

# Saturation rank for finite group schemes and varieties of elementary subalgebras

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# Abstract

Let  $G$  be a connected reductive algebraic  $\mathbb{k}$ -group over an algebraically closed field  $\mathbb{k}$  of characteristic  $p > 0$ , and  $\mathfrak{g} := \text{Lie}(G)$  be its Lie algebra. Suppose in addition that  $G$  is standard, i.e.  $p$  is good for  $G$ , the derived subgroup of  $G$  is simply connected and there exists a non-degenerate  $G$ -invariant symmetric bilinear form for  $\mathfrak{g}$ . Following [CFP] and [PS] we acquaint ourselves with the projective varieties  $\mathbb{E}(r, \mathfrak{g})$  and how they relate to the representation theory of  $\mathfrak{g}$ . It is the purpose of this thesis to investigate the geometric properties of  $\mathbb{E}(r, \mathfrak{g})$ .

By refining the definition of saturation rank in the context of finite groups schemes, we reveal parallel descriptions for constant finite group schemes and infinitesimal group schemes. When our consideration concentrates on restricted Lie algebras, i.e. infinitesimal group schemes of height  $\leq 1$ , we identify the rank with the semisimple rank of  $G$  in case of reductive Lie algebras. This characterization is dedicated to establishing the relationship between  $\mathbb{E}(r, \mathfrak{g})$  and the restricted nullcone  $V(\mathfrak{g})$ . As a supplement, we also consider the non-reductive case, i.e. the proper parabolic subgroups. In order to study more generally the rank on the  $r$ -th Frobenius kernel  $G_{(r)}$ , we specialize to  $G = \text{SL}_n$  and  $r = 2$ , revealing an interesting result.

We also identify the varieties  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  for  $G$  being simple and of classical type when  $r_{\text{smax}}$  equals  $r_{\text{max}} - 1$ , where  $r_{\text{max}}$  is the maximal dimension of elementary subalgebras of  $\mathfrak{g}$ . Our goal is to determine the irreducible components of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$ , and to pursue links and similarities with the result of Pevtsova and Stark on  $\mathbb{E}(r_{\text{max}}, \mathfrak{g})$ . Having done this, the irreducibility of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  for related types will be seen.



# Zusammenfassung

Sei  $G$  eine zusammenhängende reductive algebraische  $\mathbb{k}$ -Gruppe über einem algebraisch abgeschlossenen Körper  $\mathbb{k}$  der Charakteristik  $p > 0$  und  $\mathfrak{g} := \text{Lie}(G)$  ihre Lie-Algebra. Wir setzen zusätzlich voraus, dass  $G$  standard ist, d.h.  $p$  ist gut für  $G$ , die abgeleitete Untergruppe ist einfach zusammenhängend und es existiert eine nicht-ausgeartete  $G$ -invariante symmetrische Bilinearform auf  $\mathfrak{g}$ . Wir folgen [CFP] und [PS] und betrachten die projektiven Varietäten  $\mathbb{E}(r, \mathfrak{g})$  und ihren Zusammenhang zur Darstellungstheorie von  $\mathfrak{g}$ . Es ist Ziel dieser Arbeit, die geometrischen Eigenschaften von  $\mathbb{E}(r, \mathfrak{g})$  zu untersuchen.

Indem wir die Definition des Saturationsranges für endliche Gruppenschemata verfeinern, finden wir Beschreibungen für konstante endliche Gruppenschemata und infinitesimale Gruppenschemata. Im Falle restringierter Lie-Algebren, d.h. infinitesimaler Gruppenschemata der Höhe  $\leq 1$ , zeigen wir, dass im Falle reductiver Lie-Algebren der Rang mit dem halbeinfachen Rang von  $G$  übereinstimmt. Diese Charakterisierung hat das Ziel, einen Zusammenhang zwischen  $\mathbb{E}(r, \mathfrak{g})$  und dem restringierten Nullkegel  $V(\mathfrak{g})$  herzustellen. Wir betrachten zusätzlich den nicht-reduktiven Fall, d.h. den Fall echter parabolischer Untergruppen. Um allgemeiner den Rang des  $r$ -ten Frobeniuskerns  $G_{(r)}$  zu untersuchen, betrachten wir den Spezialfall  $G = \text{SL}_n$  und  $r = 2$ , wodurch wir zu einem interessanten Resultat kommen.

Wir bestimmen außerdem die Varietäten  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  für einfaches  $G$  von klassischem Typ, wenn  $r_{\text{smax}}$  mit  $r_{\text{max}}$  übereinstimmt, wobei  $r_{\text{max}}$  die maximale Dimension einer elementaren Unter algebra von  $\mathfrak{g}$  ist. Unser Ziel ist die Bestimmung der irreduziblen Komponenten von  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  sowie Zusammenhänge und Ähnlichkeiten mit dem Resultaten von Pevtsova und Stark über  $\mathbb{E}(r_{\text{max}}, \mathfrak{g})$ . Im Anschluss zeigen wir die Irreduzibilität von  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  für damit zusammenhängende Typen.





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# Chapter 1

## Introduction

Let  $(\mathfrak{g}, [p])$  be a finite dimensional restricted Lie algebra over an algebraically closed field  $\mathbb{k}$  of positive characteristic  $p > 0$ . A Lie subalgebra  $\mathcal{E} \subset \mathfrak{g}$  is referred to as elementary if it is abelian, and the  $p$ -map is trivial. It is the purpose of this thesis to study the projective variety  $\mathbb{E}(r, \mathfrak{g})$  of elementary subalgebras of  $\mathfrak{g}$  of dimension  $r$  and its relationship to the restricted nullcone  $V(\mathfrak{g})$ .

The investigation of  $\mathbb{E}(r, \mathfrak{g})$  is dedicated to knowing more about the representation theory of  $\mathfrak{g}$ . When  $r = 1$ ,  $\mathbb{E}(1, \mathfrak{g})$  is the projectivization of  $V(\mathfrak{g})$ , with the latter identified with the spectrum of the finitely generated commutative algebra  $H^*(U(\mathfrak{g}), \mathbb{k})$ ; see [FPar]. When  $r$  is the maximal dimension  $r_{\max}$  of elementary subalgebras of  $\mathfrak{g}$ , Carlson-Friedlander-Pevtsova have identified  $\mathbb{E}(r_{\max}, \mathfrak{g})$  for simple Lie algebras of types  $A$  and  $C$  in their paper [CFP]. Their calculation reveals interesting and varied geometric structures of these varieties. Inspired by Malcev's linear and combinatorial approach on the maximal dimensions of an abelian nilpotent subalgebra, Pevtsova and Stark subsequently identify the varieties  $\mathbb{E}(r_{\max}, \mathfrak{g})$  in [PS] for a reductive restricted Lie algebra. Except for these two numbers, it is known in general that  $\mathbb{E}(r, \mathfrak{g})$  is closely related to the variety of  $r$ -tuples of pairwise commuting  $p$ -nilpotent elements of  $\mathfrak{g}$ . The appearance of these  $r$ -tuples has given an alternative description of the affine variety of infinitesimal 1-parameter subgroups over  $\mathbb{k}$  of an infinitesimal group scheme of height  $\leq r$  in special cases; see [SFB1]. The work of Premet on nilpotent commuting varieties of reductive Lie algebra then should be emphasized,

which enables us to understand more about  $\mathbb{E}(2, \mathfrak{g})$ ; see [Pre1] for details.

In what follows, we assume that  $G$  is a connected standard reductive algebraic  $\mathbb{k}$ -group, and  $\mathfrak{g} := \text{Lie}(G)$  is the Lie algebra of  $G$ . Define

$$V_{\mathbb{E}(r, \mathfrak{g})} := \bigcup_{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{g})} \mathfrak{e}$$

which is contained in  $V(\mathfrak{g})$ . We have a natural question in the following:

**Question A.** How to characterise the number

$$\max \{r \mid V(\mathfrak{g}) = V_{\mathbb{E}(r, \mathfrak{g})}\} ?$$

This question we have proposed, concentrating on the study of the relation between  $p$ -nilpotent elements and elementary subalgebras. Dating back to Quillen's work [Qui1] [Qui2] on the maximal ideal spectrum  $V_G$  of the commutative  $\mathbb{k}$ -algebra  $H(G, \mathbb{k})$  of a finite group  $G$ , an analogous question could also be raised in terms of the elementary abelian  $p$ -subgroups of  $G$ . Similar to his work, in the context of infinitesimal group schemes Suslin, Friedlander and Bendel investigated infinitesimal 1-parameter subgroups [SFB1] [SFB2]. This work refines and generalizes the theory for restricted Lie algebra studied by Friedlander, Parshall and Jantzen. The generality allows us to consider infinitesimal group schemes of height  $\leq r$  rather than the case when  $r = 1$ . When recollecting the representation-theoretic support space served for finite group schemes, we find that it is of considerable interest to develop a formula which applies to all finite group schemes, specializing in finite groups and infinitesimal group schemes. The answer we give for Question A is:

**Theorem A.** Let  $G$  be a standard connected reductive algebraic group with  $\mathfrak{g} = \text{Lie}(G)$ . Assume that  $p \geq h$ , where  $h$  is the Coxeter number of  $G$ . Then

$$\text{rk}_{\text{ss}}(G) = \max \{r \mid V(\mathfrak{g}) = V_{\mathbb{E}(r, \mathfrak{g})}\}$$

where  $\text{rk}_{\text{ss}}(G)$  is the semisimple rank of  $G$ .

The other goal for us in this thesis is to give a description of the variety  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  of elementary subalgebras of dimension  $r_{\text{smax}} = r_{\text{max}} - 1$  when  $G$  is simple and of classical type. As a beneficiary, the direct influence comes from Pevtsova et al.'s result. When combined with Malcev's method, we have the following question:



**Question B.** How to characterise the irreducible components of the variety

$$\mathbb{E}(r_{\text{smax}}, \mathfrak{g}) ?$$

Let  $B \subset G$  be a Borel subgroup of  $G$ ,  $R_u(B) \subset B$  be the unipotent radical of the Borel subgroup  $B$  and  $\mathfrak{u}_b = \text{Lie}(R_u(B))$  be the Lie algebra of  $R_u(B)$ . The determination of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  is built on a mild restriction on  $G$ , that is the standard hypothesis as we have assumed. The reason for this requirement roots in our utilization of Lemma 2.2 in [Pre2], which helps us to reduce to the unipotent case  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)$ . When the standard condition is removed, we have to resort to Levy et al.'s result. They pointed out to us, Theorem 2.2 in [LMT] can also be applied to our reduction procedure when  $p$  is a torsion prime for  $G$ . The calculation of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  then proceeds into three steps:

- Firstly, by letting  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)_{\text{max}} \subset \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)$  consist of maximal elements, we determine it as a set. The method we employ here is the map

$$\text{LT} : \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b) \longrightarrow \text{Com}(\Phi_{r_{\text{smax}}}^+),$$

which is used in [Mal] and explained in [PS, Sect. 3.1]. It is a map that sends an elementary subalgebra to a set of commuting positive roots. Our aim is to find a suitable total ordering  $\succeq$  for each classical type such that the image of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)_{\text{max}}$  under the map LT is contained in  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$ , where  $\text{Max}(\Phi_{r_{\text{smax}}}^+) \subset \text{Com}(\Phi_{r_{\text{smax}}}^+)$  is made up of maximal sets of commuting roots. Once this is settled, we are going to show that the map

$$\text{Lie} : \text{Max}(\Phi_{r_{\text{smax}}}^+) \longrightarrow \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)_{\text{max}}$$

is surjective up to conjugacy by  $G$ .

- Secondly, observe that an element of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)$  is either maximal, i.e. belongs to  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)_{\text{max}}$  or is a proper subalgebra of an elementary subalgebra in  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)$ . We shall use results of [PS] to the set  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b) \setminus \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_b)_{\text{max}}$ .
- Finally, we use the fact that every element in  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  is conjugated to an element in  $\mathfrak{u}_b$  if  $G$  is standard.

A subsidiary result in the first step is that every maximal elementary subalgebra in  $\mathfrak{u}_{\mathfrak{b}}$  of dimension  $r_{\text{smax}}$  can be conjugated to a subalgebra which is stabilized by the Borel subgroup  $B$  under the adjoint action. This interesting fact gives the isomorphism from the  $G$ -orbit of such a subalgebra to a flag variety. The following is our answer for Question B:

**Theorem B.** Let  $G$  be a connected standard simple algebraic  $\mathbb{k}$ -group with root system  $\Phi$  of type  $A_n(n \geq 5)$ ,  $B_n(n \geq 5)$ ,  $C_n(n \geq 3)$  or  $D_n(n \geq 6)$ . Then the irreducible components of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  for each type can be characterised:

Type	Rank	Irreducible components
$A_{2n+1}$	$n \geq 2$	$G.\text{Lie}(\Phi_n^{\text{rad}}), G.\text{Lie}(\Phi_{n+2}^{\text{rad}}), G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$
$A_{2n}$	$n \geq 3$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}})), G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$
$B_n$	$n \geq 5$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(S_1))$
$C_n$	$n \geq 3$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$
$D_n$	$n \geq 6$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n-1}^{\text{rad}})), G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$

This thesis is organized as follows. In Chapter 2 we introduce representation-theoretic support spaces for finite group schemes. The reductive algebraic  $\mathbb{k}$ -group and elementary subalgebra background required for the later chapters is arranged in Chapter 3. Chapter 4 deals with the saturation rank for constant finite group schemes and infinitesimal group schemes of height  $\leq 2$  respectively. In Chapter 5 we determine the maximal set of commuting positive roots of order  $r_{\text{smax}}$ , whilst the characterisation of irreducible components of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  is found in Chapter 6.

# Chapter 2

## Representation-theoretic support space

Throughout this chapter,  $\mathbb{k}$  is assumed to be the underlying ground field. Our task in this chapter is to recollect some basic information on affine group schemes firstly, like where they arise from, their definitions, and the equivalent point of view served for them. Then we are in a position to introduce representation-theoretic support spaces for finite group schemes, which is following the work given by Friedlander and Pevtsova; see [FP] for details. On the one hand, it is known that they generalise the work of Quillen on finite groups, which will be seen subsequently. On the other hand, they show good correspondence to the theory of support varieties and give an access to the representations of finite group schemes.

### 2.1 Affine group schemes

Algebraic groups, a topic people would like to investigate, has been studied in a generalised fashion, that is affine group schemes. We adopt here a viewpoint from the language of categories, giving the definition of affine group schemes functorially. Before we start, let us look at what we are going to talk about. It is known that a one-dimensional connected linear algebraic group is isomorphic to

$\mathbb{G}_a(\mathbb{k})$ (additive  $\mathbb{k}$ -group), or  $\mathbb{G}_m(\mathbb{k})$ (multiplicative  $\mathbb{k}$ -group). The elements of  $\mathbb{G}_m(\mathbb{k})$  are given by equation  $xy - 1 = 0$  with solutions in  $\mathbb{k}$ , and there are no specific equations but  $0x = 0$  satisfied by the elements of  $\mathbb{G}_a(\mathbb{k})$ . It is the purpose to assign to an arbitrary commutative  $\mathbb{k}$ -algebra  $A$  a group  $\mathbb{G}_m(A)$ (resp.  $\mathbb{G}_a(A)$ ) such that elements in  $\mathbb{G}_m(A)$ (resp.  $\mathbb{G}_a(A)$ ) correspond to solutions in  $A$  given by the equation  $xy - 1 = 0$ (resp.  $0x = 0$ ). Then take a polynomial ring  $\mathbb{k}[t_1, t_2]$ (resp.  $\mathbb{k}[t]$ ) over  $\mathbb{k}$  with indeterminates retrieved from variables of the equation, and let  $I$  be the ideal generated by  $t_1 t_2 - 1$ (resp.  $0$ ). An inspection for the  $\mathbb{k}$ -algebra  $\mathbb{k}[t_1, t_2]/I$ (resp.  $\mathbb{k}[t]$ ) yields an isomorphism from the group of  $\mathbb{k}$ -algebras homomorphisms  $Hom_{\mathbb{k}}(\mathbb{k}[t_1, t_2]/I, A)$ (resp.  $Hom_{\mathbb{k}}(\mathbb{k}[t], A)$ ) to group  $\mathbb{G}_m(A)$  (resp.  $\mathbb{G}_a(A)$ ). In this fashion, we say  $\mathbb{G}_m$ (resp.  $\mathbb{G}_a$ ) is represented by the  $\mathbb{k}$ -algebra  $\mathbb{k}[t_1, t_2]/I$ (resp.  $\mathbb{k}[t]$ ), and we are led to give the definition of affine group schemes.

Throughout,  $Ens$ ,  $Gr$  and  $Com_{\mathbb{k}}$  will denote the categories of sets, groups and commutative  $\mathbb{k}$ -algebras respectively. We follow the definition and discussion on affine (group) schemes from [Jan1] [Wat]. Given a commutative  $\mathbb{k}$ -algebra  $A$ , we consider the  $\mathbb{k}$ -functor

$$Spec_{\mathbb{k}}(A) : Com_{\mathbb{k}} \longrightarrow Ens ; R \mapsto Hom_{\mathbb{k}}(A, R)$$

where  $Hom_{\mathbb{k}}(A, R)$  is the set of  $\mathbb{k}$ -algebra homomorphisms from  $A$  to  $R$ . These  $\mathbb{k}$ -functors, having the form  $Spec_{\mathbb{k}}(A)$  are called *affine schemes*. Additionally, if  $A$  is finitely generated as a  $\mathbb{k}$ -algebra, then the affine scheme  $Spec_{\mathbb{k}}(A)$  is called *algebraic*. Accordingly, denoting by  $f_R = f \otimes_{\mathbb{k}} 1_R \in \mathbb{k}[X_1, \dots, X_n] \otimes_{\mathbb{k}} R$  when  $f \in \mathbb{k}[X_1, \dots, X_n]$  for any commutative  $\mathbb{k}$ -algebra  $R$ . Then a  $\mathbb{k}$ -functor  $\mathbf{X}$  is called affine algebraic if and only if there exist polynomials  $f_1, \dots, f_s \in \mathbb{k}[X_1, \dots, X_n]$  such that

$$\mathbf{X}(R) = \{(x_1, \dots, x_n) \in R^n \mid f_{iR}(x_1, \dots, x_n) = 0, 1 \leq i \leq s\}.$$

**Definition 2.1.1.** A  $\mathbb{k}$ -group functor  $\mathcal{G} : Com_{\mathbb{k}} \longrightarrow Gr$  is called an *affine group scheme* if it is affine, that is  $\mathcal{G} = Spec_{\mathbb{k}}(\mathbb{k}[\mathcal{G}])$ . Then  $\mathbb{k}[\mathcal{G}]$  will be referred to as the *coordinate algebra* of  $\mathcal{G}$ . If the representing algebra  $\mathbb{k}[\mathcal{G}]$  is finitely generated, then  $\mathcal{G}$  is called *algebraic*. If  $\mathbb{k}[\mathcal{G}]$  is finite-dimensional, then  $\mathcal{G}$  is a *finite group scheme*, and further the  $\mathbb{k}$ -linear dual algebra  $\mathbb{k}\mathcal{G} := \mathbb{k}[\mathcal{G}]^*$  will be called the *group algebra*(or *algebra of measure*) of  $\mathcal{G}$ .

In what follows, we will devote to establishing the connection between the category of

affine group schemes and the category of commutative Hopf algebras, which largely depends on the following result:

**Theorem 2.1.1.** (*Yoneda's Lemma*) *Suppose that  $A, B$  are commutative  $\mathbb{k}$ -algebras. There is a bijection between the set of natural transformations  $\mathrm{Spec}_{\mathbb{k}}(A) \rightarrow \mathrm{Spec}_{\mathbb{k}}(B)$  and  $\mathrm{Hom}_{\mathbb{k}}(B, A)$ .*

*Proof.* We establish a map by assigning  $\Phi : \mathrm{Spec}_{\mathbb{k}}(A) \longrightarrow \mathrm{Spec}_{\mathbb{k}}(B)$  to  $\Phi_A(\mathrm{id}_A)$ . Since  $\Phi$  is a natural transformation, we have the following commutative diagram for every commutative  $\mathbb{k}$ -algebra  $R$  and the  $\mathbb{k}$ -algebra homomorphism  $\lambda : A \longrightarrow R$

$$\begin{array}{ccc} \mathrm{Spec}_{\mathbb{k}}(A)(A) & \longrightarrow & \mathrm{Spec}_{\mathbb{k}}(B)(A) \\ \downarrow \mathrm{Spec}_{\mathbb{k}}(A)(\lambda) & & \downarrow \mathrm{Spec}_{\mathbb{k}}(B)(\lambda) \\ \mathrm{Spec}_{\mathbb{k}}(A)(R) & \longrightarrow & \mathrm{Spec}_{\mathbb{k}}(B)(R) \end{array}$$

Then this implies  $\Phi_R(\lambda) = \lambda \circ \Phi_A(\mathrm{id}_A)$ , showing that  $\Phi_R(\lambda)$  is uniquely determined by  $\Phi_A(\mathrm{id}_A)$ , so the map is injective. On the other hand, for any  $\phi \in \mathrm{Hom}_{\mathbb{k}}(B, A)$  and commutative  $\mathbb{k}$ -algebra  $R$ , we define  $\Phi : \mathrm{Spec}_{\mathbb{k}}(A) \longrightarrow \mathrm{Spec}_{\mathbb{k}}(B)$  as  $\Phi_R(\lambda) = \lambda \circ \phi$  for  $\lambda \in \mathrm{Spec}_{\mathbb{k}}(A)(R)$ . Then  $\Phi_A(\mathrm{id}_A) = \mathrm{id}_A \circ \phi = \phi$ , which gives the surjectivity, so our result follows.  $\square$

By Theorem 2.1.1, the corresponding homomorphism  $B \longrightarrow A$  of a natural transformation  $\Phi : \mathrm{Spec}_{\mathbb{k}}(A) \longrightarrow \mathrm{Spec}_{\mathbb{k}}(B)$  will be denoted by  $\Phi^*$ , and be called the *comorphism* of  $\Phi$ .

Suppose now that  $\mathcal{G}$  is an affine group scheme. Let  $R \longrightarrow S$  be a  $\mathbb{k}$ -algebra homomorphism. One checks that the following diagram

$$\begin{array}{ccc} \mathcal{G}(R) \times \mathcal{G}(R) & \xrightarrow{\mathrm{mult}} & \mathcal{G}(R) \\ \downarrow & & \downarrow \\ \mathcal{G}(S) \times \mathcal{G}(S) & \xrightarrow{\mathrm{mult}} & \mathcal{G}(S) \end{array}$$

commutes, this implies that the multiplication  $\mathrm{mult} : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$  is a natural transformation. Similarly,  $\mathrm{unit} : \{e\} \longrightarrow \mathcal{G}$  and  $\mathrm{inv} : \mathcal{G} \longrightarrow \mathcal{G}$  are also natural

maps. By Yoneda's Lemma, that  $\mathcal{G}$  is an affine  $\mathbb{k}$ -group functor is the same as giving a  $\mathbb{k}$ -algebra  $\mathbb{k}[\mathcal{G}]$  together with three algebra homomorphisms:

$$\begin{aligned} \text{comultiplication} \quad \Delta : \mathbb{k}[\mathcal{G}] &\longrightarrow \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] \\ \text{counit} \quad \varepsilon : \mathbb{k}[\mathcal{G}] &\longrightarrow \mathbb{k} \\ \text{antipode} \quad \eta : \mathbb{k}[\mathcal{G}] &\longrightarrow \mathbb{k}[\mathcal{G}]. \end{aligned}$$

Additionally, the group axioms give rise to the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{k}[\mathcal{G}] & \xrightarrow{\Delta} & \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] & & \mathbb{k}[\mathcal{G}] & \xrightarrow{\Delta} & \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} & & \downarrow \varepsilon & & \downarrow \eta \otimes \text{id} \\ \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] & \xrightarrow{\text{id} \otimes \Delta} & \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] & & \mathbb{k} & \xrightarrow{\quad} & \mathbb{k}[\mathcal{G}] \end{array}$$

and

$$\begin{array}{ccccc} & & \mathbb{k} \otimes \mathbb{k}[\mathcal{G}] & \xleftarrow{\varepsilon \otimes \text{id}} & \mathbb{k}[\mathcal{G}] \otimes \mathbb{k}[\mathcal{G}] & \xrightarrow{\text{id} \otimes \varepsilon} & \mathbb{k}[\mathcal{G}] \otimes \mathbb{k} & & \\ & & \swarrow & & \uparrow \Delta & & \searrow & & \\ & & & & \mathbb{k}[\mathcal{G}] & & & & \end{array}$$

which endows  $(\mathbb{k}[\mathcal{G}], \Delta, \varepsilon, \eta)$  with a Hopf algebra structure.

A morphism  $\Phi : \mathcal{G} \longrightarrow \mathcal{H}$  between two affine group schemes is a homomorphism if and only if its comorphism  $\Phi^* : \mathbb{k}[\mathcal{H}] \longrightarrow \mathbb{k}[\mathcal{G}]$  satisfies

$$\Delta_{\mathcal{G}} \circ \Phi^* = (\Phi^* \otimes \Phi^*) \circ \Delta_{\mathcal{H}}, \quad \varepsilon_{\mathcal{G}} \circ \Phi^* = \varepsilon_{\mathcal{H}}, \quad \eta_{\mathcal{G}} \circ \Phi^* = \Phi^* \circ \eta_{\mathcal{H}}.$$

**Theorem 2.1.2.** *The categories of affine group schemes and commutative Hopf algebras are anti-equivalent.*

## 2.2 Representation-theoretic support space

In this section,  $\mathbb{k}$  is assumed to be an algebraically closed field of characteristic  $\text{char}(\mathbb{k}) = p > 0$ . It is our goal to introduce the space  $P(\mathcal{G})$  of  $p$ -points of a finite group scheme  $\mathcal{G}$  over  $\mathbb{k}$ . This space gives a representation-theoretic interpretation of the cohomological support variety  $V_{\mathcal{G}}$  for all finite group schemes, and specializes to finite groups and infinitesimal group schemes.

Let  $\mathcal{G} = \mathcal{G}^0 \rtimes \pi_0(\mathcal{G})$  be a finite group scheme with group of connected components  $\pi_0(\mathcal{G})$ . We denote by  $H^*(\mathcal{G}, \mathbb{k})$  the cohomology ring  $H^*(\mathcal{G}, \mathbb{k})$  of  $\mathcal{G}$  if  $\text{char}(\mathbb{k}) = 2$  and the subring  $H^{ev}(\mathcal{G}, \mathbb{k})$  of elements of even degree if  $\text{char}(\mathbb{k}) > 2$ . In [FS, 1.1], the authors have shown that  $H^*(\mathcal{G}, \mathbb{k})$  is a finitely generated commutative  $\mathbb{k}$ -algebra. We adopt the notation  $V_{\mathcal{G}} = \max(H^*(\mathcal{G}, \mathbb{k}))$  for the maximal ideal spectrum of  $H^*(\mathcal{G}, \mathbb{k})$ , which is referred to as the *cohomological variety of  $\mathcal{G}$* . Moreover, for any finitely generated  $\mathcal{G}$ -module  $M$ , we denote by  $V_{\mathcal{G}}(M) \subset V_{\mathcal{G}}$  the conical closed subset of maximal ideals that contain the annihilator  $\text{ann}_{H^*(\mathcal{G}, \mathbb{k})}(\text{Ext}_{\mathcal{G}}^*(M, M))$  of  $\text{Ext}_{\mathcal{G}}^*(M, M)$ , which is called the *support variety of  $M$* .

**Definition 2.2.1.** Let  $\mathcal{G}$  be a finite group scheme. A *p-point* of  $\mathcal{G}$  is a (left) flat map of algebras  $\alpha : \mathbb{k}\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{k}\mathcal{G}$  together with an embedding  $\text{im } \alpha \subseteq \mathbb{k}\mathcal{U}$  for some abelian unipotent subgroup  $\mathcal{U} \subseteq \mathcal{G}$ .

Two *p-points*  $\alpha, \beta : \mathbb{k}\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{k}\mathcal{G}$  are said to be equivalent ( $\alpha \sim \beta$ ) if for any finite dimensional  $\mathcal{G}$ -module  $M$ ,  $\alpha^*(M)$  is projective if and only if  $\beta^*(M)$  is projective. We denote by  $P(\mathcal{G})$  the set of equivalence classes of *p-points* of  $\mathcal{G}$  and by  $P(\mathcal{G})_M$  the set of elements of  $P(\mathcal{G})$  represented by *p-points*  $\alpha : \mathbb{k}\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{k}\mathcal{G}$  such that the pullback  $\alpha^*(M)$  of  $M$  through  $\alpha$  is not a projective module over  $\mathbb{k}\mathbb{Z}/p\mathbb{Z}$ , where  $M$  is a finite dimensional  $\mathcal{G}$ -module. It is known that  $P(\mathcal{G})$  can be endowed with a Noetherian topological structure whose closed subsets are of the form  $P(\mathcal{G})_M$  such that there is a homeomorphism

$$\Psi_{\mathcal{G}} : P(\mathcal{G}) \longrightarrow \text{Proj} V_{\mathcal{G}}$$

between these two topological spaces by sending  $[\alpha]$  to  $\Psi_{\mathcal{G}}([\alpha]) := \ker \alpha'$  where  $\alpha'$  is the map from  $H^*(\mathcal{G}, \mathbb{k})$  to  $H^*(\mathbb{Z}/\mathbb{Z}p, \mathbb{k})$  induced by pullback, satisfying the property  $\Psi_{\mathcal{G}}^{-1}(\text{Proj} V_{\mathcal{G}}(M)) = P(\mathcal{G})_M$  for every finitely generated  $\mathcal{G}$ -modules  $M$ ; see [FP, Theorem 4.6/4.8/4.10].

The support space  $P(\mathcal{G})_M$  as well as the projectivization of the support variety  $\text{Proj}(V_{\mathcal{G}}(M))$  for any finitely generated  $\mathcal{G}$ -module  $M$  satisfies several pleasing properties presented in Theorem 5.6 [PS].





# Chapter 3

## Reductive groups and varieties of elementary subalgebras

This chapter has two purposes. One is to recollect a number of results from algebraic groups which will be vital in later chapters. To present these we need have an idea of root systems. Aside from this, we introduce the notation of varieties of elementary subalgebras, interacting with the root system by considering  $T$ -invariant elementary subalgebras. Throughout this chapter we assume that  $\mathbb{k}$  is an algebraically closed field of prime characteristic  $p > 0$ .

### 3.1 Restricted Lie algebras

Let  $(\mathfrak{g}, [\cdot, \cdot])$  be a Lie algebra over  $\mathbb{k}$ , and  $R$  be a commutative  $\mathbb{k}$ -algebra. By defining  $[x \otimes r, y \otimes s] = [x, y] \otimes rs$  for all  $x, y \in \mathfrak{g}$  and  $r, s \in R$ ,  $\mathfrak{g} \otimes_{\mathbb{k}} R$  obtains the structure of a Lie algebra over  $R$ . In particular, when we take  $R = \mathbb{k}[T]$ , this gives the definition of a restricted Lie algebra.

**Definition 3.1.1.** A restricted Lie algebra  $(\mathfrak{g}, [p])$  over  $\mathbb{k}$  is a Lie algebra  $\mathfrak{g}$  over  $\mathbb{k}$  together with a map  $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}, x \mapsto x^{[p]}$ , provided the following are satisfied:

- $\text{ad } x^{[p]} = (\text{ad } x)^p \quad \forall x \in \mathfrak{g}$

- $(\alpha x)^{[p]} = \alpha^p x^{[p]} \quad \forall \alpha \in \mathbb{k}, x \in \mathfrak{g}$
- $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ , where the  $s_i(x, y)$  are given by the identity in  $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{k}[T]$

$$(\text{ad}(x \otimes T + y \otimes 1))^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i s_i(x, y) \otimes T^{i-1}.$$

A map  $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$  satisfying these three conditions is called a  $p$ -map.

If  $(\mathfrak{g}, [p])$  is a restricted Lie algebra, a subalgebra  $\mathfrak{f}$  of  $\mathfrak{g}$  (resp. an ideal  $\mathfrak{m}$  of  $\mathfrak{g}$ ) is called a  $p$ -subalgebra (resp.  $p$ -ideal) if  $x^{[p]} \in \mathfrak{f} \forall x \in \mathfrak{f}$  (resp.  $x^{[p]} \in \mathfrak{m} \forall x \in \mathfrak{m}$ ).

**Example 3.1.2.** (1) Let  $\Lambda$  be an associative  $\mathbb{k}$ -algebra. Then the commutator algebra  $(\Lambda^-, [,])$  with bracket  $[x, y] = xy - yx$  is a Lie algebra over  $\mathbb{k}$ . By taking the  $p$ -map to be the ordinary  $p$ -th power operator

$$x \mapsto x^p := \underbrace{x \circ x \circ \cdots \circ x}_p,$$

$(\Lambda^-, [p])$  is a restricted Lie algebra.

(2) Let  $\Lambda := \text{Mat}_n(\mathbb{k})$  be the algebra of  $(n \times n)$ -matrices over  $\mathbb{k}$ . Then the corresponding restricted Lie algebra  $(\Lambda^-, [p])$  will be denoted by  $\mathfrak{gl}_n(\mathbb{k})$ , which is called the *general linear Lie algebra*. Recall that the subalgebra

$$\mathfrak{sl}_n(\mathbb{k}) := \{x \in \mathfrak{gl}_n(\mathbb{k}) \mid \text{tr}(x) = 0\}$$

is a  $p$ -ideal of  $\mathfrak{gl}_n(\mathbb{k})$ , and is called the *special linear Lie algebra*.

**Definition 3.1.3.** Let  $(\mathfrak{g}, [p])$  be a restricted Lie algebra with universal enveloping algebra  $U(\mathfrak{g})$ . Let  $I$  be the two sided ideal of  $U(\mathfrak{g})$  generated by  $\{x^p - x^{[p]} \mid x \in \mathfrak{g}\}$ . Then

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/I$$

is called the *restricted enveloping algebra* of  $\mathfrak{g}$ .

## 3.2 Reductive groups and root data

Let  $G$  be an affine algebraic  $\mathbb{k}$ -group scheme. We say  $G$  is smooth if  $\dim \mathfrak{g} = \dim G$ , where  $\mathfrak{g} = \text{Lie}(G) := T_e(G)$  is the Lie algebra of  $G$ , i.e. the tangent space in the identity element  $e \in G(\mathbb{k})$ . Being a group scheme,  $G$  is smooth if and only if  $G$  is reduced; see [KMRT, 21.8/21.9]. By taking the  $\mathbb{k}$ -rational points, there is a categorical equivalence between the category of smooth affine algebraic  $\mathbb{k}$ -group schemes  $G$  and the category of linear algebraic groups  $G(\mathbb{k})$  over  $\mathbb{k}$ . We then say a smooth affine algebraic  $\mathbb{k}$ -group scheme  $G$  is reductive (resp. semisimple) if  $G(\mathbb{k})$  is a reductive linear algebraic group (resp. semisimple linear algebraic group), i.e. its unipotent radical  $R_u(G(\mathbb{k}))$  (resp. solvable radical  $R(G(\mathbb{k}))$ ) is trivial; see [Bor] and [Spr] for more details. Henceforth, without any real ambiguity, we will use the term *algebraic  $\mathbb{k}$ -group* for a smooth affine algebraic  $\mathbb{k}$ -group scheme, or simply the group of  $\mathbb{k}$ -rational points of the scheme.

A *root datum* is a quadruple  $\mathcal{R} = (X, R, X^\vee, R^\vee)$ , where  $X$  and  $X^\vee$  are free abelian groups of finite rank in duality with respect to a pairing  $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$ , and  $R$  together with  $R^\vee$  are finite subsets of  $X$  and  $X^\vee$  respectively. Additional conditions required for  $\mathcal{R}$  are: (i) there is a bijection between  $R$  and  $R^\vee$  by sending  $\alpha$  to  $\alpha^\vee$  for every  $\alpha \in R$  with the property  $\langle \alpha, \alpha^\vee \rangle = 2$ ; (ii) for each  $\alpha \in R$ , the map  $s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$  on  $X$  maps  $R$  to  $R$ .

A prototypical example of a root datum arises from a pair  $(G, T)$ , where  $T$  is a maximal torus of a connected reductive algebraic  $\mathbb{k}$ -group  $G$ . Fix a Borel subgroup  $B$  which contains the maximal torus  $T$ , as well as the unipotent radical  $R_u(B)$ . Let  $X^*(T) := \text{Hom}(T, \mathbb{G}_m)$  be the character group,  $X_*(T) := \text{Hom}(\mathbb{G}_m, T)$  be the cocharacter group,  $\Phi := \Phi(G, T)$  be the root system relative to  $(G, T)$  given by the adjoint representations and  $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$  be the corresponding coroot system. By general theory,  $\mathcal{R}(G, T) = (X^*(T), \Phi, X_*(T), \Phi^\vee)$  is a root datum.

We denote by  $\Lambda_r(\Phi) = \mathbb{Z}\Phi$  the  $\mathbb{Z}$ -sublattice of  $X^*(T)$  generated by  $\Phi$ , and by  $\Lambda_w(\Phi) = \{\lambda \in \mathbb{R}\Phi \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$  the weight lattice of  $\Phi$ . Then the lattice  $\Lambda_r(\Phi)$  is a  $\mathbb{Z}$ -sublattice of  $\Lambda_w(\Phi)$  of finite index. By the same token for  $\Phi^\vee$ , we have the  $\mathbb{Z}$ -sublattice  $\Lambda_r(\Phi^\vee)$  of  $X_*(T)$  and the weight lattice  $\Lambda_w(\Phi^\vee)$  when we consider the *dual root datum*  $\mathcal{R}^\vee(G, T) = (X_*(T), \Phi^\vee, X^*(T), \Phi)$  of  $\mathcal{R}(G, T)$ .

### 3.3 Good and non-torsion primes

Let  $\Phi$  be a reduced root system with decomposition  $\Phi = \Phi_1 \cup \cdots \cup \Phi_s$  into irreducible components. For each component  $\Phi_i$  with base  $\Delta_i$ , there exists a unique highest root, say  $\beta_i$ ,  $1 \leq i \leq s$ . Write  $\beta_i = \sum m_{i\ell} \alpha_{i\ell}$  as a linear combination of simple roots for each  $i$ , and the corresponding coroots  $\beta_i^\vee = \sum m'_{i\ell} \alpha_{i\ell}^\vee$ . We say a prime  $p$  is *bad* for  $\Phi$  if it divides any coefficient  $m_{i\ell}$  occurring in the expressions of  $\beta_i$ , and  $p$  is *torsion* for  $\Phi$  if it divides any coefficient  $m'_{i\ell}$  existing among the expressions of the  $\beta_i^\vee$ . By inspecting the coefficients of highest roots in each irreducible root systems, one can easily conclude that  $p$  is a bad prime if and only if  $p = m_{i\ell}$  for some coefficient  $m_{i\ell}$ , if and only if  $p$  is smaller than the largest coefficient; see [Bou, PLATE I-IX] for highest roots of each irreducible root system. The torsion prime for  $\Phi$  is a prime such that  $\Lambda_r(\Phi^\vee)/\Lambda_r(\Phi'^\vee)$  has  $p$ -torsion for some maximal closed subsystem  $\Phi'$  of  $\Phi$ ; see [Ste, Definition 1.3/Theorem 1.12]. Meanwhile there is accordingly a criterion for  $p$  being a torsion prime, that is  $p$  equals some coefficient or  $p$  is smaller than the largest coefficient in some  $\beta_i^\vee$ ; see [Ste, Corollary 1.10/Theorem 1.12].

**Definition 3.3.1.** Let  $G$  be a reductive algebraic  $\mathbb{k}$ -group,  $G^{(1)}$  be the derived subgroup of  $G$ ,  $T^{(1)}$  be the maximal torus of  $G^{(1)}$ , and  $\pi_1(G) := \Lambda_w(\Phi)/X^*(T^{(1)})$  be the fundamental group of  $G$ . We say that  $p$  is *good* for  $G$  if  $p$  is not a bad prime for  $\Phi$ ; that  $p$  is a *torsion prime* for  $G$  if it is a torsion prime for  $\Phi$ , or if  $p$  divides  $|\pi_1(G)|$ , otherwise  $p$  is said to be *non-torsion* for  $G$ .

### 3.4 Existence of a Chevalley Basis

Let  $G$  be a connected reductive algebraic  $\mathbb{k}$ -group,  $\mathfrak{g} = \text{Lie}(G)$  be the Lie algebra of  $G$ ,  $T$  be a maximal torus in  $G$ ,  $\mathfrak{h} = \text{Lie}(T)$  be the Lie algebra of  $T$ , and  $\Phi$  be the root system of  $G$ . We choose  $\Phi^+$  to be a set of positive roots. Set  $\mathfrak{n}^+$  equal to the sum of all  $\mathfrak{g}_\alpha$  with  $\alpha \in \Phi^+$ , and  $\mathfrak{n}^-$  equal to the sum of all  $\mathfrak{g}_\alpha$  with  $\alpha \in -\Phi^+$ . Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

We choose for each root  $\alpha$  a basis vector  $x_\alpha$  of the one-dimensional root subspace  $\mathfrak{g}_\alpha$ . Being the Lie algebra of  $G$ ,  $\mathfrak{g}$  carries a canonical  $p$ -th power map equivariant

under the adjoint action of  $G$ . We then have  $x_\alpha^{[p]} \in \mathfrak{g}_{p\alpha}$  because the adjoint action of  $T$  is compatible with the  $p$ -th power map. Since  $p\alpha$  is not a root, this gives

$$x_\alpha^{[p]} = 0 \quad \text{for all } \alpha \in \Phi.$$

Therefore, the constituents of the triangular decomposition are  $p$ -subalgebras, with  $\mathfrak{n}^+$  and  $\mathfrak{n}^-$  being unipotent. We assume now  $G$  is a semisimple algebraic  $\mathbb{k}$ -group. This implies that there are inclusions of  $\mathbb{Z}$ -lattices  $\Lambda_r(\Phi) \subset X^*(T) \subset \Lambda_w(\Phi)$  and  $\Lambda_r(\Phi^\vee) \subset X_*(T) \subset \Lambda_w(\Phi^\vee)$  respectively. By Chevalley's classification theorem, each  $\mathbb{Z}$ -lattice between  $\Lambda_r(\Phi)$  and  $\Lambda_w(\Phi)$  determines a unique semisimple algebraic  $\mathbb{k}$ -group up to isomorphism with the same root system  $\Phi$ . We denote by  $G_{ad}$  the adjoint  $\mathbb{k}$ -group relative to  $G$  if  $X^*(T) = \Lambda_r(\Phi)$ , and by  $G_{sc}$  the simply connected  $\mathbb{k}$ -group corresponding to  $G$  if  $X^*(T) = \Lambda_w(\Phi)$ . The observation from  $|\Lambda_w(\Phi)/X^*(T)| = |X_*(T)/\Lambda_r(\Phi^\vee)|$  shows that the latter condition can be replaced equivalently by  $X_*(T) = \Lambda_r(\Phi^\vee)$ .

Let  $\mathfrak{g}^{sc} := \text{Lie}(G_{sc})$  be the Lie algebra of  $G_{sc}$ ,  $x_\alpha$  be a non-zero element of  $\mathfrak{g}_\alpha^{sc}$  and  $h_\alpha := [x_\alpha, x_{-\alpha}]$  for every  $\alpha \in \Phi$ . Assume  $\Delta \subset \Phi^+$  is the set of simple roots, then  $\mathcal{B}^{sc} := \{h_\alpha, x_\gamma \mid \alpha \in \Delta, \gamma \in \Phi\}$  is referred to as a *Chevalley basis*. The  $\mathbb{Z}$ -span  $\mathfrak{g}_{\mathbb{Z}}^{sc}$  of a Chevalley basis is a  $\mathbb{Z}$ -subalgebra of  $\mathfrak{g}^{sc}$ , and  $\mathfrak{g}^{sc} = \mathfrak{g}_{\mathbb{Z}}^{sc} \otimes_{\mathbb{Z}} \mathbb{k}$ . Then  $\mathcal{B}^{sc}$  is a basis of  $\mathfrak{g}^{sc}$ , but with the structure constants reduced modulo  $p$ . In general, for a semisimple algebraic  $\mathbb{k}$ -group  $G$ , the inclusion  $\Lambda_r(\Phi) \subset X^*(T) \subset \Lambda_w(\Phi)$  would give a canonical central isogeny  $\pi : G_{sc} \longrightarrow G$ . We denoted by  $d\pi : \mathfrak{g}^{sc} \longrightarrow \mathfrak{g}$  the differential of  $\pi$  for their corresponding Lie algebras. Let  $\mathcal{B} := \{h_\alpha, x_\gamma \mid \alpha \in \Delta, \gamma \in \Phi\}$  be the image of  $\mathcal{B}^{sc}$  under the map  $d\pi$  without abuse of notations. It is a set of elements generating the commutator  $[\mathfrak{g}, \mathfrak{g}]$  of  $\mathfrak{g}$ , yet it can happen that elements of  $\{h_\alpha \mid \alpha \in \Delta\}$  are linearly dependent. In conjunction with the equality  $\dim \mathfrak{g} = |\Delta| + |\Phi| = |\mathcal{B}|$ , the following proposition reveals when  $\mathfrak{g}$  affords a Chevalley basis.

**Proposition 3.4.1.** *[Let, Chap.2] Let  $G$  be a semisimple algebraic  $\mathbb{k}$ -group with Lie algebra  $\mathfrak{g}$ . Then the following assertions are equivalent:*

- $\mathcal{B} = \{h_\alpha, x_\gamma \mid \alpha \in \Delta, \gamma \in \Phi\}$  is a basis of  $\mathfrak{g}$ .
- $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ .

- $p$  does not divide the order of the fundamental group  $\pi_1(G) := \Lambda_w(\Phi)/X^*(T)$ .
- The canonical central isogeny  $\pi : G_{sc} \rightarrow G$  is separable, i.e.  $\ker d\pi \subset \text{Lie}(\ker \pi)$ .

### 3.5 Commuting roots

Let  $G$  be a semisimple algebraic  $\mathbb{k}$ -group with Lie algebra  $\mathfrak{g}$ . Suppose that  $\mathfrak{g}$  affords a Chevalley basis  $\mathcal{B}$ . Let  $\alpha, \beta \in \Phi$  be two roots. Observe that: (i)  $[x_\alpha, x_\beta] = N_{\alpha, \beta} x_{\alpha+\beta}$ , where the structure constant  $N_{\alpha, \beta} = 0$  if  $\alpha + \beta \notin \Phi \cup \{0\}$  and  $N_{\alpha, \beta} = \pm(r+1)$  if  $\alpha + \beta \in \Phi$ , where  $r$  is retrieved from a  $\alpha$ -string  $-r\alpha + \beta, \dots, \beta, \dots, s\alpha + \beta$  through  $\beta$ ; (ii)  $[x_\alpha, x_{-\alpha}] = h_\alpha$  if  $\alpha \in \Delta$ . The sign of the structure constant is determined inductively by the sign of  $N_{\alpha, \beta}$  for  $(\alpha, \beta)$  being an extraspecial pair with respect to the ordering  $\succeq$  chosen on  $\Phi$ , and it is remarked here that our choice for extraspecial pairs is always positive; see [Car, Chap 4] for details. In the modular case, if  $p$  is good for  $G$ , then  $p$  is greater than the length of the longest root string. Under this assumption, it is readily seen that  $[x_\alpha, x_\beta] = 0$  implies  $\alpha + \beta \notin \Phi \cup \{0\}$ . Inspired by this, we are led to give the following definition, which translates the commutative property of two root vectors from that of the corresponding roots.

**Definition 3.5.1.** [PS, 2.2] Let  $\alpha, \beta$  be two roots. We say  $\alpha$  and  $\beta$  *commute* if their sum is neither zero nor a root.

Suppose in addition  $p$  is good. From the above definition we have  $\alpha$  and  $\beta$  commute if and only if  $[x_\alpha, x_\beta] = 0$ . We remark that we do not necessarily need  $\mathfrak{g}$  to afford Chevalley Basis if the consideration only focuses on positive roots, i.e. when  $\alpha, \beta \in \Phi^+$ ; see [PS, Sect. 2.1].

In what follows, additionally suppose that  $G$  is simple with irreducible root system  $\Phi$ . Let  $U_\alpha$  be the root subgroup corresponding to a root  $\alpha$ , and  $B = \langle U_\alpha, T \mid \alpha \in \Phi^+ \rangle$  be the Borel subgroup of  $G$  containing  $T$ . Initially, we have studied the Weyl group  $\mathcal{W}$  in tandem with an irreducible root system  $\Phi$ . Let  $\Delta := \{\alpha_1, \dots, \alpha_n\}$ , and  $I$  be

a subset of  $\Delta$ . We define

$$\begin{aligned}\Phi_I &:= \Phi \cap \sum_{\alpha \in I} \mathbb{Z}\alpha \\ \mathscr{W}_I &:= \langle s_\alpha \mid \alpha \in I \rangle\end{aligned}$$

to be the parabolic subsystem of roots and standard parabolic subgroup of  $\mathscr{W}$  respectively; see [MT] for details. Then subgroups of the form  $P_I := B\mathscr{W}_I B = \langle T, U_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle$  are called standard parabolic subgroups of  $G$ . The Levi decomposition  $P_I = L_I \ltimes R_u(P_I)$  decomposes  $P_I$  into a semi-direct product of its Levi factor  $L_I$  and the unipotent radical  $R_u(P_I)$ , with the latter generated by root subgroups  $\{U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_I^+\}$ . Influenced by this, we set  $S := \Delta \setminus I$  and then define

$$\Phi_S^{\text{rad}} = \Phi^+ \setminus \Phi_I^+$$

to be the set of positive roots that cannot be written as a linear combination of the simple roots not in  $S$ . If  $S = \{\alpha_i\}$ , then we will simply write  $\Phi_i^{\text{rad}}$  instead of  $\Phi_{\{\alpha_i\}}^{\text{rad}}$ .

It is known that the maximal dimension  $r_{\max}$  of an elementary subalgebra is also the maximal order among sets of commuting roots in  $\Phi^+$ ; see [PS, Sect 3.1]. We use the notation  $\text{Max}(\Phi_{r_{\max}}^+)$  to denote the set of subsets of commuting roots in  $\Phi^+$  having order  $r_{\max}$ . By inspecting the elements in  $\Phi_i^{\text{rad}}$ , one can verify that various  $\Phi_i^{\text{rad}}$  give rise to sets of commuting roots, which attain the maximal order  $r_{\max}$  for some  $i$  and some types; see [PS, Table 2] for types  $A_n, B_4, C_n, D_n, E_6, E_7$ . For type  $B_n$ , the elements of  $\text{Max}(\Phi_{r_{\max}}^+)$  are diverse, which are mainly of the forms  $S_t, S_t^*$ ; see [PS, Notation 2.8]. For the purpose of our consideration in the sequel, we list the sets of commuting roots with maximal order  $r_{\max}$  again; see Table 3.1.

### 3.6 Varieties $\mathbb{E}(r, \mathfrak{g})$ and $\mathbb{E}(r, \mathfrak{g})^T$

Let  $(\mathfrak{g}, [p])$  be a restricted Lie algebra. The variety  $\mathbb{E}(r, \mathfrak{g})$  was first studied by Carlson-Friedlander-Pevtsova in their paper [CFP], with the definition

$$\mathbb{E}(r, \mathfrak{g}) = \{\mathcal{E} \subset \mathfrak{g} \mid \mathcal{E} \text{ elementary subalgebra of dimension } r\}.$$

There they have shown that  $\mathbb{E}(r, \mathfrak{g})$  is a closed subvariety of  $\text{Gr}_r(\mathfrak{g})(\mathbb{k})$ , with the latter being the variety of  $r$ -planes of the underlying vector space of  $\mathfrak{g}$ . This shows

that it can be endowed with a projective variety structure. Suppose now  $G$  is a semisimple algebraic group,  $\mathfrak{g} = \text{Lie}(G)$  which affords a Chevalley basis, and  $p$  is good. In section 3.4, it was shown that  $x_\alpha^{[p]} = 0$  for each  $\alpha \in \Phi$ . In section 3.5, we proved that  $[x_\alpha, x_\beta] = 0$  if  $\alpha, \beta$  commute. Combining these two, we say that a set of commuting roots gives rise to an elementary subalgebra spanned by the associated root vectors. More specifically, if  $R = \{\alpha \mid \alpha \in \Phi\}$  is a set of commuting roots, then  $\text{Lie}(R) := \text{Span}\{x_\alpha \mid \alpha \in R\}$  is an elementary subalgebra of dimension  $|R|$ . From now on, we consider the set  $\text{Com}(X_r)$  of sets of commuting roots in  $X$  of order  $r$  as well as the subset

$$\text{Max}(X_r) = \{\text{maximal set of commuting roots in } X \text{ of order } r\}$$

for each  $r \in \mathbb{Z}_{\geq 1}$  and  $X \subset \Phi$ . The definition of  $\text{Lie}(R)$  provides a map  $\text{Lie} : \text{Com}(\Phi_r) \rightarrow \mathbb{E}(r, \mathfrak{g})$  naturally. A characterization of elements in  $\mathbb{E}(r, \mathfrak{g})$  arising from the map  $\text{Lie}$  will be given in terms of  $T$ -invariant subalgebras. We say a subalgebra  $\mathfrak{g}' \subset \mathfrak{g}$  is  $T$ -invariant if  $t.\mathfrak{g}' = \mathfrak{g}'$  for all  $t \in T$  and denote by

$$\mathbb{E}(r, \mathfrak{g})^T = \{\mathcal{E} \subset \mathfrak{g} \mid \mathcal{E} \text{ } T\text{-invariant elementary subalgebra of dimension } r\}$$

the subset of  $\mathbb{E}(r, \mathfrak{g})$ . The related properties of the set  $\mathbb{E}(r, \mathfrak{g})^T$  are given by the following proposition, which provides a bijective map from the set  $\text{Com}(\Phi_r)$  to the set  $\mathbb{E}(r, \mathfrak{g})^T$ .

**Proposition 3.6.1.** *Keep the notation for  $G$  and  $(\mathfrak{g}, [p])$  from above. Then  $\mathbb{E}(r, \mathfrak{g})^T$  is a closed subvariety of  $\mathbb{E}(r, \mathfrak{g})$ , with elements generated by sets of root vectors.*

*Proof.* We first assume that an element  $\mathcal{E}$  of  $\mathbb{E}(r, \mathfrak{g})$  is generated by root vectors. Then by the action of elements of torus, say  $t.x_\alpha = \alpha(t).x_\alpha$ , it is readily seen that  $\mathcal{E}$  is  $T$ -invariant. Conversely, if  $\mathcal{E}$  is a  $T$ -invariant elementary subalgebra, then  $\mathcal{E}$  affords an adjoint representation, say  $\text{Ad} : T \rightarrow \text{GL}(\mathcal{E})$ . Accordingly, there is a weight space decomposition  $\mathcal{E} = \mathcal{E}^T \oplus \sum_\alpha \mathcal{E}_\alpha$ . Note that  $\mathcal{E}^T = \mathcal{E} \cap \mathfrak{g}^T = \mathcal{E} \cap \text{Lie}(T) = \{0\}$ , and  $\mathcal{E}_\alpha = \mathfrak{g}_\alpha$  if  $\mathcal{E}_\alpha \neq \{0\}$  by general theory. Thus  $\mathcal{E}$  is generated by root vectors. From this, we have seen that  $\mathbb{E}(r, \mathfrak{g})^T = \text{Lie}(\text{Com}(\Phi_r))$ . Because  $\text{Com}(\Phi_r)$  is a finite set, then  $\mathbb{E}(r, \mathfrak{g})^T$  is a finite set, so  $\mathbb{E}(r, \mathfrak{g})^T$  is closed in  $\mathbb{E}(r, \mathfrak{g})$ , as desired.  $\square$



Table 3.1: Maximal sets of commuting roots.

Type $T$	Restrictions on rank	$\text{Max}(\Phi_{r_{\max}}^+)$	$ \text{Max}(\Phi_{r_{\max}}^+) $	$r_{\max}$
$A_{2n}$	$n \geq 1$	$\Phi_{n+1}^{\text{rad}}, \Phi_n^{\text{rad}}$	2	$n(n+1)$
$A_{2n+1}$	$n \geq 0$	$\Phi_{n+1}^{\text{rad}}$	1	$(n+1)^2$
$B_n$	$n = 2, 3$	$\Phi_1^{\text{rad}}$	1	$2n - 1$
	$n = 4$	$\Phi_1^{\text{rad}},$ $S_1, S_2, S_3, S_4,$ $S_1^*, S_2^*, S_3^*$	8	7
	$n \geq 5$	$S_t, 1 \leq t \leq n$ $S_t^*, 1 \leq t < n$	$2n - 1$	$\frac{1}{2}n(n-1) + 1$
$C_n$	$n \geq 3$	$\Phi_n^{\text{rad}}$	1	$\frac{1}{2}n(n+1)$
$D_n$	$n = 4$	$\Phi_1^{\text{rad}}, \Phi_3^{\text{rad}}, \Phi_4^{\text{rad}}$	3	6
	$n \geq 5$	$\Phi_{n-1}^{\text{rad}}, \Phi_n^{\text{rad}}$	2	$\frac{1}{2}n(n-1)$
$E_6$		$\Phi_1^{\text{rad}}, \Phi_6^{\text{rad}}$	2	16
$E_7$		$\Phi_7^{\text{rad}}$	1	27
$E_8$		none of the form $\Phi_i^{\text{rad}}$	134	36
$F_4$		none of the form $\Phi_i^{\text{rad}}$	28	9
$G_2$		none of the form $\Phi_i^{\text{rad}}$	5	3



# Chapter 4

## Saturation rank for finite group schemes

In this chapter, we are ready to give the definition of saturation rank for all finite group schemes, with an emphasis on constant finite group schemes and infinitesimal group schemes. In terms of constant finite group schemes, we show that the saturation rank is determined by their maximal elementary abelian subgroup schemes; see Theorem 4.1.2. In the context of infinitesimal group schemes, we reveal that the saturation rank is controlled by the local data; see Theorem 4.2.5. The techniques we use range from Quillen's stratification for finite groups to support varieties for infinitesimal group schemes. Before we start, we will recall the definition of the saturation rank proposed by Farnsteiner: see [Far3, Sect. 6.4].

Let  $\mathcal{G}$  be a finite group scheme. For a subgroup  $\mathcal{H} \subseteq \mathcal{G}$ , the canonical inclusion map  $\iota_{\mathcal{H}} : \mathcal{H}^{\mathbb{C}} \longrightarrow \mathcal{G}$  induces a continuous yet not necessarily injective map  $\iota_{*,\mathbb{k}\mathcal{H}} : P(\mathcal{H}) \longrightarrow P(\mathcal{G})$ . The definition of  $p$ -points ensures that

$$P(\mathcal{G}) = \bigcup_{\substack{\mathcal{U} \subseteq \mathcal{G} \\ \text{unipotent abelian}}} \iota_{*,\mathbb{k}\mathcal{U}}(P(\mathcal{U})).$$

Motivated by this, we consider the set  $\text{Max}_{au}(\mathcal{G})$  of maximal abelian unipotent subgroups of  $\mathcal{G}$  as well as the subsets

$$\text{Max}_{au}(\mathcal{G})_{\ell} := \{\mathcal{U} \in \text{Max}_{au}(\mathcal{G}) \mid \text{cx}_{\mathcal{U}}(\mathbb{k}) \geq \ell\}$$

for every  $\ell \geq 1$ . Setting  $P(\mathcal{G})_\ell := \bigcup_{\mathcal{U} \in \text{Max}_{\text{au}}(\mathcal{G})_\ell} \iota_{*,\mathbb{k}\mathcal{U}}(P(\mathcal{U}))$ , the number

$$\text{srk}(\mathcal{G}) := \max\{\ell \geq 1 \mid P(\mathcal{G}) = P(\mathcal{G})_\ell\}$$

is referred to as the *saturation rank* of  $\mathcal{G}$ .

**Remark 4.0.1.** In [Far4, Section 6.2.1], the author has proved for any abelian unipotent group scheme  $\mathcal{U} \subseteq \mathcal{G}$ , there exists a unique elementary abelian subgroup scheme  $\mathcal{E}_\mathcal{U} \subseteq \mathcal{U}$  such that  $\iota_{*,\mathbb{k}\mathcal{E}_\mathcal{U}} : P(\mathcal{E}_\mathcal{U}) \longrightarrow P(\mathcal{U})$  is a homeomorphism. We consider the set  $\text{Max}_{\text{ea}}(\mathcal{G})$  of maximal elementary abelian subgroups of  $\mathcal{G}$  and the subsets  $\text{Max}_{\text{ea}}(\mathcal{G})_\ell$  of elements in  $\text{Max}_{\text{ea}}(\mathcal{G})$  with complexity  $\geq \ell$ . Replacing  $\text{Max}_{\text{au}}(\mathcal{G})_\ell$  with  $\text{Max}_{\text{ea}}(\mathcal{G})_\ell$  in  $P(\mathcal{G})_\ell$  and redefining  $\text{srk}(\mathcal{G})$ , the homeomorphism  $\iota_{*,\mathbb{k}\mathcal{E}_\mathcal{U}}$  reveals that there is no difference on the number  $\text{srk}(\mathcal{G})$  between these two settings.

Throughout this chapter we assume that  $\mathbb{k}$  is an algebraically closed field of prime characteristic  $p > 0$ .

## 4.1 Constant finite group schemes

Let  $G$  be a finite group, then it defines a constant functor  $\mathcal{G}_G$  which assigns to each finitely generated connected commutative  $\mathbb{k}$ -algebra the group  $G$  itself. This functor is represented by  $\mathbb{k}^{\times|G|}$ , indexed by the elements of  $G$ , with its  $\mathbb{k}$ -linear dual  $\mathbb{k}G$ . We call this finite group scheme  $\mathcal{G}_G$  retrieved from  $G$  a *constant finite group scheme*.

Let  $\mathcal{G}_G$  be a constant finite group scheme with  $G = \mathcal{G}_G(\mathbb{k})$ . We denote by  $H^*(G, \mathbb{k})$  the cohomology ring  $H^*(G, \mathbb{k})$  of  $G$  if  $\text{char}(\mathbb{k}) = 2$  and the subring  $H^{\text{ev}}(G, \mathbb{k})$  of elements of even degree if  $\text{char}(\mathbb{k}) > 2$ . Evens and Venkov have proven independently that  $H^*(G, \mathbb{k})$  is a finitely generated commutative  $\mathbb{k}$ -algebra. We denote by  $V_G = \max(H^*(G, \mathbb{k}))$  the maximal ideal spectrum, an affine variety corresponding to  $H^*(G, \mathbb{k})$ . Let  $E \leq G$  be an elementary abelian  $p$ -subgroup of  $G$ . Then there is a restriction map

$$\text{res}_{G,E} : H^*(G, \mathbb{k}) \longrightarrow H^*(E, \mathbb{k})$$

which gives rise to a map of affine varieties  $\text{res}_{G,E}^* : V_E \longrightarrow V_G$ . Quillen has investigated the variety  $V_G$  and shown that it is stratified by pieces coming from

elementary abelian subgroups of  $G$ , which is known as Quillen Stratification; see [Qui1] and [Qui2] for details. A weak version of his result is the following

$$V_G = \bigcup_{\substack{E \leq G \\ \text{elemab}}} \text{res}_{G,E}^* V_E,$$

where  $\text{res}_{G,E}^* V_E = V(\ker(\text{res}_{G,E}))$  is an irreducible closed subvariety of  $V_G$ .

**Notation 4.1.1.** Keep the notation for  $H^\cdot(G, \mathbb{k})$  and  $V_G$ . We denote by  $H^\cdot(G, \mathbb{k})^\dagger$  the augmentation ideal of  $H^\cdot(G, \mathbb{k})$ . Let  $I$  be an ideal of  $H^\cdot(G, \mathbb{k})$ , then  $\text{gr}(I)$  is defined to be the unique maximal homogeneous ideal inside of  $I$ . We denote by  $\text{Proj}V_G$  the set of homogeneous ideals of  $H^\cdot(G, \mathbb{k})$  which are maximal among those homogeneous ideals other than the augmentation ideal  $H^\cdot(G, \mathbb{k})^\dagger$ . Then  $\text{Proj}V_G$  can be identified with the set of  $\text{gr}(\mathfrak{m})$  for  $\mathfrak{m} \in V_G \setminus \{H^\cdot(G, \mathbb{k})^\dagger\}$ . Let  $E$  be an elementary abelian subgroup of  $G$ . Observe that  $\text{res}_{G,E}^*(\text{gr}(\mathfrak{m})) = \text{gr}(\text{res}_{G,E}^*(\mathfrak{m}))$  for any  $\mathfrak{m} \in V_E \setminus \{H^\cdot(E, \mathbb{k})^\dagger\}$ . Then the map  $\text{res}_{G,E}^* : V_E \longrightarrow V_G$  induces a map

$$\text{res}_{G,E}^* : \text{Proj}V_E \longrightarrow \text{Proj}V_G.$$

An elementary abelian subgroup of  $\mathcal{G}_G$  is isomorphic to  $\mathcal{G}_E$  where  $E$  is an elementary abelian  $p$ -subgroup of  $G$ . Without any real ambiguity, we will use  $\mathcal{G}_E$  and  $E$  alternatively for the sake of convenience. By denoting

$$V_G(\ell) = \bigcup_{\mathcal{G}_E \in \text{Max}_{\text{ea}}(\mathcal{G}_G)_\ell} \text{res}_{G,E}^* V_E,$$

we have the following Lemma:

**Lemma 4.1.1.** *Suppose  $\mathcal{G}_G$  is a constant finite group scheme with  $\mathcal{G}_G(\mathbb{k}) = G$ . Then*

$$\text{srk}(\mathcal{G}_G) = \max \{ \ell \geq 1 \mid V_G = V_G(\ell) \}.$$

*Proof.* We utilize the homeomorphism  $\Psi_{\mathcal{G}_G} : \text{P}(\mathcal{G}_G) \longrightarrow \text{Proj}V_G$  described in Chapter 2.2 to verify this. Assume that  $V_G = V_G(\ell)$ . Let  $[\alpha] \in \text{P}(\mathcal{G}_G)$ , then  $\Psi_{\mathcal{G}_G}([\alpha]) \in \text{Proj}V_G$ . Thus there is  $\mathfrak{m} \in V_G \setminus \{H^\cdot(G, \mathbb{k})^\dagger\}$  such that  $\Psi_{\mathcal{G}_G}([\alpha]) = \text{gr}(\mathfrak{m})$ . By assumption, there exists a maximal elementary abelian subgroup scheme  $\mathcal{G}_E$  of  $\mathcal{G}_G$  with  $\text{cx}_{\mathcal{G}_E}(\mathbb{k}) \geq \ell$  such that  $\mathfrak{m} \in \text{res}_{G,E}^*(V_E \setminus \{H^\cdot(E, \mathbb{k})^\dagger\})$ . Then  $\text{gr}(\mathfrak{m}) \in \text{res}_{G,E}^* \text{Proj}V_E$ . The bijective map  $\Psi_{\mathcal{G}_E}$  ensures  $\text{gr}(\mathfrak{m}) = \text{res}_{G,E}^*(\Psi_{\mathcal{G}_E}([\beta])) = \Psi_{\mathcal{G}_G}([\iota_{*,\mathbb{k}E} \circ \beta])$  for some

$[\beta] \in P(\mathcal{G}_E)$ . This gives  $\Psi_{\mathcal{G}_G}([\alpha]) = \Psi_{\mathcal{G}_G}([\iota_{*,\mathbb{k}E} \circ \beta])$ , and then  $[\alpha] = \iota_{*,\mathbb{k}E}([\beta])$  since  $\Psi_{\mathcal{G}_G}$  is bijective. Thus, we have  $P(\mathcal{G}_G) = P(\mathcal{G}_G)_\ell$ .

On the other hand, we assume  $P(\mathcal{G}_G) = P(\mathcal{G}_G)_\ell$ . Let  $\mathfrak{m} \in V_G$ . If  $\mathfrak{m} = H(G, \mathbb{k})^\dagger$ , then  $\mathfrak{m} \in \text{res}_{G,E}^* V_E$  for any maximal elementary abelian subgroup  $\mathcal{G}_E$  of  $\mathcal{G}_G$ . So it suffices to consider  $\mathfrak{m} \neq H(G, \mathbb{k})^\dagger$ . Then  $\text{gr}(\mathfrak{m}) = \ker \alpha'$  for some  $[\alpha] \in P(\mathcal{G}_G)$ . By our assumption, there is a maximal elementary abelian subgroup  $\mathcal{G}_E$  of  $\mathcal{G}_G$  with  $\text{cx}_{\mathcal{G}_E}(\mathbb{k}) \geq \ell$  such that  $[\alpha] = \iota_{*,\mathbb{k}E}([\beta])$  where  $[\beta] \in P(\mathcal{G}_E)$ . Thus  $\ker \alpha' = \ker(\beta' \circ \text{res}_{G,E})$ , and consequently  $\ker(\text{res}_{G,E}) \subset \text{gr}(\mathfrak{m}) \subset \mathfrak{m}$ . As a result,  $\mathfrak{m} \in V(\ker(\text{res}_{G,E})) = \text{res}_{G,E}^* V_E$ . Therefore, we have  $V_G = V_G(\ell)$ .

Combining these two, it has been shown that  $P(\mathcal{G}_G) = P(\mathcal{G}_G)_\ell$  if and only if  $V_G = V_G(\ell)$  for  $\ell \geq 1$ . This ultimately gives  $\text{srk}(\mathcal{G}_G) = \max \{ \ell \geq 1 \mid V_G = V_G(\ell) \}$ .  $\square$

Notice that the number of elementary abelian subgroups of a finite group is finite. From the last Lemma, the saturation rank  $\text{srk}(\mathcal{G}_G)$  of a constant finite group scheme is clear, i.e. it is the minimal dimension of irreducible components of  $V_G$ . From now on, we call an elementary abelian subgroup  $E \leq G$  distinguished if  $E$  is not conjugate to a proper subgroup of any other elementary abelian subgroups. It is indeed a maximal one and vice versa. We then denote by  $\mathfrak{D}(G)$  the set of representatives from each conjugacy class of distinguished ones. We are going to describe the irreducible components via establishing a relation on the set  $\mathfrak{D}(G)$ .

**Theorem 4.1.2.** *Let  $\mathcal{G}_G$  be a constant finite group scheme with  $\mathcal{G}_G(\mathbb{k}) = G$ . Then  $\text{srk}(\mathcal{G}_G)$  is the minimal rank of maximal elementary abelian subgroups of  $G$ .*

*Proof.* It suffices to show  $\text{res}_{G,E}^* V_E$  is maximal for  $E \in \mathfrak{D}(G)$ . Let

$$V_E^+ := V_E - \bigcup_{\substack{F < E \\ \text{elemab}}} \text{res}_{E,F}^* V_F, \quad V_{G,E}^+ := \text{res}_{G,E}^* V_E^+$$

and  $\sigma_E = \prod_{0 \neq \zeta \in H^1(E, \mathbb{F}_p)} \beta(\zeta) \in H(G, \mathbb{k})$  where  $\beta$  denotes the Bockstein homomorphism. Then  $\bigcup_{F < E} \text{res}_{E,F}^* V_F$  is the zero-locus of  $\sigma_E$  with the complement  $V_E^+ = V_E - V(\sigma_E)$  in  $V_E$ . According to [Ben, Lemma 5.6.2], there exists  $\varrho_E \in H(G, \mathbb{k})$  such that  $\text{res}_{G,E}(\varrho_E) = (\sigma_E)^{p^\alpha}$ , this gives rise to  $V_{G,E}^+ = \text{res}_{G,E}^* V_E - V(\varrho_E)$ . Suppose that  $E_i$  and  $E_j$  are in  $\mathfrak{D}(G)$ . Because  $E_i$  cannot conjugate to any proper

subgroup of  $E_j$ , then  $\text{res}_{G,E_j}(\varrho_{E_i}) = 0$  again by [Ben, Lemma 5.6.2]. As a result, we have  $\text{res}_{G,E_j}^* V_{E_j} = V(\ker(\text{res}_{G,E_j})) \subseteq V(\varrho_{E_i})$ , this implies  $\text{res}_{G,E_i}^* V_{E_i} \not\subseteq \text{res}_{G,E_j}^* V_{E_j}$ . By the same token, we can get  $\text{res}_{G,E_j}^* V_{E_j} \not\subseteq \text{res}_{G,E_i}^* V_{E_i}$  and this shows that  $\{\text{res}_{G,E}^* V_E \mid E \in \mathfrak{D}(G)\}$  is a set of irreducible components of  $V_G$ . Finally, notice that  $\dim \text{res}_{G,E}^* V_E = \dim H^1(E, \mathbb{k}) = \text{rk}(E)$ , which completes our proof.  $\square$

## 4.2 Infinitesimal group schemes

We say a finite group scheme  $\mathcal{G}$  is *infinitesimal* if its coordinate algebra  $\mathbb{k}[\mathcal{G}]$  is a local algebra. Then the augmentation ideal  $\mathbb{k}[\mathcal{G}]^\dagger$  of  $\mathbb{k}[\mathcal{G}]$  is its unique maximal ideal. It is of height  $\leq r \in \mathbb{N}_0$  if  $x^{p^r} = 0$  for all  $x \in \mathbb{k}[\mathcal{G}]^\dagger$ . A class of infinitesimal group schemes that have served as prototypical examples arise from reduced group scheme  $G$  by taking their  $r$ -th Frobenius kernel  $G_{(r)}$  via the Frobenius map  $F^r$ . The representing algebra of  $G_{(r)}$  is then  $\mathbb{k}[G_{(r)}] = \mathbb{k}[G]/I$ , where  $I$  is the ideal generated by elements  $x^{p^r}$  for  $x \in \mathbb{k}[G_{(r)}]^\dagger$ . In particular, when  $r = 1$ , we find that  $\mathbb{k}[G_{(1)}]$  is the dual of the restricted enveloping algebra of algebraic Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ . And moreover, there is a categorical equivalence between the category of finite dimensional  $p$ -restricted Lie algebras and category of infinitesimal group schemes of height  $\leq 1$ . Henceforth, for any given finite dimensional restricted Lie algebra  $(\mathfrak{g}, [p])$ , we denote the associated infinitesimal group scheme  $\mathcal{G}_{\mathfrak{g}} := \text{Spec}((U_0(\mathfrak{g}))^*)$  by  $\underline{\mathfrak{g}}$ .

Let  $\mathcal{G}$  be an infinitesimal group scheme of height  $\leq r$ . An infinitesimal 1-parameter subgroup of  $\mathcal{G}_R$  over a commutative  $\mathbb{k}$ -algebra  $R$  is a homomorphism of  $R$ -group schemes  $\mathbb{G}_{a(r),R} \longrightarrow \mathcal{G}_R$ . Let  $V_r(\mathcal{G})$  be the functor, which sends every commutative  $\mathbb{k}$ -algebra  $A$  to the group  $V_r(\mathcal{G})(A) = \text{Hom}_{G_r/A}(\mathbb{G}_{a(r),A}, \mathcal{G}_A)$ . The functor  $V_r(\mathcal{G})$  is represented by an affine scheme of finite type over  $\mathbb{k}$ , [SFB1, Theorem 1.5]. The coordinate algebra  $\mathbb{k}[V_r(\mathcal{G})]$  of  $V_r(\mathcal{G})$  is then a graded connected algebra generated by homogeneous elements of degree  $p^i$ ,  $0 \leq i \leq r-1$ . In what follows, we will only concentrate in the  $\mathbb{k}$ -rational points of the scheme  $V_r(\mathcal{G})$ . We still use the notation  $V_r(\mathcal{G})$  to represent the affine variety of  $\mathbb{k}$ -rational points of  $V_r(\mathcal{G})$ . Note that an infinitesimal 1-parameter subgroup  $v : \mathbb{G}_{a(r)} \longrightarrow \mathcal{G}$  over  $\mathbb{k}$  may be decomposed as

$$\mathbb{G}_{a(r)} \xrightarrow{u} \mathbb{G}_{a(s)} \hookrightarrow \mathcal{G}$$

for some  $1 \leq s \leq r$ , thus the performance of elementary abelian subgroups of  $\mathcal{G}$  should be on the stage. On the one hand, one hopes to find the largest elementary abelian subgroup that  $v$  may factor through; and on the other hand, we are still wondering whether there is a number that can manipulate the complexity of elementary abelian subgroups factored through by every  $v \in V_r(\mathcal{G})$ .

An elementary abelian subgroup of  $\mathcal{G}$  is isomorphic to  $\mathbb{G}_{a(r_1)} \cdots \mathbb{G}_{a(r_n)}$  for some positive integers  $r_1, \dots, r_n$ ; see [Far4, 6.2]. Let  $\mathcal{C}_{\mathcal{G}}$  be the category whose objects are elementary abelian subgroups of  $\mathcal{G}$ , and whose morphisms are inclusions. Similarly, define  $\mathcal{C}_{\mathbb{k}\mathcal{G}}$  to be the category having commutative Hopf subalgebras of  $\mathbb{k}\mathcal{G}$  whose underlying associative algebra is isomorphic to some truncated polynomial ring  $\mathbb{k}[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$  as its objects, and morphisms are also given by inclusions. There is a categorical equivalence between  $\mathcal{C}_{\mathcal{G}}$  and  $\mathcal{C}_{\mathbb{k}\mathcal{G}}$  via the functor  $\mathcal{F} : \mathcal{C}_{\mathcal{G}} \longrightarrow \mathcal{C}_{\mathbb{k}\mathcal{G}}$  by sending  $\mathcal{E}$  to  $k\mathcal{E}$ .

Let  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G}) : \mathrm{Com}_{\mathbb{k}} \longrightarrow \mathrm{Ens}$  be the Grassmann scheme, i.e., the  $\mathbb{k}$ -functor that assign to every commutative  $\mathbb{k}$ -algebra  $R$  the set  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G})(R)$  of  $R$ -direct summands of  $\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R$  of rank  $d$ ; see [Jan1, Sect I.1.9]. We begin with the consideration of the subfunctor  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G}) \subseteq \mathrm{Gr}_d(\mathbb{k}\mathcal{G})$  which is given by

$$\mathrm{Sub}_d(\mathbb{k}\mathcal{G})(R) := \{V \in \mathrm{Gr}_d(\mathbb{k}\mathcal{G})(R) \mid V \cdot V \subseteq V, \Delta(V) \subseteq V \otimes_R V\}$$

for every commutative  $\mathbb{k}$ -algebra  $R$ . Recall that the base change  $\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R$  of the Hopf  $\mathbb{k}$ -algebra  $\mathbb{k}\mathcal{G}$  is then a Hopf  $R$ -algebra.

**Proposition 4.2.1.** *Keep the notations for  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G}), \mathrm{Sub}_d(\mathbb{k}\mathcal{G})$  as above. Then  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G})$  is a closed subfunctor of  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G})$ .*

*Proof.* Let  $\psi : R \longrightarrow S$  be a  $\mathbb{k}$ -algebra homomorphism. Then it induces a Hopf  $S$ -algebra homomorphism  $\mathrm{id} \otimes (\psi \hat{\otimes} \mathrm{id}) : (\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R) \otimes_R S \longrightarrow \mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} S$  by sending  $x \otimes r \otimes s$  to  $x \otimes \psi(r)s$ , it follows that  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G})(\psi)$  sends  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G})(R)$  to  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G})(S)$ . As a result,  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G})$  is a subfunctor of the  $\mathbb{k}$ -functor  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G})$ .

The closedness of  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G})$  is verified in this fashion: for every commutative  $\mathbb{k}$ -algebra  $A$  and every morphism  $f : \mathrm{Spec}_{\mathbb{k}}(A) \longrightarrow \mathrm{Gr}_d(\mathbb{k}\mathcal{G})$ ,  $f^{-1}(\mathrm{Sub}_d(\mathbb{k}\mathcal{G}))$  is a closed subfunctor of  $\mathrm{Spec}_{\mathbb{k}}(A)$ ; see [Jan1, I.1.12]. By Yoneda's Lemma, the morphism corresponds to an  $A$ -point  $W \in \mathrm{Gr}_d(\mathbb{k}\mathcal{G})(A)$ .



Fix a basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{k}\mathcal{G}$ . Let  $\alpha_{ij\ell}$  be the elements of  $A$  that are given by

$$v_i \cdot v_j = \sum_{\ell=1}^n \alpha_{ij\ell} v_\ell, \quad 1 \leq i, j \leq n$$

Denote by  $\{w_1, \dots, w_d\}$  a set of generators of the locally free  $A$ -module  $W$  of rank  $d$ , and define elements  $a_{ri}, b_{ri}, c_{ri} \in A$  via

$$w_r = \sum_{i=1}^n v_i \otimes a_{ri}, \quad 1 \leq r \leq d,$$

$$\Delta(w_r) = \sum_{i=1}^n \sum_{j=1}^n (v_i \otimes b_{ri}) \otimes (v_j \otimes c_{rj}), \quad 1 \leq r \leq d.$$

By definition of  $\text{Gr}_d(\mathbb{k}\mathcal{G})(A)$ , there exists an  $A$ -submodule  $W' \subseteq \mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} A$  such that

$$\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} A = W \oplus W'$$

and we denote by  $\text{pr} : \mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} A \longrightarrow W'$  the corresponding projection. This  $A$ -linear map is given by

$$\text{pr}(v_j \otimes 1) = \sum_{i=1}^n v_i \otimes \kappa_{ij}, \quad 1 \leq j \leq n.$$

We let  $I \subseteq A$  be the ideal generated by the elements for  $1 \leq i, j, t \leq n, 1 \leq r, s \leq d$

$$h_{rst} = \sum_{\ell=1}^n \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij\ell} a_{ri} a_{sj} \kappa_{t\ell} \quad ; \quad g_{trj} = \sum_{q=1}^n b_{rq} \kappa_{tq} c_{rj}, \quad ; \quad \gamma_{tri} = \sum_{q=1}^n b_{ri} \kappa_{tq} c_{rq}.$$

Now let  $\psi : A \longrightarrow R$  be a homomorphism of  $\mathbb{k}$ -algebras. Then we have

$$\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R = W_R \oplus W'_R$$

where  $X_R = X \otimes_A R$  for  $X \in \{W, W'\}$ . The corresponding projection

$$\text{pr}_R : \mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R \longrightarrow W'_R$$

is given by

$$\text{pr}_R(v_j \otimes 1) = \sum_{i=1}^n v_i \otimes \psi(\kappa_{ij}), \quad 1 \leq j \leq n.$$

In view of

$$(w_r \otimes 1) \cdot (w_s \otimes 1) = \sum_{\ell=1}^n v_\ell \otimes \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij\ell} \psi(a_{ri}) \psi(a_{sj})$$

this gives rise to

$$\mathrm{pr}_R((w_r \otimes 1) \cdot (w_s \otimes 1)) = \sum_{t=1}^n v_t \otimes \psi(h_{rst}).$$

By the same token, we have

$$\begin{aligned} (\mathrm{pr}_R \otimes \mathrm{id}_R) \circ \Delta_R(w_r \otimes 1) &= \sum_{t=1}^n \sum_{j=1}^n (v_t \otimes 1) \otimes (v_j \otimes \psi(g_{trj})), \\ (\mathrm{id}_R \otimes \mathrm{pr}_R) \circ \Delta_R(w_r \otimes 1) &= \sum_{t=1}^n \sum_{i=1}^n (v_i \otimes \psi(\gamma_{tri})) \otimes (v_t \otimes 1). \end{aligned}$$

Now suppose that  $\psi(I) = 0$ . Then the three forgoing identities imply  $W_R \cdot W_R \subseteq \ker \mathrm{pr}_R = W_R$ , as well as  $\Delta_R(W_R) \subseteq \ker(\mathrm{pr}_R \otimes \mathrm{id}_R) \cap \ker(\mathrm{id}_R \otimes \mathrm{pr}_R) = (W_R \otimes_R (\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R)) \cap ((\mathbb{k}\mathcal{G} \otimes_{\mathbb{k}} R) \otimes_R W_R) = W_R \otimes_R W_R$ , thus  $W_R \in \mathrm{Sub}_d(\mathbb{k}\mathcal{G})(R)$ . Conversely, if we have  $W_R \in \mathrm{Sub}_d(\mathbb{k}\mathcal{G})(R)$ , then  $I \subseteq \ker(\psi)$ . Observe that  $f^{-1}(\mathrm{Sub}_d(\mathbb{k}\mathcal{G}))(R) = \{\psi \in \mathrm{Spec}_{\mathbb{k}}(A)(R) \mid W_R = \mathrm{Gr}_d(\mathbb{k}\mathcal{G})(\psi)(W) \in \mathrm{Sub}_d(\mathbb{k}\mathcal{G})(R)\}$ . This shows

$$f^{-1}(\mathrm{Sub}_d(\mathbb{k}\mathcal{G}))(R) = \{\psi \in \mathrm{Spec}_{\mathbb{k}}(A)(R) \mid \psi(I) = 0\}.$$

Consequently,  $f^{-1}(\mathrm{Sub}_d(\mathbb{k}\mathcal{G}))$  is a closed subfunctor of  $\mathrm{Spec}_{\mathbb{k}}(A)$ , as desired.  $\square$

**Remark 4.2.1.** Let  $\mathrm{Ab}_d(\mathbb{k}\mathcal{G}) \subseteq \mathrm{Gr}_d(\mathbb{k}\mathcal{G})$  be the subfunctor of commutative subalgebras (contains identity element of  $\mathbb{k}\mathcal{G}$ ) of  $\mathbb{k}\mathcal{G}$ . It is a closed subfunctor, which can be proved similar to Proposition 4.2.1. The above proposition shows that the sets  $\mathrm{Sub}_d(\mathbb{k}\mathcal{G})(\mathbb{k})$ ,  $\mathrm{Ab}_d(\mathbb{k}\mathcal{G})(\mathbb{k})$  of rational  $\mathbb{k}$ -points of these functors are closed subsets of the Grassmann variety  $\mathrm{Gr}_d(\mathbb{k}\mathcal{G})(\mathbb{k})$ .

**Proposition 4.2.2.** *Let  $\mathfrak{C}(\ell, \mathcal{G})$  be the set consisting of objects of  $\mathcal{C}_{\mathcal{G}}$  having complexity  $\ell$ . Then  $\mathfrak{C}(\ell, \mathcal{G})$  is a projective variety.*

*Proof.* Let  $\mathcal{X} := \{H \in \mathrm{Gr}_{p^\ell}(\mathbb{k}\mathcal{G})(\mathbb{k}) \mid H \text{ is a commutative Hopf subalgebra of } \mathbb{k}\mathcal{G}\}$ . Then by Lemma 1.2(1) of [Far2] on Hopf algebras, we have

$$\mathcal{X} = \mathrm{Sub}_{p^\ell}(\mathbb{k}\mathcal{G})(\mathbb{k}) \cap \mathrm{Ab}_{p^\ell}(\mathbb{k}\mathcal{G})(\mathbb{k}).$$

According to our Remark 4.2.1,  $\mathcal{X}$  is a closed subvariety of  $\mathrm{Gr}_{p^\ell}(\mathbb{k}\mathcal{G})(\mathbb{k})$ . Consider the set  $\mathcal{Y} := \mathfrak{C}(p^\ell, \mathbb{k}\mathcal{G})$ , consisting of objects of  $\mathcal{C}_{\mathbb{k}\mathcal{G}}$  with dimension  $p^\ell$ . Then  $\mathcal{Y} \subset \mathcal{X}$  is a subset of  $\mathcal{X}$ . By the equivalence of categories between  $\mathcal{C}_{\mathcal{G}}$  and  $\mathcal{C}_{\mathbb{k}\mathcal{G}}$ , it suffices to endow  $\mathcal{Y}$  with a projective variety structure, i.e. to show  $\mathcal{Y}$  is closed.

Let  $\mathbb{k}\mathcal{G}_p := \{u \in \mathbb{k}\mathcal{G} \mid u^p = 0\}$  be the set of  $p$ -nilpotent elements in  $\mathbb{k}\mathcal{G}$ , then it is a closed conical subvariety. By setting  $H_p := H \cap \mathbb{k}\mathcal{G}_p$  for  $H \in \mathcal{X}$ , we are going to verify that: The underlying associative algebra of  $H$  is isomorphic to  $\mathbb{k}[x_1, \dots, x_\ell]/(x_1^p, \dots, x_\ell^p)$  if and only if  $\dim_{\mathbb{k}} H_p \geq p^\ell - 1$ . Firstly if we have such algebraic isomorphism for  $H$ , then  $H_p = \mathrm{Rad} H$  and  $\dim_{\mathbb{k}} H_p \geq p^\ell - 1$ . Conversely, if  $\dim_{\mathbb{k}} H_p \geq p^\ell - 1$  with  $\dim_{\mathbb{k}} H = p^\ell$ , then  $H = H_p \oplus \mathbb{k}1_H$  and  $H$  must be local since the identity element is the unique non-zero idempotent element in  $H$ . Notice that  $H$  is commutative, then it represents an infinitesimal group scheme. Theorem in [Wat, Sect 14.4] ensures that  $H$  has to be isomorphic to a truncated polynomial ring, say  $\mathbb{k}[x_1, \dots, x_t]/(x_1^{p^{e_1}}, \dots, x_t^{p^{e_t}})$ . By the definition of  $H$ , the  $p$ -th power of  $x_i$  for  $1 \leq i \leq t$  should be zero, i.e.  $x_i^p = 0$ . This implies all  $e_i = 1$  and further  $t = \sum_{i=1}^t e_i = \ell$  by dimension. Thus for such  $H$ , its underlying associative algebra is isomorphic to  $\mathbb{k}[x_1, \dots, x_\ell]/(x_1^p, \dots, x_\ell^p)$ . Finally, recall the following map

$$\mathcal{X} \longrightarrow \mathbb{N}_0; \quad H \mapsto \dim_{\mathbb{k}} H \cap \mathbb{k}\mathcal{G}_p$$

is upper semicontinuous [Far1, Lemma 7.3]. Thus  $\mathcal{Y} = \{H \in \mathcal{X} \mid \dim_{\mathbb{k}} H \cap \mathbb{k}\mathcal{G}_p \geq p^\ell - 1\}$  is closed.  $\square$

Suppose  $\iota_{\mathcal{E}} : \mathcal{E} \hookrightarrow \mathcal{G}$  is the canonical inclusion of an elementary abelian subgroup. Then there is a morphism  $\iota_{*, \mathcal{E}} : V_r(\mathcal{E}) \longrightarrow V_r(\mathcal{G})$  of support varieties. We set

$$\mathfrak{C}(\ell \uparrow, \mathcal{G}) := \bigcup_{r \geq \ell} \mathfrak{C}(r, \mathcal{G})$$

and let  $V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})}$  be the set which is the union of  $\iota_{*, \mathcal{E}}(V_r(\mathcal{E}))$  where  $\mathcal{E}$  range over all elements of  $\mathfrak{C}(\ell \uparrow, \mathcal{G})$ . By the decomposition of  $v$  mentioned above, we have  $V_r(\mathcal{G}) = V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})}$ . In general, we have:

**Theorem 4.2.3.** *Suppose  $\mathcal{G}$  is an infinitesimal group scheme of height  $\leq r$ . Then*

$$\mathrm{srk}(\mathcal{G}) = \max \{ \ell \mid V_r(\mathcal{G}) = V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})} \}.$$

*Proof.* Let  $\{v_0, \dots, v_{p^r-1}\}$  be the dual basis of the standard basis  $\{T^0, T^1, \dots, T^{p^r-1}\}$  of  $\mathbb{k}[\mathbb{G}_{a(r)}] = \mathbb{k}[T]/(T^{p^r})$ . Denote by  $u_i = v_{p^i}$  for  $0 \leq i < r$ , this gives  $\mathbb{k}\mathbb{G}_{a(r)} = \mathbb{k}[u_0, \dots, u_{r-1}]$ . Now we turn to verify our statement. Proposition 3.8 of [FP] gives a bijection

$$\Theta_{\mathcal{G}} : \text{Proj}(V_r(\mathcal{G})) \longrightarrow \text{P}(\mathcal{G}) ; \quad [\alpha] \mapsto [\alpha_* \circ \epsilon]$$

where  $\epsilon : \mathbb{k}[u_{r-1}] \simeq \mathbb{k}\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{k}\mathbb{G}_{a(r)}$ , and  $\alpha_* : \mathbb{k}\mathbb{G}_{a(r)} \longrightarrow \mathbb{k}\mathcal{G}$ . If  $\alpha \in \text{Pt}(\mathcal{G})$ , then it represents an equivalence class given by the map  $\Theta_{\mathcal{G}}$ , say  $[\alpha] = \Theta_{\mathcal{G}}([\beta])$  for some  $\beta \in V_r(\mathcal{G})$ . Further if we assume that  $V_r(\mathcal{G}) = V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})}$ , then there exists a maximal elementary abelian subgroup  $\mathcal{E} \leq \mathcal{G}$  with  $\text{cx}_{\mathcal{E}}(\mathbb{k}) \geq \ell$  such that  $\beta = \iota_{*, \mathcal{E}}(\gamma)$  for some  $\gamma \in V_r(\mathcal{E})$ . Therefore,  $[\alpha] = [\beta_* \circ \epsilon] = [\iota_{*, \mathbb{k}\mathcal{E}} \circ \gamma_* \circ \epsilon] = \iota_{*, \mathbb{k}\mathcal{E}}([\gamma_* \circ \epsilon])$  which lies in  $\iota_{*, \mathbb{k}\mathcal{E}}(\text{P}(\mathcal{E}))$ , and this gives  $\text{P}(\mathcal{G}) = \text{P}(\mathcal{G})_{\ell}$ . On the other hand, if  $\alpha \in V_r(\mathcal{G}) \setminus \{0\}$  and suppose  $\text{P}(\mathcal{G}) = \text{P}(\mathcal{G})_{\ell}$ , then  $\Theta_{\mathcal{G}}([\alpha])$  is an equivalence class of  $\text{P}(\mathcal{G})$ . By our assumption there exists a maximal abelian unipotent subgroup  $\mathcal{U}$  along with an elementary abelian subgroup  $\mathcal{E}_{\mathcal{U}}$  with  $\text{cx}_{\mathcal{E}_{\mathcal{U}}}(\mathbb{k}) \geq \ell$  and a  $p$ -point  $\beta \in \text{Pt}(\mathcal{E}_{\mathcal{U}})$  such that  $\Theta_{\mathcal{G}}([\alpha]) = \iota_{*, \mathbb{k}\mathcal{E}_{\mathcal{U}}}([\beta])$ ; see [Far4, Lemma 6.2.1]. Again by the bijective map  $\Theta_{\mathcal{E}_{\mathcal{U}}}$ , we have  $[\beta] = [\gamma_* \circ \epsilon]$  for some  $\gamma \in V_r(\mathcal{E}_{\mathcal{U}})$ , thus  $\Theta_{\mathcal{G}}([\alpha]) = \Theta_{\mathcal{G}}([\iota_{\mathcal{E}_{\mathcal{U}}} \circ \gamma])$  and  $\alpha = \iota_{\mathcal{E}_{\mathcal{U}}} \circ \gamma \in \iota_{*, \mathcal{E}_{\mathcal{U}}}(V_r(\mathcal{E}_{\mathcal{U}}))$ . Therefore,  $V_r(\mathcal{G}) = V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})}$ .

From the above discussion, we have proved

$$V_r(\mathcal{G}) = V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})} \text{ if and only if } \text{P}(\mathcal{G}) = \text{P}(\mathcal{G})_{\ell}.$$

Then by definition, this gives rise to  $\text{srk}(\mathcal{G}) = \max \{ \ell \mid V_r(\mathcal{G}) = V_{\mathfrak{C}(\ell \uparrow, \mathcal{G})} \}$ .  $\square$

Suppose  $\alpha \in V_r(\mathcal{G})$ . We denote by

$$\mathfrak{C}(\ell \uparrow, \mathcal{G})_{\alpha} = \{ \mathcal{E} \in \mathfrak{C}(\ell \uparrow, \mathcal{G}) \mid \alpha \in \iota_{*, \mathcal{E}}(V_r(\mathcal{E})) \}$$

and by  $r_{\alpha}^{\mathcal{G}} = \max \{ \ell \mid \mathfrak{C}(\ell \uparrow, \mathcal{G})_{\alpha} \neq \emptyset \}$ . Put  $r_{min}^{\mathcal{G}} = \min \{ r_{\alpha}^{\mathcal{G}} \mid \alpha \in V_r(\mathcal{G}) \}$  and  $\mathcal{O}_{r_{min}^{\mathcal{G}}} = \{ \alpha \in V_r(\mathcal{G}) \mid r_{\alpha}^{\mathcal{G}} = r_{min}^{\mathcal{G}} \}$ .

**Remark 4.2.2.** Let  $\mathcal{G} = G_r$  be the Frobenius kernel of a smooth group scheme  $G$ . Then  $G$  acts on  $G_r$  via the adjoint representation. There results an action of  $G$  on  $V_r(G)$  and on  $\mathfrak{C}(\ell, G_r)$ . Let  $\alpha \in V_r(G)$  and  $g \in G$ . Based on these facts, we obtain the relation  $r_{\alpha}^{G_r} = r_{g \cdot \alpha}^{G_r}$ .

**Lemma 4.2.4.** *Suppose that  $\mathcal{G}$  is an infinitesimal group schemes of height  $\leq r$ . Then  $V_{\mathfrak{C}(\ell, \mathcal{G})}$  is a closed subvariety of  $V_r(\mathcal{G})$ .*

*Proof.* We denote by  $\text{pr}_1$  the projection onto the first coordinate:

$$\text{pr}_1 : V_r(\mathcal{G}) \times \mathfrak{C}(\ell, \mathcal{G}) \longrightarrow V_r(\mathcal{G}).$$

Write  $\mathbb{k}\mathbb{G}_{a(r)} = \mathbb{k}[u_0, \dots, u_{r-1}]$  as we did in Theorem 4.2.3. Consider the set  $\mathcal{Z} = \{(\alpha_*, \mathbb{k}\mathcal{E}) \in \text{Hom}(\mathbb{k}\mathbb{G}_{a(r)}, \mathbb{k}\mathcal{G}) \times \mathfrak{C}(p^\ell, \mathbb{k}\mathcal{G}) \mid \alpha_*(u_i) \in \mathbb{k}\mathcal{E}, 0 \leq i < r\}$ , which is closed. Then by categorical equivalence  $\mathcal{Z}' := \{(\alpha, \mathcal{E}) \in V_r(\mathcal{G}) \times \mathfrak{C}(\ell, \mathcal{G}) \mid \alpha \in \iota_{*, \mathcal{G}}(V_r(\mathcal{E}))\}$  is closed in  $V_r(\mathcal{G}) \times \mathfrak{C}(\ell, \mathcal{G})$ . Proposition 4.2.2 shows  $\mathfrak{C}(\ell, \mathcal{G})$  is complete. Therefore, the image  $\text{pr}_1(\mathcal{Z}')$  of  $\mathcal{Z}'$ , i.e.  $V_{\mathfrak{C}(\ell, \mathcal{G})}$ , is closed in  $V_r(\mathcal{G})$  by general theory.  $\square$

**Theorem 4.2.5.** *Suppose that  $\mathcal{G}$  is an infinitesimal group scheme of height  $\leq r$ . Then  $\text{srk}(\mathcal{G}) = r_{\min}^{\mathcal{G}}$  and  $\mathcal{O}_{r_{\min}^{\mathcal{G}}}$  is an open subset of  $V_r(\mathcal{G})$ .*

*Proof.* Let  $s = \text{srk}(\mathcal{G})$ , then  $V_r(\mathcal{G}) = V_{\mathfrak{C}(s \uparrow, \mathcal{G})}$ , this gives  $\mathfrak{C}(s \uparrow, \mathcal{G})_\alpha \neq \emptyset$  for any  $\alpha \in V_r(\mathcal{G})$ . Thus  $r_\alpha^{\mathcal{G}} \geq s$  and  $r_{\min}^{\mathcal{G}} \geq s$  on the one hand. On the other hand, for any  $\alpha \in V_r(\mathcal{G})$ ,  $r_\alpha^{\mathcal{G}} \geq r_{\min}^{\mathcal{G}}$  gives  $\mathfrak{C}(r_{\min}^{\mathcal{G}} \uparrow, \mathcal{G})_\alpha \neq \emptyset$ . Therefore,  $V_r(\mathcal{G}) = V_{\mathfrak{C}(r_{\min}^{\mathcal{G}} \uparrow, \mathcal{G})}$  and  $s \geq r_{\min}^{\mathcal{G}}$ . By the definition of  $r_\alpha^{\mathcal{G}}$ , the function

$$r^{\mathcal{G}} : V_r(\mathcal{G}) \longrightarrow \mathbb{N}; \alpha \mapsto r_\alpha^{\mathcal{G}}$$

is upper-semicontinuous(see [Far1, Sect.1]), because  $\mathcal{B}_n^{\mathcal{G}} = \{\alpha \in V_r(\mathcal{G}) \mid r_\alpha^{\mathcal{G}} \geq n\} = V_{\mathfrak{C}(n \uparrow, \mathcal{G})} = \bigcup_{s \geq n} V_{\mathfrak{C}(s, \mathcal{G})}$  is closed for every  $n \in \mathbb{N}$ ; see Lemma 4.2.4. Thus  $\mathcal{O}_{r_{\min}^{\mathcal{G}}} = V_r(\mathcal{G}) \setminus \mathcal{B}_{\text{srk}(\mathcal{G})+1}^{\mathcal{G}}$  is open.  $\square$

### 4.3 Infinitesimal group schemes: height $\leq 1$

Let  $\mathcal{G}$  be an infinitesimal group scheme of height  $\leq 1$ . Then  $\mathcal{G} = \underline{\mathfrak{g}}$  for some finite dimensional restricted Lie algebra  $(\mathfrak{g}, [p])$ . Let  $V(\mathfrak{g})$  be the fibre of the map  $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ , which is called the restricted nullcone of  $\mathfrak{g}$ . Being the special case of arbitrary infinitesimal group schemes, the passage from  $P(\mathfrak{g})$  to  $V(\mathfrak{g})$  is essentially a translation of the illustration we did in section 2.4. Without any real ambiguity, the

saturation rank of  $\underline{\mathfrak{g}}$  will be denoted by  $\text{srk}(\underline{\mathfrak{g}})$ . To reassure the reader, we redescribe  $\text{srk}(\underline{\mathfrak{g}})$  in a more specific way. The first task for us is to recall the definition of varieties of elementary subalgebras, which has been defined in Chapter 3.6. Then we define

$$V_{\mathbb{E}(r, \underline{\mathfrak{g}})} := \bigcup_{\mathfrak{e} \in \mathbb{E}(r, \underline{\mathfrak{g}})} \mathfrak{e} \subseteq V(\underline{\mathfrak{g}}).$$

When  $\mathcal{G} = \underline{\mathfrak{g}}$ ,  $V_1(\mathcal{G})$  (resp.  $\mathfrak{E}(\ell, \underline{\mathfrak{g}})$ ) can be identified with  $V(\underline{\mathfrak{g}})$  (resp.  $\mathbb{E}(\ell, \underline{\mathfrak{g}})$ ). In view of Lemma 4.2.4,  $V_{\mathbb{E}(r, \underline{\mathfrak{g}})}$  is a closed subvariety of  $V(\underline{\mathfrak{g}})$ .

**Theorem 4.3.1.** *Suppose  $(\underline{\mathfrak{g}}, [p])$  is a restricted Lie algebra, then*

$$\text{srk}(\underline{\mathfrak{g}}) = \max\{r \mid V(\underline{\mathfrak{g}}) = V_{\mathbb{E}(r, \underline{\mathfrak{g}})}\}.$$

*Proof.* From the definition, the proof can be reduced to the following formula:

$$V(\underline{\mathfrak{g}}) = V_{\mathbb{E}(r, \underline{\mathfrak{g}})} \iff \mathbb{P}(\underline{\mathfrak{g}}) = \mathbb{P}(\underline{\mathfrak{g}})_r.$$

We denote by  $\mathbb{k}\mathbb{Z}/p\mathbb{Z} = \mathbb{k}[t]$  for  $t^p = 0$ . Recall that for a restricted Lie algebra  $(\underline{\mathfrak{g}}, [p])$ , there is a bijective map

$$\Theta_{\underline{\mathfrak{g}}} : \text{Proj}(V(\underline{\mathfrak{g}})) \longrightarrow \mathbb{P}(\underline{\mathfrak{g}}) ; [x] \mapsto [\alpha_x]$$

where  $\alpha_x$  is a representative of an equivalence class in  $\mathbb{P}(\underline{\mathfrak{g}})$  which sends  $t$  to  $x$ .

As we wish, we will first verify " $\implies$ ". Using the map  $\Theta_{\underline{\mathfrak{g}}}$ , we see that each  $[\alpha] \in \mathbb{P}(\underline{\mathfrak{g}})$  is represented by some  $\alpha_x$  for  $x \in V(\underline{\mathfrak{g}}) \setminus \{0\}$ . Now the equality  $V(\underline{\mathfrak{g}}) = V_{\mathbb{E}(r, \underline{\mathfrak{g}})}$  ensures the existence of an  $r$ -dimensional elementary subalgebra  $\mathfrak{e}_r$  such that  $x \in \mathfrak{e}_r$ . Additionally, such  $\mathfrak{e}_r$  gives rise to an abelian unipotent subgroup  $\underline{\mathfrak{e}}_r = \mathbb{G}_{a(1)}^{\times r}$  of  $\underline{\mathfrak{g}}$  with complexity  $r$ , as well as the relation  $[\alpha] = \iota_{*, \mathbb{k}\underline{\mathfrak{e}}_r}([\alpha_x])$  for  $[\alpha_x] \in \mathbb{P}(\underline{\mathfrak{e}}_r)$ , hence  $\mathbb{P}(\underline{\mathfrak{g}}) = \mathbb{P}(\underline{\mathfrak{g}})_r$ .

Conversely, if  $x \in V(\underline{\mathfrak{g}}) \setminus \{0\}$ , then it corresponds to an equivalence class  $\Theta_{\underline{\mathfrak{g}}}([x]) = [\alpha_x]$  in  $\mathbb{P}(\underline{\mathfrak{g}})$ . Since  $\mathbb{P}(\underline{\mathfrak{g}}) = \mathbb{P}(\underline{\mathfrak{g}})_r$ , there exists an abelian unipotent subgroup  $\mathcal{U} \subseteq \underline{\mathfrak{g}}$  with complexity  $\geq r$  together with  $[\beta] \in \mathbb{P}(\mathcal{U})$  such that  $[\alpha_x] = \iota_{*, \mathbb{k}\mathcal{U}}([\beta])$ . In the same fashion, the passage from  $\Theta_{\underline{\mathfrak{g}}}$  to  $\Theta_{\mathfrak{u}}$  gives  $[\beta] = [\beta_y]$  for some  $y \in V(\mathfrak{u}) \setminus \{0\}$  where  $\mathfrak{u} = \bigoplus_{i=1}^n \bigoplus_{j_i=0}^{t_i-1} \mathbb{k}x_i^{p^{j_i}}$  if we write  $\mathbb{k}\mathcal{U}$  as  $\mathbb{k}[X_1, \dots, X_n]/(X_1^{p^{t_1}}, \dots, X_n^{p^{t_n}})$  with  $n \geq$

$r$  and  $x_i = X_i + (X_1^{p^{t_1}}, \dots, X_n^{p^{t_n}})$ . Now we have  $\Theta_{\mathfrak{g}}([x]) = [\alpha_x] = \iota_{*, \mathbb{k}\mathfrak{u}}([\beta]) = \iota_{*, \mathbb{k}\mathfrak{u}}([\beta_y]) = \Theta_{\mathfrak{g}}([y])$ , this gives  $[x] = [y]$ . Notice that  $V(\mathfrak{u}) = \bigoplus_{i=0}^n \mathbb{k}x_i^{p^{t_i-1}} \subseteq \mathfrak{g}$  is an elementary subalgebra of  $\dim V(\mathfrak{u}) \geq r$  as well as  $\bigcup_{t \geq r} V_{\mathbb{E}(t, \mathfrak{g})} = V_{\mathbb{E}(r, \mathfrak{g})}$ . As a result,  $x \in V_{\mathbb{E}(r, \mathfrak{g})}$  and  $V(\mathfrak{g}) = V_{\mathbb{E}(r, \mathfrak{g})}$  consecutively which completes our proof.  $\square$

**Example 4.3.1.** We consider the  $2n + 1$ -dimensional Heisenberg algebra  $\mathfrak{h} := \bigoplus_{i=1}^n \mathbb{k}x_i \oplus \bigoplus_{j=1}^n \mathbb{k}y_j \oplus \mathbb{k}z$ , whose bracket and  $p$ -map are given by

$$\begin{aligned} x_i^{[p]} &= 0, \quad y_j^{[p]} = 0, \quad z^{[p]} = 0, \\ [x_i, x_j] &= 0 = [y_i, y_j], \quad [x_i, y_j] = \delta_{ij}z, \quad [z, \mathfrak{h}] = 0 \end{aligned}$$

respectively. Suppose that  $p \geq 3$ , then Jacobson's formula implies that  $\mathfrak{h}$  is  $[p]$ -trivial, i.e.  $V(\mathfrak{h}) = \mathfrak{h}$ . Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  and  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$  be two  $n$ -tuples in  $\mathbb{A}^n$ . An observation tells us these elementary subalgebras of  $\mathfrak{h}$  of maximal dimension  $n + 1$  are of the forms  $\mathfrak{e}_{\alpha, \beta} = \bigoplus_{i=1}^n \mathbb{k}(\alpha_i x_i + \beta_i y_i) \oplus \mathbb{k}z$  parametrized by  $\alpha$  and  $\beta$  with some mild restrictions on  $(\alpha_i, \beta_i) \in \mathbb{P}^1$  for  $1 \leq i \leq n$ . An inspection shows that each element in  $\mathfrak{h}$  could be retrieved from one such maximal elementary subalgebra. As a result,  $\text{srk}(\mathfrak{h}) = n + 1$ .

From the definition and the example, it is readily seen that the saturation rank of  $\mathfrak{g}$  is determined by the local data. It is stressed that those maximal elementary subalgebras assigned by every  $[p]$ -nilpotent element play an essential role in our study. For this reason, we now define for each element  $x \in V(\mathfrak{g})$

$$\mathbb{E}(r, \mathfrak{g})_x := \{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{g}) \mid x \in \mathfrak{e}\}$$

as a subset of  $\mathbb{E}(r, \mathfrak{g})$ , and further refer to the number

$$r_x^{\mathfrak{g}} := \max \{r \mid \mathbb{E}(r, \mathfrak{g})_x \neq \emptyset\}$$

as the *local saturation rank* of  $x$ . We also set

$$\begin{aligned} \mathbb{E}(\mathfrak{g})_x &:= \mathbb{E}(r_x^{\mathfrak{g}}, \mathfrak{g})_x; \\ r_{\min}^{\mathfrak{g}} &:= \min \{r_x^{\mathfrak{g}} \mid x \in V(\mathfrak{g})\}. \end{aligned}$$

The following short proof shows that the subset  $\mathbb{E}(r, \mathfrak{g})_x$  for  $x \in V(\mathfrak{g})$  is indeed a closed subset.

**Proposition 4.3.2.** *Given  $x \in V(\mathfrak{g})$ . The subset  $\mathbb{E}(r, \mathfrak{g})_x$  is closed in  $\mathbb{E}(r, \mathfrak{g})$  for any  $r \geq 1$ .*

*Proof.* Observe that  $\mathbb{E}(r, \mathfrak{g})_0 = \mathbb{E}(r, \mathfrak{g})$ . It suffices to consider  $\mathbb{E}(r, \mathfrak{g})_x$  for  $x \in V(\mathfrak{g}) \setminus \{0\}$ . Note that  $\mathcal{A} := \mathbb{k}x$  is a closed, conical subset of  $\mathfrak{g}$ . Since the map

$$\zeta : \mathbb{E}(r, \mathfrak{g}) \longrightarrow \mathbb{N}_0 ; \mathfrak{e} \mapsto \dim \mathfrak{e} \cap \mathcal{A}$$

is upper-semicontinuous [Far1, Lemma 7.3], which implies that the set  $\mathbb{E}(r, \mathfrak{g})_x = \{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{g}) \mid \dim \mathfrak{e} \cap \mathcal{A} \geq 1\}$  is closed.  $\square$

**Remark 4.3.2.** Let  $\mathfrak{z}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} \mid [y, x] = 0\}$  be the centralizer of  $x$  in  $\mathfrak{g}$ , and  $\mathfrak{g}_x^{\text{nil}} := V(\mathfrak{g}) \cap \mathfrak{z}_{\mathfrak{g}}(x)$  be the intersection of  $V(\mathfrak{g})$  and  $\mathfrak{z}_{\mathfrak{g}}(x)$ . For each positive integer  $r$ , we define

$$\mathcal{C}_r(\mathfrak{g}_x^{\text{nil}}) := \{(x_1, x_2, \dots, x_r) \in (\mathfrak{g}_x^{\text{nil}})^{\times r} \mid [x_i, x_j] = 0, 1 \leq i, j \leq r\}$$

to be the Zariski closed subvariety of  $r$ -tuples of pairwise commuting elements of  $\mathfrak{g}_x^{\text{nil}}$ , as well as  $\mathcal{C}_r(\mathfrak{g}_x^{\text{nil}})^\circ$  the open subset of linear independent  $r$ -tuples of  $\mathcal{C}_r(\mathfrak{g}_x^{\text{nil}})$ . Denote by  $r_* = \max\{r \mid \mathcal{C}_r(\mathfrak{g}_x^{\text{nil}})^\circ \neq \emptyset\}$ . Then for every element  $(x_1, \dots, x_{r_*})$  of  $\mathcal{C}_{r_*}(\mathfrak{g}_x^{\text{nil}})$ ,  $x \in \text{Span}_{\mathbb{k}}\{x_1, \dots, x_{r_*}\}$ . Otherwise,  $(x, x_1, \dots, x_{r_*}) \in \mathcal{C}_{r_*+1}(\mathfrak{g}_x^{\text{nil}})^\circ \neq \emptyset$ , a contradiction. Thus, each element in  $\mathcal{C}_{r_*}(\mathfrak{g}_x^{\text{nil}})^\circ$  gives rise to an elementary subalgebra which contains  $x$ . According to this, one can easily check that

$$r_x^{\mathfrak{g}} = \max\{r \mid \mathcal{C}_r(\mathfrak{g}_x^{\text{nil}})^\circ \neq \emptyset\},$$

and there is a surjective map

$$q : \mathcal{C}_{r_x^{\mathfrak{g}}}(\mathfrak{g}_x^{\text{nil}})^\circ \longrightarrow \mathbb{E}(\mathfrak{g})_x$$

with the elements of  $q^{-1}(\mathfrak{e}_x)$  for any  $\mathfrak{e}_x \in \mathbb{E}(\mathfrak{g})_x$  differ by the natural action of  $\text{GL}_{r_x}$ .

The forthcoming two lemmas illustrate how the local saturation rank performs. On the one hand, we will show that the minimal value of  $r_x^{\mathfrak{g}}$  when  $x$  run over all elements of  $V(\mathfrak{g})$  is actually our saturation rank of  $\mathfrak{g}$ . On the other hand, we will reveal that the set of elements whose local saturation rank attains  $\text{srk}(\mathfrak{g})$  is an open subset.

**Lemma 4.3.3.** *Let  $(\mathfrak{g}, [p])$  be a restricted Lie algebra. Then  $\text{srk}(\mathfrak{g}) = r_{\min}^{\mathfrak{g}}$ .*



*Proof.* Let  $r := \text{srk}(\mathfrak{g})$ . We have that for each  $x \in V(\mathfrak{g})$  an elementary subalgebra  $\mathfrak{e}_x$  such that  $x \in \mathfrak{e}_x \in \mathbb{E}(r, \mathfrak{g})$ , which shows that  $\mathbb{E}(r, \mathfrak{g})_x \neq \emptyset$  and  $r \leq r_x^{\mathfrak{g}}$ . Hence,  $r \leq r_{\min}^{\mathfrak{g}}$ .

Conversely, since  $r_x^{\mathfrak{g}} \geq r_{\min}^{\mathfrak{g}}, \forall x \in V(\mathfrak{g})$ , then  $\mathbb{E}(r_{\min}^{\mathfrak{g}}, \mathfrak{g})_x \neq \emptyset$  applies. Thus  $V(\mathfrak{g}) = V_{\mathbb{E}(r_{\min}^{\mathfrak{g}}, \mathfrak{g})}$  and  $r_{\min}^{\mathfrak{g}} \leq r$  consecutively.  $\square$

**Lemma 4.3.4.** *The function*

$$r^{\mathfrak{g}} : V(\mathfrak{g}) \longrightarrow \mathbb{N}_0 ; \quad x \mapsto r_x^{\mathfrak{g}}$$

*is upper-semicontinuous. Moreover, the set  $\mathcal{O}_{r_{\min}^{\mathfrak{g}}} := \{x \in V(\mathfrak{g}) \mid r_x^{\mathfrak{g}} = r_{\min}^{\mathfrak{g}}\}$  is open.*

*Proof.* For arbitrary  $n \in \mathbb{N}$ , the set

$$\mathcal{B}_n := \{x \in V(\mathfrak{g}) \mid r_x^{\mathfrak{g}} \geq n\} = \{x \in V(\mathfrak{g}) \mid \mathbb{E}(n, \mathfrak{g})_x \neq \emptyset\} = V_{\mathbb{E}(n, \mathfrak{g})}$$

is closed in  $V(\mathfrak{g})$ , thus the function  $r^{\mathfrak{g}}$  is upper-semicontinuous [Far1, Sect.1]. Moreover, if we take the closed subset  $\mathcal{B}_{r_{\min}^{\mathfrak{g}}+1} = \{x \in V(\mathfrak{g}) \mid r_x^{\mathfrak{g}} \geq r_{\min}^{\mathfrak{g}} + 1\}$ , then  $\mathcal{O}_{r_{\min}^{\mathfrak{g}}} = V(\mathfrak{g}) \setminus \mathcal{B}_{r_{\min}^{\mathfrak{g}}+1}$  is open in  $V(\mathfrak{g})$ .  $\square$

## 4.4 Special case in 4.3 I: reductive Lie algebras

In this section, we assume that  $\mathfrak{g}$  is the Lie algebra of a connected reductive algebraic  $\mathbb{k}$ -group  $G$ . Let  $\mathcal{N}(\mathfrak{g})$  be the nullcone of  $\mathfrak{g}$  consisting of all elements  $x$  in  $\mathfrak{g}$  which are  $[p]$ -nilpotent in the sense that  $x^{[p]^r} = 0$  for some  $r > 0$  depending on  $x$ . It is known that when  $p$  is good for  $G$ ,  $\mathcal{N}(\mathfrak{g})$  is finite union of  $G$ -orbits; see [Jan2, 2.8.Theorem 1] and it is stressed there that the number of nilpotent orbits in  $\mathfrak{g}$  is always finite regardless of  $p$ . The result also applies to  $V(\mathfrak{g})$ , by the fact that  $V(\mathfrak{g})$  is a  $G$ -stable subvariety of  $\mathcal{N}(\mathfrak{g})$ . We list the good primes and their Coxeter number for  $G$  being simple algebraic groups in the following Table 4.1. With the finiteness property behind us, the existence of three canonical nilpotent orbits should be known: the regular (or principal) orbit  $\mathcal{O}_{reg}$  (or  $\mathcal{O}_{prin}$ ), the subregular orbit  $\mathcal{O}_{subreg}$  and the minimal orbit  $\mathcal{O}_{min}$ ; see [CM, 4.1-4.3]. A connected reductive group  $G$  is said to be *standard* if it satisfies the following hypotheses (see [Jan2, 2.9]):

- The derived subgroup  $G^{(1)}$  of  $G$  is simply-connected
- $p$  is a good prime for  $G$ .
- The Lie algebra  $\mathfrak{g}$  of  $G$  has a non-degenerate symmetric bilinear  $G$ -invariant form  $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{k}$ .

Type	$A_n$	$B_n/C_n$	$D_n$	$E_6/F_4$	$E_7$	$E_8$	$G_2$
Good primes $p$	$\geq 2$	$\geq 3$	$\geq 3$	$\geq 5$	$\geq 5$	$\geq 7$	$\geq 5$
Coxeter number $h(G)$	$n + 1$	$2n$	$2n - 2$	12	18	30	6

Table 4.1: Good primes and Coxeter number

Recall that the semisimple rank  $\text{rk}_{\text{ss}}(G)$  of  $G$  is defined to be the rank of its derived subgroup  $G^{(1)}$ . The main theorem of this section is the following:

**Theorem 4.4.1.** *Let  $G$  be a standard connected reductive algebraic group with  $\mathfrak{g} := \text{Lie}(G)$ . Assume that  $p \geq h$ , where  $h$  is the Coxeter number of  $G$ . Then*

$$\text{srk}(\mathfrak{g}) = \text{rk}_{\text{ss}}(G).$$

*Proof.* Let  $G^{(1)} = [G, G]$  be the derived subgroup of  $G$  and  $\mathfrak{g}^{(1)}$  be the corresponding Lie algebra. By our assumption on  $G$ ,  $G^{(1)}$  is simply-connected and semisimple. Let  $G_1, G_2, \dots, G_m$  be the simple simply-connected normal subgroups of  $G^{(1)}$  with  $\mathfrak{g}_i = \text{Lie}(G_i)$ . Then  $G^{(1)} = G_1 \times G_2 \times \dots \times G_m$  as well as  $\mathfrak{g}^{(1)} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_m$  [Pre1, Sect.3.1]. Moreover,  $G_i$  is one of the cases:

- (i)  $G_i$  is simple, simply-connected and not of type  $A_{\mathbb{k}p-1}$  ;
- (ii)  $G_i = \text{SL}(V_i)$  and  $p \mid \dim V_i$ .

By case (ii), in accordance with the "standard" criterion, we define groups  $G'_i$  by setting

$$G'_i = \begin{cases} \text{GL}(V_i), & \text{if } G_i = \text{SL}(V_i) \text{ and } p \mid \dim V_i, \\ G_i, & \text{otherwise.} \end{cases}$$

and further denote by  $G' = G'_1 \times G'_2 \times \cdots \times G'_m$ . In this fashion, one can check that  $G'$  satisfies the three hypotheses, that is, it is standard. Let  $\mathfrak{g}'_i = \text{Lie}(G'_i)$ , and  $\mathfrak{g}' = \text{Lie}(G')$ , then we have  $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2 \oplus \cdots \oplus \mathfrak{g}'_m$ . According to [GP, 6.2], there are tori  $T_0$  and  $T_1$  with their corresponding Lie algebras  $\mathfrak{t}_0$  and  $\mathfrak{t}_1$  such that  $\mathfrak{g}' \oplus \mathfrak{t}_0 = \mathfrak{g} \oplus \mathfrak{t}_1$ . Based on the definitions, this gives  $V(\mathfrak{g}) = V(\mathfrak{g}')$  as well as  $\mathbb{E}(r, \mathfrak{g}) = \mathbb{E}(r, \mathfrak{g}')$  for any positive integer  $r$ , which implies  $\text{srk}(\mathfrak{g}) = \text{srk}(\mathfrak{g}')$ .

We are in a position to compute  $\text{srk}(\mathfrak{g}')$ . Notice that  $G$  and  $G'$  have the same Coxeter number. If  $p \geq h$ , then  $V(\mathfrak{g}') = \mathcal{N}(\mathfrak{g}') = \overline{\mathcal{O}_{reg}}$  is irreducible ensured by [Jan2, Lemma 6.2], where  $\mathcal{O}_{reg} = G'.e_{reg}$  and  $e_{reg}$  is a regular nilpotent element of  $\mathfrak{g}'$ . By Lemma 4.3.4, we have  $\mathcal{O}_{reg} \cap \mathcal{O}_{rmin}^{\mathfrak{g}'} \neq \emptyset$ . An inspection on the  $r^{\mathfrak{g}'}$ -function shows that it is invariant up to conjugation, that is  $r_x^{\mathfrak{g}'} = r_{h.x}^{\mathfrak{g}'}$  for any  $h \in G'$  and  $x \in \mathfrak{g}'$ . As a result,  $\mathcal{O}_{reg} \subseteq \mathcal{O}_{rmin}^{\mathfrak{g}'}$  and by Lemma 4.3.3, we obtain  $\text{srk}(\mathfrak{g}') = r_{min}^{\mathfrak{g}'} = r_{e_{reg}}^{\mathfrak{g}'}$ .

Write  $e_{reg} = e_1 + e_2 + \cdots + e_m, e_i \in \mathfrak{g}'_i$ . Then

$$C_{G'}(e_{reg}) = C_{G'_1}(e_1) \times C_{G'_2}(e_2) \times \cdots \times C_{G'_m}(e_m)$$

and

$$\mathfrak{z}_{\mathfrak{g}'}(e_{reg}) = \mathfrak{z}_{\mathfrak{g}'_1}(e_1) \oplus \mathfrak{z}_{\mathfrak{g}'_2}(e_2) \oplus \cdots \oplus \mathfrak{z}_{\mathfrak{g}'_m}(e_m)$$

One can check that  $e_{reg}$  is regular nilpotent in  $\mathfrak{g}'$  if and only if each  $e_i$  is regular nilpotent in  $\mathfrak{g}'_i$ . Since each  $G'_i$  is standard, there is  $\mathfrak{z}_{\mathfrak{g}'_i}(e_i) = \text{Lie } C_{G'_i}(e_i)$ . So if each  $C_{G'_i}(e_i)$  is abelian, then  $\mathfrak{z}_{\mathfrak{g}'}(e_{reg})$  is abelian. Check that each  $G'_i$  is  $D$ -standard reductive group ([McT, Definition 3.2], [McN, Remark 3], [Let, Remark 2.5.6(a)], [McT, (5.2.4)] implies the abelian property of  $\mathfrak{z}_{\mathfrak{g}'}(e_{reg})$ ). Due to the Jordan-Chevalley decomposition, there is a direct sum of Lie algebra  $\mathfrak{z}_{\mathfrak{g}'}(e_{reg}) = (\mathfrak{z}_{\mathfrak{g}'}(e_{reg}))_s \oplus (\mathfrak{z}_{\mathfrak{g}'}(e_{reg}))_n$ . Since regular nilpotent elements are distinguished (see [Hum2, Sect. 7.13]), this gives rise to  $(\mathfrak{z}_{\mathfrak{g}'}(e_{reg}))_s = \mathfrak{z}(\mathfrak{g}')$  by [Lev, Theorem 3]. At this point, the abelian property of  $(\mathfrak{z}_{\mathfrak{g}'}(e_{reg}))_n$  ensures that, it is the maximal elementary subalgebra of  $\mathfrak{g}'$  containing  $e_{reg}$  of dimension  $(\dim \mathfrak{z}_{\mathfrak{g}'}(e_{reg}) - \dim \mathfrak{z}(\mathfrak{g}'))$ . For  $G'$  and  $\mathfrak{g}'$ , we have  $\dim \mathfrak{z}_{\mathfrak{g}'}(e_{reg}) = \dim C_{G'}(e_{reg}) = \text{rk}(G')$  since  $G'$  is standard. That gives,  $\text{srk}(\mathfrak{g}') = \dim(\mathfrak{z}_{\mathfrak{g}'}(e_{reg}))_n = \text{rk}(G') - \dim \mathfrak{z}(\mathfrak{g}')$ .

If  $G'_i = G_i$ , then  $\mathfrak{g}'_i$  is simple and  $\mathfrak{z}(\mathfrak{g}'_i) = 0$ . If  $G'_i = \text{GL}(V_i)$  with  $p \mid \dim V_i$ , then  $\mathfrak{g}'_i$

has a one-dimensional center. By denoting

$$\mathrm{rk}_p(G) := \mathrm{ord} \{G_i \mid G_i = \mathrm{SL}(V_i) \text{ and } p \mid \dim V_i\}$$

we have  $\dim \mathfrak{z}(\mathfrak{g}') = \mathrm{rk}_p(G)$ . Since  $\mathrm{rk}(\mathrm{GL}(V_i)) = 1 + \mathrm{rk}(\mathrm{SL}(V_i))$ , this implies  $\mathrm{rk}(G') = \mathrm{rk}(G^{(1)}) + \mathrm{rk}_p(G)$ . As a result,

$$\mathrm{srk}(\mathfrak{g}') = \mathrm{rk}(G') - \mathrm{rk}_p(G) = (\mathrm{rk}(G^{(1)}) + \mathrm{rk}_p(G)) - \mathrm{rk}_p(G) = \mathrm{rk}_{\mathrm{ss}}(G),$$

that is our  $\mathrm{srk}(\mathfrak{g})$  and we finish the proof.  $\square$

In what follows, we will recollect some material on nilpotent orbits from the partition point of view for  $G$  being simple of classical type  $A, B, C$  or  $D$  and  $p$  being good. Let  $\mathcal{P}(N)$  be a set of partitions of  $N$  where  $N = \ell + 1$  (resp.  $2\ell + 1, 2\ell$ ) when  $\Phi = A_\ell$  (resp.  $B_\ell, C_\ell$  and  $D_\ell$ ), for which we can endow it with a dominance partial order  $\preceq$ , and say that  $\lambda$  dominates  $\mu$  if  $\mu \preceq \lambda$ . If  $\lambda = (\lambda_1, \dots, \lambda_t)$  is such a partition, then we assign to it a nilpotent matrix  $x_\lambda = \mathrm{diag}(N_1, \dots, N_t)$  with upper triangular Jordan blocks  $N_1, \dots, N_t$  with sizes  $\lambda_1 \times \lambda_1, \dots, \lambda_t \times \lambda_t$ , and put  $\mathcal{O}_\lambda = G.x_\lambda$ . It is known that the nilpotent orbits of classical Lie algebra can be described in terms of partitions, and if  $\lambda$  and  $\mu$  are the partitions of two nilpotent orbits  $\mathcal{O}_\lambda$  and  $\mathcal{O}_\mu$  in  $\mathfrak{g}$ , then  $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$  if and only if  $\mu \preceq \lambda$ ; see [CM, Theorem 5.12-5.14, 6.2.5].<sup>1</sup> First suppose that  $G$  is of type  $A_\ell$ . Write  $\ell + 1 = qp + r$  with  $0 \leq r < p$ . Denote by  $\lambda \vdash \ell + 1$  the partition with  $q$  parts of size  $p$  and 1 part of size  $r$ . Then  $\lambda$  is maximal with respect to  $\preceq$  among all partitions of  $\ell + 1$  which have parts of size at most  $p$ . As a result,  $V(\mathfrak{g}) = \overline{\mathcal{O}_\lambda}$ . Now we assume that  $G$  has type  $B, C$  or  $D$ . The natural embedding via the standard representation of  $\mathfrak{g}$  into  $\mathfrak{gl}_N(\mathbb{k})$  ( $N = 2\ell, 2\ell + 1$ ) lets us view  $\mathfrak{g}$  as a restricted subalgebra of  $\mathfrak{gl}_N(\mathbb{k})$ . Let  $\mathcal{P}_X(N) \subseteq \mathcal{P}(N)$  be the subset of partitions corresponding to the nilpotent orbits in  $\mathfrak{g}$  for  $X = B, C$  or  $D$ . If  $\lambda$  is a partition of  $N$ , then there exists a partition  $\lambda_X \in \mathcal{P}_X(N)$ , which we call  $X$ -collapse is the unique largest partition dominated by  $\lambda$ ; see [CM, Lemma 6.3.3]. Write  $N = qp + d$  with  $0 \leq d < p$  and  $\lambda = (\underbrace{p, \dots, p}_q, d) \vdash N$ , this gives  $V(\mathfrak{g}) = \overline{\mathcal{O}_{\lambda_X}}$ .

Now we concentrate on the case when  $G = \mathrm{SL}_n(\mathbb{k})$ ,  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$  and  $n \geq 3$ . From the proof of Theorem 4.4.1, we find that  $\mathrm{srk}(\mathfrak{sl}_n(\mathbb{k})) = n - 1$  whenever  $p \geq n$ . In the

<sup>1</sup>We often refer to [CM] in this paragraph. Although the results there are obtained over complex numbers, they are still valid in positive characteristic  $p$  as long as  $p$  is good. See [CLNP, Sect 3.9].

following part, the local saturation rank of the elements, especially in the subregular nilpotent orbit  $\mathcal{O}_{subreg}$  and minimal nilpotent orbit  $\mathcal{O}_{min}$  will be calculated in an attempt to describe the open set  $\mathcal{O}_{rmin}$  of  $V(\mathfrak{g})$ . Inspired by Remark 4.3.2 and the proof of Theorem 4.4.1, the determination of the centraliser of a nilpotent element plays a central role in our computation. By the natural embedding of  $\mathfrak{sl}_n(\mathbb{k})$  in  $\mathfrak{gl}(\mathbb{V})$  with  $\dim \mathbb{V} = n$ , we begin with an observation from  $\mathfrak{gl}(\mathbb{V})$ . Let  $e \in \mathfrak{gl}(\mathbb{V})$  be nilpotent with the corresponding partition  $(\lambda_1, \dots, \lambda_t)$ ,  $\mathfrak{z}(e)$  be the centraliser of  $e$  in  $\mathfrak{gl}(\mathbb{V})$ . It is assumed that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t > 0$ . Then there exist elements  $v_1, \dots, v_t \in \mathbb{V}$  such that all  $e^j.v_i$  with  $1 \leq i \leq t$  and  $0 \leq j < \lambda_i$  form a basis of  $\mathbb{V}$  together with  $e^{\lambda_i}.v_i = 0$ . Let  $\xi \in \mathfrak{z}(e)$ , then  $\xi$  is completely determined by  $\xi(v_i)$  with  $1 \leq i \leq t$  because  $\xi(e^j.v_i) = e^j.\xi(v_i)$ , but  $e^{\lambda_i}.\xi(v_i) = 0$ . One can easily check that

$$\xi(v_i) = \sum_{j=1}^t \sum_{s=\max\{\lambda_j-\lambda_i, 0\}}^{\lambda_j-1} a_{ijs} e^s.v_j$$

for some  $a_{ijs}$ , which gives the basis  $\{\xi_i^{j,s}\}$  of  $\mathfrak{z}(e)$  defined by

$$\begin{cases} \xi_i^{j,s}(v_i) = e^s.v_j, \\ \xi_i^{j,s}(v_r) = 0 \text{ for } r \neq i, \end{cases} \quad 1 \leq i, j \leq k, \quad \text{and} \quad \max\{\lambda_j - \lambda_i, 0\} \leq s < \lambda_j.$$

It is convenient to assume that  $\xi_i^{j,s} = 0$  whenever  $s$  is not within the appropriate bound.

Given two basis elements  $\xi_i^{j,s}$  and  $\xi_p^{q,r}$  of  $\mathfrak{z}(e)$ , we have the composition rule:  $\xi_i^{j,s}.\xi_p^{q,r} = \delta_{q,i}\xi_p^{j,s+r}$ , and further their bracket:

$$[\xi_i^{j,s}, \xi_p^{q,r}] = \delta_{q,i}\xi_p^{j,s+r} - \delta_{j,p}\xi_i^{q,s+r}.$$

**Lemma 4.4.2.** *Suppose  $G = \mathrm{SL}_n(\mathbb{k})$  ( $n \geq 3$ ). Let  $\tau = (n-1, 1) \vdash n$  be the partition corresponding to the subregular nilpotent orbit of  $\mathfrak{g}$ ,  $x_\tau$  be the nilpotent matrix given by the partition  $\tau$ . If  $p \geq n-1$ , then we have  $r_{x_\tau} = n-1$ .*

*Proof.* Let  $e := x_\tau$  with  $\tau = (n-1, 1)$ , a nilpotent element in  $\mathfrak{gl}(\mathbb{V})$  in a natural way. Keep the notation for  $\mathfrak{z}(e)$  as above, the basis of  $\mathfrak{z}(e)$  as described is :

$$\xi_1^{1,s}, 0 \leq s \leq n-2; \quad \xi_1^{2,0}; \quad \xi_2^{1,n-2}; \quad \xi_2^{2,0}.$$

Let  $\mathfrak{z}(e)' = \mathfrak{z}(e) \cap \mathfrak{g}$ , the centraliser of the nilpotent element  $e$  in  $\mathfrak{g}$ , and  $\xi \in \mathfrak{z}(e)'$ . Write

$$\xi = \sum_{s=0}^{n-2} a_{11s} \xi_1^{1,s} + a_{120} \xi_1^{2,0} + a_{21(n-2)} \xi_2^{1,n-2} + a_{220} \xi_2^{2,0}.$$

Since it is an element of  $\mathfrak{g}$ , this implies  $a_{220} = -(n-1)a_{110}$ . Thus the basis of  $\mathfrak{z}(e)'$  is

$$\xi_1^{1,s}, 1 \leq s \leq n-2; \quad \xi_1^{2,0}; \quad \xi_2^{1,n-2}; \quad \xi_1^{1,0} - (n-1)\xi_2^{2,0}.$$

Keep the form of  $\xi$  with the relation of  $a_{110}$  and  $a_{220}$ . We want to determine the nilpotent part  $\mathfrak{z}(e)'_n$  of  $\mathfrak{z}(e)'$ . Given by the composition rule, there are the relations: (1)  $\xi_1^{1,0} \cdot \xi_1^{1,0} = \xi_1^{1,0}$ ,  $\xi_2^{2,0} \cdot \xi_2^{2,0} = \xi_2^{2,0}$ ; (2)  $\xi_2^{1,n-2} \cdot \xi_1^{2,0} = \xi_1^{1,n-2}$ ; (3)  $(\xi_1^{1,s})^{n-1} = 0$  for  $1 \leq s \leq n-2$ , and  $(\xi_1^{2,0})^2 = (\xi_2^{1,n-2})^2 = 0$ . Depending on these rules, we conclude that if  $n > 3$  and  $p \geq n-1$ , or  $n = 3$  and  $p > n-1$ , then  $\mathfrak{z}(e)'_n$  is a vector space having the following basis

$$\xi_1^{1,s}, 1 \leq s \leq n-2; \quad \xi_1^{2,0}; \quad \xi_2^{1,n-2}.$$

Alternatively, in the case  $n = 3$  and  $p = 2$ ,  $\xi \in \mathfrak{z}(e)'_n$  with  $\xi^{p^2} = 0$  if and only if  $a_{110} = a_{220} = 0$ , with  $\xi^p = 0$  if and only if  $a_{110} = a_{220} = a_{120} \cdot a_{211} = 0$ . Now we are able to determine the maximal elementary subalgebras  $\mathfrak{e}_\tau$  that will be assigned to  $e$ . Observe that for  $1 \leq s \leq n-2$ ,  $[\xi_1^{1,s}, \xi_1^{1,s}] = \xi_1^{1,s+s} - \xi_1^{1,s+s} = 0$  for  $1 \leq s' \leq n-2$ ,  $[\xi_1^{1,s}, \xi_1^{2,0}] = -\xi_1^{2,s} = 0$  and  $[\xi_1^{1,s}, \xi_2^{1,n-2}] = \xi_2^{1,s+n-2} = 0$ . We then have  $[\xi_1^{1,s}, \xi] = 0$  for any  $\xi \in \mathfrak{z}(e)'_n$  and  $1 \leq s \leq n-2$ . Still, note that  $[\xi_1^{2,0}, \xi_2^{1,n-2}] \neq 0$ . Thus, if  $n > 3$  and  $p \geq n-1$ , or  $n = 3$  and  $p > n-1$ , then

$$\mathfrak{e}_\tau = \mathbb{k}\xi_1^{1,1} \oplus \cdots \oplus \mathbb{k}\xi_1^{1,n-2} \oplus \mathbb{k}(a\xi_1^{2,0} + b\xi_2^{1,n-2});$$

parameterized by  $(a : b) \in \mathbb{P}^1$  and if  $n = 3$  and  $p = 2$ , then

$$\mathfrak{e}_\tau = \mathbb{k}\xi_1^{1,1} \oplus \cdots \oplus \mathbb{k}\xi_1^{1,n-2} \oplus \mathbb{k}\xi^*, \text{ where } \xi^* \in \{\xi_1^{2,0}, \xi_2^{1,n-2}\}$$

Notice that all of them show  $r_e = r_{x_\tau} = n-1$ , and this completes our proof.  $\square$

We remark here that if  $p = n-1$ , then  $V(\mathfrak{g}) = \overline{\mathcal{O}_\tau}$  for  $\tau = (n-1, 1)$ . According to the procedure processed in Theorem 4.4.1 and the result of Lemma 4.4.2 we have  $\text{srk}(\mathfrak{g}) = r_{x_\tau} = n-1$ . This shows that the equality  $\text{srk}(\mathfrak{g}) = \text{rk}_{\text{ss}}(G)$  still holds for smaller  $p$ , like  $p = n-1$ .

**Lemma 4.4.3.** *Suppose  $G = \mathrm{SL}_n(\mathbb{k})$  with  $n \geq 3$ . Let  $\mathcal{O}_\lambda \subseteq \mathcal{N}(\mathfrak{g}) \setminus (\mathcal{O}_{\mathrm{reg}} \sqcup \mathcal{O}_{\mathrm{subreg}})$  be a nilpotent orbit, and  $x_\lambda \in \mathcal{O}_\lambda$  be the corresponding nilpotent element given by the partition  $\lambda$ . If  $p \geq \max\{2, n-2\}$ , then we have  $r_{x_\lambda} \geq n$ . Specifically, when  $\lambda = (2, 1^{n-2})$  is the partition corresponding to the minimal nilpotent orbit  $\mathcal{O}_\lambda$ , we have  $r_{x_\lambda} = \lfloor \frac{n^2}{4} \rfloor$ .*

*Proof.* According to the dominance order  $\trianglelefteq$ , we know that  $\mathcal{O}_\lambda \subseteq \mathcal{N}(\mathfrak{g}) \setminus (\mathcal{O}_{\mathrm{reg}} \sqcup \mathcal{O}_{\mathrm{subreg}})$  if and only if  $\lambda \trianglelefteq (n-2, 2)$ . Write  $\lambda = (\lambda_1, \dots, \lambda_t)$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t > 0$ . We are going to read off  $x_\lambda$  from the basis  $\{\xi_i^{j,s}\}$  of  $\mathfrak{gl}(\mathbb{V})$ . Let  $s = \max\{i \mid \lambda_i \geq 2\}$ . If  $1 \leq i \leq s$ , the action of  $\xi_i^{i,1}$  on  $\mathbb{V}$  gives an unique non-zero Jordan block of size  $\lambda_i \times \lambda_i$ . Then we can immediately know

$$x_\lambda = \xi_1^{1,1} + \xi_2^{2,1} + \dots + \xi_s^{s,1}.$$

In what follows, we consider three cases to estimate the local saturation rank of  $x_\lambda$ . In case of all  $\lambda_i \geq 2$ , we find there is an elementary subalgebra

$$\bigoplus_{i=1}^t \bigoplus_{r=1}^{\lambda_i-1} \mathbb{k}\xi_i^{i,r} \oplus \bigoplus_{i=1}^{t-1} \mathbb{k}\xi_i^{i+1, \lambda_{i+1}-1} \oplus \mathbb{k}\xi_t^{1, \lambda_1-1}$$

that contains  $x_\lambda$  with dimension  $\sum_{i=1}^t \lambda_i = n$ . Alternatively, there is at least one block with size  $1 \times 1$ . If there is only one, then  $s = t-1$  and  $\lambda = (\lambda_1, \dots, \lambda_s, 1)$ . Since  $\lambda \trianglelefteq (n-2, 2)$ , then  $s \geq 2$  and we find an elementary subalgebra

$$\bigoplus_{i=1}^s \bigoplus_{r=1}^{\lambda_i-1} \mathbb{k}\xi_i^{i,r} \oplus \bigoplus_{i=1}^{s-1} \mathbb{k}\xi_i^{i+1, \lambda_{i+1}-1} \oplus \mathbb{k}\xi_t^{s, \lambda_s-1} \oplus \mathbb{k}\xi_t^{1, \lambda_1-1}$$

which contains  $x_\lambda$ . Otherwise,  $\lambda$  has the form  $(\lambda_1, \dots, \lambda_s, 1^{t-s})$  for  $t-s \geq 2$ . Let  $\xi = \sum_{i=1}^{t-s-1} \xi_{s+i+1}^{s+i,0}$ . Then we assign to  $x_\lambda$  an elementary subalgebra

$$\bigoplus_{i=1}^s \bigoplus_{r=1}^{\lambda_i-1} \mathbb{k}\xi_i^{i,r} \oplus \bigoplus_{i=1}^{t-s-1} \mathbb{k}\xi_i^i \oplus \bigoplus_{i=1}^s \mathbb{k}\xi_i^{i+1, \lambda_{i+1}-1} \oplus \mathbb{k}\xi_t^{1, \lambda_1-1},$$

where  $\xi^i$  is the  $i$ -th power of  $\xi$ . One can compute that  $r_{x_\lambda} \geq n$  for both cases.

Note that the minimal nilpotent orbit  $\mathcal{O}_\lambda$  where  $\lambda = (2, 1^{n-2})$  is the  $G$ -orbit of the highest root  $E_{1n}$ . When  $n = 2m$  (resp.  $n = 2m+1$ ), Theorem 2.7 (resp. Theorem 2.8) in [CFP] shows that the elementary subalgebra which contains  $E_{1n}$  has maximal dimension  $m^2$  (resp.  $m(m+1)$ ). Therefore, we have  $r_{x_\lambda} = \lfloor \frac{n^2}{4} \rfloor$ .  $\square$

Suppose that  $n \geq 4$ . If  $p = n - 2$ , then  $V(\mathfrak{g}) = \overline{\mathcal{O}_\lambda}$  with  $\lambda = (n - 2, 2)$ . By the same token, we have  $\text{srk}(\mathfrak{g}) = r_{x_\lambda} \geq n$  according to Lemma 4.4.3. This shows that for smaller  $p < n$ , like  $p = n - 2$ , we have  $\text{srk}(\mathfrak{g}) > \text{rk}_{\text{ss}}(G)$ .

**Theorem 4.4.4.** *Suppose  $G = \text{SL}_n(\mathbb{k})$  with  $n \geq 3$ . If  $p \geq n$ , then*

$$\mathcal{O}_{rmin} = \mathcal{O}_{reg} \sqcup \mathcal{O}_{subreg}.$$

*Proof.* Theorem 4.4.1 tells us  $\mathcal{O}_{rmin} = \{x \in V(\mathfrak{g}) \mid r_x = n - 1\}$ . Consecutive applications of Lemma 4.4.2 and Lemma 4.4.3 yield  $\mathcal{O}_{rmin} = \mathcal{O}_{reg} \sqcup \mathcal{O}_{subreg}$ .  $\square$

## 4.5 Special case in 4.3 II: parabolic subalgebras

Suppose  $G$  is a connected reductive algebraic group. Fix a maximal torus  $T$  of  $G$ , and denote by  $\Phi := \Phi(G, T)$  the root system of  $G$  relative to  $T$ . Let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  be a basis of simple roots in  $\Phi$ , as well as  $\Phi^+$  be the positive root system associated with  $\Delta$ . Given a subset  $I \subseteq \{1, \dots, n\}$ , we denote by  $P_I$  (resp.  $L_I$ ) the standard parabolic (resp. Levi) subgroup of  $G$  corresponding to  $I$  with the decomposition  $P_I = L_I \ltimes R_u(P_I)$ . Then  $\Phi_I$  is a root system of  $L_I$  relative to  $T$ , and  $\Delta_I := \{\alpha_i \mid i \in I\}$  is a basis of simple roots in  $\Phi_I$ . Given two subsets  $J \subseteq I \subseteq \{1, \dots, n\}$ , the standard parabolic subgroup of  $L_I$  associated to  $J$  is denoted by  $P_{I,J}$ . There is the Levi decomposition  $P_{I,J} = L_J \ltimes R_u(P_{I,J})$  of  $P_{I,J}$  similarly. Let  $\mathfrak{p}_{I,J} = \text{Lie } P_{I,J}$ ,  $\mathfrak{l}_J = \text{Lie } L_J$  and  $\mathfrak{u}_{I,J} = \text{Lie } R_u(P_{I,J})$ . It is known that  $\mathfrak{p}_{I,J}$  has Richardson elements  $e$  such that the orbit  $(\text{Ad } P_{I,J})e$  is open in  $\mathfrak{u}_{I,J}$ ; see [Ric]. Thus the following lemma is immediately obtained.

**Lemma 4.5.1.** *Keep the notation for  $P_{I,J}, \mathfrak{u}_{I,J}$  and  $e$  as above. Suppose that  $p \geq h(L_I)$ , where  $h(L_I)$  is the Coxeter number of  $L_I$ . Then  $\text{srk}(\mathfrak{u}_{I,J}) = r_e^{\mathfrak{u}_{I,J}}$ .*

*Proof.* If  $p \geq h(L_I)$ , then  $V(\mathfrak{u}_{I,J}) = \mathfrak{u}_{I,J}$  is irreducible. Note that  $\mathcal{O}_{rmin}^{\mathfrak{u}_{I,J}}$  is open in  $V(\mathfrak{u}_{I,J}) = \mathfrak{u}_{I,J}$ , this gives  $\mathcal{O}_{rmin}^{\mathfrak{u}_{I,J}} \cap (\text{Ad } P_{I,J})e \neq \emptyset$ . Observe that  $P_{I,J}$  acts on  $R_u(P_{I,J})$ , thus  $\text{Ad } P_{I,J}$  acts on  $\mathfrak{u}_{I,J}$ . As a result,  $(\text{Ad } P_{I,J})e \subseteq \mathcal{O}_{rmin}^{\mathfrak{u}_{I,J}}$ , and this implies  $\text{srk}(\mathfrak{u}_{I,J}) = r_e^{\mathfrak{u}_{I,J}}$ .  $\square$



The above lemma puts forward the question of finding the Richardson elements in  $\mathfrak{u}_{I,J}$ . When  $I = \{1, \dots, n\}$ , there is an explicit construction of these elements in case of  $G$  being of classical type and  $\text{char}(\mathbb{k}) \neq 2$  (see [BG]), which extends work in [Bau]. For the sake of simplicity, when  $I = \{1, \dots, n\}$ , we will denote  $P_{I,J}$  by  $P_J$ , and this also applies to their corresponding Lie algebras. It is clear that, a standard parabolic subalgebra  $\mathfrak{p}_J$  has the form

$$\mathfrak{p}_J = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+ \cup \Phi_J^-} \mathbb{k}x_\alpha$$

where  $\mathfrak{h} = \text{Lie}(T)$  for any given subset  $J \subseteq \{1, \dots, n\}$ , as well as its Levi factor and nilradical

$$\mathfrak{l}_J = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_J} \mathbb{k}x_\alpha, \quad \mathfrak{u}_J = \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} \mathbb{k}x_\alpha.$$

In the sequel of this section, we consider  $G = \text{SL}_{n+1}(\mathbb{k})$ , and concentrate on computing the saturation rank of the nilradical of standard parabolic subalgebras of the special linear Lie algebra  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{k})$  in some cases. Recall that a standard parabolic subalgebra  $\mathfrak{p}_J$  of  $\mathfrak{sl}_{n+1}(\mathbb{k})$  is uniquely described by the sequence of lengths of blocks, that is the square matrices on the diagonal in the standard Levi factor  $\mathfrak{l}_J$ . Motivated by these observations, we write  $\mathbf{d} = (d_1, \dots, d_r)$  with  $\sum_i d_i = n + 1$  for the associated dimension vector of the block lengths in  $\mathfrak{l}_J$ . The following lemma shows the interrelations between  $J$  and  $\mathbf{d}$ .

**Lemma 4.5.2.** *Suppose  $\mathfrak{p}_J \subseteq \mathfrak{sl}_{n+1}(\mathbb{k})$  is a standard parabolic subalgebra defined via  $J = \{1, \dots, n\} \setminus \{r_1, \dots, r_s\}$ ,  $1 \leq r_1 < \dots < r_s \leq n$ . Then  $\mathbf{d} = (r_1, r_2 - r_1, \dots, r_s - r_{s-1}, n + 1 - r_s)$ . On the other hand, if the dimension vector of the Levi factor  $\mathfrak{l}_J$  has the form  $\mathbf{d} = (d_1, \dots, d_s)$ , then  $J = \{1, \dots, n\} \setminus \{d_1, d_1 + d_2, \dots, \sum_{i=1}^{s-1} d_i\}$ .*

*Proof.* If  $J = \{1, \dots, n\} \setminus \{r_1, \dots, r_s\}$  with  $1 \leq r_1 < \dots < r_s \leq n$ , then the associated basis of  $\Phi_J$  is  $\Delta_J = \{\alpha_i \mid i \neq r_i, 1 \leq i \leq s\}$ . Note that the subset

$$\{\alpha_{r_i+1}, \dots, \alpha_{r_{i+1}-1}\} \quad (r_0 = 0)$$

gives a block of length  $r_{i+1} - r_i$ . Thus  $\Delta_J$  gives the dimension vector  $\mathbf{d} = (r_1, r_2 - r_1, \dots, r_s - r_{s-1}, n + 1 - r_s)$ . On the other hand, if the dimension vector  $\mathbf{d} = (d_1, \dots, d_s)$ , then  $J = \{1, \dots, n\} \setminus \{r'_1, \dots, r'_{s-1}\}$  with the relations:  $r'_1 = d_1$ ,  $r'_i - r'_{i-1} = d_i$  ( $1 < i \leq s - 1$ ), and  $r'_{s-1} = n + 1 - d_s$ . Thus  $J = \{1, \dots, n\} \setminus \{d_1, d_1 + d_2, \dots, \sum_{i=1}^{s-1} d_i\}$ .  $\square$

As a benefit of Lemma 4.5.2, sometimes we write  $\mathfrak{p}(\mathbf{d})$  instead of  $\mathfrak{p}_J$ . Theorem 3.2 of [Bau] gives the construction of a Richardson element for the standard parabolic subalgebra  $\mathfrak{p}(\mathbf{d})$  through the corresponding dimension vector  $\mathbf{d}$ . More precisely, it is obtained by first constructing the horizontal line diagram  $L_h(\mathbf{d})$  and then labeling the vertices, and finally the Richardson element of  $\mathfrak{p}(\mathbf{d})$  is given by

$$X(\mathbf{d}) = X(L_h(\mathbf{d})) = \sum_{i \rightarrow j} E_{i,j},$$

where  $E_{i,j}$  is the elementary matrix in  $\mathfrak{sl}_{n+1}(\mathbb{k})$ ; see [Bau, Sect.3]. In what follows, we will focus on four special cases of  $\mathfrak{p}_J$ , i.e. when  $J = \emptyset$ ,  $|J| = 1$ ,  $|J| = n - 2$  and  $|J| = n - 1$ . We assume that  $p \geq h(L_J)$ , depending on the set  $J$  being considered.

**Case I** ( $J = \emptyset$ ). By Lemma 4.5.2,  $\mathbf{d} = \overbrace{(1, 1, \dots, 1)}^{n+1}$ . This gives  $\mathfrak{l}_J = \mathfrak{h}$ , and  $\mathfrak{p}_J = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} \mathbb{k}x_\alpha$  which is the Lie algebra  $\mathfrak{b} = \text{Lie}(B)$  of the standard Borel subgroup  $B \leq G$  with  $\mathfrak{u}_J = \mathfrak{u}_\mathfrak{b}$  as its nilradical. The labeled horizontal line diagram is then

$$1 \text{ --- } 2 \text{ --- } \cdot \quad \dots \quad \cdot \text{ --- } n + 1$$

and the given Richardson element is  $X(\mathbf{d}) = E_{1,2} + E_{2,3} + \dots + E_{n,n+1}$ . Observe that  $X(\mathbf{d})$  coincides with the regular nilpotent element  $e_{reg}$  of  $\mathfrak{g}$ , and the set of generators  $\{e_{reg}, e_{reg}^2, \dots, e_{reg}^n\}$  of the centralizer  $\mathfrak{z}_\mathfrak{g}(e_{reg})$  also lies inside of  $\mathfrak{u}_\mathfrak{b}$ . Therefore,  $\text{srk}(\mathfrak{u}_\mathfrak{b}) = r_{e_{reg}}^{\mathfrak{u}_\mathfrak{b}} = \dim \mathfrak{z}_\mathfrak{g}(e_{reg}) = n$ .

We remark here that this case has already served as an example of  $\text{srk}(\mathfrak{g}) = \text{srk}(\mathfrak{u}_\mathfrak{b})$ . Let  $G$  be a reductive algebraic  $\mathbb{k}$ -group, with standard Borel subgroup  $B$ . Either Theorem 2.2 of [LMT] when  $G$  is semisimple and  $p$  is a non-torsion prime for  $G$ , or Lemma 2.2 [Pre2] when  $G$  is standard implies that  $\mathbb{E}(r, \mathfrak{g}) = G.\mathbb{E}(r, \mathfrak{u}_\mathfrak{b})$  for  $r \geq 1$  where  $\mathfrak{u}_\mathfrak{b} = \text{Lie}(U_B)$  is the Lie algebra of the unipotent radical of  $B$ . According to this, we have

$$V(\mathfrak{g}) = G.V(\mathfrak{u}_\mathfrak{b}) = G.V_{\mathbb{E}(\text{srk}(\mathfrak{u}_\mathfrak{b}), \mathfrak{u}_\mathfrak{b})} = V_{G.\mathbb{E}(\text{srk}(\mathfrak{u}_\mathfrak{b}), \mathfrak{u}_\mathfrak{b})} = V_{\mathbb{E}(\text{srk}(\mathfrak{u}_\mathfrak{b}), \mathfrak{g})},$$

which gives  $\text{srk}(\mathfrak{g}) \geq \text{srk}(\mathfrak{u}_\mathfrak{b})$  in general.

**Case II** ( $|J| = n - 1$ ). In this case,  $P_J$  always is the maximal parabolic subgroup. We denote  $P_J$  (resp.  $R_u(P_J)$ ) by  $P_{\alpha_i}$  (resp.  $U_{\alpha_i}$ ) when  $P_J$  is maximal and  $\alpha_i \notin \Delta_J$ , as well as their Lie algebras  $\mathfrak{p}_J$  (resp.  $\mathfrak{u}_J$ ) by  $\mathfrak{p}_{\alpha_i}$  (resp.  $\mathfrak{u}_{\alpha_i}$ ). Before starting, let us first look at an example.

**Example 4.5.1.** Let  $J = \{2, \dots, n\}$ . Then  $\mathbf{d} = (1, n)$  by Lemma 4.5.2 and  $\mathfrak{p}(\mathbf{d})$  is the standard maximal parabolic subalgebra of  $\mathfrak{sl}_{n+1}(\mathbb{k})$  which is denoted by  $\mathfrak{p}_{\alpha_1}$ . We also have  $\mathfrak{p}_{\alpha_1} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi_J^- \cup \Phi^+} \mathbb{k}x_\beta$  and  $\mathfrak{u}_{\alpha_1} = \bigoplus_{\beta \in \Phi^+ \setminus \Phi_J^+} \mathbb{k}x_\beta$  consecutively. Pictorially, such  $\mathfrak{u}_{\alpha_1}$  can be represented in the following matrix form, marked by  $*$ :

$$\mathfrak{u}_{\alpha_1} : \begin{pmatrix} 0 & * & * & \dots & * & * \\ \hline 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Reading off the Richardson element from the labeled horizontal line diagram, we have  $X(\mathbf{d}) = E_{1,2}$ . Observe that  $\mathfrak{u}_{\alpha_1}$  itself is an elementary subalgebra, thus  $\text{srk}(\mathfrak{u}_{\alpha_1}) = r_{E_{1,2}}^{\mathfrak{u}_{\alpha_1}} = \dim \mathfrak{u}_{\alpha_1} = n$ .

**Theorem 4.5.3.** *Suppose that  $J = \{1, \dots, n\} \setminus \{r\}$ ,  $1 \leq r \leq n$ . Then  $\text{srk}(\mathfrak{u}_J) = r(n+1-r)$ .*

*Proof.* In this case, note that  $\mathfrak{u}_J = \mathfrak{u}_{\alpha_r} \subset \mathfrak{sl}_{n+1}(\mathbb{k})$  is the Lie subalgebra of  $(n+1) \times (n+1)$  matrices  $(a_{i,j})$  with  $a_{i,j} = 0$  except for  $1 \leq i \leq r, r+1 \leq j \leq n+1$ . It is readily seen to be an elementary subalgebra of dimension  $r(n+1-r)$ , thus  $\text{srk}(\mathfrak{u}_J) = r(n+1-r)$ .  $\square$

**Remark 4.5.2.** Theorem 4.5.3 is not only valid for  $G = \text{SL}_{n+1}(\mathbb{k})$ . Notice that in this case the nilradical  $\mathfrak{u}_J$  of  $\mathfrak{p}_J$  is abelian. As a Lie algebra of  $R_u(P_J)$ , this provides an example for the unipotent radical  $R_u(P_J)$  being abelian. Lemma 2.2 of [RRS] illustrates that if  $G$  is simple and  $P$  is a proper standard parabolic subgroup of  $G$ , the unipotent radical  $R_u(P)$  is abelian if and only if  $P$  is maximal (i.e.,  $P = P_J$  for  $|J| = |\Delta| - 1$ ) and the corresponding simple root  $\alpha_i$  ( $\alpha_i \notin \Delta_J$ ) occurs in the highest root denoted by  $\rho$  with coefficient 1. According to this Lemma, it is clear what the standard proper parabolic subgroups with abelian unipotent radical are when  $G$  is simple; see [RRS, Remark 2.3] for details. And in this situation, the saturation rank of  $\mathfrak{u}_{\alpha_i}$  is the dimension of the corresponding abelian unipotent radical  $U_{\alpha_i}$ .

**Case III**( $|J| = 1$ ). Until the end of this section for a nilradical  $\mathfrak{u}_J$  and the corresponding Richardson element  $X(\mathbf{d})$ , the centraliser  $\mathfrak{z}_{\mathfrak{u}_J}(X(\mathbf{d}))$  of  $X(\mathbf{d})$  in  $\mathfrak{u}_J$  will be denoted by  $\mathfrak{z}(\mathbf{d})$ .

**Example 4.5.3.** Consider  $\mathfrak{p}_J \subseteq \mathfrak{sl}_4(\mathbb{k})$  given by  $J = \{3\}$ . Then  $\mathbf{d} = (1, 1, 2)$  is the dimension vector by Lemma 4.5.2, and the labeled horizontal line diagram is

$$1 \text{ --- } 2 \text{ --- } 3$$

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Reading off the Richardson element, we have  $X(\mathbf{d}) = E_{1,2} + E_{2,3}$ . Direct computation shows that the centralizer  $\mathfrak{z}(\mathbf{d})$  is generated by  $X(\mathbf{d})$ ,  $E_{1,3}$  and  $E_{1,4}$ . Thus  $\mathbb{E}(\mathfrak{u}_J)_{X(\mathbf{d})} = \{\mathbb{k}X(\mathbf{d}) \oplus \mathbb{k}E_{1,3} \oplus \mathbb{k}E_{1,4}\}$  is irreducible and  $\text{srk}(\mathfrak{u}_J) = 3$ .

**Definition 4.5.4.** Suppose that  $\mathbf{d}$  is a dimension vector. For any  $k \in \mathbb{N}$ , a  $k$ -chain (or a chain of length  $k$ ) is a sequence of  $k$  connected lines in  $L_h(\mathbf{d})$ . A labeled  $k$ -chain is a  $k$ -chain in the labeled horizontal line diagram, which will be denoted by  $L_h^k(\mathbf{d} : i - j)$  where  $i$  (resp.  $j$ ) is the starting (resp. ending) point of the labeled  $k$ -chain. Now we are to define a map  $\nu$  from the set of labeled  $k$ -chains where  $k$  varies to the set of root vectors

$$\begin{aligned} \nu : \{L_h^k(\mathbf{d} : i - j)\}_{k \in \mathbb{N}} &\longrightarrow \{E_{i,j}\}_{i,j} \\ L_h^k(\mathbf{d} : i - j) &\longmapsto \nu(L_h^k(\mathbf{d} : i - j)) \end{aligned}$$

by setting  $\nu(L_h^k(\mathbf{d} : i_1 - i_{k+1})) = E_{i_1, i_{k+1}}$  if  $L_h^k(\mathbf{d} : i_1 - i_{k+1})$  is of the following form

$$i_1 \text{ --- } i_2 \text{ --- } \cdot \quad \cdots \quad \cdot \text{ --- } i_{k+1}$$

For a fixed  $k$ , we now take  $\mathcal{R}_k = \mathbb{k} \sum_{L_h^k(\mathbf{d} : i - j)} \nu(L_h^k(\mathbf{d} : i - j))$  as the one-dimensional vector space generated by the sum of images of all labeled  $k$ -chains in  $L_h(\mathbf{d})$  via the map  $\nu$ .

**Lemma 4.5.4.** *Let  $\mathfrak{p}(\mathbf{d}) \subseteq \mathfrak{sl}_{n+1}(\mathbb{k})$  be given by  $J = \{r\}$ ,  $1 \leq r \leq n$ . Then the centralizer  $\mathfrak{z}(\mathbf{d})$  of the Richardson element  $X(\mathbf{d})$  in the nilradical  $\mathfrak{u}_J$  is spanned by the generators of all  $\mathcal{R}_k$  together with one choice of root vector(s) from the set  $\{E_{1,r+1}, E_{r+1,n+1}\}$  depending on  $r$ .*



- (1) The saturation rank  $\text{srk}(\mathbf{u}_J)$  of the nilradical  $\mathbf{u}_J$  is  $n$ .
- (2) Any maximal elementary subalgebra associated to a Richardson element  $X(\mathbf{d})$  is unique or determined by  $(a : b) \in \mathbb{P}^1$ .
- (3) The variety  $\mathbb{E}(\mathbf{u}_J)_{X(\mathbf{d})}$  is irreducible of dimension  $\leq 1$ .

*Remark.* In the following, we only consider the case of  $1 < r < n$  with  $n \geq 3$ . When  $r = 1$  or  $r = n$  together with  $n \geq 2$ , their proofs are analogous and will be omitted. But the elementary subalgebra we find for the Richardson element  $X(\mathbf{d})$  is  $\mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{n-1} \oplus \mathbb{k}E_{2,n+1}$  if  $r = 1$ , and is  $\mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{n-1} \oplus \mathbb{k}E_{1,n+1}$  if  $r = n$ .

*Proof.* We prove the following statement by induction: any elementary subalgebra associated to a Richardson element in  $\mathbf{u}_J$  of  $\mathfrak{sl}_{n+1}(\mathbb{k})$  has dimension at most  $n$  and any subalgebra of this dimension is of the form  $\mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{n-1} \oplus \mathbb{k}(aE_{1,r+1} + bE_{r+1,n+1})$  parametrized by points  $(a : b) \in \mathbb{P}^1$ . This ultimately implies claims (1),(2) and (3).

The statement is clear for  $n=3$ . Assume that it is proved for  $n - 1$ . Let  $\mathfrak{e}_1$  be an elementary subalgebra containing a Richardson element  $X(\mathbf{d}) \in \mathbf{u}_J$  of  $\mathfrak{sl}_{n+1}(\mathbb{k})$ . Denote by  $\mathfrak{e}_2 = \bigoplus_{i=1}^n \mathbb{k}E_{i,n+1}$  an elementary subalgebra of  $\mathbf{u}_J$ , it is a Lie ideal in  $\mathbf{u}_J$  since  $[\mathbf{u}_J, \mathfrak{e}_2] \subseteq \mathfrak{e}_2$ .

We consider the following extension

$$(0) \longrightarrow \mathfrak{e}_2 \longrightarrow \mathbf{u}_J \xrightarrow{\text{pr}} \mathbf{u}_J/\mathfrak{e}_2 \longrightarrow (0)$$

Notice that  $\mathbf{u}_J/\mathfrak{e}_2 \simeq \mathbf{u}'_J$  is the nilradical of  $\mathfrak{p}'_J$ , where  $\mathfrak{p}'_J$  is the standard parabolic subalgebra of  $\mathfrak{sl}_n(\mathbb{k})$  given by  $J = \{r\}$ . The corresponding dimension vector of  $\mathfrak{p}'_J$  is then  $\mathbf{d}' = (\underbrace{1, \dots, 1}_{r-1}, 2, \underbrace{1, \dots, 1}_{n-r-1})$ . According to this we have  $\text{pr}(X(\mathbf{d})) = X(\mathbf{d}')$ , which is a Richardson element of  $\mathbf{u}'_J$ . By our induction hypothesis, the dimension of  $\text{pr}(\mathfrak{e}_1)$  is at most  $n - 1$ , and from this we will show that  $\mathfrak{e}_1$  has dimension at most  $n$ . We do this by considering two situations. Let  $y \in \mathfrak{e}_1 \cap \mathfrak{e}_2$ . We can write  $y = \sum_{i=1}^n a_i E_{i,n+1}$  since  $y \in \mathfrak{e}_2$ . An observation from the form of  $X(\mathbf{d})$  and the relation  $[y, X(\mathbf{d})] = 0$  implies  $a_i = 0$  for all  $i \neq 1, r + 1$ . If  $\mathbb{k}E_{1,r+1}$  appears as a direct summand in  $\text{pr}(\mathfrak{e}_1)$ , then  $E_{1,r+1} + \mathfrak{z} \in \mathfrak{e}_1$  for some  $\mathfrak{z} \in \mathfrak{e}_2$ . Then the relation  $[y, E_{1,r+1} + \mathfrak{z}] = 0$  implies  $a_{r+1} = 0$ . As a result,  $\mathfrak{e}_1 \cap \mathfrak{e}_2$  is generated by at most one element, i.e.  $E_{1,n+1}$ .

Thus, we have  $\dim \mathfrak{e}_1 \leq (n-1) + 1 = n$ . Alternatively,  $\mathbb{k}E_{1,r+1}$  will not be a direct summand of  $\text{pr}(\mathfrak{e}_1)$ . If  $r = n-1$ , then by the above remark specializing to such  $r$ , we have  $\dim \text{pr}(\mathfrak{e}_1) \leq n-2$ . If  $r < n-1$ , then  $\mathbb{k}(aE_{1,r+1} + bE_{r+1,n} + c\mathfrak{z}')$  can not be a direct summand of  $\mathfrak{e}_1$  for  $(a, b, c) \in \mathbb{k}^{\times 3}$  and  $\mathfrak{z}' \in \mathfrak{e}_2$  because  $\mathbb{k}(aE_{1,r+1} + c\mathfrak{z}')$  is not a direct summand of  $\mathfrak{e}_1$  when  $b = 0$  and  $[aE_{1,r+1} + bE_{r+1,n} + c\mathfrak{z}', X(\mathbf{d})] \neq 0$  when  $b \neq 0$ . This yields that there is no  $\mathbb{k}(aE_{1,r+1} + bE_{r+1,n})$  for  $(a : b) \in \mathbb{P}^1$  as a direct summand of  $\text{pr}(\mathfrak{e}_1)$ , which gives  $\dim \text{pr}(\mathfrak{e}_1) \leq n-2$ . Thus, we have  $\dim \mathfrak{e}_1 \leq (n-2) + 2 = n$ .

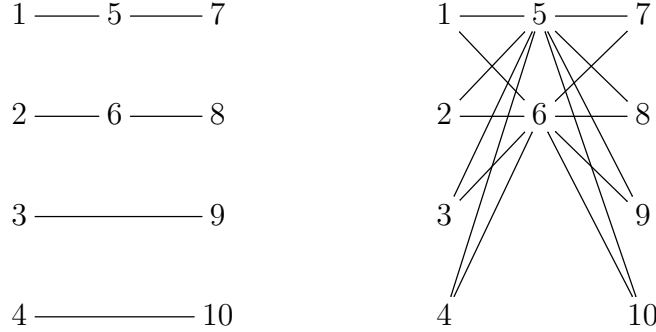
Now we assume that  $\mathfrak{e}_1$  has the maximal dimension  $n$ . We denote by  $\mathcal{R}'_i$  the one-dimensional vector space defined in Definition 4.5.4 for the dimension vector  $\mathbf{d}'$  in case of  $\mathfrak{sl}_n(\mathbb{k})$ . As discussed above, if  $\mathbb{k}E_{1,r+1}$  is a direct summand of  $\text{pr}(\mathfrak{e}_1)$ , the maximal dimension is attained if and only if  $\dim \text{pr}(\mathfrak{e}_1) = n-1$ , and by induction we have  $\text{pr}(\mathfrak{e}_1) = \mathcal{R}'_1 \oplus \cdots \oplus \mathcal{R}'_{n-2} \oplus \mathbb{k}E_{1,r+1}$ . Alternatively, for the dimension to be maximal we need  $\dim \text{pr}(\mathfrak{e}_1) = n-2$ , and by Lemma 4.5.4 we have  $\text{pr}(\mathfrak{e}_1) = \mathcal{R}'_1 \oplus \cdots \oplus \mathcal{R}'_{n-2}$ . Since  $\mathfrak{e}_1 \cap \mathfrak{e}_2 \subseteq \mathfrak{z}(\mathbf{d}) \cap \mathfrak{e}_2$ , the pre-image of  $\mathcal{R}'_i$  (resp.  $\mathbb{k}E_{1,r+1}$ ) is  $\mathcal{R}_i + \mathbb{k}E_{1,n+1} + \mathbb{k}E_{r+1,n+1}$  (resp.  $\mathbb{k}E_{1,r+1} + \mathbb{k}E_{r+1,n+1} + \mathbb{k}E_{1,n+1}$ ) by Lemma 4.5.4 for  $1 \leq i \leq n-2$ . Note that  $E_{1,n+1}$  can commute with any elements in  $\mathfrak{e}_1$  and  $\mathbb{k}E_{1,n+1} = \mathcal{R}_{n-1}$ , whence  $\mathfrak{e}_1 = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{n-1} \oplus \mathbb{k}(E_{1,r+1} + cE_{r+1,n+1})$  ( $c \in \mathbb{k}$ ) if  $\text{pr}(\mathfrak{e}_1) = \mathcal{R}'_1 \oplus \cdots \oplus \mathcal{R}'_{n-2} \oplus \mathbb{k}E_{1,r+1}$  or  $\mathfrak{e}_1 = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{n-1} \oplus \mathbb{k}E_{r+1,n+1}$  if  $\text{pr}(\mathfrak{e}_1) = \mathcal{R}'_1 \oplus \cdots \oplus \mathcal{R}'_{n-2}$ . This finishes the proof of our claims.  $\square$

**Case IV** ( $|J| = n-2$ ). The following are two examples before we start.

**Example 4.5.5.** Let  $J := \{2, \dots, n-1\}$  be the subset of the set  $\{1, \dots, n\}$  and we assign to it a standard parabolic subalgebra  $\mathfrak{p}_J$  with its nilpotent radical  $\mathfrak{u}_J$ . The Lie algebra  $\mathfrak{u}_J$  is a Heisenberg Lie algebra. Hence,  $\text{srk}(\mathfrak{u}_J) = n$  by Example 4.3.1.

**Example 4.5.6.** Let  $\mathfrak{p}_J \subseteq \mathfrak{sl}_{10}(\mathbb{k})$  be given by  $J = \{1, \dots, 9\} \setminus \{4, 6\}$ . Then the dimension vector is  $\mathbf{d} = (4, 2, 4)$ . The labeled horizontal line diagram is drawn as

follows in the first array



and we obtain  $X(\mathbf{d}) = E_{1,5} + E_{5,7} + E_{2,6} + E_{6,8} + E_{3,9} + E_{4,10}$  for the Richardson element in  $\mathfrak{u}_J$ . Note that  $\mathfrak{u}_J = \mathfrak{u}_{\alpha_4} + \mathfrak{u}_{\alpha_6}$  is a sum of two elementary subalgebras with nontrivial intersection  $\epsilon := \mathfrak{u}_{\alpha_4} \cap \mathfrak{u}_{\alpha_6} = \bigoplus kE_{i,j}$  for  $1 \leq i \leq 4$  and  $7 \leq j \leq 10$ , and the elementary matrixs  $E_{i,j} \in \mathfrak{u}_J \setminus \epsilon$  are in 1-1 correspondence to the lines  $i \text{ --- } j$  of the picture in the second array. Denote by  $\alpha_1 = E_{1,5} + E_{5,7}$ ,  $\alpha_2 = E_{1,6} + E_{5,8}$ ,  $\alpha_3 = E_{2,5} + E_{6,7}$  and  $\alpha_4 = E_{2,6} + E_{6,8}$ . Then  $X(\mathbf{d}) = \alpha_1 + \alpha_4 + E_{3,9} + E_{4,10}$ . Notice that  $\epsilon \subseteq \mathfrak{z}(\mathfrak{u}_J)$ , the center of  $\mathfrak{u}_J$ , this gives  $\mathfrak{z}(\mathbf{d}) = \mathfrak{z}_{\mathfrak{u}_J}(\alpha_1 + \alpha_4)$  and  $r_{X(\mathbf{d})} = r_{\alpha_1 + \alpha_4}$ . Thus the determination of the saturation rank  $\text{srk}(\mathfrak{u}_J)$  passes from the local saturation rank of  $X(\mathbf{d})$  to that of  $\alpha_1 + \alpha_4$ . From direct computation, it follows that  $\mathfrak{z}_{\mathfrak{u}_J}(\alpha_1 + \alpha_4)$  is generated by  $\alpha_i (1 \leq i \leq 4)$  together with the basis element of  $\epsilon$ . Let  $m = \sum_{i=1}^4 \xi_i \alpha_i$  and  $n = \sum_{j=1}^4 \ell_j \alpha_j$  with the property that both are linearly independent with  $\alpha_1 + \alpha_4$ . Now it suffices to determine the maximal elementary subalgebras containing  $\alpha_1 + \alpha_4$  and we do this by proving that the condition  $[m, n] = 0$  will imply that the vector space  $\text{Span}_{\mathbb{k}} \{m, n, \alpha_1 + \alpha_4\}$  is two-dimensional. It is easily seen that the two 4-tuples  $\xi^T := (k_1, k_2, k_3, k_4)$  and  $\ell^T := (\ell_1, \ell_2, \ell_3, \ell_4)$  satisfy the following system of equations

$$\begin{cases} (k_1 - k_4)\ell_2 = k_2(\ell_1 - \ell_4) \\ (k_1 - k_4)\ell_3 = k_3(\ell_1 - \ell_4) \\ k_2\ell_3 = k_3\ell_2 \end{cases}$$

according to  $[m, n] = 0$ . Fixing  $m$ , then  $\xi$  and  $\ell$  are the solutions of  $Ax = 0$  with coefficient matrix

$$A := \begin{pmatrix} k_2 & k_4 - k_1 & 0 & -k_2 \\ k_3 & 0 & k_4 - k_1 & -k_3 \\ 0 & k_3 & -k_2 & 0 \end{pmatrix}$$



In the following, we list all the possibilities that could happen to clarify our claim.

We write  $\alpha_1 + \alpha_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cdot (1, 0, 0, 1)^T$  for convenience.

◇ Case 1:  $k_1 = k_4, k_2k_3 = 0$ .

Since  $\xi \neq 0$ , we may assume that  $k_2 = 0 \neq k_3$  without loss of generality. Then  $\text{rk}(A) = 2$  and we get the linearly independent solutions  $\zeta = (1, 0, 0, 1)^T, \eta = (0, 0, 1, 0)^T$  of equation  $Ax = 0$ . As a result,  $\xi = k_1\zeta + k_3\eta, \ell = a\zeta + b\eta (b \neq 0)$  and  $(1, 0, 0, 1)^T, \xi, \ell$  are linearly dependent.

◇ Case 2:  $k_1 = k_4, k_2k_3 \neq 0$ .

In this case,  $\text{rk}(A) = 2$  and the linearly independent solutions are  $\zeta = (1, 0, 0, 1)^T, \eta = (0, k_2, k_3, 0)^T$ . Hence,  $\xi = k_1\zeta + \eta, \ell = a\zeta + b\eta (b \neq 0)$  and  $(1, 0, 0, 1)^T, \xi, \ell$  are linearly dependent.

◇ Case 3:  $k_1 \neq k_4, k_2 = 0$  (or  $k_3 = 0$  or  $k_2k_3 \neq 0$ ).

In this situation,  $\text{rk}(A) = 3$  and there is only one linearly independent solution, say  $\xi$ , thus  $\ell = a\xi$  for some  $a \neq 0$ . This gives  $(1, 0, 0, 1)^T, \xi, \ell$  are linearly independent.

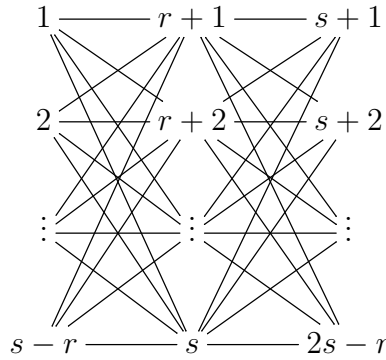
◇ Case 4:  $k_1 \neq k_4, k_2 = 0 = k_3$ .

Note that  $\text{rk}(A) = 2$  and the linearly independent solutions are  $\zeta = (1, 0, 0, 0)^T, \eta = (0, 0, 0, 1)^T$ . Thus,  $\xi = k_1\zeta + k_4\eta, \ell = a\zeta + b\eta (a \neq b)$ , this also implies that  $(1, 0, 0, 1)^T, \xi, \ell$  are linearly dependent.

Therefore, the maximal elementary subalgebra associated to  $\alpha_1 + \alpha_4$  has the form  $\mathbb{k}(\alpha_1 + \alpha_4) \oplus \mathbb{k}m \oplus \epsilon$  of dimension 18, this implies  $\text{srk}(\mathbf{u}_J) = r_{\alpha_1 + \alpha_4} = 18$ .

**Theorem 4.5.6.** *Suppose  $\mathbf{p}_J$  is given as above with  $J = \{1, \dots, n\} \setminus \{r, s\}, r < s$ . Then  $\text{srk}(\mathbf{u}_J) \geq (n + 1 - s)r + 1$ . Especially if  $s - r = \min \{r, s - r, n + 1 - s\}$ , then  $\text{srk}(\mathbf{u}_J) \geq (n - s)r + s$ .*

*Proof.* For given  $J$ , the corresponding dimension vector is  $\mathbf{d} = (r, s - r, n + 1 - s)$ , as well as the nilradical  $\mathbf{u}_J = \mathbf{u}_{\alpha_r} + \mathbf{u}_{\alpha_s}$  is the sum of two nilradicals of maximal standard parabolic subalgebras  $\mathfrak{p}_{\alpha_r}$  and  $\mathfrak{p}_{\alpha_s}$  respectively. We set  $\epsilon := \mathbf{u}_{\alpha_r} \cap \mathbf{u}_{\alpha_s}$ , then  $\epsilon$  is an elementary subalgebra with  $\epsilon \in \mathfrak{z}(\mathbf{u}_J)$ . Note that  $X(\mathbf{d}) \notin \epsilon$ , thus there is an elementary subalgebra  $\mathbb{k}X(\mathbf{d}) \oplus \epsilon$  that contains  $X(\mathbf{d})$ . This gives  $\text{srk}(\mathbf{u}_J) \geq (n + 1 - s)r + 1$ .



For the special case that  $s - r = \min \{r, s - r, n + 1 - s\}$ , we have a more explicit computation. Let  $X(\mathbf{d})' = \sum_{i=1}^{s-r} (E_{i,r+i} + E_{r+i,s+i})$ , then from the labeled horizontal line diagram we obtain  $X(\mathbf{d}) = X(\mathbf{d})'$  or  $X(\mathbf{d}) = X(\mathbf{d})' + \mathfrak{z}$  for  $\mathfrak{z} \in \mathfrak{e} \setminus \{0\}$ . Since  $\mathfrak{z}$  is a central nilpotent element, this property gives rise to  $\mathfrak{z}(\mathbf{d}) = \mathfrak{z}_{\mathbf{u}_J}(X(\mathbf{d})')$  and further  $r_{X(\mathbf{d})} = r_{X(\mathbf{d})'}$ . Denote by  $\alpha_{ij} := E_{i,r+j} + E_{r+i,s+j}$  for  $1 \leq i, j \leq s - r$  read off from the above diagram. Then  $X(\mathbf{d})' = \alpha_{11} + \cdots + \alpha_{s-r(s-r)}$  and direct computation shows that the centralizer  $\mathfrak{z}_{\mathbf{u}_J}(X(\mathbf{d})')$  is generated by all  $\alpha_{ij}$  and basis elements of  $\mathfrak{e}$ . Let  $\mathbb{V} = \text{Span}_{\mathbb{k}} \{\alpha_{ij} \mid 1 \leq i, j \leq s - r\}$  and  $\mathfrak{e}' \subseteq \mathbb{V}$  be the maximal elementary subalgebra containing  $X(\mathbf{d})'$ . The elements of set  $\{\alpha_{ii} \mid 1 \leq i \leq s - r\} \subseteq \mathbb{V}$  are linearly independent and pairwise commuting, this implies that  $\dim \mathfrak{e}' \geq s - r$ . Consequently,  $\text{srk}(\mathbf{u}_J) = r_{X(\mathbf{d})} = \dim \mathfrak{e}' + \dim \mathfrak{e} \geq (s - r) + (n + 1 - s)r = (n - s)r + s$ .  $\square$

## 4.6 Infinitesimal group schemes: height $\leq 2$

Suppose that  $\mathcal{G}$  is an infinitesimal group scheme of height  $\leq 2$ . Take  $\mathcal{G} = \text{SL}_{n(2)} \equiv \text{Ker}\{F^2 : \text{SL}_n \longrightarrow \text{SL}_n\}$  the second Frobenius kernel of  $\text{SL}_n$ , where the geometric Frobenius  $F : \text{SL}_n(R) \rightarrow \text{SL}_n(R)$  is defined by raising each matrix entry to the  $p^{\text{th}}$  power. Let  $\mathcal{N} = \mathcal{N}(\mathfrak{sl}_n(\mathbb{k}))$  be the nilpotent variety of  $\mathfrak{sl}_n(\mathbb{k})$ , and  $\mathfrak{e}^{\text{nil}}(\mathfrak{sl}_n(\mathbb{k})) := \{(x, y) \in \mathcal{N} \times \mathcal{N} \mid [x, y] = 0\}$  be the nilpotent commuting variety of  $\mathfrak{sl}_n(\mathbb{k})$ ; see [Pre1]. The variety of 1-parameter subgroup  $V_2(\text{SL}_{n(2)})$  of  $\text{SL}_{n(2)}$  has been detected in terms of 2-tuples of pairwise commuting  $p$ -nilpotent matrices; see [SFB1, Proposition 1.2/Lemma 1.8]. More specifically, if  $p \geq n$ , the element of  $V_2(\text{SL}_{n(2)})$  has uniquely

the form  $\exp_{\underline{\alpha}} : \mathbb{G}_{a(2)} \longrightarrow \mathrm{SL}_{n(2)}$  where  $\alpha = (\alpha_0, \alpha_1) \in \mathfrak{C}^{\mathrm{nil}}(\mathfrak{sl}_n(\mathbb{k}))$  by sending for any  $\mathbb{k}$ -algebra  $A$  and  $s \in A$  to  $\exp_{\underline{\alpha}}(s) = \exp(s\alpha_0) \cdot \exp(s^p\alpha_1)$ . Here for any  $p$ -nilpotent matrix  $x \in \mathfrak{sl}_n(A)$ , we set

$$\exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{p-1}}{(p-1)!} \in \mathrm{SL}_n(A).$$

Let  $e \in \mathcal{N}$  be a nilpotent element in  $\mathfrak{sl}_n(\mathbb{k})$ . As in section 3.2, we denote by  $\mathfrak{z}(e) := \mathfrak{z}_{\mathfrak{sl}_n(\mathbb{k})}(e)$  the centralizer of  $e$  in  $\mathfrak{sl}_n(\mathbb{k})$ , by  $\mathfrak{C}(e)$  the Zariski closure of  $\mathrm{SL}_n(\mathbb{k}) \cdot (e, \mathcal{N} \cap \mathfrak{z}(e))$  and by  $\mathcal{O}_e$  the  $\mathrm{SL}_n(\mathbb{k})$ -orbit of  $e$ . Consider the morphism

$$\xi : \mathrm{SL}_n(\mathbb{k}) \times (\mathcal{N} \cap \mathfrak{z}(e)) \longrightarrow \mathfrak{C}(e); \quad \xi(g, x) = (\mathrm{Ad}(g).e, \mathrm{Ad}(g).x)$$

which is dominant, the canonical embedding

$$\iota : \mathcal{N} \cap \mathfrak{z}(e) \hookrightarrow \mathrm{SL}_n(\mathbb{k}) \times (\mathcal{N} \cap \mathfrak{z}(e))$$

which maps  $x$  to  $\iota(x) = (1, x)$ , and their composition  $\xi \circ \iota : \mathcal{N} \cap \mathfrak{z}(e) \longrightarrow \mathfrak{C}(e)$ . Suppose in what follows  $p \geq n$  and  $e$  is a regular nilpotent element of  $\mathfrak{sl}_n(\mathbb{k})$ . Then  $\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}}$  is an open subset of  $\mathfrak{C}(e)$  according to Theorem 4.2.5, [Pre1, Theorem 3.7] and [Jan2, Lemma 4.1]. Therefore,  $(\xi \circ \iota)^{-1}(\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}})$  is open dense in  $\mathcal{N} \cap \mathfrak{z}(e)$ ; see [Pre1, 3.1] for the irreducibility of  $\mathcal{N} \cap \mathfrak{z}(e)$ .

**Lemma 4.6.1.**  $(\xi \circ \iota)^{-1}(\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}})$  is not empty.

*Proof.* Since  $\xi$  is a dominant morphism of irreducible affine varieties, the image  $\mathrm{SL}_{n(2)}(\mathbb{k}) \cdot (e, \mathcal{N} \cap \mathfrak{z}(e))$  contains a non-empty open subset  $U$  of  $\mathfrak{C}(e)$ . Then the non-trivial intersection  $\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}} \cap U$  implies that  $\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}}$  contains an element  $g' \cdot (e, x')$  for some  $g' \in \mathrm{SL}_{n(2)}(\mathbb{k})$  and  $x' \in \mathcal{N} \cap \mathfrak{z}(e)$ . Notice that  $\exp_{g' \cdot (e, x')} = g' \cdot \exp_{(e, x')}$ , Remark 4.2.2 ensures that  $(e, x') \in \mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}}$ . As a result,  $x' \in (\xi \circ \iota)^{-1}(\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}})$  and  $(\xi \circ \iota)^{-1}(\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}})$  is not empty, as desired.  $\square$

Since  $\mathcal{O}_e \cap \mathfrak{z}(e)$  is open in  $\mathcal{N} \cap \mathfrak{z}(e)$ , Lemma 4.6.1 ensures  $(\xi \circ \iota)^{-1}(\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}}) \cap (\mathcal{O}_e \cap \mathfrak{z}(e)) \neq \emptyset$ . Now it is our task to investigate  $\mathcal{O}_e \cap \mathfrak{z}(e)$ . It is known that  $\mathfrak{z}(e) = \mathrm{Span}_{\mathbb{k}} \{e, e^2, \dots, e^{n-1}\}$ , and a nilpotent element  $\sum_i a_i e^i$  can be conjugate to  $e$  if and only if  $a_1 \neq 0$ . We pick  $e_0 \in (\mathcal{O}_e \cap \mathfrak{z}(e)) \cap (\xi \circ \iota)^{-1}(\mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}})$ , then  $(e, e_0) = (\xi \circ \iota)(e_0) \in \mathcal{O}_{rmin}^{\mathrm{SL}_{n(2)}}$ . It follows that  $\mathrm{srk}(\mathrm{SL}_{n(2)}) = r_{(e, e_0)}^{\mathrm{SL}_{n(2)}}$ .

**Theorem 4.6.2.** *Keep the notations for  $\mathcal{G}, e$  and the assumption for  $p$  in this section. Then we have  $\mathrm{srk}(\mathrm{SL}_{n(2)}) \geq 2(n-1)$ .*

*Proof.* Let  $a = (a_1, \dots, a_{n-1}) \in \mathbb{G}_a^{\times(n-1)}$  be a  $(n-1)$ -tuple and  $e_a = \sum_{i=1}^{n-1} a_i e^i$ . We define a closed subgroup of  $\mathrm{SL}_n$ , that is  $U_e := \langle 1 + e_a \mid a \in \mathbb{G}_a^{\times(n-1)} \rangle$  generated by  $1 + e_a$  when  $a$  ranges over all elements in  $\mathbb{G}_a^{\times(n-1)}$ . Additionally  $U_e$  is an abelian unipotent subgroup of  $\mathrm{SL}_n$ . We denote by  $\mathfrak{u}_e := \mathrm{Lie}(U_e)$  the abelian Lie algebra of  $U_e$  and by  $\phi : U_e \hookrightarrow \mathrm{SL}_n$  the closed embedding with the associated morphism  $d\phi : \mathfrak{u}_e \longrightarrow \mathfrak{sl}_n(\mathbb{k})$  of corresponding restricted Lie algebras. One can easily check that for any  $\mathbb{k}$ -algebra  $A$  and every  $p$ -nilpotent element  $x \in \mathfrak{u}_e \otimes_{\mathbb{k}} A$  the homomorphism  $\exp_{d\phi(x)} : \mathbb{G}_a \otimes_{\mathbb{k}} A \longrightarrow \mathrm{SL}_n \otimes_{\mathbb{k}} A$  factors through  $U_e \otimes_{\mathbb{k}} A$ . Therefore,  $\phi$  is an embedding of exponential type. By [SFB1, Lemma 1.7], this gives  $V_2(U_e) = \{(\alpha_0, \alpha_1) \in \mathfrak{u}_e \times \mathfrak{u}_e\} = \mathfrak{u}_e \times \mathfrak{u}_e$ . We observe that  $(e, e_0) \in V_2(U_e)$ . Then  $\exp_{(e, e_0)} : \mathbb{G}_{a(2)} \longrightarrow \mathrm{SL}_{n(2)}$  factors through  $U_{e(2)}$  the second-Frobenius kernel of  $U_e$ . Take the unique elementary abelian subgroup  $\mathcal{E}_{U_{e(2)}} \subseteq U_{e(2)}$  given by [Far4, Lemma 6.2.1], then the image of map  $\exp_{(e, e_0)}$  will be contained in  $\mathcal{E}_{U_{e(2)}}$ . Notice that  $\mathrm{cx}_{\mathcal{E}_{U_{e(2)}}}(\mathbb{k}) = \mathrm{cx}_{U_{e(2)}}(\mathbb{k}) = \dim V_2(U_e) = 2 \dim \mathfrak{u}_e = 2(n-1)$ , this gives  $\mathrm{srk}(\mathrm{SL}_{n(2)}) = r_{(e, e_0)}^{\mathrm{SL}_{n(2)}} \geq 2(n-1)$ , as desired.  $\square$

**Conjecture 4.6.3.** *Let  $\mathcal{G} = G_{(r)}$  be the  $r$ -th Frobenius kernel of a reduced algebraic  $\mathbb{k}$ -group scheme  $G$ . Suppose that  $G(\mathbb{k})$  is reductive, standard and  $p \geq h(G(\mathbb{k}))$ . Then we have  $\mathrm{srk}(\mathcal{G}) \geq r \cdot \mathrm{rk}_{\mathrm{ss}}(G(\mathbb{k}))$ .*

# Chapter 5

## Maximal subsets of commuting roots

In this chapter, we will determine the set  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$  when  $\Phi$  is of type  $A_n, B_n (n \geq 2), C_n (n \geq 3)$  or  $D_n (n \geq 4)$ . It is known that, see Appendix A.1 and A.3 of [PS], all maximal subsets of commuting roots in  $\Phi$  for type  $A_{n-1}$  and  $C_n$  can be described via subsets of  $\{1, \dots, n\}$ . While for types  $B_n$  and  $D_n$ , they are unknown. As a result, more effort is needed for the set  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$  when  $\Phi = B_n, D_n$ . However, by the relation of root systems between  $B_n$  and  $D_n$  we only need to devote to computing it for  $D_n$ . This is to be done by induction on  $n$ .

**Notation 5.0.1.** Let  $\mathbb{Z}$  be the set of integers equipped with an ordering  $\leq$ , the smaller than or equal relation. Given two subsets of  $\mathbb{Z}$ , say  $A$  and  $B$ ,  $A < B$  denotes that any element in  $A$  is smaller than all the elements in  $B$ .

**Notation 5.0.2.** If  $A, B$  are two subsets of  $\Phi$ , then we use the symbol  $[A, B] = 0$  to indicate that any root in  $A$  commutes with roots in  $B$  and vice versa.

## 5.1 Type $A_n$

Suppose that  $G$  is of type  $A_n$ . The root of  $A_n$  is an integer vector in  $\mathbb{R}^{n+1}$  of length  $\sqrt{2}$  for which the coordinates sum to 0. We denote by

$$\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j, 1 \leq i, j \leq n+1\}$$

the corresponding set of roots, and by  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  the base of  $\Phi$  where  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ . There is a bijection  $\phi$  from the set of non-trivial proper subsets of  $\{1, \dots, n+1\}$  to the set of maximal subsets of commuting roots of  $\Phi$  by sending  $J$  to  $\phi(J) := \{\epsilon_i - \epsilon_j \mid i \in J, j \notin J\}$ , and the condition  $J < \{1, \dots, n+1\} \setminus J$  on  $J$  gives rise to a maximal subset of commuting roots of  $\Phi^+$ ; see [PS, A.1]. Let  $M(A) \in \text{Max}(\Phi_{r_{\max}}^+)$ . Note that the maximal dimension  $r_{\max}$  is  $(m+1)^2$  (resp.  $m(m+1)$ ) when  $n = 2m+1$  (resp.  $n = 2m$ ). If  $M(A)$  is still maximal in  $\Phi$ , then  $M(A) = \phi(J)$  for certain  $J$ . By letting  $|M(A)| = |\phi(J)| = |J|(n+1 - |J|)$  equal  $(m+1)^2 - 1$  when  $n = 2m+1$  and equal  $m(m+1) - 1$  when  $n = 2m$ , we get  $|J| = m, m+2$  for  $n = 2m+1$ , and there is no solution for  $n = 2m$ . Continuing the consideration, if  $|J| = m$  or  $m+2$  then  $M(A)$  has to be  $\Phi_m^{\text{rad}}$  or  $\Phi_{m+2}^{\text{rad}}$  respectively. Alternatively,  $M(A)$  should be contained in an element of  $\text{Max}(\Phi_{r_{\max}})$ . Inspecting the bijective map  $\phi$  especially for  $|\phi(J)| = r_{\max}$ , one gets that  $M(A)$  equals  $\Phi_{m+1, m+2}^{\text{odd}}$  (resp.  $\Phi_{m+1, m+2}^{\text{ev}}$ )  $:= \phi(J) \cap \Phi^+$  for  $J = \{1, \dots, m, m+2\}$  when  $n = 2m+1$  (resp.  $n = 2m$ ), and equals  $\Phi_{m, m+1}^{\text{ev}} := \phi(J) \cap \Phi^+$  for  $J = \{1, \dots, m-1, m+1\}$  when  $n = 2m$ .

## 5.2 Type $C_n$

Suppose that  $G$  is of type  $C_n$ . It is well known that for this type the set of positive roots  $\Phi^+$  consists of elements  $\epsilon_i - \epsilon_j$  for  $1 \leq i < j \leq n$  together with  $\epsilon_i + \epsilon_j$  for  $1 \leq i \leq j \leq n$ , and  $\Phi = \Phi^+ \cup -\Phi^+$ . Setting  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  when  $1 \leq i \leq n-1$  and consecutively  $\alpha_n = 2\epsilon_n$ , then  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  is a base of  $\Phi$ . A maximal subset of commuting roots in  $\Phi$  arises uniquely from a subset of  $\{1, \dots, n\}$ , which is given by a bijective map  $\phi$  sending  $J \subset \{1, \dots, n\}$  to  $\phi(J) := \{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j, -\epsilon_j - \epsilon_{j'} \mid i, i' \in I \text{ and } j, j' \in J\}$ ; see [PS, A.3]. Notice that  $\phi(J) \subset \Phi^+$  if and only if  $J = \{1, \dots, n\}$ , which gives rise to a unique element of  $\text{Max}(\Phi_{r_{\max}}^+)$  of order  $\frac{1}{2}n(n+1)$ , that is  $\Phi_n^{\text{rad}}$ ; see Table 3.1 for details. This fact

subsequently implies that  $M(C)$  as an element of  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$  cannot be maximal in  $\Phi$ . Let  $|\phi(J)| = \frac{1}{2}n(n+1)$ , then  $J$  can be arbitrary. Let  $|\phi(J) \cap \Phi^+| = \frac{1}{2}n(n+1) - 1$ , then  $|J| = n - 1$  and  $J = \{1, \dots, n - 1\}$  precisely. At this point, one can get  $M(C) = \phi(J) \cap \Phi^+ = \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j < n\} \cup \{\epsilon_r - \epsilon_n \mid 1 \leq r < n\}$  for  $J = \{1, \dots, n - 1\}$ , which will be denoted by  $\Phi_{n-1,n}^C$ .

### 5.3 Type $D_n$

Suppose that  $G$  is of type  $D_n (n \geq 4)$ . Let  $\Phi^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}$  be the set of positive roots of type  $D_n$ . We define  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  for  $1 \leq i \leq n - 1$  together with  $\alpha_n = \epsilon_{n-1} + \epsilon_n$ , then  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  is a base of  $\Phi$ . Before our determination of elements in  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$ , we first investigate the subsets of commuting roots of  $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ . Recall the definition for  $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ , it is a subset consisting of roots  $\{\epsilon_1 \pm \epsilon_i, \epsilon_2 \pm \epsilon_j \mid 2 \leq i \leq n, 3 \leq j \leq n\}$ . Let  $\mathcal{R} \subset \Phi_{\alpha_1, \alpha_2}^{\text{rad}}$  be a subset of commuting roots. If  $\epsilon_1 - \epsilon_2 \in \mathcal{R}$ , then  $\epsilon_2 \pm \epsilon_j \notin \mathcal{R}$  for  $3 \leq j \leq n$ , which implies the inclusion  $\mathcal{R} \subset \Phi_1^{\text{rad}} = \{\epsilon_1 \pm \epsilon_i\}$  for  $2 \leq i \leq n$ . Alternatively,  $\epsilon_1 - \epsilon_2 \notin \mathcal{R}$ . By denoting  $\mathcal{S}_a \subset \{\epsilon_1 - \epsilon_r, \epsilon_2 + \epsilon_r \mid 3 \leq r \leq n\}$  a maximal subset having the property  $\epsilon_1 - \epsilon_r \in \mathcal{S}_a$  if and only if  $\epsilon_2 + \epsilon_r \notin \mathcal{S}_a$  and  $\mathcal{S}_b \subset \{\epsilon_1 + \epsilon_r, \epsilon_2 - \epsilon_r \mid 3 \leq r \leq n\}$  a maximal subset with the property  $\epsilon_1 + \epsilon_r \in \mathcal{S}_b$  if and only if  $\epsilon_2 - \epsilon_r \notin \mathcal{S}_b$ , we have  $\mathcal{R} \subset \mathcal{S}_{ab} := \{\epsilon_1 + \epsilon_2\} \cup \mathcal{S}_a \cup \mathcal{S}_b$ .

Suppose  $n = 4$ . Let  $\Psi^+ = \Phi^+ \setminus \Phi_{\alpha_1, \alpha_2}^{\text{rad}} = \{\epsilon_3 \pm \epsilon_4\}$ . From Table 3.1, we have seen  $r_{\text{smax}} = r_{\text{max}} - 1 = 5$  for type  $D_4$ . Let  $M(D) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$ , then  $M(D) = M_a \cup M_b$ , where  $M_a \subset \Psi^+$  and  $M_b \subset \Phi_{\alpha_1, \alpha_2}^{\text{rad}}$  with  $|M_a| + |M_b| = 5$ . If  $M_a = \emptyset$ , then  $|M_b| = 5$ , and we have  $M_b \subsetneq \Phi_1^{\text{rad}}$  or  $M_b = \mathcal{S}_{ab}$ . The maximality of  $M(D)$  ensures that  $M_b$  must be  $\mathcal{S}_{ab}$ . Let  $\mathcal{S}_a^1 = \{\epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4\}$ ,  $\mathcal{S}_a^2 = \{\epsilon_1 - \epsilon_3, \epsilon_2 + \epsilon_4\}$ ,  $\mathcal{S}_a^3 = \{\epsilon_2 + \epsilon_3, \epsilon_2 + \epsilon_4\}$  and  $\mathcal{S}_a^4 = \{\epsilon_2 + \epsilon_3, \epsilon_1 - \epsilon_4\}$  be the four forms of  $\mathcal{S}_a$ . Let  $\mathcal{S}_b^1 = \{\epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4\}$ ,  $\mathcal{S}_b^2 = \{\epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_4\}$ ,  $\mathcal{S}_b^3 = \{\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4\}$  and  $\mathcal{S}_b^4 = \{\epsilon_2 - \epsilon_3, \epsilon_1 + \epsilon_4\}$  be the four forms of  $\mathcal{S}_b$ . Then there are a wide variety of possibilities of  $\mathcal{S}_{ab}$ , which is given by  $\mathcal{S}_{ab}^{ij} = \{\epsilon_1 + \epsilon_2\} \cup \mathcal{S}_a^i \cup \mathcal{S}_b^j$  for  $1 \leq i, j \leq 4$ , and that is our  $M(D)$  in this case. If  $|M_a| = 1$ , we assume that  $M_a = \{\epsilon_3 - \epsilon_4\}$  without losing any generality. By the equation  $|M_b| = 4$  and the maximality of  $M(D)$ , we have  $M_b = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_4\} \subsetneq \Phi_1^{\text{rad}}$ . Similar arguments give rise to  $M_b = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4\} \subsetneq \Phi_1^{\text{rad}}$  if  $M_a = \{\epsilon_3 + \epsilon_4\}$ .

If  $|M_a| = 2$ , then  $M_b = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3\} \subsetneq \Phi_1^{\text{rad}}$ , or  $M_b = \{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3\} \subsetneq \mathcal{S}_{ab}$  by the same arguments. Summarizing here, there are 20 possibilities of  $M(D)$ , they are  $\mathcal{S}_{ab}^{ij}$  for  $1 \leq i, j \leq 4$ ,  $\mathcal{S}_1 := \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_3 - \epsilon_4\}$ ,  $\mathcal{S}_2 := \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4, \epsilon_3 + \epsilon_4\}$ ,  $\mathcal{S}_3 := \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_3 \pm \epsilon_4\}$ , and  $\mathcal{S}_4 := \{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3, \epsilon_3 \pm \epsilon_4\}$ .

Suppose  $n = 5$ . Let  $M(D) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  with  $r_{\text{smax}} = r_{\text{max}} - 1 = 9$ . Keep the notations for  $\Psi^+$ ,  $M_a$ ,  $M_b$  and  $M(D) = M_a \cup M_b$  as above. Accordingly, we have  $\Psi^+ = \{\epsilon_3 \pm \epsilon_4, \epsilon_3 \pm \epsilon_5, \epsilon_4 \pm \epsilon_5\}$ . Notice that  $\epsilon_3 + \epsilon_4$  is the longest root in  $\Psi^+$ . If  $\epsilon_3 - \epsilon_4 \in M_a$ , then  $\epsilon_4 \pm \epsilon_5 \notin M_a$  which gives  $M_a \subset M_a^0 := \{\epsilon_3 \pm \epsilon_4, \epsilon_3 \pm \epsilon_5\}$ . Alternatively, if  $\epsilon_3 - \epsilon_4 \notin M_a$ , then one can easily check that  $M_a \subset M_a^1 := \{\epsilon_3 + \epsilon_4, \epsilon_3 + \epsilon_5, \epsilon_4 + \epsilon_5\}$ , or  $M_a \subset M_a^2 := \{\epsilon_3 + \epsilon_4, \epsilon_3 - \epsilon_5, \epsilon_4 - \epsilon_5\}$ , or  $M_a \subset M_a^3 := \{\epsilon_3 + \epsilon_4, \epsilon_3 \pm \epsilon_5\}$ , or  $M_a \subset M_a^4 := \{\epsilon_3 + \epsilon_4, \epsilon_4 \pm \epsilon_5\}$ . From the above determination for  $M_a$ , we have  $|M_a| \leq 4$ . Our goal is to give a concrete analysis for  $M(D)$  in several steps. If  $|M_a| = 4$ , then  $M_a = M_a^0 = \{\epsilon_3 \pm \epsilon_4, \epsilon_3 \pm \epsilon_5\}$ . By  $[M_a, M_b] = 0$ , it will imply  $|M_b| \leq 3$  whenever  $M_b \subset \Phi_1^{\text{rad}}$  or  $M_b \subset \mathcal{S}_{ab}$ . But  $|M_b| = 5$  if  $M(D)$  exists, so there is no  $M(D)$  in this case. If  $|M_a| = 3$ , then  $|M_b| = 6$  if  $M(D)$  exists. In this case, we first assume that  $M_b \subset \Phi_1^{\text{rad}}$ . Then  $\epsilon_1 \pm \epsilon_i$  exist for at least one choice for  $i$  from set  $\{3, 4, 5\}$ , this implies  $|M_a| \leq 2$  by  $[M_a, M_b] = 0$ , a contradiction. In the following, we assume  $M_b \subset \mathcal{S}_{ab}$ , then  $|M_b \cap (\mathcal{S}_a \cup \mathcal{S}_b)| \geq 5$ . We list several possibilities to get a contradiction in this case: (a)  $M_a = M_a^1$  or  $M_a^2$ , there is no  $M_b$  with  $|M_b| = 6$  such that  $M(D)$  is maximal; (b)  $M_a = M_a^3$  or  $M_a^4$ , then  $M_b \subset \{\epsilon_i + \epsilon_j\} \cup \{\epsilon_1 + \epsilon_2\}$  where  $1 \leq i \leq 2$  and  $3 \leq j \leq 4$ , thus  $|M_b \cap (\mathcal{S}_a \cup \mathcal{S}_b)| \leq 4$ ; (c)  $M_a \subset M_a^0$ , then  $\epsilon_3 \pm \epsilon_i$  occurs in  $M_a$  for  $i = 4$  or  $i = 5$ , which implies  $|M_b| \leq 5$  by the same reason with (c). At this point, the left case for our discussion is only when  $|M_a| = 2$ , this is because  $\Phi_1^{\text{rad}}$  is maximal in  $\Phi^+$  and of maximal order 8 among subsets of commuting roots in  $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ , so  $\Phi_1^{\text{rad}} \not\subseteq M(D)$ . We now assume  $|M_a| = 2$  and  $M(D)$  exists, then  $|M_b| = 7$ . If  $M_b \subset \Phi_1^{\text{rad}}$ , then  $M_a = \emptyset$  by  $[M_a, M_b] = 0$ , a contradiction. If  $M_b \subset \mathcal{S}_{ab}$ , then  $M_b = \{\epsilon_1 + \epsilon_2\} \cup \{\epsilon_i + \epsilon_j, \epsilon_i - \epsilon_4 \mid i = 1, 2 \text{ and } j = 3, 5\}$  with  $M_a = \{\epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_5\}$ , or  $M_b = \{\epsilon_1 + \epsilon_2\} \cup \{\epsilon_i + \epsilon_3, \epsilon_i - \epsilon_j \mid i = 1, 2 \text{ and } j = 4, 5\}$  with  $M_a = \{\epsilon_3 - \epsilon_4, \epsilon_3 - \epsilon_5\}$ . Summarizing here, by taking  $I_1 = \{1, 2, 3\}$  with  $J_1 = \{4, 5\}$  and  $I_2 = \{1, 2, 3, 5\}$  with  $J_2 = \{4\}$  there are two possibilities for  $M(D)$ , that is  $M(D) = \{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j \mid i \neq i' \in I_s \text{ and } j \in J_s\} \cap \Phi^+$  for  $s \in \{1, 2\}$ .

**Lemma 5.3.1.** *Suppose that  $G$  is of type  $D_n$  with  $n \geq 6$ . Let  $\Psi_{n-2}^{\text{rad}} := \Psi^+ \cap \Phi_n^{\text{rad}}$  and  $\Psi_{n-3}^{\text{rad}} := \Psi^+ \cap \Phi_{n-1}^{\text{rad}}$ , where  $\Psi = \Phi \setminus \pm \Phi_{\alpha_1, \alpha_2}^{\text{rad}}$  is a root system of  $D_{n-2}$ . Let*



$M(D) \in \text{Max}(\Phi_{\frac{n(n-1)}{2}-1}^+)$ . Then there is no decomposition  $M(D) = M_a \cup M_b$  with  $M_a = \Psi_{n-2}^{\text{rad}}, \Psi_{n-3}^{\text{rad}}$  and  $M_b \subset \Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ .

*Proof.* It is known from Table 3.1 that  $\frac{n(n-1)}{2} - 1$  is the submaximal order of  $D_n$ . Because  $\Psi$  is the root system of  $D_{n-2}$  and  $\frac{(n-2)(n-3)}{2}$  is the maximal order of  $D_{n-2}$ ,  $\Psi_{n-2}^{\text{rad}} := \Psi^+ \cap \Phi_n^{\text{rad}}$  and  $\Psi_{n-3}^{\text{rad}} := \Psi^+ \cap \Phi_{n-1}^{\text{rad}}$  are elements of  $\text{Max}(\Psi_{\frac{(n-2)(n-3)}{2}}^+)$ . Recall that  $\Phi_{\alpha_1, \alpha_2}^{\text{rad}} = \{\epsilon_1 \pm \epsilon_i, \epsilon_2 \pm \epsilon_j \mid 2 \leq i \leq n \text{ and } 3 \leq j \leq n\}$ . If  $M_a = \Psi_{n-2}^{\text{rad}}$ , then  $\epsilon_i - \epsilon_j \notin M_b$  for  $i = 1, 2$  and  $3 \leq j \leq n$  by  $[M_a, M_b] = 0$ , this gives rise to  $M_b \subset \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_i\}_{3 \leq i \leq n} \subset \Phi_1^{\text{rad}}$  or  $M_b \subset \{\epsilon_1 + \epsilon_i, \epsilon_2 + \epsilon_j \mid 2 \leq i \leq n \text{ and } 3 \leq j \leq n\}$ . Notice that  $|M(D)| - |M_a| = 2n - 4$ . From an inspection of these two cases it follows that  $|M_b| \leq n < 2n - 4$  or  $M_a \cup M_b \subsetneq \Phi_n^{\text{rad}}$ , from which we deduce that  $M(D)$  can not exist in this case. Alternatively, we assume that  $M_a = \Psi_{n-3}^{\text{rad}}$ . For this case, we get  $M_b \subset \{\epsilon_1 \pm \epsilon_2, \epsilon_1 - \epsilon_n, \epsilon_1 + \epsilon_j\}_{3 \leq j \leq n-1}$  or  $M_b \subset \{\epsilon_1 - \epsilon_n, \epsilon_1 + \epsilon_i\}_{2 \leq i \leq n-1} \cup \{\epsilon_2 - \epsilon_n, \epsilon_2 + \epsilon_j\}_{3 \leq j \leq n-1}$ . Moreover, we have either  $|M_b| \leq n < 2n - 4$  or  $M_a \cup M_b \subsetneq \Phi_{n-1}^{\text{rad}}$ . By the same reason as before there is no  $M(D)$ .  $\square$

Suppose  $n = 6$ . Keep the notations for  $\Psi, M_a$  and  $M_b$ . The maximal order  $r_{\max}$  is 15 for  $D_6$ , and submaximal order  $r_{\text{smax}} = r_{\max} - 1$  is 14.  $M_a$  is a subset of commuting roots of  $\Psi^+$ , thus  $|M_a| \leq 6$ .  $M_b$  is a subset of commuting roots of  $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$  and the maximal order among these subsets is 10, so  $|M_b| \leq 10$ . There are only two candidates left for our determination for  $M(D)$ , say when  $|M_a| = 6$  and  $|M_b| = 8$ , or when  $|M_a| = 5$  and  $|M_b| = 9$  by  $|M_a| \leq 6$  and  $|M_b| \leq 10$ , and  $|M_b| = 10$  cannot occur since it gives  $M_b = \Phi_1^{\text{rad}}$  which is maximal in  $\Phi^+$ . If  $|M_a| = 6$ , then  $M_a = \Psi_3^{\text{rad}}, \Psi_4^{\text{rad}}$  or  $\Psi_1^{\text{rad}}$ . By Lemma 5.3.1, it is only needed to check when  $M_a = \Psi_1^{\text{rad}}$ . Recall that  $\Psi_1^{\text{rad}} = \{\epsilon_3 \pm \epsilon_i\}_{4 \leq i \leq 6}$ , it gives  $M_b \subset \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3\}$  or  $M_b \subset \{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3\}$ , there is no  $M_b$  with  $|M_b| = 8$ . If  $|M_a| = 5$ , then  $|M_b| = 9$  if  $M(D)$  exists. If  $M_b \subset \Phi_1^{\text{rad}}$ , then every element in  $\Psi^+$  cannot commute with all of the elements of  $M_b$ , so  $M(D)$  does not exist. Otherwise, we let  $M_b \subset \mathcal{S}_{ab}$ , then it has to be  $M_b = \mathcal{S}_{ab}$  by their orders. If  $M_a$  is not maximal in  $\Psi^+$ , then  $M_a \subset \Psi_1^{\text{rad}}, \Psi_3^{\text{rad}}$  or  $\Psi_4^{\text{rad}}$ . We discuss these three possibilities to get a contradiction: (a)  $M_a \subset \Psi_1^{\text{rad}}$ , then there exists  $i_0$  where  $4 \leq i_0 \leq 6$  such that  $\epsilon_3 \pm \epsilon_{i_0} \in M_a$ . But  $\epsilon_3 \pm \epsilon_{i_0}$  can not commute with elements of  $\mathcal{S}_{ab}$  from each of the sets  $\{\epsilon_1 + \epsilon_{i_0}, \epsilon_2 - \epsilon_{i_0}\}$  and  $\{\epsilon_1 - \epsilon_{i_0}, \epsilon_2 + \epsilon_{i_0}\}$ ; (b)  $M_a \subset \Psi_3^{\text{rad}}$ , then  $\mathcal{S}_{ab}$  must be  $\{\epsilon_1 + \epsilon_i, \epsilon_2 + \epsilon_i\}_{3 \leq i \leq 5} \cup \{\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_6, \epsilon_2 - \epsilon_6\}$ , but now  $M_a \cup M_b \subsetneq \Phi_5^{\text{rad}}$ ; (c)  $M_a \subset$

$\Psi_4^{\text{rad}}$ , then  $S_{ab}$  must be  $\{\epsilon_1 + \epsilon_i\}_{2 \leq i \leq 6} \cup \{\epsilon_2 + \epsilon_r\}_{3 \leq r \leq 6}$ , but again  $M_a \cup M_b \subsetneq \Phi_6^{\text{rad}}$ . Now the consideration is only left for  $M_a$  being maximal with  $|M_a| = 5$ . Since 5 is the submaximal order for  $D_4$ ,  $M_a \in \text{Max}(\Psi_5^+)$ . There are 20 candidates but only two are suitable:  $M_a = \{\epsilon_3 + \epsilon_4, \epsilon_i - \epsilon_5, \epsilon_i - \epsilon_6\}_{3 \leq i \leq 4}$  or  $M_a = \{\epsilon_3 + \epsilon_4, \epsilon_i - \epsilon_5, \epsilon_i + \epsilon_6\}_{3 \leq i \leq 4}$ . Given by these, their corresponding  $M_b$  are  $\{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_s\}_{1 \leq i \leq 2, 3 \leq r \leq 4, 5 \leq s \leq 6}$  and  $\{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_5, \epsilon_i + \epsilon_6\}_{1 \leq i \leq 2, 3 \leq r \leq 4}$ . Summarizing here, by denoting  $I_1 = \{1, 2, 3, 4\}$  with  $J_1 = \{5, 6\}$  and  $I_2 = \{1, 2, 3, 4, 6\}$  with  $J_2 = \{5\}$  we have  $M(D) = \{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j \mid i \neq i' \in I_s \text{ and } j \in J_s\} \cap \Phi^+$  for  $s \in \{1, 2\}$ .

At this moment, for types  $D_4, D_5$  and  $D_6$  the set  $\text{Max}(\Phi_{r_{\text{max}}}^+)$  has been determined. Let  $J \subset \{1, \dots, n\}$  be a subset. We denote by

$$\phi(J) := \left\{ \epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j, -\epsilon_j - \epsilon_{j'} \mid i \neq i' \in J \text{ and } j \neq j' \notin J \right\}$$

a subset of  $\Phi$ , which is a maximal set of commuting roots (check directly). By denoting  $\Phi_{n-2, n-1}^D := \phi(J) \cap \Phi^+$  when  $J = \{1, \dots, n-2\}$  and  $\Phi_{n-1, n}^D := \phi(J) \cap \Phi^+$  when  $J = \{1, \dots, n-2, n\}$ , the elements of  $\text{Max}(\Phi_{r_{\text{max}}}^+)$  for type  $D_n$  can be written as  $\Phi_{n-2, n-1}^D$  and  $\Phi_{n-1, n}^D$  for  $n \in \{5, 6\}$  respectively. In the following, we use a theorem to reveal the general phenomenon for  $D_n$  whenever  $n \geq 5$ :

**Theorem 5.3.2.** *Suppose that  $G$  is of type  $D_n$  ( $n \geq 5$ ). Let  $M(D) \in \text{Max}(\Phi_{r_{\text{max}}}^+)$ , then  $M(D)$  is either of the form  $\Phi_{n-2, n-1}^D$  or  $\Phi_{n-1, n}^D$ .*

*Proof.* Keep the notation for  $\Psi, M_a$  and  $M_b$  again. We prove the above statement by induction. It is known that  $\Psi$  is the root system of  $D_{n-2}$  when  $\Phi$  is the root system of  $D_n$ . The statement is clear for  $n = 5$  and  $n = 6$ . Assume it is proved for  $n-2$  and  $n-1$ . We prove the statement is true for  $n$  and  $n+1$ . Let  $M(D) \in \text{Max}(\Phi_{r_{\text{max}}}^+)$  and  $M(D) = M_a \cup M_b$ . Then we get the upper bounds  $|M_a| \leq \frac{(n-2)(n-3)}{2}$ ,  $|M_b| \leq 2n-2$  and the equation  $|M_a| + |M_b| = \frac{n(n-1)}{2} - 1$ . By Lemma 5.3.1,  $|M_a| \neq \frac{(n-2)(n-3)}{2}$ . By the maximality of  $\Phi_1^{\text{rad}}$ ,  $|M_b| \neq 2n-2$  otherwise  $M_b = \Phi_1^{\text{rad}}$  but  $2n-2 < \frac{n(n-1)}{2} - 1$  when  $n \geq 5$ . The only case left for consideration is when  $|M_a| = \frac{(n-2)(n-3)}{2} - 1$  and  $|M_b| = 2n-3$ . Note that  $\frac{(n-2)(n-3)}{2} - 1$  is the submaximal order for  $\Psi$  of  $D_{n-2}$ . If  $M_a$  is maximal in  $\Psi^+$ , then by induction hypothesis  $M_a = \phi(\hat{J}) \cap \Psi^+$  for  $\hat{J} = \{3, \dots, n-2\}$ , or  $\hat{J} = \{3, \dots, n-2, n\}$ . According to this, we get  $M_b \not\subseteq \Phi_1^{\text{rad}}$  and  $M_b = S_{ab}$ . More precisely,  $M_b = \{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_{n-1}, \epsilon_i - \epsilon_n\}_{1 \leq i \leq 2, 3 \leq r \leq n-2}$  when  $\hat{J} = \{3, \dots, n-2\}$ , or  $M_b =$

$\{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_{n-1}, \epsilon_i + \epsilon_n\}_{1 \leq i \leq 2, 3 \leq r \leq n-2}$  when  $\hat{J} = \{3, \dots, n-2, n\}$ . Both of these give rise to  $M(D) = \phi(J) \cap \Phi^+$  where  $J = \hat{J} \cup \{1, 2\}$  (i.e.  $J = \{1, \dots, n-2\}$  or  $J = \{1, \dots, n-2, n\}$ ). Afterwards, let us turn to the case that  $M_a$  is not maximal. Because  $\text{Max}(\Psi_{r_{\max}}^+) = \{\Psi_{n-3}^{\text{rad}}, \Psi_{n-2}^{\text{rad}}\}$ , this gives  $M_a \subset \Psi_{n-3}^{\text{rad}}$  or  $M_a \subset \Psi_{n-2}^{\text{rad}}$ . If  $M_a \subsetneq \Psi_{n-3}^{\text{rad}}$ , then  $M_b = \{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_n\}_{1 \leq i \leq 2, 3 \leq r \leq n-1}$  by  $[M_a, M_b] = 0$ , but at the moment  $M_a \cup M_b \subsetneq \Phi_{n-1}^{\text{rad}}$ , so  $M(D)$  does not exist. If  $M_a \subsetneq \Psi_{n-2}^{\text{rad}}$ , then  $M_b = \{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r\}_{1 \leq i \leq 2, 3 \leq r \leq n}$  by  $[M_a, M_b] = 0$ , but  $M_a \cup M_b \subsetneq \Phi_n^{\text{rad}}$  which is also not maximal in  $\Phi^+$ , so there is no  $M(D)$  again in this case.  $\square$

## 5.4 Type $B_n$

Suppose that  $G$  is of type  $B_n (n \geq 2)$ . The elements of  $\Phi$  are those integer vectors in  $\mathbb{R}^n$  with length  $\sqrt{2}$  or length 1. Specifically,  $\Phi = \{\pm\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n} \cup \{\pm\epsilon_i\}_{1 \leq i \leq n}$ . We denote by  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  when  $1 \leq i \leq n-1$  together with  $\alpha_n = \epsilon_n$ , then  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  is a base of  $\Phi$ . Let  $\mathcal{K} = \{\epsilon_i \mid 1 \leq i \leq n\}$ . Then  $\Psi := \Phi \setminus \pm\mathcal{K}$  is readily seen to be a root system of  $D_n$  when  $n \geq 4$  with simple roots  $\{\alpha_1, \dots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$ . An inspection shows that  $\mathcal{K}$  is a maximal subset of non-commuting roots, it implies that any maximal subset of commuting roots of  $\Phi^+$  is the union a subset of commuting roots of  $\Psi^+$  together with at most one element from  $\mathcal{K}$ .

Suppose that  $n = 2$ . Keep the notations for  $\Psi$  and  $\mathcal{K}$ . We have  $\Psi^+ = \{\epsilon_1 \pm \epsilon_2\}$  and  $\mathcal{K} = \{\epsilon_1, \epsilon_2\}$ . From Table 3.1,  $r_{\text{smax}}(B_2) = r_{\text{max}}(B_2) - 1 = 2 \times 2 - 2 = 2$ . Note that  $\epsilon_1 + \epsilon_2$  is the longest root in  $\Phi^+$ . Let  $M(B) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$ , then  $M(B) := \mathcal{K}_2 = \{\epsilon_1 + \epsilon_2, \epsilon_2\}$ .

Suppose that  $n = 3$ . Keep the notations for  $\Psi$  and  $\mathcal{K}$ . Then  $\Psi^+ = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 \pm \epsilon_3, \epsilon_2 \pm \epsilon_3\}$ , and  $\mathcal{K} = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ . Let  $R \subset \Psi^+$  be a subset of commuting roots. If  $\epsilon_1 - \epsilon_2 \in R$ , then  $R \subset \{\epsilon_1 \pm \epsilon_2, \epsilon_1 \pm \epsilon_3\}$ . If  $\epsilon_1 - \epsilon_2 \notin R$ , then  $R \subset R_1 := \{\epsilon_1 + \epsilon_2, \epsilon_1 \pm \epsilon_3\}$ , or  $R \subset R_2 := \{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3\}$ , or  $R \subset R_3 := \{\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}$ , or  $R \subset R_4 := \{\epsilon_1 + \epsilon_2, \epsilon_2 \pm \epsilon_3\}$ . It is known that  $r_{\text{smax}}(B_3) = r_{\text{max}}(B_3) - 1 = 2 \times 3 - 2 = 4$ . Let  $M(B) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  and  $R_{ir} = R_i \cup \{\epsilon_r\}$ , then  $M(B) = R_{11}, R_{12}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{41}, R_{42}$ .

Suppose that  $n \geq 4$ . Keep the notations for  $\Psi$  and  $\mathcal{K}$ . By the fact that  $r_{\max}(B_n) = r_{\max}(D_n) + 1$  when  $n \geq 4$ ; see this from Table 3.1 we have  $r_{\text{smax}}(B_n) = r_{\max}(D_n) = r_{\text{smax}}(D_n) + 1$ . Let  $M(B) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$ . Since the element of  $\text{Max}(\Psi_{r_{\text{smax}}}^+)$  is not maximal in  $\Phi^+$ , this gives  $M(B) = \mathcal{C}(D) \cup \{\epsilon_i\}$  where  $\mathcal{C}(D) \in \text{Com}(\Psi_{r_{\text{smax}}}^+)$  for some  $1 \leq i \leq n$ . Let us first assume that  $n = 4$ . If  $\mathcal{C}(D) = M(D) \in \text{Max}(\Psi_{r_{\text{smax}}}^+)$ , we let  $\mathcal{S}_{ab}^{ijr} := \mathcal{S}_{ab}^{ij} \cup \{\epsilon_r\}$  as well as  $\mathcal{S}_{ir} = \mathcal{S}_i \cup \{\epsilon_r\}$ . Then one can easily check that  $M(B) = \mathcal{S}^{1jr}, \mathcal{S}^{22r}, \mathcal{S}^{23r}, \mathcal{S}^{33r}, \mathcal{S}^{43r}$  for  $1 \leq j \leq 4$  and  $r \in \{1, 2\}$ ;  $\mathcal{S}^{32r}, \mathcal{S}^{41r}, \mathcal{S}^{42r}$  for  $r \in \{1, 2, 3\}$ ;  $\mathcal{S}^{21r}, \mathcal{S}^{24r}, \mathcal{S}^{34r}, \mathcal{S}^{44r}$  for  $r = \{1, 2, 4\}$ ;  $\mathcal{S}^{31r}$  for  $r \in \{1, 2, 3, 4\}$ ;  $\mathcal{S}_{ir}$  for  $1 \leq i \leq 4$  and  $r \in \{1, 3\}$ ;  $\mathcal{S}_{24}$ ;  $\mathcal{S}_{42}$ . Alternatively,  $\mathcal{C}(D) \subsetneq \Psi_1^{\text{rad}}, \Psi_3^{\text{rad}}$  or  $\Psi_4^{\text{rad}}$ , but  $\mathcal{C}(D) \cup \{\epsilon_i\}$  with  $[\mathcal{C}(D), \{\epsilon_i\}] = 0$  will not be maximal. So in this case, there is no  $M(B)$ . Now we assume  $n \geq 5$ . By the same argument as  $n = 4$ , it will be clear that  $M(B) = M(D) \cup \{\epsilon_i\}$  for some  $i$ , depending on  $M(D)$ . To be more precise,  $M(B)$  has the following form:  $\mathcal{J}_t = \Phi_{n-1, n}^D \cup \{\epsilon_t\}$  with  $t = 1, \dots, n-2, n$ , and  $\mathcal{J}_t^* = \Phi_{n-2, n-1}^D \cup \{\epsilon_t\}$  with  $t = 1, 2, \dots, n-2$ .

Summarizing here for types  $A_n(n \geq 2), B_n(n \geq 2), C_n(n \geq 3)$  and  $D_n(n \geq 4)$ , we list the elements of  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$  via Table 5.1:

Type $T$	Restrictions on rank	$\text{Max}(\Phi_{r_{\text{smax}}}^+)$	$ \text{Max}(\Phi_{r_{\text{smax}}}^+) $	$r_{\text{smax}}$
$A_{2n}$	$n \geq 1$	$\Phi_{n,n+1}^{\text{ev}}, \Phi_{n+1,n+2}^{\text{ev}}$	2	$n(n+1) - 1$
$A_{2n+1}$	$n \geq 1$	$\Phi_n^{\text{rad}}, \Phi_{n+2}^{\text{rad}}, \Phi_{n+1,n+2}^{\text{odd}}$	3	$n^2 + 2n$
$B_n$	$n = 2$	$\mathcal{K}_2$	1	2
	$n = 3$	$R_{11}, R_{12},$ $R_{21}, R_{22}, R_{23}$ $R_{31}, R_{32}$ $R_{41}, R_{42}$	9	4
	$n = 4$	too many	51	6
	$n \geq 5$	$\mathcal{I}_t, 1 \leq t \neq n-1 \leq n$ $\mathcal{I}_{t^*}, 1 \leq t < n-2$	$2n-3$	$\frac{1}{2}n(n-1)$
$C_n$	$n \geq 3$	$\Phi_{n-1,n}^C$	1	$\frac{1}{2}n(n+1) - 1$
$D_n$	$n = 4$	$\mathcal{S}_{ab}^{ij}, 1 \leq i, j \leq 4$ $\mathcal{S}_i, 1 \leq i \leq 4$	20	5
	$n \geq 5$	$\Phi_{n-2,n-1}^D, \Phi_{n-1,n}^D$	2	$\frac{1}{2}n(n-1) - 1$

Table 5.1: Maximal subsets of commuting roots of order  $r_{\text{smax}}$ .



# Chapter 6

## Calculation of the variety

### $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$ for $\mathfrak{g} = \text{Lie}(G)$

In this chapter we determine the irreducible components of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$ .  $\mathbb{k}$  is assumed to be an algebraically closed field of prime characteristic  $p > 0$ .

In section 6.1-6.6 we assume that  $G$  is a classical simple algebraic  $\mathbb{k}$ -group and  $p$  is good. By introducing the subset

$$\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}} = \{\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}}); \mathcal{E} \text{ is maximal}\}$$

of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})$  we show that  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}} \subseteq G.\text{Lie}(\text{Max}(\Phi_{r_{\text{smax}}}^+))$ .

In section 6.7 we additionally assume that  $G$  is simply connected and  $\mathfrak{g}$  affords a non-degenerate  $G$ -invariant symmetric bilinear form, i.e.  $G$  is a standard classical simple algebraic group.

**Notation 6.0.1.** Let  $\succeq$  be an ordering on  $\Phi^+$ , and  $A, B$  are two subsets of  $\Phi^+$ . The notation  $A \succ B$  denotes that for any  $a, b$  with  $a \in A$  and  $b \in B$ , we have  $a \succ b$ . Let  $\mathcal{E} \subseteq \mathfrak{u}_{\mathfrak{b}}$  be an elementary subalgebra. The ordering  $\succeq$  on  $\Phi^+$  gives an ordering on the basis elements of  $\mathfrak{u}_{\mathfrak{b}}$ . We always choose the unique basis of  $\mathcal{E}$  which is in reduced echelon form with respect to this ordering and let  $\text{LT}(\mathcal{E})$  be the set of roots  $\beta$  such that the corresponding  $x_{\beta}$  are the leading terms in this reduced basis, [PS].

## 6.1 Type $A_{2m+1}$

Suppose that  $G$  is of type  $A_{2m+1}$ . Let  $\succeq$  be the reverse lexicographic ordering given by  $\alpha_{m+1} \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{2m+1}$ . Let  $\beta$  and  $\gamma$  be two positive roots, written as a linear combination of simple roots. We then say  $\beta \succeq \gamma$  if the coefficient of  $\alpha_{m+1}$  in  $\gamma$  is larger than it in  $\beta$ , or if they are equal and the coefficient of  $\alpha_1$  in  $\gamma$  is larger and so on.

**Lemma 6.1.1.** *Suppose that  $G$  is of type  $A_{2m+1}$ ,  $m \geq 1$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ , then  $\text{LT}(\mathcal{E}) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  with respect to  $\succeq$ .*

*Proof.* Assume that  $\text{LT}(\mathcal{E}) \notin \text{Max}(\Phi_{r_{\text{smax}}}^+)$ , then  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_{m+1}^{\text{rad}}$  by assumption and Table 3.1. Let  $\gamma$  be the root associated to a lower term  $x_\gamma$  of a reduced basis element. Since  $\Phi^+ \setminus \Phi_{m+1}^{\text{rad}} \succ \Phi_{m+1}^{\text{rad}}$ , it gives  $\gamma \in \Phi_{m+1}^{\text{rad}}$ . That it is reduced means  $\gamma \in \Phi_{m+1}^{\text{rad}} \setminus \text{LT}(\mathcal{E})$ , and then only one kind of possibilities left because  $|\Phi_{m+1}^{\text{rad}}| = |\text{LT}(\mathcal{E})| + 1$ . As a result,  $\mathcal{E} \not\subseteq \mathcal{E} \oplus \mathbb{k}x_\gamma$  which is not maximal, a contradiction. Therefore,  $\text{LT}(\mathcal{E}) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  if  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$  with respect to  $\succeq$ .  $\square$

**Theorem 6.1.2.** *Suppose that  $G$  is of type  $A_{2m+1}$  with  $m \geq 2$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})$  satisfies  $\text{LT}(\mathcal{E}) = \Phi_m^{\text{rad}}, \Phi_{m+2}^{\text{rad}}$  or  $\Phi_{m+1, m+2}^{\text{odd}}$  then  $\mathcal{E} = \text{Lie}(\Phi_m^{\text{rad}}), \mathcal{E} = \text{Lie}(\Phi_{m+2}^{\text{rad}})$  or  $\mathcal{E} = \text{Lie}(\Phi_{m+1, m+2}^{\text{odd}})^{\exp(\text{ad}(ax_{\alpha_{m+1}}))}$  for some  $a$ .*

*Proof.* • Case 1.  $\text{LT}(\mathcal{E}) = \Phi_m^{\text{rad}}$ . We write the reduced echelon form basis for  $\mathcal{E}$

$$\begin{aligned} x_{ij} &= x_{\epsilon_i - \epsilon_j}, 1 \leq i \leq m \text{ and } m+2 \leq j \leq 2m+2 \\ y_i &= x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^m a_{ist} x_{\epsilon_s - \epsilon_t} + \sum_{r=m+2}^{2m+2} b_{ir} x_{\epsilon_{m+1} - \epsilon_r}, 1 \leq i \leq m \end{aligned}$$

Let  $1 \leq i \leq m$  and  $2 \leq t \leq m$ , the coefficient of  $x_{\epsilon_s - \epsilon_j}$  in  $[y_i, x_{tj}]$  is  $a_{ist} N_{\epsilon_s - \epsilon_t, \epsilon_t - \epsilon_j}$ , this gives  $a_{ist} = 0$  for all  $i, s$  and  $t$ . If  $i, j \leq m$  are distinct, then the coefficient of  $x_{\epsilon_j - \epsilon_r}$  in  $[y_i, y_j]$  is  $b_{ir} N_{\epsilon_{m+1} - \epsilon_r, \epsilon_j - \epsilon_{m+1}}$ . As  $m \geq 2$ , this gives all  $b_{ir} = 0$ . Therefore, we have  $\mathcal{E} = \text{Lie}(\Phi_m^{\text{rad}})$ .



- Case 2.  $\text{LT}(\mathcal{E}) = \Phi_{m+2}^{\text{rad}}$ . The reduced echelon form basis of  $\mathcal{E}$  is of the form

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + \sum_{s=1}^{i-1} a_{ijs} x_{\epsilon_s - \epsilon_{m+2}}, 1 \leq i \leq m+1 \text{ and } m+3 \leq j \leq 2m+2$$

$$y_j = x_{\epsilon_{m+2} - \epsilon_j} + \sum_{s=1}^{m+1} \sum_{t=s+1}^{m+2} b_{jst} x_{\epsilon_s - \epsilon_t}, m+3 \leq j \leq 2m+2$$

Let  $m+3 \leq j, j' \leq 2m+2$  and  $2 \leq t \leq m+1$ . If  $j$  and  $j'$  are distinct, then the coefficient of  $x_{\epsilon_s - \epsilon_{j'}}$  in  $[y_j, x_{tj'}]$  is  $b_{jst} N_{\epsilon_s - \epsilon_t, \epsilon_t - \epsilon_{j'}}$ , it gives  $b_{jst} = 0$  for all  $j, s$  and  $t < m+2$  as  $m \geq 2$ . Then the coefficient of  $x_{\epsilon_s - \epsilon_j}$  in  $[y_j, x_{ij'}]$  is  $a_{ij's} N_{\epsilon_{m+2} - \epsilon_j, \epsilon_s - \epsilon_{m+2}}$ , this implies  $a_{ijs} = 0$  for all  $i, j, s$ . It remains to show  $b_{js(m+2)}$ . If  $m+3 \leq i, j \leq 2m+2$  are distinct, then the coefficient of  $x_{\epsilon_s - \epsilon_i}$  in  $[y_i, y_j]$  is  $b_{js(m+2)} N_{\epsilon_{m+2} - \epsilon_i, \epsilon_s - \epsilon_{m+2}}$ . According to this together with  $m \geq 2$ , we get  $b_{js(m+2)} = 0$  for all  $j$  and  $s$ . Therefore, we have  $\mathcal{E} = \text{Lie}(\Phi_{m+2}^{\text{rad}})$ .

- Case 3.  $\text{LT}(\mathcal{E}) = \Phi_{m+1, m+2}^{\text{odd}}$ . The reduced echelon form basis of  $\mathcal{E}$  consists of

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + \sum_{s=1}^{i-1} a_{ijs} x_{\epsilon_s - \epsilon_{m+2}}$$

$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^m b_{ist} x_{\epsilon_s - \epsilon_t} + \sum_{r=m+2}^{2m+2} d_{ir} x_{\epsilon_{m+1} - \epsilon_r} + \sum_{r=1}^m k_{ir} x_{\epsilon_r - \epsilon_{m+2}}$$

$$z_j = x_{\epsilon_{m+2} - \epsilon_j} + \sum_{s=1}^{m-1} \sum_{t=s+1}^m h_{jst} x_{\epsilon_s - \epsilon_t} + \sum_{r=m+2}^{2m+2} \ell_{jr} x_{\epsilon_{m+1} - \epsilon_r} + \sum_{r=1}^m \xi_{jr} x_{\epsilon_r - \epsilon_{m+2}}$$

where  $1 \leq i \leq m$  and  $m+3 \leq j \leq 2m+2$ . By the same argument as before we deduce that  $b_{ist} = h_{jst} = 0$  for all  $s$  and  $t$ . If  $i, j \leq m$  are distinct, then the coefficient of  $x_{\epsilon_j - \epsilon_r}$  in  $[y_i, y_j]$  is  $d_{ir} N_{\epsilon_{m+1} - \epsilon_r, \epsilon_j - \epsilon_{m+1}}$ . As  $m \geq 2$ , this gives  $d_{ir} = 0$  and the argument can also be applied to  $z_j$  which ensures that  $\xi_{jr} = 0$ . Let  $1 \leq i \leq m$ , if there is some  $i$  with  $k_{ii} = 0$ , then the coefficient of  $x_{\epsilon_i - \epsilon_r}$  in  $[y_i, z_j]$  is  $\ell_{jr} N_{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_r}$  which gives  $\ell_{jr} = 0$  for all  $j$  and  $r$ . Subsequently, the coefficient of  $x_{\epsilon_r - \epsilon_j}$  in  $[y_i, z_j]$  is  $k_{ir} N_{\epsilon_r - \epsilon_{m+2}, \epsilon_{m+2} - \epsilon_j}$ , this gives  $k_{ir} = 0$  for all  $i$  and  $r$ , and this also applies to  $[x_{ij}, z_{j'}]$  from which we can get  $a_{ijs} = 0$ . Alternatively, if all  $k_{ii} \neq 0$  for  $1 \leq i \leq m$ , pick an arbitrary  $i_0$  with  $1 \leq i_0 \leq m$  and use the conjugation by  $\exp(\text{ad}(\lambda_0 x_{\alpha_{m+1}}))$  to  $\mathcal{E}$  where  $\lambda_0 = -k_{i_0 i_0} N_{\epsilon_{m+1} - \epsilon_{m+2}, \epsilon_{i_0} - \epsilon_{m+1}}^{-1}$ . Observe that there is no  $x_{\epsilon_{i_0} - \epsilon_{m+2}}$  in  $\exp(\text{ad}(\lambda_0 x_{\alpha_{m+1}}))(y_{i_0})$ . Then we use the aforementioned argument to  $\exp(\text{ad}(\lambda_0 x_{\alpha_{m+1}}))(\mathcal{E})$ . As a result, we get  $\mathcal{E} = \text{Lie}(\Phi_{m+1, m+2}^{\text{odd}})^{\exp(\text{ad}(a x_{\alpha_{m+1}}))}$  where  $a = -\lambda_0$ .  $\square$

## 6.2 Type $A_{2m}$

Suppose that  $G$  is of type  $A_{2m}$ . Let  $\beta$  and  $\gamma$  be two positive roots written as a linear combinations of the simple roots  $\alpha_i$ . Define  $\beta \succeq_1 \gamma$  if the sum of the coefficients of  $\alpha_m$  and  $\alpha_{m+1}$  in the expression of  $\gamma$  is larger than or equal to the sum for  $\beta$ . Define  $\succeq_2$  to be the reverse lexicographic ordering given by  $\alpha_{m+1} \prec \alpha_m \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{2m}$ . Finally we denote by  $\succeq = (\succeq_1, \succeq_2)$  the refinement of  $\succeq_1$  by  $\succeq_2$ , and this is the ordering we choose for  $A_{2m}$ . According to this setting, one can easily check that

$$\Phi^+ \setminus (\Phi_m^{\text{rad}} \cup \Phi_{m+1}^{\text{rad}}) \succ \Phi_m^{\text{rad}} \setminus \Phi_{m+1}^{\text{rad}} \succ \Phi_{m+1}^{\text{rad}} \setminus \Phi_m^{\text{rad}} \succ \Phi_m^{\text{rad}} \cap \Phi_{m+1}^{\text{rad}}.$$

**Lemma 6.2.1.** *Suppose that  $G$  is of type  $A_{2m}$  with  $m \geq 3$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ , then  $\text{LT}(\mathcal{E}) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  with respect to  $\succeq$ .*

*Proof.* If  $\text{LT}(\mathcal{E}) \notin \text{Max}(\Phi_{r_{\text{smax}}}^+)$ , then either  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_m^{\text{rad}}$ , or  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_{m+1}^{\text{rad}}$  according to Table 3.1.

• Case 1.  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_{m+1}^{\text{rad}}$ . Then  $\Phi_{m+1}^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_u - \epsilon_v\}$  for some  $(u, v)$ . Notice that  $\Phi^+ \setminus \Phi_{m+1}^{\text{rad}} \succ \Phi_{m+1}^{\text{rad}}$ , thus the reduced echelon form basis of  $\mathcal{E}$  is as follows

$$x_{\epsilon_i - \epsilon_j} + a_{ij}x_{\epsilon_u - \epsilon_v}, \quad a_{ij} = 0 \text{ if } i < u \text{ or } i = u, j > v$$

for  $1 \leq i \leq m+1, m+2 \leq j \leq 2m+1$  and  $(i, j) \neq (u, v)$ . Then it is readily seen that  $\mathcal{E} \not\subseteq \mathcal{E} \oplus \mathbb{k}x_{\epsilon_u - \epsilon_v}$ , and the maximality of  $\mathcal{E}$  leads to a contradiction.

• Case 2.1.  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_m^{\text{rad}}$ . We may first assume that  $\Phi_m^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_u - \epsilon_v\} \subset \Phi_m^{\text{rad}} \cap \Phi_{m+1}^{\text{rad}}$ . Then the reduced basis of  $\mathcal{E}$  consists of elements for  $1 \leq i \leq m, m+2 \leq j \leq 2m+1$  and  $(i, j) \neq (u, v)$

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + a_{ij}x_{\epsilon_u - \epsilon_v}, \quad a_{ij} = 0 \text{ if } i < u \text{ or } i = u, j > v$$

$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} b_{is}x_{\epsilon_{m+1} - \epsilon_s} + d_i x_{\epsilon_u - \epsilon_v}.$$

Now we compute

$$[y_i, y_{i'}] = \sum_{s=m+2}^{2m+1} b_{i's} N_{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_s} x_{\epsilon_i - \epsilon_s} + \sum_{m+2}^{2m+1} b_{is} N_{\epsilon_{m+1} - \epsilon_s, \epsilon_{i'} - \epsilon_{m+1}} x_{\epsilon_{i'} - \epsilon_s}.$$

As  $m \geq 3$ , we take  $i \neq i'$ , this gives  $b_{is} = 0$  for all  $i$  and  $s$ . As a result, we will have  $\mathcal{E} \not\subseteq \mathcal{E} \oplus \mathbb{k}x_{\epsilon_u - \epsilon_v}$ , a contradiction.

• Case 2.2.  $\Phi_m^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_u - \epsilon_{m+1}\} \subseteq \Phi_m^{\text{rad}} \setminus \Phi_{m+1}^{\text{rad}}$ . Then the reduced echelon form basis of  $\mathcal{E}$  is  $x_{\epsilon_i - \epsilon_j}$  for  $1 \leq i \leq m$  and  $m+2 \leq j \leq 2m+1$  together with for  $1 \leq i \neq u \leq m$

$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + q_i x_{\epsilon_u - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} d_{is} x_{\epsilon_{m+1} - \epsilon_s}, \quad q_i = 0 \text{ if } i < u.$$

If  $i, i'$  are distinct, then the coefficient of  $x_{\epsilon_{i'} - \epsilon_s}$  in  $[y_i, y_{i'}]$  is  $N_{\epsilon_{m+1} - \epsilon_s, \epsilon_{i'} - \epsilon_{m+1}} d_{is}$ , so  $d_{is} = 0$  for all  $i$  and  $s$  as  $m \geq 3$ . Thus  $\mathcal{E} \not\subseteq \mathcal{E} \oplus \mathbb{k}x_{\epsilon_u - \epsilon_{m+1}}$ , a contradiction and we finish the proof.  $\square$

**Theorem 6.2.2.** *Suppose that  $G$  is of type  $A_{2m}$  with  $m \geq 3$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})$  satisfies  $\text{LT}(\mathcal{E}) = \Phi_{m,m+1}^{\text{ev}}$  or  $\Phi_{m+1,m+2}^{\text{ev}}$  then there exists some  $a$  such that  $\mathcal{E} = \text{Lie}(\Phi_{m,m+1}^{\text{ev}})^{\exp(\text{ad}(ax_{\alpha_m}))}$  or  $\mathcal{E} = \text{Lie}(\Phi_{m+1,m+2}^{\text{ev}})^{\exp(\text{ad}(ax_{\alpha_{m+1}}))}$ .*

*Proof.* • Case 1.  $\text{LT}(\mathcal{E}) = \Phi_{m,m+1}^{\text{ev}}$ . Then the reduced echelon form basis of  $\mathcal{E}$  is  $x_{\epsilon_i - \epsilon_j}$  for  $1 \leq i \leq m-1$  and  $m+2 \leq j \leq 2m+1$  and

$$y_j = x_{\epsilon_{m+1} - \epsilon_j} + \sum_{s=m+2}^{2m+1} a_{js} x_{\epsilon_m - \epsilon_s}$$

$$z_i = x_{\epsilon_i - \epsilon_m} + \sum_{u=1}^{i-1} \sum_{v=u+1}^{m-1} b_{iuv} x_{\epsilon_u - \epsilon_v} + \sum_{s=1}^m c_{is} x_{\epsilon_s - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} d_{is} x_{\epsilon_m - \epsilon_s}$$

where  $m+2 \leq j \leq 2m+1$  for  $y_j$  and  $1 \leq i < m$  for  $z_i$ . If there exists some  $j$  such that  $a_{jj} = 0$  in  $y_j$ , then we compute for  $1 \leq i < m$

$$[y_j, z_i] = \sum_{s=1}^m c_{is} N_{\epsilon_{m+1} - \epsilon_j, \epsilon_s - \epsilon_{m+1}} x_{\epsilon_s - \epsilon_j} + \sum_{t=m+2}^{2m+1} a_{jt} N_{\epsilon_m - \epsilon_t, \epsilon_i - \epsilon_m} x_{\epsilon_i - \epsilon_t}.$$

Notice that these items  $x_{\epsilon_s - \epsilon_j}$  and  $x_{\epsilon_i - \epsilon_t}$  are different, so  $c_{is} = 0$  for all  $i$  and  $s$  and further  $a_{js} = 0$  for all  $j$  and  $s$ . Otherwise, fix some  $y_j$  and denote by  $\lambda = -a_{jj} N_{\epsilon_m - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_j}^{-1}$ . Using conjugation given by  $\exp(\text{ad}(\lambda x_{\epsilon_m - \epsilon_{m+1}}))$  to  $\mathcal{E}$ , then we have explicitly

$$\exp(\text{ad}(\lambda x_{\epsilon_m - \epsilon_{m+1}}))(y_j) = x_{\epsilon_{m+1} - \epsilon_j} + \sum_{m+2 \leq s \neq j \leq 2m+1} a_{js} x_{\epsilon_m - \epsilon_s},$$

which leads us to consider  $\exp(\text{ad}(\lambda x_{\epsilon_m - \epsilon_{m+1}}))(\mathcal{E})$  or just assume there is some  $a_{jj} = 0$  for  $\mathcal{E}$  firstly without any real ambiguity. When  $1 < v < m$ , we compute

the coefficient of  $x_{\epsilon_u - \epsilon_j}$  in  $[x_{\epsilon_v - \epsilon_j}, z_i]$  for a fixed  $j$ , that is  $N_{\epsilon_v - \epsilon_j, \epsilon_u - \epsilon_v} b_{iuv}$ . This gives  $b_{iuv} = 0$  for all  $i, u$  and  $v$ . If  $1 \leq i, i' < m$  are distinct, then the coefficient of  $x_{\epsilon_i - \epsilon_s}$  in  $[z_i, z_{i'}]$  is  $N_{\epsilon_i - \epsilon_m, \epsilon_m - \epsilon_s} d_{i's}$ . As  $m \geq 3$ , this gives  $d_{is} = 0$  for all  $i, s$ , and consequently  $\mathcal{E} = \text{Lie}(\Phi_{m, m+1}^{\text{ev}})^{\exp(\text{ad}(ax_{\alpha_m}))}$  for  $a = -\lambda_0$ .

• Case 2.  $\text{LT}(\mathcal{E}) = \Phi_{m+1, m+2}^{\text{ev}}$ . We write the reduced basis for  $1 \leq i \leq m$  and  $m+3 \leq j \leq 2m+1$ .

$$\begin{aligned} x_{ij} &= x_{\epsilon_i - \epsilon_j} + \sum_{t=1}^{i-1} a_{ijt} x_{\epsilon_t - \epsilon_{m+2}}, \\ y_i &= x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} b_{is} x_{\epsilon_{m+1} - \epsilon_s} + \sum_{s=1}^m c_{is} x_{\epsilon_s - \epsilon_{m+2}}, \\ z_j &= x_{\epsilon_{m+2} - \epsilon_j} + \sum_{u=1}^{m-1} \sum_{v=u+1}^m d_{juv} x_{\epsilon_u - \epsilon_v} + \sum_{s=m+2}^{2m+1} f_{js} x_{\epsilon_{m+1} - \epsilon_s} + \sum_{s=1}^m k_{js} x_{\epsilon_s - \epsilon_{m+2}}. \end{aligned}$$

If  $m+3 \leq j \neq j' \leq 2m+1$ , choose  $1 < v < m+1$ , then we compute

$$[x_{vj}, z_{j'}] = \sum_{u=1}^{v-1} d_{j'uv} N_{\epsilon_v - \epsilon_j, \epsilon_u - \epsilon_v} x_{\epsilon_u - \epsilon_j} + \sum_{t=1}^{i-1} a_{vjt} N_{\epsilon_t - \epsilon_{m+2}, \epsilon_{m+2} - \epsilon_{j'}} x_{\epsilon_t - \epsilon_{j'}}.$$

As  $m \geq 3$ , this gives  $d_{juv} = 0$  and consequently  $a_{ijt} = 0$  by seeing the coefficient of  $x_{\epsilon_t - \epsilon_j}$  in  $[x_{ij}, z_j]$ . If  $1 \leq i \neq i' \leq m$ , then the coefficient of  $x_{\epsilon_i - \epsilon_s}$  in  $[y_i, y_{i'}]$  is  $N_{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_s} b_{i's}$ , so  $b_{is} = 0$  for all  $i$  and  $s$ . Now let  $\xi = -c_{ii} N_{\epsilon_{m+1} - \epsilon_{m+2}, \epsilon_i - \epsilon_{m+1}}^{-1}$  for some  $1 \leq i \leq m$ , conjugation given by  $\exp(\text{ad}(\xi x_{\epsilon_{m+1} - \epsilon_{m+2}}))$  lets us assume that  $c_{ii} = 0$ . Then we compute

$$[y_i, z_j] = \sum_{s=m+2}^{2m+1} f_{js} N_{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_s} x_{\epsilon_i - \epsilon_s} + \sum_{t=1}^m c_{it} N_{\epsilon_t - \epsilon_{m+2}, \epsilon_{m+2} - \epsilon_j} x_{\epsilon_t - \epsilon_j}$$

Since  $c_{ii} = 0$ , this gives that  $c_{is}$  and  $f_{js}$  are zero. Finally, the coefficient of  $x_{\epsilon_s - \epsilon_i}$  in  $[z_i, z_j]$  for  $i \neq j$  is  $N_{\epsilon_{m+2} - \epsilon_i, \epsilon_s - \epsilon_{m+2}} k_{js}$ , so  $k_{js} = 0$  for all  $j$  and  $s$ . Now we have  $\mathcal{E} = \text{Lie}(\Phi_{m+1, m+2}^{\text{ev}})^{\exp(\text{ad}(ax_{\alpha_{m+1}}))}$  for  $a = -\xi$  and complete the proof.  $\square$

### 6.3 Type $B_n$

Suppose that  $G$  is of type  $B_n$ . Let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  be the set of simple roots,

$$\alpha_{n-1} \prec \alpha_{n-2} \prec \dots \prec \alpha_1 \prec \alpha_n$$

be the ordering and  $\succeq$  be the reverse lexicographical ordering according to this. Then one can check that  $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} = N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} = 1$  holds for  $i < j < n$ .

Let  $\mathcal{H}_j = \{\epsilon_j - \epsilon_n, \epsilon_j, \epsilon_j + \epsilon_n\}$  for  $1 \leq j \leq n-1$ . Then  $\Phi^+$  can be written as the union of the sets

$$\{\alpha_n\} \succ \{\epsilon_i - \epsilon_j \mid 1 \leq i < j < n\} \succ \mathcal{H}_{n-1} \succ \cdots \succ \mathcal{H}_1 \succ \{\epsilon_i + \epsilon_j \mid 1 \leq i < j < n\}.$$

Setting

$$\mathcal{R}_1 := \{\epsilon_i + \epsilon_j \mid 1 \leq i < j < n\}$$

$$\mathcal{R}_2 := \{\epsilon_i + \epsilon_n \mid 1 \leq i < n\}$$

$$\mathcal{R}_3 := \{\epsilon_i - \epsilon_n \mid 1 \leq i < n\}$$

then the sets of commuting positive roots of maximal order  $r_{\text{max}}$  are of the forms (see Table 3.1)

$$S_t = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \{\epsilon_t\}, \quad 1 \leq t \leq n,$$

$$S_t^* = \mathcal{R}_1 \cup \mathcal{R}_3 \cup \{\epsilon_t\}, \quad 1 \leq t < n.$$

We cite the notations for the elementary subalgebras of maximal dimension from [PS, 3.5]

$$B(a_1, \dots, a_n) = \text{Span}_{\mathbb{k}} \left\{ x_\beta, \sum_{i=1}^n a_i x_{\epsilon_i} \mid \beta \in \mathcal{R}_1 \cup \mathcal{R}_2 \right\}$$

$$C(a_1, \dots, a_{n-1}) = \text{Span}_{\mathbb{k}} \left\{ x_\beta, \sum_{i=1}^{n-1} a_i x_{\epsilon_i} \mid \beta \in \mathcal{R}_1 \cup \mathcal{R}_3 \right\}$$

with  $(a_1, \dots, a_n) \neq 0$  for  $B(a_1, \dots, a_n)$  and  $(a_1, \dots, a_{n-1}) \neq 0$  for  $C(a_1, \dots, a_{n-1})$ .

**Lemma 6.3.1.** *Suppose that  $G$  is of type  $B_n$  with  $n \geq 5$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ , then  $\text{LT}(\mathcal{E}) \neq \mathcal{R}_1 \cup \mathcal{R}_2$  and  $\text{LT}(\mathcal{E}) \neq \mathcal{R}_1 \cup \mathcal{R}_3$  with respect to  $\succeq$ .*

*Proof.* As supposed, if  $\text{LT}(\mathcal{E}) = \mathcal{R}_1 \cup \mathcal{R}_3$  the reduced echelon form basis of  $\mathcal{E}$  is

$$x_{ij} = x_{\epsilon_i + \epsilon_j}, \quad 1 \leq i < j < n,$$

$$y_i = x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^i a_{is} x_{\epsilon_s} + \sum_{t=1}^i b_{it} x_{\epsilon_t + \epsilon_n}, \quad 1 \leq i < n.$$

In this case, we may assume  $a_{11} = 0$  otherwise we will use  $\exp(\text{ad}(-a_{11}N_{\epsilon_n, \epsilon_1 - \epsilon_n}^{-1}x_{\alpha_n}))$ . We prove the following statement by induction:  $b_{it} = 0$  for all  $i, t$ . The statement is clear for  $n = 5$  which is shown as follows. If  $1 < i < 5$ , the computation for  $[y_1, y_i] = 0$  yields  $b_{ii}N_{\epsilon_1 - \epsilon_5, \epsilon_i + \epsilon_5} + b_{11}N_{\epsilon_1 + \epsilon_5, \epsilon_i - \epsilon_5} = b_{ii} + b_{11} = 0$  as well as  $b_{it} = 0$  for  $1 < t < i$ . Then for  $1 < i < j < 5$  the equations  $[y_i, y_j] = 0$  yields the relations

$$\begin{aligned} b_{21} + a_{21}a_{33}N_{\epsilon_1, \epsilon_3} &= 0, \quad b_{21} + a_{21}a_{44}N_{\epsilon_1, \epsilon_4} = 0, \quad b_{31} + a_{31}a_{44}N_{\epsilon_1, \epsilon_4} = 0, \\ b_{31} + (a_{21}a_{32} - a_{22}a_{31})N_{\epsilon_1, \epsilon_2} &= 0, \quad b_{41} + (a_{21}a_{42} - a_{22}a_{41})N_{\epsilon_1, \epsilon_2} = 0, \\ b_{41} + (a_{31}a_{43} - a_{33}a_{41})N_{\epsilon_1, \epsilon_3} &= 0, \quad b_{ii} + b_{jj} + a_{ii}a_{jj}N_{\epsilon_i, \epsilon_j} = 0, \\ a_{31}a_{42} = a_{32}a_{41}, \quad a_{32}a_{43} = a_{33}a_{42}, \quad a_{32}a_{44} = a_{21}a_{43} = a_{22}a_{43} &= 0. \end{aligned}$$

We claim that  $b_{11} = 0$ . If  $b_{11} \neq 0$ , we denote by  $\varrho = a_{22}^2 N_{\epsilon_2, \epsilon_3} N_{\epsilon_2, \epsilon_4} N_{\epsilon_3, \epsilon_4}^{-1}$ , and  $\alpha = \epsilon_1 - \epsilon_5, \beta = \epsilon_1 + \epsilon_5, \gamma = \epsilon_2 - \epsilon_5, \delta = \epsilon_2$ . We pick  $t_0 \in (\beta - \alpha)^{-1}(\varrho \cdot b_{11}^{-1})$  and  $t_1 \in (\delta - \gamma)^{-1}(1)$ . Let  $H_0 := \ker(\beta - \alpha)$  and  $H_1 = \ker(\delta - \gamma)$  be two closed subgroups of  $T$ . Then it is readily seen that  $H_0 t_0 = \left\{ t \in T \mid \frac{\beta(t)}{\alpha(t)} = \varrho \cdot b_{11}^{-1} \right\}$  and  $H_1 t_1 = \left\{ t \in T \mid \frac{\delta(t)}{\gamma(t)} = 1 \right\}$  together with the relation  $H_0 t_0 \cap H_1 t_1 \neq \emptyset$  if and only if  $t_0 t_1^{-1} \in H_0 H_1$ . From inspection of the morphism of multiplication  $m : T \times T \rightarrow T$ , we know that  $H_0 H_1 = m(H_0 \times H_1)$  is a constructible subset of  $T$ , which endows itself with a closed subgroup structure. By general theory, we have  $\dim H_0 H_1 \geq \dim H_0 \geq \dim T - 1$ , this gives rise to  $H_0 H_1 = T$  (otherwise  $H_0 = H_0 H_1 = H_1$  but  $H_0 \not\subseteq H_1$  because  $\beta - \alpha$  and  $\delta - \gamma$  are not linear dependent). From this point of view,  $H_0 t_0 \cap H_1 t_1 \neq \emptyset$ , and we pick  $t^* \in H_0 t_0 \cap H_1 t_1$ . Now we apply  $t^*$  to  $\mathcal{E}$  by conjugation. By denoting  $y_i^* := (\epsilon_5 - \epsilon_i)(t^*)y_i$  for  $1 \leq i < 5$  we get

$$t^* \cdot y_i^* = x_{\epsilon_i - \epsilon_5} + \sum_{s=1}^i \widetilde{a}_{is} x_{\epsilon_s} + \sum_{t=1}^i \widetilde{b}_{it} x_{\epsilon_i + \epsilon_5}, \quad \widetilde{a}_{11} = \widetilde{b}_{11} = 0 \text{ if } 1 < t < i.$$

According to the equations  $t^* \cdot [y_i^*, y_j^*] = 0$ , we have the same relations as before:  $\widetilde{b}_{ii} + \widetilde{b}_{jj} + \widetilde{a}_{ii} \widetilde{a}_{jj} N_{\epsilon_i, \epsilon_j} = 0$  for  $1 \leq i < j < 5$ . By this, one can easily check that  $\widetilde{a}_{22}^2 N_{\epsilon_2, \epsilon_3} N_{\epsilon_2, \epsilon_4} N_{\epsilon_3, \epsilon_4}^{-1} = 2\widetilde{b}_{11}$ . Notice that the choice of  $t^*$  we take, there are  $\widetilde{b}_{11} = \frac{\beta(t^*)}{\alpha(t^*)} b_{11} = \varrho$  and  $\widetilde{a}_{22} = \frac{\delta(t^*)}{\gamma(t^*)} a_{22} = a_{22}$ . Then  $2\varrho = a_{22}^2 N_{\epsilon_2, \epsilon_3} N_{\epsilon_2, \epsilon_4} N_{\epsilon_3, \epsilon_4}^{-1} = \varrho$ , thus  $\varrho = 0$ . However, if  $b_{11} \neq 0$ , then  $a_{22} \neq 0$  because  $a_{22}^2 N_{\epsilon_2, \epsilon_3} N_{\epsilon_2, \epsilon_4} N_{\epsilon_3, \epsilon_4}^{-1} = 2b_{11}$ , and then  $\varrho \neq 0$  a contradiction, we finish our claim. Now we continue our proof. One can immediately get  $b_{ii} = a_{22}a_{33} = a_{22}a_{44} = a_{33}a_{44} = 0$  for  $1 \leq i \leq 4$  when  $b_{11} = 0$ . The remaining steps are the explicit calculations due to the relations listed above, which ultimately give rise to  $b_{i1} = 0$  for all  $i$  and also ensure the statement for  $n = 5$ .

Assume it is proved for  $n-1$ . Consider the set  $\{y_1, \dots, y_{n-2}\}$ . Then by our induction hypothesis, we have  $b_{it} = 0$  for all  $1 \leq i < n-1$  and  $t$ . Consider the coefficient of  $x_{\epsilon_1 + \epsilon_t}$  in  $[y_1, y_{n-1}]$ , it is  $b_{(n-1)t} N_{\epsilon_1 - \epsilon_n, \epsilon_t + \epsilon_n}$ , so  $b_{(n-1)t} = 0$  for  $t \neq 1$ . If  $a_{21} = a_{22} = 0$ , consider the coefficient of  $x_{\epsilon_1 + \epsilon_2}$  in  $[y_2, y_{n-1}]$ , it is  $b_{(n-1)1} N_{\epsilon_2 - \epsilon_n, \epsilon_1 + \epsilon_n}$ , this gives  $b_{(n-1)1} = 0$ . Otherwise, we assume  $a_{2r} \neq 0$  ( $r = 1$  or  $r = 2$ ). Then the coefficients of  $x_{\epsilon_r + \epsilon_{n-1}}$  and  $x_{\epsilon_r + \epsilon_{n-2}}$  in  $[y_2, y_{n-1}]$  are  $a_{2r} a_{(n-1)(n-1)} N_{\epsilon_r, \epsilon_{n-1}}$  and  $a_{2r} a_{(n-1)(n-2)} N_{\epsilon_r, \epsilon_{n-2}}$ , this gives  $a_{(n-1)(n-1)} = a_{(n-1)(n-2)} = 0$ . Then our induction hypothesis can be applied to  $\{y_1, \dots, y_{n-3}, y_{n-1}\}$ , from which we still have  $b_{(n-1)1} = 0$ , and this finally completes our induction procedure.

Our assertion for  $\text{LT}(\mathcal{E}) \neq \mathcal{R}_1 \cup \mathcal{R}_3$  is done as follows. As we wish, if  $\text{LT}(\mathcal{E}) = \mathcal{R}_1 \cup \mathcal{R}_3$ , then  $[y_i, y_j] = 0$  yields  $[\sum_{s=1}^i a_{is} x_s, \sum_{s=1}^j a_{js} x_s] = 0$  since  $b_{it} = b_{jt} = 0$  for all  $t$ . Then there exist  $r$  and  $(a_1, \dots, a_r, 0, \dots, 0) \in \mathbb{k}^{n-1} \setminus \{0\}$  such that  $\mathcal{E} \not\subseteq C(a_1, \dots, a_r, 0, \dots, 0)$  up to conjugation by elements of  $B$ , this gives a contradiction since  $\mathcal{E}$  is maximal.

To show also  $\text{LT}(\mathcal{E}) \neq \mathcal{R}_1 \cup \mathcal{R}_2$ , we write down the reduced echelon form basis of  $\mathcal{E}$  when  $\text{LT}(\mathcal{E}) = \mathcal{R}_1 \cup \mathcal{R}_2$ , that is

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j}, 1 \leq i < j < n, \\ y_i &= x_{\epsilon_i + \epsilon_n} + \sum_{s=1}^{i-1} a_{is} x_{\epsilon_s} + \sum_{t=1}^{i-1} b_{it} x_{\epsilon_t - \epsilon_n}, 1 \leq i < n. \end{aligned}$$

Methodically, as the case when  $\text{LT}(\mathcal{E}) = \mathcal{R}_1 \cup \mathcal{R}_3$ , one can check that  $b_{it} = 0$  for all  $i$  and  $t$ , and finally we can find element of  $B$  which helps to conjugate  $\mathcal{E}$  to  $B(a'_1, \dots, a'_{r'}, 0, \dots, 0)$  for some  $(a'_1, \dots, a'_{r'}, 0, \dots, 0) \in \mathbb{k}^n \setminus \{0\}$ , also a contradiction.  $\square$

**Lemma 6.3.2.** *Suppose that  $G$  is of type  $B_n$  with  $n \geq 5$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ , then we have  $\text{LT}(\mathcal{E}) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  with respect to  $\succeq$ .*

*Proof.* Suppose  $\text{LT}(\mathcal{E}) \notin \text{Max}(\Phi_{r_{\text{smax}}}^+)$ , then we have  $\text{LT}(\mathcal{E}) \not\subseteq S_t$  or  $S_t^*$  by Table 3.1 for  $B_n$ . Then by Lemma 6.3.1,  $S_t \setminus \text{LT}(\mathcal{E}) \subseteq \mathcal{R}_1 \cup \mathcal{R}_2$  if  $\text{LT}(\mathcal{E}) \not\subseteq S_t$ , and  $S_t^* \setminus \text{LT}(\mathcal{E}) \subseteq \mathcal{R}_1 \cup \mathcal{R}_3$  if  $\text{LT}(\mathcal{E}) \not\subseteq S_t^*$ .

• Case 1.1.  $S_t \setminus \text{LT}(\mathcal{E}) = \{\epsilon_q + \epsilon_n\} \subseteq \mathcal{R}_2$ . We assume  $t < n$ . The proof for  $t = n$  is

omitted. Then the reduced echelon form basis of  $\mathcal{E}$  is of the form

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j}, \quad 1 \leq i < j < n, \\ y_i &= x_{\epsilon_i + \epsilon_n} + \sum_{s=1}^{i-1} a_{is} x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{i-1} b_{is} x_{\epsilon_s} + d_{iq} x_{\epsilon_q + \epsilon_n}, \quad 1 \leq i \neq q < n, \\ z &= x_{\epsilon_t} + \sum_{s=1}^{t-1} \xi_s x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{t-1} \eta_s x_{\epsilon_s} + \gamma_q x_{\epsilon_q + \epsilon_n}, \end{aligned}$$

with  $d_{iq} = \gamma_q = 0$  if  $i, t < q$ . That it is reduced means  $b_{it} = 0$  for all  $i > t$ . The first step is to eliminate the coefficient  $d_{iq}$  for each  $i > q$  using conjugation. Notice that the action of  $\exp(\text{ad}(\lambda x_{\epsilon_q - \epsilon_i}))$  on  $\mathcal{E}$  is lower triangular with respect to  $\succeq$  for  $i > q$ . Therefore  $\text{LT}(\exp(\text{ad}(\lambda x_{\epsilon_q - \epsilon_i}))(\mathcal{E})) = \text{LT}(\mathcal{E})$  and the elements in the reduced echelon basis of  $\exp(\text{ad}(\lambda x_{\epsilon_q - \epsilon_i}))(\mathcal{E})$  with leading term  $x_{\epsilon_i + \epsilon_n}$  are  $\exp(\text{ad}(\lambda x_{\epsilon_q - \epsilon_i}))(y_i)$ . Now we choose  $\lambda_i = -d_{iq} N_{\epsilon_q - \epsilon_i, \epsilon_i + \epsilon_n}^{-1}$ , this gives

$$\exp(\text{ad}(\lambda_i x_{\epsilon_q - \epsilon_i}))(y_i) = x_{\epsilon_i + \epsilon_n} + \sum_{s=1}^{i-1} a_{is} x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{i-1} b_{is} x_{\epsilon_s},$$

and the effect given by  $\exp(\text{ad}(\lambda_i x_{\epsilon_q - \epsilon_i}))$  on the other basis elements is the disturbance on their coefficients. Thus consecutive conjugations  $\exp(\text{ad}(\lambda_{n-1} x_{\epsilon_q - \epsilon_{n-1}})) \circ \cdots \circ \exp(\text{ad}(\lambda_{q+1} x_{\epsilon_q - \epsilon_{q+1}}))$  could be conducted to  $\mathcal{E}$  which shall ultimately remove the items  $x_{\epsilon_q + \epsilon_n}$  for each  $y_i$ . Inspired by the first step, it suffices to show  $\mathcal{E}$  is not maximal when  $d_{iq} = 0$ . Our task here is to confirm that  $a_{is}, b_{is}$  and  $\xi_s$  are all zero and we analyze it by considering two cases according to  $q$ . Notice that we have if  $i < j < n$ , the coefficient of  $x_{\epsilon_j + \epsilon_s}$  in  $[y_i, y_j]$  is  $N_{\epsilon_s - \epsilon_n, \epsilon_j + \epsilon_n} a_{is}$ , this gives  $a_{is} = 0$ . If  $q \neq n - 1$ , then  $a_{is} = 0$  for all  $i \neq q < n - 1$ . And for such  $i$ , additionally if  $i \neq t$  which might happen since  $n \geq 5$ , the coefficient of  $x_{\epsilon_s + \epsilon_t}$  in  $[y_i, z]$  is  $N_{\epsilon_s, \epsilon_t} b_{is}$ , this gives  $b_{is} = 0$  and subsequently  $\xi_s = 0$  for  $s \neq i$  due to the coefficient of  $x_{\epsilon_s + \epsilon_i}$  in  $[y_i, z]$ . If  $t = n - 1$ , then  $\xi_s = 0$  for all  $s$  as  $n - 3 \geq 2$ . If  $t < n - 1$ , then we compute the coefficient of  $x_{\epsilon_s + \epsilon_{n-1}}$  in  $[y_{n-1}, z]$ , this also gives  $\xi_s = 0$  for all  $s$ . The coefficients left for elimination are  $a_{(n-1)s}, b_{t,s}$  (possibly if  $t < n - 1$ ) and  $b_{(n-1)s}$ . Note that the coefficient of  $x_{\epsilon_t + \epsilon_s}$  in  $[y_t, z]$  is  $N_{\epsilon_s, \epsilon_t} b_{ts}$  if  $t < n - 1$ , this gives  $b_{ts} = 0$ . If  $i, s < n - 1$  are distinct, the coefficient of  $x_{\epsilon_i + \epsilon_s}$  in  $[y_i, y_{n-1}]$  is  $N_{\epsilon_i + \epsilon_n, \epsilon_s - \epsilon_n} a_{(n-1)s}$ . As  $n - 3 \geq 2$ , this gives  $a_{(n-1)s} = 0$  for all  $s$ . Finally, the coefficient of  $x_{\epsilon_t + \epsilon_s}$  in  $[y_{n-1}, z]$  is  $N_{\epsilon_s, \epsilon_t} b_{(n-1)s}$ , so  $b_{(n-1)s} = 0$  and we finish the first step. Alternatively, if  $q = n - 1$ , we argue similar to the first step but we need to consider the coefficient



of  $y_{n-2}$  instead of  $y_{n-1}$ . Summarizing here, we are now in the position to say that  $\mathcal{E} \subsetneq B(\eta_1, \dots, \eta_{t-1}, 1, 0, \dots, 0)$  up to conjugation, it is not maximal.

• Case 1.2.  $\mathcal{S}_t \setminus \text{LT}(\mathcal{E}) = \{\epsilon_u + \epsilon_v\} \subseteq \mathcal{R}_1$ . Then the reduced echelon form basis of  $\mathcal{E}$  is of the form

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j} + a_{ij}x_{\epsilon_u + \epsilon_v}, \quad 1 \leq i < j < n, \\ y_i &= x_{\epsilon_i + \epsilon_n} + \sum_{s=1}^{i-1} a_{is}x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{i-1} b_{is}x_{\epsilon_s} + d_i x_{\epsilon_u + \epsilon_v}, \quad 1 \leq i < n, \\ z &= x_{\epsilon_t} + \sum_{s=1}^{t-1} \xi_s x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{t-1} \eta_s x_{\epsilon_s} + \gamma x_{\epsilon_u + \epsilon_v} \end{aligned}$$

with  $(i, j) \neq (u, v)$ . By the same token with Case 1.1, we can show that  $a_{is}, b_{is}, \xi_s$  are all zero, and this gives  $\mathcal{E} \subseteq B(\eta_1, \dots, \eta_{t-1}, 1, 0, \dots, 0)$  is not maximal.

• Case 2.1.  $\mathcal{S}_t^* \setminus \text{LT}(\mathcal{E}) = \{\epsilon_q - \epsilon_n\} \subseteq \mathcal{R}_3$ . Then the reduced echelon form basis of  $\mathcal{E}$  is the following

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j}, \quad 1 \leq i < j < n, \\ y_i &= x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^i a_{is}x_{\epsilon_s} + \sum_{s=1}^i b_{is}x_{\epsilon_s + \epsilon_n} + d_{iq}x_{\epsilon_q - \epsilon_n}, \quad 1 \leq i \neq q < n, \\ z &= x_{\epsilon_t} + \sum_{s=1}^{t-1} \xi_s x_{\epsilon_s} + \sum_{s=1}^t \eta_s x_{\epsilon_s + \epsilon_n} + \gamma x_{\epsilon_q - \epsilon_n} \end{aligned}$$

with  $d_{iq} = 0$  if  $i < q$  and  $\gamma_q = 0$  if  $t \leq q$ . That it is reduced means  $a_{it} = 0$ . If  $i > q$ , then  $\text{LT}(\exp(\text{ad}(\lambda x_{\epsilon_q - \epsilon_i}))) (\mathcal{E}) = \text{LT}(\mathcal{E})$ . By setting  $\lambda_i = -d_{iq} N_{\epsilon_q - \epsilon_i, \epsilon_i - \epsilon_n}^{-1}$ , we get

$$\begin{aligned} &\exp(\text{ad}(\lambda_i x_{\epsilon_q - \epsilon_i}))(y_i) \\ &= x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^i (a_{is} + \delta_{s,q} \lambda_i a_{ii} N_{\epsilon_q - \epsilon_i, \epsilon_i}) x_{\epsilon_s} + \sum_{s=1}^i (b_{is} + \delta_{s,q} \lambda_i b_{ii} N_{\epsilon_q - \epsilon_i, \epsilon_i + \epsilon_n}) x_{\epsilon_s + \epsilon_n}. \end{aligned}$$

Conjugation by  $\exp(\text{ad}(\lambda_{n-1} x_{\epsilon_q - \epsilon_{n-1}})) \circ \dots \circ \exp(\text{ad}(\lambda_{q+1} x_{\epsilon_q - \epsilon_{q+1}}))$  lets us assume  $d_{iq} = 0$ . Using conjugation by  $\exp(\text{ad}(-\eta_t N_{\epsilon_n, \epsilon_t}^{-1} x_{\alpha_n}))$  we assume  $\eta_t = 0$ . Due to these settings, it suffices to show  $\mathcal{E}$  is not maximal when all  $d_{iq} = 0$  and  $\eta_t = 0$  and our strategy is to verify that all  $a_{is}, b_{is}, \eta_s$  are zero. In the following, we may assume  $t > q$  because for  $t \leq q$  the situation is similar but easier. If  $i \neq t$ , then the coefficient of  $x_{\epsilon_s + \epsilon_t}$  for  $s \neq q$  in  $[z, y_i]$  is  $N_{\epsilon_t, \epsilon_s} a_{is}$ , so  $a_{is} = 0$  for all  $i \neq t$  and  $s \neq q$ . An inspection later for the coefficient of  $x_{\epsilon_i + \epsilon_s}$  in  $[z, y_i]$  for  $i \neq t$  gives  $\eta_s = 0$  for

$s \neq q$  as  $n - 3 \geq 2$ , which helps to get  $a_{ts} = 0$  for  $s \neq q$  by checking the coefficient of  $x_{\epsilon_s + \epsilon_t}$  in  $[z, y_t]$ . If  $i, j, s < n$  are distinct, then the coefficient of  $x_{\epsilon_i + \epsilon_s}$  in  $[y_i, y_j]$  is  $N_{\epsilon_i - \epsilon_n, \epsilon_s + \epsilon_n} b_{js}$ , this gives  $b_{js} = 0$  for all  $s \neq j$  because  $n \geq 5$ . Now for  $i \neq j$  the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[y_i, y_j]$  is  $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} b_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} b_{ii}$ . Thus if  $i < j < \ell < n$ , we get a system of equations

$$\begin{aligned} N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} b_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} b_{ii} &= b_{jj} + b_{ii} = 0 \\ N_{\epsilon_i - \epsilon_n, \epsilon_\ell + \epsilon_n} b_{\ell\ell} + N_{\epsilon_i + \epsilon_n, \epsilon_\ell - \epsilon_n} b_{ii} &= b_{\ell\ell} + b_{ii} = 0 \\ N_{\epsilon_j - \epsilon_n, \epsilon_\ell + \epsilon_n} b_{\ell\ell} + N_{\epsilon_j + \epsilon_n, \epsilon_\ell - \epsilon_n} b_{jj} &= b_{\ell\ell} + b_{jj} = 0 \end{aligned}$$

this gives  $b_{ii} = b_{jj} = b_{\ell\ell} = 0$ . It is straightforward to see the left coefficients are  $a_{iq}$  and  $\eta_q$ . By computing the coefficient of  $x_{\epsilon_t + \epsilon_q}$  in  $[z, y_i]$  for  $i \neq t$  yields  $a_{iq} = 0$  and further  $\eta_q = 0$  due to the coefficient of  $x_{\epsilon_i + \epsilon_q}$  in  $[z, y_i]$ . Finally, we compute  $[z, y_t]$ , and we find that the coefficient of  $x_{\epsilon_t + \epsilon_q}$  is  $a_{tq} N_{\epsilon_t, \epsilon_q}$ , this gives  $a_{tq} = 0$ . Summarizing here, we have already seen that  $\mathcal{E}$  is a subalgebra of  $C(\xi_1, \dots, \xi_{t-1}, 1, 0, \dots, 0)$  which is not maximal up to conjugation by elements in the Borel  $B$ .

• Case 2.2.  $\mathcal{S}_t^* \setminus \text{LT}(\mathcal{E}) = \{\epsilon_u + \epsilon_v\} \subseteq \mathcal{R}_1$ . The reduced echelon form basis of  $\mathcal{E}$  is

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j} + a_{ij} x_{\epsilon_u + \epsilon_v}, 1 \leq i < j < n, \\ y_i &= x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^i a_{is} x_{\epsilon_s} + \sum_{s=1}^i b_{is} x_{\epsilon_s + \epsilon_n} + d_i x_{\epsilon_u + \epsilon_v}, 1 \leq i < n, \\ z &= x_{\epsilon_t} + \sum_{s=1}^{t-1} \xi_s x_{\epsilon_s} + \sum_{s=1}^t \eta_s x_{\epsilon_s + \epsilon_n} + \gamma x_{\epsilon_u + \epsilon_v} \end{aligned}$$

with  $(i, j) \neq (u, v)$ , and  $a_{ij} = 0$  if  $j < v$  or  $j = v, i < u$ . It is readily seen that the proof for the non-maximality of  $\mathcal{E}$  is similar to Case 2.1.  $\square$

The elements of  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$  have been detected in Section 3.4, they are  $\mathcal{J}_t$  for  $1 \leq t \neq n - 1 \leq n$  and  $\mathcal{J}_t^*$  for  $1 \leq t \leq n - 2$ . In the following, by defining the

subsets of  $\mathcal{R}_1, \mathcal{R}_2$  and  $\mathcal{R}_3$

$$\begin{aligned}\mathcal{R}'_1 &= \{\epsilon_i + \epsilon_j \mid 1 \leq i < j < n-1\} \\ \mathcal{R}''_1 &= \{\epsilon_i + \epsilon_{n-1} \mid 1 \leq i < n-1\} \\ \mathcal{R}'_2 &= \{\epsilon_i + \epsilon_n \mid 1 \leq i < n-1\} \\ \mathcal{R}''_2 &= \{\epsilon_{n-1} + \epsilon_n\} \\ \mathcal{R}'_3 &= \{\epsilon_i - \epsilon_n \mid 1 \leq i < n-1\} \\ \mathcal{R}''_3 &= \{\epsilon_{n-1} - \epsilon_n\} \\ \mathcal{R}_4 &= \{\epsilon_i - \epsilon_{n-1} \mid 1 \leq i < n-1\}\end{aligned}$$

we get

$$\begin{aligned}\mathcal{J}_t : \mathcal{J}_t &= \mathcal{R}'_1 \cup \mathcal{R}'_2 \cup \mathcal{R}_4 \cup \{\epsilon_t\}, t \neq n-1, \\ \mathcal{J}_t^* : \mathcal{J}_t^* &= \mathcal{R}'_1 \cup \mathcal{R}'_3 \cup \mathcal{R}_4 \cup \{\epsilon_t\}, t \neq n-1, n.\end{aligned}$$

We also define the subalgebras given below which are elementary subalgebras of dimension  $r_{\text{smax}}$ .

$$\begin{aligned}H(a_1, \dots, a_{n-1}) &= \text{Span}_{\mathbb{k}} \left\{ x_\beta, \sum_{i=1}^{n-1} a_i x_{\epsilon_i} \mid \beta \in \mathcal{R}_1 \cup \mathcal{R}'_2 \right\} \\ L(a_1, \dots, a_{n-2}) &= \text{Span}_{\mathbb{k}} \left\{ x_\beta, \sum_{i=1}^{n-2} a_i x_{\epsilon_i} \mid \beta \in \mathcal{R}_1 \cup \mathcal{R}'_3 \right\}\end{aligned}$$

**Theorem 6.3.3.** *Suppose that  $G$  is of type  $B_n$  with  $n \geq 5$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})$  satisfies  $\text{LT}(\mathcal{E}) = \mathcal{J}_t$  or  $\mathcal{J}_t^*$  then there exist  $a_1, \dots, a_{n-1}$  such that  $\mathcal{E}$  is  $G$ -conjugate to  $H(a_1, \dots, a_{n-1})$  or  $L(a_1, \dots, a_{n-2})$  respectively.*

*Proof.* • Case 1.  $\text{LT}(\mathcal{E}) = \mathcal{J}_t$  for  $t < n-1$ . The proof for  $t = n$  is omitted. Then the reduced echelon form basis of  $\mathcal{E}$  consists of  $x_{\epsilon_i + \epsilon_j}$  where  $1 \leq i < j < n-1$  and for  $1 \leq i < n-1$  the elements

$$\begin{aligned}x_i &= x_{\epsilon_i + \epsilon_n} + \sum_{s=1}^{i-1} a_{is} x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{i-1} b_{is} x_{\epsilon_s} + \sum_{s=1}^{n-2} d_{is} x_{\epsilon_s + \epsilon_{n-1}}, \\ y_i &= x_{\epsilon_i - \epsilon_{n-1}} + \sum_{s=1}^{n-1} \xi_{is} x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{n-1} \eta_{is} x_{\epsilon_s} + \gamma_i x_{\epsilon_{n-1} + \epsilon_n} + \sum_{s=1}^{n-2} \zeta_{is} x_{\epsilon_s + \epsilon_{n-1}}, \\ z &= x_{\epsilon_t} + \sum_{s=1}^{t-1} \lambda_s x_{\epsilon_s - \epsilon_n} + \sum_{s=1}^{t-1} \varrho_s x_{\epsilon_s} + \sum_{s=1}^{n-2} \kappa_s x_{\epsilon_s + \epsilon_{n-1}}.\end{aligned}$$

That it is reduced means  $b_{it} = \eta_{it} = 0$ . By inspection of the lower triangular actions given by  $\exp(\text{ad}(-\kappa_t N_{\epsilon_{n-1}, \epsilon_t}^{-1} x_{\epsilon_{n-1}}))$  and  $\exp(\text{ad}(-d_{i_0 i_0} N_{\epsilon_{n-1} - \epsilon_n, \epsilon_{i_0} + \epsilon_n}^{-1} x_{\alpha_{n-1}}))$  with respect to  $\succeq$ , we may assume  $\kappa_t = d_{i_0 i_0} = 0$  for some  $i_0 < n-1$ . One can check that the coefficient of  $x_{\epsilon_i + \epsilon_n}$  in  $[y_i, y_j]$  is  $N_{\epsilon_i - \epsilon_{n-1}, \epsilon_{n-1} + \epsilon_n} \gamma_j$ , this gives  $\gamma_j = 0$  for all  $j < n-1$ . If  $i \neq t$ , then the coefficient of  $x_{\epsilon_s + \epsilon_t}$  in  $[z, y_i]$  is  $N_{\epsilon_t, \epsilon_s} \eta_{is}$ , so  $\eta_{is} = 0$  for all  $i \neq t$  and  $s$  which subsequently gives  $\kappa_s = 0$  for  $s \neq i$  by the coefficient of  $x_{\epsilon_s + \epsilon_i}$  in  $[z, y_i]$ . As  $n \geq 5$ , this gives  $\kappa_s = 0$  for all  $s$ , from which immediately we get  $\eta_{ts} = 0$  for all  $s$  by computing  $[z, y_t]$ . By the same token for  $[z, x_i]$ , one can compute that  $b_{is} = 0 = \lambda_s$  for all  $i$  and  $s$ . If  $i, j < n-1$  are distinct, then the coefficient of  $x_{\epsilon_i + \epsilon_s}$  in  $[x_i, y_j]$  is  $N_{\epsilon_i + \epsilon_n, \epsilon_s - \epsilon_n} \xi_{js}$  for  $s \neq i$ , and the coefficient of  $x_{\epsilon_j + \epsilon_s}$  is  $N_{\epsilon_s + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} d_{is}$  for  $j \neq s$ . As  $n \geq 5$ , this gives rise to  $\xi_{is} = d_{is} = 0$  for  $s \neq i$ . Now for  $i \neq j$ , the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[x_i, y_j]$  is  $N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} \xi_{jj} + N_{\epsilon_i + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} d_{ii}$ . Notice that there is some  $d_{i_0 i_0} = 0$ , this ultimately gives  $d_{ii} = \xi_{ii} = 0$  for all  $i$  as  $n \geq 5$ . It is straightforward to see that the left coefficients are  $a_{is}, \zeta_{is}$  and  $\varrho_s$ . By computing  $[x_i, x_j]$  for distinct  $i$  and  $j$  we can rule out all  $a_{is}$ . Choosing representatives  $\dot{s}_{n-1} \in N_G(T)$  for simple reflections  $s_{n-1} \in \mathcal{W}$ , which act on  $y_i$  by  $\dot{s}_{n-1} \cdot y_i = x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^{n-2} \zeta_{is} x_{\epsilon_s + \epsilon_n}$ . Similar to  $a_{is}$ , there will be no terms  $x_{\epsilon_s + \epsilon_n}$  in  $\dot{s}_{n-1} \cdot y_i$ . Finally, we apply  $\dot{s}_n$  to  $\dot{s}_{n-1} \mathcal{E}$ , which shows that  $\mathcal{E}$  can be  $G$ -conjugate to  $H(a_1, \dots, a_{n-1})$  for  $(a_1, \dots, a_{n-1}) \neq 0$  by  $\dot{s}_n \circ \dot{s}_{n-1} \circ \exp(\text{ad}(-d_{i_0 i_0} N_{\epsilon_{n-1} - \epsilon_n, \epsilon_{i_0} + \epsilon_n}^{-1} x_{\alpha_{n-1}})) \circ \exp(\text{ad}(-\kappa_t N_{\epsilon_{n-1}, \epsilon_t}^{-1} x_{\epsilon_{n-1}}))$ .

- Case 2.  $\text{LT}(\mathcal{E}) = \mathcal{J}_t^*$  for  $t < n-1$ . The reduced echelon form basis of  $\mathcal{E}$  then is  $x_{\epsilon_i + \epsilon_j}$  where  $1 \leq i < j < n-1$  and for  $1 \leq i < n-1$  the elements

$$\begin{aligned} x_i &= x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^i a_{is} x_{\epsilon_s} + \sum_{s=1}^i b_{is} x_{\epsilon_s + \epsilon_n} + \sum_{s=1}^{n-2} d_{is} x_{\epsilon_s + \epsilon_{n-1}}, \\ y_i &= x_{\epsilon_i - \epsilon_{n-1}} + \gamma_i x_{\epsilon_{n-1} - \epsilon_n} + \sum_{s=1}^{n-1} \xi_{is} x_{\epsilon_s} + \sum_{s=1}^{n-1} \eta_{is} x_{\epsilon_s + \epsilon_n} + \sum_{s=1}^{n-2} \zeta_{is} x_{\epsilon_s + \epsilon_{n-1}}, \\ z &= x_{\epsilon_t} + \sum_{s=1}^{t-1} \lambda_s x_{\epsilon_s} + \sum_{s=1}^t \varrho_s x_{\epsilon_s + \epsilon_n} + \sum_{s=1}^{n-2} \kappa_s x_{\epsilon_s + \epsilon_{n-1}}. \end{aligned}$$

That it is reduced means  $a_{it} = 0 = \xi_{it}$ . Observe that these terms  $x_{\epsilon_s + \epsilon_{n-1}}$  are redundant in the operation of  $[z, x_i]$ , thus up to conjugation by  $\exp(\text{ad}(-\varrho_t N_{\epsilon_n, \epsilon_t}^{-1} x_{\alpha_n}))$  if  $\varrho_t \neq 0$ , we shall have  $a_{is}, b_{is}$  and  $\varrho_s$  are zero due to [PS, Theorem 3.3]. An inspection for  $\exp(\text{ad}(-d_{i_0 i_0} N_{\epsilon_{n-1} + \epsilon_n, \epsilon_i - \epsilon_n}^{-1} x_{\alpha_{n-1} + 2\alpha_n}))$  lets us assume there is  $d_{i_0 i_0} = 0$  for some  $i_0 < n-1$ . If  $i, j, s < n-1$  are distinct, then  $x_{\epsilon_j + \epsilon_s}$  has coefficient  $N_{\epsilon_s + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} d_{is}$  and  $x_{\epsilon_i + \epsilon_s}$  has coefficient  $N_{\epsilon_i - \epsilon_n, \epsilon_s + \epsilon_n} \eta_{js}$  in  $[x_i, y_j]$ , so  $d_{is} = \eta_{is} = 0$  for all  $s \neq i$ .

Also for  $i \neq j$ , the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[x_i, y_j]$  is  $N_{\epsilon_i + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} d_{ii} + N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} \eta_{jj}$ , this gives  $d_{ii} = \eta_{ii} = 0$  for all  $i$  by  $d_{i_0 i_0} = 0$ . If  $i \neq j$ , the coefficient of  $x_{\epsilon_j - \epsilon_n}$  in  $[y_i, y_j]$  is  $N_{\epsilon_{n-1} - \epsilon_n, \epsilon_j - \epsilon_n} \gamma_i$ , so  $\gamma_i = 0$  for all  $1 \leq i < n - 1$ . Now the determination for the coefficients  $\kappa_s, \xi_{is}$  and  $\zeta_{is}$  is directly given by taking  $[z, y_i]$  and  $[y_i, y_j]$  and using  $\dot{s}_{n-1}$ . Finally, we first use simple reflection  $s_n$  and then  $s_{n-1}$ , which shows that  $\mathcal{E}$  can be  $G$ -conjugated to  $L(a_1, \dots, a_{t-1}, 1, 0, \dots, 0)$  by taking  $\dot{s}_{n-1} \circ \dot{s}_n \circ \dot{s}_{n-1} \circ \exp(\text{ad}(-d_{i_0 i_0} N_{\epsilon_{n-1} + \epsilon_n, \epsilon_i - \epsilon_n}^{-1} x_{\alpha_{n-1} + 2\alpha_n})) \circ \exp(\text{ad}(-\varrho_t N_{\epsilon_n, \epsilon_t}^{-1} x_{\alpha_n}))$  and completes the proof of the theorem.  $\square$

**Corollary 6.3.4.** *If  $n \geq 5$ , then any element of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$  is  $G$ -conjugate to  $\text{Lie}(\mathcal{J}_1)$ .*

*Proof.* The simple reflection  $s_n$  conjugates  $L(a_1, \dots, a_{n-2})$  to  $H(a_1, \dots, a_{n-2}, 0)$ . It suffices to consider  $H(a_1, \dots, a_{n-1})$ . Since  $(a_1, \dots, a_{n-1}) \neq 0$ , then by a consecutive application of simple reflection  $s_i$  for  $1 \leq i < n - 1$ ,  $H(a_1, \dots, a_{n-1})$  will be conjugate to  $H'(a_1, \dots, a_{n-2}, 1)$ , where

$$H'(a_1, \dots, a_{n-2}, 1) = \text{Span}_{\mathbb{k}} \left\{ x_\beta, \sum_{i=1}^{n-1} a_i x_{\epsilon_i} \mid \beta \in \mathcal{R}_1 \cup s_{n-2} \cdot \mathcal{R}'_2, a_{n-1} = 1 \right\}.$$

Then the element  $-a_i N_{\epsilon_i - \epsilon_{n-1}, \epsilon_{n-1}}^{-1} x_{\epsilon_i - \epsilon_{n-1}}$  for  $i < n - 1$  exponentiates to  $\mathcal{E}$  helps to remove the term  $x_{\epsilon_i}$ . After this,  $H'(a_1, \dots, a_{n-2}, 1)$  can be conjugated to  $H'(0, \dots, 0, 1)$ . Using simple reflections  $s_1, \dots, s_{n-2}$  to  $H'(0, \dots, 0, 1)$ , we conjugate  $H'(0, \dots, 0, 1)$  to  $H(1, 0, \dots, 0)$ . Finally by using conjugation  $s_{n-1} \circ s_n \circ s_{n-1}$ , this conjugates  $H(1, 0, \dots, 0, 1)$  to  $\text{Lie}(\mathcal{J}_1)$  and we finish our proof.  $\square$

## 6.4 Type $C_n$

Suppose that  $G$  is of type  $C_n$  ( $n \geq 3$ ). Let  $\succeq$  be the reverse lexicographic ordering given by  $\alpha_n \prec \alpha_{n-1} \prec \dots \prec \alpha_1$ .

**Lemma 6.4.1.** *Suppose that  $G$  is of type  $C_n$  with  $n \geq 3$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ , then  $\text{LT}(\mathcal{E}) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  with respect to  $\succeq$ .*

*Proof.* The proof is similar to Lemma 6.1.1.  $\square$

In what follows, we wish to refine the set  $\Phi^+$  by defining

$$\Phi_i^+ := \{\epsilon_1 + \epsilon_i, \epsilon_2 + \epsilon_i, \dots, 2\epsilon_i\},$$

where  $1 \leq i \leq n$  and for  $1 < i \leq n$

$$\Phi_i^- := \{\epsilon_1 - \epsilon_i, \epsilon_2 - \epsilon_i, \dots, \epsilon_{i-1} - \epsilon_i\}.$$

Then one can easily check that  $\Phi^+$  is the union of the following ordered subsets:

$$\Phi_2^- \succ \dots \succ \Phi_{n-1}^- \succ \Phi_n^- \succ \Phi_n^+ \succ \Phi_{n-1}^+ \succ \dots \succ \Phi_1^+.$$

Simultaneously, for the elements of  $\Phi_i^+$  and  $\Phi_i^-$  we have

$$\begin{aligned} 2\epsilon_i &\succ \epsilon_{i-1} + \epsilon_i \succ \dots \succ \epsilon_1 + \epsilon_i, \\ \epsilon_{i-1} - \epsilon_i &\succ \epsilon_{i-2} - \epsilon_i \succ \dots \succ \epsilon_1 - \epsilon_i. \end{aligned}$$

Recall the definition of  $\Phi_{n-1,n}^C$  for  $C_n$  in Section 5.2, we get

$$\Phi_{n-1,n}^C = \Phi_n^- \cup \bigcup_{i=1}^{n-1} \Phi_i^+.$$

**Theorem 6.4.2.** *Suppose that  $G$  is of type  $C_n$  with  $n \geq 3$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})$  satisfies  $\text{LT}(\mathcal{E}) = \Phi_{n-1,n}^C$  then  $\mathcal{E} = \text{Lie}(\Phi_{n-1,n}^C)^{\exp(\text{ad}(ax_{\alpha_n}))}$  for some  $a$ .*

*Proof.* If  $\text{LT}(\mathcal{E}) = \Phi_{n-1,n}^C$ . The reduced echelon form basis of  $\mathcal{E}$  consists of

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j}, 1 \leq i \leq j < n, \text{ and} \\ y_i &= x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^n a_{is} x_{\epsilon_s + \epsilon_n}, 1 \leq i < n. \end{aligned}$$

Our hope here is that there is  $a_{ii} = 0$  for some  $y_i$ . Otherwise we will take the action on  $\mathcal{E}$  given by  $\exp(\text{ad}(-a_{ii} N_{2\epsilon_n, \epsilon_i - \epsilon_n}^{-1} x_{\alpha_n}))$  for any  $i$ , which helps to eliminate the term  $x_{\epsilon_i + \epsilon_n}$  in  $\exp(\text{ad}(-a_{ii} N_{2\epsilon_n, \epsilon_i - \epsilon_n}^{-1} x_{\alpha_n}))(y_i)$ . So now we assume that  $a_{ii} = 0$ . If  $i, j < n$  are distinct, then the coefficient of  $x_{\epsilon_i + \epsilon_s}$  in  $[y_i, y_j]$  is  $N_{\epsilon_i - \epsilon_n, \epsilon_s + \epsilon_n} a_{js}$  if  $s \neq j$ , this gives  $a_{js} = 0$  for all  $j$  and  $s \neq j$  as  $n \geq 3$ . Also for  $i \neq j$  the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[y_i, y_j]$  is  $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} a_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} a_{ii}$ , so  $a_{jj} = 0$ . Thus  $\mathcal{E} = \text{Lie}(\Phi_{n-1,n}^C)^{\exp(\text{ad}(ax_{\alpha_n}))}$  for  $a = a_{ii} N_{2\epsilon_n, \epsilon_i - \epsilon_n}^{-1}$ .  $\square$

## 6.5 Type $D_n$

Suppose that  $G$  is of type  $D_n$ . Let  $\succeq$  be the reverse lexicographic ordering given by

$$\alpha_{n-2} \succ \cdots \alpha_2 \succ \alpha_1 \succ \alpha_{n-1} \succ \alpha_n.$$

One can compute that if  $i < j < n$ , then  $N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} = N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} = 1$ .

Let  $\mathcal{R} = \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n-1\}$ . Then

$$\begin{aligned} \Phi_n^{\text{rad}} &= \mathcal{R} \cup \{\epsilon_r + \epsilon_n \mid 1 \leq r < n\}, \text{ and} \\ \Phi_{n-1}^{\text{rad}} &= \mathcal{R} \cup \{\epsilon_r - \epsilon_n \mid 1 \leq r < n\}. \end{aligned}$$

According to  $\succeq$ , it gives rise to an refinement of  $\Phi^+$ :

$$\{\epsilon_i - \epsilon_j \mid 2 \leq i < j < n\} \succ \{\epsilon_1 - \epsilon_j \mid 2 \leq j < n\} \succ \Phi_{n-1}^{\text{rad}} \setminus \mathcal{R} \succ \Phi_n^{\text{rad}} \setminus \mathcal{R} \succ \mathcal{R}.$$

**Lemma 6.5.1.** *Suppose that  $G$  is of type  $D_n$  with  $n \geq 5$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ , then  $\text{LT}(\mathcal{E}) \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  with respect to  $\succeq$ .*

*Proof.* We prove it by assuming  $\text{LT}(\mathcal{E}) \notin \text{Max}(\Phi_{r_{\text{smax}}}^+)$ . Then  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_n^{\text{rad}}$ , or  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_{n-1}^{\text{rad}}$  by Table 3.1.

• Case 1.  $\text{LT}(\mathcal{E}) \not\subseteq \Phi_n^{\text{rad}}$ . If  $\Phi_n^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_s + \epsilon_n\}$  for some  $1 \leq s < n$ , then the reduced echelon form basis of  $\mathcal{E}$  consists of

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j}, \quad 1 \leq i < j \leq n-1, \\ y_i &= x_{\epsilon_i + \epsilon_n} + a_{is} x_{\epsilon_s + \epsilon_n}, \quad 1 \leq i \neq s < n \text{ and } a_{is} = 0 \text{ when } i < s. \end{aligned}$$

Alternatively, we have  $\Phi_n^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_s + \epsilon_t\}$  for  $1 \leq s < t \leq n-1$ . And we have the reduced echelon form basis of  $\mathcal{E}$  for  $1 \leq i < j < n$  and  $(i, j) \neq (s, t)$

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j} + a_{ij} x_{\epsilon_s + \epsilon_t}, \quad a_{ij} = 0 \text{ when } i < s \text{ or } i = s \text{ and } j < t, \\ y_k &= x_{\epsilon_k + \epsilon_n} + b_k x_{\epsilon_s + \epsilon_t}, \quad 1 \leq k < n. \end{aligned}$$

One can immediately see that, both of them yield  $\mathcal{E} \subsetneq \text{Lie}(\Phi_n^{\text{rad}})$ , a contradiction.

• Case 2.  $\text{LT}(\mathcal{E}) \subsetneq \Phi_{n-1}^{\text{rad}}$ . If  $\Phi_{n-1}^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_s - \epsilon_n\}$ , then there is the reduced echelon form basis of  $\mathcal{E}$

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j}, \quad 1 \leq i < j \leq n-1, \\ y_i &= x_{\epsilon_i - \epsilon_n} + a_{is} x_{\epsilon_s - \epsilon_n} + \sum_{t=1}^{n-1} b_{it} x_{\epsilon_t + \epsilon_n}, \quad 1 \leq i \neq s < n \text{ and } a_{is} = 0 \text{ when } i < s. \end{aligned}$$

Note that  $\exp(\text{ad}(-a_{is} N_{\epsilon_s - \epsilon_i, \epsilon_i - \epsilon_n}^{-1} x_{\epsilon_s - \epsilon_i}))$  for  $s < i$  will rule out the term  $a_{is} x_{\epsilon_s - \epsilon_n}$  in  $y_i$  and fix  $a_{js} x_{\epsilon_s - \epsilon_n}$  in  $y_j$  if  $j \neq i$ . Let  $\lambda_i = -a_{is} N_{\epsilon_s - \epsilon_i, \epsilon_i - \epsilon_n}^{-1}$ , then conjugation by  $b := \exp(\text{ad}(\lambda_{n-1} x_{\epsilon_s - \epsilon_{n-1}})) \circ \cdots \circ \exp(\text{ad}(\lambda_{s+1} x_{\epsilon_s - \epsilon_{s+1}}))$  yields the final reduced basis

$$\begin{aligned} x'_{ij} &= x_{\epsilon_i + \epsilon_j}, \quad 1 \leq i < j \leq n-1, \\ y'_i &= x_{\epsilon_i - \epsilon_n} + \sum_{t=1}^{n-1} b'_{it} x_{\epsilon_t + \epsilon_n}, \quad 1 \leq i \neq s < n. \end{aligned}$$

As  $n \geq 5$ , the proof in [PS, Theorem 3.6] shows that all  $b'_{it} = 0$ . Consequently  $\mathcal{E} \subsetneq \text{Lie}(\Phi_{n-1}^{\text{rad}})^{b^{-1}}$ , which is not maximal. Alternatively, we get  $\Phi_{n-1}^{\text{rad}} \setminus \text{LT}(\mathcal{E}) = \{\epsilon_s + \epsilon_t\}$  for  $1 \leq s < t < n$  and the reduced echelon form basis of  $\mathcal{E}$  in the following

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j} + a_{ij} x_{\epsilon_s + \epsilon_t}, \\ y_i &= x_{\epsilon_i - \epsilon_n} + \sum_{r=1}^{n-1} b_{ir} x_{\epsilon_r + \epsilon_n} + d_i x_{\epsilon_s + \epsilon_t}. \end{aligned}$$

If  $i, j, r < n$  are distinct, then the coefficient of  $x_{\epsilon_j + \epsilon_r}$  in  $[y_i, y_j]$  is  $b_{ir} N_{\epsilon_r + \epsilon_n, \epsilon_j - \epsilon_n}$ . As  $n \geq 5$ , we have  $b_{ir} = 0$  for all  $r \neq i$ . Now for  $i \neq j$  the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[y_i, y_j]$  is  $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} b_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} b_{ii}$ . Thus if  $i < j < t < n$  we have a system of equations

$$\begin{aligned} N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} b_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} b_{ii} &= b_{jj} + b_{ii} = 0 \\ N_{\epsilon_i - \epsilon_n, \epsilon_t + \epsilon_n} b_{tt} + N_{\epsilon_i + \epsilon_n, \epsilon_t - \epsilon_n} b_{ii} &= b_{tt} + b_{ii} = 0 \\ N_{\epsilon_j - \epsilon_n, \epsilon_t + \epsilon_n} b_{tt} + N_{\epsilon_j + \epsilon_n, \epsilon_t - \epsilon_n} b_{jj} &= b_{tt} + b_{jj} = 0 \end{aligned}$$

whose solution is  $b_{ii} = b_{jj} = b_{tt} = 0$ . This gives  $b_{ii} = 0$  for all  $i$ . Therefore, we have  $\mathcal{E} \subsetneq \text{Lie}(\Phi_{n-1}^{\text{rad}})$ , a contradiction.  $\square$



**Theorem 6.5.2.** *Suppose that  $G$  is of type  $D_n$  with  $n \geq 6$ . If  $\mathcal{E} \in \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})$  satisfies  $\text{LT}(\mathcal{E}) = \Phi_{n-1,n}^D$  or  $\Phi_{n-2,n-1}^D$  then  $\mathcal{E} = \text{Lie}(\Phi_{n-1,n}^D)^{\exp(\text{ad}(ax_{\alpha_{n-1}}))}$  or  $\mathcal{E} = \text{Lie}(\Phi_{n-2,n-1}^D)^{\exp(\text{ad}(ax_{\alpha_n}))}$  for some  $a$ .*

*Proof.* • Case 1.  $\text{LT}(\mathcal{E}) = \Phi_{n-1,n}^D$ . Then the reduced echelon form basis can be written as follows

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j} + \sum_{h < i} a_{ijh} x_{\epsilon_h + \epsilon_{n-1}}, \quad 1 \leq i < j < n-1, \\ y_i &= x_{\epsilon_i + \epsilon_n} + \sum_{r=1}^{n-2} b_{ir} x_{\epsilon_r + \epsilon_{n-1}}, \quad 1 \leq i < n-1 \text{ and for same } i \\ z_i &= x_{\epsilon_i - \epsilon_{n-1}} + \sum_{v < i} \sum_{v < t < n-1} c_{ivt} x_{\epsilon_v - \epsilon_t} + \sum_{r=1}^{n-1} d_{ir} x_{\epsilon_r - \epsilon_n} + k_i x_{\epsilon_{n-1} + \epsilon_n} + \sum_{s=1}^{n-2} \ell_{is} x_{\epsilon_s + \epsilon_{n-1}}. \end{aligned}$$

If  $i \geq 3$ , the coefficient of  $x_{\epsilon_1 + \epsilon_v}$  in  $[x_{1j}, z_i]$  is  $N_{\epsilon_1 + \epsilon_j, \epsilon_v - \epsilon_j} c_{ijv}$ , so  $c_{ivt} = 0$  when  $v \geq 2$ . If  $i \geq 2$ , the coefficient of  $x_{\epsilon_1 + \epsilon_2}$  in  $[x_{2j}, z_i]$  is  $N_{\epsilon_2 + \epsilon_j, \epsilon_1 - \epsilon_j} c_{i1j}$ , thus  $c_{i1t} = 0$  for  $t \geq 3$ . If  $i \neq j$ , the coefficients of  $x_{\epsilon_1 + \epsilon_j}$  in  $[x_{2j}, z_i]$  is  $N_{\epsilon_2 + \epsilon_j, \epsilon_1 - \epsilon_2} c_{i12}$ . As  $n \geq 6$ , this gives  $c_{i12} = 0$ . Now for  $i \geq 2$ , the coefficient of  $x_{\epsilon_h + \epsilon_i}$  in  $[x_{ij}, z_i]$  is  $N_{\epsilon_h + \epsilon_{n-1}, \epsilon_i - \epsilon_{n-1}} a_{ijh}$ , thus  $a_{ijh} = 0$  for all  $h < i$ . Let  $\lambda = -d_{11} N_{\epsilon_{n-1} - \epsilon_n, \epsilon_1 - \epsilon_{n-1}}^{-1}$ . Using conjugation by  $\exp(\text{ad}(\lambda x_{\epsilon_{n-1} - \epsilon_n}))$ , we may assume  $d_{11} = 0$ . If  $i, j, t < n-1$  are distinct, then we have

$$[y_i, z_j] = \sum_{t=1}^{n-1} d_{jt} x_{\epsilon_i + \epsilon_t} + \sum_{t=1}^{n-2} b_{it} x_{\epsilon_j + \epsilon_t}.$$

This gives  $d_{ir} = d_{ir} = 0$  for all  $r \neq i$ . Now for  $i \neq j$ , the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[y_i, z_j]$  is  $N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} d_{jj} + N_{\epsilon_i + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} b_{ii} = d_{jj} + b_{ii} = 0$ . As  $n \geq 6$ , this gives  $d_{ii} = d_{11} = 0$  for all  $i$  and consequently all  $b_{ii} = 0$ . For  $i \neq j$ , the coefficient of  $x_{\epsilon_j + \epsilon_n}$  in  $[z_i, z_j]$  is  $N_{\epsilon_{n-1} + \epsilon_n, \epsilon_j - \epsilon_{n-1}} k_i$ , so all  $k_i = 0$ . If  $i, j, s < n-1$  are distinct then the coefficient of  $x_{\epsilon_s + \epsilon_j}$  in  $[z_i, z_j]$  is  $N_{\epsilon_s + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} \ell_{is}$ . As  $n \geq 6$ , this gives  $\ell_{is} = 0$  for all  $s \neq i$ . Now for  $i \neq j$ , the coefficient of  $x_{\epsilon_i + \epsilon_j}$  in  $[z_i, z_j]$  is  $N_{\epsilon_i - \epsilon_{n-1}, \epsilon_j + \epsilon_{n-1}} \ell_{jj} + N_{\epsilon_i + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} \ell_{ii}$ . Thus if  $i < j < t < n-1$  are distinct, we have a system of equations

$$\begin{aligned} N_{\epsilon_i - \epsilon_{n-1}, \epsilon_j + \epsilon_{n-1}} \ell_{jj} + N_{\epsilon_i + \epsilon_{n-1}, \epsilon_j - \epsilon_{n-1}} \ell_{ii} &= \ell_{jj} + \ell_{ii} = 0 \\ N_{\epsilon_i - \epsilon_{n-1}, \epsilon_t + \epsilon_{n-1}} \ell_{tt} + N_{\epsilon_i + \epsilon_{n-1}, \epsilon_t - \epsilon_{n-1}} \ell_{ii} &= \ell_{tt} + \ell_{ii} = 0 \\ N_{\epsilon_j - \epsilon_{n-1}, \epsilon_t + \epsilon_{n-1}} \ell_{tt} + N_{\epsilon_j + \epsilon_{n-1}, \epsilon_t - \epsilon_{n-1}} \ell_{jj} &= \ell_{tt} + \ell_{jj} = 0 \end{aligned}$$

with unique solution  $\ell_{ii} = \ell_{jj} = \ell_{tt} = 0$ . This gives  $\ell_{ii} = 0$  for all  $i$  and finally yields  $\mathcal{E} = \text{Lie}(\Phi_{n-1,n}^D)^{\exp(\text{ad}(ax_{\alpha_{n-1}}))}$ .

• Case 2.  $\text{LT}(\mathcal{E}) = \Phi_{n-2,n-1}^D$  Then the reduced echelon form basis of  $\mathcal{E}$  is

$$\begin{aligned} x_{ij} &= x_{\epsilon_i + \epsilon_j} + \sum_{h < i} a_{ijh} x_{\epsilon_h + \epsilon_{n-1}}, \quad 1 \leq i < j < n-1, \\ y_i &= x_{\epsilon_i - \epsilon_n} + \sum_{p=1}^{n-1} b_{ip} x_{\epsilon_p + \epsilon_n} + \sum_{q=1}^{n-2} c_{iq} x_{\epsilon_q + \epsilon_{n-1}}, \quad 1 \leq i < n-1 \text{ and for some } i \\ z_i &= x_{\epsilon_i - \epsilon_{n-1}} + \sum_{v < i} \sum_{v < t < n-1} d_{ivt} x_{\epsilon_v - \epsilon_t} + k_i x_{\epsilon_{n-1} - \epsilon_n} + \sum_{r=1}^{n-1} f_{ir} x_{\epsilon_r + \epsilon_n} + \sum_{s=1}^{n-2} g_{is} x_{\epsilon_s + \epsilon_{n-1}}. \end{aligned}$$

By the same token, we can first show that all  $b_{ip} = 0$ . Then the calculation for the remaining coefficients is similar to Case 1, but we need to use the exponential given by  $x_{\alpha_n}$ . As a result, there exists some  $a$  such that  $\mathcal{E} = \text{Lie}(\text{LT}(\mathcal{E}))^{\exp(\text{ad}(ax_{\alpha_n}))}$ .  $\square$

## 6.6 Summarizing for unipotent case

Summarizing the above discussions for  $G$ , except for some small ranks for each type we are to give the main result. Before doing this, we recall the definition of an ideal of  $\Phi^+$ . We say  $R \subseteq \Phi^+$  is an ideal if  $\alpha + \beta \in R$  whenever  $\alpha \in \Phi^+, \beta \in R$  and  $\alpha + \beta \in \Phi^+$ ; see [PS, Definition 2.10]. A prototypical example for such an ideal arises from  $\Phi_i^{\text{rad}}$ , which plays a fundamental role in the sequel on our determination of irreducible components of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$ .

**Theorem 6.6.1.** *Suppose that  $G$  is of type  $A_n (n \geq 5), B_n (n \geq 5), C_n (n \geq 3)$  or  $D_n (n \geq 6)$ . Then*

$$\mathbb{E}(r_{\text{smax}}, \mathbf{u}_{\mathfrak{b}})_{\text{max}} \subseteq \bigcup_{\substack{R \in \text{Com}(\Phi_{r_{\text{smax}}}^+) \\ R \text{ an ideal}}} G.\text{Lie}(R),$$

and the ideals occurring here for each type are listed in the third column of the following Table 6.1.

Type	Rank	Ideal $R \in \text{Com}(\Phi_{r_{\text{smax}}}^+)$	Ideal $R \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$
$A_{2n+1}$	$n \geq 2$	$\Phi_n^{\text{rad}}, \Phi_{n+2}^{\text{rad}}, \Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\}$	$\Phi_{n+1}^{\text{rad}}$
$A_{2n}$	$n \geq 3$	$\Phi_n^{\text{rad}} \setminus \{\alpha_n\}, \Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\}$	$\Phi_n^{\text{rad}}, \Phi_{n+1}^{\text{rad}}$
$B_n$	$n \geq 5$	$S_1 \setminus \{\alpha_{n-1} + 2\alpha_n\}$	$S_1$
$C_n$	$n \geq 3$	$\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$	$\Phi_n^{\text{rad}}$
$D_n$	$n \geq 6$	$\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$	$\Phi_{n-1}^{\text{rad}}, \Phi_n^{\text{rad}}$

Table 6.1: Ideals for Theorem 6.6.1 and Corollary 6.6.2

*Proof.* Chapter 5 together with sections 6.1-6.5 ensure that

$$\mathbb{E}(r_{\text{smax}}, \mathbf{u}_b)_{\text{max}} \subseteq \bigcup_{R \in \text{Max}(\Phi_{r_{\text{smax}}}^+)} G.\text{Lie}(R).$$

If  $\dot{w} \in N_G(T)$  is a representative of an element  $w$  in Weyl group  $\mathscr{W}$ , if  $I$  and  $w.I$  are contained in  $\text{Com}(\Phi_{r_{\text{smax}}}^+)$  then  $\dot{w}.\text{Lie}(I) = \text{Lie}(w.I)$ . It suffices to show that each element of  $\text{Max}(\Phi_{r_{\text{smax}}}^+)$  can be  $\mathscr{W}$ -conjugated to an ideal of  $\Phi^+$ . For type  $A_{2n+1}$ ,  $\Phi_n^{\text{rad}}$  and  $\Phi_{n+2}^{\text{rad}}$  both are ideals, and  $\Phi_{n+1}^{\text{rad}}$  can be conjugated to  $\Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\}$  by a simple reflection  $s_{n+1}$ . For type  $A_{2n}$ ,  $\Phi_{n,n+1}^{\text{ev}}$  is conjugate to  $\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$  by  $s_n$ , and  $\Phi_{n+1,n+2}^{\text{ev}}$  is conjugate to  $\Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\}$  by  $s_{n+1}$ . For type  $B_n$ , by corollary 6.3.4 we only need to consider  $\mathscr{J}_1$  and it can be conjugate to  $S_1 \setminus \{\alpha_{n-1} + 2\alpha_n\}$ . For type  $C_n$ ,  $\Phi_{n-1,n}^{\text{C}}$  is able to be conjugate to  $\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$  by simple reflection  $s_n$ . For type  $D_n$ ,  $\Phi_{n-1,n}^{\text{D}}$  is conjugate to  $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$  by  $s_{n-1}$ , and  $\Phi_{n-2,n-1}^{\text{D}}$  is also conjugate to  $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$  by composition of simple reflections  $s_{n-1} \circ s_n$ .

Having done this, we still need to verify that  $\Phi_i^{\text{rad}} \setminus \{\alpha_i\}$  and  $S_1 \setminus \{\alpha_{n-1} + 2\alpha_n\}$  are ideals. Let  $\alpha \in \Phi^+, \beta \in \Phi_i^{\text{rad}} \setminus \{\alpha_i\}$  and  $\alpha + \beta \in \Phi^+$ , then  $\alpha + \beta \in \Phi_i^{\text{rad}}$  because  $\Phi_i^{\text{rad}}$  is an ideal. But  $\alpha + \beta \neq \alpha_i$  since  $\alpha_i$  is a simple root, this gives  $\alpha + \beta \in \Phi_i^{\text{rad}} \setminus \{\alpha_i\}$  and it is an ideal. By the same token for  $S_1 \setminus \{\alpha_{n-1} + 2\alpha_n\}$ , it suffices to show  $\alpha + \beta \neq \alpha_{n-1} + 2\alpha_n$ . If  $\alpha + \beta = \alpha_{n-1} + 2\alpha_n$ , then  $\{\alpha, \beta\} = \{\epsilon_{n-1}, \epsilon_n\}$ , it is impossible because  $\beta$  is an element of  $S_1$ .  $\square$

**Corollary 6.6.2.** *Suppose that  $G$  is of type  $A_n(n \geq 5), B_n(n \geq 5), C_n(n \geq 3)$  or  $D_n(n \geq 6)$ . Then*

$$\mathbb{E}(r_{\text{smax}}, \mathbf{u}_b) \subseteq \bigcup_{\substack{R \in \text{Com}(\Phi_{r_{\text{smax}}}^+) \\ R \text{ an ideal}}} G.\text{Lie}(R) \cup \bigcup_{\substack{R \in \text{Max}(\Phi_{r_{\text{smax}}}^+) \\ R \text{ an ideal}}} G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R)).$$

and ideals  $R \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$  are listed in the fourth column of Table 6.1 .

*Proof.* Theorem 6.6.1 has determined  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ . We still need to consider  $\mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}}) \setminus \mathbb{E}(r_{\text{smax}}, \mathfrak{u}_{\mathfrak{b}})_{\text{max}}$ . In accordance with [PS, Corollary 3.9] and an easy observation from  $\text{Gr}_{r_{\text{smax}}}(\text{Lie}(R))(\mathbb{k}) = \mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$  for  $R \in \text{Max}(\Phi_{r_{\text{smax}}}^+)$ , we arrive at the conclusion.  $\square$

## 6.7 Irreducible components of $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$

In this section, we determine the irreducible components of the variety  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  when  $G$  is a standard simple algebraic  $\mathbb{k}$ -group of classical type. According to the hypothesis for a connected reductive  $\mathbb{k}$ -group  $G$  being standard, we have (i) the fundamental group  $\pi_1(G)$  has cardinal  $n + 1$  in type  $A_n$ , 2 in types  $B_n, C_n$ , and 4 in types  $D_n$  if  $G$  is simply-connected; (ii)  $G$  cannot be of types  $A_{kp-1}$  if  $G$  affords a non-degenerate bilinear form. As a result, if  $G$  is standard of classical type then  $p$  is non-torsion for  $G$ .

**Lemma 6.7.1.** *Let  $G$  be a standard simple algebraic  $\mathbb{k}$ -group with root system  $\Phi$  of type  $A_n(n \geq 5), B_n(n \geq 5), C_n(n \geq 3)$  or  $D_n(n \geq 6)$ . Then*

$$\mathbb{E}(r_{\text{smax}}, \mathfrak{g}) = \bigcup_{\substack{R \in \text{Com}(\Phi_{r_{\text{smax}}}^+) \\ R \text{ an ideal}}} G.\text{Lie}(R) \cup \bigcup_{\substack{R \in \text{Max}(\Phi_{r_{\text{smax}}}^+) \\ R \text{ an ideal}}} G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$$

is the union of irreducible closed subsets. Moreover, the ideal orbit  $G.\text{Lie}(R)$  is isomorphic to the flag variety  $G/P_R$  for  $R \in \text{Com}(\Phi_{r_{\text{smax}}}^+)$  and  $P_R = \text{Stab}_G(\text{Lie}(R))$ ; see Table 6.1 for ideals.

*Proof.* According to Corollary 6.6.2 together with [Pre2, Lemma 2.2] or [LMT, Theorem 2.2], one can easily get the decomposition for  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$ .

Notice that the operation of the root subgroup  $U_\alpha$  associated to the root  $\alpha$  on weight space  $V_\chi$  is affirmed as  $U_\alpha.V_\chi = \sum_{m \in \mathbb{N}_0} V_{\chi+m\alpha}$ ; see [MT] for details. From the definition of ideals  $R$ , the elementary subalgebra  $\text{Lie}(R)$  will be fixed by  $B$  under the adjoint action. Thus the stabilizer  $P = \text{Stab}_G(\text{Lie}(R))$  is then a standard

parabolic subgroup. By [Hum2, Proposition 0.15], the ideal orbit  $G.\text{Lie}(R)$  is then closed and of course irreducible. Furthermore,  $G.\text{Lie}(R)$  is isomorphic to the flag variety  $G/P_R$  according to [PS, Theorem 4.9].

Now we turn to consider the space  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$ . Because  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$  is complete, it is a closed subset of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$ . Since  $R$  is an ideal of  $\Phi^+$ ,  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$  is stabilized by  $B$ . Thus,  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$  is closed in  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  again by [Hum2, Proposition 0.15]. Since  $\text{Lie}(R)$  is an elementary subalgebra,  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$  equals the Grassmannian  $\text{Gr}_{r_{\text{smax}}}(\text{Lie}(R))(\mathbb{k})$  which is irreducible. Then it is natural to get the irreducibility of  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$ .  $\square$

**Lemma 6.7.2.** *Suppose that  $\mathcal{E}$  is an elementary subalgebra of dimension  $r$ ,  $R \in \text{Com}(\Phi_r^+)$  is an ideal of  $\Phi^+$  of order  $r$ . Let  $\succeq$  be a reverse lexicographic ordering with respect to any ordering of the simple roots. Assume that there is  $g \in G$  which satisfies  $g.\mathcal{E} = \text{Lie}(R)$ . Then  $\text{LT}(\mathcal{E})$  and  $R$  are conjugate by elements of  $\mathcal{W}$ , where  $\text{LT}(\mathcal{E})$  is taken with respect to  $\succeq$ .*

*Proof.* For mentioned  $g \in G$ , there exist  $b, b' \in B$  and  $w \in \mathcal{W}$  such that  $g = bw b'$  by the Bruhat decomposition of  $G$ . Because  $g.\mathcal{E} = \text{Lie}(R)$ , it can be written as  $w b' . \mathcal{E} = b^{-1}.\text{Lie}(R)$ . Notice that  $R$  is an ideal, it will stable under the action of elements of  $B$ . Thus  $w b' . \mathcal{E} = \text{Lie}(R)$  and

$$b' . \mathcal{E} = w^{-1}.\text{Lie}(R) = \text{Lie}(w^{-1}.R).$$

Observe that: (a) if  $\alpha \in \Phi^+$ , then the action of  $U_\alpha$  on  $\mathcal{E}$  is lower triangular with respect to  $\succeq$ ; (b) the action given by elements of torus is the scale multiplication; (c)  $B = \langle U_\alpha, T \mid \alpha \in \Phi^+ \rangle$  is generated by  $U_\alpha$  for  $\alpha \in \Phi^+$  and  $T$ . Then the equality  $b' . \mathcal{E} = \text{Lie}(w^{-1}.R)$  gives  $\text{LT}(\mathcal{E}) = w^{-1}.R$ , as desired.  $\square$

**Theorem 6.7.3.** *Let  $G$  be a standard simple algebraic  $\mathbb{k}$ -group with root system  $\Phi$  of type  $A_n (n \geq 5)$ ,  $B_n (n \geq 5)$ ,  $C_n (n \geq 3)$  or  $D_n (n \geq 6)$ . Then the irreducible components of  $\mathbb{E}(r_{\text{smax}}, \mathfrak{g})$  for each type can be characterised; see Table 6.2.*

*Proof.* By Lemma 6.7.1, it suffices to check the maximality of each irreducible closed subvariety. Observe that: (1) Let  $R, R' \in \text{Com}(\Phi_{r_{\text{smax}}}^+)$ , and  $G.\text{Lie}(R) \subseteq G.\text{Lie}(R')$ . If  $R$  is an ideal, then by Lemma 6.7.2  $R$  and  $R'$  are conjugate by an element

Type	Rank	Irreducible components
$A_{2n+1}$	$n \geq 2$	$G.\text{Lie}(\Phi_n^{\text{rad}}), G.\text{Lie}(\Phi_{n+2}^{\text{rad}}), G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$
$A_{2n}$	$n \geq 3$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}})), G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$
$B_n$	$n \geq 5$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(S_1))$
$C_n$	$n \geq 3$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$
$D_n$	$n \geq 6$	$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n-1}^{\text{rad}})), G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$

Table 6.2: Irreducible components for Theorem 6.7.3

of  $\mathscr{W}$ ; (2) Let  $R \in \text{Com}(\Phi_{r_{\text{smax}}}^+)$  be an ideal,  $R' \in \text{Max}(\Phi_{r_{\text{max}}})$  and  $G.\text{Lie}(R) \subseteq G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R'))$ . Then by Lemma 6.7.2  $R$  and  $\text{LT}(\mathcal{E}(R))$  are conjugate by an element of  $\mathscr{W}$ , where  $g.\mathcal{E}(R) = \text{Lie}(R)$  for some  $g \in G$ ; (3) Let  $R \in \text{Max}(\Phi_{r_{\text{max}}}^+)$  be an ideal,  $R' \in \text{Max}(\Phi_{r_{\text{max}}}^+)$  and  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R)) \subseteq G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R'))$ . Then by Lemma 6.7.2  $R \setminus \{\gamma\}$  and  $\text{LT}(\mathcal{E}(\gamma))$  are conjugate by an element of  $\mathscr{W}$ , where  $\gamma$  is an arbitrary root in  $R$  and  $\mathcal{E}(\gamma)$  is the corresponding element in  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(R'))$  satisfying  $\text{Lie}(R \setminus \{\gamma\}) = g(\gamma).\mathcal{E}(\gamma)$  for some  $g(\gamma) \in G$ . Using these, we are now going to do the analysis for each classical type in the following.

- Type  $A_{2n+1}$ . (a)  $G.\text{Lie}(\Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\})$  is not maximal because  $\text{Lie}(\Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\})$  is an element of  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$ ; (b) If  $G.\text{Lie}(\Phi_n^{\text{rad}}) \subseteq G.\text{Lie}(\Phi_{n+2}^{\text{rad}})$  or  $G.\text{Lie}(\Phi_n^{\text{rad}}) \subseteq G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$  then  $\Phi_n^{\text{rad}}$  is  $\mathscr{W}$ -conjugate to  $\Phi_{n+2}^{\text{rad}}$  or conjugate to  $\text{LT}(\mathcal{E}(\Phi_n^{\text{rad}}))$ . Both cases are impossible when we look at [PS, Lemma 2.6], this gives  $G.\text{Lie}(\Phi_n^{\text{rad}})$  is maximal. (c)  $G.\text{Lie}(\Phi_{n+2}^{\text{rad}})$  and  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$  are maximal by the same argument of (b).

- Type  $A_{2n}$ . (a)  $G.\text{Lie}(\Phi_n^{\text{rad}} \setminus \{\alpha_n\})$  and  $G.\text{Lie}(\Phi_{n+1}^{\text{rad}} \setminus \{\alpha_{n+1}\})$  are not maximal since they are contained in  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R'))$  for  $R' = \Phi_n^{\text{rad}}, \Phi_{n+1}^{\text{rad}}$  respectively; (b)  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$  and  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}}))$  are maximal. Without loss of generality, we may assume that

$$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}})) \subseteq G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n+1}^{\text{rad}})).$$

Then  $\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$  is  $\mathscr{W}$ -conjugate to  $\Phi_{n+1}^{\text{rad}} \setminus \{\gamma\}$  where  $\gamma = \alpha_1 + \dots + \alpha_{2n}$  is the highest root, and consecutively  $\Phi_n^{\text{rad}} \setminus \{\gamma\}$  and  $\Phi_{n+1}^{\text{rad}} \setminus \{\gamma\}$  are conjugate. Notice that the Weyl group of  $A_{2n}$  is the permutation group  $\mathfrak{S}_{2n+1}$ . Let  $w.\Phi_n^{\text{rad}} \setminus \{\gamma\} = \Phi_{n+1}^{\text{rad}} \setminus \{\gamma\}$  for some  $w \in \mathscr{W}$  and  $n+1 \leq j_0 < 2n+1$ . We denote by  $w(j_0)$  the corresponding action for  $j_0$  when  $w$  acts on  $\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < n+1} \subseteq \Phi_n^{\text{rad}} \setminus \{\gamma\}$ . Then  $w.\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < n+1} \subseteq \{\epsilon_i - \epsilon_{w(j_0)}\}_{1 \leq i < n+2} \subseteq \Phi_{n+1}^{\text{rad}} \setminus \{\gamma\}$  and  $w.\{\epsilon_i - \epsilon_r\}_{1 \leq i < n+1} \not\subseteq \{\epsilon_i - \epsilon_{w(j_0)}\}_{1 \leq i < n+2}$  for

$n + 1 \leq r \neq j_0 < 2n + 1$ . As  $n \geq 3$ , there exists  $j_0$  such that  $w(j_0) \neq 2n + 1$ . Then the equality  $|\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < n+1}| < |\{\epsilon_i - \epsilon_{w(j_0)}\}_{1 \leq i < n+2}|$  shows the impossibility.

- Type  $B_n$ . Because  $\text{Lie}(S_1 \setminus \{\alpha_{n-1} + 2\alpha_n\})$  is an element of  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(S_1))$ , there is only one irreducible component, i.e.  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(S_1))$ .

- Type  $C_n$ . Because  $\text{Lie}(\Phi_n^{\text{rad}} \setminus \{\alpha_n\})$  is an element of  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$ , it is readily seen that  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$  is the only irreducible component.

- Type  $D_n$ . Because  $\text{Lie}(\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\})$  is an element of  $\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n-1}^{\text{rad}}))$ , it suffices to check the maximality of  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(R))$  for  $R = \Phi_{n-1}^{\text{rad}}$  or  $R = \Phi_n^{\text{rad}}$ .

We may assume

$$G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n-1}^{\text{rad}})) \subseteq G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}})).$$

By observation (3) when  $\gamma = \alpha_{n-1}$  we have that  $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$  and  $\text{LT}(\mathcal{E}(\alpha_{n-1}))$  are  $\mathscr{W}$ -conjugate. On the other hand, one can easily check that  $\text{LT}(\mathcal{E}(\alpha_{n-1}))$  and  $\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$  are  $\mathscr{W}$ -conjugate, so there is some  $w \in \mathscr{W}$  such that  $w.\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\} = \Phi_n^{\text{rad}} \setminus \{\alpha_n\}$ . Notice that

$$\begin{aligned} \Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\} &= \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n-1\} \cup \{\epsilon_i - \epsilon_n \mid 1 \leq i < n-1\}, \\ \Phi_n^{\text{rad}} \setminus \{\alpha_n\} &= \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n-1\} \cup \{\epsilon_i + \epsilon_n \mid 1 \leq i < n-1\}. \end{aligned}$$

Let  $A := \{\epsilon_i + \epsilon_n \mid 1 \leq i < n-1\}$ . We now consider  $w^{-1}.A$  in  $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$ . By the order of set  $A$ , one can verify that

$$w^{-1}.A = B := \{\epsilon_i + \epsilon_{n-1} \mid 1 \leq i < n-1\}$$

or

$$w^{-1}.A = C := \{\epsilon_i - \epsilon_n \mid 1 \leq i < n-1\}.$$

If  $w^{-1}.A = B$ , then by observing the action of  $w$  on  $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$ , we have  $w.C = \{\epsilon_i - \epsilon_{n-1} \mid 1 \leq i < n-1\}$ . But this is impossible since  $\{\epsilon_i - \epsilon_{n-1} \mid 1 \leq i < n-1\} \notin \Phi_n^{\text{rad}} \setminus \{\alpha_n\}$ . If  $w^{-1}.A = C$ , then we find that  $w.B = \{\epsilon_i - \epsilon_{n-1} \mid 1 \leq i < n-1\}$ . This is also impossible by the same reason. Hence, the closed subset  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n-1}^{\text{rad}}))$  is maximal. The maximality of  $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$  is verified in a similar way and omitted.  $\square$

**Remark 6.7.1.** Theorem 4.11 in [PS] has revealed that  $\mathbb{E}(r_{\text{max}}, \mathfrak{g})$  is the disjoint union of  $G$ -orbits. From our proof we have seen that for  $A_{2n+1}$  the components have empty intersection, but it is not clear for  $A_{2n}$  and  $D_n$ .





# Bibliography

- [Bau] K. Baur, *Richardson elements for classical Lie algebra*. Journal of Algebra. **297** (2006), 168–185.
- [Ben] D.J. Benson, *Representations and Cohomology*. Cambridge University Press. Vols. **I** and **II** (1991).
- [BG] K. Baur, S.M. Goodwin, *Richardson Elements for Parabolic Subgroups of Classical Groups in Positive Characteristic*. Algebr Represent Theor. **11** (2008), 275–297.
- [Bor] A. Borel, *Linear Algebraic Groups*. Graduate Texts in Mathematics. Vol. **126** (1991), Springer-Verlag, New York, second edition.
- [Bou] N. Bourbaki, *Lie groups and Lie algebras*. Elements of Mathematics(Berlin). Chapters **4–6** (2002), Springer-Verlag, Berlin.
- [Car] R.W. Carter, *Simple group of Lie Type*. Pure and Applied Math. Vol. **28** (1972), John Wiley and Sons, London-New York-sydney.
- [CFP] J. Carlson, E. Friedlander and J. Pevtsova, *Elementary subalgebras of Lie algebras*. J. Algebra **442** (2015), 155–189.
- [CLNP] J. Carlson, Z.Z. Lin, D.K. Nakano, B.J. Parshall. The Restricted Nullcone
- [CM] D.H. Collingwood, W.M. McGovern, *Nilpotent Orbits In Semisimple Lie Algebras*. New York (Van Nostrand) 1993.
- [Far1] R. Farnsteiner, *Varieties of tori and cartan subalgebras of restricted Lie algebras*. Trans of AMS. **356** (2004), no.10, 4181–4236.

- [Far2] R. Farnsteiner *Combinatorial and Geometric Aspects of the Representation Theory of Finite Group Schemes*. Lie Theory and Representation Theory. SMM **2**, 47-149
- [Far3] R. Farnsteiner, *The Jordan type of indecomposable modules of finite group schemes*. J. Eur. Math. Soc. **16** (2014), no.5, 925–989.
- [Far4] R. Farnsteiner, *Representations of finite group schemes and morphisms of projective varieties*. arXiv:1401.8083 [math.RT].
- [FPar] E.M. Friedlander, B.J. Parshall, *Support varieties for restricted Lie algebras*. Invent. Math. **86** (1986), 553–562.
- [FP] E.M. Friedlander, J. Pevtsova, *Representation-Theoretic Support Spaces For Finite Group Schemes*. American Journal of Mathematics. **127** (2005), no.2, 379–420.
- [FS] E.M. Friedlander, A. Suslin, *Cohomology of finite group scheme over a field*. Invent. Math. **127** (1997), 235–253.
- [GP] I. Gordon, A. Premet, *Block representation type of reduced enveloping algebras*. Trans. Amer. Math. Soc, **354** (2002), 1549–1581.
- [Hum1] J.E. Humphreys, *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advance Mathematics. Vol. **29**, Cambridge University Press, Cambridge, 1990.
- [Hum2] J.E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*. Mathematical Surveys and Monographs. Vol. **43**, American Mathematical Society Cambridge, 1990.
- [Jan1] J.C. Jantzen, *Representations of algebraic groups*. Math. Surveys Monogr. **107**, Amer. Math. Soc, 2003.
- [Jan2] J.C. Jantzen, *Nilpotent orbits in representation theory*. Progress in mathematics. Vol. **228**.
- [KMRT] M.A. Knus, A. Merkurjev, M. Rost, J.P. Tignol, *The book of involutions*. American Mathematical Society Colloquium Publications. Vol. **44** (1998), American Mathematical Society, Providence, RI.

- 
- [Let] E. Letellier, *Fourier transformations of invariant functions on finite reductive Lie algebras*. Springer 2005.
- [Lev] P. Levy, *Commuting Varieties of Lie Algebras over Fields of Prime Characteristic*. Journal of algebra. **250** (2002):9-10, 473–484.
- [LMT] P. Levy, G. McNinch, D.M. Testerman, *Nilpotent subalgebras of Semisimple Lie algebras*. C.R.Acad.Sci.Paris. **347** (2009):9-10, 477–482.
- [Mal] A. Malcev, *Commutative subalgebras of semisimple Lie algebras*. Bull. Acad. Sci. URSS. Ser. Math. **9** (1945):291–300, In Russian.
- [MT] G. Malle, D.M. Testerman, *Linear algebraic groups and finite groups of Lie type*. Cambridge Studies in Advanced Mathematics. Vol. **133** (2011), Cambridge University Press, Cambridge.
- [McG] W. McGovern, *The Adjoint Representation and Adjoint Action*. Invariant Theory and Algebraic Transformation Groups II, Encyclopaedia of Mathematics Sciences. Springer-Verlag, 2000.
- [McN] G.J. McNinch, *Optimal  $SL(2)$ -homomorphisms*. Comment. Math. Helv. **80** (2005), 391–426.
- [McT] G.J. McNinch, D.M. Testerman, *Nilpotent centralizers and Springer isomorphisms*. Journal of Pure and Applied Algebra **213**(2009), 1346-1363.
- [Pre1] A. Premet, *Nilpotent commuting variety of reductive Lie algebras*. Invent. math. **154** (2003), 653–683.
- [Pre2] A. Premet, *A Modular Analogue of Morozov’s Theorem on Maximal subalgebras of simple Lie algebras*. arXiv:1512.05614v4 [math. RA].
- [PS] J. Pevtsova, J. Stark, *Varieties of elementary subalgebras of maximal dimension for modular Lie algebras*. arXiv:1503.01043 [math. RT].
- [Qui1] D. Quillen, *The spectrum of an equivariant cohomology ring, I*. Ann. of Math. **94** (1971), 549–572.
- [Qui2] D. Quillen, *The spectrum of an equivariant cohomology ring, II*. Ann. of Math. **94** (1971), 573–602.

- [Ric] R.W. Richardson, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups*. Bull. London Math. Soc. **6** (1974), 21–24.
- [RRS] R. Richardson, G. Roehrl, R. Steinberg, *Parabolic subgroups with abelian unipotent radical*. Invent Math. **110** (1992), 649–671.
- [SFB1] A. Suslin, E.M. Friedlander, C.P. Bendel, *Infinitesimal 1-parameter subgroups and Cohomology*. Journal of The American Mathematical Society. 693–728, 1997.
- [SFB2] A. Suslin, E.M. Friedlander, C.P. Bendel, *Support varieties for infinitesimal group schemes*. Journal of The American Mathematical Society. 729–759, 1997.
- [Spr] T.A. Springer, *Linear algebraic group*. Progress in Mathematics. Volume **9** (1998), Birkhaeuser Boston, Inc., Boston, MA, second edition.
- [Ste] R. Steinberg, *Torsion in reductive groups*. Advances in Math. **15** (1975), 63–92.
- [Wat] W.C. Waterhouse, *Introduction to Affine Group Schemes*. Grad. Text in Math. **66**, Springer (1979). Springer-Verlag.

# Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht.

Kiel, den 26. Oktober 2016

(Yang Pan)



