

Polynomial representations of G_rT ,
simple polynomial modules and
polynomial parts of Auslander-Reiten
components

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von
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Kiel, 2016

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Tag der mündlichen Prüfung: 21.10.2016

Zum Druck genehmigt: 21.10.2016

gez. Prof. Dr. Wolfgang J. Duschl, **Dekan**

Zusammenfassung

Doty, Nakano und Peters definierten 1996 eine Familie endlich dimensionaler Algebren, die sogenannten infinitesimalen Schur-Algebren, um polynomielle Darstellungen von $G_r T$ zu untersuchen, wobei G_r der r -te Frobeniuskern der allgemeinen linearen Gruppe GL_n und $T \subseteq \mathrm{GL}_n$ der maximale Torus der Diagonalmatrizen über einem Körper k positiver Charakteristik p sind. In weiteren Arbeiten führte Doty polynomielle Darstellungen in allgemeinem Rahmen ein und nutzte dies zum Studium polynomieller Darstellungen anderer reductiver Gruppen G .

Farnsteiner folgte einem anderen Ansatz zur Untersuchung von $G_r T$ -Moduln für reductive Gruppen G , indem er sie als $X(T) \cong \mathbb{Z}^n$ -graduierte G_r -Moduln ansah, wobei $X(T)$ die Charaktergruppe des maximalen Torus T von G ist. Da die Kategorie der \mathbb{Z}^n -graduierten G_r -Moduln fast zerfallende Folgen hat, erlaubt dies das Studium unzerlegbarer $G_r T$ -Moduln und ihrer Morphismen mittels Auslander-Reiten-Theorie.

In dieser Arbeit verbinden wir beide Ansätze, indem wir die allgemeine Definition von Doty nutzen, um polynomielle Darstellungen von $G_r T$ zu betrachten, wobei G eine reductive Gruppe oder eine Borel-Untergruppe einer solchen Gruppe ist, und diese als volle Unterkategorie der Kategorie der \mathbb{Z}^n -graduierten G_r -Moduln ansehen. Es ist unser Ziel, Resultate über GL_n auf diesen Fall zu verallgemeinern und das Wechselspiel zwischen der Auslander-Reiten-Theorie von $\mathrm{mod} G_r T$ und der infinitesimalen Schur-Algebren zu untersuchen, indem wir den polynomiellen Teil von Komponenten des stabilen Auslander-Reiten-Köchers $\Gamma_s(G_r T)$ der Kategorie der endlich dimensionalen $G_r T$ -Moduln betrachten. Wir geben hinreichende Bedingungen an die Gruppe G , die eine Klassifikation der einfachen polynomiellen Moduln ähnlich wie im Fall $G = \mathrm{GL}_n$ ermöglichen und nutzen dies, um einen neuen Beweis des entsprechenden Resultats für GL_n zu geben und es auf die Gruppe der symplektischen Ähnlichkeitsabbildungen, Levi-Untergruppen von GL_n und, in schwächerer Form, die Gruppe der orthogonalen Ähnlichkeitsabbildungen in ungerader Dimension zu verallgemeinern. Wir zeigen, dass Komponenten $\Theta \subseteq \Gamma_s(G_r T)$ der Komplexität 1 nur endlich viele polynomielle Moduln enthalten. Ist G die Borel-Untergruppe der oberen Dreiecksmatrizen von GL_2 und $T \subseteq \mathrm{GL}_2$ der Torus der Diagonalmatrizen, so können wir dieses Resultat of Moduln, die ein äußeres Tensorprodukt $M \otimes_k N$ mit einem Modul M der Komplexität 1 sind, ausdehnen. In dieser Situation korrespondieren $G_r T$ -Moduln zu Moduln über elementar abelschen p -Gruppen vom Rang r . Als Anwendung zeigen wir, dass die Komponenten einfacher Moduln in diesem Fall nur endlich viele polynomielle Darstellungen enthalten. Im Fall $G = \mathrm{GL}_2$, $r = 1$ nutzen wir Premets Klassifikation der unzerlegbaren Moduln über der restringierten einhüllenden Algebra $U_0(\mathfrak{sl}_2)$ von \mathfrak{sl}_2 , um den polynomiellen Teil der Komponenten von $\Gamma_s(G_1 T)$ sowie den Auslander-Reiten-Köcher der zugehörigen infinitesimalen Schur-Algebren vollständig zu beschreiben.

Abstract

In 1996, Doty, Nakano and Peters defined a family of finite-dimensional algebras, the so-called infinitesimal Schur algebras, to study polynomial representations of G_rT , where G_r is the r -th Frobenius kernel of the general linear group GL_n and $T \subseteq \mathrm{GL}_n$ the torus of diagonal matrices over a field k of positive characteristic p . In subsequent work, Doty gave a general framework for polynomial representations and applied this to study polynomial representations of other reductive groups.

Farnsteiner followed a different approach to G_rT -modules for reductive groups G , viewing them as $X(T) \cong \mathbb{Z}^n$ -graded G_r -modules, where $X(T)$ is the character group of the maximal torus T of G . Since the category of \mathbb{Z}^n -graded G_r -modules possesses almost split sequences, this allows the study of indecomposable G_rT -modules and their morphisms via Auslander-Reiten theory.

In this thesis, we use the general framework given by Doty to consider polynomial representations of G_rT for G either a reductive group or a Borel subgroup of such a group and view the category of polynomial G_rT -modules as a full subcategory of the category of $X(T)$ -graded G_r -modules, combining the aforementioned approaches. It is our aim to extend results on GL_n to this case and to study the interplay of the Auslander-Reiten theory of $\mathrm{mod} G_rT$ and of the infinitesimal Schur algebras by considering the polynomial parts of components of the stable Auslander-Reiten quiver $\Gamma_s(G_rT)$ of the category of finite-dimensional G_rT -modules. We give sufficient conditions on G making a classification of simple polynomial G_rT -modules similar to the case $G = \mathrm{GL}_n$ possible and apply this to recover the corresponding result for GL_n with a different proof, extending it to symplectic similitude groups, Levi subgroups of GL_n and, in a weaker form, to odd orthogonal similitude groups. For components $\Theta \subseteq \Gamma_s(G_rT)$ of complexity 1, we show that Θ only contains finitely many polynomial G_rT -modules. If G is the Borel subgroup of upper triangular matrices of GL_2 and $T \subseteq G$ the torus of diagonal matrices, we extend this result to modules arising as outer tensor $M \otimes_k N$, where M has complexity 1. In this situation, G_rT -modules correspond to graded modules over an elementary abelian p -group of rank r . As an application, we show that the components of simple modules contain only finitely many polynomial modules in this case. For $G = \mathrm{GL}_2$, $r = 1$, we use Premets classification of indecomposable modules for the restricted enveloping algebra $U_0(\mathfrak{sl}_2)$ of \mathfrak{sl}_2 to fully determine the polynomial parts of the components of $\Gamma_s(G_1T)$ and the Auslander-Reiten quiver of the corresponding infinitesimal Schur algebras.

Acknowledgment

I would like to thank my advisor Rolf Farnsteiner for his constant support and all members of the representation theory group at Kiel for fruitful discussions and for providing a great working environment.

I would also like to thank Steve Doty for answering my questions at the conference on algebraic groups in Jena in 2014 and for pointing out helpful literature.

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1 Introduction and preliminaries

1.1 Introduction

When considering rational representations of $G = \mathrm{GL}_n$ over an infinite field, one can reduce many questions to polynomial representations. These representations correspond to representations of certain finite-dimensional algebras, the so-called Schur algebras (see for example [23, 29]). Letting T be a maximal torus of G and k be an infinite field of positive characteristic, one can also study the category of $G_r T$ -modules to get information on the representation theory of G (see [26, II.9]). In [9], Doty, Nakano and Peters combined these approaches and considered polynomial representations of $G_r T$ for $G = \mathrm{GL}_n$. They proved that these representations correspond to representations of the so-called infinitesimal Schur algebras which are subalgebras of the ordinary Schur algebras. In [5], Doty defined a general notion of polynomial representations and analogues of Schur algebras for algebraic groups contained in GL_n . Using his definition, one can study polynomial representations of $G_r T$ for other algebraic groups G .

In [18], Farnsteiner studied the Auslander-Reiten theory of $\mathrm{mod} G_r T$ by considering $G_r T$ -modules as $X(T)$ -graded G_r -modules, where $X(T) \cong \mathbb{Z}^n$ is the character group of T . For reductive G , he proved that every component of the stable Auslander-Reiten quiver $\Gamma_s(G_r T)$ of $\mathrm{mod} G_r T$ is isomorphic to either $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$ or $\mathbb{Z}[D_\infty]$ and that all components of complexity $\neq 2$ are of type $\mathbb{Z}[A_\infty]$.

In this thesis, we follow these approaches and study the category of polynomial representations of $G_r T$ as a full subcategory of the category of \mathbb{Z}^n -graded G_r -modules. Here, $G \subseteq \mathrm{GL}_n$ is either reductive or a Borel subgroup of a reductive group and contains the center $Z(\mathrm{GL}_n)$ of GL_n . It is our goal to extend results obtained in the case $G = \mathrm{GL}_n$ to this more general situation and to study the relative Auslander-Reiten theory of the category of polynomial $G_r T$ -modules, that is the polynomial parts of components of the stable Auslander-Reiten quiver $\Gamma_s(G_r T)$ of $\mathrm{mod} G_r T$.

If G is a reductive group and T is a maximal torus of G , general theory provides a simple $G_r T$ -module $\widehat{L}_r(\lambda)$ for every element $\lambda \in X(T)$ of the character group $X(T)$ of T and every simple $G_r T$ -module arises this way. It is a natural question for which λ these modules are polynomial $G_r T$ -modules and lift to some infinitesimal Schur algebra. For $G = \mathrm{GL}_n$, this problem was solved in [9], but the proof does not translate to other reductive groups. We give sufficient conditions on G making a classification of simple polynomial $G_r T$ -modules similar to this case possible. This allows us to rederive the corresponding result for GL_n with a different proof and extend it to symplectic similitude groups and Levi subgroups of GL_n . A weak form giving conditions on weights of simple polynomial $G_r T$ -modules is proved for odd orthogonal similitude groups. Our method

does not apply to all reductive groups and we illustrate this by considering the example of even orthogonal similitude groups. By studying orbits of the appropriate weights under the dot-action of the affine Weyl group, we construct a Morita-equivalence between certain blocks of polynomial G_rT -modules for $G = \mathrm{GL}_n$.

If $\Theta \subseteq \Gamma_s(G_rT)$ is a component of the stable Auslander-Reiten quiver of G_rT , we are interested in the shape and the size of the intersection of Θ with the category of polynomial G_rT -modules. For components of complexity 1, we show that this intersection is always finite. The restrictions of these components to G_r consist of so-called periodic modules and they occur for every reductive group G and every r , see [18, 8.1.5]. For $G = \mathrm{GL}_n$ or $G \subseteq \mathrm{GL}_n$ the Borel subgroup of upper triangular matrices, we show that if a component of complexity 1 contains a polynomial G_rT -module of sufficiently large quasi-length, then the polynomial part of the component contains a unique module of maximal quasi-length.

Since $\mathrm{mod} G_rT$ is a Frobenius category and the category of polynomial representations is not, the polynomial part of Θ also contains natural types of G_rT -modules whose position in Θ one can determine, namely modules which are Ext-projective or have finite projective dimension in the full subcategory of polynomial G_rT -modules. Furthermore, the polynomial part of a component is part of the Auslander-Reiten quiver of an infinitesimal Schur algebra. We use this approach and the classification of indecomposable $U_0(\mathfrak{sl}_2)$ -modules due to Premet ([31]) to fully determine the Auslander-Reiten quiver of the infinitesimal Schur algebras $S_d(G_1T)$ for $G = \mathrm{GL}_2$ and $T \subseteq G$ the torus of diagonal matrices. It turns out that upon deleting projective-injective modules, the underlying directed graphs of all components in this situation are isomorphic to $\mathbb{Z}[A_{2s+1}]/\langle \tau^{2s+1} \rangle$ for some $s \in \mathbb{N}$. By reducing to the case $r = 1$, we are able to determine the polynomial part of all G_rT -components of type $\mathbb{Z}[A_\infty]$ whose restriction to G_r has type $\mathbb{Z}[\tilde{A}_{12}]$ for higher r .

Let B be a Borel subgroup of a reductive group $G \subseteq \mathrm{GL}_n$. We show that in contrast to the case pertaining to GL_n (cf. [9, Section 7]), the algebras $S_d(B_rT)$ are directed quasi-hereditary algebras. For $B \subseteq \mathrm{GL}_2$ the Borel subgroup of lower triangular matrices, we can extend some results about modules of complexity 1 to modules which are outer tensor products of two modules, one of which has complexity 1. This case is of interest since representations of B_rT can be viewed as graded modules over the truncated polynomial ring $k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$, linking them to representations of elementary abelian p -groups of rank r . We also consider the algebra $S_d(B_rT)$ as a subalgebra of $S_d(G_rT)$ and describe the corresponding induction functor via several canonical isomorphisms. The thesis is subdivided as follows.

In the first chapter, we collect the necessary results on \mathbb{Z}^n -graded algebras and their connection to G_rT -modules and give the basic definitions on polynomial representations. In Section 1.3, we provide results on almost split sequences in subcategories in the more general situation of subcategories of the category of finite-dimensional \mathbb{Z}^n -graded modules over an algebra Λ closed with respect to certain operations. Since we do not give an introduction to algebraic groups and Auslander-Reiten theory, we refer the reader to [26] for representations of algebraic groups and G_rT -modules and to [1], [2] for details on Auslander-Reiten theory and representations of associative algebras.

In the second chapter, we consider simple polynomial $G_r T$ -modules and prove our classification result.

In the third chapter, we extend results from [9] to other groups and prove our results about polynomial parts of Auslander-Reiten components.

In the fourth chapter, we turn to polynomial representations of $B_r T$ for a Borel subgroup B of a reductive group $G \subseteq \mathrm{GL}_n$. We prove that $S_d(B_r T)$ is always directed quasi-hereditary and study outer tensor products for $B \subseteq \mathrm{GL}_2$ the subgroup of upper triangular matrices. The last section is independent of the rest of the thesis and deals with Auslander-Reiten components of B_r containing regular modules.

1.2 \mathbb{Z}^n -graded algebras and $G_r T$ -modules

In this section, we are going to establish basic notation and results for \mathbb{Z}^n -graded algebras and $G_r T$ -modules.

Let k be a field and Λ be a finite-dimensional k -algebra. The algebra Λ is called \mathbb{Z}^n -graded if there is a decomposition

$$\Lambda = \bigoplus_{i \in \mathbb{Z}^n} \Lambda_i$$

into k -subspaces such that $\Lambda_i \Lambda_j \subseteq \Lambda_{i+j}$ for all $i, j \in \mathbb{Z}^n$. Letting $\mathrm{mod} \Lambda$ be the category of finite-dimensional Λ -modules, $M \in \mathrm{mod} \Lambda$ is called \mathbb{Z}^n -graded if there is a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}^n} M_i$$

such that $\Lambda_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}^n$. An element $m \in M \setminus \{0\}$ is called homogeneous of degree $i \in \mathbb{Z}^n$ if $m \in M_i$. In that case, we write $\mathrm{deg}(m) = i$ for the degree of m . A Λ -submodule $N \subseteq M$ is called homogeneous if

$$N = \bigoplus_{i \in \mathbb{Z}^n} N \cap M_i.$$

In that case, N and the factor module M/N have a natural structure as \mathbb{Z}^n -graded Λ -modules. If $M = \bigoplus_{i \in \mathbb{Z}^n} M_i$, $N = \bigoplus_{i \in \mathbb{Z}^n} N_i$ are \mathbb{Z}^n -graded modules, then a morphism of \mathbb{Z}^n -graded modules $M \rightarrow N$ is a Λ -linear map $f : M \rightarrow N$ such that $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}^n$. We denote the space of \mathbb{Z}^n -graded morphisms $M \rightarrow N$ by $\mathrm{Hom}_{\mathrm{gr} \Lambda}(M, N)$ and the category of finite dimensional \mathbb{Z}^n -graded Λ -modules by $\mathrm{mod} \mathrm{gr} \Lambda$.

We denote by $F : \mathrm{mod} \mathrm{gr} \Lambda \rightarrow \mathrm{mod} \Lambda$ the forgetful functor. The modules in $F(\mathrm{mod} \mathrm{gr} \Lambda)$ are called gradable. For $i \in \mathbb{Z}^n$, there is a functor $[i] : \mathrm{mod} \mathrm{gr} \Lambda \rightarrow \mathrm{mod} \mathrm{gr} \Lambda$, where $M[i]_j := M_{j-i}$ for all $i \in \mathbb{Z}^n$ and all $M \in \mathrm{mod} \mathrm{gr} \Lambda$ and the morphisms are left unchanged. Then $[i]$ is an auto-equivalence of $\mathrm{mod} \mathrm{gr} \Lambda$ and we have $F \circ [i] = F$ for all $i \in \mathbb{Z}^n$.

The following results about \mathbb{Z}^n -graded algebras were first obtained in [21], [22] for $n = 1$

and it has been known for a long time that they hold for $n > 1$. Proofs for $n > 1$ can be found in [12].

Proposition 1.2.1. (1) *A module $M \in \text{mod gr } \Lambda$ is indecomposable iff $F(M)$ is indecomposable in $\text{mod } \Lambda$.*

(2) *If $M, N \in \text{mod gr } \Lambda$ are indecomposable such that $F(M) \cong F(N)$, there is a unique $\lambda \in \mathbb{Z}^n$ such that $M \cong N[\lambda]$.*

(3) *If $(P_n)_{n \in \mathbb{N}}$ is a minimal projective resolution of M , then $(F(P_n))_{n \in \mathbb{N}}$ is a minimal projective resolution of $F(M)$.*

Theorem 1.2.2. (1) *The category $\text{mod gr } \Lambda$ has almost split sequences.*

(2) *If $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ is an almost split sequence in $\text{mod gr } \Lambda$, then $0 \rightarrow F(M) \rightarrow F(E) \rightarrow F(N) \rightarrow 0$ is almost split in $\text{mod } \Lambda$.*

For $M \in \text{mod gr } \Lambda$, we define the support of M as $\text{supp}(M) = \{\lambda \in \mathbb{Z}^n \mid M_\lambda \neq 0\}$. If $J \subseteq \mathbb{Z}^n$, we denote by $\text{mod}_J \text{gr } \Lambda$ the full subcategory of $\text{mod gr } \Lambda$ whose objects are those $M \in \text{mod gr } \Lambda$ such that $\text{supp}(M) \subseteq J$. Note that $\text{mod}_J \text{gr } \Lambda$ is a torsion class and a torsion-free class in the sense of Section 1.3 below. The following result is useful for extending results about $\text{mod } \Lambda$ to $\text{mod gr } \Lambda$.

Proposition 1.2.3. *Let $J \subseteq \mathbb{Z}^n$ be finite. Then there is a finite-dimensional algebra A and an equivalence of categories $\text{mod}_J \text{gr } \Lambda \rightarrow \text{mod } A$.*

We illustrate the use of 1.2.3 by extending the Auslander-Reiten formula to \mathbb{Z}^n -graded modules. Note that our approach does not show functoriality in M, N of the isomorphisms. Alternatively, the result can also be proved using [28, Theorem 1.1]. We denote by $\tau_{\text{mod gr } \Lambda}$ resp. τ_Λ the Auslander-Reiten translations of $\text{mod gr } \Lambda$ resp. $\text{mod } \Lambda$, see for example [1, IV.2.3] and note that $F \circ \tau_{\text{mod gr } \Lambda} = \tau_\Lambda \circ F$ by 1.2.2.

Theorem 1.2.4. *Let $M, N \in \text{mod gr } \Lambda$. Then there exist isomorphisms*

$$\begin{aligned} \text{Ext}_{\text{mod gr } \Lambda}^1(M, N) &\cong \underline{D}\text{Hom}_{\text{gr } \Lambda}(\tau_{\text{mod gr } \Lambda}^{-1}(N), M) \\ &\cong \overline{D}\text{Hom}_{\text{gr } \Lambda}(N, \tau_{\text{mod gr } \Lambda}(M)), \end{aligned}$$

where D is the standard duality $M \mapsto M^*$.

Proof. Let $J' = \text{supp}(M) \cup \text{supp}(N) \cup \text{supp}(\tau_{\text{mod gr } \Lambda}(M)) \cup \text{supp}(\tau_{\text{mod gr } \Lambda}^{-1}(N))$. As J' is finite, 1.2.3 implies that we can identify the category $\text{mod}_{J'} \text{gr } \Lambda$ of graded modules with support in J' with the category of finite-dimensional modules over some finite-dimensional k -algebra. Thus, there are only finitely many simple modules S_1, \dots, S_l in $\text{mod}_{J'} \text{gr } \Lambda$. Let P_1, \dots, P_l be their projective covers and I_1, \dots, I_l be their injective envelopes in $\text{mod gr } \Lambda$. Set $J = J' \cup \bigcup_{i=1}^l \text{supp}(P_i) \cup \bigcup_{i=1}^l \text{supp}(I_i)$. Then J is finite and by 1.2.3, there is a finite-dimensional algebra A such that $\text{mod}_J \text{gr } \Lambda$ can be identified

with $\text{mod } A$. Since all extensions of M by N have support in J , the Auslander-Reiten formula in $\text{mod } A$ (see e.g. [1, 2.13]) implies

$$\begin{aligned} \text{Ext}_{\text{mod gr } \Lambda}^1(M, N) &\cong \text{Ext}_{\text{mod}_J \text{ gr } \Lambda}^1(M, N) \\ &\cong D\overline{\text{Hom}}_{\text{mod}_J \text{ gr } \Lambda}(\tau_{\text{mod}_J \text{ gr } \Lambda}^{-1}(N), M) \\ &\cong D\overline{\text{Hom}}_{\text{mod}_J \text{ gr } \Lambda}(N, \tau_{\text{mod}_J \text{ gr } \Lambda}(M)). \end{aligned}$$

Thus, we have to show that $\overline{\text{Hom}}_{\text{mod}_J \text{ gr } \Lambda}(\tau_{\text{mod}_J \text{ gr } \Lambda}^{-1}(N), M) \cong \overline{\text{Hom}}_{\text{gr } \Lambda}(\tau_{\text{mod gr } \Lambda}^{-1}(N), M)$ and $\overline{\text{Hom}}_{\text{mod}_J \text{ gr } \Lambda}(N, \tau_{\text{mod}_J \text{ gr } \Lambda}(M)) \cong \overline{\text{Hom}}_{\text{gr } \Lambda}(N, \tau_{\text{mod gr } \Lambda}(M))$. By definition of J , we have $\tau_{\text{mod gr } \Lambda}(M) \cong \tau_{\text{mod}_J \text{ gr } \Lambda}(M)$ in $\text{mod gr } \Lambda$, so that $\text{Hom}_{\text{mod}_J \text{ gr } \Lambda}(N, \tau_{\text{mod}_J \text{ gr } \Lambda}(M)) \cong \text{Hom}_{\text{gr } \Lambda}(N, \tau_{\text{mod gr } \Lambda}(M))$. Suppose that $f = h \circ g : N \rightarrow I \rightarrow \tau_{\text{mod gr } \Lambda}(M)$ is a graded morphism factoring through an injective graded module $I \in \text{mod gr } \Lambda$. Then f factors through the injective envelope $I' \subseteq I$ of $\text{im } g$ in $\text{mod gr } \Lambda$. As $\text{supp}(\text{im } g) \subseteq J$, the construction of J implies that $I' \in \text{mod}_J \text{ gr } \Lambda$. Since $\text{mod}_J \text{ gr } \Lambda$ is a full subcategory of $\text{mod gr } \Lambda$, I' is also the injective envelope of $\text{im } g$ in $\text{mod}_J \text{ gr } \Lambda$. Thus, f factors through an injective module in $\text{mod}_J \text{ gr } \Lambda$. If on the other hand $f = h \circ g$ factors through an injective object in $\text{mod}_J \text{ gr } \Lambda$, f factors through the injective envelope I' of $\text{im } g$ in $\text{mod}_J \text{ gr } \Lambda$. As above, the injective envelope I of $\text{im } g$ in $\text{mod gr } \Lambda$ belongs to $\text{mod}_J \text{ gr } \Lambda$, so that $I = I'$ is also injective in $\text{mod gr } \Lambda$. This shows $\overline{\text{Hom}}_{\text{mod}_J \text{ gr } \Lambda}(N, \tau_{\text{mod}_J \text{ gr } \Lambda}(M)) \cong \overline{\text{Hom}}_{\text{gr } \Lambda}(N, \tau_{\text{mod gr } \Lambda}(M))$. The other isomorphism can be shown analogously. \square

Now let k be an algebraically closed field of characteristic $p > 0$. We want to apply results about \mathbb{Z}^n -graded algebras to the category $\text{mod } G_r T$, where G is a smooth connected algebraic group scheme over k , $T \subseteq G$ is a maximal torus and G_r the r -th Frobenius kernel of G . The next proposition collects some results on this situation in a slightly more general context.

Proposition 1.2.5. *Let H be an affine group scheme, $G, T \subseteq H$ be closed subgroup schemes such that T is a torus and G is infinitesimal and normalized by T . Then the following statements hold:*

- (1) *The category $\text{mod } G \rtimes T$ is equivalent to the category of $X(T)$ -graded G -modules.*
- (2) *The category $\text{mod } GT$ is a sum of blocks of $\text{mod } G \rtimes T$. If $M \in \text{mod } GT$, $\lambda \in X(T)$, then $M[\lambda] \in \text{mod } GT$ iff λ is trivial on $G \cap T$.*
- (3) *The forgetful functor induces a morphism of stable translation quivers $\Gamma_s(GT) \rightarrow \Gamma_s(G)$ mapping components of $\Gamma_s(G_r T)$ to components of $\Gamma_s(G_r)$.*

Proof. This can be proved as [16, 2.1, 2.2]. \square

1.3 Almost split sequences in subcategories

In this section, we want to establish basic results on almost split sequences and Auslander-Reiten theory in subcategories. This allows us to treat some results in the

following sections in a uniform way. Let Λ be a finite-dimensional \mathbb{Z}^n -graded algebra and \mathcal{T}, \mathcal{F} be full subcategories of $\text{mod gr } \Lambda$ closed with respect to finite direct sums. Recall that $(\mathcal{T}, \mathcal{F})$ is called a torsion pair if $\mathcal{T} = \{V \in \text{mod gr } \Lambda \mid \text{Hom}_{\text{gr } \Lambda}(V, N) = 0 \text{ for all } N \in \mathcal{F}\}$ and $\mathcal{F} = \{V \in \text{mod gr } \Lambda \mid \text{Hom}_{\text{gr } \Lambda}(N, V) = 0 \text{ for all } N \in \mathcal{T}\}$. In that case, \mathcal{T} is called a torsion class and \mathcal{F} is called a torsion-free class in $\text{mod gr } \Lambda$. By [1, VI.1.4], \mathcal{T} is a torsion class if and only if \mathcal{T} is closed under images, finite direct sums and extensions and \mathcal{F} is a torsion-free class if and only if \mathcal{F} is closed under submodules, direct products and extensions. This is also equivalent to the existence of a subfunctor t of the identity functor $\text{mod gr } \Lambda \rightarrow \text{mod gr } \Lambda$ called the torsion radical such that $t \circ t = t$ and $\mathcal{T} = \{M \mid tM = M\}$, $\mathcal{F} = \{M \mid tM = 0\}$. Additionally, we can define a functor $u : \text{mod gr } \Lambda \rightarrow \text{mod gr } \Lambda$ mapping $M \in \text{mod gr } \Lambda$ to M/tM , the largest factor module of M belonging to \mathcal{F} .

Since not all subcategories we consider in later sections are extension-closed, we formulate the results of this section for more general subcategories \mathcal{T}, \mathcal{F} whenever possible.

We will need the following lemma and a dual version which were proved for $\text{mod } \Lambda$ in [25, Lemma 2, Lemma 3]. They can be translated to our setting using 1.2.3. We say that $M \in \mathcal{T}$ is Ext-projective resp. Ext-injective in \mathcal{T} if $\text{Ext}_{\text{mod gr } \Lambda}^1(M, -)|_{\mathcal{T}} = 0$ resp. $\text{Ext}_{\text{mod gr } \Lambda}^1(-, M)|_{\mathcal{T}} = 0$.

Lemma 1.3.1. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\text{mod gr } \Lambda$ and $V \in \mathcal{T}$ be indecomposable.*

- (1) *The module V is Ext-projective in \mathcal{T} iff $\tau_{\text{mod gr } \Lambda}(V) \in \mathcal{F}$.*
- (2) *Suppose V is not Ext-projective. Then $t(\tau_{\text{mod gr } \Lambda}(V))$ is indecomposable and if $0 \rightarrow \tau_{\text{mod gr } \Lambda}(V) \xrightarrow{f} E \xrightarrow{g} V \rightarrow 0$ is the almost split sequence in $\text{mod gr } \Lambda$ ending in V , then the induced sequence $0 \rightarrow t(\tau_{\text{mod gr } \Lambda}(V)) \rightarrow t(E) \rightarrow V \rightarrow 0$ is the almost split sequence in \mathcal{T} ending in V .*

Proof. (1) Let $V \in \mathcal{T}$ be Ext-projective and let $J = \text{supp}(V) \cup \text{supp}(\tau_{\text{mod gr } \Lambda}(V))$. Then J is finite and $(\mathcal{T} \cap \text{mod}_J \text{ gr } \Lambda, \mathcal{F} \cap \text{mod}_J \text{ gr } \Lambda)$ is a torsion pair in $\text{mod}_J \text{ gr } \Lambda$. As V is Ext-projective in $\mathcal{T} \cap \text{mod}_J \text{ gr } \Lambda$, a consecutive application of 1.2.3 and [25, Lemma 2] implies $\tau_{\text{mod gr } \Lambda}(V) = \tau_{\text{mod}_J \text{ gr } \Lambda}(V) \in \mathcal{F} \cap \text{mod}_J \text{ gr } \Lambda$.

For the other direction, let $\tau_{\text{mod gr } \Lambda}(V) \in \mathcal{F}$ and $W \in \mathcal{T}$. Let $J = \text{supp}(V) \cup \text{supp}(\tau_{\text{mod gr } \Lambda}(V)) \cup \text{supp}(W)$. Then all extensions of V by W belong to $\text{mod}_J \text{ gr } \Lambda$ and $\tau_{\text{mod}_J \text{ gr } \Lambda}(V) \in \mathcal{F} \cap \text{mod}_J \text{ gr } \Lambda$. Thus, [25, Lemma 2] yields $\text{Ext}_{\text{mod gr } \Lambda}^1(V, W) = \text{Ext}_{\text{mod}_J \text{ gr } \Lambda}^1(V, W) = 0$, so that V is Ext-projective in \mathcal{T} .

- (2) Let $X \in \mathcal{T}$ and $\alpha : X \rightarrow V$ not a split epimorphism. Set $J = \text{supp}(\tau_{\text{mod gr } \Lambda}(V)) \cup \text{supp}(V) \cup \text{supp}(X)$. Then $0 \rightarrow t(\tau_{\text{mod gr } \Lambda}(V)) \rightarrow t(E) \rightarrow V \rightarrow 0$ is the image of $0 \rightarrow \tau_{\text{mod gr } \Lambda}(V) \rightarrow E \rightarrow V \rightarrow 0$ under the torsion radical of $\mathcal{T} \cap \text{mod}_J \text{ gr } \Lambda$, so that the sequence is almost split in $\mathcal{T} \cap \text{mod}_J \text{ gr } \Lambda$ by [25, Lemma 2]. As α is not a split epimorphism in $\mathcal{T} \cap \text{mod}_J \text{ gr } \Lambda$, α factors through $t(g)$, so that $t(g)$ is right almost split in \mathcal{T} . Analogously, one shows that $t(f)$ is left almost split in \mathcal{T} , so that the sequence is almost split in \mathcal{T} .

□

If \mathcal{T} is closed with respect to submodules and factor modules, we can define functors $t, u : \text{mod gr } \Lambda \rightarrow \text{mod gr } \Lambda$ by letting $t(V)$ be the largest submodule and $u(V)$ be the largest factor module in \mathcal{T} of $V \in \text{mod gr } \Lambda$. Then standard arguments show that t is right adjoint to the inclusion functor $\mathcal{T} \rightarrow \text{mod gr } \Lambda$, so that t is left exact and maps injectives in $\text{mod gr } \Lambda$ to injective objects in \mathcal{T} , while u is left adjoint to the inclusion functor, so that u is right exact and maps projectives in $\text{mod gr } \Lambda$ to projective objects in \mathcal{T} . Thus, \mathcal{T} has enough projectives and enough injectives in this case and the notions of Ext-projective resp. -injective and projective resp. injective object in \mathcal{T} coincide. The notions of top, socle and radical in $\text{mod gr } \Lambda$ and \mathcal{T} also coincide in this case and t resp. u maps the injective envelope resp. projective cover of $V \in \text{mod gr } \Lambda$ to the injective envelope of $t(V)$ resp. projective cover of $u(V)$ in \mathcal{T} . We call $V \in \text{mod gr } \Lambda$ t -acyclic resp. u -acyclic if the higher right derived functors of t resp. left derived functors of u vanish on V . If $M \in \text{mod gr } \Lambda$ is t -acyclic, we can compute minimal injective resolutions and Heller shifts in \mathcal{T} from those in $\text{mod gr } \Lambda$.

Proposition 1.3.2. *Suppose that \mathcal{T} is closed with respect to submodules and factor modules. Let $V \in \mathcal{T}$ be t -acyclic.*

- (1) *If $0 \rightarrow V \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$ is a minimal injective resolution of V in $\text{mod gr } \Lambda$, then $0 \rightarrow V \rightarrow t(I_0) \rightarrow t(I_1) \rightarrow \dots$ is a minimal injective resolution of V in \mathcal{T} .*
- (2) *We have $\Omega_{\mathcal{T}}^{-i}(V) = t(\Omega_{\text{mod gr } \Lambda}^{-i}(V))$ for all $i \in \mathbb{N}_0$.*

Proof. This can be proved with the same arguments as [9, Theorem 7.3], noting that the authors use positive superscripts instead. \square

We leave it to the reader to formulate a dual version about projective resolutions and positive Heller shifts for a u -acyclic $V \in \mathcal{T}$. Recall that the injective resp. projective dimension of a module V is the length of a minimal injective resp. projective resolution of V , i.e. the smallest natural number i such that $\Omega^{-(i+1)}(V) = 0$ resp. $\Omega^{i+1}(V) = 0$. We write $id_{\mathcal{T}}(V)$ resp. $pd_{\mathcal{T}}(V)$ for the injective resp. projective dimension of $V \in \mathcal{T}$.

Corollary 1.3.3. *Suppose \mathcal{T} is closed with respect to submodules and factor modules. Let $M \in \mathcal{T}$ be t -acyclic such that $s = id_{\mathcal{T}}(M) < \infty$. Then $\mathcal{T} \cap \{\Omega_{\text{mod gr } \Lambda}^{-i}(M) \mid i \in \mathbb{N}_0\}$ is finite and all elements of this set have finite injective dimension in \mathcal{T} .*

Proof. By 1.3.2.(2), we have $t(\Omega_{\text{mod gr } \Lambda}^{-l}(M)) = 0$ for $l > s$, so that $\Omega_{\text{mod gr } \Lambda}^{-l}(M) \notin \mathcal{T}$ or $\Omega_{\text{mod gr } \Lambda}^{-l}(M) = 0$. Since minimal injective resolutions for M in \mathcal{T} induce minimal injective resolutions for $t(M)$ in \mathcal{T} , this also shows that all modules in $\mathcal{T} \cap \{\Omega_{\text{mod gr } \Lambda}^{-i}(M) \mid i \in \mathbb{N}_0\}$ have finite injective dimension in \mathcal{T} . \square

In the remainder of this section, let Θ be a regular component of type $\mathbb{Z}[A_{\infty}]$ of the Auslander-Reiten quiver $\Gamma(\text{mod gr } \Lambda)$ of $\text{mod gr } \Lambda$. By definition, Θ does not have any projective vertices and consequently, the arrows of Θ pointing downwards correspond to irreducible epimorphisms and the arrows pointing upwards correspond to irreducible monomorphisms. The next two results determine the position of Ext-projective modules

in \mathcal{T} inside Θ . Recall that $V \in \Theta$ is called quasi-simple if it belongs to the bottom layer of Θ . In our context, this means that the middle term of the almost split sequence starting in V is indecomposable. For each $M \in \Theta$, there is a unique quasi-simple module $N \in \Theta$ such that $N \subseteq M$, the quasi-socle of M . A path inside Θ is a sectional path if no vertex on the path is a $\tau_{\text{mod gr } \Lambda}$ -shift of another vertex on the path. For a regular component of type $\mathbb{Z}[A_\infty]$, this is equivalent to all arrows being surjective or all arrows being injective. We denote by $ql(M)$ the quasi-length of $M \in \Theta$, that is the number of vertices on a sectional path from the quasi-socle of M to M . Thus, if M is quasi-simple, we have $ql(M) = 1$.

Proposition 1.3.4. *Suppose \mathcal{T} is a torsion class and closed with respect to submodules. Let $M \in \Theta \cap \mathcal{T}$ such that M is Ext-projective in \mathcal{T} and N be the quasi-socle of M . Then all modules on the sectional path from N to M are Ext-projective in \mathcal{T} . If $ql(M) > 1$, then N is simple.*

Proof. By 1.3.1, $\tau_{\text{mod gr } \Lambda}(M)$ has no nontrivial \mathcal{T} -submodules. If M' is a predecessor of M on the sectional path from N to M , then $\tau_{\text{mod gr } \Lambda}(M')$ has no nontrivial \mathcal{T} -submodules as it embeds into $\tau_{\text{mod gr } \Lambda}(M)$. By 1.3.1, M' is Ext-projective in \mathcal{T} . Now let $ql(M) > 1$. Then there is an almost split sequence in \mathcal{T} starting in N such that the middle term of the sequence is Ext-projective in \mathcal{T} . We use arguments dual to those of [2, V.3.3] to show that N is simple. Let B be the middle term and C the right term of the almost split sequence. Without loss of generality, we may assume that N is a submodule of B . Since the sequence is almost split, N is a proper submodule of B . As B is projective indecomposable in \mathcal{T} , this implies $N \subseteq \text{Rad}(B)$. Thus, the sequence $0 \rightarrow \text{Top}(N) \rightarrow \text{Top}(B) \rightarrow \text{Top}(C) \rightarrow 0$ is not exact. Hence a dual version of [2, V.3.2] shows that N is simple. \square

Corollary 1.3.5. *Suppose \mathcal{T} is a torsion class and closed with respect to submodules. Let $M \in \Theta \cap \mathcal{T}$ such that M is Ext-projective in \mathcal{T} . Suppose there is a duality $D : \mathcal{T} \rightarrow \mathcal{T}$ such that $D(S) \cong S$ for every simple module $S \in \mathcal{T}$. Then M is quasi-simple.*

Proof. If $ql(M) > 1$, 1.3.4 provides a simple module $S \in \mathcal{T}$ which is Ext-projective in \mathcal{T} . Since $D(S) \cong S$, the remarks preceding 1.3.2 yield that S is also Ext-injective in \mathcal{T} , so the almost split sequence starting in S splits, a contradiction. \square

Lemma 1.3.6. *Let \mathcal{T} be closed with respect to submodules and factor modules. Suppose the number of quasi-simple modules in $\Theta \cap \mathcal{T}$ is finite. Then $\Theta \cap \mathcal{T}$ is finite.*

Proof. Since \mathcal{T} is closed with respect to taking submodules and factor modules, our assumption implies that the number of modules of any given quasi-length in $\Theta \cap \mathcal{T}$ is finite and that the quasi-length of modules in $\Theta \cap \mathcal{T}$ is bounded. Hence $\Theta \cap \mathcal{T}$ is finite. \square

For $M \in \Theta$ of quasi-length m , we denote the quasi-socle of M by $M(1)$ and the modules on the sectional path from $M(1)$ to M by $M(1), M(2), \dots, M(m) = M$. We denote by $\mathcal{W}(M)$ the wing of M , namely the mesh-complete full subquiver of M containing the vertices $\tau_{\text{mod gr } \Lambda}^r(M(s))$ with $1 \leq s \leq m$ and $0 \leq r \leq m - s$, see [33, 3.3] and the following figure.

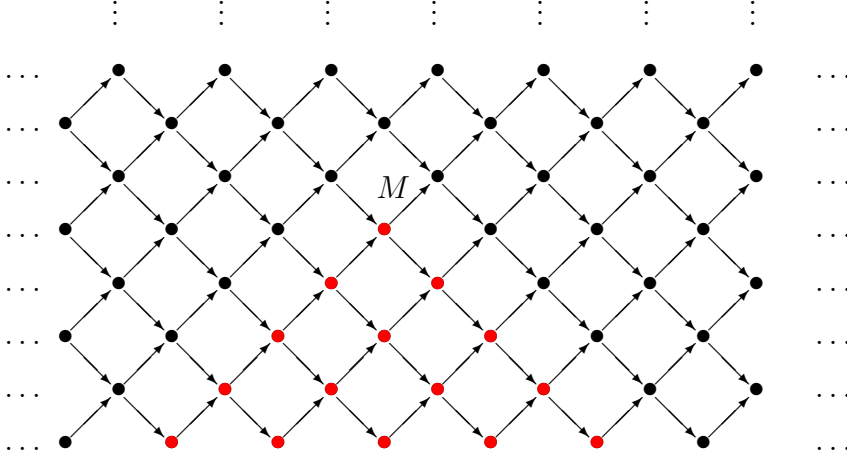


Figure 1.1: The red vertices in this $\mathbb{Z}[A_\infty]$ -component form the wing of the top red vertex M .

Lemma 1.3.7. *Suppose \mathcal{T} is closed with respect to submodules and factor modules and let $M \in \Theta \cap \mathcal{T}$. Then every module in the wing of M belongs to \mathcal{T} .*

Proof. We show this by induction on the quasi-length $ql(M)$ of M . If $ql(M) = 1$, i.e. M is quasi-simple, the statement is clear. Now let $m = ql(M) > 1$. Since \mathcal{T} is closed with respect to submodules and factor modules, $M(m-1), \tau_{\text{mod gr } \Lambda}^{-1}(M(m-1)) \in \mathcal{T}$. As the remaining vertices in $\mathcal{W}(M)$ belong to $\mathcal{W}(M(m-1)) \cup \mathcal{W}(\tau_{\text{mod gr } \Lambda}^{-1}(M(m-1)))$, the result now follows by induction. \square

The assumptions on M in the following lemma mean that two wings in $\mathcal{T} \cap \Theta$ have adjacent quasi-simple modules. We use this to show that the modules between the wings also belong to Θ and construct a module $N \in \mathcal{T} \cap \Theta$ such that the wing of N contains the wing of M and $\tau_{\text{mod gr } \Lambda}^s(M)$.

Lemma 1.3.8. *Suppose \mathcal{T} is a torsion class closed with respect to submodules. Let $s \in \mathbb{N}$ and $M \in \Theta \cap \mathcal{T}$ such that $2ql(M) > s$ and $\tau_{\text{mod gr } \Lambda}^s(M) \in \mathcal{T}$. Then $\tau_{\text{mod gr } \Lambda}^i(M) \in \mathcal{T}$ for $1 \leq i \leq s$ and there exists a module $N \in \Theta \cap \mathcal{T}$ such that $ql(N) = ql(M) + s$.*

Proof. As $ql(M) = ql(\tau_{\text{mod gr } \Lambda}^s(M)) > \frac{s}{2}$, the wings of $\tau_{\text{mod gr } \Lambda}^s(M)$ and M either intersect or the rightmost quasi-simple module Y_1 in the wing of $\tau_{\text{mod gr } \Lambda}^s(M)$ and the leftmost quasi-simple module Y_2 in the wing of M satisfy $\tau_{\text{mod gr } \Lambda}(Y_2) = Y_1$. As \mathcal{T} is extension-closed, this shows that $\tau_{\text{mod gr } \Lambda}^i(M) \in \mathcal{T}$ for $1 \leq i \leq s$.

Using again that \mathcal{T} is extension-closed, we get that in the layer above a horizontal line $A, \tau_{\text{mod gr } \Lambda}(A), \dots, \tau_{\text{mod gr } \Lambda}^l(A)$ of $(l+1)$ modules in \mathcal{T} , there is a horizontal line of the same form of l modules in \mathcal{T} . As the horizontal line in our situation has at least $(s+1)$ modules in \mathcal{T} , the existence of N now follows inductively. \square

Recall that a homogeneous tube is a component isomorphic to $\mathbb{Z}[A_\infty]/\langle \tau \rangle$. If Θ becomes a homogeneous tube upon forgetting the grading, we can determine when the almost split sequence in \mathcal{T} ending in $N \in \mathcal{T} \cap \Theta$ is also almost split in $\text{mod gr } \Lambda$.

Proposition 1.3.9. *Let $F : \text{mod gr } \Lambda \rightarrow \text{mod } \Lambda$ be the forgetful functor and \mathcal{T} be a torsion class. Suppose that $F(\Theta)$ is a homogeneous tube and $N \in \Theta \cap \mathcal{T}$ is not Ext-projective in \mathcal{T} . Let*

$$(0) \rightarrow V \rightarrow E \rightarrow N \rightarrow (0)$$

be the almost split sequence in \mathcal{T} ending in N . Then the sequence is almost split in $\text{mod gr } \Lambda$ if and only if $l(V) = l(N)$ if and only if $\dim_k V = \dim_k N$.

Proof. By 1.3.1, V is a submodule of the left term of the almost split sequence in $\text{mod gr } \Lambda$ ending in V . Since $F(\Theta)$ is a homogeneous tube, the left term of that sequence has the same length and dimension as N by 1.2.2. \square

We say that $V, W \in \Theta$ belong to the same column if there are almost split sequences ξ_1, \dots, ξ_l such that V is a non-projective direct summand of the middle term of ξ_1 , W is a non-projective direct summand of the middle term of ξ_l and for each $1 \leq i \leq l-1$, the middle terms of ξ_i and ξ_{i+1} have a common non-projective direct summand. This defines an equivalence relation on Θ . We call the equivalence classes the columns of Θ .

Proposition 1.3.10. *Suppose there is a duality $(-)^{\circ} : \text{mod gr } \Lambda \rightarrow \text{mod gr } \Lambda$. If Θ contains a module V such that $V^{\circ} \cong V$, then $N^{\circ} \cong N$ for all N in the column of V . The column of V is a symmetry axis with respect to $(-)^{\circ}$.*

Proof. As $(-)^{\circ}$ is a duality, Θ° is a component of $\Gamma(\text{mod gr } \Lambda)$ and $\Theta \cap \Theta^{\circ} = \emptyset$ or $\Theta = \Theta^{\circ}$. Thus, $V \in \Theta \cap \Theta^{\circ}$ implies $\Theta = \Theta^{\circ}$. Since $(-)^{\circ}$ does not change the quasi-length of modules, any almost split sequence in Θ such that the middle term has a self-dual summand is mapped to itself by $(-)^{\circ}$ and every direct summand of a middle term of such a sequence is self-dual. \square

1.4 Polynomial representations and infinitesimal Schur algebras

In this section, we will introduce our main objects of study. For algebraic groups, we follow the notation and terminology of [26]. For an introduction to algebraic monoids, we refer the reader to [32].

Let k be an algebraically closed field of characteristic $p > 0$ and Mat_n be the monoid scheme of $(n \times n)$ -matrices over k . Let $d \in \mathbb{N}_0$ and denote by $A(n, d)$ the space generated by all homogeneous polynomials of degree d in the coordinate ring $k[\text{Mat}_n] = k[X_{ij} \mid 1 \leq i, j \leq n]$ of Mat_n . Then $k[\text{Mat}_n] = \bigoplus_{d \geq 0} A(n, d)$ is a graded k -bialgebra with comultiplication $\Delta : k[\text{Mat}_n] \rightarrow k[\text{Mat}_n] \otimes_k k[\text{Mat}_n]$ and counit $\epsilon : k[\text{Mat}_n] \rightarrow k$ given by $\Delta(X_{ij}) = \sum_{l=1}^n X_{il} \otimes_k X_{lj}$, $\epsilon(X_{ij}) = \delta_{ij}$ and each $A(n, d)$ is a finite-dimensional subcoalgebra. As GL_n is dense in Mat_n , the canonical map $k[\text{Mat}_n] \rightarrow k[\text{GL}_n]$ is injective, so that $k[\text{Mat}_n]$ and each $A(n, d)$ can be viewed as a subcoalgebra of $k[\text{GL}_n]$. Now let G be a closed subgroup scheme of GL_n and $\pi : k[\text{GL}_n] \rightarrow k[G]$ the canonical projection. Set $A(G) = \pi(k[\text{Mat}_n])$ and $A_d(G) = \pi(A(n, d))$ as well as $S_d(G) = A_d(G)^*$. Since π is a homomorphism of Hopf algebras, $A(G)$ is a subbialgebra of $k[G]$ and each $A_d(G)$

is a subcoalgebra of $A(G)$. Thus, $S_d(G)$ is a finite-dimensional associative algebra in a natural way. Following [5], we say that G admits a graded polynomial representation theory if the sum $\sum_{d \geq 0} A_d(G)$ is direct and we say that a rational G -module V is a polynomial G -module if the corresponding comodule map $V \rightarrow V \otimes_k k[G]$ factors through $V \otimes_k A(G)$. If the comodule map factors through $V \otimes_k A_d(G)$ for some $d \in \mathbb{N}_0$, we say that V is homogeneous of degree d . Clearly, every $A_d(G)$ -comodule is an $A(G)$ -comodule and every $A(G)$ -comodule is a G -module in a natural way. As $A(G)$ is a factor bialgebra of $k[\text{Mat}_n]$, it corresponds to a closed submonoid M of Mat_n . Since $A(G) \subseteq k[G]$ is a subbialgebra, $G \subseteq M$ is a dense subscheme, so that M is the closure of G in Mat_n . Hence polynomial representations of G can be regarded as rational representations of the algebraic monoid scheme $M = \overline{G}$. Note that all these notions depend on the given embedding $G \rightarrow \text{GL}_n$.

We record some properties of polynomial representations from [5, 1.2] in the following proposition.

Proposition 1.4.1. (1) *Suppose G admits a graded polynomial representation theory and V is a polynomial G -module. Then $V = \bigoplus_{d \geq 0} V_d$, where each V_d is homogeneous of degree d . Furthermore, the category of homogeneous modules of degree d is equivalent to the category of $S_d(G)$ -modules.*

(2) *If G contains the center $Z(\text{GL}_n)$ of GL_n , then G admits a graded polynomial representation theory.*

Two important special cases are the algebras $S_d(G_r T)$ and $S_d(B_r T)$, where $G = \text{GL}_n$, $T \subseteq G$ is the maximal torus of diagonal matrices and $B \subseteq G$ is the Borel subgroup of upper triangular matrices. The algebra $S_d(G_r T)$ is the infinitesimal Schur algebra introduced in [9] and $S_d(B_r T)$ can be viewed as an infinitesimal analogue of the Borel-Schur algebra $S_d(B)$ introduced in [24]. Note that all results about $B_r T$ in this thesis can also be proved for the Borel subgroup of lower triangular matrices instead. If $D = \overline{T} \subseteq \text{Mat}_n$ is the monoid scheme of diagonal matrices, a comparison of coordinate rings shows $\overline{G_r T} = M_r D$, where $M_r D = (F_{\text{Mat}_n}^r)^{-1}(D)$ with $F_{\text{Mat}_n}^r$ being the r -th iteration of the Frobenius homomorphism on Mat_n , see [9]. By the same token, we get for $L = \overline{B} \subseteq \text{Mat}_n$ the monoid scheme of upper triangular matrices that $\overline{B_r T} = L_r D$, where $L_r D = (F_L^r)^{-1}(D)$ with F_L^r the restriction of $F_{\text{Mat}_n}^r$ to L . We have $k[M_r D] = k[X_{ij} \mid 1 \leq i, j \leq n] / (X_{ij}^{p^r} \mid i \neq j)$ and $k[L_r D] = k[X_{ij} \mid 1 \leq i \leq j \leq n] / (X_{ij}^{p^r} \mid i \neq j)$.

This can fail for general G : If G is the full orthogonal similitude group in dimension two and T the torus of diagonal matrices in G , then $G_r T = T$, but $M_r D$ defined analogously to the case $G = \text{GL}_n$ is strictly larger than $D = \overline{T}$. However, we always have $\overline{G_r T} \subseteq M_r D$. We do not know whether $M_r D = \overline{G_r T}$ e.g. for connected G .

2 Simple polynomial $G_r T$ -modules

2.1 Classification of simple polynomial $G_r T$ -modules

In this section, let $G \leq \mathrm{GL}_n$ be a closed connected reductive subgroup scheme containing $Z(\mathrm{GL}_n)$, $T \subseteq G$ be a maximal torus contained in a maximal torus $T' \subseteq \mathrm{GL}_n$, M be the closure of G in Mat_n , D be the closure of T , D' be the closure of T' and $P(D) \subseteq X(T)$ be the character monoid of D , i.e. the set of polynomial weights, W the Weyl group of G . Let R be the root system of G with respect to T and $S \subseteq R$ a set of simple roots. Denote by $\langle -, - \rangle : X(T) \times Y(T) \rightarrow \mathbb{Z}$ the canonical perfect pairing of $X(T)$ and the cocharacter group $Y(T)$. For $\alpha \in R$, let α^\vee be the coroot of α . Let $X_0(T) = \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R\}$ as well as $X_r(T) = \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p^r - 1 \text{ for all } \alpha \in S\}$.

Fix $r \in \mathbb{N}$. We want to determine the simple polynomial $G_r T$ -modules. Recall that the isomorphism classes of simple $G_r T$ -modules are parametrized by $X(T)$ via $\lambda \mapsto \widehat{L}_r(\lambda)$, where $\widehat{L}_r(\lambda)$ has highest weight λ , see [26, II.9]. As a motivation for our approach, consider the case where $G = \mathrm{GL}_n$ and $T \leq G$ is the maximal torus of diagonal matrices. According to [9, Section 3], the simple $G_r T$ -module $\widehat{L}_r(\lambda)$ is a polynomial $G_r T$ -module iff $\lambda \in P_r(D) + p^r P(D)$, where $P_r(D) = \{\lambda \in P(D) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq p^r - 1 \text{ for } 1 \leq i \leq n-1, 0 \leq \lambda_n \leq p^r - 1\}$. A character $\lambda \in X(T)$ belongs to $P(D)$ iff $\lambda_i \geq 0$ for $1 \leq i \leq n$, that is, iff $\min\{\lambda_i \mid 1 \leq i \leq n\} \in \mathbb{N}_0$. Taking this minimum is compatible with multiplication by natural numbers, in particular with multiplication by p^r . Given characters $\lambda, \mu \in X(T)$, we can of course find a permutation $w \in W \cong S_n$ such that this minimum is attained at the same coordinate for $w \cdot \lambda$ and μ , so that $\min\{(w \cdot \lambda + \mu)_i \mid 1 \leq i \leq n\} = \min\{\lambda_i \mid 1 \leq i \leq n\} + \min\{\mu_i \mid 1 \leq i \leq n\}$. Writing $d = \det|_T = (1, \dots, 1)$, we have $X_0(T) = \langle d \rangle$, $\min\{d_i \mid 1 \leq i \leq n\} = 1$ as well as $X_r(T) = \{\lambda \in X(T) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq p^r - 1 \text{ for } 1 \leq i \leq n-1\}$ and we can write $P_r(D) = \{\lambda \in X_r(T) \cap P(D) \mid \lambda - p^r d \notin P(D)\}$. We axiomatize these properties of $\det|_T$ and the function $X(T) \rightarrow \mathbb{Z}, \lambda \mapsto \min\{\lambda_i \mid 1 \leq i \leq n\}$ for general G in the following

Assumption 2.1.1. *Suppose there exists a function $\varphi : X(T) \rightarrow \mathbb{Z}^l$ with the following properties:*

- (1) $\forall \lambda \in X(T) : \varphi(\lambda) \in \mathbb{N}_0^l \iff \lambda \in P(D)$,
- (2) $\forall \lambda \in X(T) : \varphi(p^r \lambda) = p^r \varphi(\lambda)$,
- (3) $\forall \lambda, \lambda' \in X(T) : \exists w \in W : \varphi(w \cdot \lambda + \lambda') = \varphi(\lambda) + \varphi(\lambda')$,
- (4) $\varphi|_{X_0(T)} : X_0(T) \rightarrow \mathbb{Z}^l$ is bijective.

Since W acts trivially on $X_0(T)$, (1), (3) and (4) imply that the restriction to $X_0(T)$ of any function φ as in 2.1.1 is an isomorphism mapping $P(D) \cap X_0(T)$ to \mathbb{N}_0^l . It will turn out that 2.1.1 is sufficient for a classification similar to the case $G = \mathrm{GL}_n$. We first show that such a function is essentially unique.

Proposition 2.1.2. *If $\varphi, \psi : X(T) \rightarrow \mathbb{Z}^l$ are functions as in 2.1.1, there is a permutation of coordinates $\sigma : \mathbb{Z}^l \rightarrow \mathbb{Z}^l$ such that $\varphi = \sigma \circ \psi$.*

Proof. Let e_1, \dots, e_l be the standard basis of \mathbb{Z}^l and d_1, \dots, d_l resp. d'_1, \dots, d'_l be the preimages of the e_i in $X_0(T)$ for φ resp. ψ . Then every element of $X_0(T) \cap P(D)$ can be written as a linear combination in the d_i with non-negative coefficients. Writing the d'_i as such a linear combination and then writing the d_i as such a linear combination in the d'_i , we see that the base change matrix for the two bases is a permutation matrix. Thus, there is $\sigma \in S_l$ such that $d_i = d'_{\sigma(i)}$ for $1 \leq i \leq l$. Now let $\lambda \in X(T)$ and set $\tilde{\lambda} = \lambda - \sum_{i=1}^l \varphi(\lambda)_i d_i$. Using (3) and that W acts trivially on $X_0(T)$ for the first equality and the fact that $\psi|_{X_0(T)}$ is an isomorphism for the second equality, we get $\psi(\tilde{\lambda}) = \psi(\lambda) + \psi(-\sum_{i=1}^l \varphi(\lambda)_i d_i) = \psi(\lambda) - \sum_{i=1}^l \varphi(\lambda)_i \psi(d_i) = \psi(\lambda) - \sum_{i=1}^l \varphi(\lambda)_i \psi(d'_{\sigma(i)}) = \psi(\lambda) - \sum_{i=1}^l \varphi(\lambda)_i e_{\sigma(i)}$. Applying the same arguments to φ , we get $\varphi(\tilde{\lambda}) = 0$ and $\varphi(\tilde{\lambda} - d_i) = -e_i$ for $1 \leq i \leq l$. Thus, $\tilde{\lambda} \in P(D)$ and $\tilde{\lambda} - d_i = \tilde{\lambda} - d'_{\sigma(i)} \notin P(D)$ for $1 \leq i \leq l$, so that $\psi(\tilde{\lambda}) \in \mathbb{N}_0^l$ and $\psi(\tilde{\lambda}) - e_i \notin \mathbb{N}_0^l$. This shows $\psi(\tilde{\lambda}) = 0$, so that $\psi(\lambda) = \sum_{i=1}^l \varphi(\lambda)_i e_{\sigma(i)}$, hence $\varphi(\lambda)_i = \psi(\lambda)_{\sigma(i)}$. \square

Definition 2.1.3. *Suppose φ is a function as in 2.1.1 for G . Letting e_1, \dots, e_l be the standard basis of \mathbb{Z}^l and d_1, \dots, d_l be the preimages of the e_i in $X_0(T)$ with respect to φ , we set $P_r(D) = \{\lambda \in P(D) \cap X_r(T) \mid \forall i \in \{1, \dots, l\} : \lambda - p^r d_i \notin P(D)\}$.*

It follows from the proof of 2.1.2 that $P_r(D)$ does not depend on the choice of φ .

Lemma 2.1.4. *Let $\lambda = \lambda_0 + p^r \tilde{\lambda} = \lambda'_0 + p^r \tilde{\lambda}'$ with $\lambda_0, \lambda'_0 \in X_r(T)$, $\tilde{\lambda}, \tilde{\lambda}' \in X(T)$. Then there is $\mu \in X_0(T)$ such that $\lambda_0 = \lambda'_0 + p^r \mu$, $\tilde{\lambda}' = \tilde{\lambda} + \mu$.*

Proof. We have $\lambda_0 - \lambda'_0 = p^r \mu$, where $\mu = \tilde{\lambda}' - \tilde{\lambda}$. For all $\alpha \in R$, we have $p^r \langle \mu, \alpha^\vee \rangle = \langle \lambda_0 - \lambda'_0, \alpha^\vee \rangle = \langle \lambda_0, \alpha^\vee \rangle - \langle \lambda'_0, \alpha^\vee \rangle$. As $\lambda_0, \lambda'_0 \in X_r(T)$, we have $-(p^r - 1) \leq \langle \lambda_0, \alpha^\vee \rangle - \langle \lambda'_0, \alpha^\vee \rangle \leq p^r - 1$, forcing $\langle \mu, \alpha^\vee \rangle = 0$. Thus, $\mu \in X_0(T)$ and the result follows. \square

Theorem 2.1.5. *Suppose 2.1.1 holds for G . If $\lambda \in X_r(T) + p^r X(T)$, then there is a unique decomposition $\lambda = \lambda_0 + p^r \tilde{\lambda}$ with $\lambda_0 \in P_r(D)$, $\tilde{\lambda} \in X(T)$. Furthermore, if all weights of the simple $G_r T$ -module $\widehat{L}_r(\lambda)$ belong to $P(D)$, then $\lambda \in P_r(D) + p^r P(D)$.*

Proof. Let $\lambda = \lambda_0 + p^r \tilde{\lambda}$ with $\lambda_0 \in X_r(T)$, $\tilde{\lambda} \in X(T)$. As $d_i \in X_0(T)$, we get that for each $a \in \mathbb{Z}^l$, $\lambda_0 + p^r \sum_{i=1}^l a_i d_i + p^r (\tilde{\lambda} - \sum_{i=1}^l a_i d_i)$ is another decomposition such that $\lambda_0 + p^r \sum_{i=1}^l a_i d_i \in X_r(T)$. Since d_1, \dots, d_l generate $X_0(T)$, 2.1.4 shows that every decomposition of λ arises in this fashion. By (1), (3), (4) and since W acts trivially on $X_0(T)$, there is a unique $a \in \mathbb{Z}^l$ such that $\lambda_0 + p^r \sum_{i=1}^l a_i d_i \in P_r(D)$. Thus, there is a unique decomposition $\lambda = \lambda_0 + p^r \tilde{\lambda}$ such that $\lambda_0 \in P_r(D)$. We show by contraposition

that if all weights of $\widehat{L}_r(\lambda)$ are polynomial, then $\tilde{\lambda} \in P(D)$. Suppose that $\tilde{\lambda} \notin P(D)$, so that $\varphi(\tilde{\lambda}) \notin \mathbb{N}_0^l$. Then there is $i \in \{1, \dots, l\}$ such that $\varphi(\tilde{\lambda})_i < 0$. Since $\lambda_0 - p^r d_i \notin P(D)$, (1), (3), (4) yield $\varphi(\lambda_0)_i < p^r$. Using (2) and (3), we see that there is $w \in W$ such that $\varphi(w \cdot \lambda_0 + p^r \tilde{\lambda})_i = \varphi(\lambda_0)_i + p^r \varphi(\tilde{\lambda})_i < 0$, so that $w \cdot \lambda_0 + p^r \tilde{\lambda} \notin P(D)$ by (1). Since $\widehat{L}_r(\lambda_0)$ lifts to a simple G -module by [26, II.9.6.(g)], its character is W -invariant, so that $w \cdot \lambda_0$ is a weight of $\widehat{L}_r(\lambda_0)$ and $w \cdot \lambda_0 + p^r \tilde{\lambda}$ is a weight of $\widehat{L}_r(\lambda_0 + p^r \tilde{\lambda}) \cong \widehat{L}_r(\lambda_0) \otimes_k p^r \tilde{\lambda}$, so that not all weights of $\widehat{L}_r(\lambda)$ are polynomial. \square

For $G = \mathrm{GL}_n$, it was shown in the Appendix of [30] that a $G_r T$ -module such that all of its weights are polynomial is a polynomial $G_r T$ -module. It is not known for which general G a similar statement holds. However, if M is normal as a variety, we can at least prove this for some simple modules.

Proposition 2.1.6. *Suppose that M is a normal variety. If $\lambda = \lambda_0 + p^r \tilde{\lambda} \in (X_r(T) \cap P(D)) + p^r P(D)$, the simple $G_r T$ -module $\widehat{L}_r(\lambda)$ is a polynomial $G_r T$ -module.*

Proof. We have $\widehat{L}_r(\lambda) \cong \widehat{L}_r(\lambda_0) \otimes_k p^r \tilde{\lambda}$ and $\widehat{L}_r(\lambda_0)$ lifts to G by [26, II.9.6]. Since M is normal and λ_0 is a dominant weight contained in $P(D)$, $\widehat{L}_r(\lambda_0)$ lifts to M by [6, 3.5], so that $\widehat{L}_r(\lambda_0)$ is a polynomial $G_r T$ -module. As $\tilde{\lambda}$ is a D -module, $p^r \tilde{\lambda}$ is an $F^{-r}(D)$ -module, where F^r is the r -th iteration of the Frobenius morphism, and thus a polynomial $G_r T$ -module by restriction. Hence $\widehat{L}_r(\lambda)$ is a polynomial $G_r T$ -module. \square

Corollary 2.1.7. *Let M be normal and suppose that 2.1.1 holds for G . If $\lambda \in X_r(T) + p^r X(T)$, then the simple module $\widehat{L}_r(\lambda)$ is a polynomial $G_r T$ -module iff $\lambda \in P_r(D) + p^r P(D)$.*

Proof. This follows directly from 2.1.5 and 2.1.6. \square

Corollary 2.1.8. *Let the derived subgroup of G be simply connected and M be normal. Suppose 2.1.1 holds for G . Then a system of representatives of isomorphism classes of simple polynomial $G_r T$ -modules is given by $\{\widehat{L}_r(\lambda) \mid \lambda \in P_r(D) + p^r P(D)\}$.*

Proof. Let $\lambda \in X(T)$. Since the derived subgroup of G is simply connected, $X_r(T)$ contains a set of representatives of $X(T)/p^r X(T)$, so that we have $\lambda \in X_r(T) + p^r X(T)$. By 2.1.5, all weights of $\widehat{L}_r(\lambda)$ are polynomial iff $\lambda \in P_r(D) + p^r P(D)$ and by 2.1.6, $\widehat{L}_r(\lambda)$ lifts to $G_r T$ in this case. Since all weights of polynomial $G_r T$ -modules are polynomial and a polynomial module is a simple polynomial $G_r T$ -module iff it is simple as a $G_r T$ -module, the result follows. \square

We now give an explicit construction for a function as in 2.1.1 under certain assumptions.

Theorem 2.1.9. *Suppose there is an action of W on $X(T')$ such that the canonical projection $\pi : X(T') \rightarrow X(T)$ is W -equivariant and there are $b_1, \dots, b_s \in \pi^{-1}(X_0(T)) \cap P(D')$ such that*

$$(a) \quad \forall i \in \{1, \dots, s\}, j \in \{1, \dots, n\} : (b_i)_j \in \{0, 1\},$$

(b) the sets $M_i = \{j \in \{1, \dots, n\} \mid (b_i)_j = 1\}$ form a partition of $\{1, \dots, n\}$,

(c) the image of W in $S_{X(T')}$ is contained in the image of $W' \cong S_n$ and contains the subgroup $S_{M_1} \times \dots \times S_{M_s}$,

(d) $\exists d_1, \dots, d_l \in \{b_1, \dots, b_s\} : \pi(d_1), \dots, \pi(d_l)$ is a basis of $X_0(T)$ such that each $\pi(b_i)$ is a linear combination of the $\pi(d_j)$ with non-negative coefficients.

Then there is a function $\varphi : X(T) \rightarrow \mathbb{Z}^l$ with the properties of 2.1.1.

Proof. For every $i \in \{1, \dots, s\}$, write $\pi(b_i) = \sum_{j=1}^l n_{ij} \pi(d_j)$ with $n_{ij} \in \mathbb{N}_0$. Define $\varphi : X(T') \rightarrow \mathbb{Z}^l$, $\lambda \mapsto \sum_{j=1}^l \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} n_{ij} e_j$. We show that φ induces a function $\bar{\varphi} : X(T) \rightarrow \mathbb{Z}^l$ with the properties of 2.1.1. For this, we show the following statements (i) – (v).

(i) $\forall \lambda \in X(T'), \mu \in \ker(\pi) : \varphi(\lambda + \mu) = \varphi(\lambda) :$

Let $\mu \in \ker(\pi)$. Since the sets M_i are pairwise disjoint, (a) implies that the element $\mu' = \mu - \sum_{i=1}^s \min\{\mu_a \mid a \in M_i\} b_i$ belongs to $P(D')$ and that for every i , there is $a \in M_i$ such that $\mu'_a = 0$. By 2.1.10 below and (a), we get $\mu'_l = 0$ for all $l \in M_i$. As the M_i form a partition of $\{1, \dots, n\}$, we get $\mu' = 0$, so that $\mu = \sum_{i=1}^s \min\{\mu_a \mid a \in M_i\} b_i$. Thus, (a) and the definition of the M_i imply that for each i and each $\lambda \in X(T')$, the minima $\min\{\lambda_a \mid a \in M_i\}$ and $\min\{(\lambda + \mu)_a \mid a \in M_i\}$ are attained at the same coordinate $a \in M_i$. By definition of φ , we get $\varphi(\lambda + \mu) = \varphi(\lambda) + \varphi(\mu)$ for all $\lambda \in X(T')$. As $\mu \in \ker(\pi)$, we have

$$\begin{aligned} 0 = \pi(\mu) &= \sum_{i=1}^s \min\{\mu_a \mid a \in M_i\} \pi(b_i) \\ &= \sum_{j=1}^l \left(\sum_{i=1}^s n_{ij} \min\{\mu_a \mid a \in M_i\} \right) \pi(d_j), \end{aligned}$$

so that $\sum_{i=1}^s n_{ij} \min\{\mu_a \mid a \in M_i\} = 0$ for every j and $\varphi(\mu) = 0$. Hence (i) holds.

(ii) $\forall \lambda \in X(T') : \varphi(p^r \lambda) = p^r \varphi(\lambda) :$

Clear by definition.

(iii) $\forall \lambda \in X(T') : \varphi(\lambda) \in \mathbb{N}_0^l \iff \lambda \in P(D) + \ker(\pi) :$

Suppose $\varphi(\lambda) \in \mathbb{N}_0^l$. We have $\lambda - \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} b_i \in P(D')$, so that $\pi(\lambda) - \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} \pi(b_i) \in \pi(P(D'))$. Hence $\pi(\lambda) \in \pi(P(D')) + \sum_{j=1}^l \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} n_{ij} \pi(d_j)$. By assumption, the coefficients in the linear combination are non-negative, so that this set is contained in $\pi(P(D'))$ and $\lambda \in P(D') + \ker(\pi)$. Now let $\lambda = \lambda' + \mu \in P(D') + \ker(\pi)$. Since $\varphi(\lambda) = \varphi(\lambda')$ by (i) and since λ' has no negative coordinates and $n_{ij} \geq 0$, we have $\varphi(\lambda) \in \mathbb{N}_0^l$.

(iv) $\forall \lambda, \lambda' \in X(T') : \exists w \in W : \varphi(w \cdot \lambda + \lambda') = \varphi(\lambda) + \varphi(\lambda') :$

Using (c), we permute the coordinates in each M_i for λ in such a way that the lowest coordinate for $w \cdot \lambda$ and λ' in each M_i occurs at the same place. Then clearly $\varphi(w \cdot \lambda + \lambda') = \varphi(\lambda) + \varphi(\lambda')$.

(v) $\varphi(d_i) = e_i$:

This follows directly from (d) and the definition of φ .

The statement (i) implies that there is a function $\bar{\varphi} : X(T) \rightarrow \mathbb{Z}^l$ such that $\varphi = \bar{\varphi} \circ \pi$. Now (ii) – (v) imply that $\bar{\varphi}$ has the properties of 2.1.1. □

Lemma 2.1.10. *In the situation of 2.1.9, let $i \in \{1, \dots, s\}$ and $\mu \in \ker(\pi)$. Then $\mu_k = \mu_l$ for all $l, k \in M_i$.*

Proof. Let $\mu \in \ker(\pi)$ and $i \in \{1, \dots, s\}$. Suppose there are $k, l \in M_i$ such that $\mu_k \neq \mu_l$. Let $w = (k \ l)$ be the transposition with respect to k and l . We have $w \in W$ by (c). Then $\mu - w.\mu = (\mu_k - \mu_l)(e_k - e_l) \in \ker(\pi)$. As the restriction of this element is zero in $X(T)$ and $X(T)$ is torsion-free, $(e_k - e_l)|_T = 0$, so that $e_k|_T = e_l|_T$. It follows that w acts trivially on T , a contradiction. □

The function φ defined in the proof of 2.1.9 provides an easy way to check whether the restriction of a character $\lambda \in X(T')$ to T is a polynomial weight. We can also use it to determine whether the restrictions of two characters of T' to T coincide.

Proposition 2.1.11. *In the situation of 2.1.9, let $\lambda, \mu \in X(T')$. Then the following statements are equivalent:*

(1) $\lambda|_T = \mu|_T$,

(2) $\varphi(\lambda) = \varphi(\mu)$ and $\lambda - \mu$ is a linear combination of the b_i ,

(3) $\varphi(\lambda) = \varphi(\mu)$ and $w.(\lambda - \mu) = \lambda - \mu$ for all $w \in S_{M_1} \times \dots \times S_{M_s}$.

Proof. (1) \Rightarrow (2): We have $\varphi(\lambda) = \bar{\varphi} \circ \pi(\lambda) = \bar{\varphi} \circ \pi(\mu) = \varphi(\mu)$. By assumption, we have $\pi(\lambda - \mu) = 0$, so that the second part follows from 2.1.10.

(2) \Rightarrow (3): Clear.

(3) \Rightarrow (2): Clear.

(2) \Rightarrow (1): By assumption, we may write $\lambda = \mu + \sum_{i=1}^s a_i b_i$ for some $a_i \in \mathbb{Z}$. By definition of φ , we have $\varphi(\lambda) = \varphi(\mu) + \varphi(\sum_{i=1}^s a_i b_i)$, so that $\varphi(\sum_{i=1}^s a_i b_i) = 0$. As $\varphi = \bar{\varphi} \circ \pi$, we get $\bar{\varphi} \circ \pi(\sum_{i=1}^s a_i b_i) = 0$. By definition of the b_i , $\pi(\sum_{i=1}^s a_i b_i) \in X_0(T)$. As $\bar{\varphi}|_{X_0(T)}$ is an isomorphism, we get $\pi(\sum_{i=1}^s a_i b_i) = 0$. Hence $\pi(\lambda) = \pi(\mu)$. □

We now apply our results to several examples. The first part of the following corollary is [10, Corollary 3.2]. For the orthogonal and symplectic similitude groups GO_{2l+1} and GSp_{2l} , we adopt the definition and notation from [5, Sections 5, 7]. Recall that with suitable choice of defining form, the maximal torus of diagonal matrices in GSp_{2l} resp. GO_{2l+1} is given by $\{\text{diag}(t_1, \dots, t_{2l}) | t_{ii}t_{i'i'} = t_{jj}t_{j'j'} \text{ for } 1 \leq i, j \leq l\}$ resp. $\{\text{diag}(t_1, \dots, t_{2l+1}) | t_{ii}t_{i'i'} = t_{jj}t_{j'j'} \text{ for } 1 \leq i, j \leq l+1\}$, where $i' = 2l - i + 1$ resp. $i' = 2l + 1 - i + 1$ for $1 \leq i \leq 2l$ resp. $1 \leq i \leq 2l + 1$.

Corollary 2.1.12. (a) For $G = \mathrm{GL}_n, T$ the torus of diagonal matrices, a complete set of representatives of simple polynomial $G_r T$ -modules is given by $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) + p^r P(D)\}$, where $P_r(D) = \{\lambda \in P(D) | 0 \leq \lambda_i - \lambda_{i+1} \leq p^r - 1 \text{ for } 1 \leq i \leq n-1, 0 \leq \lambda_n \leq p^r - 1\}$.

(b) Let $G = \mathrm{GSp}_{2l}, T$ the torus of diagonal matrices in G . Then a complete set of representatives of the isomorphism classes of simple polynomial $G_r T$ -modules is given by $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) + p^r P(D)\}$, where $P_r(D) = \{\lambda \in P(D) \cap X_r(T) | \lambda - p^r d \notin P(D)\}$ and d is the character of T given by $t \mapsto t_{11} t_{2l2l}$.

(c) Let $G = \mathrm{GO}_{2l+1}, T$ the torus of diagonal matrices in G . If $\lambda \in X_r(T) + p^r X(T)$ and all weights of the module $\widehat{L}_r(\lambda)$ are polynomial, then $\lambda \in P_r(D) + p^r P(D)$, where $P_r(D) = \{\lambda \in P(D) \cap X_r(T) | \lambda - p^r d \notin P(D)\}$ and d is the character of T given by $t \mapsto t_{l+1}$.

(d) Let $n_i \in \mathbb{N}$ such that $n = \sum_{i=1}^l n_i$ and embed $G = \mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_l}$ into GL_n as a Levi subgroup in the obvious way. Let $M_i = \{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$ and $d_i = \sum_{j \in M_i} e_j$ for $1 \leq i \leq l$. Then a complete set of representatives of isomorphism classes of simple polynomial $G_r T$ -modules is given by $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) + p^r P(D)\}$, where $P_r(D) = \{\lambda \in X_r(T) \cap P(D) | \lambda - p^r d_i \notin P(D) \text{ for } 1 \leq i \leq l\}$.

Proof. (a) Let $d = \det|_T = (1, \dots, 1), M = \{1, \dots, n\}$. Clearly d and M have the properties of 2.1.9 and $P_r(D)$ has the required form. As the derived subgroup of GL_n is simply connected and Mat_n is normal, the result follows from 2.1.8.

(b) We have $W \cong S_l \ltimes (\mathbb{Z}/(2))^l$. We can identify W with a subgroup of $W' \cong S_{2l}$ by letting $\sigma \in S_l$ act on $\lambda \in X(T')$ via $(\sigma.\lambda)_i = \lambda_{\sigma^{-1}(i)}, (\sigma.\lambda)_{i'} = \lambda_{(\sigma^{-1}(i))'}$ for $1 \leq i \leq l$ and letting the i -th unit vector of $(\mathbb{Z}/(2))^l$ act as the transposition (ii') . With this action, the canonical projection $\pi : X(T') \rightarrow X(T)$ is W -equivariant. For $1 \leq i \leq l$, let $d_i = e_i + e_{i'}$ and $M_i = \{i, i'\}$. Then $\pi(d_i) = \pi(d_j)$ for $1 \leq i, j \leq l$, the character $d = \pi(d_1)$ is a basis of $X_0(T)$ and the M_i form a partition of $\{1, \dots, 2l\}$. Inside $W \cong S_l \ltimes (\mathbb{Z}/(2))^l$, $(\mathbb{Z}/(2))^l$ corresponds to $S_{M_1} \times \dots \times S_{M_l}$. Thus, 2.1.9 yields that G has the properties of 2.1.1 and $P_r(D)$ has the desired form. The derived subgroup of G is Sp_{2l} , hence simply connected, and by the remark below 2.11 in [7], \overline{G} is normal. The result now follows from 2.1.8.

(c) For $1 \leq i \leq l$, let $d_i = e_i + e_{i'}$, $M_i = \{i, i'\}$ and let $d_{l+1} = e_{l+1}$, $M_{l+1} = \{l+1\}$. Then the M_i form a partition of $\{1, \dots, 2l+1\}$, the character $d = \pi(d_{l+1})$ is a basis of $X_0(T)$ and $\pi(d_i) = 2\pi(d)$ for $1 \leq i \leq l$. The other properties of 2.1.9 can be checked as in the proof of (b), so that G has the properties of 2.1.1 and $P_r(D)$ has the desired form. The result now follows from 2.1.5.

(d) The M_i form a partition of $\{1, \dots, n\}$. By definition of G , we have $W = S_{M_1} \times \dots \times S_{M_l} \subseteq W' = S_n$ and the d_i form a basis of $X_0(T)$. Thus, 2.1.9 shows that G has the properties of 2.1.1, so that the result follows from 2.1.8. \square

We give an example showing that our approach does not work for the connected component G of the even orthogonal similitude group GO_{2l} . Let $l = 4$ and choose p, r such that $4|p^r - 1$. Let T, T', d be as in 2.1.12.(b). Let $\varphi : X(T') \rightarrow \mathbb{Z}$, $t \mapsto \sum_{i=1}^l \min\{t_i, t_{i'}\}$. The same proof shows that with the exception of (iv), all statements in the proof of 2.1.9 hold for φ and d . Let $\lambda_0, \tilde{\lambda} \in X(T')$ be the characters given by $\lambda_0 = (\frac{p^r-1}{2}, \frac{p^r-1}{2}, \frac{p^r-1}{2}, \frac{p^r-1}{2}, \frac{p^r-1}{4}, \frac{p^r-1}{4}, \frac{p^r-1}{4}, \frac{p^r-1}{4})$ and $\tilde{\lambda} = (0, 0, 0, 0, 1, 1, -1, 1)$. Then $\varphi(\lambda_0) = p^r - 1 > 0$, $\varphi(\lambda_0 - p^r d) = -1 < 0$ and $\lambda_0|_T \in X_r(T)$, so that $\lambda_0|_T \in P_r(D)$. In $W \cong S_l \times (\mathbb{Z}/(2))^{l-1}$, S_l acts on $X(T')$ as in the proof of 2.1.12.(b) while $(\mathbb{Z}/(2))^{l-1}$ acts as the subgroup $\{x \in (\mathbb{Z}/(2))^l | x_i \neq 0 \text{ for an even number of } i\}$ of $(\mathbb{Z}/(2))^l$. Thus, λ_0 is invariant under S_l and we have $\varphi(w \cdot \lambda_0|_T + p^r \tilde{\lambda}|_T) > 0$ and hence $w \cdot \lambda_0|_T + p^r \tilde{\lambda}|_T \in P(D)$ for all $w \in (\mathbb{Z}/(2))^{l-1}$. Hence the arguments in the proof of 2.1.5 are not applicable in this case. As the statement $w \cdot \lambda_0|_T + p^r \tilde{\lambda}|_T \in P(D)$ does not depend on φ and there is no other choice for d in this case, we cannot find other φ or d in this example to salvage the proof of 2.1.5. However, since we have not computed the weights of $\widehat{L}_r(\lambda_0|_T + p^r \tilde{\lambda}|_T)$, we do not know whether all weights of this module are in fact polynomial.

2.2 Equivalences between blocks of polynomial representations

We adopt the notation of the previous section. Suppose that G has simply connected derived subgroup and that there is $0 \neq b \in X_0(T) \cap P(D)$ such that $\langle b \rangle = X_0(T)$. By [26, II.1.18], we have $X(G) \cong X_0(T)$ via restriction, so that we can regard b as a character of G and have a shift functor $[b] : \text{mod } G_r T \rightarrow \text{mod } G_r T, V \mapsto V[b] = V \otimes_k k_b$, where k_b is the one-dimensional G -module induced by b . This shift functor is an autoequivalence of $\text{mod } G_r T$ with inverse $[-b]$, hence it induces equivalences between blocks of $\text{mod } G_r T$. Since $-b$ is not a polynomial weight, $[-b]$ does not restrict to an autoequivalence of the category of polynomial $G_r T$ -modules. However, we will see in this section that the functor $[b]$ sometimes still induces an equivalence between blocks of this category. If $\alpha \in \Delta$ and ω_α is a representative of the corresponding fundamental dominant weight of the derived subgroup of G , we can choose a preimage ω'_α of ω_α in $X(T')$. Then there is $l \in \mathbb{N}$ such that $\omega'_\alpha + l \det|_{T'} \in P(D')$, so that $\omega_\alpha + \det|_T \in P(D)$. There also is $l \in \mathbb{N}$ such that the $Z(\text{GL}_n)$ -weight of $\omega'_\alpha - l \det|_{T'}$ is negative, so that $\omega_\alpha - l \det|_T \notin P(D)$. As b generates $X_0(T)$, there is $n \in \mathbb{N}$ such that $nb = \det|_T$. Thus, we can choose representatives $(\omega_\alpha)_{\alpha \in \Delta}$ of the fundamental dominant weights of the derived subgroup such that for all $\alpha \in \Delta$, $\omega_\alpha \in P(D)$ and $\omega_\alpha - b \notin P(D)$. Then the ω_α together with b form a basis of $X(T)$. Let $\varphi : X(T) \rightarrow \mathbb{Z}$ be the projection on the coordinate corresponding to b and define $P_r(D) = \{\lambda \in X_r(T) \cap P(D) | 0 \leq \varphi(\lambda) \leq p^r - 1\}$. As the derived subgroup of G is simply connected, each $\lambda \in X(T)$ can be written uniquely as $\lambda = \lambda_0 + p^r \tilde{\lambda}$ with $\lambda_0 \in X_r(T), \tilde{\lambda} \in X(T)$. Since the sum of an element of $X_r(T)$ and $X_0(T)$ is again an element of $X_r(T)$, subtracting a suitable multiple of $p^r b$ from λ_0 and adding this multiple to $p^r \tilde{\lambda}$ yields a decomposition $\lambda = \lambda_0 + p^r \tilde{\lambda}$ with $\lambda_0 \in P_r(D), \tilde{\lambda} \in X(T)$. As $\langle b \rangle = X_0(T)$, 2.1.4 implies that such a decomposition is unique. Let $W_p \cong W \times p\mathbb{Z}R$ be the affine

Weyl group of G , let R^+ be a set of positive roots and $\rho = \sum_{\alpha \in R^+} \frac{1}{2}\alpha$ be the half-sum of positive roots. We denote by $w \bullet \lambda = w(\lambda + \rho) - \rho$ the dot-action of $w \in W_p$ on $\lambda \in X(T)$. For $a \in \mathbb{Z}$, if $a = qp + r$ for $q, r \in \mathbb{Z}$ such that $0 \leq r \leq p - 1$, we write $r = a \bmod p$.

Lemma 2.2.1. *Let the derived subgroup of G be simply connected. Let $\lambda \in P_r(D) + p^r P(D)$ and $a = \max_{w \in W} \{\varphi(w \cdot \lambda + w \cdot \rho - \rho) \bmod p\}$. Then for $1 \leq i \leq p - a - 1$, the shift $[ib]$ defines a bijection*

$$W_p \bullet \lambda \cap (P_r(D) + p^r P(D)) \rightarrow W_p \bullet (\lambda + ib) \cap (P_r(D) + p^r P(D)).$$

Proof. Let $(w, pl\mu) \in W_p$. We show that $(w, pl\mu) \bullet \lambda \in P_r(D) + p^r P(D)$ iff $(w, pl\mu) \bullet (\lambda + ib) \in P_r(D) + p^r P(D)$. We have $(w, pl\mu) \bullet \lambda = w \cdot \lambda + w \cdot \rho - \rho + pl\mu$ and $(w, pl\mu) \bullet (\lambda + ib) = (w, pl\mu) \bullet \lambda + ib$ by definition of the dot-action and since $w \cdot b = b$. Writing $w \cdot \lambda + w \cdot \rho - \rho + pl\mu = \nu_0 + p^r \tilde{\nu}$ with $\nu_0 \in P_r(D)$ and $\tilde{\nu} \in X(T)$, we have $\varphi(\nu_0) \bmod p = \varphi(w \cdot \lambda + w \cdot \rho - \rho) \bmod p \leq a$. It follows that $(\varphi(\nu_0) \bmod p) + i \leq a + p - a - 1 = p - 1$. Thus, $\varphi(\nu_0 + ib) \leq p^r - 1$, so that $\nu_0 + ib \in P_r(D)$. By unicity of decomposition, we get $(w, pl\mu) \bullet \lambda \in P_r(D) + p^r P(D) \iff \tilde{\nu} \in P(D) \iff (w, pl\mu) \bullet (\lambda + ib) \in P_r(D) + p^r P(D)$. \square

For $G = \mathrm{GL}_n$, we can choose $b = \det$ and have $a = \max_{w \in W} \{(w \cdot \lambda)_n + (w \cdot \rho)_n - \rho_n \bmod p\}$. We also have that every $G_r T$ -module V such that all weights of V are polynomial lifts to $\overline{G_r T} = M_r D$ by Jantzen's result in [30, 5.4], implying $\mathrm{Ext}_{G_r T}^1(V, W) \cong \mathrm{Ext}_{M_r D}^1(V, W)$ for all $V, W \in \mathrm{mod} M_r D$. If $p^r > d$, we have $A_d(G_r T) \cong A_d(G)$ by definition. Thus, for large r , the infinitesimal Schur algebra $S_d(G_r T)$ is isomorphic to $S_d(G)$, so that the following proposition can also be applied to blocks of the ordinary Schur algebra of GL_n .

Proposition 2.2.2. *Let $G = \mathrm{GL}_n, T$ the torus of diagonal matrices. Let $\lambda \in P_r(D) + p^r P(D)$ and $a = \max_{w \in W} \{(w \cdot \lambda)_n + (w \cdot \rho)_n - \rho_n \bmod p\}$. Then for $1 \leq i \leq p - a - 1$, the functor $[i \det]$ defines an equivalence between the blocks of $\mathrm{mod} M_r D$ containing $\widehat{L}_r(\lambda)$ and $\widehat{L}_r(\lambda + i \det)$.*

Proof. Let B be the block containing λ and B' be the block containing $\lambda + i \det$. As $[i \det]$ maps nonsplit extensions in $\mathrm{mod} M_r D$ to nonsplit extensions in $\mathrm{mod} M_r D$, we have $B[i \det] \subseteq B'$. Let $\mu \in B'$. Suppose there is $\nu \in B[i \det]$ such that $\mathrm{Ext}_{M_r D}^1(\widehat{L}_r(\mu), \widehat{L}_r(\nu)) \neq 0$. Then $\mu \in W_p \bullet (\lambda + i \det)$ since $B' \subseteq W_p \bullet (\lambda + i \det)$ by [26, II.9.19]. Now 2.2.1 yields $\mu - i \det \in P_r(D) + p^r P(D)$. Applying $[-i \det]$ to a nonzero element of $\mathrm{Ext}_{M_r D}^1(\widehat{L}_r(\mu), \widehat{L}_r(\nu))$, we get $\mathrm{Ext}_{G_r T}^1(\widehat{L}_r(\mu - i \det), \widehat{L}_r(\nu - i \det)) \neq 0$. Since both simple modules lift to $M_r D$, Jantzen's result yields $\mathrm{Ext}_{G_r T}^1(\widehat{L}_r(\mu - i \det), \widehat{L}_r(\nu - i \det)) \cong \mathrm{Ext}_{M_r D}^1(\widehat{L}_r(\mu - i \det), \widehat{L}_r(\nu - i \det)) \neq 0$, so that $\mu - i \det \in B$ and $\mu \in B[i \det]$. \square

We do not know whether an analogue of [30, 5.4] and hence a generalization of the foregoing result holds for other reductive group schemes.

3 Polynomial representations of $G_r T$

3.1 Contravariant duality

In this section, let $M \subseteq \text{Mat}_n$ be a normal algebraic monoid with reductive unit group G containing $Z(\text{GL}_n)$, $T \subseteq G$ a maximal torus, $B, B^- \subseteq G$ opposite Borel subgroups of G containing T and $D = \overline{T}$. We denote by $P(D)$ the monoid of characters of D and identify $P(D)$ with a submonoid of the character group $X(T)$ of T . We want to define an analogue of contravariant duality (see [10, 2.1]) in this situation.

Proposition 3.1.1. *There is an anti-automorphism $\eta : M \rightarrow M$ such that $\eta|_D = \text{id}_D$, $\eta^2 = \text{id}_M$ and $\eta(\overline{G_r T}) = \overline{G_r T}$, $\eta(\overline{B_r T}) = \overline{B_r^- T}$.*

Proof. By [32, Theorem 5.2], the anti-automorphism $G \rightarrow G$ from [26, II.1.16] can be extended to an anti-homomorphism $\eta : M \rightarrow M$ such that $\eta|_D = \text{id}_D$. As $\eta^2|_G = \text{id}_G$, unicity of extension yields $\eta^2 = \text{id}_M$. As $\eta(G_r T) = G_r T$, we have $\overline{G_r T} \subseteq \eta(\overline{G_r T})$. Using $\eta^2 = \text{id}_M$, we get $\eta(\overline{G_r T}) \subseteq \overline{G_r T}$. Analogous arguments show $\eta(\overline{B_r T}) = \overline{B_r^- T}$. \square

Definition 3.1.2. *The anti-automorphism of 3.1.1 induces an involutory anti-automorphism of $S_d(G_r T)$ mapping $S_d(B_r T)$ to $S_d(B_r^- T)$ and restricting to the identity on $S_d(T)$ for every $d \in \mathbb{N}_0$, resulting in duality functors $\text{mod } S_d(G_r T) \rightarrow \text{mod } S_d(G_r T)$ and $\text{mod } S_d(B_r T) \rightarrow S_d(B_r^- T)$ which by abuse of notation will be denoted by $(-)^{\circ}$. We call these functors **contravariant dualities**.*

By [26, II.9.6(13)], we have $\widehat{L}_r(\lambda)^{\circ} \cong \widehat{L}_r(\lambda)$ for all $\lambda \in X(T)$ and the simple module $\widehat{L}_r(\lambda)$. Taking $G = \text{GL}_n$ and η to be the transposition map in the proof of 3.1.1, we recover the contravariant duality functor from [10, 2.1].

Now let $X \in \{B_r T, G_r T\}$. For $\mathcal{T} = \text{mod } \overline{X}$, we let $\mathfrak{F}_{\overline{X}} = t$, $\mathcal{G}_{\overline{X}} = u$ be the functors defined above 1.3.2. Applying the canonical isomorphism $(V^{\circ})^{\circ} \cong V$ to the submodule $(\mathcal{G}_{\overline{X}}(V^{\circ}))^{\circ}$ of $(V^{\circ})^{\circ}$, we get the following connection between $\mathfrak{F}_{\overline{X}}$ and $\mathcal{G}_{\overline{X}}$.

Proposition 3.1.3. *There is an isomorphism $\mathfrak{F}_{\overline{X}}(V) \cong (\mathcal{G}_{\overline{X}}(V^{\circ}))^{\circ}$ natural in $V \in \text{mod } X$, where we abuse notation for $X = B_r T$ by letting $\mathcal{G}_{\overline{X}}(V^{\circ})$ denote the largest $\overline{B_r^- T}$ -factor module of V° .*

Let C be a finite-dimensional k -coalgebra and $A = C^*$ be the dual associative algebra. If V is a finite-dimensional right C -comodule with comodule map $\rho : V \rightarrow V \otimes C$, then V obtains the structure of a left A -module via

$$f.v = \mu_V \circ (\text{id}_V \otimes f) \circ \rho(v)$$

for all $v \in V, f \in A$, where $\mu_V : V \otimes_k k \rightarrow V$ is the scalar multiplication, see [35, Proposition 2.2]. In particular, the comultiplication $C \rightarrow C \otimes_k C$ induces the structure of a left A -module on C .

Lemma 3.1.4. *Let C be a finite-dimensional k -coalgebra, $A = C^*$ be the dual associative algebra and $\eta : A \rightarrow A$ be an involutory anti-automorphism. Then the regular module A is isomorphic to the contravariant dual of C with respect to η as an A -module.*

Proof. We define

$$(-, -) : A \times C \rightarrow k, (f, c) \mapsto \eta(f)(c).$$

By general theory, we have to show that $(-, -)$ is a non-degeneratve contravariant form. Since η is bijective, $(-, -)$ is non-degenerate. Now let $f, g \in A, c \in C$ and let μ_k be the multiplication map $k \otimes_k k \rightarrow k$, μ_C the scalar multiplication $C \otimes_k k \rightarrow C$. We have

$$(f.g, c) = \eta(fg)(c) = \mu_k \circ (\eta(g) \otimes \eta(f)) \circ \Delta(c)$$

and

$$(g, \eta(f).c) = \eta(g) \circ \mu_C \circ (\text{id}_C \otimes \eta(f)) \circ \Delta(c) = \mu_k \circ (\eta(g) \otimes \eta(f)) \circ \Delta(c),$$

so that $(f.g, c) = (g, \eta(f).c)$ and $(-, -)$ is contravariant. \square

We want to generalize [9, Theorem 5.4]. To formulate an analogous statement, we first need idempotents in $S_d(G_r T)$ with a connection to characters of T .

Lemma 3.1.5. *Let χ_1, \dots, χ_n be a basis of $A_d(T)$ consisting of characters and $\chi_1^*, \dots, \chi_n^*$ the dual basis.*

(1) *The χ_i^* form a complete set of primitive orthogonal idempotents.*

(2) *An element $f \in S_d(T)$ is idempotent iff f is a sum of some of the χ_i^* .*

Proof. (1) As the χ_i are group-like, we have by definition

$$\chi_i^* \chi_j^*(\chi_l) = \mu \circ \chi_i^* \otimes \chi_j^* \circ \Delta(\chi_l) = \delta_{il} \delta_{jl} = \chi_i^*(\chi_l) \chi_j^*(\chi_l),$$

so that the χ_i^* are orthogonal idempotents. Since the χ_i^* form a basis, they are primitive.

(2) Clear. \square

If $\lambda \in P(D)$, we denote the element dual to λ in $S_d(T) = A_d(T)^* = kP(D)_d^*$ by ξ_λ . Through the embedding $S_d(T) \rightarrow S_d(G_r T)$ induced by the projection $A_d(G_r T) \rightarrow A_d(T)$, we can view ξ_λ as an element of $S_d(G_r T)$ which is then an idempotent (not necessarily primitive) in $S_d(G_r T)$.

Induction for algebraic monoids is defined in [11, Section 2.1]. Standard properties of this notion are proved in [6, Section 2].

Proposition 3.1.6. *Let $\lambda \in P(D)$. There is an isomorphism of $S_d(G_rT)$ -modules $(\text{ind}_D^{\overline{G_rT}} \lambda)^o \cong S_d(G_rT)\xi_\lambda$.*

Proof. Let $(-, -) : S_d(G_rT) \times A_d(G_rT) \rightarrow k, (\xi, c) \mapsto \eta(\xi)(c)$. Then $(-, -)$ is a non-degenerate contravariant form by the proof of 3.1.4. We show that the restriction to $S_d(G_rT)\xi_\lambda \times \text{ind}_D^{\overline{G_rT}} \lambda$ is non-degenerate. Let $\xi \in S_d(G_rT), c \in A_d(G_rT)$ and let μ denote the multiplication map $k \otimes_k k \rightarrow k$. Then

$$(\xi\xi_\lambda, c) = \eta(\xi\xi_\lambda)(c) = (\xi_\lambda\eta(\xi))(c) = \mu \circ (\xi_\lambda \otimes \eta(\xi)) \circ \Delta(c)$$

as $\eta(\xi_\lambda) = \xi_\lambda$. Let $\pi : k[G_rT] \rightarrow k[T]$ be the canonical projection. By definition, $\text{ind}_D^{\overline{G_rT}} \lambda$ is the λ -weight space of $A(G_rT)$ for the right action of left translation by elements of D . Thus, if $\pi : A_d(G_rT) \rightarrow A_d(T)$ is the canonical projection and $c \in \text{ind}_D^{\overline{G_rT}} \lambda \setminus \{0\}$, we have $(\pi \otimes \text{id}_{A_d(G_rT)}) \circ \Delta(c) = \lambda \otimes c$. By definition of ξ_λ , we have $(\xi_\lambda \otimes \eta(\xi)) \circ \Delta(c) = (\xi_\lambda \circ \pi_{A_d(G_rT)} \otimes \eta(\xi)) \circ \Delta(c) = 1 \otimes \eta(\xi)(c)$. Choosing $\xi \in S_d(G_rT)$ such that $\eta(\xi)(c) \neq 0$, we get $(\xi\xi_\lambda, c) \neq 0$, so that the restriction of $(-, -)$ is non-degenerate. By general theory (see for example [29, 3.4.5]), we get $(\text{ind}_D^{\overline{G_rT}} \lambda)^o \cong S_d(G_rT)\xi_\lambda$. \square

If V, W are G_rT -modules with V indecomposable, we write $[W : V]$ for the multiplicity of V as a direct summand of W . For $\mu \in P(D)$ such that the simple G_rT -modules $\widehat{L}_r(\lambda)$ lifts to $\text{mod } \overline{G_rT}$, we denote by $\widehat{P}_r(\mu)$ and $\widehat{I}_r(\mu)$ the projective cover and the injective hull of $\widehat{L}_r(\mu)$ in $\text{mod } \overline{G_rT}$, respectively. The following is a generalization of [9, Theorem 5.4] to our setting.

Corollary 3.1.7. *Let $\lambda \in P(D)$ be a weight of degree d . If $\mu \in P(D)$ such that the simple G_rT -module $\widehat{L}_r(\mu)$ is a polynomial G_rT -module, we have*

$$[\text{ind}_D^{\overline{G_rT}} \lambda : \widehat{I}_r(\mu)] = \dim_k \widehat{L}_r(\mu)_\lambda = [S_d(G_rT)\xi_\lambda : \widehat{P}_r(\mu)].$$

Proof. Using 3.1.4 and 3.1.6, we can employ the same proof as in [9, Theorem 5.4]. \square

3.2 Representations of infinitesimal Schur algebras

We fix an embedding $G \subseteq \text{GL}_n$ of a smooth connected algebraic group scheme and let $T \subseteq G$ be a maximal torus. Our first lemma shows that the category of polynomial G_rT -modules is closed with respect to canonical operations.

Lemma 3.2.1. *The category of polynomial G_rT -modules is closed with respect to submodules, factor modules and finite direct sums.*

Proof. Let V be a polynomial G_rT -module and $W \subseteq V$ be a G_rT -submodule. Then the comodule map $\rho : V \rightarrow V \otimes_k k[G_rT]$ factors through $V \otimes_k A(G_rT)$. Since W is a submodule of V , we get $\rho(W) \subseteq W \otimes_k k[G_rT] \cap V \otimes_k A(G_rT) = W \otimes_k A(G_rT)$, so that W is a polynomial G_rT -module. Since V/W has a natural structure as an $A(G_rT)$ -comodule, the category of polynomial G_rT -modules is closed with respect to factor modules. The statement about direct sums can be proved similarly. \square

From now on, we always assume $Z(\mathrm{GL}_n) \subseteq G$. As any $S_d(G_rT)$ -comodule is also a $\overline{G_rT}$ -comodule, 1.4.1 yields

Corollary 3.2.2. *If $V \in \mathrm{mod} \overline{G_rT}$ is indecomposable, then $V = V_d$ for some $d \in \mathbb{N}_0$.*

The preceding result and 1.4.1 show that the blocks of $\mathrm{mod} \overline{G_rT}$ are just the blocks of the $S_d(G_rT)$ for $d \in \mathbb{N}_0$. Thus, injective and projective indecomposables as well as almost split sequences in $\mathrm{mod} S_d(G_rT)$ coincide with those in $\mathrm{mod} \overline{G_rT}$. In particular, $\mathrm{mod} \overline{G_rT}$ has almost split sequences.

We call the elements of $X(Z(\mathrm{GL}_n))$ central characters. As $Z(\mathrm{GL}_n) \subseteq G$ is a torus contained in the center of G_rT , the $Z(\mathrm{GL}_n)$ -weight spaces of a G_rT -module V are G_rT -submodules of V . This shows that $Z(\mathrm{GL}_n)$ acts via a single central character on indecomposable G_rT -modules V and that indecomposable modules affording different central characters belong to different blocks. If V is a module for $\overline{G_rT}$, these submodules are just the homogeneous components of 1.4.1. We say that $V \in \mathrm{mod} G_rT$ is homogeneous of degree d if $Z(\mathrm{GL}_n)$ acts on V via the central character given by $d \in \mathbb{Z}$. If $V = V_d, W = W_{d'} \in \mathrm{mod} G_rT$ are homogeneous such that $d \neq d'$, then any exact sequence starting in V and ending in W splits as the middle term is the direct sum of its homogeneous constituents. As almost split sequences are non-split, we get

Proposition 3.2.3. *Let Θ be a component of the stable Auslander-Reiten quiver of $\mathrm{mod} G_rT$. Then there is $d \in \mathbb{Z}$ such that all modules in Θ are homogeneous of degree d .*

For $G \neq \mathrm{GL}_n$ with maximal torus $T \subseteq G$, it is not known whether a G_rT -module V such that all T -weights of V are polynomial is a polynomial G_rT -module. However, for $G = \mathrm{GL}_n$, this was proved by Jantzen in [30, Appendix]. We can also prove it for B_rT , where $B \subseteq \mathrm{GL}_n$ is the Borel subgroup of lower triangular matrices. In the remainder of this section, let $G = \mathrm{GL}_n, T \subseteq \mathrm{GL}_n$ the torus of diagonal matrices, $B \subseteq \mathrm{GL}_n$ the Borel subgroup of lower triangular matrices, $U \subseteq B$ the subgroup of unipotent lower triangular matrices and $X \in \{G_rT, B_rT\}$, $D \subseteq \mathrm{Mat}_n$ the monoid scheme of diagonal matrices and $P(D)$ the character monoid of D . We identify $P(D)$ with a submonoid of $X(T)$ and call this submonoid the set of polynomial weights.

Theorem 3.2.4. *If V is an X -module such that all T -weights of V are polynomial, then V lifts to \overline{X} .*

Proof. For $X = G_rT$, this is Jantzen's result in [30, Appendix]. Let $X = B_rT = U_r \rtimes T$, $\rho : V \rightarrow V \otimes_l k[B_rT]$ be the comodule map and $v \in V$. Since all T -weights of V are polynomial, $V|_T$ lifts to D , so there are $f_i \in k[D], v_i \in V$ such that

$$t.(v \otimes 1) = \sum_{i=1}^s v_i \otimes f_i(t)$$

for all $t \in T(A)$ and all commutative k -algebras A . Since $B_rT = U_rT = U_r \rtimes T$, there are $g_{ij} \in k[U_r], v_{ij} \in V$ such that

$$(ut).(v \otimes 1) = u.(t.(v \otimes 1)) = \sum_{i=1}^s \sum_{j=1}^l v_{ij} \otimes g_{ij}(u) f_i(t)$$

for all $t \in T(A), u \in U_r(A)$ and all commutative k -algebras A . Applying this to $A = k[B_r T]$ and $\text{id}_A \in B_r T(A)$, we see that

$$\rho(v) = \sum_{i=1}^s \sum_{j=1}^l v_{ij} \otimes g_{ij} \otimes f_i \in V \otimes_k k[U_r] \otimes_k k[D] = V \otimes_k k[L_r D],$$

so that V lifts to $L_r D$. \square

Viewing X -modules as \mathbb{Z}^n -graded G_r resp. B_r -modules, the theorem shows that $V \in \text{mod } \overline{X}$ if and only if $V \in \text{mod } X$ and $\text{supp}(V) \subseteq \mathbb{N}_0^n$.

Corollary 3.2.5. *The category $\text{mod } \overline{X}$ is extension-closed in $\text{mod } X$.*

Thus, $\text{mod } \overline{X}$ is a torsion class and a torsion-free class, so that all results of Section 1.3 can be applied to $\text{mod } \overline{X}$ and we have $t = \mathfrak{F}_{\overline{X}}, u = \mathcal{G}_{\overline{X}}$ for the torsion radical t and the functor $u(V) = V/t(V)$.

In the remainder of this section, let Θ be a regular component of type $\mathbb{Z}[A_\infty]$ of the stable Auslander-Reiten quiver $\Gamma_s(G_r T)$ of $G_r T$. By [26, II.9.6(13)], we have $\widehat{L}_r(\lambda)^\circ \cong \widehat{L}_r(\lambda)$ for all $\lambda \in X(T)$ and the simple module $\widehat{L}_r(\lambda)$. We can use this to determine the position of indecomposable Ext-projective modules $V \in \text{mod } M_r D$ in Θ .

Proposition 3.2.6. *Let $V \in \Theta \cap \text{mod } M_r D$ be Ext-projective in $\text{mod } M_r D$. Then V is quasi-simple and lies at the left side of a wing in $\Theta \cap \text{mod } M_r D$.*

Proof. This follows directly from 1.3.1, 1.3.5 and 1.3.7. \square

Since $\text{mod } G_r T$ is a Frobenius category, there are no non-projective $G_r T$ -modules of finite projective dimension. However, there are $M_r D$ -modules with finite projective dimension.

Corollary 3.2.7. *Let $V \in \Theta \cap \text{mod } M_r D$ and P be the projective cover of V in $\text{mod } G_r T$. Suppose $P \in \text{mod } M_r D$ and $\text{pd}_{M_r D}(V) = 1$. Then V is quasi-simple and $\tau_{G_r T}(V) \notin \text{mod } M_r D$.*

Proof. We have $\tau_{G_r T}(V) = \Omega_{G_r T}^2(V)$ by [18, 7.2.3]. Since $\text{pd}_{M_r D}(V) = 1$ and $P \in M_r D$, we have $\Omega_{G_r T}^1(V) = \Omega_{M_r D}^1(V)$ projective in $\text{mod } M_r D$. In particular, the projective cover of $\Omega_{G_r T}(V)$ in $\text{mod } G_r T$ is not an $M_r D$ -module, so that $\Omega_{G_r T}^2(V) \notin \text{mod } M_r D$ by 3.2.4. As $\Omega_{G_r T}^1(\Theta)$ is a component isomorphic to Θ by [2, p. 338], 3.2.6 implies that $\Omega_{G_r T}^1(V)$ is quasi-simple. Hence V is also quasi-simple. \square

Proposition 3.2.8. *Suppose $V \in \text{mod } M_r D$ is indecomposable and $\mathcal{G}_{M_r D}$ -acyclic such that $\text{pd}_{M_r D}(V) < \infty$. Then there are only finitely many $M_r D$ -modules in $\{\tau_{G_r T}^i(V) \mid i \in \mathbb{N}_0\}$ and they all have finite projective dimension.*

Proof. By [18, 7.2.3], we have $\tau_{G_r T} \cong \Omega_{G_r T}^2$. Setting $s = \text{pd}_{M_r D}(V)$, a dual version of 1.3.2 shows that $\mathcal{G}_{M_r D}(\tau_{G_r T}^l(V)) = \mathcal{G}_{M_r D}(\Omega_{G_r T}^{2l}(V)) = 0$ for $l > \frac{s}{2}$, so that $\tau_{G_r T}^l(V) \notin \text{mod } M_r D$. Since minimal projective resolutions for V in $\text{mod } G_r T$ induce minimal projective resolutions for $\mathcal{G}_{M_r D}(V)$ in $\text{mod } M_r D$, this also shows that all $M_r D$ -modules in $\{\tau_{G_r T}^i(V) \mid i \in \mathbb{N}_0\}$ have finite projective dimension. \square

Corollary 3.2.9. *Suppose Θ contains a $\mathcal{G}_{M_r D}$ -acyclic $M_r D$ -module V which is quasi-simple such that $\text{pd}_{M_r D}(V) < \infty$ and a self-dual module N . Then Θ contains only finitely many $M_r D$ -modules.*

Proof. Since $X \in \text{mod } M_r D$ iff $X^\circ \in \text{mod } M_r D$ for all $X \in \text{mod } G_r T$, this follows from 3.2.8 and 1.3.10. \square

Proposition 3.2.10. *Let $V \in \text{mod } M_r D$ be $\mathcal{G}_{M_r D}$ -acyclic and indecomposable such that $\text{pd}_{M_r D}(V) \leq 2n$ and $\tau_{G_r T}^n(V) \in \text{mod } M_r D$. Then $\tau_{G_r T}^n(V)$ is projective in $\text{mod } M_r D$. If moreover $V \in \Theta$, then V is quasi-simple.*

Proof. Let $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow V \rightarrow 0$ be a minimal projective resolution in $\text{mod } G_r T$. As V is $\mathcal{G}_{M_r D}$ -acyclic, a dual version of 1.3.2 yields a minimal projective resolution $\dots \rightarrow \mathcal{G}_{M_r D}(P_1) \rightarrow \mathcal{G}_{M_r D}(P_0) \rightarrow V \rightarrow 0$ in $\text{mod } M_r D$. Since $\text{pd}_{M_r D}(V) \leq 2n$, we get $\mathcal{G}_{M_r D}(P_{2n+1}) = 0$. Now $\mathfrak{F}_{M_r D}(\tau_{G_r T}^{n+1}(V)) \cong \mathfrak{F}_{M_r D}(\Omega_{G_r T}^{2n+2}(V)) \subseteq \mathfrak{F}_{M_r D}(P_{2n+1}) \cong \mathcal{G}_{M_r D}(P_{2n+1})^\circ = 0$ by 3.1.3 since $P_{2n+1}^\circ \cong P_{2n+1}$. By 1.3.1, $\tau_{G_r T}^n(V)$ is projective in $\text{mod } M_r D$. If $V \in \Theta$, 3.2.6 shows that V is quasi-simple. \square

In the next result, we adapt a standard Morita-equivalence between blocks of $\text{mod } G_r T$ to blocks of $\text{mod } M_r D$. Later we will also need that this equivalence is compatible with the restriction functor to $(\text{SL}_n)_r$. We denote by $St_s \cong \widehat{Z}_s((p^s - 1)\rho)$ the s -th Steinberg module, where ρ is the half-sum of positive roots and we choose the root system of GL_n as in [26, I.1.21]. If $V \in \text{mod } G_r T$, we denote by $V^{[s]}$ the module obtained by composing the module structure on V with the s -th iteration of the Frobenius morphism and by $k_{1/2(p^s-1)(n-1)\det|_{G_r T}}$ the 1-dimensional module given by the character $\frac{1}{2}(p^s - 1)(n - 1)\det|_{G_r T}$ of $G_r T$.

Proposition 3.2.11. *Suppose $p \geq 3$. Let b be a block of $M_{r-s}D$. The functor $A : \text{mod } G_{r-s}T \rightarrow \text{mod } G_r T, V \mapsto St_s \otimes_k V^{[s]} \otimes_k k_{1/2(p^s-1)(n-1)\det|_{G_r T}}$ commutes with the forgetful functors $\text{mod } G_{r-s}T \rightarrow \text{mod } (\text{SL}_n)_{r-s}, \text{mod } G_r T \rightarrow \text{mod } (\text{SL}_n)_r$ and induces a Morita-equivalence from b to a block of $M_r D$.*

Proof. We have $\text{Ext}_{M_r D}^1(V_1, V_2) \cong \text{Ext}_{G_r T}^1(V_1, V_2)$ for all $V_1, V_2 \in \text{mod } M_r D$ by 3.2.4, so there is a block b' of $\text{mod } G_{r-s}T$ containing b . By [26, II.10.5], $V \mapsto St_s \otimes_k V^{[s]}$ induces a Morita-equivalence between b' and a block of $G_r T$. Since shifting with a character of $G_r T$ is an equivalence of categories, A also induces such an equivalence. We now show that $V \in \text{mod } M_{r-s}D$ iff $A(V) \in \text{mod } M_r D$. Since $St_s \cong \widehat{Z}_s((p^s - 1)\rho)$, so that the set of weights of St_s is $W \cong S_n$ -invariant by [26, II.9.16(1)] and $-(n - 1)$ occurs as a coordinate of 2ρ , we have that for every $i \in \{1, \dots, n\}$, there is a weight λ of St_s such that $\lambda_i = -\frac{1}{2}(p^s - 1)(n - 1)$. Suppose there is a weight μ of St_s with

a smaller coordinate. Using the action of the Weyl group, we may assume that $\mu_n < -\frac{1}{2}(p^s - 1)(n - 1)$. Then [26, II.9.2(6)] implies that $(p^s - 1)\rho - \mu$ is a sum of positive roots. As $((p^s - 1)\rho - \mu)_n > 0$ and there is no positive root with a positive n -th coordinate, this is a contradiction. Thus, there is no weight μ of St_s with $\mu_i < -\frac{1}{2}(p^s - 1)(n - 1)$. As the grading on $V^{[s]}$ arises by multiplying all degrees in the grading of V by p^s , we get that V has a weight with a negative coordinate iff $A(V)$ has a weight with a negative coordinate by the definition of the grading on a tensor product. Thus, $V \in \text{mod } M_{r-s}D$ iff $A(V) \in \text{mod } M_rD$. Hence A maps the class of simple $M_{r-s}D$ -modules of the $G_{r-s}T$ -block containing b to the class of simple M_rD -modules in the block of G_rT containing $A(b)$. As $\text{Ext}_{M_rD}^1(V_1, V_2) \cong \text{Ext}_{G_rT}^1(V_1, V_2)$ for all $V_1, V_2 \in \text{mod } M_rD$, we see that V_1, V_2 belong to the same block of $M_{r-s}D$ iff $A(V_1), A(V_2)$ belong to the same block of M_rD , so that A induces a Morita-equivalence from b to a block of M_rD . Since \det vanishes on SL_n , the functor A commutes with the forgetful functors. \square

3.3 Modules of complexity one

In this section, let $\text{char}(k) \geq 3$. Let $G \subseteq \text{GL}_n$ be a reductive algebraic group over k with $Z(\text{GL}_n) \subseteq G$, $T \subseteq G$ a maximal torus, R the root system of G relative to T and $B = U \rtimes T \subseteq G$ a Borel subgroup containing T with unipotent radical U . Let $R^+ \subseteq R$ be the set of positive roots determined by B . Recall that the complexity $cx(V)$ of a module V is the polynomial rate of growth of a minimal projective resolution of V . Let $X \in \{G_rT, B_rT\}$ and Θ be a component of $\Gamma_s(X)$. By [15, Section 1] and [18, 8.1], the complexity of a module coincides with the dimension of its rank variety and we have $cx_{G_rT}(V_1) = cx_{G_rT}(V_2)$ for all $V_1, V_2 \in \Theta$, so that we may define $cx_{G_rT}(\Theta) = cx_{G_rT}(V_1)$. As in the previous section, we denote by \overline{X} the closure of X in Mat_n . In this section, we will show that if $cx_{G_rT}(\Theta) = 1$, then $\Theta \cap \text{mod } \overline{X}$ is finite.

Lemma 3.3.1. *Let $X \in \{B_rT, G_rT\}$ and $cx_X(\Theta) = 1$.*

- (1) *If $X = G_rT$, there are $\alpha \in R$, $0 \leq s \leq r - 1$ such that $\tau_X^{p^s}(V) = V[p^r\alpha]$ for all $V \in \Theta$.*
- (2) *If $X = B_rT$, there are $\alpha \in R^+ \cup \{0\}$, $0 \leq s \leq r - 1$ such that $\tau_X^{p^s}(V) = V[p^r\alpha - 2p^s(p^r - 1)\rho]$ for all $V \in \Theta$, where $\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta$ is the half-sum of positive roots.*

Proof. (1) This follows directly from [18, 6.1.2, 7.2.3, 8.1.2], noting that the proof of [18, 8.1.2] only depends on the support variety and hence on the component of V .

- (2) Let $X = B_rT$. Then [18, 7.2.2] yields that $\mathcal{N}(V) = V \otimes_k k_{\lambda_B|_{B_rT}}[-p^r\lambda_B|_T]$ for the Nakayama functor \mathcal{N} of $\text{mod } B_rT$, where λ_B is the character of B given by $\lambda_B = \det \circ \text{Ad}$. By the remarks preceding [18, 7.2.2], $\lambda_B|_T = 2\rho$. As $B_rT \cong U_r \rtimes T$ with U_r unipotent, tensoring with $k_{\lambda_B|_{B_rT}}$ only changes the T -action on a module, so that $V \otimes_k k_{\lambda_B|_{B_rT}} \cong V[2\rho]$ and $\mathcal{N}(V) = V[-2(p^r - 1)\rho]$. By [18, 6.1.2, 8.1.1],

there exist $\alpha \in R^+ \cup \{0\}$ and $s \in \mathbb{N}_0$ such that $\Omega_{B_r T}^{2p^s}(V) \cong V \otimes_k p^r \alpha$, so that $\tau^{p^s}(V) \cong \mathcal{N}^{p^s}(V[p^r \alpha]) \cong V[p^r \alpha - 2p^s(p^r - 1)\rho]$. \square

Lemma 3.3.2. *Let $X \in \{B_r T, G_r T\}$, V be an indecomposable X -module such that $cx_X(V) = 1$. Then there are only finitely many polynomial X -modules in the τ_X -orbit of V .*

Proof. Let $X = B_r T$ and $R^+ \subseteq R$ be the set of roots of B . By 3.3.1, (2), we have $\tau_{B_r T}^{p^s}(V) = V[p^r \alpha - 2p^s(p^r - 1)\rho]$ for some $\alpha \in R^+ \cup \{0\}$, $0 \leq s \leq r - 1$. It follows that

$$\text{supp}(\tau_{B_r T}^{ip^s}(V)) = \text{supp}(V) + ip^r \alpha - 2ip^s(p^r - 1)\rho$$

for all $i \in \mathbb{Z}$. Since $p^r \alpha \neq 2p^s(p^r - 1)\rho$, we have $\text{supp}(\tau_{B_r T}^{ip^s}(V)) \neq \text{supp}(\tau_{B_r T}^{jp^s}(V))$ for $i \neq j$. As $Z(\text{GL}_n) \subseteq G$, the degree d weights of a maximal torus of GL_n containing T surject onto the degree d weights of T . As all weights in the sets $\text{supp}(\tau_{B_r T}^{ip^s}(V))$ have degree d and there are only finitely many polynomial weights of a given degree d for GL_n , only finitely many of these sets consist entirely of polynomial weights. Thus, the $\tau_{B_r T}^{p^s}$ -orbit of M contains only finitely many polynomial $B_r T$ -modules. As the $\tau_{B_r T}$ -orbit of V is the union of the $\tau_{B_r T}^{p^s}$ -orbits of $V, \tau_{B_r T}(V), \dots, \tau_{B_r T}^{p^s-1}(V)$, the result follows. For $X = G_r T$, apply 3.3.1, (1), and use analogous arguments. \square

We denote by F the forgetful functor $\text{mod } G_r T \rightarrow \text{mod } G_r$ resp. $\text{mod } B_r T \rightarrow \text{mod } U_r$.

Proposition 3.3.3. *Let $X \in \{B_r T, G_r T\}$ and V be an indecomposable polynomial X -module such that $cx_X(V) = 1$ and Θ be the component of $\Gamma_s(X)$ containing V . Then Θ contains only finitely many polynomial X -modules.*

Proof. We first show that Θ is either regular or has only finitely many τ_X -orbits. Let F be the forgetful functor. If $X = G_r T$, then Θ is regular; else there would be a projective indecomposable $G_r T$ -module P such that $\text{Rad}(P) \in \Theta$ by [2, V.5.5], so that $cx_{G_r T}(\text{Rad}(P)) = 1$ and the simple $G_r T$ -module $S = \Omega_{G_r T}^{-1}(\text{Rad}(P))$ would also have complexity 1. But then $cx_{G_r}(F(S)) = 1$, a contradiction to [19, Lemma 2.2]. If $X = B_r T = U_r \rtimes T$ and Θ is a non-regular component of complexity 1, the above arguments show that there is a simple $B_r T$ -module k_λ of complexity 1, so that $cx_{U_r}(F(k_\lambda)) = cx_{U_r}(k) = 1$. As the rank variety of $(U_r)_2$ is contained in that of U_r , we get $cx_{(U_r)_2} \leq 1$ and [20, Theorem 2.7] shows that U_r has finite representation type. Thus, $F(\Theta)$ is finite and [15, 5.6] implies $F(\Theta) \cong \mathbb{Z}[A_{p^l}]/(\tau)$ for some $l \in \mathbb{N}$, so that there are two distinct vertices in $F(\Theta)$ with only one successor and that all other vertices in $F(\Theta)$ have exactly two predecessors. Since $N \in \text{mod } B_r T$ is projective iff $F(N) \in \text{mod } U_r$ is projective by 1.2.1, it follows from 1.2.2 that there are two distinct $\tau_{B_r T}$ -orbits in Θ such that every element of these orbits has exactly one predecessor and all other orbits consist of vertices with exactly two predecessors. Thus, Θ does not have tree class $A_\infty, A_\infty^\infty$ or D_∞ . Then [19, 3.4] (which does not depend on the group being reductive) implies that Θ has only finitely many $\tau_{B_r T}$ -orbits.

Thus, Θ is either regular or has only finitely many τ_X -orbits. In the second case, the

result follows from 3.3.2. In the first case, [15, 5.6] and [18, 8.2.2] imply $\Theta \cong \mathbb{Z}[A_\infty]$. Letting $\Theta \cong \mathbb{Z}[A_\infty]$ and $N \in \Theta$ be quasi-simple, the set of quasi-simple modules in Θ is the τ_X -orbit of N . Thus, 3.3.2 shows that there are only finitely many quasi-simple polynomial X -modules in Θ , so that the result follows from 1.3.6. \square

In the remainder of this section, let $G = \mathrm{GL}_n$ with the notation of Section 3.2, $T \subseteq G$ the maximal torus of diagonal matrices and $B \subseteq G$ the Borel subgroup of upper triangular matrices.

Proposition 3.3.4. *Let $F(\Theta) = \mathbb{Z}[A_\infty]/\langle \tau^{p^t} \rangle$ and $V \in \Theta \cap \mathrm{mod} \bar{X}$ such that $ql(V) > \frac{3p^t-2}{2}$. Then the following statements hold:*

- (1) *For every $N \in \Theta \cap \mathrm{mod} \bar{X}$ such that $ql(N) > \frac{p^t}{2}$, there is an undirected path from V to N in $\Theta \cap \mathrm{mod} \bar{X}$.*
- (2) *$\Theta \cap \mathrm{mod} \bar{X}$ contains a unique module of maximal quasi-length.*

Proof. We first show:

$$(*) \text{ If } \tau_X^{ip^t}(N) \notin \mathrm{mod} \bar{X}, \text{ then } \tau_X^{jp^t}(N) \notin \mathrm{mod} \bar{X} \text{ for all } j > i.$$

Let $X = B_r T$. Since $F(\Theta) \cong \mathbb{Z}[A_\infty]/\langle \tau^{p^t} \rangle$, we get that for each $N \in \Theta$, there is $\lambda_N \in X(T)$ such that $\tau_X^{p^t}(N) = N[\lambda]$. As shifting with λ maps almost split sequences to almost sequences, we get $\lambda_N = \lambda_{N'}$ for every N' which is a successor, predecessor or element of the τ_X -orbit of N . Since Θ is a connected component, we inductively get that $\lambda_N = \lambda_{N'}$ for all $N, N' \in \Theta$. Thus, $(\tau_X|_\Theta)^{p^t} = [\lambda]$ for some $\lambda \in X(T)$. By 3.3.1, (2), we also find suitable $\alpha \in R^+ \cup \{0\}, s \in \mathbb{N}_0$ such that $(\tau_X|_\Theta)^{p^s} = [p^r \alpha - 2p^s(p^r - 1)\rho]$ for all $V \in \Theta$. As $G = \mathrm{GL}_n$ and we have chosen the roots as in [26, II.1.21], the first coordinate of $p^r \alpha - 2p^s(p^r - 1)\rho$ is negative and the last coordinate is positive, so that the same holds for λ . We get that if $N \in \mathrm{mod} \bar{X} \cap \Theta$ and $\tau_X^{p^t}(N) \notin \mathrm{mod} \bar{X}$, then $\tau_X^{jp^t}(N) \notin \mathrm{mod} \bar{X}$ for all $j \in \mathbb{N}$, showing the statement for $B_r T$. For $X = G_r T$, use 3.3.1, (1), and argue as above.

Analogously, one shows: If $\tau_X^{-ip^t}(N) \notin \mathrm{mod} \bar{X}$, then $\tau_X^{-jp^t}(N) \notin \mathrm{mod} \bar{X}$ for all $j > i$.

- (1) Since every module on the downward sectional path starting in N or V is also an \bar{X} -module, we may assume $ql(N) = \frac{p^t+1}{2}$ and $ql(V) = \frac{3p^t-1}{2}$ (Bemerkung in Klammer gelöscht.). By following the downward sectional path starting in V for p^t-1 steps, we get a module $N' \in \Theta \cap \mathrm{mod} \bar{X}$ such that $ql(N') = ql(V) - (p^t-1) = ql(N)$. Then the modules $\tau_{G_r T}^i(N')$ belong to the wing of V for $1 \leq i \leq p^t-1$, so that they belong to $\mathrm{mod} \bar{X}$ by 1.3.7. Thus, there are $l \in \mathbb{N}_0$ and $0 \leq i \leq p^t-1$ such that $N = \tau_X^{i+lp^t}(N')$. By (*), $\tau_X^{lp^t}(\tau_X^i(N')) \in \mathrm{mod} \bar{X}$ implies $\tau_X^{kp^t}(\tau_X^i(N')) \in \mathrm{mod} \bar{X}$ for $1 \leq k \leq l$. Since $ql(N) = \frac{p^t+1}{2}$, 1.3.8 shows that $\tau_X^j(N') \in \mathrm{mod} \bar{X}$ for $1 \leq j \leq i+lp^t$. As $\mathrm{mod} \bar{X}$ is extension-closed, the middle terms of the almost split sequences defined by these modules also belong to $\mathrm{mod} \bar{X}$, so that we get an undirected path in $\Theta \cap \mathrm{mod} \bar{X}$ from N' to N . As there is also such a path from V to N' , the result follows.

- (2) If $N, N' \in \Theta \cap \text{mod } \overline{X}$ have maximal quasi-length and $N \neq N'$, $(*)$ and 1.3.8 imply the existence of a module with greater quasi-length in $\Theta \cap \text{mod } \overline{X}$, a contradiction. \square

Corollary 3.3.5. *Let $X \in \{B_r T, G_r T\}$. Suppose that $F(\Theta)$ is a homogeneous tube and $\Theta \cap \text{mod } \overline{X} \neq \emptyset$. Then there is a unique module $N \in \Theta \cap \text{mod } \overline{X}$ such that $\Theta \cap \text{mod } \overline{X}$ consists of the wing of N .*

Proof. Taking $t = 0$ in 3.3.4, we see that $\Theta \cap \text{mod } \overline{X}$ is connected. Since $\text{mod } \overline{X}$ is extension-closed by 3.2.4, we get that $\Theta \cap \text{mod } \overline{X}$ is equal to the wing of a module $V \in \Theta \cap \text{mod } \overline{X}$ with maximal quasi-length. \square

Note that for $X = G_r T$ and $r = 1$, $F(\Theta)$ always is a homogeneous tube by 3.3.1.

3.4 The case $n = 2, r = 1$

In this section, let $p \geq 3$, $G = \text{GL}_2$, $T \subseteq G$ be the torus of diagonal matrices. In [10], quiver and relations for the blocks of $S_d(G_1 T)$ as well as the number of blocks were determined and it was shown that all blocks in this case are representation-finite. Our aim in this section is to determine the Auslander-Reiten quiver for the algebras $S_d(G_1 T)$ by first considering the position of the relevant modules in the stable Auslander-Reiten quiver of $G_1 T$. Since $G_1 T = (\text{SL}_2)_1 T$ and modules for $(\text{SL}_2)_1$ correspond to modules for the restricted enveloping algebra $U_0(\mathfrak{sl}_2)$, we have a restriction functor $F : \text{mod } G_1 T \rightarrow \text{mod } U_0(\mathfrak{sl}_2)$ and the results of Section 1.2 apply to this functor. Contrary to the \mathbb{Z} -grading obtained from the usual theory of $(\text{SL}_2)_1(T \cap \text{SL}_2)$ -modules, we obtain a \mathbb{Z}^2 -grading on \mathfrak{sl}_2 and $U_0(\mathfrak{sl}_2)$ given by $\deg(e) = (1, -1)$, $\deg(f) = (-1, 1)$ and $\deg(h) = (0, 0)$ for the standard basis e, f, h of \mathfrak{sl}_2 . The indecomposable $U_0(\mathfrak{sl}_2)$ -modules were classified by Premet in [31]. For $d \geq 0$, let $V(d)$ be the Weyl module of highest weight d for the group scheme SL_2 . Then $V(d)$ has a basis v_0, \dots, v_d such that for the standard basis $\{e, f, h\} \subseteq \mathfrak{sl}_2$, we have

$$e.v_i = (i+1)v_{i+1}, f.v_i = (d-i+1)v_{i-1}, h.v_i = (2i-d)v_i$$

For $d = sp + a$, where $s \geq 1$ and $0 \leq a \leq p-2$, the space

$$W(d) := \bigoplus_{i=a+1}^d kv_i \subseteq V(d)$$

is a maximal $U_0(\mathfrak{sl}_2)$ -submodule of $V(d)$. We record the classification of indecomposable $U_0(\mathfrak{sl}_2)$ -modules in the following theorem:

Theorem 3.4.1. *The following statements hold:*

- (1) *Let $C \subseteq \text{SL}_2$ be a complete set of coset representatives of SL_2/B , where $B \subseteq \text{SL}_2$ is the Borel subgroup of upper triangular matrices. Then any nonprojective indecomposable $U_0(\mathfrak{sl}_2)$ -module is isomorphic to exactly one of the modules of the following list:*

(i) $V(d), V(d)^*$ for $d \geq p, d \not\equiv -1 \pmod{p}$;

(ii) $V(r) =: L(r)$ for $0 \leq r \leq p-1$;

(iii) $g.W(d)$ for $g \in C$ and $d = sp + a$ with $s \geq 1$ and $0 \leq a \leq p-2$.

In particular, the modules appearing in the list are pairwise nonisomorphic.

(2) Up to isomorphism, every indecomposable $U_0(\mathfrak{sl}_2)$ -module N is uniquely determined by the triple $(\dim_k N, \text{Soc}_{\mathfrak{sl}_2}(N), V_{\mathfrak{sl}_2}(N))$, where $V_{\mathfrak{sl}_2}(N)$ is the $U_0(\mathfrak{sl}_2)$ -rank variety of N .

The stable Auslander-Reiten quiver of $U_0(\mathfrak{sl}_2)$ is the disjoint union of $p-1$ components of type $\mathbb{Z}[\tilde{A}_{12}]$ and infinitely many homogeneous tubes. The modules $g.W(d)$ lie in homogeneous tubes and $ql(g.W(d)) = s$, where $d = sp + a$ ([17, 4.1.2]). We first determine all indecomposable homogeneous submodules and factor modules of $V(sp+a)$. By definition (see for example [17, Section 4.1]), the \mathfrak{sl}_2 -structure on $V(sp+a)$ comes from a twist of the dual of the d -th symmetric power of the natural SL_2 -module $L(1) = k^2$. This module is a GL_2 -module in a natural way and the differential of the GL_2 -action restricts to the \mathfrak{sl}_2 -action on this module. Thus, we get an $X(T)$ -grading given by $\deg(v_i) = (i, sp + a - i)$ compatible with the \mathfrak{sl}_2 -action. Applying contravariant duality, we see that $V(sp+a)^\circ$ has a basis with the same weights since the anti-automorphism η of 3.1.1 is just transposition of matrices in this case. Letting $w_0 \in G$ be a representative of the nontrivial element of the Weyl group W of G , the submodules $W(sp+a)$ resp. $w_0.W(sp+a)$ of $V(sp+a)$ have bases v_{a+1}, \dots, v_{sp+a} resp. v_0, \dots, v_{sp-1} , so they are homogeneous. Other modules of the form $g.W(d)$ don't have T -invariant rank varieties and are thus not G_1T -modules. We have $w_0.W(sp+a) = W(sp+a)^{w_0}$, the twist of $W(sp+a)$ by w_0 . Since $w_0.(t_1, t_2) = (t_2, t_1)$ for all $t_1, t_2 \in T$, the grading on $w_0.V$ is given by interchanging the coordinates of the grading of V for all $V \in \text{mod } G_1T$. We denote the G_1T -structures defined above on $V(sp+a), V(sp+a)^\circ, W(sp+a)$ and $W(sp+a)^{w_0}$ by $\widehat{V}(sp+a), \widehat{V}(sp+a)^\circ, \widehat{W}(sp+a)$ and $\widehat{W}(sp+a)^{w_0}$, respectively. For $0 \leq r \leq p-1$, we also write $\widehat{L}(r) := \widehat{V}(r)$. By the classification in 3.4.1, we have $V(d) \cong V(d)^* \cong V(d)^\circ$ for $0 \leq d \leq p-1$. This shows that $F(\widehat{V}(sp+a)^\circ) \cong V(sp+a)^*$.

Lemma 3.4.2. *Let V be a nonprojective indecomposable G_1T -module. Then there are $x, y \in \mathbb{Z}$ and a basis v_0, \dots, v_l of V such that $\deg(v_i) = (x+i, y+l-i)$.*

Proof. By 3.4.1 and its succeeding remarks, $F(V)$ is isomorphic to either of $V(sp+a) \cong F(\widehat{V}(sp+a)), V(sp+a)^* \cong F(\widehat{V}(sp+a)^\circ)$ with $s \geq 0$ or $W(sp+a) \cong F(\widehat{W}(sp+a))$ or $W(sp+a)^{w_0} \cong F(\widehat{W}(sp+a)^{w_0})$ with $s \geq 1$. Let $F(V) \cong V(sp+a) \cong F(\widehat{V}(sp+a))$. An application of 1.2.1 and 1.2.5 yields $\lambda \in X(T)$ such that $\lambda|_{T \cap (\text{SL}_2)_1} = 1$ and $V \cong \widehat{V}(sp+a)[\lambda]$. By the remarks above, $\widehat{V}(sp+a)$ has a basis v_0, \dots, v_{sp+a} such that $\deg(v_i) = (i, sp+a-i)$. In $\widehat{V}(sp+a)[\lambda]$, this basis has degrees $\deg(v_i) = (\lambda_1 + i, \lambda_2 + sp+a-i)$, so that $(x, y) = (\lambda_1, \lambda_2)$ yields the claim in this case. As $\widehat{V}(sp+a)^\circ$ has a basis with the same degrees, the preceding arguments also apply in the case $F(V) \cong F(\widehat{V}(sp+a)^\circ)$. In the remaining two cases, we apply the same arguments to the bases v_{a+1}, \dots, v_{sp+a}

resp. v_0, \dots, v_{sp-1} of $\widehat{W}(sp+a)$ resp. $\widehat{W}(sp+a)^{w_0}$, setting $(x, y) = (\lambda_1 + a + 1, \lambda_2)$ resp. $(x, y) = (\lambda_1, \lambda_2 + a + 1)$. \square

Using the lemma, we can now determine the M_1D -parts of the G_1T -components containing $\widehat{W}(sp+a)$, $\widehat{W}(sp+a)^{w_0}$, $\widehat{V}(sp+a)$ and $\widehat{V}(sp+a)^o$.

Proposition 3.4.3. *Let Θ be the component of the stable Auslander-Reiten quiver of $\text{mod } G_1T$ containing the module $\widehat{W}(sp+a)$ resp. $\widehat{W}(sp+a)^{w_0}$. Then Θ is a $\mathbb{Z}[A_\infty]$ -component such that $\tau_{G_1T}|_\Theta = [(p, -p)]$ resp. $\tau_{G_1T}|_\Theta = [(-p, p)]$ and $\Theta \cap M_1D$ is the wing of $\widehat{W}(sp+a)$ resp. $\widehat{W}(sp+a)^{w_0}$. The i -th predecessor of $\widehat{W}(sp+a)$ resp. $\widehat{W}(sp+a)^{w_0}$ on the sectional path starting in its quasi-socle is $\widehat{W}((s-i)p+a)[(ip, 0)]$ resp. $\widehat{W}((s-i)p+a)^{w_0}[(0, ip)]$. The quasi-socle $\widehat{W}(p+a)[(s-1)p, 0]$ resp. $\widehat{W}(p+a)^{w_0}[(0, (s-1)p)]$ is projective in $\text{mod } M_1D$.*

Proof. We show the result for $\widehat{W}(sp+a)$. For $\widehat{W}(sp+a)^{w_0}$, the result then follows since twisting with w_0 is an equivalence of categories interchanging the two coordinates of the grading. Since $F(\Theta)$ is a homogeneous tube, we have $cx_{G_1T}(\Theta) = cx_{U_0(\mathfrak{sl}_2)}(F(\Theta)) = 1$, so that Θ is of type $\mathbb{Z}[A_\infty]$ by [18, 8.2.2]. We first consider the case $s > 1$. In $\text{mod } U_0(\mathfrak{sl}_2)$, we have an almost split sequence

$$\xi : (0) \rightarrow W((s-1)p+a) \rightarrow W(sp+a) \oplus W((s-2)p+a) \rightarrow W((s-1)p+a) \rightarrow (0)$$

by [17, 4.1.2], where $W(a) = (0)$. Since $F(\widehat{W}((s-1)p+a)) \cong W((s-1)p+a)$, $F(\widehat{W}((s-2)p+a)) \cong W((s-2)p+a)$ and $F(\widehat{W}(sp+a)) \cong W(sp+a)$, 1.2.2 and 1.2.1 show that there is an almost split sequence

$$\xi' : (0) \rightarrow \widehat{W}((s-1)p+a)[\lambda] \rightarrow \widehat{W}(sp+a) \oplus \widehat{W}((s-2)p+a)[\nu] \rightarrow \widehat{W}((s-1)p+a)[\mu] \rightarrow (0)$$

in $\text{mod } G_1T$ with $\lambda, \nu, \mu \in X(T)$, where $\widehat{W}(a) := 0$. Thus, $\tau_{G_1T}(\widehat{W}((s-1)p+a)[\mu]) \cong \widehat{W}((s-1)p+a)[\lambda]$. On the other hand, 3.3.1 yields $\alpha \in R$ such that $\tau_{G_1T}|_\Theta = [p\alpha]$. Hence $\tau_{G_1T}|_\Theta = [p\alpha] = [\lambda - \mu]$ and 1.2.1, (2) yields $p\alpha = \lambda - \mu$. In order to determine λ, μ , we determine all submodules and factor modules of $\widehat{W}(sp+a)$ isomorphic to a shift of $\widehat{W}((s-1)p+a)$. Let A be such a submodule. In the canonical basis v_{a+1}, \dots, v_{sp+a} of $\widehat{W}(sp+a)$, each v_i has degree $(i, sp+a-i)$, so that the subspace $\widehat{W}(sp+a)_{(i, sp+a-i)}$ is one-dimensional with basis v_i . Hence any G_1T -submodule of $\widehat{W}(sp+a)$ is the span of a subset of the v_i . As $\widehat{W}((s-1)p+a)$ is indecomposable and has dimension $(s-1)p$, 3.4.2 provides a basis $v'_0, \dots, v'_{(s-1)p-1}$ of A and $x, y \in \mathbb{Z}$ such that $\deg(v'_i) = (x+i, y+sp-1-i)$. Comparing the degrees of these basis elements with the degrees of the v_i shows that there is $0 \leq i \leq p$ such that A is spanned by $v_{a+1+i}, \dots, v_{(s-1)p+a+i}$. Suppose first that $0 \leq i \leq p-a-2$. Then $e.v_{(s-1)p+a+i} = ((s-1)p+a+i+1)v_{(s-1)p+a+i+1}$ yields $v_{(s-1)p+a+i+1} \in A$, a contradiction. Thus, $i \geq p-a-1$. If $p-a-1 \leq i \leq p-1$, then $f.v_{a+1+i} = (sp+i)v_{a+i}$ yields $v_{a+i} \in A$, a contradiction. As a result, we have $i = p$ and A is spanned by $v_{p+a+1}, \dots, v_{sp+a}$. Comparing the degrees of this basis of A with the degrees of the canonical basis of $\widehat{W}((s-1)p+a)$, we see that $A \cong \widehat{W}((s-1)p+a)[(p, 0)]$,

so that $\lambda = (p, 0)$.

As $[\lambda - \mu] = [p\alpha]$ and GL_n has roots $(1, -1), (-1, 1)$, we have $\mu = (2p, -p)$ or $\mu = (0, p)$. Since $\widehat{W}((s-1)p+a)[\mu] \in \mathrm{mod} M_1D$, $\mu \neq (2p, -p)$; else the last base vector of the canonical basis of $\widehat{W}((s-1)p+a)[\mu]$ would have a negative coordinate. Thus, $\mu = (0, p)$. As a result, we have $\tau_{G_1T}|_{\Theta} = [(p, -p)]$. By considering the degrees of the canonical basis of $\widehat{W}(sp+a)$ again, we see that not all weights of $\widehat{W}(sp+a)[(p, -p)], \widehat{W}(sp+a)[(-p, p)]$ are polynomial, so that $\tau_{G_1T}(\widehat{W}(sp+a)), \tau_{G_1T}^{-1}(\widehat{W}(sp+a)) \notin \mathrm{mod} M_1D$. Now 3.3.5 shows that $\Theta \cap \mathrm{mod} M_1D$ is equal to the wing of $\widehat{W}(sp+a)$. Now let $s = 1$. The arguments above in particular yield an almost split sequence

$$(0) \rightarrow \widehat{W}((p+a)[(p, 0)]) \rightarrow \widehat{W}(2p+a) \rightarrow \widehat{W}(p+a)[(0, p)] \rightarrow (0).$$

Applying the shift functor $[(-p, 0)]$ to this sequence, we see that $\tau_{G_1T}(\widehat{W}(p+a)[(-p, p)]) = \widehat{W}(p+a)$, so that $\tau_{G_1T}|_{\Theta} = [(p, -p)]$ also holds in this case. Considering the degrees of the canonical basis for $\widehat{W}(p+a)[(-p, p)], \widehat{W}(p+a)[(p, -p)]$, we see that not all weights of these modules are polynomial, so that $\Theta \cap \mathrm{mod} M_1D$ is the wing of $\widehat{W}(p+a)$.

For $1 \leq s \leq 2$, the almost split sequences computed above now show that the left-most quasi-simple module in the wing of $\widehat{W}(sp+a)$ is $\widehat{W}(p+a)[((s-1)p, 0)]$. For $s > 2$, one shows by induction that the i -th predecessor of $\widehat{W}(sp+a)$ on the sectional path starting in the quasi-socle of $\widehat{W}(sp+a)$ is $W((s-i)p+a)[(ip, 0)]$ by shifting the almost split sequence starting in $\widehat{W}((s-i)p+a)[(p, 0)]$ by $[((i-1)p, 0)]$. The definition of the \mathfrak{sl}_2 -action on the basis vectors v_{a+1}, \dots, v_{p+a} of $\widehat{W}(p+a)$ shows that v_{a+1}, \dots, v_{p-1} span a simple submodule of $\widehat{W}(p+a)$ while each of the remaining basis vectors generates $\widehat{W}(p+a)$ as an \mathfrak{sl}_2 -module, so that the socle of $\widehat{W}(p+a)$ is spanned by v_{a+1}, \dots, v_{p-1} . As $\deg(v_{a+1}) = (p-1, a+1)$, we get $\mathfrak{F}_{M_rD}(\mathrm{Soc}(\widehat{W}(p+a)[sp, -p])) = 0$, so that $\mathfrak{F}_{M_rD}(\widehat{W}(p+a)[sp, -p]) = 0$ and $W(p+a)[((s-1)p, 0)]$ is projective in $\mathrm{mod} M_1D$ by 1.3.1. \square

For an illustration of the following proposition, see Figure 3.1 below.

Proposition 3.4.4. *Let Θ be the component of the stable Auslander-Reiten quiver of $\mathrm{mod} G_1T$ containing the module $\widehat{V}(sp+a)[(i, i)]$ for $0 \leq i \leq p-1$. Then Θ is a $\mathbb{Z}[A_{\infty}^{\infty}]$ -component such that $\Theta = \Theta^{\circ}$. There is a column of simple module in Θ which is a symmetry axis with respect to $(-)^{\circ}$ and $\Theta \cap \mathrm{mod} M_1D$ consists of the modules on directed paths between $\widehat{V}(sp+a)[(i, i)]$ and $\widehat{V}(sp+a)^{\circ}[(i, i)]$. We also have $\tau_{G_1T}(\widehat{V}(sp+a)[(i, i)]) = \widehat{V}((s+2)p+a)[(i-p, i-p)]$.*

Proof. By [17, Section 4.1], $F(\Theta)$ is of type $\mathbb{Z}[\tilde{A}_{12}]$ and contains exactly one simple module $L(a)$. By 1.2.5, F induces a morphism of stable translation quivers $\Theta \rightarrow F(\Theta)$. Since $F(\Theta)$ has no quasi-simple modules and Θ is isomorphic to either $\mathbb{Z}[A_{\infty}], \mathbb{Z}[A_{\infty}^{\infty}]$ or $\mathbb{Z}[D_{\infty}]$ by [16, 3.4], this shows that Θ is of type $\mathbb{Z}[A_{\infty}^{\infty}]$. We show that there is an almost

split sequence

$$\begin{aligned} \zeta : (0) &\rightarrow \widehat{V}(sp+a) \rightarrow \widehat{V}((s-1)p+a)[(p,0)] \oplus \widehat{V}((s-1)p+a)[(0,p)] \\ &\rightarrow \widehat{V}((s-2)p+a)[(p,p)] \rightarrow (0) \end{aligned}$$

for $s > 1$ while the almost split sequence starting in $\widehat{V}(p+a)$ is

$$\begin{aligned} \zeta' : (0) &\rightarrow \widehat{V}(p+a) \rightarrow \widehat{L}(a)[(p,0)] \oplus \widehat{L}(a)[(0,p)] \oplus \widehat{P} \\ &\rightarrow \widehat{V}(p+a)^o \rightarrow (0) \end{aligned}$$

with \widehat{P} projective indecomposable. By [2, V.5.5], there is an almost split sequence

$$\xi : (0) \rightarrow \text{Rad}(P) \rightarrow \text{Rad}(P)/\text{Soc}(P) \oplus P \rightarrow P/\text{Soc}(P) \rightarrow (0)$$

in $\text{mod } U_0(\mathfrak{sl}_2)$, where P is the projective cover of $L(p-a-2)$. By [13, IV.3.8.3] in conjunction with [19, 2.4], we have $\text{Rad}(P)/\text{Soc}(P) \cong S \oplus S$ for some simple $U_0(\mathfrak{sl}_2)$ -module S such that $S \not\cong L(p-a-2)$ and $S, L(p-a-2)$ are the only simple modules in the block of S . As $L(a)$ belongs to the block of $L(p-a-2)$, we get $S \cong L(a)$. Hence $\text{Rad}(P)$ has dimension $p+a+1$ and $\text{Soc}(\text{Rad}(P)) \cong L(p-a-2)$, so that 3.4.1 yields $\text{Rad}(P) \cong V(p+a)$. By 1.2.2 in combination with 1.2.5, the almost split sequence in $\text{mod } G_1T$ starting in $\widehat{V}(p+a)$ is mapped to ξ by F . Thus, the non-projective part of the middle term is a sum of two shifts of $\widehat{L}(a)$ and is obtained by factoring out the socle of $\widehat{V}(p+a)$. For the canonical basis v_0, \dots, v_{p+a} of $\widehat{V}(p+a)$, the action of \mathfrak{sl}_2 on the basis vectors shows that the socle is spanned by v_{a+1}, \dots, v_{p-1} and the direct summands of the factor module are spanned by the images of v_0, \dots, v_a and v_p, \dots, v_{p+a} , respectively. Comparing the degrees of these vectors with the degrees in the canonical basis of $\widehat{L}(a)$, we see that $\widehat{V}(p+a)/\text{Soc}(\widehat{V}(p+a)) \cong \widehat{L}(a)[(p,0)] \oplus \widehat{L}(a)[(0,p)]$. As the middle term of the almost split sequence starting in $\widehat{V}(p+a)$ is self-dual with respect to $(-)^o$, the right term of this sequence is $\widehat{V}(p+a)^o$, so that ζ' is almost split. By applying the shift functors $[(0,p)], [(p,0)]$ to ζ' , we see that $\widehat{V}(p+a)[(p,0)], \widehat{V}(p+a)[(0,p)]$ are predecessors of $\widehat{V}(a)[(p,p)]$ in $\Gamma_s(G_1T)$. As Θ is of type $\mathbb{Z}[A_\infty^\infty]$, this shows that the middle term of the almost split sequence ending in $\widehat{V}(p+a)[(p,p)]$ has the claimed form. Inductively, to show that the almost split sequence for $s > 1$ has the claimed form, it suffices to show that if

$$\begin{aligned} \xi' : (0) &\rightarrow X_s \rightarrow \widehat{V}((s-1)p+a)[(p,0)] \oplus \widehat{V}((s-1)p+a)[(0,p)] \\ &\rightarrow \widehat{V}((s-2)p+a)[(p,p)] \rightarrow (0) \end{aligned}$$

is almost split in $\text{mod } G_1T$, then $X_s \cong \widehat{V}(sp+a)$. As above, $F(\xi')$ is almost split in $\text{mod } U_0(\mathfrak{sl}_2)$ and by exactness, we have $\dim_k X_s = sp+a+1$. Thus, 3.4.1 yields that $F(X_s) \cong V(sp+a) \cong F(\widehat{V}(sp+a))$ or $F(X_s) \cong V(sp+a)^* \cong F(\widehat{V}(sp+a)^o)$, so that X_s is isomorphic to a shift of $\widehat{V}(sp+a)$ or $\widehat{V}(sp+a)^o$ by 1.2.1. We show that $\widehat{V}(sp+a)$ does not have a submodule isomorphic to a shift of $\widehat{V}((s-1)p+a)^o$, so that $\widehat{V}(sp+a)^o$

does not have a factor module isomorphic to a shift of $\widehat{V}((s-1)p+a)$ and the first alternative has to apply. Suppose A is a submodule of $\widehat{V}(sp+a)$ isomorphic to a shift of $\widehat{V}((s-1)p+a)^\circ$. Taking the canonical basis v_0, \dots, v_{sp+a} of $\widehat{V}(sp+a)$ and comparing the degrees of this basis with the degrees of the basis of A provided by 3.4.2, we see that A is spanned by $v_i, \dots, v_{(s-1)p+a+i}$ for some $0 \leq i \leq p$. Using the action of $U_0(\mathfrak{sl}_2)$ on the v_j , we get the following statements:

- (1) If $v_j \in A$ for some $0 \leq j \leq a$, then $v_0, \dots, v_a \in A$,
- (2) if $v_j \in A$ for some $a+1 \leq j \leq p-1$, then $v_{a+1}, \dots, v_{p-1} \in A$,
- (3) if $v_j \in A$ for some $(s-1)p+a+1 \leq j \leq sp-1$, then $v_{(s-1)p+a+1}, \dots, v_{sp-1} \in A$,
- (4) if $v_j \in A$ for some $sp \leq j \leq sp+a$, then $v_{sp}, \dots, v_{sp+a} \in A$.

Thus, A is spanned by either $v_0, \dots, v_{(s-1)p+a}$ or by v_p, \dots, v_{sp+a} . However, we have $e.v_{(s-1)p+a} = ((s-1)p+a+1)v_{(s-1)p+a+1}$ and $f.v_p = ((s-1)p+a+1)v_{p-1}$, so that in both cases, A is not a submodule, a contradiction. Hence $X_s \cong \widehat{V}(sp+a)[\lambda]$ for some $\lambda \in X(T)$. Since ξ' is almost split, $\widehat{V}(sp+a)[\lambda]$ has factor modules $A_1 \cong \widehat{V}((s-1)p+a)[(p,0)]$, $A_2 \cong \widehat{V}((s-1)p+a)[(0,p)]$. Comparing degrees of bases and applying the arguments on submodules from above to factor modules and kernels of the canonical projections, we see that A_1 resp. A_2 are spanned by the images of the basis vectors v_p, \dots, v_{sp+a} resp. $v_0, \dots, v_{(s-1)p+a}$ under the canonical projections and that $\lambda = 0$, so that $X_s \cong \widehat{V}(sp+a)$ and ζ is almost split. The statement about τ_{G_1T} now follows by shifting the sequence for $s+2$ by $[(i-p, i-p)]$. By applying the shift functor $[(lp, l'p)]$ to the almost split sequences ζ, ζ' , we see that for all $l, l' \in \mathbb{Z}$, the element directly above resp. below $\widehat{V}(sp+a)[(lp, l'p)]$ in the column of $\widehat{V}(sp+a)[(lp, l'p)]$ is $\widehat{V}(sp+a)[((l+1)p, (l'-1)p)]$ resp. $\widehat{V}(sp+a)[((l-1)p, (l'+1)p)]$. Thus, if $0 \leq j \leq s$ and we follow the upper resp. lower sectional path starting in $\widehat{V}(sp+a)$ for j steps, we arrive at the module $\widehat{V}((s-j)p+a)[(jp, 0)]$ resp. $\widehat{V}((s-j)p+a)[(0, jp)]$. The modules above $\widehat{V}((s-j)p+a)[(jp, 0)]$ have the form $\widehat{V}((s-j)p+a)[((j+l)p, -lp)]$ while the modules below $\widehat{V}((s-j)p+a)[(0, jp)]$ have the form $\widehat{V}((s-j)p+a)[(-lp, (j+l)p)]$ for some $l \in \mathbb{N}$. Thus, the degree of either v_0 or $v_{(s-j)p+a}$ in their canonical bases has a coordinate which is a negative multiple of p , so that these modules and their shifts by $[(i, i)]$ are not polynomial G_1T -modules. All modules between $\widehat{V}((s-j)p+a)[(jp, 0)]$ and $\widehat{V}((s-j)p+a)[(0, jp)]$ in their column can be reached via directed paths starting in $\widehat{V}(sp+a)$. As all arrows on these paths correspond to irreducible epimorphisms, all modules obtained this way are polynomial G_1T -modules by 3.2.1. The arguments above show in particular that there is a column of simple modules in Θ which is equal to $\{\widehat{L}(a)[(sp+jp, -jp)] \mid j \in \mathbb{Z}\}$ and that a module to the left of this column belongs to $\text{mod } M_1D$ iff it lies on a directed path from $\widehat{V}(sp+a)$ to this column. Since $S^\circ \cong S$ for every simple G_1T -module S by [26, II.6.9(13)], we have $\Theta^\circ = \Theta$ and the column of simple modules in Θ is a symmetry axis with respect to $(-)^\circ$. As $V \in \text{mod } G_1T$ is a polynomial G_1T -module iff V° is a polynomial G_1T -module, we get that any module to

the right of the column of simple modules belongs to $\text{mod } M_1D$ iff it lies on a directed path from the column of simple modules ending in $\widehat{V}(sp+a)^\circ$. As a result, $\Theta \cap \text{mod } M_1D$ consists of all modules on directed paths from $\widehat{V}(sp+a)$ to $\widehat{V}(sp+a)^\circ$ for $i = 0$. For $1 \leq i \leq p-1$, the statement follows by applying the functor $[(i, i)]$ to the component containing $\widehat{V}(sp+a)$, noting that the above arguments about negative coordinates also apply in this case. \square

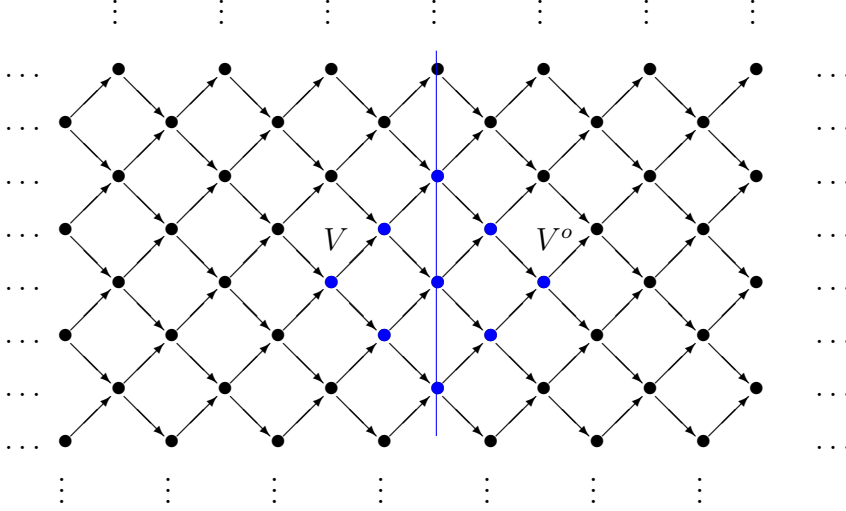


Figure 3.1: Component of $\Gamma_s(G_1T)$ containing the module $V = \widehat{V}(2p+a)$. The blue vertices are M_1D -modules and the blue line is the column of simple modules.

By 1.2.5, the shifts of $\widehat{V}(sp+a)$ with a G_1T -structure are exactly those $\widehat{V}(sp+a)[\lambda]$ with $\lambda \in X(T)$ such that $\lambda|_{(\text{SL}_2)_1 \cap T} = 0$. These are the elements of $pX(T) + \mathbb{Z} \det|_T$. The proof of 3.4.4 shows that all these modules belong to components of the type described in 3.4.4. As modules of the form $V(sp+a)$ and their duals are the only $U_0(\mathfrak{sl}_2)$ -modules contained in components of type $\mathbb{Z}[\tilde{A}_{12}]$ by [17, Section 4], we have found all of these modules for $r = 1$.

Since the restriction functor $\text{mod } M_1D \rightarrow \text{mod } G_1T$ is fully faithful, almost split sequences in $\text{mod } G_1T$ such that all constituents lift to M_1D are also almost split in $\text{mod } M_1D$. In the following, we determine the almost split sequences in $\text{mod } M_1D$ which are not almost split in $\text{mod } G_1T$.

Lemma 3.4.5. *There are almost split sequences*

$$\begin{aligned} \xi_1 : (0) &\rightarrow \widehat{W}((s-l)p+a)[(0, lp)] \\ &\rightarrow \widehat{V}((s-l)p+a)[(0, lp)] \oplus \widehat{W}((s-l-1)p+a)[(0, (l+1)p)] \\ &\rightarrow \widehat{V}((s-l-1)p+a)[(0, (l+1)p)] \rightarrow (0) \end{aligned}$$

for $0 \leq l \leq s - 1$ and

$$\begin{aligned} \xi_2 : (0) &\rightarrow \widehat{V}((s-l)p - a - 2)[(a+1+lp, a+1)]^\circ \\ &\rightarrow \widehat{W}((s-l)p + a)[(lp, 0)] \\ &\oplus \widehat{V}((s-l+1)p - a - 2)[(a+1+(l-1)p, a+1)]^\circ \\ &\rightarrow \widehat{W}((s-l+1)p + a)[((l-1)p, 0)] \rightarrow (0) \end{aligned}$$

for $s > 1$ and $1 \leq l \leq s - 1$ in $\text{mod } M_1D$, where $\widehat{W}(a) = 0$.

Proof. We have almost split sequences

$$\begin{aligned} \xi'_1 : (0) &\rightarrow \widehat{V}((s-l+1)p + a)[(-p, lp)] \\ &\rightarrow \widehat{V}((s-l)p + a)[(0, lp)] \oplus \widehat{V}((s-l)p + a)[(-p, (l+1)p)] \\ &\rightarrow \widehat{V}((s-l-1)p + a)[(0, (l+1)p)] \rightarrow (0) \end{aligned}$$

and

$$\begin{aligned} \xi'_2 : (0) &\rightarrow \widehat{W}((s-l+1)p + a)[(lp, -p)] \\ &\rightarrow \widehat{W}((s-l)p + a)[(lp, 0)] \oplus \widehat{W}((s-l+2)p + a)[((l-1)p, -p)] \\ &\rightarrow \widehat{W}((s-l+1)p + a)[((l-1)p, 0)] \rightarrow (0) \end{aligned}$$

in $\text{mod } G_1T$ by the preceding results and their proofs. We want to prove $\mathfrak{F}_{M_rD}(\xi'_1) = \xi_1$ and $\mathfrak{F}_{M_rD}(\xi'_2) = \xi_2$. We first show $A := \mathfrak{F}_{M_rD}(\widehat{V}((s-l+1)p + a)[(-p, lp)]) = \langle v_{p+a+1}, \dots, v_{(s-l+1)p+a} \rangle$, where $v_0, \dots, v_{(s-l+1)p+a}$ is the canonical basis of $\widehat{V}((s-l+1)p + a)[(-p, lp)]$. As every weight space of $\widehat{V}((s-l+1)p + a)[(-p, lp)]$ is one-dimensional and A is a homogeneous submodule, A is spanned by a subset of the v_i . Since the first coordinates of their weights are negative, we have to delete at least v_0, \dots, v_{p-1} , and since the $U_0(\mathfrak{sl}_2)$ -module spanned by each of the vectors v_p, \dots, v_{p+a} contains v_{a+1}, \dots, v_{p-1} , we have to delete those as well, so that only the basis vectors spanning $\langle v_{p+a+1}, \dots, v_{(s-l+1)p+a} \rangle$ remain. As $\langle v_{p+a+1}, \dots, v_{(s-l+1)p+a} \rangle$ is stable under the \mathfrak{sl}_2 -action on $\widehat{V}((s-l+1)p + a)[(-p, lp)]$ and spanned by homogeneous elements, it is a homogeneous submodule. Since all weights of the basis vectors occurring are polynomial, we get $A = \langle v_{p+a+1}, \dots, v_{(s-l+1)p+a} \rangle$. In particular, 1.3.1,(1) shows that the right-hand term of ξ'_1 is not Ext-projective in $\text{mod } M_1D$. Hence 1.3.1,(2) implies that A is indecomposable, so that $F(A)$ is indecomposable by 1.2.1. Since $v_{p+a+1}, \dots, v_{2p-1}$ span a simple \mathfrak{sl}_2 -submodule which is isomorphic to $L(p-a-2)$ by 3.4.1, $L(p-a-2)$ occurs in the socle of $F(A)$. As $\dim_k A = (s-l)p$, 3.4.1 now yields $F(A) \cong W((s-l)p+a)$ or $F(A) \cong W((s-l)p+a)^{w_0}$. Using the action of f on A , we see that for $0 \leq i \leq s-l-1$, the vectors $v_{p+a+1+ip}, \dots, v_{2p+a+ip}$ span a p -dimensional $U_0(\langle f \rangle) \cong k[f]/(f^p)$ -module which is generated by $v_{2p+a+ip}$ and is hence projective indecomposable for $U_0(\langle f \rangle)$. As $A|_{U_0(\langle f \rangle)}$ is the direct sum of these modules, $A|_{U_0(\langle f \rangle)}$ is projective, so that f does not belong to the rank variety of $F(A)$. Since f belongs to the

rank variety of $W((s-l)p+a)^{w_0}$, we get $F(A) \cong W((s-l)p+a)$. Thus, A is isomorphic to a shift of $\widehat{W}((s-l)p+a)$ and comparing the degrees in the canonical basis of $\widehat{W}((s-l)p+a)$ to those of $v_{p+a+1}, \dots, v_{(s-l+1)p+a}$ yields $A \cong \widehat{W}((s-l)p+a)[(0, lp)]$. This also shows that $\mathfrak{F}_{M_r D}(\widehat{V}((s-l)p+a)[(-p, (l+1)p)]) \cong \widehat{W}((s-l-1)p+a)[(0, (l+1)p)]$. Now 1.3.1 shows that ξ_1 is almost split in mod $M_1 D$.

We now compute $B := \mathfrak{F}_{M_r D}(\widehat{W}(s-l+1)p+a)[(lp, -p)]$. As above, B is spanned by a subset of the v_i as a vector space. Since their weights have a negative second coordinate, we have to delete at least the last p base vectors $v_{(s-l)p+a+1}, \dots, v_{(s-l+1)p+a}$. The $U_0(\mathfrak{sl}_2)$ -module spanned by each of the vectors $v_{(s-l)p+1}, \dots, v_{(s-l)p+a}$ contains the last p basis vectors, so they also have to be deleted. As the remaining vectors $v_{a+1}, \dots, v_{(s-l)p}$ span a homogeneous submodule of $W(s-l+1)p+a[(lp, -p)]$, we get $B = \langle v_{a+1}, \dots, v_{(s-l)p} \rangle$. As above, B is indecomposable. As $\dim_k(B) = (s-l+1)p - a - 1$, 3.4.1 shows that $F(B) \cong V((s-l+1)p-a-2)$ or $F(B) \cong V((s-l+1)p-a-2)^\circ$. As $\langle v_{a+1}, \dots, v_{p-1} \rangle$ is a simple \mathfrak{sl}_2 -submodule isomorphic to $L(p-a-2)$, we get $F(B) \cong V((s-l+1)p-a-2)^\circ$ since $L(p-a-2)$ does not occur in the socle of $V((s-l+1)p-a-2)$. Now 1.2.1 yields $B \cong \widehat{V}((s-l+1)p-a-2)^\circ[\lambda]$ for some $\lambda \in X(T)$. Comparing the degrees of $v_{a+1}, \dots, v_{(s-l)p}$ with those of the canonical basis of $\widehat{V}((s-l+1)p-a-2)^\circ$ yields $\lambda = (a+1+lp, a+1)$. Thus, $B \cong \widehat{V}((s-l+1)p-a-2)^\circ[(a+1+lp, a+1)]$. This also shows $\mathfrak{F}_{M_r D}(\widehat{W}((s-l+2)p+a)[((l-1)p, -p)]) \cong \widehat{V}((s-l)p-a-2)[(a+1+(l-1)p, a+1)]^\circ$. Now 1.3.1 shows that ξ_2 is almost split in mod $M_1 D$. \square

Lemma 3.4.6. *There are almost split sequences*

$$\begin{aligned} \xi_1 : (0) &\rightarrow \widehat{W}((s-l)p+a)^{w_0}[(lp, 0)] \\ &\rightarrow \widehat{V}((s-l)p+a)[(lp, 0)] \oplus \widehat{W}((s-l-1)p+a)^{w_0}[((l+1)p, 0)] \\ &\rightarrow \widehat{V}((s-l-1)p+a)[((l+1)p, 0)] \rightarrow (0) \end{aligned}$$

for $0 \leq l \leq s-1$ and

$$\begin{aligned} \xi_2 : (0) &\rightarrow \widehat{V}((s-l)p-a-2)[(a+1, a+1+lp)]^\circ \\ &\rightarrow \widehat{W}((s-l)p+a)^{w_0}[(0, lp)] \\ &\oplus \widehat{V}((s-l+1)p-a-2)[(a+1, a+1+(l-1)p)]^\circ \\ &\rightarrow \widehat{W}((s-l+1)p+a)^{w_0}[(0, (l-1)p)] \rightarrow (0) \end{aligned}$$

for $s > 1$ and $1 \leq l \leq s-1$ in mod $M_1 D$, where $\widehat{W}(a) = 0$.

Proof. Twisting with w_0 is an auto-equivalence of mod $M_1 D$ mapping the almost split sequences from 3.4.5 to ξ_1, ξ_2 . The result now follows. \square

Lemma 3.4.7. *There is an almost split sequence*

$$\begin{aligned} (0) &\rightarrow \widehat{V}(sp-a-2)[(a+1, a+1)]^\circ \\ &\rightarrow \widehat{W}(sp+a) \oplus \widehat{W}(sp+a)^{w_0} \\ &\rightarrow \widehat{V}(sp+a) \rightarrow (0) \end{aligned}$$

in mod M_1D for $s \geq 1$.

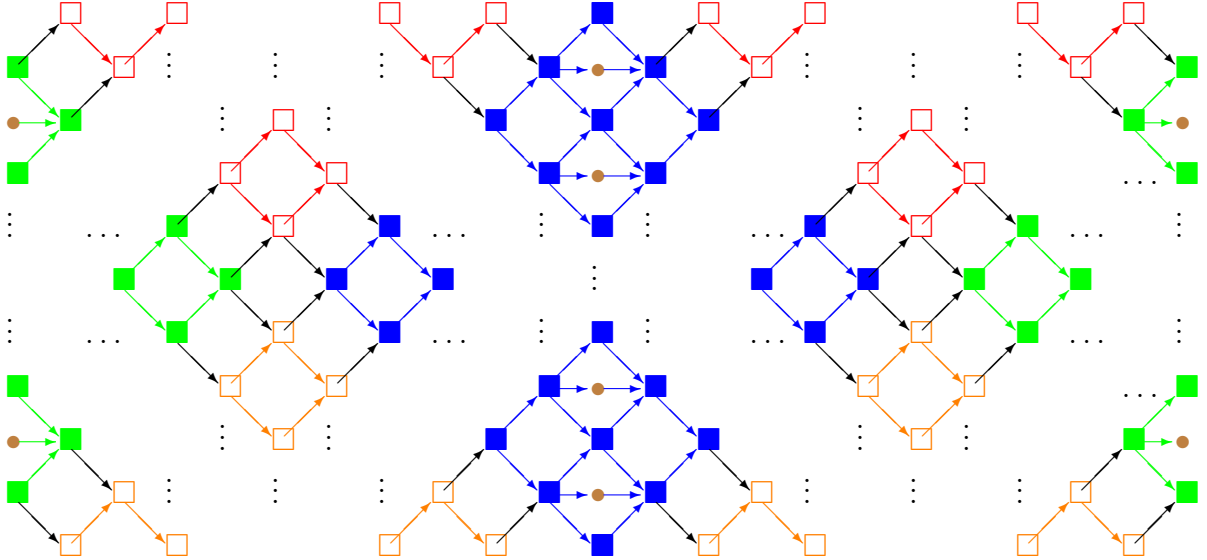
Proof. By the proof of 3.4.4, there is an almost split sequence

$$\begin{aligned} (0) &\rightarrow \widehat{V}((s+2)p+a)[-(p,p)] \\ &\rightarrow \widehat{V}((s+1)p+a)[(-p,0)] \oplus \widehat{V}((s+1)p+a)[(0,-p)] \\ &\rightarrow \widehat{V}(sp+a) \rightarrow (0) \end{aligned}$$

in mod G_rT . In the proof of 3.4.5, it was shown that $\mathfrak{F}_{M_rD}(\widehat{V}((s+1)p+a)[(-p,0)]) \cong \widehat{W}(sp+a)$. Since $\widehat{V}((s+1)p+a)[(-p,0)]^{w_0} \cong \widehat{V}((s+1)p+a)[(0,-p)]$, this also shows $\mathfrak{F}_{M_rD}(\widehat{V}((s+1)p+a)[(0,-p)]) \cong \widehat{W}(sp+a)^{w_0}$. We compute $C = \mathfrak{F}_{M_rD}(\widehat{V}((s+2)p+a)[-(p,p)])$. Taking the canonical basis $v_0, \dots, v_{(s+2)p+a}$ of $\widehat{V}((s+2)p+a)[-(p,p)]$, the \mathfrak{sl}_2 -action on the base vectors shows that C is spanned by $v_{p+a+1}, \dots, v_{(s+1)p-1}$ as all other base vectors generate submodules which are not polynomial. Since $\dim_k(C) = sp - a - 1$ and $v_{p+a+1}, \dots, v_{2p-1}$ span an \mathfrak{sl}_2 -module isomorphic to $L(p-a-2)$, 3.4.1 yields $F(C) \cong V(sp-a-2)^* \cong F(\widehat{V}(sp-a-2)^o)$. An application of 1.2.1 and a comparison of degrees of the basis $v_{p+a+1}, \dots, v_{(s+1)p-1}$ of C and the canonical basis of $\widehat{V}(sp-a-2)^o$ yields $C \cong \widehat{V}(sp-a-2)[(a+1, a+1)]^o$. The claim now follows from 1.3.1. \square

By using the results about stable AR-components of mod G_1T and almost split sequences in mod M_1D , we are able to determine the Auslander-Reiten quiver for the blocks of $S_{sp+a}(G_1T)$ containing $\widehat{V}(sp+a)$.

Theorem 3.4.8. *The component of the AR-quiver of $S_{sp+a}(G_1T)$ for $s \geq 1$ and $0 \leq a \leq p-2$ containing $\widehat{V}(sp+a)$ has the following form:*



Here, the leftmost and the rightmost column are identified, the brown dots are projective-injective, the blue squares are the M_1D -part of the G_1T -component containing $\widehat{V}(sp+a)$, the green squares are the M_1D -part of the G_1T -component containing $\widehat{V}(sp-a-2)[(a+1, a+1)]$, the red squares on the left side are the M_1D -part of the G_1T -component containing $\widehat{W}(sp+a)$, the orange squares on the left side are the M_1D -part of the G_1T -component containing $\widehat{W}(sp+a)^{w_0}$ and the red and orange squares on the right side are the duals of those on the left side. All colored arrows represent morphisms which are irreducible in $\text{mod } G_1T$, while the black arrows represent morphisms which are irreducible in $\text{mod } M_1D$, but not in $\text{mod } G_1T$. Thus, upon deleting projective-injective vertices, the underlying directed graph of the component is isomorphic to $\mathbb{Z}[A_{2s+1}]/\langle \tau^{2s+1} \rangle$.

Proof. For the left part of the quiver, one uses the shape of the M_rD -part of the G_1T -components containing $\widehat{W}(sp+a)$, $\widehat{V}(sp+a)$, $\widehat{W}(sp+a)^{w_0}$ and $\widehat{V}(sp-a-2)[(a+1, a+1)]$ and the almost split sequences from 3.4.5, 3.4.7 and 3.4.6. The right part is now obtained from the left part by contravariant duality. \square

Remark. Using calculations similar to those in the proofs of this section, one can show that $\widehat{W}(sp+a)^o \cong \widehat{W}((s+1)p-a-2)^{w_0}[(a+1, a+1-p)]$ and $(\widehat{W}(sp+a)^{w_0})^o \cong \widehat{W}((s+1)p-a-2)[(a+1-p, a+1)]$ and use this to give a different description of the modules in the right part of the component given above.

3.2.11 shows that components of this shape also occur for $r > 1$. We will show that all components of $S_d(G_1T)$ are shifts of a component of this shape.

For this, we first show that some shifts induce Morita-equivalences between blocks of different $S_d(G_1T)$.

Proposition 3.4.9. *Let $s \geq 0$ and $0 \leq a \leq p-2$. Then for $1 \leq i \leq p-a-2$, the shift functor $[(i, i)]$ restricts to an equivalence of categories between the categories of finite dimensional modules belonging to the block of $S_{sp+a}(G_1T)$ containing $\widehat{V}(sp+a)$ and a block of $S_{sp+a+2i}(G_1T)$.*

Proof. Since the block contains the simple module of highest weight $(sp+a, 0)$, $\rho = (\frac{1}{2}, -\frac{1}{2})$, $W = \{1, w_0\}$ and $\det|_T = (1, 1)$, this follows from an application of 2.2.2 to our situation. \square

By counting the number of shifts and comparing this to the number of non-semisimple blocks of $S_{sp+a}(G_1T)_1$ determined in [10], we see that every non-semisimple block arises this way.

Corollary 3.4.10. *Let b be a non-simple block of $S_d(G_1T)$. Then b is the shift of the block of $S_d(G_1T)$ containing $\widehat{V}(d')$ for some $d' \leq d$ such that $d' = sp+a$, $0 \leq a \leq p-2$.*

Proof. Write $d = sp+a$ with $0 \leq a \leq p-1$ and $s > 1$. We first consider the case $a \neq p-1$. By 3.4.9, the shifts by $[(i, i)]$ define a Morita-equivalence of the block of $S_{sp+a-2i}(G_1T)$ containing the module $\widehat{V}(sp+a-2i)$ to a block of $S_d(G_1T)$ for $0 \leq i \leq \lfloor \frac{a}{2} \rfloor$. By the

same token, the shifts by $[(a+i+1, a+i+1)]$ define a Morita-equivalence of the block of $S_{(s-1)p+(p-a-2)-2i}(G_1T)$ containing $\widehat{V}((s-1)p+(p-a-2)-2i)$ to a block of $S_d(G_1T)$ for $1 \leq i \leq \lfloor \frac{p-a-2}{2} \rfloor$. These blocks are pairwise distinct since the indecomposable modules of largest dimension belonging to them are $\widehat{V}(sp+a-2i)$ with dimension $sp+a-2i+1$ resp. $\widehat{V}((s-1)p+(p-a-2)-2i)$ with dimension $(s-1)p+(p-a-2)-2i+1$. For $a = p-1$, only the first kind of blocks occurs with $1 \leq i \leq \frac{p-1}{2}$. Thus, we have found $\frac{p-1}{2}$ distinct blocks in both cases. By [10, 2.2], we have found all blocks not associated to a shift of a Steinberg module, hence all non-simple blocks. If now $d = a$ with $0 \leq a \leq p-1$, then $S_d(G_1T)$ is semisimple by [8, Theorem 3]. If $d = p+a$ for $0 \leq a \leq p-1$, then we get all blocks not associated to Steinberg modules as above, but only the first kind of blocks is non-simple since the second kind of blocks is a shift of a block of $S_{d'}(G_1T)$ with $d' \leq p-1$. \square

Now let r, n be arbitrary again. By reducing to the case $r = 1$, we are able to determine the M_rD -parts of G_rT -components Θ whose restrictions to G_r have type $\mathbb{Z}[\tilde{A}_{12}]$. Since the restriction to $(\mathrm{SL}_n)_r$ induces a morphism of stable translation quivers by 1.2.5, so that the restriction of Θ is a component of $\Gamma_s((\mathrm{SL}_n)_r)$, [15, 4.1] implies that the restriction of Θ to $(\mathrm{SL}_n)_r$ is also of type $\mathbb{Z}[\tilde{A}_{12}]$. Now [15, 4.2] implies $n = 2$ as SL_n is almost simple.

Corollary 3.4.11. *Suppose that $F(\Theta)$ is of type $\mathbb{Z}[\tilde{A}_{12}]$ and $\Theta \cap \mathrm{mod} M_rD \neq \emptyset$. Then Θ is of type $\mathbb{Z}[A_\infty^\infty]$, $\Theta^\circ = \Theta$ and there is a column of simple modules in Θ which is a symmetry axis with respect to $(-)^{\circ}$. There is an M_rD -module $V \in \Theta$ such that $\mathrm{mod} M_rD \cap \Theta$ consists of all modules on directed paths from V to V° .*

Proof. Note that taking the Morita-equivalence in 3.2.11 followed by the forgetful functor $\mathrm{mod} G_rT \rightarrow \mathrm{mod} (\mathrm{SL}_2)_r$ is the same as taking the forgetful functor followed by the Morita-equivalence of [18, 5.1]. As the Morita-equivalence of 3.2.11 maps the $M_{r-s}D$ -modules in a $G_{r-s}T$ -block to the M_rD -modules in a G_rT -block and commutes with the forgetful functors, we can use the arguments of [18, 5.4] to reduce to the case $r = 1$. The result now follows from 3.4.4 and its succeeding remarks. \square

4 Polynomial representations of $B_r T$

4.1 Structure and induction

In this section, let $G \subseteq \mathrm{GL}_n$ be a closed connected reductive subgroup containing $Z(\mathrm{GL}_n)$, $T \subseteq G$ be a maximal torus and $B, B^- \subseteq G$ be opposite Borel subgroups containing T , $D = \overline{T}$ the closure of T in Mat_n , $U \subseteq B$ the unipotent radical of B and $U^- \subseteq B^-$ the unipotent radical of B^- . In contrast to the ordinary Schur algebra, the algebra $S_d(G_r T)$ is not in general quasi-hereditary, see [9, Section 7]. However, as we show below, the algebras $S_d(B_r T)$ are directed, i.e. quasi-hereditary with simple standard modules. See [4] for the definition of a quasi-hereditary algebra.

Proposition 4.1.1. *Let $G \subseteq \mathrm{GL}_n$ be a reductive group, $T \subseteq G$ a maximal torus and B a Borel subgroup of G containing T . Then the algebra $S_d(B_r T)$ is quasi-hereditary and directed.*

Proof. Let λ be a polynomial weight for T of degree d . By [26], II.9.5 and the remark before II.9.3, all weights μ of the projective cover $\widehat{Z}_r(\lambda)$ of k_λ in $\mathrm{mod} B_r T$ satisfy $\mu \leq \lambda$ and the weight space for the weight λ is 1-dimensional. Since every module with top k_λ is a factor module of $\widehat{Z}_r(\lambda)$, the ordering is adapted (see the remarks below [4, Lemma 1.2] for the definition of adapted ordering). As $\mathcal{G}_{\overline{B_r T}}(\widehat{Z}_r(\lambda))$ is projective in $\mathrm{mod} \overline{B_r T}$ and has simple top k_λ , it is the projective cover of k_λ in $\mathrm{mod} \overline{B_r T}$ and we get $\mathcal{G}_{\overline{B_r T}}(\widehat{Z}_r(\lambda)) \cong \Delta(\lambda)$ by definition, see [4, Section 1]. As the λ -weight space of $\mathcal{G}_{\overline{B_r T}}(\widehat{Z}_r(\lambda))$ is one-dimensional, k_λ occurs only once as a composition factor of $\mathcal{G}_{\overline{B_r T}}(\widehat{Z}_r(\lambda))$, so that $\mathcal{G}_{\overline{B_r T}}(\widehat{Z}_r(\lambda))$ has endomorphism ring isomorphic to k by [4, 1.3]. Now [4, Theorem 1] implies that $S_d(B_r T)$ is quasi-hereditary. Relative to the reverse ordering, $S_d(B_r T)$ obtains the structure of a quasi-hereditary algebra with simple standard modules $\Delta(\lambda) = k_\lambda$, so that $S_d(B_r T)$ is directed. \square

Example. Let $r = 1$, $B = U \rtimes T \subseteq \mathrm{GL}_n$ the Borel subgroup of upper triangular matrices with $T \subseteq B$ the torus of diagonal matrices and $U \subseteq B$ the subgroup of upper triangular unipotent matrices and $L_1 D = \overline{B_1 T}$. Then $\mathrm{mod} B_1 T = \mathrm{mod} U_1 \rtimes T$ is the category of \mathbb{Z}^2 -graded modules for the Nakayama algebra $kU_r \cong k[X]/(X^p)$, where the grading on $k[X]/(X^p)$ is given by $\deg(x) = (1, -1)$, where $x = X + (X^p)$. In the following, we will compute the almost split sequences and the Auslander-Reiten quiver of $B_1 T$, the Auslander-Reiten quiver of the algebras $S_d(B_1 T)$ and their quiver and relations.

(1) By [1, V.4.1] combined with the classification of indecomposable $k[X]/(X^p)$ -modules,

the almost split sequences for $k[X]/(X^p)$ have the form

$$(0) \rightarrow (x)/(x^{t+1}) \xrightarrow{\begin{bmatrix} q \\ i \end{bmatrix}} (x)/(x^t) \oplus (x)/(x^{t+2}) \xrightarrow{\begin{bmatrix} -j & p \end{bmatrix}} (x)/(x^{t+1}) \rightarrow (0)$$

with $1 \leq t \leq p-1$, where q, p are the canonical epimorphisms and i, j are the canonical inclusions and where $(x)/(x^{p+1}) = k[X]/(X^p)$. Thus, the stable Auslander-Reiten quiver of kU_1 is isomorphic to $\mathbb{Z}[A_{p-1}]/\langle \tau \rangle$ and the directed graph underlying the (non-stable) Auslander-Reiten quiver is $\mathbb{Z}[A_p]/\langle \tau \rangle$. Since the image of an almost split sequence in $\text{mod } B_r T$ under the forgetful functor is an almost split sequence in $\text{mod } kU_1$ by 1.2.2, we can find the almost split sequences in $\text{mod } B_1 T$ by shifting the canonical grading on almost split sequences in $\text{mod } kU_1$ in such a way that the canonical maps occurring in the sequence become morphisms of graded modules. Here, we identify $(x)/(x^t)$ with the homogeneous submodule (x^{p-t}) of $k[X]/(X^p)$ with the grading given by $\deg(x) = (1, -1)$. Since the projections p, q correspond to multiplication with x , we get that almost split sequences in $\text{mod } B_1 T$ are shifts of sequences of the form

$$(0) \rightarrow (x)/(x^{t+1}) \xrightarrow{\begin{bmatrix} q \\ i \end{bmatrix}} (x)/(x^t)[(-1, 1)] \oplus (x)/(x^{t+2}) \xrightarrow{\begin{bmatrix} -j & p \end{bmatrix}} (x)/(x^{t+1})[(-1, 1)] \rightarrow (0)$$

for $1 \leq t \leq p-1$, so that $\tau_{B_1 T} = [(1, -1)]$ and all stable components of $\Gamma(B_1 T)$ are isomorphic to $\mathbb{Z}[A_{p-1}]$ while the underlying graph of the (non-stable) quiver $\Gamma(B_r T)$ is $\mathbb{Z}[A_p]$, where the bottom layer consists of simple modules and the top layer consists of shifts of the projective module $k[X]/(X^p)$.

- (2) The Auslander-Reiten quiver of $S_d(B_1 T)$ consists of the $L_1 D$ -part of the component of $\Gamma(B_1 T)$ containing $k_{(d,0)}$ and contains the simple modules $k_{(d,0)}, k_{(d-1,1)}, \dots, k_{(0,d)}$. Since $\text{mod } L_r D$ is closed with respect to extensions, submodules and factor modules, the $L_1 D$ -part of a component of $B_1 T$ consists of all indecomposable extensions of these quasi-simple modules. For $d \leq p$, this means that $\Gamma(S_d(B_1 T))$ is the wing of $(x)/(x^{d+1})$. For $d \geq p+1$, $\Gamma(S_d(B_1 T))$ is the union of the wings of $k[X]/(X^p)[(d-p, p)], k[X]/(X^p)[(d-p-1, p+1)], \dots, k[X]/(X^p)[(0, d)]$. Since $\mathfrak{F}_{L_r D}(k_{(d+1,-1)}) = 0 = \mathcal{G}_{L_r D}(k_{(-1,d+1)})$, 1.3.1 and a dual version of that result imply that all modules in the leftmost part of the leftmost wing are projective in $\text{mod } S_d(B_1 T)$ while all modules in the rightmost part of the rightmost wing are injective.
- (3) Reading off the extensions of the simple modules in the bottom row of the Auslander-Reiten quiver, we see that the Ext-quiver of $S_d(B_1 T)$ is of the form A_{d+1} with all arrows oriented in the same direction. As every projective indecomposable $S_d(B_1 T)$ -module is a factor module of a graded projective indecomposable $k[X]/(X^p)$ -module such that the submodule structures of the module in both categories coincide, the fact that $k[X]/(X^p)$ is a Nakayama algebra implies that the same holds for $S_d(B_1 T)$. The form of the Auslander-Reiten quiver now shows that for $d \leq p-1$, $S_d(B_1 T)$ is

hereditary, so that we have $S_d(B_1T) \cong kA_{d+1}$ in that case. Now let $d \geq p$ and I an admissible ideal of kA_{d+1} such that $S_d(B_1T) \cong kA_{d+1}/I$. By definition (see [1, II.2.3, II.2.9]), the relations generating I consist of linear combinations of paths with the same source and target. Since there is at most one path in A_{d+1} for every source and target, all relations for $S_d(B_1T)$ are given by paths. As all indecomposable $S_d(B_1T)$ -modules have dimension $\leq p$, all paths of length p are contained in I . For $d \geq p$, $S_d(B_1T)$ has $d - p + 1$ pairwise non-isomorphic indecomposable modules of dimension p by (2). As there are $d - p + 1$ paths of length $p - 1$ in A_{d+1} , I cannot contain a path of length smaller than p ; else there would be fewer projective indecomposable modules of dimension p in kA_{d+1}/I . Thus, $S_d(B_1T)$ is isomorphic to the bound quiver algebra kA_{d+1}/I with relations given by all paths of length p .

From now on, we suppose that the closure of G in Mat_n is normal. Since U, U^- are generated by the root subgroups $U_\alpha, U_{-\alpha}$ for $\alpha \in R^+$, see [26, II.1.7, II.1.8], the construction in [26, II.1.16] implies that we can choose the anti-automorphism η of 3.1.1 in such a way that $\eta(B^-) = B$. Note that each $\lambda \in P(D)$ can be extended to a character of \overline{B} by [6, Lemma 3.3], which we will also denote by λ , so that k_λ has a natural structure as a $\overline{B_rT}$ -module. By abuse of notation, we will also denote the character of $\overline{B^-}$ obtained this way by λ and the corresponding one-dimensional $\overline{B_r^-T}$ -module by k_λ .

Proposition 4.1.2. *For every $\lambda \in P(D)$ of degree d , there is an isomorphism of $S_d(G_rT)$ -modules*

$$S_d(G_rT) \otimes_{S_d(\overline{B_r^-T})} k_\lambda \cong (\text{ind}_{\overline{B_rT}}^{\overline{G_rT}} k_\lambda)^\circ.$$

Proof. By [34, 7.1], there is an isomorphism

$$S_d(G_rT) \otimes_{S_d(\overline{B_r^-T})} k_\lambda \cong (\text{Hom}_{S_d(\overline{B_rT})}(S_d(G_rT), k_\lambda^\circ))^\circ$$

of $S_d(G_rT)$ -modules, where k_λ° is the one-dimensional $S_d(\overline{B_rT})$ -module induced by λ . By definition, $\text{ind}_{\overline{B_rT}}^{\overline{G_rT}} k_\lambda = \{f \in k[\overline{G_rT}] \mid f(bx) = \lambda(b)f(x) \text{ for all } x \in \overline{G_rT}(A), b \in \overline{B_rT}(A)\}$ and all commutative k -algebras A . Since λ has degree d and $A_d(G_rT)$ is the d -weight space for the action of $Z(\text{GL}_n)$ on $k[\overline{G_rT}]$, this is contained in $A_d(G_rT)$ and we also have $\lambda \in A_d(\overline{B_rT})$. Denote by $\rho : A_d(G_rT) \rightarrow A_d(\overline{B_rT}) \otimes_k A_d(G_rT)$ the left $A_d(\overline{B_rT})$ -comodule structure on $A_d(G_rT)$ and by $\xi.f$ the induced right action of $\xi \in S_d(\overline{B_rT})$ on $f \in A_d(G_rT)$. Then for all $f \in \text{ind}_{\overline{B_rT}}^{\overline{G_rT}} k_\lambda$, we have $\rho(f) = \lambda \otimes f$, so that for all $\xi \in S_d(\overline{B_rT})$, we get

$$\xi.f = \mu \circ (\xi \otimes \text{id}_{A_d(G_rT)}) \circ \rho(f) = \xi(\lambda)f = \lambda'(\xi)f,$$

where $\lambda' : S_d(\overline{B_rT}) \rightarrow k$ is the representation of $S_d(\overline{B_rT})$ on k induced by the $\overline{B_r^-T}$ -module k_λ and μ is the multiplication map. Thus, $f \in \{g \in A_d(G_rT) \mid \forall \xi \in S_d(\overline{B_rT}) : \xi.g = \lambda'(\xi)g\}$. Conversely, let $f \in \{g \in A_d(G_rT) \mid \forall \xi \in S_d(\overline{B_rT}) : \xi.g = \lambda'(\xi)g\}$ and extend f to a basis $f = f_1, \dots, f_l$ of $A_d(G_rT)$. Then there are $g_i \in S_d(\overline{B_rT})$ such that $\rho(f) = \sum_{i=1}^l g_i \otimes f_i$. For all $\xi \in S_d(\overline{B_rT})$, we have

$$\xi(\lambda)f = \lambda'(\xi)f = \xi.f = \sum_{i=1}^l \xi(g_i)f_i$$

by definition, so that comparing coefficients yields $\xi(g_i) = 0$ for $i \neq 1$ and $\xi(g_1) = \xi(\lambda)$. This shows $g_i = 0$ for $i \neq 1$ and $g_1 = \lambda$, so that $\rho(f) = \lambda \otimes f$ and $f \in \text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda$. Hence $\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda = \{g \in A_d(G_r T) \mid \forall \xi \in S_d(B_r T) : \xi \cdot g = \lambda'(\xi)g\}$. By 3.1.4, the canonical isomorphism of vector spaces

$$\Phi : A_d(G_r T) \rightarrow \text{Hom}_k(S_d(G_r T), k) = S_d(G_r T)^o, f \mapsto (\varphi \mapsto \varphi(f))$$

is an isomorphism of $S_d(G_r T)$ -modules. We show that its restriction to $\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda$ is the required isomorphism. Let $f \in \text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda$. Then for all $\xi \in S_d(B_r T), \varphi \in S_d(G_r T)$, we have

$$\begin{aligned} \Phi(f)(\xi \cdot \varphi) &= (\xi \cdot \varphi)(f) = \sum \xi(f_{(1)})\varphi(f_{(2)}) = \varphi(\xi \cdot f) \\ &= \varphi(\lambda'(\xi)f) = \lambda'(\xi)\varphi(f) = \lambda'(\xi)\Phi(f)(\varphi), \end{aligned}$$

so that $\Phi(f) \in \text{Hom}_{S_d(B_r T)}(S_d(G_r T), k_\lambda^o)$. Conversely, if $\Phi(f) \in \text{Hom}_{S_d(B_r T)}(S_d(G_r T), k_\lambda^o)$ and $\xi \in S_d(B_r T)$, then we have

$$\varphi(\xi \cdot f) = \sum \xi(f_{(1)})\varphi(f_{(2)}) = \Phi(f)(\xi \cdot \varphi) = \xi \cdot \Phi(f)(\varphi) = \varphi(\lambda'(\xi)f)$$

for all $\varphi \in S_d(G_r T) = A_d(G_r T)^*$, so that $\xi \cdot f = \lambda'(\xi)f$ and $f \in \text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda$. Thus, $\Phi(\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda) = \text{Hom}_{S_d(B_r T)}(S_d(G_r T), k_\lambda^o)$. \square

Corollary 4.1.3. *We have isomorphisms of $S_d(G_r T)$ -modules*

$$\begin{aligned} S_d(G_r T) \otimes_{S_d(B_r^- T)} k_\lambda &\cong (\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda)^o \cong (\mathfrak{F}_{\overline{G_r T}}(\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda))^o \\ &\cong \mathcal{G}_{\overline{G_r T}}((\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda)^o) \cong \mathcal{G}_{\overline{G_r T}}(\widehat{Z}_r(\lambda)^o) \\ &\cong \mathcal{G}_{\overline{G_r T}}(\widehat{Z}'_r(\lambda)) \cong \mathcal{G}_{\overline{G_r T}}(kG_r \otimes_{kB_r^-} k_\lambda). \end{aligned}$$

Proof. For the second isomorphism, one shows $\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda \cong \mathfrak{F}_{\overline{G_r T}}(\text{ind}_{\overline{B_r T}}^{\overline{G_r T}} k_\lambda)$ as in the proof of [9, Proposition 5.1]. Then the third isomorphism follows from 3.1.3, the fourth one from the definition, the fifth one from [26, II.9.3(5)] and the last one from [26, I.8.20] combined with the definition in [26, I.8.14]. \square

Proposition 4.1.4. *Multiplication induces an epimorphism of vector spaces $S_d(G_r T) \rightarrow S_d(B_r^- T) \otimes_k S_d(U_r)$.*

Proof. By [26, II.9.7], we have an isomorphism of schemes $U_r^- \times T \times U_r \rightarrow G_r T$ induced by multiplication. Since $U_r^- \times T \cong B_r^- T$ via multiplication, we get that $\mu \circ (i_{B_r^- T} \times i_{U_r}) : B_r^- T \times U_r \rightarrow G_r T$ is an isomorphism, where μ is multiplication in $G_r T$ and $i_{B_r^- T}, i_{U_r}$ are the canonical inclusions. Taking comorphisms, we see that $(\pi_{B_r^- T} \otimes \pi_{U_r}) \circ \Delta : k[G_r T] \rightarrow k[B_r^- T] \otimes_k k[U_r]$ is an isomorphism of k -algebras. Since $\Delta(A_d(G_r T)) \subseteq A_d(G_r T) \otimes_k A_d(G_r T)$ and $\pi_{B_r^- T}(A_d(G_r T)) = A_d(B_r^- T)$, $\pi_{U_r}(A_d(G_r T)) = A_d(U_r)$ by definition, we get a monomorphism $A_d(G_r T) \rightarrow A_d(B_r^- T) \otimes_k A_d(U_r)$. Dualizing this, we get the claimed epimorphism. \square

It follows that as a vector space, $S_d(G_r T)$ is generated by $S_d(B_r^- T)S_d(B_r T)$. The analogue of this result for the ordinary Schur algebra was shown in [24, Section 4].

Corollary 4.1.5. *We have $S_d(G_r T) = \langle S_d(B_r^- T)S_d(B_r T) \rangle$.*

Corollary 4.1.6. *For all $M, N \in \text{mod } S_d(G_r T)$, we have $\text{Hom}_{S_d(G_r T)}(M, N) = \text{Hom}_{S_d(B_r^- T)}(M|_{S_d(B_r^- T)}, N|_{S_d(B_r^- T)}) \cap \text{Hom}_{S_d(B_r T)}(M|_{S_d(B_r T)}, N|_{S_d(B_r T)})$.*

4.2 Outer tensor products

In this section, let $p \geq 3$. Let $T \subseteq \text{GL}_2$ be the torus of diagonal matrices, $U \subseteq \text{GL}_2$ be the group of upper triangular unipotent matrices, $B = U \rtimes T \subseteq \text{GL}_2$ be the subgroup of upper triangular matrices and $L_r D = \overline{B_r T}$. It is well known that $U \cong G_a$, $k[U_r] \cong k[X]/(X^{p^r})$ and $kU_r \cong k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$, see for example [18, Section 2]. By considering the action of T on U_r and the induced actions on $k[U_r]$ and kU_r , we see that the $\mathbb{Z}^2 \cong X(T)$ -grading on $k[U_r]$ is given by $\deg(x_i) = p^{i-1}(1, -1)$, where x_i is the residue class of X_i in $k[U_r]$. Then $\text{mod } B_r T = \text{mod } U_r \rtimes T$ is the category of \mathbb{Z}^2 -graded kU_r -modules. In this section, we consider all algebras $k[X_i, \dots, X_j]/(X_i^p, \dots, X_j^p)$, $1 \leq i \leq j \leq r$, to be \mathbb{Z}^2 -graded with these degrees. In this case, we can extend our results about modules of complexity one to tensor products of these modules with other indecomposable modules.

If M, N as in the next result are indecomposable, general theory ([3, I.10E]) in combination with 1.2.1 shows that $M \otimes_k N$ is an indecomposable $B_r T$ -module as k is algebraically closed.

Proposition 4.2.1. *Let $M \in \text{mod } B_{r-h} T$ be indecomposable such that $cx_{B_{r-h} T}(M) = 1$ and let N be a finite-dimensional non-projective indecomposable graded module over the algebra $k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)$. Suppose the component $\Theta \subseteq \Gamma_s(B_r T)$ containing $M \otimes_k N$ is regular. Then $\Theta \cong \mathbb{Z}[A_\infty]$ and the quasi-length of elements of $\Theta \cap \text{mod } L_r D$ is bounded. If $M \otimes_k N$ is quasi-simple, $\Theta \cap \text{mod } L_r D$ is finite.*

Proof. As Θ is regular, we have $cx_{k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)}(N) \geq 1$ and thus $cx_{B_r T}(M \otimes_k N) = cx_{B_{r-h} T}(M) + cx_{k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)}(N) \geq 2$ (see for example [27, 3.2.15]), so that Θ is not periodic. Now [14, Theorem 1] implies $\Theta \cong \mathbb{Z}[A_\infty]$ as $k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$ has wild representation type for $r > 1$ since $p \geq 3$.

We show that the $\tau_{B_r T}$ -orbit of $M \otimes_k N$ contains only finitely many $L_r D$ -modules. As $\text{mod } L_r D$ is closed with respect to submodules and factor modules, the result then follows from 1.3.6.

Let $P = (P_l)_{l \in \mathbb{N}_0}$ be a minimal projective resolution of M as a graded $k[X_1, \dots, X_{r-h}]/(X_1^p, \dots, X_{r-h}^p)$ -module and $Q = (Q_l)_{l \in \mathbb{N}_0}$ be a minimal projective resolution of N as a graded $k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)$ -module. Then the complex $P \otimes_k Q$ is a minimal projective resolution of $M \otimes_k N$ as a $B_r T$ -module, where

$$(P \otimes_k Q)_l = \bigoplus_{i+j=l} P_i \otimes_k Q_j$$

and

$$d_{P \otimes_k Q, l} = \bigoplus_{i+j=l} d_{P, i} \otimes \text{id}_{Q, j} + (-1)^i \text{id}_{P, i} \otimes d_{Q, j}$$

for all $l > 0$. Thus, for all $i > 0$, we get $\Omega_{B_{r-h}T}^i(M) \otimes_k \Omega_{k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)}^1(N) \subseteq \Omega_{B_rT}^i(M \otimes_k N)$. Let $\lambda \in \mathbb{Z}^2$ such that $\Omega_{k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)}^1(N)_\lambda \neq 0$. By definition of the grading on the tensor product of modules, we get $\text{supp}(\Omega_{B_{r-h}T}^i(M[\lambda])) \subseteq \text{supp}(\Omega_{B_rT}^i(M \otimes_k N))$ for all $i > 0$. Let $\alpha = (1, -1)$ be the positive root of G relative to B . By the proof of 3.3.1, the Nakayama functor \mathcal{N} of $\text{mod } B_rT$ is just the shift by $-(p^r - 1)\alpha$. For every $s \in \mathbb{N}_0$, we get

$$\text{supp}(\tau_{B_rT}^{p^s}(M \otimes_k N)) \supseteq \text{supp}(\Omega_{B_{r-h}T}^{2p^s}(M[\lambda])[-p^s(p^r - 1)\alpha]).$$

By [18, 8.1.1], there is $s \in \mathbb{N}_0$ such that

$$\Omega_{B_{r-h}T}^{2p^s}(M[\lambda])[-p^s(p^r - 1)\alpha] = M[\lambda][-p^s(p^r - 1)\alpha + p^{r-h}\alpha]$$

or

$$\Omega_{B_{r-h}T}^{2p^s}(M[\lambda])[-p^s(p^r - 1)\alpha] = M[\lambda][-p^s(p^r - 1)\alpha].$$

Since similar computations can also be applied to $\tau_{B_rT}^i(M \otimes_k N) \cong \mathcal{N}^i \circ \Omega_{B_rT}^{2i}(M \otimes_k N)$ for $1 \leq i \leq p^s - 1$, arguments analogous to those in the proof of 3.3.2 show that the τ_{B_rT} -orbit of $M \otimes_k N$ is finite. \square

Note that if N in the situation of the previous result is projective, then $cx_{B_rT}(M \otimes_k N) = 1$ and $\Theta \cap L_rD$ is finite by 3.3.3. If the component containing $M \otimes_k N$ is not regular, we have $cx_{U_r}(M \otimes_k N) = r$ and the Künneth-formula yields $h = r - 1$ and $cx_{k[X_{r-h+1}, \dots, X_r]/(X_{r-h+1}^p, \dots, X_r^p)}(N) = h$.

Corollary 4.2.2. *Let $\lambda \in \mathbb{Z}^2$, $\Theta \subseteq \Gamma_s(B_rT)$ be the component containing k_λ . Then $\Theta \cap \text{mod } L_rD$ is finite.*

Proof. For $r = 1$, this follows from 3.3.3 since $cx_{B_rT}(k_\lambda) = 1$. For $r \geq 2$, $cx_{B_rT}(k_\lambda) = r > 1$, so that k_λ is not periodic. Now [15, 5.6] shows $F(\Theta) \cong \mathbb{Z}[A_\infty]$, so that k_λ is a quasi-simple module in a $\mathbb{Z}[A_\infty]$ -component of $\Gamma_s(B_rT)$. Suppose Θ is not regular. Then $F(\Theta)$ is not regular and the standard almost split sequence [2, V.5.5] shows that $\Omega_{U_r}(k) \in F(\Theta)$. As Ω_{U_r} is an auto-equivalence of the stable module category, this shows that $\Omega_{U_r}(F(\Theta)) = F(\Theta)$, so that Ω_{U_r} defines an automorphism of $F(\Theta)$. In particular, k and $\Omega_{U_r}(k)$ have the same quasi-length, so that $\tau_{U_r} = \Omega_{U_r}^2$ implies $\Omega(k)_{U_r}^{2l+1}(k) = k$ for some $l \in \mathbb{Z}$. Thus, k is periodic, a contradiction. Hence Θ is regular. Since U_r acts trivially on k_λ , we have $k_\lambda \cong k_{\lambda|B_1T} \otimes_k k$, where we regard k as the trivial module for $k[X_2, \dots, X_r]/(X_2^p, \dots, X_r^p)$. The result now follows from 4.2.1. \square

4.3 B_r -components containing regular modules

In this section, we show that certain components of $\Gamma_s(B_r)$, where B is a connected solvable group scheme, are of type $\mathbb{Z}[A_\infty]$. We have $B_r = U_r \rtimes T_r$, where U is the unipotent radical and T is a maximal torus of B . By [15, 5.6], every infinite component of $\Gamma_s(U_r)$ has tree class A_∞ . We want to use methods from [17] to extend this to components of B_r containing so-called regular modules defined there. Let $F : \text{mod } B_r \rightarrow \text{mod } U_r$ be the canonical restriction functor.

Lemma 4.3.1. *Let $p \geq 5$, $\Theta \subseteq \Gamma_s(B_r)$ be a component containing a regular module. Then $F(\Theta)$ is a component of $\Gamma_s(U_r)$.*

Proof. We use results from [17] to imitate the proof of [16, 2.2(2)]. By [17, 3.1.5], $F : \Theta \rightarrow \Gamma_s(U_r)$ is a morphism of stable translation quivers. Hence there exists a unique component $\Psi \subset \Gamma_s(U_r)$ such that $F(\Theta) \subset \Psi$. By [17, 3.1.1], $F(\Theta)$ is $\tau_{U_r}^n$ -invariant for all $n \in \mathbb{Z}$, so we only have to show that each neighbor of an element of $F(\Theta)$ also belongs to $F(\Theta)$. Let $M \in \Theta$ and

$$\xi : (0) \longrightarrow \tau_{U_r}(F(M)) \longrightarrow \bigoplus_{i=1}^n m_i E_i \longrightarrow F(M) \longrightarrow (0)$$

be almost split in $\text{mod } U_r$, E_i the indecomposable predecessors of $F(M)$ in Ψ . As every module in Θ is regular, [17, 3.1.4] yields that ξ is isomorphic to

$$(0) \longrightarrow F(\tau_{B_r}(M)) \longrightarrow F(X) \longrightarrow F(M) \longrightarrow (0),$$

where

$$(0) \longrightarrow \tau_{B_r}(M) \longrightarrow X \longrightarrow M \longrightarrow (0)$$

is the almost split sequence in $\text{mod } B_r$ ending in M . Let $X = \bigoplus_{i=1}^l s_i X_i$ be a decomposition of X into indecomposable summands. As F is additive and sends indecomposables to indecomposables by [17, 2.1.2], we get that $\bigoplus_{i=1}^n m_i E_i \cong \bigoplus_{i=1}^l s_i F(X_i)$ with $F(X_i), E_i$ indecomposable. By the Krull-Remak-Schmidt Theorem, for every $1 \leq i \leq n$, there is some j such that $E_i \cong F(X_j)$, so that all predecessors of $F(M)$ also belong to $F(\Theta)$. Using the bijectivity of τ_{U_r} , one shows the same for all successors of $F(M)$. \square

Remark. The proof works in the more general setting of [17, 3.1.5].

Proposition 4.3.2. *Let $p \geq 5$ and $\Theta \subseteq \Gamma_s(B_r)$ be an infinite component containing a regular module. Then Θ has tree class A_∞ .*

Proof. By [15, 5.6] and 4.3.1, $F(\Theta)$ is a component of $\Gamma_s(U_r)$ and has tree class A_∞ . Hence $F(\Theta)$ and Θ have infinitely many τ -orbits. By [17, 3.1.4], the number of summands of a middle term of an almost split sequence in Θ is at most two and there are almost split sequences whose terms lie in Θ with indecomposable middle term. As a component whose tree class is a finite Dynkin diagram has finitely many τ -orbits, the list of possible components in [15, 1.3] shows that Θ has tree class A_∞ . \square

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Einige der Hauptergebnisse der Dissertation sind in folgenden Publikationen enthalten:
Drenkhahn, Christian. *On simple polynomial G_rT -modules*. Preprint. Zur Publikation eingereicht bei *Journal of Algebra*.

Drenkhahn, Christian. *Almost split sequences for polynomial G_rT -modules and polynomial parts of Auslander-Reiten components*. Preprint. Zur Publikation eingereicht bei *Journal of Pure and Applied Algebra*.

Kiel, den 01. August 2016

(Christian Drenkhahn)