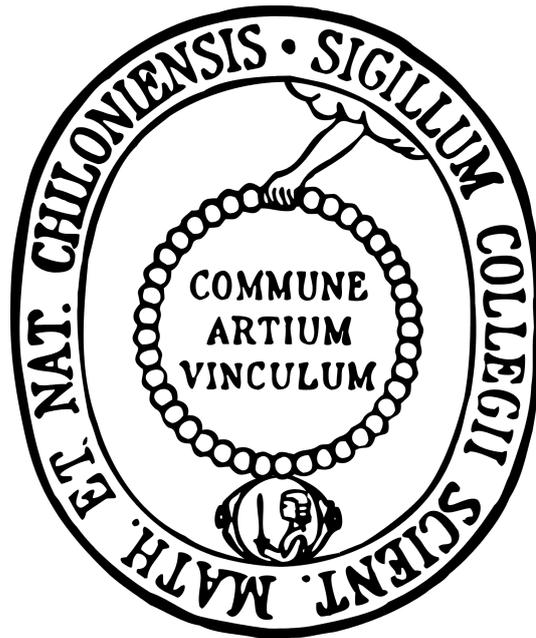


Affine Structures of Convex Bodies and Duality Aspects



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Zusammenfassung

In der vorliegenden Arbeit behandeln wir verschiedene Themen aus der affinen Konvexgeometrie und untersuchen insbesondere Dualitätsaspekte.

Im ersten Teil der Arbeit widmen wir uns dem Studium affin invarianter Punkte. Wir beweisen eine Vermutung von B. Grünbaum, die eine Verbindung zwischen der Menge der affin invarianten Punkte eines Konvexkörpers und den Symmetrien herstellt. Ausgehend von diesem Resultat behandeln wir verschiedene Fragestellungen, die in Arbeiten von B. Grünbaum, M. Meyer, C. Schütt und E. Werner zu affin invarianten Punkten und Symmetriemaßen aufgeworfen werden. Wir geben einen alternativen Beweis für die Tatsache, dass der Raum der affin invarianten Punkte unendlich-dimensional ist. Wir beweisen, dass die Menge der affin invarianten Punkte, die keinen dualen affin invarianten Punkt besitzen, von zweiter Kategorie sind. Ferner geben wir ein Beispiel an, das zeigt, dass John- und Löwner-Punkt eines Konvexkörpers weit auseinander liegen können.

Im zweiten Teil der Arbeit behandeln wir die Frage, inwieweit Schwimm- und Illuminationskörper zueinander duale Konzepte sind. Wir untersuchen dieses Problem, indem wir den Abstand von der Polaren des Schwimmkörpers zum Illuminationskörper der Polaren eines Konvexkörpers abschätzen. Wir geben für ℓ_p^d -Kugeln und für zentralsymmetrische Ovaloide Abschätzungen des Abstandes im Fall an, dass der Parameter des Schwimmkörpers gegen 0 geht. Die Abschätzungen sind im Fall $p = 1$ und im Fall $2 < p < \infty$ sowie im Fall der zentralsymmetrischen Ovaloide optimal. Es stellt sich heraus, dass im Fall $p = 1$ und $p = \infty$, also in den Polytopfällen, der Abstand groß ist und im Fall $2 \leq p < \infty$ und für Ovaloide der Abstand verhältnismäßig klein ist. Ferner wird gezeigt, dass Gleichheit von der Polaren des Schwimmkörpers und vom Illuminationskörpers der Polaren im Fall zentralsymmetrischer Ovaloide Ellipsen charakterisiert.

Abstract

In this thesis we treat different topics from affine convex geometry and in particular, we investigate duality aspects.

The first part is dedicated to the study of affine invariant points. We prove a conjecture of B. Grünbaum that links the set of affine invariant points of a convex body to its symmetries. Based on this result we treat several questions from papers of B. Grünbaum, M. Meyer, C. Schütt and E. Werner about affine invariant points and symmetry measures. We give an alternative proof of the fact that the space of affine invariant points is infinite dimensional. We show that the set of affine invariant points which do not possess a dual affine invariant point is of second category. Furthermore, we give an example that demonstrates that the John and Löwner point of a convex body can be far apart.

In the second part of this thesis we examine the question to what extent the floating and illumination bodies are dual notions. We investigate this problem by estimating the distance of the polar of the floating body and the illumination body of the polar of a convex body. We provide estimates for the distance for ℓ_p^d -balls and for centrally symmetric ovaloids in the case that the parameter of the floating body tends to 0. The estimates are optimal for the cases $p = 1$ and $2 < p < \infty$ and in the case of centrally symmetric ovaloids. It turns out that the distance is large for the case $p = 1$ and $p = \infty$, i.e., in the polytope cases, and relatively small for the case $2 \leq p < \infty$ and for ovaloids. Furthermore, we prove that equality of the floating body and the illumination body of the polar characterizes ellipsoids among centrally symmetric convex bodies.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

Affine notions play an important role in the study of convex bodies. An elementary example is the Banach-Mazur distance on the set of convex bodies which identifies two convex bodies if one can be transformed into the other by an affine linear map. We may give more advanced examples of affine structures later. Another central concept in convex geometry is the duality structure on the set of convex bodies which is given by the polar body. If C is a convex body with the origin in its interior the polar body C° is defined by $\{y \in \mathbb{R}^d : \forall x \in C : \langle x, y \rangle \leq 1\}$. The polar is a central notion in local Banach space theory because if $(\mathbb{R}^d, \|\cdot\|)$ is a normed (Banach) space and $(\mathbb{R}^d, \|\cdot\|')$ is the dual space then the unit ball of $(\mathbb{R}^d, \|\cdot\|)$ is a centrally symmetric convex body and the polar body is the unit ball of $(\mathbb{R}^d, \|\cdot\|')$. One of the most famous examples in convex geometry where polarity is studied is the notion of volume products. For a convex body C the volume product is defined by $\mathcal{P}(C) = \inf_{z \in \text{int}(C)} |C|_d |(C - z)^\circ|_d$ where $|\cdot|_d$ is the d -dimensional Lebesgue measure. The volume product links the polar with the affine structure of convex bodies since the volume product does not change under affine transformations of the convex body C . The well-known Blaschke-Santaló inequality states that the maximum of \mathcal{P} is attained for the euclidean ball (see [Bl], [Sa]). The Mahler conjecture states that the minimum of \mathcal{P} on the set of convex bodies is attained by simplices and on the set of centrally symmetric convex bodies by the cube. By a result of Santaló (see [Sa]) there is a unique point $s(C) \in \text{int}(C)$ for every convex body C – also called the Santaló point – such that $|(C - s(C))^\circ|_d = \inf_{z \in \text{int}(C)} |(C - z)^\circ|_d$. For every convex body C and every affine linear and invertible map T we have the equivariance relation $s(T(C)) = Ts(C)$. The Santaló point is an example for an affine invariant point. Affine invariant points were first introduced

by Grünbaum in his seminal paper on symmetry measures of convex bodies [G]. Recently, M. Meyer, C. Schütt and E. Werner answered some of his questions in [G] and extended the theory of affine invariant points (cf. [MSW1], [MSW2] and [MSW3]). Chapter 2 is dedicated to the study of affine invariant points and this part is essentially contained in [M].

Denote by \mathcal{K}_d the set of convex body in \mathbb{R}^d . An affine invariant point is a map $p : \mathcal{K}_d \rightarrow \mathbb{R}^d$ such that for every convex body C and every invertible affine map T we have

$$p(T(C)) = T(p(C)) \quad .$$

We also require p to be continuous with respect to the Hausdorff distance and the euclidean norm. Along with the Santaló point s there are three other examples of affine invariant points in [G], namely the centroid g (i.e. center of gravity), the center of the John ellipsoid j and the center of the Löwner ellipsoid l . Other examples of affine invariant points were constructed in [MSW2] and [MSW3].

In section 2.1 we answer a question of Grünbaum whether we can relate the symmetry structure of a convex body to the set of affine invariant points which gives us enough freedom to construct new classes of affine invariant points. Denote by

$$\mathfrak{P}_d(C) = \{p(C) \in \mathbb{R}^d : p \text{ affine invariant point}\}$$

and by

$$\mathfrak{F}_d(C) = \{x \in \mathbb{R}^d : Tx = x \text{ for every } T \text{ affine linear with } T(C) = C\}$$

then we have:

Theorem 1.1.1 *For every $C \in \mathcal{K}_d$, we have*

$$\mathfrak{P}_d(C) = \mathfrak{F}_d(C) \quad .$$

The idea of $\mathfrak{F}_d(C)$ following [G] is essentially that C is more symmetric the smaller the set $\mathfrak{F}_d(C)$ is. There have been some major contributions to prove this conjecture. In [MSW2], there is a proof for the case that the affine dimension of $\mathfrak{P}_d(C)$ equals to $d - 1$. In 2014 I. Iurchenko showed that this conjecture is true for the case $\mathfrak{F}_d(C) = \mathbb{R}^d$ (see [I]). Recently, Jonard-Pérez and Iurchenko provided alternative proofs for Theorem 1.1.1 (see [JP], [I2]). In [K1] (see also [K2] for an English translation), P. Kuchment proved that the conjecture holds for classes of points $p : \mathcal{K}_d \rightarrow \mathbb{R}^d$ with $p(T(C)) = Tp(C)$ for every affine map T such that the linear part is an element of a fixed compact subgroup of $GL(d)$. He obtained an even stronger variant of this conjecture for points with $p(C) \in \text{int}(C)$ which we prove analogously for affine invariant points (see Proposition 2.4.1). He also proved the conjecture for similarity invariant points by reducing the problem to convex bodies

with volume 1 which leads to the compact case. The proof of Theorem 1.1.1 is based on ideas of [K1]: We reduce the problem to convex bodies in John position which leads also to just consider the compact subgroup $O(d) \subseteq GL(d)$. We decided to avoid the fibre bundle language of [K1] and to replace the short and elegant but abstract topological arguments by explicit computations in order to make the proof more transparent.

The set \mathfrak{P}_d of all affine invariant points of \mathcal{K}_d is an affine subspace of $C(\mathcal{K}_d, \mathbb{R}^d)$, the space of continuous functions from \mathcal{K}_d to \mathbb{R}^d , since one can show that every affine combination of affine invariant points is itself an affine invariant point. The following theorem answers a question of Grünbaum and is proven in [MSW2].

Theorem 1.1.2 *The affine dimension of \mathfrak{P}_d is infinite.*

In section 2.2 we use Theorem 1.1.1 in order to provide an alternative proof of this theorem in a more concise way without using the technique of floating bodies.

In section 2.3 we apply Theorem 1.1.1 to duality questions about convex bodies. It is a classical fact that the centroid and the Santaló point as well as the Löwner and John point have a duality property in the following sense. L.A. Santaló proved in [Sa] that we always have the identity $s((C - g(C))^o) = 0$. For the Löwner and John point we have $j((C - l(C))^o) = 0$ as well. Let p be an affine invariant point such that $p(C) \in \text{int}(C)$ for every convex body C . A general definition for duality following [MSW3] is that an affine invariant point q is dual to p if and only if

$$q((C - p(C))^o) = 0$$

for every $C \in \mathcal{K}_d$. Note that q is dual to p if and only if p is dual to q and we refer the reader to [MSW3] for further properties on duality. A central result of this paper is the fact that there are affine invariant points with no dual. With Theorem 1.1.1, we are able to show something even stronger. We equip \mathfrak{P}_d with the following distance

$$\text{dist}(p_1, p_2) = \sup_{B_2^d \subseteq C \subseteq dB_2^d} \|p_1(C) - p_2(C)\|_2$$

(see also [MSW2], section 3.2) and we then have

Theorem 1.1.3 *There exists an open and dense set $W \subseteq \mathfrak{P}_d$ of affine points with no dual.*

In section 2.4 we consider symmetry measures for convex bodies involving affine invariant points. In [MSW1], the following class of symmetry measures is introduced. For $p_1, p_2 \in \mathfrak{P}_d$ such that we have $p_1(C), p_2(C) \in C$ for every convex body C , put

$$\phi_{p_1, p_2}(C) = \begin{cases} 1 & \text{if } p_1(C) = p_2(C) \\ 1 - \frac{\|p_1(C) - p_2(C)\|_2}{\text{vol}_1(a \cap C)} & \text{else} \end{cases}$$

where a is the line through $p_1(C)$ and $p_2(C)$. The idea is that a convex body lacks symmetry if ϕ is close to 0. The concern of [MSW1] is to give extremal cases for this class of symmetry measures. We give a positive answer to the question if $\phi_{p_1, p_2}(C) = 0$ can happen for some p_1, p_2 and we improve a result for the case $p_1 = j$ and $p_2 = l$.

Chapter 3 deals with duality questions concerning floating and illumination bodies. Floating and illumination bodies are an important tool in affine convex geometry. A notion of floating bodies and floating surfaces appeared already in the work of C. Dupin [D] in 1822. In 1990, a new definition was given by Schütt and Werner [SW1] and independently by Bárány and Larman [BL]. The illumination body was introduced in [W1]. If C is a convex body and $\delta \geq 0$ then the illumination body C^δ is defined by

$$C^\delta = \{x \in \mathbb{R}^d : |\text{conv}[C, x]|_d \leq (1 + \delta)|C|_d\}$$

and the floating body C_δ is defined by

$$C_\delta = \bigcap_{|C \cap H^-|_d \leq \delta |C|_d} H^+$$

where the H 's are hyperplanes and H^+, H^- are the corresponding half-spaces. Please note that C^δ is always a convex body and that C_δ might be empty or only a singleton if δ is not small enough. We have the nice equivariance property that for affine linear and invertible maps T we have $(T(C))^\delta = T(C^\delta)$ and $(T(C))_\delta = T(C_\delta)$. Floating and illumination bodies were used in [MSW2] and [MSW3] for the construction of new affine invariant points. Even more important is the fact that floating and illumination bodies can be used to define the affine and even the p -affine surface area (see [SW1], [W1] and [W2]). It was noted in [[W3], p.8] that floating and illumination bodies are somewhat dual in the following sense. If C is a convex body with the origin in its interior, $z \in \mathbb{R}^d$ and $H^+ = \{x \in \mathbb{R}^d : \langle x, z \rangle \leq 1\}$ then $\text{conv}[C, z]^\circ = C^\circ \cap H^+$ and $(C \cap H^+)^\circ = \text{conv}[C^\circ, z]$. Hence, since C_δ is the intersection of half-spaces the polar of C_δ is the union of vectors corresponding to those half-spaces, and since the illumination body of C^δ is the union of vectors the polar of C^δ is the intersection of half-spaces corresponding to those vectors. This leads to the idea that for $\delta > 0$ there should be a $\delta' > 0$ such that $(C_\delta)^\circ \approx (C^\circ)^{\delta'}$. It was also noted in [W3] that we do not have equality in general since for positive $\delta > 0$ the floating body is

always strictly convex but the illumination body of a polytope is always a polytope and therefore the polar is also a polytope.

Up to our knowledge there is no systematic study of this topic in the literature and we like to shed some light on this question. We consider this question for different classes of centrally symmetric convex bodies. Since we cannot hope to have $(C_\delta)^\circ = (C^\circ)^{\delta'}$ in general we need a measure of approximation. We define a distance \mathbf{d} (not to be confused with the dimension d) on the set of centrally symmetric convex bodies by

$$\mathbf{d}(S_1, S_2) = \inf\{a \geq 1 : \frac{1}{a}S_1 \subseteq S_2 \subseteq aS_1\} \quad .$$

For a centrally symmetric body S we define $\mathbf{d}_S(\delta)$ to be the best approximation of $(S_\delta)^\circ$ by $(S^\circ)^{\delta'}$, i.e.

$$\mathbf{d}_S(\delta) = \inf_{\delta' \geq 0} \mathbf{d}\left((S_\delta)^\circ, (S^\circ)^{\delta'}\right) = \inf_{\delta' \geq 0} \mathbf{d}\left(S_\delta, \left[(S^\circ)^{\delta'}\right]^\circ\right) \quad .$$

In sections 3.1, 3.2 and 3.3 we consider the class of ℓ_p^d -balls and in section 3.4 we consider the class of centrally symmetric convex bodies with everywhere positive Gauss curvature. The main result of chapter 3 is the following.

Theorem 1.1.4 *Let $d \geq 2$. Then the following holds.*

1. If $p = 1$ then

$$\lim_{\delta \rightarrow 0} \frac{\mathbf{d}_{B_1^d}(\delta) - 1}{\delta^{1/d}} = \frac{2^{1/d}}{2}.$$

2. For every $1 < p < 2$ there is a positive constant $c(p, d) > 0$ such that

$$\liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_{B_p^d}(\delta) - 1}{\delta^{\frac{p}{d-1+p}}} \geq c(p, d) \quad .$$

3. For every $2 < p < \infty$ there is a positive constant $c(p, d)$ such that

$$\lim_{\delta \rightarrow 0} \frac{\mathbf{d}_{B_p^d}(\delta) - 1}{\delta^{\frac{2}{d+1}}} = c(p, d) \quad .$$

4. If $p = \infty$ then

$$\frac{2\sqrt[d]{\gamma_d}}{1 + 2^{d-1}} \leq \liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_{B_\infty^d}(\delta) - 1}{\delta^{1/d}} \leq \limsup_{\delta \rightarrow 0} \frac{\mathbf{d}_{B_\infty^d}(\delta) - 1}{\delta^{1/d}} \leq \frac{2^d \sqrt[d]{\gamma_d}}{1 + 2^{d-1}}$$

where $\gamma_d = \frac{d!}{d^d}$.

5. If $C \subseteq \mathbb{R}^d$ is a centrally symmetric convex body with everywhere positive Gauss curvature then

$$\lim_{\delta \rightarrow 0} \frac{\mathfrak{d}_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} = G_{\max} - G_{\min}$$

where G_{\max} and G_{\min} are defined as the maximum and minimum of the function $G : \partial C \rightarrow \mathbb{R}_{\geq 0}$,

$$G(x) = \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle}$$

and $\kappa(x)$ is the Gauss curvature and $N(x)$ the outer normal at x .

The theorem shows that the convergence rate is good in the cases where the convex body has a C^2 boundary, i.e., in the case $2 < p < \infty$ and in the case of everywhere positive Gauss curvature and bad if we are in a polytope case, i.e., if $p = 1$ or $p = \infty$.

In section 3.4 we prove a corollary of this theorem stating that for a centrally symmetric convex body with everywhere positive Gauss curvature the condition $\mathfrak{d}_C(\delta) = 0$ for every $\delta \geq 0$ is only true for ellipsoids.

The second part of this thesis is joint work with Elisabeth Werner and we point out her contributions at the beginning of chapter 3.

1.2 Preliminaries

In this section we only define basic notions we need throughout the whole thesis. Notations and definitions which are only important for the chapter itself will be defined at the beginning of each chapter. We denote by \mathbb{N} the set of positive integers. Let $d \in \mathbb{N}$ and equip \mathbb{R}^d with the euclidean norm $\|\cdot\|_2$. We denote by $|\cdot|_k$ the k -dimensional Hausdorff measure on \mathbb{R}^d . In particular, $|\cdot|_d$ is the Lebesgue measure on \mathbb{R}^d . We also use $|\cdot|$ for the absolute value since this will not cause any confusion. We denote by $\|\cdot\|_{O_p}$ the spectral norm on $\mathbb{R}^{d \times d}$. We denote by $\{e_1, \dots, e_d\}$ the canonical basis of \mathbb{R}^d , i.e., e_i is the vector where the i -th entry is 1 and the other entries are 0. The boundary of a set $M \subseteq \mathbb{R}^d$ is denoted by ∂M , the interior by $\text{int}(M)$ and the closure by \bar{M} .

Definition 1.2.1 *Let $C \subseteq \mathbb{R}^d$. We call C a d -dimensional convex body if C is convex, compact and has non-empty interior. We denote by \mathcal{K}_d the set of d -dimensional convex bodies.*

A classical example for symmetric convex bodies are the l_p^d -balls which are defined by

$$B_p^d = \left\{ x \in \mathbb{R}^d : \sum_{k=1}^d |x_k|^p \leq 1 \right\}$$

for $p \in [1, \infty)$ and

$$B_\infty^d = \{x \in \mathbb{R}^d : \max_{k=1\dots d} |x_k| \leq 1\} \quad .$$

We denote the convex hull by conv . For $x, y \in \mathbb{R}^d$ the line segment between x and y is defined by $[x, y] = \text{conv}[x, y]$. For $a \in \mathbb{R}$ and $C_1, C_2 \in \mathcal{K}_d$ we define $aC_1 = \{ax \in \mathbb{R}^d : x \in C_1\}$ and $C_1 + C_2 = \{x + y \in \mathbb{R}^d : x \in C_1 \text{ and } y \in C_2\}$. The *centroid* g of a convex body C is defined by

$$g(C) = \frac{1}{|C|_d} \int_C x dx \quad .$$

We make use of the Landau symbols $o(\cdot)$ and $O(\cdot)$. Throughout this thesis we refer by d to the dimension (of the convex body or the underlying space \mathbb{R}^d) and we may always assume from now on that $d \geq 2$ since our results become either trivial or false for the case $d = 1$.

Chapter 2

Affine Invariant Points

For background information on affine invariant points, we refer the reader to [G], [K1], [K2], [MSW1], [MSW2] and [MSW3]. We follow the presentation of [MSW2], [MSW3]. To be self-contained, we recall some notions of these these papers.

We equip \mathcal{K}_d with the Hausdorff distance d_H which for $C_1, C_2 \in \mathcal{K}_d$ is defined by

$$d_H(C_1, C_2) = \min\{\alpha \geq 0 : C_1 \subseteq C_2 + \alpha B_2^d \text{ and } C_2 \subseteq C_1 + \alpha B_2^d\} \quad .$$

We denote by $GL(d)$ the group of linear, invertible operators from \mathbb{R}^d to \mathbb{R}^d and by $O(d) \subseteq GL(d)$ the subgroup of orthogonal operators. Let

$$\mathcal{AT}_d = \{T : T = L + b, L \in GL(d) \text{ and } b \in \mathbb{R}^d\}$$

be the set of affine linear and invertible transformations of \mathbb{R}^d where $L + b$ should be understood as the map $L + b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $(L + b)(x) = L(x) + b$. From now on, we omit the d if it is clear from the context.

Definition 2.0.1 *Let $p : \mathcal{K}_d \rightarrow \mathbb{R}^d$ be a map which is continuous with respect to the Hausdorff distance and the euclidean norm. We call p an affine invariant point if*

$$p(T(C)) = T(p(C))$$

holds for every $C \in \mathcal{K}_d$ and $T \in \mathcal{AT}$.

This notion was introduced by Grünbaum in [G] and we adapt it here. It should be noted that it would be more adequate to call those maps affine equivariant points and throughout this thesis, we refer by equivariance to the identity $p(T(C)) = Tp(C)$. Also, we call affine invariant points more shortly affine points.

Let \mathfrak{P}_d be the set of affine points. This is an affine subspace of the vector space $C(\mathcal{K}_d, \mathbb{R}^d)$ of continuous functions from \mathcal{K}_d to \mathbb{R}^d . We put $\mathfrak{P}_d(C) = \{p(C) : p \in \mathfrak{P}_d\}$ and

$$\mathfrak{F}_d(C) = \{x \in \mathbb{R}^d : T(x) = x \text{ for every } T \in \mathcal{AT} \text{ with } T(C) = C\}$$

which are both affine subspaces of \mathbb{R}^d . This means $\mathfrak{F}_d(C)$ is the affine subspace of all points which do not change under symmetries of C , i.e. under $T \in \mathcal{AT}$ with $T(C) = C$. For $p \in \mathfrak{P}_d$, $C \in \mathcal{K}_d$ and $T \in \mathcal{AT}$ with $T(C) = C$ we have

$$p(C) = p(T(C)) = T(p(C))$$

and hence, $p(C) \in \mathfrak{F}_d(C)$, i.e., we always have $\mathfrak{P}_d(C) \subseteq \mathfrak{F}_d(C)$. For $M \subseteq \mathcal{AT}$ and $X \subseteq \mathcal{K}_d$, we put

$$M(X) = \{T(C) : T \in M \text{ and } C \in X\} \quad .$$

For $t \in \mathbb{R}$ and $C \in \mathcal{K}_d$, we put $(t, C) = \{(t, x) : x \in C\}$.

It is a classical fact of convex geometry that for every convex body C there exists a unique ellipsoid $\mathcal{J}(C)$ of maximal volume inside C , called the John ellipsoid, and a unique ellipsoid $\mathcal{L}(C)$ of minimal volume including C , called the Löwner ellipsoid. The maps $\mathcal{J} : \mathcal{K}_d \rightarrow \mathcal{K}_d$ and $\mathcal{L} : \mathcal{K}_d \rightarrow \mathcal{K}_d$ are continuous with respect to the Hausdorff distance and we have $\mathcal{J}(T(C)) = T(\mathcal{J}(C))$ and $\mathcal{L}(T(C)) = T(\mathcal{L}(C))$ for every $T \in \mathcal{AT}$ and $C \in \mathcal{K}_d$. We say that a convex body is in John position (Löwner position) if the euclidean ball is the John ellipsoid (Löwner ellipsoid). The following theorem is due to F. John and K. Ball (see [Ba], [J]).

Theorem 2.0.1 *Let C be a convex body with $B_2^d \subseteq C$. Then C is in John position if and only if there exist contact points $v_1, \dots, v_m \in \partial C \cap \partial B_2^d$ and weights $c_1, \dots, c_m > 0$ with $\sum_{i=1}^m c_i v_i v_i^{tr} = \text{Id}_d$ and $\sum_{i=1}^m c_i v_i = 0_d$.*

An analogue result is also true for the Löwner position. The following corollary is found in [J].

Corollary 2.0.2 *Let $C \in \mathcal{K}_d$ be a convex body in John position. Then we have $C \subseteq dB_2^d$.*

2.1 Proof of Grünbaum's conjecture

Let $\mathcal{K}_d^J = \{K \in \mathcal{K}_d : B_2^d \subseteq K \text{ John-ellipsoid of } K\}$ be the set of convex bodies in John position. \mathcal{K}_d^J is an $O(d)$ -invariant subset of \mathcal{K}_d , i.e.

$O(d)(\mathcal{K}_d^J) \subseteq \mathcal{K}_d^J$ (and obviously, we may replace ' \subseteq ' by ' $=$ '). Moreover, \mathcal{K}_d^J is a closed subset of \mathcal{K}_d . Let $\tilde{p} : \mathcal{K}_d^J \rightarrow \mathbb{R}^d$ be a continuous map with $\tilde{p}(L(C)) = L\tilde{p}(C)$ for every $C \in \mathcal{K}_d^J$ and $L \in O(d)$. Then we may extend \tilde{p} (in a unique way) to an affine point. We remark that for every $C \in \mathcal{K}_d$ there are $T \in \mathcal{AT}$ and $K \in \mathcal{K}_d^J$ with $C = T(K)$.

Lemma 2.1.1 *The map $p : \mathcal{K}_d \rightarrow \mathbb{R}^d$, $p(TK) = T\tilde{p}(K)$ for $K \in \mathcal{K}_d^J$ and $T \in \mathcal{AT}$ is an affine point.*

Proof. First, we show that p is well-defined. Let $K, K' \in \mathcal{K}_d^J$ be two convex bodies in John position and let T, T' be two invertible affine maps such that $T(K) = T'(K')$. We show $T\tilde{p}(K) = T'\tilde{p}(K')$. From $T'^{-1}T(K) = K'$, we deduce $B_2^d = \mathcal{J}(K') = \mathcal{J}(T'^{-1}T(K)) = T'^{-1}T\mathcal{J}(K) = T'^{-1}TB_2^d$ and hence, $T'^{-1}T \in O(d)$. It follows

$$T'\tilde{p}(K') = T'\tilde{p}((T'^{-1}T)(K)) = T'(T'^{-1}T)\tilde{p}(K) = T\tilde{p}(K)$$

Next, we show that p is continuous. We show this by proving that for every convergent sequence $\lim_{n \rightarrow \infty} C_n = C$, there is a subsequence $(C_{n_m})_m$ such that $\lim_{m \rightarrow \infty} p(C_{n_m}) = p(C)$. Let $C_n = L_n(K_n) + b_n$ with $K_n \in \mathcal{K}_d^J$, $L_n \in GL(d)$ and $b_n \in \mathbb{R}^d$ for $n \in \mathbb{N}$ and $C = L(K) + b$ with L, K, b , accordingly. Since the John point $j(C_n) = b_n$ is continuous as an affine point we have $\lim_{n \rightarrow \infty} b_n = b$ and hence, $\lim_{n \rightarrow \infty} L_n(K_n) = L(K)$. There are constants $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 B_2^d \subseteq L(K) \subseteq \alpha_2 B_2^d$. Let $0 < \varepsilon < \alpha_1$ be fixed. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$(\alpha_1 - \varepsilon)B_2^d \subseteq L_n(K_n) \subseteq (\alpha_2 + \varepsilon)B_2^d \quad .$$

We have $B_2^d \subseteq M \subseteq dB_2^d$ for $M \in \mathcal{K}_d^J$ and we conclude that for $n \geq N$, we have

$$(\alpha_1 - \varepsilon)B_2^d \subseteq dL_n(B_2^d) \text{ and } L_n(B_2^d) \subseteq (\alpha_2 + \varepsilon)B_2^d \quad ,$$

i.e., the singular values of L_n are bounded from below by $\frac{\alpha_1 - \varepsilon}{d}$ and from above by $\alpha_2 + \varepsilon$. We consider the set of all operators with singular values bounded from below by $\frac{\alpha_1 - \varepsilon}{d}$ and from above by $\alpha_2 + \varepsilon$:

$$\{UDV \in GL(d) : U, V \in O(d) \text{ and } D = \text{diag}(s_1, \dots, s_d), \frac{\alpha_1 - \varepsilon}{d} \leq s_i \leq \alpha_2 + \varepsilon\}$$

where $\text{diag}(a_1, \dots, a_d)$ denotes the diagonal matrix with diagonal entries a_1, \dots, a_d . This set is compact as the image of the compact set

$$O(d) \times \{\text{diag}(s_1, \dots, s_2) \in GL(d) : \frac{\alpha_1 - \varepsilon}{d} \leq s_i \leq \alpha_2 + \varepsilon\} \times O(d)$$

with respect to the continuous map $\omega : GL(d) \times GL(d) \times GL(d) \rightarrow GL(d)$, $\omega(A, B, C) = ABC$. Therefore, there exists a convergent subsequence $(L_{n_m})_m$ with limit L' in $GL(d)$. From $\lim_{m \rightarrow \infty} L_{n_m}(K_{n_m}) = L(K)$ it follows $\lim_{m \rightarrow \infty} K_{n_m} = L'^{-1}L(K)$ and since \mathcal{K}_d^J is closed, we have $L'^{-1}L(K) \in \mathcal{K}_d^J$ and hence, $L'^{-1}L \in O(d)$. This yields:

$$\begin{aligned} \lim_{m \rightarrow \infty} p(L_{n_m}(K_{n_m})) &= \lim_{m \rightarrow \infty} L_{n_m} \tilde{p}(K_{n_m}) = L' \tilde{p}((L'^{-1}L)(K)) \\ &= L'(L'^{-1}L) \tilde{p}(K) = L \tilde{p}(K) = p(L(K)) \end{aligned}$$

The proof of $Tp(C) = p(T(C))$ is straight-forward. □

We can now prove Grünbaum's conjecture. We only have to show that for every $x_0 \in \mathfrak{F}_d(K)$ there is an affine point p such that $p(K) = x_0$ since we always have $\mathfrak{P}_d(K) \subseteq \mathfrak{F}_d(K)$.

Theorem 2.1.2 *Let $K \in \mathcal{K}_d$ be a convex body and $x_0 \in \mathfrak{F}_d(K)$. Then there is an affine point $p : \mathcal{K}_d \rightarrow \mathbb{R}^d$ such that $p(K) = x_0$.*

Proof. By an affine transformation of K and x_0 we may assume without loss of generality that K is in John position. Using the preceding lemma it is sufficient to construct a continuous and $O(d)$ -equivariant map $\tilde{p} : \mathcal{K}_d^J \rightarrow \mathbb{R}^d$ such that $\tilde{p}(K) = x_0$. We construct \tilde{p} by an averaging argument. We start with the map $\theta : O(d)(K) \rightarrow \mathbb{R}^d$

$$\theta(L(K)) = L(x_0) \quad .$$

This map is well-defined and continuous: Suppose $L(K) = L'(K)$, i.e. $L'^{-1}L(K) = K$. Since $x_0 \in \mathfrak{F}_d(K)$ it follows $L'^{-1}L(x_0) = x_0$, i.e. $L(x_0) = L'(x_0)$ and hence θ is well-defined.

For the continuity we show that every convergent sequence in $O(d)(K)$ has a subsequence such that the image of this subsequence converges to the image of the limit. Suppose $(L_n(K))_{n \in \mathbb{N}}$ converges to an $L(K) \in O(d)(K)$. By similar arguments as the ones in Lemma 2.1.1, we may extract a convergent subsequence $(L_{n_m})_{m \in \mathbb{N}}$ with limit say L' . First, note that $\lim_{m \rightarrow \infty} L_{n_m}(x_0) = L'(x_0)$. From

$$L(K) = \lim_{m \rightarrow \infty} L_{n_m}(K) = L'(K)$$

we conclude that $L'^{-1}L(K) = K$ and therefore, $L'(x_0) = L(x_0) = \theta(L(K))$. We remark that $O(d)(K)$ is compact since $\iota : O(d) \rightarrow \mathcal{K}_d$, $\iota(L) = L(K)$ is continuous and therefore, $O(d)(K)$ is compact as an image of a compact set, and in particular, closed in \mathcal{K}_d^J . By the Tietze-Urysohn lemma, there is a continuous extension of θ to a function θ' on \mathcal{K}_d^J . Let μ be the normalized (i.e. probability) Haar measure on $O(d)$. We put $\tilde{p} : \mathcal{K}_d^J \rightarrow \mathbb{R}^d$

$$\tilde{p}(K) = \int_{O(d)} L^{-1}\theta'(L(K))d\mu(L)$$

and show that \tilde{p} is continuous, $O(d)$ -equivariant and $\tilde{p}(K) = x_0$. Indeed,

$$\tilde{p}(K) = \int_{O(d)} L^{-1}\theta'(L(K))d\mu(L) = \int_{O(d)} L^{-1}L(x_0)d\mu(L) \int_{O(d)} x_0d\mu(L) = x_0 \quad .$$

For every $\Lambda \in O(d)$ and $C \in \mathcal{K}_d^J$ we have using the invariance of the Haar measure (for $O(d)$, left- and right-invariance are the same):

$$\begin{aligned} \tilde{p}(\Lambda(C)) &= \int_{O(d)} L^{-1}\theta'(L\Lambda(C))d\mu(L) = \int_{O(d)} (L\Lambda^{-1})^{-1}\theta'((L\Lambda^{-1})k(C))d\mu(L) \\ &= \Lambda \int_{O(d)} L^{-1}\theta'(L(C))d\mu(L) = \Lambda\tilde{p}(C) \end{aligned}$$

As for the continuity suppose the sequence $(C_n)_{n \in \mathbb{N}}$ converges to C in \mathcal{K}_d^J . We have pointwise convergence of the integrand, i.e. $L^{-1}\theta'(L(C_n))$ converges to $L^{-1}\theta'(L(C))$ for every $L \in O(d)$. Furthermore, by compactness of \mathcal{K}_d^J , we have

$$B = \sup_{M \in \mathcal{K}_d^J} \|\theta'(M)\|_2 < \infty$$

and we conclude $\lim_{n \rightarrow \infty} \tilde{p}(C_n) = \tilde{p}(C)$, using the theorem of dominated convergence. □

2.2 The affine dimension of \mathfrak{P}_d is infinite

We show a slight extension of Theorem 2.1.2 in order to prove infinite dimensionality. We call a convex body C_1 affinely equivalent to a convex body C_2 if there is a $T \in \mathcal{AT}$ with $T(C_1) = C_2$. One checks immediately, that this is an equivalence relation and from now on, we just say C_1 and C_2 are affinely equivalent.

Corollary 2.2.1 *Let K_1, \dots, K_m be convex bodies which are pairwise not affinely equivalent, and let $x_i \in \mathfrak{F}(K_i)$ for $1 \leq i \leq m$. Then there is an affine point p such that $p(K_i) = x_i$ for $1 \leq i \leq m$.*

Proof. The proof relies on similar techniques as the proof of Theorem 2.1.2. Assume without loss of generality that all convex bodies concerned are in John position. Put

$$\theta : \bigcup_{i=1}^m O(d)(K_i) \rightarrow \mathbb{R}^d$$

$$L(K_j) \mapsto L(x_j) \quad .$$

The sets $O(d)(K_i)$ are pairwise disjoint, since no two of the K_i 's are affinely equivalent. By compactness of the $O(d)(K_i)$, these sets have positive distance with respect to the Hausdorff distance and hence, we can conclude from the continuity of $\theta|_{O(d)(K_i)}$ that θ is itself continuous. Using this θ , we now proceed as in the proof of Theorem 2.1.2. \square

Using this corollary, we can construct a sequence of affine points which are linearly independent and hence, also affinely independent.

Theorem 2.2.2 *The affine dimension of \mathfrak{P}_d is infinite.*

Proof. Assume, we have a sequence of convex bodies K_k , $k = 1, 2, 3, \dots$ which are pairwise not affinely equivalent and such that $\mathfrak{F}_d(K_k)$ does not reduce to one point for every k . Start with an arbitrary affine point p_1 . The sequence only containing p_1 is of course linearly independent. Now, assume we have a sequence p_1, \dots, p_n of linearly independent affine points. For every choice of $x_k \in \mathfrak{F}_d(K_k)$, $k = 1, \dots, n+1$, we find by Theorem 2.2.1 an affine point p with $p(K_k) = x_k$ for $k = 1, \dots, n+1$. In particular, we find an affine point p_{n+1} linearly independent to the sequence p_1, \dots, p_n . Hence, we obtain a sequence of infinite many linearly independent affine points and therefore, \mathfrak{P}_d is infinite dimensional. The only problem left open is the question if such sequences of K_k 's exist. This should be clear at least by the fact, that polytopes with different numbers of vertices cannot be affinely equivalent because affine, invertible operators always map vertices to vertices. However, we work out an explicit construction at least for the case $d \geq 3$. For $k = 1, 2, 3, \dots$, put

$$K_k = \text{conv} \left[\{0_d\}, (1, B_{k+1}^{d-1}) \right]$$

where $0_d \in \mathbb{R}^d$ is the origin. The K_k 's are cones with base B_{k+1}^{d-1} and the construction of a double cone over a $(d-1)$ -dimensional convex body was

already considered in [[K1], Theorem 5] for a somewhat related problem on similarity invariant points. For no two k, k' with $k \neq k'$, the cones K_k and $K_{k'}$ are affine equivalent. The set of extreme points of K_k is $\{0_d\} \cup \{0\} \times \partial B_{k+1}^{d-1}$ (note, that $k+1 > 1$). Since every affine invertible operator maps extreme points to extreme points, we conclude that every affine operator T with $T(K_k) = K_{k'}$ maps 0_d to $0_{d'}$ and $\{1\} \times \partial B_{k+1}^{d-1}$ to $\{1\} \times \partial B_{k'+1}^{d'-1}$ and hence, T does not change the first coordinate. Hence, we deduce that $\mathfrak{F}_d(K_k) \supseteq \mathbb{R} \times \{0_{d-1}\}$. K_k is invariant under permutations of the last $d-1$ coordinates and that yields $\mathfrak{F}_{d+1}(K_k) = \mathbb{R} \times \{0_{d-1}\}$. □

2.3 Dual affine invariant points

We call an affine point p proper if for every $C \in \mathcal{K}_d$, we have $p(C) \in \text{int}(C)$. Let p, q be two proper affine points. We say that q is dual to p if $q((C - p(C))^{\circ}) = 0$ for every $C \in \mathcal{K}_d$. In [MSW3], it was pointed out that there are proper affine points with no dual. Here, we want to show that the set of affine points with no dual is of second category with respect to some distance. More precisely, we show that there is an open and dense set of affine points with no dual. We make use of the following fact (cf. [MSW3], Lemma 3.6, Theorem 4.3):

Theorem 2.3.1 *A proper affine point has a dual if and only if for every $C \in \mathcal{K}_d$, there is at most one $z \in \text{int}(C)$ with $p((C - z)^{\circ}) = 0$.*

From [[MSW3], Proposition 3.13] we know that there is always at least one z with $p((C - z)^{\circ}) = 0$.

We use the following metric to measure the distance of two affine points which was introduced in [[MSW2], section 3.2]:

$$\text{dist}(p, q) = \sup_{B_2^d \subseteq C \subseteq dB_2^d} \|p(C) - q(C)\|_2$$

With respect to this distance we have the following topological result on proper affine points with no dual.

Theorem 2.3.2 *There exists an open and dense set $W \subseteq \mathfrak{P}_d$ of affine points with no dual.*

Proof. Let p be an affine point having a dual and $\varepsilon > 0$. We construct an affine point q such that $\text{dist}(p, q) < \varepsilon$ and such that there is a neighbourhood V of q of affine points with no dual. We will construct V in such a way that for every $q' \in V$ there is a $z_0 \in \text{int}(B_1^d) \setminus \{0\}$ with $q'((B_1^d - z_0)^{\circ}) = 0$. Since we always have $q'(B_{\infty}^d) = 0_d$ we conclude from the preceding discussion that q' has no dual.

Since p is continuous there exists a $\eta > 0$ such that for every $C \in \mathcal{K}_d$ with $d_H(C, B_\infty^d) < \eta$ we have $\|p(C)\|_2 < \frac{\varepsilon}{4d}$. For every $\delta \in (-1, 1)$ we can choose l_δ , c_δ and a_δ such that $T_\delta((B_1^d - \delta e_1)^\circ)$ is in John position where T_δ is the affine map

$$T_\delta = \begin{pmatrix} l_\delta & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & c_\delta \cdot I_{d-1} & \\ 0 & & & \end{pmatrix} + \begin{pmatrix} a_\delta \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

We have $\lim_{\delta \rightarrow 0} l_\delta = 1$, $\lim_{\delta \rightarrow 0} c_\delta = 1$ and $\lim_{\delta \rightarrow 0} a_\delta = 0$, moreover

$$\lim_{\delta \rightarrow 0} T_\delta \left((B_1^d - \delta e_1)^\circ \right) = B_\infty^d .$$

Therefore, we may choose sufficiently small $0 < \delta_1 < \delta_2 < 1$ such that for $i = 1, 2$ we have

$$\left| \frac{(-1)^i l_{\delta_i} \cdot \varepsilon}{5d} + a_{\delta_i} \right| < \frac{\varepsilon}{4d}$$

and

$$d_H(B_\infty^d, T_{\delta_i} \left((B_1^d - \delta_i e_1)^\circ \right)) < \eta .$$

We put $C_i = T_{\delta_i} \left((B_1^d - \delta_i e_1)^\circ \right)$. We have $\mathfrak{F}_d((B_1^d - \delta_i e_1)^\circ) = \mathbb{R} \times \{0_{d-1}\}$. Note that B_∞^d , $(B_1^d - \delta_1 e_1)^\circ$ and $(B_1^d - \delta_2 e_1)^\circ$ are pairwise not affinely equivalent, i.e. there exists no affine map which transforms one convex body into another. By Corollary 2.2.1 we can choose an affine point r such that

$$r(B_\infty^d) = 0_d, \quad r((B_1^d - \delta_1 e_1)^\circ) = -\frac{\varepsilon}{5d} e_1 \text{ and } r((B_1^d - \delta_2 e_1)^\circ) = \frac{\varepsilon}{5d} e_1 .$$

It follows that

$$\|r(C_i)\|_2 = \left| \frac{(-1)^i l_{\delta_i} \cdot \varepsilon}{5d} + a_{\delta_i} \right| < \frac{\varepsilon}{4d} .$$

Since the compact sets $\{B_\infty^d\}$, $O(d)(C_1)$ and $O(d)(C_2)$ are pairwise disjoint we conclude

$$\alpha_i = \inf_{L \in O(d)} d_H(L(C_i), B_\infty^d) > 0 .$$

We put $\alpha = \min[\alpha_1, \alpha_2]$. Let $U_i \subseteq \mathcal{K}_d^J$ be relatively open neighbourhoods of C_i such that

$$\|r(K)\|_2 < \frac{\varepsilon}{4d}, \quad d_H(K, B_\infty^d) < \eta \text{ and } d_H(K, C_i) < \frac{\alpha}{2}$$

for every $K \in U_i$ and put $U = U_1 \cup U_2$. We conclude that for every $K \in U_i$ and $L \in O(d)$ we have by the reverse triangle inequality

$$\begin{aligned} d_H(B_\infty^d, L(K)) &\geq d_H(B_\infty^d, L(C_i)) - d_H(L(C_i), L(K)) \\ &= d_H(B_\infty^d, L(C_i)) - d_H(C_i, K) > \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} . \end{aligned}$$

By the Tietze-Urysohn lemma there is a continuous map $\phi : \mathcal{K}_d^J \rightarrow [0, 1]$ with $\phi(L(C_i)) = 1$ for $L \in O(d)$ and $\phi(K) = 0$ for every $K \in (O(d)(U))^c$. Put $\Phi : \mathcal{K}_d^J \rightarrow [0, 1]$

$$\Phi(K) = \int_{O(d)} \phi(L(K)) d\mu(L)$$

where μ is the normalized Haar measure on $O(d)$. Put $\tilde{q} : \mathcal{K}_d^J \rightarrow \mathbb{R}^d$, $\tilde{q}(K) = \Phi(K)r(K) + (1 - \Phi(K))p(K)$ which is continuous as a combination of continuous maps and $O(d)$ -equivariant. Denote by q the unique extension of \tilde{q} to an affine point. For $K \in \mathcal{K}_d^J \setminus O(d)(U)$ we have $\|q(K) - p(K)\|_2 = 0$ and for $K \in U$ and $L \in O(d)$ we have

$$\begin{aligned} \|q(L(K)) - p(L(K))\|_2 &= \Phi(L(K))\|r(L(K)) - p(L(K))\|_2 \leq \|L(r(K) - p(K))\|_2 \\ &= \|r(K) - p(K)\|_2 \leq \|r(K)\|_2 + \|p(K)\|_2 \\ &< \frac{\varepsilon}{4d} + \frac{\varepsilon}{4d} = \frac{\varepsilon}{2d} . \end{aligned}$$

Let $C \in \mathcal{K}_d$ be an arbitrary convex body with the property $B_2^d \subseteq C \subseteq dB_2^d$. Then there is an affine transformation $T = L + b$, $L \in GL(d)$, $b \in \mathbb{R}^d$ and a convex body $K \in \mathcal{K}_d^J$ with $C = T(K)$. We have the following estimate:

$$\begin{aligned} \|q(C) - p(C)\|_2 &= \|q(T(K)) - p(T(K))\|_2 = \|L(q(K) - p(K))\|_2 \\ &\leq \|L\|_{O_p} \|q(K) - p(K)\|_2 < \|L\|_{O_p} \frac{\varepsilon}{2d} \end{aligned}$$

where $\|\cdot\|_{O_p}$ denotes the spectral norm. We have $b = b + L(0_d) \in b + L(K) \subseteq dB_2^d$ and hence, $dB_2^d - b \subseteq 2dB_2^d$, and we conclude

$$L(B_2^d) \subseteq L(K) \subseteq dB_2^d - b \subseteq 2dB_2^d$$

which leads to the bound of the spectral norm of L by $2d$ and therefore,

$$\|q(C) - p(C)\|_2 < \|L\|_{Op} \frac{\varepsilon}{2d} \leq \varepsilon$$

or $\text{dist}(q, p) < \varepsilon$, respectively. Put

$$V = \{q' \in \mathfrak{P}_d : q'((B_1^d - \delta_1 e_1)^\circ)_1 < 0 \text{ and } q'((B_1^d - \delta_2 e_1)^\circ)_1 > 0\}$$

where $q'(\dots)_1$ is the first component of $q'(\dots)$. The set V is an open neighbourhood of q . Note that for every $q' \in V$ the map

$$\psi : [\delta_1, \delta_2] \rightarrow \mathbb{R} \times \{0_{d-1}\}, \quad \psi(\delta) = q'((B_1^d - \delta e_1)^\circ)$$

is continuous and hence, for every q' there is a $\delta' \in [\delta_1, \delta_2]$ with

$$q'((B_1^d - \delta' e_1)^\circ) = 0e_1 = 0_d$$

by the intermediate value theorem. □

2.4 Symmetry measures and the maximal distance of John and Löwner point

When we talk about symmetry measures, we mean the class introduced in [MSW1]. Consider two affine points p_1, p_2 with $p_1(C), p_2(C) \in C$ for every $C \in \mathcal{K}_d$. We put $\delta_{p_1, p_2} : \mathcal{K}_d \rightarrow [0, 1]$

$$\begin{aligned} \delta_{p_1, p_2}(C) &= 0, \text{ if } p_1(C) = p_2(C) \\ \delta_{p_1, p_2}(C) &= \frac{\|p_1(C) - p_2(C)\|}{|a \cap C|_1} \end{aligned}$$

where a denotes the line through $p_1(C), p_2(C)$ and define the symmetry measure as $\phi_{p_1, p_2} = 1 - \delta_{p_1, p_2}$. Morally, if the value of this symmetry measure is close to 1, we have high symmetry. For example this value is always 1 for the simplex and for symmetric convex bodies. In [MSW1], the question is raised if $\phi_{p_1, p_2}(C) = 0$ can occur for some p_1, p_2, C . In part 1, we answer this question in the affirmative for every dimension $d \geq 2$. In part two, we treat the special case where $p_1 = j$ and $p_2 = l$, i.e. the case of the John and Löwner point.

2.4.1 Extremal p_1, p_2 for ϕ

We start with two propositions how to construct proper affine points and more generally, affine points with $p(C) \in C$ for every $C \in \mathcal{K}_d$. The proof of the first proposition is almost analogous to the proof of the same statement about different classes of equivariant points considered in [K1] but we repeat the arguments in our setting for the sake of completeness.

Proposition 2.4.1 *Let $K \in \mathcal{K}_d$ be a convex body and $x_0 \in \mathfrak{F}_d(K) \cap \text{int}(K)$. Then there is a proper affine point p with $p(K) = x_0$.*

Proof. Without loss of generality we assume K to be in the John position. By Theorem 2.1.2, we have an $O(d)$ -equivariant, continuous map $\tilde{p} : \mathcal{K}_d^J \rightarrow \mathbb{R}^d$ with $\tilde{p}(K) = x_0$. By continuity, we have an open neighbourhood U of K in \mathcal{K}_d^J with $\tilde{p}(C) \in \text{int}(C)$ for every $C \in U$. The same holds for the open $O(d)$ -invariant set

$$W = \bigcup_{l \in O(d)} l(U) \quad .$$

By the Tietze-Urysohn lemma, there is a continuous function $\phi : \mathcal{K}_d^J \rightarrow [0, 1]$ such that

$$\phi(C) = \begin{cases} 1 & \text{if } C \in O(d)(K) \\ 0 & \text{if } C \in W^c \end{cases} \quad .$$

Let μ be the normalized Haar measure on $O(d)$ and define $\Phi : \mathcal{K}_d^J \rightarrow [0, 1]$ by

$$\Phi(C) = \int_{O(d)} \phi(L(K)) d\mu(L) \quad .$$

Obviously, Φ is a continuous function with 1 on $O(d)(K)$ and 0 on W^c . We put $\tilde{q} : \mathcal{K}_d^J \rightarrow \mathbb{R}^d$

$$\tilde{q}(C) = \Phi(C)\tilde{p}(C) + (1 - \Phi(C))g(C)$$

where g is the centroid. Then, \tilde{q} is continuous, $O(d)$ -equivariant and we have $\tilde{q}(C) \in \text{int}(C)$ for every $C \in \mathcal{K}_d^J$. The same holds if we extend \tilde{q} to an affine point q . \square

Proposition 2.4.2 *Let K be a convex body and $x_0 \in \mathfrak{F}_d(K) \cap \partial K$. Then there exists an affine point p with $p(K) = x_0$ and $p(C) \in C$ for every convex body C .*

Proof. Since $\mathfrak{F}_d(K)$ is an affine subspace of \mathbb{R}^d and $\frac{1}{2}(g(K) + x_0) \in \text{int}(K)$ we conclude from Proposition 2.4.1 that there is a proper affine point q with $q(K) = \frac{1}{2}(g(K) + x_0)$. We put $\gamma_q : \mathcal{K}_d \rightarrow [0, 1]$

$$\gamma_q(C) = \begin{cases} 0 & \text{if } q(C) = g(C) \\ (\sup\{\lambda > 0 : g(C) + \lambda(q(C) - g(C)) \in C\})^{-1} & \text{else} \end{cases}$$

We omit the proof that γ_q does not change under affine transformations, i.e. $\gamma_q(T(C)) = \gamma_q(C)$ for affine maps T , and γ_q is continuous. The function γ_q has the nice property that for every C and γ with $\gamma \geq \gamma_q(C) > 0$ we have $g(C) + \frac{1}{\gamma}(q(C) - g(C)) \in C$ and especially, $g(K) + \frac{1}{\gamma_q(K)}(q(K) - g(K)) \in \partial K$. Put $\psi : [0, 1] \rightarrow [\frac{1}{2}, 1]$ with $\psi(s) = 1 - s$ for $s \leq \frac{1}{2}$ and $\psi(s) = s$ for $s \geq \frac{1}{2}$ and define $\Gamma_q = \psi \circ \gamma_q$. Γ_q is itself continuous as a composition of continuous functions and does not change under affine transformations. We can then define the desired affine invariant point p as $p(C) = g(C) + \frac{1}{\Gamma_q(C)}(q(C) - g(C))$ for $C \in \mathcal{K}_d$. □

Remark 2.4.1 *Using the centroid g in the proof is not crucial. Actually, we could have taken any other proper affine point instead.*

Theorem 2.4.3 *For every $d \geq 2$ there exists a symmetry measure ϕ_{p_1, p_2} and a convex body $C \in \mathcal{K}_d$ with $\phi_{p_1, p_2}(C) = 0$.*

Proof: Choose a convex body $K \in \mathcal{K}_d$ such that the affine dimension of $\mathfrak{P}_d(K)$ is at least 1. Those convex bodies do always exist because the set of convex bodies C with $\mathfrak{P}_d(C) = \mathbb{R}^d$ is dense by [[MSW2], Theorem 3]. If we want to avoid this implicit argument, one can choose as C for example the cone

$$\text{conv} \left[\{0_d\}, (1, B_2^{d-1}) \right]$$

for the case $d \geq 3$. Since the affine point $g(K)$ is an element of $\text{int}(K)$, the set $\mathfrak{P}_d(K) \cap \partial K$ has at least two distinct points, say x_1, x_2 . By Proposition 2.4.2, we find two affine points p_1, p_2 with $p_1(C), p_2(C) \in C$ for every $C \in \mathcal{K}_d$ and $p_1(K) = x_1$ and $p_2(K) = x_2$. This yields $\phi_{p_1, p_2}(C) = 0$. □

2.4.2 An extremal case for the Löwner and John point

What we will prove here is that

$$\frac{2}{d+1} \leq \inf_{C \in \mathcal{K}_d} \phi_{j,l}(C) = \frac{2}{d+1}(1 + o(1)) \quad . \quad (2.4.1)$$

In [MSW1], we find the estimate

$$\frac{2}{d+1} \leq \inf_{C \in \mathcal{K}_d} \phi_{j,l}(C) \leq \frac{1}{2} \quad . \quad (2.4.2)$$

It turns out that, even though the right-hand side of (2.4.2) is not the right order of magnitude, we only have to modify the construction of [MSW1] slightly to obtain the right order $\frac{2}{d+1}$. We start with the left-hand side of (2.4.1). Actually, you can find a proof in [[MSW1], section 3, remark]. We recall it for two reasons. The first reason is that this proof is done for the Santaló point and the centroid (although the arguments work analogously for the John and Löwner point). Secondly, there is a minor mistake in the proof.

Lemma 2.4.4 *Let $d \geq 2$. For every convex body C we have $\phi_{l,j}(C) \geq \frac{2}{d+1}$.*

Proof. Let \mathcal{E} be the John ellipsoid of $C - j(C)$. By Corollary 2.0.2, we have $C - j(C) \subseteq d\mathcal{E}$. We conclude $C - j(C) \subseteq d\mathcal{E} = -d\mathcal{E} \subseteq d(j(C) - C)$. Similarly, we have $C - l(C) \subseteq d(l(C) - C)$. Let a be an arbitrary chord in $C - j(C)$ passing through the origin. Then a is split by the origin into two line segments a_1, a_2 . From $C - j(C) \subseteq d(j(C) - C)$, we deduce that $a_1 \subseteq -da_2$ and hence,

$$\frac{|a_2|_1}{|a_1|_1} \geq \frac{1}{d} \quad \text{or equivalently,} \quad \frac{|a_2|_1}{|a_1|_1} \geq \frac{2}{d+1} \quad .$$

The same holds for every chord through the Löwner point. Assume $l(C) \neq j(C)$ and let now a be the chord of C through $j(C)$ and $l(C)$. By the preceding discussion, the portion of the part of the chord which is not covered by the line segment between $l(C), j(C)$ is at least $\frac{1}{d+1} + \frac{1}{d+1} = \frac{2}{d+1}$ of the length of a . □

Now, we establish the right-hand side of 2.4.1. We need the following lemma.

Lemma 2.4.5 *There is a sequence $(\varepsilon_d)_{d \in \mathbb{N}}$ with $\varepsilon_d = \frac{1}{d} + O(\frac{1}{d^2})$ such that the Löwner ellipsoid of*

$$K_d = \text{conv} \left[\left(-\varepsilon_d, \sqrt{1 - \varepsilon_d^2} B_2^d \right), \left(\sqrt{1 - \frac{1}{d}}, \frac{1}{\sqrt{d}} B_2^d \right) \right]$$

is B_2^{d+1} .

Proof. For fixed $0 < \varepsilon < 1$ and d consider

$$K = \text{conv} \left[\left(-\varepsilon, \sqrt{1 - \varepsilon^2} B_2^d \right), \left(\sqrt{1 - \frac{1}{d}}, \frac{1}{\sqrt{d}} B_2^d \right) \right] .$$

For $2 \leq i \leq d + 1$ put

$$v_i^\pm = \pm \sqrt{1 - \varepsilon^2} e_i - \varepsilon e_1 \text{ and } w_i^\pm = \pm \frac{1}{\sqrt{d}} e_i + \sqrt{1 - \frac{1}{d}} e_1$$

where e_i is the i -th unit coordinate vector. We note that $v_i^\pm, w_i^\pm \in \partial K \cap \partial B_2^{d+1}$ and

$$V = \sum_{o \in \{\pm\}, i} v_i^o v_i^{o\text{tr}} = 2 \begin{pmatrix} d\varepsilon^2 & & & \mathbf{0} \\ & 1 - \varepsilon^2 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 - \varepsilon^2 \end{pmatrix}$$

$$W = \sum_{o \in \{\pm\}, i} w_i^o w_i^{o\text{tr}} = 2 \begin{pmatrix} d(1 - \frac{1}{d}) & & & \mathbf{0} \\ & \frac{1}{d} & & \\ & & \ddots & \\ \mathbf{0} & & & \frac{1}{d} \end{pmatrix} .$$

If we find $t_1, t_2 > 0$ such that $t_1 V + t_2 W = \text{Id}_{d+1}$ and

$$t_1 \left(\sum_{o \in \{\pm\}, i} v_i^o \right) + t_2 \left(\sum_{o \in \{\pm\}, i} w_i^o \right) = 0 \quad ,$$

we may conclude that B_2^{d+1} is the Löwner ellipsoid of K (see Theorem 2.0.1). These two equalities are equivalent to the following three equations:

$$t_1(1 - \varepsilon^2) + t_2 \frac{1}{d} = \frac{1}{2}, \quad t_1 d \varepsilon^2 + t_2 d(1 - \frac{1}{d}) = \frac{1}{2} \text{ and } t_1 \varepsilon - t_2 \sqrt{1 - \frac{1}{d}} = 0 \quad .$$

From $t_1(1 - \varepsilon^2) + t_2 \frac{1}{d} = t_1 d \varepsilon^2 + t_2 d(1 - \frac{1}{d})$ we get

$$\frac{t_1}{t_2} = \frac{d - 1 - \frac{1}{d}}{1 - (d + 1)\varepsilon^2}$$

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and the third equation is equivalent to

$$\frac{t_1}{t_2} = \frac{\sqrt{1 - \frac{1}{d}}}{\varepsilon} .$$

We obtain that

$$\sqrt{1 - \frac{1}{d}} (1 - (d+1)\varepsilon^2) = (d - 1 - \frac{1}{d})\varepsilon$$

and the only positive solution of this quadratic equation in ε is

$$\begin{aligned} \varepsilon_d &= \frac{\sqrt{1 + 4(d+1)\frac{1-\frac{1}{d}}{d^2-d-1}} - 1}{2(d+1)\frac{\sqrt{1-\frac{1}{d}}}{d-1-\frac{1}{d}}} = \frac{\sqrt{1 + \frac{4}{d} + O(\frac{1}{d^2})} - 1}{2 + O(\frac{1}{d})} \\ &= \frac{1}{d} + O\left(\frac{1}{d^2}\right) \end{aligned} \quad (2.4.3)$$

With $\varepsilon = \varepsilon_d$ we verify that we have positive solutions t_1, t_2 . □

Consider the convex body

$$C_d = \text{conv} \left[\left(-\varepsilon_d, \sqrt{1 - \varepsilon_d^2} \Delta_d \right), \left(\sqrt{1 - \frac{1}{d}}, \frac{1}{\sqrt{d}} B_2^d \right) \right]$$

where Δ_d is the regular d -dimensional simplex inscribed in B_2^d .

Theorem 2.4.6 *Asymptotically in d we have the estimate $\phi_{j,l}(C_d) \leq \frac{2}{d+1}(1+o(1))$.*

Proof. The convex body C_d is similar to the construction in [[MSW1], section 5]. The only difference is another scaling factor for Δ_d . We also use the same arguments as in [[MSW1], proof of Proposition 15] to show that the Löwner ellipsoid of C_d is the same as the Löwner ellipsoid of K_d as defined in Lemma 2.4.5 and hence, the Löwner ellipsoid is B_2^{d+1} . We can also argue in the same way as in [[MSW1], Proposition 15] that the John ellipsoid of C_d is included in

$$\text{conv} \left[\left(-\varepsilon_d, \frac{\sqrt{1 - \varepsilon_d^2}}{d} B_2^d \right), \left(\sqrt{1 - \frac{1}{d}}, \frac{1}{\sqrt{d}} B_2^d \right) \right]$$

and hence, they share the same John ellipsoid. This is also the John ellipsoid of

$$\text{conv} \left[(-\varepsilon_d, \sqrt{1 - \varepsilon_d^2} \Delta_d), (\sqrt{1 - \frac{1}{d}}, \sqrt{d} \Delta_d) \right] .$$

This set is the frustum or truncation of a simplex. The exact expression for this simplex is

$$\tilde{\Delta}_{d+1} = \text{conv} \left[-\rho_d e_1, (\sqrt{1 - \frac{1}{d}}, \sqrt{d} \Delta_d) \right]$$

where we put

$$\rho_d = \frac{\varepsilon_d \sqrt{d} + \sqrt{1 - \varepsilon_d^2} \sqrt{1 - \frac{1}{d}}}{\sqrt{d} - \sqrt{1 - \varepsilon_d^2}} .$$

We apply (2.4.3) to ρ_d and we get

$$\frac{\varepsilon_d \sqrt{d} + \sqrt{1 - \varepsilon_d^2} \sqrt{1 - \frac{1}{d}}}{\sqrt{d} - \sqrt{1 - \varepsilon_d^2}} = \frac{O(\frac{1}{\sqrt{d}}) + 1 + O(\frac{1}{d})}{\sqrt{d} + O(1)} = \frac{1}{\sqrt{d}} + O\left(\frac{1}{d}\right) .$$

Simplices have only one affine point and hence, the John point and the centroid of $\tilde{\Delta}_{d+1}$ coincide. It is a classical result for simplices that a line segment from a vertex to the opposite base through the centroid is cut by the centroid in the ratio dimension to 1. Therefore, the centroid of $\tilde{\Delta}_{d+1}$ is

$$\left(\frac{1}{d+2} (-\rho_d) + \frac{d+1}{d+2} \sqrt{1 - \frac{1}{d}} \right) e_1 = \left(\left(1 - \frac{1}{d+2} \right) \sqrt{1 - \frac{1}{d}} + O\left(\frac{1}{d^{3/2}}\right) \right) e_1 .$$

The John ellipsoid is symmetric about the centroid and we conclude that the John ellipsoid is included in

$$\text{conv} \left[(-\varepsilon_d, \sqrt{1 - \varepsilon_d^2} \Delta_d), (\sqrt{1 - \frac{1}{d}}, \sqrt{d} \Delta_d) \right] \quad (2.4.4)$$

if d is large enough. Hence, it is also the John ellipsoid of (2.4.4), by the preceding discussion also of K_d and C_d . We are ready to calculate $\phi_{j,l}(C_d)$

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$$\begin{aligned}
 1 - \phi_{j,l}(C_d) &= \frac{\|j(C_d) - l(C_d)\|_2}{|a \cap C_d|_1} = \frac{\left(1 - \frac{1}{d+2}\right) \sqrt{1 - \frac{1}{d}} + O\left(\frac{1}{d^{3/2}}\right)}{\sqrt{1 - \frac{1}{d}} + \varepsilon_d} \\
 &= \frac{\left(1 - \frac{1}{d+2}\right) \left(1 - \frac{1}{2d} + O\left(\frac{1}{d^2}\right)\right) + O\left(\frac{1}{d^{3/2}}\right)}{1 - \frac{1}{2d} + O\left(\frac{1}{d^2}\right) + \frac{1}{d} + O\left(\frac{1}{d^2}\right)} \\
 &= \frac{1 - \frac{3}{2d} + O\left(\frac{1}{d^{3/2}}\right)}{1 + \frac{1}{2d} + O\left(\frac{1}{d^2}\right)} = 1 - \frac{2}{d} + O\left(\frac{1}{d^{3/2}}\right) \\
 &= 1 - \frac{2}{d+1} (1 + o(1))
 \end{aligned}$$

and this shows the right-hand side of inequality (2.4.1). □

Chapter 3

Duality of Floating and Illumination Bodies

We define a distance on the set of centrally symmetric convex bodies:

Definition 3.0.1 *Let $S_1, S_2 \subseteq \mathbb{R}^d$ be centrally symmetric convex bodies. We define*

$$d(S_1, S_2) = \inf \left\{ a \geq 1 : \frac{1}{a}S_1 \subseteq S_2 \subseteq aS_1 \right\} .$$

Note that $\log d(\cdot, \cdot)$ is a metric.

Definition 3.0.2 *Let $C \subseteq \mathbb{R}^d$ be a convex body and let $\delta \geq 0$. The (proportional, convex) floating body C_δ is defined by*

$$C_\delta = \bigcap_{|C \cap H^-|_d \leq \delta |C|_d} H^+$$

where H are hyperplanes and H^+, H^- are the corresponding closed half-spaces. The (proportional) illumination body is defined by

$$C^\delta = \{x \in \mathbb{R}^d : |\text{conv}(C, x)|_d \leq (1 + \delta)|C|_d\} .$$

The definition of the convex floating body is due to Schütt/Werner (see [SW1]) and Bárány/Larman (see [BL]) and the definition of the illumination body is due to Werner (see [W1]). For a convex body C and $x \in \partial C$ we denote by $N_C(x)$ the outer normal at x if the outer normal is unique. In most cases we omit the C in N_C since the convex body involved is usually clear from the argument we are applying N to. Note that the illumination body is always convex (see [W3]). This can be seen by the formula

$$|\text{conv}[C, x]|_d = \frac{1}{2} \left(|C|_d + \frac{1}{d} \int_{\partial C} \langle x - y, N(y) \rangle |d\mu(y)| \right)$$

where μ is the surface measure of C and $N(y)$ is defined almost everywhere. Obviously, the expression on the right-hand side is convex in x . For centrally symmetric bodies S we put

$$\langle S \rangle^\delta = \left[(S^\circ)^\delta \right]^\circ.$$

Definition 3.0.3 Let $S \subseteq \mathbb{R}^d$ be a centrally symmetric convex body and $0 \leq \delta < \frac{1}{2}$. We define

$$\mathbf{d}_S(\delta) = \inf_{\delta' > 0} \mathbf{d} \left(S_\delta, \langle S \rangle^{\delta'} \right)$$

and we put $\mathbf{d}_p(\delta) = \mathbf{d}_{B_p^d}(\delta)$.

The aim of this chapter is to estimate $\mathbf{d}_S(\delta)$ for different classes of centrally symmetric convex bodies, namely ℓ_p^d -balls and centrally symmetric ovaloids. Since for any two centrally symmetric convex bodies $S_1, S_2 \subseteq \mathbb{R}^d$ the equality $\mathbf{d}(S_1, S_2) = \mathbf{d}(S_1^\circ, S_2^\circ)$ holds it follows that for every centrally symmetric convex body $S \subseteq \mathbb{R}^d$ that

$$\mathbf{d}_S(\delta) = \inf_{\delta' > 0} \mathbf{d} \left(S_\delta, \langle S \rangle^{\delta'} \right) = \inf_{\delta' > 0} \mathbf{d} \left((S_\delta)^\circ, (S^\circ)^{\delta'} \right).$$

Note also that $\mathbf{d}_{L(S)}(\delta) = \mathbf{d}_S(\delta)$ for every linear invertible map L . In other words, the problem of estimating \mathbf{d}_S is invariant under linear transformations of S .

Definition 3.0.4 Let $\Omega \subseteq \mathbb{R}^{d-1}$ be an open neighbourhood of 0 and let $f : \Omega \rightarrow \mathbb{R}$ be a convex function of class C^2 with $f(0) = 0$ and $\nabla f(0) = 0$. The principal curvatures at 0 are the eigenvalues of the Hessian $Hf(0)$ and the Gauss curvature at 0 is the product of the principal curvatures, i.e., $\det Hf(0)$.

Let $C \subseteq \mathbb{R}^d$ be a convex body, let $x \in \partial C$ and assume that there is a unique outer normal $N(x)$. Let $L \in O(d)$ be a rotation such that $L(N(x)) = -e_d$ and put $C' = L(C - x)$. Then there is a $\gamma > 0$ and a convex function $f_x : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ with $f(0) = 0$ such that the boundary of C' is locally around the origin given by the graph of f_x . We call f_x a *parametrization* of C at x . If C is differentiable at 0 then $\nabla f_x(0) = 0$. If f_x is C^2 on a neighbourhood of 0 the *principal curvatures* and the *Gauss curvature* of C at x are defined by the principal curvatures and the Gauss curvature of f_x at 0. These definitions are independent of the choice of f_x (see also the introduction of [RSW] for the definition of curvature for convex bodies). We denote the Gauss curvature at x by $\kappa_C(x)$ and we omit C in κ_C in most cases since the convex body involved will be usually clear from the context. We say that $y \in \partial C$ corresponds to $(z, f_x(z))$, $z \in \gamma B_2^{d-1}$, or vice versa, $(z, f_x(z))$ corresponds to y if $L(y - x) = (z, f_x(z))$.

The results of the following sections are joint work with Elisabeth Werner. The original question in what sense the illumination body and floating body are dual notions is due to her. She provided the general idea to compute crucial points on the boundary of the floating and illumination bodies in order to get good estimates. She also pointed out to me the results of the paper [H] which are used in section 3.4. The computations in the following sections were carried out by me.

3.1 Cube

In this section we prove the following theorem:

Theorem 3.1.1 *For every $d \geq 2$ we have*

$$\frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}} \leq \liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_\infty(\delta) - 1}{\delta^{1/d}} \leq \limsup_{\delta \rightarrow 0} \frac{\mathbf{d}_\infty(\delta) - 1}{\delta^{1/d}} \leq \frac{2^d \sqrt[d]{\gamma_d}}{1+2^{d-1}}$$

where $\gamma_d = \frac{d!}{d^d}$.

Note that $\lim_{n \rightarrow \infty} \sqrt[n]{\gamma_n} = \frac{1}{e}$. We start with some results on the floating body of B_∞^d and the illumination body of B_1^d .

Lemma 3.1.2 *For every $\delta \geq 0$ we have:*

$$\frac{1}{1+2^d\delta} B_\infty^d \subseteq \langle B_\infty^d \rangle^\delta \subseteq \frac{1}{1+2\delta} B_\infty^d$$

Proof. We show that $(B_1^d)^\delta \subseteq (1+2^d\delta)B_1^d$. Let $x \in \partial[(1+2^d\delta)B_1^d]$, i.e. $\sum_{i=1}^d |x_i| = 1+2^d\delta$. We may assume without loss of generality that $x_i \geq 0$, $i = 1, \dots, d$. We start showing that $|\text{conv}[B_1^d, x]|_d - |B_1^d|_d \geq \delta |B_1^d|_d$. Note that

$$|\text{conv}[B_1^d, x]|_d - |B_1^d|_d \geq |\text{conv}[e_1, \dots, e_d, x]|_d$$

and the set on the right-hand side is a cone with base $\text{conv}[e_1, \dots, e_d]$ and apex x . The $(d-1)$ -dimensional volume of $\text{conv}[e_1, \dots, e_d]$ is given by $\frac{\sqrt{d}}{(d-1)!}$. Since $N := \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i$ is the outer normal of the facet $\text{conv}[e_1, \dots, e_d]$ the distance from x to $\text{conv}[e_1, \dots, e_d]$ is given by

$$\langle x, N \rangle - \langle e_1, N \rangle = \frac{1+2^d\delta}{\sqrt{d}} - \frac{1}{\sqrt{d}} = \frac{2^d\delta}{\sqrt{d}}.$$

Hence,

$$|\text{conv}[e_1, \dots, e_d, x]|_d = \frac{1}{d} \frac{\sqrt{d}}{(d-1)!} \frac{2^d\delta}{\sqrt{d}} = \delta \frac{2^d}{d!} = \delta |B_1^d|_d$$

and we conclude that $(B_1^d)^\delta \subseteq (1+2^d\delta)B_1^d$ which is equivalent to $\frac{1}{1+2^d\delta}B_\infty^d \subseteq \langle B_\infty^d \rangle^\delta$.

The second step is to prove that $\pm(1+2\delta)e_j \in \partial [(B_1^d)^\delta]$ for $j = 1, \dots, d$. Thus, $(1+2\delta)B_1 \subseteq (B_1^d)^\delta$ which is equivalent to $\langle B_\infty^d \rangle^\delta \subseteq \frac{1}{1+2\delta}B_\infty^d$. Let $h > 0$ be such that $(1+h)e_1 \in \partial [(B_1^d)^\delta]$. Since

$$\begin{aligned} & \text{conv}[B_1^d, (1+h)e_1] \\ &= \text{conv}[-e_1, \pm e_2, \pm e_3, \dots, \pm e_d] \cup \text{conv}[(1+h)e_1, \pm e_2, \pm e_3, \dots, \pm e_d] \end{aligned}$$

and the union in the last line is disjoint up to a null set. It follows that

$$\begin{aligned} (1+\delta)\frac{2^d}{d!} &= |\text{conv}[B_1^d, (1+h)e_1]|_d = \frac{1}{d}|B_1^{d-1}|_{d-1} + \frac{1}{d}(1+h)|B_1^{d-1}|_{d-1} \\ &= \left(1 + \frac{h}{2}\right)\frac{2^d}{d!} \end{aligned}$$

hence, $h = 2\delta$. By symmetry we conclude that for every $j = 1, \dots, d$ we have $\pm(1+h)e_j \in \partial [(B_1^d)^\delta]$. □

Lemma 3.1.3 For $0 \leq \delta < \frac{1}{d!}$ we have

$$(1 - 2\sqrt[d]{\gamma_d}\delta^{1/d})B_\infty^d \subseteq (B_\infty^d)_\delta \subseteq (1 - 2\delta)B_\infty^d$$

and $(1 - 2\sqrt[d]{\gamma_d}\delta^{1/d})\sum_{i=1}^d e_i \in \partial [(B_\infty^d)_\delta]$ where $\gamma_d = \frac{d!}{d^d}$.

Proof. We start with $(B_\infty^d)_\delta \subseteq (1 - 2\delta)B_\infty^d$. Since for every $j = 1, \dots, d$ and every $\sigma \in \{-1, 1\}$ we have

$$\left| \{x \in B_\infty^d : \sigma x_j \geq 1 - 2\delta\} \right|_d = 2^d \delta$$

it follows that

$$\begin{aligned} (B_\infty^d)_\delta &\subseteq \bigcap_{j \in \{1, \dots, d\}, \sigma = \pm 1} \{x \in \mathbb{R}^d : \sigma x_j \leq 1 - 2\delta\} \\ &= \bigcap_{j \in \{1, \dots, d\}} \{x \in \mathbb{R}^d : -(1 - 2\delta) \leq x_j \leq 1 - 2\delta\} = (1 - 2\delta)B_\infty^d \quad . \end{aligned}$$

We show that for every $s = (s_1, \dots, s_d) \in \{-1, 1\}^d$ that

$$(1 - 2\sqrt[d]{\gamma_d}\delta^{1/d})\sum_{i=1}^d e_i \in \partial [(B_\infty^d)_\delta] \quad .$$

By symmetry we just need to consider the case $s = (1, \dots, 1)$. Let $h = 2\sqrt[d]{d!}\delta^{1/d}$ and observe that $h \leq 2$ since $\delta \leq \frac{1}{d!}$. Let H be the hyperplane

through the d points $\sum_{i=1}^d e_i - he_j$, $j = 1, \dots, d$ and let H^+ be the closed half-space including $\sum_{i=1}^d e_i$. Since $h \leq 2$ we have

$$B_\infty^d \cap H^+ = \text{conv} \left[\sum_{i=1}^d e_i, \sum_{i=1}^d e_i - he_1, \dots, \sum_{i=1}^d e_i - he_d \right] .$$

This set is a rotation and translation of the simplex $\text{conv}[0, he_1, \dots, he_d]$ and the volume of this simplex is equal to

$$|\text{conv}[0, he_1, \dots, he_d]|_d = \frac{h^d}{d!} = 2^d \delta = \delta |B_\infty^d|_d .$$

By a result of M. Meyer and S. Reisner (see [MR]) the floating body of Dupin exists for centrally symmetric convex bodies. It follows that H touches the boundary of $(B_\infty^d)_\delta$ at the $(d-1)$ -dimensional centroid of $H \cap C$ which is an equilateral $(d-1)$ -dimensional simplex. Hence, the centroid of $C \cap H$ is given by the average of its vertices, i.e.

$$\frac{1}{d} \sum_{j=1}^d \left(\sum_{i=1}^d e_i - he_j \right) = \left(1 - \frac{h}{d} \right) \sum_{i=1}^d e_i = \left(1 - 2 \sqrt[d]{\gamma_d} \delta^{1/d} \right) \sum_{i=1}^d e_i .$$

Therefore,

$$\left(1 - 2 \sqrt[d]{\gamma_d} \delta^{1/d} \right) B_\infty^d \subseteq (B_\infty^d)_\delta .$$

□

Lemma 3.1.4 *For every $d \geq 2$ the following inequality holds:*

$$\limsup_{\delta \rightarrow 0} \frac{\mathbf{d}_\infty(\delta) - 1}{\delta^{1/d}} \leq \frac{2^d \sqrt[d]{\gamma_d}}{1 + 2^{d-1}}$$

Proof. Assume $0 \leq \delta \leq \frac{1}{d!}$. Put $\delta' = \frac{\sqrt[d]{\gamma_d}}{1+2^{d-1}} \delta^{1/d}$ and

$$a_\delta = \max \left[(1 - 2\delta)(1 + 2^d \delta'), \frac{1}{(1 + 2\delta')(1 - 2 \sqrt[d]{\gamma_d} \delta^{1/d})} \right] .$$

The constant a_δ is chosen in such a way that $\frac{1}{a_\delta} \frac{1}{(1-2\delta)} B_\infty^d \subseteq \frac{1}{1+2^d \delta'} B_\infty^d$ and $\frac{1}{1+2\delta} B_\infty^d \subseteq a_\delta (1 - 2 \sqrt[d]{\gamma_d} \delta^{1/d}) B_\infty^d$. It follows by Lemma 3.1.2 and Lemma 3.1.3 that

$$\frac{1}{a_\delta} (B_\infty^d)_\delta \subseteq \left\langle B_\infty^d \right\rangle^{\delta'} \subseteq a_\delta (B_\infty^d)_\delta .$$

Since

$$\lim_{\delta \rightarrow 0} \frac{(1 - 2\delta)(1 + 2^d \delta') - 1}{\delta^{1/d}} = \frac{2^d \sqrt[d]{\gamma_d}}{1 + 2^{d-1}}$$

and

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta^{1/d}} \left(\frac{1}{(1+2\delta')(1-2\sqrt[d]{\gamma_d}\delta^{1/d})} - 1 \right) = \frac{2^d \sqrt[d]{\gamma_d}}{1+2^{d-1}}$$

it follows that

$$\begin{aligned} \limsup_{\delta \rightarrow 0} \frac{\mathbf{d}_\infty(\delta) - 1}{\delta^{1/d}} &\leq \limsup_{\delta \rightarrow 0} \frac{\mathbf{d} \left((B_\infty^d)_\delta, \langle B_\infty^d \rangle^{\delta'} \right) - 1}{\delta^{1/d}} \\ &\leq \lim_{\delta \rightarrow 0} \frac{a_\delta - 1}{\delta^{1/d}} = \frac{2^d \sqrt[d]{\gamma_d}}{1+2^{d-1}} . \end{aligned}$$

□

Lemma 3.1.5 *For every $d \geq 2$ the following inequality holds:*

$$\frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}} \leq \liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_\infty(\delta) - 1}{\delta^{1/d}}$$

Proof. Assume that $0 \leq \delta \leq \frac{1}{d!}$. Let $a \geq 1$ be such that $\frac{1}{a}(B_\infty^d)_\delta \subseteq \langle B_\infty^d \rangle^{\delta'} \subseteq a(B_\infty^d)_\delta$. By Lemma 3.1.2 and Lemma 3.1.3 a necessary condition to have $\frac{1}{a}(B_\infty^d)_\delta \subseteq \langle B_\infty^d \rangle^{\delta'}$ is that

$$a \geq (1+2\delta')(1-2\delta) . \quad (3.1.1)$$

By Lemma 3.1.3 the vector $(1-2\sqrt[d]{\gamma_d}\delta^{1/d}) \sum_{i=1}^d e_i$ lies on the boundary of $(B_\infty^d)_\delta$. Using Lemma 3.1.2 we conclude that a necessary condition for $\langle B_\infty^d \rangle^{\delta'} \subseteq a(B_\infty^d)_\delta$ is that

$$a \geq \frac{1}{(1+2^d\delta')(1-2\sqrt[d]{\gamma_d}\delta^{1/d})} . \quad (3.1.2)$$

The assumption $\delta' \geq \frac{\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d}$ together with (3.1.1) yields that

$$a \geq \left(1 + \frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d} \right) (1-2\delta) = 1 + \frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d} + o(\delta^{1/d})$$

and the assumption $\delta' \leq \frac{\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d}$ together with (3.1.2) yields

$$a \geq \left[\left(1 + \frac{2^d \sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d} \right) (1-2\sqrt[d]{\gamma_d}\delta^{1/d}) \right]^{-1} = 1 + \frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d} + o(\delta^{1/d}) .$$

thus

$$\liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_\infty(\delta) - 1}{\delta^{1/d}} \geq \lim_{\delta \rightarrow 0} \frac{1 + \frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d} + o(\delta^{1/d}) - 1}{\delta^{1/d}} = \frac{2\sqrt[d]{\gamma_d}}{1+2^{d-1}}\delta^{1/d} .$$

□

3.2 Crosspolytope

In this section we prove the following theorem.

Theorem 3.2.1 *For every $d \geq 2$ we have*

$$\lim_{\delta \rightarrow 0} \frac{d_1(\delta) - 1}{\delta^{1/d}} = \frac{2^{1/d}}{2} .$$

We start with several lemmas about the shape of $\langle B_1^d \rangle^\delta$ and $(B_1^d)_\delta$.

Lemma 3.2.2 *Let $\delta > 0$. Then we have:*

$$\langle B_1^d \rangle^\delta = \text{conv} \left[\frac{1}{k + 2d\delta} \sum_{i \in J} \pm e_i : J \subseteq \{1, \dots, d\}, \quad |J| = k, \quad 1 \leq k \leq d \right]$$

Proof. We compute the illumination body of the cube. Let $J \subseteq \{1, \dots, d\}$, $J \neq \emptyset$ and $s \in \{-1, 1\}^J$. Put

$$\mathcal{M}_s^J = \{x \in \mathbb{R}^d : s_i x_i \geq 1 \text{ for every } i \in J \text{ and } -1 \leq x_j \leq 1 \text{ for every } j \in J^c\} .$$

and

$$v_s^J = \frac{1}{|J| + 2d\delta} \sum_{i \in J} s_i e_i .$$

The complement of $\text{int} B_\infty^d$ is completely covered by these sets \mathcal{M}_s^J . We show that the boundary points of $(B_\infty^d)^\delta$ contained in \mathcal{M}_s^J lie on the hyperplane

$$H_s^J = \{x \in \mathbb{R}^d : \langle x, v_s^J \rangle = 1\}$$

from which follows that

$$(B_\infty^d)^\delta = \bigcap_{J,s} \left\{ x \in \mathbb{R}^d : \langle x, v_s^J \rangle \leq 1 \right\} .$$

Thus,

$$\langle B_1^d \rangle^\delta = \text{conv} [v_s^J : J \subseteq \{1, \dots, d\}, J \neq \emptyset \text{ and } s \in \{-1, 1\}^J] .$$

Let $\xi \in \mathbb{R}^d$ be a boundary point of $(B_\infty^d)^\delta$ and let J, s be such that $\xi \in \mathcal{M}_s^J$. Put $k = |J|$. By symmetry we may assume that $J = \{1, \dots, k\}$ and $s = (1, \dots, 1)$. The $2d$ facets of B_∞^d are given by $F_i^\sigma = \{x \in B_\infty^d : x_i = \sigma\}$, $1 \leq i \leq d$, $\sigma \in \{-1, 1\}$. The volume of $\text{conv}[B_\infty^d, \xi] \setminus B_\infty^d$ is given by (cf. the proof of Proposition 2 in [W3]):

$$\begin{aligned} & \frac{1}{d} \sum_{1 \leq i \leq d, \sigma = \pm 1} |F_i^\sigma|_{d-1} \max[\langle \xi - \sigma e_i, \sigma e_i \rangle, 0] \\ &= \frac{2^{d-1}}{d} \sum_{1 \leq i \leq d, \sigma = \pm 1} \max[\sigma \xi_i - 1, 0] \\ &= \frac{2^{d-1}}{d} \sum_{i=1}^k (\xi_i - 1) \end{aligned} \tag{3.2.1}$$

Since $2^d \delta = |\text{conv}[B_\infty^d, \xi]|_d - |B_\infty^d|$, the term in (3.2.1) yields $\langle v_s^J, \xi \rangle = 1$. \square

Corollary 3.2.3 *For every $0 \leq \delta \leq \frac{1}{2d}$, $a \geq 1$ and $s \in [0, \frac{1}{3}]$ the vector*

$$s e_1 + \left(\frac{a}{1+2d\delta} - \frac{s}{1+2d\delta} \right) e_2$$

lies on the boundary of $a \cdot \langle B_1^d \rangle^\delta$.

Proof. First, we show that the line segment between $\frac{1}{1+2d\delta} e_2$ and $\frac{1}{2+2d\delta} (e_1 + e_2)$ lies on the boundary of $\langle B_1^d \rangle^\delta$. Put $v = \frac{1}{1+2d\delta} e_1 + e_2$. Note that $\langle v, \frac{1}{1+2d\delta} e_2 \rangle = \frac{1}{1+2d\delta}$ and $\langle v, \frac{1}{2+2d\delta} (e_1 + e_2) \rangle = \frac{1}{1+2d\delta}$. Hence, it suffices to show that $\langle B_1^d \rangle^\delta \subseteq \{x \in \mathbb{R}^d : \langle v, x \rangle \leq \frac{1}{1+2d\delta}\}$. We have

$$\left\langle v, \frac{1}{|J|+2d\delta} \sum_{i \in J} e_i \right\rangle = \frac{1}{|J|+2d\delta} \left(\frac{1}{1+2d\delta} \chi_J(1) + \chi_J(2) \right)$$

where χ denotes the characteristic function. Considering the two cases $|J| = 1$, $|J| \geq 2$ separately one gets that the right-hand side is smaller or equal to $\frac{1}{1+2d\delta}$. We conclude that the line segment $\frac{1}{1+2d\delta} e_2$ and $\frac{1}{2+2d\delta} (e_1 + e_2)$ lies on the boundary of $a \cdot \langle B_1^d \rangle^\delta$, i.e., for every $\lambda \in [0, 1]$ we have

$$\frac{a(1-\lambda)}{2+2d\delta} (e_1 + e_2) + \frac{a\lambda}{1+2d\delta} e_2 \in a \langle B_1^d \rangle^\delta .$$

Put $s = \frac{a(1-\lambda)}{2+2d\delta}$ then $\lambda = 1 - \frac{2(1+d\delta)}{a} s$ and hence,

$$\left(\frac{a(1-\lambda)}{2+2d\delta} + \frac{a\lambda}{1+2d\delta} \right) e_2 = \left(\frac{a}{1+2d\delta} - \frac{s}{1+2d\delta} \right) e_2 .$$

For every $s \in [0, \frac{1}{3}]$ the following holds:

$$1 \geq 1 - \frac{2(1+d\delta)}{a} s = \lambda \geq 1 - 2(1+d\delta)s \geq 1 - 2 \left(1 + \frac{d}{2d} \right) \cdot \frac{1}{3} = 0$$

We conclude that for every $s \in [0, \frac{1}{3}]$ that

$$s e_2 + \left(\frac{a}{1+2d\delta} - \frac{s}{1+2d\delta} \right) e_2 \in a \langle B_1^d \rangle^\delta .$$

\square

Lemma 3.2.4 *Let $1 \leq k \leq d$. Then there exists a constant $c_{d,k}$ and a function $s_{d,k}$ with $\lim_{\delta \rightarrow 0} s_{d,k}(\delta) = 0$ such that for every $J \subseteq \{1, \dots, d\}$ with $|J| = k$ we have:*

$$\left(1 - c_{d,k}(1 + s_{d,k}(\delta))\delta^{\frac{1}{d+1-k}}\right) \frac{1}{k} \sum_{i \in J} \pm e_i \in \partial \left(B_1^d\right)_\delta$$

where

$$c_{d,k} = \left(k^{\frac{d-k}{2}} \frac{2^k}{\binom{d}{k-1}}\right)^{\frac{1}{d+1-k}}.$$

Proof. Let $1 \leq k \leq d$ be fix and let $\Delta \geq 0$ be such that $\frac{1-\Delta}{k} \sum_{i=1}^k e_i \in (B_1^d)_\delta$. By symmetry we know that $v_k = \frac{1}{\sqrt{k}} \sum_{i=1}^k e_i$ is normal to $(B_1^d)_\delta$ at $\frac{1-\Delta}{k} \sum_{i=1}^k e_i$ and hence, we have that

$$\left|B_1^d \cap \left\{x \in \mathbb{R}^d : \left\langle x, \sum_{i=1}^k e_i \right\rangle \geq 1 - \Delta\right\}\right|_d = \delta |B_1^d|_d.$$

We compute

$$\begin{aligned} & \left|B_1^d \cap \left\{x \in \mathbb{R}^d : \left\langle x, \sum_{i=1}^k e_i \right\rangle \geq 1 - \Delta\right\}\right|_d \\ &= \int_{\frac{1-\Delta}{\sqrt{k}}}^{\frac{1}{\sqrt{k}}} \left|B_1^d \cap \left\{x \in \mathbb{R}^d : \langle x, v_k \rangle = s\right\}\right|_{d-1} ds. \end{aligned}$$

We compute the integrand:

$$\begin{aligned} & \left|B_1^d \cap \left\{x \in \mathbb{R}^d : \langle x, v_k \rangle = s\right\}\right|_{d-1} \\ &= \left|\left\{x \in \mathbb{R}^d : \sum_{i=1}^k x_i = \sqrt{k}s \wedge \sum_{i=1}^k |x_i| \leq 1 \wedge \sum_{i=k+1}^d |x_i| \leq 1 - \sum_{i=1}^k |x_i|\right\}\right|_{d-1} \\ &= \left|\left\{x \in \mathbb{R}_+^k \times \mathbb{R}^{d-k} : \sum_{i=1}^k x_i = \sqrt{k}s \wedge \sum_{i=k+1}^d |x_i| \leq 1 - \sqrt{k}s\right\}\right|_{d-1} \\ &+ \left|\left\{x \in \mathbb{R}^d \setminus (\mathbb{R}_+^k \times \mathbb{R}^{d-k}) : \sum_{i=1}^k x_i = \sqrt{k}s \wedge \sum_{i=k+1}^d |x_i| \leq 1 - \sum_{i=1}^k |x_i|\right\}\right|_{d-1} \\ &= (\sqrt{k}s)^{k-1} |\text{conv}[e_1, \dots, e_k]|_{k-1} \cdot (1 - \sqrt{k}s)^{d-k} |B_1^{d-k}|_{d-k} \\ &+ \left|\left\{x \in \mathbb{R}^d \setminus (\mathbb{R}_+^k \times \mathbb{R}^{d-k}) : \sum_{i=1}^k x_i = \sqrt{k}s \wedge \sum_{i=k+1}^d |x_i| \leq 1 - \sum_{i=1}^k |x_i|\right\}\right|_{d-1} \end{aligned} \tag{3.2.2}$$

and applying the volume formulas for the simplex and the ℓ_1^d -balls we obtain

$$\begin{aligned} & (\sqrt{ks})^{k-1} |\text{conv}[e_1, \dots, e_k]|_{k-1} \cdot (1 - \sqrt{ks})^{d-k} |B_1^{d-k}|_{d-k} \\ &= (\sqrt{ks})^{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \quad . \end{aligned}$$

We show that the second summand in (3.2.2) is relatively small if Δ is small. Note first that by the triangle inequality we have

$$1 - \sum_{i=1}^k |x_i| \leq 1 - \left| \sum_{i=1}^k x_i \right| = 1 - \sqrt{ks}$$

and hence,

$$\begin{aligned} & \left| \left\{ x \in \mathbb{R}^d \setminus (\mathbb{R}_+^k \times \mathbb{R}^{d-k}) : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=k+1}^d |x_i| \leq 1 - \sum_{i=1}^k |x_i| \right\} \right|_{d-1} \\ & \leq \left| \left\{ x \in \mathbb{R}^d \setminus (\mathbb{R}_+^k \times \mathbb{R}^{d-k}) : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=1}^k |x_i| \leq 1 \wedge \sum_{i=k+1}^d |x_i| \leq 1 - \sqrt{ks} \right\} \right|_{d-1} \\ & = \left| \left\{ \bar{x} \in \mathbb{R}^k \setminus \mathbb{R}_+^k : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=1}^k |x_i| \leq 1 \right\} \right|_{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k}}{(d-k)!} \quad . \end{aligned}$$

Let $s = \frac{1-\varepsilon}{\sqrt{k}}$ for some $0 \leq \varepsilon \leq \Delta$. Then $\sqrt{ks} = 1 - \varepsilon$. We now show that

$$\left\{ y \in \mathbb{R}^k : \sum_{i=1}^k y_i = 1 - \varepsilon \wedge \sum_{i=1}^k |y_i| \leq 1 \right\}$$

is contained in

$$\left\{ (1 - \varepsilon) \sum_{i=1}^k (\lambda \mu_i - \frac{\lambda}{k} + \frac{1}{k}) e_i : \sum_{i=1}^k \mu_i = 1 \wedge \forall_{1 \leq i \leq k} : \mu_i \geq 0 \right\}$$

where $\lambda = 1 + \frac{\varepsilon k}{1-\varepsilon}$. Let $y \in \mathbb{R}^k$ be such that $\sum_{i=1}^k y_i = 1 - \varepsilon$, $\sum_{i=1}^k |y_i| \leq 1$ and $y_i = (1 - \varepsilon)(\lambda \mu_i - \frac{\lambda}{k} + \frac{1}{k})$ for $1 \leq i \leq k$. Put $\mu_i = \frac{1}{\lambda}(\frac{y_i}{1-\varepsilon} - \frac{1}{k} + \frac{\lambda}{k})$, $s \leq i \leq k$. Then

$$\sum_{i=1}^k \mu_i = \frac{1}{\lambda} \left(\frac{\sum_{i=1}^k y_i}{1-\varepsilon} - 1 + \lambda \right) = 1 \quad .$$

An easy computation, which we postpone, shows that $y_i \geq -\varepsilon$ for all $1 \leq i \leq k$ and this yields

$$\mu_i = \frac{1}{\lambda} \left(\frac{y_i}{1-\varepsilon} - \frac{1}{k} + \frac{\lambda}{k} \right) \geq \frac{1}{\lambda} \left(-\frac{\varepsilon}{1-\varepsilon} - \frac{1}{k} + \frac{1 + \frac{\varepsilon k}{1-\varepsilon}}{k} \right) = 0 \quad .$$

The inequality $y_i \geq -\varepsilon$ follows for the case $i = 1$ (and analogously for the other cases) from

$$1 - \varepsilon - y_1 = \sum_{i=2}^k y_i \leq \sum_{i=1}^k |y_i| \leq 1 \quad .$$

It follows that

$$\begin{aligned} & \left| \left\{ \bar{x} \in \mathbb{R}^k \setminus \mathbb{R}_+^k : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=1}^k |x_i| \leq 1 \right\} \right|_{k-1} \\ &= \left| \left\{ \bar{x} \in \mathbb{R}^k : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=1}^k |x_i| \leq 1 \right\} \right|_{k-1} \\ & \quad - \left| \left\{ \bar{x} \in \mathbb{R}_+^k : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=1}^k |x_i| \leq 1 \right\} \right|_{k-1} \\ &\leq \left(\left(1 + \frac{\sqrt{k} - ks}{s} \right)^{k-1} - 1 \right) \left| \left\{ \bar{x} \in \mathbb{R}_+^k : \sum_{i=1}^k x_i = \sqrt{ks} \wedge \sum_{i=1}^k |x_i| \leq 1 \right\} \right|_{k-1} \quad . \end{aligned}$$

Hence, we have

$$\begin{aligned} & (1 - \Delta)^{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \\ &\leq (\sqrt{ks})^{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \\ &\leq \left| B_1^d \cap \left\{ x \in \mathbb{R}^d : \langle x, v_k \rangle = s \right\} \right|_{d-1} \\ &\leq \left(1 + \frac{\sqrt{k} - ks}{s} \right)^{k-1} (\sqrt{ks})^{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \\ &\leq \left(1 + \frac{k\Delta}{1-\Delta} \right)^{k-1} (\sqrt{ks})^{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \\ &\leq \left(1 + \frac{k\Delta}{1-\Delta} \right)^{k-1} (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \quad . \end{aligned}$$

This means that

$$\left| B_1^d \cap \left\{ x \in \mathbb{R}^d : \langle x, v_k \rangle = s \right\} \right|_{d-1} = (1 + O(\Delta)) (1 - \sqrt{ks})^{d-k} \frac{2^{d-k} \sqrt{k}}{(d-k)!(k-1)!} \quad .$$

We compute

$$\int_{\frac{1-\Delta}{\sqrt{k}}}^{\frac{1}{\sqrt{k}}} (1 - \sqrt{ks})^{d-k} ds = \int_0^{\frac{\Delta}{\sqrt{k}}} t^{d-k} dt = \frac{1}{d+1-k} \left(\frac{\Delta}{\sqrt{k}} \right)^{d+1-k} \quad ,$$

hence,

$$\begin{aligned} & \int_{\frac{1-\Delta}{\sqrt{k}}}^{\frac{1}{\sqrt{k}}} \left| B_1^d \cap \left\{ x \in \mathbb{R}^d : \langle x, v_k \rangle = s \right\} \right|_{d-1} ds \\ &= (1 + O(\Delta)) \frac{2^{d-k}}{(d+1-k)!(k-1)!} \frac{1}{\sqrt{k}^{d-k}} \Delta^{d+1-k} \end{aligned}$$

and we obtain

$$\Delta = \left(\sqrt{k}^{d-k} \frac{2^k}{\binom{d}{k-1}} \right)^{\frac{1}{d+1-k}} (1 + o_\delta(1)) \delta^{\frac{1}{d+1-k}} =: c_{d,k} (1 + s_{d,k}(\delta)) \delta^{\frac{1}{d+1-k}}$$

where the index δ in $o_\delta(1)$ means that the expression tends to 0 when δ tends to 0. □

Lemma 3.2.5 *Let $0 < \nu, \mu \leq 1$ and let $0 < \delta \leq \frac{1}{2}$ be such that*

$$\frac{(\mu\nu)^{d-1}}{(\mu+\nu)^{d-2}} = 2^{3-d}\delta \quad .$$

Then we have

$$\frac{\mu-\nu}{d} e_1 + \left(1 - \frac{(\mu-\nu)^2}{d(\mu+\nu)} - \frac{2\mu\nu}{\mu+\nu} \right) e_2 \in (B_1^d)_\delta \quad .$$

Proof. W.l.o.g. let $0 < \nu \leq \mu \leq 1$ and let H be the hyperplane given by the line through $e_2 + \nu(-e_1 - e_2)$ and $e_2 + \mu(e_1 - e_2)$ and the span of the vectors e_3, \dots, e_d , i.e. the hyperplane consisting of vectors of the form

$$(\alpha\mu - (1-\alpha)\nu, 1 - \alpha\mu - (1-\alpha)\nu, x_3, \dots, x_d)$$

with $\alpha, x_3, \dots, x_d \in \mathbb{R}$. We compute the volume of the intersection of B_1^d with the closed half-space H^- having H as boundary and not containing the origin. The hyperplane H and the line $\mathbb{R}e_2$ intersect each other in a single point which is given by $\left(1 - \frac{2\mu\nu}{\mu+\nu}\right) e_2$. The set $C = \text{conv}[e_1, e_2, \pm e_3, \dots, \pm e_d]$ has volume $\frac{2^{d-2}}{d!}$. Consider the linear map $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$L(x) = \mu \langle e_1, x \rangle e_1 + \left(\frac{2\mu\nu}{\mu+\nu} - \mu \right) \langle e_1, x \rangle e_2 + \sum_{i=3}^d \frac{2\mu\nu}{\mu+\nu} \langle e_i, x \rangle e_i \quad .$$

The determinant of L is $\mu \left(\frac{2\mu\nu}{\mu+\nu} \right)^{d-1}$ since L is a lower triangular matrix with one time μ and $d-1$ times $\frac{2\mu\nu}{\mu+\nu}$ on its diagonal. Since

$$L(C - e_2) + e_2 = \text{conv}[e_2 + \mu(e_1 - e_2), e_2, e_2 + \vartheta(\pm e_3 - e_2), \dots, e_2 + \vartheta(\pm e_d - e_2)]$$

where $\vartheta = \frac{2\mu\nu}{\mu+\nu}$ we get that the volume of the above set is

$$\mu \left(\frac{2\mu\nu}{\mu+\nu} \right)^{d-1} \frac{2^{d-2}}{d!} .$$

Analogously, we have that the volume of the set

$$\text{conv}[e_2 + \nu(-e_1 - e_2), e_2, e_2 + \vartheta(\pm e_3 - e_2), \dots, e_2 + \vartheta(\pm e_d - e_2)]$$

is

$$\nu \left(\frac{2\mu\nu}{\mu+\nu} \right)^{d-1} \frac{2^{d-2}}{d!} .$$

Hence, the volume of $B_1^d \cap H^-$ is

$$\mu \left(\frac{2\mu\nu}{\mu+\nu} \right)^{d-1} \frac{2^{d-2}}{d!} + \nu \left(\frac{2\mu\nu}{\mu+\nu} \right)^{d-1} \frac{2^{d-2}}{d!} = \frac{(\mu\nu)^{d-1}}{(\mu+\nu)^{d-2}} \frac{2^{2d-3}}{d!} .$$

We conclude that the hyperplane H cuts off the volume $\delta|B_1^d|_d$ if

$$\frac{(\mu\nu)^{d-1}}{(\mu+\nu)^{d-2}} = 2^{3-d}\delta .$$

Since B_1^d is centrally symmetric we know from [MR] that H is a support hyperplane of $(B_1^d)_\delta$ touching the body exactly at the centroid of the $(d-1)$ -dimensional convex body $(B_1^d)_\delta \cap H$. We determine the centroid. We start with the centroid of

$$C_1 = \text{conv}[e_2 + \mu(e_1 - e_2), e_2 + \vartheta(\pm e_3 - e_2), \dots, e_2 + \vartheta(\pm e_d - e_2)] .$$

This is a pyramid with apex $e_2 + \mu(e_1 - e_2)$ and base $\text{conv}[e_2 + \vartheta(\pm e_3 - e_2), \dots, e_2 + \vartheta(\pm e_d - e_2)]$, and the centroid is given by

$$g_1 = \frac{d-1}{d}(1-\vartheta)e_2 + \frac{1}{d}(e_2 + \mu(e_1 - e_2)) = \frac{\mu}{d}e_1 + \left(1 - \frac{\mu^2 + (2n-1)\mu\nu}{d(\mu+\nu)}\right)e_2 .$$

Analogously, the centroid of

$$C_2 = \text{conv}[e_2 + \mu(-e_1 - e_2), e_2 + \vartheta(\pm e_3 - e_2), \dots, e_2 + \vartheta(\pm e_d - e_2)] .$$

is

$$g_2 = -\frac{\nu}{d}e_1 + \left(1 - \frac{\nu^2 + (2n-1)\mu\nu}{d(\mu+\nu)}\right)e_2 .$$

It is easy to see that the volume ratio of C_1 to C_2 is $\frac{\mu}{\nu}$ and therefore, the centroid g of $B_1^d \cap H$ is given by

$$g = \frac{1}{\mu+\nu}(\mu g_1 + \nu g_2) = \frac{\mu-\nu}{d}e_1 + \left(1 - \frac{(\mu-\nu)^2}{d(\mu+\nu)} - \frac{2\mu\nu}{\mu+\nu}\right)e_2 .$$

□

3.2.1 Upper Bound

We start deducing an upper bound using the preceding lemmas.

Proposition 3.2.6 *For every $d \geq 2$ the inequality*

$$\limsup_{\delta \rightarrow 0} \frac{d_1(\delta) - 1}{\delta^{1/d}} \leq \frac{2^{1/d}}{2}$$

holds.

Proof. Put $\delta' = \frac{2^{1/d}}{4d} \delta^{1/d}$. Let $a \geq 1$ be such that

$$\frac{1}{a} \langle B_1^d \rangle^{\delta'} \subseteq (B_1^d)_\delta \subseteq a \langle B_1^d \rangle^{\delta'} .$$

For the first inclusion it is sufficient by Lemma 3.2.2 and Lemma 3.2.4 that

$$a \geq \frac{1}{(1 + 2\frac{d}{k}\delta')(1 - c_{d,k}(1 + s_{d,k}(\delta))\delta^{\frac{1}{d+1-k}})}$$

for $k = 1, \dots, d$. Since δ' is of order $\delta^{1/d}$ the right-hand side is smaller than 1 for all k but $k = 1$ if δ is sufficiently close to 0. Hence, we only need to take care of the case $k = 1$, i.e., when

$$a \geq \frac{1}{(1 + 2d\delta')(1 - 2^{1/d}(1 + s_{d,1}(\delta))\delta^{1/d})} .$$

From Lemma 3.2.4 it follows that

$$\begin{aligned} (B_1^d)_\delta &\subseteq \bigcap_{v = \frac{1}{d} \sum_{i=1}^d \pm e_i} \left[\left(1 - \frac{2^d}{d} (1 + s_{d,d}(\delta)) \delta \right) v + \{x \in \mathbb{R}^d : \langle x, v \rangle \leq 0\} \right] \\ &= \left(1 - \frac{2^d}{d} (1 + s_{d,d}(\delta)) \delta \right) B_1^d \end{aligned}$$

and from Lemma 3.2.2 it follows that

$$\frac{1}{1 + 2d\delta'} B_1^d \subseteq \langle B_1^d \rangle^{\delta'} .$$

Hence, for the second inclusion we only need that

$$a \geq \left(1 - \frac{2^d}{d} (1 + s_{d,d}(\delta)) \delta \right) (1 + 2d\delta') .$$

If δ converges to 0 then

$$\frac{1}{(1 + 2d\delta')(1 - 2^{1/d}(1 + s_{d,1}(\delta))\delta^{1/d})} - 1$$

$$\frac{1}{\delta^{1/d}}$$

as well as

$$\frac{(1 - \frac{2^d}{d}(1 + s_{d,d}(\delta))\delta)(1 + 2d\delta') - 1}{\delta^{1/d}}$$

converge to $\frac{2^{1/d}}{2}$. We conclude that

$$\limsup_{\delta \rightarrow 0} \frac{\mathbf{d}_1(\delta) - 1}{\delta^{1/d}} \leq \frac{2^{1/d}}{2} .$$

□

3.2.2 Lower Bound

We prove the following lower bound which yields together with the upper bound from the previous section Theorem 3.2.1

Proposition 3.2.7 *For every $d \geq 2$ the inequality*

$$\frac{2^{1/d}}{2} \leq \liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_1(\delta) - 1}{\delta^{1/d}}$$

holds.

Proof. We may assume that $\frac{2^{1/d}}{8d}\delta^{1/d} \leq \delta' \leq 2^{1/d}\delta^{1/d}$:

Assume that $\delta' \geq 2^{1/d}\delta^{1/d}$. By Lemma 3.2.2 we have the inclusion $\langle B_1^d \rangle^{\delta'} \subseteq \frac{1}{1+2\delta'} B_1^d$. Hence, $(B_1^d)_\delta \subseteq a \langle B_1^d \rangle^{\delta'}$ and Lemma 3.2.4 implies that

$$\begin{aligned} a &\geq (1 + 2\delta')(1 - c_{d,1}(1 + s_{d,1})\delta^{1/d}) \geq (1 + 2 \cdot 2^{1/d}\delta^{1/d}) \left(1 - 2^{1/d}(1 + s_{d,1}(\delta))\delta^{1/d}\right) \\ &> 1 + \frac{2^{1/d}}{2}\delta^{1/d} \end{aligned}$$

for sufficiently small $\delta > 0$.

Suppose that $\delta' \leq \frac{2^{1/d}}{8d}\delta^{1/d}$. Hence, $\langle B_1^d \rangle^{\delta'} \subseteq a(B_1^d)_\delta$. Moreover, Lemma 3.2.2 and Lemma 3.2.4 imply that

$$\begin{aligned} a &\geq \frac{1}{(1 + 2d\delta')(1 - 2^{1/d}(1 + s_{d,1}(\delta))\delta^{1/d})} \\ &\geq \frac{1}{(1 + \frac{2^{1/d}}{4}\delta^{1/d})(1 - 2^{1/d}(1 + s_{d,1}(\delta))\delta^{1/d})} \\ &> 1 + \frac{2^{1/d}}{2}\delta^{1/d} \end{aligned}$$

if $\delta > 0$ is sufficiently small.

For $0 \leq c \leq (2 \cdot 3^{1/d}\delta)^{-\frac{1}{d-1}} - 1$ put

$$\nu = \frac{(1 + \frac{c}{2})^{1-\frac{2}{d}}}{(1+c)^{1-\frac{1}{d}}} 2^{1/d}\delta^{1/d}$$

and $\mu = (1+c)\nu$. Then $\frac{(\mu\nu)^{d-1}}{(\mu+\nu)^{d-2}} = 2^{3-d}\delta$ and

$$\begin{aligned}\mu &= (1+c)\nu = \left(1 + \frac{c}{2}\right)^{1-\frac{2}{d}} (1+c)^{\frac{1}{d}} 2^{1/d} \delta^{1/d} \leq (1+c)^{1-\frac{1}{d}} 2^{1/d} \delta^{1/d} \\ &\leq \left(1 + \left(2 \cdot 3^{1/d} \delta\right)^{-\frac{1}{d-1}} - 1\right)^{1-\frac{1}{d}} 2^{1/d} \delta^{1/d} = \frac{1}{3}\end{aligned}$$

and we may apply Lemma 3.2.5. Put $s = \frac{\mu-\nu}{d}$. Since $0 \leq \frac{\mu-\nu}{d} \leq \frac{\mu}{d} \leq \frac{1}{2}$ we may apply Corollary 3.2.3 for sufficiently small δ . We obtain as a necessary condition for $(B_1^d)_\delta \subseteq a \left[(B_\infty^d)^{\delta'} \right]^\circ$ that for every $c \leq (2 \cdot 3^{1/d} \delta)^{-\frac{1}{d-1}} - 1$ we have

$$\frac{a}{1+2d\delta'} - \frac{s}{1+2d\delta'} \geq 1 - \frac{(\mu-\nu)^2}{d(\mu+\nu)} - \frac{2\mu\nu}{\mu+\nu}$$

where $\nu = \frac{(1+\frac{c}{2})^{1-\frac{2}{d}}}{(1+c)^{1-\frac{1}{d}}} 2^{1/d} \delta^{1/d}$, $\mu = (1+c)\nu$ and $s = \frac{\mu-\nu}{d} = \frac{c\nu}{d}$. First, eliminating μ from the inequality we obtain

$$\begin{aligned}a &\geq (1+2d\delta') \left(1 + \frac{c\nu}{d(1+2d\delta')} - \frac{c^2\nu}{d(2+c)} - \frac{2(1+c)\nu}{(2+c)}\right) \\ &= 1+2d\delta' + \left(\frac{c}{d(1+2d\delta')} - \frac{c}{d} \frac{c}{2+c} - \frac{2(1+c)}{2+c}\right) \nu(1+2d\delta') \quad .\end{aligned}$$

If $\delta > 0$ is sufficiently small we may put $c = \delta^{-1/d} \leq (2 \cdot 3^{1/d} \delta)^{-\frac{1}{d-1}} - 1$ and, since $\frac{c}{d} \frac{c}{2+c} \leq \frac{c}{d}$ and $\frac{2(1+c)}{2+c} \leq 3$, we compute

$$\begin{aligned}\frac{c}{d} \frac{1}{1+2d\delta'} - \frac{c}{d} \frac{2}{2+c} - \frac{2(1+c)}{2+c} &\geq \frac{c}{d} \frac{1}{1+2d\delta'} - \frac{c}{d} - 3 \\ &\geq -2\delta'c - 3 = -2\delta'\delta^{-1/d} - 3 \quad .\end{aligned}$$

We have the following upper bound on ν :

$$\begin{aligned}\nu &= \frac{(1+\frac{c}{2})^{1-\frac{2}{d}}}{(1+c)^{1-\frac{1}{d}}} 2^{1/d} \delta^{1/d} \leq \frac{(1+c)^{1-\frac{2}{d}}}{(1+c)^{1-\frac{1}{d}}} 2^{1/d} \delta^{1/d} = \left(\frac{2\delta}{1+c}\right)^{1/d} \\ &\leq \left(\frac{2\delta}{c}\right)^{1/d} = 2^{1/d} \delta^{\frac{1}{d} + \frac{1}{d^2}}\end{aligned}$$

and therefore,

$$\begin{aligned}&\left(\frac{c}{d} \frac{1}{1+2d\delta'} - \frac{c}{d} \frac{2}{2+c} - \frac{2(1+c)}{2+c}\right) \nu(1+2d\delta') \\ &\geq -2^{1/d} (1+2d\delta') (2\delta' + 3\delta^{1/d}) \delta^{\frac{1}{d^2}} \quad .\end{aligned}$$

The assumption $\delta' \leq 2^{1/d}\delta^{1/d}$ yields

$$\begin{aligned} 2^{1/d}(1+2d\delta')(2\delta'+3\delta^{1/d})\delta^{\frac{1}{a^2}} &\leq \sqrt{2}(1+2\sqrt{2}d)(2\sqrt{2}+3)\delta^{\frac{1}{a}+\frac{1}{a^2}} \\ &\leq \sqrt{2}\left(\frac{1}{2}+2\sqrt{2}\right)(2\sqrt{2}+3)d\delta^{\frac{1}{a}+\frac{1}{a^2}} \leq 30d\delta^{\frac{1}{a}+\frac{1}{a^2}} \end{aligned}$$

or, respectively,

$$a \geq 1 + 2d\delta' - 30d\delta^{\frac{1}{a}+\frac{1}{a^2}} .$$

Together with the condition $a \geq \frac{1}{(1+2d\delta')(1-2^{1/d}\delta^{1/d})}$ we have that

$$\liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_1(\delta) - 1}{\delta^{1/d}} \geq \frac{2^{1/d}}{2} .$$

□

3.3 The ℓ_p^d -case for $1 < p < \infty$

In this section we prove the following.

Theorem 3.3.1 *For every $d \geq 2$ and $2 < p < \infty$ there exists a constant $c(p, d) > 0$ such that*

$$\lim_{\delta \rightarrow 0} \frac{\mathbf{d}_p(\delta) - 1}{\delta^{\frac{2}{d+1}}} = c(p, d) .$$

Furthermore, we give a lower bound for the case $1 < p < 2$. First, we need some preparatory definitions and lemmas.

Definition 3.3.1 *Let $C \subseteq \mathbb{R}^d$ be a convex body and let $x \in \partial C$ be a boundary point with unique outer normal N . We denote by $\Delta_x(\delta) > 0$ the unique value such that*

$$|\{y \in C : \langle y, N(x) \rangle \geq \langle x, N(x) \rangle - \Delta_x(\delta)\}|_d = \delta|C|_d .$$

The convex body C to which $\Delta_x(\delta)$ is referring to should always be clear from the context.

Definition 3.3.2 *Let $1 \leq p \leq \infty$. The polar cap C_p^d of B_p^d of height $0 \leq \Delta \leq 1$ is defined by*

$$C_p^d(\Delta) = \{x \in B_p^d : x_1 \geq 1 - \Delta\}.$$

The polar hat $H_p^d(\Delta)$ of height $\Delta \geq 0$ is defined by

$$H_p^d(\Delta) = \text{conv}[B_p^d, (1 + \Delta)e_1] \setminus B_p^d.$$

Lemma 3.3.2 *Let $1 \leq p < \infty$ and $0 \leq \Delta \leq 1$. Then*

$$|C_p^d(\Delta)|_d = \frac{1}{d-1+p} |B_p^{d-1}|_{d-1} (p\Delta)^{\frac{d-1+p}{p}} \phi_p^d(\Delta) \quad ,$$

where ϕ_p^d is a function with $\lim_{\Delta \rightarrow 0} \phi_p^d(\Delta) = 1$.

Proof. By Taylor's theorem we have that for every $0 < s < 1$ there is $0 < \sigma(s) < s$ such that $(1-s)^p = 1 - p(1-\sigma(s))^{p-1}s$. This yields

$$\begin{aligned} \int_0^\Delta (1-(1-s)^p)^{\frac{d-1}{p}} ds &= \int_0^\Delta (p(1-\sigma(s))^{p-1}s)^{\frac{d-1}{p}} ds \\ &\leq \int_0^\Delta (ps)^{\frac{d-1}{p}} ds = \frac{1}{d-1+p} (p\Delta)^{\frac{d-1+p}{p}} \end{aligned}$$

and

$$\begin{aligned} \int_0^\Delta (p(1-\sigma(s))^{p-1}s)^{\frac{d-1}{p}} ds &\geq (1-\Delta)^{\frac{p-1}{p}(d-1)} \int_0^\Delta (ps)^{\frac{d-1}{p}} ds \\ &= (1-\Delta)^{\frac{p-1}{p}(d-1)} \frac{1}{d-1+p} (p\Delta)^{\frac{d-1+p}{p}} \quad . \end{aligned}$$

Hence, we obtain by Cavalieri's principle

$$|C_p^d(\Delta)|_d = \frac{1}{d-1+p} |B_p^{d-1}|_{d-1} (p\Delta)^{\frac{d-1+p}{p}} \phi_p^d(\Delta)$$

with $(1-\Delta)^{\frac{p-1}{p}(d-1)} \leq \phi_p^d(\Delta) \leq 1$. □

Lemma 3.3.3 *Let $1 < p < \infty$ and $0 \leq \Delta \leq 1$ then*

$$|H_p^d(\Delta)|_d = \frac{p-1}{d(d-1+p)} |B_p^{d-1}|_{d-1} \left(\frac{p}{p-1} \Delta \right)^{\frac{d-1+p}{p}} \psi_p^d(\Delta) \quad ,$$

where ψ_p^d is a function with $\lim_{\Delta \rightarrow 0} \psi_p^d(\Delta) = 1$.

Proof. Put $f(t) = (1-t^p)^{\frac{1}{p}}$, $-1 < t < 1$. The tangential function $T_{t_0}(t)$ of f at t_0 is given by:

$$T_{t_0}(t) = f(t_0) + f'(t_0)(t-t_0) = (1-t_0^p)^{\frac{1}{p}-1} (1-t_0^{p-1}t)$$

If $T_{t_0}(1+\Delta) = 0$, then $t_0 = \frac{1}{(1+\Delta)^{\frac{1}{p-1}}}$. It follows that the boundary of the cap of B_p^d touches the ℓ_p^d -ball at height $\frac{1}{(1+\Delta)^{\frac{1}{p-1}}}$ with respect to the

direction e_1 . The hat volume is given by the difference of the cone volume and the corresponding cap volume:

$$\begin{aligned} |H_p^d(\Delta)|_d &= \frac{1}{d} \left(1 + \Delta - \frac{1}{(1 + \Delta)^{\frac{1}{p-1}}} \right) \left(1 - \frac{1}{(1 + \Delta)^{\frac{p}{p-1}}} \right)^{\frac{d-1}{p}} |B_p^{d-1}|_{d-1} \\ &\quad - \left| C_p^d \left(1 - \frac{1}{(1 + \Delta)^{\frac{1}{p-1}}} \right) \right|_d \end{aligned}$$

and hence,

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta^{\frac{d-1+p}{p}}} |H_p^d(\Delta)|_d &= \frac{1}{d} \left(\frac{p}{p-1} \right)^{\frac{d-1+p}{p}} |B_p^{d-1}|_{d-1} \\ &\quad - \frac{1}{d-1+p} \left(\frac{p}{p-1} \right)^{\frac{d-1+p}{p}} |B_p^{d-1}|_{d-1} \\ &= \frac{p-1}{d(d-1+p)} \left(\frac{p}{p-1} \right)^{\frac{d-1+p}{p}} |B_p^{d-1}|_{d-1} \quad . \end{aligned}$$

□

The following formulas for the Gauss curvature and the normal directions of the ℓ_p^d -balls can be found in [[KRZ], p. 367-368].

Lemma 3.3.4 *The Gauss curvature at $x \in \partial B_p^d$ is given by*

$$\kappa(x) = (p-1)^{d-1} \frac{\prod_{i=1}^d |x_i|^{p-2}}{\left(\sum_{i=1}^d |x_i|^{2p-2} \right)^{\frac{d+1}{2}}}$$

and the normal is given by

$$N(x) = \frac{1}{\left(\sum_{i=1}^d |x_i|^{2p-2} \right)^{\frac{d+1}{2}}} \left(\text{sign}(x_i) |x_i|^{p-1} \right)_{i=1}^d$$

if $2 \leq p < \infty$ or if $1 < p < 2$ and all components of x are not equal to zero.

Lemma 3.3.5 *Let $1 < p < \infty$. There are constants c_j and functions s_j ,*

$1 \leq j \leq 4$, depending only on d and p such that $\lim_{\delta \rightarrow \infty} s_j(\delta) = 0$ and

$$\begin{aligned} & \pm(1 - c_1(1 + s_1(\delta))\delta^{\frac{p}{d-1+p}})e_i \in \partial(B_p^d)_\delta, \quad 1 \leq i \leq d, \\ & \left(1 - c_2(1 + s_2(\delta))\delta^{\frac{2}{d+1}}\right) \frac{1}{\sqrt[d]{d}} \sum_{i=1}^d \pm e_i \in \partial(B_p^d)_\delta \\ & \pm(1 - c_3(1 + s_3(\delta'))\delta'^{\frac{p}{d-1+p}})e_i \in \partial \langle B_{p'}^d \rangle^{\delta'}, \quad 1 \leq i \leq d, \\ & \left(1 - c_4(1 + s_4(\delta'))\delta'^{\frac{2}{d+1}}\right) \frac{1}{\sqrt[d]{d}} \sum_{i=1}^d \pm e_i \in \partial \langle B_{p'}^d \rangle^{\delta'} . \end{aligned}$$

In order to prove this lemma we need two results. The first lemma is an immediate consequence of Lemma 7 and 10 of [SW1]:

Lemma 3.3.6 *Let $C \subseteq \mathbb{R}^d$ be a convex body with 0 in its interior and $x \in \partial C$ such that the Gauss curvature $\kappa(x)$ exists. Then the following holds:*

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta^{\frac{2}{d+1}}} \cdot \frac{\|x\|_2 - \|x_\delta\|_2}{\|x\|_2} = \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle}$$

where x_δ is the unique point lying in the intersection of ∂C_δ with the line segment $[0, x]$.

The second lemma is a direct consequence of Lemma 3 of [W1]:

Lemma 3.3.7 *Let $C \subseteq \mathbb{R}^d$ be a convex body with 0 in its interior and $x \in \partial C$ such that the Gauss curvature $\kappa(x)$ exists. Then the following holds:*

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta^{\frac{2}{d+1}}} \cdot \frac{\|x^\delta\|_2 - \|x\|_2}{\|x\|_2} = \frac{d^{\frac{2}{d+1}}(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle}$$

where x^δ is the unique point on the boundary of C^δ such that x lies on the line segment $[0, x^\delta]$.

Proof of Lemma 3.3.5. Let $\delta > 0$ be fix and $\Delta > 0$ such that

$$|B_p^d \cap \{x \in \mathbb{R}^d : x_1 \geq 1 - \Delta\}|_d = \delta |B_p^d|_d .$$

Since B_p^d is centrally symmetric, the hyperplane $H = \{x \in \mathbb{R}^d : x_1 = 1 - \Delta\}$ is a support hyperplane of $(B_p^d)_\delta$ touching the boundary at $(1 - \Delta)e_1$. From Lemma 3.3.3 it follows that

$$\frac{1}{d-1+p} |B_p^{d-1}|_{d-1} (p\Delta)^{\frac{d-1+p}{p}} \phi_p^d(\Delta) = \delta |B_p^d|_d$$

and hence,

$$\Delta(\phi_p^d(\Delta))^{\frac{p}{d-1+p}} = \frac{(d-1+p)^{\frac{p}{d-1+p}}}{p} \left(\frac{|B_p^d|_d}{|B_p^{d-1}|_{d-1}} \right)^{\frac{p}{d-1+p}} \delta^{\frac{p}{d-1+p}} .$$

Put

$$c_1 = \frac{(d-1+p)^{\frac{p}{d-1+p}}}{p} \left(\frac{|B_p^d|_d}{|B_p^{d-1}|_{d-1}} \right)^{\frac{p}{d-1+p}} .$$

Since $\lim_{\Delta \rightarrow 0} \phi_p^d(\Delta) = 1$ we conclude that there is a function s_1 with $\lim_{\delta \rightarrow 0} s_1(\delta) = 0$ such that

$$\Delta = c_1(1 + s_1(\delta))\delta^{\frac{p}{d-1+p}} .$$

Analogously, we conclude with Lemma 3.3.2 that there is a constant c_3 depending on d and p and a function \tilde{s}_3 with $\lim_{\delta' \rightarrow 0} \tilde{s}_3(\delta') = 0$ and

$$(1 + c_3(1 + \tilde{s}_3(\delta'))\delta'^{\frac{p}{d-1+p}})e_1 \in \partial \left[(B_p^d)^{\delta'} \right] .$$

Since e_1 is an outer normal of $(B_p^d)^{\delta'}$ at $(1 + c_3(1 + \tilde{s}_3(\delta'))\delta'^{\frac{p}{d-1+p}})e_1$ we conclude that

$$\frac{1}{1 + c_3(1 + \tilde{s}_3(\delta'))\delta'^{\frac{p}{d-1+p}}}e_1 \in \partial \left[(B_p^d)^{\delta'} \right]^\circ = \partial \langle B_{p'}^d \rangle^{\delta'}$$

i.e., there is a function s_3 with $\lim_{\delta' \rightarrow 0} s_3(\delta') = 0$ such that

$$(1 - c_3(1 + s_3(\delta'))\delta'^{\frac{p}{d-1+p}})e_1 \in \partial \left[(B_p^d)^{\delta'} \right]^\circ = \langle B_{p'}^d \rangle^{\delta'} .$$

From Lemma 3.3.6 we obtain that there is a constant c_2 and a function s_2 only depending on d and p such that $\lim_{\delta \rightarrow 0} s_2(\delta) = 0$ and

$$\left(1 - c_2(1 + s_2(\delta))\delta^{\frac{2}{d+1}} \right) \frac{1}{\sqrt[d]{d}} \sum_{i=1}^d \pm e_i \in \partial(B_p^d)_\delta .$$

Similarly, we derive from Lemma 3.3.7 that there is a constant c_4 and a function \tilde{s}_4 only depending on d and p such that $\lim_{\delta \rightarrow 0} \tilde{s}_4(\delta) = 0$ and

$$\left(1 + c_4(1 + \tilde{s}_4(\delta'))\delta'^{\frac{2}{d+1}} \right) \frac{1}{\sqrt[d]{d}} \sum_{i=1}^d \pm e_i \in \partial(B_p^d)^{\delta'} .$$

For symmetry reasons $\frac{1}{\sqrt[d]{d}} \sum_{i=1}^d \pm e_i$ is an outer normal of $(B_p^d)^{\delta'}$ at

$$\left(1 - c_4(1 + \tilde{s}_4(\delta'))\delta'^{\frac{2}{d+1}} \right) \frac{1}{\sqrt[d]{d}} \sum_{i=1}^d \pm e_i .$$

Hence,

$$\frac{1}{1 - c_4(1 + \tilde{s}_4(\delta'))\delta'^{\frac{2}{d+1}}} \cdot \frac{1}{\sqrt[p']{d}} \sum_{i=1}^d \pm e_i \in \partial \langle B_{p'}^d \rangle^{\delta'}$$

and thus, there is a function s_4 only depending on p and d with $\lim_{\delta' \rightarrow 0} s_4(\delta') = 0$ such that

$$\left(1 - c_4(1 + s_4(\delta'))\delta'^{\frac{2}{d+1}}\right) \frac{1}{\sqrt[p']{d}} \sum_{i=1}^d \pm e_i \in \partial \langle B_{p'}^d \rangle^{\delta'}. \quad \square$$

3.3.1 Lower Bound

We provide lower bounds for \mathbf{d}_p in the cases $1 < p < 2$ and $2 < p < \infty$.

Corollary 3.3.8 *Let $2 < p < \infty$ and let c_2 be the constant from Lemma 3.3.5. Then we have*

$$\liminf_{\delta \rightarrow 0} \frac{d_p(\delta) - 1}{\delta^{\frac{2}{d+1}}} \geq c_2.$$

Proof. Fix $\delta' \geq 0$ and let $a_\delta \geq 1$ be such that

$$\frac{1}{a_\delta} (B_p^d)_\delta \subseteq \langle B_p^d \rangle^{\delta'} \subseteq a_\delta (B_p^d)_\delta.$$

From Lemma 3.3.5 we deduce that

$$a_\delta \geq \max \left[\frac{1 - c_4 \delta'^{\frac{2}{d+1}} (1 + s_4(\delta'))}{1 - c_2 \delta^{\frac{2}{d+1}} (1 + s_2(\delta))}, \frac{1 - c_1 \delta^{\frac{p}{d-1+p}} (1 + s_1(\delta))}{1 - c_3 \delta'^{\frac{p'}{d-1+p'}} (1 + s_3(\delta'))} \right]$$

Since $\frac{p'}{d-1+p'} < \frac{2}{d+1} < \frac{p}{d-1+p}$ we may choose a fixed constant $1 < \alpha < \frac{2}{d+1} \cdot \frac{d-1+p'}{p'}$ and we get the following. If $\delta' \geq \delta^\alpha$ it follows that

$$\begin{aligned} a_\delta &\geq \frac{1 - c_1 \delta^{\frac{p}{d-1+p}} (1 + s_1(\delta))}{1 - c_3 \delta^\alpha \delta'^{\frac{p'}{d-1+p'}} (1 + s_3(\delta'))} = 1 + c_3 \delta^\alpha \delta'^{\frac{p'}{d-1+p'}} (1 + o_\delta(1) + o_{\delta'}(1)) \\ &\geq 1 + c_2 \delta^{\frac{2}{d+1}} (1 + o_\delta(1) + o_{\delta'}(1)) \end{aligned}$$

since $\alpha \frac{p'}{d-1+p'} < \frac{2}{d+1}$. If $\delta' \leq \delta^\alpha$ it follows that

$$a_\delta \geq \frac{1 - c_4 \delta^\alpha \delta'^{\frac{2}{d+1}} (1 + s_4(\delta'))}{1 - c_2 \delta^{\frac{2}{d+1}} (1 + s_2(\delta))} = 1 + c_2 \delta^{\frac{2}{d+1}} (1 + o_\delta(1) + o_{\delta'}(1)).$$

We may assume without loss of generality that δ' tends to zero as δ tends to zero and hence,

$$\lim_{\delta \rightarrow 0} \frac{a_\delta - 1}{\delta^{\frac{2}{d+1}}} \geq c_2. \quad \square$$

Corollary 3.3.9 *Let $1 < p < 2$ and let c_1 be the constant from Lemma 3.3.5. We have*

$$\liminf_{\delta \rightarrow 0} \frac{d_p(\delta) - 1}{\delta^{\frac{p}{d-1+p}}} \geq c_1 \quad .$$

Proof. Similarly as in the previous corollary one may argue that

$$a_\delta \geq \max \left[\frac{1 - c_2 \delta^{\frac{2}{d+1}} (1 + o_\delta(1))}{1 - c_4 \delta'^{\frac{2}{d+1}} (1 + o_{\delta'}(1))}, \frac{1 - c_3 \delta'^{\frac{p'}{d-1+p'}}}{1 - c_1 \delta^{\frac{p}{d-1+p}}} \right] \quad .$$

Note that we have $\frac{p}{d-1+p} < \frac{2}{d+1} < \frac{p'}{d-1+p'}$. We fix an $\alpha > 0$ with

$$\frac{p}{d-1+p} \cdot \frac{d-1+p'}{p'} < \alpha < \frac{p}{d-1+p} \cdot \frac{d+1}{2}$$

and we argue as in the previous proof by considering the two cases $\delta' \leq \delta^\alpha$ and $\delta' \geq \delta^\alpha$. □

3.3.2 Upper Bound

We provide an upper estimate for \mathbf{d}_p in the case $2 < p < \infty$ which yields together with the lower estimate Theorem 3.3.1.

Proposition 3.3.10 *Let $2 < p < \infty$. Then*

$$\limsup_{\delta \rightarrow 0} \frac{\mathbf{d}_p(\delta) - 1}{\delta^{\frac{2}{d+1}}} \leq c_2$$

where c_2 is the constant of Lemma 3.3.5.

Before we start proving this proposition we need some preparatory steps. Put $\lambda(\delta) = \max_{x \in \partial B_p^d \cap (\mathbb{R}_{\geq 0})^d} \frac{\Delta_x(\delta)}{\langle N(x), x \rangle}$. Then the following holds.

Lemma 3.3.11 *For $p > 2$,*

$$(B_p^d)_\delta \supseteq (1 - \lambda(\delta)) B_p^d \quad .$$

Proof. We start with the following fact.

(a) **Claim** *Let $C \subseteq \mathbb{R}^d$ be a convex body with $0 \in \text{int}(C)$, $x \in \partial C$, N an outer normal at x and $\alpha \geq 0$ such that*

$$0 \in \{y \in \mathbb{R}^d : \langle y - x, N \rangle \leq -\alpha\} \quad .$$

Then

$$\left(1 - \frac{\alpha}{\langle x, N \rangle}\right) C \subseteq \{y \in \mathbb{R}^d : \langle y - x, N \rangle \leq -\alpha\} \quad .$$

Proof of the claim: Let $\xi \in C$. Then $\langle \xi - x, N \rangle \leq 0$ and $1 - \frac{\alpha}{\langle x, N \rangle} \geq 0$ since $0 \in x + \{y \in \mathbb{R}^d : \langle y, N \rangle \leq -\alpha\}$. It follows that

$$\left\langle \left(1 - \frac{\alpha}{\langle x, N \rangle}\right) \xi - x, N \right\rangle = \left(1 - \frac{\alpha}{\langle x, N \rangle}\right) \langle \xi - x, N \rangle - \alpha \leq -\alpha \quad .$$

(b) Since

$$(B_p^d)_\delta = \bigcap_{x \in \partial B_p^d} \{y \in \mathbb{R}^d : \langle y - x, N(x) \rangle \leq -\Delta_x(\delta)\}$$

Lemma 3.3.11 follows immediately from (a). □

Let $C_p^d = \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|B_p^d|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}}$, let $\kappa(x)$ be the Gauss curvature at x and $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_{d-1}(x)$ be the principal curvatures at x and put

$$F : \left(\partial B_p^d \cap (\mathbb{R}_{\geq 0})^d \right) \times [0, 1] \rightarrow \mathbb{R}_{\geq 0}$$

$$(x, \phi) \mapsto C_p^d \frac{\prod_{i=1}^{d-1} (\lambda_i(x) + \phi)^{\frac{1}{d+1}}}{\langle N(x), x \rangle}$$

We show:

Lemma 3.3.12 1. We have $F(x, 0) \leq F((d^{-1/p}, \dots, d^{-1/p}), 0)$ with equality if and only if $x = (d^{-1/p}, \dots, d^{-1/p})$.

2. For every $\varepsilon > 0$ there is a $\delta_0 > 0$ such that for every $0 < \delta \leq \delta_0$ we have

$$\frac{\Delta_x(\delta)}{\langle N(x), x \rangle} \leq C_p^d \frac{(\kappa(d^{-1/p}, \dots, d^{-1/p})^{1/(d-1)} + \varepsilon)^{\frac{d-1}{d+1}}}{\langle N(d^{-1/p}, \dots, d^{-1/p}), (d^{-1/p}, \dots, d^{-1/p}) \rangle} \delta^{\frac{2}{d+1}} \quad .$$

Proof. Part 1.: Applying $\kappa(x)^{\frac{1}{d+1}} = \prod_{i=1}^{d-1} \lambda_i(x)^{\frac{1}{d+1}} = (p-1)^{\frac{d-1}{d+1}} \frac{\prod_{i=1}^d x_i^{\frac{p-2}{d+1}}}{(\sum_{i=1}^d x_i^{2p-2})^{1/2}}$ to $F(x, 0)$ we have

$$F(x, 0) = C_p^d (p-1)^{\frac{d-1}{d+1}} \prod_{i=1}^d x_i^{\frac{p-2}{d+1}}$$

and the inequality of arithmetic and geometric means yields

$$\prod_{i=1}^d x_i^{\frac{p-2}{d+1}} = \left(\sqrt[d]{\prod_{i=1}^d x_i^p} \right)^{\frac{d}{d+1} \cdot \frac{p-2}{p}} \leq \left(\frac{1}{d} \left(\sum_{i=1}^d x_i^p \right) \right)^{\frac{d}{d+1} \cdot \frac{p-2}{p}} = d^{-\frac{d}{d+1} \cdot \frac{p-2}{p}}$$

with equality if and only if all x_i are equal to $d^{-1/p}$.

Part 2.: Since F is continuous and $\partial B_p^d \cap (\mathbb{R}_{\geq 0})^d$ is compact, there is a $\phi_0 > 0$ such that for every $0 \leq \phi \leq \phi_0$ and every $x \in \partial B_p^d \cap (\mathbb{R}_{\geq 0})^d$ the inequality

$$F(x, \phi) \leq F((d^{-1/p}, \dots, d^{-1/p}), \varepsilon)$$

holds. □

Proof of Proposition 3.3.10 Apply Corollary A.0.3 to $\eta = \frac{\phi_0}{2}$ and let $\gamma > \theta > 0$ be chosen accordingly to this η . Let $Hf_x(0) = \sum_{i=1}^{d-1} \lambda_i(x) v_i \otimes v_i$, and put $\tilde{\lambda}_i(x) = \max[\lambda_i(x), \frac{\phi_0}{2}]$, $\tilde{H}_x = \sum_{i=1}^{d-1} \tilde{\lambda}_i(x) v_i \otimes v_i$. For $w \in \mathbb{S}^{d-2}$ we get

$$\begin{aligned} & \{s \in [-\gamma, \gamma] : f_x(sw) \leq t\} \supseteq \{s \in [-\theta, \theta] : f_x(sw) \leq t\} \\ & \supseteq \{s \in [-\theta, \theta] : \frac{s^2}{2} \left(\langle Hf_x(0)w, w \rangle + \frac{\phi_0}{2} \right) \leq t\} \\ & \supseteq \{s \in [-\theta, \theta] : \frac{s^2}{2} \left(\langle \tilde{H}_x w, w \rangle + \frac{\phi_0}{2} \right) \leq t\} \\ & = \left\{ s \in [-\theta, \theta] : |s| \leq \sqrt{\frac{2t}{\langle \tilde{H}_x w, w \rangle + \frac{\phi_0}{2}}} \right\} \end{aligned}$$

We may assume that $t \leq \frac{\phi_0}{4} \theta^2$. Note that $s \leq \theta$ follows automatically since $\langle \tilde{H}_x w, w \rangle + \frac{\phi_0}{2} \leq \frac{\phi_0}{2} \|w\|_2^2 + \frac{\phi_0}{2} = \phi_0$. This yields

$$\{z \in \gamma B_2^{d-1} : f_x(z) \leq t\} \supseteq \sqrt{2t} \left\{ y \in \mathbb{R}^{d-1} : \left\langle \left(\tilde{H}_x + \frac{\phi_0}{2} I \right) y, y \right\rangle \leq 1 \right\}$$

and the set on the right-hand side is an ellipsoid with principle axes

$$\left(\tilde{\lambda}_i(x) + \frac{\phi_0}{2} \right)^{-1/2} v_i \quad .$$

Note that

$$\left(\tilde{\lambda}_i(x) + \frac{\phi_0}{2} \right)^{-1/2} \geq (\lambda_i(x) + \phi_0)^{-1/2} \quad .$$

This yields for $\Delta \leq \frac{\phi_0}{4} \theta^2$ that

$$\begin{aligned} & \int_0^\Delta |\{z \in \gamma B_2^{d-1} : f_x(z) \leq t\}|_{d-1} dt \\ & \geq \int_0^\Delta (2t)^{\frac{d-1}{2}} \left| \left\{ y \in \mathbb{R}^{d-1} : \left\langle \left(\tilde{H}_x + \frac{\phi_0}{2} I \right) y, y \right\rangle \leq 1 \right\} \right|_{d-1} dt \\ & \geq \frac{(2\Delta)^{\frac{d+1}{2}}}{d+1} \prod_{i=1}^{d-1} \frac{1}{(\lambda_i(x) + \phi_0)^{1/2}} |B_2^{d-1}|_{d-1} \quad . \end{aligned} \tag{3.3.1}$$

Since B_p^d has a C^2 -boundary there is a $\Lambda > 0$ such that for all $x \in \partial B_p^d$ and all $1 \leq i \leq d-1$ we have $\lambda_i(x) \leq \Lambda$. Hence,

$$\frac{(2\Delta)^{\frac{d+1}{2}}}{d+1} \prod_{i=1}^{d-1} \frac{1}{(\lambda_i(x) + \phi_0)^{1/2}} |B_2^{d-1}|_{d-1} \geq \frac{|B_2^{d-1}|_{d-1}}{d+1} \left(\frac{2\Delta}{\Lambda + \phi_0} \right)^{\frac{2}{d+1}}.$$

Choose $\delta_0 > 0$ such that for all $\delta_0 \geq \delta > 0$ the inequality

$$\delta |B_p^d|_d \geq \frac{|B_2^{d-1}|_{d-1}}{d+1} \left(\frac{2\Delta}{\Lambda + \phi_0} \right)^{\frac{2}{d+1}}$$

implies $\Delta \leq \frac{\phi_0}{4} \theta^2$. Therefore, if $0 < \delta \leq \delta_0$, the equality

$$\delta |B_p^d|_d = \int_0^\Delta |\{z \in \gamma B_2^{d-1} : f_x(z) = t\}|_{d-1} dt$$

implies $\Delta \leq \frac{\phi_0}{4} \theta^2$ since $\Delta > \frac{\phi_0}{4} \theta^2$ implies

$$\begin{aligned} & \int_0^\Delta |\{z \in \gamma B_2^{d-1} : f_x(z) = t\}|_{d-1} dt > \int_0^{\frac{\phi_0}{4} \theta^2} |\{z \in \gamma B_2^{d-1} : f_x(z) = t\}|_{d-1} dt \\ & \geq \frac{(\phi_0/2)^{\frac{d+1}{2}}}{d+1} \prod_{i=1}^{d-1} \frac{1}{(\lambda_i(x) + \phi_0)^{1/2}} |B_2^{d-1}|_{d-1} \geq \frac{|B_2^{d-1}|_{d-1}}{d+1} \left(\frac{\phi_0/2}{\Lambda + \phi_0} \right)^{\frac{2}{d+1}} \\ & = \delta_0 |B_p^d|_d. \end{aligned}$$

Since $\Delta_x(\delta)$ is by definition such that

$$\delta |B_p^d|_d = \int_0^{\Delta_x(\delta)} |\{z \in \gamma B_2^{d-1} : f_x(z) = t\}|_{d-1} dt$$

it follows from (3.3.1) that

$$\begin{aligned} \Delta_x(\delta) & \leq \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|B_p^d|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \prod_{i=1}^{d-1} (\lambda_i(x) + \phi_0)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}} \\ & = C_p^d \prod_{i=1}^{d-1} (\lambda_i(x) + \phi_0)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}}, \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{\Delta_x(\delta)}{\langle N(x), x \rangle} & \leq F(x, \phi_0) \delta^{\frac{2}{d+1}} \leq F((d^{-1/p}, \dots, d^{-1/p}), \varepsilon) \delta^{\frac{2}{d+1}} \\ & = C_p^d (\kappa(d^{-1/p}, \dots, d^{-1/p})^{1/(d-1)} + \varepsilon)^{\frac{d-1}{d+1}} \delta^{\frac{2}{d+1}}. \end{aligned}$$

which was the last step to prove. \square

3.4 Ovaloid

An *ovaloid* is a convex body with C_+^2 boundary, i.e. with C^2 boundary such that the Gauss curvature is strictly positive in every point of the boundary. In the following we consider centrally symmetric ovaloids. Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid. The function

$$G : \partial C \rightarrow \mathbb{R}_{>0}$$

$$x \mapsto \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle}$$

is continuous with respect to the euclidean distance. Put $G_{\min} = \min_{x \in \partial C} G(x)$ and $G_{\max} = \max_{x \in \partial C} G(x)$. We have the following result.

Theorem 3.4.1 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid, $d \geq 2$. Then*

$$\lim_{\delta \rightarrow 0} \frac{\mathbf{d}_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} = G_{\max} - G_{\min} \quad .$$

By a theorem of Petty (see [P]) a centrally symmetric ovaloid C is an ellipsoid if and only if there is a constant α_C such that $\alpha_C = \frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle}$. An immediate consequence of this fact and Theorem 3.4.1 is the following corollary.

Corollary 3.4.2 *For every centrally symmetric ovaloid $C \subseteq \mathbb{R}^d$ there exists a constant δ_0 such that for every $0 < \delta < \delta_0$ and every $\delta' > 0$ the identity*

$$C_\delta = \langle C \rangle^{\delta'}$$

implies that C is an ellipsoid.

This corollary supports the conjecture that equality of floating body and the dual of the illumination body of the dual characterizes ellipsoids. Note that in [St] and [WY] similar theorems for the homothety conjecture also make use of Petty's lemma as a crucial step in their proofs.

3.4.1 Lower Bound

We show:

Proposition 3.4.3 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid, $d \geq 2$. Then*

$$G_{\max} - G_{\min} \leq \liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} \quad .$$

We need the following lemma.

Lemma 3.4.4 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid. Then for every $x \in \partial C$ the following holds:*

1. *There is a function s_x with $\lim_{\delta \rightarrow 0} s_x(\delta) = 0$ such that*

$$\left(1 - (1 + s_x(\delta))G(x)\delta^{\frac{2}{d+1}}\right) x \in \partial C_\delta \quad .$$

2. *There is a function s'_x with $\lim_{\delta \rightarrow 0} s'_x(\delta) = 0$ such that*

$$\left(1 - (1 + s'_x(\delta))h_d(C^\circ) \frac{\langle x, N(x) \rangle}{\kappa(x)^{\frac{1}{d+1}}} \delta^{\frac{2}{d+1}}\right) x \in \partial \left[\langle C \rangle^\delta \right]$$

$$\text{where } h_d(C^\circ) = \frac{1}{2} \left[d(d+1) |C^\circ|_d / |B_2^{d-1}|_{d-1} \right]^{2/(d+1)} .$$

Proof. Part 1.: From Lemma 3.3.6 it follows immediately that for every $x \in \partial C$ there is a function s_x such that $\lim_{\delta \rightarrow 0} s_x(\delta) = 0$ and

$$\left(1 - (1 + s_x(\delta))G(x)\delta^{\frac{2}{d+1}}\right) x \in \partial C_\delta \quad .$$

Part 2.: By a result of D. Hug (see [H]) the polar body C° has C^2 -boundary with everywhere positive Gauss curvature, too and for every $x \in \partial C$ there is a unique dual point $y \in \partial C^\circ$, i.e. $\langle x, y \rangle = 1$. In this case we have

$$\frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle} = \left[\frac{\kappa(y)^{\frac{1}{d+1}}}{\langle y, N(y) \rangle} \right]^{-1} .$$

By Lemma 3.3.7 we have for the body C° that

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta^{\frac{2}{d+1}}} \frac{\|y^\delta\| - \|y\|}{\|y\|} = h_d(C^\circ) \frac{\kappa(y)^{\frac{1}{d+1}}}{\langle y, N(y) \rangle} = h_d(C^\circ) \frac{\langle x, N(x) \rangle}{\kappa(x)^{\frac{1}{d+1}}} .$$

It follows that there is a function \tilde{s}'_y with $\lim_{\delta' \rightarrow 0} \tilde{s}'_y(\delta') = 0$ such that

$$\left(1 + (1 + \tilde{s}'_y(\delta'))h_d(C^\circ) \frac{\langle x, N(x) \rangle}{\kappa(x)^{\frac{1}{d+1}}} \delta^{\frac{2}{d+1}}\right) y \in (C^\circ)^\delta \quad .$$

We show that we may choose a function s'_x with $\lim_{\delta' \rightarrow 0} s'_x(\delta) = 0$ such that

$$\left(1 - (1 + s'_x(\delta))h_d(C^\circ) \frac{\langle x, N(x) \rangle}{\kappa(x)^{\frac{1}{d+1}}} \delta^{\frac{2}{d+1}}\right) x \in \partial \left[(C^\circ)^\delta \right]^\circ = \langle C \rangle^\delta \quad .$$

For a fixed $x \in \partial C$ let $y \in \partial C^\circ$ be the unique vector such that $\langle x, y \rangle = 1$. Translate and rotate C° to a convex body K such that y is mapped to the origin and the outer normal is $N_K(0) = -e_d$. Let $f : \gamma B_2^{d-1} \rightarrow \mathbb{R}$ be

a parametrization of a part of the boundary of K around the origin. Let $\varepsilon > 0$ and choose $\eta > 0$ such that

$$\frac{1}{d}(1-\eta)^{d-1} - \frac{1}{d+1}(1+\eta)^{d-1} \geq \frac{1}{d(d+1)} \frac{1}{(1+\varepsilon)^{\frac{2}{d+1}}} \quad .$$

Then there exists $\Delta_0 > 0$ such that for every $0 \leq \Delta \leq \Delta_0$ we have

$$(1-\eta)\sqrt{2\Delta}\mathcal{E} \subseteq \{z \in \gamma B_2^{d-1} : f(z) \leq \Delta\} \subseteq (1+\eta)\sqrt{2\Delta}\mathcal{E}$$

where $\mathcal{E} = \{z \in \mathbb{R}^{d-1} : \langle Hf(0)z, z \rangle \leq 1\}$ is the indicatrix of Dupin (see for example [RSW]). We conclude that for every $\zeta \in \mathbb{R}^{d-1}$

$$\text{conv}[K, \zeta - \Delta e_d] \setminus K \supseteq \text{conv}[\Delta e_d + (1-\eta)\sqrt{2\Delta}\mathcal{E}, \zeta - \Delta e_d] \setminus K \quad .$$

We compute

$$\begin{aligned} |\text{conv}[K, \zeta - \Delta e_d] \setminus K|_d &\geq \left| \text{conv}[\Delta e_d + (1-\eta)\sqrt{2\Delta}\mathcal{E}, \zeta - \Delta e_d] \setminus K \right|_d \\ &= |\text{conv}[\Delta e_d + (1-\eta)\sqrt{2\Delta}\mathcal{E}, \zeta - \Delta e_d]|_d - |\{v \in K : v_d \leq \Delta\}|_d \quad . \end{aligned}$$

We have

$$|\text{conv}[\Delta e_d + (1-\eta)\sqrt{2\Delta}\mathcal{E}, \zeta - \Delta e_d]|_d = (1-\eta)^{d-1} \frac{(2\Delta)^{\frac{2}{d+1}}}{d} |\mathcal{E}|_{d-1}$$

and

$$\begin{aligned} |\{v \in K : v_d \leq \Delta\}|_d &\leq \int_0^\Delta |\{z \in \mathbb{R}^{d-1} : f(z) \leq t\}|_{d-1} dt \\ &\leq (1+\eta)^{d-1} \int_0^\Delta (2t)^{\frac{d-1}{2}} |\mathcal{E}|_{d-1} dt = (1+\eta)^{d-1} \frac{(2\Delta)^{\frac{d+1}{2}}}{d+1} |\mathcal{E}|_{d-1} \end{aligned}$$

and therefore,

$$\begin{aligned} |\text{conv}[K, \zeta - \Delta e_d] \setminus K|_d &\geq (2\Delta)^{\frac{d+1}{2}} |\mathcal{E}|_{d-1} \left(\frac{(1-\eta)^{d-1}}{d} - \frac{(1+\eta)^{d-1}}{d+1} \right) \\ &\geq \frac{(2\Delta)^{\frac{d+1}{2}}}{d(d+1)} |\mathcal{E}|_{d-1} \frac{1}{(1+\varepsilon)^{\frac{d+1}{2}}} \quad . \end{aligned}$$

Suppose that $\delta_0 > 0$ is sufficiently small such that for every $0 \leq \delta \leq \delta_0$ we have

$$\frac{(2\Delta_0)^{\frac{d+1}{2}}}{d(d+1)} |\mathcal{E}|_{d-1} \frac{1}{(1+\varepsilon)^{\frac{d+1}{2}}} \geq \delta |K|_d \quad .$$

Hence, if $0 \leq \delta \leq \delta_0$ and $\Delta \geq 0$ is chosen such that

$$|\text{conv}[K, \zeta - \Delta e_d] \setminus K|_d = \delta |K|_d$$

it follows that $\Delta \leq \Delta_0$ and we conclude that

$$\delta|K| \geq \frac{(2\Delta)^{\frac{d+1}{2}}}{d(d+1)} |\mathcal{E}|_{d-1} \frac{1}{(1+\varepsilon)^{\frac{d+1}{2}}} \quad ,$$

i.e.

$$\Delta \leq (1+\varepsilon) \frac{1}{2} \left[d(d+1) \frac{|K|_d}{|B_2^{d-1}|_{d-1}} \right]^{\frac{2}{d+1}} \kappa_K(0)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}} \quad .$$

Hence,

$$K^\delta \subseteq \left\{ \xi \in \mathbb{R}^d : \xi_d \geq -(1+\varepsilon) \frac{1}{2} \left[d(d+1) \frac{|K|_d}{|B_2^{d-1}|_{d-1}} \right]^{\frac{2}{d+1}} \kappa_K(0)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}} \right\}$$

thus,

$$(C^\circ)^\delta \subseteq \{v \in \mathbb{R}^d : \langle v - y, N(y) \rangle \leq (1+\varepsilon) h_d(C^\circ) \kappa(y)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}}\} \quad .$$

Let $x \in \partial C$ be the unique point such that $\langle x, y \rangle = 1$ and let $\lambda(\delta)$ be such that $\lambda(\delta)x \in \langle C \rangle^{\delta'}$. It follows for $0 \leq \delta \leq \delta_0$ that

$$\begin{aligned} \lambda(\delta) &\geq \left(1 + (1+\varepsilon) h_d(C^\circ) \frac{\kappa(y)^{\frac{1}{d+1}}}{\langle y, N(y) \rangle} \delta^{\frac{2}{d+1}} \right)^{-1} \\ &\geq 1 - (1+\varepsilon) h_d(C^\circ) \frac{\kappa(y)^{\frac{1}{d+1}}}{\langle y, N(y) \rangle} \delta^{\frac{2}{d+1}} = 1 - (1+\varepsilon) h_d(C^\circ) \frac{\langle x, N(x) \rangle}{\kappa(x)^{\frac{1}{d+1}}} \delta^{\frac{2}{d+1}} \end{aligned}$$

which proves that we may choose a function s'_x such that $\lim_{\delta \rightarrow 0} s'_x(\delta) = 0$ and

$$\left(1 - (1 + s'_x(\delta)) h_d(C^\circ) \frac{\langle x, N(x) \rangle}{\kappa(x)^{\frac{1}{d+1}}} \delta^{\frac{2}{d+1}} \right) x \in \partial \langle C \rangle^{\delta'} \quad .$$

□

Proof of Proposition 3.4.3. For $x \in \partial C$ apply Lemma 3.4.4 and put $c_d(C) = \frac{1}{2} \left((d+1) |C|_d / |B_2^{d-1}|_{d-1} \right)^{\frac{2}{d+1}}$ and $\phi = (h_d(C^\circ) c_d(C))^{\frac{d+1}{2}} \delta'$, i.e.

$$(1 - (1 + t'_x(\phi)) G(x)^{-1} \phi^{\frac{2}{d+1}}) x \in \partial \langle C \rangle^{\delta'}$$

where $t'_x(\phi) = s'_x(\delta')$. Let $x_0 \in \partial C$ be such that $G_{\min} = \min_{x \in \partial C} G(x) = G(x_0)$ and let $x_1 \in \partial C$ be such that $G_{\max} = \max_{x \in \partial C} G(x) = G(x_1)$. Let $a \geq 1$ be such that $\frac{1}{a} C_\delta \subseteq \left[(C^\circ)^{\delta'} \right]^\circ \subseteq a C_\delta$. We conclude that

$$a \geq \frac{1 - (1 + s_{x_0}(\delta)) G_{\min} \delta^{\frac{2}{d+1}}}{1 - (1 + t'_{x_0}(\phi)) G_{\min}^{-1} \phi^{\frac{2}{d+1}}} \quad (3.4.1)$$

and

$$a \geq \frac{1 - (1 + t'_{x_1}(\phi))G_{\max}^{-1}\phi^{\frac{2}{d+1}}}{1 - (1 + s_{x_1}(\delta))G_{\max}\delta^{\frac{2}{d+1}}} . \quad (3.4.2)$$

The assumption that $\phi \geq (G_{\min}G_{\max})^{\frac{d+1}{2}}\delta$ and inequality (3.4.1) lead to $\liminf_{\delta>0} \frac{a-1}{\delta^{\frac{2}{d+1}}} \geq G_{\max} - G_{\min}$. The assumption $\phi \leq (G_{\min}G_{\max})^{\frac{d+1}{2}}\delta$ and (3.4.2) lead to $\liminf_{\delta>0} \frac{a-1}{\delta^{\frac{2}{d+1}}} \geq G_{\max} - G_{\min}$ as well. Hence, we conclude that

$$\liminf_{\delta \rightarrow 0} \frac{\mathbf{d}_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} \geq G_{\max} - G_{\min} .$$

□

3.4.2 Upper Bound

We deduce an upper bound which yields together with Proposition 3.4.3 that Theorem 3.4.1 holds. Before we start to deduce an upper bound we need several lemmas in order to determine an upper and a lower bound for the illumination body of an ovaloid.

Lemma 3.4.5 *Let $C \subseteq \mathbb{R}^d$ be a convex body and $x \in \mathbb{R}^d$. Then*

$$\text{conv}[C, x] = \{\lambda x + (1 - \lambda)y : y \in C, \lambda \in [0, 1]\} =: M .$$

Proof. The inclusion ' \supseteq ' is obvious. For the inclusion ' \subseteq ' note first that C and x are included in M , and therefore, the only thing we need to show is that M is convex. Let $(1 - \lambda_1)x + \lambda_1 y_1, (1 - \lambda_2)x + \lambda_2 y_2 \in M$ with $\lambda_1, \lambda_2 \in [0, 1]$ and $y_1, y_2 \in C$ and let $\mu \in [0, 1]$. Put $z = \frac{\mu\lambda_1}{\mu\lambda_1 + (1-\mu)\lambda_1}y_1 + \frac{(1-\mu)\lambda_2}{\mu\lambda_1 + (1-\mu)\lambda_2}y_2 \in C$. Then

$$\begin{aligned} & \mu((1 - \lambda_1)x + \lambda_1 y_1) + (1 - \mu)((1 - \lambda_2)x + \lambda_2 y_2) \\ &= (1 - \mu\lambda_1 - (1 - \mu)\lambda_2)x + (\mu\lambda_1 + (1 - \mu)\lambda_2)z \in M . \end{aligned}$$

□

Lemma 3.4.6 *Let $C \subseteq \mathbb{R}^d$ be a convex body and $x \in \mathbb{R}^d$. Then*

$$\partial(\text{conv}[C, x]) \subseteq \{\lambda x + (1 - \lambda)y : y \in \partial C, \lambda \in [0, 1]\} .$$

Proof. From Lemma 3.4.5 we know that

$$\partial(\text{conv}[C, x]) \subseteq \{\lambda x + (1 - \lambda)y : y \in C, \lambda \in [0, 1]\}$$

Assume that $y \in \text{int}(C)$. Then there is an $\varepsilon > 0$ such that $y + \varepsilon B_2^d \subseteq C$. Then for every $\lambda \in [0, 1]$ we have

$$(1 - \lambda)y + \lambda x \in \text{int}\left(\text{conv}[y + \varepsilon B_2^d, x]\right) \subseteq \text{int}[C, x] .$$

Note that x lies on the line segment between x and an arbitrary point of the boundary of C and hence, we draw our inclusion. \square

We call $y \in \partial C$ a *touching point* of x with C if the line segment $[y, x]$ lies in a support hyperplane of C at y . The following lemma tells us that $\text{conv}[C, x]$ depends only on the touching points of x with C .

Lemma 3.4.7 *Let $C \subseteq \mathbb{R}^d$ be a convex body and let $x \in \mathbb{R}^d \setminus C$. Then*

$$\begin{aligned} & \partial(\text{conv}[C, x]) \\ & \subseteq \partial C \cup \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1], y \in \partial C \text{ is a touching point}\} \end{aligned}$$

Proof. Let $z \in \partial(\text{conv}[C, x])$ and assume that $z \neq x$ and $z \notin \partial C$. By Lemma 3.4.6 there are $\lambda \in (0, 1)$ and $y \in \partial C$ such that $z = \lambda y + (1 - \lambda)x$. Let H be a support hyperplane of $\text{conv}[C, x]$ at z and let H^+ be the corresponding closed half-space including $\text{conv}[C, x]$. Then $y, x \in H^+$, hence, $y, x \in H$ and it follows that H is also a support hyperplane of $\text{conv}[C, x]$ at y . \square

Lemma 3.4.8 *Let $\gamma > 0$ and let $f : (-\gamma, \gamma) \rightarrow \mathbb{R}_{\geq 0}$ be convex, of class C^2 and such that $f(0) = 0$, $f'(0) = 0$ and $f''(0) > 0$. Let $\frac{1}{3} > \eta \geq 0$ and let $\gamma > \theta > 0$ such that $|t| \leq \theta$ implies $(1 - \eta)f''(0) \leq f''(z) \leq (1 + \eta)f''(0)$. If $\Delta > 0$ is such that $\Delta < \frac{(1-3\eta)f''(0)}{2}\theta^2$ there is some $0 < t_0 < \theta$ such that the line through $-\Delta e_2$ and $(t_0, f(t_0))$ touches the graph of f , i.e. is the tangent of f at $(t_0, f(t_0))$. In this case $f(t_0) \leq \frac{1+\eta}{1-3\eta}\Delta$.*

Proof. For every $s \in (0, \gamma)$ the line l_s through $-\Delta e_2$ and $(s_0, f(s_0))$ is given by $l_s(t) = \frac{f(s)+\Delta}{s}t$ and l_s touches the graph of f if and only if $f'(s) = \frac{f(s)+\Delta}{s}$. By Taylor's theorem there are $\xi_1(s), \xi_2(s) \in (0, s)$ such that

$$\begin{aligned} f'(s)s - f(s) &= (f'(0) + f''(\xi_1(s))s)s - f(0) - f'(0)s - \frac{1}{2}f''(\xi_2(s))s^2 \\ &= \left(f''(\xi_1(s)) - \frac{1}{2}f''(\xi_2(s)) \right) s^2 \quad . \end{aligned}$$

If $s = \sqrt{\frac{2\Delta}{(1-3\eta)f''(0)}}$ then $s \leq \theta$ since $\Delta \leq \frac{(1-3\eta)f''(0)}{2}\theta^2$ and hence,

$$f'(s)s - f(s) \geq \left((1 - \eta)f''(0) - \frac{1}{2}(1 + \eta)f''(0) \right) \frac{2\Delta}{(1 - 3\eta)f''(0)} = \Delta \quad .$$

If $s = \sqrt{\frac{2\Delta}{(1+3\eta)f''(0)}} \leq \sqrt{\frac{2\Delta}{(1-3\eta)f''(0)}} \leq \theta$ then

$$f'(s)s - f(s) \leq \left((1 + \eta)f''(0) - \frac{1}{2}(1 - \eta)f''(0) \right) \frac{2\Delta}{(1 + 3\eta)f''(0)} = \Delta$$

and hence, there is $\sqrt{\frac{2\Delta}{(1+3\eta)f''(0)}} \leq t_0 \leq \sqrt{\frac{2\Delta}{(1-3\eta)f''(0)}} \leq \theta$ such that $f'(t_0) = \frac{f(t_0)+\Delta}{t_0}$, i.e. the line through $-\Delta e_2$ and $(t_0, f(t_0))$ touches the graph of f at $(t_0, f(t_0))$. Since f is monotonously increasing we have by Taylor's theorem:

$$f(t_0) \leq f\left(\sqrt{\frac{2\Delta}{(1-3\eta)f''(0)}}\right) \leq \frac{1}{2}(1+\eta)f''(0)\frac{2\Delta}{(1-3\eta)f''(0)} \leq \frac{1+\eta}{1-3\eta}\Delta \quad .$$

□

We obtain the following immediate generalization for higher dimensions:

Corollary 3.4.9 *Let $\gamma > 0$ and let $f : \text{int}(\gamma B_2^{d-1}) \rightarrow \mathbb{R}_{\geq 0}$ be convex, of class C^2 such that $f(0) = 0$, $\nabla f(0) = 0$ and the smallest eigenvalue λ_{d-1} of $Hf(0)$ is positive. Let $\frac{1}{3} > \eta \geq 0$ and let $\gamma > \theta > 0$ be such that $\|z\|_2 \leq \theta$ implies $(1-\eta)Hf(0) \leq Hf(z) \leq (1+\eta)Hf(0)$. If $\Delta > 0$ is such that $\Delta < \frac{(1-3\eta)\lambda_{d-1}\theta^2}{2}$ then for every $v \in \mathbb{S}^{d-2}$ there is some $0 < t_0(v) < \theta$ such that the line through $-\Delta e_d$ and $(t_0(v)v, f(t_0(v)v))$ touches the graph of f , i.e. the line lies in the tangent hyperplane of f at $(t_0(v)v, f(t_0(v)v))$. In this case $f(t_0(v)v) \leq \frac{1+\eta}{1-3\eta}\Delta$.*

Corollary 3.4.10 *Let $C \subseteq \mathbb{R}^d$ be an ovaloid. Then for every $\varepsilon > 0$ there is some $\Delta_0 > 0$ such that for every $0 \leq \Delta \leq \Delta_0$ and every $x \in \partial C$ the following holds: Let*

$$C_1 = C \cap \{\xi \in \mathbb{R}^d : \langle \xi - x, N(x) \rangle \leq -(1+\varepsilon)\Delta\}$$

and

$$C_2 = C \cap \{\xi \in \mathbb{R}^d : \langle \xi - x, N(x) \rangle \geq -(1+\varepsilon)\Delta\} \quad .$$

Then

$$\text{conv}[C, x + \Delta N(x)] = C_1 \cup \text{conv}[C_2, x + \Delta N(x)] \quad .$$

Proof. Let $\varepsilon > 0$ and $\frac{1}{3} > \eta > 0$ with $\frac{1+\eta}{1-3\eta} \leq 1+\varepsilon$. By Corollary A.0.4 we can choose some $\gamma > 0$ and some $\gamma > \theta > 0$ such that for every $x \in \partial C$ the following holds:

There is a parametrization $f_x : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ with $(1-\eta)Hf_x(0) \leq Hf_x(z) \leq (1+\eta)Hf_x(0)$ if $\|z\|_2 \leq \theta$. Put $\Delta_0 = \frac{(1-3\eta)\kappa_{\min}\theta}{3}$ where κ_{\min} is the smallest principal curvature which occurs on C .

Let $x_0 \in \partial C$ and $0 < \Delta < \Delta_0$. By Lemma 3.4.7 it is sufficient to show that the touching points of $x_0 + \Delta N(x_0)$ with C lie in C_2 . If $y \in \partial C$ is a touching point then there is some $w \in \mathbb{S}^{d-1}$ orthonormal to $N(x_0)$ such that

$$y \in \partial C \cap \{x_0 + \mu_1 N(x_0) + \mu_2 w : \mu_1, \mu_2 \in \mathbb{R}, \mu_2 \geq 0\} \quad .$$

It is obvious that y is the unique touching point which lies on the half-plane $\{x_0 + \mu_1 N(x_0) + \mu_2 w : \mu_1, \mu_2 \in \mathbb{R}, \mu_2 \geq 0\}$. Let $f_{x_0} : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ be a parametrization of the boundary at x_0 and let $v \in \mathbb{S}^{d-2}$ be the corresponding vector to w . By Corollary 3.4.9 there is a $t_0(v)$ such that the line through $-\Delta e_d$ and $(t_0(v)v, f_{x_0}(t_0(v)))$ is tangential to the graph of f_{x_0} . Hence, $(t_0(v)v, f_{x_0}(t_0(v)))$ corresponds to the touching point y and we have the estimate $f_{x_0}(t_0(v)v) \leq \frac{1+\eta}{1-3\eta}\Delta \leq (1+\varepsilon)\Delta$. It follows that

$$y \in \{\xi \in \mathbb{R}^d : \langle \xi, N(x_0) \rangle \geq \langle x_0, N(x_0) \rangle - (1+\varepsilon)\Delta\} \quad .$$

□

We use Corollary 3.4.10 in order to obtain an upper volume estimate for the convex hull of an ovaloid with a point. From now on, C is always an ovaloid and $\gamma = \gamma(C) > 0$, $f_x : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ accordingly to Corollary A.0.4. We may assume without loss of generality that γ is chosen sufficiently small such that for every $z \in \gamma B_2^{d-1}$ we already have $\frac{1}{2}Hf_x(0) \leq Hf_x(z) \leq \frac{3}{2}Hf_x(0)$. Let $\kappa_{\min} > 0$ be the smallest and $\kappa_{\max} > 0$ be the largest principal curvature which can occur on C . Note that for every $v \in \mathbb{S}^{d-2}$ we have

$$\begin{aligned} f_x(\gamma v) &= \int_0^\gamma \int_0^s \langle Hf_x(rv)v, v \rangle dr ds \geq \int_0^\gamma \int_0^s \left\langle \frac{1}{2}Hf_x(0)v, v \right\rangle dr ds \\ &\geq \int_0^\gamma \int_0^s \frac{\kappa_{\min}}{2} dr ds = \frac{\kappa_{\min}\gamma^2}{4} =: T_0(C) \end{aligned}$$

and hence, the part of the boundary of C lying in the half-space $\{\xi \in \mathbb{R}^d : \langle \xi, N(x) \rangle \geq \langle x, N(x) \rangle - T_0(C)\}$ is completely parametrized by f_x , i.e. for every

$$y \in \partial C \cap \{\xi \in \mathbb{R}^d : \langle \xi, N(x) \rangle \geq \langle x, N(x) \rangle - T_0(C)\}$$

is a $z \in \gamma B_2^{d-1}$ such that $(z, f_x(z))$ corresponds to y .

Lemma 3.4.11 *Let $C \subseteq \mathbb{R}^d$ be an ovaloid. Then there are non-negative functions ϕ, ψ with $\lim_{\Delta \rightarrow 0} \phi(\Delta) = 0$ and $\lim_{\Delta \rightarrow 0} \psi(\Delta) = 0$ such that for every $x \in \partial C$ the following holds:*

1.

$$\begin{aligned} \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d+1} \frac{1}{\kappa(x)^{\frac{1}{2}}} (1 - \phi(\Delta)) &\leq |\{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta\}|_d \\ &\leq \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d+1} \frac{1}{\kappa(x)^{\frac{1}{2}}} (1 + \phi(\Delta)) \end{aligned}$$

2. For every $y \in \mathbb{R}^d$ such that $\langle y, N(x) \rangle \geq \langle x, N(x) \rangle + \Delta$ we have

$$\frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d(d+1)} \frac{1}{\kappa(x)^{\frac{1}{2}}} (1 - \psi(\Delta)) \leq |\text{conv}[C, y]|_d - |C|_d \quad .$$

3.

$$|\text{conv}[C, x + \Delta N(x)]|_d - |C|_d \leq \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d(d+1) \kappa(x)^{\frac{1}{2}}} (1 + \psi(\Delta))$$

Proof. Before we start to show the volume estimates we establish some general facts. Let $0 < \varepsilon < 1$ and $\Delta_0 > 0$ be chosen according to Corollary 3.4.10. Let $0 < \theta < \gamma$ be such that for every $z \in \theta B_2^{d-1}$ and every $x \in \partial C$ we have $(1 - \varepsilon)Hf_x(0) \leq Hf_x(z) \leq (1 + \varepsilon)Hf_x(0)$. Assume without loss of generality that we also have $(1 + \varepsilon)\Delta_0 \leq T_0(C)$ and $\Delta_0 \leq \frac{(1-\varepsilon)\kappa_{\min}\theta^2}{2}$. Let $0 \leq t \leq (1 + \varepsilon)\Delta_0$. We show that for every $x \in \partial C$ we have

$$\sqrt{\frac{2t}{1+\varepsilon}} \mathcal{E}_x \subseteq \{z \in \gamma B_2^{d-1} : f_x(z) \leq t\} \subseteq \sqrt{\frac{2t}{1-\varepsilon}} \mathcal{E}_x \quad (3.4.3)$$

where $\mathcal{E}_x = \{\zeta \in \mathbb{R}^{d-1} : \langle Hf_x(0)\zeta, \zeta \rangle \leq 1\}$ is the indicatrix of Dupin at x . For $z \in \gamma B_2^{d-1} \setminus \{0\}$ put $z' = \frac{\tilde{z}}{\|z\|_2}$ then

$$f_x(z) = \int_0^{\|z\|_2} \int_0^s \langle Hf_x(rz')z', z' \rangle dr ds$$

and it follows for $z \in \theta B_2^{d-1}$ that

$$\begin{aligned} \frac{(1-\varepsilon)\kappa_{\min}\|z\|_2^2}{2} &\leq \frac{1-\varepsilon}{2} \langle Hf_x(0)z, z \rangle \leq f_x(z) \leq \frac{1+\varepsilon}{2} \langle Hf_x(0)z, z \rangle \\ &\leq \frac{(1+\varepsilon)\kappa_{\max}\|z\|_2^2}{2} . \end{aligned}$$

Since $\sigma \mapsto f_x(\sigma v)$ is strictly monotonously increasing on $[0, \gamma]$ for $v \in \mathbb{S}^{d-2}$ we conclude for $\|z\|_2 > \theta$ and $z' = \frac{\tilde{z}}{\|z\|_2}$ that

$$f_x(z) > f_x(\theta z') \geq \frac{(1-\varepsilon)\kappa_{\min}\theta^2}{2} \geq \Delta_0 \geq t.$$

Therefore,

$$\begin{aligned} \{z \in \gamma B_2^{d-1} : f_x(z) \leq t\} &= \{z \in \theta B_2^{d-1} : f_x(z) \leq t\} \\ &\subseteq \{z \in \theta B_2^{d-1} : \frac{1-\varepsilon}{2} \langle Hf_x(0)z, z \rangle \leq t\} \end{aligned}$$

and since for $\|z\|_2 > \theta$ we have $\frac{1-\varepsilon}{2} \langle Hf_x(0)z, z \rangle > \frac{(1-\varepsilon)\kappa_{\min}\theta^2}{2} \geq t$ it follows that

$$\begin{aligned} \left\{ z \in \theta B_2^{d-1} : \frac{1-\varepsilon}{2} \langle Hf_x(0)z, z \rangle \leq t \right\} &= \left\{ z \in \mathbb{R}^{d-1} : \frac{1-\varepsilon}{2} \langle Hf_x(0)z, z \rangle \leq t \right\} \\ &= \sqrt{\frac{2t}{1-\varepsilon}} \mathcal{E}_x . \end{aligned}$$

We conclude in a similar way that

$$\{z \in \gamma B_2^{d-1} : f_x(z) \leq t\} \supseteq \sqrt{\frac{2t}{1+\varepsilon}} \mathcal{E}_x \quad .$$

Proof of 1. For every $x \in \partial C$ the following holds. Since $\Delta_0 \leq T_0(C)$ the part of the boundary of C lying in the half-space $\{\xi \in \mathbb{R}^d : \langle \xi - x, N(x) \rangle \geq -\Delta_0\}$ is completely parametrized by f_x and hence, for every $0 \leq \Delta \leq \Delta_0$ the volume of $\{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta\}$ is given by the volume of

$$\{(z, t) \in \mathbb{R}^d : f_x(z) \leq t \leq \Delta\} \quad .$$

Cavalieri's principle and the right-hand side of (3.4.3) yield

$$\begin{aligned} |\{(z, t) \in \mathbb{R}^d : f_x(z) \leq t \leq \Delta\}|_d &= \int_0^\Delta |\{z \in \gamma B_2^{d-1} : f_x(z) \leq t\}|_{d-1} dt \\ &\leq \int_0^\Delta \left| \sqrt{\frac{2t}{1-\varepsilon}} \mathcal{E}_x \right|_{d-1} dt = \int_0^\Delta (2t)^{\frac{d-1}{2}} dt \frac{|\mathcal{E}_x|_{d-1}}{(1-\varepsilon)^{\frac{d-1}{2}}} \\ &= \frac{(2\Delta)^{\frac{d+1}{2}} |\mathcal{E}_x|_{d-1}}{d+1} (1-\varepsilon)^{-\frac{d-1}{2}} \quad . \end{aligned}$$

Similarly, using the left-hand side of (3.4.3) one has

$$|\{(z, t) \in \mathbb{R}^d : f_x(z) \leq t \leq \Delta\}|_d \geq \frac{(2\Delta)^{\frac{d+1}{2}} |\mathcal{E}_x|_{d-1}}{d+1} (1+\varepsilon)^{-\frac{d-1}{2}} \quad .$$

Since $\det(Hf_x(0)) = \kappa(x)$ it follows that $|\mathcal{E}_x|_{d-1} = \kappa(x)^{-1/2} |B_2^{d-1}|_{d-1}$ which shows the first part of the lemma.

Proof of 2. Let $0 \leq \Delta \leq \Delta_0$, $y \in \{\xi \in \mathbb{R}^d : \langle \xi - x, N(x) \rangle \geq \Delta\}$ and put $B_x(\Delta) = \{\xi \in C : \langle \xi - x, N(x) \rangle = -\Delta\}$. The set $\text{conv}[C, y] \setminus C$ includes the set

$$\text{conv}[B_x(\Delta), y] \setminus \{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta\} \quad .$$

The height of the cone $\text{conv}[B_x(\Delta), y]$ is at least 2Δ . Using part 1. of this lemma, a lower estimate for the volume of $\text{conv}[C, y] \setminus C$ is

$$\begin{aligned} &|\text{conv}[B_x(\Delta), y]|_d - |\{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta\}|_d \\ &\geq \frac{2\Delta |B_x(\Delta)|_{d-1}}{d} - \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d+1} \frac{1}{\kappa(x)^{\frac{1}{2}}} (1 + \phi(\Delta)) \quad . \end{aligned}$$

Since $\Delta_0 \leq T_0(C)$ we get by a discussion similar to the one in the proof of part 1. that the volume of $|B_x(\Delta)|_{d-1}$ is given by

$$|\{z \in \gamma B_2^{d-1} : f_x(z) \leq \Delta\}|_{d-1} \quad .$$

Using the left-hand side of (3.4.3), we obtain

$$|B_x(\Delta)|_{d-1} \geq \frac{(2\Delta)^{\frac{d-1}{2}} |\mathcal{E}_x|_{d-1}}{(1+\varepsilon)^{\frac{d-1}{2}}} = \frac{(2\Delta)^{\frac{d-1}{2}} |B_2^{d-1}|_{d-1}}{\kappa(x)^{1/2}} (1+\varepsilon)^{-\frac{d-1}{2}}$$

which yields

$$\begin{aligned} & \frac{2\Delta |B_x(\Delta)|_{d-1}}{d} - \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d+1} \frac{1}{\kappa(x)^{\frac{1}{2}}} (1+\phi(\Delta)) \\ & \geq \frac{(2\Delta)^{\frac{2+1}{2}} |B_2^{d-1}|_{d-1}}{\kappa(x)^{1/2}} \left(\frac{(1+\varepsilon)^{-\frac{d-1}{2}}}{d} - \frac{1+\phi(\Delta)}{d+1} \right) . \end{aligned}$$

The expression

$$\frac{(1+\varepsilon)^{-\frac{d-1}{2}}}{d} - \frac{1+\phi(\Delta)}{d+1}$$

is arbitrarily close to $\frac{1}{d(d+1)}$ if $\Delta_0 > 0$ is small enough to guarantee that $\phi(\Delta)$ is sufficiently small and it is possible to choose $\varepsilon > 0$ sufficiently small.

Proof of 3. For every $x \in \partial C$ the following holds. Since Δ_0 is chosen accordingly to Corollary 3.4.10 we have that for every $0 < \Delta \leq \Delta_0$ the volume of $\text{conv}[C, x + \Delta N(x)] \setminus C$ is the same as the volume of $\text{conv}[C_2, x + \Delta N(x)] \setminus C_2$ where

$$C_2 = \{\xi \in C : \langle \xi - x, N(x) \rangle \geq -(1+\varepsilon)\Delta\} .$$

Since $(1+\varepsilon)\Delta_0 \leq T_0(C)$ the volume of $\text{conv}[C_2, x + \Delta N(x)]$ is given by the volume of $\text{conv}[C'_2, -\Delta e_d]$ where

$$C'_2 = \{(z, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : f_x(z) \leq t \leq (1+\varepsilon)\Delta\} .$$

Let $F_x(\Delta) \subseteq \mathbb{R}^d$ be the cone with base $G_x(\Delta) = \sqrt{\frac{\Delta}{2(1-\varepsilon)}}(2+\varepsilon)\mathcal{E}_x \times \{(1+\varepsilon)\Delta\}$ and apex $-\Delta e_d$, i.e. $F_x(\Delta) = \text{conv}[G_x(\Delta), -\Delta e_d]$. It follows from the right-hand side of (3.4.3) that for every $0 \leq t \leq (1+\varepsilon)\Delta$ that

$$\{z \in \mathbb{R}^{d-1} : (z, t) \in C'_2\} \subseteq \{z \in \mathbb{R}^{d-1} : (z, t) \in F_x(\Delta)\},$$

thus, $C'_2 \subseteq F_x(\Delta)$ and $\text{conv}[C'_2, -\Delta e_d] \subseteq F_x(\Delta)$. Since the height of $F_x(\Delta)$ is $(2+\varepsilon)\Delta$ the volume of $F_x(\Delta)$ is given by

$$\frac{1}{d}(2+\varepsilon) \left[\sqrt{\frac{2\Delta}{(1-\varepsilon)}}(2+\varepsilon) \right]^{d-1} |\mathcal{E}_x|_{d-1} = \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{\kappa(x)^{1/2}} \frac{(1+\frac{\varepsilon}{2})^d}{d(1-\varepsilon)^{\frac{d-1}{2}}} .$$

By part 1. the volume of C_2 can be bounded from below by

$$\frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1} (1 - \phi((1+\varepsilon)\Delta)) (1+\varepsilon)^{\frac{d+1}{2}}}{\kappa(x)^{\frac{1}{2}} d+1} .$$

Arguing in a similar way as in the proof of part 2. one can also prove the claim of the third part. \square

Proposition 3.4.12 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid. Then there is a non-negative function Φ with $\lim_{\delta \rightarrow 0} \Phi(\delta) = 0$ such that*

$$\left(1 - (1 + \Phi(\delta))G_{\max}\delta^{\frac{2}{d+1}}\right) C \subseteq C_\delta \subseteq \left(1 - (1 - \Phi(\delta))G_{\min}\delta^{\frac{2}{d+1}}\right) C \quad (3.4.4)$$

Proof. For $x \in \partial C$ and $0 \leq \delta < \frac{1}{2}$ we define $\Delta(x, \delta) \geq 0$ by

$$|\{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta(x, \delta)\}|_d = \delta|C|_d \quad .$$

Let $1 > \varepsilon > 0$ and let ϕ be the function of Lemma 3.4.11. Let $\Delta_0 = \Delta_0(\varepsilon) > 0$ be sufficiently small such that for every $0 \leq \Delta \leq \Delta_0$ we have $\phi(\Delta) \leq \varepsilon$. Let $\Omega = \max_{x \in \partial C} \kappa(x)$ and let $\delta_0 > 0$ be such that

$$\frac{(2\Delta_0)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d+1} \frac{1}{\Omega^{1/2}} (1 - \varepsilon) = \delta_0|C|_d \quad .$$

If $\delta \leq \delta_0$ then $\Delta(x, \delta) \leq \Delta_0$ because if $\Delta(x, \delta) > \Delta_0$ then

$$\begin{aligned} & |\{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta(x, \delta)\}|_d = \delta|C|_d \\ & > |\{\xi \in C : \langle \xi - x, N(x) \rangle \geq -\Delta_0\}|_d = \delta|C|_d \\ & \geq \frac{(2\Delta_0)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d+1} \frac{1}{\Omega^{1/2}} (1 - \varepsilon) = \delta_0|C|_d \quad . \end{aligned}$$

Hence, if we suppose that $\delta \leq \delta_0$ the following holds. We start with the second inclusion in (3.4.4). Let $x \in \partial C$ and let x_δ be the boundary point of C_δ lying on the line segment $[0, x]$. Since

$$x_\delta \in \{\xi \in C : \langle \xi - x, N(x) \rangle \leq -\Delta(x, \delta)\}$$

it follows that

$$\frac{\|x_\delta\|_2}{\|x\|_2} = \frac{\langle x_\delta, N(x) \rangle}{\langle x, N(x) \rangle} \leq \frac{\langle x - \Delta(x, \delta)N(x), N(x) \rangle}{\langle x, N(x) \rangle} = 1 - \frac{\Delta(x, \delta)}{\langle x, N(x) \rangle} \quad .$$

Moreover, since $\Delta(x, \delta) \leq \Delta_0$, we obtain the estimate

$$\Delta(x, \delta) \geq \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \kappa(x)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}} (1 + \varepsilon)^{-\frac{2}{d+1}} \quad .$$

This yields

$$\frac{\|x_\delta\|_2}{\|x\|_2} \leq 1 - (1 + \varepsilon)^{-\frac{2}{d+1}} G(x) \delta^{\frac{2}{d+1}} \leq 1 - (1 + \varepsilon)^{-\frac{2}{d+1}} G_{\min} \delta^{\frac{2}{d+1}}$$

which proves the inclusion on the right-hand side of (3.4.4).

We prove the left-hand side of (3.4.4). Let again $x \in \partial C$ and let x_δ be defined as above. Then there is a support hyperplane of C_δ at x_δ cutting of exactly the volume $\delta|C|_d$ and this support hyperplane is given by

$$\{\xi \in \mathbb{R}^d : \langle \xi - x', N(x') \rangle = -\Delta(x', \delta)\}$$

for some $x' \in \partial C$. Since

$$x_\delta, x \in \{\xi \in \mathbb{R}^d : -\Delta(x', \delta) \leq \langle \xi - x', N(x') \rangle \leq 0\}$$

we obtain that

$$\frac{\|x_\delta\|}{\|x\|} = \frac{\langle x_\delta, N(x') \rangle}{\langle x, N(x') \rangle} \geq 1 - \frac{\Delta(x', \delta)}{\langle x', N(x') \rangle} .$$

Since $\Delta(x', \delta) \leq \Delta_0$ it follows that

$$\Delta(x', \delta) \leq \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \kappa(x)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}} (1-\varepsilon)^{-\frac{2}{d+1}}$$

and we may deduce the inclusion on the left-hand side of (3.4.4) in a similar way to the inclusion on the right-hand side. \square

For a centrally symmetric ovaloid define $H : \partial C \rightarrow \mathbb{R}_{\geq 0}$,

$$H(x) = \frac{[d(d+1)]^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|} \right)^{\frac{2}{d+1}} \frac{\kappa(x)^{\frac{1}{d+1}}}{\langle x, N(x) \rangle} ,$$

i.e. $H(x) = d^{\frac{2}{d+1}} G(x)$, and put $H_{\min} = \min_{x \in \partial C} H(x)$ and $H_{\max} = \max_{x \in \partial C} H(x)$.

Proposition 3.4.13 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid. Then there is a non-negative function $\tilde{\Psi}$ with $\lim_{\delta \rightarrow 0} \tilde{\Psi}(\delta) = 0$ such that*

$$\left(1 + (1 - \tilde{\Psi}(\delta))H_{\min}\delta^{\frac{2}{d+1}}\right) C \subseteq C^\delta \subseteq \left(1 + (1 + \tilde{\Psi}(\delta))H_{\max}\delta^{\frac{2}{d+1}}\right) C \quad (3.4.5)$$

Proof. For $x \in \partial C$ let $\Delta(x, \delta) \geq 0$ be defined as the value such that

$$|\text{conv}[C, x + \Delta(x, \delta)N(x)]|_d - |C|_d = \delta|C|_d .$$

Let $\varepsilon > 0$ and let $\Delta_0 > 0$ be such that for every $0 \leq \Delta \leq \Delta_0$ the function ψ of Lemma 3.4.11 is smaller than or equal to ε . We show that there is a δ_0 such that for every $0 \leq \delta \leq \delta_0$ and every $y \in \mathbb{R}^d$ with $|\text{conv}[C, y]|_d - |C|_d = \delta|C|_d$

it follows for every $x \in \partial C$ that $\langle y-x, N(x) \rangle \leq \Delta_0$, especially, $\Delta(x, \delta) \leq \Delta_0$. Put $\Omega = \max_{x \in \partial C} \kappa(x)$ and let $\delta_0 > 0$ be such that

$$\frac{(2\Delta_0)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d(d+1) \Omega^{1/2}} (1 - \varepsilon) = \delta_0 |C|_d \quad .$$

Then a discussion similar to the one in Proposition 3.4.12 yields that δ_0 has the desired properties. For every $0 \leq \delta \leq \delta_0$ the following holds. Let $x \in \partial C$ and x^δ be the point on the boundary of C^δ such that x lies on the line segment $[0, x^\delta]$.

We start with the inclusion on the right-hand side of (3.4.5). Since $|\text{conv}[C, x^\delta]|_{d-1} = \delta |C|_d$ it follows that $\langle x^\delta - x, N(x) \rangle =: \Delta \leq \Delta_0$ and we conclude with Lemma 3.4.11, 2., that

$$\delta |C|_d \geq \frac{(2\Delta)^{\frac{d+1}{2}} |B_2^{d-1}|_{d-1}}{d(d+1) \kappa(x)^{1/2}} (1 - \varepsilon) \quad ,$$

i.e.

$$\Delta \leq \frac{[d(d+1)]^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \kappa(x)^{\frac{1}{d+1}} \delta^{\frac{2}{d+1}} (1 - \varepsilon)^{-\frac{2}{d+1}} \quad .$$

Hence,

$$\begin{aligned} \frac{\|x^\delta\|_2}{\|x\|_2} &= \frac{\langle x^\delta, N(x) \rangle}{\langle x, N(x) \rangle} = 1 + \frac{\Delta}{\langle x, N(x) \rangle} \leq 1 + H(x) \delta^{\frac{2}{d+1}} (1 - \varepsilon)^{-\frac{2}{d+1}} \\ &\leq 1 + H_{\max} \delta^{\frac{2}{d+1}} (1 - \varepsilon)^{-\frac{2}{d+1}} \end{aligned}$$

which proves the right-hand side of (3.4.5).

For the left-hand side inclusion of (3.4.5) let again $x \in \partial C$ and let x^δ be defined as above. Then there is a $x' \in \partial C$ such that $x^\delta = x' + \Delta(x', \delta) N(x')$, i.e. x' is the point of C with minimal distance to x^δ which is unique since C is strictly convex. Note that $\langle x - x', N(x') \rangle \leq 0$, i.e. $\langle x, N(x') \rangle \leq \langle x', N(x') \rangle$ and therefore,

$$\frac{\|x^\delta\|_2}{\|x\|_2} = \frac{\langle x^\delta, N(x') \rangle}{\langle x, N(x') \rangle} \geq \frac{\langle x' + \Delta(x', \delta) N(x'), N(x') \rangle}{\langle x', N(x') \rangle} = 1 + \frac{\Delta(x', \delta)}{\langle x', N(x') \rangle} \quad .$$

Since $\Delta(x', \delta) \leq \Delta_0$ Lemma 3.4.11 yields

$$\frac{\Delta(x', \delta)}{\langle x', N(x') \rangle} \geq H(x') \delta^{\frac{2}{d+1}} (1 + \varepsilon)^{-\frac{2}{d+1}} \geq H_{\min} \delta^{\frac{2}{d+1}} (1 + \varepsilon)^{-\frac{2}{d+1}}$$

and this establishes the inclusion on the left-hand side of (3.4.5). \square

Note that if C is a centrally symmetric ovaloid then C° is also a centrally symmetric ovaloid (see [H]). An immediate corollary of this proposition is the following.

Corollary 3.4.14 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid and let H_{\min}, H_{\max} be defined with respect to C° . Then there is a non-negative function Ψ with $\lim_{\delta \rightarrow 0} \Psi(\delta) = 0$ such that*

$$\left(1 - (1 + \Psi(\delta))H_{\max}\delta^{\frac{2}{d+1}}\right) C \subseteq \langle C \rangle^\delta \subseteq \left(1 - (1 - \Psi(\delta))H_{\min}\delta^{\frac{2}{d+1}}\right) C$$

We are able to prove the last step in order to establish $\lim_{\delta \rightarrow 0} \frac{d_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} = G_{\max} - G_{\min}$.

Proposition 3.4.15 *Let $C \subseteq \mathbb{R}^d$ be a centrally symmetric ovaloid. Then,*

$$\limsup_{\delta \rightarrow 0} \frac{d_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} \leq G_{\max} - G_{\min} \quad .$$

Proof. Let G be defined with respect to C , i.e.

$$G(x) = \frac{(d+1)^{\frac{2}{d+1}}}{2} \left(\frac{|C|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \frac{\kappa_C(x)^{\frac{1}{d+1}}}{\langle x, N_C(x) \rangle} \quad .$$

For $x \in \partial C$ and let H be defined with respect to C° , i.e.

$$H(y) = \frac{[d(d+1)]^{\frac{2}{d+1}}}{2} \left(\frac{|C^\circ|_d}{|B_2^{d-1}|_{d-1}} \right)^{\frac{2}{d+1}} \frac{\kappa_{C^\circ}(y)^{\frac{1}{d+1}}}{\langle y, N_{C^\circ}(y) \rangle}$$

for $y \in \partial C^\circ$. Note that by a result of D. Hug (see [H]) for every $x \in \partial C$ there is a unique $y \in \partial C^\circ$ with $\langle x, y \rangle = 1$ and in this case we have

$$\frac{\kappa_C(x)^{\frac{1}{d+1}}}{\langle x, N_C(x) \rangle} = \left[\frac{\kappa_{C^\circ}(y)^{\frac{1}{d+1}}}{\langle y, N_{C^\circ}(y) \rangle} \right]^{-1} \quad .$$

It follows that there is a constant $\Gamma(C, d)$ only depending on the dimension d and the convex body C such that $G(x) = \Gamma(C, d)H(y)^{-1}$ for such dual pairs (x, y) with $\langle x, y \rangle = 1$. In particular, $G_{\max} = \Gamma(C, d)H_{\min}^{-1}$ and $G_{\min} = \Gamma(C, d)H_{\max}^{-1}$. For $\delta > 0$ put $\delta' > 0$ such that

$$\delta'^{\frac{2}{d+1}} = \Gamma(C, d)G_{\max}G_{\min}\delta^{\frac{2}{d+1}} \quad .$$

Let $a_\delta \geq 1$ be defined by

$$a_\delta = \max \left[\frac{1 - (1 - \Phi(\delta))G_{\min}\delta^{\frac{2}{d+1}}}{1 - (1 + \Psi(\delta'))H_{\max}\delta'^{\frac{2}{d+1}}}, \frac{1 - (1 - \Psi(\delta'))H_{\min}\delta'^{\frac{2}{d+1}}}{1 - (1 + \Phi(\delta))G_{\max}\delta^{\frac{2}{d+1}}} \right]$$

where Φ is the function of Proposition 3.4.12 and Ψ is the function of Corollary 3.4.14. We obtain by Proposition 3.4.12 and Corollary 3.4.14 that $\frac{1}{a_\delta}C_\delta \subseteq \langle C \rangle^{\delta'} \subseteq a_\delta C$. Furthermore, $H_{\max}\delta'^{\frac{2}{d+1}} = G_{\max}\delta^{\frac{2}{d+1}}$ and

$H_{\min}\delta^{\frac{2}{d+1}} = G_{\min}\delta^{\frac{2}{d+1}}$. Hence, $\lim_{\delta \rightarrow 0} \frac{a_\delta - 1}{\delta^{\frac{2}{d+1}}} = G_{\max} - G_{\min}$ and we conclude that

$$\limsup_{\delta \rightarrow 0} \frac{d_C(\delta) - 1}{\delta^{\frac{2}{d+1}}} \leq \lim_{\delta \rightarrow 0} \frac{a_\delta - 1}{\delta^{\frac{2}{d+1}}} = G_{\max} - G_{\min} \quad .$$

□

Appendix A

Appendix for Chapter 3

We prove some results of differential geometric nature which are important for sections 3.3 and 3.4.

Lemma A.0.1 *Let $c > 0$ and let $g : cB_2^{d-1} \rightarrow \mathbb{R}$ be a convex function, C^2 in its interior, such that $g(0) = 0$, $\nabla g(0) = 0$. Then, there is a constant $0 < c_1 < c$ such that for $c_2 = c_1/2$ and $c_3 = c_1/4$ the following assertions hold:*

1. $\|\nabla g(x)\|_2 \leq \frac{1}{4}$ for every $x \in c_1 B_2^{d-1}$
2. For every $x \in \text{int}(c_2 B_2^{d-1})$ the map

$$y_x : \text{int}(c_1 B_2^{d-1}) \rightarrow \text{Im}(y_x) \subseteq v^\perp$$

$$y_x(z) = P_{v^\perp}((z, g(z))^t - (x, g(x))^t)$$

is a C^2 -diffeomorphism with $c_3 B_2^d \cap v^\perp \subseteq \text{Im}(y_x)$ where

$$v = \frac{1}{(1 + \|\nabla g(x)\|_2^2)^{1/2}} \begin{pmatrix} -\nabla g(x) \\ 1 \end{pmatrix}$$

and P_{v^\perp} is the orthogonal projection onto v^\perp .

Hence, for every $x \in \text{int}(c_2 B_2^{d-1})$ the function

$$g^x : c_3 B_2^d \cap v^\perp \rightarrow \mathbb{R}$$

$$g^x(\zeta) = \left\langle \begin{pmatrix} y_x^{-1}(\zeta) \\ g(y_x^{-1}(\zeta)) \end{pmatrix}, v \right\rangle - \left\langle \begin{pmatrix} x \\ g(x) \end{pmatrix}, v \right\rangle$$

is a well-defined reparametrization of g at x which is convex, C^2 in its interior, such that 0 corresponds to x , $g^x(0) = 0$ and $\nabla g^x(0) = 0$.

Proof. Put c_1 such that for every $x \in c_1 B_2^{d-1}$ we have the bound $\|\nabla g(x)\| \leq \frac{1}{4}$. We want to apply the inverse mapping theorem. First, we show that y_x is injective: Let $z_1, z_2 \in c_1 B_2^{d-1}$ and $v = \begin{pmatrix} \bar{v} \\ v_d \end{pmatrix}$. It follows:

$$\begin{aligned}
\|y_x(z_1) - y_x(z_2)\|_2 &= \left\| \begin{pmatrix} z_1 - z_2 \\ g(z_1) - g(z_2) \end{pmatrix} - \left\langle \begin{pmatrix} z_1 - z_2 \\ g(z_1) - g(z_2) \end{pmatrix}, v \right\rangle v \right\|_2 \\
&\geq \left\| \begin{pmatrix} z_1 - z_2 \\ g(z_1) - g(z_2) \end{pmatrix} \right\|_2 - \left| \left\langle \begin{pmatrix} z_1 - z_2 \\ g(z_1) - g(z_2) \end{pmatrix}, v \right\rangle \right| \\
&\geq \|z_1 - z_2\|_2 - \|\bar{v}\|_2 \|z_1 - z_2\|_2 - |v_d| \|g(z_1) - g(z_2)\|_2 \\
&\geq \|z_1 - z_2\|_2 - \frac{1}{4} \|z_1 - z_2\|_2 - 1 \cdot \sup_{\|\zeta\| \leq c_1} \|\nabla g(\zeta)\|_2 \|z_1 - z_2\|_2 \geq \frac{1}{2} \|z_1 - z_2\|_2
\end{aligned} \tag{A.0.1}$$

and this proves injectivity. We need additionally, that $(Dy_x)(z) : \mathbb{R}^{d-1} \rightarrow v^\perp$ is invertible for every $z \in \text{int}(c_1 B_2^{d-1})$. Note that $(Dy_x)(z) = P_{v^\perp} \begin{pmatrix} I_{d-1} \\ \nabla g(z)^t \end{pmatrix}$ and hence, we deduce for every $\xi \in \mathbb{S}^{d-2}$:

$$\begin{aligned}
\|(Dy_x)(z)\xi\|_2 &= \left\| \begin{pmatrix} \xi \\ \langle \nabla g(z), \xi \rangle \end{pmatrix} - (\langle \xi, \bar{v} \rangle + v_d \langle \nabla g(z), \xi \rangle) v \right\|_2 \\
&\geq \|\xi\|_2 - \|\bar{v}\|_2 |\langle \xi, \bar{v} \rangle| + |v_d| |\langle \nabla g(z), \xi \rangle| = 1 - \frac{1}{4} \left(\frac{1}{4} + \frac{1}{4} \right) = \frac{7}{8}
\end{aligned}$$

By the inverse mapping theorem y_x is a bijective map onto its image and the inverse map is C^2 again. For every $x \in \text{int}(c_2 B_2^{d-1})$ the inclusion $x + c_2 B_2^{d-1} \subseteq c_1 B_2^{d-1}$ is true and since y_x is a homeomorphism and $\|y_x(z)\|_2 = \|y_x(z) - y_x(x)\|_2 \geq \frac{1}{2} \|z - x\|_2$ a topological argument shows that $c_3 B_2^{d-1} \subseteq \text{Im}(y_x)$. \square

Lemma A.0.2 *Let $c, c_1, c_2, c_3 > 0$, y_x, g, g^x and v be as in the previous lemma. Then for every $\eta > 0$ there is some $0 < \vartheta < c_3$ such that $\|\zeta\|_2 \leq \vartheta$ and $x \in \text{int}(c_2 B_2^{d-1})$ implies $\|Hg^x(\zeta) - Hg^x(0)\|_{O_p} \leq \eta$.*

Proof. We need the following derivatives:

$$\begin{aligned}
(Dy_x)(z) &= P_{v^\perp} \begin{pmatrix} I_{d-1} \\ \nabla g(z)^t \end{pmatrix} \\
D(g^x \circ y_x)(z) &= v^t (I_{d-1}, \nabla g(z)) \\
(D^2 y_x)(z)(\cdot, \cdot) &= \langle Hg(z) \cdot, \cdot \rangle P_{v^\perp}(e_d) \\
H(g^x \circ y_x)(z) &= v_d^2 Hg(z)
\end{aligned}$$

$$\begin{aligned}
(Hg^x)(y_x(z)) &= (Dy_x)^{\prime-1}(z) \left[H(g^x \circ y_x)(z) \right. \\
&\quad \left. - D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}(D^2y_x)(z) \right] (Dy_x)^{-1}(z) \\
&= (Dy_x)^{\prime-1}(z) \left[H(g^x \circ y_x)(z) \right. \\
&\quad \left. - (D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}P_{v^\perp}e_d) Hg(z) \right] (Dy_x)^{-1}(z)
\end{aligned}$$

where A' denotes the (real) adjoint of the matrix A .

We compute Hg^x . We start with $D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}(D^2y_x)(z)$:

$$\begin{aligned}
&D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}(D^2y_x)(z) \\
&= (D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}P_{v^\perp}e_d) Hg(z) \\
&= \left\langle \left(\begin{array}{c} I_{d-1} \\ \nabla g(z)^t \end{array} \right) (Dy_x)(z)^{-1}P_{v^\perp}e_d, v \right\rangle Hg(z)
\end{aligned}$$

Let $w \in \mathbb{R}^{d-1}$ be the unique element such that

$$P_{v^\perp}e_d = (Dy_x)(z)w = P_{v^\perp} \left(\begin{array}{c} w \\ \langle \nabla g(z), w \rangle \end{array} \right)$$

If some $\xi \in \mathbb{R}^d$ satisfies $P_{v^\perp}\xi = P_{v^\perp}e_d$ it follows that there is a $\lambda \in \mathbb{R}$ with

$$\xi = e_d - \lambda v \quad .$$

Hence, solving the equation

$$e_d - \lambda v = \left(\begin{array}{c} w \\ \langle \nabla g(z), w \rangle \end{array} \right)$$

leads to $w = -\lambda \bar{v}$ and therefore, $1 - \lambda v_d = \langle \nabla g(z), w \rangle = -\lambda \langle \nabla g(z), \bar{v} \rangle$, i.e., $\lambda = \frac{1}{v_d - \langle \nabla g(z), \bar{v} \rangle}$. We note that the denominator is never zero since $\|\nabla g(z)\| \leq \frac{1}{4}$. We compute

$$\left\langle \left(\begin{array}{c} I_{d-1} \\ \nabla g(z)^t \end{array} \right) w, v \right\rangle = -\lambda \|\bar{v}\|_2^2 - \lambda v_d \langle \nabla g(z), \bar{v} \rangle = -\frac{v_d \langle \nabla g(z), \bar{v} \rangle + \|\bar{v}\|_2^2}{v_d - \langle \nabla g(z), \bar{v} \rangle} \quad .$$

We show that $z \mapsto D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}(D^2y_x)(z)$ is uniformly continuous and bounded independently of x where we mean by 'uniformly continuous independently of x ' that the corresponding family of functions indexed by x is equicontinuous. Put $\alpha_x(z) = v_d \langle \nabla g(z), \bar{v} \rangle + \|\bar{v}\|_2^2$, $\beta_x(z) = v_d - \langle \nabla g(z), \bar{v} \rangle$ and $\omega(z) = Hg(z)$. Since Hg is a continuous matrix valued function on a compactum there is an $\Omega > 0$ such that $\|\omega(z)\|_{Op} \leq \Omega$ for every $z \in c_1 B_2^{d-1}$. Note that $|\alpha_x(z)| \leq 1 + \|\nabla g(z)\|_2 \leq \frac{5}{4}$ and

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} \leq |v_d| - |\langle \nabla g(z), \bar{v} \rangle| \leq |\beta_x(z)| \quad .$$

If $z_1, z_2 \in c_1 B_2^{d-1}$ we have

$$|\alpha_x(z_1) - \alpha_x(z_2)| = |v_d| |\langle \nabla g(z_1) - \nabla g(z_2), \bar{v} \rangle| \leq \|\nabla g(z_1) - \nabla g(z_2)\|_2$$

and

$$|\beta_x(z_1) - \beta_x(z_2)| \leq \|\nabla g(z_1) - \nabla g(z_2)\|_2 \quad .$$

Since firstly, $\alpha_x(z)$ is uniformly continuous independently of x and $|\alpha_x(z)| \leq \frac{5}{4}$, secondly, $\beta_x(z)$ is uniformly continuous and $|\beta_x(z)| \geq \frac{1}{4}$ and thirdly, $\|\omega(z)\|_{Op} \leq \Omega$ and uniformly continuous, it follows that

$$z \mapsto D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}(D^2 y_x)(z) = \frac{\alpha_x(z)}{\beta_x(z)} \omega(z)$$

is bounded and uniformly continuous independently of x .

The function $H(g^x \circ y_x) = v_d^2 Hg$ is also bounded and uniformly continuous independently of x and hence,

$$H(g^x \circ y_x)(z) - D(g^x \circ y_x)(z)(Dy_x)(z)^{-1}(D^2 y_x)(z)$$

is bounded and uniformly continuous independently of x . We show that $(Dy_x)^{-1}$ is uniformly continuous and bounded on $c_1 B_2^{d-1}$ independently of x . Therefore, the same is true for the adjoint. It was shown in the proof of the last lemma that $\|Dy_x(z)^{-1}\|_{Op} \leq \frac{7}{8}$. It follows for $z_1, z_2 \in c_1 B_2^{d-1}$ that

$$\begin{aligned} & \|(Dy_x)^{-1}(z_1) - (Dy_x)^{-1}(z_2)\|_{Op} \\ &= \|(Dy_x)^{-1}(z_1)((Dy_x)(z_1) - (Dy_x)(z_2))(Dy_x)^{-1}(z_2)\|_{Op} \\ &\leq \|(Dy_x)^{-1}(z_1)\|_{Op} \|(Dy_x)^{-1}(z_2)\|_{Op} \|(Dy_x)(z_1) - (Dy_x)(z_2)\|_{Op} \\ &\leq \left(\frac{8}{7}\right)^2 \|(Dy_x)(z_1) - (Dy_x)(z_2)\|_{Op} \leq \left(\frac{8}{7}\right)^2 \|\nabla g(z_1) - \nabla g(z_2)\|_2 \end{aligned}$$

An elementary argument shows that $H(g^x)(y_x(z))$ is uniformly continuous independently of x as a composition of operator valued functions which are uniformly continuous and bounded independently of x . It was shown in the proof of the last lemma that $2\|y_x(z_1) - y_x(z_2)\|_2 \geq \|z_1 - z_2\|_2$ and hence, Hg_x is uniformly continuous independently of x . □

Corollary A.0.3 *Let $C \subseteq \mathbb{R}^d$ be a convex body with C^2 boundary. Then there is some $\gamma > 0$ such that for every $\xi \in \partial C$ there is a parametrization $f_\xi : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ of C at ξ such that for every $\eta > 0$ there is a $\gamma > \theta > 0$ independent of ξ such that $\|z\|_2 \leq \theta$ implies $\|Hf_\xi(z) - Hf_\xi(0)\|_{Op} \leq \theta$.*

Proof. For every $\xi \in \partial C$ let $\tilde{f}_\xi : c(\xi)B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ be a parametrization of C at ξ and let $c_1(\xi), c_2(\xi), c_3(\xi)$ be as in Lemma A.0.1. Since the boundary of C is compact there are $\xi_1, \dots, \xi_n \in \partial C$ such that every point of the boundary

is parametrized by some $\tilde{f}_{x_i}|_{c_2(\xi)B_2^{d-1}}$. Put $\gamma = \min_{i=1}^n c_3(\xi_i)$. Let $\xi_0 \in \partial C$ then there is an $1 \leq i \leq n$ and a $x_0 \in c_3(\xi_i)B_2^{d-1}$ such that ξ_0 corresponds to $(x_0, \tilde{f}_{\xi_i}(x_0))$. Put $g = \tilde{f}_{\xi_i}$ and $f_{\xi_0} : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$, $f_{\xi_0}(z) = g^{x_0}(z)$. Then f_{ξ_0} is a parametrization of the boundary of C at ξ_0 with the desired properties. For every $\eta > 0$ there are $0 < \vartheta(\xi) < c_3(\xi)$ as in Lemma A.0.2. Then $0 < \theta = \min_{i=1}^n \vartheta(\xi_i) < \gamma$ has the desired property that $\|Hf_{\xi_0}(z) - Hf_{\xi_0}(0)\|_{Op} \leq \eta$ if $\|z\|_2 \leq \theta$. □

For an ovaloid $C \subseteq \mathbb{R}^d$ let $0 < \kappa_{\min} \leq \kappa_{\max} < \infty$ be the smallest and largest principal curvature which occur on C . It follows that for a parametrization f_ξ we have $\kappa_{\min}I_{d-1} \leq Hf_\xi(0) \leq \kappa_{\max}I_{d-1}$ where ' $M_1 \leq M_2$ ' means that $M_2 - M_1$ is positive semidefinite. An immediate consequence of Corollary A.0.3 is the following.

Corollary A.0.4 *Let $C \subseteq \mathbb{R}^d$ be an ovaloid. Then there is some $\gamma > 0$ such that for every $\xi \in \partial C$ there is a parametrization $f_\xi : \gamma B_2^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ of C at ξ such that for every $1 > \eta > 0$ there is a $\gamma > \theta > 0$ independent of ξ such that $\|z\|_2 \leq \theta$ implies $(1 - \eta)Hf_\xi(0) \leq Hf_\xi(z) \leq (1 + \eta)Hf_\xi(0)$.*

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung meines Betreuers und den Beiträgen meiner Koautorin im dritten Kapitel meiner Arbeit, inhaltlich und der Form nach unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe. Die Anteile meiner Koautorin sind an entsprechender Stelle kenntlich gemacht. Die wesentlichen Teile zu affin invarianten Punkten sind in [M] enthalten. Die Arbeit oder Auszüge derselben wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt.

Kiel, den 12.12.2016

(Olaf Mordhorst)