \in P_m^{-1}(I(f))$. Then $P_m(x) = y$ for some $y \in I(f)$. Thus, $\|f^n(y)\| \rightarrow \infty$ and

$$\|f^n(y)\| = \|f^n \circ P_m(x)\| = \|P_m \circ g^n(x)\|.$$

Since P_m does not have any poles, this implies that $\|g^n(x)\| \rightarrow \infty$ for $n \rightarrow \infty$ and hence $x \in I(g)$. \square

Remark 4.17. P_m is uniformly quasiregular.

Proof. By Lemma 4.15, we have

$$P_m \circ \text{Cos} = \text{Cos} \circ \mathcal{M}_m.$$

Just like in the proof of Lemma 4.16, we obtain

$$P_m^k \circ \text{Cos} = \text{Cos} \circ \mathcal{M}_m^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{M}_m^k(x) = m^k x$ is 1-quasiregular, P_m^k is $(K(\text{Cos}))^2$ -quasiregular for all $k \in \mathbb{N}$. \square

Distortion of cubes

Next, we show that the branches Ψ_r are Lipschitz continuous away from zero. For that, we first prove the following estimate:

Lemma 4.18. *Let $\|y\| \geq \|x\|$ and $0 < \alpha < 1$. Then*

$$\|y\|^\alpha - \|x\|^\alpha \leq \alpha \|y - x\| \cdot \|x\|^{\alpha-1}.$$

Proof.

$$\begin{aligned} \|y\|^\alpha - \|x\|^\alpha &= \int_{\|x\|}^{\|y\|} \frac{d}{dt} t^\alpha dt \\ &\leq \|y\| - \|x\| \alpha \max_{t \in [\|x\|, \|y\|]} t^{\alpha-1} \\ &\leq \alpha \|y - x\| \cdot \|x\|^{\alpha-1}. \end{aligned}$$

□

Now let $r \in R$ with $0 \leq r_j \leq m - 1$, $1 \leq j \leq d - 1$ and consider the inverse branch

$$\Psi_r : \mathbb{H}^\pm \setminus B(0, e^{m-1}) \rightarrow \Omega(r) \setminus B(0, 1), \quad y \mapsto e^{-\frac{m-1}{m}} \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right).$$

Then we get the following Lipschitz estimate:

Lemma 4.19. *Let $\tau \geq 2e^{m-1}$, $\frac{\tau}{2} \leq \|x\| \leq \|y\|$, where x and y are both in \mathbb{H}^+ or both in \mathbb{H}^- . Then the branches Ψ_r , with $\sigma(r)$ even or $\sigma(r)$ odd, respectively, satisfy*

$$\|\Psi_r(y) - \Psi_r(x)\| \leq \frac{C_2}{\tau^{\frac{m-1}{m}}} \|y - x\|,$$

where $C_2 := \left(\frac{2}{e}\right)^{\frac{m-1}{m}} \left(2L_{\tilde{\Psi}} e^{m-1} + \frac{1}{m}\right)$.

Proof. Since $\|y\| \geq \|x\| \geq \frac{\tau}{2} \geq e^{m-1}$, we can write Ψ_r as an angular function times a scaling function. The triangle inequality then yields

$$\begin{aligned} \|\Psi_r(y) - \Psi_r(x)\| &= \left\| e^{-\frac{m-1}{m}} \left(\|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) - \|x\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) \right) \right\| \\ &\leq e^{-\frac{m-1}{m}} \left\| \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) - \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) \right\| \\ &\quad + e^{-\frac{m-1}{m}} \left\| \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) - \|x\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) \right\|. \end{aligned}$$

To estimate the first summand, we use that $\tilde{\Psi}_r$ is bi-Lipschitz with Lipschitz constant $L_{\tilde{\Psi}}$ by Remark 4.13. Hence,

$$\left\| \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{y}{\|y\|} e^{m-1} \right) - \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) \right\| \leq \|y\|^{\frac{1}{m}} L_{\tilde{\Psi}} \left\| \frac{y}{\|y\|} - \frac{x}{\|x\|} \right\| e^{m-1}.$$

Again by the triangle inequality, we obtain

$$\begin{aligned} \|y\|^{\frac{1}{m}} \left\| \frac{y}{\|y\|} - \frac{x}{\|x\|} \right\| &\leq \|y\|^{\frac{1}{m}} \left(\left\| \frac{y}{\|y\|} - \frac{x}{\|y\|} \right\| + \left\| \frac{x}{\|y\|} - \frac{x}{\|x\|} \right\| \right) \\ &= \|y\|^{\frac{1}{m}} \left(\frac{1}{\|y\|} \|y - x\| + \|x\| \left\| \frac{\|x\| - \|y\|}{\|y\| \|x\|} \right\| \right) \\ &\leq \|y\|^{\frac{1}{m}} \frac{2}{\|y\|} \|y - x\| \\ &= 2 \|y\|^{\frac{1}{m}-1} \|y - x\|. \end{aligned}$$

To estimate the second summand, we note that the range of $\tilde{\Psi}_r$ is a subset of the unit sphere. Using Lemma 4.18, we thus obtain

$$\begin{aligned} \left\| \|y\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) - \|x\|^{\frac{1}{m}} \tilde{\Psi}_r \left(\frac{x}{\|x\|} e^{m-1} \right) \right\| &= \left| \|y\|^{\frac{1}{m}} - \|x\|^{\frac{1}{m}} \right| \\ &\leq \frac{1}{m} \|y - x\| \|x\|^{\frac{1}{m}-1}. \end{aligned}$$

Since $\frac{\tau}{2} \leq \|x\| \leq \|y\|$, we finally conclude that

$$\begin{aligned} \|\Psi_r(y) - \Psi_r(x)\| &\leq e^{-\frac{m-1}{m}} \left(2L_{\tilde{\Psi}} e^{m-1} \|y\|^{\frac{1}{m}-1} \|y - x\| + \frac{1}{m} \|y - x\| \|x\|^{\frac{1}{m}-1} \right) \\ &\leq e^{-\frac{m-1}{m}} \left(2L_{\tilde{\Psi}} e^{m-1} \left(\frac{\tau}{2} \right)^{\frac{1}{m}-1} \|y - x\| + \frac{1}{m} \|y - x\| \left(\frac{\tau}{2} \right)^{\frac{1}{m}-1} \right) \\ &= \left(\frac{2}{e} \right)^{\frac{m-1}{m}} \left(2L_{\tilde{\Psi}} e^{m-1} + \frac{1}{m} \right) \frac{1}{\tau^{\frac{m-1}{m}}} \|y - x\| \\ &= \frac{C_2}{\tau^{\frac{m-1}{m}}} \|y - x\|. \end{aligned}$$

□

We have established Lipschitz estimates for the branches Ψ_r , where the Lipschitz constant gets smaller if we move away from the origin. Next, we use that Lipschitz maps have bounded distortion in the sense of Theorem 2.5.

By Remark 4.3, the non-escaping set of $m \text{Cos}(x)$ has finite measure in each tract $T(r)$. By covering $T(r)$ with cubes of side length 2, we obtain a summable sequence of the measure of the non-escaping set in these cubes. Formally, let $m_1, \dots, m_d \in \mathbb{Z}$ and put

$$\tilde{W}(m_1, \dots, m_d) := \{x \in \mathbb{R}^d : |2m_j + 1 - x_j| \leq 1\}.$$

Now with $m_j = r_j$ for $1 \leq j \leq d-1$, we note that either $\widetilde{W}(m_1, \dots, m_d) \subset T(r)$ or $\widetilde{W}(m_1, \dots, -m_d) \subset T(r)$. Let $r_d = 1$ and $m_d \geq 0$. Now put

$$\tilde{\varepsilon}_{m_d} := \text{meas} \left(\widetilde{W}(m_1, \dots, m_d) \cap I^c(m \text{ Cos}) \right).$$

Note that by symmetry of the quasiregular cosine, $\tilde{\varepsilon}_{m_d}$ does not depend on r . Since the measure of the non-escaping set is finite in tracts, we obtain

$$\sum_{m_d=0}^{\infty} \tilde{\varepsilon}_{m_d} < \infty.$$

Now we pull these cubes back by using \mathcal{M}_m^{-1} and put

$$\begin{aligned} W(m_1, \dots, m_d) &:= \frac{1}{m} \widetilde{W}(m_1, \dots, m_d) \\ &= \left\{ x \in \mathbb{R}^d : \left| \frac{2m_j + 1}{m} - x_j \right| \leq \frac{1}{m} \right\}. \end{aligned}$$

Thus, $W(m_1, \dots, m_d)$ denotes the axis-aligned cube with centre $(\frac{2m_1+1}{m}, \dots, \frac{2m_d+1}{m})$ and side length $\frac{2}{m}$. By conjugacy, we have

$$I^c(\text{Cos} \circ \mathcal{M}_m) = \frac{1}{m} I^c(\mathcal{M}_m \circ \text{Cos}).$$

Thus,

$$\begin{aligned} \varepsilon_{m_d} &:= \text{meas} (W(m_1, \dots, m_d) \cap I^c(\text{Cos} \circ \mathcal{M}_m)) \\ &= \text{meas} \left(\frac{1}{m} \widetilde{W}(m_1, \dots, m_d) \cap \frac{1}{m} I^c(m \text{ Cos}) \right) \\ &= \frac{1}{m^d} \tilde{\varepsilon}_{m_d}. \end{aligned}$$

Then again

$$\sum_{m_d=0}^{\infty} \varepsilon_{m_d} < \infty.$$

Lemma 4.20. *Let $(m_1, \dots, m_d) \in \mathbb{Z}^d$ such that*

$$\left\| \left(\frac{2m_1 + 1}{m}, \dots, \frac{2m_d + 1}{m} \right) \right\| \geq \max \left\{ \frac{2\sqrt{d}}{m}, 2e^{m-1} \right\}.$$

Furthermore, let $r \in R$ with $0 \leq r_j \leq m-1$ for $1 \leq j \leq d-1$. Then

$$\text{meas}(\Psi_r(W(m_1, \dots, m_d) \cap I^c(\text{Cos} \circ \mathcal{M}_m))) \leq \frac{m^{\frac{(m-1)d}{m}} \cdot C_2^d}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{2}}} \cdot \varepsilon_{m_d},$$

where C_2 is the constant from Lemma 4.19.

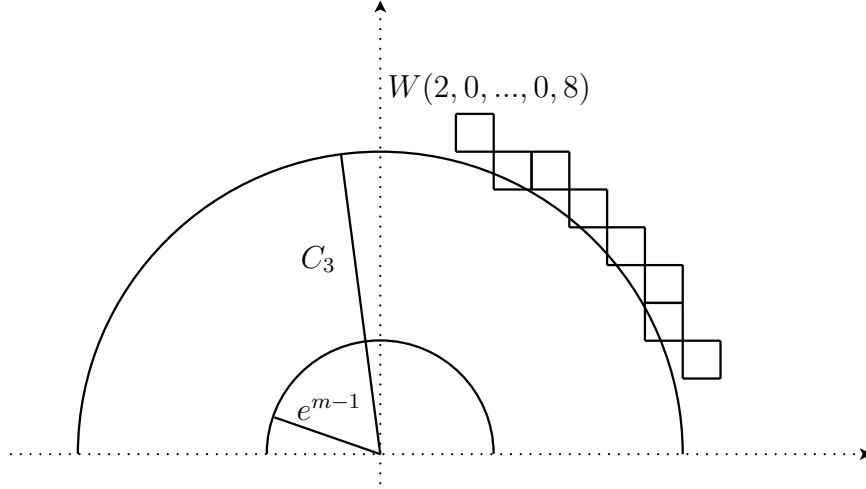


Figure 4.6: Illustrating Lemma 4.20.

Proof. Without loss of generality, let $m_d \geq 0$ and $\sigma(r)$ be even. Put

$$\varrho := \left\| \left(\frac{2m_1 + 1}{m}, \dots, \frac{2m_d + 1}{m} \right) \right\|.$$

Since

$$\text{diam } W(m_1, \dots, m_d) = \frac{2\sqrt{d}}{m},$$

the condition

$$\varrho \geq \max \left\{ \frac{2\sqrt{d}}{m}, 2e^{m-1} \right\}$$

ensures that

$$W(m_1, \dots, m_d) \subset \left\{ x \in \mathbb{R}^d : \frac{\varrho}{2} \leq \|x\|, x_d \geq 0 \right\}$$

as well as $\frac{\varrho}{2} \geq e^{m-1}$. By Lemma 4.19, the branch Ψ_r is Lipschitz on the set

$$\left\{ x \in \mathbb{R}^d : \frac{\varrho}{2} \leq \|x\|, x_d \geq 0 \right\}$$

with Lipschitz constant $\frac{C_2}{\varrho^{\frac{m-1}{m}}}$. Put

$$V(m_1, \dots, m_d) := W(m_1, \dots, m_d) \cap I^c(\text{Cos} \circ \mathcal{M}_m).$$

Then Remark 2.7 and the definition of ε_{m_d} yield

$$\begin{aligned}
\text{meas}(\Psi_r(V(m_1, \dots, m_d))) &\leq \left(\frac{C_2}{\rho^{\frac{m-1}{m}}}\right)^d \text{meas}(V(m_1, \dots, m_d)) \\
&= \left(\frac{C_2}{\rho^{\frac{m-1}{m}}}\right)^d \text{meas}(W(m_1, \dots, m_d) \cap I^c(\text{Cos} \circ \mathcal{M}_m)) \\
&= \frac{C_2^d}{\left\| \left(\frac{2m_1+1}{m}, \dots, \frac{2m_d+1}{m} \right) \right\|^{\frac{(m-1)d}{m}}} \cdot \varepsilon_{m_d} \\
&= \frac{C_2^d}{m^{\frac{(m-1)d}{m}} \cdot C_2^d} \cdot \varepsilon_{m_d} \\
&= \frac{C_2^d}{((2m_1+1)^2 + \dots + (2m_d+1)^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d} \\
&\leq \frac{m^{\frac{(m-1)d}{m}} \cdot C_2^d}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d}.
\end{aligned}$$

□

To proceed, we need the following lemma:

Lemma 4.21. *The sum*

$$\sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ (m_1, \dots, m_d) \neq 0}} \frac{1}{(m_1^2 + \dots + m_d^2)^s}$$

converges if and only if $s > \frac{d}{2}$.

Proof. Using the identity

$$\sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ (m_1, \dots, m_d) \neq 0}} \frac{1}{(m_1^2 + \dots + m_d^2)^s} = \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ (m_1, \dots, m_d) \neq 0}} \frac{1}{\|(m_1^2, \dots, m_d^2)\|^{2s}},$$

we see that the sum converges if and only if the integral

$$\int_{\mathbb{R}^d \setminus B(0,1)} \frac{1}{\|x\|^{2s}} dx$$

exists. Using spherical coordinates, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus B(0,1)} \frac{1}{\|x\|^{2s}} dx &= \int_1^\infty \int_0^{2\pi} \int_0^\pi \dots \int_0^\pi \frac{1}{r^{2s}} r^{d-1} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{d-2} \theta_{d-1} d\theta_{d-1} \dots d\theta_1 dr \\
&= \omega \cdot \int_1^\infty \frac{1}{r^{2s-d+1}} dr,
\end{aligned}$$

where ω denotes again the surface area of \mathbb{S} . The integral on the right hand side exists if and only if $2s - d + 1 > 1$ and thus $s > \frac{d}{2}$. □

It remains to prove the final lemma:

Lemma 4.22. *Let $m > d$. Then*

$$\text{meas}(P_m^{-1}(I^c(\text{Cos} \circ \mathcal{M}_m))) < \infty.$$

Proof. Put

$$C_3 := \max \left\{ \frac{2\sqrt{d}}{m}, 2e^{m-1} \right\}.$$

Let $(m_1, \dots, m_d) \in \mathbb{Z}^d$ such that

$$\left\| \left(\frac{2m_1 + 1}{m}, \dots, \frac{2m_d + 1}{m} \right) \right\| \geq C_3.$$

Without loss of generality, let $m_d \geq 0$. Then the cubes $W(m_1, \dots, m_d)$ cover the upper half-space \mathbb{H}^+ except for some bounded set around zero. Since P_m has degree m^{d-1} and by Lemma 4.20, we obtain

$$\begin{aligned} \text{meas}(P_m^{-1}(V(m_1, \dots, m_d))) &\leq m^{d-1} \frac{m^{\frac{(m-1)d}{m}} \cdot C_2^d}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d} \\ &=: \frac{C_4}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d}. \end{aligned}$$

With $\varrho = \left\| \left(\frac{2m_1+1}{m}, \dots, \frac{2m_d+1}{m} \right) \right\|$, we obtain

$$\begin{aligned} \text{meas} \left(\bigcup_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} P_m^{-1}(V(m_1, \dots, m_d)) \right) &= \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} \text{meas}(P_m^{-1}(V(m_1, \dots, m_d))) \\ &\leq \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} \frac{C_4}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d} \\ &\leq \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, m_d \geq 0, \\ (m_1, \dots, m_d) \neq 0}} \frac{C_4}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d}. \end{aligned}$$

With

$$\begin{aligned} \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, m_d \geq 0 \\ (m_1, \dots, m_d) \neq 0}} \frac{C_4}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d} &\leq \sum_{m_d \geq 0} \varepsilon_{m_d} \sum_{\substack{m_1, \dots, m_{d-1} \in \mathbb{Z}, \\ (m_1, \dots, m_{d-1}) \neq 0}} \frac{C_4}{(m_1^2 + \dots + m_{d-1}^2)^{\frac{m-1}{m} \frac{d}{2}}} \\ &\quad + \sum_{m_d \geq 1} \frac{C_4}{m_d^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d}, \end{aligned}$$

it remains to show that the two sums on the right hand side converge. Since $m > d$, we note that $\frac{m-1}{m} \frac{d}{2} > \frac{d-1}{2}$. Thus, Lemma 4.21 yields the convergence of

$$\sum_{\substack{m_1, \dots, m_{d-1} \in \mathbb{Z}, \\ (m_1, \dots, m_{d-1}) \neq 0}} \frac{1}{(m_1^2 + \dots + m_{d-1}^2)^{\frac{m-1}{m} \frac{d}{2}}}.$$

Since

$$\sum_{m_d=0}^{\infty} \varepsilon_{m_d}$$

converges as well, we obtain the convergence of

$$\sum_{m_d \geq 0} \varepsilon_{m_d} \sum_{\substack{m_1, \dots, m_{d-1} \in \mathbb{Z}, \\ (m_1, \dots, m_{d-1}) \neq 0}} \frac{C_4}{(m_1^2 + \dots + m_{d-1}^2)^{\frac{m-1}{m} \frac{d}{2}}}.$$

For m_d large, we obtain

$$\frac{C_4}{m_d^{\frac{m-1}{m} \frac{d}{2}}} \varepsilon_{m_d} \leq \varepsilon_{m_d}.$$

Thus, the second sum converges as well. We conclude that

$$\text{meas} \left(P_m^{-1}(\mathbb{H}^+ \setminus B(0, 2C_3)) \cap I^c(\text{Cos} \circ \mathcal{M}_m) \right) < \infty$$

and hence

$$\text{meas} \left(P_m^{-1}(\mathbb{H}^+ \cap I^c(\text{Cos} \circ \mathcal{M}_m)) \right) < \infty.$$

By symmetry, we obtain

$$\text{meas} \left(P_m^{-1}(\mathbb{H}^- \cap I^c(\text{Cos} \circ \mathcal{M}_m)) \right) < \infty$$

as well, finishing the proof. \square

Theorem 3 now follows from Lemma 4.16 and Lemma 4.22.

Remark 4.23. Given any fixed $\lambda > 0$, we can choose a bi-Lipschitz map in the construction of $\text{Cos}(x)$ such that $\lambda \text{Cos}(x)$ has an attractive fixed point in the unit ball. This immediately implies that the measure of the non-escaping set of $\lambda \text{Cos}(x)$ is positive in each tract and hence that the measure of the whole non-escaping set of $\lambda \text{Cos}(x)$ is infinite. Thus, for every $m \geq d+1$, there exists an initial bi-Lipschitz map such that the corresponding map $\text{Cos}(x)$ has the following properties: $\text{meas}(I^c(m \text{Cos})) = \infty$ and thus $\text{meas}(I^c(\text{Cos} \circ P_m)) > 0$, but $\text{meas}(I^c(\text{Cos} \circ P_m)) < \infty$.

4.4 Sharpness

In this section, we will see that the condition $m > d$ in Theorem 3 is sharp.

Theorem 4.24. *Let $m \leq d$. Then either*

$$\text{meas}(I^c(\text{Cos} \circ P_m)) = \infty$$

or

$$\text{meas}(I^c(\text{Cos} \circ P_m)) = 0,$$

depending on the initial bi-Lipschitz map in the construction of $\text{Cos}(x)$.

We first prove the following Lipschitz estimate for P_m :

Lemma 4.25. *Let $x, y \in \mathbb{R}^d$ such that $\|x\| \geq \|y\| \geq 1$.*

$$\|P_m(x) - P_m(y)\| \leq (2L_{\tilde{\Phi}} + e^{m-1}) \|x\|^{m-1} \|x - y\|,$$

where $\tilde{\Phi} = P_m|_{\mathbb{S}}$ and $L_{\tilde{\Phi}}$ denotes the Lipschitz constant of $\tilde{\Phi}$.

Proof. Just like in the proof of Lemma 4.19, the triangle inequality yields

$$\begin{aligned} \|P_m(x) - P_m(y)\| &= \left\| \|x\|^m \tilde{\Phi} \left(\frac{x}{\|x\|} \right) - \|y\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) \right\| \\ &\leq \left\| \|x\|^m \tilde{\Phi} \left(\frac{x}{\|x\|} \right) - \|x\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) \right\| \\ &\quad + \left\| \|x\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) - \|y\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) \right\|. \end{aligned}$$

Since $\tilde{\Phi}$ is Lipschitz continuous with Lipschitz constant $L_{\tilde{\Phi}}$, and again by the triangle inequality, we obtain

$$\begin{aligned} \left\| \|x\|^m \tilde{\Phi} \left(\frac{x}{\|x\|} \right) - \|x\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) \right\| &\leq \|x\|^m L_{\tilde{\Phi}} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \\ &\leq \|x\|^m L_{\tilde{\Phi}} \left(\left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| + \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\| \right) \\ &= \|x\|^m L_{\tilde{\Phi}} \left(\frac{\|x - y\|}{\|x\|} + \left| \frac{\|y\| - \|x\|}{\|x\|\|y\|} \right| \|y\| \right) \\ &\leq 2\|x\|^{m-1} L_{\tilde{\Phi}} \|x - y\|. \end{aligned}$$

With $\left\| \tilde{\Phi} \left(\frac{z}{\|z\|} \right) \right\| = e^{m-1}$ for all $z \in \mathbb{R}^d \setminus \{0\}$, we get the following estimate for the second summand:

$$\begin{aligned} \left\| \|x\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) - \|y\|^m \tilde{\Phi} \left(\frac{y}{\|y\|} \right) \right\| &\leq e^{m-1} \left| \|x\|^m - \|y\|^m \right| \\ &= e^{m-1} (\|x\| - \|y\|) \sum_{j=0}^{m-1} \|x\|^j \|y\|^{m-1-j} \\ &\leq e^{m-1} \|x - y\| \|x\|^{m-1} \cdot m. \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} \|P_m(x) - P_m(y)\| &\leq 2\|x\|^{m-1} L_{\tilde{\Phi}} \|x - y\| + m e^{m-1} \|x - y\| \|x\|^{m-1} \\ &= (2L_{\tilde{\Phi}} + m e^{m-1}) \|x\|^{m-1} \|x - y\|. \end{aligned}$$

□

We get the following estimate for the branches of the inverse map:

Lemma 4.26. *Let $\tau \geq \frac{2}{3}e^{m-1}$ and $r \in R$ with $0 \leq r_j \leq m - 1$ for $1 \leq j \leq d - 1$. Furthermore, let $y_1, y_2 \in \mathbb{H}^+$ or $y_1, y_2 \in \mathbb{H}^-$ and $\sigma(r)$ be even or odd, respectively. Then for $e^{m-1} \leq \|y_1\| \leq \|y_2\| \leq \frac{3\tau}{2}$,*

$$\frac{C_5}{\tau^{\frac{m-1}{m}}} \|y_1 - y_2\| \leq \|\Psi_r(y_1) - \Psi_r(y_2)\|$$

holds with some positive constant C_5 .

Proof. Put $x_1 := \Psi_r(y_1)$ and $x_2 := \Psi_r(y_2)$. Since $\|y_2\| \geq \|y_1\| \geq e^{m-1}$, Lemma 4.12 implies that

$$\begin{aligned} \|x_1\| &= e^{-\frac{m-1}{m}} \|y_1\|^{\frac{1}{m}} \\ &\leq e^{-\frac{m-1}{m}} \|y_2\|^{\frac{1}{m}} \\ &= \|x_2\|. \end{aligned}$$

By Lemma 4.25, we obtain

$$\|P_m(x_1) - P_m(x_2)\| \leq (2L_{\tilde{\Phi}} + e^{m-1}) \|x_2\|^{m-1} \|x_1 - x_2\|.$$

Since $\|y\| \leq \frac{3\tau}{2}$, we have

$$\begin{aligned} \|x_2\| &= e^{-\frac{m-1}{m}} \|y_2\|^{\frac{1}{m}} \\ &\leq e^{-\frac{m-1}{m}} \left(\frac{3\tau}{2} \right)^{\frac{1}{m}}. \end{aligned}$$

With $y_1 = P_m(x_1)$, $y_2 = P_m(x_2)$ we thus obtain

$$\begin{aligned} \|y_1 - y_2\| &\leq (2L_{\tilde{\Phi}} + me^{m-1}) \|x_2\|^{m-1} \|\Psi_r(y_1) - \Psi_r(y_2)\| \\ &\leq (2L_{\tilde{\Phi}} + me^{m-1}) \left(e^{-\frac{m-1}{m}} \left(\frac{3\tau}{2} \right)^{\frac{1}{m}} \right)^{m-1} \|\Psi_r(y_1) - \Psi_r(y_2)\|. \end{aligned}$$

Thus,

$$\frac{C_5}{\tau^{\frac{m-1}{m}}} \|y_1 - y_2\| \leq \|\Psi_r(y_1) - \Psi_r(y_2)\|,$$

where

$$C_5 := \frac{\left(\left(\frac{2}{3} \right)^{\frac{1}{m}} e^{\frac{m-1}{m}} \right)^{m-1}}{2L_{\tilde{\Phi}} + me^{m-1}}. \quad \square$$

Remark 4.27. Putting Lemma 4.19 and Lemma 4.26 together, we obtain

$$\frac{C_5}{\tau^{\frac{m-1}{m}}} \|y_1 - y_2\| \leq \|\Psi_r(y_1) - \Psi_r(y_2)\| \leq \frac{C_2}{\tau^{\frac{m-1}{m}}} \|y_1 - y_2\|$$

for $e^{m-1} \leq \frac{\tau}{2} \leq \|y_1\| \leq \|y_2\| \leq \frac{3\tau}{2}$.

The following lemma is the counterpart of Lemma 4.20.

Lemma 4.28. *Let $(m_1, \dots, m_d) \in \mathbb{Z}^d$ with $m_d \geq 0$ such that*

$$\left\| \left(\frac{2m_1 + 1}{m}, \dots, \frac{2m_d + 1}{m} \right) \right\| \geq C_3,$$

where again $C_3 = \max \left\{ \frac{2\sqrt{d}}{m}, 2e^{m-1} \right\}$. Furthermore, let $r \in R$, $0 \leq r_j \leq m - 1$ for $0 \leq j \leq d - 1$, and $\sigma(r)$ even. Then

$$\text{meas}(\Psi_r(W(m_1, \dots, m_d) \cap I^c(\text{Cos} \circ \mathcal{M}_m))) \geq \frac{C_5^d}{(3\sqrt{d})^{\frac{(m-1)d}{m}} (m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d}.$$

Proof. Put again

$$\varrho = \left\| \left(\frac{2m_1 + 1}{m}, \dots, \frac{2m_d + 1}{m} \right) \right\|.$$

Since

$$\text{diam } W(m_1, \dots, m_d) = \frac{2\sqrt{d}}{m}$$

and $\varrho \geq C_3$, we obtain $\frac{\varrho}{2} \geq e^{m-1}$ as well as

$$W(m_1, \dots, m_d) \subset \left\{ x \in \mathbb{R}^d : \|x\| \leq \frac{3\varrho}{2}, x_d \geq 0 \right\}.$$

By Lemma 4.26, Ψ_r is Lipschitz from below on $\{x \in \mathbb{R}^d : \|x\| \leq \frac{3\rho}{2}, x_d \geq 0\}$ with Lipschitz constant $\frac{C_5}{\rho^{\frac{m-1}{m}}}$. Lemma 2.8 yields

$$\begin{aligned} \text{meas}(\Psi_r(V(m_1, \dots, m_d))) &\geq \left(\frac{C_5}{\rho^{\frac{m-1}{m}}}\right)^d \text{meas}(V(m_1, \dots, m_d)) \\ &= \frac{C_5^d}{\left(\left(\frac{2m_1+1}{m}\right)^2 + \dots + \left(\frac{2m_d+1}{m}\right)^2\right)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d} \\ &\geq \frac{C_5^d}{((2m_1+1)^2 + \dots + (2m_d+1)^2)^{\frac{m-1}{m} \frac{d}{2}}} \cdot \varepsilon_{m_d}. \end{aligned}$$

Since $\rho > 0$, we have $m_j \neq 0$ for some j . Thus,

$$\begin{aligned} \frac{C_5^d \cdot \varepsilon_{m_d}}{((2m_1+1)^2 + \dots + (2m_d+1)^2)^{\frac{m-1}{m} \frac{d}{2}}} &\geq \frac{C_5^d \cdot \varepsilon_{m_d}}{(d((2m_1+m_1)^2 + \dots + (2m_d+m_d)^2))^{\frac{m-1}{m} \frac{d}{2}}} \\ &= \frac{C_5^d \cdot \varepsilon_{m_d}}{(3\sqrt{d})^{\frac{(m-1)d}{m}} (m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}}. \end{aligned}$$

□

Lemma 4.29. *Let $m \leq d$. Then the sum*

$$\sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, m_d \geq 0, \\ (m_1, \dots, m_d) \neq 0}} \frac{\varepsilon_{m_d}}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}}$$

either diverges or has the value 0.

Proof. We have to consider two cases:

First, assume that $\varepsilon_{m_d} = 0$ for all $m_d \geq 0$. This implies that

$$\text{meas}(I^c(\text{Cos} \circ \mathcal{M}_m)) = 0.$$

Thus, by Lemma 4.15, we obtain

$$\text{meas}(I^c(\text{Cos} \circ P_m)) = 0.$$

Now assume that $\varepsilon_j > 0$ for some $j \geq 1$. Then

$$\begin{aligned} \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, m_d \geq 0, \\ (m_1, \dots, m_d) \neq 0}} \frac{\varepsilon_{m_d}}{(m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} &\geq \varepsilon_j \sum_{m_1, \dots, m_{d-1} \in \mathbb{Z}} \frac{1}{(m_1^2 + \dots + m_{d-1}^2 + j^2)^{\frac{m-1}{m} \frac{d}{2}}} \\ &\geq \frac{\varepsilon_j}{j^{\frac{d(m-1)}{m}}} \sum_{m_1, \dots, m_{d-1} \in \mathbb{Z}} \frac{1}{(m_1^2 + \dots + m_{d-1}^2 + 1)^{\frac{m-1}{m} \frac{d}{2}}} \\ &\geq \frac{\varepsilon_j}{(\sqrt{2}j)^{\frac{d(m-1)}{m}}} \sum_{\substack{m_1, \dots, m_{d-1} \in \mathbb{Z} \\ (m_1, \dots, m_{d-1}) \neq 0}} \frac{1}{(m_1^2 + \dots + m_{d-1}^2)^{\frac{m-1}{m} \frac{d}{2}}}. \end{aligned}$$

Since $m \leq d$, we obtain $\frac{m-1}{m} \frac{d}{2} \leq \frac{d-1}{2}$. Lemma 4.21 now yields the divergence of the sum on the right hand side. A similar estimate for $j = 0$ finishes the proof. \square

The proof of Theorem 4.24 now follows from the lemmas above:

Proof of Theorem 4.24. With $\varrho = \left\| \left(\frac{2m_1+1}{m}, \dots, \frac{2m_d+1}{m} \right) \right\|$, the cubes $W(m_1, \dots, m_d)$ with $\varrho \geq C_3$ cover the upper half-space except for some bounded set. Now by Lemma 4.28, and since $m \leq d$, we obtain

$$\begin{aligned} \text{meas} \left(\bigcup_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} P_m^{-1}(V(m_1, \dots, m_d)) \right) &= \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} \text{meas} (P_m^{-1}(V(m_1, \dots, m_d))) \\ &\geq \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} \frac{m^{d-1} C_5^d \cdot \varepsilon_{m_d}}{\left(3\sqrt{d} \right)^{\frac{(m-1)d}{m}} (m_1^2 + \dots + m_d^2)^{\frac{m-1}{m} \frac{d}{2}}} \\ &\geq \sum_{\substack{m_1, \dots, m_d \in \mathbb{Z}, \\ m_d \geq 0, \varrho \geq C_3}} \frac{m^{d-1} C_5^d \cdot \varepsilon_{m_d}}{\left(3\sqrt{d} \right)^{\frac{(m-1)d}{m}} (m_1^2 + \dots + m_d^2)^{\frac{d-1}{2}}}. \end{aligned}$$

By Lemma 4.29, the last sum either diverges or has value zero. In the case of divergence, we have

$$\text{meas} (P_m^{-1}(I^c(\text{Cos} \circ \mathcal{M}_m) \cap \mathbb{H}^+)) = \infty.$$

Lemma 4.16 then implies that

$$\text{meas}(I^c(\text{Cos} \circ P_m)) = \infty.$$

The case that the sum has value zero required $\varepsilon_{m_d} = 0$ for all m_d , thus the measure of the non-escaping set of $\text{Cos} \circ \mathcal{M}_m$ has measure zero in each cube $W(m_1, \dots, m_d)$ and hence measure zero in \mathbb{R}^d . Again, Lemma 4.16 implies that in this case

$$\text{meas}(I^c(\text{Cos} \circ P_m)) = 0. \quad \square$$

Bibliography

- [AB12] Magnus Aspenberg and Walter Bergweiler. Entire functions with Julia sets of positive measure. *Math. Ann.*, 352(1):27–54, 2012.
- [AE01] Herbert Amann and Joachim Escher. *Analysis. III. Grundstudium Mathematik*. [Basic Study of Mathematics]. Birkhäuser Verlag, Basel, 2nd edition, 2001.
- [Bak75] I. N. Baker. The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):277–283, 1975.
- [BDF14] Walter Bergweiler, David Drasin, and Alastair Fletcher. The fast escaping set for quasiregular mappings. *Anal. Math. Phys.*, 4(1-2):83–98, 2014.
- [BE11] Walter Bergweiler and Alexandre Eremenko. Dynamics of a higher dimensional analog of the trigonometric functions. *Ann. Acad. Sci. Fenn. Math.*, 36(1):165–175, 2011.
- [Ber93] Walter Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N.S.)*, 29(2):151–188, 1993.
- [Ber10a] Walter Bergweiler. Iteration of quasiregular mappings. *Comput. Methods Funct. Theory*, 10(2):455–481, 2010.
- [Ber10b] Walter Bergweiler. Karpińska’s paradox in dimension 3. *Duke Math. J.*, 154(3):599–630, 2010.
- [Ber13] Walter Bergweiler. Fatou-Julia theory for non-uniformly quasiregular maps. *Ergodic Theory Dynam. Systems*, 33(1):1–23, 2013.
- [BFLM09] Walter Bergweiler, Alastair Fletcher, Jim Langley, and Janis Meyer. The escaping set of a quasiregular mapping. *Proc. Amer. Math. Soc.*, 137(2):641–651, 2009.
- [BH99] Walter Bergweiler and A. Hinkkanen. On semiconjugation of entire functions. *Math. Proc. Cambridge Philos. Soc.*, 126(3):565–574, 1999.

- [BN14] Walter Bergweiler and Daniel A. Nicks. Foundations for an iteration theory of entire quasiregular maps. *Israel J. Math.*, 201(1):147–184, 2014.
- [DH84] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie I*, volume 84 of *Publications Mathématiques d’Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.
- [DH85] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes Partie II*, volume 85 of *Publications Mathématiques d’Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.
- [DK84] Robert L. Devaney and Michał Krych. Dynamics of $\exp(z)$. *Ergodic Theory Dynam. Systems*, 4(1):35–52, 1984.
- [DT86] Robert L. Devaney and Folkert Tangerman. Dynamics of entire functions near the essential singularity. *Ergodic Theory Dynam. Systems*, 6(4):489–503, 1986.
- [EL92] A. È. Erëmenko and M. Yu. Lyubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble)*, 42(4):989–1020, 1992.
- [Erë89] A. È. Erëmenko. On the iteration of entire functions. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 339–345. PWN, Warsaw, 1989.
- [Fal86] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.
- [Fat19] P. Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47:161–271, 1919.
- [Fat26] P. Fatou. Sur l’itération des fonctions transcendentes entières. *Acta Math.*, 47(4):337–370, 1926.
- [FL80] Wolfgang Fischer and Ingo Lieb. *Funktionentheorie*, volume 47 of *Vieweg Studium: Aufbaukurs Mathematik [Vieweg Studies: Mathematics Course]*. Friedr. Vieweg & Sohn, Braunschweig, 7th edition, 1980. Aufbaukurs Mathematik.

- [FN11] Alastair N. Fletcher and Daniel A. Nicks. Quasiregular dynamics on the n -sphere. *Ergodic Theory Dynam. Systems*, 31(1):23–31, 2011.
- [FN13] Alastair N. Fletcher and Daniel A. Nicks. Chaotic dynamics of a quasiregular sine mapping. *J. Difference Equ. Appl.*, 19(8):1353–1360, 2013.
- [Guz81] Miguel de Guzman. *Real variable methods in Fourier analysis / Miguel de Guzman*. North-Holland Pub. Co. ; sole distributors for the U.S.A. and Canada, Elsevier North-Holland Amsterdam ; New York : New York, 1981.
- [Hem05] Jan-Martin Hemke. Recurrence of entire transcendental functions with simple post-singular sets. *Fund. Math.*, 187(3):255–289, 2005.
- [IM01] Tadeusz Iwaniec and Gaven Martin. *Geometric function theory and non-linear analysis*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.
- [Jul18] Gaston Julia. Sur l'itération des fonctions rationnelles. *J. Math. Pures Appl.*, 1918.
- [Kar99a] Bogusława Karpińska. Area and Hausdorff dimension of the set of accessible points of the Julia sets of λe^z and $\lambda \sin z$. *Fund. Math.*, 159(3):269–287, 1999.
- [Kar99b] Bogusława Karpińska. Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(11):1039–1044, 1999.
- [Lav38] M. Lavrentiev. Sur un critère différentiel des transformations homéomorphes des domaines à trois dimensions. *Dokl. Akad. Nauk*, 22:241–242, 1938.
- [Mat95] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
- [May97] Volker Mayer. Uniformly quasiregular mappings of Lattès type. *Conform. Geom. Dyn.*, 1:104–111 (electronic), 1997.
- [McM87] Curt McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300(1):329–342, 1987.
- [Min82] Ruth Miniowitz. Normal families of quasimeromorphic mappings. *Proc. Amer. Math. Soc.*, 84(1):35–43, 1982.
- [MS75] Olli Martio and Uri Srebro. Periodic quasimeromorphic mappings in n . *Journal d'Analyse Mathématique*, 28(1):20–40, 1975.

- [Reš66] Ju. G. Rešetnjak. Estimates of the modulus of continuity for certain mappings. *Sibirsk. Mat. Ž.*, 7:1106–1114, 1966.
- [Ric80] Seppo Rickman. On the number of omitted values of entire quasiregular mappings. *J. Analyse Math.*, 37:100–117, 1980.
- [Ric93] Seppo Rickman. *Quasiregular mappings*, volume 26 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.
- [RS86] S. Rickman and U. Srebro. Remarks on the local index of quasiregular mappings. *J. Analyse Math.*, 46:246–250, 1986.
- [RS05] P. J. Rippon and G. M. Stallard. On questions of Fatou and Eremenko. *Proc. Amer. Math. Soc.*, 133(4):1119–1126 (electronic), 2005.
- [Sch07] Dierk Schleicher. The dynamical fine structure of iterated cosine maps and a dimension paradox. *Duke Math. J.*, 136(2):343–356, 2007.
- [Sch08] Hendrik Schubert. Area of Fatou sets of trigonometric functions. *Proc. Amer. Math. Soc.*, 136(4):1251–1259, 2008.
- [Sul83] Dennis Sullivan. Conformal dynamical systems. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 725–752. Springer, Berlin, 1983.
- [Sul85] Dennis Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math. (2)*, 122(3):401–418, 1985.
- [Väi80] Jussi Väisälä. A survey of quasiregular maps in \mathbf{R}^n . In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 685–691. Acad. Sci. Fennica, Helsinki, 1980.
- [Vuo88] Matti Vuorinen. *Conformal geometry and quasiregular mappings*, volume 1319 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- [Zor67] V. A. Zorič. M. A. Lavrent’ev’s theorem on quasiconformal space maps. *Mat. Sb. (N.S.)*, 74 (116):417–433, 1967.

Nomenclature

\mathcal{B}	Eremenko-Lyubich class, page 10
$B(u, r_n)$	Largest ball in $\varphi_n(Q(v))$ with centre u , page 35
B_f	Branch set of f , page 17
β	Positive constant, page 22
C_1	Positive constant in Lemma 3.1 depending on d , page 29
C_2	Positive constant in Lemma 4.19, page 67
C_3	Positive constant, page 72
C_4	Positive constant, page 72
C_5	Positive constant, page 76
$\text{Cos}(x)$	Quasiregular cosine, page 55
$\text{dens}(A, B)$	Density of A in B , page 7
$Df(x)$	Derivative of f at x , page 13
$\Delta_n(w_d)$	Rapidly decreasing sequence, page 29
$\delta(t)$	Decreasing function, page 32
$\tilde{\delta}(t)$	Decreasing function, page 30
$E_{\frac{1}{2}}(t)$	Exponential map of $\frac{1}{2}t$, page 28
ε_{m_d}	Measure of $I^c(\text{Cos} \circ \mathcal{M}_m)$ in the cube $W(m_1, \dots, m_d)$, page 69
η	Positive constant, page 41
$\tilde{\eta}$	Positive constant, page 38
f^n	n th iterate of f , page 7

f_d^n	d th component of the n th iterate of f , page 28
$\mathbb{H}^+, \mathbb{H}^-$	Upper and lower half-space, page 19
h	Angular map of $\text{Sin}(x)$, page 21
h	Angular map of $\text{Cos}(x)$, page 55
$I(f)$	Escaping set f , page 10
$I^c(f)$	Non-escaping set of f , page 27
$i(f, x)$	Local index of f at x , page 18
$J_f(x)$	Jacobian of f at x , page 13
K	Positive constant, page 38
L_h	Bi-Lipschitz constant of h , page 21
L	Set of points, where the modulus of the d th component of f is “large”, page 28
$\ell(Df(x))$	Like operator norm of $Df(x)$, but with infimum instead of supremum, page 14
Λ^r	Inverse branch of $\text{Sin}(x)$ with respect to r , page 22
\mathcal{M}_m	Multiplication by the natural number m , page 56
ν	Measure of the d -dimensional unit ball, page 7
ω	Surface area of the unit-sphere in \mathbb{R}^d , page 7
$\Omega(r)$	Sector with label r , page 60
$\tilde{\Omega}(r)$	Intersection of $\Omega(r)$ with the unit sphere, page 60
$p(x)$	The projection from \mathbb{R}^d onto \mathbb{R}^{d-1} , page 28
P_m	Quasiregular power mapping based on $\text{Cos}(x)$, page 56
φ_n	Pullback with respect to external address, page 35
$\tilde{\Phi}$	P_m restricted to the unit-sphere, page 74
Ψ_r	Inverse branch of P_m from \mathbb{H}^\pm onto the sector $\Omega(r)$, page 61

$\tilde{\Psi}_r$	Inverse branch of $P_m _S$ from $e^{m-1}\mathbb{S}^\pm$ onto $\tilde{\Omega}(r)$, page 61
$Q(x)$	Axis-aligned cube with centre x and side length of modulus x_d , page 28
R	Set of labels for tracts, page 22
ϱ	Norm of the centre of $W(m_1, \dots, m_d)$, page 70
S	Set of points, where the modulus of f_d is “small”, page 28
$\text{Sin}(x)$	Quasiregular sine, page 21
\underline{s}	External address, page 23
$\sigma(r)$	Counts the number of reflections to obtain $T(r)$, page 22
\mathbb{S}	Unit-sphere in \mathbb{R}^d , page 7
\mathbb{S}^\pm	Upper or lower half-sphere of radius 1, page 61
\mathbb{S}_b	Spherical cap of height $1 - b$, page 29
\mathbb{S}_b^c	Cupped unit-sphere of height b , page 29
\mathcal{T}	Set of points which stay in L under iteration, page 28
\mathcal{T}_n	Set of points which stay in L for n iterations, page 28
$T(r)$	Tract with label r , page 22
$T(r_0)$	The initial tract in the construction of the quasiregular sine, page 22
u	Point in \mathcal{T}_{n-1} , page 34
v	n th forward image of u under f , page 34
$V(m_1, \dots, m_d)$	The cube $W(m_1, \dots, m_d)$ intersected by $I^c(\text{Cos} \circ \mathcal{M}_m)$, page 71
$\widetilde{W}(m_1, \dots, m_d)$	Unit cube around (m_1, \dots, m_d) , page 69
$W(m_1, \dots, m_d)$	The cube $\widetilde{W}(m_1, \dots, m_d)$ scaled by $1/m$, page 69
$x_{0,n}$	n th iterate of x_0 under $E_{\frac{1}{2}}$, page 38

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Abhandlung - abgesehen von der Beratung durch meinen Betreuer - inhaltlich und der Form nach unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe.

Diese Arbeit hat weder ganz, noch in Teilen einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen und wurde weder veröffentlicht, noch zur Veröffentlichung eingereicht.

Kiel, den

Sebastian Vogel