

Varieties of elementary Lie algebras

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Abstract

Friedlander and Parshall developed a theory of support varieties and rank varieties for finite-dimensional restricted Lie algebras over a field k of characteristic $p > 0$. In particular, for any restricted Lie algebra \mathfrak{g} , the rank variety can be identified with its nullcone $V(\mathfrak{g})$ which is a conical closed subvariety of \mathfrak{g} . By a result of Carlson, the projectivization of $V(\mathfrak{g})$ is connected. Recently, Carlson, Friedlander and Pevtsova introduced the elementary subalgebras of restricted Lie algebras. The varieties of elementary subalgebras are natural varieties in which to define generalized support varieties for restricted representations of \mathfrak{g} . Farnsteiner defined an invariant for modules of restricted Lie algebras, called j -degrees. The invariant, together with the constant j -rank property can be linked by the two-dimensional elementary Lie subalgebras.

The aim of this thesis is to understand the geometric structure of the variety $\mathbb{E}(2, \mathfrak{g})$. We first investigate the non-empty property of $\mathbb{E}(2, \mathfrak{g})$. By using the sandwich elements, we give a complete description of the restricted Lie algebras which have no two-dimensional elementary subalgebras. Moreover, we show that the variety $\mathbb{E}(2, \mathfrak{g})$ is always connected whenever \mathfrak{g} is centerless and $p > 5$. This is a generalization of Carlson's result, which claims that $\mathbb{E}(1, \mathfrak{g})$ is connected.

We also study the degree functions \deg_M^j for centerless restricted Lie algebra. We show that the function \deg_M^j is constant on $\mathbb{E}(2, \mathfrak{g})$ if M is a restricted \mathfrak{g} -module of constant j -rank.

In the last, we will endow the set $\mathbb{E}(2, \mathfrak{g})$ with a graph structure which plays an important role in the investigation of the equal images property. We give a sufficient condition for the connectedness of the graph $\mathbb{E}(2, \mathfrak{g})$. As a consequence, the graph $\mathbb{E}(2, W(n))$ is connected, where $W(n)$ is the n -th Witt-Jacobson algebra.

Zusammenfassung

Friedlander und Parshall haben eine Theorie der Trägervarietäten und Rangvarietäten für endlich-dimensionale restringierte Lie-Algebren über einem Körper k der Charakteristik $p > 0$ entwickelt. Insbesondere kann die Rangvarietät einer Lie-Algebra \mathfrak{g} mit dem Nullkegel $V(\mathfrak{g})$ identifiziert werden, der eine kegelförmige Untervarietät von \mathfrak{g} ist. Die Projektivierung von $V(\mathfrak{g})$ ist nach einem Resultat von Carlson zusammenhängend. Kürzlich haben Carlson, Friedlander und Pevtsova den Begriff der elementaren Unter algebra einer restringierten Lie-Algebra eingeführt. Die Varietäten elementarer Unter algebren sind natürliche Varietäten, die man als verallgemeinerte Trägervarietäten für restringierte Darstellungen von \mathfrak{g} definieren kann. Farnsteiner hat eine Invariante für Moduln restringierter Lie-Algebren definiert, den sogenannten j -Grad. Durch zweidimensionale elementare Unter algebren lässt sich ein Zusammenhang zwischen dieser Invariante und der konstanten j -Rang-Eigenschaft herstellen.

Es ist das Ziel dieser Arbeit, die geometrische Struktur der Varietät $\mathbb{E}(2, \mathfrak{g})$ zu verstehen. Wir untersuchen zunächst, wann $\mathbb{E}(2, \mathfrak{g})$ nicht leer ist. Durch Benutzung sogenannter Sandwich-Elemente geben wir eine vollständige Beschreibung derjenigen restringierten Lie-Algebren, die keine zweidimensionalen elementaren Unter algebren haben. Außerdem zeigen wir, dass die Varietät $\mathbb{E}(2, \mathfrak{g})$ zusammenhängend ist, wenn \mathfrak{g} kein Zentrum hat und $p > 5$. Dies ist eine Verallgemeinerung eines Resultates von Carlson, das besagt, dass $\mathbb{E}(1, \mathfrak{g})$ zusammenhängend ist.

Wir untersuchen außerdem die Betragsfunktion \deg_M^j für restringierte Lie-Algebren ohne Zentrum. Wir zeigen, dass die Funktion \deg_M^j auf $\mathbb{E}(2, \mathfrak{g})$ konstant ist, wenn M ein restringierter \mathfrak{g} -Modul von konstantem j -Rang ist.

Im letzten Kapitel versehen wir $\mathbb{E}(2, \mathfrak{g})$ mit der Struktur eines Graphen, die eine wichtige Rolle bei der Untersuchung der Equal-Images-Eigenschaft spielt. Wir geben eine hinreichende Bedingung dafür, dass der Graph von $\mathbb{E}(2, \mathfrak{g})$ zusammenhängend ist. Als eine Folgerung erhalten wir, dass der Graph von $\mathbb{E}(2, W(n))$ zusammenhängend ist, wobei $W(n)$ die n -te Witt-Jacobson-Algebra ist.

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Chapter 1

Introduction and notation

1.1 Introduction

Addressing the study the representations of a finite-dimensional restricted Lie algebra over an algebraically closed field k of characteristic $p > 0$, Friedlander and Parshall introduced the rank varieties for the restricted Lie algebra ([14],[15]). More precisely, let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra and M be a restricted \mathfrak{g} -module. Denote by $V(\mathfrak{g})$ the nullcone of \mathfrak{g} . Then the rank variety of M is defined via:

$$V_{\mathfrak{g}}(M) := \{x \in V(\mathfrak{g}); M|_{k[x]} \text{ is not free} \} \cup \{0\}.$$

This is a conical closed subvariety of $V(\mathfrak{g})$. Let $\text{Proj}(V_{\mathfrak{g}}(M))$ be its projectivization. By a result of Carlson, the variety $\text{Proj}(V_{\mathfrak{g}}(M))$ is connected provided M is indecomposable. In particular, $\text{Proj}(V(\mathfrak{g}))$ is connected(see Proposition 1.3.1).

Recently, Carlson, Friedlander and Pevtsova introduced the notion of so-called elementary subalgebras (see [6]). The subset

$$\mathbb{E}(r, \mathfrak{g}) := \{\mathfrak{e} \in \text{Gr}_r(\mathfrak{g}); \mathfrak{e} \text{ is an elementary subalgebra}\}.$$

has a natural projective variety structure(Proposition 2.1.1). The varieties of elementary subalgebras are natural varieties in which to define generalized support varieties for restricted representations of \mathfrak{g} . In particular, $\mathbb{E}(1, \mathfrak{g}) = \text{Proj}(V(\mathfrak{g}))$.

In this paper, we consider the projective variety $\mathbb{E}(2, \mathfrak{g})$. We first investigate the non-empty property of $\mathbb{E}(2, \mathfrak{g})$. It is well-known that $\mathbb{E}(1, \mathfrak{g}) = \emptyset$ if and only if \mathfrak{g} is a torus([7]). We prove:

Theorem A. Suppose that $p \geq 5$. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra such that $\mathbb{E}(2, \mathfrak{g}) = \emptyset$. Then we have

$$\mathfrak{g} \cong \mathfrak{sl}_2, \mathfrak{b}_{\mathfrak{sl}_2}, (0),$$

where $\mathfrak{b}_{\mathfrak{sl}_2}$ is the Borel subalgebra of \mathfrak{sl}_2 .

Furthermore, we would like to determine the geometric structure of $\mathbb{E}(2, \mathfrak{g})$. Recall that $\mathbb{E}(1, \mathfrak{g})$ is always a connected variety, but in general this is not true for $\mathbb{E}(2, \mathfrak{g})$ (see Example 1.3.2).

Theorem B. Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.

In [10], Farnsteiner defined the invariants for the modules of restricted Lie algebras, called j -degrees. Let M be a $U_0(\mathfrak{g})$ -module of constant j -rank. Suppose that $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$. Then we can restrict the module to the two-dimensional elementary subalgebra. This results a function:

$$\deg_M^j : \mathbb{E}(2, \mathfrak{g}) \rightarrow \mathbb{N}_0; \quad \mathfrak{e} \mapsto \deg^j(M|_{\mathfrak{e}})$$

is called the j -degree function of M .

Theorem C. Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra. If M is a $U_0(\mathfrak{g})$ -module of constant j -rank, then the function \deg_M^j is constant.

We thus show that the degree does not depend on the choice of the two-dimensional elementary subalgebra.

This thesis is organized as follows: In Chapter 2, we introduce the sandwich elements and give some properties. We give a necessary and sufficient condition for $\mathbb{E}(2, \mathfrak{g}) = \emptyset$ when \mathfrak{g} is centerless. As an application, we describe the structure $V(\mathfrak{g})$ when $\dim V(\mathfrak{g}) = 2$. We will investigate the geometric properties of $\mathbb{E}(2, \mathfrak{g})$ in Chapter 3. In Chapter 4, we study the degree functions and prove Theorem C. Last, we endow the $\mathbb{E}(2, \mathfrak{g})$ with the structure of a graph and give a sufficient condition such that the graph $\mathbb{E}(2, \mathfrak{g})$ is connected.

1.2 Notations and Preliminaries

Throughout, k denotes an algebraically closed field, $\text{char}(k) = p > 0$. All the vector spaces over k will be assumed to be finite dimensional. We are interested in a particular class of Lie algebras, called restricted (or Lie p -algebras). This was introduced by Jacobson [19]. The reader is referred to [31, Chapter II] for basic facts on restricted Lie algebras.

Definition 1.2.1. A Lie algebra \mathfrak{g} over k is said to be *restricted* if there exists a map, $[p] : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

1. $\text{ad}(x^{[p]}) = \text{ad}(x)^p$ for all $x \in \mathfrak{g}$,
2. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for all $\lambda \in k, x \in \mathfrak{g}$,
3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x, y)}{i}$ for all $x, y \in \mathfrak{g}$, where $s_i(x, y)$ is the coefficient of t^{i-1} in the formal expression $\text{ad}(tx + y)^{p-1}(x)$.

Example 1.2.2. 1. If G is a linear algebraic group, then the Lie algebra of G is a restricted Lie algebra, we will denote by $\mathfrak{g} := \text{Lie}(G)$ the Lie algebra of G .

2. We can endow any abelian Lie algebra \mathfrak{g} with trivial p -map structure, i.e., $x^{[p]} = 0$ for any $x \in \mathfrak{g}$. Then \mathfrak{g} becomes a restricted Lie algebra.

We call a restricted Lie algebra $(\mathfrak{g}, [p])$ is *p -trivial*, provided $[p] = 0$. If $(\mathfrak{g}, [p])$ is a restricted Lie algebra, an element $x \in \mathfrak{g}$ is said to be *semisimple* if $x \in \text{span}\{x^{[p]}, x^{[p]^2}, \dots\}$ and is said to be *nilpotent* if $x^{[p]^n} = 0$ for some n . A *torus* \mathfrak{t} is a subalgebra (necessarily abelian) such that every $x \in \mathfrak{t}$ is semisimple. An element $x \in \mathfrak{g}$ is called *toral* if $x^{[p]} = x$. Any torus $\mathfrak{t} \subseteq \mathfrak{g}$ has a basis consisting of toral elements (see [31, Page 82]).

The following theorem is due to Seligman [29].

Theorem 1.2.1. (*Jordan-Chevalley-Seligman Decomposition*). *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k and $x \in \mathfrak{g}$. Then there exist unique elements $x_s, x_n \in \mathfrak{g}$ satisfying:*

$$(a) \quad x = x_s + x_n,$$

(b) x_s is semisimple and x_n is nilpotent,

(c) $[x_s, x_n] = 0$.

Let \mathfrak{g} be a Lie algebra. The sequence $(\mathfrak{g}^n)_{n \in \mathbb{N}}$ of ideals of \mathfrak{g} given by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^{n+1} := [\mathfrak{g}^n, \mathfrak{g}]$ is referred to as the *descending central series* of \mathfrak{g} . Abelian Lie algebras are described by the condition $\mathfrak{g}^2 = 0$. \mathfrak{g} is called *nilpotent* if there is $s \in \mathbb{N}$ such that $\mathfrak{g}^s = 0$.

Let V be a finite-dimensional vector space over k . Let $S \subseteq \text{End}_k(V)$ be a subset. S is called *strictly triangulable on V* if there exists a finite sequence $(V_i)_{0 \leq i \leq m}$ of subspaces of V such that

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V; \quad f(V_i) \subset V_{i-1}, \quad \forall f \in S.$$

Hence, any Lie algebra \mathfrak{g} is nilpotent if and only if $\text{ad}(\mathfrak{g})$ is strictly triangulable on \mathfrak{g} . A subset of $S \neq \emptyset$ of an associative algebra is called *Lie set* if it is closed under Lie multiplication.

Theorem 1.2.2. (*Engel-Jacobson Theorem*). *Let V be a vector space and Let $S \subseteq \text{End}_k(V)$ be a Lie subset of $\text{End}_k(V)$ such that*

(a) s is nilpotent for any $s \in S$.

(b) $\mathfrak{g} := \langle S \rangle$ is finite dimensional

Then \mathfrak{g} is strictly triangulable on V .

1.3 Nullcones of restricted Lie algebras

In this section we consider restricted representations of restricted Lie algebras $(\mathfrak{g}, [p])$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra of \mathfrak{g} and recall that $U(\mathfrak{g})$ is a cocommutative Hopf algebra, whose comultiplication $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ and counit $\varepsilon : U(\mathfrak{g}) \rightarrow k$ are determined by

$$\Delta(x) = 1 \otimes x + x \otimes 1; \quad \varepsilon(x) = 0 \quad \forall x \in \mathfrak{g},$$

respectively. The ideal $I := (\{x^p - x^{[p]}; x \in \mathfrak{g}\})$ satisfies

$$\Delta(I) \subseteq I \otimes_k U(\mathfrak{g}) + U(\mathfrak{g}) \otimes_k I \quad \text{and} \quad \varepsilon(I) = (0).$$

This implies that the *restricted enveloping algebra*

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/I$$

of $(\mathfrak{g}, [p])$ inherits from $U(\mathfrak{g})$ the structure of a Hopf algebra. By the Theorem of Poincaré-Birkhoff-Witt, we have $\dim_k U_0(\mathfrak{g}) = p^{\dim_k \mathfrak{g}}$. We let

$$V(\mathfrak{g}) := \{x \in \mathfrak{g}; x^{[p]} = 0\}$$

be the *nullcone* of \mathfrak{g} . This is a conical, Zariski closed subset of \mathfrak{g} . By definition, a restricted Lie algebra $(\mathfrak{g}, [p])$ is p -trivial if and only if $\mathfrak{g} = V(\mathfrak{g})$.

At the end of their paper, Friedlander and Parshall [15, Section 3] posed the following question.

Problem 1.3.1. For which restricted Lie algebras, is $V(\mathfrak{g})$ irreducible?

Moreover, it is known that $V(\mathfrak{g})$ is indeed irreducible for reductive Lie algebra ([24], see [32] for the bad characteristic).

Example 1.3.2. Let $\widetilde{\mathfrak{sl}}_2 := \mathfrak{sl}_2 \oplus k$ be a central extension of \mathfrak{sl}_2 , which splits as an extension of ordinary Lie algebras, and whose p -map is defined via the p -semilinear map

$$\psi : \mathfrak{sl}_2 \rightarrow k; \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto a^p. \quad (1.1)$$

This means that $(x, \alpha)^{[p]} = (x^{[p]}, \psi(x))$ for all $(x, \alpha) \in \widetilde{\mathfrak{sl}}_2$. Then

$$V(\widetilde{\mathfrak{sl}}_2) = (\ker \psi \cap V(\mathfrak{sl}_2)) \times k = \left(k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times k \right) \cup \left(k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times k \right)$$

is a two-dimensional, reducible variety.

Notation 1.3.3. If V is a conical, closed subset of \mathfrak{g} , then we denote by $\text{Proj}(V)$ the projectivation of V .

Let M be a $U_0(\mathfrak{g})$ -module. Friedlander and Parshall [14] introduced the *rank variety*

$$V(\mathfrak{g})_M := \{x \in V(\mathfrak{g}); M_{k[x]} \text{ is not free}\} \cup \{0\}.$$

The definition of $V(\mathfrak{g})_M$ parallels that of Carlson's rank varieties (see [4]). It is easy to see that $V(\mathfrak{g})_M$ is a conical, closed subset of $V(\mathfrak{g})$. Furthermore, the variety gives a geometric characterization of $U_0(\mathfrak{g})$ -modules. We have the following ([15, Corollary 1.4, Theorem 2.2 and Corollary 2.3], see also [5]):

Proposition 1.3.1. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and let M be a $U_0(\mathfrak{g})$ -module.*

- (a) *M is a projective module if and only if the rank variety $V(\mathfrak{g})_M = \{0\}$,*
- (b) *If M is an indecomposable $U_0(\mathfrak{g})$ -module, then the projective variety $\text{Proj}(V(\mathfrak{g})_M)$ is connected.*

In particular, the projective variety $\text{Proj}(V(\mathfrak{g}))$ is connected.

A \mathfrak{g} -module M is called *restricted* if, for all $x \in \mathfrak{g}$,

$$(x^p - x^{[p]}) \cdot M = (0).$$

By the definition, the restricted \mathfrak{g} -modules can be identified with the $U_0(\mathfrak{g})$ -modules. For instance, consider the adjoint representation, \mathfrak{g} is a restricted \mathfrak{g} -module (see (1.2.1)). In fact, we have the element $x^p - x^{[p]} \in Z(\mathfrak{g})$ for all $x \in \mathfrak{g}$, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ (cf. [20, Section 2])

Chapter 2

Elementary subalgebras

In this chapter, we will introduce the elementary subalgebras which defined by Carlson, Friedlander and Pevtsova in [6]. These algebras form a closed subsets of certain Grassmann varieties and hence are projective varieties. We study these varieties and give a necessary and sufficient condition for the variety to be non-empty set. The non-empty property plays an important role in determining some invariants of restricted \mathfrak{g} -modules (see [10]).

2.1 Definition and basic properties

Definition 2.1.1. [6, Definition 1.2]. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. An *elementary subalgebra* $\mathfrak{e} \subseteq \mathfrak{g}$ of dimension r is a Lie subalgebra of dimension r which is abelian and has a trivial p -map.

For an arbitrary $r \in \mathbb{N}$, we denote by $\text{Gr}_r(\mathfrak{g})$ the *Grassmann variety* of r -dimensional subspaces of \mathfrak{g} . We define the subset

$$\mathbb{E}(r, \mathfrak{g}) := \{\mathfrak{e} \in \text{Gr}_r(\mathfrak{g}); \mathfrak{e} \text{ is an elementary subalgebra}\}.$$

We denote by $C_r(V(\mathfrak{g})) \subseteq (V(\mathfrak{g}))^{\times r}$ the variety of r -tuples of p -nilpotent, pairwise commuting elements of \mathfrak{g} , and by $C_r(V(\mathfrak{g}))^\circ \subseteq C_r(V(\mathfrak{g}))$ the subset of linearly independent r -tuples of p -nilpotent, pairwise commuting elements of \mathfrak{g} . Since the condition for linear dependence is Zariski closed, it follows that $C_r(V(\mathfrak{g}))^\circ \subseteq C_r(V(\mathfrak{g}))$ is

an open subvariety. Let $(v_1, \dots, v_r) \in C_r(V(\mathfrak{g}))^\circ$. Since the v_i 's commute pairwise, there is a natural action of GL_r on $C_r(V(\mathfrak{g}))^\circ$ via

$$g \cdot (v_1, \dots, v_r) := (v_1, \dots, v_r)g^{-1}; \quad \forall (v_1, \dots, v_r) \in C_r(V(\mathfrak{g}))^\circ, g \in \mathrm{GL}_r.$$

By the definition, there is a natural map $\pi : C_r(V(\mathfrak{g}))^\circ \rightarrow \mathbb{E}(r, \mathfrak{g})$. Then we have

Proposition 2.1.1. [6, Proposition 1.4]. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then $\mathbb{E}(r, \mathfrak{g})$ is a closed subvariety of $\mathrm{Gr}_r(\mathfrak{g})$. In particular, $\mathbb{E}(r, \mathfrak{g})$ is a projective variety.*

Remark 2.1.2. (a) According to the proof of [6, Proposition 1.4], the map $\pi : C_r(V(\mathfrak{g}))^\circ \rightarrow \mathbb{E}(r, \mathfrak{g})$ is GL_r -torsors locally trivial for the Zariski topology. We have

$$\dim \mathbb{E}(r, \mathfrak{g}) = \dim C_r(V(\mathfrak{g}))^\circ - \dim \mathrm{GL}_r = \dim C_r(V(\mathfrak{g}))^\circ - r^2$$

whenever $\mathbb{E}(r, \mathfrak{g}) \neq \emptyset$.

(b) If $r = 1$, then $\mathbb{E}(1, \mathfrak{g}) = \mathrm{Proj}(V(\mathfrak{g}))$. The theorem by Carlson (Proposition 1.3.1) asserts that $\mathbb{E}(1, \mathfrak{g})$ is connected.

2.2 Sandwich elements

Let \mathfrak{g} be a finite dimensional Lie algebra over k . A non-zero element $c \in \mathfrak{g}$ is called an *absolute zero-divisor* if $(\mathrm{ad} c)^2 = 0$ (see [21]). In more recent terminology c is sometimes referred to as a *sandwich element*. Since Lie algebras containing sandwich elements have a degenerate Killing form, Kostrikin referred to them as *algebras with strong degeneration*. We let

$$\mathrm{Sw}(\mathfrak{g}) := \{c \in \mathfrak{g}; (\mathrm{ad} c)^2 = 0\}$$

be the closed conical subset of sandwich elements of \mathfrak{g} .

We denote by $\mathrm{Aut}(\mathfrak{g})$ the group of automorphism of \mathfrak{g} and $\mathfrak{z}(\mathfrak{g})$ the *center* of \mathfrak{g} . We call \mathfrak{g} *centerless* if $\mathfrak{z}(\mathfrak{g}) = \{0\}$. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We denote by $\mathrm{Aut}_p(\mathfrak{g})$ the automorphism group of $(\mathfrak{g}, [p])$. Being the stabilizer of the p -map,

$\text{Aut}_p(\mathfrak{g})$ is a closed subgroup of $\text{Aut}(\mathfrak{g})$. It is clear that the group action $\text{Aut}_p(\mathfrak{g})$ on \mathfrak{g} keeps the p -map, so $\text{Aut}_p(\mathfrak{g})$ has a natural action on the variety $\mathbb{E}(r, \mathfrak{g})$ via

$$g \cdot \mathfrak{e} := \{g \cdot x; x \in \mathfrak{e}\}, \quad \mathfrak{e} \in \mathbb{E}(r, \mathfrak{g}).$$

We let $\text{Aut}_p(\mathfrak{g})^\circ$ be the identity component of $\text{Aut}_p(\mathfrak{g})$. Note that $\text{Aut}(\mathfrak{g}) = \text{Aut}_p(\mathfrak{g})$ if \mathfrak{g} is centerless. The map

$$\exp : V(\mathfrak{g}) \rightarrow \text{GL}(\mathfrak{g}); \quad x \mapsto \sum_{i=0}^{p-1} \frac{(\text{ad } x)^i}{i!}$$

is a morphism of affine varieties.

Lemma 2.2.1. *Suppose that $\text{char}(k) = p \geq 3$. Let \mathfrak{g} be a Lie algebra.*

- (1) *Let $c \in \text{Sw}(\mathfrak{g})$. Then $[[c, x], [c, y]] = 0$ for all $x, y \in \mathfrak{g}$.*
- (2) *The set $\text{Sw}(\mathfrak{g})$ is closed under Lie multiplication and generated a nilpotent subalgebra of \mathfrak{g} .*
- (3) *If $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra, then $\text{Sw}(\mathfrak{g}) \subseteq V(\mathfrak{g})$ and $\exp(c) \in \text{Aut}_p(\mathfrak{g})^\circ$ for every $c \in \text{Sw}(\mathfrak{g})$.*

Proof. (1) The Leibniz formula tells us that

$$(\text{ad } c)^2([x, y]) = [(\text{ad } c)^2(x), y] + 2[(\text{ad } c)(x), (\text{ad } c)(y)] + [x, (\text{ad } c)^2(y)].$$

Now our assertion follows from $(\text{ad } c)^2 = 0$ and $p \neq 2$.

First, it is easy to check that $\exp(c) \in \text{Aut}(\mathfrak{g})$ for all $c \in \text{Sw}(\mathfrak{g})$. Since the set $\text{Sw}(\mathfrak{g})$ is stable under $\text{Aut}(\mathfrak{g})$, it follows that $\exp(c)(\text{Sw}(\mathfrak{g})) \subseteq \text{Sw}(\mathfrak{g})$ for all $c \in \text{Sw}(\mathfrak{g})$. Given $c_1, c_2 \in \text{Sw}(\mathfrak{g})$, the map

$$f_{c_1, c_2} : k \rightarrow \mathfrak{g}; \alpha \mapsto \alpha c_1 + [c_1, c_2]$$

is a morphism such that $f(k \setminus \{0\}) \subseteq \text{Sw}(\mathfrak{g})$. Hence,

$$f(k) = \overline{f(k \setminus \{0\})} \subseteq \overline{f(k \setminus \{0\})} \subseteq \overline{\text{Sw}(\mathfrak{g})} = \text{Sw}(\mathfrak{g}).$$

In particular, $f(0) = [c_1, c_2] \in \text{Sw}(\mathfrak{g})$. Consider the adjoint representation, by definition, $\text{ad } c$ is nilpotent for all $c \in \text{Sw}(\mathfrak{g})$. Hence, Engel-Jacobson Theorem

implies that $\langle \text{Sw}(\mathfrak{g}) \rangle$ is strictly triangulable on \mathfrak{g} (see Theorem 1.2.2). It follows that $\langle \text{Sw}(\mathfrak{g}) \rangle$ is a nilpotent subalgebra of \mathfrak{g} , this proved (2).

(3) Since \mathfrak{g} is centerless, it is clear that $\text{Sw}(\mathfrak{g}) \subseteq V(\mathfrak{g})$. Furthermore, we have $\text{Aut}(\mathfrak{g}) = \text{Aut}_p(\mathfrak{g})$. Let $c \in \text{Sw}(\mathfrak{g})$. Then the map

$$\varphi : k \rightarrow \text{Aut}_p(\mathfrak{g}); \lambda \mapsto \exp(\lambda c) = \text{id}_{\mathfrak{g}} + \lambda \text{ad } c$$

is a morphism. Since k is connected, it follows that $\varphi(k)$ is connected. It is obvious that $\text{id}_{\mathfrak{g}} \in \varphi(k)$. Consequently, $\text{id}_{\mathfrak{g}} + \lambda \text{ad } c \in \text{Aut}_p(\mathfrak{g})^\circ$ for any $\lambda \in k$. \square

Let \mathfrak{g} be a simple Lie algebra over complex number \mathbb{C} , $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra, $\Phi := \Phi(\mathfrak{g}, \mathfrak{h})$ the corresponding root system, and $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a basis of simple roots in Φ . Then the Lie algebra \mathfrak{g} has a Chevelley basis:

$$\mathfrak{B} := \{e_\alpha; \alpha \in \Phi\} \cup \{h_i; 1 \leq i \leq l\}.$$

The \mathbb{Z} -span $\mathfrak{g}_{\mathbb{Z}}$ of \mathfrak{B} is a \mathbb{Z} -form in \mathfrak{g} closed under taking Lie multiplications. Therefore, $\mathfrak{g}_k := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ is a Lie algebra over k and structure constants obtained from those for $\mathfrak{g}_{\mathbb{Z}}$ by reducing modulo p . For $p > 3$, the Lie algebra \mathfrak{g}_k fails to be simple if and only if the root system Φ has type A_l where $l = mp - 1$ for some $m \in \mathbb{N}$. If Φ has type A_{mp-1} , then $\mathfrak{g}_k \cong \mathfrak{sl}_{mp}$ has a one-dimensional center (consisting of scalar matrices) and the Lie algebra $\mathfrak{g}_k/\mathfrak{z}(\mathfrak{g}_k) \cong \mathfrak{psl}_{mp}$ is simple, where $\mathfrak{z}(\mathfrak{g}_k)$ is the center of \mathfrak{g}_k . The simple Lie algebras over k thus obtained are called *classical* (see [28, (3.1)]).

A finite-dimensional Lie algebra over k which is representable as a sum of simple ideals of classical type is called a *classical semisimple Lie algebra*. Following [26], we will say that \mathfrak{g} is an *almost classical semisimple Lie algebra*, if there exists a classical semisimple Lie algebra \mathfrak{L} such that

$$\text{Der } \mathfrak{L} \supseteq \mathfrak{g} \supseteq \text{ad } \mathfrak{L}.$$

The following useful theorem was proved by Premet ([26] and [27]):

Theorem 2.2.2. *Suppose that \mathfrak{g} is a finite-dimensional Lie algebra over k and $\text{char}(k) = p \geq 5$. The following conditions are equivalent:*

$$(i) \text{ Sw}(\mathfrak{g}) = \{0\}$$

(ii) \mathfrak{g} is an almost classical semisimple Lie algebra.

Furthermore, any Lie algebra that is equal to its derived algebra and contains no sandwich elements is a sum of simple ideals of classical type.

Notation 2.2.1. We set

$$r_{\max}(\mathfrak{g}) := \max\{r \in \mathbb{N}_0; \mathbb{E}(r, \mathfrak{g}) \neq \emptyset\}$$

By the definition, a restricted Lie algebra \mathfrak{g} is a torus if and only if the nullcone $V(\mathfrak{g}) = \{0\}$, hence $\mathbb{E}(1, \mathfrak{g}) = \emptyset$. So $r_{\max}(\mathfrak{g}) = 0$ if and only if \mathfrak{g} is a torus. The inequality $r_{\max}(\mathfrak{g}) \leq \dim V(\mathfrak{g})$ is obvious. The succeeding example shows that the inequality may be strict.

Example 2.2.2. Suppose that $p \geq 3$. Let $\mathfrak{g} = \mathfrak{sl}_2$ with usual basis x, h, y . The bracket and p -map is given by

$$[h, x] = 2x, [h, y] = -2y, [x, y] = h; x^{[p]} = y^{[p]} = 0, h^{[p]} = h,$$

respectively. We claim that $\mathbb{E}(2, \mathfrak{g}) = \emptyset$. Let $\mathfrak{e} := ka \oplus kb \in \mathbb{E}(2, \mathfrak{g})$ be an elementary subalgebra. By a SL_2 -conjugation, there exists some element $g \in SL_2$ such that $\text{Ad}(g).a = x$. Let $\text{Ad}(g).b = c$. Consequently, $\text{Ad}(g).\mathfrak{e} = kx \oplus kc \in \mathbb{E}(2, \mathfrak{g})$. The Lie bracket rule implies that $\mathfrak{z}_{\mathfrak{g}}(x) = kx$. We thus obtain a contradiction.

Remark 2.2.3. Recently, the number $r_{\max}(\mathfrak{g})$ was determined for classical simple Lie algebra (see [6] for type A, C and [25] for other types).

Lemma 2.2.3. *Suppose that $\text{char}(k) = p \geq 3$. If $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra such that $\dim V(\mathfrak{g}) \geq 2$ and $r_{\max}(\mathfrak{g}) \leq 1$, then $\text{Sw}(\mathfrak{g}) = \{0\}$.*

Proof. Suppose that $\text{Sw}(\mathfrak{g}) \neq \{0\}$. We fix a non-zero element $c \in \text{Sw}(\mathfrak{g})$. Since \mathfrak{g} is centerless, it follows that $c \in V(\mathfrak{g})$. The condition $\dim V(\mathfrak{g}) \geq 2$ ensures that we can find an element $b \in V(\mathfrak{g})$ such that b and c are linearly independent. The assumption $r_{\max}(\mathfrak{g}) \leq 1$ implies that $[c, b] \neq 0$. Thanks to Lemma 2.2.1, the element $\exp(\lambda c) = \text{id}_{\mathfrak{g}} + \lambda \text{ad } c \in \text{Aut}_p(\mathfrak{g})^\circ$ for every $\lambda \in k$. Thus, the set $\{b + \lambda[c, b]; \lambda \in k\} \subseteq V(\mathfrak{g})$. Consequently, the map

$$f : k \rightarrow \mathfrak{g}; \alpha \mapsto \alpha b + [c, b]$$

is a morphism such that $f(k \setminus \{0\}) \subseteq V(\mathfrak{g})$. Hence $f(k) = f(\overline{k \setminus \{0\}}) \subseteq \overline{f(k \setminus \{0\})} \subseteq \overline{V(\mathfrak{g})} = V(\mathfrak{g})$, so that $[c, b] = f(0) \in V(\mathfrak{g})$. Since $(\text{ad } c)^2(b) = [c, [c, b]] = 0$ and $r_{\max}(\mathfrak{g}) \leq 1$, we have $[c, b] = \mu c$ for some $\mu \in k \setminus \{0\}$. This contradicts the fact that $\text{ad } b$ is nilpotent. \square

The proof of the foregoing Lemma also yields:

Corollary 2.2.4. *Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra such that $\text{Sw}(\mathfrak{g}) \neq \{0\}$ and $\dim V(\mathfrak{g}) \geq 2$. Then there exists an elementary subalgebra $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$ such that $\mathfrak{e} \cap \text{Sw}(\mathfrak{g}) \neq \{0\}$.*

Corollary 2.2.5. *Suppose that $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra. Let $c \in \text{Sw}(\mathfrak{g})$ and $x \in V(\mathfrak{g})$. Then the subspace $V := \langle x, [c, x] \rangle$ is contained in $V(\mathfrak{g})$, where $\langle x, [c, x] \rangle$ is the linear span of x and $[c, x]$*

Proof. If $[c, x] = 0$, then there is nothing to prove. If $[c, x] \neq 0$, then the proof of Lemma 2.2.3 ensures that $\lambda x + \mu [c, x] \in V(\mathfrak{g})$ for any $\lambda, \mu \in k$, so that $V \subseteq V(\mathfrak{g})$. \square

Lemma 2.2.6. *Let $(\mathfrak{g}, [p])$ be an almost classical semisimple Lie algebra. Then the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of \mathfrak{g} is a classical semisimple p -ideal and $V([\mathfrak{g}, \mathfrak{g}]) = V(\mathfrak{g})$.*

Proof. Since \mathfrak{g} is almost classical semisimple, there exists a classical semisimple Lie algebra \mathfrak{L} such that

$$\text{Der } \mathfrak{L} \supseteq \mathfrak{g} \supseteq \text{ad } \mathfrak{L}.$$

Writing $\mathfrak{L} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_l$, where the \mathfrak{a}_i are simple ideals of classical type. Let $d \in \text{Der } \mathfrak{L}$. We have

$$d(\mathfrak{a}_i) = d([\mathfrak{a}_i, \mathfrak{a}_i]) = [d(\mathfrak{a}_i), \mathfrak{a}_i] + [\mathfrak{a}_i, d(\mathfrak{a}_i)] \subseteq \mathfrak{a}_i.$$

One considers the map:

$$\text{Der } \mathfrak{L} \rightarrow \bigoplus_{i=1}^l \text{Der } \mathfrak{a}_i; \quad d \mapsto (d|_{\mathfrak{a}_1}, \dots, d|_{\mathfrak{a}_l}).$$

It is obvious that the map is a well-defined isomorphism. Consequently,

$$\text{Der } \mathfrak{L} \cong \text{Der } \mathfrak{a}_1 \oplus \cdots \oplus \text{Der } \mathfrak{a}_l$$

is a direct sum of restricted Lie algebras. It is well-known that for a classical simple Lie algebra \mathfrak{a} the inequality $\text{ad } \mathfrak{a} \neq \text{Der } \mathfrak{a}$ holds if and only if $\mathfrak{a} \cong \tilde{A}_{mp-1}$, where

$\tilde{A}_{mp-1} = \mathfrak{sl}_{mp}/k$, $m \in \mathbb{N}$. In this situation, $\text{Der } \tilde{A}_{mp-1} \cong \text{Lie}(\text{PGL}_{mp}) = \mathfrak{pgl}_{mp}$ (cf. [1, Lemma 2.7]). It is easy to check that

$$[\text{Der } \tilde{A}_{mp-1}, \text{Der } \tilde{A}_{mp-1}] = \text{ad } \tilde{A}_{mp-1}$$

is a p -subalgebra of $\text{Der } \tilde{A}_{mp-1}$ and $V(\text{Der } \tilde{A}_{mp-1}) = V(\text{ad } \tilde{A}_{mp-1})$. The observation also implies,

$$[\text{Der } \mathfrak{L}, \text{Der } \mathfrak{L}] = \text{ad } \mathfrak{L},$$

so that $\text{ad } \mathfrak{L} = [\text{ad } \mathfrak{L}, \text{ad } \mathfrak{L}] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq [\text{Der } \mathfrak{L}, \text{Der } \mathfrak{L}] = \text{ad } \mathfrak{L}$ (cf. [26, p.160]). Accordingly, $[\mathfrak{g}, \mathfrak{g}] = \text{ad } \mathfrak{L}$ is a p -subalgebra of $\text{Der } \mathfrak{L}$, whose p -th power map is denoted by $x \mapsto x^p$.

Given $x \in [\mathfrak{g}, \mathfrak{g}]$, we consider the derivation $d := x^{[p]} - x^p \in \text{Der } \mathfrak{L}$. Then

$$0 = [d, [\mathfrak{g}, \mathfrak{g}]] = [d, \text{ad } \mathfrak{L}] = \text{ad } d(\mathfrak{L}),$$

so that $d(\mathfrak{L}) \subseteq \mathfrak{z}(\mathfrak{L}) = (0)$. Thus $x^{[p]} = x^p \in [\mathfrak{g}, \mathfrak{g}]$, showing that $[\mathfrak{g}, \mathfrak{g}]$ is a p -ideal of \mathfrak{g} . Note that $\mathfrak{pgl}_{mp}/\mathfrak{psl}_{mp}$ is a one-dimensional torus. So that, in view of the above, we have $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \subseteq \text{Der } \mathfrak{L}/\text{ad } \mathfrak{L}$ is a torus. Consequently, $V(\mathfrak{g}) = V([\mathfrak{g}, \mathfrak{g}])$. \square

We denote by $\mathfrak{b}_{\mathfrak{sl}_2}$ the two-dimensional non-abelian restricted Lie algebra. If $p \geq 3$, this algebra is isomorphic to the Borel subalgebra of \mathfrak{sl}_2 .

Theorem 2.2.7. *Suppose that $p \geq 5$. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra such that $r_{\max}(\mathfrak{g}) \leq 1$. Then we have*

$$\mathfrak{g} \cong \mathfrak{sl}_2, \mathfrak{b}_{\mathfrak{sl}_2}, (0).$$

Proof. Suppose that $r_{\max}(\mathfrak{g}) \leq 1$. If $\dim V(\mathfrak{g}) \geq 2$, it follows from Lemma 2.2.3 that \mathfrak{g} is a Lie algebra without strong degeneration, that is $\text{Sw}(\mathfrak{g}) = \{0\}$. According to Theorem 2.2.2, \mathfrak{g} is an almost classical semisimple Lie algebra. Lemma 2.2.6 ensures that $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is a p -subalgebra of \mathfrak{g} satisfying $V(\mathfrak{g}) = V(\mathfrak{g}')$. Consequently,

$$\mathbb{E}(2, \mathfrak{g}) = \mathbb{E}(2, \mathfrak{g}'),$$

and we can assume that \mathfrak{g} is a classical semisimple restricted Lie algebra. Accordingly, we write

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_l,$$

where the \mathfrak{a}_i are simple ideals of classical type. Suppose that $l \leq 2$. Since $[\mathfrak{a}_i, f\mathfrak{a}_j] = (0)$ for $i \neq j$, we have $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$. Hence, the assumption $r_{\max}(\mathfrak{g}) \leq 1$ implies that $l = 1$, so that \mathfrak{g} is classical simple. Let Φ^+ be the set of positive roots (relative to some maximal torus \mathfrak{t}). We claim that there exists only one element in Φ^+ . Otherwise, let β be the longest element in Φ^+ and α be another element. We choose the root vectors $x_\beta \in \mathfrak{g}_\beta$ and $x_\alpha \in \mathfrak{g}_\alpha$. Since β is the longest positive root, it follows that $kx_\alpha \oplus kx_\beta$ is a two-dimensional elementary subalgebra. This, however, contradicts the fact $r_{\max}(\mathfrak{g}) \leq 1$. Consequently, $\mathfrak{g} \cong \mathfrak{sl}_2$.

It remains to consider the case $\dim V(\mathfrak{g}) \leq 1$. If $\dim V(\mathfrak{g}) = 0$, then \mathfrak{g} is a torus, so that $\mathfrak{g} = (0)$. Alternatively, let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal torus. According to [13, Lemma 3.1], the root space decomposition relative to \mathfrak{t} consists a single root space of dimension one. We can write

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) + \mathfrak{g}_\alpha,$$

where $\dim_k \mathfrak{g}_\alpha = 1$. Since \mathfrak{g} is centerless, this yields $\alpha \neq 0$ and $\dim_k \mathfrak{t} = 1$. Furthermore, we claim that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$. Otherwise, let $0 \neq x \in V(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}))$. We assume that $\mathfrak{g}_\alpha = kx_\alpha$, it follows that $[x, x_\alpha] = \lambda x_\alpha$ for some $\lambda \in k$. As x is nilpotent, this ensures that $\lambda = 0$, so that $x \in \mathfrak{z}(\mathfrak{g})$. We have arrived at a contradiction, thereby proving that $\mathfrak{g} = \mathfrak{t} + \mathfrak{g}_\alpha$ is a two-dimensional non-abelian restricted Lie algebra, hence $\mathfrak{g} \cong \mathfrak{b}_{\mathfrak{sl}_2}$. \square

2.3 Applications

Recall that $\mathbb{E}(1, \mathfrak{g}) = \text{Proj}(V(\mathfrak{g}))$ is connected (see Proposition 1.3.1). It follows that the nullcone $V(\mathfrak{g})$ is irreducible if $\dim V(\mathfrak{g}) = 1$. In this section, we discuss the case $\dim V(\mathfrak{g}) = 2$ (see Problem 1.3.1).

We call a variety *equidimensional* if all its irreducible components have the same dimension.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We define

$$V_{\mathbb{E}(r, \mathfrak{g})} := \bigcup_{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{g})} \mathfrak{e}.$$

Proposition 2.3.1. *Suppose that $p \geq 5$. Let \mathfrak{g} be a centerless restricted Lie algebra such that $\dim V(\mathfrak{g}) = 2$. Then $V(\mathfrak{g})$ is equidimensional.*

Proof. Suppose that $\text{Sw}(\mathfrak{g}) = \{0\}$. By virtue of Theorem 2.2.2, \mathfrak{g} is an almost classical semisimple Lie algebra. Lemma 2.2.6 ensures that $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is a p -subalgebra of \mathfrak{g} satisfying $V(\mathfrak{g}) = V(\mathfrak{g}')$. We can assume that \mathfrak{g} is a classical semisimple restricted Lie algebra. Accordingly, we write

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_l,$$

where the \mathfrak{a}_i are simple ideals of classical type. According to the proof of Theorem 2.2.7, we know that $l = 1$. Moreover, the nullcone $V(\mathfrak{g})$ is irreducible ([24] and [32]). Root space considerations imply that $\mathfrak{g} \cong \mathfrak{sl}_2$.

Suppose that $\text{Sw}(\mathfrak{g}) \neq \{0\}$. Let $0 \neq c \in \text{Sw}(\mathfrak{g})$. Since every irreducible component Z of $V(\mathfrak{g})$ is a cone, we have $0 \in Z$ and hence $Z \neq \{0\}$. We therefore only have to exclude the case $\dim Z = 1$. If $\dim Z = 1$, there is $z \in Z \setminus \{0\}$ such that $Z = kz$. We claim that $[c, z] = 0$. [18, (8.2)] ensures that $\text{Aut}_p(\mathfrak{g})^\circ$ stabilizes Z . It follows from Lemma 2.2.1 that the set

$$\{z + \lambda[c, z]; \lambda \in k\} \subseteq Z.$$

By a direct computation, we have $[c, z] = 0$, whence $kc \oplus kz \subseteq V(\mathfrak{g})$. This contradicts our assumption, that $Z = kz$ is a component. \square

Proposition 2.3.2. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then $V_{\mathbb{E}(r, \mathfrak{g})}$ is a conical, closed subvariety of $V(\mathfrak{g})$.*

Proof. It is clear that $V_{\mathbb{E}(r, \mathfrak{g})}$ is a conical set. Furthermore, it is well-known that the set

$$Z := \{(x, \mathfrak{e}) \in V(\mathfrak{g}) \times \mathbb{E}(r, \mathfrak{g}); x \in \mathfrak{e}\}$$

is closed in $V(\mathfrak{g}) \times \mathbb{E}(r, \mathfrak{g})$. Thanks to Proposition 2.1.1, $\mathbb{E}(r, \mathfrak{g})$ is a projective variety. Let

$$\pi_1 : V(\mathfrak{g}) \times \mathbb{E}(r, \mathfrak{g}) \rightarrow V(\mathfrak{g}); (x, \mathfrak{e}) \mapsto x$$

denote the canonical projection. Hence π_1 is a closed map. Consequently, $\pi_1(Z) = V_{\mathbb{E}(r, \mathfrak{g})}$ is closed. \square

A torus $\mathfrak{t} \subseteq \mathfrak{g}$ is said to be *self-centralizing* if the centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$. If \mathfrak{g} admits a self-centralizing torus, then every torus of \mathfrak{g} of maximal dimension is self-centralizing (see [9, Corollary 3.8]).

- Remark 2.3.1.** 1. Let $p \geq 3$. Suppose that $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra such that $\dim V(\mathfrak{g}) = 2$ and admits a self-centralizing torus. In this case, a result by Farnsteiner and Voigt ([12, Theorem 4.2]) gives a complete classification that ensures $V(\mathfrak{g})$ is always irreducible.
2. The variety $V(\mathfrak{g})$ is not always irreducible even when $\dim V(\mathfrak{g}) = 2$ (see Example 1.3.2).
3. In Corollary 3.4.11, we can prove that $V(\mathfrak{g})$ is irreducible when \mathfrak{g} is centerless and $p \geq 7$.

Chapter 3

Connectedness of $\mathbb{E}(2, \mathfrak{g})$

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. A theorem by Carlson asserts that $\mathbb{E}(1, \mathfrak{g})$ is always connected (Proposition 1.3.1). In this Chapter, we would like to consider the variety $\mathbb{E}(2, \mathfrak{g})$ and give a sufficient condition such that $\mathbb{E}(2, \mathfrak{g})$ is connected. We should notice that the variety $\mathbb{E}(2, \mathfrak{g})$ is not always connected. It can be easily verified that $\mathbb{E}(2, \widetilde{\mathfrak{sl}}_2)$ consists of only two elements (see Example 1.3.2).

3.1 Notations and general facts

Recall that a map $f : X \rightarrow \mathbb{N}_0$ originating in a topological space X is called *upper semicontinuous* if $f^{-1}(\{n \in \mathbb{N}_0; n \geq d\})$ is closed for every $d \in \mathbb{N}_0$.

Lemma 3.1.1. *[9, Lemma 7.3]. Let \mathfrak{g} be a k -vector space, and $Y \subseteq \mathfrak{g}$ be a closed, conical subvariety. Then the map*

$$\mathrm{Gr}_r(\mathfrak{g}) \rightarrow \mathbb{N}_0; X \mapsto \dim(X \cap Y)$$

is upper semicontinuous.

Let V_1, V_2 be k -vector spaces. Then the map

$$s : \mathrm{Proj}(V_1) \times \mathrm{Proj}(V_2) \rightarrow \mathrm{Proj}(V_1 \otimes V_2); (\langle v_1 \rangle, \langle v_2 \rangle) \mapsto \langle v_1 \otimes v_2 \rangle$$

is a morphism, and its image is closed. The map s is called *Serge embedding* (e.g. [17, 3.3.3]). Let V be a k -vector space. For $0 \leq r \leq \dim_k V$, we denote by $\Lambda^r V$ the

r -th exterior power. Let $U \subseteq V$ be a r -dimensional subspace. Then choose a basis of U , say $\{v_1, \dots, v_r\}$, and consider $\Lambda^r U := v_1 \wedge \dots \wedge v_r \in \Lambda^r V$ is a one-dimensional subspace of $\Lambda^r V$. Then the map

$$\text{pl}_V : \text{Gr}_r(V) \rightarrow \text{Proj}(\Lambda^r V); U \mapsto \langle v_1 \wedge \dots \wedge v_r \rangle \quad (3.1)$$

is a well-defined morphism, and its image is a closed subset of $\text{Proj}(\Lambda^r V)$. The set $\text{Gr}_r(V)$ can be viewed as a projective variety via structure transports. This map is called *Plücker embedding* (e.g. [17, 3.3.7]). Note that a map $f : X \rightarrow \text{Gr}_r(V)$ is a morphism if and only if $\text{pl}_V \circ f$ enjoys this property.

Lemma 3.1.2. *If V_1 and V_2 are vector spaces, and $0 \leq d_i \leq \dim_k V_i$ for $i = 1, 2$, then the map*

$$\sigma : \text{Gr}_{d_1}(V_1) \times \text{Gr}_{d_2}(V_2) \rightarrow \text{Gr}_{d_1+d_2}(V_1 \oplus V_2); (U_1, U_2) \mapsto U_1 \oplus U_2$$

is a morphism of varieties.

Proof. Let

$$s : \text{Proj}(\Lambda^{d_1} V_1) \times \text{Proj}(\Lambda^{d_2} V_2) \rightarrow \text{Proj}(\Lambda^{d_1} V_1 \otimes \Lambda^{d_2} V_2)$$

be the map given by $(\langle a \rangle, \langle b \rangle) \mapsto \langle a \otimes b \rangle$. s is the Serge embedding. On the other hand, there is a canonical isomorphism

$$\Lambda^{d_1+d_2}(V_1 \oplus V_2) \cong \bigoplus_{p+q=d_1+d_2} \Lambda^p V_1 \otimes \Lambda^q V_2.$$

Hence, $\Lambda^{d_1} V_1 \otimes \Lambda^{d_2} V_2$ is a subspace of $\Lambda^{d_1+d_2}(V_1 \oplus V_2)$. We have an embedding:

$$\iota : \text{Proj}(\Lambda^{d_1} V_1 \otimes \Lambda^{d_2} V_2) \hookrightarrow \text{Proj}(\Lambda^{d_1+d_2}(V_1 \oplus V_2)).$$

Set $s' := \iota \circ s$ the composition map. Then, we have

$$s' \circ (\text{pl}_{V_1} \times \text{pl}_{V_2}) = \text{pl}_{V_1 \oplus V_2} \circ \sigma,$$

so that the right-hand side is a morphism. By the Plücker embedding (see (3.1)), we have σ is a morphism □

Lemma 3.1.3. *Let V be a k -vector space. Take any linear independent triple $x, y, z \in V$, then the map*

$$\varphi : \text{Proj}(ky \oplus kz) \rightarrow \text{Gr}_2(V); \langle x' \rangle \mapsto kx \oplus kx'$$

is a morphism of varieties.

Proof. Let U be the complementary space of $ky \oplus kz$ in V such that $kx \subseteq U$. Note that our map is the composite of σ with the map φ

$$\mathrm{Proj}(ky \oplus kz) \rightarrow \mathrm{Proj}(ky \oplus kz) \times \mathrm{Proj}(U); \langle x' \rangle \mapsto (\langle x' \rangle, \langle x \rangle),$$

where σ is the map defined in Lemma 3.1.2. □

As φ is a morphism of projective varieties, the image is closed. We will denote by $kx \oplus \mathrm{Proj}(ky \oplus kz)$ the image of φ . It is a connected variety.

Lemma 3.1.4. *Let $f : V \rightarrow W$ be a linear transformation of vector spaces, and $0 < d \leq \min(\dim_k V, \dim_k W)$. Then the following statements holds:*

- (1) $U := \{X \in \mathrm{Gr}_d(V); X \cap \ker(f) = (0)\} \subseteq \mathrm{Gr}_d(V)$ is an open subset of $\mathrm{Gr}_d(V)$.
- (2) The map $\bar{f} : U \rightarrow \mathrm{Gr}_d(W); X \mapsto f(X)$ is a morphism of varieties.

Proof. (1) is a direct consequence of Lemma 3.1.1.

(2) Let $X \in U$. Since $X \cap \ker(f) = (0)$, the map is well-defined. We denote by $\Lambda^d(f) : \Lambda^d V \rightarrow \Lambda^d W$ the linear transformation such that $\Lambda^d(f)(v_1 \wedge \cdots \wedge v_d) = f(v_1) \wedge \cdots \wedge f(v_d)$ for all $v_1, \dots, v_d \in V$. According to [10, Lemma 1.3.1], the set

$$\mathcal{O}_{\Lambda^d(f)} := \{[v] \in \mathrm{Proj}(\Lambda^d V); \Lambda^d(f)(v) \neq 0\}$$

is an open subset of $\mathrm{Proj}(\Lambda^d V)$. Furthermore, the map

$$\overline{\Lambda^d(f)} : \mathcal{O}_{\Lambda^d(f)} \rightarrow \mathrm{Proj}(\Lambda^d W); [v] \mapsto [\Lambda^d(f)(v)]$$

is a morphism of varieties. By a direct computation, we obtain the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\bar{f}} & \mathrm{Gr}_d(W) \\ \downarrow \mathrm{pl}_V & & \downarrow \mathrm{pl}_W \\ \mathcal{O}_{\Lambda^d(f)} & \xrightarrow{\overline{\Lambda^d(f)}} & \mathrm{Proj}(\Lambda^d W) \end{array}$$

So that $\mathrm{pl}_W \circ \bar{f} = \overline{\Lambda^d(f)} \circ \mathrm{pl}_V$ is a morphism. Hence \bar{f} is a morphism. □

3.2 Nilpotent strong degeneration

Let $(\mathfrak{g}, [p])$ is a restricted Lie algebra. Recall that

$$\text{Sw}(\mathfrak{g}) = \{c \in \mathfrak{g}; (\text{ad } c)^2 = 0\}$$

the closed conical subset of sandwich elements of \mathfrak{g} . We set

$$\mathbb{E}(2, \mathfrak{g})^* := \{\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g}); \mathfrak{e} \cap \text{Sw}(\mathfrak{g}) \neq \{0\}\}.$$

Lemma 3.2.1. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. The subset $\mathbb{E}(2, \mathfrak{g})^*$ is a closed subvariety of $\mathbb{E}(2, \mathfrak{g})$.*

Proof. Since $\text{Sw}(\mathfrak{g})$ is a closed conical variety, we have

$$\mathbb{E}(2, \mathfrak{g})^* = \{\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g}); \dim(\mathfrak{e} \cap \text{Sw}(\mathfrak{g})) \geq 1\}.$$

Note that $\mathbb{E}(2, \mathfrak{g})$ is a closed subvariety of $\text{Gr}_2(\mathfrak{g})$ (see Proposition 2.1.1). According to Lemma 3.1.1, the map

$$f : \mathbb{E}(2, \mathfrak{g}) \rightarrow \mathbb{N}_0; \mathfrak{e} \mapsto \dim(\mathfrak{e} \cap \text{Sw}(\mathfrak{g})),$$

is upper semicontinuous. Consequently,

$$\mathbb{E}(2, \mathfrak{g})^* = f^{-1}(\{n \in \mathbb{N}_0; n \geq 1\})$$

is closed in $\mathbb{E}(2, \mathfrak{g})$. □

Since $V(\mathfrak{g})$ is a conical variety, it is smooth if and only if it is a subspace of \mathfrak{g} . If \mathfrak{g} is nilpotent Lie algebra, then $r := \min\{t; \mathfrak{g}^t = 0\}$ is rederred to as the *nilpotent length* of \mathfrak{g} .

Example 3.2.1. 1 . If $(\mathfrak{g}, [p])$ is a restricted nilpotent Lie algebra of nilpotency length $\leq p$, given $x, y \in V(\mathfrak{g})$, Jacobson's formula (1.2.1(3)) yields:

$$(x + y)^{[p]} = 0,$$

so that $V(\mathfrak{g})$ is smooth.

2 . If \mathfrak{g} is p -trivial, then $V(\mathfrak{g}) = \mathfrak{g}$ is also smooth.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and $V(\mathfrak{g})$ its nullcone. We call \mathfrak{g} with *nilpotent strong degeneration* if $\text{Sw}(\mathfrak{g}) \neq \{0\}$ and $\text{Sw}(\mathfrak{g}) \subseteq V(\mathfrak{g})$. For instance, if \mathfrak{g} is a p -trivial, then \mathfrak{g} is a Lie algebra with nilpotent strong degeneration. Also, if $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra, then it is clear that $\text{Sw}(\mathfrak{g}) \subseteq V(\mathfrak{g})$.

Proposition 3.2.2. *Suppose that $p \geq 3$. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with nilpotent strong degeneration. Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace. Then the variety $\mathbb{E}(2, \mathfrak{g})^*$ is a connected closed subvariety of $\mathbb{E}(2, \mathfrak{g})$. Furthermore, $\mathbb{E}(2, \mathfrak{g})^*$ is non-empty if $\dim V(\mathfrak{g}) \geq 2$.*

Proof. $\mathbb{E}(2, \mathfrak{g})^*$ is closed follows from Lemma 3.2.1. It is clear that $r_{\max}(\mathfrak{g}) \leq 1$ if $\dim V(\mathfrak{g}) \leq 1$. Hence there is nothing to prove in this case. So that we can assume that $\dim V(\mathfrak{g}) \geq 2$.

Firstly, we claim that $\mathbb{E}(2, \mathfrak{g})^* \neq \emptyset$. By definition, we have

$$\mathfrak{z}(\mathfrak{g}) \subseteq \text{Sw}(\mathfrak{g}) \subseteq V(\mathfrak{g}),$$

where $\mathfrak{z}(\mathfrak{g})$ is the center of \mathfrak{g} . Suppose that $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$. Let $z \in \mathfrak{z}(\mathfrak{g})$ be a non-zero element. Since $\dim V(\mathfrak{g}) \geq 2$, we can find some element $x \in V(\mathfrak{g})$ such that $kz \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$. If \mathfrak{g} is centerless, then non-empty property of $\mathbb{E}(2, \mathfrak{g})^*$ directly follows from Corollary 2.2.4.

We have proved that $\mathbb{E}(2, \mathfrak{g})^*$ is non-empty. Let $\mathfrak{e} := kc \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$ be an elementary subalgebra. Let $Z \subseteq \mathbb{E}(2, \mathfrak{g})^*$ be a connected component such that $\mathfrak{e} \in Z$. To prove that $\mathbb{E}(2, \mathfrak{g})^*$ is connected, it is equivalent to show that $Z = \mathbb{E}(2, \mathfrak{g})^*$. Let $c_1 \in \text{Sw}(\mathfrak{g}) \setminus \{0\}$ and $\mathfrak{e}_1 := kc_1 \oplus ky \in \mathbb{E}(2, \mathfrak{g})^*$. We divide our discussion into the following two cases:

Case 1: $kc = kc_1$. It is clear that x, y, c are linearly independent. Otherwise, $\mathfrak{e}_1 = \mathfrak{e} \in Z$. Since $V(\mathfrak{g})$ is a vector space, it follows that

$$U := kc \oplus \text{Proj}(kx \oplus ky) \subseteq \mathbb{E}(2, \mathfrak{g})^*.$$

Note that U is connected (Lemma 3.1.3) and $\mathfrak{e} \in U \cap Z$. This implies that $U \subseteq Z$. In particular, $\mathfrak{e}_1 \in U \subseteq Z$. We obtain a contradiction.

Case 2: c and c_1 are linearly independent. Suppose that $[c, c_1] = 0$. As before, we have $kc \oplus \text{Proj}(kx \oplus kc_1) \subseteq Z$. In particular, $kc \oplus kc_1 \in Z$. It follows that $kc_1 \oplus \text{Proj}(ky \oplus kc) \subseteq Z$. Hence, we obtain a contradiction: $\mathfrak{e}_1 = kc_1 \oplus ky \in Z$.

Assume that $[c, c_1] \neq 0$. According to Lemma 2.2.1, $c_2 := [c, c_1] \in \text{Sw}(\mathfrak{g})$. By definition, we have $[c_2, c_1] = [c_2, c] = 0$. It follows that

$$U_1 := kc \oplus \text{Proj}(kc_2 \oplus kx) \subseteq Z.$$

Note that

$$U_2 := kc_2 \oplus \text{Proj}(kc \oplus kc_1) \subseteq \mathbb{E}(2, \mathfrak{g})^*$$

is connected and $U_1 \cap U_2 \neq \emptyset$, so that $U_2 \subseteq Z$. Set

$$U_3 := kc_1 \oplus \text{Proj}(kc_2 \oplus ky).$$

Since $kc_1 \oplus kc_2 \in U_2 \cap U_3$, it follows that $U_3 \subseteq Z$. In particular, $\mathfrak{e}_1 \in U_3 \subseteq Z$. We have arrived at a contradiction, thereby proving that $\mathbb{E}(2, \mathfrak{g})^*$ is connected. \square

Theorem 3.2.3. *Suppose that $p \geq 3$. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with nilpotent strong degeneration. Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace. If $\mathfrak{z}_{\mathfrak{g}}(x) \cap \text{Sw}(\mathfrak{g}) \neq \{0\}$ for any $x \in V(\mathfrak{g})$, then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. Thanks to Proposition 3.2.2, we can find a connected component $Z \subseteq \mathbb{E}(2, \mathfrak{g})$ such that $\mathbb{E}(2, \mathfrak{g})^* \subseteq Z$. Let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$ an arbitrary elementary subalgebra. It suffices to prove that $\mathfrak{e} \in Z$. We write $\mathfrak{e} = kx \oplus ky$. If $\mathfrak{e} \cap \text{Sw}(\mathfrak{g}) \neq \{0\}$, then $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})^* \subseteq Z$. We can assume that $\mathfrak{e} \notin \mathbb{E}(2, \mathfrak{g})^*$. By assumption, we can find some non-zero element $c \in \text{Sw}(\mathfrak{g})$ such that $[c, x] = 0$, so that $kx \oplus kc \in \mathbb{E}(2, \mathfrak{g})^*$. We claim that $x \notin kc \oplus ky$. Otherwise, $\mathfrak{e} = kc \oplus ky \in \mathbb{E}(2, \mathfrak{g})^*$, this is a contradiction. Note that $[x, \lambda c + \mu y] = 0$ for any $\lambda, \mu \in k$, this yields $kx \oplus \text{Proj}(kc \oplus ky) \subseteq Z$. In particular, $\mathfrak{e} = kx \oplus ky \in Z$, as desired. \square

Corollary 3.2.4. *Suppose that $p \geq 3$. Let $(\mathfrak{g}, [p])$ be a p -trivial restricted Lie algebra. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. We can assume that $\dim V(\mathfrak{g}) \geq 2$. \mathfrak{g} is p -trivial implies that $\mathfrak{g} = V(\mathfrak{g})$. Engel's theorem ensures that the center $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$. Note that $\mathfrak{z}(\mathfrak{g}) \subseteq \text{Sw}(\mathfrak{g}) \subseteq V(\mathfrak{g})$. For any $x \in V(\mathfrak{g})$ and $z \in \mathfrak{z}(\mathfrak{g})$, we have $[x, z] = 0$. Now our assertion follows from Theorem 3.2.3. \square

Example 3.2.2. Suppose that $p \geq 5$. Let $W(1)$ be the rank one Jacobson-Witt algebra and $W(1)$ is a restricted simple Lie algebra (e.g. see [31]), which has basis

$$\{e_i; -1 \leq i \leq p-2\}.$$

The bracket and p -map is given by

$$[e_i, e_j] = (j - i)e_{i+j}, e_i^{[p]} = \delta_{i,0}e_i.$$

Let $\mathfrak{g} := \bigoplus_{i \geq 1} ke_i \subseteq W(1)$. Then it is easy to check that \mathfrak{g} is a p -trivial subalgebra of $W(1)$. Moreover, we claim that $\mathbb{E}(2, W(1)) = \mathbb{E}(2, \mathfrak{g})$. Otherwise, let $\mathfrak{e} = kx \oplus ky \in \mathbb{E}(2, W(1))$ be a two-dimensional elementary subalgebra and $\mathfrak{e} \not\subseteq \mathfrak{g}$. Without loss of generality, we can assume that $x \in W(1) \setminus \mathfrak{g}$ and $x^{[p]} = 0$. Then x is G -conjugate to e_{-1} , where $G := \text{Aut}_p(W(1))$ is the automorphism group of $W(1)$ (see [8, Page 235]). Hence, there exists some $g \in \text{Aut}_p(W(1))$ such that

$$g \cdot \mathfrak{e} = ke_1 \oplus ky \in \mathbb{E}(2, W(1)).$$

On the other hand, by the Lie multiplication rule, we can easily get $\mathfrak{z}_{W(1)}(e_{-1}) = ke_{-1}$. This is a contradiction. Hence the Corollary 3.2.4 implies $\mathbb{E}(2, W(1))$ is a connected variety.

3.3 Enough strong degeneration

Throughout this section, we always assume that $\text{char}(k) = p \geq 3$ and consider a centerless restricted Lie algebra \mathfrak{g} with strong degeneration, i.e., $\text{Sw}(\mathfrak{g}) \neq \{0\}$. Since $p \geq 3$ and \mathfrak{g} is centerless, Lemma 2.2.1 implies that $\exp(c) \in \text{Aut}_p(\mathfrak{g})$ for every $c \in \text{Sw}(\mathfrak{g})$. This results the following definition.

Definition 3.3.1. The subgroup $\text{Exp}(\mathfrak{g}) \subseteq \text{Aut}_p(\mathfrak{g})$ is generated by $\exp(\text{Sw}(\mathfrak{g}))$ is called the *exponential group* of \mathfrak{g} .

Lemma 3.3.1. *Let $(\mathfrak{g}, [p])$ be a centerless Lie algebra. Then the exponential group $\text{Exp}(\mathfrak{g})$ is a connected unipotent algebraic group such that $\text{ad}(\langle \text{Sw}(\mathfrak{g}) \rangle) \subseteq \text{Lie}(\text{Exp}(\mathfrak{g}))$, where $\langle \text{Sw}(\mathfrak{g}) \rangle$ is the linear span of the sandwich elements of \mathfrak{g} .*

Proof. In view of Lemma 2.2.1, $\text{Sw}(\mathfrak{g})$ is a Lie subset, that acts via the adjoint representation on \mathfrak{g} by nilpotent transformations. Thus the linear span $\langle \text{Sw}(\mathfrak{g}) \rangle$ is a Lie subalgebra of \mathfrak{g} , and the Engel-Jacobson Theorem (see Theorem 1.2.2) provides a basis of \mathfrak{g} , such that for each element $x \in \langle \text{Sw}(\mathfrak{g}) \rangle$ the transformation $\text{ad } x$ is represented by strictly upper triangular matrices. Consequently, $\text{Exp}(\mathfrak{g})$ is a

subgroup of the group of upper triangular unipotent matrices and thus in particular unipotent.

According to Lemma 2.2.1, each sandwich element $c \in \text{Sw}(\mathfrak{g})$ gives rise to a one-parameter subgroup

$$\varphi_c : k \rightarrow \text{Aut}_p(\mathfrak{g})^\circ; \lambda \mapsto \exp(\lambda c).$$

As $\text{Exp}(\mathfrak{g})$ is generated by all $\varphi_c(k)$, general theory (cf. [17, Theorem 2.4.6]) provides $c_1, \dots, c_n \in \text{Sw}(\mathfrak{g})$ such that

$$\text{Exp}(\mathfrak{g}) = \varphi_{c_1}(k) \cdots \varphi_{c_n}(k) \tag{3.2}$$

for some $n \geq 0$. Hence $\text{Exp}(\mathfrak{g})$ is a closed and connected subgroup of $\text{Aut}_p(\mathfrak{g})^\circ$. Since the differential $d(\varphi_c)(0) = \text{ad } c$ for all $c \in \text{Sw}(\mathfrak{g})$, we obtain $\text{ad}(\langle \text{Sw}(\mathfrak{g}) \rangle) \subseteq \text{Lie}(\text{Exp}(\mathfrak{g}))$. \square

A non-zero element $c_0 \in \text{Sw}(\mathfrak{g})$ is called *special sandwich element* if $[c_0, c] = 0$ for any $c \in \text{Sw}(\mathfrak{g})$.

Lemma 3.3.2 (Borel's fixed point theorem). *If G is a connected, solvable, algebraic group acting regularly on a non-empty, complete algebraic variety V over an algebraically closed field k , then there is a G fixed-point of V .*

Lemma 3.3.3. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then there exists a sandwich element $c_0 \in \text{Sw}(\mathfrak{g})$ such that c_0 is special.*

Proof. Suppose that the center $\mathfrak{z}(\mathfrak{g}) \neq (0)$. Note that $\mathfrak{z}(\mathfrak{g}) \subseteq \text{Sw}(\mathfrak{g})$. Let c_0 be a nonzero element in $\mathfrak{z}(\mathfrak{g})$. Then c_0 is special.

We assume that \mathfrak{g} is centerless. Let $G := \text{Aut}_p(\mathfrak{g})$ be the automorphism group of $(\mathfrak{g}, [p])$. By definition, the set $\text{Sw}(\mathfrak{g})$ is a closed, conical subvariety of \mathfrak{g} . Let $\text{Proj}(\text{Sw}(\mathfrak{g}))$ be its projectivization. Since the set $\text{Sw}(\mathfrak{g})$ is invariant under the group action of G , the exponential group $\text{Exp}(\mathfrak{g})$ naturally acts on the projective variety $\text{Proj}(\text{Sw}(\mathfrak{g}))$. According to Lemma 3.3.1, the group $\text{Exp}(\mathfrak{g})$ is solvable and connected, Borel's fixed point theorem provides an element $\langle c_0 \rangle \in \text{Proj}(\text{Sw}(\mathfrak{g}))$ such that $\text{Exp}(\mathfrak{g}).\langle c_0 \rangle = \{\langle c_0 \rangle\}$. Let $0 \neq c \in \text{Sw}(\mathfrak{g})$. Then there exists some non-zero element $\lambda \in k$ such that $(1 + \text{ad } c)(c_0) = \lambda c_0$. Direct computation implies that $[c, c_0] = 0$, so that c_0 is special. \square

We say that the Lie algebra \mathfrak{g} has *enough strong degeneration*, if $\text{Sw}(\mathfrak{g}) \neq \{0\}$ and $\text{Sw}(\mathfrak{g}) \neq kc$ for every $c \in \text{Sw}(\mathfrak{g})$.

Proposition 3.3.4. *Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with enough strong degeneration. Then the variety $\mathbb{E}(2, \mathfrak{g})^*$ is a non-empty connected closed subvariety of $\mathbb{E}(2, \mathfrak{g})$.*

Proof. $\mathbb{E}(2, \mathfrak{g})^*$ is closed follows from Lemma 3.2.1. Thanks to Lemma 3.3.3, there exists a special sandwich element c_0 . By assumption, we can find another sandwich element $c_1 \in \text{Sw}(\mathfrak{g})$ such that c_0 and c_1 are linearly independent. Since \mathfrak{g} is centerless, the subspace $kc_0 \oplus kc_1$ is an elementary subalgebra. Hence $\mathbb{E}(2, \mathfrak{g})^*$ is non-empty. We denote by $\mathfrak{e}_0 := kc_0 \oplus kc_1$.

Let $Z \subseteq \mathbb{E}(2, \mathfrak{g})^*$ be a connected component such that $\mathfrak{e}_0 \in Z$. To prove that $\mathbb{E}(2, \mathfrak{g})^*$ is connected, it is equivalent to show that $Z = \mathbb{E}(2, \mathfrak{g})^*$. Let $c \in \text{Sw}(\mathfrak{g}) \setminus \{0\}$ and $\mathfrak{e}_1 := kc \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$. We can assume that $\mathfrak{e}_1 \notin Z$. We divide our discussion into the following two cases:

Case 1: Suppose that $c \in \mathfrak{e}_0$. We can rewrite $\mathfrak{e}_0 = kc \oplus kc'$, where $c' \in \text{Sw}(\mathfrak{g}) \cap \mathfrak{e}_0$.

◇ If $[c', x] = 0$, then $U := kc \oplus \text{Proj}(kc' \oplus kx) \subseteq \mathbb{E}(2, \mathfrak{g})^*$ is connected. Furthermore, $\mathfrak{e}_0 \in U$, it follows that $U \subseteq Z$. In particular, $\mathfrak{e}_1 \in U \subseteq Z$, this is a contradiction.

◇ Assume that $[c', x] \neq 0$. It follows from Corollary 2.2.5 that $[c', x] \in V(\mathfrak{g})$, so that the subalgebra $kc' \oplus k[c', x] \in \mathbb{E}(2, \mathfrak{g})^*$. Assume that $kc = k[c', x]$. As $[c, c'] = [c, x] = 0$, the Jacobson's formula ensures that $kc' \oplus kx \subseteq V(\mathfrak{g})$. Hence, $kc \oplus \text{Proj}(kc' \oplus kx) \subseteq \mathbb{E}(2, \mathfrak{g})^*$ is connected. It follows that $\mathfrak{e}_1 \in Z$. Hence, we can assume that c and $[c', x]$ are linearly independent. Since $[c, c'] = [c, x] = 0$, we have the subspace $kc \oplus k[c', x] \subseteq V(\mathfrak{g})$. It follows that $kc' \oplus \text{Proj}(kc \oplus k[c', x]) \subseteq Z$. In particular,

$$kc' \oplus k[c', x] \in Z \quad (3.3)$$

Furthermore, $k[c', x] \oplus \text{Proj}(kc \oplus kc') \subseteq Z$. It follows that

$$kc \oplus k[c', x] \in Z \quad (3.4)$$

On the other hand, Corollary 2.2.5 implies that $kx \oplus k[c', x] \subseteq V(\mathfrak{g})$, so that $kc \oplus \text{Proj}(kx \oplus k[c', x]) \subseteq \mathbb{E}(2, \mathfrak{g})^*$ is connected. Hence (3.4) ensures that $\mathfrak{e}_1 \in Z$, we arrive at a contradiction.

Case 2: Suppose that $c \notin \mathfrak{e}_0$.

As c_0 is special, we obtain $[c_0, c] = 0$.

◇ If $[c, c_1] = 0$, then $kc_1 \oplus \text{Proj}(kc_0 \oplus kc) \subseteq Z$. In particular, $kc \oplus kc_1 \in Z$. Now we can replace \mathfrak{e}_0 by $kc \oplus kc_1$. Now $c \in \mathfrak{e}_0$ so that we can use the same argument as in Case 1.

◇ Suppose that $[c, c_1] \neq 0$. Assume that $kc_0 = k[c, c_1]$. As $[c_0, c] = [c_0, c_1] = 0$, the Jacobson's formula implies that $kc \oplus kc_1 \subseteq V(\mathfrak{g})$. Hence, $kc_0 \oplus \text{Proj}(kc_1 \oplus kc) \subseteq Z$. In particular, $kc_0 \oplus kc \in Z$. As above, we replace \mathfrak{e}_0 by $kc_0 \oplus kc$. Now $c \in \mathfrak{e}_0$, we can use the discussion in Case 1. Assume that c_0 and $[c, c_1]$ are linearly independent. We claim that $kc_0 \oplus k[c, c_1] \in Z$. Moreover, we can assume that $c_0 \notin kc_1 \oplus k[c, c_1]$. Otherwise, $kc_0 \oplus k[c, c_1] = kc_0 \oplus kc_1 \in Z$. Consequently, $kc_0 \oplus k[c, c_1] \in Z$ since $kc_0 \oplus \text{Proj}(kc_1 \oplus k[c, c_1]) \subseteq Z$. It follows that $k[c, c_1] \oplus \text{Proj}(kc_0 \oplus kc) \subseteq Z$, so that $kc \oplus k[c, c_1] \in Z$. We can replace \mathfrak{e}_0 by $kc \oplus k[c, c_1]$, now also $c \in \mathfrak{e}_0$. \square

Lemma 3.3.5. *Suppose that $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra. Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, \mathfrak{g})$ and $c \in \text{Sw}(\mathfrak{g})$ such that $\mathfrak{z}_{\mathfrak{g}}(c) \cap \mathfrak{e} = \{0\}$. Then $\mathfrak{e}' := k[c, x] \oplus k[c, y]$ is a two-dimensional elementary and \mathfrak{e} and \mathfrak{e}' lie in the same irreducible (connected) component of $\mathbb{E}(2, \mathfrak{g})$.*

Proof. By assumption, $[c, x]$ and $[c, y]$ are linearly independent elements in \mathfrak{g} . It follows from Corollary 2.2.5 that both of them are contained in $V(\mathfrak{g})$. Moreover, we have $[[c, x], [c, y]] = 0$ (Lemma 2.2.1). So that \mathfrak{e}' is an elementary subalgebra of \mathfrak{g} . Lemma 2.2.1 also implies that $\exp(\lambda c) \in \text{Aut}_p(\mathfrak{g})^\circ$ for any $\lambda \in k$. Let $X \subseteq \mathbb{E}(2, \mathfrak{g})$ be an irreducible component which contains \mathfrak{e} . [18, (8.2)] ensures that $\text{Aut}_p(\mathfrak{g})^\circ$ stabilizes X . It follows that

$$\exp(\lambda c) \cdot \mathfrak{e} = k \exp(\lambda c)(x) \oplus k \exp(\lambda c)(y) \in X.$$

By a direct computation, this implies

$$k(\lambda x + [c, x]) \oplus k(\lambda y + [c, y]) \in X.$$

for any $\lambda \in k \setminus \{0\}$. The map:

$$f : k \rightarrow \mathbb{E}(2, \mathfrak{g}); \lambda \mapsto k(\lambda x + [c, x]) \oplus k(\lambda y + [c, y])$$

is a morphism such that $f(k \setminus \{0\}) \subseteq X$. Then $f(k) \subseteq X$, so that $f(0) = \mathfrak{e}' \in X$. \square

Theorem 3.3.6. *Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with enough strong degeneration. Then $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. Thanks to Proposition 3.3.4, we can find a connected component $Z \subseteq \mathbb{E}(2, \mathfrak{g})$ such that $Z \supseteq \mathbb{E}(2, \mathfrak{g})^*$. Let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$ an arbitrary elementary subalgebra. It suffices to prove that $\mathfrak{e} \in Z$. We write $\mathfrak{e} = kx \oplus ky$. If $\mathfrak{e} \cap \text{Sw}(\mathfrak{g}) \neq \{0\}$, then $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})^* \subseteq Z$. We fix a non-zero element $c \in \text{Sw}(\mathfrak{g})$ and assume that $\mathfrak{e} \notin \mathbb{E}(2, \mathfrak{g})^*$.

◇ Suppose that there exists some non-zero element $z_1 \in \mathfrak{e}$ such that $[c, z_1] = 0$. Then we can rewrite $\mathfrak{e} = kz_1 \oplus kz_2$. Moreover, if $[c, z_2] = 0$, then the variety $kz_1 \oplus \text{Proj}(kz_2 \oplus kc) \subseteq Z$, this implies that $\mathfrak{e} = kz_1 \oplus kz_2 \in Z$. If $[c, z_2] \neq 0$ and $kz_1 = k[c, z_2]$, then $\mathfrak{e} = k[c, z_2] \oplus kz_2$. By Jacobson's formula (Definition 1.2.1), we have $kc \oplus kz_2 \subseteq V(\mathfrak{g})$. It follows that $k[c, z_2] \oplus \text{Proj}(kc \oplus kz_2) \subseteq Z$. In particular, $\mathfrak{e} \in Z$. Assume that $[c, z_2] \neq 0$ and $kz_1 \neq k[c, z_2]$. It follows from Corollary 2.2.5 that $kz_2 \oplus k[c, z_2] \subseteq V(\mathfrak{g})$. Moreover, we can assume that z_1 is not contained in either the subspace $z_1 \notin kz_2 \oplus k[c, z_2]$ or $z_1 \notin kc \oplus k[c, z_2]$. Otherwise, $kz_1 \oplus k[c, z_2] = \mathfrak{e} \in Z$. Then we consider the connected varieties

$$U_1 := kz_1 \oplus \text{Proj}(kz_2 \oplus k[c, z_2]) \subseteq \mathbb{E}(2, \mathfrak{g})$$

and

$$U_2 := kz_1 \oplus \text{Proj}(kc \oplus k[c, z_2]) \subseteq \mathbb{E}(2, \mathfrak{g}).$$

On the other hand, $kz_1 \oplus kc \in \mathbb{E}(2, \mathfrak{g})^*$, it follows that $U_2 \subseteq Z$. Note that $kz_1 \oplus k[c, z_2] \in U_1 \cap U_2$. Thus, U_1 is also contained in Z . It follows that $\mathfrak{e} \in U_2 \subseteq Z$. We obtain a contradiction.

◇ Suppose that there is no non-zero element $z \in \mathfrak{e}$ such that $[c, z] = 0$. This means $\mathfrak{z}_{\mathfrak{g}}(c) \cap \mathfrak{e} = \{0\}$. Then we denote by $\mathfrak{e}' := k[c, x] \oplus k[c, y]$. According to Lemma 3.3.5, we see that $\mathfrak{e}' \in \mathbb{E}(2, \mathfrak{g})$. Consider the three-dimensional subspace $kc \oplus \mathfrak{e}'$. It is clear $kc \oplus \mathfrak{e}' \in \mathbb{E}(3, \mathfrak{g})$ is a three-dimensional elementary subalgebra. It follows that $\text{Gr}_2(kc \oplus \mathfrak{e}') \subseteq Z$. In particular, $\mathfrak{e}' \in Z$. Furthermore, Lemma 3.3.5 implies that \mathfrak{e} and \mathfrak{e}' lie in the same connected component of $\mathbb{E}(2, \mathfrak{g})$, so that $\mathfrak{e} \in Z$. \square

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and V be a faithful irreducible \mathfrak{g} -module with $\dim_k V \geq 2$. We construct a new restricted Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \ltimes V$, whose bracket and p -map are given by

$$[(x_1, v_1), (x_2, v_2)] = ([x_1, x_2], x_1.v_2 - x_2.v_1); \quad (x, v)^{[p]} = (x^{[p]}, x^{p-1}.v)$$

for any $x_1, x_2, x \in \mathfrak{g}$ and $v_1, v_2, v \in V$. Since V is an abelian ideal of $\tilde{\mathfrak{g}}$, we have $V \subseteq \text{Sw}(\tilde{\mathfrak{g}})$.

Proposition 3.3.7. *Keep the notations as above. The variety $\mathbb{E}(2, \tilde{\mathfrak{g}})$ is connected.*

Proof. Thanks to Theorem 3.3.6, it is sufficient to prove that $\tilde{\mathfrak{g}}$ is centerless. Let $c = (x, v) \in \mathfrak{z}(\tilde{\mathfrak{g}})$. For any $v' \in V$, we have

$$(0, 0) = [c, (0, v')] = [(x, v), (0, v')] = (0, x.v')$$

Since V is faithful, it follows that $x = 0$. Moreover, for any $y \in \mathfrak{g}$, we have

$$[(y, 0), c] = [(y, 0), (0, v)] = (0, y.v) = (0, 0).$$

Hence kv forms a one-dimensional trivial \mathfrak{g} -submodule of V . This is a contradiction since V is irreducible. This ensures that $c = (x, v) = 0$, as required. \square

Remark 3.3.2. Following the above proof, if $V^{\mathfrak{g}} := \{v \in V; x.v = 0, \forall x \in \mathfrak{g}\} = \{0\}$ and \mathfrak{g} is centerless, then Proposition 3.3.7 also holds.

3.4 The case $\text{Sw}(\mathfrak{g}) = kc$

In the last section, we discussed the case where $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra with enough strong degeneration. Hence, we still need to consider the Lie algebra with strong degeneration but not enough, i.e., $\text{Sw}(\mathfrak{g}) = kc$ for some $c \in \text{Sw}(\mathfrak{g})$. we always assume that $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra.

First, we recall the definition of Jacobson-Witt algebra (e.g. see [31]). Let B_n be the truncated polynomial ring in variables:

$$B_n := k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p) \quad (n \geq 1).$$

Let $W(n) := \text{Der}(B_n)$ be the algebra of derivations of B_n . The Lie algebra $W(n)$ is called the n -th *Jacobson-Witt algebra*. It is a restricted simple Lie algebra and its p -map is the standard p -th power of linear operators. Let

$$x_i := X_i + (X_1^p, \dots, X_n^p) \in B_n.$$

Denote by $\partial_i = \partial/\partial x_i \in W(n)$ the partial derivative with respect to the variable x_i . Then $\{\partial_1, \dots, \partial_n\}$ is a basis of the B_n -module $W(n)$, so that $\dim_k W(n) = np^n$.

Here and further, we use the standard multi-index notation and write

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{and} \quad |\alpha| := \sum_{i=1}^n \alpha_i \quad \text{for} \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n.$$

Setting $\tau := (p-1, \dots, p-1)$, the Lie algebra $W(n)$ obtains a \mathbb{Z} -grading $W(n) = \bigoplus_{j=-1}^{|\tau|-1} W(n)_j$ via

$$W(n)_j := \sum_{i=1}^n \sum_{|\alpha|=j+1} kx^\alpha \partial_i.$$

The grading is also a *restricted grading*, i.e., $W(n)_j^{[p]} \subseteq W(n)_{pj}$.

For $n = 0$, we put $B_0 = k$ and $W(0) = (0)$. Given a finite-dimensional Lie algebra S , we put $S(n) := S \otimes_k B_n$ for $n \geq 0$. General theory yields (see [2, Page 1089]):

$$\text{Der } S(n) = (\text{Der } S) \otimes_k B_n + \text{id}_S \otimes W(n).$$

We say that a Lie algebra \mathfrak{g} is *semisimple* if \mathfrak{g} has no non-zero abelian ideals.

The *socle* of a semisimple Lie algebra \mathfrak{g} is the sum ΣI_i of all minimal ideals I_i of \mathfrak{g} . We will denote by $\text{Soc}(\mathfrak{g})$ the socle of \mathfrak{g} . In particular, these ideals are irreducible \mathfrak{g} -modules. Therefore every I_i is a $\text{Der } I_i$ -simple Lie algebra, and $\text{Der } I_i$ is closed under associative p -th powers.

The following theorem is due to Block [2, Theorem 7] (see also [30, Corollary 3.3.6]).

Theorem 3.4.1. *Let \mathfrak{g} be a finite dimensional semisimple Lie algebra. Then there are simple Lie algebras S_i and truncated polynomial rings B_{n_i} such that $\bigoplus_{i=1}^r S_i \otimes_k B_{n_i}$ is the socle of \mathfrak{g} and*

$$\bigoplus_{i=1}^r S_i \otimes_k B_{n_i} \subseteq \mathfrak{g} \subseteq \bigoplus_{i=1}^r ((\text{Der } S_i) \otimes_k B_{n_i} + \text{id}_{S_i} \otimes W(n_i)).$$

Lemma 3.4.2. *Suppose that $(\mathfrak{g}, [p])$ is semisimple restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ for some non-zero $c \in \text{Sw}(\mathfrak{g})$. Then there exist simple Lie algebras S_1, \dots, S_r such that*

$$S_1 \oplus \dots \oplus S_r \subseteq \mathfrak{g} \subseteq \text{Der } S_1 \oplus \dots \oplus \text{Der } S_r.$$

Proof. According to Theorem 3.4.1, we have $\text{Soc}(\mathfrak{g}) = S_1 \otimes_k B_{n_1} \oplus \dots \oplus S_r \otimes_k B_{n_r}$. Suppose that $n_k > 0$ for some $k \in \{1, \dots, r\}$. We choose a non-zero element $f \in B_{n_k}$ such that $f^2 = 0$, then [26, Lemma 4] ensures that $S_k \otimes_k kf \subseteq \text{Sw}(\mathfrak{g}) = kc$. This is a contradiction. Consequently, all n_i ($1 \leq i \leq r$) are equal to zero. \square

Lemma 3.4.3. *Let $p \geq 7$. Suppose that $(\mathfrak{g}, [p])$ is a semisimple restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ for some non-zero $c \in \text{Sw}(\mathfrak{g})$. Then there exists an elements c' such that $kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})^*$ and $(\text{ad } c')^3 = 0$.*

Proof. First, we have $\text{Der Soc}(\mathfrak{g}) \supseteq \mathfrak{g} \supseteq \text{Soc}(\mathfrak{g})$. Moreover, according to Lemma 3.4.2, there exist simple Lie algebras $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ such that

$$\text{Soc}(\mathfrak{g}) = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r.$$

Since \mathfrak{g} acts faithfully on the socle, it follows from [26, Lemma 6] that we have an inclusion $\text{Sw}(\mathfrak{g}) \subseteq \text{Soc}(\mathfrak{g})$. Hence there are $c_i \in \mathfrak{a}_i$ such that $c = c_1 + \cdots + c_r$. Since $c \neq 0$, it follows that $c_k \neq 0$ for some $k \in \{1, \dots, r\}$. Consequently, $c_k \in \text{Sw}(\mathfrak{a}_k)$ and \mathfrak{a}_k is a simple Lie algebra with strong degeneration. According to [22, Page 27], the Lie algebra \mathfrak{a}_k affords a filtration

$$\mathfrak{a}_k = (\mathfrak{a}_k)_{-1} \supsetneq (\mathfrak{a}_k)_0 \supsetneq \cdots (\mathfrak{a}_k)_r \supsetneq (0)$$

of depth $r \geq p-3$. In view of $p \geq 7$, we obtain \mathfrak{a}_k is a Lie algebra with enough strong degeneration. According to Lemma 3.3.3, we can find some element $c' \in \text{Sw}(\mathfrak{a}_k)$ such that $[c, c'] = 0$. As \mathfrak{a}_k is an ideal of \mathfrak{g} , we obtain $(\text{ad } c')^3(\mathfrak{g}) = 0$. Hence $\mathfrak{e} := kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})^*$. \square

Proposition 3.4.4. *Let $p \geq 7$. Suppose that $(\mathfrak{g}, [p])$ is a semisimple restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ for some non-zero $c \in \text{Sw}(\mathfrak{g})$. Then the variety $\mathbb{E}(2, \mathfrak{g})^*$ is a non-empty connected closed subvariety of $\mathbb{E}(2, \mathfrak{g})$.*

Proof. According to Lemma 3.4.3, we can find some c' such that $kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})^*$ and $(\text{ad } c')^3 = 0$. We denote by $\mathfrak{e} := kc \oplus kc'$.

Now suppose that $\mathbb{E}(2, \mathfrak{g})^*$ is disconnected. There exist non-empty closed subsets $Z_1, Z_2 \subseteq \mathbb{E}(2, \mathfrak{g})^*$ such that $\mathbb{E}(2, \mathfrak{g})^* = Z_1 \cup Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Without loss of generality, we can assume that $\mathfrak{e} \in Z_1$. Consider the connected solvable group \mathfrak{C}' generated by $\exp(\lambda \text{ad } c')$ with $\lambda \in k$. Since $(\text{ad } c')^3 = 0$, it follows that $\exp(\lambda \text{ad } c') \in \text{Aut}_p(\mathfrak{g})^\circ$ for any $\lambda \in k$. As Z_2 is the union of connected components, it follows that Z_2 is \mathfrak{C}' -stable. Borel's fixed point theorem (Lemma 3.3.2) provides an element $\mathfrak{e}_x := kc \oplus kx \in Z_2$ such that \mathfrak{e}_x is \mathfrak{C}' -stable. Hence

$$\exp(\text{ad } c')(x) = x + [c', x] + \frac{1}{2}[c', [c', x]] \in kc \oplus kx.$$

We get

$$[c', [c', x]] \in (\text{ad } c')(kc \oplus kx) = k[c', x].$$

Thus, $[c', [c', x]] = 0$ so that $[c', x] \in kc \oplus kx$. We consider the vectors c, c' and x . By virtue if $\mathfrak{e} \neq \mathfrak{e}_x$, these vectors are linear independent. Because of $[c', x] \in kc \oplus kx$, we

have $[x, [c', x]] = 0$ and thus $kc' \oplus kx \subseteq V(\mathfrak{g})$. This gives $kc \oplus \text{Proj}(kc' \oplus kx) \subseteq Z_1 \cap Z_2$. Consequently, $\mathbb{E}(2, \mathfrak{g})^*$ is connected. \square

Theorem 3.4.5. *Let $p \geq 7$. Suppose that $(\mathfrak{g}, [p])$ is a semisimple restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ for some non-zero $c \in \text{Sw}(\mathfrak{g})$. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. Thanks to Proposition 3.4.4, we can find a connected component $Z \subseteq \mathbb{E}(2, \mathfrak{g})$ such that $Z \supseteq \mathbb{E}(2, \mathfrak{g})^*$. Let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$ an arbitrary elementary subalgebra. It suffices to prove that $\mathfrak{e} \in Z$. We write $\mathfrak{e} = kx \oplus ky$.

\diamond Suppose that there exists some non-zero element $z \in \mathfrak{e}$ such that $[c, z] = 0$. Write $\mathfrak{e} = kz \oplus kz'$. If $[c, z'] = 0$, then the variety $kz \oplus \text{Proj}(kz' \oplus kc) \subseteq Z$, this implies that $\mathfrak{e} = kz \oplus kz' \in Z$. Assume that $[c, z'] \neq 0$ and $k[c, z'] \neq kz$. According to Corollary 2.2.5, we know that $kz' \oplus k[c, z'] \subseteq V(\mathfrak{g})$. Consider the connected varieties $kz \oplus \text{Proj}(kz' \oplus k[c, z'])$ and $kz \oplus \text{Proj}(kc \oplus k[c, z'])$. They have non-trivial intersection and note that $kz \oplus \text{Proj}(kc \oplus k[c, z']) \subseteq Z$. It follows that $kz \oplus \text{Proj}(kz' \oplus k[c, z']) \subseteq Z$. In particular, $\mathfrak{e} \in Z$. If $[c, z'] \neq 0$ and $kz = k[c, z']$, then $\mathfrak{e} = k[c, z'] \oplus kz'$. By the Jacobson's formula (see 1.2.1), we have $kc \oplus kz' \subseteq V(\mathfrak{g})$. It follows that $k[c, z'] \oplus \text{Proj}(kc \oplus kz') \subseteq Z$. In particular, $\mathfrak{e} \in Z$.

\diamond Suppose that $[c, z] \neq 0$ for any $0 \neq z \in \mathfrak{e}$. Note that $\mathfrak{e}' := k[c, x] \oplus k[c, y] \in \mathbb{E}(2, \mathfrak{g})$ and \mathfrak{e}' lie in the same connected component with \mathfrak{e} (see Remark 3.3.5). So we can replace \mathfrak{e} by \mathfrak{e}' and use the argument as above. \square

Remark 3.4.1. Suppose that \mathfrak{g} contains an elementary subalgebra $\mathfrak{e} := kc \oplus kc'$, where $c \in \text{Sw}(\mathfrak{g})$ and $(\text{ad } c')^3 = 0$. According to the proof of Proposition 3.4.4 and Theorem 3.4.5, we only need $p \geq 5$.

Let \mathfrak{g} be a Lie algebra. A subspace $V \subseteq \mathfrak{g}$ is called an *inner ideal* of the Lie algebra \mathfrak{g} if $[V, [V, \mathfrak{g}]] \subseteq V$. Any non-zero element $c \in \text{Sw}(\mathfrak{g})$ generates a one-dimensional inner ideal kc .

Theorem 3.4.6. *[26, Theorem 1][27]. Suppose that $\text{char}(k) = p \geq 5$. Any finite-dimensional Lie algebra over k contains a one-dimensional inner ideal.*

If B is an inner ideal of a Lie algebra \mathfrak{g} with $\dim_k B = 1$, and $b \in B \setminus \{0\}$, then there exists a linear function $\beta \in \mathfrak{g}^*$ such that $[b, [b, x]] = \beta(x)b$ for any $x \in \mathfrak{g}$.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$. If \mathfrak{a} is an abelian ideal, then $(\text{ad } a)^2 = 0$ for any $a \in \mathfrak{a}$, so that $\mathfrak{a} \subseteq \text{Sw}(\mathfrak{g}) = kc$. Hence, if \mathfrak{g} is not semisimple, then $\text{Sw}(\mathfrak{g}) = kc$ is the only non-zero abelian ideal of \mathfrak{g} .

Theorem 3.4.7. *Let $p \geq 7$. Suppose that $(\mathfrak{g}, [p])$ is centerless restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ and \mathfrak{g} is not semisimple. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. First, we know that $\text{Sw}(\mathfrak{g}) = kc$ is the only abelian ideal of \mathfrak{g} . Let $\mathfrak{g}' := \mathfrak{z}_{\mathfrak{g}}(c)/kc$ be the quotient algebra. According to Theorem 3.4.6, there exists some element $c' \in \mathfrak{z}_{\mathfrak{g}}(c)$ such that $k\bar{c}'$ is an inner ideal of \mathfrak{g}' , where $\bar{c}' = c' + kc$. By definition, $[\bar{c}', [\bar{c}', \bar{x}]] \in k\bar{c}'$ for any element $x \in \mathfrak{z}_{\mathfrak{g}}(c)$. Since c lies in the center of $\mathfrak{z}_{\mathfrak{g}}(c)$, it follows that $(\text{ad } c')^3(\mathfrak{z}_{\mathfrak{g}}(c)) = 0$ and $[c', [c', x]] \in kc \oplus kc'$ for any $x \in \mathfrak{z}_{\mathfrak{g}}(c)$. As kc is an ideal of \mathfrak{g} , it follows that $[c', \mathfrak{g}] \subseteq \mathfrak{z}_{\mathfrak{g}}(c)$, hence $(\text{ad } c')^4 = 0$. Since \mathfrak{g} is centerless, it follows that $c' \in V(\mathfrak{g})$ and $kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})$. Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, \mathfrak{g})$ and $c \notin \mathfrak{e}$. Since $[c, z] = 0$ for any $z \in V(\mathfrak{g})$, it follows that $kc \oplus \mathfrak{e} \subseteq V(\mathfrak{g})$ is a three-dimensional elementary subalgebra of \mathfrak{g} . Hence, it suffices to prove that

$$\mathbb{E}(2, \mathfrak{g})^* := \{\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g}); c \in \mathfrak{e}\}$$

is connected.

Now suppose that $\mathbb{E}(2, \mathfrak{g})^*$ is disconnected. There exist non-empty closed subsets $Z_1, Z_2 \subseteq \mathbb{E}(2, \mathfrak{g})^*$ such that $\mathbb{E}(2, \mathfrak{g})^* = Z_1 \cup Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Without loss of generality, we can assume that $\mathfrak{e} := kc \oplus kc' \in Z_1$. As $p \geq 7$, consider the connected solvable group \mathfrak{C}' generated by $\exp(\lambda \text{ad } c')$ with $\lambda \in k$. Since $(\text{ad } c')^4 = 0$, it follows that $\exp(\lambda \text{ad } c') \in \text{Aut}_p(\mathfrak{g})^\circ$ for any $\lambda \in k$. As Z_2 is the union of connected components, it follows that Z_2 is \mathfrak{C}' -stable. Borel's fixed point theorem (Lemma 3.3.2) provides an element $\mathfrak{e}_x := kc \oplus kx \in Z_2$ such that \mathfrak{e}_x is \mathfrak{C}' -stable. Note that $(\text{ad } c')^3(\mathfrak{z}_{\mathfrak{g}}(c)) = 0$. Hence

$$\exp(\text{ad } c')(x) = x + [c', x] + \frac{1}{2}[c', [c', x]] \in kc \oplus kx.$$

We get

$$[c', [c', x]] \in (\text{ad } c')(kc \oplus kx) = k[c', x].$$

Thus, $[c', [c', x]] = 0$ so that $[c', x] \in kc \oplus kx$. We consider the vectors c, c' and x . By virtue if $\mathfrak{e} \neq \mathfrak{e}_x$, these vectors are linear independent. Because of $[c', x] \in kc \oplus kx$, we have $[x, [c', x]] = 0$ and thus $kc' \oplus kx \subseteq V(\mathfrak{g})$. This gives $kc \oplus \text{Proj}(kc' \oplus kx) \subseteq Z_1 \cap Z_2$. Consequently, $\mathbb{E}(2, \mathfrak{g})^*$ is connected. \square

Theorem 3.4.8. *Let $p \geq 7$. If $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra with strong degeneration, then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. This is a direct consequence of Theorem 3.3.6, Theorem 3.4.5 and Theorem 3.4.7. \square

In recent preprint, Farnsteiner proved the following theorem:

Theorem 3.4.9. *[11] Suppose that $p \geq 7$. Let G be a connected algebraic group with Lie algebra $\mathfrak{g} := \text{Lie}(G)$. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Combine with Theorem 3.4.9, we have:

Theorem 3.4.10. *Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. If $\text{Sw}(\mathfrak{g}) \neq \{0\}$, then our assertion follows from Theorem 3.4.8. If $\text{Sw}(\mathfrak{g}) = \{0\}$, then \mathfrak{g} is an almost classical semisimple Lie algebra (Theorem 2.2.2). Lemma 2.2.6 ensures that $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is a p -subalgebra of \mathfrak{g} satisfying $V(\mathfrak{g}) = V(\mathfrak{g}')$. Consequently,

$$\mathbb{E}(2, \mathfrak{g}) = \mathbb{E}(2, \mathfrak{g}'),$$

and we may assume that \mathfrak{g} classical semisimple restricted Lie algebra. Accordingly, we write

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_l,$$

where the \mathfrak{a}_i are simple ideals of classical type. Note that $[\mathfrak{a}_i, \mathfrak{a}_j] = (0)$ for all $i \neq j$. We claim that if all $\mathbb{E}(2, \mathfrak{a}_i)$ are connected, then $\mathbb{E}(2, \mathfrak{g})$ is connected. If $l = 1$, then there is nothing to prove. Suppose that $l \geq 2$. Furthermore, we can assume that $\mathbb{E}(2, \mathfrak{a}_1) \neq \emptyset$. According to Theorem 2.2.7, we know that $\mathbb{E}(2, \mathfrak{a}_i) = \emptyset$ if and only if $\mathfrak{a}_i \cong \mathfrak{sl}_2$. Suppose that all $\mathbb{E}(2, \mathfrak{a}_i) = \emptyset$. Then \mathfrak{g} is an algebraic Lie algebra. Theorem 3.4.9 ensures that $\mathbb{E}(2, \mathfrak{g})$ is connected. So that we assume that $\mathbb{E}(2, \mathfrak{a}_1) \neq \emptyset$. Let $Z \subseteq \mathbb{E}(2, \mathfrak{g})$ be a connected component such that $\mathbb{E}(2, \mathfrak{a}_1) \subseteq Z$. Let $kx \oplus ky \in \mathbb{E}(2, \mathfrak{g})$ be an elementary subalgebra. We write

$$x = x_1 + \cdots + x_l, \quad y = y_1 + \cdots + y_l$$

the direct sum decomposition. By direct computation, we know that

$$x_i^{[p]} = y_i^{[p]} = [x_i, y_j] = 0.$$

Since $\mathbb{E}(2, \mathfrak{a}_1) \neq \emptyset$, we let $\mathfrak{e}_0 = kx_0 \oplus ky_0 \in \mathbb{E}(2, \mathfrak{a}_1)$. We denote by

$$U := \langle x_0, y_0, x_i, y_i; i \geq 2 \rangle,$$

and

$$V := \langle x_i, y_i; i \geq 1 \rangle.$$

The above arguments yield $U \subseteq V(\mathfrak{g})$ and $V \subseteq V(\mathfrak{g})$. Note that $\mathrm{Gr}_2(U) \subseteq \mathbb{E}(2, \mathfrak{g})$ and $\mathfrak{e}_0 \in Z \cap \mathrm{Gr}_2(U)$, so that $\mathrm{Gr}_2(U) \subseteq Z$. Moreover, $\mathrm{Gr}_2(U) \cap \mathrm{Gr}_2(V) \neq \emptyset$, we have $\mathrm{Gr}_2(V) \subseteq Z$. In particular, $kx \oplus ky \in Z$.

Now, we can assume that \mathfrak{g} is a simple Lie algebra of classical type. It is well-known that \mathfrak{g} is an algebraic Lie algebra except $\mathfrak{g} \cong \mathfrak{psl}(mp)$. Theorem 3.4.9 implies that we only need consider the case $\mathfrak{g} \cong \mathfrak{psl}_{mp}$. We denote by $\pi : \mathfrak{sl}_{mp} \rightarrow \mathfrak{psl}_{mp}$ the canonical projection map. Let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{sl}_{mp})$. We have $\mathfrak{e} \cap \ker(\pi) = (0)$. In view of Lemma 3.1.4, we have the induced map

$$\bar{\pi} : \mathbb{E}(2, \mathfrak{sl}_{mp}) \rightarrow \mathbb{E}(2, \mathfrak{psl}_{mp})$$

is a morphism. Thanks to [12, Lemma 3.1], $\bar{\pi}$ is injective. Hence we can identify $\mathbb{E}(2, \mathfrak{sl}_{mp})$ as a subvariety of $\mathbb{E}(2, \mathfrak{psl}_{mp})$. Note that \mathfrak{sl}_{mp} is an algebraic Lie algebra, hence $\mathbb{E}(2, \mathfrak{sl}_{mp})$ is connected (Theorem 3.4.9). We can find a connected component $Z \subseteq \mathbb{E}(2, \mathfrak{psl}_{mp})$ such that $Z \supseteq \mathbb{E}(2, \mathfrak{sl}_{mp})$. Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, \mathfrak{psl}_{mp})$, it suffices to prove that $\mathfrak{e} \in Z$. If $[x, y] = 0$, then $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{sl}_{mp}) \subseteq Z$. Otherwise, we have $k[x, y] = k$. By a conjugation, we can assume that x has the form of standard Jordan block. Let $\mathfrak{sl}_{mp} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ be its standard decomposition, then $x \in \mathfrak{n}^+$. Let z be the root vector corresponding to the highest positive root. We have $[x, z] = 0$, in particular $kx \oplus kz \in Z$.

◇ If $[z, y] = 0$, then $kx \oplus \mathrm{Proj}(ky \oplus kz) \subseteq Z$. In particular, $\mathfrak{e} \in Z$.

◇ If $[z, y] \neq 0$, then it is easy to check that $[z, [z, y]] = 0$. Also note that $\exp(\lambda z) \in \mathrm{Aut}_p(\mathfrak{psl}_{mp})^\circ$ for any $\lambda \in k$. Corollary 2.2.5 implies that $ky \oplus k[z, y] \subseteq V(\mathfrak{psl}_{mp})$. Moreover, $kx \oplus \mathrm{Proj}(kz \oplus k[z, y]) \subseteq Z$. It follows that $kx \oplus \mathrm{Proj}(ky \oplus k[z, y]) \subseteq Z$, so that $\mathfrak{e} \in Z$. \square

A Lie algebra \mathfrak{g} is called *perfect* if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Remark 3.4.2. In Theorem 3.4.7, it is obvious that \mathfrak{g} can not be a perfect restricted Lie algebra since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}_{\mathfrak{g}}(c)$.

Combine with Proposition 2.3.1, we readily obtain:

Corollary 3.4.11. *Suppose that $p \geq 7$. Let \mathfrak{g} be a centerless restricted Lie algebra such that $\dim V(\mathfrak{g}) = 2$. Then $V(\mathfrak{g})$ is irreducible and $\text{Proj}(V(\mathfrak{g})) \cong \mathbb{P}^1$.*

Proof. If $\text{Sw}(\mathfrak{g}) = \{0\}$, then $\mathfrak{g} \cong \mathfrak{sl}_2$. Hence, the Veronese embedding:

$$\zeta : \mathbb{P}^1 \rightarrow \text{Proj}(\mathfrak{sl}_2); (x : y) \mapsto \left[\begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix} \right]$$

is a morphism such that $\text{im}\zeta = \text{Proj}(V(\mathfrak{sl}_2))$ and $\mathbb{P}^1 \rightarrow \text{Proj}(V(\mathfrak{sl}_2))$ is an isomorphism.

Assume that $\text{Sw}(\mathfrak{g}) \neq \{0\}$. According to Theorem 3.4.10, we know that $\mathbb{E}(2, \mathfrak{g})$ contains only one element. Let

$$V(\mathfrak{g}) = Z_1 \cup \cdots \cup Z_s$$

be the decomposition of $V(\mathfrak{g})$ into irreducible components. Suppose that \mathfrak{g} has enough strong degeneration. According to Proposition 3.3.4, there exists an elementary subalgebra $\mathfrak{e} = kc_1 \oplus kc_2$ such that $c_i \in \text{Sw}(\mathfrak{g})$ for $i = 1, 2$. Assume that $s \geq 2$. Let $x \in Z_s \setminus (\cup_{i \neq s} Z_i)$. By the same argument as in Proposition 2.3.1, we know that both $[c_1, x]$ and $[c_2, x]$ are contained in Z_s . Since $\mathbb{E}(2, \mathfrak{g}) = \{kc_1 \oplus kc_2\}$, we have $[c_1, x] \neq 0$ and $[c_2, x] \neq 0$. Hence, we obtain $Z_s = kx \oplus k[c_1, x] = kx \oplus k[c_2, x]$. So that $k[c_1, x] = k[c_2, x]$. We can find another elementary subalgebra, this is a contradiction. Suppose that \mathfrak{g} is semisimple with $\text{Sw}(\mathfrak{g}) = kc$. Lemma 3.4.3 implies that $\mathbb{E}(2, \mathfrak{g}) = \{kc \oplus kc'\}$. We can use the same argument as above. Suppose that $\text{Sw}(\mathfrak{g}) = kc$ and \mathfrak{g} is not semisimple. Then kc is the only non-zero abelian ideal of \mathfrak{g} , thus $[c, x] = 0$ for any $x \in V(\mathfrak{g})$. This contradicts the fact, that $\mathbb{E}(2, \mathfrak{g})$ has only one element. \square

Proposition 3.4.12. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Suppose that $\dim \text{Sw}(\mathfrak{g}) = 1$. The following statement holds:*

1. $[c, c'] = 0$ for any $c, c' \in \text{Sw}(\mathfrak{g})$.
2. If the center $\mathfrak{z}(\mathfrak{g}) \neq (0)$, then $\text{Sw}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = kc$ for some $c \in \mathfrak{z}(\mathfrak{g})$.

Proof. If the center $\mathfrak{z}(\mathfrak{g}) \neq (0)$, then we claim that $\text{Sw}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = kc$ for some $c \in \mathfrak{z}(\mathfrak{g})$. Assume that we can find another sandwich element c' such that c and c' are linearly independent. Then it is easy to check that the subspace $kc \oplus kc'$ is contained in $\text{Sw}(\mathfrak{g})$. This is a contradiction since $\dim \text{Sw}(\mathfrak{g}) = 1$. Suppose that \mathfrak{g} is centerless. We write

$$\text{Sw}(\mathfrak{g}) = kc_1 \cup \cdots \cup kc_s$$

as a union of its irreducible components. Thanks to Lemma 2.2.1, $\exp(c_i) \in \text{Aut}_p(\mathfrak{g})^\circ$ for every c_i ($1 \leq i \leq s$). Note that all irreducible components kc_i are $\text{Aut}_p(\mathfrak{g})^\circ$ -stable (see [18, (8.2)]). It follows that $\exp(c_i)(c_j) = \lambda_{ij}c_j$ for some $\lambda_{ij} \in k$. Hence $[c_i, c_j] = (\lambda_{ij} - 1)c_j$. As $(\text{ad } c_i)^2 = 0$, this ensures that $\lambda_{ij} = 1$. \square

Proposition 3.4.13. *Let G be a connected algebraic group with Lie algebra $\mathfrak{g} := \text{Lie}(G)$. Suppose that $\dim \text{Sw}(\mathfrak{g}) = 1$. Then $\text{Sw}(\mathfrak{g}) = kc$ is a one-dimensional ideal of \mathfrak{g} .*

Proof. Similar to the proof of Proposition 3.4.12, We write

$$\text{Sw}(\mathfrak{g}) = kc_1 \cup \cdots \cup kc_s$$

as a union of its irreducible components. We consider the action of G on $\text{Sw}(\mathfrak{g})$. As G is connected, for each c_i , we obtain a character

$$\lambda_i : G \rightarrow k \setminus \{0\}$$

such that $g.c_i = \lambda_i(g)c_i$ for all $g \in G$. We denote by

$$d\lambda_i : \mathfrak{g} \rightarrow k$$

the tangent map of λ_i . By abuse of notation, we replace $d\lambda_i$ by λ_i . It follows that $[x, c_i] = \lambda_i(x)c_i$ for any $x \in \mathfrak{g}$. Suppose that $s \geq 2$. Then

$$[[x, ac_1 + bc_2], ac_2 + bc_2] = [a\lambda_1(x)c_1 + b\lambda_2(x)c_2, ac_2 + bc_2] = 0$$

for all $x \in \mathfrak{g}$ and $(a, b) \in k^2$, where the second equality holds by Proposition 3.4.12(1). Hence the space $kc_1 \oplus kc_2 \subseteq \text{Sw}(\mathfrak{g})$. We arrive at a contradiction. \square

Conjecture 3.4.14. *Suppose that $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra with strong degeneration. Let $\text{Sw}(\mathfrak{g})$ be the set of sandwich elements and $\text{Proj}(\text{Sw}(\mathfrak{g}))$ its projectivization. Then the variety $\text{Proj}(\text{Sw}(\mathfrak{g}))$ is connected.*

3.5 Applications

We call a restricted Lie algebra $(\mathfrak{g}, [p])$ admit a *restricted \mathbb{Z} -grading* if

$$\mathfrak{g} = \bigoplus_{i=-r}^s \mathfrak{g}_i, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \quad \mathfrak{g}_i^{[p]} \subseteq \mathfrak{g}_{pi} \quad (r, s \geq 1),$$

where $\mathfrak{g}_i := (0)$ for $i \notin \{-r, \dots, s\}$.

Proposition 3.5.1. *Suppose that $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra. If $\mathfrak{g} = \bigoplus_{i=-r}^s \mathfrak{g}_i$ is a restricted \mathbb{Z} -grading and \mathfrak{g} satisfies one of the following condition:*

(a) $s > r + 2,$

(b) $p \geq 7$ and $r \neq s,$

then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.

Proof. Since $s > r + 2$, $2(s - 1) - r > s$, this yields that

$$\mathfrak{g}_{s-1} \oplus \mathfrak{g}_s \subseteq \text{Sw}(\mathfrak{g}),$$

so that \mathfrak{g} is a centerless Lie algebra with enough strong degeneration. The connectedness of $\mathbb{E}(2, \mathfrak{g})$ follows from Theorem 3.3.6.

Suppose that $r \neq s$, we may assume that $s > r$. Direct computation implies that $\mathfrak{g}_s \subseteq \text{Sw}(\mathfrak{g})$. Now our asseration follows from Theorem 3.4.8. \square

Remark 3.5.1. Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ be a restricted simple Lie algebra. The Block-Wilson-Strade-Premet classification Theorem of simple Lie algebra (cf.[30]) implies that \mathfrak{g} is of classical or Cartan type (see also [3] for $p > 7$). Let $(\mathfrak{g}, [p])$ be a restricted simple Lie algebra of Cartan type. According to Theorem 2.2.2, \mathfrak{g} is a Lie algebra with strong degeneration. Otherwise, we assume that $\text{Sw}(\mathfrak{g}) = \{0\}$. Then \mathfrak{g} is an almost classical semisimple. But \mathfrak{g} is simple, thus \mathfrak{g} is of classical type. This is a contradiction, since the simple classical Lie algebras do not contain sandwich element. Hence, we can directly show that $\mathbb{E}(2, \mathfrak{g})$ is connected when $\mathfrak{g} = W, S, H, K$ (for further details, we refer to [31, Chapter IV]).

Corollary 3.5.2. *Suppose that $p \geq 5$. Let $(\mathfrak{g}, [p])$ be a restricted simple Lie algebra of Cartan type or restricted Melikian algebra. Then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. It is well-known that restricted simple Lie algebra of Cartan type and restricted Melikian algebra possess a restricted \mathbb{Z} -grading:

$$\mathfrak{g} = \bigoplus_{i=-r}^s \mathfrak{g}_i.$$

Furthermore, the grading satisfies the condition (a) in Proposition 3.5.1 except $p = 5$ and $\mathfrak{g} \cong W(1)$ (see [31, Chapter VI] and [23] for Melikian algebra). But Example 3.2.2) ensures that $\mathbb{E}(2, W(1))$ is also connected. \square

Remark 3.5.2. Let $p = 5$. We consider the restricted simple Lie algebra $\mathfrak{g} := W(1)$. There is a \mathbb{Z} -grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_3,$$

and $\text{Sw}(\mathfrak{g}) = \mathfrak{g}_3 = ke_3$ for some $0 \neq e_3 \in \mathfrak{g}_3$. The variety $\mathbb{E}(2, \mathfrak{g})$ is connected (see Example 3.2.2). It seems that Theorem 3.4.5 still works when $p = 5$.

Suppose that \mathfrak{g} is a centerless Lie algebra, we will identify \mathfrak{g} with the Lie algebra of inner derivations $\text{ad } \mathfrak{g} \subseteq \text{Der } \mathfrak{g}$. Let \mathfrak{g}_p be the p -closure of \mathfrak{g} , i.e., the smallest p -subalgebra of $\text{Der } \mathfrak{g}$ containing $\text{ad } \mathfrak{g}$.

Proposition 3.5.3. *Let $p \geq 5$. Suppose that \mathfrak{g} is a centerless perfect Lie algebra with enough strong degeneration. Then the variety $\mathbb{E}(2, \mathfrak{g}_p)$ is connected.*

Proof. If \mathfrak{g} is restricted, then $\mathfrak{g} = \mathfrak{g}_p$. According to Theorem 3.3.6, the variety $\mathbb{E}(2, \mathfrak{g})$ is connected. Suppose that \mathfrak{g} is not restricted. We claim that \mathfrak{g}_p is centerless with strong degeneration. Let $d \in \mathfrak{z}(\mathfrak{g}_p)$. It follows that

$$0 = [d, \text{ad } x] = \text{ad } d(x)$$

for any $x \in \mathfrak{g}$, so that $d(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g}) = (0)$. This implies that $\mathfrak{z}(\mathfrak{g}_p) = (0)$.

Assume that $\text{Sw}(\mathfrak{g}_p) = \{0\}$. According to Theorem 2.2.2, \mathfrak{g}_p is an almost classical semisimple Lie algebra. Thanks to Lemma 2.2.6, $[\mathfrak{g}_p, \mathfrak{g}_p]$ is a classical semisimple restricted Lie subalgebra of \mathfrak{g}_p . Since it is well-known $[\mathfrak{g}_p, \mathfrak{g}_p] \subseteq \mathfrak{g}$ (e.g. see [31]), so that

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g}_p, \mathfrak{g}_p] \subseteq \mathfrak{g}.$$

This is a contradiction since \mathfrak{g} is not restricted.

If \mathfrak{g}_p has enough strong degeneration, then our assertion follows from Theorem 3.3.6. So we can assume that $\text{Sw}(\mathfrak{g}_p) = kc$ for some non-zero element $c \in \text{Sw}(\mathfrak{g}_p)$. By virtue of [26, Lemma 6], we have $c \in \text{Sw}(\mathfrak{g})$. We claim that c is a special sandwich element of \mathfrak{g} , i.e., $[c, c'] = 0$ for any $c' \in \text{Sw}(\mathfrak{g})$. Suppose that $[c, c'] \neq 0$ for some $c' \in \text{Sw}(\mathfrak{g})$. It can be easily checked that $(\text{ad}[c, c'])^2(\mathfrak{g}_p) = 0$. This contradicts $\text{Sw}(\mathfrak{g}_p) = kc$. Since \mathfrak{g} has enough strong degeneration, we can find $c' \in \text{Sw}(\mathfrak{g})$ such that $kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})$. Note that $(\text{ad} c')^3(\mathfrak{g}_p) = (0)$. Then we use the same argument as in Proposition 3.4.4 and Theorem 3.4.5 (see Remark 3.4.1). \square

Recall that $\mathbb{E}(2, \mathfrak{g})^*$ is a closed subvariety of $\mathbb{E}(2, \mathfrak{g})$. Also, both $\mathbb{E}(2, \mathfrak{g})^*$ and $\mathbb{E}(2, \mathfrak{g})$ are connected projective varieties when $(\mathfrak{g}, [p])$ is a centerless restricted Lie with strong degeneration (see Proposition 3.3.4, Proposition 3.4.4 and Theorem 3.4.8). They are the closed subvarieties of $\text{Gr}_2(\mathfrak{g})$. We introduce the following subset of $\text{Gr}_2(\mathfrak{g})$

$$\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})} := \{V \in \text{Gr}_2(\mathfrak{g}); V \subseteq V(\mathfrak{g})\}.$$

As $V(\mathfrak{g})$ is conical, closed subvariety of \mathfrak{g} , the following Lemma is a direct consequence of Lemma 3.1.1.

Lemma 3.5.4. $\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$ is a closed subvariety of $\text{Gr}_2(\mathfrak{g})$.

Obviously, if $(\mathfrak{g}, [p])$ is a restricted Lie algebra, then

$$\mathbb{E}(2, \mathfrak{g})^* \subseteq \mathbb{E}(2, \mathfrak{g}) \subseteq \text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}.$$

By virtue of Lemma 3.5.4, $\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$ is a projective variety. Furthermore, we have the following theorem:

Theorem 3.5.5. Suppose that $p \geq 5$. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with enough strong degeneration. Then the variety $\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$ is connected.

Proof. Thanks to Theorem 3.3.6, we can find a connected component Z of $\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$ such that $Z \supseteq \mathbb{E}(2, \mathfrak{g})$. Let $V = kx \oplus ky \in \text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$. It is enough to prove that $V \in Z$. Fix an non-zero element $c \in \text{Sw}(\mathfrak{g})$.

\diamond Suppose that $c \in V$. We can write $V = kc \oplus kv$. If $[c, v] = 0$, then $V \in \mathbb{E}(2, \mathfrak{g}) \subseteq Z$. If $[c, v] \neq 0$, then we claim that the space $W := \langle c, v, [c, v] \rangle$ is contained in $V(\mathfrak{g})$. In face, since $\lambda_1 c + \lambda_2 v \in V(\mathfrak{g})$ for any $\lambda_1, \lambda_2 \in k$, it follows from Corollary 2.2.5 that

$$(1 + \lambda_3 \text{ad } c)(\lambda_1 c + \lambda_2 v) = \lambda_1 c + \lambda_2 v + \lambda_3 \lambda_2 [c, v] \in V(\mathfrak{g}). \quad (3.5)$$

Let $x := a_1c + a_2v + a_3[c, v]$ be an element in W . If $a_2 \neq 0$, then (3.5) ensures that $x \in V(\mathfrak{g})$. Suppose that $a_2 = 0$. In view of Corollary 2.2.5, we also obtain $x \in V(\mathfrak{g})$, so that $W \subseteq V(\mathfrak{g})$.

Note that $kc \oplus k[c, v] \in \mathbb{E}(2, \mathfrak{g}) \cap \text{Gr}_2(W)$ and $\text{Gr}_2(W)$ is connected, it follows that $\text{Gr}_2(W) \subseteq Z$, in particular, $V = kc \oplus kv \in Z$.

◇ Suppose that $c \notin V$. If there exists some nonzero element $w_0 \in V$ such that $[c, w_0] = 0$, then we may write $V = kw_0 \oplus kw_1$. Moreover, if $[c, w_1] = 0$, then $W' := \langle c, w_0, w_1 \rangle \subseteq V(\mathfrak{g})$. As $kc \oplus kw_0 \in \mathbb{E}(2, \mathfrak{g}) \cap \text{Gr}_2(W')$, it follows that $\text{Gr}_2(W') \subseteq Z$, so that $V \in Z$. If $[c, w_1] \neq 0$ and $k[c, w_1] \neq kw_0$, then we set

$$X := kw_0 \oplus \text{Proj}(kw_1 \oplus k[c, w_1])$$

and

$$Y := kw_0 \oplus \text{Proj}(kc \oplus k[c, w_1]).$$

We claim that $X \cup Y \subseteq V(\mathfrak{g})$. Let $\lambda_1w_0 + \lambda_2w_1 \in V$. It follows from Corollary 2.2.5 that

$$(1 + \lambda_3 \text{ad } c)(\lambda_1w_0 + \lambda_2w_1) = \lambda_1w_0 + \lambda_2w_1 + \lambda_3\lambda_2[c, w_1] \in V(\mathfrak{g}).$$

In particular, $kw_0 \oplus k[c, w_1] \subseteq V(\mathfrak{g})$. As $[c, w_0] = 0$, we have $Y \subseteq V(\mathfrak{g})$. Since $X \cap Y \neq \emptyset$ and both of them are contained in $\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$, also $kc \oplus kw_0 \in \mathbb{E}(2, \mathfrak{g}) \cap Y$, it follows that $X \subseteq Z$. So that $V = kw_0 \oplus kw_1 \in Z$. If $[c, w_1] \neq 0$ and $kw_0 = k[c, w_1]$, then we can write $V = k[c, w_1] \oplus kw_1$. Since \mathfrak{g} has enough strong degeneration, we can find some non-zero $c' \in \text{Sw}(\mathfrak{g})$ such that $[c, c'] = 0$ and $kc \neq kc'$ (see Lemma 3.3.3). If $[c', w_1] = 0$ or $[c', [c, w_1]] = 0$, then we can use the same argument as above. If $[c', w_1] \neq 0$ and $[c', [c, w_1]] \neq 0$, we claim that $\mathfrak{z}_{\mathfrak{g}}(c') \cap V = \{0\}$, we can reduce to the last case. In fact if $[c', \lambda w_1 + \mu[c, w_1]] = 0$ for $\lambda, \mu \in k$, then

$$\lambda[c', w_1] + \mu[c, [c', w_1]] = 0,$$

this is a contradiction (see Lemma 2.2.1).

◇ We only need to prove that the last case, i.e., $\mathfrak{z}_{\mathfrak{g}}(c) \cap V = \{0\}$. It follows from Lemma 3.3.5, we have $\mathfrak{e} := k[c, x] \oplus k[c, y] \in \mathbb{E}(2, \mathfrak{g})$. Meanwhile, V and \mathfrak{e} lie in the same connected component of $\text{Gr}_2(\mathfrak{g})^{V(\mathfrak{g})}$. This implies $V \in Z$. \square

Given a p -subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we let

$$\text{Nor}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g}; [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

be the *normalizer* of \mathfrak{h} in \mathfrak{g} . The $\text{Nor}_{\mathfrak{g}}(\mathfrak{h})$ is a p -subalgebra of \mathfrak{g} .

Proposition 3.5.6. *Suppose that $p \geq 3$. If $\mathfrak{e}_0 \in \mathbb{E}(2, \mathfrak{g})$, then the variety $\mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0))$ is connected.*

Proof. Note that $\mathfrak{e}_0 \in \text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0)$ is an ideal of $\text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0)$. If $\mathfrak{e} \in \mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0))$, then \mathfrak{e}_0 is an \mathfrak{e} -module and Engel's Theorem implies

$$(\text{ad } x) \circ (\text{ad } y)(\mathfrak{e}_0) = (0), \quad \forall x, y \in \mathfrak{e}.$$

As $p \geq 3$ Jacobson's formula yields: $\mathfrak{e}_0 + \mathfrak{e} \subseteq V(\mathfrak{g})$. Hence $\mathfrak{e}_0 + \mathfrak{e}$ is a p -trivial restricted subalgebra of $\text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0)$. Let Z be the connected component containing \mathfrak{e}_0 . By virtue of Corollary 3.2.4 we obtain $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{e}_0 + \mathfrak{e}) \subseteq Z$, so that $Z = \mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0))$. \square

We readily obtain a corollary:

Corollary 3.5.7. *Suppose that $p \geq 3$. If $\mathfrak{e}_0 \in \mathbb{E}(2, \mathfrak{g})$ is an ideal of \mathfrak{g} , then the variety $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Example 3.5.3. Suppose that $p \geq 5$. Let $W(1)$ the Witt algebra. As the discussion in Example 3.2.2, we still denote by $\mathfrak{g} := \bigoplus_{i \geq 1} k e_i \subseteq W(1)$. We have $\mathbb{E}(2, W(1)) = \mathbb{E}(2, \mathfrak{g})$.

On the other hand, we denote by $\mathfrak{e}_0 := k e_{p-3} \oplus k e_{p-2}$. It is easy to prove that \mathfrak{e}_0 is a p -ideal of \mathfrak{g} and $\mathfrak{e}_0 \in \mathbb{E}(2, \mathfrak{g})$. So it follows from Corollary 3.5.7 that the variety

$$\mathbb{E}(2, W(1)) = \mathbb{E}(2, \mathfrak{g}) = \mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{e}_0))$$

is connected.

If $p = 3$, then $W(1) \cong \mathfrak{sl}_2$. Hence $\mathbb{E}(2, W(1)) \cong \mathbb{E}(2, \mathfrak{sl}_2) = \emptyset$ (see Example 2.2.2).

Chapter 4

Degree functions

Recently, in [10], the author introduced the invariants for modules of restricted Lie algebras, called j -degrees. Let M be a $U_0(\mathfrak{g})$ -module. The j -degree $\deg^j(M)$ of M can be linked with j -rank. We will restrict these modules to the two-dimensional elementary subalgebras and consider the j -degree functions. Throughout, $(\mathfrak{g}, [p])$ is assumed to be a restricted Lie algebra. In the sequel, we shall be concerned with modules for restricted Lie algebras \mathfrak{g} .

4.1 j -degree functions

Let M be a finite-dimensional $U_0(\mathfrak{g})$ -module, and let $x \in V(\mathfrak{g})$, we denote by

$$x_M : M \rightarrow M; \quad m \mapsto x.m$$

the operator associated to x . Given $j \in \{1, \dots, p-1\}$, we define the *generic j -rank* of M via

$$\mathrm{rk}^j(M) := \max\{\mathrm{rk}(x_M^j); x \in V(\mathfrak{g})\},$$

and we say that M has *constant j -rank* provided $\mathrm{rk}(x_M^j) = \mathrm{rk}^j(M)$ for any non-zero element $x \in V(\mathfrak{g})$ (see [16, Definition 3.7]).

Suppose that $V(\mathfrak{g})$ is a subspace of \mathfrak{g} . Then the morphism

$$\mathrm{pl}_M \circ \mathrm{im}_M^j : \mathrm{Proj}(V(\mathfrak{g})) \rightarrow \mathrm{Proj}\left(\bigwedge^{\mathrm{rk}^j(M)} (M)\right); [x] \mapsto \left[\bigwedge^{\mathrm{rk}^j(M)} (\mathrm{im}x_M^j)\right]$$

is homogeneous.

Definition 4.1.1. [10, Definition 4.1]. Suppose that $V(\mathfrak{g})$ is a subspace of \mathfrak{g} , $j \in \{1, \dots, p-1\}$. Let M be a $U_0(\mathfrak{g})$ -module of generic j -rank $\text{rk}^j(M) > 0$. Then

$$\text{deg}^j(M) := \text{deg}(\text{pl}_M \circ \text{im}_M^j)$$

is called the j -degree of M . We write $\text{deg}(M) := \text{deg}^1(M)$ refer to $\text{deg}(M)$ as the degree of M .

If $\text{rk}^j(M) = 0$, then we define $\text{deg}^j(M) := 0$.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra such that $r_{\max}(\mathfrak{g}) \geq 2$, i.e., $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$, M be a $U_0(\mathfrak{g})$ -module of constant j -rank. Then

$$\text{deg}_M^j : \mathbb{E}(2, \mathfrak{g}) \rightarrow \mathbb{N}_0; \quad \mathfrak{e} \mapsto \text{deg}^j(M|_{\mathfrak{e}})$$

is called the j -degree function of M .

4.2 Constant j -degree function

We will frequently use the following theorem to investigate the property of the function deg_M^j .

Theorem 4.2.1. [10, Theorem 4.1.2]. Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a p -subalgebra such that $\dim V(\mathfrak{h}) \geq 2$. If M is a $U_0(\mathfrak{g})$ -module of constant j -rank, then we have $\text{deg}^j(M) = \text{deg}^j(M|_{\mathfrak{h}})$.

According to the above theorem, the map deg_M^j is constant whenever the nullcone $V(\mathfrak{g})$ is a subspace of \mathfrak{g} .

Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with strong degeneration. Recall that

$$\mathbb{E}(2, \mathfrak{g})^* = \{\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g}); \mathfrak{e} \cap \text{Sw}(\mathfrak{g}) \neq \{0\}\}$$

is a closed subvariety of $\mathbb{E}(2, \mathfrak{g})$ (see Lemma 3.2.1).

Proposition 4.2.2. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with enough strong degeneration. then the function deg_M^j is constant on $\mathbb{E}(2, \mathfrak{g})^*$.

Proof. Since \mathfrak{g} has enough strong degeneration, thanks to Lemma 3.3.3, we can find non-zero elements $c_1, c_2 \in \text{Sw}(\mathfrak{g})$ such that $[c_1, c_2] = 0$ and $kc_1 \neq kc_2$. Moreover, since \mathfrak{g} is centerless, the space $\mathfrak{e}_0 := kc_1 \oplus kc_2$ is an elementary subalgebra and contained in $\mathbb{E}(2, \mathfrak{g})^*$. Let $\mathfrak{e} := kc \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$ with $c \in \text{Sw}(\mathfrak{g})$.

(1). Suppose that $c \in \mathfrak{e}_0$, we can rewrite $\mathfrak{e}_0 = kc \oplus kc'$. By construction, the space $\text{Proj}(\text{Sw}(\mathfrak{g}) \cap \mathfrak{e}_0)$ contains at least two points, hence we can assume that $kc' \in \text{Proj}(\text{Sw}(\mathfrak{g}) \cap \mathfrak{e}_0) \setminus \{kc\}$. If $[c', x] = 0$, then the subspace spanned by $\langle c, c', x \rangle$ is contained in $V(\mathfrak{g})$. According to Theorem 4.2.1, we have $\deg_M^j(\mathfrak{e}_0) = \deg_M^j(\mathfrak{e})$. If $[c', x] \neq 0$, then we have $kc' \oplus k[c', x] \in \mathbb{E}(2, \mathfrak{g})^*$.

◇ $k[c', x] \neq kc$.

$$\deg_M^j(\mathfrak{e}_0) = \deg_M^j(kc' \oplus k[c', x]) = \deg_M^j(kc \oplus k[c', x]) = \deg_M^j(\mathfrak{e}).$$

The last equality holds because the space spanned by \mathfrak{e} and $[c', x]$ is contained in $V(\mathfrak{g})$.

◇ $k[c', x] = kc$.

Note that $[c, c'] = [c, x] = 0$. It follows that the subspace $kc' \oplus kx \subseteq V(\mathfrak{g})$. Hence $\deg_M^j(\mathfrak{e}_0) = \deg_M^j(\mathfrak{e})$.

(2). Suppose that $c \notin \mathfrak{e}_0$. If $[c, c_1] \neq 0$, then

$$\deg_M^j(\mathfrak{e}_0) = \deg_M^j(kc_1 \oplus k[c, c_1]) = \deg_M^j(kc \oplus k[c, c_1]).$$

If $[c, c_1] = 0$, then we can replace \mathfrak{e}_0 by $kc \oplus kc_1$ since $\deg_M^j(\mathfrak{e}_0) = \deg_M^j(kc \oplus kc_1)$. In sum, we can always find some elementary subalgebra \mathfrak{e}' such that $\deg_M^j(\mathfrak{e}_0) = \deg_M^j(\mathfrak{e}')$ and $c \in \mathfrak{e}'$. By Case (1), we know $\deg_M^j(\mathfrak{e}) = \deg_M^j(\mathfrak{e}')$, so that $\deg_M^j(\mathfrak{e}_0) = \deg_M^j(\mathfrak{e})$. □

Now we are going to prove our main theorem in this section.

Theorem 4.2.3. *Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with enough strong degeneration. Then the function \deg_M^j is constant on $\mathbb{E}(2, \mathfrak{g})$.*

Proof. We fix a non-zero element $c \in \text{Sw}(\mathfrak{g})$.

Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, \mathfrak{g})$. Firstly, we assume that $\mathfrak{e} \notin \mathbb{E}(2, \mathfrak{g})^*$.

(1) Suppose that there exists some non-zero element $z_1 \in \mathfrak{e}$ such that $[c, z_1] = 0$. Then we can rewrite $\mathfrak{e} = kz_1 \oplus kz_2$. Moreover, if $[c, z_2] = 0$, then $\deg_M^j(\mathfrak{e}) = \deg_M^j(kc \oplus kz_1)$. If $[c, z_2] \neq 0$ and $kz_1 \neq k[c, z_2]$, then

$$\deg_M^j(\mathfrak{e}) = \deg_M^j(kz_1 \oplus k[c, z_2]) = \deg_M^j(kc \oplus k[c, z_2]).$$

If $[c, z_2] \neq 0$ and $kz_1 = k[c, z_2]$, then $kc \oplus kz_2 \subseteq V(\mathfrak{g})$. It follows that

$$\deg_M^j(\mathfrak{e}) = \deg_M^j(kc \oplus kz_1).$$

(2) Suppose that there is no non-zero element $z \in \mathfrak{e}$ such that $[c, z] = 0$. We claim that the subspace V spanned by $V := \langle x, y, [c, x + y] \rangle$ is contained in $V(\mathfrak{g})$. It is equivalent to say that $\lambda_1 x + \lambda_2 y + \lambda_3 [c, x + y] \in V(\mathfrak{g})$ for any $\lambda_1, \lambda_2, \lambda_3 \in k$. According to Lemma 2.2.1, we have

$$\exp(\mu c)(\lambda_1 x + \lambda_2 y) = \lambda_1 x + \lambda_2 y + \mu(\lambda_1 + \lambda_2)[c, x + y] \in V(\mathfrak{g}).$$

Hence, if $\lambda_1 + \lambda_2 \neq 0$, then we can find some $\mu \in k$ such that $\mu(\lambda_1 + \lambda_2) = \lambda_3$. Therefore, the set

$$\{\lambda_1 x + \lambda_2 y + \lambda_3 [c, x + y]; \lambda_1 + \lambda_2 \neq 0\}$$

is contained in $V(\mathfrak{g})$. Note that $\lambda_1 + \lambda_2 \neq 0$ is an open condition and $V(\mathfrak{g})$ is closed variety, so that $V \subseteq V(\mathfrak{g})$. Furthermore, we prove that the subspace W spanned by $W := \langle x, [c, x], [c, x + y] \rangle$ is contained in $V(\mathfrak{g})$. For any $\mu_1, \mu_2 \in k$, by the above discussion, we have $\mu_1 x + \mu_2 [c, x + y] \in V(\mathfrak{g})$. Hence

$$\exp(\lambda c)(\mu_1 x + \mu_2 [c, x + y]) = \mu_1 x + \mu_2 [c, x + y] + \lambda \mu_1 [c, x] \in V(\mathfrak{g}).$$

So the set

$$\{\mu_1 c + \mu_2 [c, x + y] + \mu_3 [c, x]; \mu_1 \neq 0\}$$

is contained in $V(\mathfrak{g})$. It follows that $W \subseteq V(\mathfrak{g})$. It should be noticed that $\mathfrak{e} \subseteq V$. On the other hand, the subspace $k[c, x] \oplus k[c, x + y]$ is an elementary subalgebra and contained in W . Moreover, we have the space $kx \oplus k[c, x + y] \subseteq V \cap W$, so that $\dim_k(V \cap W) \geq 2$. Note that $k[c, x] \oplus k[c, y] \subseteq W$. Hence the Theorem 4.2.1 implies that $\deg_M^j(\mathfrak{e}) = \deg_M^j(k[c, x] \oplus k[c, y]) = \deg_M^j(kc \oplus k[c, x])$. Also, $kc \oplus k[c, x] \in \mathbb{E}(2, \mathfrak{g})^*$.

In sum, we can always find some elementary subalgebra $\mathfrak{e}' \in \mathbb{E}(2, \mathfrak{g})^*$ such that

$$\deg_M^j(\mathfrak{e}) = \deg_M^j(\mathfrak{e}').$$

Now Proposition 4.2.2 ensures that the function \deg_M^j is always constant. \square

Corollary 4.2.4. *Suppose that $p \geq 5$. Let $(\mathfrak{g}, [p])$ be a restricted simple Lie algebra of Cartan type or restricted Melikian algebra. Let M be a $U_0(\mathfrak{g})$ -module of constant j -rank, then \deg_M^j is constant.*

Proof. The proof is similar to Corollary 3.5.2. These Lie algebras have enough strong degeneration except $p = 5$ and $\mathfrak{g} \cong W(1)$. Hence Theorem 4.2.3 implies that the function \deg_M^j is constant.

For the Witt algebra $W(1)$, we denote by $\mathfrak{g} := \bigoplus_{i \geq 1} ke_i \subseteq W(1)$, then $\mathbb{E}(2, W(1)) = \mathbb{E}(2, \mathfrak{g})$ (Example 3.2.2). Then it is obvious that the function \deg_M^j is constant since $V(\mathfrak{g})$ is a subspace of \mathfrak{g} (Theorem 4.2.1). \square

Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with strong degeneration. We still need to discuss the case $\text{Sw}(\mathfrak{g}) = kc$ for some non-zero element $c \in \text{Sw}(\mathfrak{g})$.

Lemma 4.2.5. *Let $p \geq 7$. Suppose that $(\mathfrak{g}, [p])$ is a semisimple restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ for some non-zero $c \in \text{Sw}(\mathfrak{g})$. Then the function \deg_M^j is constant on $\mathbb{E}(2, \mathfrak{g})^*$.*

Proof. Thanks to Lemma 3.4.3, we can find some element c' such that $\mathfrak{e}_0 := kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})^*$ and $(\text{ad } c')^3(\mathfrak{g}) = 0$. Let $\mathfrak{e} = kc \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$.

(1). Suppose that $[c', x] = 0$. It is clear that $\deg_M^j(\mathfrak{e}) = \deg_M^j(\mathfrak{e}_0)$ since the subspace spanned by $\langle c, c', x \rangle$ is contained in $V(\mathfrak{g})$.

(2). Suppose that $[c', x] \neq 0$ and $[c', [c', x]] = 0$. Since $\exp(c') \in \text{Aut}_p(\mathfrak{g})$, it follows that $kc' \oplus k[c', x] \in \mathbb{E}(2, \mathfrak{g})^*$.

$\diamond k[c', x] \neq kc$.

$$\deg_M^j(\mathfrak{e}_0) = \deg_M^j(kc' \oplus k[c', x]) = \deg_M^j(kc \oplus k[c', x]) = \deg_M^j(\mathfrak{e}).$$

$\diamond k[c', x] = kc$.

Note that $[c, c'] = [c, x] = 0$, it follows that $kc' \oplus kx \subseteq V(\mathfrak{g})$. So that

$$\deg_M^j(\mathfrak{e}_0) = \deg_M^j(\mathfrak{e}).$$

(3). Suppose that $[c', [c', x]] \neq 0$. First, we claim that $x, [c', x], [c', [c', x]]$ are linear independent. It is clear $x, [c', x]$ are linear independent. Assume that

$$[c', [c', x]] = ax + b[c', x]$$

for some $(a, b) \in k^2 \setminus \{(0, 0)\}$. So that $0 = a[c', x] + b[c', [c', x]]$. This is a contradiction. Furthermore, if $c \in \langle x, [c', x], [c', [c', x]] \rangle$, then there exists $(\alpha, \beta, \gamma) \in k^3 \setminus \{(0, 0, 0)\}$ such that

$$c = \alpha x + \beta[c', x] + \gamma[c', [c', x]]. \quad (4.1)$$

In this situation, we claim that $\alpha = \beta = 0$ and $\gamma \neq 0$.

$$0 = [c', c] = \alpha[c', x] + \beta[c', [c', x]].$$

This is also a contradiction.

Since

$$\exp(\text{ad } \lambda c')(x) = x + \lambda[c', x] + \frac{1}{2}\lambda^2[c', [c', x]] \in V(\mathfrak{g}) \quad (4.2)$$

for any $\lambda \in k$. This implies that

$$2\alpha^2 x + 2\alpha[c', x] + [c', [c', x]] \in V(\mathfrak{g}) \quad (4.3)$$

for any $\alpha \in k \setminus \{0\}$. By taking α goes to 0, we have $[c', [c', x]] \in V(\mathfrak{g})$. Note that $[c, c'] = [c, x] = 0$, so that the subspace spanned by $\langle c, c', [c', [c', x]] \rangle$ is contained in $V(\mathfrak{g})$.

◇ $c \notin \langle x, [c', x], [c', [c', x]] \rangle$.

$$\deg_M^j(\mathfrak{e}_0) = \deg_M^j(kc \oplus k[c', [c', x]]) = \deg_M^j(\mathfrak{e}).$$

• The last equality is non-trivial, I will explain it as follows: We define a map:

$$\varphi : k^3 \rightarrow \mathfrak{g}; \quad (\alpha, \beta, \gamma) \mapsto \alpha^2 x + \alpha\beta[c', x] + \frac{1}{2}\beta^2[c', [c', x]] + \gamma^2 c.$$

φ is a homogeneous morphism of degree 2. Hence φ induces a morphism

$$\omega : \mathbb{P}^2 \rightarrow \text{Proj}(\mathfrak{g}); \quad (\alpha : \beta : \gamma) \mapsto \langle \alpha^2 x + \alpha\beta[c', x] + \frac{1}{2}\beta^2[c', [c', x]] + \gamma^2 c \rangle$$

of degree $\deg(\omega) = 2$. According to (4.2), ω factors through $\text{Proj}(V(\mathfrak{g}))$. We denote by $\mathfrak{e}' := kc \oplus k[c', [c', x]]$. It follows that

$$\eta_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^2; \quad (m_1; m_2) \mapsto (0 : m_1 : m_2)$$

is a morphism of degree 1 such that $\text{im } \eta_1 \subseteq \omega^{-1}(\text{Proj}(\mathfrak{e}'))$. Similarly, the map

$$\eta_2 : \mathbb{P}^1 \rightarrow \mathbb{P}^2; \quad (n_1; n_2) \mapsto (n_1 : 0 : n_2)$$

is a morphism of degree 1 such that $\text{im } \eta_2 \subseteq \omega^{-1}(\text{Proj}(\mathfrak{e}))$.

Let $d := \deg(\text{pl}_M \circ \text{im}_M^j \circ \omega)$. Thanks to [10, (1.1.5)], we have

$$d = \deg((\text{pl}_M \circ \text{im}_M^j \circ \omega) \circ \eta_1) = \deg((\text{pl}_M \circ \text{im}_M^j)|_{\text{Proj}(\mathfrak{e}')} \circ (\omega \circ \eta_1)) = 2 \deg_M^j(\mathfrak{e}').$$

Also,

$$d = \deg((\text{pl}_M \circ \text{im}_M^j \circ \omega) \circ \eta_2) = \deg((\text{pl}_M \circ \text{im}_M^j)|_{\text{Proj}(\mathfrak{e})} \circ (\omega \circ \eta_2)) = 2 \deg_M^j(\mathfrak{e}).$$

So that $\deg_M^j(\mathfrak{e}) = \deg_M^j(kc \oplus k[c', [c', x]])$.

$\diamond c \in \langle x, [c', x], [c', [c', x]] \rangle$. According to (4.1), we know $kc = k[c', [c', x]]$.

Note that $[c, x] = [c, [c', x]] = 0$, so that

$$(x + \lambda[c', x] + \frac{1}{2}\lambda^2[c', [c', x]])^{[p]} = (x + \lambda[c', x])^{[p]} = 0.$$

Hence, $x + \lambda[c', x] \in V(\mathfrak{g})$ for any $\lambda \in k$. It follows that $k[c', x] \oplus kx \subseteq V(\mathfrak{g})$. Hence $\deg_M^j(\mathfrak{e}_0) = \deg_M^j(\mathfrak{e})$. \square

Lemma 4.2.6. *Let $p \geq 7$. Suppose that $(\mathfrak{g}, [p])$ is centerless restricted Lie algebra with $\text{Sw}(\mathfrak{g}) = kc$ and \mathfrak{g} is not semisimple. Then the function \deg_M^j is constant on $\mathbb{E}(2, \mathfrak{g})^*$.*

Proof. According to Theorem 3.4.7, we can find some non-zero element $c' \in V(\mathfrak{g})$ such that $[c', [c', x]] \in kc \oplus kc'$ for any $x \in \mathfrak{z}_{\mathfrak{g}}(c)$ and $\mathfrak{e}_0 := kc \oplus kc' \in \mathbb{E}(2, \mathfrak{g})^*$. Let $\mathfrak{e} := kc \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$. Note that $(\text{ad } c')^3(x) = 0$ for any $x \in V(\mathfrak{g})$. In the proof of Lemma 4.2.5, we only need the element c and c' . so that we can use the same argument. \square

Theorem 4.2.7. *Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra with strong degeneration. If M is a $U_0(\mathfrak{g})$ -module of constant j -rank, then the function \deg_M^j is constant.*

Proof. According to Proposition 4.2.2, Lemma 4.2.5 and Lemma 4.2.6, the function \deg_M^j is constant on $\mathbb{E}(2, \mathfrak{g})^*$. Let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$. Note that the proof of Theorem 4.2.3 only need one sandwich element. Hence, we can use the same argument. Then we can find some element $\mathfrak{e}_0 \in \mathbb{E}(2, \mathfrak{g})^*$ such that $\deg_M^j(\mathfrak{e}) = \deg_M^j(\mathfrak{e}_0)$, so that the function is constant. \square

Theorem 4.2.8. [11]. *Suppose that $p \geq 7$. Let G be a connected algebraic group with Lie algebra $\mathfrak{g} := \text{Lie}(G)$ and M a $U_0(\mathfrak{g})$ -module of constant j -rank. Then deg_M^j is constant*

Theorem 4.2.9. *Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra. If M is a $U_0(\mathfrak{g})$ -module of constant j -rank, then the function deg_M^j is constant.*

Proof. Our discussion is similar to the proof the Theorem 3.4.10. We only consider the case $\mathfrak{g} \cong \mathfrak{psl}_{mp}$. The canonical projection $\pi : \mathfrak{sl}_{mp} \rightarrow \mathfrak{psl}_{mp}$ induces a morphism (see the proof of Theorem 3.4.10)

$$\bar{\pi} : \mathbb{E}(2, \mathfrak{sl}_{mp}) \rightarrow \mathbb{E}(2, \mathfrak{psl}_{mp}).$$

Thanks to [12, Lemma 3.1], π is injective. Hence we can identify $\mathbb{E}(2, \mathfrak{sl}_{mp})$ as a subset of $\mathbb{E}(2, \mathfrak{psl}_{mp})$. Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, \mathfrak{psl}_{mp})$. We assume that $\mathfrak{e} \notin \mathbb{E}(2, \mathfrak{sl}_{mp})$. So that $k[x, y] = k$. Let $\mathfrak{sl}_{mp} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ be its standard decomposition. Without loss of generality, we can assume that $x \in \mathfrak{n}^+$ (Otherwise, we can reverse the conjugation). Let z be the root vector corresponding to the highest positive root. We have $[x, z] = 0$. In particular $kx \oplus kz \in \mathbb{E}(2, \mathfrak{sl}_{mp})$.

◇ If $[z, y] = 0$, then $\text{deg}_M^j(\mathfrak{e}) = \text{deg}_M^j(kz \oplus kx)$.

◇ If $[z, y] \neq 0$, then it is easy to check that $[z, [z, y]] = 0$. Also note that $\exp(\lambda z) \in \text{Aut}_p(\mathfrak{g})^\circ$ for any $\lambda \in k$. Corollary 2.2.5 implies $ky \oplus k[z, y] \subseteq V(\mathfrak{g})$. We have

$$\text{deg}_M^j(\mathfrak{e}) = \text{deg}_M^j(kx \oplus k[z, y]) = \text{deg}_M^j(kz \oplus k[z, y]).$$

In sum, we can always find some elementary subalgebra $\mathfrak{e}' \in \mathbb{E}(2, \mathfrak{sl}_{mp})$ such that $\text{deg}_M^j(\mathfrak{e}) = \text{deg}_M^j(\mathfrak{e}')$. Note that \mathfrak{sl}_{mp} is an algebraic Lie algebra, hence deg_M^j is constant on $\mathbb{E}(2, \mathfrak{sl}_{mp})$ (Theorem 4.2.8). So that deg_M^j is constant. \square

Definition 4.2.1. A $U_0(\mathfrak{g})$ -module M is said to have the *equal j -images property*, provided there exists a subspace $V^j \subseteq M$ such that $\text{im} x_M^j = V^j$ for all $x \in V(\mathfrak{g}) \setminus \{0\}$.

Corollary 4.2.10. *Suppose that $p \geq 7$. Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra. Let M be a $U_0(\mathfrak{g})$ -module of constant j -rank. If there exists $\mathfrak{e}_0 \in \mathbb{E}(2, \mathfrak{g})$ such that $M|_{\mathfrak{e}_0} \in \text{EIP}^j(\mathfrak{e}_0)$, then $M \in \text{EIP}^j(\mathfrak{e})$, $\forall \mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$.*

Proof. By assumption, we have $\text{deg}^j(M|_{\mathfrak{e}_0}) = 0$. The assertion now follows from Theorem 4.2.9. \square

Recall Theorem 3.4.10 and Theorem 4.2.9. It seems that there is some connection between connectedness of $\mathbb{E}(2, \mathfrak{g})$ and constant property of \deg_M^j . The following example shows that $\mathbb{E}(2, \mathfrak{g})$ is not connected even when the \deg_M^j is constant.

Example 4.2.2. Suppose that $p \geq 3$. We construct a 8-dimensional nilpotent restricted Lie algebra $\mathfrak{g} := kx_1 \oplus kx_2 \oplus ky_1 \oplus ky_2 \oplus kt_1 \oplus kt_2 \oplus kt_3 \oplus kt_4$, whose Lie bracket and p -map are given by

$$[x_1, x_2] = [y_1, y_2] = 0, [x_1, y_1] = t_1, [x_1, y_2] = t_2, [x_2, y_1] = t_3, [x_2, y_2] = t_4,$$

$$x_i^{[p]} = y_i^{[p]} = 0, t_i^{[p]} = t_i, i = 1, 2, 3, 4,$$

where the center $\mathfrak{z}(\mathfrak{g}) = kt_1 \oplus kt_2 \oplus kt_3 \oplus kt_4$. Since all $t_i \in \mathfrak{z}(\mathfrak{g})$, it follows (and can be easily verified) that the nullcone $V(\mathfrak{g}) = kx_1 \oplus kx_2 \oplus ky_1 \oplus ky_2$ is a vector space. We would like to compute the variety $\mathbb{E}(2, \mathfrak{g})$. Given a non-zero element $x := a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2 \in V(\mathfrak{g})$. It is sufficient to compute $V(\mathfrak{z}_{\mathfrak{g}}(x))$. Let $y := c_1x_1 + c_2x_2 + d_1y_1 + d_2y_2 \in V(\mathfrak{z}_{\mathfrak{g}}(x))$. By the Lie bracket rule, we have:

$$a_1d_1 = b_1c_1, a_1d_2 = b_2c_1, a_2d_1 = b_1c_2, a_2d_2 = b_2c_2. \quad (4.4)$$

Consider the matrix:

$$A := \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \\ b_1 & d_1 \\ b_2 & d_2 \end{pmatrix}$$

Denote by

$$v_1 := (a_1, c_1), v_2 := (a_2, c_2), v_3 := (b_1, d_1), v_4 := (b_2, d_2)$$

the row vectors and

$$w_1 := (a_1, a_2, b_2, b_2), w_2 := (c_1, c_2, d_1, d_2)$$

the column vectors. By assumption, x and y should be linearly independent. This is equivalent to say w_1 and w_2 are linearly independent. Hence the rank $\text{rk}(A) = 2$. We shall write $v \sim w$ to indicate v and w are linearly dependent. (4.4) implies that

$$v_1 \sim v_3, v_1 \sim v_4, v_2 \sim v_3, v_2 \sim v_4.$$

Hence, the only possibilities are the following:

$$v_1, v_2 \text{ linearly independent and } v_3 = v_4 = (0),$$

or

$$v_3, v_4 \text{ linearly independent and } v_1 = v_2 = (0).$$

Hence, we obtain that there are only two elementary subalgebra $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathbb{E}(2, \mathfrak{g})$, where $\mathfrak{e}_1 := kx_1 \oplus kx_2$ and $\mathfrak{e}_2 := ky_1 \oplus ky_2$. The variety is consisting of two isolated points, i.e.,

$$\mathbb{E}(2, \mathfrak{g}) = \{\mathfrak{e}_1, \mathfrak{e}_2\},$$

hence is disconnected. Let M be a $U_0(\mathfrak{g})$ -module of constant j -rank. Since $V(\mathfrak{g})$ is a vector space, the function \deg_M^j is constant (see Theorem 4.2.1).

Chapter 5

The graph $\mathbb{E}(2, \mathfrak{g})$

In this Chapter, we will endow the set $\mathbb{E}(2, \mathfrak{g})$ with a graph structure which plays an important role in the investigation of equal images property.

5.1 Graph structure

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We endow the $\mathbb{E}(2, \mathfrak{g})$ with the structure of a graph whose set of bonds is

$$\{(\mathfrak{e}, \mathfrak{e}') \in \mathbb{E}(2, \mathfrak{g}) \times \mathbb{E}(2, \mathfrak{g}); \dim_k(\mathfrak{e} \cap \mathfrak{e}') \geq 1\}.$$

If $\dim_k(\mathfrak{e} \cap \mathfrak{e}') \geq 1$, then we will denote by $\mathfrak{e} - \mathfrak{e}'$. We say that the set $\mathbb{E}(2, \mathfrak{g})$ is *graph connected* if for any $\mathfrak{e}, \mathfrak{e}' \in \mathbb{E}(2, \mathfrak{g})$, then we can find a *path*

$$\mathfrak{e} - \mathfrak{e}_1 - \cdots - \mathfrak{e}_n - \mathfrak{e}'.$$

A graph that is not connected is said to be *disconnected*.

Example 5.1.1. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. If $V(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g}) \neq (0)$, then the graph $\mathbb{E}(2, \mathfrak{g})$ is connected. We assume that $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$. Let $\mathfrak{e} = kx_1 \oplus kx_2$ and $\mathfrak{e}' = ky_1 \oplus ky_2$ be two elementary subalgebras. Since $V(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g}) \neq (0)$, we can find some non-zero element $z \in V(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g})$. We have the following path:

$$\mathfrak{e} - kx_1 \oplus kz - kz \oplus ky_1 - \mathfrak{e}'.$$

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We know

$$V_{\mathbb{E}(r, \mathfrak{g})} := \bigcup_{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{g})} \mathfrak{e}.$$

is a closed conical subvariety of $V(\mathfrak{g})$ (Lemma 2.3.2). We say a restricted Lie algebra \mathfrak{u} is *unipotent* if every element $x \in \mathfrak{u}$ is annihilated by some iterate of the p -map.

Lemma 5.1.1. *Let \mathfrak{u} be a unipotent restricted Lie algebra. Then the graph $\mathbb{E}(2, \mathfrak{u})$ is connected. Furthermore, if $\mathbb{E}(2, \mathfrak{u}) \neq \emptyset$, then $V(\mathfrak{u}) = V_{\mathbb{E}(2, \mathfrak{u})}$.*

Proof. We assume that $\mathbb{E}(2, \mathfrak{u}) \neq \emptyset$. Then there exists a non-zero element $z \in V(\mathfrak{z}(\mathfrak{u}))$. Now our assertion follows from Example 5.1.1. \square

Proposition 5.1.2. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and $\mathfrak{u} \subseteq \mathfrak{g}$ a non-zero unipotent subalgebra of \mathfrak{g} . Then the graph $\mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{u}))$ is connected. Furthermore, if $\mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{u})) \neq \emptyset$, then $V(\text{Nor}_{\mathfrak{g}}(\mathfrak{u})) = V_{\mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{u}))}$.*

Proof. Let $\mathfrak{e}_0 := kx \oplus ky \in \mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{u}))$. Thanks to Lemma 5.1.1, the subgraph $\mathbb{E}(2, \mathfrak{u})$ is connected. We assume that $\mathbb{E}(2, \mathfrak{u}) \neq \emptyset$. Moreover, we can assume that $x \notin \mathfrak{u}$. The operator $\text{ad } x$ acts on the space \mathfrak{u} via nilpotent transformation. Hence, there exists an element $z \in V(\mathfrak{u})$. It follows that $x \in \mathfrak{e} := kx \oplus kz \subseteq V_{\mathbb{E}(2, \text{Nor}_{\mathfrak{g}}(\mathfrak{u}))}$. We have a path:

$$\mathfrak{e}_0 - \mathfrak{e}.$$

Note that $\dim_k(\mathfrak{e} \cap \mathfrak{u}) \geq 1$, our assertion follows from Lemma 5.1.1. Suppose that $\mathbb{E}(2, \mathfrak{u}) = \emptyset$. Given two elementary subalgebras $\mathfrak{e}_i := kx_i \oplus ky_i; i = 1, 2$. We have the following path:

$$\mathfrak{e}_1 - kx_1 \oplus kz - kz \oplus kx_2 - \mathfrak{e}_2,$$

where $z \in \mathfrak{u}$ is the central p -nilpotent element. \square

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We denote by $T(\mathfrak{g})$ the *toral radical* of \mathfrak{g} , that is, the largest toral ideal of \mathfrak{g} . It is easy to check that $T(\mathfrak{g})$ is contained in the center $\mathfrak{z}(\mathfrak{g})$ of \mathfrak{g} . We denote by $\mu(\mathfrak{g})$ the maximal dimension of all tori $\mathfrak{t} \subseteq \mathfrak{g}$.

In Lemma 5.1.1, we consider the special nilpotent Lie algebra, the unipotent Lie algebra. It should be notice that a nilpotent Lie algebra \mathfrak{g} is unipotent if and only if $T(\mathfrak{g}) = (0)$. Now we are going to extend our results to more general nilpotent Lie algebra.

Lemma 5.1.3. [12, Lemma 3.1]. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and $T(\mathfrak{g})$ the toral radical of \mathfrak{g} . For any subtorus $\mathfrak{t} \subseteq T(\mathfrak{g})$. The canonical projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{t}$ induces a bijective morphism $V(\mathfrak{g}) \rightarrow V(\mathfrak{g}/\mathfrak{t})$ of affine varieties.

Lemma 5.1.4. Let \mathfrak{g} be a nilpotent restricted Lie algebra and $T(\mathfrak{g})$ the toral radical of \mathfrak{g} . Let $\mathfrak{t} \subseteq T(\mathfrak{g})$ be a subtorus of $T(\mathfrak{g})$. Then $T(\mathfrak{g}/\mathfrak{t}) = T(\mathfrak{g})/\mathfrak{t}$.

Proof. Since \mathfrak{g} is nilpotent, $T(\mathfrak{g})$ is the unique maximal torus of \mathfrak{g} . According to [13, Lemma 3.3],

$$\mu(\mathfrak{g}/\mathfrak{t}) = \mu(\mathfrak{g}) - \mu(\mathfrak{t}) = \dim_k T(\mathfrak{g}) - \dim_k \mathfrak{t} = \dim_k(T(\mathfrak{g})/\mathfrak{t}).$$

Since $\mathfrak{g}/\mathfrak{t}$ is also a nilpotent Lie algebra, it follows that $T(\mathfrak{g})/\mathfrak{t}$ is the unique maximal torus of $\mathfrak{g}/\mathfrak{t}$ and equals to $T(\mathfrak{g}/\mathfrak{t})$. \square

Proposition 5.1.5. Let $(\mathfrak{g}, [p])$ be a nilpotent restricted Lie algebra. and $r := \dim_k(T(\mathfrak{g}))$. Suppose that $\mathfrak{e} \cap V_{\mathbb{E}(r+1, \mathfrak{g})} \neq \{0\}$ for every $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$. Then the graph $\mathbb{E}(2, \mathfrak{g})$ is connected.

Proof. If $r = 0$, then \mathfrak{g} is unipotent. Our assertion follows from Lemma 5.1.1.

We assume that $r \geq 1$. We can furthermore assume that $\mathfrak{z}(\mathfrak{g}) = T(\mathfrak{g})$. Otherwise $\mathfrak{z}(\mathfrak{g}) \cap V(\mathfrak{g}) \neq \{0\}$, then Example 5.1.1 implies that the graph is connected. Since $\dim_k T(\mathfrak{g}) = r$, we can find some toral elements $\{t_1, \dots, t_r\} \subseteq T(\mathfrak{g})$ such that $T(\mathfrak{g}) = \langle t_1, \dots, t_r \rangle$ (see Section 1.2). We assume that $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$. We denote by

$$\pi : \mathfrak{g} \rightarrow \mathfrak{g}/T(\mathfrak{g})$$

the canonical projection. According to Lemma 5.1.4, the Lie algebra $\mathfrak{g}/T(\mathfrak{g})$ is unipotent, there exists $0 \neq z \in V(\mathfrak{z}(\mathfrak{g}/T(\mathfrak{g})))$. By virtue of Lemma 5.1.3, by abuse of notation, we still denote by $z := \pi^{-1}(z)$. Given vertices

$$\mathfrak{e}_1 := kx_1 \oplus ky_1, \quad \mathfrak{e}_2 := kx_2 \oplus ky_2 \in \mathbb{E}(2, \mathfrak{g}).$$

By assumption, we can find $z_i \in \mathfrak{e}_i$ and $\mathfrak{e}'_i \in \mathbb{E}(r+1, \mathfrak{g})$ such that $z_i \in \mathfrak{e}'_i$ for $i = 1, 2$. Since $\dim_k T(\mathfrak{g}) = r$ and $z \in V(\mathfrak{z}(\mathfrak{g}/T(\mathfrak{g})))$, so that we can find $c_i \in \mathfrak{e}'_i$ such that $[z, c_i] = 0$. We have the following path:

$$\mathfrak{e}_1 - kz_1 \oplus kc_1 - kz \oplus kc_1 - kz \oplus kc_2 - kc_2 \oplus kz_2 - \mathfrak{e}_2.$$

Consequently, the graph $\mathbb{E}(2, \mathfrak{g})$ is connected. \square

Example 5.1.2. Suppose that $p \geq 3$. We construct a 6-dimensional nilpotent restricted Lie algebra $\mathfrak{g} := kx_1 \oplus kx_2 \oplus ky_1 \oplus ky_2 \oplus kt_1 \oplus kt_2$, whose Lie bracket and p -map are given by

$$[x_1, x_2] = [y_1, y_2] = 0, [x_1, y_1] = [x_1, y_2] = t_1, [x_2, y_2] = [x_2, y_1] = t_2,$$

$$x_i^{[p]} = y_i^{[p]} = 0, t_i^{[p]} = t_i, i = 1, 2,$$

where the toral radical $T(\mathfrak{g}) = kt_1 \oplus kt_2$. It is easy to check that the nullcone $V(\mathfrak{g}) = kx_1 \oplus kx_2 \oplus ky_1 \oplus ky_2$ (see Example 4.2.2). Let $\mathfrak{e} := kx_1 \oplus kx_2$ and $\mathfrak{e}' := ky_1 \oplus ky_2$. We have a path:

$$\mathfrak{e} - k(x_1 + x_2) \oplus k(y_1 - y_2) - \mathfrak{e}'.$$

It should be notice that $kx_1 \oplus kx_2 \oplus k(y_1 - y_2) \in \mathbb{E}(3, \mathfrak{g})$.

Let \mathfrak{g} be the nilpotent restricted Lie algebra in Example 4.2.2, we know that $\dim_k(T(\mathfrak{g})) = 4$ and $V(\mathfrak{g})$ is a 4-dimensional vector space. The variety is not connected. The dimension of toral radial is too large, the Lie algebra will be very complicated.

5.2 General case

In this section, we would like to give some criteria for detecting the graph connect-
edness.

Let $(\mathfrak{g}, [p])$ be an abelian restricted Lie algebra. According to the Jordan-Chevalley-Seligman decomposition (Theorem 1.2.1), we have $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_n$, where $\mathfrak{g}_s = T(\mathfrak{g})$ is the set of semisimple elements of \mathfrak{g} and \mathfrak{g}_n is the set of nilpotent elements of \mathfrak{g} .

Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with strong degeneration. Recall that the following closed subset (see Section 3.2)

$$\mathbb{E}(2, \mathfrak{g})^* = \{\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g}); \mathfrak{e} \cap \text{Sw}(\mathfrak{g}) \neq \{0\}\}.$$

Lemma 5.2.1. *Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with strong degeneration. Then the subgraph $\mathbb{E}(2, \mathfrak{g})^*$ is connected.*

Proof. Assume that $\mathbb{E}(2, \mathfrak{g})^* \neq \emptyset$. Thanks to Lemma 3.3.3, there exists a special sandwich element $c_0 \in \text{Sw}(\mathfrak{g})$ such that $[c_0, c] = 0$ for any $c \in \text{Sw}(\mathfrak{g})$. On the other hand, Corollary 2.2.4 implies that we can find some element $x_0 \in V(\mathfrak{g})$ such that

$\mathfrak{e}_0 := kc_0 \oplus kx_0 \in \mathbb{E}(2, \mathfrak{g})^*$. Given an elementary subalgebra $\mathfrak{e} := kc \oplus kx \in \mathbb{E}(2, \mathfrak{g})^*$. If $kc_0 = kc$, then there is a path:

$$\mathfrak{e} - \mathfrak{e}_0.$$

If c_0 and c are linearly independent, then we have:

$$\mathfrak{e} - kc \oplus kc_0 - \mathfrak{e}_0.$$

In sum, we always can find a path between \mathfrak{e} and \mathfrak{e}_0 for every $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})^*$. Thus, the subgraph $\mathbb{E}(2, \mathfrak{g})^*$ is connected. \square

Theorem 5.2.2. *Let $(\mathfrak{g}, [p])$ be a centerless restricted Lie algebra with strong degeneration. Suppose that for any elementary subalgebra $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$, there exists a non-zero element $z \in \mathfrak{e}$ such that $\mathfrak{z}_{\mathfrak{g}}(z) \cap \text{Sw}(\mathfrak{g}) \neq \{0\}$. Then the graph $\mathbb{E}(2, \mathfrak{g})$ is connected.*

Proof. Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, \mathfrak{g})$. By assumption, there exists some element $z \in \mathfrak{e}$ such that $[z, c] = 0$ for some non-zero element $c \in \text{Sw}(\mathfrak{g})$. Since \mathfrak{g} is centerless, c is p -nilpotent. Hence there is a path:

$$\mathfrak{e} - kz \oplus kc.$$

Note that $kz \oplus kc \in \mathbb{E}(2, \mathfrak{g})^*$, now our assertion directly follows from Lemma 5.2.1. \square

In Section 3.4, we introduced the Jacobson-Witt algebra $W(n)$. In the following we will investigate the graph structure of $\mathbb{E}(2, W(n))$, we will assume that $p \geq 5$. Let $G := \text{Aut}_p(W(n))$ be the automorphism group of the restricted Lie algebra $W(n)$. It is well-known that any automorphism of $W(n)$ is induced by an automorphism of B_n (cf.[8]). We have:

$$\text{Aut}(B_n) \cong \text{Aut}_p(W(n)).$$

Let ϕ be an automorphism of B_n . This results an automorphism $\tilde{\phi}$ of $W(n)$, given by

$$\tilde{\phi} = \phi \circ D \circ \phi^{-1}, \quad \forall D \in W(n).$$

According to [33, Theorem 2], G is a connected algebraic group. Furthermore, let \mathfrak{m} be the unique maximal ideal of B_n . The group G is of dimension $np^n - n$ and

$$\text{Lie}(G) = \left\{ \sum f_i \partial_i; f_i \in \mathfrak{m} \right\}. \quad (5.1)$$

Recall that the Lie algebra $W(n)$ is a restricted \mathbb{Z} -graded Lie algebra. We denote by $W(n)_{(i)} = \sum_{j \geq i} W(n)_j$. Hence, $\text{Lie}(G) = W(n)_{(0)}$. Also, it should be noted that G preserves the $W(n)_{(i)}$ for every i (see [33]).

Lemma 5.2.3. *Let $x \in V(W(n))$ and $x \notin W(n)_{(0)}$. Then x is conjugate to ∂_1 .*

Proof. This is a direct consequence of [8, Page 235]. \square

Suppose that $n = 1$. According to Example 3.2.2, $\mathbb{E}(2, W(1)) = \mathbb{E}(2, W(1)_{(1)})$ and $W(1)_{(1)}$ is a p -unipotent Lie algebra, so that the graph $\mathbb{E}(2, W(1))$ is connected (see Lemma 5.1.1).

Recall that $W(n) = \bigoplus_{j=-1}^{|\tau|-1} W(n)_j$.

Lemma 5.2.4. *Suppose that $n \geq 2$. Then $V(W(n)) = V_{\mathbb{E}(2, W(n))}$.*

Proof. Let $x \in V(W(n))$. Assume that $x \notin W(n)_{(0)}$. Thanks to Lemma 5.2.3, x is conjugate to ∂_1 . Since $n \geq 2$, note that $[\partial_1, \partial_2] = 0$, so that $x \in V_{\mathbb{E}(2, W(n))}$. Let $c := x_1^{p-1} \cdots x_n^{p-1} \partial_n$. Then c lies in the highest \mathbb{Z} -graded part and $c \in \text{Sw}(W(n))$ is a sandwich element. Note that $c \in W(n)_{|\tau|-1}$. If $x \in W(n)_{(1)}$, then

$$[x, c] \in W(n)_{1+|\tau|-1} = W(n)_{|\tau|} = (0).$$

Hence, $x \in kx \oplus kc$ and $x \in V_{\mathbb{E}(2, W(n))}$. We consider the last case. Suppose that $x \in V(W(n)_{(0)} \setminus W(n)_{(1)})$. Since $W(n)_{(0)}$ is an algebraic Lie algebra (see (5.1)), x is conjugate to $x' = x_0 + x_1$, where $x_0 = \sum_{j>i} a_{ij} x_i \partial_j$ and $x_1 \in W(n)_{(1)}$. By a direct computation, we get $[c, x_0] = 0$. On the other hand, it is clear that $[c, x_1] = 0$. Hence $x' \in kx' \oplus kc$, so that $x \in V_{\mathbb{E}(2, W(n))}$. \square

Proposition 5.2.5. *Let $W(n)$ be the n -th Jacobson-Witt algebra. Then the graph $\mathbb{E}(2, W(n))$ is connected.*

Proof. The case $n = 1$ we have explained in the above, so that we assume that $n \geq 2$. Let $\mathfrak{e} := kx \oplus ky \in \mathbb{E}(2, W(n))$ be an elementary subalgebra.

\diamond Suppose that there exists some element $z \in \mathfrak{e}$ such that $z \in W(n) \setminus W(n)_{(0)}$. Thanks to Lemma 5.2.3, z is conjugate to ∂_1 , i.e., there is some $g \in G$ such that $g.z = \partial_1$. We denote by $c' := x_2^{p-1} \cdots x_n^{p-1} \partial_1$. It is easy to check that $c' \in V(W(n))$ and $[\partial_1, c'] = 0$. It is obvious that $c' \in W(n)_{(1)}$. Since G preserves the filtration, it

follows that $c'' := g^{-1}.c' \in W(n)_{(1)}$. Let $c := x_1^{p-1} \cdots x_n^{p-1} \partial_n$. Since $[c, c''] = 0$, we have a path:

$$\mathfrak{e} - kz \oplus kc'' - kc'' \oplus kc.$$

◇ Suppose that $\mathfrak{e} \subseteq W(n)_{(0)}$ and there exists some element $z \in \mathfrak{e}$ such that $z \in V(W(n)_{(0)} \setminus W(n)_{(1)})$. We still denote by $c := x_1^{p-1} \cdots x_n^{p-1} \partial_n$. According to the proof of Lemma 5.2.4, there exists some $g \in G$ such that $[g.z, c] = 0$. Since G preserves the filtration, it follows $c' := g^{-1}.c$ also lies in the highest \mathbb{Z} -grading part, so that $[c', c] = 0$. Hence, there results a path:

$$\mathfrak{e} - kz \oplus kc' - kc' \oplus kc.$$

◇ Suppose that $\mathfrak{e} \subseteq W(n)_{(1)}$. By the same argument as before, we immediately obtain a path:

$$\mathfrak{e} - kx \oplus kc.$$

In sum, we can always find some elementary subalgebra \mathfrak{e}' with $c \in \mathfrak{e}'$ such that there is a path from \mathfrak{e} to \mathfrak{e}' . Hence, the graph $\mathbb{E}(2, \mathfrak{g})$ is connected. \square

Remark 5.2.1. If we denote by $\mathfrak{g} := W(n)_0$, then the connectedness of graph $\mathbb{E}(2, \mathfrak{g})$ is a direct consequence of Theorem 5.2.2 (see the proof of Proposition 5.2.5).

In the last, we present an example which shows that the graph $\mathbb{E}(2, \mathfrak{g})$ is not connected even when the topological space $\mathbb{E}(2, \mathfrak{g})$ is irreducible (connected).

Example 5.2.2. Let $p \geq 3$. We consider the vector space

$$\mathfrak{g} := \mathfrak{e} \oplus \mathfrak{f} \oplus T(\mathfrak{g}),$$

where

$$\mathfrak{e} := kx_1 \oplus kx_2; \quad \mathfrak{f} := ky_1 \oplus ky_2; \quad T(\mathfrak{g}) := kz_1 \oplus kz_2 \oplus kz_3.$$

The Lie bracket and p -map are defined via:

$$[T(\mathfrak{g}), \mathfrak{g}] = (0) = [\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}]; \quad [x_1, y_1] = z_1; \quad [x_1, y_2] = z_2 = [x_2, y_1]; \quad [x_2, y_2] = z_3$$

and

$$\mathfrak{e}^{[p]} = \{0\} = \mathfrak{f}^{[p]}; \quad z_i^{[p]} = z_i,$$

respectively.

Since $p \geq 3$ and $[\mathfrak{g}, \mathfrak{g}] = T(\mathfrak{g})$, it follows that $V(\mathfrak{g})$ is a subspace of \mathfrak{g} containing $\mathfrak{e} \oplus \mathfrak{f}$. Since $T(\mathfrak{g})$ is the toral radical, it follows that $V(\mathfrak{g}) = \mathfrak{e} \oplus \mathfrak{f}$. Given $\mathfrak{a} \in \mathbb{E}(2, \mathfrak{g})$, we let $\mathcal{C}(\mathfrak{a})$ be the connected component of the graph $\mathbb{E}(2, \mathfrak{g})$ that contains \mathfrak{a} .

We prove the following assertions:

1. We have $\mathcal{C}(\mathfrak{a}) = \mathfrak{a}$ for all $\mathfrak{a} \in \mathbb{E}(2, \mathfrak{g})$.
2. The graph $\mathbb{E}(2, \mathfrak{g})$ has infinitely many connected components.
3. The variety $\mathbb{E}(2, \mathfrak{g})$ is irreducible.

Proof. (1) We consider the map

$$\mu : \mathfrak{e} \otimes_k \mathfrak{f} \rightarrow T(\mathfrak{g}); \quad a \otimes b \mapsto [a, b].$$

Then $\ker \mu = k(x_1 \otimes y_2 - x_2 \otimes y_1)$ does not contain a non-zero simple tensor. Let $x \in V(\mathfrak{g}) \setminus \{0\}$. Writing $x = x_{\mathfrak{e}} + x_{\mathfrak{f}}$, with $x_{\mathfrak{e}} \in \mathfrak{e}$ and $x_{\mathfrak{f}} \in \mathfrak{f}$, we obtain

$$[x, V(\mathfrak{g})] = [x_{\mathfrak{e}}, \mathfrak{f}] + [x_{\mathfrak{f}}, \mathfrak{e}].$$

By the observation above, the maps $\text{ad } x_{\mathfrak{e}} : \mathfrak{f} \rightarrow T(\mathfrak{g})$ and $\text{ad } x_{\mathfrak{f}} : \mathfrak{e} \rightarrow T(\mathfrak{g})$ are injective whenever $x_{\mathfrak{e}} \neq 0 \neq x_{\mathfrak{f}}$, so that $\dim_k(\text{ad } x)(V(\mathfrak{g})) \geq 2$. Consequently,

$$\dim_k \mathfrak{z}_{V(\mathfrak{g})}(x) = 4 - \dim_k \text{ad } x(V(\mathfrak{g})) \leq 2.$$

Now let $\mathfrak{a} \in \mathbb{E}(2, \mathfrak{g})$. If $\mathcal{C}(\mathfrak{a}) \setminus \{\mathfrak{a}\} \neq \emptyset$, then there is $\mathfrak{a}' \in \mathcal{C}(\mathfrak{a}) \setminus \{\mathfrak{a}\}$ such that $\mathfrak{a} \cap \mathfrak{a}' \neq (0)$. Accordingly, there is $z \in V(\mathfrak{g}) \setminus \{0\}$ such that $\mathfrak{a} + \mathfrak{a}' \subseteq \mathfrak{z}_{V(\mathfrak{g})}(z)$. Since the latter space is at most two-dimensional, we conclude that $\mathfrak{a} = \mathfrak{a}'$, a contradiction.

(2) For $(\alpha : \beta) \in \mathbb{P}^1$, we consider $\mathfrak{a}_{(\alpha:\beta)} := k(\alpha x_1 + \beta y_1) \oplus k(\alpha x_2 + \beta y_2)$. Since

$$[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2] = \alpha\beta z_2 - \alpha\beta z_2 = 0,$$

we have $\mathfrak{a}_{(\alpha:\beta)} \in \mathbb{E}(2, \mathfrak{g})$.

Suppose that $\mathfrak{a}_{(\alpha:\beta)} = \mathfrak{a}_{(\gamma:\delta)}$. Then we have

$$0 = [\alpha x_1 + \beta y_1, \gamma x_2 + \delta y_2] = (\alpha\delta - \beta\gamma)z_2,$$

so that $\alpha\delta = \beta\gamma$. It follows that $\mathfrak{a}_{(1:\beta)} \neq \mathfrak{a}_{(1:\delta)}$, whenever $\beta \neq \delta$.

(3) First, we define the following subvariety of $\text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})}) \times \mathbb{E}(2, \mathfrak{g})$:

$$X := \{([x], \mathfrak{e}) \in \text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})}) \times \mathbb{E}(2, \mathfrak{g}); x \in \mathfrak{e}\}.$$

Consider the composition map:

$$\text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})}) \hookrightarrow X \xrightarrow{\text{Pr}} \mathbb{E}(2, \mathfrak{g}),$$

where Pr is the canonical projection map. Let $x \in V_{\mathbb{E}(2, \mathfrak{g})}$ be a non-zero element. In view of (1), there exists a unique elementary subalgebra $\mathfrak{e} = \mathfrak{z}_{V(\mathfrak{g})}(x)$ which contains x . This results a surjective morphism:

$$\mathfrak{z}_{V(\mathfrak{g})} : \text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})}) \rightarrow \mathbb{E}(2, \mathfrak{g}); [z] \mapsto \mathfrak{z}_{V(\mathfrak{g})}(z).$$

Since the map is a morphism, it follows that $\mathbb{E}(2, \mathfrak{g})$ is irreducible if $\text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})})$ enjoys this property.

Let $z := \alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2$ be an element of $V(\mathfrak{g})$. Then we have

$$[z, x_1] = -\gamma z_1 - \delta z_2; [z, x_2] = -\gamma z_2 - \delta z_3; [z, y_1] = \alpha z_1 + \beta z_2; [z, y_2] = \alpha z_2 + \beta z_3.$$

Hence $\text{ad } z(V(\mathfrak{g}))$ coincides with the image of the linear map $f : k^4 \rightarrow T(\mathfrak{g})$, which is given with respect to the canonical basis of k^4 and $\{z_1, z_2, z_3\}$ by the matrix

$$A(z) := \begin{pmatrix} -\gamma & 0 & \alpha & 0 \\ -\delta & -\gamma & \beta & \alpha \\ 0 & -\delta & 0 & \beta \end{pmatrix}.$$

Let $A_j(z)$ be the (3×3) -submatrix obtained by deleting the j -th column. Setting $a := \beta\gamma - \alpha\delta$, we obtain

$$\det(A_1(z)) = -\alpha a; \det(A_2(z)) = -\beta a; \det(A_3(z)) = \gamma a; \det(A_4(z)) = -\delta a.$$

Hence we obtain $[z] \in \text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})})$ if and only if $\dim_k \text{ad } z(V(\mathfrak{g})) = 2$ if and only if $\det(A_j(z)) = 0$ for $1 \leq j \leq 4$. As a result,

$$\text{Proj}(V_{\mathbb{E}(2, \mathfrak{g})}) \cong \{(\alpha : \beta : \gamma : \delta) \in \mathbb{P}^3; \beta\gamma - \alpha\delta = 0\}.$$

Since the determinant is an irreducible polynomial, the latter is irreducible. \square

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht.

Kiel, den 28.06.2016

(Hao Chang)

