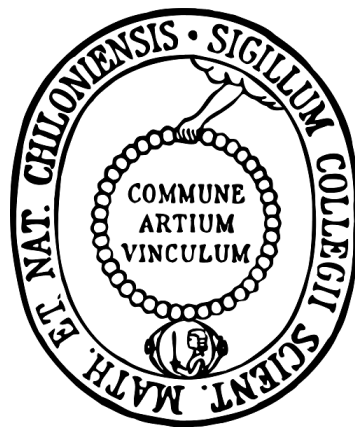


Optimal investment and utility indifference pricing in the presence of small fixed transaction costs

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Zusammenfassung

Diese Arbeit beschäftigt sich mit der Erwartungsnutzenmaximierung des Endvermögens unter festen Transaktionskosten.

Wir betrachten dabei einen Investor mit konstanter absoluter Risikoaversion, der in einem Markt, bestehend aus einer risikolosen Anleihe und einer riskanten Aktie, handelt. Die Aktie wird dabei als allgemeiner Itô-Prozess modelliert. Wir nehmen an, dass der Investor für jede Transaktion einen festen Betrag ε unabhängig vom Transaktionsvolumen zahlen muss.

Mit einem nicht-markowschen Prinzip der dynamischen Programmierung bestimmen wir eine Kandidaten-Strategie. Diese Strategie ist dadurch gekennzeichnet, dass sie stets in einem zufälligen und zeitabhängigen Intervall um den friktionslosen Optimierer liegt, beim Erreichen der Intervallgrenzen auf diesen Optimierer springt und alle Aktiendpositionen liquidiert, sobald das Vermögen eine vorgegebene (stochastische) Schwelle unterschreitet. Die asymptotische Optimalität dieser Kandidatenstrategie zeigen wir unter geeigneten Regularitätsbedingungen an die Dynamiken der Aktie und des friktionslosen Optimierers. Zwei Beispiele zeigen, dass die Regularitätsbedingungen in konkreten Modellen erfüllt werden. Außerdem wenden wir unsere Ergebnisse auf die Nutzenindifferenzbewertung bei festen Transaktionskosten an. Durch eine Abschwächung der Regularitätsannahmen des Optimierungsproblems können wir den Nutzenindifferenzpreis einer Put-Option unter festen Kosten im Black-Scholes-Modell asymptotisch berechnen.

Unsere Resultate beweisen die Heuristik von (Korn, 1998, Abschnitt 5) in der Abwesenheit von proportionalen Kosten, aber für allgemeine Itô-Prozesse. Im Gegensatz zu Altarovici et al. (2015a) in einem etwas anderen Ausgangsproblem basieren unsere Ergebnisse auf Martingalmethoden, (Semi-)Martingalungleichungen und Hilfsmitteln aus der Stochastischen Analysis wie zum Beispiel aus Peskir (2007) statt Homogenisierung und Viskositätslösungen.

Abstract

This thesis deals with utility maximization from terminal wealth under fixed transaction costs.

We consider an investor with constant absolute risk aversion trading in a market consisting of one safe and one risky asset with general Itô dynamics. We assume that she has to pay a fixed transaction cost ε for each trade regardless of its size.

Using a non-Markovian dynamic programming approach, we derive candidate strategy for optimality. This strategy lies in a random and time-dependent interval around the frictionless optimizer, changes to the latter once the boundaries of this interval are breached and liquidates all stock positions if the corresponding wealth falls below a given (stochastic) threshold. We verify the (almost) optimality of the candidate under suitable regularity assumptions. Furthermore, we give two examples of models fulfilling these assumptions and present an application to utility indifference pricing. After weakening the regularity assumptions, we derive a pricing formula for the European put option in the Black-Scholes model under fixed transaction costs.

Our results verify the heuristics of (Korn, 1998, Section 5) in the absence of proportional costs, but for general Itô dynamics. Contrary to the related study of Altarovici et al. (2015a) in a different setup, our derivation and verification rely on martingale methods and tools from stochastic calculus like the change-of-variable formula from Peskir (2007) rather than homogenization and viscosity solutions.

Contents

1. Introduction	3
1.1. Optimization with frictions in Mathematical Finance	3
1.2. Outline of this thesis	5
1.3. Notation	5
2. Inputs from the frictionless problem	7
2.1. Introduction	7
2.2. Trading in a frictionless market	7
2.3. Optimal expected utility from terminal wealth	8
2.4. Different methods to solve the optimization problems	9
2.4.1. Martingale approach	9
2.4.2. Dynamic programming	11
2.4.3. Connection between both approaches	15
3. Fixed transaction costs	17
3.1. Introduction	17
3.2. Trading with fixed transaction costs	17
3.3. Optimality and dynamic programming	20
3.4. Main result and interpretation	21
3.5. Heuristics	22
3.5.1. Ansatz for value process	22
3.5.2. Dynamics	25
3.5.3. Identifying of the correct scaling	27
3.5.4. Derivation of the optimal strategy	28
3.5.5. Modification of the value process	30
3.6. Existence of involved processes and regularity assumptions	31
3.7. Upper bound for maximal expected utility	33
3.7.1. On a subclass of admissible strategies	33
3.7.2. On the larger set of admissible strategies	38
3.8. Lower bound for maximal expected utility and almost optimality of the candidate	47
3.8.1. Properties of the candidate and drift of its value process	47
3.8.2. Lower bound	50
3.8.3. Optimality	54
4. Extensions	57
4.1. Introduction	57

Contents

4.2. Truncated portfolio gamma and fewer regularity assumptions	57
4.2.1. Truncated no-trade corridor	57
4.2.2. Upper bound	60
4.2.3. Lower bound	68
4.2.4. Optimality	71
4.3. Truncated and frozen portfolio gamma with growth conditions	71
4.3.1. Truncated and frozen no-trade corridor	71
4.3.2. Upper bound	75
4.3.3. Lower bound	80
4.3.4. Optimality	82
5. Examples	83
5.1. Black-Scholes model	83
5.2. Stochastic volatility model	89
6. Application: Utility indifference pricing	91
6.1. General framework	91
6.2. Black-Scholes model	92
6.2.1. For regular option payoffs	92
6.2.2. For the put option	94
7. Fixed and proportional costs	103
7.1. Trading with fixed and proportional costs	103
7.2. Heuristics	103
7.3. An outlook to the verification	108
8. Conclusion and future research	111
A. Analyzing error terms	113
A.1. For Chapter 3	113
A.2. For Section 4.2	122
B. Tools from stochastic calculus	133
B.1. Change-of-variable formulas	133
B.1.1. For piecewise twice continuously differentiable functions with smooth fit	133
B.1.2. For linear combinations of convex functions	135
B.2. (Semi-)Martingale Inequalities	136

1. Introduction

1.1. Optimization with frictions in Mathematical Finance

According to an old German stock market wisdom¹, frictions in financial markets make high frequent investment strategies gainless. These frictions can have several forms. For example, they can be taxes and the gains from trading, losses due to bid-ask spreads in the stock market, brokerage fees for the reallocation of a portfolio and other small imperfections.

Since research on frictionless optimization² suggests high frequent optimal strategies, this business wisdom is probably one of the reasons why utility maximization (and pricing) under transaction costs has become quite popular in Mathematical Finance. In the literature regarding this topic there are basically three main types of cost structures: proportional, quasi fixed and fixed transaction costs.

The by far largest group of these three is the one on proportional costs. Optimally behaving investors do not trade while their holdings lie inside an interval around the frictionless optimizer and use minimal effort to stay inside this so called no-trade region once their holdings reach the boundaries (Magill and Constantinides (1976); Constantinides (1986); Davis and Norman (1990); Dumas and Luciano (1991); Shreve and Soner (1994)). Even in the Black-Scholes model, the size of the no-trade region and the corresponding utility loss due to proportional costs have to be determined numerically (Davis and Norman (1990)) or asymptotically at the leading order for small costs (Shreve and Soner (1994); Janeček and Shreve (2004)). Thus, analyzing the asymptotic influence of proportional costs (in more complicated models) seems reasonable: Formal results were obtained by Goodman and Ostrov (2010); Martin (2012); Kallsen and Muhle-Karbe (2013); Soner and Touzi (2013); Kallsen and Muhle-Karbe (2015). Rigorous proofs can be found in Soner and Touzi (2013) for general utility and Markovian asset dynamics, in Possamaï et al. (2015) for higher-dimensional asset price processes and in Kallsen and Li (2013) for general Itô dynamics. With an asymptotic dynamic programming approach Melnyk and Seifried (2016) construct almost optimal strategies under proportional costs, logarithmic utility and Markovian asset price dynamics.

When introducing fixed or quasi fixed costs, stochastic control problems become impulse control problems: in contrast to proportional costs, fixed and quasi fixed costs only allow for trading strategies changing finitely often on bounded time intervals. Nevertheless, the (almost) optimal strategies lie in a no-trade region around the frictionless target. In models with quasi fixed costs as in Morton and Pliska (1995) and

¹”Hin und her macht Taschen leer”, Schwarzer (18th April, 2012)

²The literature on frictionless utility maximization is discussed in Chapter 2.

1. Introduction

Atkinson and Wilmott (1995) the investor has to pay transaction costs proportional to current wealth, whereas in models with fixed costs a fixed amount $\varepsilon > 0$ has to be paid for each transaction that is done. The first study considering fixed transaction costs is Eastham and Hastings (1988). In a model with Markovian asset dynamics, finite time horizon, both fixed and proportional transaction costs they show that, if the so called quasi-variational inequalities have a smooth solution³, this solution is the value function and the related control is optimal. Similar results can be found in parts of Korn (1998) with weaker regularity assumptions for the value function and infinite time horizon³ and in Øksendal and Sulem (2002), where consumption is not only allowed at the transaction times of a trading strategy. Furthermore, the last study shows that the value function is a possibly discontinuous viscosity solution of the dynamic programming equation. Schroder (1995) and Liu (2004) numerically investigate the optimal trading strategies. Asymptotic results without rigorous proofs can be found in Korn (1998); Lo et al. (2004); Alcala and Fahim (2013). The most recent studies on fixed costs are Altarovici et al. (2015a), Altarovici et al. (2015b) and Belak and Christensen (2016). In a multidimensional Black-Scholes model with constant relative risk aversion and infinite time horizon Altarovici et al. (2015a) determine the asymptotic expansion of the value function and derive the almost optimal strategy in this setup. This is done by means of homogenization and viscosity solutions being the first result with rigorous verifications for the asymptotics of fixed costs. However, their optimal strategy can only be determined explicitly up to some stopping time that depends on wealth. With the same asset price dynamics, general utility and a combination of fixed and proportional costs Altarovici et al. (2015b) prove a weak dynamic programming principle and a comparison result for solutions of the dynamic programming equation. Furthermore, numerical results are used for illustrative purposes. Belak and Christensen (2016) show that the value function is a pointwise infimum of suitable superharmonic functions and a continuous viscosity solution of the corresponding quasi-variational inequalities. With these characterization they construct optimal strategies. This is done in a multidimensional factor model with both fixed and proportional costs. Cai et al. (2015, 2016) consider quadratic tracking problems with general cost structures and a target with Itô-dynamics. In (Cai et al., 2015, section 5) the authors show a relation between optimal tracking and utility maximization under transaction costs.

Our work is the second study with rigorous verifications for the asymptotics of fixed costs and the first one considering a finite time horizon and constant absolute risk aversion. Our results verify the heuristics of (Korn, 1998, Section 5c) in the absence of proportional costs. In contrast to most of the other studies on fixed costs we do not require Markovian asset price dynamics; thus, rather complicated analytical methods like viscosity solutions and homogenization techniques can be avoided. The (almost) optimal trading strategy can be constructed under suitable regularity assumptions on the underlying model, and the frictionless optimizer is known on the whole time horizon. As in the literature it is kept inside some (stochastic) no-trade region around the frictionless target, but just sells all stocks if wealth falls under a given thresh-

³According to Korn (1998) this smoothness assumption can only be fulfilled if the optimal strategy is constant, i.e. if it never hits the boundaries of the not trade region.

old depending on the transaction cost ε . Furthermore, the no-trade region and the corresponding asymptotic utility or certainty equivalent loss can easily be described in terms of the frictionless quantities, the degree of risk aversion and - of course - the transaction cost. As in other studies (Janeček and Shreve (2004); Martin (2012); Kallsen and Muhle-Karbe (2013); Kallsen and Li (2013); Ahrens (2015); Kallsen and Muhle-Karbe (2015)), one important factor is the so called portfolio gamma of the frictionless optimizer.

Similar to Ahrens (2015) dealing with proportional costs, our results can directly be applied to utility indifference pricing under fixed transaction costs in the Black-Scholes model. After weakening the assumed regularity assumptions this can also be applied to put options.

1.2. Outline of this thesis

This thesis is organized as follows: In Chapter 2 we summarize the necessary inputs from the frictionless expected exponential utility maximization problem from terminal wealth: we require the two main approaches to solve the optimization problems in Mathematical Finance: martingale methods and dynamic programming. Furthermore, we derive a connection between both approaches.

Chapter 3 is the main part of this thesis: the derivation of a trading strategy that almost maximizes the expected exponential utility from terminal wealth under fixed transaction costs. We heuristically derive a solution and then state rigorous verification theorems under suitable regularity assumptions on the frictionless optimization problem.

In Chapter 4 we extend the results from Chapter 3: we present two verifications with alternative regularity assumptions.

Chapter 5 analyzes two tractable examples for stock price models and their resulting almost optimal strategies: the Black-Scholes model and a stochastic volatility model. Chapter 6 deals with utility indifference pricing under fixed transaction costs. We derive utility indifference prices in the Black-Scholes model for option prices being regular enough and use the extended verifications from Chapter 4 to obtain a utility indifference price for a European put in the Black-Scholes model in the presence of fixed transaction costs.

Furthermore, in Chapter 7, we consider a combination of both fixed and proportional costs and heuristically derive the strategy that we believe is almost optimal.

1.3. Notation

Most processes under consideration are Itô processes or at least continuous semimartingales plus a left continuous jump term. The definitions of these can be found in subsection B.1.1. For a continuous semimartingale X , V^X denotes its bounded variation part and M^X its local martingale part. Furthermore, its quadratic variation is denoted by $[X]$ or $[X, X]$, respectively. For an Itô process X , the drift V^X is pathwise absolutely continuous and we write b^X for its local drift. The local quadratic variation of X is denoted by c^X or $c^{X,X}$. For two Itô processes X, Y , the processes $[X, Y]$

1. Introduction

and $c^{X,Y}$ denote the corresponding objects for the quadratic covariation. If we change the underlying probability measure, then we write $V^{X,Q}$ for the drift part under the probability measure Q , $b^{X,Q}$ for the Q local drift and $M^{X,Q}$ for the martingale part of X under Q . For all other notation we refer to the list of symbols on page 145.

2. Inputs from the frictionless problem

2.1. Introduction

The maximization of the expected utility from terminal wealth in frictionless markets has been investigated extensively in Mathematical Finance starting with the work of Merton (1969). In this chapter we collect the necessary results on the optimization problems in frictionless markets with one riskless bond and one risky asset that are needed for the following chapters. After a brief introduction to the mathematical model we define our optimality criterion: the maximization of the expected exponential utility from terminal wealth. In the literature regarding optimization in Mathematical Finance two types of approaches can be found: martingale methods and dynamic programming. These are discussed in section 2.4.

2.2. Trading in a frictionless market

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ with \mathcal{F}_0 being generated by all P -nullsets and $(\mathcal{F}_t)_{t \geq 0}$ being right continuous.¹ For a fixed time horizon $T \in (0, \infty)$ our market model consists of a riskless bond $S^0 = 1$ and a risky asset as a positive Itô process

$$S_t = S_0 + \int_0^t b_s^S ds + \int_0^t \sigma_s^S dW_s.$$

Here W is a one-dimensional standard Brownian motion $b^S \in L(I), \sigma^S \in L(W)$ and $S_0 > 0$. Examples for specific stock price models will be considered in section 2.4. We state the definitions of trading strategies in frictionless markets:

Definition 2.2.1 (Trading strategies). *A trading strategy is an \mathbb{R}^2 -valued predictable process $(\varphi_t^0, \varphi_t)_{t \in [0, T]}$ such that $\varphi \in L(S)$.*

Here φ_t^0 and φ_t denote the number of shares held in the bank account and in the stock at time t , respectively. As usual in utility maximization we only consider self-financing strategies:

Definition 2.2.2 (Self-financing strategies). *A trading strategy (φ^0, φ) is called self-financing iff*

$$\varphi^0 + \varphi S = \varphi_0^0 + \varphi_0 S_0 + \int_0^\cdot \varphi_s dS_s$$

¹These assumptions imply that the filtration fulfills the so called usual conditions and that every \mathcal{F}_0 -measurable random variable is constant almost surely.

2. Inputs from the frictionless problem

holds.

Except for the initial allocation (φ_0^0, φ_0) we can identify a self-financing strategy (φ^0, φ) with its number of shares held in the stock φ .

Definition 2.2.3 (Portfolio process). For φ as in Definition 2.2.1 and initial capital $x \in \mathbb{R}$ we define the portfolio process $V(\varphi)$ via

$$V_t(\varphi) := x + \int_0^t \varphi_s dS_s.$$

Remark 2.2.4. For $\varphi^0 = V(\varphi) - \varphi S$ we see that (φ^0, φ) is self-financing in the sense of Definition 2.2.2 and that

$$V(\varphi) = \varphi^0 + \varphi S.$$

Thus, $V(\varphi)$ is the sum of the money invested in the bond and the money invested in the stock, which is a common definition for the portfolio process².

2.3. Optimal expected utility from terminal wealth

Definition 2.3.1 (Exponential utility functions). The function $U : \mathbb{R} \rightarrow (-\infty, 0]$ with

$$U(y) = -\exp(-py)$$

is the so-called exponential utility function with risk aversion parameter $p > 0$.

Exponential utility functions have a so called constant absolute risk aversion (CARA). This means that their Arrow-Pratt measure

$$ARA(y) := -\frac{U''(y)}{U'(y)}$$

introduced by Pratt (1964); Arrow (1965) is constant and equals the risk aversion parameter p .

We specify the meaning of optimality. Here, optimal means that the expected utility from terminal wealth is maximized. For that purpose, we fix an initial allocation of bonds $x^B \in \mathbb{R}$ and stocks $x^S \in \mathbb{R}$. Then, the total initial wealth is given by $x = x^B + x^S S_0$. Optimality will be defined on arbitrary subsets \mathfrak{B} of $L(S)$ such that the initial number of stocks equals the given value: $\varphi(0) = x^S$ for all $\varphi \in \mathfrak{B}$.

Definition 2.3.2 (Optimality). A predictable process $\varphi \in \mathfrak{B}$ is called optimal on the set \mathfrak{B} iff

$$\mathbb{E}(U(V_T(\varphi))) = \sup_{\psi \in \mathfrak{B}} \mathbb{E}(U(V_T(\psi))).$$

If the strategy φ with $\varphi_t = 0$ for $t \in (0, T]$, i.e. the strategy that keeps all money in the bond, is in \mathfrak{B} the right-hand side of the above optimality condition is bounded from below by $U(x)$. Thus it is strictly larger than $-\infty$, and the optimization problem is not trivial.

²See e.g. (Irlle, 2012, section 12.2).

Remark 2.3.3. *The assumption that the bond S^0 is constant equal to 1 is not a restriction as long as it is a deterministic and nonnegative process $S^0 = S_0^0 + \int_0^\cdot b_s^{S^0} ds$. In this case we take $V(\varphi^0, \varphi) = \varphi^0 S^0 + \varphi S$ as the definition for the portfolio process, which equals $V(\varphi^0, \varphi) = V_0(\varphi^0, \varphi) + \int_0^\cdot \varphi^0 dS_s^0 + \int_0^\cdot \varphi dS_s$ for a self-financing strategy. Then, the discounted portfolio process $\widehat{V}(\varphi) = \frac{V(\varphi^0, \varphi)}{S^0}$ does not depend on φ^0 and fulfills $\widehat{V}(\varphi) = \widehat{V}_0(\varphi) + \int_0^\cdot \varphi_s d\widehat{S}_s$, where $\widehat{S} = \frac{S}{S^0}$ is an Itô process. So the strategy that maximizes the expected utility from discounted terminal wealth $\widehat{V}_T(\varphi)$ in the market $(1, \widehat{S})$ at risk aversion $\widehat{p} = pS_T^0$ is then also a maximizer for the expected utility from wealth $V_T(\varphi)$ in the market (S^0, S) at risk aversion p .*

2.4. Different methods to solve the optimization problems

The literature on the solution of optimization problems in Mathematical Finance can basically be divided into two groups: *martingale methods* and approaches using *dynamic programming*. We will discuss both in the following:

2.4.1. Martingale approach

As the name suggests martingale methods rely on well-known concepts from Mathematical Finance and stochastic integration and are based on a relation between optimal trading strategies and equivalent martingale measures. This *duality* usually requires results from convex analysis (Rockafellar (1997)) like the convex dual in (Karatzas and Shreve, 1998, section 3.3.4) and can be treated in complete markets where there is only one equivalent martingale measure (Pliska (1986); Karatzas et al. (1987); Cox and Huang (1989, 1991)), and with more effort also in incomplete markets (He and Pearson (1991); Karatzas et al. (1991); Cvitanic and Karatzas (1992); Kramkov and Schachermayer (1999); Schachermayer (2001); Kramkov and Schachermayer (2003)). These approaches are quite general, they do not require a Markovian structure of the asset price process and even work in the general semimartingale case with general utility functions.

We state one possible version of the so called *fundamental theorem of utility maximization*. For that purpose we require the following assumption:

Assumption 2.4.1. *Assume the existence of $\varphi^* \in L(S)$ such that the probability measure Q given by*

$$\frac{dQ}{dP} = \frac{U'(V_T(\varphi^*))}{\mathbb{E}(U'(V_T(\varphi^*)))}$$

is an equivalent martingale measure and $V(\varphi^)$ is a Q -martingale.*

Since this property does not change when the initial value of φ^* is changed, we assume that φ_0^* equals x^S . This implies that φ^* is in the following class of admissible strategies:

Definition 2.4.2 (Admissible strategies). *An admissible strategy is a predictable process φ as in Definition 2.2.1 such that $\varphi(0) = x^S$ and $V(\varphi)$ is a Q -supermartingale. We denote this set of admissible strategies by \mathfrak{A}_2 .*

2. Inputs from the frictionless problem

Remark 2.4.3. *One could also work on the smaller class of admissible strategies where the corresponding portfolio process is bounded from below:*

$$\mathfrak{A}_1 := \{\varphi \in L(S) : \varphi(0) = x^S \wedge (\exists K > 0 \forall t \in [0, T] V_t(\varphi) \geq -K)\}.$$

Observe that for each $\varphi \in \mathfrak{A}_1$ the portfolio process $V(\varphi)$ is a local Q -martingale being bounded from below. Thus, it is a Q -supermartingale. This shows that \mathfrak{A}_1 is contained in \mathfrak{A}_2 .

We are now able to state and prove the following theorem.

Theorem 2.4.4. *The strategy φ^* from Assumption 2.4.1 is optimal on the set \mathfrak{A}_2 .*

Proof. For $\varphi \in \mathfrak{A}_2$ the portfolio process $V(\varphi)$ is a Q -supermartingale. Hence, $V(\varphi) - V(\varphi^*)$ is also a Q -supermartingale with initial value equal to 0. By concavity of U we obtain

$$\begin{aligned} \mathbb{E}(U(V_T(\varphi))) - \mathbb{E}(U(V_T(\varphi^*))) &\leq \mathbb{E}(U'(V_T(\varphi^*))(V_T(\varphi) - V_T(\varphi^*))) \\ &= \mathbb{E}(U'(V_T(\varphi^*))) \underbrace{\mathbb{E}_Q(V_T(\varphi) - V_T(\varphi^*))}_{\leq 0} \leq 0, \end{aligned}$$

i.e. $\mathbb{E}(U(V_T(\varphi)))$ is dominated by $\mathbb{E}(U(V_T(\varphi^*)))$ for every arbitrary $\varphi \in \mathfrak{A}_2$. Since $V(\varphi^*)$ is assumed to be a Q -martingale, this yields the optimality of φ^* . \square

The most famous example of stock price models is the Black-Scholes model, where the optimizer φ^* can be determined explicitly.

Example 2.4.5. *If the stock S is a geometric Brownian motion, i.e.*

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

for a standard Brownian motion W , $\mu \in \mathbb{R}$ and $\sigma > 0$, we know from (Pliska, 1986, Equation (21)) or (Kallsen and Li, 2013, Theorem A.1) that the strategy φ^* from Theorem 2.4.4 equals

$$\varphi_t^* = \frac{\mu}{\sigma^2 p} \frac{1}{S_t}, \quad t \in (0, T].$$

This implies that φ^* is also a geometric Brownian motion.

Another tractable model is a stochastic volatility model with coefficients being independent from the underlying Brownian motion.

Example 2.4.6. *As in (Kallsen and Li, 2013, section 4.2) we consider the stochastic volatility model*

$$dS_t = S_t (\mu(Z_t)dt + \sigma(Z_t)dW_t),$$

where μ and σ are real-valued continuous functions, Z is an Itô process being independent of W and the underlying filtration is generated by Z and W . If $\frac{\mu}{\sigma}$ is bounded, the strategy φ^* given by

$$\varphi_t^* = \frac{\pi_t}{S_t} \quad \text{with } \pi_t := \frac{\mu(Z_t)}{p\sigma^2(Z_t)}$$

2.4. Different methods to solve the optimization problems

for $t \in (0, T]$ is the optimizer from Theorem 2.4.4, and the probability measure Q given by the density

$$\frac{dQ}{dP} = \frac{U'(V_T(\varphi^*))}{\mathbb{E}(U'(V_T(\varphi^*)))}$$

has the density process

$$\mathbb{E} \left(\frac{dQ}{dP} \middle| \mathcal{F}_t \right) = \frac{\tilde{Z}_t}{\tilde{Z}_0} \mathcal{E} \left(\int_0^t \frac{\mu(Z_s)}{\sigma(Z_s)} dW_s \right)_t$$

with

$$\tilde{Z}_t = \mathbb{E} \left(\exp \left(-\frac{1}{2} \int_0^t \left(\frac{\mu(Z_s)}{\sigma(Z_s)} \right)^2 ds \right) \middle| \mathcal{F}_t \right)$$

and is an equivalent martingale measure. $W^Q = W + \int_0^\cdot \frac{\mu(Z_s)}{\sigma(Z_s)} ds$ is a Q -standard Brownian motion. See (Kallsen and Li, 2013, Theorem A.1) for more details.

Except for these two examples it is in general quite difficult to guess a candidate for the optimal strategy or the corresponding equivalent martingale measure. This is one reason why another approach being discussed below is used.

2.4.2. Dynamic programming

In contrast to martingale methods, approaches with dynamic programming try to avoid equivalent martingale measures and usually rely on Markov process theory and stochastic optimal control. By heuristic arguments one can often obtain a candidate for optimality: deriving a partial differential equation for the value function of the utility maximization problem, one then needs analytical methods to prove the existence and uniqueness of a solution to this equation. Doing so, Merton (1969) determined this so-called *dynamic programming equation* and was able to solve it for logarithmic, power and exponential utility. In (Korn, 2014, sections 5.4 and 5.5), a detailed introduction to Markovian stochastic optimal control and its application to utility maximization can be found. Fleming and Soner (2006) provide the corresponding theory of viscosity solutions for the dynamic programming equation.

But one can also do without these more analytical procedures and work with the following non-Markovian dynamic programming approach from El Karoui (1981) and (Kallsen, 2016, Chapter II.7) with a value process instead of a value function. This can also be done on subsets of the slightly larger class of admissible strategies $\mathfrak{B} \subseteq \mathfrak{A}_2$, that are stable under bifurcation, i.e. for any stopping time τ and any $\varphi, \psi \in \mathfrak{B}$ with $\varphi_{\cdot \wedge \tau} = \psi_{\cdot \wedge \tau}$ and any $B \in \mathcal{F}_\tau$, the process $\mathbf{1}_B \varphi + \mathbf{1}_{B^c} \psi$ is in \mathfrak{B} .

Definition 2.4.7 (Value process). *The value process for the optimization problem on $\mathfrak{B} \subseteq \mathfrak{A}_2$ as above is the family of (adapted) processes such that*

$$v_t(\varphi) := \text{esssup} \{ \mathbb{E}(U(V_T(\tilde{\varphi})) | \mathcal{F}_t) : \tilde{\varphi} \in \mathfrak{B}, \tilde{\varphi}_{t \wedge \cdot} = \varphi_{t \wedge \cdot} \}$$

for all $t \in [0, T]$ and $\varphi \in \mathfrak{B}$.

2. Inputs from the frictionless problem

Observe that $v_0 := \bar{v}_0(\varphi) = \sup_{\varphi \in \mathfrak{B}} E(U(V_T(\varphi)))$ does not depend on $\varphi \in \mathfrak{A}_2$ and is the optimal value of our utility maximization problem. The \mathcal{F}_t -measurable random variable $v_t(\varphi)$ describes the optimal value of the utility maximization starting at time t having wealth $V_t(\varphi)$. This is of course a rather abstract mathematical object, but it can help to characterize optimality or suboptimality in terms of martingale and supermartingale properties on subsets $\mathfrak{B} \subseteq \mathfrak{A}_2$ with $0 \in \mathfrak{B}$. The following theorem is based on (Kallsen, 2016, Theorem II.7.3).

Theorem 2.4.8. (1) For any strategy $\varphi \in \mathfrak{B}$ with $U(V_T(\varphi)) \in L_1(P)$, $v(\varphi)$ is a supermartingale with terminal value $v_T(\varphi) = U(V_T(\varphi))$. If $\bar{\varphi} \in \mathfrak{B}$ is an optimal strategy, then $v(\bar{\varphi})$ is a martingale.

(2) Suppose that $(\bar{v}(\varphi))_{\varphi \in \mathfrak{B}}$ is a family of processes such that the following properties hold:

(i) $\bar{v}_0 := \bar{v}_0(\varphi)$ does not depend on $\varphi \in \mathfrak{B}$.

(ii) $\bar{v}(\varphi)$ is a supermartingale with terminal value $\bar{v}_T(\varphi) = U(V_T(\varphi))$ for all $\varphi \in \mathfrak{B}$ with $U(V_T(\varphi)) \in L_1(P)$.

(iii) $\bar{v}(\bar{\varphi})$ is a martingale for some strategy $\bar{\varphi} \in \mathfrak{B}$.

Then $\bar{\varphi}$ is optimal on \mathfrak{B} and $\bar{v}_t(\bar{\varphi}) = v_t(\bar{\varphi})$ for all $t \in [0, T]$.

(3) If furthermore $(\bar{v}(\varphi))_{\varphi \in \mathfrak{B}}$ is of the form

$$\bar{v}_t(\varphi) = \mathbb{E} \left(U \left(V_t(\varphi) + \int_t^T \bar{\varphi}_s dS_s \right) \middle| \mathcal{F}_t \right),$$

then $\bar{v}_t(\varphi) = v_t(\varphi)$ for all $\varphi \in \mathfrak{B}$ and $t \in [0, T]$.

Proof. (1) (i) Adaptedness and terminal value follow from Definition 2.4.7.

(ii) Fix $t \in [0, T]$. We will show that the set $M := \{\mathbb{E}(U(V_T(\tilde{\varphi})) | \mathcal{F}_t) : \tilde{\varphi}_{t \wedge \cdot} = \varphi_{t \wedge \cdot}\}$ has the so-called lattice property, i.e. for $X, Y \in M$ we can find $Z \in M$ with $X \vee Y \leq Z$: Let $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ be in \mathfrak{B} with $\tilde{\varphi}_{t \wedge \cdot}^1 = \tilde{\varphi}_{t \wedge \cdot}^2 = \varphi_{t \wedge \cdot}$. Define the set

$$B := \{\mathbb{E}(U(V_T(\tilde{\varphi}^1)) | \mathcal{F}_t) \geq \mathbb{E}(U(V_T(\tilde{\varphi}^2)) | \mathcal{F}_t)\} \in \mathcal{F}_t$$

and the process $\tilde{\varphi}^3 = \tilde{\varphi}^1 \mathbf{1}_B + \tilde{\varphi}^2 \mathbf{1}_{B^c}$. Then $\tilde{\varphi}^3$ is in \mathfrak{B} and

$$\begin{aligned} \mathbb{E}(U(V_T(\tilde{\varphi}^1)) | \mathcal{F}_t) \vee \mathbb{E}(U(V_T(\tilde{\varphi}^2)) | \mathcal{F}_t) &= \mathbb{E}(U(V_T(\tilde{\varphi}^1)) | \mathcal{F}_t) \mathbf{1}_B + \mathbb{E}(U(V_T(\tilde{\varphi}^2)) | \mathcal{F}_t) \mathbf{1}_{B^c} \\ &= \mathbb{E}(U(V_T(\tilde{\varphi}^1)) \mathbf{1}_B + U(V_T(\tilde{\varphi}^2)) \mathbf{1}_{B^c} | \mathcal{F}_t) \\ &= \mathbb{E}(U(V_T(\tilde{\varphi}^3)) | \mathcal{F}_t). \end{aligned}$$

Due to this lattice property we can conclude that there exists a sequence of strategies $(\tilde{\varphi}^n)_{n \in \mathbb{N}}$ in \mathfrak{B} with $\tilde{\varphi}_{t \wedge \cdot}^n = \varphi_{t \wedge \cdot}$ and

$$\mathbb{E}(U(V_T(\tilde{\varphi}^n)) | \mathcal{F}_t) \nearrow v_t(\varphi), \quad (2.4.1)$$

see (Lamberton, 2009, Proposition 1.1.3) for more details. For the supermartingale property let $s \in [0, t]$. Since $\tilde{\varphi}_{s \wedge \cdot}^n = \varphi_{s \wedge \cdot}$ we see that

$$\mathbb{E}(\mathbb{E}(U(V_T(\tilde{\varphi}^n)) | \mathcal{F}_t) | \mathcal{F}_s) = \mathbb{E}(U(V_T(\tilde{\varphi}^n)) | \mathcal{F}_s) \leq v_s(\varphi),$$

2.4. Different methods to solve the optimization problems

which implies

$$\mathbb{E}(v_t(\varphi)|\mathcal{F}_s) \leq v_s(\varphi)$$

by the monotone convergence theorem for conditional expectations. If $\bar{\varphi}$ is optimal, we obtain

$$v_0(\bar{\varphi}) = \sup_{\varphi \in \mathfrak{B}} \mathbb{E}(U(V_T(\varphi))) = \mathbb{E}(U(V_T(\bar{\varphi}))) = \mathbb{E}(v_T(\bar{\varphi})).$$

In conjunction with the supermartingale property of $v(\varphi)$ this yields that $v(\varphi)$ is a martingale.

(2) Since $\bar{v}(\varphi)$ is assumed to be a supermartingale, we have the inequality

$$\mathbb{E}(U(V_T(\varphi))) = \mathbb{E}(\bar{v}_T(\varphi)) \leq \bar{v}_0(\varphi) = \bar{v}_0 = \bar{v}_0(\bar{\varphi}) = \mathbb{E}(U(V_T(\bar{\varphi})))$$

for all $\varphi \in \mathfrak{B}$, so $\bar{\varphi}$ is optimal. Moreover, $\bar{v}(\bar{\varphi})$ and $v(\bar{\varphi})$ are martingales with the same terminal value $\bar{v}_T(\bar{\varphi}) = U(V_T(\bar{\varphi})) = v_T(\bar{\varphi})$. By taking the conditional expectation we obtain $\bar{v}_t(\bar{\varphi}) = v_t(\bar{\varphi})$ for all $t \in [0, T]$.

(3) For $\tilde{\varphi} \in \mathfrak{B}$ with $\tilde{\varphi}_{\cdot \wedge t} = \varphi_{\cdot \wedge t}$ we have

$$\mathbb{E}(U(V_T(\tilde{\varphi}))|\mathcal{F}_t) = \mathbb{E}(\bar{v}_T(\tilde{\varphi})|\mathcal{F}_t) \leq \bar{v}_t(\tilde{\varphi}) = \bar{v}_t(\varphi)$$

and passing to the essential supremum yields

$$v_t(\varphi) \leq \bar{v}_t(\varphi).$$

For the other inequality observe that for $t \in [0, T]$ the strategy $\hat{\varphi}^t := \varphi \mathbf{1}_{[0, t]} + \bar{\varphi} \mathbf{1}_{(t, T]}$ is in \mathfrak{B} . Thus we have

$$\bar{v}_t(\varphi) = \mathbb{E}(U(V_T(\hat{\varphi}^t))|\mathcal{F}_t) \leq v_t(\varphi)$$

due to the definition of the value process. This implies the equality of $v_t(\varphi)$ and $\bar{v}_t(\varphi)$. \square

As for the martingale methods, it seems quite difficult to come up with a candidate for optimality in this rather abstract view on dynamic programming. Furthermore, the natural question arises whether there is a relation between the value process here and the value function in Markovian type models that we show in our setup for the Black-Scholes model. This will be related to (Kallsen, 2016, Example II.9.3) dealing with power utility. For that purpose, we change the bookkeeping to the money invested in the stock. i.e. we consider $\Pi = \varphi S$ for $\varphi \in \mathfrak{A}_2$. We then obtain $V_t(\varphi) = x + \int_0^t \Pi_s \frac{dS_s}{S_s}$ for the corresponding portfolio process and observe that in the Black-Scholes model

$$dV_t(\varphi) = \Pi_t(\mu dt + \sigma dW_t) = \tilde{\mu}(\Pi_t)dt + \tilde{\sigma}(\Pi_t)dW_t$$

with $\tilde{\mu}(y) = \mu y$ and $\tilde{\sigma}(y) = \sigma y$. We now heuristically determine the value function of our problem. In order to do so, we suppose that there exists a $C^{1,2}$ -function $\bar{v} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that the value process can be written as

$$v_t(\varphi) = \bar{v}(t, V_t(\varphi)),$$

2. Inputs from the frictionless problem

which is usually true for Markovian type models³. Then, by applying Itô's formula we obtain that the local drift of $v(\varphi)$ is given by

$$\partial_1 \bar{v}(t, V_t(\varphi)) + \partial_2 \bar{v}(t, V_t(\varphi)) \tilde{\mu}(\varphi_t S_t) + \frac{1}{2} \partial_{22} \bar{v}(t, V_t(\varphi)) \tilde{\sigma}^2(\varphi_t S_t). \quad (2.4.2)$$

Now, according to Theorem 2.4.8 we want that $v(\varphi)$ is a supermartingale for an arbitrary admissible strategy φ and a martingale if the optimizer is used. So we require that the local drift is non-positive and vanishes for the optimal strategy, i.e. we would like to have for the function \bar{v}

$$\sup_{y \in \mathbb{R}} \left(\partial_1 \bar{v}(t, z) + \partial_2 \bar{v}(t, z) \tilde{\mu}(y) + \frac{1}{2} \partial_{22} \bar{v}(t, z) \tilde{\sigma}^2(y) \right) = 0 \quad (2.4.3)$$

for all $(t, z) \in (0, T) \times \mathbb{R}$. Furthermore, at the terminal time $\bar{v}(T, z) = U(z)$ should hold. We suppose that the dependence on time t and wealth x separates in the following way:

$$\bar{v}(t, z) = g(t)h(z)$$

for functions $g : [0, T] \rightarrow \mathbb{R}_{\geq 0}$, $h : (0, \infty) \rightarrow \mathbb{R}_{\leq 0}$. The terminal conditions implies

$$U(z) = g(T)h(z),$$

thus $g(T) = 1$ and $h = U$ is a suitable choice. Inserting the ansatz into (2.4.3) yields

$$\begin{aligned} g'(t)U(z) &= - \sup_{y \in \mathbb{R}} \left(g(t)U'(z)\mu y + \frac{1}{2}g(t)U''(z)y^2\sigma^2 \right) \\ &= -g(t)U(z) \inf_{y \in \mathbb{R}} \left(\mu y(-p) + \frac{p^2}{2}y^2\sigma^2 \right). \end{aligned}$$

The infimum is attained for $y = y^* := \frac{\mu}{p\sigma^2}$, yielding that

$$\begin{aligned} g'(t) &= -g(t) \left(\mu y^*(-p) \frac{p^2}{2} (y^*)^2 \sigma^2 \right) \\ &= -g(t) \left(-\frac{\mu^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} \right) \\ &= \frac{\mu^2}{2\sigma^2} g(t). \end{aligned}$$

Together with $g(T) = 1$ this implies

$$g(t) = \exp \left(-\frac{\mu^2}{2\sigma^2} (T - t) \right).$$

So we have obtained a candidate and we will verify its optimality on the set $\mathfrak{A}_1 \cup \{\varphi^*\}$ below.

³See (Kallsen, 2016, section II.9.1)

2.4. Different methods to solve the optimization problems

Lemma 2.4.9. (1) *There exists a $C^{1,2}$ -function $\bar{v} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$\bar{v}(T, z) = U(z)$$

and (2.4.3) holds. *This function is given by*

$$\bar{v}(t, z) = U(z) \exp\left(-\frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 (T - t)\right).$$

(2) *Let $\varphi^* \in \mathfrak{A}_2$ be the strategy keeping the money invested in the stock equal to $y^* = \frac{\mu}{p\sigma^2}$. Then,*

$$\left(\partial_1 \bar{v}(t, z) + \partial_2 \bar{v}(t, z) \tilde{\mu}(y^*) + \frac{1}{2} \partial_{22} \bar{v}(t, z) \tilde{\sigma}^2(y^*)\right) = 0$$

for all $(t, x) \in (0, T) \times \mathbb{R}$ and φ^* is optimal. Furthermore, $(\bar{v}(\cdot, V(\varphi)))_{\varphi \in \mathfrak{A}_1 \cup \{\varphi^*\}}$ is the value process ⁴ for the frictionless problem in the Black-Scholes model.

Proof. For the proof of the first part we can just insert the solution and verify that (2.4.3) and the terminal condition hold. For the second part recall that the local drift of $\bar{v}(\cdot, V(\varphi))$ is given by (2.4.2), so it is non-positive for an arbitrary admissible strategy and 0 for $\varphi = \varphi^*$. This implies that $\bar{v}(\cdot, V(\varphi))$ is a local supermartingale⁵ and a local martingale for $\varphi = \varphi^*$. For $\varphi \in \mathfrak{A}_1$, $\bar{v}(\cdot, V(\varphi))$ is bounded from below, which yields that it is a true supermartingale. Furthermore, $\bar{v}(\cdot, V(\varphi^*))$ is a true martingale due to Novikov's condition. So φ^* is optimal by the second assertion of the theorem and we can rewrite \bar{v} to obtain

$$\bar{v}(t, V_t(\varphi)) = \mathbb{E} \left(U \left(V_t(\varphi) + \int_t^T \varphi_s^* dS_s \right) \middle| \mathcal{F}_t \right).$$

This yields $\bar{v}(t, V_t(\varphi)) = v_t(\varphi)$ by the third assertion of the theorem. □

2.4.3. Connection between both approaches

Another natural question is whether there is a relation between martingale methods and dynamic programming approaches. Rewriting the result from Lemma 2.4.9 as in its proof yields that the value process equals

$$v_t(\varphi) = \mathbb{E} \left(U \left(V_t(\varphi) + \int_t^T \varphi_s^* dS_s \right) \middle| \mathcal{F}_t \right)$$

for $\varphi^* = \frac{\mu}{\sigma^2 p S}$. We will generalize this result and hence show a connection between both approaches:

Proposition 2.4.10. *If φ^* is chosen as in Theorem 2.4.4, then the value process is given by*

$$v_t(\varphi) = \mathbb{E} \left(U \left(V_t(\varphi) + \int_t^T \varphi_s^* dS_s \right) \middle| \mathcal{F}_t \right) \quad \text{for all } \varphi \in \mathfrak{A}_2. \quad (2.4.4)$$

⁴More precisely, $\bar{v}(\cdot, V(\varphi))$ is a modification of $v(\varphi)$ for each $\varphi \in \mathfrak{A}_1 \cup \{\varphi^*\}$.

⁵In the sense of (Kallsen, 2016, section II.6.1).

2. Inputs from the frictionless problem

Proof. Define $\bar{v}(\varphi)$ as the right hand side of (2.4.4) and let $\varphi \in \mathfrak{A}_2$ with $U(V_T(\varphi)) \in L_1(P)$.

- (i) We see that $\bar{v}_0(\varphi) = \mathbb{E}(U(x + \int_0^T \varphi_s^* dS_s)) = \mathbb{E}(U(V_T(\varphi^*)))$ does not depend on φ .
- (ii) Observe that

$$\begin{aligned}\bar{v}_t(\varphi) &= \exp((-p)(V_t(\varphi) - V_t(\varphi^*))) \mathbb{E}(U(V_T(\varphi^*)) | \mathcal{F}_t) \\ &= X_t(\varphi) D_t \frac{(-1)\mathbb{E}(U(V_T(\varphi^*)))}{p}\end{aligned}$$

with

$$X_t(\varphi) := \exp((-p)(V_t(\varphi) - V_t(\varphi^*))) \quad \text{and} \quad D_t = \frac{\mathbb{E}(U'(V_T(\varphi^*)) | \mathcal{F}_t)}{\mathbb{E}(U'(V_T(\varphi^*)))}.$$

Since D is the density process of the EMM Q , the P -supermartingale property of $\bar{v}(\varphi)$ is equivalent to the Q -submartingale property of $X(\varphi)$, which we will show in the following: We observe that

$$\mathbb{E}_Q(X_T(\varphi)) = \mathbb{E}(X_T(\varphi) D_T) = \frac{\mathbb{E}(U'(V_T(\varphi)))}{\mathbb{E}(U'(V_T(\varphi^*)))} < \infty$$

and thus $X_T(\varphi) \in L_1(Q)$. Together with the Q -supermartingale property of $V(\varphi) - V(\varphi^*)$ and according to Jensen's inequality for conditional expectations from (Klenke, 2008, Proposition 8.19) we have for $s \in [0, T]$

$$\mathbb{E}_Q(X_T(\varphi) | \mathcal{F}_s) \geq X_s(\varphi)$$

and taking expectations yields

$$\infty > \mathbb{E}_Q(X_T(\varphi)) \geq \mathbb{E}_Q(X_s(\varphi)).$$

We now obtain the Q -submartingale property of $X(\varphi)$ with the same arguments.

- (iii) We see

$$\bar{v}_t(\varphi^*) = \mathbb{E}(U(V_T(\varphi^*)) | \mathcal{F}_t) = -\frac{E(U'(V_T(\varphi^*)))}{p} D_t,$$

which is a martingale. Thus $(\bar{v}(\varphi))_{\varphi \in \mathfrak{A}_2}$ is the value process up to some modification issues according to Theorem 2.4.8. \square

We now see that the value process can also be understood as a *stochastic value function* in the following sense: We observe that for all $(\omega, t) \in \Omega \times [0, T]$

$$v_t(\varphi)(\omega) = U(V_t(\varphi)(\omega)) \mathbb{E}\left(\exp\left(-p \int_t^T \varphi_s^* dS_s\right) \middle| \mathcal{F}_t\right)(\omega) = G(\omega, t, V_t(\varphi)(\omega))$$

with $G : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_{\leq 0}$ and

$$G(\omega, t, z) = U(z) \mathbb{E}\left(\exp\left(-p \int_t^T \varphi_s^* dS_s\right) \middle| \mathcal{F}_t\right)(\omega).$$

Coming back to the Black Scholes model we see that G coincides with \bar{v} from Lemma 2.4.9. Furthermore, the optimality of φ^* in Proposition 2.4.10 holds on \mathfrak{A}_2 and not only on $\mathfrak{A}_1 \cup \{\varphi^*\}$.

3. Fixed transaction costs

3.1. Introduction

This chapter is the main part of this thesis and is organized as follows: in section 3.2 we define how trading is modeled in a market with fixed transaction costs and in section 3.3 we state the dynamic programming approach for the frictional case. For the convenience of the reader we summarize the main result of this chapter in section 3.4. Furthermore we give an interpretation of the asymptotic no-trade region and the corresponding certainty equivalent loss when trading almost optimally in the market with frictions. In section 3.5 we choose a candidate for the value process at transaction costs and heuristically derive a trading strategy that we suppose is almost optimal. In section 3.6 we state the regularity assumptions of the frictionless problem that have to be fulfilled for the rigorous verification. We construct an asymptotic upper bound for the maximal expected utility in section 3.7. However, at first sight, the verification only works on a subclass of admissible strategies (subsection 3.7.1). Using a modified candidate value process from subsection 3.5.5 we can show that the value from subsection 3.7.1 is also an asymptotic upper bound on the prior set of admissible strategies. The proofs rely on the fact that the local drift of the value process ansatzes under the equivalent martingale measure Q will be negative or of small order in a proper sense. Due to the construction, the jumps of the value processes will be negative as well. Then, using a localization argument we obtain that the initial value of the ansatz is an asymptotic upper bound for maximal expected utility. Finally, in section 3.8 we show that the ansatz for the value process of the heuristically determined strategy from subsection 3.5.4 is almost a martingale and that this strategy is almost optimal if one sells all stocks when wealth falls below some given stochastic threshold.

3.2. Trading with fixed transaction costs

In the market model with bond $S^0 = 1$ and risky asset

$$S_t = S_0 + \int_0^t b_s^S ds + \int_0^t \sigma_s^S dW_s$$

from section 2.2 we introduce fixed transaction costs: each transaction reduces the investor's wealth by a fixed amount $\varepsilon > 0$. Due to these costs we only allow for trading strategies that change their values finitely many times on the interval $[0, T]$. As in the cited literature on fixed costs, this leads to the following definition.

3. Fixed transaction costs

Definition 3.2.1 (Trading strategies). *A trading strategy is an \mathbb{R}^2 -valued (predictable) process $(\varphi_t^0, \varphi_t)_{t \in [0, T]}$ such that there exist a strictly increasing sequence of stopping times $(\tau_i)_{i \in \mathbb{N}}$ and sequences of random variables $(n_i)_{i \in \mathbb{N}}$ and $(n_i^0)_{i \in \mathbb{N}}$ with*

$$(i) \quad P(\exists i \in \mathbb{N} : \tau_i \geq T) = 1,$$

$$(ii) \quad P(n_i = n_i^0 = 0) = 0 \text{ for all } i \in \mathbb{N},$$

$$(iii) \quad n_i \text{ and } n_i^0 \text{ are } \mathcal{F}_{\tau_i}\text{-measurable for all } i \in \mathbb{N},$$

(iv) *the number of stocks and bonds fulfill*

$$\varphi_t = \varphi_0 + \sum_{i \in \mathbb{N}} n_i \mathbf{1}_{\{\tau_i < t\}} \quad \text{and} \quad (3.2.1)$$

$$\varphi_t^0 = \varphi_0^0 + \sum_{i \in \mathbb{N}} n_i^0 \mathbf{1}_{\{\tau_i < t\}} \quad \text{for all } t \in [0, T]. \quad (3.2.2)$$

Here φ_t^0 and φ_t denote the number of shares held in the bank account and in the stock at time t , respectively. The stopping times $(\tau_i)_{i \in \mathbb{N}}$ describe the times where the transactions take place, and the random variables $(n_i)_{i \in \mathbb{N}}$ and $(n_i^0)_{i \in \mathbb{N}}$ describe the transfers measured in absolute numbers of shares at the transaction times.

As usual in utility maximization we only consider self-financing portfolios. When dealing with fixed transaction costs, it seems reasonable to assume that the monetary transfers of shares in the bank account have to equal the negative monetary transfer in the stock minus the transaction cost ε . With equations (3.2.1) and (3.2.2) this leads to the definition of self-financing strategies.

Definition 3.2.2 (Self-financing strategies). *A trading strategy (φ^0, φ) is called self-financing iff*

$$n_i^0 = -S_{\tau_i} n_i - \varepsilon \quad \text{for all } i \in \mathbb{N}$$

with the notation from Definition 3.2.1.

Except for the initial allocation (φ_0^0, φ_0) , we can identify a self-financing strategy (φ^0, φ) with its number of shares φ held in the stock.

Definition 3.2.3 (Portfolio process). *For φ as in Definition 3.2.1 and initial capital $x \in \mathbb{R}$ we define the portfolio process $V^\varepsilon(\varphi)$ via*

$$V_t^\varepsilon(\varphi) = x + \int_0^t \varphi_s dS_s - \varepsilon |\{i \in \mathbb{N} : \tau_i < t\}|.$$

Remark 3.2.4. *Due to the local boundedness of φ , the above stochastic integral is well-defined. Since φ is left-continuous and of bounded variation, one also knows how to define the integral $\int_0^t S_s d\varphi_s$, namely*

$$\int_0^t S_s d\varphi_s := \sum_{i \in \mathbb{N}} S_{\tau_i} n_i \mathbf{1}_{\{\tau_i < t\}}.$$

We can directly verify a corresponding integration by parts rule¹:

$$\begin{aligned}
 \int_0^t \varphi_s dS_s + \int_0^t S_s d\varphi_s &= \int_0^t \left(\varphi_0 + \sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < s\}} \right) dS_s + \sum_{i \in \mathbb{N}} S_{\tau_i} n_i \mathbf{1}_{\{\tau_i < t\}} \\
 &= \varphi_0(S_t - S_0) + \sum_{i \in \mathbb{N}} n_i \int_0^t \mathbf{1}_{\{\tau_i < s\}} dS_s + \sum_{i \in \mathbb{N}} S_{\tau_i} n_i \mathbf{1}_{\{\tau_i < t\}} \\
 &= \varphi_0(S_t - S_0) + \sum_{i \in \mathbb{N}} n_i (S_t - S_{t \wedge \tau_i} + S_{\tau_i} \mathbf{1}_{\{\tau_i < t\}}) \\
 &= \varphi_0(S_t - S_0) + \sum_{i \in \mathbb{N}} n_i S_t \mathbf{1}_{\{\tau_i < t\}} \\
 &= S_t \varphi_t - S_0 \varphi_0.
 \end{aligned}$$

With $\mathcal{N}_t(\varphi) := |\{i \in \mathbb{N} : \tau_i < t\}|$ this implies

$$\begin{aligned}
 V_t^\varepsilon(\varphi) &= x + \int_0^t \varphi_s dS_s - \varepsilon \mathcal{N}_t(\varphi) = x + \varphi_t S_t - \varphi_0 S_0 - \int_0^t S_s d\varphi_s - \varepsilon \mathcal{N}_t(\varphi) \\
 &= \varphi_t S_t - \varphi_0 S_0 + x - \sum_{i \in \mathbb{N}} (S_{\tau_i} n_i + \varepsilon) \mathbf{1}_{\{\tau_i < t\}}.
 \end{aligned}$$

If we set $\varphi_0^0 = x - \varphi_0 S_0$ and $n_i^0 = -S_{\tau_i} m_i - \varepsilon$ as in the self-financing condition, we obtain

$$V_t^\varepsilon(\varphi) = \varphi_t S_t + \varphi_t^0 = \varphi_t S_t + \varphi_t^0 S_t^0,$$

so $V_t^\varepsilon(\varphi)$ is the money invested in the stock plus the money invested in the bond as in the frictionless case in remark 2.2.4.

Again, we fix an initial allocation of bonds $x^B \in \mathbb{R}$ and stocks $x^S \in \mathbb{R}$ leading to a total initial capital $x = x^B + x^S S_0$. In the following we suppose that Assumption 2.4.1 holds.

Definition 3.2.5 (Admissible strategies). *An admissible strategy is a predictable process φ as in Definition 3.2.1 with the following property:*

1. $\varphi_0 = x^S$.
2. $\int_0^\cdot \varphi_s dS_s$ is a Q -supermartingale.

We write \mathfrak{A}^ε for the set of all admissible strategies.

Here, Q denotes the equivalent martingale measure from Assumption 2.4.1.

Remark 3.2.6. *Let $\varphi \in L(S)$ be a process satisfying (ii)-(iv) of Definition 3.2.1. Then, φ might have countably many trading times before T due to a cluster point. This cannot happen if the associated portfolio process is bounded from below, i.e. $V_t^\varepsilon(\varphi) \geq -K$ for all $t \in [0, T]$ for some $K > 0$, because this implies*

$$-K \leq V_t^\varepsilon(\varphi) = x + \int_0^t \varphi_s dS_s - \varepsilon \mathcal{N}_t(\varphi),$$

¹One could also use the extended Stieltjes integral from (Muhle-Karbe, 2009, section 8.2).

3. Fixed transaction costs

Then, $\int_0^\cdot \varphi_s dS_s$ is also bounded by $-K$ and a Q -local martingale. So, $\int_0^\cdot \varphi_s dS_s$ is a Q -supermartingale, and by taking Q -expectations we get

$$-K \leq x + \mathbb{E}_Q \left(\int_0^T \varphi_s dS_s \right) - \varepsilon \mathbb{E}_Q (\mathcal{N}_T(\varphi)) \leq x - \varepsilon \mathbb{E}_Q (\mathcal{N}_T(\varphi)),$$

i.e. $\mathcal{N}_T(\varphi)$ has a finite Q -expectation and has to be finite almost surely yielding condition (i) of Definition 3.2.1.

3.3. Optimality and dynamic programming

As in section 2.3, we want to maximize the expected utility from terminal wealth:

Definition 3.3.1 (Optimality). *A trading strategy $\varphi \in \mathfrak{A}^\varepsilon$ is called optimal iff*

$$\mathbb{E} (U(V_T^\varepsilon(\varphi))) = \sup_{\psi \in \mathfrak{A}^\varepsilon} \mathbb{E} (U(V_T^\varepsilon(\psi))).$$

Here, U is the exponential utility function from Definition 2.3.1.

Parallel to frictionless value process from section 2.4.2 we define a version for fixed transaction costs on \mathfrak{A}^ε .

Definition 3.3.2 (Value process). *The value process for the optimization problem under fixed transaction costs is a the family of (adapted) processes such that*

$$v_t^\varepsilon(\varphi) := \text{esssup} \{ \mathbb{E} (U(V_T^\varepsilon(\tilde{\varphi})) | \mathcal{F}_t) : \varphi \in \mathfrak{A}^\varepsilon, \tilde{\varphi}_{t \wedge \cdot} = \varphi_{t \wedge \cdot} \}$$

for all $t \in [0, T]$ and $\varphi \in \mathfrak{A}^\varepsilon$.

We state the dynamic programming version for the optimization problem at transaction costs which can be verified with the arguments from the frictionless counterpart from section 2.4.2.

Theorem 3.3.3. (1) *For any admissible strategy $\varphi \in \mathfrak{A}^\varepsilon$, $v^\varepsilon(\varphi)$ is a supermartingale with terminal value $v_T^\varepsilon(\varphi) = U(V_T^\varepsilon(\varphi))$. If $\bar{\varphi}$ is an optimal strategy, then $v^\varepsilon(\varphi)$ is a martingale.*

(2) *Suppose that $(\bar{v}(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ is a family of processes such that the following properties hold:*

(i) $\bar{v}_0 := \bar{v}_0(\varphi)$ does not depend on $\varphi \in \mathfrak{A}^\varepsilon$.

(ii) $\bar{v}(\varphi)$ is a supermartingale with terminal value $\bar{v}_T(\varphi) = U(V_T^\varepsilon(\varphi))$.

(iii) $\bar{v}(\bar{\varphi})$ is a martingale for some admissible strategy $\bar{\varphi}$.

Then $\bar{\varphi}$ is optimal and $\bar{v}_t(\bar{\varphi}) = v_t^\varepsilon(\bar{\varphi})$ for all $t \in [0, T]$.

(3) *If furthermore $(\bar{v}(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ is of the form*

$$\bar{v}_t(\varphi) = \mathbb{E} \left(U \left(V_t(\varphi) + \int_t^T \bar{\varphi}_s dS_s \right) \middle| \mathcal{F}_t \right),$$

then $\bar{v}_t(\varphi) = v_t^\varepsilon(\varphi)$ for all $\varphi \in \mathfrak{A}^\varepsilon$ and $t \in [0, T]$.

Proof. We obtain the desired result by replacing the frictionless portfolio process $V(\varphi)$ with the portfolio process at transaction costs $V^\varepsilon(\varphi)$ and the set \mathfrak{B} with \mathfrak{A}^ε in the proof of Theorem 2.4.8 since \mathfrak{A}^ε is stable under bifurcation. \square

3.4. Main result and interpretation

Having collected the necessary inputs from the frictionless problem in Chapter 2, we are now able to formulate the main result of this chapter: the determination of an almost optimal trading strategy. For that purpose we require some regularity assumptions that can be found in section 3.6.

Theorem 3.4.1. *Define $\Delta\varphi^+ := \sqrt[4]{\frac{12\varepsilon}{p}} \sqrt[4]{\frac{c^{\varphi^*}}{c^S}}$ and the no-trade region*

$$NT^\varepsilon := [\varphi^* - \Delta\varphi^+, \varphi^* + \Delta\varphi^+].$$

Let φ^ε be the strategy that does not trade while φ^ε lies in NT^ε and jumps to the frictionless optimizer φ^ when it hits the boundaries of NT^ε . Then the admissible strategy $\tilde{\varphi}^\varepsilon$ that coincides with φ^ε up to the stopping time*

$$\tau^\varepsilon = \inf \left\{ t \in [0, T] : V_t^\varepsilon(\varphi^\varepsilon) - V_t(\varphi^*) < -1 \text{ or } V_t^\varepsilon(\varphi^\varepsilon) < -\frac{1}{\varepsilon} \right\} \wedge T$$

and sells all stocks afterwards is optimal at the leading order $\varepsilon^{1/2}$, i.e.

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E}(U(V_T^\varepsilon(\varphi))) = \mathbb{E}(U(V_T^\varepsilon(\tilde{\varphi}^\varepsilon))) + o(\varepsilon^{1/2}).$$

Furthermore, the optimal certainty equivalent equals

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} U^{-1}(\mathbb{E}(U(V_T^\varepsilon(\varphi)))) = U^{-1}(\mathbb{E}(U(V_T(\varphi^*)))) - \gamma_0 + o(\varepsilon^{1/2})$$

with

$$\gamma_0 = \sqrt{\frac{\varepsilon p}{3}} \mathbb{E}_Q \left(\int_0^T \sqrt{c_s^{\varphi^*} c_s^S} ds \right) = \frac{p}{6} \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^2 d[S]_s \right).$$

Examples for specific stock price models are discussed in Chapter 5. In this section we discuss our result for the case of general Itô dynamics. As in studies on proportional costs (Martin (2012); Kallsen and Muhle-Karbe (2013); Kallsen and Li (2013); Kallsen and Muhle-Karbe (2015)) and on fixed costs (Altarovici et al. (2015a)), the halfwidth of the (stochastic) no-trade region depends on the so-called (squared) *portfolio gamma*

$$\frac{d[\varphi^*]_t}{d[S]_t} = \frac{c_t^{\varphi^*}}{c_t^S}$$

times the inverse risk aversion parameter² $\frac{1}{p}$ times the transaction cost ε , but in contrast to the results on proportional costs with a different power and a different constant. Interpreting $\frac{1}{p}$ as the risk tolerance, our formula for the halfwidth of the no-trade region coincides with (Altarovici et al., 2015a, Equation 2.6). The dependence on the above quantities can be understood as in Kallsen and Muhle-Karbe (2013):

²Observe that $\frac{1}{p}$ is the inverse Arrow-Pratt measure of the exponential utility function $-\frac{U'(x)}{U''(x)}$ as well as the indirect risk tolerance process of the frictionless problem from (Kallsen and Muhle-Karbe (2013)).

3. Fixed transaction costs

Obviously, a higher value of the transaction cost ε that has to be paid for each transaction should lead to a wider no-trade region to save money due to these payments. The portfolio gamma is the local activity rate of the frictionless optimizer per local activity rate of the market. A high portfolio gamma implies that the frictionless target is rapidly moving compared to the asset. A wide no-trade region and thus few transactions are required to avoid too high transaction costs. On the other hand, highly volatile asset prices require small corridors around the frictionless optimizer to avoid a too high displacement loss.

Since the frictionless target may also depend on the degree of risk aversion p , one requires this dependence in order to analyze the influence of the risk aversion on the no-trade corridor. We refer to Chapter 5 where this is discussed for the Black-Scholes model. For the interpretation of the asymptotic certainty equivalent loss γ_0 we refer to (Kallsen and Muhle-Karbe, 2013, section 3.2 (i)). A large no-trade region does not automatically imply a large certainty equivalent due to the averaging factor $d[S]_t$ that is small in calm times of the market. Conversely, close tracking does not imply a small certainty equivalent loss.

3.5. Heuristics

3.5.1. Ansatz for value process

We now look for a strategy φ^ε with $\varphi_0^\varepsilon = \varphi_0^* = x^S$ that is almost optimal in the sense

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E}(U(V_T^\varepsilon(\varphi))) = \mathbb{E}(U(V_T^\varepsilon(\varphi^\varepsilon))) + o(\varepsilon^\delta),$$

where $\delta \in (0, 1)$ has to be determined. For an admissible strategy φ , we define

$$v_t(\varphi) := \mathbb{E} \left(U \left(V_t^\varepsilon(\varphi) + \int_t^T \varphi_s^* dS_s \right) \middle| \mathcal{F}_t \right)$$

which is the expected utility at time t of a strategy that coincides with φ up to time t in the market with transaction costs and switches to the optimal strategy φ^* in the market without transactions costs after time t . Although we use the same notation for it, this is not completely the same as the value process of the frictionless problem from section 2.4.2, but we observe that

$$\begin{aligned} v_t(\varphi) &= U(V_t^\varepsilon(\varphi)) \mathbb{E} \left(\exp \left(-p \int_t^T \varphi_s^* dS_s \right) \middle| \mathcal{F}_t \right) \\ &= G(t, V_t^\varepsilon(\varphi)) \end{aligned}$$

for the function $G : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_{\leq 0}$ from section 2.4.3 that has the terminal condition $G(T, y) = U(y)$ for all $y \in \mathbb{R}$. So $v_t(\varphi)$ can be understood as the value process in the frictionless market at time t with frictional portfolio process $V_t^\varepsilon(\varphi)$. Hence one could call it *semi-frictionless* value process. Since $(v(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ does not consider the remaining trading time in the frictional market, it will not be a good approximation

to the value process $(v^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ at costs. But we suppose that our ansatz $(\tilde{v}^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ for the value process in the presence of transaction costs has a similar dependence structure on time and current wealth and also depends on the current stock position, more specifically on the difference to the frictionless optimizer $\Delta\varphi_t := \varphi_t - \varphi_t^*$:

$$\tilde{v}_t^\varepsilon(\varphi) = \tilde{G}^\varepsilon(t, V_t^\varepsilon(\varphi), \Delta\varphi_t)$$

with $\tilde{G}^\varepsilon : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\leq 0}$. By definition of the value process, \tilde{G}^ε should almost fulfill the terminal condition $\tilde{G}^\varepsilon(\omega, T, y, \delta) = U(y)$ for all $y, \delta \in \mathbb{R}$ and

$$\tilde{G}^\varepsilon(\omega, t, y, \delta) \geq \sup_{\tilde{\delta} \in \mathbb{R}} \tilde{G}^\varepsilon(\omega, t, y - \varepsilon, \tilde{\delta}) \text{ for all } y, \delta \in \mathbb{R}.$$

Furthermore, we suppose that there exists a random and time dependent interval $NT^\varepsilon = [\varphi^* + \Delta\varphi^-, \varphi^* + \Delta\varphi^*]$ around the frictionless optimizer called *no-trade region* where it is (almost) optimal for the investor to stay inactive. In the complement of this no-trade region we suppose that an immediate bulk trade to the frictionless optimizer is the best choice to do. Therefore we impose on \tilde{G}^ε

- the terminal condition

$$\tilde{G}^\varepsilon(\omega, T, y, \delta) \approx U(y) \text{ for all } y, \delta \in \mathbb{R}$$

and the following conditions that are supposed to hold for all $(\omega, t) \in \Omega \times [0, T]$ and $y \in \mathbb{R}$:

- Monotonicity and optimality of the frictionless optimizer: for $\delta_1, \delta_2 \in \mathbb{R}$ with $|\delta_1| \leq |\delta_2|$

$$\tilde{G}^\varepsilon(\omega, t, y, \delta_1) \geq \tilde{G}^\varepsilon(\omega, t, y, \delta_2)$$

and thus

$$\tilde{G}^\varepsilon(\omega, t, y, 0) = \sup_{\delta \in \mathbb{R}} \tilde{G}^\varepsilon(\omega, t, y, \delta). \quad (3.5.3)$$

- Relation between trade and no-trade region:

$$\tilde{G}^\varepsilon(\omega, t, y, \delta) = \tilde{G}^\varepsilon(\omega, t, y - \varepsilon, 0) \text{ for } \delta \in (NT^\varepsilon(\omega, t))^c. \quad (3.5.4)$$

- Smoothness: we assume that $\delta \mapsto \tilde{G}^\varepsilon(\omega, t, y, \delta)$ is twice continuously differentiable on $\mathbb{R} \setminus \partial NT^\varepsilon(\omega, t)$ and continuously differentiable on \mathbb{R} . This implies that we have a smooth fit at the boundaries of the no-trade region:

$$\begin{aligned} \lim_{\delta \nearrow \varphi_t^*(\omega) + \Delta\varphi_t^-(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^\varepsilon(\omega, t, y, \delta) &= \lim_{\delta \searrow \varphi_t^*(\omega) + \Delta\varphi_t^-(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^\varepsilon(\omega, t, y, \delta), \\ \lim_{\delta \nearrow \varphi_t^*(\omega) + \Delta\varphi_t^+(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^\varepsilon(\omega, t, y, \delta) &= \lim_{\delta \searrow \varphi_t^*(\omega) + \Delta\varphi_t^+(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^\varepsilon(\omega, t, y, \delta). \end{aligned}$$

- Symmetry around 0:

$$\begin{aligned} \Delta\varphi^- &= -\Delta\varphi^+ \\ \tilde{G}^\varepsilon(\omega, t, y, -\delta) &= \tilde{G}^\varepsilon(\omega, t, y, \delta) \text{ for all } \delta \in \mathbb{R}. \end{aligned}$$

3. Fixed transaction costs

The optimality of φ^* in (3.5.3) implies that

$$0 = \frac{\partial}{\partial \delta} \widetilde{G}^\varepsilon(\omega, t, y, 0) \quad \text{for all } y \in \mathbb{R}.$$

Together with the assumed smoothness and (3.5.4) we obtain

$$0 = \frac{\partial}{\partial \delta} \widetilde{G}^\varepsilon(\omega, t, y, \varphi_t^*(\omega) \pm \Delta\varphi_t^\pm(\omega)).$$

We now transform the ansatz $\widetilde{v}^\varepsilon$. We use

$$\widetilde{CE}_t^\varepsilon(\varphi) = CE_t(\varphi) + F_t^{CE}(\varphi)$$

with a suitable correction term $F^{CE}(\varphi)$ as an approximation to the certainty equivalent process

$$CE_t^\varepsilon(\varphi) = U^{-1}(v_t^\varepsilon(\varphi))$$

of the true value process $v^\varepsilon(\varphi)$. Here $CE_t(\varphi) = U^{-1}(v_t(\varphi))$ denotes the certainty equivalent of the semi-frictionless value process. The correction term $F^{CE}(\varphi)$ is supposed to be of the form

$$F_t^{CE}(\varphi) = \widetilde{g}^\varepsilon(t, \Delta\varphi_t)$$

for some function $\widetilde{g}^\varepsilon : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_{\leq 0}$. Then we work with

$$\widetilde{v}_t^\varepsilon(\varphi) = U(\widetilde{CE}_t^\varepsilon(\varphi))$$

as a candidate for the value process. The conditions for $\widetilde{G}^\varepsilon$ can be transformed in conditions for $\widetilde{g}^\varepsilon$:

$$\widetilde{g}^\varepsilon(\omega, T, \delta) = 0, \tag{3.5.5}$$

$$\widetilde{g}^\varepsilon(\omega, t, 0) = \sup_{\delta \in \mathbb{R}} \widetilde{g}^\varepsilon(\omega, t, \delta), \tag{3.5.6}$$

$$\widetilde{g}^\varepsilon(\omega, t, \delta) = \widetilde{g}^\varepsilon(\omega, t, 0) - \varepsilon \quad \text{for all } \delta \in (\text{NT}^\varepsilon(\omega, t))^c, \tag{3.5.7}$$

$$0 = \frac{\partial}{\partial \delta} \widetilde{g}^\varepsilon(\omega, t, 0) = \frac{\partial}{\partial \delta} \widetilde{g}^\varepsilon(\omega, t, \varphi_t^*(\omega) \pm \Delta\varphi_t^\pm(\omega)), \tag{3.5.8}$$

$$\widetilde{g}^\varepsilon(\omega, t, -\delta) = \widetilde{g}^\varepsilon(\omega, t, \delta) \quad \text{for all } \delta \in \mathbb{R}. \tag{3.5.9}$$

We see that all requirements are essentially met if we choose a polynomial of order 4 with stochastic coefficients inside the no-trade region³

$$\widetilde{g}^\varepsilon(\omega, t, \delta) = \alpha_t(\omega)\delta^4 - \beta_t(\omega)\delta^2 - \gamma_t(\omega) \quad \text{for } \delta \in \text{NT}^\varepsilon(\omega, t),$$

where the coefficient β has to fulfill $\beta^2 = 4\alpha\varepsilon$. In the following, the coefficients α, β and γ are assumed to be nonnegative Itô processes. The terminal condition (3.5.5) implies that we should choose $\gamma_T = 0$. Then $-\varepsilon \leq F_T^{CE}(\varphi) \leq 0$, i.e. $F_T^{CE}(\varphi)$ is almost equal to 0. We now see that we can write

$$F_t^{CE}(\varphi) = f^{CE}(\alpha_t, \Delta\varphi_t) - \gamma_t$$

³Similar polynomials also occur in (Soner and Touzi, 2013, section 4.1), (Altarovici et al., 2015a, section 3.5) and (Cai et al., 2015, Example 4.5).

with the deterministic function

$$f^{CE}(a, \delta) = \begin{cases} a\delta^4 - 2\sqrt{\varepsilon a}\delta^2, & \text{if } |\delta| \leq \delta^+(a), \\ -\varepsilon, & \text{otherwise,} \end{cases}$$

where $(a, \delta) \in \mathbb{R}_{>0} \times \mathbb{R}$, and $\delta^+(a) = \sqrt[4]{\frac{\varepsilon}{a}}$ being the positive solution of

$$a\delta^4 - 2\sqrt{\varepsilon a}\delta^2 + \varepsilon = 0.$$

This function f^{CE} is smooth enough for the version of Itô's formula from Theorem B.1.1. Now, the boundaries of the no-trade region NT^ε are given by $\varphi^* - \delta^+(\alpha)$ and

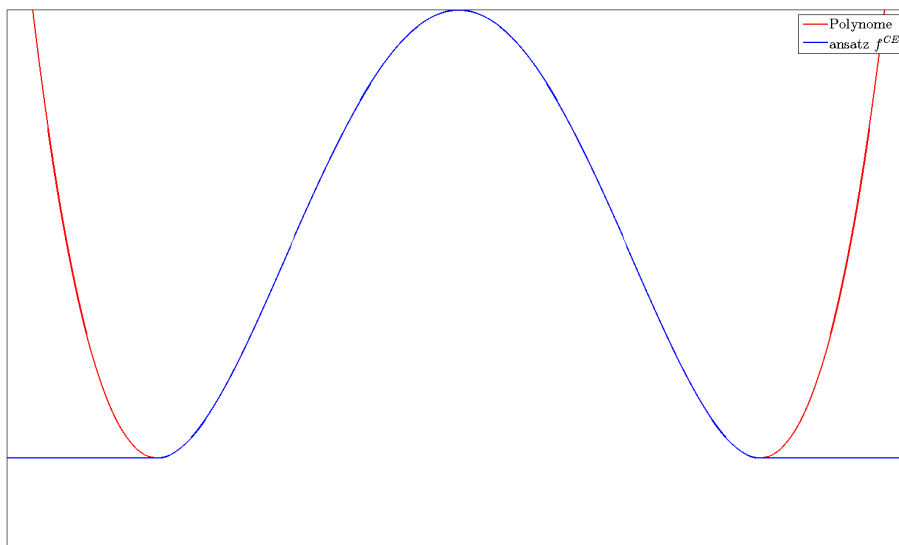


Figure 3.1.: Correction term f^{CE} as a function of the deviation from the frictionless target

$\varphi^* + \delta^+(\varphi)$. Recall that φ^ε always lies in the interval NT^ε and jumps to the frictionless optimizer φ^* once φ^ε hits the boundaries of NT^ε . This strategy implies an indifference principle for the approximated certainty equivalent $\widetilde{CE}^\varepsilon(\varphi^\varepsilon)$: Each bulk trade reduces the process $U^{-1}(v(\varphi^\varepsilon))$ by ε , but trading at the boundaries increases $F^{CE}(\varphi^\varepsilon)$ by ε . Thus, there are no jumps in $\widetilde{CE}^\varepsilon(\varphi^\varepsilon)$ and hence $\widetilde{v}^\varepsilon(\varphi^\varepsilon)$ when following φ^ε .

3.5.2. Dynamics

We are left with the determination of the processes α and γ . Here, the supermartingale/martingale characterization of the true value process $(v^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ from section 3.3 comes into play. Using this principle of dynamic programming we want to achieve that $\widetilde{v}^\varepsilon(\varphi)$ is asymptotically a martingale for the candidate φ^ε - i.e. a martingale up to some *small* drift terms that tend to 0 when the transactions costs ε tend to 0 - and asymptotically a supermartingale for each admissible strategy in the same sense.

3. Fixed transaction costs

For that purpose we have to compute the drift and martingale parts of the involved processes. Due to the exponential utility function we see that

$$v_t(\varphi^\varepsilon) = \underbrace{\exp(-p(V_t^\varepsilon(\varphi^\varepsilon) - V_t(\varphi^*)))}_{:=X_t(\varphi^\varepsilon)} \underbrace{\mathbb{E}(U(V_T(\varphi^*))|\mathcal{F}_t)}_{:=M_t}.$$

Since $U' = (-p)U$ holds, the process M is the density process of the EMM Q up to a negative constant. The process $\tilde{v}^\varepsilon(\varphi^\varepsilon)$ can be rewritten as

$$\tilde{v}_t^\varepsilon(\varphi^\varepsilon) = v_t(\varphi^\varepsilon)F_t^U(\varphi) = M_tX_t(\varphi^\varepsilon)F_t^U(\varphi) = M_t\tilde{X}_t(\varphi^\varepsilon)$$

with

$$F_t^U(\varphi) := \exp(-pF_t^{CE}(\varphi)) \quad \text{and} \quad \tilde{X}_t(\varphi) := X_t(\varphi^\varepsilon)F_t^U(\varphi).$$

In order for $\tilde{v}^\varepsilon(\varphi)$ to be a P -martingale (or a P -supermartingale) asymptotically, the process $\tilde{X}(\varphi)$ needs to be an asymptotic Q -martingale (or Q -submartingale). For reasons of simplicity we will work with $\tilde{X}(\varphi)$. Since all processes fulfill the assumptions of Theorem B.1.1, we have

$$\begin{aligned} \tilde{X}_t(\varphi) &= \tilde{X}_0(\varphi) + \int_0^t \tilde{X}_s(\varphi) \left((-p)b_s^{F^{CE}(\varphi),Q} + \frac{p^2}{2} c_{\int_0^s \Delta\varphi_u dS_u + F^{CE}(\varphi), \int_0^s \Delta\varphi_u dS_u + F^{CE}(\varphi)} \right) ds \\ &\quad + \int_0^t \tilde{X}_s(\varphi) (-p) \left(\Delta\varphi_s dS_s + dM_s^{F^{CE}(\varphi),Q} \right) + \sum_{0 \leq s < t} \Delta^+ \tilde{X}_s(\varphi). \end{aligned} \quad (3.5.10)$$

For the Q -dynamics of the candidate's correction term $F^{CE}(\varphi^\varepsilon)$ we obtain by the version of Itô's formula allowing for left continuous jumps from remark B.1.2

$$\begin{aligned} F_t^{CE}(\varphi^\varepsilon) &= F_0^{CE}(\varphi^\varepsilon) + \int_0^t [(\Delta\varphi_s^\varepsilon)^4 b_s^{\alpha,Q} + (\Delta\varphi_s^\varepsilon)^3 ((-4)(\alpha_s b_s^{\varphi^*,Q} + c_s^{\alpha,\varphi^*})) \\ &\quad + (\Delta\varphi_s^\varepsilon)^2 ((-1)b_s^{\beta,Q} + 6\alpha_s (c_s^{\varphi^*,\varphi^*})) + (\Delta\varphi_s^\varepsilon) (2\beta_s b_s^{\varphi^*,Q} + 2c_s^{\beta,\varphi^*}) \\ &\quad - (b_s^{\gamma,Q} + \beta_s c_s^{\varphi^*,\varphi^*})] ds \\ &\quad + \int_0^t (\Delta\varphi_s^\varepsilon)^4 dM_s^{\alpha,Q} - \int_0^t (\Delta\varphi_s^\varepsilon)^2 dM_s^{\beta,Q} - \int_0^t (4\alpha_s (\Delta\varphi_s^\varepsilon)^3 - 2\beta_s \Delta\varphi_s^\varepsilon) dM_s^{\varphi^*,Q} \\ &\quad - M_t^{\gamma,Q} + \sum_{0 \leq s < t} \Delta^+ F_s^{CE}(\varphi^\varepsilon). \end{aligned}$$

Some computations show that

$$\begin{aligned} &c_s^{F^{CE}(\varphi^\varepsilon), F^{CE}(\varphi^\varepsilon)} \\ &= (\Delta\varphi_s^\varepsilon)^8 c_s^{\alpha,\alpha} - (\Delta\varphi_s^\varepsilon)^7 8\alpha_s c_s^{\alpha,\varphi^*} + (\Delta\varphi_s^\varepsilon)^6 (16\alpha_s^2 c_s^{\varphi^*,\varphi^*} - 2c_s^{\alpha,\beta}) \\ &\quad + (\Delta\varphi_s^\varepsilon)^5 (2\beta_s c_s^{\alpha,\varphi^*} + 8\alpha_s c_s^{\beta,\varphi^*}) + (\Delta\varphi_s^\varepsilon)^4 (c_s^{\beta,\beta} - 16\alpha_s \beta_s c_s^{\varphi^*,\varphi^*} - 2c_s^{\alpha,\gamma}) \\ &\quad + (\Delta\varphi_s^\varepsilon)^3 ((-4)\beta_s c_s^{\beta,\varphi^*} + 8\alpha_s c_s^{\varphi^*,\gamma}) + (\Delta\varphi_s^\varepsilon)^2 (4\beta_s^2 c_s^{\varphi^*,\varphi^*} + 2c_s^{\beta,\gamma}) \\ &\quad + (\Delta\varphi_s^\varepsilon) ((-2)\beta_s c_s^{\varphi^*,\gamma}) + c_s^{\gamma,\gamma} \end{aligned}$$

and

$$\begin{aligned} &\Delta\varphi_s^\varepsilon c_s^{S, F^{CE}(\varphi)} \\ &= (\Delta\varphi_s^\varepsilon)^5 c_s^{S,\alpha} - (\Delta\varphi_s^\varepsilon)^4 \alpha_s c_s^{S,\varphi^*} - (\Delta\varphi_s^\varepsilon)^3 c_s^{S,\beta} + (\Delta\varphi_s^\varepsilon)^2 2\beta_s c_s^{S,\varphi^*} - \Delta\varphi_s^\varepsilon c_s^{S,\gamma}. \end{aligned}$$

Thus, we can compute the Q -dynamics of $\tilde{X}(\varphi^\varepsilon)$ from (3.5.10) and rearrange the terms:

$$\begin{aligned}
\tilde{X}_t(\varphi^\varepsilon) &= \tilde{X}_0(\varphi^\varepsilon) + \int_0^t \tilde{X}_s(\varphi^\varepsilon) \left[(\Delta\varphi_s^\varepsilon)^8 \frac{p^2}{2} c_s^{\alpha,\alpha} - (\Delta\varphi_s^\varepsilon)^7 p^2 4\alpha_s c_s^{\alpha,\varphi^*} \right. \\
&\quad + (\Delta\varphi_s^\varepsilon)^6 p^2 (8\alpha_s^2 c_s^{\varphi^*,\varphi^*} - c_s^{\alpha,\beta}) \\
&\quad + (\Delta\varphi_s^\varepsilon)^5 (p^2 (\beta_s c_s^{\alpha,\varphi^*} + 4\alpha_s c_s^{\beta,\varphi^*}) + p^2 c_s^{S,\alpha}) \\
&\quad + (\Delta\varphi_s^\varepsilon)^4 \left((-p) b_s^{\alpha,Q} + \frac{p^2}{2} (c_s^{\beta,\beta} - 16\alpha_s \beta_s c_s^{\varphi^*,\varphi^*} - 2c_s^{\alpha,\gamma}) - p^2 \alpha_s c_s^{S,\varphi^*} \right) \\
&\quad + (\Delta\varphi_s^\varepsilon)^3 (p^2 ((-2)\beta_s c_s^{\beta,\varphi^*} + 4\alpha_s c_s^{\varphi^*,\gamma}) - p^2 c_s^{S,\beta} + 4p(\alpha_s b_s^{\varphi^*,Q} + c_s^{\alpha,\varphi^*})) \\
&\quad + (\Delta\varphi_s^\varepsilon)^2 \left(p (b_s^{\beta,Q} - 6\alpha_s c_s^{\varphi^*,\varphi^*}) + \frac{p^2}{2} c_s^{S,S} + \frac{p^2}{2} (4\beta_s^2 c_s^{\varphi^*,\varphi^*} + 2c_s^{\beta,\gamma}) + 2p^2 \beta_s c_s^{S,\varphi^*} \right) \\
&\quad + (\Delta\varphi_s^\varepsilon) ((-p)(2\beta_s b_s^{\varphi^*,Q} + 2c_s^{\beta,\varphi^*}) - p^2 (\beta_s c_s^{\varphi^*,\gamma}) + p c_s^{S,\gamma}) \\
&\quad + p(b_s^{\gamma,Q} + \beta_s c_s^{\varphi^*,\varphi^*}) + \frac{p^2}{2} c_s^{\gamma,\gamma} \Big] ds \\
&\quad + \int_0^t \tilde{X}_s(\varphi^\varepsilon) (-p) \left[dM_s^{F^{CE}(\varphi),Q} + (-p)\Delta\varphi_s^\varepsilon dS_s \right]. \tag{3.5.11}
\end{aligned}$$

3.5.3. Identifying of the correct scaling

We adapt the heuristic derivation of the correct scaling in (Altarovici et al., 2015a, Section 3.3) for our purposes: as pointed out by Altarovici et al. (2015a) in the case of fixed costs and Janeček and Shreve (2004); Rogers (2004) for proportional costs, the utility loss due to transaction costs can be divided into two effects: the displacement loss due to deviations from the frictionless target and the direct costs that have to be paid when trading. According to Taylor's formula, sufficient differentiability implies that for small δ

$$\begin{aligned}
\tilde{G}^\varepsilon(\omega, t, y, \delta) &\approx \tilde{G}^\varepsilon(\omega, t, y, 0) + \frac{\partial}{\partial \delta} \tilde{G}^\varepsilon(\omega, t, y, 0) \delta + \frac{\partial^2}{\partial \delta^2} \tilde{G}^\varepsilon(\omega, t, y, 0) \delta^2 \\
&= \tilde{G}^\varepsilon(\omega, t, y, 0) + \frac{\partial}{\partial \delta^2} \tilde{G}^\varepsilon(\omega, t, y, 0) \delta^2.
\end{aligned}$$

Thus, an admissible strategy with $|\Delta\varphi| \leq \delta$ should cause a displacement loss of order δ^2 .

If we assume that this strategy φ trades to the frictionless optimizer when $\Delta\varphi$ hits the boundaries $-\delta$ and δ , the times between trades can be approximated by the first exit time of a Brownian motion from the interval $[-\delta, \delta]$ for sufficiently small δ . This first exit time scales with δ^2 : Let τ denote this first exit time, i.e.

$$\tau = \inf\{t > 0 : |W_t| \geq \delta\}$$

3. Fixed transaction costs

for some standard Brownian motion W . We observe that we can rewrite τ as

$$\begin{aligned}\tau &= \inf \left\{ t > 0 : \left| \frac{W_t}{\delta} \right| \geq 1 \right\} \\ &= \inf \left\{ s\delta^2 : s > 0, \left| \frac{W_{\delta^2 s}}{\delta} \right| \geq 1 \right\} \\ &= \delta^2 \inf \left\{ s > 0 : \left| \widehat{W}_s \right| \geq 1 \right\},\end{aligned}$$

where $\widehat{W} = (\delta^{-1}W_{\delta^2 s})_{s \geq 0}$ is also a standard Brownian motion. After each transaction this procedure is repeated. Thus, the number of transactions should scale with $1/\delta^2$. Summing up both effects shows that the total effect of fixed transaction costs should be of the order

$$K\delta^2 + \frac{\varepsilon}{\delta^2}$$

for some constant K . As in Altarovici et al. (2015a) minimizing this with respect to δ leads to an optimal value of $\delta = \varepsilon^{1/4}$ and a total effect of $\varepsilon^{1/2}$.

3.5.4. Derivation of the optimal strategy

From the above derivations we expect that the halfwidth of the no-trade region is of the order $O(\varepsilon^{1/4})$ and the utility loss due to transaction costs is of the order $O(\varepsilon^{1/2})$. So in (3.5.11), we neglect the terms scaling with $(\Delta\varphi_s^\varepsilon)^k$, where k is strictly larger than 2. The terms scaling with $(\Delta\varphi_s^\varepsilon)^2$ contain the local drift of the process $X(\varphi^\varepsilon)$. In order to obtain the order $o(\varepsilon^{1/2})$ in these terms when following φ^ε (or being in the no-trade region respectively) this suggests

$$(-p)6\alpha_s c_s^{\varphi^*, \varphi^*} + \frac{p^2}{2} c_s^{S,S} = 0, \quad (3.5.12)$$

which leads to

$$\alpha_t = \frac{p}{12} \frac{c_t^{S,S}}{c_t^{\varphi^*, \varphi^*}}$$

and from the definitions of β and $\Delta\varphi^+$

$$\begin{aligned}\Delta\varphi_t^+ &= \sqrt[4]{\frac{\varepsilon}{\alpha_t}} = \sqrt[4]{\frac{12\varepsilon}{p}} \sqrt[4]{\frac{c_t^{\varphi^*, \varphi^*}}{c_t^{S,S}}}, \\ \beta_t &= 2(\Delta\varphi_t^+)^2 \alpha_t = \sqrt{\frac{\varepsilon p}{3}} \sqrt{\frac{c_t^{S,S}}{c_t^{\varphi^*, \varphi^*}}}.\end{aligned} \quad (3.5.13)$$

Assuming that γ , b^γ and $c^{\cdot\gamma}$ are of order $O(\varepsilon^{1/2})$ we see that all remaining terms except for $(b_s^{\gamma,Q} + \beta_s c_s^{\varphi^*, \varphi^*})$ are of order $o(\varepsilon^{1/2})$. Thus, we require

$$b_s^{\gamma,Q} = -\beta_s c_s^{\varphi^*, \varphi^*} = -\sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*, \varphi^*} c_s^{S,S}}. \quad (3.5.14)$$

By assumption γ has the representation

$$\gamma_t = \gamma_0 + \int_0^t b_s^{\gamma, Q} ds + M_t^{\gamma, Q},$$

where $M^{\gamma, Q}$ is a continuous Q -local martingale with $M_0^{\gamma, Q} = 0$. The condition $\gamma_T = 0$ and (3.5.14) give us

$$\begin{aligned} \gamma_t &= \gamma_0 + \int_0^t b_s^{\gamma, Q} ds + M_t^{\gamma, Q} \\ &= \gamma_0 - \int_0^t \left(\sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*, \varphi^*} c_s^{S, S}} \right) ds + M_t^{\gamma, Q} - \gamma_T \\ &= \gamma_0 - \int_0^t \left(\sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*, \varphi^*} c_s^{S, S}} \right) ds + M_t^{\gamma, Q} \\ &\quad - \left(\gamma_0 - \int_0^T \left(\sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*, \varphi^*} c_s^{S, S}} \right) ds + M_T^{\gamma, Q} \right) \\ &= \int_t^T \left(\sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*, \varphi^*} c_s^{S, S}} \right) ds + M_t^{\gamma, Q} - M_T^{\gamma, Q}. \end{aligned}$$

If $M^{\gamma, Q}$ is a Q -martingale we end up with

$$\gamma_t = \sqrt{\frac{\varepsilon p}{3}} \mathbb{E}_Q \left(\int_t^T \sqrt{c_s^{\varphi^*, \varphi^*} c_s^{S, S}} ds \middle| \mathcal{F}_t \right).$$

All processes have been determined and Q -drift coefficient of $\tilde{X}(\varphi^\varepsilon)$ is now given by

$$\begin{aligned} &b_s^{\tilde{X}(\varphi^\varepsilon), Q} \\ &= \tilde{X}_s(\varphi^\varepsilon) \left[(\Delta\varphi_s^\varepsilon)^8 \frac{p^2}{2} c_s^{\alpha, \alpha} - (\Delta\varphi_s^\varepsilon)^7 p^2 4\alpha_s c_s^{\alpha, \varphi^*} \right. \\ &\quad + (\Delta\varphi_s^\varepsilon)^6 p^2 (8\alpha_s^2 c_s^{\varphi^*, \varphi^*} - c_s^{\alpha, \beta}) + (\Delta\varphi_s^\varepsilon)^5 p^2 (\beta_s c_s^{\alpha, \varphi^*} + 4\alpha_s c_s^{\beta, \varphi^*} + c_s^{S, \alpha}) \\ &\quad + (\Delta\varphi_s^\varepsilon)^4 \left((-p)b_s^{\alpha, Q} + \frac{p^2}{2} (c_s^{\beta, \beta} - 16\alpha_s \beta_s c_s^{\varphi^*, \varphi^*} - 2c_s^{\alpha, \gamma}) - p^2 \alpha_s c_s^{S, \varphi^*} \right) \\ &\quad + (\Delta\varphi_s^\varepsilon)^3 (p^2 ((-2)\beta_s c_s^{\beta, \varphi^*} + 4\alpha_s c_s^{\varphi^*, \gamma}) - p^2 c_s^{S, \beta} + 4p(\alpha_s b_s^{\varphi^*, Q} + c_s^{\alpha, \varphi^*})) \\ &\quad + (\Delta\varphi_s^\varepsilon)^2 \left(p b_s^{\beta, Q} + \frac{p^2}{2} (4\beta_s^2 c_s^{\varphi^*, \varphi^*} + 2c_s^{\beta, \gamma}) + 2p^2 \beta_s c_s^{S, \varphi^*} \right) \\ &\quad \left. + (\Delta\varphi_s^\varepsilon) ((-p)(2\beta_s b_s^{\varphi^*, Q} + 2c_s^{\beta, \varphi^*}) - p^2 \beta_s c_s^{\varphi^*, \gamma} + p c_s^{S, \gamma}) + \frac{p^2}{2} c_s^{\gamma, \gamma} \right] \\ &=: \tilde{X}_s(\varphi^\varepsilon) \sum_{i=0}^8 (\Delta\varphi_s^\varepsilon)^i H_s^i. \end{aligned} \tag{3.5.15}$$

With the ansatz $(\tilde{v}^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$ for the value process we can do parts of the verification of the almost optimality of the candidate strategy φ^ε . However, problems occur when considering competing strategies from \mathfrak{A}^ε .

3.5.5. Modification of the value process

It will turn out that we can only compute an asymptotic upper bound for the maximal expected utility on a smaller set of competing strategies due to the asymptotic error terms introduced by our ansatz $\tilde{v}^\varepsilon(\varphi)$ and $\tilde{X}(\varphi)$, respectively. To overcome this problem we use the following modifications: Define

$$\hat{X}(\varphi) := X(\varphi) \exp\left(-p\hat{F}^{CE}(\varphi)\right),$$

where

$$\hat{F}_t^{CE}(\varphi) := \hat{f}^{CE}(\hat{\alpha}_t, \Delta\varphi_t) - \gamma_t + \delta_t(\varphi)$$

and

$$\delta_t(\varphi) := c_1\varepsilon^{1/2} \exp(-c_2(\Delta V_t^\varepsilon(\varphi))) \wedge \gamma_t.$$

The constants $c_1, c_2 > 0$ will be powers of ε

$$c_1 = c_{1,\varepsilon} = \varepsilon^{\eta_1} \quad \text{for } \eta_1 > 0,$$

$$c_2 = c_{2,\varepsilon} = \varepsilon^{\eta_2} \quad \text{for } \eta_2 < 0$$

and have to be determined explicitly in the end. The idea behind this approach is the following: seen as a function of the difference of portfolio values $\Delta V_t^\varepsilon(\varphi)$, the new ansatz $\hat{v}^\varepsilon(\varphi)$ will be close to the old one $\tilde{v}^\varepsilon(\varphi)$, but at some point, when the portfolio process is too negative compared to the portfolio value of the frictionless optimizer, it will almost coincide with the value process for the frictionless optimization problem. This level of portfolio value (or more precisely portfolio value difference to

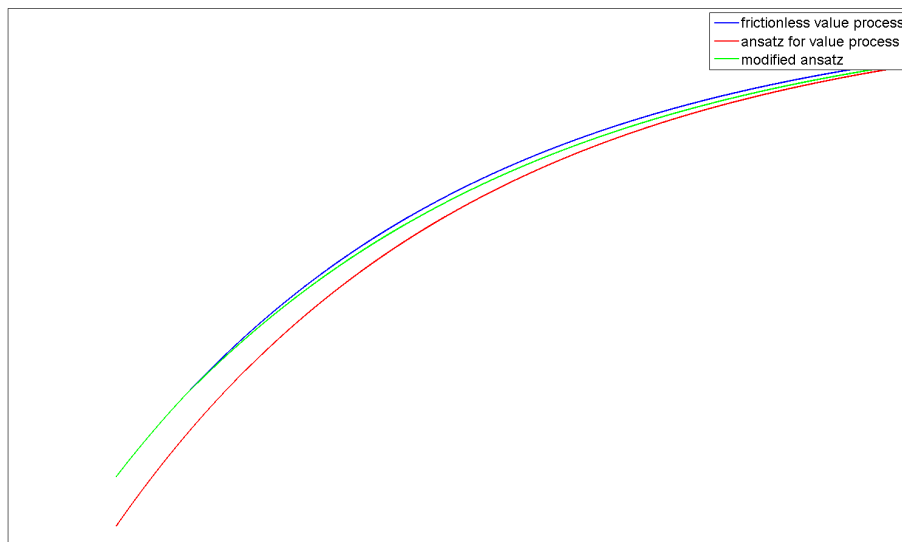


Figure 3.2.: Value processes as functions of difference in portfolio values

(the frictionless optimizer) will then be a stochastic process, but, as we will see in

3.6. Existence of involved processes and regularity assumptions

section 3.7.2, its dependence on ε can only come from the constants c_1 and c_2 due to the scaling factor $\varepsilon^{1/2}$ in $\delta_t(\varphi)$.

Since $\delta(\varphi)$ introduces new jumps terms we have to modify f^{CE} , α and β in a way that the jumps of $\widehat{X}(\varphi)$ stay positive. We choose

$$\widehat{f}^{CE}(a, \delta) = \begin{cases} a\delta^4 - b\delta^2, & \text{if } |\delta| \leq \widehat{\delta}^+(a), \\ -\bar{c}_1\varepsilon, & \text{otherwise,} \end{cases}$$

with $\bar{c}_1 = 1 - c_1$, $b^2 = 4a\bar{c}_1\varepsilon$ and $\widehat{\delta}^+(a) = \sqrt[4]{\frac{\bar{c}_1\varepsilon}{a}}$ as the positive solution of $a\delta^4 - 2\sqrt{a\bar{c}_1\varepsilon}\delta^2 + \bar{c}_1\varepsilon = 0$. Using the same calculations as before leads to

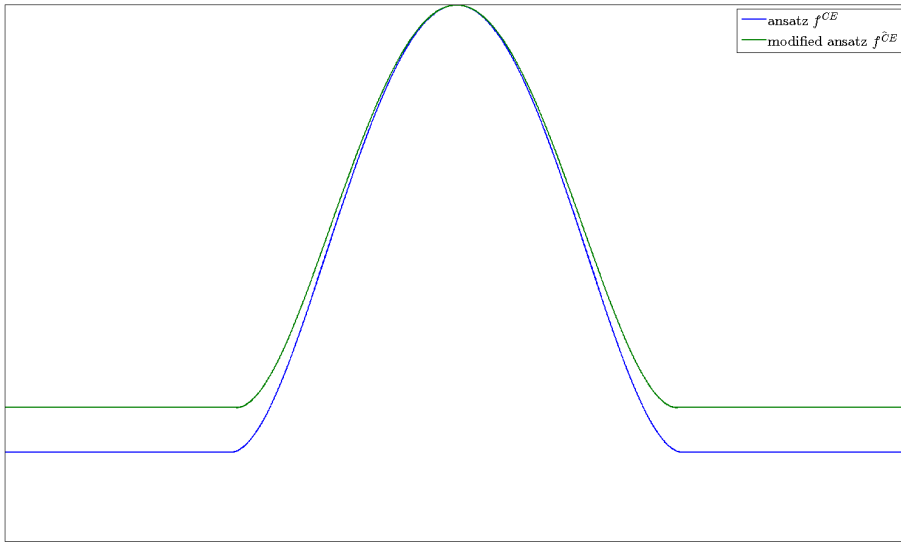


Figure 3.3.: Comparison of error terms f^{CE} and its modified version \widehat{f}^{CE}

$$\widehat{\alpha} = \alpha, \quad \Delta\widehat{\varphi}^+ = \sqrt[4]{\frac{\bar{c}_1\varepsilon}{\alpha}}, \quad \widehat{\beta} = \sqrt{\bar{c}_1 \frac{\varepsilon p}{3}} \sqrt{\frac{c^{S,S}}{c^{\varphi^*, \varphi^*}}}. \quad (3.5.16)$$

As one can see from (3.5.16) and figure 3.3, the modified halfwidth $\Delta\widehat{\varphi}^+$ is smaller than $\Delta\varphi^+$. Keeping γ as before will introduce an additional drift term that we will deal with in section 3.7.2.

3.6. Existence of involved processes and regularity assumptions

In the following sections we give rigorous proofs for the almost optimality of the candidate φ^ε . Therefore we have to make an existence assumption on the frictionless optimizer φ^* that recalls Assumption 2.4.1, but also requires that the frictionless optimizer φ^* is an Itô process.

3. Fixed transaction costs

Assumption 3.6.1. *We assume that the trading strategy $\varphi^* \in \mathfrak{A}_2$ from Assumption 2.4.1 exists and that this strategy is an Itô process.*

According to Theorem 2.4.4, this strategy φ^* is the frictionless optimizer. Moreover, due to the continuity of φ^* , we have $x^S = \varphi_0^* = \varphi_{0+}^*$. Thus, there is no bulk trade of the frictionless optimizer at the initial point in time.

As a first result we can find an Itô process with the desired Q -local drift (3.5.14) and terminal value 0. Therefore we recall that every martingale has a modification which is right continuous with left hand limits (RCLL) due to (Karatzas and Shreve, 1991, Theorem 3.13).

Proposition 3.6.2. *Define the stochastic process $A := \sqrt{c^{\varphi^*, \varphi^*} c^{S, S}}$ and assume that $\mathbb{E}_Q \left(\int_0^T A_s ds \right) < \infty$. Define*

$$\bar{Y}_t := \mathbb{E}_Q \left(\int_t^T A_s ds \mid \mathcal{F}_t \right), \quad \bar{M}_t^1 := \mathbb{E}_Q \left(\int_0^T A_s ds \mid \mathcal{F}_t \right) - \bar{Y}_0. \quad (3.6.17)$$

Assume that the RCLL modification of the martingale M^1 of \bar{M}^1 is continuous. Then, there exists a modification Y of \bar{Y} which is an Itô process with

$$M_t^{Y, Q} = M_t^1 \quad \text{and} \quad b_t^{Y, Q} = -A_t. \quad (3.6.18)$$

Proof. We see that

$$\begin{aligned} \bar{Y}_t &= \mathbb{E}_Q \left(\int_t^T A_s ds \mid \mathcal{F}_t \right) = \mathbb{E}_Q \left(\int_0^T A_s ds - \int_0^t A_s ds \mid \mathcal{F}_t \right) \\ &= \mathbb{E}_Q \left(\int_0^T A_s ds \mid \mathcal{F}_t \right) - \int_0^t A_s ds = \bar{Y}_0 - \int_0^t A_s ds + \bar{M}_t^1. \end{aligned}$$

Now define $Y := \bar{Y}_0 - \int_0^\cdot A_s ds + M^1$. This process is an Itô process and fulfills the desired property. \square

We state an additional assumption on the process Y which will turn out to be useful in subsection 3.7.2.

Assumption 3.6.3. *The random variable $\int_0^T A_s ds$ from Proposition 3.6.2 is supposed to be bounded and the RCLL modification of $\mathbb{E}_Q \left(\int_0^T A_s ds \mid \mathcal{F}_\cdot \right)$ is supposed to be continuous.*

If Assumption 3.6.3 holds, then the process Y is bounded by a constant κ_Y . It is of course fulfilled if the process A is bounded. However, in the application in Chapter 6 we will see an example, where A is unbounded, but Y is bounded. Similar to the work on proportional costs by Kallsen and Li (2013) we need some regularity conditions on the stock price model and its corresponding frictionless optimal strategy. As mentioned above, Assumption 3.6.3 is only required for section 3.7.2, but stating it now will reduce the necessary assumptions on the other involved processes. These assumptions are collected below:

Assumption 3.6.4. Assume that $C := \frac{c^{S,S}}{c^{\varphi^*,\varphi^*}}$ is an Itô process with $C > 0$ and that the following conditions hold:

- $\exp(\pm 8pV_T(\varphi^*)) \in L_1(Q)$,
- $[S]_T, [C]_T, \int_0^T |b_s^{C,Q}| ds \in L_2(Q)$,
- $\|\varphi^*\|_{H_8(Q)} < \infty$,
- $\|C^{-1}\|_{S_4(Q)} < \infty$.

Here $\|\cdot\|_{S_q(Q)}$ denotes the $S_q(Q)$ -norm and $\|\cdot\|_{H_q(Q)}$ denotes the $H_q(Q)$ -norm of Itô processes that are explained in section B.2.

3.7. Upper bound for maximal expected utility

3.7.1. On a subclass of admissible strategies

In this subsection we want to determine an asymptotic upper bound for the maximal expected utility from terminal wealth in the presence of fixed transaction costs. However, using the ansatz $\tilde{v}^\varepsilon(\varphi)$ from the previous section we have to restrict the class of admissible strategies to the set $\mathfrak{A}^{\varepsilon,K} := \{\varphi \in \mathfrak{A}^\varepsilon | V_t^\varepsilon(\varphi) - V_t(\varphi^*) \geq -K \text{ for all } t \in [0, T]\}$ for a given constant $K > 0$.⁴

In the following proposition we want to show that the drift of the value process ansatz of a strategy φ is of small order when the strategy lies in the no-trade region. We define $\gamma := \sqrt{\frac{\varepsilon p}{3}} Y$ and $\alpha, \beta, v(\varphi), F^{CE}(\varphi), F^U(\varphi), \tilde{v}^\varepsilon(\varphi), \tilde{X}(\varphi), H^0, \dots, H^8$ and φ^ε as in the previous section.

Proposition 3.7.1 (No-trade region). *We have*

$$\sup_{\sigma \in \mathcal{T}} \sup_{\varphi \in \mathfrak{A}^{\varepsilon,K}} \mathbb{E}_Q \left(\left| \int_0^\sigma b_s^{\tilde{X}(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\varphi_s^+\}} ds \right| \right) = O(\varepsilon^{3/4}).$$

Proof. (i) By the computations of section 3.5 we know that

$$\alpha_t = \frac{p}{12} C_t \quad \text{and} \quad \beta_t = \sqrt{\frac{\varepsilon p}{3}} \sqrt{C_t}.$$

Applying Itô's formula to $\sqrt{C_t}$ gives us

$$\begin{aligned} \sqrt{C_t} &= \sqrt{C_0} + \int_0^t \frac{1}{2C_s^{1/2}} dC_s - \int_0^t \frac{1}{4C_s^{3/2}} d[C]_s \\ &= \sqrt{C_0} + \int_0^t \frac{1}{2C_s^{1/2}} (b_s^{C,Q} ds + dM_s^{C,Q}) - \int_0^t \frac{1}{4C_s^{3/2}} c_s^{C,C} ds \\ &= \sqrt{C_0} + \int_0^t \left(\frac{1}{2C_s^{1/2}} b_s^{C,Q} - \frac{1}{4C_s^{3/2}} c_s^{C,C} \right) ds + \int_0^t \frac{1}{2C_s^{1/2}} dM_s^{C,Q} \end{aligned}$$

⁴This subclass corresponds to the compact set in (Bichuch, 2012, Theorem 3.5).

3. Fixed transaction costs

So the local drift coefficients and the martingale parts of α and β are given by

$$\begin{aligned} b_t^{\alpha,Q} &= \frac{p}{12} b_t^{C,Q}, & M_t^{\alpha,Q} &= \frac{p}{12} M_t^{C,Q}, \\ b_t^{\beta,Q} &= \sqrt{\frac{\varepsilon p}{3}} \left(\frac{1}{2C_s^{1/2}} b_s^{C,Q} - \frac{1}{4C_s^{3/2}} c_t^{C,C} \right), & M_t^{\beta,Q} &= \int_0^t \frac{\sqrt{\varepsilon p}}{2\sqrt{3}C_s^{1/2}} dM_s^{C,Q}. \end{aligned} \quad (3.7.19)$$

(ii) From line (3.5.10) in section 3.5 we know that the Q -local drift parameter of $\tilde{X}(\varphi)$ is given by

$$b^{\tilde{X}(\varphi),Q} = \tilde{X}_s(\varphi) \left((-p)b_s^{F^{CE}(\varphi),Q} + \frac{p^2}{2} c_{\int_0^s \Delta\varphi_u dS_u + F^{CE}(\varphi), \int_0^s \Delta\varphi_u dS_u + F^{CE}(\varphi)} \right).$$

Furthermore, $X_s(\varphi)$ is bounded by $\exp(pK)$ and $F_s^{CE}(\varphi) \geq -\varepsilon - \gamma_s$. This implies

$$\begin{aligned} & \left| \int_0^\sigma b_s^{\tilde{X}(\varphi),Q} \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\varphi_s^+\}} ds \right| \\ & \leq \int_0^\sigma \tilde{X}_s(\varphi) \left| \sum_{k=0}^8 (\Delta\varphi_s)^k H_{k,s} \right| \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\varphi_s^+\}} ds \\ & \leq \int_0^\sigma \exp(pK) \exp(-pF_s^{CE}(\varphi)) \left(\sum_{k=0}^8 |(\Delta\varphi_s)|^k |H_{k,s}| \right) \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\varphi_s^+\}} ds \\ & \leq \exp(p(K + \varepsilon)) \int_0^\sigma \exp(p\gamma_s) \left(\sum_{k=0}^8 |(\Delta\varphi_s^+)|^k |H_{k,s}| \right) \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\varphi_s^+\}} ds \\ & \leq \exp(p(K + \varepsilon + \sqrt{\varepsilon p/3}\kappa_Y)) \sum_{k=0}^8 \left(\int_0^T |(\Delta\varphi_s^+)|^k |H_{k,s}| ds \right), \end{aligned}$$

which does not depend on φ nor σ . Hence, we only need to show

$$\mathbb{E}_Q \left(\int_0^T |(\Delta\varphi_s^+)|^k |H_{k,s}| ds \right) = O(\varepsilon^{3/4}) \quad \text{for all } k \in \{0, \dots, 8\}, \quad (3.7.20)$$

i.e. that the Q -expectation of the above expression is of order $O(\varepsilon^{3/4})$. For the convenience of the reader, these calculations are skipped here and can be found in section A.1 of the appendix. \square

We now come to the trade region, which is the region where we expect a submartingale behavior of the process $\tilde{X}(\varphi)$.

Proposition 3.7.2 (Trade region). *We have*

$$\inf_{\varphi \in \mathfrak{A}^\varepsilon} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_Q \left(\int_0^\sigma b_s^{\tilde{X}(\varphi),Q} \mathbf{1}_{\{|\Delta\varphi_s| \geq \Delta\varphi_s^+\}} ds \right) \geq o(\varepsilon^{1/2}).$$

Proof. (i) In the trade region $\{|\Delta\varphi_s| > \Delta\varphi_s^+\}$ the function f^{CE} is constant equal to $-\varepsilon$. So the local drift and local quadratic variation of $F^{CE}(\varphi)$ under Q equal those of

3.7. Upper bound for maximal expected utility

$-\gamma$. With this, the representation of $b^{\gamma, Q}$ in (3.5.14) and the corridor $\Delta\varphi^+$ in (3.5.13) we have on $\{|\Delta\varphi_s| > \Delta\varphi_s^+\}$

$$\begin{aligned}
& (-p)b_s^{F^{CE}(\varphi), Q} + \frac{p^2}{2} c_s^{\int_0^\cdot \Delta\varphi_u dS_u + F^{CE}(\varphi), \int_0^\cdot \Delta\varphi_u dS_u + F^{CE}(\varphi)} \\
&= pb_s^{\gamma, Q} + \frac{p^2}{2} c_s^{\int_0^\cdot \Delta\varphi_u dS_u + F^{CE}(\varphi), \int_0^\cdot \Delta\varphi_u dS_u + F^{CE}(\varphi)} \\
&= -p \left(\frac{\varepsilon p}{3}\right)^{1/2} \underbrace{(c_s^{S, S} c_s^{\varphi^*, \varphi^*})^{1/2}}_{=(\Delta\varphi_s^+)^2 \left(\frac{12\varepsilon}{p}\right)^{-1/2} c_s^{S, S}} + \frac{p^2}{2} \left((\Delta\varphi_s)^2 c_s^{S, S} + 2\Delta\varphi_s c_s^{S, F^{CE}(\varphi)} + c_s^{F^{CE}(\varphi), F^{CE}(\varphi)} \right) \\
&= -\frac{p^2}{6} \underbrace{(\Delta\varphi_s^+)^2}_{\leq (\Delta\varphi_s)^2} c_s^{S, S} + \frac{p^2}{2} \left((\Delta\varphi_s)^2 c_s^{S, S} + 2\Delta\varphi_s c_s^{S, F^{CE}(\varphi)} + c_s^{F^{CE}(\varphi), F^{CE}(\varphi)} \right) \\
&\geq \frac{p^2}{3} (\Delta\varphi_s)^2 c_s^{S, S} + p^2 \Delta\varphi_s c_s^{S, F^{CE}(\varphi)} + \frac{p^2}{2} c_s^{F^{CE}(\varphi), F^{CE}(\varphi)} \\
&= p^2 \left(\frac{1}{3} (\Delta\varphi_s)^2 c_s^{S, S} + \Delta\varphi_s c_s^{S, F^{CE}(\varphi)} + \frac{1}{2} c_s^{F^{CE}(\varphi), F^{CE}(\varphi)} \right) \\
&\quad + p^2 \left(\frac{\sqrt{3}}{2} c_s^{F^{CE}(\varphi), F^{CE}(\varphi)} - \frac{\sqrt{3}}{2} c_s^{F^{CE}(\varphi), F^{CE}(\varphi)} \right) \\
&= p^2 \left(c_s^{\frac{1}{\sqrt{3}} \int_0^\cdot \Delta\varphi_u dS_u + \frac{\sqrt{3}}{2} F^{CE}(\varphi)} + \frac{1 - \sqrt{3}}{2} c_s^{F^{CE}(\varphi)} \right) \\
&\geq \frac{1 - \sqrt{3}}{2} p^2 c_s^{\gamma, \gamma}
\end{aligned}$$

using that the quadratic variation of the sum of two processes equals $[X] + [Y] + 2[X, Y]$. This yields a lower bound for the Q -local drift of $\tilde{X}(\varphi)$ outside the no-trade region according to (3.5.10). So we obtain

$$\begin{aligned}
\int_0^\sigma b_s^{\tilde{X}(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| > \Delta\varphi_s^+\}} ds &\geq \underbrace{\frac{1 - \sqrt{3}}{2}}_{< 0} p^2 \int_0^\sigma \tilde{X}_s(\varphi) c_s^{\gamma, \gamma} \mathbf{1}_{\{|\Delta\varphi_s| > \Delta\varphi_s^+\}} ds \\
&\geq \frac{1 - \sqrt{3}}{2} p^2 \exp(p(K + \varepsilon)) \int_0^\sigma \exp(p\gamma_s) c_s^{\gamma, \gamma} ds.
\end{aligned}$$

Analyzing this term yields

$$\begin{aligned}
& \left| \mathbb{E}_Q \left(\frac{1 - \sqrt{3}}{2} p^2 \exp(p(K + \varepsilon)) \int_0^\sigma \exp(p\gamma_s) c_s^{\gamma, \gamma} ds \right) \right| \\
&\leq \frac{\sqrt{3} - 1}{2} p^2 \exp(p(K + \varepsilon)) \mathbb{E}_Q \left(\int_0^T \exp(p\gamma_s) c_s^{\gamma, \gamma} ds \right) \\
&\leq \frac{\sqrt{3} - 1}{2} p^2 \exp(p(K + \varepsilon + \kappa_Y \sqrt{p\varepsilon/3})) \mathbb{E}_Q \left(\int_0^T c_s^{\gamma, \gamma} ds \right) \\
&\leq \frac{\sqrt{3} - 1}{2} p^2 \exp(p(K + 1 + \kappa_Y \sqrt{p\varepsilon/3})) \varepsilon \kappa_Y^2 \frac{p}{3} \mathbb{E}_Q([Y]_T).
\end{aligned}$$

3. Fixed transaction costs

For this expression we see that by Theorem B.2.2

$$\mathbb{E}_Q([Y]_T) \leq \text{const.} \mathbb{E}_Q \left(\sup_{t \in [0, T]} \left(M_t^{Y, Q} \right)^2 \right), \quad (3.7.21)$$

which is finite since $\int_0^T A_s ds$ and thus $M^{Y, Q}$ are bounded, see line (3.6.17). Hence, the term under consideration is of order $O(\varepsilon)$ uniformly in σ and φ .

(ii) At the boundaries of the no-trade region we see that with Theorem B.1.1

$$\begin{aligned} b_s^{\tilde{X}(\varphi), Q} &= \frac{1}{2} \tilde{X}_s(\varphi) \left((-p) b_s^{\alpha(\Delta\varphi)^4 - \beta(\Delta\varphi)^2 - \gamma, Q} + \frac{p^2}{2} c_s^{\int_0^i \Delta\varphi_u dS_u + \alpha(\Delta\varphi)^4 - \beta(\Delta\varphi)^2 - \gamma} \right) \\ &+ \frac{1}{2} \tilde{X}_s(\varphi) \left((-p) b_s^{-\gamma, Q} + \frac{p^2}{2} c_s^{\int_0^i \Delta\varphi_u dS_u - \gamma} \right). \end{aligned}$$

With the proof of Proposition 3.7.1 and part (i) we obtain

$$\inf_{\varphi \in \mathfrak{A}^\varepsilon} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_Q \left(\int_0^\sigma b_s^{\tilde{X}(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| = \Delta\varphi_s^+\}} ds \right) \geq o(\varepsilon^{1/2}).$$

□

We now determine the asymptotic upper bound. The proof will require the above results on the asymptotics of the drift and a localization argument for the local martingale part of $\tilde{X}(\varphi)$ under Q .

Theorem 3.7.3. *We have*

$$\sup_{\varphi \in \mathfrak{A}^{\varepsilon, K}} \mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}).$$

Proof. Let φ an admissible strategy from $\mathfrak{A}^{\varepsilon, K}$. From the definition of $\tilde{v}^\varepsilon(\varphi)$ we obtain

$$\begin{aligned} U(V_T^\varepsilon(\varphi)) &= v_T(\varphi) F_T^U(\varphi) F_T^U(\varphi)^{-1} = \tilde{v}_T^\varepsilon(\varphi) F_T^U(\varphi)^{-1} \\ &= \tilde{v}_T^\varepsilon(\varphi) \exp(p F_T^{CE}(\varphi)) \leq \tilde{v}_T^\varepsilon(\varphi) \exp(-p\varepsilon) \end{aligned}$$

since $\tilde{v}_T^\varepsilon(\varphi)$ is negative. This inequality yields

$$\mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq \mathbb{E}(\tilde{v}_T^\varepsilon(\varphi) \exp(-p\varepsilon)). \quad (3.7.22)$$

Furthermore, we have

$$\mathbb{E}(\tilde{v}_T^\varepsilon(\varphi)) = \mathbb{E}\left(M_T \tilde{X}_T(\varphi)\right) = M_0 \mathbb{E}_Q\left(\tilde{X}_T(\varphi)\right) \quad (3.7.23)$$

and recall that the Q -dynamics of $\tilde{X}(\varphi)$ are given by

$$b_s^{\tilde{X}(\varphi), Q} = \tilde{X}_s(\varphi) \left((-p) b_s^{F^{CE}, Q} + \frac{p^2}{2} \tilde{X}_s(\varphi) c_s^{\int_0^i \Delta\varphi_u dS_u + F^{CE}(\varphi)} \right), \quad (3.7.24)$$

$$M_t^{\tilde{X}(\varphi), Q} = \int_0^t (-p) \tilde{X}_s(\varphi) dM_s^{F^{CE}(\varphi), Q} + \int_0^t (-p) \tilde{X}_s(\varphi) \Delta\varphi_s dS_s. \quad (3.7.25)$$

3.7. Upper bound for maximal expected utility

Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for the local martingale $M^{\tilde{X}(\varphi), Q}$ and define $Y_n := \tilde{X}_{T \wedge \tau_n}(\varphi)$ for all $n \in \mathbb{N}$. Then we have

$$\lim_{n \rightarrow \infty} Y_n = \tilde{X}_T(\varphi)$$

and

$$\begin{aligned} 0 \leq Y_n &= \exp \left(- \underbrace{p(V_{T \wedge \tau_n}^\varepsilon(\varphi) - V_{T \wedge \tau_n}(\varphi^*))}_{\geq -K} + \underbrace{F_{T \wedge \tau_n}^{CE}(\varphi)}_{\geq -\varepsilon - \gamma_{T \wedge \tau_n}} \right) \\ &\leq \exp(p(K + \varepsilon)) \exp(p\gamma_{T \wedge \tau_n}) \\ &\leq \exp(p(K + \varepsilon)) \exp(p\gamma_{T \wedge \tau_n}). \end{aligned}$$

Since Y and thus γ are assumed to be bounded, $\{Y_n, n \in \mathbb{N}\}$ is a bounded set in $L^2(Q)$, which implies that it is uniformly integrable. As a consequence,

$$\mathbb{E}_Q \left(\tilde{X}_T(\varphi) \right) = \lim_{n \rightarrow \infty} \mathbb{E}_Q(Y_n). \quad (3.7.26)$$

For the right-hand side we see that

$$\begin{aligned} \mathbb{E}_Q(Y_n) &= \mathbb{E}_Q \left(\tilde{X}_0(\varphi) + \int_0^{T \wedge \tau_n} b_s^{\tilde{X}(\varphi), Q} ds + M_{T \wedge \tau_n}^{\tilde{X}(\varphi), Q} + \sum_{0 \leq s < T \wedge \tau_n} \Delta^+ \tilde{X}_s(\varphi) \right) \\ &\geq \tilde{X}_0(\varphi) + \mathbb{E}_Q \left(\int_0^{T \wedge \tau_n} b_s^{\tilde{X}(\varphi), Q} ds \right) + \mathbb{E}_Q \left(M_{T \wedge \tau_n}^{\tilde{X}(\varphi), Q} \right), \end{aligned} \quad (3.7.27)$$

since the jumps of $\tilde{X}(\varphi)$ are always positive:

$$\begin{aligned} \Delta^+ \tilde{X}_s(\varphi) &= \tilde{X}_{s+}(\varphi) - \tilde{X}_s(\varphi) \\ &= \exp((-p)(V_{s+}^\varepsilon(\varphi) - V_s(\varphi^*) + F_{s+}^{CE}(\varphi))) \\ &\quad - \exp((-p)(V_s^\varepsilon(\varphi) - V_s(\varphi^*) + F_s^{CE}(\varphi))). \end{aligned}$$

At a transaction time of φ we have $V_{s+}^\varepsilon(\varphi) = V_s^\varepsilon(\varphi) - \varepsilon$ and $F_{s+}^{CE}(\varphi) \leq F_s^{CE}(\varphi) + \varepsilon$, so

$$V_{s+}^\varepsilon(\varphi) + F_{s+}^{CE}(\varphi) \leq V_s^\varepsilon(\varphi) + F_s^{CE}(\varphi).$$

Since jumps only occur at the transaction times of φ , this shows $\Delta^+ \tilde{X}_s(\varphi) \geq 0$. In view of equation (3.7.27) we have to analyze the drift and martingale components.

Since the stopped local martingale $M_{\cdot \wedge \tau_n}^{\tilde{X}(\varphi), Q}$ is a true Q -martingale, we have

$$\mathbb{E}_Q \left(M_{T \wedge \tau_n}^{\tilde{X}(\varphi), Q} \right) = M_0^{\tilde{X}(\varphi), Q} = 0.$$

For the drift component we can use Propositions 3.7.1 and 3.7.2 which imply

$$\mathbb{E}_Q \left(\int_0^{T \wedge \tau_n} b_s^{\tilde{X}(\varphi), Q} ds \right) \geq o(\varepsilon^{1/2}),$$

where the $o(\varepsilon^{1/2})$ term does not depend on φ nor n . In view of equation (3.7.23) we end up with

$$M_0 \mathbb{E}_Q(Y_n) \leq M_0 \exp(p(\gamma_0 - \varepsilon)) + o(\varepsilon^{1/2}).$$

The right-hand side is independent of φ and n . Together with (3.7.22) and the Taylor series of the exponential function we obtain the claimed upper bound for the maximal expected utility. \square

3. Fixed transaction costs

3.7.2. On the larger set of admissible strategies

In this section we want to extend the results of the previous section to the set \mathfrak{A}^ε of admissible strategies. For this purpose we use the modified processes from subsection 3.5.5.

In the first step we consider the drift of the modified value process $\widehat{v}^\varepsilon(\varphi)$, or to be more precise its transformation $\widehat{X}(\varphi)$. As before, the drift is supposed to be of small order inside the no-trade region and greater than or equal to a value of small order in the trade region. Thus, the proofs for the modified value process ansatz are based on the proofs from subsection 3.7.1. However, in this section we only consider the drift up to the stopping time

$$\tau(\varphi) := \inf\{t \in [0, T] : V_t^\varepsilon(\varphi) - V_t(\varphi^*) < -K_t\} \wedge T$$

that is defined for each strategy $\varphi \in \mathfrak{A}^\varepsilon$. Here, the process K is given by

$$K_t = \frac{1}{c_2} \log\left(\frac{\gamma_t}{\varepsilon^{1/2} c_1}\right).$$

A simple calculation reveals that K fulfills $c_1 \varepsilon^{1/2} \exp(c_2 K_t) = \gamma_t$. Before this stopping time we know for the portfolio process

$$V_t^\varepsilon(\varphi) - V_t(\varphi^*) \geq -K_t \quad \text{on } [0, \tau(\varphi)],$$

which will be useful in the proofs of the subsequent propositions. The first proposition deals with the drift in the interior of the modified no-trade region.

Proposition 3.7.4 (Modified no-trade region). *We have*

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \sup_{\sigma \in \mathcal{T}} \left| \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\widehat{\varphi}_s^+\}} ds \right) \right| \leq (1 - \sqrt{c_1}) \kappa_{c_1, c_2} \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T A_s ds \right) + \kappa_{c_1, c_2} (1 + c_2 + c_2^2) \cdot O(\varepsilon^{3/4}),$$

where κ_{c_1, c_2} is a constant that only depends on ε via c_1 and c_2 and fulfills

$$\limsup_{c \rightarrow 0} \kappa_{c, 1/c} < \infty.$$

Proof. (i) Similar calculations as before show that the Q -local drift of $\widehat{X}(\varphi)$ is given by

$$b_s^{\widehat{X}(\varphi), Q} = \widehat{X}_s(\varphi) \left((-p) b_s^{\widehat{F}^{CE}(\varphi), Q} + \frac{p^2}{2} c_s^2 \int_0^{\cdot} \Delta\varphi_u dS_u + \widehat{F}^{CE}(\varphi), \int_0^{\cdot} \Delta\varphi_u dS_u + \widehat{F}^{CE}(\varphi) \right).$$

In the interior of the modified no-trade region $\{|\Delta\varphi_s| < \Delta\widehat{\varphi}_s^+\}$ this leads to

$$\begin{aligned} b_s^{\widehat{X}(\varphi), Q} &= \widehat{X}_s(\varphi) \left((-p) b^{\widehat{F}^{CE}(\alpha, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2} c_s^2 \widehat{f}^{CE}(\alpha, \Delta\varphi) - \gamma \right) \\ &\quad + \widehat{X}_s(\varphi) \left((-p) b_s^{\delta(\varphi), Q} + \frac{p^2}{2} c_s^2 \delta(\varphi) + p^2 c_s^2 \widehat{f}^{CE}(\alpha, \Delta\varphi) - \gamma, \delta(\varphi) \right). \end{aligned} \quad (3.7.28)$$

3.7. Upper bound for maximal expected utility

For the left summand observe that it has a similar representation as the drift of the process $\tilde{X}(\varphi)$ from subsection 3.7.1 in the no-trade region, namely

$$\begin{aligned} & \widehat{X}_s(\varphi) \left((-p)b^{\widehat{f}^{CE}(\alpha, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2}c_s^{\widehat{f}^{CE}(\alpha, \Delta\varphi) - \gamma} \right) \\ &= \widehat{X}_s(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^i + p \left(b_s^{\gamma, Q} + \widehat{\beta}_s c_s^{\varphi^*} \right) \right) \\ &= \widehat{X}_s(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^i + p(\sqrt{\bar{c}_1} - 1) \sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*} c_s^S} \right) \end{aligned}$$

using the representation of $b^{\gamma, Q}$ from (3.5.14) and the modified $\widehat{\beta}$ from (3.5.16), where the processes \widehat{H}^i can be obtained from e.g. line (3.5.15) by replacing β with $\widehat{\beta} = \sqrt{\bar{c}_1}\beta$. This implies

$$\begin{aligned} & \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s(\varphi) \left((-p)b^{\widehat{f}^{CE}(\alpha, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2}c_s^{\widehat{f}^{CE}(\alpha, \Delta\varphi) - \gamma} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^+\}} ds \right| \\ &\leq \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s(\varphi) \left(\sum_{i=0}^8 |\Delta\widehat{\varphi}_s^+|^i |\widehat{H}_s^i| + p(\sqrt{\bar{c}_1} - 1) \sqrt{\frac{\varepsilon p}{3}} \sqrt{c_s^{\varphi^*} c_s^S} \right) ds \\ &\leq \sum_{i=0}^8 \int_0^T \exp(p(K_s + \gamma_s + \bar{c}_1\varepsilon)) |\Delta\widehat{\varphi}_s^+|^i |\widehat{H}_s^i| ds \\ &\quad + p(\sqrt{\bar{c}_1} - 1) \int_0^T \exp(p(K_s + \gamma_s + \bar{c}_1\varepsilon)) \sqrt{c_s^{\varphi^*} c_s^S} ds \end{aligned} \tag{3.7.29}$$

since

$$\widehat{X}_s(\varphi) = \exp \left((-p) \underbrace{(\Delta V_s^\varepsilon(\varphi))}_{\geq -K_s} + \underbrace{\widehat{F}^{CE}(\varphi)}_{\geq -\gamma_s - \bar{c}_1\varepsilon} \right) \leq \exp(p(K_s + \gamma_s + \bar{c}_1\varepsilon)) \quad \text{on } [0, \tau(\varphi)]. \tag{3.7.30}$$

For the right summand of (3.7.28) we require the Q -local drift and the local covariation of $\delta(\varphi)$. On $[0, \tau(\varphi)]$ they are given by

$$\begin{aligned} b^{\delta(\varphi), Q} &= \frac{c_2^2}{2} \delta(\varphi) (\Delta\varphi)^2 c^S, \\ c^{\delta(\varphi), \cdot} &= (-c_2) \delta(\varphi) \Delta\varphi c^S. \end{aligned} \tag{3.7.31}$$

3. Fixed transaction costs

Inserting these leads to

$$\begin{aligned}
& \widehat{X}_s(\varphi) \left((-p)b_s^{\delta(\varphi),Q} + \frac{p^2}{2}c_s^{\delta(\varphi)} + p^2c_s^{\widehat{f}^{CE}(\alpha,\Delta\varphi)-\gamma,\delta(\varphi)} \right) \\
&= \widehat{X}_s(\varphi) \left((-p)\frac{c_2^2}{2}\delta_s(\varphi)(\Delta\varphi_s)^2c_s^S + \frac{p^2}{2}c_2^2\delta_s(\varphi)^2(\Delta\varphi_s)^2c_s^S \right. \\
&\quad \left. + \frac{p^2}{2}(-c_2)\delta_s(\varphi)\Delta\varphi_s \underbrace{c_s^{\widehat{f}^{CE}(\alpha,\Delta\varphi)-\gamma,S}}_{=(\Delta\varphi)^4c_s^{\alpha,S}-(\Delta\varphi_s)^2c_s^{\widehat{\beta},S}-(4\alpha_s(\Delta\varphi_s)^3-2\widehat{\beta}_s\Delta\varphi_s)c_s^{\varphi^*,S}-c_s^{\gamma,S}} \right) \\
&= \widehat{X}_s(\varphi) \left((-p)\frac{c_2^2}{2}\delta_s(\varphi)(\Delta\varphi_s)^2c_s^S + \frac{p^2}{2}c_2^2\delta_s(\varphi)^2(\Delta\varphi_s)^2c_s^S + \frac{p^2}{2}(-c_2)\delta_s(\varphi)((\Delta\varphi)^5c_s^{\alpha,S} \right. \\
&\quad \left. - (\Delta\varphi_s)^3c_s^{\widehat{\beta},S} - (4\alpha_s(\Delta\varphi_s)^4 - 2\widehat{\beta}_s(\Delta\varphi_s)^2)c_s^{\varphi^*,S} - \Delta\varphi_sc_s^{\gamma,S} \right)
\end{aligned}$$

and by the upper bound for $\widehat{X}_s(\varphi)$ from (3.7.30) and $\delta_s(\varphi) \leq \gamma_s$ for $s \leq \tau(\varphi)$ we obtain

$$\begin{aligned}
& \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s(\varphi) \left((-p)b_s^{\delta(\varphi),Q} + \frac{p^2}{2}c_s^{\delta(\varphi)} + p^2c_s^{\widehat{f}^{CE}(\alpha,\Delta\varphi)-\gamma,\delta(\varphi)} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^+\}} ds \right| \\
&\leq \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s(\varphi) c_2^2 (\Delta\varphi_s)^2 c_s^S \left| \frac{p^2}{2} \delta_s(\varphi)^2 - \frac{p}{2} \delta_s(\varphi) \right| \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^+\}} ds \\
&\quad + \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s(\varphi) c_2 \delta_s(\varphi) \left(\sum_{i=0}^8 |\Delta\varphi_s|^k |\widehat{H}_s^i| \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^+\}} ds \\
&\leq \int_0^T \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) c_2^2 (\Delta\widehat{\varphi}_s^+)^2 c_s^S \left(\frac{p^2}{2} \gamma_s^2 + \frac{p}{2} \gamma_s \right) ds \\
&\quad + \int_0^T \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) c_2 \gamma_s \left(\sum_{i=0}^8 |\Delta\widehat{\varphi}_s^+|^k |\widehat{H}_s^i| \right) ds.
\end{aligned}$$

So the Q -expectation of the Q -drift up to the stopping time $\sigma \wedge \tau(\varphi)$ in absolute values

3.7. Upper bound for maximal expected utility

has the upper bound

$$\begin{aligned}
& \left| \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^+\}} ds \right) \right| \\
& \leq \sum_{i=0}^8 \mathbb{E}_Q \left(\int_0^T \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) |\Delta\varphi_s|^k |\widehat{H}_s^i| ds \right) \\
& \quad + (1 - \sqrt{\overline{c}_1}) \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) \sqrt{c_s^{\varphi^*} c_s^S} ds \right) \\
& \quad + c_2^2 \mathbb{E}_Q \left(\int_0^T \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) (\Delta\widehat{\varphi}_s^+)^2 c_s^S \left(\frac{p^2}{2} \gamma_s^2 + \frac{p}{2} \gamma_s \right) ds \right) \\
& \quad + c_2 \sum_{i=0}^8 \mathbb{E}_Q \left(\int_0^T \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) c_2 \gamma_s \left(|\Delta\widehat{\varphi}_s^+|^i |\widehat{H}_s^i| ds \right) \right) \\
& \leq \kappa_{c_1, c_2} \left(\left(c_2 + \kappa_Y \sqrt{\frac{p\varepsilon}{3}} \right) \sum_{i=0}^8 \mathbb{E}_Q \left(\int_0^T |\Delta\widehat{\varphi}_s^+|^i |\widehat{H}_s^i| ds \right) \right. \\
& \quad + (1 - \sqrt{\overline{c}_1}) \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T A_s ds \right) \\
& \quad \left. + c_2^2 \left(\frac{p^3}{6} \varepsilon \kappa_Y^2 + \frac{p^{3/2}}{2\sqrt{3}} \sqrt{\varepsilon} \kappa_Y \right) \mathbb{E}_Q \left(\int_0^T (\Delta\widehat{\varphi}_s^+)^2 c_s^S ds \right) \right), \tag{3.7.32}
\end{aligned}$$

using that for $\varepsilon \leq 1$

$$\begin{aligned}
& \exp(p(K_s + \gamma_s + \overline{c}_1 \varepsilon)) \\
& = \exp(p\overline{c}_1 \varepsilon) \exp(p\gamma_s) \underbrace{\exp\left(\frac{p}{c_2} \log\left(\frac{\gamma_s}{\varepsilon^{1/2} c_1}\right)\right)}_{=\left(\frac{\gamma_s}{\varepsilon^{1/2} c_1}\right)^{\frac{p}{c_2}}} \\
& = \exp(p\overline{c}_1 \varepsilon) \exp(p\gamma_s) \left(\frac{\gamma_s}{\varepsilon^{1/2} c_1}\right)^{\frac{p}{c_2}} \\
& \leq \exp(p\overline{c}_1) \exp\left(p\sqrt{\frac{p}{3}} \kappa_Y\right) \left(\frac{\sqrt{p} \kappa_Y}{\sqrt{3} c_1}\right)^{\frac{p}{c_2}} =: \kappa_{c_1, c_2}, \tag{3.7.33}
\end{aligned}$$

since Y is bounded by κ_Y . We can conclude for $c \in (0, 1)$

$$\kappa_{c, 1/c} \leq \exp\left(p\left(1 + \sqrt{\frac{p}{3}} \kappa_Y\right)\right) \underbrace{\left(\sqrt{\frac{p}{3}} \kappa_Y\right)^{pc}}_{\rightarrow 1 \text{ if } c \rightarrow 0} \underbrace{\left(\frac{1}{c}\right)^{pc}}_{\rightarrow 1 \text{ if } c \rightarrow 0}.$$

So the limes superior of $\kappa_{c, 1/c}$ is bounded. Due to $|\Delta\widehat{\varphi}_s^+| \leq |\Delta\varphi_s^+|$ and $\widehat{H}^i \leq \sum_{j=1}^i |H_j^i|$, the terms $\mathbb{E}_Q \left(\int_0^T |\Delta\widehat{\varphi}_s^+|^i |\widehat{H}_s^i| ds \right)$ as well as the remaining term in line (3.7.32) are of order $O(\varepsilon^{3/4})$, since $\int_0^T (\Delta\varphi_s^+)^2 c_s^S ds$ is a bounded process by Assumption 3.6.3. This shows the assertion. \square

3. Fixed transaction costs

We come to the drift in the trade region, to be more precise to the trade region and the boundaries.

Proposition 3.7.5 (Modified trade region). *If $c_2^2 = o(\varepsilon^{-1/2})$ we have*

$$\begin{aligned}
& \inf_{\varphi \in \mathfrak{A}^\varepsilon} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}(\varphi), Q} ds \right) \\
& \geq -\frac{3}{2} \kappa_{c_1, c_2} \left(\left(c_2 + \kappa_Y \sqrt{\frac{p\varepsilon}{3}} \right) \sum_{i=0}^8 \mathbb{E}_Q \left(\int_0^T |\Delta \widehat{\varphi}_s^+|^i |\widehat{H}_s^i| ds \right) \right. \\
& \quad + (1 - \sqrt{c_1}) \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T A_s ds \right) \\
& \quad + c_2^2 \left(\frac{p^3}{6} \varepsilon \kappa_Y^2 + \frac{p^{3/2}}{2\sqrt{3}} \sqrt{\varepsilon} \kappa_Y \right) \mathbb{E}_Q \left(\int_0^T (\Delta \widehat{\varphi}_s^+)^2 c_s^S ds \right) \Big) \\
& \quad - \frac{3p^2}{2} \kappa_\varepsilon^{c_2} \kappa_{c_1, c_2} \mathbb{E}_Q([\gamma]_T)
\end{aligned}$$

where $\kappa_\varepsilon^{c_2}$ is a nonnegative constant such that $\kappa_\varepsilon^{c_2} = O(1) + c_2^2 O(\varepsilon)$.

Proof. (i) In $\{|\Delta \varphi_s| > \Delta \widehat{\varphi}_s^+\}$ the Q -local drift of $\widehat{X}(\varphi)$ is given by

$$b_s^{\widehat{X}(\varphi), Q} = \widehat{X}_s(\varphi) \left((-p)(-b_s^{\gamma, Q} + b_s^{\delta(\varphi), Q}) + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi) - \gamma + \delta(\varphi)} \right).$$

Using the fact that the quadratic variation fulfills $[X + Y + Z] = [X] + [Y] + [Z] + 2[X, Y] + 2[X, Z] + 2[Y, Z]$ for some Itô processes X, Y, Z , we can rewrite $b_s^{\widehat{X}(\varphi), Q}$ as

$$\begin{aligned}
b_s^{\widehat{X}(\varphi), Q} &= \widehat{X}_s(\varphi) \left(pb_s^{\gamma, Q} - pb_s^{\delta(\varphi), Q} + \frac{p^2}{2} (c_s^{\Delta V^\varepsilon(\varphi)} - 2c_s^{\Delta V^\varepsilon(\varphi), \gamma} + c_s^\gamma + 2c_s^{\Delta V^\varepsilon(\varphi), \delta(\varphi)} \right. \\
& \quad \left. - 2c_s^{\gamma, \delta(\varphi)} + c_s^{\delta(\varphi)}) \right).
\end{aligned}$$

A similar argument as in the proof of proposition 3.7.2 shows

$$\begin{aligned}
pb_s^{\gamma, Q} + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi)} &= -\frac{p^2}{6} (\Delta \varphi_s^+)^2 c_s^S + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi)} \\
&= -\frac{p^2}{6\overline{c}_1^{1/2}} (\Delta \widehat{\varphi}_s^+)^2 c_s^S + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi)} \\
&\geq -\frac{p^2}{6\overline{c}_1^{1/2}} (\Delta \varphi_s)^2 c_s^S + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi)} \\
&= p^2 \left(\frac{1}{2} - \frac{1}{6\overline{c}_1^{1/2}} \right) c_s^{\Delta V^\varepsilon(\varphi)}
\end{aligned}$$

in the modified trade region. Recalling this and the dynamics of $\delta(\varphi)$ up to the stopping time $\tau(\varphi)$ in the modified trade region yields

$$\begin{aligned}
b_s^{\widehat{X}(\varphi), Q} &\geq \widehat{X}_s(\varphi) \left(p^2 \left(\frac{1}{2} - \frac{1}{6\overline{c}_1^{1/2}} \right) c_s^{\Delta V^\varepsilon(\varphi)} - \frac{p}{2} \delta_s(\varphi) c_2^2 c_s^{\Delta V^\varepsilon(\varphi)} + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi)} - p^2 c_s^{\Delta V^\varepsilon(\varphi), \gamma} \right. \\
& \quad \left. + \frac{p^2}{2} c_s^\gamma + p^2 c_s^{\Delta V^\varepsilon(\varphi)} \delta_s(\varphi) (-c_2) - p^2 c_s^{\Delta V^\varepsilon(\varphi), \gamma} \delta_s(\varphi) (-c_2) + \frac{p^2}{2} c_2^2 \delta_s(\varphi)^2 c_s^{\Delta V^\varepsilon(\varphi)} \right) \\
&= \widehat{X}_s(\varphi) (a_s c_s^{\Delta V^\varepsilon(\varphi)} + b_s c_s^{\Delta V^\varepsilon(\varphi), \gamma} + p^2 c_s^\gamma), \tag{3.7.34}
\end{aligned}$$

3.7. Upper bound for maximal expected utility

where

$$\begin{aligned} a_s &= p^2 \left(\frac{1}{2} - \frac{1}{6\bar{c}_1^{1/2}} \right) - \frac{p}{2} \delta_s(\varphi) c_2^2 - \frac{p^2}{2} \delta_s(\varphi) c_2 + \frac{p^2}{2} c_2 \delta_s(\varphi)^2, \\ b_s &= p^2 (\delta_s(\varphi) c_2 - 1). \end{aligned}$$

Since Y is assumed to be a bounded process and c_2^2 is of order $o(\varepsilon^{-1/2})$, we see that

$$\begin{aligned} a_s &\geq p^2 \left(\frac{1}{2} - \frac{1}{6\bar{c}_1^{1/2}} \right) - \delta_s(\varphi) \frac{p^2}{2} c_2 \left(\frac{c_2}{p} + 1 \right) \\ &\geq p^2 \left(\frac{1}{2} - \frac{1}{6\bar{c}_1^{1/2}} \right) - \gamma_s \frac{p^2}{2} c_2 \left(\frac{c_2}{p} + 1 \right) \\ &\geq p^2 \left(\frac{1}{2} - \frac{1}{6\bar{c}_1^{1/2}} \right) - \sqrt{\frac{p\varepsilon}{3}} \kappa_Y \frac{p^2}{2} c_2 \left(\frac{c_2}{p} + 1 \right) > 0 \end{aligned}$$

for ε being small enough. Thus, we can rewrite (3.7.34) as

$$\begin{aligned} &\widehat{X}_s(\varphi) \left(a_s c_s^{\Delta V^\varepsilon(\varphi)} + \frac{2b_s \sqrt{a_s}}{2\sqrt{a_s}} c_s^{\Delta V^\varepsilon(\varphi), \gamma} + p^2 c_s^\gamma \right) \\ &= \widehat{X}_s(\varphi) \left(c^{\int_0^s \sqrt{a_u} dS_u + \int_0^s \frac{b_u}{2\sqrt{a_u}} d\gamma_u} + \left(p^2 - \frac{b_s^2}{4a_s} \right) c_s^\gamma \right) \\ &\geq \widehat{X}_s(\varphi) \left(p^2 - \frac{b_s^2}{4a_s} \right) c_s^\gamma \end{aligned}$$

with

$$\begin{aligned} \frac{b_s^2}{4a_s} &\leq \frac{p^4 (\delta_s(\varphi) c_2 - 1)^2}{\frac{p^2}{3} - \sqrt{\frac{\varepsilon p}{3}} \kappa_Y} = p^2 \frac{c_2^2 \delta_s(\varphi)^2 - 2c_2 \delta_s(\varphi) + 1}{\frac{1}{3} - \sqrt{\varepsilon} 3p^3 \kappa_Y} \\ &\leq p^2 \frac{c_2^2 \delta_s(\varphi)^2 + 1}{\frac{1}{3} - \sqrt{\varepsilon} 3p^3 \kappa_Y} \leq p^2 \frac{c_2^2 \varepsilon^{\frac{p}{3}} \kappa_Y^2 + 1}{\frac{1}{3} - \sqrt{\varepsilon} 3p^3 \kappa_Y} := p^2 (1 + \kappa_\varepsilon^{c_2}). \end{aligned} \quad (3.7.35)$$

This implies for sufficiently small ε

$$\begin{aligned} &\mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}(\varphi), Q} \mathbf{1}_{\{|\Delta \varphi_s| > \Delta \widehat{\varphi}_s^+\}} ds \right) \\ &\geq p^2 \underbrace{(-\kappa_\varepsilon^{c_2})}_{<0} \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s(\varphi) c_s^\gamma \mathbf{1}_{\{|\Delta \varphi_s| > \Delta \widehat{\varphi}_s^+\}} ds \right) \\ &\geq p^2 (-\kappa_\varepsilon^{c_2}) \mathbb{E}_Q \left(\int_0^T \exp(p(K_s + \bar{c}_1 \varepsilon + \gamma_s)) c_s^\gamma ds \right) \\ &\geq p^2 (-\kappa_\varepsilon^{c_2}) \kappa_{c_1, c_2} \mathbb{E}_Q([\gamma]_T) \end{aligned} \quad (3.7.36)$$

by using the upper bound for $\widehat{X}(\varphi)$ on $[0, \tau(\varphi)]$. Furthermore, we see that $\kappa_\varepsilon^{c_2}$ is nonnegative and

$$\kappa_\varepsilon^{c_2} = \frac{2 + \sqrt{\varepsilon} 9p^3 \kappa_Y + c_2^2 \varepsilon \kappa_Y}{1 - \sqrt{\varepsilon} 9p^3 \kappa_Y} = O(1) + c_2^2 O(\varepsilon).$$

3. Fixed transaction costs

(ii) At the boundaries of the modified no-trade region $\{|\Delta\varphi_s| = \Delta\widehat{\varphi}_s^+\}$ we have

$$\begin{aligned}
b_s^{\widehat{X}(\varphi), Q} &= \widehat{X}_s(\varphi) \left((-p)b_s^{\widehat{F}^{CE}(\varphi), Q} + \frac{p^2}{2} c_s \int_0^{\cdot} \Delta\varphi_u dS_u + \widehat{F}^{CE}(\varphi) \right) \\
&= \frac{1}{2} \widehat{X}_s(\varphi) \left((-p)b^{\alpha(\Delta\varphi)^4 - \widehat{\beta}(\Delta\varphi)^2 - \gamma + \delta(\varphi), Q} + \frac{p^2}{2} c_s \int_0^{\cdot} \Delta\varphi_u dS_u + \alpha(\Delta\varphi)^4 - \widehat{\beta}(\Delta\varphi)^2 - \gamma + \delta(\varphi) \right) \\
&\quad + \frac{1}{2} \widehat{X}_s(\varphi) \left((-p)b^{-\overline{c}_1 \varepsilon - \gamma + \delta(\varphi), Q} + \frac{p^2}{2} c_s \int_0^{\cdot} \Delta\varphi_u dS_u - \overline{c}_1 \varepsilon - \gamma + \delta(\varphi) \right) \\
&= -\frac{1}{2} \widehat{X}_s(\varphi) \left((-p)b^{\alpha(\Delta\varphi)^4 - \widehat{\beta}(\Delta\varphi)^2 - \gamma + \delta(\varphi), Q} + \frac{p^2}{2} c_s \int_0^{\cdot} \Delta\varphi_u dS_u + \alpha(\Delta\varphi)^4 - \widehat{\beta}(\Delta\varphi)^2 - \gamma + \delta(\varphi) \right).
\end{aligned} \tag{3.7.37}$$

So we obtain from lines (3.7.37), (3.7.32) and (3.7.36)

$$\begin{aligned}
&\mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| = \Delta\widehat{\varphi}_s^+\}} ds \right) \\
&\geq -\frac{1}{2} \left[\kappa_{c_1, c_2} \left(\left(c_2 + \kappa_Y \sqrt{\frac{p\varepsilon}{3}} \right) \sum_{i=0}^8 \mathbb{E}_Q \left(\int_0^T |\Delta\widehat{\varphi}_s^+|^i |\widehat{H}_s^i| ds \right) \right. \right. \\
&\quad \left. \left. + (1 - \sqrt{\overline{c}_1}) \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T A_s ds \right) \right. \right. \\
&\quad \left. \left. + c_2^2 \left(\frac{p^3}{6} \varepsilon \kappa_Y^2 + \frac{p^{3/2}}{2\sqrt{3}} \sqrt{\varepsilon} \kappa_Y \right) \mathbb{E}_Q \left(\int_0^T (\Delta\widehat{\varphi}_s^+)^2 c_s^S ds \right) \right) \right] \\
&\quad + \frac{p^2}{2} (1 - \kappa_\varepsilon^{c_2}) \kappa_{c_1, c_2} \mathbb{E}_Q([\gamma]_T).
\end{aligned}$$

Combining this with the result of Proposition 3.7.4 yields the assertion. \square

In order to determine an asymptotic upper bound for the maximal expected utility in the presence of fixed transaction costs, we exploit another property of the stopping time $\tau(\varphi)$: at this stopping time the value process ansatz $\widehat{v}^\varepsilon(\varphi)$ (or the transformed version $\widehat{X}(\varphi)$) almost coincides with the semi frictionless value process $v(\varphi)$ (or $X(\varphi)$), which is a supermartingale (submartingale). Then, from 0 to $\tau(\varphi)$ we can use the asymptotics from the previous propositions and between $\tau(\varphi)$ and T we can use the submartingale property of $X(\varphi)$. The precise formulation and verification can be found in the following.

Theorem 3.7.6. *If $c_2 = 1/c_1$ and $c_2 = o(\varepsilon^{-1/4})$, then we have*

$$\begin{aligned}
&\sup_{\varphi \in \mathfrak{A}^\varepsilon(x)} \mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq M_0 \exp(p(\gamma_0 - c_1 \varepsilon^{1/2})) \\
&\quad + (1 - \sqrt{1 - c_1}) \kappa_{c_1, c_2} \varepsilon^{1/2} + \kappa_{c_1, c_2} (1 + c_2 + c_2^2) O(\varepsilon^{3/4}) + \left(\frac{1}{c_1} \right)^{1/c_2} O(\varepsilon). \tag{3.7.38}
\end{aligned}$$

Proof. (i) Similar to Theorem 3.7.3 we have

$$\mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq \exp(-p\varepsilon) M_0 \mathbb{E}_Q(\widehat{X}_T(\varphi)).$$

3.7. Upper bound for maximal expected utility

Observe that the stopping time $\tau(\varphi)$

$$\Delta V_{\tau(\varphi)}^\varepsilon(\varphi) \leq \Delta V_{\tau(\varphi)+}^\varepsilon(\varphi) + \varepsilon < -K_{\tau(\varphi)} + \varepsilon,$$

which implies for the process $\delta(\varphi)$

$$\begin{aligned} \delta_{\tau(\varphi)}(\varphi) &= c_1 \varepsilon^{1/2} \exp(-c_2 \Delta V_{\tau(\varphi)}^\varepsilon(\varphi)) \\ &\geq c_1 \varepsilon^{1/2} \exp(-c_2(-K_{\tau(\varphi)} + \varepsilon)) \\ &= \gamma_{\tau(\varphi)} \exp(-c_2 \varepsilon). \end{aligned}$$

So we get the following upper bound for the transformed value process ansatz

$$\begin{aligned} \widehat{X}_{\tau(\varphi)}(\varphi) &= X_{\tau(\varphi)}(\varphi) \exp(-p \widehat{F}_{\tau(\varphi)}^{CE}(\varphi)) \\ &= X_{\tau(\varphi)}(\varphi) \exp(-p \widehat{f}^{CE}(\alpha_{\tau(\varphi)}, \Delta \varphi_{\tau(\varphi)}) + p \gamma_{\tau(\varphi)} - p \delta_{\tau(\varphi)}(\varphi)) \\ &\leq X_{\tau(\varphi)}(\varphi) \exp(p \varepsilon + p \gamma_{\tau(\varphi)} (1 - \exp(-c_2 \varepsilon))) \\ &\leq X_{\tau(\varphi)}(\varphi) \exp(p \varepsilon + p \kappa_Y (\varepsilon p / 3)^{1/2} (1 - \exp(-c_2 \varepsilon))). \end{aligned}$$

Denoting the factor on the right by h_ε , this yields

$$\begin{aligned} \mathbb{E}_Q \left(\widehat{X}_T(\varphi) \right) &= \mathbb{E}_Q \left(\widehat{X}_T(\varphi) - \widehat{X}_{\tau(\varphi)}(\varphi) \right) + \mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}(\varphi) \right) \\ &\geq \mathbb{E}_Q \left(X_T(\varphi) - X_{\tau(\varphi)}(\varphi) h_\varepsilon \right) + \mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}(\varphi) \right) \\ &= \mathbb{E}_Q \left(X_T(\varphi) - X_{\tau(\varphi)}(\varphi) \right) + (1 - h_\varepsilon) \mathbb{E}_Q \left(X_{\tau(\varphi)}(\varphi) \right) + \mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}(\varphi) \right) \\ &\geq (1 - h_\varepsilon) \mathbb{E}_Q \left(X_{\tau(\varphi)}(\varphi) \right) + \mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}(\varphi) \right) \end{aligned} \quad (3.7.39)$$

due to the submartingale property of $X(\varphi)$.

(ii) Since $\Delta V_s^\varepsilon(\varphi) \geq -K_s$ on $[0, \tau(\varphi)]$, we obtain

$$\begin{aligned} \mathbb{E}_Q \left(X_{\tau(\varphi)}(\varphi) \right) &\leq \mathbb{E}_Q \left(\exp(p K_s) \right) = \mathbb{E}_Q \left(\left(\frac{\gamma_s}{\varepsilon^{1/2} c_1} \right)^{1/c_2} \right) \\ &\leq \left(\frac{1}{c_1} \right)^{1/c_2} \left(\frac{p}{3} \right)^{1/(2c_2)} \kappa_Y^{1/c_2}. \end{aligned}$$

So

$$\mathbb{E}_Q \left(X_{\tau(\varphi)}(\varphi) \right) \leq \left(\frac{1}{c_1} \right)^{1/c_2} O(1). \quad (3.7.40)$$

Furthermore, h_ε is of order $O(\varepsilon) + O(\varepsilon^{3/2} c_2) = O(\varepsilon)$ due to the assumed property of c_2 .

(iii) We repeat the localization argument from the proof of Theorem 3.7.3: Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for the local martingale $M^{\widehat{X}(\varphi), Q}$ and define $Y_n := \widehat{X}_{\tau(\varphi) \wedge \tau_n}(\varphi)$ for all $n \in \mathbb{N}$. Then we have

$$\lim_{n \rightarrow \infty} Y_n = \widehat{X}_{\tau(\varphi)}(\varphi)$$

3. Fixed transaction costs

and

$$\begin{aligned}
0 \leq Y_n &= \exp \left(- \underbrace{p(V_{\tau(\varphi) \wedge \tau_n}^\varepsilon(\varphi) - V_{\tau(\varphi) \wedge \tau_n}(\varphi^*))}_{\geq -K_{\tau(\varphi) \wedge \tau_n}} + \underbrace{\widehat{F}_{\tau(\varphi) \wedge \tau_n}^{CE}(\varphi)}_{\geq -\bar{c}_1 \varepsilon - \gamma_{\tau(\varphi) \wedge \tau_n}} \right) \\
&\leq \exp(p(K_{\tau(\varphi) \wedge \tau_n} + \bar{c}_1 \varepsilon) + \gamma_{\tau(\varphi) \wedge \tau_n}) \\
&\leq \exp(p(K_{\tau(\varphi) \wedge \tau_n} + \bar{c}_1 \varepsilon + \gamma_{\tau(\varphi) \wedge \tau_n})),
\end{aligned}$$

which is bounded by κ_{c_1, c_2} . Hence $\{Y_n, n \in \mathbb{N}\}$ is a bounded set in $L^2(Q)$, which implies that it is uniformly integrable. As a consequence,

$$\mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}(\varphi) \right) = \lim_{n \rightarrow \infty} \mathbb{E}_Q(Y_n). \quad (3.7.41)$$

(iv) For the right-hand side we see that

$$\begin{aligned}
\mathbb{E}_Q(Y_n) &= \mathbb{E}_Q \left(\widehat{X}_0(\varphi) + \int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}(\varphi), Q} ds + M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}(\varphi), Q} + \sum_{0 \leq s < \tau(\varphi) \wedge \tau_n} \Delta^+ \widehat{X}_s(\varphi) \right) \\
&\geq \widetilde{X}_0(\varphi) + \mathbb{E}_Q \left(\int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}(\varphi), Q} ds \right) + \mathbb{E}_Q \left(M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}(\varphi), Q} \right), \quad (3.7.42)
\end{aligned}$$

since the jumps of $\widehat{X}(\varphi)$ are always positive:

$$\begin{aligned}
\Delta^+ \widehat{X}_s(\varphi) &= \widehat{X}_{s+}(\varphi) - \widehat{X}_s(\varphi) \\
&= \exp((-p)(V_{s+}^\varepsilon(\varphi) - V_s(\varphi^*) + \widehat{F}_{s+}^{CE}(\varphi))) \\
&\quad - \exp((-p)(V_s^\varepsilon(\varphi) - V_s(\varphi^*) + \widehat{F}_s^{CE}(\varphi))).
\end{aligned}$$

At a transaction time of φ we have $V_{s+}^\varepsilon(\varphi) = V_s^\varepsilon(\varphi) - \varepsilon$ and

$$\begin{aligned}
\Delta^+ \delta_s(\varphi) &= c_1 \varepsilon^{1/2} \left(\exp(-c_2 \Delta V_{s+}^\varepsilon(\varphi)) - \exp(-c_2 \Delta V_s^\varepsilon(\varphi)) \right) \\
&= c_1 \varepsilon^{1/2} \underbrace{\exp(-c_2 \Delta V_s^\varepsilon(\varphi))}_{=\delta_s(\varphi)} (\exp(c_2 \varepsilon) - 1) \\
&\leq \gamma_s (\exp(p c_2 \varepsilon) - 1) \\
&\leq \frac{\sqrt{p} \kappa_Y \exp(c_2 \varepsilon) - 1}{\sqrt{3} c_1 \varepsilon^{1/2}} c_1 \varepsilon = O \left(\frac{c_2}{c_1} \varepsilon^{1/2} \right) c_1 \varepsilon \leq c_1 \varepsilon.
\end{aligned}$$

for a sufficiently small ε , which implies for the modified correction term

$$\begin{aligned}
\widehat{F}_{s+}^{CE}(\varphi) &= \underbrace{\widehat{f}^{CE}(\alpha_s, \Delta \varphi_{s+})}_{\leq \widehat{f}^{CE}(\alpha_s, \Delta \varphi_s) + \bar{c}_1 \varepsilon} - \gamma_s + \underbrace{\delta_{s+}(\varphi)}_{\leq \delta_s(\varphi) + c_1 \varepsilon} \\
&\leq \widehat{F}_s^{CE}(\varphi) + \varepsilon,
\end{aligned}$$

so

$$V_{s+}^\varepsilon(\varphi) + \widehat{F}_{s+}^{CE}(\varphi) \leq V_s^\varepsilon(\varphi) + \widehat{F}_s^{CE}(\varphi).$$

3.8. Lower bound for maximal expected utility and almost optimality of the candidate

Since jumps only occur at the transaction times of φ , this shows $\Delta^+ \widehat{X}_s(\varphi) \geq 0$.
(v) In the next steps we analyze the drift and martingale components in equation (3.7.42). Since the stopped local martingale $M_{\cdot \wedge \tau_n}^{\widehat{X}(\varphi), Q}$ is a true Q -martingale, we have

$$\mathbb{E}_Q \left(M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}(\varphi), Q} \right) = M_0^{\widehat{X}(\varphi), Q} = 0.$$

For the drift component we can use Propositions 3.7.4 and 3.7.5 which imply

$$\begin{aligned} & \mathbb{E}_Q \left(\int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}(\varphi), Q} ds \right) \\ & \geq \frac{3}{2} (\sqrt{c_1} - 1) \kappa_{c_1, c_2} \varepsilon^{3/2} + \frac{3}{2} \kappa_{c_1, c_2} (1 + c_2 + c_2^2) O(\varepsilon^{3/4}). \end{aligned}$$

In view of equation (3.7.42) we end up with

$$\begin{aligned} M_0 \mathbb{E}_Q (Y_n) & \leq M_0 \exp(p(\gamma_0 - c_1 \varepsilon^{1/2})) \\ & \quad + \frac{3}{2} (\sqrt{c_1} - 1) \kappa_{c_1, c_2} \varepsilon^{3/2} + \frac{3}{2} \kappa_{c_1, c_2} (1 + c_2 + c_2^2) O(\varepsilon^{3/4}), \end{aligned}$$

where the right-hand side does not depend on φ nor n . Together with (3.7.39), (3.7.40) and (3.7.41) we obtain the claimed upper bound for the maximal expected utility. \square

3.8. Lower bound for maximal expected utility and almost optimality of the candidate

3.8.1. Properties of the candidate and drift of its value process

As a next step, we need two technical lemmas in order to prove the almost optimality of the candidate. The first lemma deals with an upper bound for the deviation from gains of trading of the candidate and the frictionless optimizer $\int_0^t (\varphi_s^\varepsilon - \varphi_s^*) dS_s$. It relies on the Burkholder-Davis-Gundy inequalities from Theorem B.2.2.

Lemma 3.8.1. *For every $\kappa > 0$ we have*

$$Q \left(\sup_{t \in [0, T]} \left| \int_0^t \Delta \varphi_s^\varepsilon dS_s \right| \geq \kappa \right) \leq \frac{12\varepsilon C_2}{p\kappa^4} \mathbb{E}_Q \left\| \int_0^T A_s ds \right\|_{L_2(Q)}^2 < \infty,$$

where $C_2 > 0$ is the constant from Theorem B.2.2.

Proof. Applying the inequalities of Markov from (Klenke, 2008, Satz 5.11) and Burkholder-

3. Fixed transaction costs

Davis-Gundy (Theorem B.2.2) and $|\Delta\varphi^\varepsilon| \leq \Delta\varphi^+$ delivers

$$\begin{aligned}
& Q \left(\sup_{t \in [0, T]} \left| \int_0^t \Delta\varphi_s^\varepsilon dS_s \right| \geq \kappa \right) \\
& \leq \frac{1}{\kappa^4} \mathbb{E}_Q \left(\sup_{t \in [0, T]} \left(\int_0^t \Delta\varphi_s^\varepsilon dS_s \right)^4 \right) \leq \frac{C_2}{\kappa^4} \left\| \int_0^T (\Delta\varphi_s^\varepsilon)^2 d[S]_s \right\|_{L^2(Q)}^2 \\
& \leq \frac{C_2}{\kappa^4} \left\| \int_0^T \underbrace{(\Delta\varphi_s^+)^2}_{=\sqrt{\frac{12\varepsilon}{p}} \sqrt{\frac{c_s^{\varphi^*, \varphi^*}}{c_s^{S, S}}}} d[S]_s \right\|_{L^2(Q)}^2 = \frac{C_2}{\kappa^4} \left(\frac{12\varepsilon}{p} \right) \left\| \int_0^T (c_s^{\varphi^*, \varphi^*} c_s^{S, S})^{1/2} ds \right\|_{L^2(Q)}^2 \\
& = \frac{12\varepsilon C_2}{p\kappa^4} \mathbb{E}_Q \left\| \int_0^T A_s ds \right\|_{L^2(Q)}^2.
\end{aligned}$$

Since $\int_0^T A_s ds$ is bounded, this yields the assertion. \square

The second Lemma yields an upper bound for distribution of the candidate's number of trades during the time horizon $[0, T]$.

Lemma 3.8.2 (Number of transactions). *For every $\kappa > 0$ we have*

$$Q(\mathcal{N}_T(\varphi^\varepsilon) \geq \kappa) \leq \frac{\text{const.}}{\varepsilon\kappa^2} \|C\|_{S_2(Q)} \|\varphi^* - \varphi_0^*\|_{H_8(Q)}^4 < \infty \quad (3.8.43)$$

where the constant does not depend on κ .

Proof. Rewriting the set $\{\mathcal{N}_T(\varphi) \geq \kappa\}$ shows

$$\begin{aligned}
Q(\mathcal{N}_T(\varphi^\varepsilon) \geq \kappa) &= Q \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \geq \kappa \right) \\
&= Q \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\frac{|\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*|}{\Delta\varphi_{\tau_i}^+} \varepsilon^{1/4} \right)^2 \geq \kappa \varepsilon^{1/2} \right) \\
&= Q \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\frac{|\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*|}{\left(\frac{12}{p}\right)^{1/2} C_{\tau_i}^{-1/2}} \right)^2 \geq \kappa \varepsilon^{1/2} \right) \\
&= Q \left(\left(\frac{p}{12}\right)^{1/2} \sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} C_{\tau_i}^{1/2} (\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*)^2 \geq \kappa \varepsilon^{1/2} \right).
\end{aligned}$$

By passing to the pathwise supremum of C and using Markov's and Hölder's inequal-

3.8. Lower bound for maximal expected utility and almost optimality of the candidate

ities we obtain

$$\begin{aligned}
& Q(\mathcal{N}_T(\varphi^\varepsilon) \geq \kappa) \\
& \leq Q\left(\left(\frac{p}{12}\right)^{1/2} \sup_{t \in [0, T]} C_t^{1/2} \sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*\right)^2 \geq \kappa \varepsilon^{1/2}\right) \\
& \leq \varepsilon^{-1} \kappa^{-2} \mathbb{E}_Q \left(\frac{p}{12} \sup_{t \in [0, T]} C_t \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*\right)^2 \right)^2 \right) \\
& \leq \frac{p}{12 \varepsilon \kappa^2} \left\| \sup_{t \in [0, T]} C_t \right\|_{L^2(Q)} \mathbb{E}_Q \left(\left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*\right)^2 \right)^4 \right)^{1/2}.
\end{aligned}$$

We see that with $\bar{\tau}_i := \tau_i \wedge T$ and two applications of Jensen's inequality to the functions $y \mapsto y^2$ and $y \mapsto y^4$

$$\begin{aligned}
& \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*\right)^2 \right)^4 \\
& = \left(\sum_{i \in \mathbb{N}} \left(\int_{\bar{\tau}_{i-1}}^{\bar{\tau}_i} b_s^{\varphi^*, Q} ds + M_{\bar{\tau}_i}^{\varphi^*, Q} - M_{\bar{\tau}_{i-1}}^{\varphi^*, Q} \right)^2 \right)^4 \\
& \leq \left(\sum_{i \in \mathbb{N}} 2 \left(\int_{\bar{\tau}_{i-1}}^{\bar{\tau}_i} |b_s^{\varphi^*, Q}| ds \right)^2 + 2 \sum_{i \in \mathbb{N}} \left(M_{\bar{\tau}_i}^{\varphi^*, Q} - M_{\bar{\tau}_{i-1}}^{\varphi^*, Q} \right)^2 \right)^4 \\
& \leq 2^7 \left(\sum_{i \in \mathbb{N}} \left(\int_{\bar{\tau}_{i-1}}^{\bar{\tau}_i} |b_s^{\varphi^*, Q}| ds \right)^2 \right)^4 + 2^7 \left(\sum_{i \in \mathbb{N}} \left(M_{\bar{\tau}_i}^{\varphi^*, Q} - M_{\bar{\tau}_{i-1}}^{\varphi^*, Q} \right)^2 \right)^4
\end{aligned}$$

almost surely. For the left term we observe that

$$\sum_{i \in \mathbb{N}} \left(\int_{\bar{\tau}_{i-1}}^{\bar{\tau}_i} |b_s^{\varphi^*, Q}| ds \right)^2 \leq \left(\sum_{i \in \mathbb{N}} \int_{\bar{\tau}_{i-1}}^{\bar{\tau}_i} |b_s^{\varphi^*, Q}| ds \right)^2 = \left(\int_0^T |b_s^{\varphi^*, Q}| ds \right)^2,$$

which implies

$$\left(\sum_{i \in \mathbb{N}} \left(\int_{\bar{\tau}_{i-1}}^{\bar{\tau}_i} |b_s^{\varphi^*, Q}| ds \right)^2 \right)^4 \leq \left(\int_0^T |b_s^{\varphi^*, Q}| ds \right)^8.$$

For the term on the right-hand side we use the discrete version of the Burkholder-Davis-Gundy inequalities: Define $\widehat{M} := (M_{\tau_i}^{\varphi^*, Q})_{i \in \mathbb{N}}$. Then \widehat{M} is a Q -martingale with respect to the filtration $(\mathcal{F}_{\tau_i})_{i \in \mathbb{N}}$ and according to (Burkholder, 1973, Theorem 15.1) we have

$$\mathbb{E}_Q \left(\left(\sum_{i \in \mathbb{N}} \left(\widehat{M}_i - \widehat{M}_{i-1} \right)^2 \right)^4 \right) \leq \text{const.} \mathbb{E}_Q \left(\sup_{i \in \mathbb{N}} \widehat{M}_i^8 \right) \leq \text{const.} \|M^{\varphi^*, Q}\|_{S_8(Q)}^8$$

3. Fixed transaction costs

and by the Burkholder-Davis-Gundy inequalities for the continuous case (Theorem B.2.2) we can conclude that

$$\mathbb{E}_Q \left(\left(\sum_{i \in \mathbb{N}} (\widehat{M}_i - \widehat{M}_{i-1})^2 \right)^4 \right) \leq \text{const.} \cdot \mathbb{E}_Q ([\varphi^*]_T^4).$$

So we end up with

$$\begin{aligned} & \mathbb{E}_Q \left(\left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} (\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*)^2 \right)^4 \right)^{1/2} \\ & \leq \left(2^7 T^4 \mathbb{E}_Q \left(\int_0^T |b_s^{\varphi^*, Q}| ds \right)^8 + \text{const.} \cdot 2^7 \mathbb{E}_Q ([\varphi^*]_T^4) \right)^{1/2}. \end{aligned}$$

These expressions are finite by assumption. Theorem B.2.3 now states that

$$\|C\|_{S_2(Q)} \leq \text{const.} \cdot \|C\|_{H_2(Q)},$$

which is finite by the definition of the H_8 -norm from section B.2 and Assumption 3.6.4. This gives the desired inequality. \square

For the drift of the candidate's value process we also require that the corresponding portfolio wealth is not too far away from the frictionless portfolio wealth. For that reason we only consider its drift up to the stopping time

$$\tau^{\varepsilon,1} := \inf \{t \in [0, T] : \Delta V_t^\varepsilon(\varphi^\varepsilon) < -1\} \wedge T.$$

Proposition 3.8.3. *The drift of the candidate fulfills*

$$\sup_{\sigma \in \mathcal{T}} \mathbb{E}_Q \left(\left| \int_0^{\tau^{\varepsilon,1} \wedge \sigma} b_s^{\widetilde{X}(\varphi^\varepsilon), Q} ds \right| \right) = O(\varepsilon^{3/4}).$$

Proof. Apply the Itô formula from Remark B.1.2 and the same calculations as in the proof of Proposition 3.7.1 using $V_s^\varepsilon(\varphi) - V_s(\varphi^*) \geq -1$ on $[0, \tau^{\varepsilon,1}]$. This yields

$$\left| \int_0^{\tau^{\varepsilon,1} \wedge \sigma} b_s^{\widetilde{X}(\varphi^\varepsilon), Q} ds \right| \leq \exp(p(1 + \varepsilon + \sqrt{\varepsilon p / 3\kappa_Y})) \sum_{i=0}^8 \left(\int_0^T |(\Delta \varphi_s^+)|^i |H_s^i| ds \right).$$

We have already shown that the expectation of the right hand side is of order $O(\varepsilon^{3/4})$. \square

3.8.2. Lower bound

From the definition of φ^ε it is not obvious whether it belongs to the set of admissible strategies \mathfrak{A}^ε . Therefore, we define the stopping times

$$\begin{aligned} \tau^{\varepsilon,2} & := \inf \left\{ t \in [0, T] : V_t^\varepsilon(\varphi^\varepsilon) < -\frac{1}{\varepsilon} \right\} \wedge T \quad \text{and} \\ \tau^\varepsilon & := \tau^{\varepsilon,1} \wedge \tau^{\varepsilon,2} = \inf \left\{ t \in [0, T] : V_t^\varepsilon(\varphi^\varepsilon) < \max \left\{ -\frac{1}{\varepsilon}, V_t(\varphi^*) - 1 \right\} \right\} \wedge T \end{aligned}$$

3.8. Lower bound for maximal expected utility and almost optimality of the candidate

and pass to the strategy $\widetilde{\varphi}^\varepsilon := \varphi^\varepsilon \mathbf{1}_{[0, \tau^\varepsilon]}$ that sells all stocks directly after τ^ε . According to remark 3.2.6, $\widetilde{\varphi}^\varepsilon$ is an admissible strategy. We will show that $M_0 \exp(p\gamma_0)$ is an asymptotic lower bound for the associated expected utility of $\widetilde{\varphi}^\varepsilon$.

Theorem 3.8.4. *The stopped candidate strategy φ^ε fulfills*

$$\mathbb{E} \left(U(V_T^\varepsilon(\widetilde{\varphi}^\varepsilon)) \right) = \mathbb{E} \left(U(V_{\tau^\varepsilon+}^\varepsilon(\varphi^\varepsilon)) \right) \geq M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}).$$

Proof. (i) Since $\widetilde{v}_T^\varepsilon(\widetilde{\varphi}^\varepsilon) \leq v_T(\widetilde{\varphi}^\varepsilon)$, we have

$$\mathbb{E} \left(U(V_T^\varepsilon(\widetilde{\varphi}^\varepsilon)) \right) \geq \mathbb{E}_Q \left(\widetilde{v}_T^\varepsilon(\widetilde{\varphi}^\varepsilon) \right) = M_0 \mathbb{E}_Q \left(\widetilde{X}_T(\widetilde{\varphi}^\varepsilon) \right)$$

and

$$\begin{aligned} \widetilde{X}_T(\widetilde{\varphi}^\varepsilon) &= \widetilde{X}_{\tau^\varepsilon}(\widetilde{\varphi}^\varepsilon) + \left(\widetilde{X}_T(\widetilde{\varphi}^\varepsilon) - \widetilde{X}_{\tau^\varepsilon}(\widetilde{\varphi}^\varepsilon) \right) \\ &= \widetilde{X}_0(\widetilde{\varphi}^\varepsilon) + \int_0^{\tau^\varepsilon} b_s^{\widetilde{X}(\widetilde{\varphi}^\varepsilon), Q} ds + M_{\tau^\varepsilon}^{\widetilde{X}(\widetilde{\varphi}^\varepsilon), Q} + \widetilde{X}_T(\widetilde{\varphi}^\varepsilon) - \underbrace{\widetilde{X}_{\tau^\varepsilon}(\widetilde{\varphi}^\varepsilon)}_{\geq X_{\tau^\varepsilon}(\varphi^\varepsilon)} \\ &\geq \widetilde{X}_0(\widetilde{\varphi}^\varepsilon) + \int_0^{\tau^\varepsilon} b_s^{\widetilde{X}(\widetilde{\varphi}^\varepsilon), Q} ds + M_{\tau^\varepsilon}^{\widetilde{X}(\widetilde{\varphi}^\varepsilon), Q} + \widetilde{X}_T(\widetilde{\varphi}^\varepsilon) - X_{\tau^\varepsilon}(\varphi^\varepsilon). \end{aligned}$$

(ii) By proposition 3.7.1 we know that

$$\mathbb{E}_Q \left(\int_0^{\tau^\varepsilon} b_s^{\widetilde{X}(\varphi^\varepsilon), Q} ds \right) = O(\varepsilon^{3/4}), \quad (3.8.44)$$

so it is enough to show

$$\mathbb{E}_Q \left(M_{\tau^\varepsilon}^{\widetilde{X}(\widetilde{\varphi}^\varepsilon), Q} \right) = 0, \quad (3.8.45)$$

$$\mathbb{E}_Q \left(\widetilde{X}_T(\widetilde{\varphi}^\varepsilon) - X_{\tau^\varepsilon}(\varphi^\varepsilon) \right) = O(\varepsilon^{3/4}). \quad (3.8.46)$$

(iii) From line (3.7.25) we know

$$M_t^{\widetilde{X}(\varphi^\varepsilon), Q} = \int_0^t (-p) \widetilde{X}_s(\varphi^\varepsilon) dM_s^{F^{CE}(\varphi^\varepsilon), Q} + \int_0^t (-p) \widetilde{X}_s(\varphi^\varepsilon) \Delta \varphi_s^\varepsilon dS_s.$$

By the convexity of the quadratic variation we obtain

$$\begin{aligned} \left[M_{\tau^\varepsilon \wedge \cdot}^{\widetilde{X}(\varphi^\varepsilon), Q} \right]_T &= \left[M^{\widetilde{X}(\varphi^\varepsilon), Q} \right]_{\tau^\varepsilon} \\ &\leq 2p^2 \left(\int_0^{\tau^\varepsilon} \underbrace{(\widetilde{X}_s(\varphi^\varepsilon))^2}_{\leq \exp(2p)} d[M^{F^{CE}(\varphi^\varepsilon), Q}]_s + \int_0^{\tau^\varepsilon} \underbrace{(\widetilde{X}_s(\varphi^\varepsilon))^2}_{\leq \exp(2p)} \underbrace{(\Delta \varphi_s^\varepsilon)^2}_{\leq (\Delta \varphi_s^+)^2} d[S]_s \right) \\ &\leq 2p^2 \exp(2p) \left([M^{F^{CE}(\varphi^\varepsilon), Q}]_T + \int_0^T (\Delta \varphi_s^+)^2 d[S]_s \right). \end{aligned}$$

3. Fixed transaction costs

Since

$$[M^{F^{CE}(\varphi^\varepsilon), Q}]_T = \int_0^T c_s^{F^{CE}(\varphi^\varepsilon)} ds \leq \sum_{i=0}^8 \int_0^T |\Delta\varphi_s^+|^i |H_s^k| ds,$$

we know $[M^{F^{CE}(\varphi^\varepsilon), Q}]_T \in L_1(Q)$ by equation (3.7.20). Furthermore, $\int_0^T (\Delta\varphi_s^+)^2 c_s^{S, S} ds$ is in $L_1(Q)$. So $M_{\tau^\varepsilon}^{\tilde{X}(\varphi^\varepsilon), Q}$ is a Q -martingale, which implies

$$\mathbb{E}_Q \left(M_{\tau^\varepsilon}^{F^{CE}(\varphi^\varepsilon), Q} \right) = 0.$$

(iv) We have

$$\begin{aligned} & \mathbb{E}_Q \left(\tilde{X}_T(\tilde{\varphi}^\varepsilon) - X_{\tau^\varepsilon}(\varphi^\varepsilon) \right) \\ &= \mathbb{E}_Q \left(X_T(\tilde{\varphi}^\varepsilon) F_T^U(\tilde{\varphi}^\varepsilon) - X_{\tau^\varepsilon}(\varphi^\varepsilon) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) \\ &= \mathbb{E}_Q \left(X_{\tau^\varepsilon}(\varphi^\varepsilon) \left(F_T^U(\tilde{\varphi}^\varepsilon) \frac{X_T(\tilde{\varphi}^\varepsilon)}{X_{\tau^\varepsilon}(\varphi^\varepsilon)} - 1 \right) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) \\ &= \mathbb{E}_Q \left(X_{\tau^\varepsilon}(\varphi^\varepsilon) \left(F_T^U(\tilde{\varphi}^\varepsilon) \exp \left(p \int_{\tau^\varepsilon}^T \varphi_s^* dS_s \right) - 1 \right) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right), \end{aligned}$$

because

$$\frac{X_T(\tilde{\varphi}^\varepsilon)}{X_{\tau^\varepsilon}(\varphi^\varepsilon)} = \exp \left(-p \left(\int_{\tau^\varepsilon}^T \Delta\tilde{\varphi}_s^\varepsilon dS_s \right) \right) = \exp \left(p \left(\int_{\tau^\varepsilon}^T \varphi_s^* dS_s \right) \right).$$

Since $X_{\tau^\varepsilon}(\varphi^\varepsilon) \leq \exp(p)$ and $F_T^U(\tilde{\varphi}^\varepsilon) \leq \exp(p\varepsilon)$, this gives us

$$\begin{aligned} & \left| \mathbb{E}_Q \left(\tilde{X}_T(\tilde{\varphi}^\varepsilon) - X_{\tau^\varepsilon}(\varphi^\varepsilon) \right) \right| \\ & \leq \mathbb{E}_Q \left(X_{\tau^\varepsilon}(\varphi^\varepsilon) F_T^U(\tilde{\varphi}^\varepsilon) \exp(p(N_T - N_{\tau^\varepsilon})) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) + \mathbb{E}_Q \left(X_{\tau^\varepsilon}(\varphi^\varepsilon) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) \\ & \leq \exp(p(1 + \varepsilon)) \left(\mathbb{E}_Q \left(\exp(p(N_T - N_{\tau^\varepsilon})) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) + Q(\tau^\varepsilon < T) \right), \end{aligned} \quad (3.8.47)$$

where $N := \int_0^\cdot \varphi_s^* dS_s$ is a Q -martingale. Set $a = 8$ and $b = 4/3$. Then $2/a + 1/b = 1$ and by Hölder's inequality we obtain

$$\begin{aligned} & \mathbb{E}_Q \left(\exp(p(N_T - N_{\tau^\varepsilon})) \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) \\ &= \mathbb{E}_Q \left(\frac{\exp(pN_T)}{\exp(pN_{\tau^\varepsilon})} \mathbf{1}_{\{\tau^\varepsilon < T\}} \right) \\ & \leq \left\| \exp(pN_T) \right\|_{L_a(Q)} \left\| \exp(-pN_{\tau^\varepsilon}) \right\|_{L_a(Q)} Q(\tau^\varepsilon < T)^{1/b} \\ & \leq \left\| \exp(pN_T) \right\|_{L_a(Q)} \left\| \exp(-pN_T) \right\|_{L_a(Q)} Q(\tau^\varepsilon < T)^{1/b}, \end{aligned} \quad (3.8.48)$$

since $\exp(-apN)$ is a Q -submartingale.

3.8. Lower bound for maximal expected utility and almost optimality of the candidate

(v) We will show that the Q -probability of $\{\tau^\varepsilon < T\}$ is of order $O(\varepsilon)$.

$$\begin{aligned}
& Q(\tau^{\varepsilon,1} < T) \\
&= Q\left(\sup_{t \in [0, T]} |V_t^\varepsilon(\varphi^\varepsilon)| \geq \frac{1}{\varepsilon}\right) \\
&= Q\left(\sup_{t \in [0, T]} \left| \int_0^t \varphi_s^\varepsilon dS_s + \varepsilon \mathcal{N}_t(\varphi^\varepsilon) \right| \geq \frac{1}{\varepsilon}\right) \\
&\leq Q\left(\sup_{t \in [0, T]} \left| \int_0^t \varphi_s^\varepsilon dS_s \right| \geq \frac{1}{2\varepsilon}\right) + Q\left(\varepsilon \mathcal{N}_T(\varphi^\varepsilon) \geq \frac{1}{2\varepsilon}\right) \\
&= Q\left(\sup_{t \in [0, T]} \left| \int_0^t \varphi_s^\varepsilon dS_s \right| \geq \frac{1}{2\varepsilon}\right) + Q\left(\mathcal{N}_T(\varphi^\varepsilon) \geq \frac{1}{2\varepsilon^2}\right).
\end{aligned}$$

For the left term of the sum we use the inequalities of Markov and Burkholder-Davis-Gundy (Theorem B.2.2) and $\varphi^\varepsilon = \varphi^* + \Delta\varphi^\varepsilon$ to obtain

$$\begin{aligned}
& Q\left(\sup_{t \in [0, T]} \left| \int_0^t \varphi_s^\varepsilon dS_s \right| \geq \frac{1}{2\varepsilon}\right) \\
&\leq 2\varepsilon \mathbb{E}_Q\left(\sup_{t \in [0, T]} \left| \int_0^t \varphi_s^\varepsilon dS_s \right|\right) \leq \text{const.} \varepsilon \mathbb{E}_Q\left(\int_0^T (\varphi_s^\varepsilon)^2 d[S]_s\right) \\
&\leq \text{const.} \varepsilon \left(\mathbb{E}_Q\left(\int_0^T (\varphi_s^*)^2 c_s^{S,S} ds\right) + \mathbb{E}_Q\left(\int_0^T (\Delta\varphi_s^+)^2 c_s^{S,S} ds\right)\right) \\
&= \text{const.} \varepsilon \mathbb{E}_Q\left(\int_0^T (\varphi_s^*)^2 c_s^{S,S} ds\right) + \text{const.} \varepsilon^{3/2} \mathbb{E}_Q\left(\int_0^T (c_s^{\varphi^*, \varphi^*})^{1/2} ds\right),
\end{aligned}$$

and for the right term we use Lemma 3.8.2 with $K = 1/(2\varepsilon)$ to see that

$$Q\left(\mathcal{N}_T(\varphi^\varepsilon) \geq \frac{1}{2\varepsilon^2}\right) = O(\varepsilon^3).$$

This gives the desired order of convergence for $Q(\tau^{\varepsilon,1} < T)$. For the probability of the event $\{\tau^{\varepsilon,2} < T\}$ we obtain

$$\begin{aligned}
Q(\tau^{\varepsilon,2} < T) &\leq Q\left(\sup_{t \in [0, T]} |V_t^\varepsilon(\varphi^\varepsilon) - V_t(\varphi^*)| \geq 1\right) \\
&\leq Q\left(\sup_{t \in [0, T]} \left| \int_0^t \Delta\varphi_s dS_s \right| \geq 1/2\right) + Q(\varepsilon \mathcal{N}_T(\varphi^\varepsilon) \geq 1/2),
\end{aligned}$$

and with Lemma 3.8.1 and Lemma 3.8.2 these are of the correct order. Since $Q(\tau^\varepsilon < T)$ is bounded by $Q(\tau^{\varepsilon,1} < T) + Q(\tau^{\varepsilon,2} < T)$, the above inequalities give

$$Q(\tau^\varepsilon < T) = O(\varepsilon) \quad \text{and} \quad Q(\tau^\varepsilon < T)^{1/b} = O(\varepsilon^{3/4}),$$

and with (3.8.47) and (3.8.48) we obtain (3.8.46). \square

3. Fixed transaction costs

3.8.3. Optimality

As a corollary we obtain the almost optimality of the stopped candidate $\widetilde{\varphi}^\varepsilon$ and the leading order certainty equivalent loss.

Corollary 3.8.5. *We have*

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E} (U(V_T^\varepsilon(\varphi))) = M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}) = \mathbb{E} (U(V_T^\varepsilon(\widetilde{\varphi}^\varepsilon))) + o(\varepsilon^{1/2}).$$

and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \left(\sup_{\varphi \in \mathfrak{A}^\varepsilon(x)} U^{-1} (\mathbb{E} (U(V_T^\varepsilon(\varphi)))) - U^{-1} (\mathbb{E} (U(V_T(\varphi^*)))) \right) \\ &= \sqrt{\frac{p}{3}} \mathbb{E}_Q \left(\int_0^T \sqrt{c_s^{\varphi^*} c_s^S} ds \right). \end{aligned}$$

Proof. According to Theorem 3.7.6 we have to make the correct choice of the constants c_1 and c_2 . We choose

$$c_1 = \varepsilon^{1/16} \quad \text{and} \quad c_2 = \frac{1}{c_1}.$$

These values also fulfill the condition from Theorem 3.7.6. Since $1 - \sqrt{1 - c_1} = O(\varepsilon^{1/16})$, $(\frac{1}{c_1})^{c_1} = O(1)$ and $\kappa_{c_1, 1/c_1} = O(1)$ as ε tends to 0, we obtain from equation (3.7.38)

$$\begin{aligned} \sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E} (U(V_T^\varepsilon(\varphi))) &\leq M_0 \exp(p(\gamma_0 - \varepsilon^{9/16})) + o(\varepsilon^{1/2}) \\ &= M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}) \end{aligned} \tag{3.8.49}$$

due to the Taylor series of the exponential function. With this, Theorem 3.8.4 and the fact that $\widetilde{\varphi}^\varepsilon$ is admissible we obtain

$$\begin{aligned} M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}) &\geq \sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E} (U(V_T^\varepsilon(\varphi))) \\ &\geq \mathbb{E} (U(V_T^\varepsilon(\widetilde{\varphi}^\varepsilon))) \\ &\geq M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}). \end{aligned}$$

For the leading order certainty equivalent loss we denote by a_ε the difference of $\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E} (U(V_T^\varepsilon(\varphi)))$ and $M_0 \exp(p\gamma_0)$ being an $o(\varepsilon^{1/2})$ - term. Then we get

$$\begin{aligned} & \sup_{\varphi \in \mathfrak{A}^\varepsilon} U^{-1} (\mathbb{E} (U(V_T^\varepsilon(\varphi)))) - U^{-1} (\mathbb{E} (U(V_T(\varphi^*)))) \\ &= U^{-1}(M_0 \exp(p\gamma_0) + a_\varepsilon) - U^{-1}(M_0) \\ &= -\frac{1}{p} \log \left(\exp(p\gamma_0) + \frac{a_\varepsilon}{M_0} \right) \\ &= -\gamma_0 + \log \left(1 + \frac{a_\varepsilon}{\exp(p\gamma_0 M_0)} \right). \end{aligned}$$

3.8. Lower bound for maximal expected utility and almost optimality of the candidate

Due to the Taylor series of the logarithm around 1, that has convergence radius 1, we know

$$\log(1 + y) = O(y), \quad (y \rightarrow 0),$$

so

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \log \left(1 + \frac{a_\varepsilon}{\exp(p(\gamma_0 M_0))} \right) = 0.$$

This shows that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \left(\sup_{\varphi \in \mathfrak{A}^\varepsilon(x)} U^{-1} (\mathbb{E} (U(V_T^\varepsilon(\varphi)))) - U^{-1} (\mathbb{E} (U(V_T(\varphi^*)))) \right) \\ &= \sqrt{\frac{p}{3}} \mathbb{E}_Q \left(\int_0^T \sqrt{c_s^{\varphi^*} c_s^S} ds \right). \end{aligned}$$

□

4. Extensions

4.1. Introduction

In this chapter we present two extensions of the verifications from Chapter 3 under weaker regularity assumptions. In section 4.2 we truncate the portfolio gamma $\rho = \frac{c^{\varphi^*}}{c^S}$ from below and above by two constants k_1 and k_2 . Hereby, one will avoid the conditions on the inverse portfolio gamma $C = \rho^{-1}$ and only require a condition on the Q -local drift of ρ , $b^{\rho, Q}$. In the verification, these constants will then be chosen as powers of the transaction cost ε , where the small value k_1 is supposed to converge to 0 and the large value k_2 will tend to infinity. Although the regularity assumptions are quite simple, they are not fulfilled for the application to utility indifference pricing of a European put option in the Black-Scholes model in subsection 6.2.2.

For that reason we consider a truncated portfolio gamma in section 4.3 as well, but it is also stopped if some process related to $b^{\rho, Q}$ reaches a certain threshold. In contrast to section 4.2, this threshold and the truncation boundaries of the portfolio gamma will depend on just one value k that is supposed to tend to infinity when the costs ε converge to 0. Furthermore, the regularity assumptions will slightly differ from the ones in section 4.2.

4.2. Truncated portfolio gamma and fewer regularity assumptions

4.2.1. Truncated no-trade corridor

Assumption 3.6.4 restricts the stock price models and corresponding frictionless optimizers to those with a strictly positive no-trade corridor $\Delta\varphi^+$. In applications, e.g. Chapter 6, one would like to overcome this restriction, but this would lead to strategies trading infinitely many times shortly before or when the corridor width becomes 0.¹ Furthermore it is not obvious how to apply the ansatz from section 3.5. Therefore, we introduce a truncated portfolio gamma

$$\rho_t^k := \left(\frac{c_t^{\varphi^*}}{c_t^S} \vee k_1 \right) \wedge k_2$$

and the corresponding truncated no-trade corridor

$$\Delta\varphi_t^{+,k} := \sqrt[4]{\frac{12\varepsilon}{p}} \sqrt[4]{\rho_t^k}, t \in [0, T]$$

¹Except for the case when the optimizer φ^* is constant.

4. Extensions

for some $k = (k_1, k_2) \in (0, 1) \times (1, \infty)$ that will again be some powers of the transaction costs ε and similarly

$$\begin{aligned}\alpha_t^k &:= \frac{\varepsilon}{(\Delta\varphi_t^{+,k})^4}, & \beta_t^k &:= 2\alpha_t^k(\Delta\varphi_t^{+,k})^2, \\ C_t^k &:= \left(\frac{c_t^S}{c_t^{\varphi^*}} \wedge \frac{1}{k_1} \right) \vee \frac{1}{k_2} = (\rho_t^k)^{-1}.\end{aligned}$$

Since γ is the process being responsible for the certainty equivalent loss, we do not change it. We also define the counterparts for the modified value process from section 3.5.5

$$\begin{aligned}\Delta\widehat{\varphi}^{+,k} &:= \sqrt[4]{\frac{12\varepsilon}{c_1}} \frac{1}{p} \sqrt[4]{\left(\frac{c_t^{\varphi^*}}{c_t^S} \vee k_1 \right) \wedge k_2}, \\ \widehat{\alpha}^k &:= \alpha^k, & \widehat{\beta}^k &:= 2\alpha^k (\Delta\widehat{\varphi}^{+,k})^2.\end{aligned}$$

For the constants we choose

$$k_1 = \varepsilon^{1/28}, \quad k_2 = \frac{1}{k_1}, \quad c_1 = \varepsilon^{1/16} \quad \text{and} \quad c_2 = \frac{1}{c_1}.$$

These choices will be justified in the proofs below. Then we use as candidates for the (transformed) value processes

$$\widetilde{X}^k(\varphi) := \exp\left((-p) \left(\Delta V^\varepsilon(\varphi) + F_t^{CE,k}(\varphi) \right)\right)$$

with $F^{CE,k}(\varphi) = f^{CE}(\alpha^k, \Delta\varphi) - \gamma$ to obtain a lower bound and

$$\widehat{X}^k(\varphi) := \exp\left((-p) \left(\Delta V^\varepsilon(\varphi) + \widehat{F}^{CE,k}(\varphi) \right)\right)$$

with $\widehat{F}^{CE,k}(\varphi) = \widehat{f}^{CE}(\alpha^k, \Delta\varphi) - \gamma + \delta(\varphi)$ for an upper bound of maximal expected utility.

Remark 4.2.1. *With the change-of-variable formula from Theorem B.1.3 and assuming that $\rho := \frac{c^{\varphi^*}}{c^S}$ is an Itô process, we see that $\rho^k := (\rho \vee k_1) \wedge k_2$ is a continuous semimartingale with Q -dynamics*

$$\begin{aligned}V_t^{\rho^k, Q} &= \int_0^t \mathbf{1}_{\{\rho_s \in (k_1, k_2)\}} b_s^{\rho, Q} ds + L_t(k_1) - L_t(k_2), \\ M_t^{\rho^k, Q} &= \int_0^t \mathbf{1}_{\{\rho_s \in (k_1, k_2)\}} dM_s^{\rho, Q},\end{aligned}$$

where L denotes a semimartingale local time of ρ . Thus, α^k, β^k and C^k are also continuous semimartingales.

The next lemma shows that the function $a \mapsto \mathbb{E}_Q(L_T(a))$ is bounded, which will be useful for the asymptotic analysis.

4.2. Truncated portfolio gamma and fewer regularity assumptions

Lemma 4.2.2 (Local time). *If $\int_0^T |b_s^{\rho, Q}| ds \in L_1(Q)$ we have*

$$\sup_{a \in (0, \infty)} \mathbb{E}_Q(L_T(a)) < \infty.$$

Proof. Applying the change-of-variable formula from Theorem B.1.3 to the process $\rho_a := \rho \wedge a$ yields

$$\begin{aligned} L_t(a) &= - \left(\rho_{a,t} - \rho_{a,0} - \int_0^t \mathbf{1}_{\{\rho_s \leq a\}} b_s^{\rho, Q} ds - \int_0^t \mathbf{1}_{\{\rho_s \leq a\}} dM_s^{\rho, Q} \right) \\ &= -\rho_{a,t} + \rho_{a,0} + \int_0^t \mathbf{1}_{\{\rho_s \leq a\}} b_s^{\rho, Q} ds + \int_0^t \mathbf{1}_{\{\rho_s \leq a\}} dM_s^{\rho, Q}, \end{aligned}$$

which implies

$$0 \leq L_t(k) \leq \rho_0 + \int_0^T |b_s^{\rho, Q}| ds + \int_0^t \mathbf{1}_{\{\rho_s \leq a\}} dM_s^{\rho, Q} \quad (4.2.1)$$

and hence

$$-\rho_0 - \int_0^T |b_s^{\rho, Q}| ds \leq \int_0^t \mathbf{1}_{\{\rho_s \leq a\}} dM_s^{\rho, Q}.$$

So $\int_0^t \mathbf{1}_{\{\rho_s \leq a\}} dM_s^{\rho, Q}$ is a supermartingale, and we see that

$$\mathbb{E}_Q \left(\int_0^T \mathbf{1}_{\{\rho_s \leq a\}} dM_s^{\rho, Q} \right) \leq 0.$$

Taking expectations in (4.2.1) for $t = T$ leads to

$$\mathbb{E}_Q(L_T(a)) \leq \rho_0 + \mathbb{E}_Q \left(\int_0^T |b_s^{\rho, Q}| ds \right) < \infty,$$

where the right-hand side does not depend on a . □

Assumptions 3.6.1 and 3.6.3 are still supposed to hold, but we replace Assumption 3.6.4 with the following:

Assumption 4.2.3. *Assume that ρ is an Itô process and that the following conditions hold:*

- $\exp(\pm 8pV_T(\varphi^*)) \in L_1(Q)$,
- $[S]_T \in L_2(Q)$, $\int_0^T |b_s^{\rho, Q}| ds \in L_1(Q)$ and
- $\|\varphi^*\|_{H_8(Q)} < \infty$.

Comparing the new assumptions to the old ones reveals that the new ones are weaker: all conditions on the inverse portfolio gamma $C = \rho^{-1}$ have been replaced by just one assumption on $b^{\rho, Q}$. By applying Itô's formula to $\rho = C^{-1}$, one obtains that the old conditions on C imply the new condition on $b^{\rho, Q}$. However, they are not required anymore since the truncated portfolio gamma is bounded from above and below by some suitable powers of the transaction cost ε that will not destroy the asymptotics. The following remark gives further explanations.

4. Extensions

Remark 4.2.4. From Remark 4.2.1 we know that

$$[\rho^k]_T = [M^{\rho^k, Q}]_T = \int_0^T \mathbf{1}_{\{\rho_s \in (k_1, k_2]\}} d[M^{\rho, Q}]_s.$$

According to Theorem B.1.3 this can be rewritten as

$$[\rho^k]_T = 2 \int_{-\infty}^{\infty} \mathbf{1}_{(k_1, k_2]}(a) L_T(a) da = 2 \int_{k_1}^{k_2} L_T(a) da.$$

Taking expectations and using Lemma 4.2.2 yields

$$\mathbb{E}_Q([\rho^k]_T) = 2 \int_{k_1}^{k_2} \mathbb{E}_Q(L_T(a)) da \leq \text{const.} (k_2 - k_1).$$

Applying Itô's formula to the process C^k shows

$$\begin{aligned} dC_s^k &= ((\rho_s^k)^{-2} b_s^{\rho, Q} + (\rho_s^k)^{-3} c_s^\rho) \mathbf{1}_{\{\rho_s \in (k_1, k_2]\}} ds + (-1)(\rho_s^k)^{-2} \mathbf{1}_{\{\rho_s \in (k_1, k_2]\}} dM_s^{\rho, Q} \\ &\quad + (-1)(\rho_s^k)^{-2} d(L_s(k_1) - L_s(k_2)) \end{aligned}$$

and this yields

$$\mathbb{E}_Q([C^k]_T) \leq \text{const.} \frac{k_2}{k_1^4}, \quad (4.2.2)$$

$$\mathbb{E}_Q \left(\int_0^T |b_s^{C^k, Q}| ds \right) \leq \text{const.} k_2 \left(\frac{1}{k_1^3} + \frac{1}{k_1^2} \right), \quad (4.2.3)$$

which will be required later.

4.2.2. Upper bound

Recall the process K

$$K_t = \frac{1}{c_2} \log \left(\frac{\gamma_t}{\varepsilon^{1/2} c_1} \right),$$

and the stopping time

$$\tau(\varphi) := \inf\{t \in [0, T] : \Delta V_t^\varepsilon(\varphi) < -K_t\} \wedge T,$$

that was defined for each admissible strategy $\varphi \in \mathfrak{A}^\varepsilon$ from subsection 3.7.2. As in this subsection, they will be helpful for the asymptotics of the Q -local drift $b^{\widehat{X}^k(\varphi), Q}$ of the transformed value process; hence, this section is structured in the same way: Asymptotics of the no-trade and trade regions followed by the construction of the asymptotic upper bound for maximal expected utility from terminal wealth.

Proposition 4.2.5 (No-trade region). *We have*

$$\sup_{\sigma \in \mathcal{T}} \sup_{\varphi \in \mathfrak{A}^\varepsilon} \left| \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta \varphi_s| < \Delta \widehat{\varphi}_s^{+, k}\}} ds \right) \right| = o(\varepsilon^{1/2}).$$

4.2. Truncated portfolio gamma and fewer regularity assumptions

Proof. (i) Similar calculations as in Propositions 3.7.1 and 3.7.4 show that the Q -local drift of $\widehat{X}^k(\varphi)$ is given by

$$b_s^{\widehat{X}^k(\varphi),Q} = \widehat{X}_s^k(\varphi) \left((-p)b_s^{\widehat{F}^{CE,k}(\varphi),Q} + \frac{p^2}{2} c_s^{\int_0^\cdot \Delta\varphi_u dS_u + \widehat{F}^{CE,k}(\varphi), \int_0^\cdot \Delta\varphi_u dS_u + \widehat{F}^{CE,k}(\varphi)} \right).$$

In the (interior of the) modified truncated no-trade region $\{|\Delta\varphi| < \Delta\widehat{\varphi}^{+,k}\}$ this leads to

$$\begin{aligned} b_s^{\widehat{X}^k(\varphi),Q} &= \widehat{X}_s^k(\varphi) \left((-p)b_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2} c_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma} \right) \\ &\quad + \widehat{X}_s^k(\varphi) \left((-p)b_s^{\delta(\varphi), Q} + \frac{p^2}{2} c_s^{\delta(\varphi)} + p^2 c_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma, \delta(\varphi)} \right). \end{aligned} \quad (4.2.4)$$

For the left summand observe that it can be written as

$$\begin{aligned} &\widehat{X}_s^k(\varphi) \left((-p)b_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2} c_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma} \right) \\ &= \widehat{X}_s^k(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^{k,i} + p \left(b_s^{\gamma, Q} + \widehat{\beta}_s^k c_s^{\varphi^*} \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \\ &= \widehat{X}_s^k(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^{k,i} + p(\sqrt{c_1} - 1) \sqrt{\frac{\varepsilon p}{3}} A_s \mathbf{1}_{\{\rho_s \in (k_1, k_2)\}} \right. \\ &\quad \left. + p \left(\widehat{\beta}_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) \mathbf{1}_{\{\rho_s \notin (k_1, k_2)\}} + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \mathbf{1}_{\{\rho_s \notin (k_1, k_2)\}} \right) \end{aligned}$$

in the modified no-trade region, where the processes $\widehat{H}^{k,j}$ can be obtained from e.g. line (3.5.15) by replacing α with α^k , β with $\widehat{\beta}^k$.

(ii) For a part of this left summand we obtain the following upper bound by using $\Delta\widehat{\varphi}^{+,k} \leq \Delta\varphi^{+,k}$:

$$\begin{aligned} G_1 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^{k,i} + p(\sqrt{c_1} - 1) \sqrt{\frac{\varepsilon p}{3}} A_s \mathbf{1}_{\{\rho_s \in (k_1, k_2)\}} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right| \\ &\leq \kappa_{c_1, c_2} \sum_{i=0}^8 \int_0^T |\Delta\varphi_s^{+,k}|^i |\widehat{H}_s^{k,i}| ds + p(\sqrt{c_1} - 1) \kappa_{c_1, c_2} \sqrt{\frac{\varepsilon p}{3}} \int_0^T A_s ds, \end{aligned}$$

since

$$\begin{aligned} \widehat{X}_s^k(\varphi) &= \exp \left((-p) \underbrace{(\Delta V_s^\varepsilon(\varphi))}_{\geq -K_s} + \underbrace{\widehat{F}^{CE,k}(\varphi)}_{\geq -\gamma_s - \bar{c}_1 \varepsilon} \right) \\ &\leq \exp(p(K_s + \gamma_s + \bar{c}_1 \varepsilon)) \leq \kappa_{c_1, c_2}, \quad \text{on } [0, \tau(\varphi)], \end{aligned}$$

which can be concluded from and (3.7.33). Replacing $\alpha, \widehat{\beta}$ and C with $\alpha^k, \widehat{\beta}^k$ and C^k in the proof of Proposition 3.7.4 yields

$$\begin{aligned} \mathbb{E}_Q(G_1) &= \kappa_{c_1, c_2} (O(k_1^{-2} k_2^{5/4} \varepsilon^{3/4}) + O(k_1^{-4} k_2^3 \varepsilon)) + \kappa_{c_1, c_2} p (1 - \sqrt{c_1}) \sqrt{\frac{\varepsilon p}{3}} Y_0 \\ &= \kappa_{c_1, c_2} (O(k_1^{-2} k_2^{5/4} \varepsilon^{3/4}) + O(k_1^{-4} k_2^3 \varepsilon) + O(c_1 \varepsilon^{1/2})) = o(\varepsilon^{1/2}) \end{aligned} \quad (4.2.5)$$

4. Extensions

uniformly in φ and σ . A more detailed argument can be found in section A.2 of the Appendix. For the remaining term observe that

$$\begin{aligned}
G_2 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(p \left(\widehat{\beta}_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \right. \\
&\quad \left. \cdot \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}, \rho_s \leq k_1\}} ds \right| \\
&\leq \kappa_{c_1, c_2} \left[\left(\frac{4}{\sqrt{3}} p^{3/2} \frac{\varepsilon^{1/2}}{k_1^{1/2}} \right) \left(\int_0^T c_s^{\varphi^*} \mathbf{1}_{\{\rho_s \leq k_1\}} ds \right) + \frac{p^{3/2} \varepsilon^{1/2}}{\sqrt{3}} \int_0^T (c_s^{\varphi^*} c_s^S)^{1/2} \mathbf{1}_{\{\rho_s \leq k_1\}} ds \right. \\
&\quad \left. + \sqrt{3} p^{3/2} (\varepsilon k_1)^{1/2} \left(\int_0^T c_s^S \mathbf{1}_{\{\rho_s \leq k_1\}} ds \right) \right]
\end{aligned}$$

since $\widehat{\beta}^k \leq \beta^k$. From $c_s^{\varphi^*} \leq c_s^S k_1$ on $\{\rho \leq k_1\}$ we can conclude

$$E_Q(G_2) \leq \kappa_{c_1, c_2} O(k_1^{1/2} \varepsilon^{1/2}) = o(\varepsilon^{1/2}). \quad (4.2.6)$$

Furthermore, we obtain the same estimates on $\{\rho > k_2\}$:

$$\begin{aligned}
G_3 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(p \left(\widehat{\beta}_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \right. \\
&\quad \left. \cdot \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}, \rho_s > k_2\}} ds \right| \\
&\leq \kappa_{c_1, c_2} \left[\frac{4}{\sqrt{3}} p^{3/2} \frac{\varepsilon^{1/2}}{k_2^{1/2}} \left(\int_0^T c_s^{\varphi^*} \mathbf{1}_{\{\rho_s > k_2\}} ds \right) \frac{p^{3/2} \varepsilon^{1/2}}{\sqrt{3}} \int_0^T (c_s^{\varphi^*} c_s^S)^{1/2} \mathbf{1}_{\{\rho_s > k_2\}} ds \right. \\
&\quad \left. + \sqrt{3} p^{3/2} (\varepsilon k_2)^{1/2} \left(\int_0^T c_s^S \mathbf{1}_{\{\rho_s > k_2\}} ds \right) \right].
\end{aligned}$$

On $\{\rho > k_2\}$, the inequality $c^S < \frac{c^{\varphi^*}}{k_2}$ holds, which then implies

$$\mathbb{E}_Q(G_3) \leq \kappa_{c_1, c_2} O(k_2^{-1/2} \varepsilon^{1/2}) = o(\varepsilon^{1/2}) \quad (4.2.7)$$

for ε tending to 0.

(iii) For the right summand of (4.2.4) we need the Q -dynamics of $\delta(\varphi)$ on $[0, \tau(\varphi)]$,

4.2. Truncated portfolio gamma and fewer regularity assumptions

that were computed in (3.7.31). Inserting these leads to

$$\begin{aligned}
& \widehat{X}_s^k(\varphi) \left((-p)b_s^{\delta(\varphi),Q} + \frac{p^2}{2}c_s^{\delta(\varphi)} + p^2c_s^{\widehat{f}^{CE}(\alpha^k, \Delta\varphi) - \gamma, \delta(\varphi)} \right) \\
&= \widehat{X}_s^k(\varphi) \left((-p)\frac{c_2^2}{2}\delta_s(\varphi)(\Delta\varphi_s)^2c_s^S + \frac{p^2}{2}c_2^2\delta_s(\varphi)^2(\Delta\varphi_s)^2c_s^S \right. \\
&\quad \left. + \frac{p^2}{2}(-c_2)\delta_s(\varphi)\Delta\varphi_s \underbrace{c_s^{\widehat{f}^{CE}(\alpha^k, \Delta\varphi) - \gamma, S}}_{=(\Delta\varphi_s)^4c_s^{\alpha^k, S} - (\Delta\varphi_s)^2c_s^{\widehat{\beta}^k, S} - (4\alpha_s^k(\Delta\varphi_s)^3 - 2\widehat{\beta}_s^k\Delta\varphi_s)c_s^{\varphi^*, S} - c_s^{\gamma, S}} \right) \\
&= \widehat{X}_s^k(\varphi) \left((-p)\frac{c_2^2}{2}\delta_s(\varphi)(\Delta\varphi_s)^2c_s^S + \frac{p^2}{2}c_2^2\delta_s(\varphi)^2(\Delta\varphi_s)^2c_s^S \right. \\
&\quad \left. + \frac{p^2}{2}(-c_2)\delta_s(\varphi)((\Delta\varphi_s)^5c_s^{\alpha^k, S} - (\Delta\varphi_s)^3c_s^{\widehat{\beta}^k, S} \right. \\
&\quad \left. - (4\alpha_s^k(\Delta\varphi_s)^4 - 2\widehat{\beta}_s^k(\Delta\varphi_s)^2)c_s^{\varphi^*, S} - \Delta\varphi_sc_s^{\gamma, S} \right)
\end{aligned}$$

in the modified truncated no-trade region $\{|\Delta\varphi| < \Delta\widehat{\varphi}^{+,k}\}$. Observe that with $\delta_s(\varphi) \leq \gamma_s \leq \sqrt{\frac{\varepsilon p}{3}}\kappa_Y$

$$\begin{aligned}
G_4 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left((-p)b_s^{\delta(\varphi),Q} + \frac{p^2}{2}c_s^{\delta(\varphi)} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right| \\
&\leq \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) c_2^2 (\Delta\varphi_s)^2 c_s^S \left| \frac{p^2}{2}\delta_s(\varphi)^2 - \frac{p}{2}\delta_s(\varphi) \right| \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \\
&\leq \int_0^T \kappa_{c_1, c_2} c_2^2 (\Delta\widehat{\varphi}_s^{+,k})^2 c_s^S \left(\frac{p^2}{2}(\gamma_s)^2 + \frac{p}{2}\gamma_s \right) ds \\
&\leq \kappa_{c_1, c_2} c_2^2 \left(\frac{p^2}{2} \frac{\varepsilon p}{3} (\kappa_Y)^2 + \frac{p}{2} \sqrt{\frac{\varepsilon p}{3}} \kappa_Y \right) \int_0^T (\Delta\widehat{\varphi}_s^{+,k})^2 c_s^S ds.
\end{aligned}$$

Taking expectations and using $\Delta\widehat{\varphi}_s^{+,k} \leq \left(\frac{12k_2\varepsilon}{p}\right)^{1/4}$ yields

$$\begin{aligned}
\mathbb{E}_Q(G_4) &\leq \kappa_{c_1, c_2} c_2^2 \left(\frac{p^2}{2} \frac{\varepsilon p}{3} (\kappa_Y)^2 + \frac{p}{2} \sqrt{\frac{\varepsilon p}{3}} \kappa_Y \right) \sqrt{\frac{12k_2\varepsilon}{p}} \mathbb{E}_Q([S]_T) \\
&= \kappa_{c_1, c_2} O(c_2^2 k_2^{1/2} \varepsilon) = o(\varepsilon^{1/2}). \tag{4.2.8}
\end{aligned}$$

We consider the remaining term that is given by

$$\begin{aligned}
G_5 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(p^2 c_s^{\widehat{f}^{CE}(\alpha^k, \widehat{\beta}^k, \Delta\varphi) - \gamma, \delta(\varphi)} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right| \\
&\leq \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) c_2 \delta_s(\varphi) \left(\sum_{j=0}^8 |\Delta\varphi_s|^j |\widehat{H}_{j,s}^k| \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \\
&\leq \int_0^T \kappa_{c_1, c_2} c_2 \gamma_s \left(\sum_{j=0}^8 |\Delta\widehat{\varphi}_s^{+,k}|^j |\widehat{H}_{j,s}^k| \right) ds \\
&\leq \kappa_{c_1, c_2} c_2 \sqrt{\frac{\varepsilon p}{3}} \kappa_Y \int_0^T \left(\sum_{j=0}^8 |\Delta\widehat{\varphi}_s^{+,k}|^j |\widehat{H}_{j,s}^k| \right) ds
\end{aligned}$$

4. Extensions

and taking expectations gives as for G_1 in (4.2.5)

$$\mathbb{E}_Q(G_5) = \kappa_{c_1, c_2} (O(c_2 k_1^{-2} \varepsilon^{5/4}) + O(c_2 k_1^{-4} k_2^3 \varepsilon^{3/2})) = o(\varepsilon^{1/2}). \quad (4.2.9)$$

(iv) Summing up inequalities (4.2.5), (4.2.6), (4.2.7), (4.2.8) and (4.2.9) yields the desired result. \square

Proposition 4.2.6 (Trade region). *We have*

$$\inf_{\varphi \in \mathfrak{A}^\varepsilon} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| \geq \Delta\widehat{\varphi}_s^{+, k}\}} ds \right) \geq o(\varepsilon^{1/2}).$$

Proof. (i) In $\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+, k}\}$ the Q -local drift of $\widehat{X}^k(\varphi)$ is given by

$$\begin{aligned} b_s^{\widehat{X}^k(\varphi), Q} &= \widehat{X}_s^k(\varphi) \left((-p)(-b_s^{\gamma, Q} + b_s^{\delta(\varphi), Q}) + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi) - \gamma + \delta(\varphi)} \right) \\ &= \widehat{X}_s^k(\varphi) \left(p b_s^{\gamma, Q} - p b_s^{\delta(\varphi), Q} + \frac{p^2}{2} (c_s^{\Delta V^\varepsilon(\varphi)} - 2c_s^{\Delta V^\varepsilon(\varphi), \gamma} + c_s^\gamma \right. \\ &\quad \left. + 2c_s^{\Delta V^\varepsilon(\varphi, \delta(\varphi))} - 2c_s^{\gamma, \delta(\varphi)} + c_s^{\delta(\varphi)}) \right) \end{aligned}$$

and in $\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+, k}, \rho_s \leq k_2\}$ we get the following lower bound for $p b_s^{\gamma, Q}$

$$\begin{aligned} p b_s^{\gamma, Q} &= -\frac{p^2}{6} (\Delta\varphi_s^+)^2 c_s^S = -\frac{p^2}{6\bar{c}_1^{1/2}} (\Delta\widehat{\varphi}_s^+)^2 c_s^S \\ &\geq -\frac{p^2}{6\bar{c}_1^{1/2}} (\Delta\widehat{\varphi}_s^{+, k})^2 c_s^S \\ &\geq -\frac{p^2}{6\bar{c}_1^{1/2}} (\Delta\varphi_s)^2 c_s^S \end{aligned}$$

since the truncated corridor $\Delta\widehat{\varphi}_s^{+, k}$ is always larger than the modified corridor $\Delta\widehat{\varphi}_s^+$ from the original problem on this set. This implies

$$p b_s^{\gamma, Q} + \frac{p^2}{2} c_s^{\Delta V^\varepsilon(\varphi)} \geq p^2 \left(\frac{1}{2} - \frac{1}{6\bar{c}_1^{1/2}} \right) c_s^{\Delta V^\varepsilon(\varphi)},$$

which then yields in $\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+, k}, \rho_s \leq k_2\}$ the lower bound

$$b_s^{\widehat{X}^k(\varphi), Q} \geq \widehat{X}_s^k(\varphi) (a_s c_s^{\Delta V^\varepsilon(\varphi)} + b_s c_s^{\Delta V^\varepsilon(\varphi), \gamma} + p^2 c_s^\gamma), \quad (4.2.10)$$

where a and b are as in (3.7.34). Recalling that $a_s > 0$ for sufficiently small ε , we

4.2. Truncated portfolio gamma and fewer regularity assumptions

obtain

$$\begin{aligned}
& \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+,k}, \rho_s \leq k_2\}} ds \right) \\
& \geq p^2 \underbrace{(-\kappa_\varepsilon^{c_2})}_{<0} \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) c_s^\gamma \mathbf{1}_{\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+,k}, \rho_s \leq k_2\}} ds \right) \\
& \geq p^2 (-\kappa_\varepsilon^{c_2}) \mathbb{E}_Q \left(\int_0^T \exp(p(K_s + \bar{c}_1 \varepsilon + \gamma_s)) c_s^\gamma ds \right) \\
& \geq p^2 (-\kappa_\varepsilon^{c_2}) \kappa_{c_1, c_2} \mathbb{E}_Q([\gamma]_T) \\
& = \frac{p^3}{3} (-\kappa_\varepsilon^{c_2}) \kappa_{c_1, c_2} \varepsilon \mathbb{E}_Q([Y]_T) = o(\varepsilon^{1/2})
\end{aligned} \tag{4.2.11}$$

by recalling that $\kappa_\varepsilon^{c_2}$ is nonnegative and

$$\kappa_\varepsilon^{c_2} = \frac{2 + \sqrt{\varepsilon} 9p^3 \kappa_Y + c_2^2 \varepsilon \kappa_Y}{1 - \sqrt{\varepsilon} 9p^3 \kappa_Y} = O(1) + c_2^2 O(\varepsilon).$$

In $\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+,k}, \rho_s > k_2\}$ we know that $c_s^S < k_2^{-1} c_s^{\varphi^*}$, so

$$pb_s^{\gamma, Q} = -p \sqrt{\frac{p\varepsilon}{3}} (c_s^S c_s^{\varphi^*})^{1/2} \geq -\frac{p^{3/2} \varepsilon^{1/2}}{\sqrt{3}} k_2^{-1} c_s^{\varphi^*},$$

and this leads to the lower bound for the drift of $b^{\widehat{X}^k(\varphi), Q}$

$$b_s^{\widehat{X}^k(\varphi), Q} \geq \widehat{X}_s^k(\varphi) (\widehat{a}_s c_s^{\Delta V^\varepsilon(\varphi)} + b_s c_s^{\Delta V^\varepsilon(\varphi), \gamma} + p^2 c_s^\gamma) - \widehat{X}_s^k(\varphi) \frac{p^{3/2} \varepsilon^{1/2}}{\sqrt{3}} k_2^{-1} c_s^{\varphi^*}$$

with \widehat{a}_s only differing from a_s by an additive constant:

$$\widehat{a}_s = \frac{p^2}{2} - \frac{p}{2} \delta_s(\varphi) c_2^2 - \frac{p^2}{2} \delta_s(\varphi) c_2 + \frac{p^2}{2} c_2 \delta_s(\varphi)^2.$$

Then, $\widehat{a}_s > 0$ for a sufficiently small ε and by repeating the above argument with \widehat{a}_s instead of a_s for the case $\{\rho > k_2\}$ we obtain

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| > \Delta\widehat{\varphi}_s^{+,k}, \rho_s > k_2\}} ds \right) & \geq O(\varepsilon) + O(c_2^2 \varepsilon^2) + O(k_2^{-1/2} \varepsilon^{1/2}) \\
& = o(\varepsilon^{1/2}).
\end{aligned}$$

(ii) At the boundaries of the modified truncated no-trade region $\{|\Delta\varphi_s| = \Delta\widehat{\varphi}_s^{+,k}\}$ we have

$$\begin{aligned}
& b_s^{\widehat{X}^k(\varphi), Q} \\
& = \widehat{X}_s^k(\varphi) \left((-p) b_s^{\widehat{F}^{CE, k}(\varphi), Q} + \frac{p^2}{2} c_s^\gamma \int_0^{\cdot} \Delta\varphi_u dS_u + \widehat{F}^{CE, k}(\varphi) \right) \\
& = \frac{1}{2} \widehat{X}_s^k(\varphi) \left((-p) b^{\alpha^k(\Delta\varphi)^4 - \widehat{\beta}^k(\Delta\varphi)^2 - \gamma + \delta(\varphi), Q} + \frac{p^2}{2} c_s^\gamma \int_0^{\cdot} \Delta\varphi_u dS_u + \alpha^k(\Delta\varphi)^4 - \widehat{\beta}^k(\Delta\varphi)^2 - \gamma + \delta(\varphi) \right) \\
& \quad + \frac{1}{2} \widehat{X}_s^k(\varphi) \left((-p) b^{-\gamma + \delta(\varphi), Q} + \frac{p^2}{2} c_s^\gamma \int_0^{\cdot} \Delta\varphi_u dS_u - \gamma + \delta(\varphi) \right).
\end{aligned} \tag{4.2.12}$$

4. Extensions

So we obtain from (4.2.12), Proposition 4.2.5 and (4.2.11)

$$\begin{aligned} & \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| = \widehat{\Delta\varphi_s^+}\}} ds \right) \\ & \geq -\kappa_{c_1, c_2} \left[O(k_1^{-2} k_2^{5/4} \varepsilon^{3/4}) + O(k_1^{-4} k_2^3 \varepsilon) + O(c_1 \varepsilon^{1/2}) + O(k_1^{-1/4} \varepsilon^{3/4}) + O(k_1^{1/2} \varepsilon^{1/2}) \right. \\ & \quad + O(k_2^{-1/2} \varepsilon^{1/2}) + O(c_2^2 k_2^{1/2} \varepsilon) + O(c_2 k_1^{-4} k_2^3 \varepsilon^{3/2}) + O(c_2 \varepsilon^2) + O(c_2 k_2^{1/4} \varepsilon^{3/4}) \\ & \quad \left. + O(\varepsilon) + O(c_2^2 \varepsilon^2) + O(k_2^{-1/2} \varepsilon^{1/2}) \right] = o(\varepsilon^{1/2}). \end{aligned}$$

□

Theorem 4.2.7. *We have*

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon(x)} \mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq M_0 \exp(p(\gamma_0 - c_1 \varepsilon^{1/2})) + o(\varepsilon^{1/2}). \quad (4.2.13)$$

Proof. (i) Similar to Theorem 3.7.3 we have

$$\mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq \exp(-p\varepsilon) M_0 \mathbb{E}_Q(\widehat{X}_T^k(\varphi)).$$

and as in (3.7.39)

$$\mathbb{E}_Q(\widehat{X}_T^k(\varphi)) \geq (1 - h_\varepsilon) \mathbb{E}_Q(X_{\tau(\varphi)}(\varphi)) + \mathbb{E}_Q(\widehat{X}_{\tau(\varphi)}^k(\varphi)). \quad (4.2.14)$$

(ii) From (3.7.40) we recall that

$$\mathbb{E}_Q(X_{\tau(\varphi)}(\varphi)) \leq \left(\frac{1}{c_1}\right)^{1/c_2} O(1). \quad (4.2.15)$$

(iii) We can repeat the localization argument from Theorem 3.7.3: Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for the local martingale $M^{\widehat{X}^k(\varphi), Q}$ and define $Y_n := \widehat{X}_{\tau(\varphi) \wedge \tau_n}^k(\varphi)$ for all $n \in \mathbb{N}$. Then $\{Y_n, n \in \mathbb{N}\}$ is a bounded set in $L_2(Q)$, which implies that it is uniformly integrable. As in the proof of Theorems 3.7.3 and 3.7.6 we then can show

$$\mathbb{E}_Q(\widehat{X}_{\tau(\varphi)}^k(\varphi)) = \lim_{n \rightarrow \infty} \mathbb{E}_Q(Y_n). \quad (4.2.16)$$

(iv) Applying Theorem B.1.1 on Y_n and taking expectations yields

$$\begin{aligned} & \mathbb{E}_Q(Y_n) \\ & = \mathbb{E}_Q \left(\widehat{X}_0^k(\varphi) + \int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}^k(\varphi), Q} ds + M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q} + L_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi)} + \sum_{0 \leq s < \tau(\varphi) \wedge \tau_n} \Delta^+ \widehat{X}_s^k(\varphi) \right), \end{aligned}$$

where $L^{\widehat{X}^k(\varphi)}$ denotes the term introduced by the local time L of ρ . We obtain

$$\mathbb{E}_Q(Y_n) \geq \widetilde{X}_0^k(\varphi) + \mathbb{E}_Q \left(\int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}^k(\varphi), Q} ds \right) + \mathbb{E}_Q \left(M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q} \right) + \mathbb{E}_Q \left(L_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi)} \right), \quad (4.2.17)$$

4.2. Truncated portfolio gamma and fewer regularity assumptions

since the jumps of $\widehat{X}^k(\varphi)$ are always positive, which is shown as in part (iv) of the proof of Theorem 3.7.6:

$$\begin{aligned}\Delta^+ \widehat{X}_s^k(\varphi) &= \widehat{X}_{s+}^k(\varphi) - \widehat{X}_s^k(\varphi) \\ &= \exp((-p)(V_{s+}^\varepsilon(\varphi) - V_s(\varphi^*) + \widehat{F}_{s+}^{CE,k}(\varphi))) \\ &\quad - \exp((-p)(V_s^\varepsilon(\varphi) - V_s(\varphi^*) + \widehat{F}_s^{CE,k}(\varphi))).\end{aligned}$$

At a transaction time of φ we have $V_{s+}^\varepsilon(\varphi) = V_s^\varepsilon(\varphi) - \varepsilon$ and

$$\Delta^+ \delta_s(\varphi) \leq c_1 \varepsilon$$

for a sufficiently small ε , which implies for the modified error term

$$\begin{aligned}\widehat{F}_{s+}^{CE,k}(\varphi) &= \underbrace{\widehat{f}^{CE}(\alpha_s^k, \Delta\varphi_{s+})}_{\leq \widehat{f}^{CE}(\alpha_s^k, \Delta\varphi_s) + c_1 \varepsilon} - \gamma_s + \underbrace{\delta_{s+}(\varphi)}_{\leq \delta_s(\varphi) + c_1 \varepsilon} \leq \widehat{F}_{s+}^{CE,k}(\varphi) + \varepsilon,\end{aligned}$$

so

$$V_{s+}^\varepsilon(\varphi) + \widehat{F}_{s+}^{CE,k}(\varphi) \leq V_s^\varepsilon(\varphi) + \widehat{F}_s^{CE,k}(\varphi).$$

Since jumps only occur at the transaction times of φ , this shows $\Delta^+ \widehat{X}_s^k(\varphi) \geq 0$.

(v) In the next steps we analyze the drift and martingale components in equation (4.2.17). Since the stopped local martingale $M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q}$ is a true Q -martingale, we have

$$\mathbb{E}_Q \left(M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q} \right) = M_0^{\widehat{X}^k(\varphi), Q} = 0.$$

For the drift component we can use Propositions 4.2.5 and 4.2.6, which imply

$$\mathbb{E}_Q \left(\int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}^k(\varphi), Q} ds \right) \geq o(\varepsilon^{1/2})$$

independent of φ and n .

(vi) Due to remark 4.2.4 and Theorem B.1.1 we see that the term introduced by the local time of ρ equals

$$\begin{aligned}L_t^{\widehat{X}^k(\varphi)} &= \int_0^t \widehat{X}_s^k(\varphi) \left((\Delta\varphi_s)^4 \frac{p}{12} (-1)(\rho_s^k)^{-2} + (\Delta\varphi_s)^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} (\rho_s^k)^{-3/2} \right) \\ &\quad \cdot \left(\mathbf{1}_{\{|\Delta\varphi_s| < \Delta\widehat{\varphi}_s^{+,k}\}} + \frac{1}{2} \mathbf{1}_{\{|\Delta\varphi_s| = \Delta\widehat{\varphi}_s^{+,k}\}} \right) d(L_s(k_1) - L_s(k_2)),\end{aligned}$$

which implies

$$\begin{aligned}\left| L_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi)} \right| &\leq \frac{3}{2} \kappa_{c_1, c_2} \int_0^{\tau(\varphi) \wedge \tau_n} \left((\Delta\varphi_s^{+,k})^4 \frac{p}{12} (\rho_s^k)^{-2} + (\Delta\varphi_s^{+,k})^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} (\rho_s^k)^{-3/2} \right) \\ &\quad d(L_s(k_1) + L_s(k_2)) \\ &= \frac{3}{2} \kappa_{c_1, c_2} \int_0^{\tau(\varphi) \wedge \tau_n} \left(\frac{12\varepsilon}{p} \rho_s^k \frac{p}{12} (\rho_s^k)^{-2} + \left(\frac{12\varepsilon}{p} \rho_s^k \right)^{1/2} \frac{1}{2} \left(\frac{\varepsilon p}{3} \right)^{1/2} (\rho_s^k)^{-3/2} \right) \\ &\quad d(L_s(k_1) + L_s(k_2)) \\ &= 3\kappa_{c_1, c_2} \int_0^{\tau(\varphi) \wedge \tau_n} \varepsilon (\rho_s^k)^{-1} d(L_s(k_1) + L_s(k_2)) \\ &\leq 3\kappa_{c_1, c_2} \frac{\varepsilon}{k} (L_{\tau(\varphi) \wedge \tau_n}(k_1) + L_{\tau(\varphi) \wedge \tau_n}(k_2)) \leq 3 \exp(p) \frac{\varepsilon}{k_1} (L_T(k_1) + L_T(k_2)),\end{aligned}$$

4. Extensions

since $L(a)$ is an increasing process. By Lemma 4.2.2 we know

$$\sup_{a \in (0, \infty)} \mathbb{E}_Q(L_T(a)) < \infty.$$

This implies

$$\mathbb{E}_Q \left(\left| L_{\tau^k(\varphi)}^{\widehat{X}^k(\varphi)} \right| \right) = \kappa_{c_1, c_2} O(k_1^{-1} \varepsilon) = o(\varepsilon^{1/2}), \quad (4.2.18)$$

which is dominated by the $O(k_1^{-4} k_2^3 \varepsilon)$ term in the drift component.

(vii) In view of equation (4.2.17) we end up with

$$M_0 \mathbb{E}_Q(Y_n) \leq M_0 \exp(p(\gamma_0 - c_1 \varepsilon^{1/2})) + o(\varepsilon^{1/2}),$$

where the right-hand side does not depend on φ nor n . Together with (4.2.14), (4.2.16), (4.2.15) and (4.2.18) we obtain the claimed upper bound for the maximal expected utility. \square

4.2.3. Lower bound

We want to show that the asymptotic upper bound $M_0 p \gamma_0$ is also an asymptotic lower bound for the maximal expected utility to verify the asymptotic optimality of the candidate under the weaker regularity assumptions. In order to prove this we modify the proof of Theorem 3.8.4. Furthermore, similar to section 3.8, we define the stopping times

$$\begin{aligned} \tau^{\varepsilon, 1, k} &:= \inf\{t \in [0, T] : \Delta V_t^\varepsilon(\varphi^{\varepsilon, k}) > -1\} \wedge T, \\ \tau^{\varepsilon, 2, k} &:= \inf\left\{t \in [0, T] : V_t^\varepsilon(\varphi^{\varepsilon, k}) < -\frac{1}{\varepsilon}\right\} \wedge T \quad \text{and} \\ \tau^{\varepsilon, k} &:= \tau^{\varepsilon, 1, k} \wedge \tau^{\varepsilon, 2, k} \end{aligned}$$

and pass to the strategy $\widetilde{\varphi}^{\varepsilon, k} = \varphi^{\varepsilon, k} \mathbf{1}_{[0, \tau^{\varepsilon, k}]}$.

Theorem 4.2.8. *The candidate strategy $\widetilde{\varphi}^{\varepsilon, k}$ fulfills*

$$\mathbb{E}(U(V_T^\varepsilon(\widetilde{\varphi}^{\varepsilon, k}))) \geq M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}).$$

Proof. (i) As in Theorem 3.8.4 we have

$$\mathbb{E}(U(V_T^\varepsilon(\widetilde{\varphi}^{\varepsilon, k}))) \geq \mathbb{E}\left(\widetilde{v}_T^{\varepsilon, k}(\widetilde{\varphi}^{\varepsilon, k})\right) = M_0 \mathbb{E}_Q\left(\widetilde{X}_T^k(\widetilde{\varphi}^{\varepsilon, k})\right)$$

and similarly (except for the process introduced by the local time of ρ)

$$\begin{aligned} \widetilde{X}_T^k(\widetilde{\varphi}^{\varepsilon, k}) &\geq \widetilde{X}_0^k(\widetilde{\varphi}^{\varepsilon, k}) + \int_0^{\tau^{\varepsilon, k}} b_s^{\widetilde{X}^k(\widetilde{\varphi}^{\varepsilon, k}), Q} ds + M_{\tau^{\varepsilon, k}}^{\widetilde{X}^k(\widetilde{\varphi}^{\varepsilon, k}), Q} + L_{\tau^{\varepsilon, k}}^{\widetilde{X}^k(\varphi)} \\ &\quad + \widetilde{X}_T^k(\widetilde{\varphi}^{\varepsilon, k}) - X_{\tau^{\varepsilon, k}}(\varphi^{\varepsilon, k}). \end{aligned}$$

4.2. Truncated portfolio gamma and fewer regularity assumptions

As before we analyze all these terms separately. For the local Q -drift of the process $\tilde{X}^k(\varphi^{\varepsilon,k})$ we know that

$$b_s^{\tilde{X}^k(\varphi^{\varepsilon,k}),Q} = \tilde{X}_s^k(\varphi^{\varepsilon,k}) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i H_s^k + \left(p \left(\beta_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \mathbf{1}_{\{\rho_s \notin (k_1, k_2]\}} \right),$$

where the processes H^k can be obtained from line (3.5.15) by replacing α and β with α^k and β^k . Furthermore, on $[0, \tau^{\varepsilon,1,k}]$ we have $\tilde{X}_s^k(\varphi^{\varepsilon,k}) \leq \exp(p(1 + \varepsilon\sqrt{\varepsilon p/3\kappa_Y}))$. Combining this with the arguments for $E_Q(G_1)$ in the proof of Proposition 4.2.5 in line (4.2.5) and Proposition 3.8.3 we can conclude

$$\begin{aligned} \mathbb{E}_Q \left(\left| \int_0^{\tau^{\varepsilon,1,k} \wedge \sigma} \tilde{X}_s^k(\varphi^{\varepsilon,k}) \left(\sum_{i=0}^8 (\Delta\varphi_s^{\varepsilon,k})^i H_s^k \right) ds \right| \right) &= O(k_1^{-2} k_2^{5/4} \varepsilon^{3/4}) + O(k_1^{-4} k_2^3 \varepsilon) \\ &= o(\varepsilon^{1/2}) \end{aligned} \quad (4.2.19)$$

uniformly in $\sigma \in \mathcal{T}$.

(ii) For the remaining term we can use the same arguments as in Proposition 4.2.5 and obtain

$$\begin{aligned} \mathbb{E}_Q \left(\left| \int_0^{\tau^{\varepsilon,1,k} \wedge \sigma} \tilde{X}_s^k(\varphi^{\varepsilon,k}) \left(p \left(\beta_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s^{\varepsilon,k})^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \cdot \mathbf{1}_{\{\rho_s \notin (k_1, k_2]\}} ds \right| \right) \\ = O(k_1^{1/2} \varepsilon^{1/2}) + O(k_2^{-1/2} \varepsilon^{1/2}) = o(\varepsilon^{1/2}). \end{aligned}$$

(iii) From line (3.7.25) we know

$$M_t^{\tilde{X}^k(\varphi^{\varepsilon,k}),Q} = \int_0^t (-p) \tilde{X}_s^k(\varphi^{\varepsilon,k}) dM_s^{F^{CE,k}(\varphi^{\varepsilon,k}),Q} + \int_0^t (-p) \tilde{X}_s^k(\varphi^{\varepsilon,k}) \Delta\varphi_s^{\varepsilon,k} dS_s.$$

As in the proof of theorem 3.8.4 we obtain

$$\left[M^{\tilde{X}^k(\varphi^{\varepsilon,k}),Q} \right]_{\tau^{\varepsilon,k}} \leq 2p^2 \exp(2p) \left(\left[M^{F^{CE,k}(\varphi^{\varepsilon,k}),Q} \right]_T + \int_0^T (\Delta\varphi_s^{+,k})^2 d[S]_s \right).$$

One shows similarly to this proof $[M^{F^{CE,k},Q}]_T, \int_0^T (\Delta\varphi_s^{+,k})^2 d[S]_s \in L_1(Q)$, which then implies that the Q -expectation of $M_{\tau^{\varepsilon,k}}^{\tilde{X}^k(\varphi^{\varepsilon,k}),Q}$ vanishes.

(iv) With remarks 4.2.4 and B.1.2 we see that the term introduced by the local time of ρ equals

$$L_t^{\tilde{X}^k(\varphi^{\varepsilon,k})} = \int_0^t \tilde{X}_s^k(\varphi^{\varepsilon,k}) \left((\Delta\varphi_s^{\varepsilon,k})^4 \frac{p}{12} (-1)(\rho_s^k)^{-2} + (\Delta\varphi_s^{\varepsilon,k})^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} (\rho_s^k)^{-3/2} \right) d(L_s(k_1) - L_s(k_2)),$$

4. Extensions

which implies as in the proof of Theorem 4.2.7

$$\left| L_{\tau^{\varepsilon,k}}^{\tilde{X}^k(\varphi^{\varepsilon,k})} \right| \leq 2 \exp(p) \frac{\varepsilon}{k_1} L_{\tau^{\varepsilon,k}}(k) \leq 2 \exp(p) \frac{\varepsilon}{k_1} (L_T(k_1) + L_T(k_2))$$

since $L(a)$ is an increasing process. Recalling that $\mathbb{E}_Q(L_T(a))$ is bounded in a by Lemma 4.2.2, we obtain

$$\mathbb{E}_Q \left(\left| L_{\tau^{\varepsilon,k}}^{\tilde{X}^k(\varphi^{\varepsilon,k})} \right| \right) = O(k_1^{-1} \varepsilon). \quad (4.2.20)$$

This is dominated by the $O(k_1^{-4} k_2^3 \varepsilon)$ term of the drift.

(v) We will show that the difference $\tilde{X}_T^k(\tilde{\varphi}^{\varepsilon,k}) - X_{\tau^{\varepsilon,k}}(\varphi^{\varepsilon,k})$ is of small order in expectation. As in equations (3.8.47) and (3.8.48) we see that

$$\begin{aligned} & \left| \mathbb{E}_Q \left(\tilde{X}_T^k(\tilde{\varphi}^{\varepsilon,k}) - X_{\tau^{\varepsilon,k}}(\varphi^{\varepsilon,k}) \right) \right| \\ & \leq \exp(p(1 + \varepsilon)) \left(\left\| \exp(pN_T) \right\|_{L_a(Q)} \left\| \exp(-pN_T) \right\|_{L_a(Q)} Q(\tau^{\varepsilon,k} < T) \right)^{1/b} + Q(\tau^{\varepsilon,k} < T). \end{aligned}$$

It remains to show that $Q(\tau^{\varepsilon,k} < T)$ is of the correct order. Similar to Lemma 3.8.1 we have for all $\kappa > 0$

$$\begin{aligned} Q \left(\sup_{t \in [0, T]} \left| \int_0^t \Delta \varphi_s^{\varepsilon,k} dS_s \right| \geq \kappa \right) & \leq \frac{1}{\kappa^4} \mathbb{E}_Q \left(\sup_{t \in [0, T]} \left(\int_0^t \Delta \varphi_s^{\varepsilon,k} dS_s \right)^4 \right) \\ & \leq \frac{\text{const.}}{\kappa^4} \left\| \int_0^T (\Delta \varphi_s^{\varepsilon,k})^2 c_s^S ds \right\|_{L_2(Q)}^2 \\ & \leq \frac{\text{const.}}{\kappa^4} \varepsilon k_2 \mathbb{E}_Q([S]_T^2). \end{aligned}$$

For a similar statement as in Lemma 3.8.2 observe that $\mathcal{N}_T(\varphi^{\varepsilon,k}) \leq \mathcal{N}_T(\bar{\varphi}^k)$, where $\bar{\varphi}^k$ denotes the strategy, such that $\Delta \bar{\varphi}^k$ always lies in $I := \left[-\sqrt[4]{\frac{12\varepsilon k_1}{p}}, \sqrt[4]{\frac{12\varepsilon k_1}{p}} \right]$ and trades to φ^* once $\Delta \bar{\varphi}^k$ hits the boundaries of I . With Markov's inequality we obtain for all $\kappa > 0$

$$\begin{aligned} Q(\mathcal{N}_T(\bar{\varphi}^k) \geq \kappa) & = Q \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \geq \kappa \right) = Q \left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} \left(\frac{\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*}{\left(\frac{12\varepsilon k_1}{p} \right)^{1/4}} \right)^2 \geq \kappa \right) \\ & = Q \left(\left(\frac{p}{12k_1} \right)^{1/2} \sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} (\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*)^2 \geq \kappa \varepsilon^{1/2} \right) \\ & \leq \frac{p}{12\varepsilon k_1 \kappa^2} \mathbb{E}_Q \left(\left(\sum_{i \in \mathbb{N}} \mathbf{1}_{\{\tau_i < T\}} (\varphi_{\tau_i}^* - \varphi_{\tau_{i-1}}^*)^2 \right)^2 \right), \end{aligned}$$

where $(\tau_i)_{i \in \mathbb{N}}$ denote the trading times of $\bar{\varphi}^k$. This implies

$$\begin{aligned} Q(\tau^{\varepsilon,1,k} < T) & \leq Q \left(\sup_{t \in [0, T]} \left| \int_0^t \varphi_s^{\varepsilon,k} dS_s \right| \geq \frac{1}{2\varepsilon} \right) + Q \left(\mathcal{N}_T(\varphi^{\varepsilon,k}) \geq \frac{1}{2\varepsilon^2} \right) \\ & = O(k_2 \varepsilon) + \frac{1}{k_1} O(\varepsilon^3). \end{aligned}$$

4.3. Truncated and frozen portfolio gamma with growth conditions

For $\tau^{\varepsilon,2,k}$ we obtain the upper bound

$$\begin{aligned} Q(\tau^{\varepsilon,2,k} < T) &\leq Q\left(\sup_{t \in [0,T]} |V_t^\varepsilon(\varphi^{\varepsilon,k}) - V_t(\varphi^*)| \geq 1\right) \\ &\leq Q\left(\sup_{t \in [0,T]} \left|\int_0^t \Delta\varphi_s^{\varepsilon,k} dS_s\right| \geq \frac{1}{2}\right) + Q\left(\varepsilon \mathcal{N}_t(\varphi^{\varepsilon,k}) \geq \frac{1}{2}\right) \\ &= O(k_2\varepsilon), \end{aligned}$$

and this shows for $\tau^{\varepsilon,k}$

$$Q(\tau^{\varepsilon,k} < T) = \frac{1}{k_1}O(\varepsilon) + \frac{1}{k_1}O(\varepsilon^3) + O(k_2\varepsilon) = O(k_2\varepsilon)$$

as ε tends to 0. This implies

$$\left|\mathbb{E}_Q\left(\tilde{X}_T^k(\tilde{\varphi}^{\varepsilon,k}) - X_{\tau^{\varepsilon,k}}(\varphi^{\varepsilon,k})\right)\right| = O(k_2^{3/4}\varepsilon^{3/4}) = o(\varepsilon^{1/2}). \quad (4.2.21)$$

Summing up (4.2.19) to (4.2.21) yields the result. \square

4.2.4. Optimality

As a corollary we obtain the almost optimality of the stopped candidate $\tilde{\varphi}^{\varepsilon,k}$ for the choices of the parameters (k_1, k_2, c_1, c_2) from the beginning of the section.

Corollary 4.2.9. *We have*

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E}(U(V_T^\varepsilon(\varphi))) = M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}) = \mathbb{E}\left(U(V_T^\varepsilon(\tilde{\varphi}^{\varepsilon,(\varepsilon^{1/28}, \varepsilon^{-1/28})}))\right) + o(\varepsilon^{1/2})$$

and

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \left(\sup_{\varphi \in \mathfrak{A}^\varepsilon(x)} U^{-1}(\mathbb{E}(U(V_T^\varepsilon(\varphi)))) - U^{-1}(\mathbb{E}(U(V_T(\varphi^*)))) \right) \\ &= \sqrt{\frac{p}{3}} \mathbb{E}_Q \left(\int_0^T \sqrt{c_s^{\varphi^*} c_s^S} ds \right). \end{aligned}$$

Proof. According to Theorems 4.2.7 and 4.2.8 the correct choices for the parameters (k_1, k_2, c_1, c_2) yield the assertion as in Corollary 3.8.5. \square

4.3. Truncated and frozen portfolio gamma with growth conditions

4.3.1. Truncated and frozen no-trade corridor

An alternative reduction of the model regularity is the double limit technique presented in this section. Instead of Assumption 4.2.3 we replace Assumption 3.6.4 with the following:

4. Extensions

Assumption 4.3.1. *Assume that ρ is an Itô process and that the following conditions hold:*

- $\exp(\pm 8pV_T(\varphi^*)) \in L_1(Q)$,
- $[S]_T \in L_2(Q)$, $\int_0^T (c_s^S)^{1+\eta_1} ds \in L_1(Q)$ for some $\eta_1 > 0$,
- $\|\varphi^*\|_{H_8(Q)} < \infty$, $\int_0^T (c_s^{\varphi^*})^{1+\eta_2} ds \in L_1(Q)$ for some $\eta_2 > 0$ and
- *there exist nonnegative and adapted processes $(a_j)_{j \in \{0, \dots, m\}}$ and b such that b is increasing and*
 - $|b^{\rho, Q}| \leq \sum_{j=0}^m a_j b^j$,
 - $\mathbb{E}_Q \left(\int_0^T a_{j,s} ds \right) < \infty$ and
 - $\mathbb{E}_Q \left(\int_0^T b_s^{\eta_3} ds \right) < \infty$ for a suitable $\eta_3 > 0$.

The last condition can be interpreted as a kind of integrability condition on the Q -local drift $b^{\rho, Q}$ of the portfolio gamma ρ and is weaker than the counterpart from Assumption 4.2.3. The constant η_3 is supposed to be small, but the process b has at least some regularity. The conditions on c^S and c^{φ^*} being slightly stronger than the conditions from Assumption 4.2.3 are required to determine the right order of convergence at some point in the verification. Motivated by the results from section 4.2 we replace the constants c_1, c_2, k_1, k_2 by k^{-1}, k, k^{-1}, k depending only on one value k that is supposed to be a negative power of ε :

$$k := k_\varepsilon = \varepsilon^{-\eta_4}.$$

So this value will converge to infinity when the transaction costs tend to 0. We will choose $\eta_4 \in (0, \frac{1}{14} \wedge \frac{1}{2(m+1)})$. Since we dropped the integrability condition on $b^{\rho, Q}$ from Assumption 4.2.3, we introduce another stopping time depending on the process b . Denote by $\bar{\eta}$ the conjugate index of $1 + \delta_1$, which is given by $\bar{\delta} := \frac{1+\delta_1}{\delta_1}$. For $k \in (1, \infty)$ and $c := \frac{\bar{\eta}}{\eta_3}$ define the stopping time

$$\tau_k := \inf\{t \in [0, T] : |b_t| > k^c\} \wedge T$$

and the stopped and truncated portfolio gamma process

$$\rho^k := (\rho_{\cdot \wedge \tau_k} \vee k^{-1}) \wedge k,$$

which is the portfolio gamma ρ stopped at τ_k and cut off from below by k^{-1} and from above by k . Accordingly, the halfwidth of the no-trade region

$$\Delta\varphi_t^{+,k} := \sqrt[4]{\frac{12\varepsilon}{p}} \sqrt[4]{\rho_t^k},$$

and all other objects are defined:

$$\alpha_t^k := \frac{\varepsilon}{(\Delta\varphi_t^{+,k})^4}, \quad \beta_t^k := 2\alpha_t^k (\Delta\varphi_t^{+,k})^2 \quad \text{and} \quad C_t^k := (\rho_t^k)^{-1}.$$

4.3. Truncated and frozen portfolio gamma with growth conditions

We also define the counterparts for the modified value process from section 3.5.5, but we replace c_1 and c_2 with k^{-1} and k

$$\Delta\widehat{\varphi}^{+,k} := \sqrt[4]{k^{-1}} \frac{12\varepsilon}{p} \sqrt[4]{\rho_t^k}, \quad \widehat{\alpha}^k := \alpha^k \quad \text{and} \quad \widehat{\beta}^k := 2\alpha^k (\Delta\widehat{\varphi}^{+,k})^2.$$

Then we use as candidates for the (transformed) value processes

$$\widetilde{X}^k(\varphi) := \exp\left((-p) (\Delta V^\varepsilon(\varphi) + F^{CE,k}(\varphi))\right)$$

with $F^{CE,k}(\varphi) = f^{CE}(\alpha^k, \Delta\varphi) - \gamma$ to obtain a lower bound and

$$\widehat{X}^k(\varphi) := \exp\left((-p) (\Delta V^\varepsilon(\varphi) + \widehat{F}^{CE,k}(\varphi))\right)$$

with $\widehat{F}^{CE,k}(\varphi) = \widehat{f}^{CE}(\alpha^k, \Delta\varphi) - \gamma + \delta(\varphi)$ for an upper bound of maximal expected utility.

Remark 4.3.2. *With the change-of-variable formula from Theorem B.1.3 and assuming that ρ is an Itô process we see that ρ^k is a continuous semimartingale with Q -dynamics*

$$\begin{aligned} V_t^{\rho^k, Q} &= \int_0^t \mathbf{1}_{\{\rho_s \in (k^{-1}, k], s \leq \tau_k\}} b_s^{\rho^k, Q} ds + L_t^k(k^{-1}) - L_t^k(k), \\ M_t^{\rho^k, Q} &= \int_0^t \mathbf{1}_{\{\rho_s \in (k^{-1}, k], s \leq \tau_k\}} dM_s^{\rho^k, Q}, \end{aligned}$$

where L^k denotes a semimartingale local time of $\rho_{\tau_k \wedge \cdot}$. Thus α^k, β^k and C^k are also continuous semimartingales.

Lemma 4.3.3 (Local time). *We have*

$$\sup_{y \in (0, \infty)} \mathbb{E}_Q(L_T^k(y)) \leq \sum_{j=0}^n \mathbb{E}_Q \left(\int_0^T a_{j,s} ds \right) k^j + \rho_0.$$

Proof. Applying the change-of-variable formula from Theorem B.1.3 on the process $\rho_y^k := \rho_{\tau_k \wedge \cdot} \wedge y$ yields

$$\begin{aligned} L_t^k(y) &= - \left(\rho_{y,t}^k - \rho_{y,0}^k - \int_0^t \mathbf{1}_{\{\rho_s^k \leq y\}} b_s^{\rho^k, Q} ds - \int_0^t \mathbf{1}_{\{\rho_s^k \leq y\}} dM_s^{\rho^k, Q} \right) \\ &= -\rho_{y,t}^k + \rho_{y,0}^k + \int_0^t \mathbf{1}_{\{\rho_s^k \leq y\}} b_s^{\rho^k, Q} ds + \int_0^t \mathbf{1}_{\{\rho_s^k \leq y\}} dM_s^{\rho^k, Q}, \end{aligned}$$

which implies

$$0 \leq L_t^k(y) \leq \rho_0 + \int_0^T |b_s^{\rho^k, Q}| ds + \int_0^t \mathbf{1}_{\{\rho_s^k \leq a\}} dM_s^{\rho^k, Q} \quad (4.3.22)$$

and hence

$$-\rho_0 - \int_0^T |b_s^{\rho^k, Q}| ds \leq \int_0^t \mathbf{1}_{\{\rho_s^k \leq y\}} dM_s^{\rho^k, Q}.$$

4. Extensions

So $\int_0^\cdot \mathbf{1}_{\{\rho_s^k \leq y\}} dM_s^{\rho^k, Q}$ is a supermartingale and we see that

$$\mathbb{E}_Q \left(\int_0^T \mathbf{1}_{\{\rho_s^k \leq y\}} dM_s^{\rho^k, Q} \right) \leq 0.$$

Furthermore, we obtain for the bounded variation part of $\rho_{\tau_k \wedge \cdot}$.

$$\begin{aligned} \mathbb{E}_Q \left(\int_0^T |b_s^{\rho^k, Q}| ds \right) &= \mathbb{E}_Q \left(\int_0^T |b_s^{\rho^k, Q}| \mathbf{1}_{\{\tau_k \geq s\}} ds \right) \\ &\leq \sum_{j=0}^m \mathbb{E}_Q \left(\int_0^T a_{j,s} b_s^j \mathbf{1}_{\{\tau_k \geq s\}} ds \right) \\ &\leq \sum_{j=0}^m \mathbb{E}_Q \left(\int_0^T a_{j,s} ds \right) k^j. \end{aligned}$$

Taking expectations in (4.3.22) for $t = T$ leads to

$$\mathbb{E}_Q (L_T^k(y)) \leq \sum_{j=0}^m \mathbb{E}_Q \left(\int_0^T a_{j,s} ds \right) k^j + \rho_0,$$

where the right hand side does not depend on y . □

Remark 4.3.4. From Remark 4.3.2 we know that

$$[\rho^k]_T = [M^{\rho^k, Q}]_T = \int_0^{\tau_k} \mathbf{1}_{\{\rho_s \in (k^{-1}, k]\}} d[M^{\rho^k, Q}]_s.$$

According to Theorem B.1.3 this can be rewritten as

$$[\rho^k]_T = 2 \int_{-\infty}^{\infty} \mathbf{1}_{(k^{-1}, k]}(a) L_T^k(a) da = 2 \int_{k^{-1}}^k L_T^k(a) da.$$

Taking expectations and using Lemma 4.2.2 yields

$$\mathbb{E}_Q([\rho^k]_T) = 2 \int_{k^{-1}}^k \mathbb{E}_Q(L_T(a)) da \leq O(k^{m+1})$$

Applying Itô's formula to the process C^k shows

$$\begin{aligned} dC_s^k &= ((\rho_s^k)^{-2} b_s^{\rho^k, Q} + (\rho_s^k)^{-3} c_s^\rho) \mathbf{1}_{\{\rho_s \in (k^{-1}, k], s \leq \tau_k\}} ds + (-1)(\rho_s^k)^{-2} \mathbf{1}_{\{\rho_s \in (k^{-1}, k], s \leq \tau_k\}} dM_s^{\rho^k, Q} \\ &\quad + (-1)(\rho_s^k)^{-2} d(L_s^k(k^{-1}) - L_s^k(k)) \end{aligned}$$

and this yields for the quadratic variation and the absolute continuous local drift of C^k

$$\mathbb{E}_Q([C^k]_T) \leq O(k^{m+5}), \tag{4.3.23}$$

$$\mathbb{E}_Q \left(\int_0^T |b_s^{C^k, Q}| ds \right) \leq O(k^{m+4}), \tag{4.3.24}$$

which will be required later.

4.3.2. Upper bound

For the asymptotic upper bound we recall the process K and the stopping time $\tau(\varphi)$ from subsection 3.7.2.

Proposition 4.3.5 (No-trade region). *We have*

$$\sup_{\sigma \in \mathcal{T}} \sup_{\varphi \in \mathfrak{A}^\varepsilon} \left| \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right) \right| = o(\varepsilon^{1/2}).$$

Proof. (i) Similar calculations as in Propositions 3.7.1 and 3.7.4 show that the Q -local drift of $\widehat{X}^k(\varphi)$ is given by

$$b_s^{\widehat{X}^k(\varphi), Q} = \widehat{X}_s^k(\varphi) \left((-p)b_s^{\widehat{F}^{CE,k}(\varphi), Q} + \frac{p^2}{2} c_s^{\int_0^{\cdot} \Delta\varphi_u dS_u + \widehat{F}^{CE,k}(\varphi), \int_0^{\cdot} \Delta\varphi_u dS_u + \widehat{F}^{CE,k}(\varphi)} \right).$$

In the (interior of the) modified minimal no-trade region $\{|\Delta\varphi| < \Delta\widehat{\varphi}^{+,k}\}$ this leads to

$$\begin{aligned} b_s^{\widehat{X}^k(\varphi), Q} &= \widehat{X}_s^k(\varphi) \left((-p)b_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2} c_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma} \right) \\ &\quad + \widehat{X}_s^k(\varphi) \left((-p)b_s^{\delta(\varphi), Q} + \frac{p^2}{2} c_s^{\delta(\varphi)} + p^2 c_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma, \delta(\varphi)} \right). \end{aligned} \quad (4.3.25)$$

For the left summand observe that it can be written as

$$\begin{aligned} &\widehat{X}_s^k(\varphi) \left((-p)b_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma, Q} + \frac{p^2}{2} c_s^{\widehat{F}^{CE}(\alpha^k, \Delta\varphi) - \gamma} \right) \\ &= \widehat{X}_s^k(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^{k,i} + p \left(b_s^{\gamma, Q} + \widehat{\beta}_s^k c_s^{\varphi^*} \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \\ &= \widehat{X}_s^k(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^{k,i} + p(\sqrt{c_1} - 1) \sqrt{\frac{\varepsilon p}{3}} A_s \mathbf{1}_{\{\rho_s \in (k^{-1}, k], s \leq \tau_k\}} \right. \\ &\quad \left. + \left(p \left(\widehat{\beta}_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \mathbf{1}_{\{\rho_s \notin (k^{-1}, k] \cup (\tau_k, T]\}}(s) \right) \end{aligned}$$

in the modified truncated no-trade region, where the processes $\widehat{H}^{k,j}$ can be obtained from e.g. line (3.5.15) by replacing α with α^k and β with $\widehat{\beta}^k$.

(ii) For a part of this left summand we obtain the following upper bound by using $\Delta\widehat{\varphi}^{+,k} \leq \Delta\varphi^{+,k}$:

$$\begin{aligned} G_1 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i \widehat{H}_s^{k,i} + p(\sqrt{c_1} - 1) \sqrt{\frac{\varepsilon p}{3}} A_s \mathbf{1}_{\{\rho_s \in (k^{-1}, k]\}} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right| \\ &\leq \kappa_{k^{-1}, k} \sum_{i=0}^8 \int_0^T |\Delta\varphi_s^{+,k}|^i |\widehat{H}_s^{k,i}| ds + p(\sqrt{k^{-1}} - 1) \kappa_{k^{-1}, k} \sqrt{\frac{\varepsilon p}{3}} \int_0^T A_s ds, \end{aligned}$$

4. Extensions

since $\widehat{X}_s^k(\varphi)$ is bounded by $\kappa_{k^{-1},k}$ on $[0, \tau(\varphi)]$ by (3.7.33). Replacing $\alpha, \widehat{\beta}$ and C with $\alpha^k, \widehat{\beta}^k$ and C^k in the proof of Proposition 3.7.4 yields

$$\begin{aligned}\mathbb{E}_Q(G_1) &= \kappa_{k^{-1},k}(O(k^2 k^{5/4} \varepsilon^{3/4}) + O(k^4 k^3 \varepsilon)) + \kappa_{k^{-1},k} p(1 - \sqrt{k^{-1}}) \sqrt{\frac{\varepsilon p}{3}} Y_0 \\ &= \kappa_{k^{-1},k}(O(k^{13/4} \varepsilon^{3/4}) + O(k^7 \varepsilon) + O(k^{-1} \varepsilon^{1/2})) = o(\varepsilon^{1/2})\end{aligned}\quad (4.3.26)$$

uniformly in φ and σ . For the remaining term observe that the complement of $M := \{\rho^{\tau_k} \in (k^{-1}, k]\} \cap [0, \tau_k]$ can be rewritten in the following way

$$\begin{aligned}M^c &= (\{\rho \notin (k^{-1}, k]\} \cap [0, \tau_k]) \cup (\{\rho^\tau \in (k^{-1}, k]\} \cap (\tau_k, T]) \\ &:= M_1 \cup M_2,\end{aligned}$$

where M_1 and M_2 are disjoint sets. We consider both sets separately. On M_1 we define

$$G_2 := \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(p \left(\widehat{\beta}_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \cdot \mathbf{1}_{\{|\Delta\varphi_s| < \Delta\widehat{\varphi}_s^{+,k}\} \cap M_1} ds \right|.$$

As for G_2 and G_3 from the proof of proposition 4.2.5 we obtain

$$\mathbb{E}_Q(G_2) \leq \kappa_{k^{-1},k} O(k^{-1/2} \varepsilon^{1/2}),$$

which implies

$$\mathbb{E}_Q(G_2) = o(\varepsilon^{1/2}). \quad (4.3.27)$$

Furthermore we obtain on M_2 :

$$\begin{aligned}G_3 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(p \left(\widehat{\beta}_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \cdot \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}^{+,k}\} \cap M_2} ds \right| \\ &\leq \kappa_{k^{-1},k} \left[\left(\frac{p\varepsilon}{3} \right)^{1/2} \int_0^T |\rho_{\tau_k}^{-1/2} c_s^{\varphi^*} - A_s| \mathbf{1}_{M_2}(s) ds \right. \\ &\quad \left. + (1 - k^{-1})^{1/2} \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{p^2}{2} \int_0^T \rho_{\tau_k}^{1/2} |c_s^S - \rho_{\tau_k}^{-1} c_s^{\varphi^*}| \mathbf{1}_{M_2}(s) ds \right] \\ &\leq \kappa_{k^{-1},k} \left[\left(\frac{p\varepsilon}{3} \right)^{1/2} \left(k^{1/2} \int_0^T c_s^{\varphi^*} \mathbf{1}_{(\tau_k, T]}(s) ds + \int_0^T A_s \mathbf{1}_{(\tau_k, T]}(s) ds \right) \right. \\ &\quad \left. + (1 - k^{-1})^{1/2} \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{p^2}{2} k^{1/2} \left(\int_0^T c_s^S \mathbf{1}_{(\tau_k, T]}(s) ds + \int_0^T c_s^{\varphi^*} \mathbf{1}_{(\tau_k, T]}(s) ds \right) \right].\end{aligned}$$

4.3. Truncated and frozen portfolio gamma with growth conditions

We will show that

$$\mathbb{E}_Q \left(\int_0^T c_s^S \mathbf{1}_{(\tau_k, T]}(s) ds \right) = o(k^{-1/2}), \quad (4.3.28)$$

$$\mathbb{E}_Q \left(\int_0^T c_s^{\varphi^*} \mathbf{1}_{(\tau_k, T]}(s) ds \right) = o(k^{-1/2}). \quad (4.3.29)$$

By Hölder's inequality this then yields

$$\mathbb{E}_Q \left(\int_0^T A_s \mathbf{1}_{(\tau_k, T]}(s) ds \right) = o(k^{-1/2})$$

since $A = (c^{\varphi^*} c^S)^{1/2}$ and hence

$$\mathbb{E}_Q(G_3) = o(\varepsilon^{1/2}). \quad (4.3.30)$$

Recall that $\bar{\eta}$ is the conjugate index of $1 + \eta_1$. By Hölder's inequality applied pathwise we have

$$\int_0^T c_s^S \mathbf{1}_{(\tau_k, T]}(s) ds \leq \left(\int_0^T (c_s^S)^{1+\eta_1} ds \right)^{1/(1+\eta_1)} \left(\int_0^T \mathbf{1}_{(\tau_k, T]}(s) ds \right)^{1/\bar{\eta}}.$$

For the right term of the product we use Markov's inequality pathwise to obtain for all $\omega \in \Omega$

$$\begin{aligned} \int_0^T \mathbf{1}_{(\tau_k, T]}(s) ds(\omega) &= \text{Leb}(\{s \in [0, T] : b_s(\omega) > k^c\}) \\ &= \text{Leb}(\{s \in [0, T] : b_s^{\eta_3}(\omega) > k^{c\eta_3}\}) \\ &\leq k^{-c\eta_3} \int_0^T b_s^{\eta_3} ds(\omega). \end{aligned}$$

This implies

$$\int_0^T c_s^S \mathbf{1}_{(\tau_k, T]}(s) ds \leq k^{-(c\eta_3)/\bar{\eta}} \left(\int_0^T (c_s^S)^{1+\eta_1} ds \right)^{1/(1+\eta_1)} \left(\int_0^T b_s^{\eta_3} ds \right)^{1/\bar{\eta}}$$

and by using Hölders inequality

$$\mathbb{E}_Q \left(\int_0^T c_s^S \mathbf{1}_{(\tau_k, T]}(s) ds \right) \leq k^{-(c\eta_3)/\bar{\eta}} \mathbb{E}_Q \left(\int_0^T (c_s^S)^{1+\eta_1} ds \right)^{1/(1+\eta_1)} \mathbb{E}_Q \left(\int_0^T b_s^{\eta_3} ds \right)^{1/\bar{\eta}}.$$

Since $\frac{c\eta_3}{\bar{\eta}} > \frac{1}{2}$, we obtain the desired convergence rate for c^S in line (4.3.28). For line (4.3.29) we can use the same arguments.

(iii) For the right summand of (4.3.25) we insert the Q -dynamics of $\delta(\varphi)$ on $[0, \tau(\varphi)]$ in the modified no-trade region $\{|\Delta\varphi| < \Delta\widehat{\varphi}^{+,k}\}$ and observe that as for G_4 in (iii) of the proof of Proposition 4.2.5

$$\begin{aligned} G_4 &:= \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left((-p)b_s^{\delta(\varphi), Q} + \frac{p^2}{2} c_s^{\delta(\varphi)} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right| \\ &\leq \kappa_{k^{-1}, k} k^2 \left(\frac{p^2}{2} \frac{\varepsilon p}{3} (\kappa_Y)^2 + \frac{p}{2} \sqrt{\frac{\varepsilon p}{3}} \kappa_Y \right) \int_0^T (\Delta\widehat{\varphi}_s^{+,k})^2 c_s^S ds. \end{aligned}$$

4. Extensions

Taking expectations and using $\Delta\widehat{\varphi}_s^{+,k} \leq \left(\frac{12k\varepsilon}{p}\right)^{1/4}$ yields

$$\begin{aligned}\mathbb{E}_Q(G_4) &\leq \kappa_{k^{-1},k} k^2 \left(\frac{p^2}{2} \frac{\varepsilon p}{3} (\kappa_Y)^2 \frac{p}{2} + \sqrt{\frac{\varepsilon p}{3}} \kappa_Y \right) \sqrt{\frac{12k\varepsilon}{p}} \mathbb{E}_Q([S]_T) \\ &= \kappa_{k^{-1},k} O(k^{5/2}\varepsilon)\end{aligned}$$

leading to

$$\mathbb{E}_Q(G_4) = o(\varepsilon^{1/2}). \quad (4.3.31)$$

We consider the remaining term that is given by

$$G_5 := \left| \int_0^{\sigma \wedge \tau(\varphi)} \widehat{X}_s^k(\varphi) \left(p^2 c_s^{\widehat{f}^{CE}(\alpha^k, \widehat{\beta}^k, \Delta\varphi) - \gamma, \delta(\varphi)} \right) \mathbf{1}_{\{|\Delta\varphi| < \Delta\widehat{\varphi}_s^{+,k}\}} ds \right|.$$

For this term we can find the upper bound

$$G_5 \leq \kappa_{k^{-1},k} k \sqrt{\frac{\varepsilon p}{3}} \kappa_Y \int_0^T \left(\sum_{j=0}^8 |\Delta\widehat{\varphi}_s^+|^j |\widehat{H}_{j,s}^k| \right) ds$$

from the upper bound of G_5 in Proposition 4.2.5 and taking expectations gives as for G_1 in (4.3.26)

$$\mathbb{E}_Q(G_5) = \kappa_{k^{-1},k} (O(k^3 \varepsilon^{5/4}) + O(k^8 \varepsilon^{3/2})),$$

which implies

$$\mathbb{E}_Q(G_5) = o(\varepsilon^{1/2}). \quad (4.3.32)$$

(iv) Summing up inequalities (4.3.26), (4.3.27), (4.3.30), (4.3.31) and (4.3.32) yields the desired result. \square

Proposition 4.3.6 (Trade region). *We have*

$$\inf_{\varphi \in \mathfrak{A}^\varepsilon} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_Q \left(\int_0^{\sigma \wedge \tau(\varphi)} b_s^{\widehat{X}^k(\varphi), Q} \mathbf{1}_{\{|\Delta\varphi_s| \geq \Delta\widehat{\varphi}_s^{+,k}\}} ds \right) \geq o(\varepsilon^{1/2}).$$

Proof. We use the same lower bound estimates as in the proof of Proposition 4.2.6 to obtain the desired result. \square

Theorem 4.3.7. *We have*

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E}(U(V_T(\varphi))) \leq \mathbb{E}(U(V_T(\varphi^*))) \exp(p\gamma_0) + o(\varepsilon^{1/2}).$$

Proof. (i) Similar to Theorem 3.7.3 we have

$$\mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq \exp(-p\varepsilon) M_0 \mathbb{E}_Q \left(\widehat{X}_T^k(\varphi) \right).$$

4.3. Truncated and frozen portfolio gamma with growth conditions

Observe that with (4.2.14)

$$\mathbb{E}_Q \left(\widehat{X}_T^k(\varphi) \right) \geq (1 - h_\varepsilon) \mathbb{E}_Q \left(X_{\tau(\varphi)}(\varphi) \right) + \mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}^k(\varphi) \right). \quad (4.3.33)$$

and from (4.2.15)

$$\mathbb{E}_Q \left(X_{\tau(\varphi)}(\varphi) \right) \leq k^{1/k} O(1). \quad (4.3.34)$$

(ii) Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for the local martingale $M^{\widehat{X}^k(\varphi), Q}$ and define $Y_n := \widehat{X}_{\tau(\varphi) \wedge \tau_n}^k(\varphi)$ for all $n \in \mathbb{N}$. Then $\{Y_n, n \in \mathbb{N}\}$ is a bounded set in $L_2(Q)$, which implies that it is uniformly integrable. As in the proof of Theorems 3.7.3 and 3.7.6 we then can show

$$\mathbb{E}_Q \left(\widehat{X}_{\tau(\varphi)}^k(\varphi) \right) = \lim_{n \rightarrow \infty} \mathbb{E}_Q(Y_n). \quad (4.3.35)$$

(iii) Applying Theorem B.1.1 on Y_n and taking expectations yields

$$\begin{aligned} & \mathbb{E}_Q(Y_n) \\ &= \mathbb{E}_Q \left(\widehat{X}_0^k(\varphi) + \int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}^k(\varphi), Q} ds + M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q} + L_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi)} + \sum_{0 \leq s < \tau(\varphi) \wedge \tau_n} \Delta^+ \widehat{X}_s^k(\varphi) \right), \end{aligned}$$

where $L^{\widehat{X}^k(\varphi)}$ denotes the term introduced by the local time L^k of $\rho_{\cdot \wedge \tau_k}$. We obtain

$$\mathbb{E}_Q(Y_n) \geq \widetilde{X}_0^k(\varphi) + \mathbb{E}_Q \left(\int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}^k(\varphi), Q} ds \right) + \mathbb{E}_Q \left(M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q} \right) + \mathbb{E}_Q \left(L_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi)} \right), \quad (4.3.36)$$

since the jumps of $\widehat{X}^k(\varphi)$ are always positive, which can be shown as in the proof of Theorem 4.2.7.

(iv) In the next steps we analyze the drift and martingale components in equation (4.2.17). Since the stopped local martingale $M_{\cdot \wedge \tau_n}^{\widehat{X}^k(\varphi), Q}$ is a true Q -martingale, we have

$$\mathbb{E}_Q \left(M_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi), Q} \right) = M_0^{\widehat{X}^k(\varphi), Q} = 0.$$

For the drift component we can use Propositions 4.3.5 and 4.3.6 which imply

$$\inf_{n \in \mathbb{N}} \mathbb{E}_Q \left(\int_0^{\tau(\varphi) \wedge \tau_n} b_s^{\widehat{X}^k(\varphi), Q} ds \right) \geq o(\varepsilon^{1/2}).$$

(v) We see that the term introduced by the local time of $\rho_{\cdot \wedge \tau_k}$ equals

$$\begin{aligned} L_t^{\widehat{X}^k(\varphi)} &= \int_0^t \widehat{X}_s^k(\varphi) \left((\Delta \varphi_s)^4 \frac{p}{12} (-1) (\rho_s^k)^{-2} + (\Delta \varphi_s)^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} (\rho_s^k)^{-3/2} \right) \\ &\quad \cdot \left(\mathbf{1}_{\{|\Delta \varphi_s| < \Delta \widehat{\varphi}_s^{+, k}\}} + \frac{1}{2} \mathbf{1}_{\{|\Delta \varphi_s| = \Delta \widehat{\varphi}_s^{+, k}\}} \right) d(L_s^k(k^{-1}) - L_s^k(k)), \end{aligned}$$

4. Extensions

which implies as in the proof of Theorem 4.2.7

$$\left| L_{\tau(\varphi) \wedge \tau_n}^{\widehat{X}^k(\varphi)} \right| \leq 3\kappa_{k^{-1},k}\varepsilon k \left(L_{\tau(\varphi) \wedge \tau_n}^k(k^{-1}) + L_{\tau(\varphi) \wedge \tau_n}^k(k) \right) \leq 3 \exp(p)\varepsilon k \left(L_T^k(k^{-1}) + L_T^k(k) \right),$$

since $L(y)$ is an increasing process. By Lemma 4.3.3 we know that $\sup_{y \in (0, \infty)} \mathbb{E}_Q(L_T(y))$ is of order $O(k^m)$. This implies

$$\sup_{n \in \mathbb{N}} \mathbb{E}_Q \left(\left| L_{\tau^k(\varphi)}^{\widehat{X}^k(\varphi)} \right| \right) = \kappa_{k^{-1},k} O(k^{m+1}\varepsilon) = o(\varepsilon^{1/2}). \quad (4.3.37)$$

(vii) By letting n go to infinity and using (4.3.33), (4.3.35), (4.3.34) and (4.3.37) we obtain

$$\mathbb{E}(U(V_T^\varepsilon(\varphi))) \leq M_0 \exp(p(\gamma_0 - k^{-1}\varepsilon^{1/2} - \varepsilon)) + o(\varepsilon^{1/2}).$$

This completes the proof. \square

4.3.3. Lower bound

Similar to section 3.8 we define the stopping times

$$\begin{aligned} \tau^{\varepsilon,1,k} &:= \inf \{ t \in [0, T] : V_t^\varepsilon(\varphi^{\varepsilon,k}) < -1 + V_t(\varphi^*) \} \wedge T, \\ \tau^{\varepsilon,2,k} &:= \inf \left\{ t \in [0, T] : V_t^\varepsilon(\varphi^{\varepsilon,k}) < -\frac{1}{\varepsilon} \right\} \wedge T \quad \text{and} \\ \tau^{\varepsilon,k} &:= \tau^{\varepsilon,1,k} \wedge \tau^{\varepsilon,2,k}. \end{aligned}$$

Theorem 4.3.8. *The candidate strategy $\widetilde{\varphi}^{\varepsilon,k}$ fulfills*

$$\mathbb{E}(U(V_T^\varepsilon(\widetilde{\varphi}^{\varepsilon,k}))) \geq M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}).$$

Proof. (i) As in Theorem 3.8.4 we have

$$\mathbb{E}(U(V_T^\varepsilon(\widetilde{\varphi}^{\varepsilon,k}))) \geq E \left(\widetilde{v}_T^{\varepsilon,k}(\widetilde{\varphi}^{\varepsilon,k}) \right) = M_0 E_Q \left(\widetilde{X}_T^k(\widetilde{\varphi}^{\varepsilon,k}) \right)$$

and similarly (except for the process introduced by the local time of ρ)

$$\begin{aligned} \widetilde{X}_T^k(\widetilde{\varphi}^{\varepsilon,k}) &\geq \widetilde{X}_0^k(\widetilde{\varphi}^{\varepsilon,k}) + \int_0^{\tau^{\varepsilon,k}} b_s^{\widetilde{X}^k(\widetilde{\varphi}^{\varepsilon,k}),Q} ds + M_{\tau^{\varepsilon,k}}^{\widetilde{X}^k(\widetilde{\varphi}^{\varepsilon,k}),Q} + L_{\tau^{\varepsilon,k}}^{\widetilde{X}^k(\varphi)} \\ &\quad + \widetilde{X}_T^k(\widetilde{\varphi}^{\varepsilon,k}) - X_{\tau^{\varepsilon,k}}(\varphi^{\varepsilon,k}). \end{aligned}$$

As before we analyze all these terms separately. For the local Q -drift of the process $\widetilde{X}^k(\varphi^{\varepsilon,k})$ we know that

$$\begin{aligned} &b_s^{\widetilde{X}^k(\varphi^{\varepsilon,k}),Q} \\ &= \widetilde{X}_s^k(\varphi^{\varepsilon,k}) \left(\sum_{i=0}^8 (\Delta\varphi_s)^i H_s^k + \left(p \left(\beta_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s)^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \right. \\ &\quad \left. \cdot \mathbf{1}_{\{\rho^{\tau_k} \in (k_1, k_2]\} \cap [0, \tau_k]^c} \right), \end{aligned}$$

4.3. Truncated and frozen portfolio gamma with growth conditions

where the processes H^k can be obtained from line (3.5.15) by replacing α and β with α^k and β^k . Furthermore, on $[0, \tau^{\varepsilon, 1, k}]$ we have $\tilde{X}_s^k(\varphi^{\varepsilon, k}) \leq \exp(p(1 + \varepsilon\sqrt{\varepsilon p/3\kappa_Y}))$. Combining this with the arguments for $E_Q(G_1)$ in the proof of Proposition 4.3.5 in line (4.3.26) and Proposition 3.8.3 we can conclude

$$\mathbb{E}_Q \left(\left| \int_0^{\tau^{\varepsilon, 1, k} \wedge \sigma} \tilde{X}_s^k(\varphi^{\varepsilon, k}) \left(\sum_{i=0}^8 (\Delta\varphi_s^{\varepsilon, k})^i H_s^k \right) ds \right| \right) = O(k^{13/4} \varepsilon^{3/4}) + O(k^7 \varepsilon) = o(\varepsilon^{1/2}). \quad (4.3.38)$$

(ii) For the remaining term we can use the same arguments as in Proposition 4.3.5 and obtain

$$\begin{aligned} & \mathbb{E}_Q \left(\left| \int_0^{\tau^{\varepsilon, 1, k} \wedge \sigma} \tilde{X}_s^k(\varphi^{\varepsilon, k}) \left(p \left(\beta_s^k c_s^{\varphi^*} - \sqrt{\frac{p\varepsilon}{3}} A_s \right) + (\Delta\varphi_s^{\varepsilon, k})^2 \left(\frac{p^2}{2} c_s^S - 6p\alpha_s^k c_s^{\varphi^*} \right) \right) \right. \right. \\ & \quad \left. \left. \cdot \mathbf{1}_{\{\rho^{\tau_k} \in (k_1, k_2]\} \cap [0, \tau_k]^c} ds \right| \right) \\ & = o(\varepsilon^{1/2}). \end{aligned}$$

(iii) From line (3.7.25) we know

$$M_t^{\tilde{X}^k(\varphi^{\varepsilon, k}), Q} = \int_0^t (-p) \tilde{X}_s^k(\varphi^{\varepsilon, k}) dM_s^{F^{CE, k}(\varphi^{\varepsilon, k}), Q} + \int_0^t (-p) \tilde{X}_s^k(\varphi^{\varepsilon, k}) \Delta\varphi_s^{\varepsilon, k} dS_s.$$

As in the proof of theorem 3.8.4 we obtain

$$\left[M^{\tilde{X}^k(\varphi^{\varepsilon, k}), Q} \right]_{\tau^{\varepsilon, k}} \leq 2p^2 \exp(2p) \left(\left[M^{F^{CE, k}(\varphi^{\varepsilon, k}), Q} \right]_T + \int_0^T (\Delta\varphi_s^{\varepsilon, k})^2 d[S]_s \right).$$

One shows similarly to this proof $[M^{F^{CE, k}(\varphi^{\varepsilon, k}), Q}]_T, \int_0^T (\Delta\varphi_s^{\varepsilon, k})^2 d[S]_s \in L_1(Q)$, which then implies that the Q -expectation of $M_{\tau^{\varepsilon, k}}^{\tilde{X}^k(\varphi^{\varepsilon, k}), Q}$ vanishes.

(iv) With the same arguments as in the proof of Theorem 4.3.7 show that the term introduced by the local time of ρ^{τ_k} fulfills

$$\mathbb{E}_Q \left(\left| L_{\tau^{\varepsilon, k}}^{\tilde{X}^k(\varphi^{\varepsilon, k})} \right| \right) = O(k\varepsilon) = o(\varepsilon^{1/2}). \quad (4.3.39)$$

(v) As in equations (3.8.47) and (3.8.48) we see that

$$\begin{aligned} & \left| \mathbb{E}_Q \left(\tilde{X}_T^k(\tilde{\varphi}^{\varepsilon, k}) - X_{\tau^{\varepsilon, k}}(\varphi^{\varepsilon, k}) \right) \right| \\ & \leq \exp(p(1 + \varepsilon)) \left(\left\| \exp(pN_T) \right\|_{L_a(Q)} \left\| \exp(-pN_T) \right\|_{L_a(Q)} Q(\tau^{\varepsilon, k} < T)^{1/b} + Q(\tau^{\varepsilon, k} < T) \right) \end{aligned}$$

and it remains to show that $Q(\tau^{\varepsilon, k} < T)$ is of the correct order, which is done in the same way as in the proof of Theorem 4.2.8. This then implies

$$\left| \mathbb{E}_Q \left(\tilde{X}_T^k(\tilde{\varphi}^{\varepsilon, k}) - X_{\tau^{\varepsilon, k}}(\varphi^{\varepsilon, k}) \right) \right| = O(k^{3/4} \varepsilon^{3/4}) = o(\varepsilon^{1/2}). \quad (4.3.40)$$

Summing up (4.3.38) to (4.3.40) yields the result. \square

4. Extensions

4.3.4. Optimality

As a corollary we obtain the almost optimality of the stopped candidate $\tilde{\varphi}^{\varepsilon,k}$ in the following sense.

Corollary 4.3.9. *We have*

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E} (U(V_T^\varepsilon(\varphi))) = M_0 \exp(p\gamma_0) + o(\varepsilon^{1/2}) = \mathbb{E} \left(U(V_T^\varepsilon(\tilde{\varphi}^{\varepsilon,\varepsilon^{-\eta_4}})) \right) + o(\varepsilon^{1/2})$$

and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \left(\sup_{\varphi \in \mathfrak{A}^\varepsilon(x)} U^{-1} (\mathbb{E} (U(V_T^\varepsilon(\varphi)))) - U^{-1} (\mathbb{E} (U(V_T(\varphi^*)))) \right) \\ &= \sqrt{\frac{p}{3}} \mathbb{E}_Q \left(\int_0^T \sqrt{c_s^{\varphi^*} c_s^S} ds \right). \end{aligned}$$

Proof. We can use the same arguments as in the proof of Corollary 4.2.9. □

5. Examples

In this chapter we study two examples of models that fulfill the regularity assumptions of chapter 3.

5.1. Black-Scholes model

As in Example 2.4.5 we consider the Black-Scholes model

$$S_t = S_0 + \int_0^t S_s \mu ds + \int_0^t S_s \sigma dW_s.$$

Since the frictionless optimizer $\varphi^* = \frac{\mu}{\sigma^2 p} \frac{1}{S}$ is an Itô process, we can compute in the case $\mu \neq 0$:

$$\begin{aligned} c_t^{\varphi^*, \varphi^*} &= \left(\frac{\mu}{\sigma p} \right)^2 \frac{1}{S_t^2}, & b_t^{\varphi^*, Q} &= \sigma^2 \varphi_t^*, & C_t &= \left(\frac{\sigma^2 p}{\mu} \right)^2 S_t^4, \\ \gamma_t &= \left(\frac{\varepsilon}{3p} \right)^{1/2} |\mu| (T - t), & \Delta \varphi_t^+ &= \frac{\varepsilon^{1/4} (12)^{1/4} |\mu|^{1/2}}{p^{3/4} \sigma S_t}. \end{aligned}$$

Thus, Assumptions 3.6.1, 3.6.4 and 3.6.3 are fulfilled. The utility loss due to transactions costs is

$$M_0 p \gamma_0 = U(x) \exp \left(-\frac{T \mu^2}{2 \sigma^2} \right) |\mu| T \left(\frac{\varepsilon}{3p} \right)^{1/2}$$

and the certainty equivalent loss equals

$$-\gamma_0 = |\mu| T \left(\frac{\varepsilon}{3p} \right)^{1/2}.$$

Denote by $(\tau_i)_{i \in \mathbb{N}}$ the trading times of φ^ε with $\tau_0 = 0$. With some rewriting we can obtain the distribution of the trading times: Observe that

$$\begin{aligned} \tau_i &= \inf \left\{ t > \tau_{i-1} : |\varphi_t^* - \varphi_{\tau_{i-1}}^*| = \Delta \varphi_t^+ \right\} \\ &= \inf \left\{ t > \tau_{i-1} : \left| \frac{\mu}{\sigma^2 p S_t} - \frac{\mu}{\sigma^2 p S_{\tau_{i-1}}} \right| = \frac{\varepsilon^{1/4} (12)^{1/4} |\mu|^{1/2}}{p^{3/4} \sigma S_t} \right\} \\ &= \inf \left\{ t > \tau_{i-1} : \frac{S_t}{S_{\tau_{i-1}}} = 1 \pm \frac{\varepsilon^{1/4} \sigma p^{1/4} (12)^{1/4}}{|\mu|^{1/2}} \right\} \\ &= \tau_{i-1} + \inf \left\{ s > 0 : \frac{S_{\tau_{i-1}+s}}{S_{\tau_{i-1}}} = 1 \pm \frac{\varepsilon^{1/4} \sigma p^{1/4} (12)^{1/4}}{|\mu|^{1/2}} \right\}. \end{aligned}$$

5. Examples

We see that $\sigma_i = \inf \left\{ s > 0 : \frac{S_{\tau_{i-1}+s}}{S_{\tau_{i-1}}} = 1 \pm \frac{\varepsilon^{1/4} \sigma p^{1/4} (12)^{1/4}}{|\mu|^{1/2}} \right\}$ is a first exit time of the geometric Brownian motion

$$\frac{S_{\tau_{i-1}+s}}{S_{\tau_{i-1}}} = \exp \left((\mu - \sigma^2/2)s + \sigma W_s^i \right)$$

with $W^i = W_{\cdot+\tau_{i-1}} - W_{\tau_{i-1}}$ being a standard Brownian motion. With formulas from Borodin and Salminen (2015) we obtain the distribution of these exit times from the interval

$$[a, b] := \left[1 - \frac{\varepsilon^{1/4} \sigma p^{1/4} (12)^{1/4}}{|\mu|^{1/2}}, 1 + \frac{\varepsilon^{1/4} \sigma p^{1/4} (12)^{1/4}}{|\mu|^{1/2}} \right].$$

Proposition 5.1.1. *The exit times $(\sigma_i)_{i \in \mathbb{N}}$ have the probability density function*

$$\begin{aligned} f(y) = & a^\nu \exp \left(-\frac{\nu^2 \sigma^2 y}{2} \right) g \left(y, \frac{\log(b)}{\sigma}, \frac{\log(b/a)}{\sigma} \right) \\ & + b^\nu \exp \left(-\frac{\nu^2 \sigma^2 y}{2} \right) g \left(y, \frac{\log(1/a)}{\sigma}, \frac{\log(b/a)}{\sigma} \right) \end{aligned}$$

for $y > 0$, where $\nu = \frac{\mu}{\sigma^2} - \frac{1}{2}$ and the function g is given by

$$g(y, w, z) = \sum_{j=-\infty}^{\infty} \frac{z - w + 2jz}{\sqrt{2\pi} y^{3/2}} \exp \left(-\frac{(z - w + 2jz)^2}{2y} \right).$$

Proof. (Borodin and Salminen, 2015, Formulas 9.3.0.6(a)+(b) and Appendix 2.11) \square

This quite complicated formula could be evaluated numerically to determine distribution based values like the mean of the exit times. With results from Karlin and Taylor (1981) we can find a simpler formula for the mean:

Proposition 5.1.2. *We have*

$$\mathbb{E}(\tau_1) = \begin{cases} -\frac{2}{\sigma^2} \log(b) \log(a), & \text{if } \nu = 0, \\ \frac{2}{\nu \sigma^2} \left[\frac{a^{-2\nu} - 1}{a^{-2\nu} - b^{-2\nu}} \left(\frac{b^{-2\nu} - 1}{2\nu} + \log(b) \right) + \frac{1 - b^{-2\nu}}{a^{-2\nu} - b^{-2\nu}} \left(\frac{a^{-2\nu} - 1}{2\nu} + \log(a) \right) \right], & \text{otherwise.} \end{cases}$$

Proof. According to (Karlin and Taylor, 1981, equation 3.12 in section 15.3) the expectation of the first exit time(s) is given by

$$\mathbb{E}(\tau_1) = 2 \left(p_{a,b} \int_1^b (s(b) - s(y)) m(y) dy + (1 - p_{a,b}) \int_a^1 (s(y) - s(a)) m(y) dy \right),$$

where s is the scale function and m the density of the speed measure of a geometric Brownian motion. From (Borodin and Salminen, 2015, Appendix I.1.20) we know that they are given by

$$\begin{aligned} s(y) &= \begin{cases} -\frac{y^{-2\nu}}{2\nu}, & \text{if } \nu \neq 0 \\ \log(y), & \text{if } \nu = 0, \end{cases} \\ m(y) &= \frac{2}{\sigma^2} y^{2\nu-1}. \end{aligned}$$

Furthermore $p_{a,b}$ denotes the probability that the geometric Brownian motion hits b before a , which is given by

$$p_{a,b} = \frac{s(1) - s(a)}{s(b) - s(a)}$$

according to (Karlin and Taylor, 1981, equation (3.10) in section 15.3). In the case $\nu = 0$ we observe that

$$p_{a,b} = -\frac{\log(a)}{\log(b) - \log(a)} \quad \text{and} \quad 1 - p_{a,b} = \frac{\log(b)}{\log(b) - \log(a)}$$

and for the integrals

$$\begin{aligned} \int_1^b (s(b) - s(y))m(y)dy &= \int_1^b (\log(b) - \log(y)) \frac{2}{\sigma^2} y^{-1} dy \\ &= \frac{2}{\sigma^2} \log(b) \int_1^b y^{-1} dy - \frac{2}{\sigma^2} \int_1^b \log(y) y^{-1} dy \\ &= \frac{1}{\sigma^2} \log(b)^2. \end{aligned}$$

Similarly, one shows

$$\int_a^1 (s(y) - s(a))m(y)dy = \frac{1}{\sigma^2} \log(a)^2.$$

This yields

$$\begin{aligned} \mathbb{E}(\tau_1) &= 2 \left(\frac{-\log(a)}{\log(b) - \log(a)} \frac{1}{\sigma^2} \log(b)^2 + \frac{\log(b)}{\log(b) - \log(a)} \frac{1}{\sigma^2} \log(a)^2 \right) \\ &= \frac{2}{\sigma^2 (\log(b) - \log(a))} (\log(b) \log(a)^2 - \log(a) \log(b)^2) \\ &= -\frac{2}{\sigma^2} \log(b) \log(a). \end{aligned}$$

For the case $\nu \neq 0$ we have

$$p_{a,b} = \frac{a^{-2\nu} - 1}{a^{-2\nu} - b^{-2\nu}}$$

and for the integrals

$$\begin{aligned} &\int_1^b (s(b) - s(y))m(y)dy \\ &= \int_1^b \left(-\frac{b^{-2\nu}}{2\nu} + \frac{y^{-2\nu}}{2\nu} \right) \frac{2}{\sigma^2} y^{2\nu-1} dy \\ &= -\frac{b^{-2\nu}}{2\nu} \frac{2}{\sigma^2} \int_1^b y^{2\nu-1} dy + \frac{2}{2\nu\sigma^2} \int_1^b y^{-1} dy \\ &= \frac{b^{-2\nu} - 1}{2\nu^2\sigma^2} + \frac{1}{\nu\sigma^2} \log(b). \end{aligned}$$

5. Examples

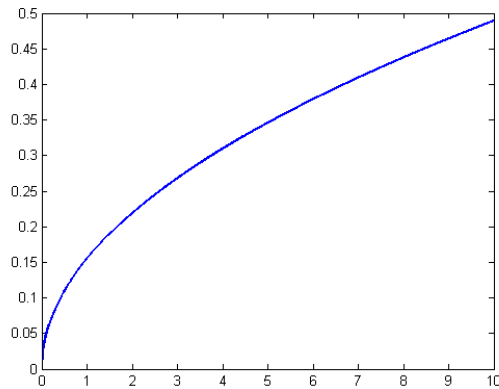
Similarly,

$$\int_a^1 (s(y) - s(a))m(y)dy = \frac{a^{-2\nu} - 1}{2\nu^2\sigma^2} + \frac{1}{\nu\sigma^2} \log(a).$$

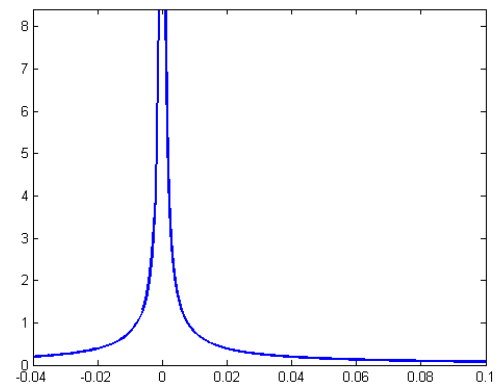
Inserting everything yields the result. \square

We close this section with a few words on the dependence of the mean time to the next transaction, the no-trade region and the certainty equivalent loss on the transaction cost ε and the parameters μ , σ and p : For the parameters $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$ and $\varepsilon = 1\text{€}$ we obtain a mean time to the next trade of 0.1549.¹ If $\mu - \frac{\sigma^2}{2}$ and σ are the yearly expected log return and volatility and we assume 250 trading days a year, then this number implies that a trade roughly occurs every 39 trading days. With higher transaction costs the no-trade region and thus the mean time to the next transaction become larger (figure 5.1a). For example, at a transaction cost

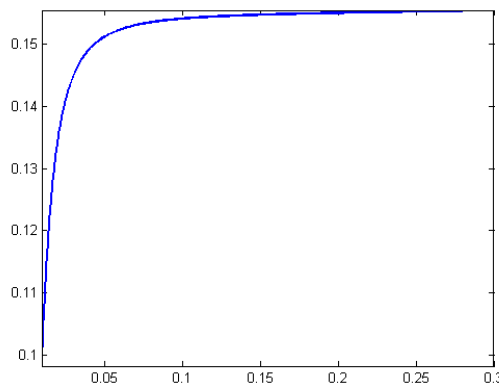
¹The risk aversion is chosen in a way that $\varphi_0^* S_0 = \frac{\mu}{p\sigma^2} = 10^6\text{€}$.



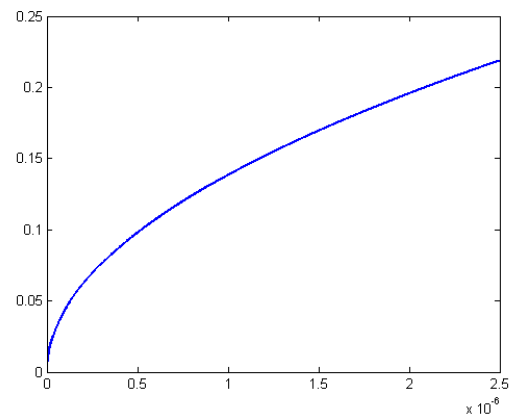
(a) Mean time as a function of costs



(b) Mean time as a function of drift rate μ



(c) Mean time as a function of volatility σ



(d) Mean time as a function of risk aversion p

Figure 5.1.: Mean time to the next transaction. The fixed parameters are given by $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$ and $\varepsilon = 1\text{€}$.

level of $\varepsilon = 5\text{€}$ a trade occurs every 85 trading days or roughly three to four times a year on average.

The dependence on the drift and the volatility is not as obvious since the geometric Brownian motion under consideration also depends on these parameters, but with the use of Proposition 5.1.2 their influence on the mean time can be found out: In the case of a positive drift rate μ an increase in the latter will decrease the exit times and thus increase the number of transactions up to time T (figure 5.1b). The converse is true for the volatility σ , the risk aversion parameter p and negative drift rates: An increase of these parameters will lead to fewer transactions (figures 5.1b, 5.1c and 5.1d). Although an increase in volatility or risk aversion will reduce the corridor width, the target itself is lowered (in absolute values) implying that less money (in absolute values) is invested in the stock. The latter is the dominating effect here. In figure 5.1b we observe a singularity in 0: If the drift equals 0, then the optimal strategy in the frictionless market is to keep all the money in the bank account. This is an admissible strategy in the market with frictions that does not trade at all and corresponds to an infinitely large no-trade region.

Furthermore, the asymptotic certainty equivalent loss depends linearly on the absolute value of the drift rate μ and is decreasing in the risk aversion parameter. Although the width of the no-trade region becomes smaller for higher risk aversion, the certainty equivalent loss is lowered. The reason for this phenomenon was already discussed above: More risk averse investors do not only have a closer tracking to the target, but also the number of stocks is closer to 0 dominating the smaller corridor. This also explains the remarkable fact that the asymptotic certainty equivalent loss does not depend on the volatility parameter. Here, the contrary effects of closer tracking and fewer stock investments cancel out each other. In our numerical example with 1€ (5€) transaction costs the asymptotic certainty equivalent loss over a period of one year equals 25.82€ (57.74€). Here, the frictionless certainty equivalent minus initial wealth $\frac{T\mu^2}{2p\sigma^2}$ is 25000€ .

As already pointed out, the formula of the no-trade region coincides with the one from (Altarovici et al., 2015a, Equation 2.6) up to the different risk tolerance process.² Moreover, for the Black-Scholes model we can see that the formula also coincides with the one from (Korn, 1998, Proposition 5.1(iii)) with the same preferences and finite time horizon in the absence of proportional costs.

²See (Kallsen and Muhle-Karbe, 2013, section 2.2).

5. Examples

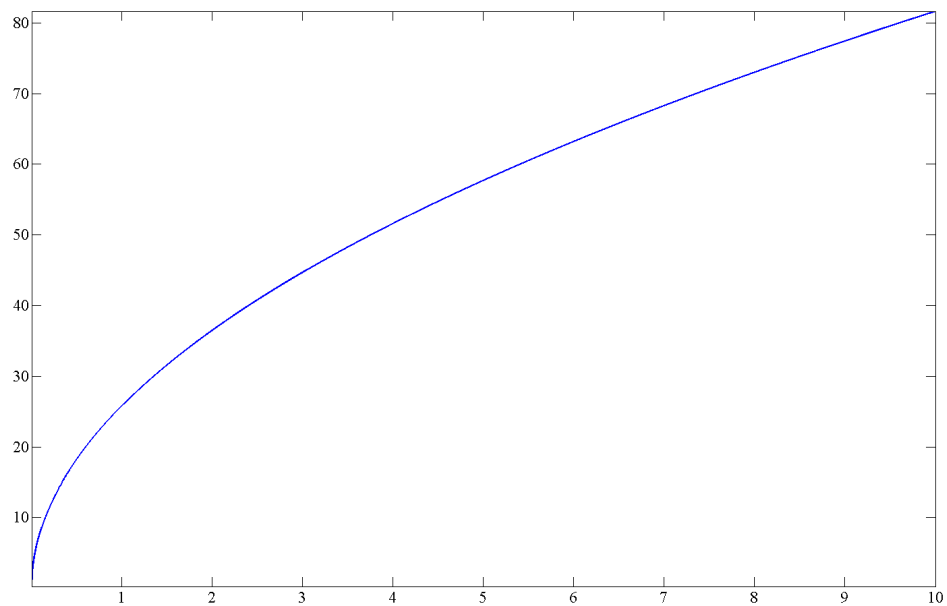


Figure 5.2.: Certainty equivalent loss as a function of the transaction costs with parameters $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$ and $T = 1$

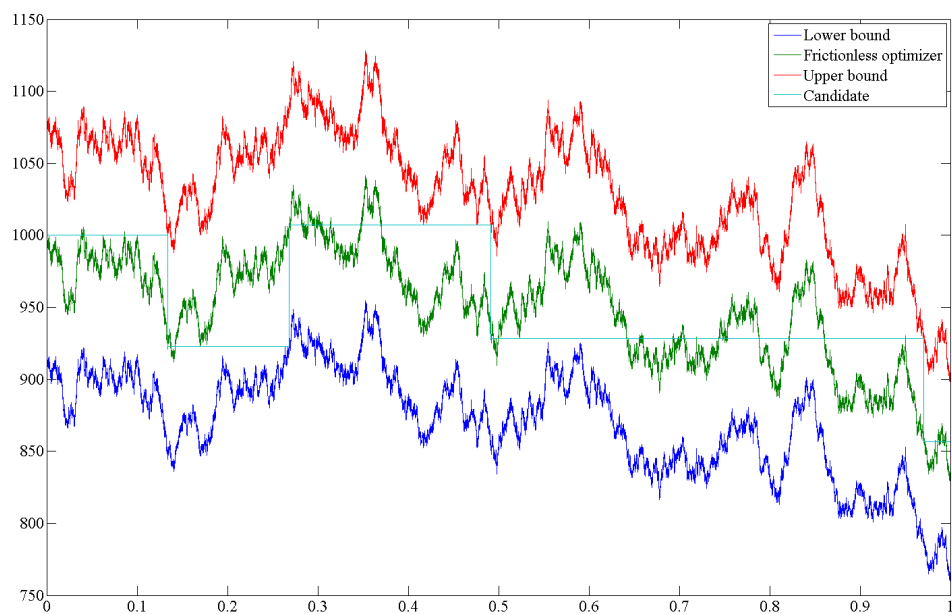


Figure 5.3.: Simulation in time of the candidate strategy with parameters $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$, $T = 1$, $S_0 = 1000\text{€}$ and $\varepsilon = 5\text{€}$

5.2. Stochastic volatility model

Another Example and an extension of the Black-Scholes model is the stochastic volatility model

$$dS_t = S_t (\mu(Z_t)dt + \sigma(Z_t)dW_t)$$

from Example 2.4.6. We will state some regularity assumptions for this model that have to be fulfilled in order to find the almost optimal strategy. Assuming that $\pi = \frac{\mu(Z)}{p\sigma^2(Z)}$ is an Itô process we will see in the subsequent proof how to compute the quadratic variation of φ^* , namely

$$c_t^{\varphi^*} = \frac{\pi_t^2 \sigma^2(Z_t) + c_t^\pi}{S_t^2},$$

this implies for the process C

$$C_t = S_t^4 \underbrace{\frac{1}{\pi_t^2 + \frac{c_t^\pi}{\sigma(Z_t)}}}_{:=J_t} = S_t^4 J_t.$$

Proposition 5.2.1. *Suppose that the stochastic volatility model fulfills the following regularity assumptions:*

1. μ and σ are bounded and bounded away from 0.
2. π and J are Itô processes such that $b^{\pi,Q}$, c^π , $b^{J,Q}$ and c^J are bounded.

Then, the model satisfies Assumptions 3.6.1, 3.6.3 and 3.6.4, i.e. $\widetilde{\varphi}^\varepsilon$ from section 3.8 is the almost optimal strategy. The certainty equivalent loss due to fixed transaction costs amounts to

$$\gamma_0 = \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T (\mu(Z_s)^2 + c_s^\pi \sigma^2(Z_s))^{1/2} ds \right).$$

Proof. (i) From the proof of proposition 2.3.2 in Ahrens (2015) we know that $[f(S), X] = 0$ for every twice continuously differentiable function f and every Itô process X being adapted to the filtration generated by Z . Using this fact will make it easier to compute the dynamics of the involved processes.

(ii) As a next step we will show that

$$\mathbb{E}_Q \left(\int_0^T S_s^q ds \right) < \infty \text{ for all } q \in \mathbb{R}. \quad (5.2.1)$$

Since S is a Q -martingale, S^q is a Q -submartingale by Jensen's inequality; hence, we obtain

$$\mathbb{E}_Q \left(\int_0^T S_s^q ds \right) \leq T \mathbb{E}_Q (S_T^q),$$

5. Examples

where S_T^q can be rewritten as

$$\begin{aligned} S_T^q &= S_0^q \exp \left(- \int_0^T q \frac{\sigma^2(Z_s)}{2} ds + \int_0^T q \sigma(Z_s) dW_s^Q \right) \\ &= S_0^q \exp \left(\int_0^T (q^2 - q) \frac{\sigma^2(Z_s)}{2} ds \right) \mathcal{E} \left(\int_0^T q \sigma(Z_s) dW_s^Q \right)_T. \end{aligned}$$

Since σ is bounded, $\mathcal{E} \left(\int_0^T q \sigma(Z_s) dW_s^Q \right)$ is a true Q -martingale by the Novikov condition, and we obtain the desired result.

(iii) We will verify the conditions in Assumptions 3.6.1, 3.6.3 and 3.6.4: By Itô's formula the process S^{-1} has the dynamics

$$b_t^{S^{-1}, Q} = S_t^{-1} \sigma^2(Z_t) \quad \text{and} \quad c_t^{S^{-1}} = S_t^{-2} \sigma^2(Z_t).$$

Using the property in (i) and integration by parts implies for the dynamics of φ^*

$$\begin{aligned} b_t^{\varphi^*, Q} &= \frac{\pi_t}{S_t} \sigma^2(Z_t) + \frac{1}{S_t} b_t^{\pi, Q} \\ c_t^{\varphi^*} &= \frac{\pi_t^2}{S_t^2} \sigma^2(Z_t) + \frac{c_t^\pi}{S_t^2}, \end{aligned}$$

which yield the desired properties for $[\varphi^*]_T$ and $\int_0^T |b_s^{\varphi^*, Q}| ds$. The process A is bounded since

$$A_t^2 = c_t^{\varphi^*} c_t^S = \pi_t^2 \sigma^4(Z_t) + c_t^\pi \sigma^2(Z_t).$$

Computing the Q -dynamics of C with integration by parts and Itô's formula yields

$$\begin{aligned} b_t^{C, Q} &= S_t^4 b_t^{J, Q} + 6 J_s S_t^4 \sigma^2(Z_t), \\ M_t^{C, Q} &= \int_0^t S_s^4 dM_s^{J, Q} + \int_0^t 4 J_s S_s^4 \sigma(Z_s) dW_s^Q. \end{aligned}$$

Again by the property in (i) this implies

$$[C]_T = \int_0^T S_s^8 c_s^J ds + \int_0^T 16 J_s S_s^8 \sigma^2(Z_s) ds,$$

so by the boundedness of the involved processes we obtain

$$\mathbb{E}_Q([C]_T^2) \leq \text{const.} \mathbb{E}_Q \left(\int_0^T S_s^{16} ds \right) < \infty$$

and similarly $\int_0^T |b^{C, Q}| ds, [S]_T \in L_2(Q)$. Since the process J^{-1} is bounded, we obtain by Theorem B.2.3

$$\begin{aligned} \|C^{-1}\|_{S_4(Q)} &\leq \text{const.} \quad \|S^{-1}\|_{S_4(Q)} \leq \text{const.} \quad \|S^{-1}\|_{H_4(Q)} \\ &\leq \text{const.} \left(S_0^{-1} + \left\| \int_0^T \frac{\sigma^2(Z_s)}{S_s} ds \right\|_{L_4(Q)} + \left\| \sqrt{\int_0^T \frac{\sigma^2(Z_s)}{S_s^2} ds} \right\|_{L_4(Q)} \right) \\ &\leq \text{const.} \left(S_0^{-1} + \mathbb{E}_Q \left(\int_0^T S_s^{-4} ds \right)^{1/4} \right), \end{aligned}$$

which is finite by (ii). □

6. Application: Utility indifference pricing

6.1. General framework

Similar to Whalley and Wilmott (1997), Barles and Soner (1998), Bichuch (2014) and Ahrens (2015) we can apply our results from utility maximization on pricing a contingent claim with the utility indifference argument in the presence of *fixed* instead of proportional transaction costs. Therefore, we adapt the terminology for proportional transaction costs from (Ahrens, 2015, section 2.2.2): Let H be a bounded, non-negative and \mathcal{F}_T -measurable random variable representing the payoff of a contingent claim with maturity T . Then the utility indifference price of H under fixed transaction costs is given by

$$\pi^\varepsilon(H) := \sup_{\varphi \in \mathfrak{A}^\varepsilon} U^{-1}(\mathbb{E}(U(V_T^\varepsilon(\varphi)))) - \sup_{\varphi \in \mathfrak{A}^\varepsilon} U^{-1}(\mathbb{E}(U(V_T^\varepsilon(\varphi) - H))).$$

The first supremum can be determined asymptotically with corollary 3.8.5. For the second one observe that changing the probability measure to P^H given by

$$\frac{dP^H}{dP} := \frac{e^{pH}}{\mathbb{E}(e^{pH})}$$

leads to

$$\sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E}(U(V_T^\varepsilon(\varphi) - H)) = \mathbb{E}(e^{pH}) \sup_{\varphi \in \mathfrak{A}^\varepsilon} \mathbb{E}_{P^H}(U(V_T^\varepsilon(\varphi)))$$

due to exponential utility. Again, we can apply the previous results from Chapter 3 but now under the new measure P^H . This leads to a new frictionless optimizer φ^H instead of φ^* and a new EMM Q^H satisfying Assumption 3.6.1 with P^H instead of P . We obtain the following result which is the counterpart of (Ahrens, 2015, Corollary 2.25) for fixed costs.

Corollary 6.1.1. *Suppose that Assumption 3.6.1 holds for P^H instead of P and denote by φ^H and Q^H the frictionless optimizer under P^H and the corresponding EMM. Additionally, suppose that Assumptions 3.6.3 and 3.6.4 hold for the processes φ^H , $A^H = (c^{\varphi^H} c^S)^{1/2}$ and $C^H = \frac{c^S}{c^{\varphi^H}}$. Then, the exponential utility indifference price of H in the presence of fixed transaction costs is given by*

$$\begin{aligned} \pi^\varepsilon(H) &= \pi(H) \\ &+ \sqrt{\frac{p\varepsilon}{3}} \left(\mathbb{E}_{Q^H} \left(\int_0^T (c_t^S c_t^{\varphi^H})^{1/2} dt \right) - \mathbb{E}_Q \left(\int_0^T (c_t^S c_t^{\varphi^*})^{1/2} dt \right) \right) + o(\varepsilon^{1/2}). \end{aligned}$$

6. Application: Utility indifference pricing

Here $\pi(H)$ denotes the frictionless utility indifference price. Furthermore, relating $\varphi^{H,\varepsilon}$ and $\tilde{\varphi}^{H,\varepsilon}$ in the same way as φ^ε and $\tilde{\varphi}^\varepsilon$, we obtain that $\tilde{\varphi}^{H,\varepsilon}$ is the optimal strategy for the utility indifference price $\pi^\varepsilon(H)$ at leading order $\varepsilon^{1/2}$.

Remark 6.1.2. If Assumption 3.6.4 does not hold for one of the optimization problems under P or P^H , a way out is to use one of the generalized versions from Chapter 4. We will do so when considering the European put option in the Black-Scholes model in subsection 6.2.2.

In the case of a complete market, we can simplify the formula for the asymptotic utility indifference price as in (Ahrens, 2015, Remark 2.2.6).

Remark 6.1.3. If the frictionless market is complete, we have a unique EMM, i.e. $Q^H = Q$. In this case the frictionless utility indifference price is the Q -expectation of the payoff: $\pi(H) = \mathbb{E}_Q(H)$. The frictionless optimal strategy for the utility indifference price is given by $\varphi^H = \varphi^* + \Delta$, where Δ is the replicating strategy for H . The formula for the utility indifference price in the presence of transaction costs then simplifies to

$$\pi^\varepsilon(H) = \mathbb{E}_Q(H) + \sqrt{\frac{p\varepsilon}{3}} \mathbb{E}_Q \left(\int_0^T (c_t^S)^{1/2} \left((c_t^{\varphi^H})^{1/2} - (c_t^{\varphi^*})^{1/2} \right) dt \right) + o(\varepsilon^{1/2})$$

in the situation of Corollary 6.1.1.

6.2. Black-Scholes model

We now focus on the Black-Scholes model from Example 2.4.5 and present two results for the utility indifference pricing in the presence of fixed transaction costs. The first one is for regular option payoffs in the sense of Bichuch (2014) and Ahrens (2015), the second for the European put option.

6.2.1. For regular option payoffs

Proposition 6.2.1. Assume that $\mu > 0$ and that we have a payoff $H = g(S_T)$ with some nonnegative, measurable function g , such that

1. g is four times continuously differentiable and the mappings $s \mapsto s^i g^{(i)}(s)$ for $i \in \{0, 1, 2, 3, 4\}$ are bounded.
2. $\frac{\mu}{p\sigma^2} - \sup_{s \in (0, \infty)} |s^2 g''(s)| > 0$.

Denote by V the solution $V : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ of the Black-Scholes partial differential equation

$$\partial_2 V(s, t) + \frac{\sigma^2 s^2}{2} \partial_{11}^2 V(s, t) = 0, \quad V(s, T) = g(s).$$

Then the utility indifference price of H is given by

$$\pi^\varepsilon(H) = \mathbb{E}_Q(H) + \sqrt{\frac{p\varepsilon}{3}} 2\mathbb{E}_Q \left(\int_0^T \partial_2 V(S_t, t) dt \right) + o(\varepsilon^{1/2}).$$

Here, Q denotes the unique EMM in the Black-Scholes model.

Proof. (i) From (Bichuch, 2014, Lemma 9) we recall that for $i = 0, \dots, 4$ the functions $(s, t) \mapsto s^i \partial_1^i V(S, t)$ are bounded by $\sup_{x>0} x^i |g^{(i)}(x)| =: C_i$.

(ii) From Remark 6.1.3 and the proof of Proposition 2.3.1 in Ahrens (2015) we know that the frictionless optimal strategy for the contingent claim liability φ^H is given by

$$\varphi_t^H = \varphi_t^* + \partial_1 V(S_t, t).$$

Due to the first assumption, Itô's formula can now be applied delivering the Q -local drift and covariation:

$$\begin{aligned} b_t^{\varphi^H, Q} &= \sigma^2 \varphi_t^* + \partial_{12}^2 V(S_t, t) + \frac{\sigma^2}{2} S_t^2 \partial_{111}^3 V(S_t, t) \\ &= \sigma^2 \varphi_t^* - \sigma^2 S_t \partial_{11}^2 V(S_t, t) \\ c_t^{\varphi^H} &= (\sigma S_t \partial_{11}^2 V(S_t, t) - \sigma \varphi_t^*)^2, \\ &= \frac{1}{\sigma^2 S_t^2} \left(2\partial_2 V(S_t, t) + \frac{\mu}{p} \right)^2. \end{aligned}$$

With (i) we obtain the following upper bounds for these processes

$$|b_t^{\varphi^H, Q}| \leq \sigma^2 \varphi_t^* + \sigma^2 C_2 S_t^{-1}$$

and

$$\begin{aligned} c_t^{\varphi^*} &\leq 2 (\sigma S_t \partial_{11}^2 V(S_t, t))^2 + 2\sigma^2 \varphi_t^2 \\ &\leq 2C_2^2 \sigma^2 S_t^{-2} + 2\sigma^2 (\varphi_t^*)^2 \end{aligned}$$

which fulfill the desired integrability conditions. The above calculation implies for the process $A^H = \sqrt{c^{\varphi^H} c^S}$:

$$(A_t^H)^2 = \left(\frac{\mu}{p} - \sigma^2 S_t^2 \partial_{11}^2 V(S_t, t) \right)^2 = \left(\frac{\mu}{p} + 2\partial_2 V(S_t, t) \right)^2,$$

being a bounded process. According to the second assumption the constant C_2 is strictly smaller than $\frac{\mu}{p\sigma^2}$; hence, $\frac{\mu}{p} - \sigma^2 S_t^2 \partial_{11}^2 V(S, \cdot)$ does not vanish. So we obtain that $C^H = \frac{c^S}{c^{\varphi^H}}$ is well-defined and

$$C_t^H = \frac{\sigma^4 S_t^4}{\left(2\partial_2 V(S_t, t) + \frac{\mu}{p} \right)^2} = \left(\partial_{11} V(S_t, t) + \frac{\mu}{p\sigma^2 S_t^2} \right)^{-2}.$$

From the differentiability assumptions we know that C^H is an Itô process, and its dynamics contain the partial derivatives $\partial_1^i V$ for $i = 2, 3, 4$ of the option price function. Since these are bounded by $C_i S^{-i}$ and $\partial_2 V + \frac{\mu}{p}$ is bounded away from 0, we can conclude that $b^{C^H, Q}$ and $[C^H]$ fulfill Assumption 3.6.4. The boundedness of $\partial_2 V = -\frac{\sigma^2}{2} S^2 \partial_{11} V(S, \cdot)$ shows the remaining property $\left\| (C^H)^{-1} \right\|_{S_4(Q)} < \infty$.

6. Application: Utility indifference pricing

(iii) Due to the second property of g and (i) the process $\partial_2 V + \frac{\mu}{p}$ is nonnegative and thus A^H is equal to the latter. So by $A = |\frac{\mu}{p}| = \frac{\mu}{p}$ according to section 5.1 we have

$$\mathbb{E}_Q \left(\int_0^T A_t^H dt \right) - \mathbb{E}_Q \left(\int_0^T A_t dt \right) = 2\mathbb{E}_Q \left(\int_0^T \partial_2 V(S_t, t) dt \right)$$

yielding the asymptotic expansion of the utility indifference price. \square

Remark 6.2.2. *The conditions for g in Proposition 6.2.1 are the ones in (Bichuch, 2014, Assumptions 2 and 3) and (Ahrens, 2015, (C1) and (C2) in Proposition 2.3.1). However, the classical examples for contingent claims, namely call and put, do not fulfill these assumptions. A nontrivial Example for an option payoff fulfilling these conditions is to take the call option price at some fixed time before maturity as an option payoff, which is done in (Bichuch, 2014, Example 10).*

6.2.2. For the put option

We want to extend the above result to the case of a European put option in the Black-Scholes model. However, this will require the extended result on optimization from section 4.3.

Theorem 6.2.3. *The utility indifference price for the put $H = (K - S_T)^+$ with strike $K > 0$ in the Black-Scholes model under fixed transaction costs is given by*

$$\begin{aligned} \pi^\varepsilon(H) &= K\Phi_{0,1}(-d_2(S_0, 0)) - S_0\Phi_{0,1}(-d_1(S_0, 0)) \\ &\quad + \sqrt{\frac{p\varepsilon}{3}} 2\mathbb{E}_Q \left(\int_0^T \left(\left| \partial_2 V(S_s, s) + \frac{\mu}{p} \right| - \frac{|\mu|}{p} \right) ds \right) + o(\varepsilon^{1/2}). \end{aligned}$$

Here, the functions d_1 and d_2 are given by $d_1(s, t) = \frac{\log(s/K) + \sigma^2/2(T-t)}{\sigma\sqrt{T-t}}$ and $d_2(s, t) = d_1(s, t) - \sigma\sqrt{T-t}$. In the case $\mu = 0$ this can be simplified to

$$\begin{aligned} \pi^\varepsilon(H) &= K\Phi_{0,1}(-d_2(S_0, 0)) - S_0\Phi_{0,1}(-d_1(S_0, 0)) \\ &\quad + \sqrt{\frac{p\varepsilon}{3}} K\sigma T^{1/2} \varphi_{0,1}(d_2(S_0, 0)) + o(\varepsilon^{1/2}). \end{aligned} \quad (6.2.1)$$

Proof. We have to verify Assumptions 3.6.3 and 4.2.3 for the optimization problem under P^H .

(i) We see that $A^H = |\frac{\mu}{p} + 2\partial_2 V(S, \cdot)|$ as in the proof of Proposition 6.2.1, which implies

$$\int_0^T A_s^H ds \leq \left| \frac{\mu}{p} \right| T + 2 \int_0^T |\partial_2 V(S_s, s)| ds \quad (6.2.2)$$

with

$$\partial_2 V(s, t) = -\frac{s\varphi_{0,1}(d_1(s, t))\sigma}{2\sqrt{T-t}}, \quad d_1(s, t) = \frac{\log(s/K) + \sigma^2/2(T-t)}{\sigma\sqrt{T-t}}.$$

We see that

$$\begin{aligned} 0 \leq s\varphi_{0,1}(d_s(s, t)) &= K \exp\left(\sigma\sqrt{T-t}d_1(s, t) - \frac{\sigma^2}{2}(T-t)\right) \exp\left(-\frac{d_1(s, t)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \\ &= \frac{K}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(d_1(s, t) - \sigma\sqrt{T-t})^2\right) \leq \frac{K}{\sqrt{2\pi}}. \end{aligned}$$

So $(s, t) \mapsto s\varphi_{0,1}(d_1(s, t))$ is a bounded function, and we obtain

$$\int_t^T |\partial_2 V(S_u, u)| du \leq \text{const.} \int_t^T \frac{1}{\sqrt{T-u}} du \leq \text{const.} \int_0^T \frac{1}{\sqrt{T-u}} du < \infty.$$

Hence, by (6.2.2) the random variable $\int_0^T A_u^H du$ is bounded.

(ii) We see that

$$V_T(\varphi^H) = V_T(\varphi^*) + \int_0^T \partial_1 V(S_u, u) dS_u = V_T(\varphi^*) + \int_0^T S_u (\Phi_{0,1}(d_1(S_u, u)) - 1) \sigma dW_u^Q.$$

Define the function $h : (s, t) \mapsto s(\Phi_{0,1}(d_1(s, t)) - 1)$. We obtain

$$\begin{aligned} \exp\left(\int_0^T 16h(S_u, u) \sigma dW_u^Q\right) &= \mathcal{E}\left(\int_0^T 16h(S_u, u) dW_u^Q\right) \exp\left(\frac{1}{2} \left[\int_0^T 16h(S_u, u) \sigma dW_u^Q\right]^2\right) \\ &= \mathcal{E}\left(\int_0^T 16h(S_u, u) dW_u^Q\right) \exp\left(128 \int_0^T h(S_u, u)^2 \sigma^2 du\right). \end{aligned}$$

Since h and thus $\exp(128 \int_0^T h(S_u, u)^2 \sigma^2 du)$ are bounded, we obtain that the random variable $\exp\left(\int_0^T 16h(S_u, u) \sigma dW_u^Q\right)$ is in $L_1(Q)$ due to Novikov's condition and hence

$$\begin{aligned} \mathbb{E}_Q(\exp(8V_T(\varphi^H))) &\leq (\mathbb{E}_Q(\exp(16V_T(\varphi^*)))^{1/2} \mathbb{E}_Q\left(\exp\left(\int_0^T 16h(S_u, u) \sigma dW_u^Q\right)\right)^{1/2} \\ &< \infty \end{aligned}$$

by Hölder's inequality. The same arguments show $\exp(-8pV_T(\varphi^H)) < \infty$.

(iii) We will show that $\int_0^T |b_s^{\varphi^H, Q}| ds \in L_8(Q)$ and $[\varphi^H]_T \in L_4(Q)$ hold: We see that

$$b^{\varphi^H, Q} = b^{\varphi^*, Q} + b^{\partial_1 V(S, \cdot), Q}$$

and we know $\int_0^T |b_s^{\varphi^*, Q}| ds \in L_8(Q)$. Furthermore, due to Itô's formula and the Black-Scholes PDE

$$\begin{aligned} b^{\partial_1 V(S, \cdot), Q} &= \partial_{12}^2 V(S, \cdot) + \frac{\sigma^2}{2} S^2 \partial_{111}^3 V(S, \cdot) \\ &= \partial_1(\partial_2 V(S, \cdot)) + \frac{\sigma^2}{2} S^2 \partial_{111}^3 V(S, \cdot) \\ &= \partial_1\left(-\frac{\sigma^2}{2} S^2 \partial_{11}^2 V(S, \cdot)\right) + \frac{\sigma^2}{2} S^2 \partial_{111}^3 V(S, \cdot) \\ &= (-1)\left(\frac{\sigma^2}{2} S^2 \partial_{111}^3 V(S, \cdot) + \sigma^2 S \partial_{11}^2 V(S, \cdot)\right) + \frac{\sigma^2}{2} S^2 \partial_{111}^3 V(S, \cdot) \\ &= (-1)\sigma^2 S \partial_{11}^2 V(S, \cdot) = -\sigma \frac{\varphi_{0,1}(d_1(S, \cdot))}{\sqrt{T-t}}, \end{aligned}$$

6. Application: Utility indifference pricing

so

$$\int_0^T |b_t^{\partial_1 V(S, \cdot), Q}| dt \leq \sigma \int_0^T \frac{1}{(T-t)^{1/2}} dt < \infty$$

is a bounded random variable, which implies $\int_0^T |b_t^{\partial_1 V(S, \cdot), Q}| dt \in L_8(Q)$. Due to $\varphi^H = \varphi^* + \partial_1 V(S, \cdot)$, we have with the inequality of Kunita-Watanabe (Theorem B.2.1)

$$\begin{aligned} [\varphi^H]_T &= [\varphi^*]_T + [\partial_1 V(S, \cdot)]_T + 2[\varphi^*, \partial_1 V(S, \cdot)]_T \\ &\leq \left([\varphi^*]_T^{1/2} + [\partial_1 V(S, \cdot)]_T^{1/2} \right)^2 \\ &\leq 2[\varphi^*]_T + 2[\partial_1 V(S, \cdot)]_T. \end{aligned}$$

Since $[\varphi^*]_T \in L_4(Q)$, we are left with $[\partial_1 V(S, \cdot)]_T \in L_4(Q)$: By the Burkholder-Davis-Gundy inequalities (Theorem B.2.2) we get

$$\mathbb{E}_Q \left([\partial_1 V(S, \cdot)]_T^4 \right)^{1/8} \leq \text{const.} \|M^{\partial_1 V(S, \cdot), Q}\|_{S_8(Q)}$$

and

$$\begin{aligned} \|M^{\partial_1 V(S, \cdot), Q}\|_{S_8(Q)} &= \left\| \partial_1 V(S, \cdot) - \partial_1 V(S_0, 0) - \int_0^\cdot b_t^{\partial_1 V(S, \cdot), Q} dt \right\|_{S_8(Q)} \\ &\leq \|\partial_1 V(S, \cdot)\|_{S_8(Q)} + |\partial_1 V(S_0, 0)| + \left\| \int_0^\cdot b_t^{\partial_1 V(S, \cdot), Q} dt \right\|_{S_8(Q)}. \end{aligned}$$

Since $\partial_1 V(S, \cdot)$ is bounded, $\|\partial_1 V(S, \cdot)\|_{S_8(Q)}$ is finite. Furthermore,

$$\left\| \int_0^\cdot b_s^{\partial_1 V(S, \cdot), Q} ds \right\|_{S_8(Q)} \leq \left\| \int_0^\cdot |b_s^{\partial_1 V(S, \cdot), Q}| ds \right\|_{S_8(Q)} = \left\| \int_0^T |b_s^{\partial_1 V(S, \cdot), Q}| ds \right\|_{L_8(Q)} < \infty.$$

(iv) We have to verify the remaining conditions from Assumption 4.3.1. The condition on $c^S = \sigma^2 S^2$ is fulfilled for $\eta_1 = 2$. For c^{φ^H} we observe that with $\eta_2 := 1/4$

$$\begin{aligned} (c_t^{\varphi^H})^{1+\eta_2} &= \frac{1}{\sigma^{2(1+\eta_2)}} \left(2\partial_2 V(S_t, t) + \frac{\mu}{p} \right)^{2(1+\eta_2)} \\ &\leq \text{const.} \left(\frac{\varphi_{0,1}(d_1(S_t, t))}{\sqrt{T-t}} \right)^{2(1+\eta_2)} + \text{const.} \frac{1}{S_t^{2(1+\eta_2)}} \end{aligned}$$

by the convexity of the mapping $y \mapsto y^{2(1+\eta_2)}$. The term $\frac{1}{S_t^{2(1+\eta_2)}}$ is integrable, so we have to consider $\left(\frac{\varphi_{0,1}(d_1(S_t, t))}{\sqrt{T-t}} \right)^{2(1+\eta_2)}$:

$$\begin{aligned} &\mathbb{E}_Q \left(\int_0^T \left(\frac{\varphi_{0,1}(d_1(S_t, t))}{\sqrt{T-t}} \right)^{2(1+\eta_2)} dt \right) \\ &= \int_0^T \mathbb{E}_Q \left(\left(\frac{\varphi_{0,1}(d_1(S_t, t))}{\sqrt{T-t}} \right)^{2(1+\eta_2)} \right) dt \\ &= \int_0^T \frac{1}{(T-t)^{1+\eta_2}} \mathbb{E}_Q \left((\varphi_{0,1}(d_1(S_t, t)))^{2(1+\eta_2)} \right) dt. \end{aligned}$$

We will show that $\mathbb{E}_Q \left((\varphi_{0,1}(d_1(S_t, t)))^{2(1+\eta_2)} \right)$ is bounded by a constant times $\sqrt{T-t}$: with $q := 2(1 + \eta_2)$ we obtain

$$\mathbb{E}_Q \left((\varphi_{0,1}(d_1(S_t, t)))^q \right) = \frac{1}{(2\pi)^{q/2}} \mathbb{E}_Q \left(\exp \left(-\frac{q}{2} \frac{(\log(S_t/K) + \sigma^2/2(T-t))^2}{\sigma^2(T-t)} \right) \right).$$

Since $\log(S_t/K) = \log(S_0/K) - t\sigma^2/2 + \sigma W_t^Q$, this becomes

$$\begin{aligned} & \mathbb{E}_Q \left((\varphi_{0,1}(d_1(S_t, t)))^q \right) \\ &= \frac{1}{(2\pi)^{q/2}} \mathbb{E}_Q \left(\exp \left(-\frac{q}{2\sigma^2(T-t)} \left(\log(S_0/K) + T\sigma^2/2 - \sigma^2 t + \sigma W_t^Q \right)^2 \right) \right) \\ &= \frac{1}{(2\pi)^{(q+1)/2}} \int_{-\infty}^{\infty} \exp \left(-\frac{q}{2\sigma^2(T-t)} \left(a_t + \sigma\sqrt{t}y \right)^2 - \frac{1}{2}y^2 \right) dy \end{aligned}$$

where $a_t := \log(S_0/K) + T\sigma^2/2 - \sigma^2 t$. We rewrite the exponential distribution inside the integral as the density of some normal distribution. Observe that

$$\begin{aligned} & -\frac{q}{2\sigma^2(T-t)} \left(a_t + \sigma\sqrt{t}y \right)^2 - \frac{1}{2}y^2 \\ &= -\frac{q}{2\sigma^2(T-t)} \left(a_t^2 + 2a_t\sigma\sqrt{t}y + \sigma^2ty^2 \right) - \frac{1}{2}y^2 \\ &= -\frac{1}{2} \left[\left(\frac{qt}{T-t} + 1 \right) y^2 + 2\frac{qa_t\sigma\sqrt{t}}{\sigma^2(T-t)}y + \frac{q}{\sigma^2(T-t)}a_t^2 \right] \\ &= -\frac{1}{2} \left[b_{1,t}^2 y^2 + 2b_{1,t}b_{2,t}y + \frac{q}{\sigma^2(T-t)}a_t^2 \right] \end{aligned}$$

with

$$b_{1,t} := \left(\frac{qt}{T-t} + 1 \right)^{1/2}, \quad b_{2,t} := \frac{qa_t\sigma\sqrt{t}}{\sigma^2(T-t)b_{1,t}}.$$

Completing the square then leads to

$$\begin{aligned} -\frac{q}{2\sigma^2(T-t)} \left(a_t + \sigma\sqrt{t}y \right)^2 - \frac{1}{2}y^2 &= -\frac{1}{2} \left[(b_{1,t}y + b_{2,t})^2 - b_{2,t}^2 + \frac{qa_t^2}{\sigma^2(T-t)} \right] \\ &= -\frac{(y - \mu_{1,t})^2}{2(\sigma_{1,t}^2)} + \frac{b_{2,t}^2}{2} - \frac{qa_t^2}{2\sigma^2(T-t)} \end{aligned}$$

with $\mu_{1,t} := -\frac{b_{2,t}}{b_{1,t}}$ and $\sigma_{1,t} := \frac{1}{b_{1,t}}$. Then, the expectation under consideration becomes

$$\begin{aligned} & \mathbb{E}_Q \left((\varphi_{0,1}(d_1(S_t, t)))^q \right) \\ &= \frac{1}{(2\pi)^{(q+1)/2}} \int_{-\infty}^{\infty} \exp \left(-\frac{(y - \mu_{1,t})^2}{2\sigma_{1,t}^2} \right) \exp \left(\frac{b_{2,t}^2}{2} - \frac{qa_t^2}{2\sigma^2(T-t)} \right) dy \\ &= \frac{1}{(2\pi)^{q/2}} \sigma_{1,t} \exp \left(\frac{b_{2,t}^2}{2} - \frac{qa_t^2}{2\sigma^2(T-t)} \right) \int_{-\infty}^{\infty} \varphi_{\mu_{1,t}, \sigma_{1,t}^2}(y) dy \\ &= \frac{1}{(2\pi)^{q/2}} \sigma_{1,t} \exp \left(\frac{b_{2,t}^2}{2} - \frac{qa_t^2}{2\sigma^2(T-t)} \right) \\ &\leq \text{const.} \sqrt{T-t} \end{aligned}$$

6. Application: Utility indifference pricing

since

$$\sigma_{1,t} = \left(\frac{q}{T-t} + 1 \right)^{-1/2} \leq q^{-1/2} (T-t)^{1/2}$$

and

$$\begin{aligned} \frac{b_{2,t}^2}{2} - \frac{qa_t^2}{2\sigma^2(T-t)} &= \frac{1}{2} \left(\left(\frac{qa_t\sigma\sqrt{t}}{\sigma^2(T-t)} \right)^2 \left(\frac{q}{T-t} + 1 \right)^{-1} - \frac{qa_t^2}{\sigma^2(T-t)} \right) \\ &= \frac{1}{2} \left(\frac{q^2 a_t^2 \sigma^2 t}{\sigma^4 (T-t)^2} \frac{1}{\frac{qt}{T-t} + 1} - \frac{qa_t^2}{\sigma^2(T-t)} \right) \\ &= \frac{qa_t^2}{2\sigma^2(T-t)} \left(\frac{qt}{T-t} \frac{1}{\frac{qt}{T-t} + 1} - 1 \right) \leq 0. \end{aligned}$$

We obtain the following upper bound for the considered expectation

$$\mathbb{E}_Q \left(\int_0^T \left(\frac{\varphi_{0,1}(d_1(S_t, t))}{\sqrt{T-t}} \right)^{2(1+\eta_2)} dt \right) \leq \text{const.} \int_0^T (T-t)^{1/2-1-\eta_2} dt.$$

From the choice of η_2 we obtain $1/2 - 1 - \eta_2 = -3/4$. This shows that the above expectation is finite.

(v) We will show the growth condition for $b^{\rho, Q}$. Recall that

$$\begin{aligned} \rho_t^H &= \frac{\left(2\partial_2 V(S_t, t) + \frac{\mu}{p} \right)^2}{\sigma^4 S_t^4} \\ &= \frac{1}{\sigma^4} \left(2\partial_2 V(S_t, t) S_t^{-2} + \frac{\mu}{p} S_t^{-2} \right)^2 \\ &= \frac{1}{\sigma^4} \left(-\partial_{11} V(S_t, t) + \frac{\mu}{p} S_t^{-2} \right)^2 \\ &= \frac{1}{\sigma^4} \left(-\Gamma(S_t, t) + \frac{\mu}{p} S_t^{-2} \right)^2 := f(S_t, t) \end{aligned}$$

with $\Gamma := \partial_{11} V$ and

$$f(s, t) = \frac{1}{\sigma^4} \left(-\Gamma(s, t) + \frac{\mu}{p} s^{-2} \right)^2.$$

We compute the partial derivatives of f :

$$\begin{aligned} \partial_1 f(s, t) &= 2 \left(-\Gamma(s, t) + \frac{\mu}{p\sigma^2 s^2} \right) \left(-\partial_1 \Gamma(s, t) + \frac{\mu(-2)}{p\sigma^2 s^3} \right) \\ &= (-2) \left(\frac{\mu}{p\sigma^2 s^2} - \Gamma(s, t) \right) \left(\partial_1 \Gamma(s, t) + \frac{2\mu}{p\sigma^2 s^3} \right) \\ \partial_2 f(s, t) &= 2 \left(-\Gamma(s, t) + \frac{\mu}{p\sigma^2 s^2} \right) (-\partial_2 \Gamma(s, t)) \\ \partial_{11} f(s, t) &= (-2) \left[\left(\frac{\mu(-2)}{p\sigma^2 s^3} - \partial_1 \Gamma(s, t) \right) \left(\partial_1 \Gamma(s, t) + \frac{2\mu}{p\sigma^2 s^3} \right) \right. \\ &\quad \left. + \left(\frac{\mu}{p\sigma^2 s^2} - \Gamma(s, t) \right) \left(\partial_{11} \Gamma(s, t) + \frac{2\mu(-3)}{p\sigma^2 s^4} \right) \right]. \end{aligned}$$

Due to Itô's formula, $b^{\rho, Q}$ is given by $g(S, \cdot)$, where

$$\begin{aligned} g(s, t) &:= \partial_2 f(s, t) + \frac{\sigma^2}{2} s^2 \partial_{11} f(s, t) \\ &= 2 \left(-\Gamma(s, t) + \frac{\mu}{p\sigma^2 s^2} \right) (-\partial_2 \Gamma(s, t)) \\ &\quad - \sigma^2 s^2 2 \left[(-1) \left(\frac{2\mu}{p\sigma^2 s^3} + \partial_1 \Gamma(s, t) \right) \left(\partial_1 \Gamma(s, t) + \frac{2\mu}{p\sigma^2 s^3} \right) \right. \\ &\quad \left. + \left(\frac{\mu}{p\sigma^2 s^2} - \Gamma(s, t) \right) \left(\partial_{11} \Gamma(s, t) - \frac{6\mu}{p\sigma^2 s^4} \right) \right]. \end{aligned}$$

We replace $\partial_2 \Gamma$ using the Black-Scholes PDE:

$$\begin{aligned} \partial_2 \Gamma(s, t) &= \partial_2 \partial_{11} V(s, t) = \partial_{11} \partial_2 V(s, t) \\ &= \partial_{11} \left(-\frac{\sigma^2 s^2}{2} \Gamma(s, t) \right) = -\frac{\sigma^2}{2} \partial_{11} (s^2 \Gamma(s, t)) \\ &= -\frac{\sigma^2}{2} \partial_1 (2s \Gamma(s, t) + s^2 \partial_1 \Gamma(s, t)) \\ &= -\frac{\sigma^2}{2} (2\Gamma(s, t) + 4s \partial_1 \Gamma(s, t) + s^2 \partial_{11} \Gamma(s, t)) \\ &= -\sigma^2 \left(\Gamma(s, t) + 2s \partial_1 \Gamma(s, t) + \frac{s^2}{2} \partial_{11} \Gamma(s, t) \right). \end{aligned}$$

So by inserting this representation and rearranging the terms we obtain for g

$$\begin{aligned} g(s, t) &= (-4) \frac{\mu}{p\sigma^2 s^2} \Gamma(s, t) + 8 \frac{\mu}{p\sigma^2} \frac{\partial_1 \Gamma(s, t)}{s} - \frac{2}{\sigma^2} \Gamma(s, t) \\ &\quad - 4s (\partial_1 \Gamma(s, t)) \Gamma(s, t) + 10 \frac{\mu^2}{p\sigma^2} \frac{1}{s^4} + s^2 (\partial_1 \Gamma(s, t))^2. \end{aligned}$$

With this we can find the following upper bound for the absolute value of $g(s, t)$:

$$\begin{aligned} &|g(s, t)| \\ &\leq \text{const.} \left(\frac{\Gamma(s, t)}{s^2} + \frac{|\partial_1 \Gamma(s, t)|}{s} + \Gamma(s, t) + |\partial_1 \Gamma(s, t)| \Gamma(s, t) + \frac{1}{s^4} + s^2 (\partial_1 \Gamma(s, t))^2 \right). \end{aligned} \tag{6.2.3}$$

Recall that

$$\Gamma(s, t) = \frac{\varphi_{0,1}(d_1(s, t))}{\sigma s (T - t)^{1/2}}.$$

Thus, the partial derivative of Γ with respect to s is given by

$$\partial_1 \Gamma(s, t) = -\Gamma(s, t) \left(\frac{d_1(s, t)}{s\sigma(T-t)^{1/2}} + \frac{1}{s} \right).$$

6. Application: Utility indifference pricing

Hence, we obtain the following upper bounds for Γ and $|\partial_1\Gamma|$:

$$\begin{aligned}\Gamma(s, t) &\leq \frac{1}{\sigma s(T-t)^{1/2}}, \\ |\partial_1\Gamma(s, t)| &\leq \frac{1}{\sigma s(T-t)^{1/2}} \left(\frac{|\log(s/K) + \sigma^2/2(T-t)|}{s\sigma^2(T-t)} + \frac{1}{s\sigma} \right) \\ &\leq \text{const.} \left(\frac{|\log(s/K)|}{s^2(T-t)^{3/2}} + \frac{1}{s^2(T-t)^{1/2}} \right).\end{aligned}$$

With these estimates and (6.2.3) we obtain

$$|g(s, t)| \leq \sum_{j=0}^6 a_j(s, t)b^j(t),$$

with some suitable functions $a_j : (0, \infty) \times [0, T] \rightarrow [0, \infty)$ such that

$$\mathbb{E}_Q \left(\int_0^T a_j(S_t, t) dt \right) < \infty$$

and $b(t) = (T-t)^{-1/2}$. This yields the growth condition for $b^{\rho, Q}$, and the optimizer φ^H as well as the portfolio gamma ρ^H thus fulfill Assumption 4.3.1.

(vi) We are left with the computation of the formula of the asymptotic utility indifference price in the case $\mu = 0$. Recall that

$$s\varphi_{0,1}(d_1(s, t)) = K\varphi_{0,1}(d_2(s, t)).$$

This yields

$$\partial_2 V(s, t) = -\frac{K\sigma\varphi_{0,1}(d_2(s, t))}{2\sqrt{T-t}},$$

and we thus obtain for the leading order term

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T |\partial_2 V(S_t, t)| dt \right) &= \int_0^T \mathbb{E}_Q (|\partial_2 V(S_t, t)|) dt \\ &= \frac{K\sigma}{2} \int_0^T \frac{1}{(T-t)^{1/2}} \mathbb{E}_Q(\varphi_{0,1}(d_2(S_t, t))) dt.\end{aligned}$$

We will compute the expectation inside the integral. Similarly to the calculations for d_1 , we obtain

$$\mathbb{E}_Q(\varphi_{0,1}(d_2(S_t, t))) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2\sigma^2(T-t)} \left(a + \sigma\sqrt{t}y \right)^2 - \frac{1}{2}y^2 \right) dy$$

with $a := \log(S_0/K) - T\sigma^2/2$. As before we can write

$$-\frac{q}{2\sigma^2(T-t)} \left(a + \sigma\sqrt{t}y \right)^2 - \frac{1}{2}y^2 = -\frac{1}{2} \left[\widehat{b}_{1,t}^2 y^2 + 2\widehat{b}_{1,t}\widehat{b}_{2,t} + \frac{1}{\sigma^2(T-t)} a^2 \right]$$

with

$$\widehat{b}_{1,t} := \left(\frac{t}{T-t} + 1 \right)^{1/2}, \quad \widehat{b}_{2,t} := \frac{a\sigma\sqrt{t}}{\sigma^2(T-t)\widehat{b}_{1,t}}$$

and by completing the square

$$-\frac{1}{2\sigma^2(T-t)} \left(a + \sigma\sqrt{t}y \right)^2 - \frac{1}{2}y^2 = -\frac{(y - \widehat{\mu}_{1,t})^2}{2(\widehat{\sigma}_{1,t}^2)} + \frac{\widehat{b}_{2,t}^2}{2} - \frac{a^2}{2\sigma^2(T-t)}$$

with $\widehat{\mu}_{1,t} := -\frac{\widehat{b}_{2,t}}{\widehat{b}_{1,t}}$ and $\widehat{\sigma}_{1,t} := \frac{1}{\widehat{b}_{1,t}}$. This can be simplified since

$$\begin{aligned} \frac{\widehat{b}_{2,t}^2}{2} - \frac{a^2}{2\sigma^2(T-t)} &= \frac{a^2\sigma^2t}{2\sigma^4(T-t)^2\widehat{b}_{1,t}^2} - \frac{a^2}{2\sigma^2(T-t)} \\ &= \frac{a^2}{2\sigma^2(T-t)} \left(\frac{t}{(T-t)\widehat{b}_{1,t}^2} - 1 \right) \\ &= \frac{a^2}{2\sigma^2(T-t)} \left(\frac{t}{(T-t)\left(\frac{t}{T-t} + 1\right)} - 1 \right) \\ &= \frac{a^2}{2\sigma^2(T-t)} \left(\frac{t}{T} - 1 \right) \\ &= -\frac{a^2}{2\sigma^2T}. \end{aligned}$$

So for the expectation we obtain

$$\begin{aligned} \mathbb{E}_Q(\varphi_{0,1}(d_2(S_t, t))) &= \frac{1}{(2\pi)^{1/2}} \widehat{\sigma}_{1,t} \exp\left(-\frac{a^2}{2\sigma^2T}\right) \\ &= \frac{(T-t)^{1/2}}{(2T\pi)^{1/2}} \exp\left(-\frac{a^2}{2\sigma^2T}\right), \end{aligned}$$

which implies for the leading order term

$$\begin{aligned} \mathbb{E}_Q\left(\int_0^T |\partial_2 V(S_t, t)| dt\right) &= \frac{K\sigma}{2} \int_0^T \frac{1}{(2T\pi)^{1/2}} \exp\left(-\frac{a^2}{2\sigma^2T}\right) dt \\ &= \frac{K\sigma T^{1/2}}{2(2\pi)^{1/2}} \exp\left(-\frac{a^2}{2\sigma^2T}\right) \\ &= \frac{K\sigma T^{1/2}}{2} \varphi_{0,1}(d_2(S_0, 0)). \end{aligned}$$

Inserting this yields the desired formula for the asymptotic utility indifference price for the put option and thus completes the proof. \square

In the case $\mu = 0$ the optimization problem under P is trivial, because one chooses the strategy that keeps all the money in the bond according to Example 2.4.5. Then

6. Application: Utility indifference pricing

$\varphi^H = \partial_1 V(S, \cdot)$ is the replicating strategy in the frictionless market. Furthermore the halfwidth of the no-trade region is given by

$$\Delta\varphi_t^{H,+,k} = \sqrt[4]{\frac{12\varepsilon}{p}} \sqrt[4]{(\rho_{\tau_k \wedge t}^H) \vee k},$$

where the portfolio gamma ρ^H for the optimization under P^H was computed in (v) of the above proof. So one does not only obtain an asymptotic price for the put option, but also an asymptotic hedging strategy $\tilde{\varphi}^{H,\varepsilon,k}$.

A question might be how to choose the degree of risk aversion for utility indifference prices. We take $p = 1.25 \cdot 10^{-4}$ being higher than the value from section 5.1 and work with $S_0 = 100, K = 100, \sigma = 0.2$ and $\varepsilon = 5\text{€}$. This yields an asymptotic utility indifference price equal to 8.02 compared the the frictionless price 7.97. As one can see from the formula in (6.2.1) and figure 6.1 the asymptotic utility indifference price is always above the frictionless price. In contrast to the asymptotics for utility indifference pricing under proportional costs as in Bichuch (2014) and Ahrens (2015), this total premium scales linearly with the number of options traded: j put options with strike K on the asset with initial value S_0 have the same frictionless price as one put option with strike jK and initial asset value jS_0 . According to (6.2.1) the premium is j times the premium of the single put option. However, the number of transactions will increase: The frictionless target φ^{jH} is j times φ^H , whereas the halfwidth of the no-trade region scales with \sqrt{j} .

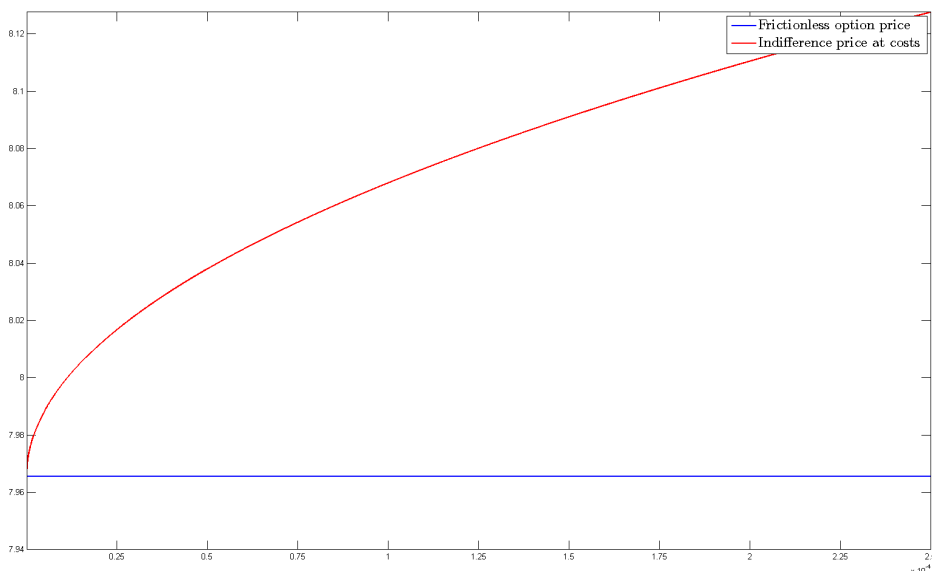


Figure 6.1.: Asymptotic utility indifference price and frictionless option price of a put option in the Black-Scholes model with $S_0 = 100, K = 100, \sigma = 0.2$ and $\varepsilon = 5\text{€}$ as a function of risk aversion

7. Fixed and proportional costs

7.1. Trading with fixed and proportional costs

What changes if one considers a combination of fixed transaction fees ε and proportional fees $\lambda \in (0, \infty)$? Then an investor has to pay $\lambda S_{\tau_i} |n_i| + \varepsilon$ when buying n_i stocks at time τ_i . As before, we only allow for trading strategies as in Definition 3.2.1, and we use the following intuitive notion of self-financing strategies.

Definition 7.1.1 (Self-financing strategies). *A trading strategy (φ^0, φ) is called self-financing iff*

$$n_i^0 = -(1 + \lambda)S_{\tau_i} n_i^+ + (1 - \lambda)S_{\tau_i} n_i^- - \varepsilon \quad \text{for all } i \in \mathbb{N}$$

with the notation from Definition 3.2.1.

Again, except for the initial allocation (φ_0^0, φ_0) , we can identify a self-financing strategy (φ^0, φ) with its number of shares held in the stock φ . We define the corresponding portfolio process and the set of admissible strategies:

Definition 7.1.2 (Portfolio process). *For φ as in Definition 3.2.1 we define the portfolio process $V^{\varepsilon, \lambda}(\varphi)$ via*

$$V_t^{\varepsilon, \lambda}(\varphi) = V_0^{\varepsilon, \lambda}(\varphi) + \int_0^t \varphi_s dS_s - \varepsilon |\{i \in \mathbb{N} : \tau_i < t\}| - \lambda \sum_{i \in \mathbb{N}} S_{\tau_i} |n_i| \mathbf{1}_{\{\tau_i < t\}}.$$

Definition 7.1.3 (Admissible strategies). *An admissible strategy with initial capital $x \in \mathbb{R}$ is a predictable process φ as in Definition 3.2.1 with the following properties:*

1. $\varphi_0 = \varphi_0^*$.
2. $\int_0^\cdot \varphi_s dS_s$ is a Q -supermartingale.

We write $\mathfrak{A}^{\varepsilon, \lambda}$ for the set of all admissible strategies.

7.2. Heuristics

In this section we will heuristically derive a candidate for optimality as in the absence of proportional costs in section 3.5 and similar to (Korn, 1998, section 5). Again, we want to find an approximation to the value process $(v^{\varepsilon, \lambda}(\varphi))_{\varphi \in \mathfrak{A}^{\varepsilon, \lambda}}$ for both fixed and proportional costs, which is defined analogously to Definition 3.3.2. Theorem 3.3.3 for $(v^{\varepsilon, \lambda}(\varphi))_{\varphi \in \mathfrak{A}^{\varepsilon, \lambda}}$ can be shown in the same way. Thus, the ansatz $\tilde{v}^{\varepsilon, \lambda}(\varphi)$ for fixed and proportional costs has to be asymptotically a supermartingale for an arbitrary

7. Fixed and proportional costs

admissible strategy $\varphi \in \mathfrak{A}^{\varepsilon, \lambda}$ and asymptotically a martingale for a candidate strategy. As in subsection 3.5.1 we suppose that our ansatz $(\tilde{v}^{\varepsilon, \lambda}(\varphi))_{\varphi \in \mathfrak{A}^{\varepsilon, \lambda}}$ is of the form

$$\tilde{v}_t^{\varepsilon, \lambda}(\varphi) = \tilde{G}^{\varepsilon, \lambda}(t, V_t^\varepsilon(\varphi), \Delta\varphi_t)$$

with a function $\tilde{G}^{\varepsilon, \lambda} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\leq 0}$. We suppose again that there exists a random and time dependent interval $NT^{\varepsilon, \lambda} = [\varphi^* - \Delta\varphi^-, \varphi^* + \Delta\varphi^+]$ around the frictionless optimizer, where it is (almost) optimal for the investor to stay inactive. In the complement of this no-trade region we suppose that the investor now trades to some yet unknown process $\bar{\varphi}^- \in L(S)$ with values in $[\varphi^* + \Delta\varphi^-, \varphi^*]$ if the current number of stocks is below the no-trade region and a process $\bar{\varphi}^+ \in L(S)$ with values in $[\varphi^*, \varphi^* + \Delta\varphi^+]$ if it is above the no-trade region. Therefore, we impose on $\tilde{G}^{\varepsilon, \lambda}$

- the terminal condition

$$\tilde{G}^{\varepsilon, \lambda}(\omega, T, y, \delta) \approx U(y) \quad \text{for all } y, \delta \in \mathbb{R}$$

and the following conditions that are supposed to hold for all $(\omega, t) \in \Omega \times [0, T)$ and $y \in \mathbb{R}$:

- Monotonicity: for $\delta_1, \delta_2 \in \mathbb{R}$ with $|\delta_1| \leq |\delta_2|$

$$\tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta_1) \geq \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta_2)$$

and thus

$$\tilde{G}^{\varepsilon, \lambda}(\omega, t, y, 0) = \sup_{\delta \in \mathbb{R}} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta).$$

- Relation between trade and no-trade region optimality of the processes $\Delta\bar{\varphi}^-, \Delta\bar{\varphi}^+$:

$$\begin{aligned} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta) &= \sup_{\tilde{\delta} \in \mathbb{R}} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y - \varepsilon - \lambda(\delta - \tilde{\delta})S_t(\omega), \tilde{\delta}) \\ &= \tilde{G}^{\varepsilon, \lambda}(\omega, t, y - \varepsilon - \lambda(\delta - \Delta\bar{\varphi}_t^+(\omega))S_t(\omega), \Delta\bar{\varphi}_t^+(\omega)) \\ &\quad \text{for } \delta > \Delta\varphi_t^+(\omega), \\ \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta) &= \sup_{\tilde{\delta} \in \mathbb{R}} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y - \varepsilon - \lambda(\tilde{\delta} - \delta)S_t(\omega), \tilde{\delta}) \\ &= \tilde{G}^{\varepsilon, \lambda}(\omega, t, y - \varepsilon - \lambda(\Delta\bar{\varphi}_t^-(\omega) - \delta)S_t(\omega), \Delta\bar{\varphi}_t^-(\omega)) \\ &\quad \text{for } \delta < \Delta\varphi_t^-(\omega). \end{aligned}$$

- Smoothness: we assume that $\delta \mapsto \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta)$ is twice continuously differentiable on $\mathbb{R} \setminus \partial NT^\varepsilon(\omega, t)$ and continuously differentiable on \mathbb{R} . This implies that we have a smooth fit at the boundaries of the no-trade region:

$$\begin{aligned} \lim_{\delta \nearrow \varphi_t^*(\omega) + \Delta\varphi_t^-(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta) &= \lim_{\delta \searrow \varphi_t^*(\omega) + \Delta\varphi_t^-(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^\varepsilon(\omega, t, y, \delta), \\ \lim_{\delta \nearrow \varphi_t^*(\omega) + \Delta\varphi_t^+(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta) &= \lim_{\delta \searrow \varphi_t^*(\omega) + \Delta\varphi_t^+(\omega)} \frac{\partial}{\partial \delta} \tilde{G}^{\varepsilon, \lambda}(\omega, t, y, \delta). \end{aligned}$$

- Symmetry around 0:

$$\begin{aligned}\Delta\varphi^- &= -\Delta\varphi^+ \\ \Delta\bar{\varphi}^- &= -\Delta\bar{\varphi}^+ \\ \tilde{G}^{\varepsilon,\lambda}(\omega, t, y, -\delta) &= \tilde{G}^\varepsilon(\omega, t, y, \delta) \quad \text{for all } \delta \in \mathbb{R}.\end{aligned}$$

Let $F^{CE,\varepsilon,\lambda}$ be the corresponding correction term for the certainty equivalent in the presence of both fixed and proportional costs. As before, we assume

$$F_t^{CE,\varepsilon,\lambda}(\varphi)(\omega) = \tilde{g}^{\varepsilon,\lambda}(\omega, t, \Delta\varphi_t(\omega))$$

with $\tilde{g}^{\varepsilon,\lambda} : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_{\leq 0}$ and the transformed conditions

$$\begin{aligned}\tilde{g}^{\varepsilon,\lambda}(\omega, T, \delta) &\approx 0, \\ \tilde{g}^{\varepsilon,\lambda}(\omega, t, \delta_1) &\geq \tilde{g}^{\varepsilon,\lambda}(\omega, t, \delta_2) \quad \text{for } |\delta_1| \leq |\delta_2|, \\ \tilde{g}^{\varepsilon,\lambda}(\omega, t, \delta) &= \sup_{\tilde{\delta} \in \mathbb{R}} \tilde{g}^{\varepsilon,\lambda}(\omega, t, \tilde{\delta}) - \varepsilon + \lambda S_t(\omega)(\delta - \tilde{\delta}) \\ &= \tilde{g}^{\varepsilon,\lambda}(\omega, t, \Delta\bar{\varphi}_t^-(\omega)) - \varepsilon + \lambda S_t(\omega)(\delta - \Delta\bar{\varphi}_t^-(\omega)) \quad \text{for } \delta < \Delta\varphi_t^-(\omega),\end{aligned}\tag{7.2.1}$$

$$\begin{aligned}\tilde{g}^{\varepsilon,\lambda}(\omega, t, \delta) &= \sup_{\tilde{\delta} \in \mathbb{R}} \tilde{g}^{\varepsilon,\lambda}(\omega, t, \tilde{\delta}) - \varepsilon - \lambda S_t(\omega)(\delta - \tilde{\delta}) \\ &= \tilde{g}^{\varepsilon,\lambda}(\omega, t, \Delta\bar{\varphi}_t^+(\omega)) - \varepsilon - \lambda S_t(\omega)(\delta - \Delta\bar{\varphi}_t^+(\omega)) \quad \text{for } \delta > \Delta\varphi_t^+(\omega)\end{aligned}\tag{7.2.2}$$

and the smoothness condition at the boundaries of the no-trade region. Conditions (7.2.1) and (7.2.2) imply that $\tilde{g}^{\varepsilon,\lambda}$ is an affine function outside the no-trade region. Thus, the smoothness properties become

$$\begin{aligned}\lambda S_t(\omega) &= \lim_{\delta \searrow \Delta\varphi_t^-(\omega)} \frac{\partial}{\partial \delta} \tilde{g}^{\varepsilon,\lambda}(\omega, t, \delta), \\ -\lambda S_t(\omega) &= \lim_{\delta \nearrow \Delta\varphi_t^+(\omega)} \frac{\partial}{\partial \delta} \tilde{g}^{\varepsilon,\lambda}(\omega, t, \delta).\end{aligned}$$

Furthermore, the optimality of $\Delta\bar{\varphi}^+$ and $\Delta\bar{\varphi}^-$ in (7.2.1) and (7.2.2) yield that

$$\begin{aligned}\lambda S_t(\omega) &= \frac{\partial}{\partial \delta} \tilde{g}^{\varepsilon,\lambda}(\omega, t, \Delta\bar{\varphi}_t^-(\omega)), \\ -\lambda S_t(\omega) &= \frac{\partial}{\partial \delta} \tilde{g}^{\varepsilon,\lambda}(\omega, t, \Delta\bar{\varphi}_t^+(\omega)),\end{aligned}$$

i.e. the derivatives of $\tilde{g}^{\varepsilon,\lambda}$ with respect to δ at the boundaries equal those at the optimal trading points. As for the case of purely fixed costs, we choose a polynomial of order 4 with stochastic coefficients inside the no-trade region

$$\tilde{g}^\varepsilon(\omega, t, \delta) = \bar{\alpha}_t(\omega)\delta^4 - \bar{\beta}_t(\omega)\delta^2 - \bar{\gamma}_t(\omega) \quad \text{for } \delta \in \text{NT}^\varepsilon(\omega, t).$$

Then, the above smoothness and the optimality conditions become

$$\mp \lambda S_t(\omega) = 4\bar{\alpha}_t(\omega)(\Delta\bar{\varphi}_t(\omega)^\pm)^3 - 2\bar{\beta}_t(\omega)\Delta\bar{\varphi}_t^\pm(\omega)\tag{7.2.3}$$

7. Fixed and proportional costs

and

$$\mp \lambda S_t(\omega) = 4\bar{\alpha}_t(\omega)(\Delta\varphi_t(\omega)^\pm)^3 - 2\bar{\beta}_t(\omega)\Delta\varphi_t^\pm(\omega). \quad (7.2.4)$$

Subtracting equations (7.2.3) and (7.2.4) then yields

$$0 = 4\bar{\alpha}_t(\omega)(\Delta\bar{\varphi}_t(\omega)^\pm)^3 - (\Delta\varphi_t(\omega)^\pm)^3 - 2\bar{\beta}_t(\omega)(\Delta\bar{\varphi}_t^\pm(\omega) - \Delta\varphi_t^\pm(\omega)),$$

which, in turn, leads to a representation of $\bar{\beta}$ in terms of $\bar{\alpha}$ and the boundaries

$$\bar{\beta}_t(\omega) = 2\bar{\alpha}_t(\omega) \left((\Delta\varphi_t^\pm(\omega))^2 + (\Delta\bar{\varphi}_t(\omega)^\pm)^2 + \Delta\bar{\varphi}_t(\omega)^\pm \Delta\varphi_t^\pm(\omega) \right).$$

Inserting this representation into equation (7.2.4) then gives

$$\begin{aligned} -\lambda S_t(\omega) &= 4\bar{\alpha}_t(\omega)(\Delta\varphi_t^+(\omega))^3 - 2\Delta\varphi_t^+(\omega)2\bar{\alpha}_t(\omega) \left((\Delta\varphi_t^+(\omega))^2 + (\Delta\bar{\varphi}_t(\omega)^+)^2 \right. \\ &\quad \left. + \Delta\bar{\varphi}_t(\omega)^+ \Delta\varphi_t^+(\omega) \right) \\ &= 4\bar{\alpha}_t(\omega) \left((\Delta\varphi_t^+(\omega))^3 - \Delta\varphi_t^+(\omega) \left((\Delta\varphi_t^+(\omega))^2 + (\Delta\bar{\varphi}_t(\omega)^+)^2 \right. \right. \\ &\quad \left. \left. + \Delta\bar{\varphi}_t(\omega)^+ \Delta\varphi_t^+(\omega) \right) \right) \\ &= -4\bar{\alpha}_t(\omega) \left(\Delta\varphi_t^+(\omega)(\Delta\bar{\varphi}_t(\omega)^+)^2 + (\Delta\varphi_t^+(\omega))^2 \Delta\bar{\varphi}_t(\omega)^+ \right). \end{aligned}$$

With the use of equation (7.2.1) with $\delta = \Delta\varphi_t^+(\omega)$ we see that

$$\begin{aligned} &\bar{\alpha}_t(\omega)(\Delta\varphi_t^+(\omega))^4 - \bar{\beta}_t(\omega)(\Delta\varphi_t^+(\omega))^2 \\ &= \bar{\alpha}_t(\omega)(\Delta\bar{\varphi}_t(\omega)^+)^4 - \bar{\beta}_t(\omega)(\Delta\bar{\varphi}_t(\omega)^+)^2 - \varepsilon - \lambda|\Delta\bar{\varphi}_t(\omega)^+ - \Delta\varphi_t^+(\omega)|S_t(\omega). \end{aligned}$$

By inserting everything we end up with the two nonlinear equations

$$\lambda S_t(\omega) = 4\bar{\alpha}_t(\omega)\Delta\varphi_t^+(\omega)\Delta\bar{\varphi}_t(\omega)^+(\Delta\bar{\varphi}_t(\omega)^+ + \Delta\varphi_t^+(\omega)), \quad (7.2.5)$$

$$\varepsilon = \bar{\alpha}_t(\omega)(\Delta\varphi_t^+(\omega) - \Delta\bar{\varphi}_t(\omega)^+)^3(\Delta\varphi_t^+(\omega) + \Delta\bar{\varphi}_t(\omega)^+). \quad (7.2.6)$$

According to (Korn, 1998, Proposition 5,1) equations (7.2.5) and (7.2.6) have a unique solution $(\delta^+(\bar{\alpha}_t(\omega), S_t(\omega)), \bar{\delta}^+(\bar{\alpha}_t(\omega), S_t(\omega)))$ with $0 < \bar{\delta}^+(a, s) < \delta^+(a, s)$ for all $a, s > 0$. However, an explicit representation of these values seems unavailable. As in the case of fixed costs we can write

$$\tilde{g}^{\varepsilon, \lambda}(\omega, t, \Delta\varphi_t(\omega)) = f^{CE, \varepsilon, \lambda}(\bar{\alpha}_t(\omega), \Delta\varphi_t(\omega), S_t(\omega)) - \bar{\gamma}_t(\omega)$$

with the deterministic function

$$f^{CE, \varepsilon, \lambda}(a, \delta, s) = \begin{cases} a\delta^4 - b(a, s)\delta^2, & \text{if } \delta^-(a, s) \leq \delta \leq \delta^+(a, s) \\ (\bar{\delta}^+(a, s))^4 - b(a, s)(\bar{\delta}^+(a, s))^2 + \lambda s(\delta + \bar{\delta}^+(a, s)) - \varepsilon, & \text{if } \delta < \delta^-(a, s) \\ (\bar{\delta}^+(a, s))^4 - b(a, s)(\bar{\delta}^+(a, s))^2 - \lambda s(\delta - \bar{\delta}^+(a, s)) - \varepsilon, & \text{if } \delta > \delta^+(a, s), \end{cases}$$

where

$$b(a, s) = 2a((\delta^+(a, s))^2 + (\bar{\delta}^+(a, s))^2 + \delta^+(a, s)\bar{\delta}^+(a, s)).$$

This coincides with the function f^{CE} if the proportional costs λ vanish.

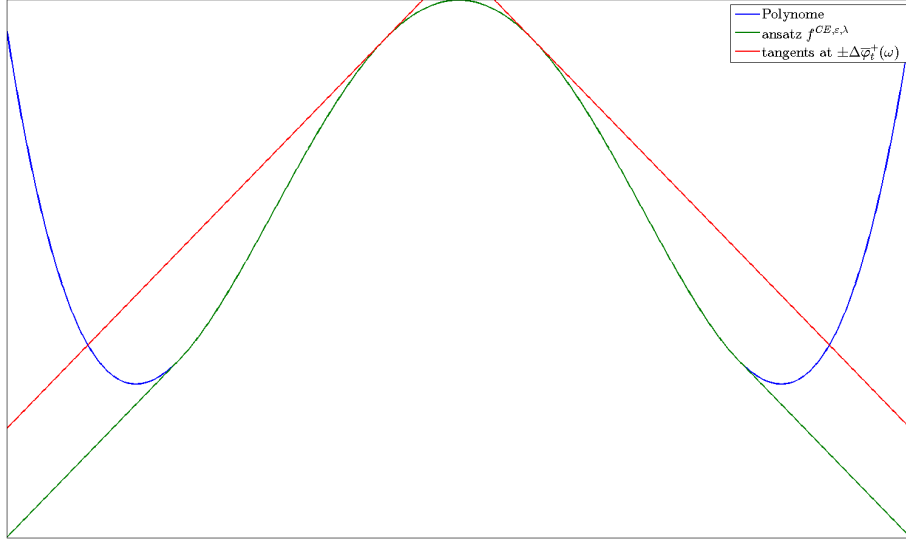


Figure 7.1.: Correction term $f^{CE,\varepsilon,\lambda}$ as a function of the deviation from the frictionless target

For $\tilde{X}^{\varepsilon,\lambda}(\varphi) = \exp((-p)(V_t^{\varepsilon,\lambda}(\varphi) - V_t(\varphi^*) + F_t^{CE,\varepsilon,\lambda}(\varphi)))$ we can obtain the Q -dynamics in the case $\varphi = \varphi^{\varepsilon,\lambda}$ as in section 3.5:

$$\begin{aligned}
\tilde{X}_t^{\varepsilon,\lambda}(\varphi^{\varepsilon,\lambda}) &= \tilde{X}_0^{\varepsilon,\lambda}(\varphi^{\varepsilon,\lambda}) + \int_0^t \tilde{X}_s^{\varepsilon,\lambda}(\varphi^{\varepsilon,\lambda}) \left[(\Delta\varphi_s^{\varepsilon,\lambda})^8 \frac{p^2}{2} c_s^{\bar{\alpha}} - (\Delta\varphi_s^{\varepsilon,\lambda})^7 p^2 4\bar{\alpha}_s c_s^{\bar{\alpha},\varphi^*} \right. \\
&\quad + (\Delta\varphi_s^{\varepsilon,\lambda})^6 p^2 (8\bar{\alpha}_s^2 c_s^{\varphi^*,\varphi^*} - c_s^{\alpha,\beta}) \\
&\quad + (\Delta\varphi_s^{\varepsilon,\lambda})^5 (p^2 (\bar{\beta}_s c_s^{\bar{\alpha},\varphi^*} + 4\alpha_s c_s^{\beta,\varphi^*}) + p^2 c_s^{S,\alpha}) \\
&\quad + (\Delta\varphi_s^{\varepsilon,\lambda})^4 \left((-p) b_s^{\bar{\alpha},Q} + \frac{p^2}{2} (c_s^{\bar{\beta}} - 16\bar{\alpha}_s \bar{\beta}_s c_s^{\varphi^*,\varphi^*} - 2c_s^{\bar{\alpha},\bar{\gamma}}) - p^2 \bar{\alpha}_s c_s^{S,\varphi^*} \right) \\
&\quad + (\Delta\varphi_s^{\varepsilon,\lambda})^3 \left(p^2 ((-2)\bar{\beta}_s c_s^{\bar{\beta},\varphi^*} + 4\bar{\alpha}_s c_s^{\varphi^*,\bar{\gamma}}) - p^2 c_s^{S,\bar{\beta}} + 4p(\bar{\alpha}_s b_s^{\varphi^*,Q} + c_s^{\bar{\alpha},\varphi^*}) \right) \\
&\quad + (\Delta\varphi_s^{\varepsilon,\lambda})^2 \left(p (b_s^{\bar{\beta},Q} - 6\bar{\alpha}_s c_s^{\varphi^*,\varphi^*}) + \frac{p^2}{2} c_s^{S,S} + \frac{p^2}{2} (4\bar{\beta}_s^2 c_s^{\varphi^*,\varphi^*} + 2c_s^{\bar{\beta},\bar{\gamma}}) \right. \\
&\quad \left. + 2p^2 \bar{\beta}_s c_s^{S,\varphi^*} \right) \\
&\quad + (\Delta\varphi_s^{\varepsilon,\lambda}) \left((-p)(2\bar{\beta}_s b_s^{\varphi^*,Q} + 2c_s^{\bar{\beta},\varphi^*}) - p^2 (\bar{\beta}_s c_s^{\varphi^*,\bar{\gamma}}) + p c_s^{S,\bar{\gamma}} \right) \\
&\quad \left. + p(b_s^{\bar{\gamma},Q} + \beta_s c_s^{\varphi^*,\varphi^*}) + \frac{p^2}{2} c_s^{\bar{\gamma}} \right] ds \\
&\quad + \int_0^t \tilde{X}_s^{\varepsilon,\lambda}(\varphi^{\varepsilon,\lambda}) (-p) \left[dM_s^{F^{CE,\varepsilon,\lambda}(\varphi^{\varepsilon,\lambda}),Q} + (-p)\Delta\varphi_s^{\varepsilon,\lambda} dS_s \right].
\end{aligned}$$

7. Fixed and proportional costs

For the asymptotic analysis both costs should depend on one small parameter. Since the utility loss due to proportional costs¹ is usually of order $\lambda^{2/3}$, we suppose that the proportional costs are of the form $\lambda = \text{const. } \varepsilon^{3/4}$. Then, the effects of the individual costs are both of order $\varepsilon^{1/2}$, and we believe that this is also true for the case of mixed costs. Furthermore, we suppose that the no-trade region also scales with $\varepsilon^{1/4}$. As in subsection 3.5.4, we then assume that

$$\bar{\alpha}_t = \alpha_t = \frac{p}{12} \frac{c_t^S}{c_t^{\varphi^*}}$$

in order to obtain the correct scaling of the dynamics of $\tilde{X}^{\varepsilon, \lambda}(\varphi^{\varepsilon, \lambda})$. For the process $\bar{\gamma}$ we get

$$\begin{aligned} b_t^{\bar{\gamma}, Q} &= -\bar{\beta}_t c_t^{\varphi^*} \\ &= -b(\bar{\alpha}_t, S_t) c_t^{\varphi^*} \\ &= -\frac{p}{6} \left((\Delta\varphi_t^+)^2 + (\Delta\bar{\varphi}_t^+)^2 + \Delta\varphi_t^+ \Delta\bar{\varphi}_t^+ \right) c_t^S, \end{aligned}$$

which does not seem to have a nice representation in terms of the frictionless quantities as in Chapter 3.

7.3. An outlook to the verification

Despite this fact, all objects have been determined and a verification similar to chapter 3 seems possible if one is able to find out the required regularity assumptions. Since the no-trade boundaries and the optimal transaction points are not given explicitly, this seems rather difficult at first sight. However, by looking at equation 7.2.5 and using that the optimal transaction points lie inside the no-trade region, we obtain

$$\lambda S_t \geq 8\alpha_t (\Delta\bar{\varphi}_t^+)^3,$$

so for the optimal transaction points we have the upper bound

$$\Delta\bar{\varphi}_t^+ \leq \frac{1}{2} \left(\frac{\lambda S_t}{\alpha_t} \right)^{1/3}.$$

Writing $Z_t := \Delta\varphi_t^+ - \Delta\bar{\varphi}_t^+$ we obtain from (7.2.6)

$$\varepsilon = \alpha_t Z_t^3 (Z_t + 2\Delta\bar{\varphi}_t^+) \geq \alpha_t Z_t^4,$$

so Z_t has the upper bound $\left(\frac{\varepsilon}{\alpha_t} \right)^{1/4}$. Then, the halfwidth of the no-trade region $\Delta\varphi_t^+$ is bounded by

$$\Delta\varphi_t^+ \leq \frac{1}{2} \left(\frac{\lambda S_t}{\alpha_t} \right)^{1/3} + \left(\frac{\varepsilon}{\alpha_t} \right)^{1/4}. \quad (7.3.7)$$

¹See the literature references on proportional costs in section 1.1.

The process $(\frac{\varepsilon}{\alpha})^{1/4}$ is the halfwidth of the no-trade region in the absence of proportional costs (see 3.5.13). So the regularity assumption 3.6.4 are required when analyzing the drift of the value process. Additionally, we will have to find out the suitable regularity assumptions that are needed due to the presence of the process $\frac{S}{\alpha}$ in (7.3.7). Once this is achieved, we believe that the construction of the asymptotic lower bound from section 3.8 may work similarly: we will have to add a third lemma in subsection 3.8.1 dealing with the Q - probability of the event $\{\sum_{i \in \mathbb{N}} S_{\tau_i} |\varphi_{\tau_i+}^{\varepsilon, \lambda} - \varphi_{\tau_i}^{\varepsilon, \lambda}| \mathbf{1}_{\{\tau_i < t\}} \geq K\}$, since the difference of the portfolio process $V^{\varepsilon, \lambda}(\varphi^{\varepsilon, \lambda})$ and the frictionless portfolio process $V(\varphi^*)$ of the optimal strategy φ^* is given by

$$\Delta V_t^{\varepsilon, \lambda}(\varphi^{\varepsilon, \lambda}) = \int_0^t (\varphi_s^{\varepsilon, \lambda} - \varphi_s^*) dS_s - \varepsilon \mathcal{N}_t(\varphi) - \lambda \sum_{i \in \mathbb{N}} S_{\tau_i} |\varphi_{\tau_i+}^{\varepsilon, \lambda} - \varphi_{\tau_i}^{\varepsilon, \lambda}| \mathbf{1}_{\{\tau_i < t\}}.$$

We believe that Proposition 3.8.3 and Theorem 3.8.4 can then be proved in the same way for both fixed and proportional costs. However, it will become more involved for competing admissible strategies $\varphi \in \mathfrak{A}^{\varepsilon, \lambda}$. One reason for this is the correction term $F^{CE, \varepsilon, \lambda}$ or, more precisely, the function $f^{CE, \varepsilon, \lambda}$. In the absence of proportional costs it is bounded by $-\varepsilon$ from below. Furthermore, its derivatives vanish in the trade region. These properties were used in the proofs of section 3.7, and they do not hold in the presence of proportional costs. The derivatives of $f^{CE, \varepsilon, \lambda}$ with respect to a and s will require the partial derivatives (or at least suitable upper bounds) of $\bar{\delta}^+$, which are not known explicitly. Moreover, it is not obvious whether $\bar{\delta}^+$ is smooth enough. In the trade region a well behaving lower bound for $f^{CE, \varepsilon, \lambda}$ has not been found yet. To summarize, the verification for both fixed and proportional costs still has a few challenging open questions.

8. Conclusion and future research

This thesis ends with a few concluding remarks regarding our results and possible topics for future research.

We have seen that in a relatively general Itô process model, the problem of exponential utility maximization under fixed transaction costs can be solved asymptotically. Using a dynamic programming approach, a candidate strategy was derived: It lies in a random and time dependent interval around the frictionless optimizer, changes to the latter once the boundaries of this interval are breached and sells all stocks at the right point in time. The halfwidth of this so-called no trade region can be determined in terms of the transaction cost, the degree of risk aversion and the portfolio gamma of the frictionless optimizer. Under suitable regularity assumptions on the dynamics of the stock price and the frictionless optimizer, that hold in the Black-Scholes model, we were able to verify the almost optimality of the candidate strategy. This involved quite technical remainder estimates and required a modification of the value process ansatz when considering competing admissible strategies.

We applied our results on utility indifference pricing under fixed transaction costs and after we weakened the regularity assumptions of the optimization problem, we were able to derive a utility indifference price for the European put in the Black-Scholes model.

Finally, we heuristically derived a candidate strategy maximizing the expected exponential utility under both fixed and proportional costs. The rigorous verification is left to future research.

Other interesting questions are whether the heuristics and verifications also work for other than exponential utility functions, for multidimensional asset price processes and for stochastic and time-dependent transaction costs.

Furthermore, it might be interesting to find out whether the methods for asymptotic dynamic programming developed in this thesis can be applied to other optimization problems in Mathematical Finance or related subjects.

A. Analyzing error terms

A.1. For Chapter 3

In this section we do the tedious calculations in order to prove equation (3.7.20) in Proposition 3.7.1. Therefore, recall the definitions of the processes H^i , that can be found in equation (3.5.15).

$i = 0$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^0 |H_s^0| ds \right) &\leq E_Q \left(\int_0^T \frac{p^2}{2} c_s^{\gamma, \gamma} ds \right) \\ &= \frac{p^3}{6} \varepsilon \mathbb{E}_Q ([Y]_T)\end{aligned}$$

$i = 1$:

$$\begin{aligned}H_s^1 &= (-p)(2\beta_s b_s^{\varphi^*, Q} + 2c_s^{\beta, \varphi^*}) + \frac{p^2}{2}(-2)\beta_s c_s^{\varphi^*, \gamma} - p^2 c_s^{S, \gamma} \\ &= 2(-p)\sqrt{\frac{\varepsilon p}{3}}\sqrt{C_s} b_s^{\varphi^*, Q} + (-p)\sqrt{\frac{\varepsilon p}{3}}\frac{1}{\sqrt{C_s}} c_s^{C, \varphi^*} - p^2 \frac{\varepsilon p}{3} \sqrt{C_s} c_s^{\varphi^*, Y} \\ &\quad - p^2 \sqrt{\frac{\varepsilon p}{3}} c_s^{S, Y} \\ &=: \sum_{j=1}^4 H_s^{1, j}\end{aligned}$$

$H^{1,1}$:

$$\begin{aligned}&\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+) |H_s^{1,1}| ds \right)^{1/2} \\ &= \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} C_s^{-1/4} \right) \left| (-p) 2 \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{1/2} b_s^{\varphi^*, Q} \right| ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/4} 2p \left(\frac{\varepsilon p}{3} \right) \mathbb{E}_Q \left(\int_0^T C_s^{-1/4} C_s^{1/2} |b_s^{\varphi^*, Q}| ds \right) \\ &= 2\sqrt{6} p^{3/4} \varepsilon^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{1/4} |b_s^{\varphi^*, Q}| ds \right)\end{aligned}$$

A. Analyzing error terms

$H^{1,2}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+) |H_s^{1,2}| ds \right) &= \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} C_s^{-1/4} \right) \left((-p) \frac{\varepsilon p^{1/2}}{3C_s} |c_s^{C,\varphi^*}| \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/4} \left(\frac{\varepsilon p}{3} \right)^{1/2} p \mathbb{E}_Q \left(\int_0^T C_s^{-1/4} C_s^{-1/2} |c_s^{C,\varphi^*}| ds \right) \\ &= \frac{2^{1/2}}{3^{1/4}} \varepsilon^{3/4} p^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C,\varphi^*}| ds \right)\end{aligned}$$

$H^{1,3}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+) |H_s^{1,3}| ds \right) &= \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} C_s^{-1/4} \right) \left(p^2 \frac{\varepsilon p}{3} C_s^{1/2} |c_s^{\varphi^*,Y}| \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/4} p^2 \frac{\varepsilon p}{3} \mathbb{E}_Q \left(\int_0^T C_s^{-1/4} C_s |c_s^{\varphi^*,Y}| ds \right) \\ &= \frac{2^{1/2}}{3^{3/4}} p^{11/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{1/4} |c_s^{\varphi^*,Y}| ds \right)\end{aligned}$$

$H^{1,4}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+) |H_s^{1,4}| ds \right) &= \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} C_s^{-1/4} \right) \left(p^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} |c_s^{S,Y}| \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/4} \frac{p^{5/2} \varepsilon^{1/2}}{3^{1/2}} \mathbb{E}_Q \left(\int_0^T C_s^{-1/4} |c_s^{S,Y}| ds \right) \\ &= \frac{2^{1/2}}{3^{3/4}} p^{9/4} \varepsilon^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{-1/4} |c_s^{S,Y}| ds \right)\end{aligned}$$

$i = 2$:

$$\begin{aligned}H_s^2 &= (-p) \left(-b_s^{\beta,Q} + \frac{p^2}{2} (4\beta_s^2 c_s^{\varphi^*,\varphi^*} + 2c_s^{\varphi^*,\gamma}) + p2p^2 \beta_s c_s^{S,\varphi^*} \right) \\ &= p \left(\frac{\varepsilon p}{3} \right)^{1/2} \left(\frac{b_s^{C,Q}}{2C_s^{1/2}} - \frac{c_s^{C,C}}{4C_s^{3/2}} \right) - \frac{p^3}{2} 4 \frac{\varepsilon p}{3} C_s c_s^{\varphi^*,\varphi^*} - \frac{p^3 p}{2(3^{1/2})} 2 \frac{(\varepsilon p)^{1/2}}{2(3)^{1/2} C_s^{1/2}} \varepsilon^{1/2} c_s^{C,Y} \\ &\quad + 2p \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{1/2} c_s^{S,\varphi^*} \\ &= \frac{p}{2} \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{b_s^{C,Q}}{C_s^{1/2}} - \frac{p}{4} \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{c_s^{C,C}}{C_s^{3/2}} - \frac{2}{3} p^4 \varepsilon C_s c_s^{\varphi^*,\varphi^*} - \frac{p^{7/2}}{6} \varepsilon C_s^{-1/2} c_s^{C,Y} \\ &\quad + 2p \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{1/2} c_s^{S,\varphi^*} \\ &=: \sum_{j=1}^5 H_s^{2,j}\end{aligned}$$

$H^{2,1}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^2 |H_s^{2,1}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{p}{2} \left(\frac{\varepsilon p}{3} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1/2} \frac{|b_s^{C,Q}|}{C_s^{1/2}} ds \right) \\ &= p\varepsilon \mathbb{E}_Q \left(\int_0^T \frac{|b_s^{C,Q}|}{C_s} ds \right)\end{aligned}$$

 $H^{2,2}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^2 |H_s^{2,2}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1/2} \left(\frac{p}{4} \left(\frac{\varepsilon p}{3} \right)^{1/2} C^{-3/2} c_s^{C,C} \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{p}{4} \left(\frac{\varepsilon p}{3} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1/2} C^{-3/2} c_s^{C,C} ds \right) \\ &= \frac{\varepsilon p}{2} \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C,C} ds \right)\end{aligned}$$

 $H^{2,3}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^2 |H_s^{2,3}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1/2} \left(\frac{2}{3} p^4 \varepsilon C_s c_s^{\varphi^*, \varphi^*} \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{2}{3} p^4 \varepsilon \mathbb{E}_Q \left(\int_0^T C_s^{1/2} c_s^{\varphi^*, \varphi^*} \right) \\ &= \frac{4}{3^{1/2}} p^{7/2} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T A_s ds \right)\end{aligned}$$

 $H^{2,4}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^2 |H_s^{2,4}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1/2} \left| \frac{p^4}{6} \varepsilon C_s^{-1/2} c_s^{S, \varphi^*} \right| ds \right) \\ &= \frac{p^{7/2}}{3^{1/2}} \varepsilon \mathbb{E}_Q \left(\int_0^T C_s^{-1} |c_s^{S, \varphi^*}| ds \right)\end{aligned}$$

 $H^{2,5}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^2 |H_s^{2,5}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1/2} \left| 2p \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{1/2} c_s^{S, \varphi^*} \right| ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/2} 2p \left(\frac{\varepsilon p}{3} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T |c_s^{S, \varphi^*}| ds \right) \\ &= 4p\varepsilon \mathbb{E}_Q \left(\int_0^T |c_s^{S, \varphi^*}| ds \right)\end{aligned}$$

A. Analyzing error terms

$i = 3$:

$$\begin{aligned}
H_s^3 &= \frac{p^2}{2} \left((-4)\beta_s c_s^{\beta, \varphi^*} + 8\alpha_s c_s^{\varphi^*, \gamma} \right) - p^2 c_s^{S, \beta} + 4p\alpha_s b_s^{\varphi^*, Q} + 4p c_s^{\alpha, \varphi^*} \\
&= -2p^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{1/2} \frac{1}{2} \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{-1/2} c_s^{C, \varphi^*} + 4p^2 \frac{p}{12} C_s \left(\frac{\varepsilon p}{3} \right)^{1/2} c_s^{\varphi^*, Y} \\
&\quad - \frac{p^2}{2} \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{-1/2} c_s^{S, C} + 4p \frac{p}{12} C_s b_s^{\varphi^*, Q} + 4p \frac{p}{12} c_s^{C, \varphi^*} \\
&= -\frac{p^3}{3} \varepsilon c_s^{C, \varphi^*} + \frac{p^{7/2}}{3^{3/2}} \varepsilon^{1/2} C_s c_s^{\varphi^*, Y} - \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{1/2} c_s^{S, C} + \frac{p^2}{3} C_s b_s^{\varphi^*, Q} + \frac{p^2}{3} c_s^{C, \varphi^*} \\
&=: \sum_{j=1}^5 H_s^{3, j}
\end{aligned}$$

$H^{3,1}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^3 |H_s^{3,1}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} \left(\frac{p^3}{3} \varepsilon |c_s^{C, \varphi^*}| \right) ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^3}{3} \varepsilon \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/2}} p^{9/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right)
\end{aligned}$$

$H^{3,2}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^3 |H_s^{3,2}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} \left| \frac{p^{7/2}}{3^{3/2}} \varepsilon^{1/2} C_s c_s^{\varphi^*, Y} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^{7/2}}{3^{3/2}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} C_s |c_s^{\varphi^*, Y}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/4}} p^{11/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{1/4} |c_s^{\varphi^*, Y}| ds \right)
\end{aligned}$$

$H^{3,3}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^3 |H_s^{3,3}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} \left| \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{1/2} c_s^{S, C} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{S, C}| ds \right) \\
&= 2^{1/2} 3^{1/4} p^{7/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{S, C}| ds \right)
\end{aligned}$$

$H^{3,4}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^3 |H_s^{3,4}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} \left| \frac{p^2}{3} C_s b_s^{\varphi^*, Q} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^2}{3} \mathbb{E}_Q \left(\int_0^T C_s^{1/4} |b_s^{\varphi^*, Q}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/4}} p^{5/4} \varepsilon^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{1/4} |b_s^{\varphi^*, Q}| ds \right)
\end{aligned}$$

 $H^{3,5}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^3 |H_s^{3,5}|^2 ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} \left| \frac{p^2}{3} c_s^{C, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^2}{3} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/4}} p^{5/4} \varepsilon^{3/4} \frac{p^2}{3} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right)
\end{aligned}$$

 $i = 4$:

$$\begin{aligned}
H_s^4 &= (-p)b_s^\alpha + \frac{p^2}{2} (c_s^{\beta, \beta} + 16\alpha_s \beta_s c_s^{\varphi^*, \varphi^*} - 2c_s^{\alpha, \beta}) - p^2 \alpha_s c_s^{S, \varphi^*} \\
&= (-p) \frac{p}{12} b_s^C + \frac{p^2}{2} \frac{\varepsilon p}{12} \frac{c_s^{C, C}}{C_s} - \frac{p^2}{2} 16 \frac{p}{12} C_s \left(\frac{\varepsilon p}{3} \right)^{1/2} C_s^{1/2} c_s^{\varphi^*, \varphi^*} \\
&\quad - \frac{p^2}{2} 2 \frac{p}{12} \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} C_s^{-1} c_s^{C, C} - \frac{p^2}{2} 2 \frac{p}{12} C_s c_s^{\varphi^*, S} \\
&= -\frac{p^2}{12} b_s^C + \frac{p^3}{24} \varepsilon C_s^{-1} c_s^{C, C} - \frac{2p^{7/2}}{3\sqrt{3}} \varepsilon^{1/2} C_s^{3/2} c_s^{\varphi^*, \varphi^*} - \frac{p^{7/2}}{24\sqrt{3}} \varepsilon^{1/2} C_s^{-1} c_s^{C, C} - \frac{p^3}{12} C_s c_s^{S, \varphi^*} \\
&:= \sum_{j=1}^5 H_s^{4, j}
\end{aligned}$$

 $H^{4,1}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^4 |H_s^{4,1}| ds \right) &= \mathbb{E}_Q \left(\int_0^T \left(\frac{12\varepsilon}{p} \right) C_s^{-1} \left| \frac{p^2}{12} b_s^{C, Q} \right| ds \right) \\
&= \frac{12\varepsilon}{p} \frac{p^2}{12} \mathbb{E}_Q \left(\int_0^T C_s^{-1} |b_s^{C, Q}| ds \right) \\
&= p\varepsilon \mathbb{E}_Q \left(\int_0^T \left| \frac{b_s^{C, Q}}{C_s^2} \right| ds \right)
\end{aligned}$$

A. Analyzing error terms

$H^{4,2}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^4 |H_s^{4,2}| ds \right) &= \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T C_s^{-1} \left(\frac{p^3}{24} \varepsilon C_s^{-1} c_s^{C,C} \right) ds \right) \\ &= \frac{12\varepsilon}{p} \frac{p^3}{24} \varepsilon \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C,C} ds \right) \\ &= \frac{p^2}{2} \varepsilon^2 \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C,C} ds \right)\end{aligned}$$

$H^{4,3}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^4 |H_s^{4,3}| ds \right) &= \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T C_s^{-1} \left(\frac{2p^{7/2}}{3\sqrt{3}} \varepsilon^{1/2} C_s^{3/2} c_s^{\varphi^*,\varphi^*} \right) ds \right) \\ &= \frac{12\varepsilon}{p} \frac{2p^{7/2}}{3\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1} (C_s^{3/2} c_s^{\varphi^*,\varphi^*}) ds \right) \\ &= \frac{8p^{5/2}}{3^{1/2}} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T C_s^{1/2} c_s^{\varphi^*,\varphi^*} ds \right)\end{aligned}$$

$H^{4,4}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^4 |H_s^{4,4}| ds \right) &= \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T C_s^{-1} \left(\frac{p^{7/2}}{24\sqrt{3}} \varepsilon^{1/2} C_s^{-1} c_s^{C,C} \right) ds \right) \\ &= \frac{12\varepsilon}{p} \frac{p^{7/2}}{24\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-1} (C_s^{-1} c_s^{C,C}) ds \right) \\ &= \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C,C} ds \right)\end{aligned}$$

$H^{4,5}$:

$$\begin{aligned}\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^+)^4 |H_s^{4,5}| ds \right) &= \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T C_s^{-1} \left| \frac{p^3}{12} C_s c_s^{S,\varphi^*} \right| ds \right) \\ &= \frac{12\varepsilon}{p} \frac{p^3}{12} \mathbb{E}_Q \left(\int_0^T |c_s^{S,\varphi^*}| ds \right) \\ &= p^2 \varepsilon \mathbb{E}_Q \left(\int_0^T |c_s^{S,\varphi^*}| ds \right)\end{aligned}$$

$i = 5$:

$$\begin{aligned}
H_s^5 &= \frac{p^2}{2} (2\beta_s c_s^{\alpha, \varphi^*} 8\alpha_s c_s^{\beta, \varphi^*}) + p^2 c_s^{S, \alpha} \\
&= p^2 \beta_s c_s^{\alpha, \varphi^*} + 4p^2 \alpha_s c_s^{\beta, \varphi^*} + p^2 c_s^{S, \alpha} \\
&= p^2 \left(\frac{\varepsilon p}{3}\right)^{1/2} C_s^{1/2} \frac{p}{12} c_s^{C, \varphi^*} + 4p^2 \frac{p}{12} C_s \frac{1}{2} C_s^{-1/2} \left(\frac{\varepsilon p}{3}\right)^{1/2} c_s^{C, \varphi^*} + p^2 \frac{p}{12} c_s^{S, C} \\
&= \frac{p^{7/2}}{12\sqrt{3}} \varepsilon^{1/2} C_s^{1/2} c_s^{C, \varphi^*} + \frac{p^{7/2}}{6\sqrt{3}} \varepsilon^{1/2} C_s^{1/2} c_s^{C, \varphi^*} + \frac{p^3}{12} c_s^{S, C} \\
&=: \sum_{j=1}^3 H_s^{5, j}
\end{aligned}$$

 $H^{5,1}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^5 |H_s^{5,1}| ds \right) &= \left(\frac{12\varepsilon}{p}\right)^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} \left(\frac{p^{7/2}}{12\sqrt{3}} \varepsilon^{1/2} C_s^{1/2} |c_s^{C, \varphi^*}| \right) ds \right) \\
&= \left(\frac{12\varepsilon}{p}\right)^{5/4} \frac{p^{7/2}}{12\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} (C_s^{1/2} |c_s^{C, \varphi^*}|) ds \right) \\
&= \frac{2^{1/2}}{3^{1/4}} p^{9/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right)
\end{aligned}$$

 $H^{5,2}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^5 |H_s^{5,2}| ds \right) &= \left(\frac{12\varepsilon}{p}\right)^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} \left| \frac{p^{7/2}}{6\sqrt{3}} \varepsilon^{1/2} C_s^{1/2} c_s^{C, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p}\right)^{5/4} \frac{p^{7/2}}{6\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} |C_s^{1/2} c_s^{C, \varphi^*}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/4}} p^{9/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right)
\end{aligned}$$

 $H^{5,3}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^5 |H_s^{5,3}| ds \right) &= \left(\frac{12\varepsilon}{p}\right)^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} \left| \frac{p^3}{12} c_s^{S, C} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p}\right)^{5/4} \frac{p^3}{12} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} |c_s^{S, C}| ds \right) \\
&= (12)^{1/4} p^{17/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T C_s^{-5/4} |c_s^{S, C}| ds \right)
\end{aligned}$$

A. Analyzing error terms

$i = 6$:

$$\begin{aligned}
H_s^6 &= \frac{p^2}{2} 16\alpha_s^2 c_s^{\varphi^*, \varphi^*} - \frac{p^2}{2} 2c_s^{\alpha, \beta} \\
&= 8p^2 \left(\frac{p}{12} C_s \right)^2 c_s^{\varphi^*, \varphi^*} - p^2 \frac{p}{12} \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} C_s^{-1/2} c_s^{C, C} \\
&= \frac{1}{18} p^4 C_s^2 c_s^{\varphi^*, \varphi^*} - \frac{1}{24\sqrt{3}} p^{7/2} \varepsilon^{1/2} C_s^{-1/2} c_s^{C, C} \\
&=: H_s^{6,1} + H_s^{6,2}
\end{aligned}$$

$H^{6,1}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^6 |H_s^{6,1}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/2} \mathbb{E}_Q \left(\int_0^T C_s^{-3/2} \left| \frac{1}{18} p^4 C_s^2 c_s^{\varphi^*, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/2} \frac{1}{18} p^4 \mathbb{E}_Q \left(\int_0^T C_s^{-3/2} |C_s^2 c_s^{\varphi^*, \varphi^*}| ds \right) \\
&= \frac{4}{3^{1/2}} p^{5/2} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T C_s^{1/2} c_s^{\varphi^*, \varphi^*} ds \right)
\end{aligned}$$

$H^{6,2}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^6 |H_s^{6,2}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/2} \mathbb{E}_Q \left(\int_0^T C_s^{-3/2} \left| \frac{1}{24\sqrt{3}} p^{7/2} \varepsilon^{1/2} C_s^{-1/2} c_s^{C, C} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/2} \frac{1}{24\sqrt{3}} p^{7/2} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T C_s^{-3/2} |C_s^{-1/2} c_s^{C, C}|^2 ds \right) \\
&= p^2 \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C, C} ds \right)
\end{aligned}$$

$i = 7$:

$$H_s^7 = -\frac{p^2}{2} 8\alpha_s c_s^{\alpha, \varphi^*} = -4p^2 \frac{p}{12} C_s \frac{p}{12} c_s^{C, \varphi^*} = -\frac{p^3}{36} C_s c_s^{C, \varphi^*}$$

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^7 |H_s^7| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{7/4} \mathbb{E}_Q \left(\int_0^T C_s^{-7/4} \left| \frac{p^3}{36} C_s c_s^{C, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{7/4} \frac{p^3}{36} \mathbb{E}_Q \left(\int_0^T C_s^{-7/4} |C_s c_s^{C, \varphi^*}| ds \right) \\
&= \frac{4}{3^{1/4}} p^{5/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T C_s^{-3/4} |c_s^{C, \varphi^*}| ds \right)
\end{aligned}$$

$i=8$:

$$H_s^8 = \frac{p^2}{2} c_s^{\alpha, \alpha} = \frac{p^2}{2} \left(\frac{p}{12} \right)^2 c_s^{C, C} = \frac{p^4}{288} c_s^{C, C}$$

$$\begin{aligned} \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^+)^8 |H_s^8| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^2 \mathbb{E}_Q \left(\int_0^T C_s^{-2} \left| \frac{p^4}{288} c_s^{C, C} \right| ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^2 \frac{p^4}{288} \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C, C} ds \right) \\ &= \frac{p^2}{2} \varepsilon^2 \mathbb{E}_Q \left(\int_0^T C_s^{-2} c_s^{C, C} ds \right) \end{aligned}$$

For H_0 we can use (3.7.21). For $H_{2,1}$, applying Hölders inequality yields that the above expectation is bounded by

$$\mathbb{E}_Q \left(\sup_{t \in [0, T]} C_t^{-2} \right)^{1/2} \mathbb{E}_Q \left(\left(\int_0^T |b_s^{C, Q}| ds \right)^2 \right)$$

which is finite by assumption. $H_{3,4}$ is analogous. Since A is bounded, the expression in $H_{2,3}$ is finite. All remaining expectations above are of the form

$$\mathbb{E}_Q \left(\int_0^T C_s^q |c_s^{X, \tilde{X}}| ds \right)$$

for $X, \tilde{X} \in \{C, \varphi^*, S, Y\}$ and $q \in [-2, 1/2]$. By applying Hölders inequality we can obtain an upper bound for these expressions, namely:

$$\mathbb{E}_Q \left(\sup_{t \in [0, T]} C_s^{2q} \right)^{1/2} \mathbb{E}_Q \left(V([X, \tilde{X}]_T^2) \right)^{1/2},$$

where $V(f)$ is the total variation of a function $f : [0, T] \rightarrow \mathbb{R}$ being of bounded variation. Using the relation between the L_q - norms shows that

$$\mathbb{E}_Q \left(\sup_{t \in [0, T]} C_s^{2q} \right)^{1/2} = \begin{cases} \|C\|_{S_{2q}(Q)}^q, & \text{if } q \in [0, 1/2] \\ \|C^{-1}\|_{S_{2|q|}(Q)}^{|q|}, & \text{if } q \in [-2, 0) \end{cases} \leq \begin{cases} \|C\|_{S_1(Q)}^q, & \text{if } q \in [0, 1/2] \\ \|C^{-1}\|_{S_4(Q)}^{|q|}, & \text{if } q \in [-2, 0). \end{cases}$$

For the case $q \in [0, 1/2]$ we can use $\|\cdot\|_{S_1(Q)} \leq \|\cdot\|_{S_2(Q)} \leq \text{const.}$ $\| \cdot \|_{H_2(Q)}$ by Theorem B.2.3 to see that $\|C\|_{S_1(Q)}$ is finite. The finiteness in the case $q \in [-2, 0)$ is due to Assumption 3.6.4. Using the Kunita-Watanabe inequality (B.2.1) we obtain

$$\begin{aligned} \left\| V(X, \tilde{X})_T \right\|_{L_2(Q)} &\leq \left\| [X]_T^{1/2} [\tilde{X}]_T^{1/2} \right\|_{L_2(Q)} \\ &\leq \left\| [X]_T \right\|_{L_2(Q)}^{1/2} \left\| [\tilde{X}]_T \right\|_{L_2(Q)}^{1/2}, \end{aligned}$$

which are all finite by Assumption 3.6.4 and (3.7.21).

A.2. For Section 4.2

$i = 0$:

$$\begin{aligned} \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^0 \left| \widehat{H}_s^{k,0} \right| ds \right) &\leq E_Q \left(\int_0^T \frac{p^2}{2} c_s^{\gamma^k, \gamma^k} ds \right) \\ &= \frac{p^3}{6} \varepsilon \mathbb{E}_Q ([Y^k]_T) \end{aligned}$$

$i = 1$:

$$\begin{aligned} \widehat{H}_s^{k,s} &= (-p)(2\widehat{\beta}_s^k b_s^{\varphi^*,Q} + 2c_s^{\widehat{\beta}^k, \varphi^*}) + \frac{p^2}{2}(-2)\widehat{\beta}_s^k c_s^{\varphi^*, \gamma^k} - p^2 c_s^{S, \gamma^k} \\ &= 2(-p)\sqrt{\frac{\varepsilon p c_1}{3}} \sqrt{C_s^k} b_s^{\varphi^*,Q} + (-p)\sqrt{\frac{\varepsilon p c_1}{3}} \frac{1}{\sqrt{C_s^k}} c_s^{C^k, \varphi^*} - p^2 \frac{\varepsilon p}{3} \sqrt{C_s^k} c_s^{\varphi^*, Y^k} \\ &\quad - p^2 \sqrt{\frac{\varepsilon p}{3}} c_s^{S, Y^k} \\ &:= \sum_{j=1}^4 \widehat{H}_s^{k,1,j} \end{aligned}$$

$\widehat{H}^{k,1,1}$:

$$\begin{aligned} &\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k}) \left| \widehat{H}_s^{k,1,1} \right| ds \right) \\ &\leq \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} (C_s^k)^{-1/4} \right) \left| (-p) 2 \left(\frac{\varepsilon p}{3} \right)^{1/2} (C_s^k)^{1/2} b_s^{\varphi^*,Q} \right| ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/4} 2p \left(\frac{\varepsilon p}{3} \right) \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/4} (C_s^k)^{1/2} |b_s^{\varphi^*,Q}| ds \right) \\ &= 2\sqrt{6} p^{3/4} \varepsilon^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/4} |b_s^{\varphi^*,Q}| ds \right) \\ &\leq \text{const. } \varepsilon^{3/4} k_1^{-1/4} \end{aligned}$$

$\widehat{H}^{k,1,2}$:

$$\begin{aligned} &\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k}) \left| \widehat{H}_s^{k,1,2} \right| ds \right) \\ &\leq \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} (C_s^k)^{-1/4} \right) \left((-p) \frac{\varepsilon p}{3(C_s^k)}^{1/2} |c_s^{C^k, \varphi^*}| \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/4} \left(\frac{\varepsilon p}{3} \right)^{1/2} p \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/4} (C_s^k)^{-1/2} |c_s^{C^k, \varphi^*}| ds \right) \\ &= \frac{2^{1/2}}{3^{1/4}} \varepsilon^{3/4} p^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\ &\leq \text{const. } \varepsilon^{3/4} k_2^{5/4} k_1^{-2} \end{aligned}$$

$\widehat{H}^{k,1,3}$:

$$\begin{aligned}
& \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k}) \left| \widehat{H}_s^{k,1,3} \right| ds \right) \\
& \leq \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} (C_s^k)^{-1/4} \right) \left(p^2 \frac{\varepsilon p}{3} (C_s^k)^{1/2} |c_s^{\varphi^*, Y^k}| \right) ds \right) \\
& = \left(\frac{12\varepsilon}{p} \right)^{1/4} p^2 \frac{\varepsilon p}{3} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/4} (C_s^k) |c_s^{\varphi^*, Y^k}| ds \right) \\
& = \frac{2^{1/2}}{3^{3/4}} p^{11/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/4} |c_s^{\varphi^*, Y^k}| ds \right) \\
& \leq \text{const. } \varepsilon^{5/4} k_1^{-1/4}
\end{aligned}$$

 $\widehat{H}^{k,1,4}$:

$$\begin{aligned}
& \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k}) \left| \widehat{H}_s^{k,1,4} \right| ds \right) \\
& \leq \mathbb{E}_Q \left(\int_0^T \left(\left(\frac{12\varepsilon}{p} \right)^{1/4} (C_s^k)^{-1/4} \right) \left(p^2 \left(\frac{\varepsilon p}{3} \right)^{1/2} |c_s^{S, Y^k}| \right) ds \right) \\
& = \left(\frac{12\varepsilon}{p} \right)^{1/4} \frac{p^{5/2} \varepsilon^{1/2}}{3^{1/2}} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/4} |c_s^{S, Y^k}| ds \right) \\
& = \frac{2^{1/2}}{3^{3/4}} p^{9/4} \varepsilon^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/4} |c_s^{S, Y^k}| ds \right) \\
& \leq \text{const. } \varepsilon^{3/4} k_2^{1/4}
\end{aligned}$$

 $i = 2$:

$$\begin{aligned}
\widehat{H}_s^{k,2} & = (-p) \left(-b_s^{\widehat{\beta}^k, Q} + \frac{p^2}{2} \left(4(\widehat{\beta}_s^k)^2 c_s^{\varphi^*, \varphi^*} + 2c_s^{\varphi^*, \gamma^k} \right) + p2p^2 \widehat{\beta}_s^k c_s^{S, \varphi^*} \right) \\
& = p \left(\frac{\varepsilon p \overline{c_1}}{3} \right)^{1/2} \left(\frac{b_s^{C^k, Q}}{2(C_s^k)^{1/2}} - \frac{c_s^{C^k, C^k}}{4(C_s^k)^{3/2}} \right) - \frac{p^3}{2} 4 \frac{\varepsilon p \overline{c_1}}{3} C_s^k c_s^{\varphi^*, \varphi^*} \\
& \quad - \frac{p^3 p}{2(3^{1/2})^2} 2 \frac{(\varepsilon p)^{1/2}}{2(3)^{1/2} (C_s^k)^{1/2}} \varepsilon^{1/2} c_s^{C, Y^k} + 2p \left(\frac{\varepsilon p \overline{c_1}}{3} \right)^{1/2} (C_s^k)^{1/2} c_s^{S, \varphi^*} \\
& = \frac{p}{2} \left(\frac{\varepsilon p \overline{c_1}}{3} \right)^{1/2} \frac{b_s^{C^k, Q}}{(C_s^k)^{1/2}} - \frac{p}{4} \left(\frac{\varepsilon p \overline{c_1}}{3} \right)^{1/2} \frac{c_s^{C^k, C^k}}{(C_s^k)^{3/2}} - \frac{2}{3} p^4 \varepsilon C_s^k c_s^{\varphi^*, \varphi^*} \\
& \quad - \frac{p^{7/2}}{6} \varepsilon (C_s^k)^{-1/2} c_s^{C^k, Y} + 2p \left(\frac{\varepsilon p \overline{c_1}}{3} \right)^{1/2} (C_s^k)^{1/2} c_s^{S, \varphi^*} \\
& =: \sum_{j=1}^5 \widehat{H}_s^{k,2,j}
\end{aligned}$$

A. Analyzing error terms

$\widehat{H}^{k,2,1}$:

$$\begin{aligned} \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^2 |H_s^{k,2,1}| ds \right) &\leq \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{p}{2} \left(\frac{\varepsilon p}{3} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/2} \frac{|b_s^{C^k, Q}|}{(C_s^k)^{1/2}} ds \right) \\ &= p\varepsilon \mathbb{E}_Q \left(\int_0^T \frac{|b_s^{C^k, Q}|}{C_s^k} ds \right) \\ &\leq \text{const. } \varepsilon k_2^2 k_1^{-3} \end{aligned}$$

$\widehat{H}^{k,2,2}$:

$$\begin{aligned} &\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^2 |H_s^{k,2,2}| ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/2} \left(\frac{p}{4} \left(\frac{\varepsilon p}{3} \right)^{1/2} (C_s^k)^{-3/2} c_s^{C^k} \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{p}{4} \left(\frac{\varepsilon p}{3} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/2} (C_s^k)^{-3/2} c_s^{C^k, C^k} ds \right) \\ &= \frac{\varepsilon p}{2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\ &\leq \text{const. } \varepsilon k_2^2 k_1^{-4} \end{aligned}$$

$\widehat{H}^{k,2,3}$:

$$\begin{aligned} \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^2 |\widehat{H}_s^{k,2,3}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/2} \left(\frac{2}{3} p^4 \varepsilon C_s^k c_s^{\varphi^*, \varphi^*} \right) ds \right) \\ &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \frac{2}{3} p^4 \varepsilon \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/2} c_s^{\varphi^*, \varphi^*} \right) \\ &= \frac{4}{3^{1/2}} p^{7/2} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T A_s ds \right) = \text{const. } \varepsilon^{3/2} \end{aligned}$$

$\widehat{H}^{k,2,4}$:

$$\begin{aligned} \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^2 |\widehat{H}_s^{k,2,4}| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/2} \left| \frac{p^4}{6} \varepsilon (C_s^k)^{-1/2} c_s^{S, \varphi^*} \right| ds \right) \\ &= \frac{p^{7/2}}{3^{1/2}} \varepsilon \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} |c_s^{S, \varphi^*}| ds \right) \\ &\leq \text{const. } \varepsilon k_2 \end{aligned}$$

$\widehat{H}^{k,2,5}$:

$$\begin{aligned}
 & \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^2 \left| \widehat{H}_s^{k,2,5} \right| ds \right) \\
 &= \left(\frac{12\varepsilon}{p} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1/2} \left| 2p \left(\frac{\varepsilon p}{3} \right)^{1/2} (C_s^k)^{1/2} c_s^{S,\varphi^*} \right| ds \right) \\
 &= \left(\frac{12\varepsilon}{p} \right)^{1/2} 2p \left(\frac{\varepsilon p}{3} \right)^{1/2} \mathbb{E}_Q \left(\int_0^T |c_s^{S,\varphi^*}| ds \right) \\
 &= 4p\varepsilon \mathbb{E}_Q \left(\int_0^T |c_s^{S,\varphi^*}| ds \right) = \text{const. } \varepsilon
 \end{aligned}$$

$i = 3$:

$$\begin{aligned}
 \widehat{H}_s^{k,3} &= \frac{p^2}{2} \left((-4)\widehat{\beta}_s^k c_s^{\widehat{\beta}^k, \varphi^*} + 8\alpha_s^k c_s^{\varphi^*, \gamma^k} \right) - p^2 c_s^{S, \widehat{\beta}^k} + 4p\alpha_s^k b_s^{\varphi^*, Q} + 4pc_s^{\alpha^k, \varphi^*} \\
 &= -2p^2 \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} (C_s^k)^{1/2} \frac{1}{2} \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} (C_s^k)^{-1/2} c_s^{C^k, \varphi^*} + 4p^2 \frac{p}{12} C_s^k \left(\frac{\varepsilon p}{3} \right)^{1/2} c_s^{\varphi^*, Y^k} \\
 &\quad - \frac{p^2}{2} \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} (C_s^k)^{-1/2} c_s^{S, C^k} + 4p \frac{p}{12} C_s^k b_s^{\varphi^*, Q} + 4p \frac{p}{12} c_s^{C^k, \varphi^*} \\
 &= -\frac{p^3}{3} \varepsilon \bar{c}_1 c_s^{C^k, \varphi^*} + \frac{p^{7/2}}{3^{3/2}} \varepsilon^{1/2} \bar{c}_1^{-1/2} C_s^k c_s^{\varphi^*, Y^k} - \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{1/2} c_s^{S, C^k} + \frac{p^2}{3} C_s^k b_s^{\varphi^*, Q} + \frac{p^2}{3} c_s^{C^k, \varphi^*} \\
 &=: \sum_{j=1}^5 \widehat{H}_s^{k,3,j}
 \end{aligned}$$

$\widehat{H}^{k,3,1}$:

$$\begin{aligned}
 \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^3 \left| \widehat{H}_s^{k,3,1} \right| ds \right) &\leq \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} \left(\frac{p^3}{3} \varepsilon |c_s^{C^k, \varphi^*}| \right) ds \right) \\
 &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^3}{3} \varepsilon \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\
 &= \frac{2^{3/2}}{3^{1/2}} p^{9/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\
 &\leq \varepsilon^{7/4} k_2^{5/4} k_1^{-2}
 \end{aligned}$$

$\widehat{H}^{k,3,2}$:

$$\begin{aligned}
 \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^3 \left| \widehat{H}_s^{k,3,2} \right| ds \right) &\leq \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} \left| \frac{p^{7/2}}{3^{3/2}} \varepsilon^{1/2} C_s^k c_s^{\varphi^*, Y^k} \right| ds \right) \\
 &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^{7/2}}{3^{3/2}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} C_s^k |c_s^{\varphi^*, Y^k}| ds \right) \\
 &= \frac{2^{3/2}}{3^{1/4}} p^{11/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/4} |c_s^{\varphi^*, Y^k}| ds \right) \\
 &\leq \text{const. } \varepsilon^{5/4} k_1^{-5/4}
 \end{aligned}$$

A. Analyzing error terms

$\widehat{H}^{k,3,3}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^3 \left| \widehat{H}_s^{k,3,3} \right| ds \right) &\leq \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} \left| \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{1/2} c_s^{S,C^k} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{S,C^k}| ds \right) \\
&= 2^{1/2} 3^{1/4} p^{7/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{S,C^k}| ds \right) \\
&\leq \varepsilon^{5/4} k_2^{5/4} k_1^{-2}
\end{aligned}$$

$\widehat{H}^{k,3,4}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^3 \left| \widehat{H}_s^{k,3,4} \right| ds \right) &\leq \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} \left| \frac{p^2}{3} C_s^k b_s^{\varphi^*,Q} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^2}{3} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/4} |b_s^{\varphi^*,Q}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/4}} p^{5/4} \varepsilon^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/4} |b_s^{\varphi^*,Q}| ds \right) \\
&\leq \text{const. } \varepsilon^{3/4} k_1^{-1/4}
\end{aligned}$$

$\widehat{H}^{k,3,5}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^3 |H_s^{k,3,5}|^2 ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{3/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} \left| \frac{p^2}{3} c_s^{C^k, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/4} \frac{p^2}{3} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\
&= \frac{2^{3/2}}{3^{1/4}} p^{5/4} \varepsilon^{3/4} \frac{p^2}{3} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\
&\leq \text{const. } \varepsilon^{3/4} k_2^{5/4} k_1^{-2}
\end{aligned}$$

$i = 4$:

$$\begin{aligned}
 \widehat{H}_s^{k,4} &= (-p)b_s^{\alpha^k,Q} + \frac{p^2}{2} \left(c_s^{\widehat{\beta}^k} - 16\alpha_s^k \widehat{\beta}_s^k c_s^{\varphi^*} - 2c_s^{\alpha^k, \widehat{\beta}^k} \right) - p^2 \alpha_s^k c_s^{S, \varphi^*} \\
 &= (-p) \frac{p}{12} b_s^{C^k, Q} + \frac{p^2 \varepsilon p}{2} \frac{c_s^{C^k}}{12 C_s^k} - \frac{p^2}{2} 16 \frac{p}{12} C_s^k \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} (C_s^k)^{1/2} c_s^{\varphi^*, \varphi^*} \\
 &\quad - \frac{p^2}{2} 2 \frac{p}{12} \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} \frac{1}{2} (C_s^k)^{-1} c_s^{C^k} - \frac{p^2}{2} 2 \frac{p}{12} C_s^k c_s^{\varphi^*, S} \\
 &= -\frac{p^2}{12} b_s^{C^k, Q} + \frac{p^3}{24} \varepsilon \bar{c}_1 (C_s^k)^{-1} c_s^{C^k} - \frac{2p^{7/2}}{3\sqrt{3}} \varepsilon^{1/2} \bar{c}_1^{-1/2} (C_s^k)^{3/2} c_s^{\varphi^*, \varphi^*} \\
 &\quad - \frac{p^{7/2}}{24\sqrt{3}} \varepsilon^{1/2} \bar{c}_1^{-1/2} (C_s^k)^{-1} c_s^{C^k} - \frac{p^3}{12} (C_s^k) c_s^{S, \varphi^*} \\
 &:= \sum_{j=1}^5 \widehat{H}_s^{k,4,j}
 \end{aligned}$$

$\widehat{H}^{k,4,1}$:

$$\begin{aligned}
 \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^4 \left| \widehat{H}_s^{k,4,1} \right| ds \right) &\leq \mathbb{E}_Q \left(\int_0^T \left(\frac{12\varepsilon}{p} \right) (C_s^k)^{-1} \left| \frac{p^2}{12} b_s^{C^k, Q} \right| ds \right) \\
 &= \frac{12\varepsilon}{p} \frac{p^2}{12} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left| b_s^{C^k, Q} \right| ds \right) \\
 &= p\varepsilon \mathbb{E}_Q \left(\int_0^T \left| \frac{b_s^{C^k, Q}}{(C_s^k)^2} \right| ds \right) \\
 &\leq \text{const. } \varepsilon k_2^3 k_1^{-3}
 \end{aligned}$$

$\widehat{H}^{k,4,2}$:

$$\begin{aligned}
 \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^4 \left| \widehat{H}_s^{k,4,2} \right| ds \right) &\leq \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left(\frac{p^3}{24} \varepsilon (C_s^k)^{-1} c_s^{C^k} \right) ds \right) \\
 &= \frac{12\varepsilon}{p} \frac{p^3}{24} \varepsilon \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\
 &= \frac{p^2}{2} \varepsilon^2 \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\
 &\leq \text{const. } \varepsilon^2 k_2^3 k_1^{-4}
 \end{aligned}$$

$\widehat{H}^{k,4,3}$:

$$\begin{aligned}
 \mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^4 \left| \widehat{H}_s^{k,4,3} \right| ds \right) &\leq \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left(\frac{2p^{7/2}}{3\sqrt{3}} \varepsilon^{1/2} (C_s^k)^{3/2} c_s^{\varphi^*, \varphi^*} \right) ds \right) \\
 &= \frac{12\varepsilon}{p} \frac{2p^{7/2}}{3\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left((C_s^k)^{3/2} c_s^{\varphi^*, \varphi^*} \right) ds \right) \\
 &= \frac{8p^{5/2}}{3^{1/2}} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/2} c_s^{\varphi^*, \varphi^*} ds \right) \\
 &\leq \text{const. } \varepsilon^{3/2} k_1^{-1/2}
 \end{aligned}$$

A. Analyzing error terms

$\widehat{H}^{k,4,4}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^4 \left| \widehat{H}_s^{k,4,4} \right| ds \right) &\leq \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left(\frac{p^{7/2}}{24\sqrt{3}} \varepsilon^{1/2} (C_s^k)^{-1} c_s^{C^k} \right) ds \right) \\
&= \frac{12\varepsilon}{p} \frac{p^{7/2}}{24\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left((C_s^k)^{-1} c_s^{C^k} \right) ds \right) \\
&= \frac{p^{5/2}}{2\sqrt{3}} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\
&\leq \varepsilon^{3/2} k_2^3 k_1^{-4}
\end{aligned}$$

$\widehat{H}^{k,4,5}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^4 \left| \widehat{H}_s^{k,4,5} \right| ds \right) &\leq \frac{12\varepsilon}{p} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-1} \left| \frac{p^3}{12} C_s^k c_s^{S,\varphi^*} \right| ds \right) \\
&= \frac{12\varepsilon}{p} \frac{p^3}{12} \mathbb{E}_Q \left(\int_0^T |c_s^{S,\varphi^*}| ds \right) \\
&= p^2 \varepsilon \mathbb{E}_Q \left(\int_0^T |c_s^{S,\varphi^*}| ds \right) \\
&\leq \text{const. } \varepsilon
\end{aligned}$$

$i = 5$:

$$\begin{aligned}
\widehat{H}_s^{k,5} &= \frac{p^2}{2} \left(2\widehat{\beta}_s^k c_s^{\alpha^k, \varphi^*} 8\alpha_s^k \widehat{\beta}_s^k c_s^{\varphi^*} \right) + p^2 c_s^{S, \alpha^k} \\
&= p^2 \widehat{\beta}_s^k c_s^{\alpha^k, \varphi^*} + 4p^2 \alpha_s^k \widehat{\beta}_s^k c_s^{\varphi^*} + p^2 c_s^{S, \alpha^k} \\
&= p^2 \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} (C_s^k)^{1/2} \frac{p}{12} c_s^{C^k, \varphi^*} + 4p^2 \frac{p}{12} C_s^k \frac{1}{2} (C_s^k)^{-1/2} \left(\frac{\varepsilon p \bar{c}_1}{3} \right)^{1/2} c_s^{C^k, \varphi^*} \\
&\quad + p^2 \frac{p}{12} c_s^{S, C^k} \\
&= \frac{p^{7/2}}{12\sqrt{3}} \varepsilon^{1/2} (\bar{c}_1)^{1/2} (C_s^k)^{1/2} c_s^{C^k, \varphi^*} + \frac{p^{7/2}}{6\sqrt{3}} \varepsilon^{1/2} (\bar{c}_1)^{1/2} (C_s^k)^{1/2} c_s^{C^k, \varphi^*} + \frac{p^3}{12} c_s^{S, C^k} \\
&=: \sum_{j=1}^3 \widehat{H}_s^{k,5,j}
\end{aligned}$$

$\widehat{H}^{k,5,1}$:

$$\begin{aligned}
& \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^5 \left| \widehat{H}_s^{k,5,1} \right| ds \right) \\
& \leq \left(\frac{12\varepsilon}{p} \right)^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} \left(\frac{p^{7/2}}{12\sqrt{3}} \varepsilon^{1/2} (C_s^k)^{1/2} |c_s^{C^k, \varphi^*}| \right) ds \right) \\
& = \left(\frac{12\varepsilon}{p} \right)^{5/4} \frac{p^{7/2}}{12\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} \left((C_s^k)^{1/2} |c_s^{C^k, \varphi^*}| \right) ds \right) \\
& = \frac{2^{1/2}}{3^{1/4}} p^{9/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\
& \leq \text{const. } \varepsilon^{7/4} k_2^{5/4} k_1^{-2}
\end{aligned}$$

 $\widehat{H}^{k,5,2}$:

$$\begin{aligned}
& \mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^5 \left| \widehat{H}_s^{k,5,2} \right| ds \right) \\
& \leq \left(\frac{12\varepsilon}{p} \right)^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} \left| \frac{p^{7/2}}{6\sqrt{3}} \varepsilon^{1/2} (C_s^k)^{1/2} c_s^{C^k, \varphi^*} \right| ds \right) \\
& = \left(\frac{12\varepsilon}{p} \right)^{5/4} \frac{p^{7/2}}{6\sqrt{3}} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} \left| (C_s^k)^{1/2} c_s^{C^k, \varphi^*} \right| ds \right) \\
& = \frac{2^{3/2}}{3^{1/4}} p^{9/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} |c_s^{C^k, \varphi^*}| ds \right) \\
& \leq \text{const. } \varepsilon^{7/4} k_2^{5/4} k_1^{-2}
\end{aligned}$$

 $\widehat{H}^{k,5,3}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^5 \left| \widehat{H}_s^{k,5,3} \right| ds \right) & \leq \left(\frac{12\varepsilon}{p} \right)^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} \left| \frac{p^3}{12} c_s^{S, C^k} \right| ds \right) \\
& = \left(\frac{12\varepsilon}{p} \right)^{5/4} \frac{p^3}{12} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} |c_s^{S, C^k}| ds \right) \\
& = (12)^{1/4} p^{17/4} \varepsilon^{5/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-5/4} |c_s^{S, C^k}| ds \right) \\
& \leq \text{const. } \varepsilon^{5/4} k_2^{7/4} k_1^{-2}
\end{aligned}$$

 $i = 6$:

$$\begin{aligned}
\widehat{H}_s^{k,6} & = \frac{p^2}{2} 16\alpha_s^2 c_s^{\varphi^*, \varphi^*} - \frac{p^2}{2} 2c_s^{\alpha^k, \widehat{\beta}^k} \\
& = 8p^2 \left(\frac{p}{12} C_s^k \right)^2 c_s^{\varphi^*, \varphi^*} - p^2 \frac{p}{12} \left(\frac{\varepsilon p}{3} \right)^{1/2} \frac{1}{2} (C_s^k)^{-1/2} (\bar{c}_1)^{1/2} c_s^{C^k} \\
& = \frac{1}{18} p^4 (C_s^k)^2 c_s^{\varphi^*, \varphi^*} - \frac{1}{24\sqrt{3}} p^{7/2} \varepsilon^{1/2} (\bar{c}_1)^{1/2} (C_s^k)^{-1/2} c_s^{C^k} \\
& =: \widehat{H}_s^{k,6,1} + \widehat{H}_s^{k,6,2}
\end{aligned}$$

A. Analyzing error terms

$\widehat{H}^{k,6,1}$:

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^6 \left| \widehat{H}_s^{k,6,1} \right| ds \right) &\leq \left(\frac{12\varepsilon}{p} \right)^{3/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/2} \left| \frac{1}{18} p^4 (C_s^k)^2 c_s^{\varphi^*, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/2} \frac{1}{18} p^4 \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/2} \left| (C_s^k)^2 c_s^{\varphi^*, \varphi^*} \right| ds \right) \\
&= \frac{4}{3^{1/2}} p^{5/2} \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{1/2} c_s^{\varphi^*, \varphi^*} ds \right) \\
&\leq \text{const. } \varepsilon^{3/2} k_1^{-1/2}
\end{aligned}$$

$\widehat{H}^{k,6,2}$:

$$\begin{aligned}
&\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^6 \left| \widehat{H}_s^{k,6,1} \right| ds \right) \\
&\leq \left(\frac{12\varepsilon}{p} \right)^{3/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/2} \left| \frac{1}{24\sqrt{3}} p^{7/2} \varepsilon^{1/2} (C_s^k)^{-1/2} c_s^{C^k} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{3/2} \frac{1}{24\sqrt{3}} p^{7/2} \varepsilon^{1/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/2} \left| (C_s^k)^{-1/2} c_s^{C^k} \right|^2 ds \right) \\
&= p^2 \varepsilon^{3/2} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\
&\leq \text{const. } \varepsilon^{3/2} k_2^3 k_1^{-4}
\end{aligned}$$

$i = 7$:

$$\widehat{H}_s^{k,7} = -\frac{p^2}{2} 8 \alpha_s^k c_s^{\alpha^k, \varphi^*} = -4p^2 \frac{p}{12} C_s^k \frac{p}{12} c_s^{C^k, \varphi^*} = -\frac{p^3}{36} C_s^k c_s^{C^k, \varphi^*}$$

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta\varphi_s^{+,k})^7 \left| \widehat{H}_s^{k,7} \right| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^{7/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-7/4} \left| \frac{p^3}{36} C_s^k c_s^{C^k, \varphi^*} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^{7/4} \frac{p^3}{36} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-7/4} \left| C_s^k c_s^{C^k, \varphi^*} \right| ds \right) \\
&= \frac{4}{3^{1/4}} p^{5/4} \varepsilon^{7/4} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-3/4} \left| c_s^{C^k, \varphi^*} \right| ds \right) \\
&\leq \text{const. } \varepsilon^{7/4} k_2^{5/4} k_1^{-2}
\end{aligned}$$

$i = 8$:

$$\widehat{H}_s^{k,8} = \frac{p^2}{2} c_s^{\alpha^k} = \frac{p^2}{2} \left(\frac{p}{12} \right)^2 c_s^{C^k} = \frac{p^4}{288} c_s^{C^k}$$

$$\begin{aligned}
\mathbb{E}_Q \left(\int_0^T (\Delta \varphi_s^{+,k})^8 \left| \widehat{H}_s^{k,8} \right| ds \right) &= \left(\frac{12\varepsilon}{p} \right)^2 \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} \left| \frac{p^4}{288} c_s^{C^k} \right| ds \right) \\
&= \left(\frac{12\varepsilon}{p} \right)^2 \frac{p^4}{288} \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\
&= \frac{p^2}{2} \varepsilon^2 \mathbb{E}_Q \left(\int_0^T (C_s^k)^{-2} c_s^{C^k} ds \right) \\
&\leq \text{const. } \varepsilon^2 k_2^3 k_1^{-4}
\end{aligned}$$

For the calculations we used

$$k_2^{-1} \leq C_s^k \leq k_1^{-1} \quad \text{for all } s \in [0, T]$$

and for $i = 0$ we make use of Proposition 3.6.2. For $\widehat{H}_{k,2,1}$ we use Equation 4.2.2 in Remark 4.2.4. $\widehat{H}_{k,3,4}$ uses Assumption 3.6.3. Since A is bounded, the expression in $\widehat{H}_{k,2,3}$ is finite as before. For the remaining terms we used the same combination of the Kunita-Watanabe-inequality (Theorem B.2.1) and Hölder's inequality from section A.1 but, in contrast, we only require the $L_1(Q)$ -norm of $V([X, \widetilde{X}]_T)$ for $X, \widetilde{X} \in \{S, \varphi^*, C^k, Y^k\}$:

$$\begin{aligned}
\left\| V(X, \widetilde{X})_T \right\|_{L_1(Q)} &\leq \left\| [X]_T^{1/2} [\widetilde{X}]_T^{1/2} \right\|_{L_1(Q)} \\
&\leq \left\| [X]_T \right\|_{L_1(Q)}^{1/2} \left\| [\widetilde{X}]_T \right\|_{L_1(Q)}^{1/2},
\end{aligned}$$

which then can be analyzed with the integrability assumptions in Assumption 3.6.3 and equation 4.2.3 in Remark 4.2.4.

B. Tools from stochastic calculus

We require two extended versions of Itô's formula that allow for functions not being twice continuously differentiable.

B.1. Change-of-variable formulas

B.1.1. For piecewise twice continuously differentiable functions with smooth fit

The verifications of the asymptotic upper bound of maximal expected utility from terminal wealth in section 3.7 and subsections 4.2.2 and 4.3.2 make use of a change-of-variable formula, where the function under consideration is just twice continuously differentiable between some surfaces and fulfill some smooth fit on the surfaces. In our case, these surfaces are the boundaries of the no trade region.

Since the portfolio process of admissible strategy in the market with frictions has left-continuous jumps at the transaction times of the strategy, we have to allow for processes of that kind. So we require the notion of a left-continuous jump process J , that is a process of the form

$$J = J_0 + \sum_{i=1}^{\infty} Y_n \mathbf{1}_{(\tau_n, \infty)},$$

where $(\tau_n)_{n \in \mathbb{N}}$ is a sequence of strictly increasing stopping times with $\lim_{n \rightarrow \infty} \tau_n = \infty$ and $(Y_n)_{n \in \mathbb{N}}$ is a sequence of random variables such that Y_n is \mathcal{F}_{τ_n} -measurable. Furthermore, we recall from (Karatzas and Shreve, 1991, Definition III.3.1) that an adapted process Z with decomposition

$$Z = Z_0 + V^Z + M^Z$$

with M^Z being a continuous local martingale and V^Z being a continuous and adapted process of bounded variation is called continuous semimartingale. All considered processes are the sum of the two types from above.

Theorem B.1.1. *Let $X = (X^1, \dots, X^n)$ be an \mathbb{R}^d -valued process, such that each X^i is of the form*

$$X^i = Z^i + J^i$$

where Z^i is a continuous semimartingale and J^i is a left continuous jump process. Let $b : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, $F_1, F_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice continuously differentiable functions with the following smooth fit conditions:

$$\begin{aligned} \partial_n F_1(x_1, \dots, x_n+) &= \partial_n F_2(x_1, \dots, x_n-) && \text{for } x \in \mathbb{R}^n \text{ with } x_n = b(x_1, \dots, x_{n-1}) \text{ and} \\ \partial_n F_1(x_1, \dots, x_n-) &= \partial_n F_2(x_1, \dots, x_n+) && \text{for } x \in \mathbb{R}^n \text{ with } x_n = -b(x_1, \dots, x_{n-1}). \end{aligned}$$

B. Tools from stochastic calculus

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$F(x) = \begin{cases} F_1(x), & \text{if } |x_n| > b(x_1, \dots, x_{n-1}), \\ F_2(x), & \text{if } |x_n| \leq b(x_1, \dots, x_{n-1}) \end{cases}$$

be a continuous function. Then, the following version of the change-of-variable formula holds:

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=1}^n \int_0^t \frac{1}{2} (\partial_i F(X_s^1, \dots, X_s^n +) + \partial_i F(X_s^1, \dots, X_s^n -)) dZ_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{1}{2} (\partial_{ij}^2 F(X_s^1, \dots, X_s^n +) + \partial_{ij}^2 F(X_s^1, \dots, X_s^n -)) d[Z^i, Z^j]_s \\ &\quad + \sum_{0 \leq s < t} (F(X_{s+}) - F(X_s)). \end{aligned}$$

Proof. (i) In the case $J^i = 0$ for all $i = 1, \dots, n$ we can apply (Peskir, 2007, Remark 2.2) choosing $m = 2, b_1 = -b, b_2 = b, F_3 = F_1$. (The local time terms vanish due to the assumed boundary behavior of $\partial_n F_1$ and $\partial_n F_2$.)

(ii) Otherwise we use a similar argument as in (Irle, 2012, section 16.14): Let $(T_k)_{k \in \mathbb{N}}$ be the jump times of the process (J^1, \dots, J^n) . By applying (Peskir, 2007, Remark 2.2) on (Z^1, \dots, Z^n) we obtain for $T_{k-1} < s < t < T_k$

$$\begin{aligned} F(X_t) - F(X_s) &= \sum_{i=1}^n \int_s^t \frac{1}{2} (\partial_i F(X_u^1, \dots, X_u^n +) + \partial_i F(X_u^1, \dots, X_u^n -)) dZ_u^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_s^t \frac{1}{2} (\partial_{ij}^2 F(X_u^1, \dots, X_u^n +) + \partial_{ij}^2 F(X_u^1, \dots, X_u^n -)) d[Z^i, Z^j]_u. \end{aligned}$$

For s as above we have $X_{s+} = X_s$. Together with the right-continuity of $X_{\cdot+}$ we obtain

$$\begin{aligned} &F(X_t) - F(X_{T_{k-1}+}) \\ &= \sum_{i=1}^n \int_{T_{k-1}}^t \frac{1}{2} (\partial_i F(X_u^1, \dots, X_u^n +) + \partial_i F(X_u^1, \dots, X_u^n -)) dZ_u^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_{T_{k-1}}^t \frac{1}{2} (\partial_{ij}^2 F(X_u^1, \dots, X_u^n +) + \partial_{ij}^2 F(X_u^1, \dots, X_u^n -)) d[Z^i, Z^j]_u \end{aligned}$$

and due to the left-continuity of X the same formula for $t = T_k$. With $T_0 = 0$ this

yields for $t \in (T_{k-1}, T_k)$:

$$\begin{aligned}
 & F(X_t) - F(X_0) \\
 &= F(X_t) - F(X_{T_{k-1}+}) + \sum_{j=1}^{k-1} (F(X_{T_j+}) - F(X_{T_{j-1}+})) + F(X_{0+}) - F(X_0) \\
 &= F(X_t) - F(X_{T_{k-1}+}) + \sum_{j=1}^{k-1} (F(X_{T_j}) - F(X_{T_{j-1}+})) \\
 &\quad + \sum_{j=1}^{k-1} (F(X_{T_j+}) - F(X_{T_j})) + F(X_{0+}) - F(X_0) \\
 &= \sum_{i=1}^n \int_0^t \frac{1}{2} (\partial_i F(X_s^1, \dots, X_s^n+) + \partial_i F(X_s^1, \dots, X_s^n-)) dZ_s^i \\
 &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{1}{2} (\partial_{ij}^2 F(X_s^1, \dots, X_s^n+) + \partial_{ij}^2 F(X_s^1, \dots, X_s^n-)) d[Z^i, Z^j]_s \\
 &\quad + \sum_{0 \leq s < t} (F(X_{s+}) - F(X_s)).
 \end{aligned}$$

So we obtain the desired formula for all $t \in \cup_{k \in \mathbb{N}} (T_{k-1}, T_k)$ and due to the left continuity of both sides for all $t \geq 0$. \square

Remark B.1.2. *With a regular version of Itô's formula from (Karatzas and Shreve, 1991, Theorem III.3.3) and the same argument from the previous proof one obtains a change-of-variable formula for a twice continuously differentiable function F and a process $X = Z + J$ from above:*

$$\begin{aligned}
 F(X_t) &= F(X_0) + \sum_{i=1}^n \int_0^t \partial_i F(X_s^1, \dots, X_s^n) dZ_s^i \\
 &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \partial_{ij}^2 F(X_s^1, \dots, X_s^n) d[Z^i, Z^j]_s + \sum_{0 \leq s < t} (F(X_{s+}) - F(X_s)).
 \end{aligned}$$

B.1.2. For linear combinations of convex functions

Another generalization of Itô's formula is a version for convex functions which are not necessarily twice differentiable. Therefore, we introduce the left-derivative for a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$D^- f(x) := \lim_{h \searrow 0} \frac{1}{h} (f(x+h) - f(x)),$$

which exists and is finite for all $x \in \mathbb{R}$ according to (Karatzas and Shreve, 1991, Problem 3.6.19). This definition can, of course, be extended to linear combinations of convex functions. Furthermore, we require the second derivative measure μ^f on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\mu^f([a, b]) := D^- f(b) - D^- f(a), \quad -\infty < a < b < \infty,$$

B. Tools from stochastic calculus

that fulfills $d\mu^f(x) = f''(x)dx$ if f is twice differentiable (see (Karatzas and Shreve, 1991, Problem 3.6.21)). Again, this definition can be extended to linear combinations of convex functions, but then μ^f is a signed measure being finite on each bounded interval according to (Karatzas and Shreve, 1991, Corollary 3.7.2). Now, we can state the generalized change of variable formula for continuous semimartingales:

Theorem B.1.3. *Let X be a continuous semimartingale. There exists a semimartingale local time for X , i.e. a family $(L(a))_{a \in \mathbb{R}}$ of nonnegative stochastic processes on $\Omega \times [0, T]$ with the following properties:*

1. *The mapping $(t, a, \omega) \mapsto L_t(a, \omega)$ is measurable, and $L(a)$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ for all $a \in \mathbb{R}$.*
2. *For all $a \in \mathbb{R}$, $L(a)$ is a nondecreasing and continuous process moving only on the set $\{X = a\}$, i.e. more precisely*

$$\int_0^T \mathbf{1}_{\mathbb{R} \setminus \{a\}}(X_t) dL_t(a) = 0 \quad \text{almost surely.}$$

3. *For every Borel-measurable function $h : \mathbb{R} \rightarrow [0, \infty)$ the formula*

$$\int_0^t h(X_s) d[X]_s = 2 \int_{-\infty}^{\infty} h(a) L_t(a) da$$

holds almost surely for all $t \in [0, T]$.

4. *For every linear combination of convex functions $f : \mathbb{R} \rightarrow \mathbb{R}$ we have*

$$f(X_t) = f(X_0) + \int_0^t D^- f(X_s) dX_s + \int_{-\infty}^{\infty} L_t(a) d\mu^f(a)$$

for all $t \in [0, T]$ almost surely.

Proof. (Karatzas and Shreve, 1991, Theorem 3.7.1) and (Karatzas and Shreve, 1991, Corollary 3.7.2) □

According to this theorem, $f(X)$ is again a semimartingale if f is a linear combination of convex functions and X is a semimartingale. Then, its drift is equal to $V^{f(X)} = \int_0^t D^- f(X_s) dV_s^X + \int_{-\infty}^{\infty} L_t(a) d\mu^f(a)$ and the martingale part is $M^{f(X)} = \int_0^t D^- f(X_s) dM_s^X$. If X is an Itô process, we write $b^{f(X)} = \int_0^t D^- f(X_s) b_s^X ds$ for the absolutely continuous part of the drift.

B.2. (Semi-)Martingale Inequalities

Theorem B.2.1 (Kunita-Watanabe Inequality). *Let M and N be square integrable martingales, $X \in L(M)$ and $Y \in L(N)$. Then*

$$\int_0^t |X_s Y_s| dV_s([M, N]) \leq \left(\int_0^t X_s^2 d[M]_s \right)^{1/2} \left(\int_0^t Y_s^2 d[N]_s \right)^{1/2}, t \geq 0.$$

Proof. (Karatzas and Shreve, 1991, Proposition 2.14). \square

Theorem B.2.2 (Burkholder-Davis-Gundy Inequalities). *Let M be a continuous local martingale. For every $m > 0$ there exist universal constants c_m, C_m (depending only on m), such that*

$$c_m \mathbb{E}([M]_\tau^m) \leq \mathbb{E} \left(\sup_{s \in [0, \tau]} |M_s|^{2m} \right) \leq C_m \mathbb{E}([M]_\tau^m)$$

holds for all stopping times τ .

Proof. (Karatzas and Shreve, 1991, Theorem 3.3.28). \square

A useful tool are the S - and H -norms of semimartingales from (Protter, 2004, section V.2)¹, that we require for Itô processes: Let $X = X_0 + \int_0^\cdot b_s^X ds + M^X$ be an Itô process. Then for $p \in [1, \infty)$ we call

$$\|X\|_{S_p(P)} := \left\| \sup_{s \in [0, T]} |X_s| \right\|_{L_p(P)}$$

S_p -norm of X and

$$\|X\|_{H_p(P)} := |X_0| + \left\| \int_0^T |b_s^X| ds \right\|_{L_p(P)} + \left\| [M^X]_T^{1/2} \right\|_{L_p(P)}$$

the H_p -norm of X . There is a relation between these norms that can be verified with Theorem B.2.2.

Theorem B.2.3 (S- and H-Norms). *Let X be an Itô process. For every $p \in [1, \infty)$ there exists a universal constant $\widetilde{C}_p > 0$ (depending only on p) such that*

$$\|X\|_{S_p(P)} \leq \widetilde{C}_p \|X\|_{H_p(P)}.$$

Proof. According to Theorem B.2.2 we have

$$\begin{aligned} \|X\|_{S_p(P)} &= \|X - X_0 + X_0\|_{S_p(P)} \leq \|X - X_0\|_{S_p(P)} + |X_0| \\ &\leq \left\| \sup_{t \in [0, T]} \left| \int_0^t b_s^X ds \right| + \sup_{t \in [0, T]} |M_t^X| \right\|_{L_p(P)} + |X_0| \\ &\leq \left\| \sup_{t \in [0, T]} \left| \int_0^t b_s^X ds \right| \right\|_{L_p(P)} + \left\| \sup_{t \in [0, T]} |M_t^X| \right\|_{L_p(P)} + |X_0| \\ &\leq \left\| \int_0^T |b_s^X| ds \right\|_{L_p(P)} + C_{p/2}^{1/p} \left\| [M^X]_T^{1/2} \right\|_{L_p(P)} + |X_0| \leq \widetilde{C}_p \|X\|_{H_p(P)} \end{aligned}$$

for the constant $\widetilde{C}_p = \max\{C_{p/2}, 1\}$. \square

¹In this reference, a slightly different H -norm is used which is equivalent to the H -norm used here.

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List of Figures

3.1.	Correction term f^{CE} as a function of the deviation from the frictionless target	25
3.2.	Value processes as functions of difference in portfolio values	30
3.3.	Comparison of error terms f^{CE} and its modified version \hat{f}^{CE}	31
5.1.	Mean time to the next transaction. The fixed parameters are given by $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$ and $\varepsilon = 1\text{€}$	86
5.2.	Certainty equivalent loss as a function of the transaction costs with parameters $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$ and $T = 1$	88
5.3.	Simulation in time of the candidate strategy with parameters $\mu = 0.05$, $\sigma = 0.2$, $p = 1.25 \cdot 10^{-6}$, $T = 1$, $S_0 = 1000\text{€}$ and $\varepsilon = 5\text{€}$	88
6.1.	Asymptotic utility indifference price and frictionless option price of a put option in the Black-Scholes model with $S_0 = 100$, $K = 100$, $\sigma = 0.2$ and $\varepsilon = 5\text{€}$ as a function of risk aversion	102
7.1.	Correction term $f^{CE,\varepsilon,\lambda}$ as a function of the deviation from the frictionless target	107

List of Symbols

\mathbb{N}	the integers
\mathbb{R}	the real numbers
$x \wedge y, x \vee y$	$\min\{x, y\}, \max\{x, y\}$ for $x, y \in \mathbb{R}$
\mathcal{F}, \mathcal{G}	σ -fields
$(\mathcal{F}_t)_{t \geq 0}, (\mathcal{G}_t)_{t \geq 0}$	filtrations
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$	filtered probability space
$\mathbb{E}(X), \mathbb{E}_{\tilde{P}}(X)$	expectation of X under P , expectation under a probability measure \tilde{P}
$\mathbb{E}(X \mathcal{G}), \mathbb{E}_{\tilde{P}}(X \mathcal{G})$	conditional expectation (under \tilde{P}) of X given \mathcal{G}
$L_p(\tilde{P})$	the set of random variables with $\mathbb{E}_{\tilde{P}}(X ^p) < \infty$ for $p \in [1, \infty)$
P, P^H	(physical) probability measures
Q, Q^H	equivalent martingale measures
$\frac{d\tilde{P}}{dP}$	the Radon-Nikodym derivative of \tilde{P} w.r.t. P
$L(X)$	The set of integrands for the stochastic integral with respect to a continuous semimartingale X
$V^X, V^{X, \tilde{P}}$	the drift part (under \tilde{P}) of a continuous semimartingale X
$b^X, b^{X, \tilde{P}}$	the (\tilde{P}) -local drift of an Itô-process X
$M^X, M^{X, \tilde{P}}$	the martingale part (under \tilde{P}) of a continuous semimartingale X
$[X, Y]$	the quadratic covariation of two continuous semimartingales X, Y
$[X]$	the quadratic variation of a continuous semimartingale X

List of Figures

$c^{X,Y}$	the local quadratic covariation of two continuous semimartingales X, Y
c^X	the local quadratic variation of a continuous semimartingales X
$\mathcal{E}(X)$	the stochastic exponential $\exp(X - X_0 - \frac{1}{2}[X])$ of a continuous semimartingale X
$\Delta^+ X$	the left jump of a process: $\Delta^+ X_t = X_{t+} - X_t$
$(\tilde{v}^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}, (\widehat{v}^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$	the approximation to the value process with frictions
$(v(\varphi))_{\varphi \in \mathfrak{A}_2}$	the value process for the frictionless problem (Chapter 2) and the semi-frictional value process in Chapter 3
$(v^\varepsilon(\varphi))_{\varphi \in \mathfrak{A}^\varepsilon}$	the value process for the market with frictions
$\Delta\varphi^+, \Delta\widehat{\varphi}^+, \Delta\varphi^{+,k}, \Delta\widehat{\varphi}^{+,k}$	halfwidths of the (modified and truncated) no-trade regions
$F^{CE}(\varphi), \widehat{F}^{CE}(\varphi)$	correction terms in the presence of transaction costs in Chapter 3
$F^{CE,k}(\varphi), \widehat{F}^{CE,k}(\varphi)$	correction terms in the presence of transaction costs in Chapter 4
\mathfrak{A}_1	admissible strategies in the frictionless market (smaller class)
\mathfrak{A}_2	admissible strategies in the frictionless market (larger class)
\mathfrak{A}^ε	admissible strategies in the presence of fixed transaction costs

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation – abgesehen von der Beratung durch meinen Betreuer Herrn Prof. Dr. Jan Kallsen – nach Inhalt und Form eigenständig angefertigt habe. Dabei habe ich die Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft eingehalten. Die Arbeit hat weder ganz noch zum Teil einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen und ist weder ganz noch zum Teil veröffentlicht oder zur Veröffentlichung eingereicht worden.

Kiel, den 28.07.2016