

Representations of regular trees and wild subcategories for generalized Kronecker quivers

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von
Daniel Bissinger

Kiel, 2017

1. Gutachter: Prof. Dr. Rolf Farnsteiner

2. Gutachter: Priv.-Doz. Dr. Amir Džambić

Tag der mündlichen Prüfung: 19.10.2017

Zum Druck genehmigt: 19.10.2017

gez. Prof. Dr. Natascha Oppelt, **Dekanin**

Zusammenfassung

Seien $r \geq 2$, k ein algebraisch abgeschlossener Körper der Charakteristik $p > 0$ und E_r eine p -elementar abelsche Gruppe von Rang r . Die Kategorie der endlich-dimensionalen kE_r -Moduln $\text{mod } kE_r$ ist wild genau dann, wenn $p \geq 3$ oder $p = 2$ und $r > 2$ gilt. Deshalb werden in diesen Fällen Unterklassen von Moduln mit restriktiveren Eigenschaften studiert.

Im Jahr 2008 definierten Carlson, Friedlander und Suslin die Klasse von Moduln von konstantem Rang und noch restriktiver Moduln mit gleichen Bildern bzw. gleichen Kernen.

Mithilfe eines treuen und exakten Funktors $\mathfrak{F}_n: \text{mod } B(n, r) \rightarrow \text{mod } kE_r$ von der Modulkategorie der verallgemeinerten Beilinson-Algebra $B(n, r)$ auf n Knoten und r Pfeilen in die Modulkategorie der Gruppenalgebra kE_r definierte und studierte Worch die Moduln mit gleichen Bildern und Moduln mit gleichen Kernen in $\text{mod } B(n, r)$.

Wir vertiefen diesen Zugang für den Fall $n = 2$ und $r \geq 3$. Dann ist $B(n, r)$ isomorph zur verallgemeinerten Kronecker-Algebra $\mathcal{K}_r = k\Gamma_r$.

Wir verwenden dabei bekannte Methoden für die erbliche Kategorie $\text{mod } \mathcal{K}_r$, die in $\text{mod } kE_r$ nicht zu Verfügung stehen, um neue Einsichten in die Kategorie $\text{mod } kE_r$ zu gewinnen.

Als Analogon zu Moduln, die vor kurzem von Carlson, Friedlander und Pevtsova eingeführt wurden, definieren wir Moduln von konstantem d -Radikal Rank und Moduln von konstantem d -Sockel Rank in $\text{mod } \mathcal{K}_r$ für $1 \leq d \leq r - 1$. Unter Verwendung der Ringelschen Vereinfachungsmethode weisen wir die Wildheit der vollen Unterkategorien nach, die durch Moduln mit gleichen d -Radikalen und Moduln mit gleichen d -Sockeln gegeben sind. Danach ziehen wir mithilfe des Funktors $\mathfrak{F}_2: \text{mod } \mathcal{K}_r \rightarrow \text{mod } kE_r$ Schlüsse für $\text{mod } kE_r$.

Anschließend nutzen wir die entwickelten Methoden, um die Kategorie $\text{mod } \mathcal{K}_r$ zu studieren. Dafür greifen wir auf den Auslander-Reiten-Köcher $\Gamma_{\mathcal{K}_r}$ von \mathcal{K}_r zurück und setzen zwei verschiedene Invarianten regulärer Komponenten in Beziehung.

Für eine reguläre Komponente \mathcal{C} des Auslander-Reiten-Köchers bezeichnen wir mit $\text{rk}(\mathcal{C}) \in \mathbb{Z}_{\leq 1}$ den Quasi-Rang von \mathcal{C} und zeigen, dass dieser in enger Beziehung zu $\mathcal{W}(\mathcal{C}) \in \mathbb{N}_0$ steht. Dabei gibt $\mathcal{W}(\mathcal{C})$ die Distanz zwischen den beiden disjunkten Kegeln in \mathcal{C} an, die durch Moduln mit gleichen Bildern und gleichen Kernen gegeben sind. Präziser ausgedrückt zeigen wir

$$-\mathcal{W}(\mathcal{C}) \leq \text{rk}(\mathcal{C}) \leq -\mathcal{W}(\mathcal{C}) + 3.$$

Unter Verwendung von Überlagerungstheorie konstruieren wir für jedes $n \in \mathbb{N}_0$ eine bijektive Abbildung φ_n zwischen dem Körper k und der Menge $\{\mathcal{C} \mid \mathcal{C} \text{ reguläre Komponente, } \mathcal{W}(\mathcal{C}) = n\}$. Daraus ergeben sich insbesondere neue Aussagen über die Anzahl von regulären Komponenten von festem Quasi-Rang.

Weiter studieren wir die innere Struktur von regulären Komponenten mithilfe des Gabriel-Roiter-Maßes μ und untersuchen das Limes-Verhalten von τ -Bahnen bezüglich μ . Anschließend verallgemeinern wir die erzielten Ergebnisse auf wilde Köcher-Algebren mit einer "ähnlichen" Gabriel-Roiter-Theorie.

Abstract

Let $r \geq 2$, k be an algebraically closed field of characteristic $p > 0$ and E_r be a p -elementary abelian group of rank r . It is well known that the category of finite dimensional kE_r -modules $\text{mod } kE_r$ is of wild type, whenever $p \geq 3$ or $p = 2$ and $r > 2$. Therefore subclasses with more restrictive properties have been studied.

In 2008 Carlson, Friedlander and Suslin introduced the subclass of modules of constant rank and modules with even more restrictive properties, called equal images property and equal kernels property.

In her PhD thesis Worch uses a faithful exact functor $\mathfrak{F}_n: \text{mod } B(n, r) \rightarrow \text{mod } kE_r$ from the module category of the generalized Beilinson algebra $B(n, r)$ on n vertices into the module category of the group algebra kE_r to define the equal images and equal kernels property in the framework of the Beilinson algebra. We follow this approach in the special case $n = 2$ and $r \geq 3$, i.e. $B(n, r)$ is the generalized Kronecker $\mathcal{K}_r = k\Gamma_r$ algebra with $r \geq 3$ arrows.

Inspired by recent work of Carlson, Friedlander and Pevtsova concerning modules E_r , we introduce the notions of modules with constant d -radical rank and modules with constant d -socle rank for \mathcal{K}_r and $1 \leq d \leq r - 1$. We study subcategories given by modules with the equal d -radical property and the equal d -socle property. Utilizing the Simplification method due to Ringel, we prove that these subcategories in $\text{mod } \mathcal{K}_r$ are of wild type. Then we use the functor $\mathfrak{F}_2: \text{mod } \mathcal{K}_r \rightarrow \text{mod } kE_r$ to transfer our results to $\text{mod } kE_r$.

This thesis is also motivated by the aim to give new insights into the wild category $\text{mod } \mathcal{K}_r$. Therefore we consider the Auslander-Reiten quiver $\Gamma_{\mathcal{K}_r}$ of \mathcal{K}_r and show how two different invariants of regular components relate to each other.

Given a regular component \mathcal{C} of the Auslander-Reiten quiver of \mathcal{K}_r , we prove that the quasi-rank $\text{rk}(\mathcal{C}) \in \mathbb{Z}_{\leq 1}$ can be described almost exactly as the distance $\mathcal{W}(\mathcal{C}) \in \mathbb{N}_0$ between the two non-intersecting cones in \mathcal{C} , given by modules with the equal images and the equal kernels property; more precisely, we show that the two numbers are linked by the inequality

$$-\mathcal{W}(\mathcal{C}) \leq \text{rk}(\mathcal{C}) \leq -\mathcal{W}(\mathcal{C}) + 3.$$

Utilizing covering theory, we construct for each $n \in \mathbb{N}_0$ a bijection φ_n between the field k and $\{\mathcal{C} \mid \mathcal{C} \text{ regular component, } \mathcal{W}(\mathcal{C}) = n\}$. As a consequence, we get new results about the number of regular components of a fixed quasi-rank.

Furthermore, we use the theory of Gabriel-Roiter measures to study the inner structure of regular components by considering the limiting behavior in τ -orbits. We use the Kronecker algebra as a prototypical example and extend our findings to other families of wild hereditary algebras with a "similar" Gabriel-Roiter theory.

Danksagung

Mein herzlicher Dank gilt Rolf Farnsteiner für die gute Betreuung. Die richtungsweisenden Diskussionen, die Freiheit bei der Themenwahl und die Möglichkeit meinen eigenen Forschungsfragen nachzugehen haben diese Arbeit erst ermöglicht.

Besonders danken möchte ich Claus Michael Ringel für seine Gesprächsbereitschaft während meiner Besuche in Bielefeld und für seine anschaulichen und kompakten Erläuterungen zur Überlagerungstheorie, von denen ich sehr profitiert habe.

Außerdem danke ich Otto Kerner und Bo Chen für die Beantwortung meiner Fragen zu erblichen Algebren und Gabriel-Roiter-Maßen.

Allen ehemaligen und aktuellen Mitgliedern der Arbeitsgruppe Darstellungstheorie möchte ich für die angenehme Arbeitsatmosphäre danken.

Ich danke Patrick Comdühr und Christian Drenkhahn für das Korrekturlesen von Teilen dieser Arbeit.

Meiner Familie danke ich für die fortwährende Unterstützung hinweg durch mein gesamtes Studium.

Hanne, danke für die geduldige Unterstützung und deinen großen Rückhalt.

Contents

Zusammenfassung	v
Abstract	vii
Danksagung	ix
1 Introduction	1
2 Preliminaries	5
2.1 Notation	5
2.2 Quiver, quiver algebras and representations of quivers	5
2.3 Representation type	7
2.4 Auslander-Reiten theory	12
2.5 Tilting theory for quiver algebras	15
2.6 The Kronecker algebra	16
3 Modules of constant radical and socle rank	19
3.1 Elementary modules of small dimension	19
3.2 An algebraic family of test-modules	21
3.3 Modules for the generalized Kronecker algebra	23
3.4 Modules with the equal c -images and equal c -kernels property	27
4 Process of Simplification and Applications	30
4.1 Representation type	30
4.2 Passage between \mathcal{K}_r and \mathcal{K}_s	31
4.3 Numerous components lying in CSR_d	33
4.4 Components lying almost completely in CSR_d	34
5 Wild representation type	36
5.1 Wildness of strata	36
5.2 $\text{ESP}_2 \cap \text{ERP}_2$ is wild	38
5.3 The module category of E_r	40
6 The connection between $\text{rk}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C})$	42
7 Covering Theory	44
7.1 General Theory	44
7.2 Duality	46

8	The categories EKP and EIP for $\text{rep}(C_r)$	47
9	Considerations in the universal covering	52
9.1	Indecomposable representations arising from extensions	53
9.2	Small representations and trees	55
9.3	The Main Theorem	56
10	Applications and further considerations	58
10.1	Regular components for every width	58
10.2	Counting regular components of fixed width	61
10.2.1	\mathcal{S}_r -stability	62
10.2.2	Automorphisms of trees	62
10.2.3	The number of regular components in $\text{rep}(\Gamma_r)$	64
10.3	Thin representations	65
10.4	Representations with different quasi-lengths	67
10.5	The $*$ -construction revisited	68
11	Gabriel-Roiter Theory for quivers	73
11.1	A connection to modules with the equal d -socle property	75
12	Limiting behavior	78
12.1	Existence of wild quivers with $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$	80
12.2	Morphisms between regular modules	83
12.3	Limiting behavior for wild algebras with $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$	84
13	Central quivers	87
13.1	Wild quivers with $l \geq 3$ vertices	89
13.2	A counter example	91
14	Elementary Representations for Γ_r	93
14.1	Consequences of Westwick's Theorem	93
14.2	A fundamental domain of dimension vectors	95
14.3	Restricting x and y	97
14.4	Elementary representations for Γ_3	98
14.5	Elementary representations for Γ_4	98
	Bibliography	101
	Erklärung	104

1 Introduction

Let $r \geq 2$, k be an algebraically closed field of characteristic $p > 0$ and E_r be a p -elementary abelian group of rank r . It is well known that the category of finite dimensional kE_r -modules $\text{mod } kE_r$ is of wild type whenever $p \geq 3$ or $p = 2$ and $r > 2$. Therefore subclasses with more restrictive properties have been studied; in [10], the subclass of modules of constant rank $\text{CR}(E_r)$ and modules with even more restrictive properties, called equal images property and equal kernels property, were introduced.

In [48] the author defined analogous categories CR, EIP and EKP in the context of the generalized Kronecker algebra \mathcal{K}_r , and in more generality for the generalized Beilinson algebra $B(n, r)$. Using a natural functor $\mathfrak{F}: \text{mod } \mathcal{K}_r \rightarrow \text{mod } kE_r$ with nice properties, she gave new insights into the categories of equal images and equal kernels modules for $\text{mod } kE_r$ of Loewy length ≤ 2 . A crucial step is the description of CR, EIP and EKP in homological terms, involving a \mathbb{P}^{r-1} -family of regular "test"-modules.

Building on this approach, we show that the recently introduced ([9]) modules of constant socle rank and constant radical rank can be described in the same fashion. For $1 \leq d < r$, we introduce modules of constant d -radical rank CRR_d and constant d -socle rank CSR_d in $\text{mod } \mathcal{K}_r$. More restrictive - and also easier to handle - are modules with the equal d -radical property ERP_d and the equal d -socle property ESP_d . For $d = 1$, we have $\text{ESP}_1 = \text{EKP}$, $\text{ERP}_1 = \text{EIP}$ and $\text{CSR}_1 = \text{CR} = \text{CRR}_1$. Studying these classes in the hereditary module category $\text{mod } \mathcal{K}_r$ allows us to use tools not available in $\text{mod } kE_r$.

As a first step, we establish a homological characterization of CSR_d , CRR_d , ESP_d and ERP_d . We denote by $\text{Gr}_{d,r}$ the Grassmannian of d -dimensional subspaces of k^r . In generalization of [48], we define a $\text{Gr}_{d,r}$ -family of "test"-modules $(X_U)_{U \in \text{Gr}_{d,r}}$ and show that this family can be described in purely combinatorial terms by being indecomposable of dimension vector $(1, r-d)$. This allows us to construct many examples of modules of equal socle rank in $\text{mod } \mathcal{K}_s$ for $s \geq 3$ by considering pullbacks along natural embeddings $\mathcal{K}_r \rightarrow \mathcal{K}_s$.

Since \mathcal{K}_r is a wild algebra for $r > 2$ and every regular component in the Auslander-Reiten quiver of \mathcal{K}_r is of type $\mathbb{Z}A_\infty$, it is desirable to find invariants that give more specific information about the regular components. It is shown in [48] that there are uniquely determined quasi-simple modules $M_{\mathcal{C}}$ and $W_{\mathcal{C}}$ in \mathcal{C} such that the cone $(M_{\mathcal{C}} \rightarrow)$ of all successors of $M_{\mathcal{C}}$ satisfies $(M_{\mathcal{C}} \rightarrow) = \text{EKP} \cap \mathcal{C}$ and the cone $(\rightarrow W_{\mathcal{C}})$ of all predecessors of $W_{\mathcal{C}}$ satisfies $(\rightarrow W_{\mathcal{C}}) = \text{EIP} \cap \mathcal{C}$. Using results on elementary modules, we generalize this statement for ESP_d and ERP_d by proving:

Theorem A. *Let $r \geq 3$ and \mathcal{C} be a regular component of the Auslander-Reiten quiver of \mathcal{K}_r .*

- (a) *For each $1 \leq i < r$ there exists a unique quasi-simple module M_i in \mathcal{C} such that $\text{ESP}_i \cap \mathcal{C} = (M_i \rightarrow)$.*
- (b) *For each $1 \leq i < r$ the category $\Delta_i := \text{ESP}_i \setminus \text{ESP}_{i-1}$ is wild, where $\text{ESP}_0 := \emptyset$.*
- (c) *There exists at most one number $1 < m(\mathcal{C}) < r$ such that $\Delta_{m(\mathcal{C})} \cap \mathcal{C}$ is non-empty. If such a number exists, $\Delta_{m(\mathcal{C})} \cap \mathcal{C}$ is the ray starting in $M_{m(\mathcal{C})}$.*

If there is no such number as in (c) we set $m(\mathcal{C}) = 1$. An immediate consequence of (a) and (c) is that for $1 \leq i \leq j < r$ we have $(M_i \rightarrow) = (M_j \rightarrow)$ or $\tau M_i = M_j$, where τ denotes the Auslander-Reiten translation of \mathcal{K}_r . Moreover, statement (b) shows the existence of a lot of AR-components such that $\Delta_i \cap \mathcal{C}$ is a ray and for every such component we have $m(\mathcal{C}) = i$. With the dual result for ERP, we assign an invariant $1 \leq w(\mathcal{C}) < r$ to each regular component \mathcal{C} giving us the possibility to distinguish $(r - 1)^2$ different types of regular components.

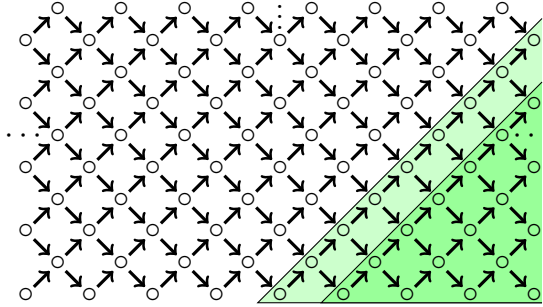


Figure 1.1: Illustration of a regular component with $m(\mathcal{C}) \neq 1$.

To prove statement (b) we exploit the fact that every regular module M in $\text{mod } \mathcal{K}_r$ has self-extensions with $\dim_k \text{Ext}_{\mathcal{K}_r}(M, M) \geq 2$ in conjunction with the Process of Simplification. This method was introduced in [34] and produces extension closed subcategories, whose objects may be filtered by pairwise orthogonal bricks. For p -elementary abelian groups, $\text{mod } kE_r$ is the only subcategory of such type. But we can use the functor $\mathfrak{F}: \text{mod } \mathcal{K}_r \rightarrow \text{mod } kE_r$, whose essential image consists of all modules of Loewy length ≤ 2 , to transfer our results to $\text{mod } kE_r$. We denote with $\text{ESP}_{2,d}(E_r)$ the category of modules in $\text{mod } kE_r$ of Loewy length ≤ 2 with the equal d -socle property and arrive at:

Corollary. *Let $\text{char}(k) > 0$, $r \geq 3$ and $1 \leq d < r$. Then $\text{ESP}_{2,d}(E_r) \setminus \text{ESP}_{2,d-1}(E_r)$ has wild representation type, where $\text{ESP}_{2,0}(E_r) := \emptyset$.*

For $r = 2$, we consider the Beilinson algebra $B(3, 2)$. The fact that $B(3, 2)$ is a concealed algebra of type $Q = 1 \rightarrow 2 \rightrightarrows 3$ allows us to apply the Simplification process in $\text{mod } kQ$. We find a wild subcategory in the category of all modules in $\text{mod } B(3, 2)$ with the equal kernels property and conclude:

Corollary. *Assume that $\text{char}(k) = p > 2$, then the full subcategory of modules with the equal kernels property in $\text{mod } kE_2$ and Loewy length ≤ 3 has wild representation type.*

In particular, we generalize results by Benson [5], and Bondarenko and Lytvynchuk [6] concerning the wildness of various subcategories of kE_r -modules. We also construct examples of regular components \mathcal{C} such that each module in \mathcal{C} has constant d -socle rank, but no module in \mathcal{C} is GL_r -stable in the sense of [9].

We devote the second part of this thesis to the extensive study of two specific invariants of regular components for the generalized Kronecker algebra.

Let \mathcal{C} be a regular component of the Auslander-Reiten quiver of \mathcal{K}_r . The integer $\mathcal{W}(\mathcal{C})$, introduced in [48], is defined as the unique number with $\tau^{\mathcal{W}(\mathcal{C})+1}(M_{\mathcal{C}}) = W_{\mathcal{C}}$. The other invariant,

introduced in [21] (for any wild hereditary algebra), is the quasi-rank $\text{rk}(\mathcal{C}) \in \mathbb{Z}$ of a regular component \mathcal{C} . For a quasi-simple module X in \mathcal{C} , $\text{rk}(\mathcal{C})$ is defined as

$$\text{rk}(\mathcal{C}) := \min\{l \in \mathbb{Z} \mid \forall m \geq l: \text{rad}_{\mathcal{K}_r}(X, \tau^m X) \neq 0\},$$

where $\text{rad}_{\mathcal{K}_r}(X, Y)$ is the space of non-invertible homomorphisms from X to Y .

Utilizing the homological descriptions of EKP and EIP involving the family of elementary modules $(X_U)_{U \in \text{Gr}_{1,r}}$, we show that the two invariants $\text{rk}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C})$ are linked by the inequality

$$-\mathcal{W}(\mathcal{C}) \leq \text{rk}(\mathcal{C}) \leq -\mathcal{W}(\mathcal{C}) + 3.$$

Motivated by this connection, we construct for each $n \in \mathbb{N}$ a regular component \mathcal{C} with $\mathcal{W}(\mathcal{C}) = n$. In order to do so, we consider representations over the universal covering C_r of Γ_r .

We define classes Inj, Surj of representations over C_r such that $M \in \text{rep}(C_r)$ is in Inj (resp. Surj) if and only if for each arrow $\delta: x \rightarrow y$ of C_r the linear map $M(\delta): M_x \rightarrow M_y$ is injective (resp. surjective).

Let $\pi_\lambda: \text{rep}(C_r) \rightarrow \text{rep}(\Gamma_r)$ be the push-down functor [19, 2.7] and $M \in \text{rep}(C_r)$ be indecomposable. We prove that M is in Inj (resp. Surj) if and only if $\pi_\lambda(M)$ is in EKP (resp. EIP). Since a component \mathcal{D} of the Auslander-Reiten quiver of C_r which is taken to a regular component $\mathcal{C} := \pi_\lambda(\mathcal{D})$ is also of type $\mathbb{Z}A_\infty$, we can lift the definition of $\mathcal{W}(\mathcal{C})$ to \mathcal{D} . We define $\mathcal{W}_\mathcal{C}(\mathcal{D}) \in \mathbb{N}_0$ as the distance between the cones $\text{Surj} \cap \mathcal{D}$ and $\text{Inj} \cap \mathcal{D}$ and show that $\mathcal{W}_\mathcal{C}(\mathcal{D}) = \mathcal{W}(\mathcal{C})$.

For $X \in \mathcal{D}$, we denote by $\text{ql}(X)$ its quasi-length. If X has certain properties, we show how to construct a short exact sequence $\delta_X = 0 \rightarrow Y \rightarrow E \rightarrow X \rightarrow 0$ with indecomposable middle term E in a component \mathcal{E} such that

$$(*) \quad \mathcal{W}_\mathcal{C}(\mathcal{E}) = \mathcal{W}_\mathcal{C}(\mathcal{D}) + \text{ql}(X) - 1.$$

The construction of δ_X relies on the fact that C_r is an infinite r -regular tree with bipartite orientation. Using (*), we construct for each $n \in \mathbb{N}$ a component \mathcal{D}_n with $\mathcal{W}_\mathcal{C}(\mathcal{D}_n) = n$.

In conjunction with the natural action of $\text{GL}_r(k)$ on $\text{rep}(\Gamma_r)$, we arrive at:

Theorem B. *Let $n \in \mathbb{N}_0$. There is a bijection $k \rightarrow \{\mathcal{C} \mid \mathcal{C} \text{ regular component of } \Gamma_r, \mathcal{W}(\mathcal{C}) = n\}$.*

As an immediate consequence, we get the following statements, which are generalizations of results by Kerner and Lukas [25, 3.1], [25, 5.2] for the Kronecker algebra.

Corollary. *Let $r \geq 3$, then for each $n \in \mathbb{N}$ there are exactly $|k|$ regular components with quasi-rank in $\{-n, -n + 1, -n + 2, -n + 3\}$.*

Corollary. *Assume that k is uncountable and $q \in \mathbb{N}$. The set of components of quasi-rank $\leq -q$ is uncountable.*

Another tool we use to get a better understanding of the internal structure of regular components is the Gabriel-Roiter measure introduced in [37]. This measure is a method which was used by Roiter in his proof of the first Brauer-Thrall conjecture and was formalized by Gabriel in [18]. The GR measure can be seen as an assignment that associates to every object M of $\text{mod } \mathcal{K}_r$ an element in a totally ordered set, with the property that $U \subseteq V$ implies $\mu(U) \leq \mu(V)$ and $\mu(\bigoplus_{i=1}^n M_i) = \max_{i=1}^n \mu(M_i)$ for all indecomposable modules M_1, \dots, M_n .

We show that for X in $\text{mod } \mathcal{K}_r$ regular indecomposable the sequence $(\tau^{-n} X)_{n \in \mathbb{N}}$ has an infimum that is independent of X . Thus all sequences have the same limiting behavior. Since our proof relies on the fact that $\mu(P) < \mu(R)$ for each preprojective module P and each regular module

R , we search for wild quivers Q with a Gabriel-Roiter theory "similar" to Γ_r . We call these quivers central and show the existence of infinitely many wild central quivers with n vertices for each $n \geq 2$.

Using that all wild quivers with $n \geq 3$ vertices have regular tilting modules, we can prove:

Theorem C. *Let Q be a central quiver, $|Q_0| \geq 3$ and $X, Y \in \text{mod } kQ$ regular indecomposable with X quasi-simple. There is $n \in \mathbb{N}$ such that for every $S \in (\tau_{kQ}^{-n} X \rightarrow)$ we have $\mu(S) < \mu(Y)$.*

Since $\mu(-)$ behaves well under monomorphisms, we can combine our result with results of Kerner and Lukas and get:

Corollary. *Let Q be a central quiver, $|Q_0| \geq 3$ and $X, Y \in \text{mod } kQ$ regular indecomposable with X quasi-simple. There is $n \in \mathbb{N}$ such that $\mu(S) < \mu(Y) < \mu(T)$ for every S in $(\tau_{kQ}^{-n} X \rightarrow)$ and every T in $\{\tau_{kQ}^m X[i] \mid m \geq n, i \in \mathbb{N}\}$.*

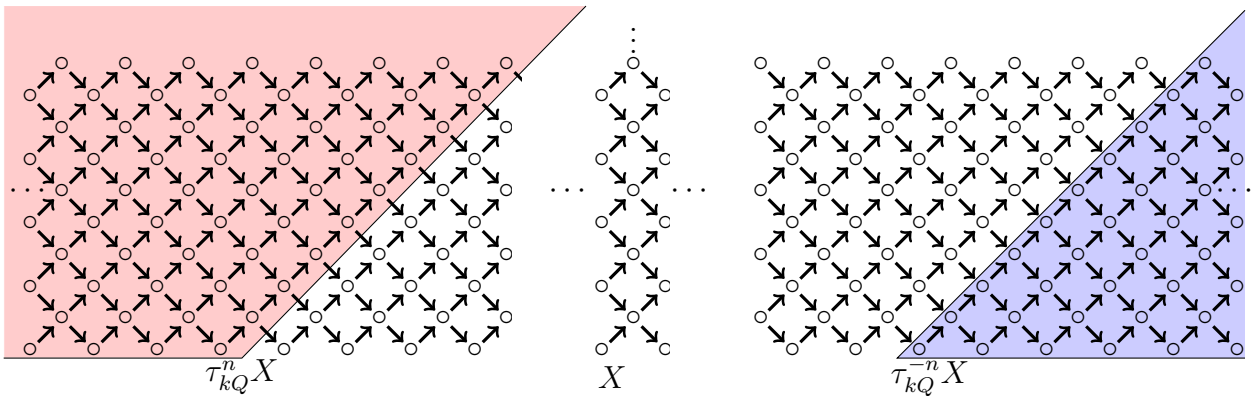
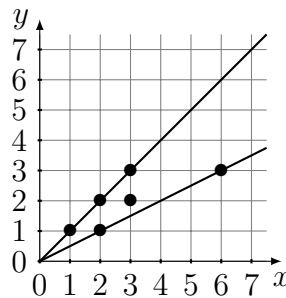


Figure 1.2: Illustration of the set $(\tau_{kQ}^{-n} X \rightarrow) \cup \{\tau_{kQ}^m X[i] \mid m \geq n, i \in \mathbb{N}\}$.

In the last part of this thesis, we take a closer look at elementary modules of \mathcal{K}_r which play an essential role in all of our considerations. A non-zero regular module E is called elementary if there is no short exact sequence $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ with A, B regular and non-zero.

Ringel recently proved that the Coxeter-orbits of dimension vectors of elementary modules for the 3-Kronecker quiver are $(1, 1), (1, 2), (2, 2)$ and $(2, 4)$. We generalize some of the used methods for arbitrary r . To do so we use a result of Westwick on spaces of matrices of fixed rank to conclude:

Proposition. *If $1 \leq d < r$, $M \in \text{ESP}_d$ and $\dim_k M_1 \neq 0$, then $d \cdot \dim_k M_2 - \dim_k M_1 \geq r - d$.*



As an application, we show that the number of Coxeter-orbits for $r = 4$ is 5 or 6. The existence of an elementary module with dimension vector $(6, 3)$ is still an open question. The other five dots in the above figure show representatives for the Coxeter-orbits that exist.

2 Preliminaries

2.1 Notation

Throughout this thesis, k denotes an algebraically closed field and if not stated otherwise, k is of arbitrary characteristic. By a k -algebra we always mean an associative, finite dimensional, unitary and connected k -algebra. For a (not necessarily finite dimensional) k -algebra Λ we denote by $\text{mod } \Lambda$ the category of finitely generated left modules over Λ and by $\text{Mod } \Lambda$ the category of all left modules over Λ . We denote the category of finite dimensional Λ -modules by $\text{fin } \Lambda$.

For $M, N \in \text{Mod } \Lambda$ we denote by $\text{Hom}_\Lambda(M, N)$ the vector space of Λ -linear maps from M to N . The k -linear maps from M to N are always denoted by $\text{Hom}_k(M, N)$. We denote by \mathbb{N} the natural numbers $1, 2, 3, \dots$ and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.2 Quiver, quiver algebras and representations of quivers

A quiver $Q = (Q_0, Q_1, s, t)$ consists of two sets Q_0, Q_1 with $Q_0 \neq \emptyset$ and maps $s, t: Q_1 \rightarrow Q_0$. An element in Q_0 is called vertex and an element in Q_1 is called arrow. If α is an arrow, we say that α starts in $s(\alpha)$ and ends in $t(\alpha)$, and just write $\alpha: s(\alpha) \rightarrow t(\alpha)$. If $\alpha: x \rightarrow y$ is an arrow, then by definition $s(\alpha) = x$ and $t(\alpha) = y$. Given a quiver Q , we define the underlying graph $\overline{Q} = (Q_0, Q_1, e)$ of Q , where e is the map $e: Q_1 \rightarrow \mathcal{P}^{Q_0}, \alpha \mapsto \{s(\alpha), t(\alpha)\}$ and \mathcal{P}^{Q_0} is the set of all subsets of Q_0 . That means we do no longer distinguish the orientation of arrows. For a vertex $x \in Q_0$ let

$$x^+ := \{y \in Q_0 \mid \exists \alpha \in Q_1: t(\alpha) = y\} \text{ and } x^- := \{y \in Q_0 \mid \exists \alpha \in Q_1: s(\alpha) = y\}.$$

We define the neighbourhood $n(x)$ of x as $n(x) := x^+ \cup x^-$. For $x, y \in Q_0$ a path from x to y of length $n \in \mathbb{N}$ is a sequence of arrows $\alpha_n \cdots \alpha_1$ with $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < n$. The trivial arrow in x is denoted by ϵ_x and is by definition the unique path of length 0 starting (ending) in x . A path $\alpha_n \cdots \alpha_1$ of length $n \geq 1$ is called cycle if $t(\alpha_n) = s(\alpha_1)$. Q is called acyclic if Q has no cycles. Q is called connected if the underlying graph \overline{Q} is connected. Q is called a tree if \overline{Q} is a tree, i.e. connected and acyclic as an unoriented graph. If Q is a tree, then Q has bipartite orientation if for each vertex x we have $x^+ = \emptyset$ or $x^- = \emptyset$.

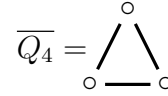
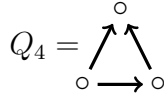
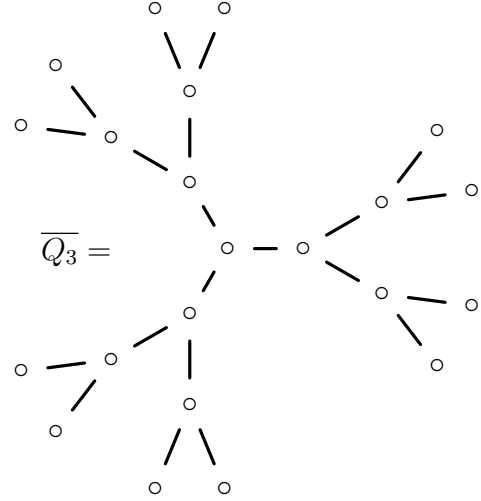
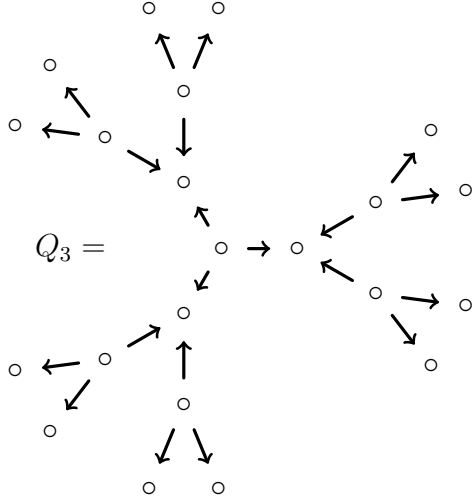
Definition. A quiver Q is called

- (a) locally finite if $n(x)$ is finite for each $x \in Q_0$,
- (b) of bounded length if for each $x \in Q_0$ there is $n_x \in \mathbb{N}$ such that each path in Q which starts or ends in x is of length $\leq n_x$,
- (c) locally bounded if Q is locally finite and of bounded length.

Examples. The quivers Q_2, Q_3 and Q_4 are locally bounded, Q_1 is locally finite and not locally bounded. The quivers Q_1, Q_2, Q_3 are trees, but only Q_2 and Q_3 have bipartite orientation. Quiver Q_4 is connected and acyclic but not a tree since $\overline{Q_4}$ is not acyclic as a graph.

$$Q_1 = \cdots \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \cdots$$

$$Q_2 = \cdots \circ \leftarrow \circ \rightarrow \circ \leftarrow \circ \rightarrow \circ \leftarrow \circ \rightarrow \circ \leftarrow \circ \rightarrow \circ \leftarrow \circ \rightarrow \cdots$$



From now on we assume that every quiver Q is locally bounded. Note that this implies that Q is acyclic. Moreover, we assume that Q is connected. A finite dimensional representation $M = ((M_x)_{x \in Q_0}, (M(\alpha))_{\alpha \in Q_1})$ over Q consists of vector spaces M_x and linear maps $M(\alpha): M_{s(\alpha)} \rightarrow M_{t(\alpha)}$ such that $\dim_k M := \sum_{x \in Q_0} \dim_k M_x$ is finite. A morphism $f: M \rightarrow N$ between representations is a collection of linear maps $(f_z)_{z \in Q_0}$ such that for each arrow $\alpha: x \rightarrow y$ there is a commutative diagram

$$\begin{array}{ccc} M_x & \xrightarrow{M(\alpha)} & M_y \\ \downarrow f_x & & \downarrow f_y \\ N_x & \xrightarrow{N(\alpha)} & N_y. \end{array}$$

The category of finite dimensional representations over Q is denoted by $\text{rep}(Q)$. If moreover Q_0 is a finite set, we denote by kQ the path algebra of Q with idempotents $e_x, x \in Q_0$. In this case kQ is a finite dimensional, associative, basic and connected k -algebra. Given $M \in \text{mod } kQ$, we let $M_x := e_x M$. The categories $\text{mod } kQ$ and $\text{rep}(Q)$ are equivalent (see for example [1, III.1.6]). We will therefore switch freely between representations of Q and modules of kQ , if one of the approaches seems more convenient for us. Recall the definition of the dimension function

$$\underline{\dim}: \text{mod } kQ \rightarrow \mathbb{Z}^{Q_0}, M \mapsto (\dim_k M_x)_{x \in Q_0}.$$

If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence, then $\underline{\dim} A + \underline{\dim} C = \underline{\dim} B$. A finite quiver Q defines a (non-symmetric) bilinear form $\langle -, - \rangle_Q: \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$, given by $((x_i), (y_j)) \mapsto \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}$, which coincides with the Euler-Ringel form [34] on the Grothendieck group $K_0(kQ) \cong \mathbb{Z}^{Q_0}$, i.e. for $M, N \in \text{mod } kQ$ we have

$$\langle \underline{\dim} M, \underline{\dim} N \rangle_Q = \dim_k \text{Hom}_{kQ}(M, N) - \dim_k \text{Ext}_{kQ}(M, N),$$

where $\text{Ext}_{kQ}(M, N) := \text{Ext}_{kQ}^1(M, N)$.

By $q_Q: \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}, x \mapsto \langle x, x \rangle_Q$ we denote the corresponding quadratic form, also called Tits form.

2.3 Representation type

One natural question that comes up when considering a k -algebra Λ is how complicated the task of determining all indecomposable modules in $\text{mod } \Lambda$ and all morphisms between such modules is. Quite often we deal with the case that such a classification of $\text{mod } \Lambda$ is considered hopeless. In this section, we give a short introduction to a common approach to measure the complexity of the module category and specify the term "hopeless" in this context.

Definition. [4, 4.4.1] Let Λ be a finite dimensional k -algebra.

- The algebra Λ has finite representation type if there are only finitely many isomorphism classes of indecomposable Λ -modules.
- The algebra Λ has tame representation type if it does not have finite representation type, and for any dimension n , there is a finite set of Λ - $k[T]$ -bimodules M_i which are free as right $k[T]$ -modules, with the property that all but a finite number of indecomposable Λ -modules of dimension n are of the form $M_i \otimes_{k[T]} M$ for some i and for some indecomposable $k[T]$ -module M .

The following definitions are based on the definitions of [43, 2.4].

Definition. Let Λ be a (not necessarily finite dimensional) k -algebra and let \mathcal{C} be an additive exact k -subcategory of the module category $\text{Mod } \Lambda$. The category \mathcal{C} is defined to be

1. k -wild if there exists an exact k -linear functor $T: \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C}$ which reflects isomorphisms and preserves indecomposables.
2. fully k -wild if there exists a fully faithful exact k -linear functor $T': \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C}$.

Remark. Observe that a fully k -wild category is k -wild, since a fully faithful exact functor reflects isomorphisms and preserves indecomposables [3, XIX.1.2].

One reason for calling a category \mathcal{C} k -wild is the following result.

Theorem 2.1. [3, XIX.1.7] *Let B be a finite dimensional k -algebra. Then there exists a fully faithful and exact k -linear functor $\text{Mod } B \rightarrow \text{Mod } k\langle t_1, t_2 \rangle$, which restricts to a functor $\text{mod } B \rightarrow \text{fin } k\langle t_1, t_2 \rangle$.*

Thus, if \mathcal{C} is k -wild one finds for each finite dimensional k -algebra B an injection from the isomorphism classes of indecomposable modules of B into the isomorphism classes of indecomposable objects in \mathcal{C} . Hence, the classification of all indecomposable modules in B can be obtained from the classification of \mathcal{C} , if one knows the injection. Furthermore, the theorem shows that if \mathcal{C} is fully k -wild, the category $\text{mod } B$ is equivalent to a full subcategory of \mathcal{C} . So in this case the classification of all B -linear maps between indecomposable modules can also be obtained from the classification of \mathcal{C} .

We summarize the quoted results in the following theorem.

Theorem 2.2. *Let Λ be a (not necessarily finite dimensional) k -algebra and let \mathcal{C} be an additive exact k -subcategory of the module category $\text{Mod } \Lambda$.*

1. *The following statements are equivalent:*
 - (a) *The category \mathcal{C} is k -wild.*
 - (b) *For each finite dimensional k -algebra B there exists a k -linear functor $H: \text{mod } B \rightarrow \mathcal{C}$ which is exact, reflects isomorphisms and preserves indecomposables.*
2. *The following statements are equivalent:*
 - (a) *The category \mathcal{C} is fully k -wild.*
 - (b) *For each finite dimensional k -algebra B there exists a fully faithful exact k -linear functor $H: \text{mod } B \rightarrow \mathcal{C}$.*

Proof. In both cases it remains to prove (b) \Rightarrow (a). By [3, XIX.1.9] there is a fully faithful exact functor $L: \text{fin } k\langle t_1, t_2 \rangle \rightarrow \text{mod } B$, where B is the 5-dimensional Kronecker algebra $k\Gamma_3$ with 3 arrows, which we will introduce later. Now $H \circ L: \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C}$ has the desired properties. \square

Lemma 2.3. (*Wildness Correction Lemma*) [44, 2.6] *Suppose that A is a finitely generated k -algebra such that there exists a functor $G': \text{fin } k\langle t_1, t_2 \rangle \rightarrow \text{fin } A$ which is exact, reflects isomorphisms and preserves indecomposables. Then there exists a A - A -bimodule ${}_A M_A$ such that the endofunctor $F = M \otimes_A -: \text{Mod } A \rightarrow \text{Mod } A$ satisfies the following conditions:*

- (a) *F is a faithful exact and induces a functor $F': \text{fin } A \rightarrow \text{fin } A$, which is exact, reflects isomorphisms and preserves indecomposables. If in addition G' is full, then F' is also full.*
- (b) *M_A viewed as a right A -module is finitely generated and free.*
- (c) *If Λ is a (not necessarily finite dimensional) k -algebra and $T: \text{fin } A \rightarrow \text{Mod } \Lambda$ is an exact functor such that the composed functor $T \circ F': \text{fin } A \rightarrow \text{Mod } \Lambda$ is non-zero, then $T \circ F'$ is of the form $T \circ F' \cong M' \otimes_A -$, where ${}_A M'_A$ is a Λ - A -bimodule which is free as a right A -module. If in addition $T(\text{fin } A) \subseteq \text{fin } \Lambda$, then the right A -module M'_A is finitely generated free.*

Proof. We clarify a step in the proof of (c) given in [44]. Let B be the 3-Kronecker algebra and $G: \text{mod } B \rightarrow \text{fin } \Lambda$ be a k -linear functor that is exact, preserves indecomposables and reflects isomorphisms. Let $T: \text{fin } A \rightarrow \text{Mod } \Lambda$ be exact such that $T \circ G: \text{fin } A \rightarrow \text{Mod } \Lambda$ is non-zero. It follows from Watt's Theorem that $T \circ G \cong {}_\Lambda N_B \otimes_B -$, where $N_B = T(G(B))$. Hence, $\text{Tor}_B^1(M, N_B) = 0$ for every module in $\text{mod } B$. Since Tor^1 interchanges with direct limits [41, 7.8] and every module in $\text{Mod } B$ is the direct limit of its finitely generated submodules [41, 5.32], we conclude that $\text{Tor}_B^1(-, N_B)$ vanishes on $\text{Mod } B$, so N_B is flat. Since $\text{Mod } B$ has enough projectives, the Theorem of Bass (see section 4.6 in [41]) implies that N_B is projective. If, in addition, $T(\text{fin } A) \subseteq \text{fin } \Lambda$ then $T \circ G(\text{mod } B) \subseteq \text{fin } \Lambda$ and therefore N_B is a finite-dimensional B -module. Now proceed as in the proof of [44, 2.6]. \square

Corollary 2.4. *Let Λ be a (not necessarily finite dimensional) k -algebra and let \mathcal{C} be a k -wild subcategory of $\text{fin } \Lambda$ and $A := k\langle t_1, t_2 \rangle$. Then there exists a Λ - A -bimodule ${}_{\Lambda}M_A$ such that ${}_{\Lambda}M_A \otimes_A -: \text{mod } A \rightarrow \mathcal{C}$ is exact, reflects isomorphisms and preserves indecomposables; moreover, ${}_{\Lambda}M_A$ is a free right A -module of finite rank. If \mathcal{C} is fully k -wild, then the functor is exact and fully faithful.*

Proof. Since \mathcal{C} is (fully) k -wild, we find a k -linear functor $T: \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C} \subseteq \text{fin } \Lambda$ which is exact, preserves indecomposables and reflects isomorphisms (resp. exact and fully faithful). Consider the identity functor $G': \text{fin } k\langle t_1, t_2 \rangle \rightarrow \text{fin } A$. Let $F := {}_A M_A \otimes -: \text{Mod } A \rightarrow \text{Mod } A$ be the endofunctor from 2.3. Then it follows from 2.3(c) that $T \circ F': \text{fin } A \rightarrow \mathcal{C}$ is of the desired form. \square

Since the implication (c) \Rightarrow (a) in the following result is trivial, we have (see also [43, 2.5]):

Corollary 2.5. *Let Λ be a (not necessarily finite dimensional) k -algebra and let \mathcal{C} be an additive exact k -subcategory of the module category $\text{fin } \Lambda$.*

1. *The following statements are equivalent:*
 - (a) *The category \mathcal{C} is k -wild.*
 - (b) *For each finite dimensional k -algebra A there exists a k -linear functor $H: \text{mod } A \rightarrow \mathcal{C}$ which is exact, reflects isomorphism and preserves indecomposables.*
 - (c) *There exists a finitely generated Λ - $k\langle t_1, t_2 \rangle$ -bimodule ${}_{\Lambda}M_{k\langle t_1, t_2 \rangle}$ that is free of finite rank as a $k\langle t_1, t_2 \rangle$ -module such that $M \otimes_{k\langle t_1, t_2 \rangle} -: \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C}$ is exact, reflects isomorphisms and preserves indecomposables.*
2. *The following statements are equivalent:*
 - (a) *The category \mathcal{C} is fully k -wild.*
 - (b) *For each finite dimensional k -algebra A there exists a fully faithful exact k -linear functor $H: \text{mod } A \rightarrow \mathcal{C}$.*
 - (c) *There exists a Λ - $k\langle t_1, t_2 \rangle$ -bimodule ${}_{\Lambda}M_{k\langle t_1, t_2 \rangle}$ that is free of finite rank as a $k\langle t_1, t_2 \rangle$ -module such that $M \otimes_{k\langle t_1, t_2 \rangle} -: \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C}$ is exact and fully faithful.*

Definition. Let Λ be a (not necessarily finite dimensional) k -algebra and $\mathcal{C} \subseteq \text{fin } \Lambda$ an additive exact k -subcategory. We say that \mathcal{C} has wild representation type (or is wild) if \mathcal{C} is k -wild. The category has strictly wild representation type (or is strictly wild) if \mathcal{C} is fully k -wild.

Observe that if $\mathcal{C} \subseteq \text{fin } \Lambda$ is k -wild, then $\text{fin } \Lambda$ is k -wild and Λ is of wild representation type. Moreover, note that if \mathcal{C} is fully k -wild, the category $\text{fin } \Lambda$ is in general not strictly wild, since a fully faithful functor $T': \text{fin } k\langle t_1, t_2 \rangle \rightarrow \mathcal{C}$ might not be full considered as a functor $T': \text{fin } k\langle t_1, t_2 \rangle \rightarrow \text{fin } \Lambda$.

We now consider algebras that play an important role in this thesis and clarify the definitions.

- Examples.**
- (a) Let Λ be a finite dimensional k -algebra that has strictly wild representation type, then Λ is not local and not commutative (see [3, XIX.1.16]).
 - (b) Let $\text{char}(k) = p > 0$ and $r \geq 2$. Let E_r be an p -elementary abelian group of rank r . Then $\text{mod } kE_r$ is of wild representation type if and only if $r \geq 3$ or $r = 2$ and $p > 2$ (see [4, 4.4.4]). Since kE_r is commutative, the algebra is not strictly wild.
 - (c) Let $k\langle x, y \rangle$ be the free associative algebra in two generators and I be the ideal generated by $x^2, y^3, xy - yx, xy^2$. Then $k\langle x, y \rangle / I$ is a 5-dimensional algebra and $\text{mod } k\langle x, y \rangle / I$ is wild (see [39]) but not strictly wild, since $k\langle x, y \rangle / I$ is commutative.

(d) Let $l \geq 2$ and $k\langle\langle t_1, \dots, t_l \rangle\rangle$ be the k -algebra of power-series in l non-commuting variables t_1, \dots, t_l . Consider the k -algebra epimorphism $\varphi: k\langle\langle t_1, \dots, t_l \rangle\rangle \rightarrow k\langle\langle x, y \rangle\rangle/I$ given by $t_1 \mapsto x + I, t_2 \mapsto y + I$ and $t_i \mapsto 0$ for $i > 2$. Then each finite dimensional module of $k\langle\langle x, y \rangle\rangle/I$ obtains the structure of a $k\langle\langle t_1, \dots, t_l \rangle\rangle$ -module via pullback along φ . The k -linear pullback functor $\varphi^*: \text{mod } k\langle\langle x, y \rangle\rangle/I \rightarrow \text{fin } k\langle\langle t_1, \dots, t_l \rangle\rangle$ is faithful and exact and also full since φ is surjective. Hence, φ^* is exact, reflects isomorphisms and preserves indecomposables. By definition, the category $\text{fin } k\langle\langle t_1, \dots, t_l \rangle\rangle$ is k -wild and $k\langle\langle t_1, \dots, t_l \rangle\rangle$ has wild representation type.

Definition. A module $M \in \text{mod } \Lambda$ is called a brick if $\text{End}_\Lambda(M) \cong k$.

Lemma 2.6. *Let Λ be a (not necessarily finite dimensional) local k -algebra. Up to isomorphism there exists a unique simple Λ -module S , and a module $M \in \text{fin } \Lambda$ is a brick if and only if $M \cong S$.*

Proof. Since Λ is local, there is a unique maximal left-ideal I in Λ . Let $S \in \text{mod } \Lambda$ be simple, $s \in S \setminus \{0\}$ and consider the Λ -linear map $m_s: \Lambda \rightarrow S; x \mapsto x.s$. Then m_s is non-zero and surjective since S is simple. Hence, $\Lambda/\ker m_s$ is simple, $\ker m_s \cong I$ and $S \cong \Lambda/I$.

Let M be a brick in $\text{fin } \Lambda$ and $0 = M_n \subset \dots \subset M_2 \subset M_1 \subset M_0 = M$ be a filtration of M with M_i/M_{i+1} simple for all $0 \leq i \leq n-1$. Let $\pi: M \rightarrow M_0/M_1$ be the natural projection and $\iota: M_{n-1} \rightarrow M$ be the natural injection. Since M_{n-1} and M_0/M_1 are simple, we find an isomorphism $M_0/M_1 \rightarrow M_{n-1}$ and conclude that $f: M \xrightarrow{\pi} M_0/M_1 \rightarrow M_{n-1} \xrightarrow{\iota} M$ is non-zero. Since M is a brick, f is an isomorphism. In particular, π is injective and $M_1 = \{0\}$. Hence, $n = 1$ and M is simple, i.e. $M \cong S$. \square

Corollary 2.7. *Let $l \geq 2$ and $k\langle\langle t_1, \dots, t_l \rangle\rangle$ be the k -algebra of power-series in l non-commuting variables t_1, \dots, t_l . The algebra $k\langle\langle t_1, \dots, t_l \rangle\rangle$ has wild representation type but not strictly wild representation type.*

Proof. It remains to show that $k\langle\langle t_1, \dots, t_l \rangle\rangle$ is not strictly wild. It is proven in [3, XIX.1.16] that for a finite dimensional algebra Λ of strictly wild representation type and each $d > 0$ there exists $M \in \text{fin } \Lambda$ such that $\text{End}_\Lambda(M) \cong k$ and $\dim_k M \geq d$.

Observe that the proof extends to the case of Λ being of infinite dimension by the characterization (c) in Corollary 2.5. In view of 2.6, it suffices to show that $k\langle\langle t_1, \dots, t_l \rangle\rangle$ is local. By [8, IV 4.4.6], an element $f \in k\langle\langle t_1, \dots, t_l \rangle\rangle$ is invertible if and only if the constant term of f is non-zero. Hence, $\{f \in k\langle\langle t_1, \dots, t_l \rangle\rangle \mid f \text{ non-invertible}\}$ is the unique maximal left-ideal and $k\langle\langle t_1, \dots, t_l \rangle\rangle$ is local. \square

In 1977 Drozd proved the following remarkable theorem.

Theorem 2.8. (*Trichotomy Theorem*) [4, 4.4.2] *Let k be an algebraically closed field. Then every finite dimensional k -algebra has finite, tame or wild representation type, and these types are mutually exclusive.*

Hence, each finite dimensional algebra that is not of finite or tame representation type is already wild and by 2.2 the complete understanding of its module category is as difficult as the classification of the module category of every finite dimensional k -algebra.

We are interested in the case of Λ being a quiver algebra kQ with finite, connected and acyclic quiver Q . In this setting, the quiver Q already determines the representation type of kQ ; Gabriel has shown (see [1, VII.5.10]) that the algebra kQ has finite representation type if and only if \overline{Q} is a simply-laced Dynkin diagram. Donovan and Freislich have shown (see [15])

that kQ has tame representation type if and only if \overline{Q} is a simply-laced Euclidean diagram. By the Trichotomy theorem, the algebra kQ has wild representation type in all other cases. The families of simply-laced Dynkin and Euclidean diagrams are listed in Figure 2.1. The simply-laced Dynkin diagrams are the graphs of type A_n, D_n, E_6, E_7, E_8 and shown on the left-hand side. Recall that for $Z \in \{A, D, E\}$ the Dynkin diagram Z_n has n vertices and the Euclidean diagram \tilde{Z}_n has $n + 1$ vertices.

Definition. Let Q be a finite, connected and acyclic quiver. Then Q is a wild quiver, if \overline{Q} is not a simply-laced Dynkin or Euclidean diagram.

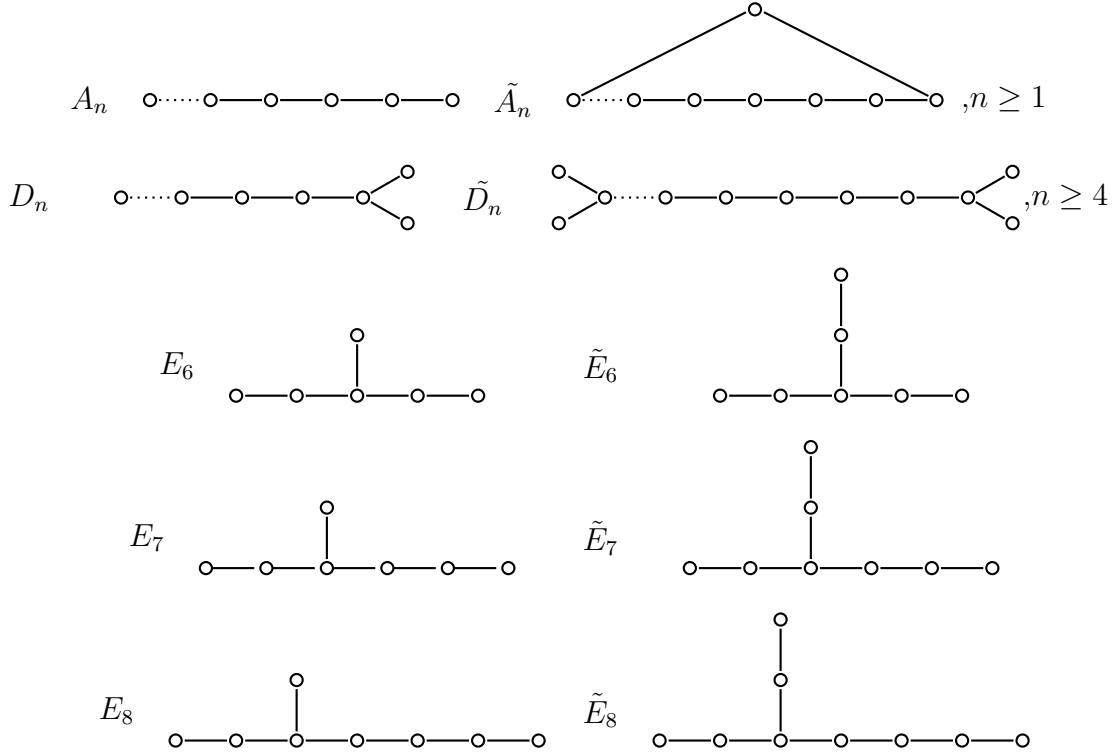


Figure 2.1: List of simply-laced Dynkin and Euclidean diagrams.

As we have seen in Example (b) and Corollary 2.7, not every algebra of wild representation type has strictly wild representation type. However, for quiver algebras these conditions are equivalent:

Theorem 2.9. [3, XIX.1.11], [4, 4.4.3], [27, 17.22] *Let Q be a finite, connected and acyclic quiver. The following statements are equivalent:*

- (a) Q is a wild quiver.
- (b) kQ has strictly wild representation type.
- (c) kQ has wild representation type.
- (d) There exists a k -linear fully faithful and exact functor $\text{mod } k\Gamma_3 \rightarrow \text{mod } kQ$.
- (e) The theory of finite dimensional kQ -modules is undecidable.

For details on (e) we refer the reader to [27, 17.4] and [4, 4.4.3].

2.4 Auslander-Reiten theory

We introduce the notion of Auslander-Reiten sequences for finite dimensional k -algebras. For a more general introduction needed for locally bounded categories we refer to [7] and [19].

Let Λ be a finite dimensional k -algebra.

Definition. Let M and N in $\text{mod } \Lambda$. A Λ -linear map $f: M \rightarrow N$ is called irreducible, provided:

- (a) f is neither a section nor a retraction and
- (b) if $f = f_1 f_2$, either f_1 is a retraction or f_2 is a section.

We denote by $\text{rad}_\Lambda(-, -)$ the radical in $\text{mod } \Lambda$ and let

$$\text{Irr}_\Lambda(M, N) := \text{rad}_\Lambda(M, N) / \text{rad}_\Lambda^2(M, N).$$

If M and N are indecomposable, [1, IV.1.6] shows that $f: M \rightarrow N$ is irreducible if and only if $f \in \text{rad}_\Lambda(M, N) \setminus \text{rad}_\Lambda^2(M, N)$.

Definition. Let $\delta: 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence in $\text{mod } \Lambda$ with A and C indecomposable. Then δ is called almost split or Auslander-Reiten sequence if f and g are irreducible.

Theorem 2.10. [1, IV.3.1,4.4] *Let X, Y be indecomposable Λ -modules.*

- (a) *If Y is non-projective, then there exists an almost split sequence*

$$0 \rightarrow Z \rightarrow E \rightarrow Y \rightarrow 0,$$

which is unique up to equivalence of short exact sequences.

- (b) *If X is non-injective, then there exists an almost split sequence*

$$0 \rightarrow X \rightarrow E \rightarrow Z \rightarrow 0,$$

which is unique up to equivalence of short exact sequences.

- (c) *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an Auslander-Reiten sequence and $B = \bigoplus_{i=1}^m (B_i)^{n_i}$ with $n_j \geq 1$ and $B_i \not\cong B_j$ indecomposable for all $i, j \in \{1, \dots, m\}$ and $i \neq j$. Then $\dim_k \text{Irr}_\Lambda(A, B_i) = n_i = \dim_k \text{Irr}_\Lambda(B_i, C)$ for all $1 \leq i \leq m$.*

Definition. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an Auslander-Reiten sequence. Then we set $\tau_\Lambda C := \tau_\Lambda(C) := A$ and $\tau_\Lambda^{-1} A := \tau_\Lambda^{-1}(A) := C$.

Remark. Moreover one can show that the assignments τ_Λ and τ_Λ^{-1} obtain the structure of functors when one defines them as $D \text{Tr}$ and $\text{Tr } D$ (see [1, VI.2.3]).

Definition. The Auslander-Reiten quiver Γ_Λ of Λ is the translation quiver with vertices corresponding to isomorphism classes $[M]$ of indecomposable modules M in $\text{mod } \Lambda$. The number of arrows from $[M] \rightarrow [N]$ is given by $\dim_k \text{Irr}(M, N)$. Finally, the translation τ is just the Auslander-Reiten translation defined on all vertices not given by projective modules, i.e. $\tau[M] = [\tau_A M]$ with inverse $\tau^{-1}[N] = [\tau_A^{-1} N]$.

From now on, we will not distinguish between an indecomposable module M , its isomorphism class and the corresponding vertex in Γ_Λ . Moreover, we will also write τ_Λ for the translation automorphism of Γ_Λ . If M is an indecomposable projective (injective) module we define $\tau_\Lambda M$ (resp. $\tau_\Lambda^{-1} M$) to be the zero-module. If X_1, \dots, X_n are indecomposable Λ modules we set $\tau_\Lambda(\bigoplus_{i=1}^n X_i) := \bigoplus_{i=1}^n \tau_\Lambda X_i$ and $\tau_\Lambda^{-1}(\bigoplus_{i=1}^n X_i) := \bigoplus_{i=1}^n \tau_\Lambda^{-1} X_i$

A very powerful tool that we will often use is the Auslander-Reiten formula which we formulate in the framework of quiver algebras without relations.

Theorem 2.11. [22, 1.3] *Let Q be a quiver and $X, Y \in \text{mod } kQ$, then there are functorial isomorphisms*

$$\text{Ext}_{kQ}(X, Y) \cong \text{Hom}_{kQ}(X, \tau Y)^* \cong \text{Hom}_{kQ}(\tau^{-1}Y, X)^*,$$

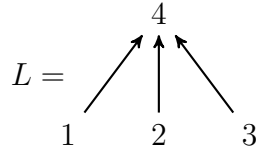
where $*$ denotes the standard duality on vector spaces.

Definition. An indecomposable module $X \in \text{mod } kQ$ is called

- (a) preprojective, if there exists $n \in \mathbb{N}_0$ such that $\tau_{kQ}^n X$ is projective.
- (b) preinjective, if there exists $n \in \mathbb{N}_0$ such that $\tau_{kQ}^{-n} X$ is injective.
- (c) regular, if X is not preprojective or preinjective.

By [1, VIII.2.1], an indecomposable module is preprojective (resp. preinjective) if and only if it is contained in the unique component $\mathcal{P}(Q)$ (resp. $\mathcal{I}(Q)$) which contains all projective (injective) indecomposable modules. Every other component of Γ_{kQ} is called regular. We define $\mathcal{R}(Q) := \{\mathcal{C} \mid \mathcal{C} \text{ regular component of } \Gamma_{kQ}\}$. Note that a component is regular if and only if τ_Λ and τ_Λ^{-1} are defined on every vertex.

Examples. 1. Consider the quiver



The underlying graph \bar{L} is the Dynkin diagram D_4 . By [1, VII.5.10] $\text{mod } kL$ is representation finite and there are 12 indecomposable modules in $\text{mod } kL$ up to isomorphism. The Auslander-Reiten quiver of kL is shown in Figure 2.2, cf. [1, VII.5.15]. If M is a module of kQ with dimension vector $\underline{\dim} M = (\dim_k M_1, \dim_k M_2, \dim_k M_3, \dim_k M_4) = (a, b, c, d)$ we write $\begin{smallmatrix} d \\ abc \end{smallmatrix}$.

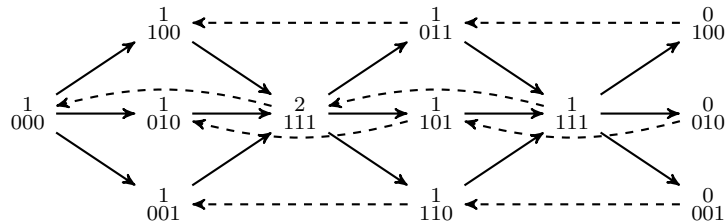
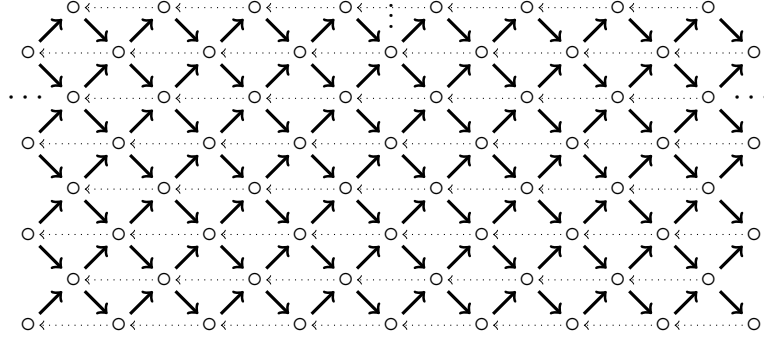


Figure 2.2: Illustration of the Auslander-Reiten quiver of kL .

If C in $\text{mod } kL$ is indecomposable and not projective, then $\tau_{kL} C$ is the module that ends in the dashed arrow starting in C . Clearly every indecomposable module in kL is preprojective and preinjective and there are no regular modules, i.e. $\mathcal{P}(L) = \mathcal{I}(L)$ and $\mathcal{R}(L) = \emptyset$.

- 2. Important components in this thesis are components of type $\mathbb{Z}A_\infty$. These are of the form



An irreducible morphism in a component of type $\mathbb{Z}A_\infty$ is injective if the corresponding arrow is uprising and surjective otherwise. A module M in a $\mathbb{Z}A_\infty$ component is called quasi-simple if the AR sequence terminating in M has an indecomposable middle term. These modules are the modules in the bottom layer of the component. If M is quasi-simple in a component \mathcal{C} of type $\mathbb{Z}A_\infty$, then there is an infinite chain (ray) of irreducible monomorphisms (resp. epimorphisms)

$$M = M[1] \rightarrow M[2] \rightarrow M[3] \rightarrow \cdots \rightarrow M[l] \rightarrow \cdots$$

$$\cdots (l)M \rightarrow \cdots \rightarrow (3)M \rightarrow (2)M \rightarrow (1)M = M,$$

and for each indecomposable module X in \mathcal{C} there are unique quasi-simple modules N, M and $l \in \mathbb{N}$ with $(l)M = X = N[l]$. The number $\text{ql}(X) := l$ is called the quasi-length of X .

It follows from [1, VIII.2.9] and Gabriel's theorem that the set $\mathcal{R}(Q)$ is non-empty whenever Q is not a simply-laced Dynkin diagram. Moreover, one can show the following.

Theorem 2.12. [1, VIII.2.3] *Let Q be a quiver. The following statements are equivalent:*

- (a) Q is not a simply-laced Dynkin diagram.
- (b) The set $\mathcal{R}(Q)$ is non-empty.

In this case there is a bijection between k and $\mathcal{R}(Q)$, $\mathcal{P}(Q)$ is of type $\mathbb{N}Q$ and $\mathcal{I}(Q)$ of type $-\mathbb{N}Q$. In particular, $\text{mod } kQ$ contains infinitely many non-isomorphic indecomposable modules.

Proposition 2.13. [3, XVIII.1.8], [1, VIII.2.13] *Let $\mathcal{R}(Q)$ be non-empty, $\mathcal{C} \in \mathcal{R}(Q)$ be a regular component, $P \in \mathcal{P}(Q)$, $I \in \mathcal{I}(Q)$ and $R \in \mathcal{C}$. Then $\text{Hom}_{kQ}(I, R) = 0 = \text{Hom}_{kQ}(I, P)$ and $\text{Hom}_{kQ}(R, P) = 0$.*

This is illustrated by Figure 2.3 with non-zero morphisms only going from left to right.

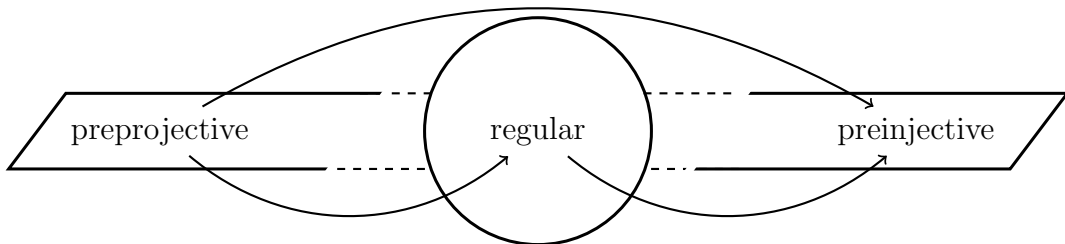


Figure 2.3: Illustration of morphisms in the Auslander-Reiten quiver.

To emphasise this fact later on we just write

$$\mathrm{Hom}_{kQ}(\mathcal{R}(Q), \mathcal{P}(Q)) = 0 = \mathrm{Hom}_{kQ}(\mathcal{I}(Q), \mathcal{R}(Q)) = \mathrm{Hom}_{kQ}(\mathcal{I}(Q), \mathcal{P}(Q)).$$

When \overline{Q} is an Euclidean diagram, then $\mathcal{R}(Q) \neq \emptyset$ and by [2, XI.2.8] for each two regular components \mathcal{C}, \mathcal{D} with $\mathcal{C} \neq \mathcal{D}$ we have $\mathrm{Hom}_{kQ}(\mathcal{C}, \mathcal{D}) = 0 = \mathrm{Hom}_{kQ}(\mathcal{D}, \mathcal{C})$, i.e. \mathcal{C} and \mathcal{D} are Hom-orthogonal. The next result shows that the situation is quite different if Q is a wild quiver. For $(x_i), (y_i) \in \mathbb{Z}^{Q_0}$ we write $(x_i) \leq (y_i)$, if $x_i \leq y_i$ for all $i \in Q_0$.

Let us mention the dual version of a lemma by Kerner, that will be needed later on.

Proposition 2.14. [23, 4.6] *Let Q be a wild quiver and X, Y be non-zero regular modules. There exists an integer $n \in \mathbb{N}$ with $\mathrm{Hom}_{kQ}(Z, \tau_{kQ}^m Y) \neq 0$ for all regular modules Z with $\underline{\dim} Z \leq \underline{\dim} X$ and all $m > n$.*

Definition. We call an arbitrary module preprojective (resp. preinjective, regular) if all its indecomposable direct summands are preprojective (res. preinjective, regular).

Let $\mathrm{mod}_{pf} kQ$ be the subcategory of all modules without non-zero projective direct summands and $\mathrm{mod}_{if} kQ$ the subcategory of all modules without non-zero injective summands. Since kQ is a hereditary algebra, the Auslander-Reiten translation τ_{kQ} , on the level of functors, is given by $\tau_{kQ} = D \mathrm{Tr} \cong \mathrm{Ext}_{kQ}^1(M, kQ)^*$ [1, VII.1.9] and induces an equivalence from $\mathrm{mod}_{pf} kQ$ to $\mathrm{mod}_{if} kQ$, that will often be used later on without further notice. Ringel has proven (see [30, 2.3]) that if Q is a wild quiver then every component in $\mathcal{R}(Q)$ is of type $\mathbb{Z}A_\infty$.

2.5 Tilting theory for quiver algebras

We recall well-known definitions and results of tilting theory for quiver algebras. Let Q be a finite, connected and acyclic quiver.

Definition. Let $l \in \mathbb{N}$. A module $T = T_1 \oplus \dots \oplus T_l \in \mathrm{mod} kQ$ is called partial tilting module, provided that

- (a) T_i is indecomposable for all $i \in \{1, \dots, l\}$,
- (b) $T_i \not\cong T_j$ for $i \neq j$,
- (c) $\mathrm{Ext}_{kQ}(T_i, T_j) = 0$ for all i, j .

A partial tilting module is called tilting module if $l = |Q_0|$.

Ringel has shown the following result.

Theorem 2.15. [38] *Let Q be a wild quiver with $|Q_0| \geq 3$, then there exists a regular tilting module T for kQ .*

We recall an important result on tilting modules.

Theorem 2.16. [1, VI.2.5] *Let T be a partial tilting module. The following statements are equivalent:*

- (a) T is a tilting module.
- (b) $\mathrm{Gen} T = \mathcal{T}(T) := \{M \in \mathrm{mod} kQ \mid \mathrm{Ext}_{kQ}(T, M) = 0\}$.
- (c) $\mathrm{Cogen} \tau_{kQ} T = \mathcal{F}(T) := \{M \in \mathrm{mod} kQ \mid \mathrm{Hom}_{kQ}(T, M) = 0\}$.

Here $\mathrm{Gen} T$ denotes the class of modules in $\mathrm{mod} kQ$ that are generated by T and $\mathrm{Cogen} \tau_{kQ} T$ is the class of modules in $\mathrm{mod} kQ$ that are cogenerated by $\tau_{kQ} T$.

The following result, known as Bongartz's lemma, shows that a module $M \in \text{mod } kQ$ with $\text{Ext}_{kQ}(M, M) = 0$ can always be extended to a tilting module.

Lemma 2.17. [1, VI.2.4] *Let T be a partial tilting module. There exists a kQ -module E such that $T \oplus E$ is a tilting module.*

2.6 The Kronecker algebra

As mentioned before, we are especially interested in the family of Kronecker quivers. For $r \geq 2$, Γ_r denotes the r -Kronecker quiver, given by two vertices 1, 2 and arrows $\gamma_1, \dots, \gamma_r: 1 \rightarrow 2$. For $r = 2$ we have $\overline{\Gamma}_2 = \tilde{A}_1$, i.e. $k\Gamma_2$ has tame representation type and $k\Gamma_r$ is (even strictly) wild for $r \geq 3$.

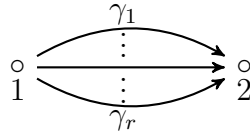


Figure 2.4: The Kronecker quiver Γ_r .

We set $\mathcal{K}_r := k\Gamma_r$ and $P_1 := \mathcal{K}_r e_2$, $P_2 := \mathcal{K}_r e_1$. The modules P_1 and P_2 are the indecomposable projective modules of $\text{mod } \mathcal{K}_r$, $\dim_k \text{Hom}_{\mathcal{K}_r}(P_1, P_2) = r$ and $\dim_k \text{Hom}_{\mathcal{K}_r}(P_2, P_1) = 0$. As Figure 2.4 suggests, we write $\underline{\dim} M = (\dim_k M_1, \dim_k M_2)$. For example, $\underline{\dim} P_1 = (0, 1)$ and $\underline{\dim} P_2 = (1, r)$.

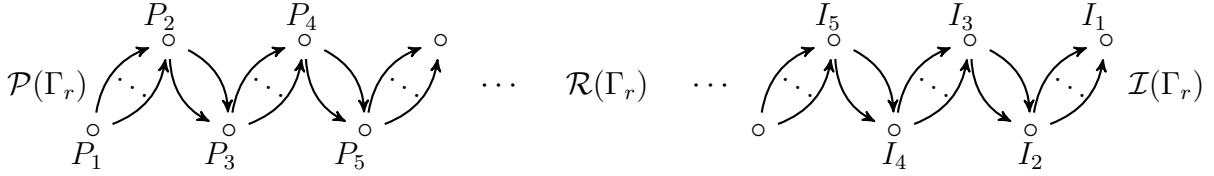


Figure 2.5: Auslander-Reiten quiver of Γ_r .

Figure 2.5 shows the notation we use for the components $\mathcal{P}(\Gamma_r), \mathcal{I}(\Gamma_r)$ in the Auslander-Reiten quiver of \mathcal{K}_r which contain the indecomposable projective modules P_1, P_2 and indecomposable injective modules I_1, I_2 . By τ we denote the Auslander-Reiten translation $\tau_{\mathcal{K}_r}$.

We introduce the duality $D_{\Gamma_r}: \text{rep}(\Gamma_r) \rightarrow \text{rep}(\Gamma_r)$ by setting $(D_{\Gamma_r} M)_x := (M_{\psi(x)})^*$ and $(D_{\Gamma_r} M)(\gamma_i) := (M(\gamma_i))^*$ for each $M \in \text{rep}(\Gamma_r)$, where $\psi: \{1, 2\} \rightarrow \{1, 2\}$ is the permutation of order 2. Note that $D_{\Gamma_r}(P_i) = I_i$ for all $i \in \mathbb{N}$. Using the equivalence of categories $\text{mod } \mathcal{K}_r \cong \text{rep}(\Gamma_r)$, we get a duality $D := D_{\mathcal{K}_r}: \text{mod } \mathcal{K}_r \rightarrow \text{mod } \mathcal{K}_r$.

We state a simplified version of Kac's Theorem [20, 1.10] for the Kronecker algebra in combination with results on the quadratic form proven in [34, 2.3]:

Theorem 2.18. *Let $r \geq 2$ and $d \in \mathbb{N}_0^2$.*

- If $d = \underline{\dim} M$ for some indecomposable module M , then $q_{\Gamma_r}(d) \leq 1$.*
- If $q_{\Gamma_r}(d) = 1$, then there exists a unique indecomposable module X with $\underline{\dim} X = d$. In this case, X is preprojective or preinjective and X is preprojective if and only if $\dim_k X_1 < \dim_k X_2$.*

- (c) If $q_{\Gamma_r}(d) \leq 0$, then there exist infinitely many indecomposable modules Y with $\underline{\dim} Y = d$ and each Y is regular. In this case, d is called imaginary root of q_{Γ_r} .

Since there is no pair $(a, b) \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$ satisfying $a^2 + b^2 - rab = q_{\Gamma_r}(a, b) = 0$ for $r \geq 3$, we conclude together with [1, VIII.2.7]:

Corollary 2.19. *Let M be an indecomposable \mathcal{K}_r -module. Then $\text{Ext}_{\mathcal{K}_r}(M, M) = 0$ if and only if M is preprojective or preinjective. If $r \geq 3$ and M is regular, then $\dim_k \text{Ext}_{\mathcal{K}_r}(M, M) \geq 2$.*

This result and Theorem 2.15 show that the family $(\Gamma_r)_{r \geq 3}$ of quivers is quite special among wild quivers; in particular, the theory on regular tilting modules is not applicable in $\text{mod } \mathcal{K}_r$. However, we will see later on that Corollary 2.19 is valuable in conjunction with the Simplification method for constructing k -wild subcategories in given torsion classes (resp. torsion free classes) of $\text{mod } \mathcal{K}_r$.

Let us end this section by recalling some important definitions and results from [48]. We use a slightly different notation, since we are only interested in the case $B(2, r) = \mathcal{K}_r$. For $\alpha \in k^r \setminus \{0\}$ and M in $\text{mod } \mathcal{K}_r$ we define $x_\alpha := \alpha_1 \gamma_1 + \dots + \alpha_r \gamma_r$ and denote with $x_\alpha^M : M \rightarrow M$ the linear operator associated to x_α .

Definition. Let M be in $\text{mod } \mathcal{K}_r$. The module M has

- (a) the equal images property if $\text{im}(x_\alpha^M)$ is independent of the choice of $\alpha \in k^r \setminus \{0\}$.
- (b) the equal kernels property if $\text{ker}(x_\alpha^M)$ is independent of the choice of $\alpha \in k^r \setminus \{0\}$.

Lemma 2.20. *An indecomposable non-simple module $M \in \text{mod } \mathcal{K}_r$ has the equal images property if and only if $\text{im}(x_\alpha^M) = M_2$ for all $\alpha \in k^r \setminus \{0\}$ and the equal kernels property if and only if $\text{ker}(x_\alpha^M) = M_2$.*

Definition. (a) $\text{EIP} := \{M \in \text{mod } \mathcal{K}_r \mid \forall \alpha \in k^r \setminus \{0\} : \text{im}(x_\alpha^M) = M_2\}$.
(b) $\text{EKP} := \{M \in \text{mod } \mathcal{K}_r \mid \forall \alpha \in k^r \setminus \{0\} : \text{ker}(x_\alpha^M) = M_2\}$.
(c) $\text{CR} := \{M \in \text{mod } \mathcal{K}_r \mid \exists c \in \mathbb{N}_0 \forall \alpha \in k^r \setminus \{0\} : \text{rk}(x_\alpha^M) = c\}$.

Remark. Note that I_1 has the equal kernels property but is by definition not in EKP. It is easy to see that each module in EKP has the equal kernels property and every module M with the equal kernels property is in EKP if and only if I_1 is not a direct summand of M . When considering EIP just replace I_1 by P_1 .

Theorem 2.21. *Let $r \geq 2$. For each $\alpha \in k^r \setminus \{0\}$, there exists a family of indecomposable regular representations $(X_\alpha)_{\alpha \in k^r \setminus \{0\}}$ such that*

- (a) $\text{EIP} = \{M \in \text{mod } \mathcal{K}_r \mid \forall \alpha \in k^r \setminus \{0\} : \text{Ext}_{\mathcal{K}_r}(X_\alpha, M) = 0\}$.
- (b) $\text{EKP} = \{M \in \text{mod } \mathcal{K}_r \mid \forall \alpha \in k^r \setminus \{0\} : \text{Hom}_{\mathcal{K}_r}(X_\alpha, M) = 0\}$.
- (c) $\text{CR} = \{M \in \text{mod } \mathcal{K}_r \mid \exists c \in \mathbb{N}_0 \forall \alpha \in k^r \setminus \{0\} : \dim_k \text{Hom}_{\mathcal{K}_r}(X_\alpha, M) = c\}$.

Definition. Let Λ be a finite-dimensional k -algebra. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod } \Lambda$ is called a torsion pair if the following conditions are satisfied:

- (a) $\text{Hom}_\Lambda(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$,
- (b) $\text{Hom}_\Lambda(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$,
- (c) $\text{Hom}_\Lambda(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

Using the Auslander-Reiten formula one concludes:

Proposition 2.22. *The category EIP is a torsion class \mathcal{T} of a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod } \mathcal{K}_r$ with $\text{EKP} \subseteq \mathcal{F}$ that is closed under the Auslander-Reiten translate τ and which contains all preinjective modules, in particular $\text{EIP} \cap \text{EKP} = \{0\}$.*

Theorem 2.23. *Let $r \geq 3$. For each regular component \mathcal{C} there exist uniquely determined quasi-simple modules $M_{\mathcal{C}}$ and $W_{\mathcal{C}}$ in \mathcal{C} such that $(M_{\mathcal{C}} \rightarrow) = \text{EKP} \cap \mathcal{C}$ and $(\rightarrow W_{\mathcal{C}}) = \text{EIP} \cap \mathcal{C}$.*

Definition. Let \mathcal{C} be a regular component. Then $\mathcal{W}(\mathcal{C})$ is the unique integer with $\tau^{\mathcal{W}(\mathcal{C})+1} M_{\mathcal{C}} = W_{\mathcal{C}}$, which is a natural number since $\text{EKP} \cap \text{EIP} = \{0\}$.

Lemma 2.24. [48, 4.10] *If a regular component \mathcal{C} contains a brick, then $\mathcal{W}(\mathcal{C}) \leq 2$.*

Moreover, it is shown in [48] that $X_{\alpha} \cong X_{\beta}$ if and only if $k\alpha = k\beta$. Hence, the non-isomorphic representatives of the family are parametrized by the projective space \mathbb{P}^{r-1} . In the next chapter we generalize this result by constructing analogous families $(X_U)_{U \in \text{Gr}_{d,r}}$ for each $1 \leq d \leq r-1$, where $\text{Gr}_{d,r}$ denotes the Grassmannian of d -dimensional subspaces of k^r , the case $d=1$ being $\text{Gr}_{1,r} \cong \mathbb{P}^{r-1}$. Then we show that the recently introduced families of [9] can be realized using similar methods to those in [48].

3 Modules of constant radical and socle rank

3.1 Elementary modules of small dimension

Let $r \geq 3$. The module family $(X_\alpha)_{\alpha \in k^r \setminus \{0\}}$ in Theorem 2.21 has the property that for each $\alpha \in k^r \setminus \{0\}$ every proper submodule Y of X_α is preprojective. We will see later on that each non-zero proper submodule of X_α is isomorphic to a finite number of copies of P_1 and X_α itself is regular. In particular, we do not find a short exact sequence $0 \rightarrow A \rightarrow X_\alpha \rightarrow B \rightarrow 0$ such that A and B are regular and non-zero. In the language of wild quiver algebras, we therefore deal with elementary modules:

Definition. [24, 1] Let Q be a wild quiver. A non-zero regular module $E \in \text{mod } kQ$ is called elementary, if there is no short exact sequence $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ with A and B regular non-zero.

Elementary modules are analogues of quasi-simple modules in the tame hereditary case ($r = 2$). If X is a regular module, then X has a filtration $0 = X_0 \subset X_1 \subset \dots \subset X_s = X$ such that X_i/X_{i-1} is elementary for all $1 \leq i \leq s$ and the elementary modules are the smallest class with that property. For basic results on elementary modules, used in this section, we refer to [24].

Lemma 3.1. [24, 1.3], [36, A1] *Let E be a non-zero indecomposable regular kQ -module. The following statements are equivalent:*

- (a) E is elementary.
- (b) There exists an integer n such that $\tau_{kQ}^l E$ has no non-trivial regular factor module for all $l \geq n$.
- (c) There exists an integer m such that $\tau_{kQ}^{-l} E$ has no non-trivial regular submodules for all $l \geq m$.
- (d) If $Y \neq 0$ is a regular submodule of E then E/Y is preinjective.
- (e) If $X \neq E$ is a submodule of E with E/X regular, then X is preprojective.
- (f) If $X \neq E$ is a submodule of E that is not preprojective, then E/X is preinjective.

We are grateful to Otto Kerner for pointing out the following helpful lemma.

Lemma 3.2. *Let Q be a wild quiver. Let E be an elementary module and X, Y regular with non-zero morphisms $f: X \rightarrow E$ and $g: E \rightarrow Y$. Then $g \circ f \neq 0$. In particular $\text{End}_{kQ}(E) = k$.*

Proof. Consider the short exact sequence

$$0 \rightarrow \text{im } f \rightarrow E \xrightarrow{\pi} \text{coker } f \rightarrow 0.$$

Since X is regular, f is non-zero and $\text{Hom}_{kQ}(\mathcal{R}(Q), \mathcal{P}(Q)) = 0$ every direct summand of $\text{im } f$ is regular or preinjective. Since $\text{Hom}_{kQ}(\mathcal{I}(Q), \mathcal{R}(Q)) = 0$ and E is regular, we conclude that

$\text{im } f$ is a regular non-zero submodule of E . Consequently, since E is elementary, $\text{coker } f$ is preinjective. Now assume that $g \circ f = 0$. Then the universal property of $\text{coker } f$ yields $h: \text{coker } f \rightarrow Y$ with $h \circ \pi = g$. Now $\text{Hom}_{kQ}(\mathcal{I}(Q), \mathcal{R}(Q)) = 0$ yields $h = 0$ and $g = 0$, a contradiction. Hence $g \circ f \neq 0$. \square

We now use the theory on elementary modules to generalize [48, 2.7] in the hereditary case.

Proposition 3.3. *Let \mathcal{E} be a family of elementary modules, $c \in \mathbb{N}$, $\dim_k E \leq c$ for all $E \in \mathcal{E}$ and $\mathcal{T}(\mathcal{E}) := \{M \in \text{mod } \mathcal{K}_r \mid \forall E \in \mathcal{E} : \text{Ext}_{\mathcal{K}_r}(E, M) = 0\}$. Then the following statements hold.*

- (1) $\mathcal{T}(\mathcal{E})$ is closed under extensions, images and τ .
- (2) $\mathcal{T}(\mathcal{E})$ contains all preinjective modules.
- (3) For each regular component \mathcal{C} , the set $\mathcal{T}(\mathcal{E}) \cap \mathcal{C}$ forms a non-empty cone in \mathcal{C} , which consists of the predecessors of a uniquely determined quasi-simple module in \mathcal{C} .

Proof. Since $\text{Ext}_{\mathcal{K}_r}^2 = 0$, the category is closed under images. Let M be indecomposable with $\text{Ext}_{\mathcal{K}_r}(E, M) = 0$ for all $E \in \mathcal{E}$. Then the Auslander-Reiten formula yields for all $E \in \mathcal{E}$ that $0 = \dim_k \text{Hom}_{\mathcal{K}_r}(M, \tau E)$. Assume that $P = M$ is preprojective, let $l \in \mathbb{N}_0$ such that $\tau^l P$ is projective, then $\tau^l P = P_i$ for an $i \in \{1, 2\}$ and

$$0 = \dim_k \text{Hom}_{\mathcal{K}_r}(P, \tau E) = \dim_k \text{Hom}_{\mathcal{K}_r}(\tau^l P, \tau^{l+1} E) = (\underline{\dim}(\tau^{l+1} E))_{3-i},$$

since τ induces an equivalence between $\text{mod}_{p_f} kQ$ and $\text{mod}_{i_f} kQ$. This is a contradiction, since every \mathcal{K}_r -module L with $\underline{\dim} L \in \{(x, 0), (0, x) \mid x \in \mathbb{N}\}$ is semisimple. Hence, M is regular or preinjective.

We are going to show that $\text{Ext}_{\mathcal{K}_r}(E, \tau M) = 0$. In view of the Auslander-Reiten formula we get $\dim_k \text{Ext}_{\mathcal{K}_r}(E, \tau M) = \dim_k \text{Hom}_{\mathcal{K}_r}(M, E)$. Let $f: M \rightarrow E$ be a morphism and assume that f is non-zero. Since $0 = \text{Hom}_{\mathcal{K}_r}(\mathcal{I}(\Gamma_r), \mathcal{R}(\Gamma_r))$, the module M is not preinjective and hence regular. As a regular module E has self-extensions (see 2.19) we conclude $E \notin \mathcal{T}(\mathcal{E})$. Therefore $0 \neq \dim_k \text{Ext}_{\mathcal{K}_r}(E, E) = \dim_k \text{Hom}_{\mathcal{K}_r}(E, \tau E)$ and we find $0 \neq g \in \text{Hom}_{\mathcal{K}_r}(E, \tau E)$. Lemma 3.2 provides a non-zero morphism

$$M \xrightarrow{f} E \xrightarrow{g} \tau E.$$

We conclude $0 \neq \dim_k \text{Hom}_{\mathcal{K}_r}(M, \tau E) = \dim_k \text{Ext}_{\mathcal{K}_r}(E, M) = 0$, a contradiction. Now note that I_1, I_2 are contained in $\mathcal{T}(\mathcal{E})$. The existence of the cones can be shown as in [48, 3.3] using the upper bound and [23, 10.7]. \square

The next result follows by the Auslander-Reiten formula and duality since $D(E)$ is elementary if and only if E is elementary.

Proposition 3.4. *Let \mathcal{E} be a family of elementary modules, $c \in \mathbb{N}$, $\dim_k E \leq c$ for all $E \in \mathcal{E}$ and $\mathcal{F}(\mathcal{E}) := \{M \in \text{mod } \mathcal{K}_r \mid \forall E \in \mathcal{E} : \text{Hom}_{\mathcal{K}_r}(E, M) = 0\}$. Then the following statements hold.*

- (1) $\mathcal{F}(\mathcal{E})$ is closed under extensions, submodules and τ^{-1} .
- (2) $\mathcal{F}(\mathcal{E})$ contains all preprojective modules.
- (3) For each regular component \mathcal{C} the set $\mathcal{F}(\mathcal{E}) \cap \mathcal{C}$ forms a non-empty cone in \mathcal{C} , which consists of the successors of a uniquely determined quasi-simple module in \mathcal{C} .

Observe that $\mathcal{F}(\mathcal{E})$ is a torsion free class class of some torsion pair $(\mathcal{T}, \mathcal{F}(\mathcal{E}))$ and $\mathcal{T}(\mathcal{E})$ is the torsion class of some torsion pair $(\mathcal{T}(\mathcal{E}), \mathcal{F})$.

Lemma 3.5. *Let M, N be indecomposable modules with $\underline{\dim} M = (c, 1)$, $\underline{\dim} N = (1, c)$, $1 \leq c < r$. Then the following statements hold.*

- (a) $\tau^z M$ and $\tau^z N$ are elementary for all $z \in \mathbb{Z}$. Moreover, every proper factor of M is injective and every proper submodule of N is projective.
- (b) Every proper factor module of $\tau^l M$ is preinjective and every proper submodule of $\tau^{-l} N$ is preprojective for $l \in \mathbb{N}_0$.

Proof. We will give the proofs for M . The statements for N follow by duality.

- (a) By [24, 1.1] it suffices to show that M is elementary. Since $q_{\Gamma_r}(c, 1) < 0$, the module M is regular. Now let $0 \subset X \subset M$ be a proper submodule with dimension vector $\underline{\dim} X = (a, b)$. Then $b = 1$ since $\text{Hom}_{\mathcal{K}_r}(\mathcal{I}(\Gamma_r), \mathcal{R}(\Gamma_r)) = 0$. Hence, $\underline{\dim} M/X = (c-a, 0)$ and M/X is injective.
- (b) Let $l \geq 1$. Since a proper factor $\tau^l M \rightarrow X$ with X regular induces a proper regular factor $M \rightarrow \tau^{-l} X$ the statement follows by (a). \square

3.2 An algebraic family of test-modules

Let $r \geq 2$. Now we take a closer look at the modules $(X_\alpha)_{\alpha \in k^r \setminus \{0\}}$. We start with the definition.

Definition. For $\alpha \in k^r \setminus \{0\}$, the map $\bar{\alpha}: \langle e_2 \rangle_k = P_1 \rightarrow P_2, e_2 \mapsto \alpha_1 \gamma_1 + \dots + \alpha_r \gamma_r = x_\alpha$ defines an embedding of \mathcal{K}_r -modules. We now set $X_\alpha := \text{coker } \bar{\alpha}$.

Observe that $\text{im } \bar{\alpha}$ is a 1-dimensional submodule of P_2 contained inside the radical $\text{rad}(P_2)$ of the local module P_2 . From the definition we get an exact sequence $0 \rightarrow P_1 \rightarrow P_2 \rightarrow X_\alpha \rightarrow 0$ and $\underline{\dim} X_\alpha = (1, r) - (0, 1) = (1, r-1)$. Since P_2 is local with semisimple radical $\text{rad}(P_2) = P_1^r$ it now seems natural to study embeddings $P_1^d \rightarrow P_2$ for $1 \leq d < r$ and the corresponding cokernels. This motivates the next definition. We restrict ourselves to $d < r$ since otherwise the cokernel is the simple injective module.

Definition. Let and $1 \leq d < r$. For $T = (\alpha_1, \dots, \alpha_d) \in (k^r)^d$ we define $\bar{T}: (P_1)^d \rightarrow P_2$ as the \mathcal{K}_r -linear map

$$\bar{T}(x) = \sum_{i=1}^d \bar{\alpha}_i \circ \pi_i(x),$$

where $\pi_i: (P_1)^d \rightarrow P_1$ denotes the projection onto the i -th coordinate.

The map \bar{T} is injective if and only if T is linearly independent; then we have $\underline{\dim} \text{coker } \bar{T} = \underline{\dim} P_2 - d \underline{\dim} P_1 = (1, r-d)$ and $\text{coker } \bar{T}$ is indecomposable, because P_2 is local. Moreover, $(1, r-d)$ is an imaginary root of q_{Γ_r} and therefore $\text{coker } \bar{T}$ is regular indecomposable and by 3.5 elementary. We define $\langle T \rangle := \langle \alpha_1, \dots, \alpha_d \rangle_k$.

Lemma 3.6. *Let $T, S \in (k^r)^d$ be linearly independent, then $\text{coker } \bar{T} \cong \text{coker } \bar{S}$ if and only if $\langle T \rangle = \langle S \rangle$.*

Proof. If $\langle T \rangle = \langle S \rangle$, then the definition of \bar{T} and \bar{S} implies that $\text{im } \bar{T} = \text{im } \bar{S}$. Hence, $\text{coker } \bar{T} = P_2 / \text{im } \bar{T} = \text{coker } \bar{S}$. Now let $\langle S \rangle \neq \langle T \rangle$ and assume that $0 \neq \varphi: \text{coker } \bar{T} \rightarrow \text{coker } \bar{S}$ is \mathcal{K}_r -linear. Since $\text{coker } \bar{S}$ is local with radical P_1^{r-d} and $\text{Hom}_{\mathcal{K}_r}(\mathcal{R}(\Gamma_r), \mathcal{P}(\Gamma_r)) = 0$ the map φ is surjective and therefore injective. Recall that P_2 has $\{e_1, \gamma_1, \dots, \gamma_r\}$ as a basis. Let $x \in P_2$ such that $\varphi(e_1 + \text{im } \bar{T}) = x + \text{im } \bar{S}$. Since φ is \mathcal{K}_r -linear we get

$$x + \text{im } \bar{S} = e_1 \varphi(e_1 + \text{im } \bar{T}) = e_1 x + \text{im } \bar{S},$$

and hence $x - e_1x \in \text{im } \bar{S}$. Write $x = \mu e_1 + \sum_{i=1}^r \mu_i \gamma_i$, then $x - \mu e_1 = \sum_{i=1}^r \mu_i \gamma_i = x - e_1x \in \text{im } \bar{S}$ and $x + \text{im } \bar{S} = \mu e_1 + \text{im } \bar{S}$.

The assumption $\langle S \rangle \neq \langle T \rangle$ yields $y \in \text{im } \bar{S} \setminus \text{im } \bar{T} \subseteq \langle \gamma_1, \dots, \gamma_r \rangle_k$. Then $y + \text{im } \bar{T} \neq 0$ and

$$\varphi(y + \text{im } \bar{T}) = y\varphi(e_1 + \text{im } \bar{T}) = \mu y(e_1 + \text{im } \bar{S}) = \mu y + \text{im } \bar{S} = \text{im } \bar{S},$$

a contradiction to the injectivity of φ . Hence, $\text{Hom}_{\mathcal{K}_r}(\text{coker } \bar{T}, \text{coker } \bar{S}) = 0$ and $\text{coker } \bar{T} \not\cong \text{coker } \bar{S}$. \square

Definition. Let $r \geq 2$, $1 \leq d < r$ and $\text{Gr}_{d,r}$ be the Grassmannian of d -dimensional subspaces of k^r . For $U \in \text{Gr}_{d,r}$ with basis $T = (u_1, \dots, u_d)$ we define $X_U := \text{coker } \bar{T}$.

Remark. By the above considerations X_U is well defined (up to isomorphism) with dimension vector $\dim X_U = (1, r - d)$, X_U is elementary for $r \geq 3$ and quasi-simple for $r = 2$.

For a module X we define $\text{add } X$ as the category of summands of finite direct sums of X and Q^d denotes the set of isomorphism classes $[M]$ of indecomposable modules M with dimension vector $(1, r - d)$ for $1 \leq d < r$.

Proposition 3.7. *Let M be indecomposable.*

- (a) *If $[M] \in Q^d$, then there exists $U \in \text{Gr}_{d,r}$ with $M \cong X_U$.*
- (b) *The map $\varphi: \text{Gr}_{d,r} \rightarrow Q^d; U \mapsto [X_U]$, is bijective.*
- (c) *Let $1 \leq c \leq d < r$ and $[M] \in Q^d$. There is $[N] \in Q^c$ and an epimorphism $\pi: N \rightarrow M$.*

Proof. (a) Let $0 \subsetneq X \subsetneq M$ be a submodule of M . Then $X \subseteq \text{rad}(M) = P_1^{r-d}$ and X is in $\text{add } P_1$. It is $1 = \dim_k M_1 = \dim_k \text{Hom}_{\mathcal{K}_r}(\mathcal{K}_r e_1, M) = \dim_k \text{Hom}_{\mathcal{K}_r}(P_2, M)$, so we find a non-zero map $\pi: P_2 \rightarrow M$. Since every proper submodule of M is in $\text{add } P_1$ and $\text{Hom}_{\mathcal{K}_r}(P_2, P_1) = 0$, the map $\pi: P_2 \rightarrow M$ is surjective and yields an exact sequence $0 \rightarrow P_1^d \xrightarrow{\iota} P_2 \xrightarrow{\pi} M \rightarrow 0$. For $1 \leq i \leq d$ there exist uniquely determined elements $\beta_i, \alpha_i^1, \dots, \alpha_i^r \in k$ such that $\iota(g_i(e_2)) = \beta_i e_1 + \alpha_i^1 \gamma_1 + \dots + \alpha_i^r \gamma_r \in P_2 = \langle \gamma_i, e_1 \mid 1 \leq i \leq r \rangle_k$, where $g_i: P_1 \rightarrow P_1^d$ denotes the embedding into the i -th coordinate. Since e_2 is an idempotent with $e_2 \gamma_j = \gamma_j$ ($1 \leq j \leq r$) and $e_2 e_1 = 0$ we get

$$\alpha_i^1 \gamma_1 + \dots + \alpha_i^r \gamma_r = e_2(\iota \circ g_i(e_2)) = (\iota \circ g_i)(e_2 \cdot e_2) = (\iota \circ g_i)(e_2) = \beta_i e_1 + \alpha_i^1 \gamma_1 + \dots + \alpha_i^r \gamma_r.$$

Hence, $\beta_i = 0$. Now define $\alpha_i := (\alpha_i^1, \dots, \alpha_i^r)$, $T := (\alpha_1, \dots, \alpha_d)$ and $U := \langle T \rangle$. It is $\iota = \bar{T}$ and by the injectivity of ι we conclude that T is linearly independent and therefore $U \in \text{Gr}_{d,r}$. Now we conclude $X_U = \text{coker } \bar{T} = \text{coker } \iota = M$.

(b) This is an immediate consequence of (a).

(c) By (a) we find U in $\text{Gr}_{d,r}$ with basis $T = (u_1, \dots, u_d)$ such that $X_U \cong M$. Let V be the subspace with basis $S = (u_1, \dots, u_c)$. Then $\text{im } \bar{S} \subseteq \text{im } \bar{T}$ and we get an epimorphism $\pi: X_V = P_2/\text{im } \bar{S} \rightarrow P_2/\text{im } \bar{T} = X_U, x + \bar{S} \rightarrow x + \bar{T}$ with $\underline{\dim} X_V = (1, r - c)$. \square

As a generalization of $x_\alpha^M: M \rightarrow M$ we introduce maps $x_T^M: M \rightarrow M^d$ and $y_T^M: M^d \rightarrow M$ for $1 \leq d < r$ and $T \in (k^r)^d$.

Definition. Let $1 \leq d < r$ and $T = (\alpha_1, \dots, \alpha_d) \in (k^r)^d$. We denote by x_T^M and y_T^M the operators

$$\begin{aligned} x_T^M: M &\rightarrow M^d, m \mapsto (x_{\alpha_1}^M(m), \dots, x_{\alpha_d}^M(m)), \\ y_T^M: M^d &\rightarrow M, (m_1, \dots, m_d) \mapsto x_{\alpha_1}^M(m_1) + \dots + x_{\alpha_d}^M(m_d). \end{aligned}$$

Note that $x_T^M = y_T^M$ if and only if $d = 1$. It is $\text{im } x_T^M \subseteq M_2 \oplus \dots \oplus M_2$, $M_2 \oplus \dots \oplus M_2 \subseteq \ker y_T^M$ and $(x_T^M)^* = y_T^{DM}$ since for $f = (f_1, \dots, f_d) \in (DM)^d$ and $m \in M$ we have

$$\begin{aligned} (x_T^M)^*(f)(m) &= (x_T^M)^*(f_1, \dots, f_d)(m) = \sum_{i=1}^d (f_i \circ x_{\alpha_i}^M)(m) \\ &= \sum_{i=1}^d f_i(x_{\alpha_i} \cdot m) = \sum_{i=1}^d (x_{\alpha_i} \cdot f_i)(m) \\ &= y_T^{DM}(f_1, \dots, f_d)(m) = y_T^{DM}(f)(m). \end{aligned}$$

Lemma 3.8. *Let $1 \leq d < r$ and $U \in \text{Gr}_{d,r}$. Every non-zero quotient Q of X_U is indecomposable. Q is preinjective (injective) if $\underline{\dim} Q = (1, 0)$ and regular otherwise.*

Proof. Since X_U is regular, we conclude with $\text{Hom}_{\mathcal{K}_r}(\mathcal{R}(\Gamma_r), \mathcal{P}(\Gamma_r)) = 0$, that every indecomposable non-zero quotient of X_U is preinjective or regular. Let Q be such a quotient with $\underline{\dim} Q = (a, b)$ and $Q \neq X_U$. Since X_U is local with radical P_1^{r-d} and $\underline{\dim} = (1, r-d)$, it follows $(1, r-d) = (a, b) + (0, c)$ for some $c > 0$. Hence, $a = 1$ and Q is an injective module if $b = 0$. Otherwise Q is also indecomposable since $b > 0$ and $Q = A \oplus B$ with $A, B \neq 0$ imply w.l.o.g $(\underline{\dim} B)_1 = 0$. Hence, $B \in \text{add } P_1$ which is a contradiction to $\text{Hom}_{\mathcal{K}_r}(\mathcal{R}(\Gamma_r), \mathcal{P}(\Gamma_r)) = 0$. \square

3.3 Modules for the generalized Kronecker algebra

In the following we will give the definition of \mathcal{K}_r -modules ($r \geq 2$) with constant radical rank and constant socle rank.

Definition. Let M be in $\text{mod } \mathcal{K}_r$ and $1 \leq d < r$.

(a) M has constant d-radical rank if the dimension of

$$\text{Rad}_U(M) := \sum_{u \in U} x_u^M(M) \subseteq M_2$$

is independent of the choice of $U \in \text{Gr}_{d,r}$.

(b) M has constant d-socle rank if the dimension of

$$\text{Soc}_U(M) := \{m \in M \mid \forall u \in U : x_u^M(M) = 0\} = \bigcap_{u \in U} \ker(x_u^M) \supseteq M_2$$

is independent of the choice of $U \in \text{Gr}_{d,r}$.

(c) M has the equal d-radical property if $\text{Rad}_U(M)$ is independent of the choice of $U \in \text{Gr}_{d,r}$.

(d) M has the equal d-socle property if $\text{Soc}_U(M)$ is independent of the choice of $U \in \text{Gr}_{d,r}$.

Lemma 3.9. *Let M be indecomposable and not simple. Then*

(a) M has the equal d-socle property if and only if $M_2 = \text{Soc}_U(M)$ for all $U \in \text{Gr}_{d,r}$.

(b) M has the equal d-radical property if and only if $M_2 = \text{Rad}_U(M)$ for all $U \in \text{Gr}_{d,r}$.

Proof. (a) Assume that M is in ESP_d and let $W := \text{Soc}_U(M)$ for $U \in \text{Gr}_{d,r}$. Denote by $e_1, \dots, e_r \in k^r$ the canonical basis vectors. Since M is not simple it is ([17, 5.1.1]) $\bigcap_{i=1}^r \ker(x_{e_i}^M) = M_2$. Denote by $S(d)$ the set of all subsets of $\{1, \dots, r\}$ of cardinality d . Then

$$\bigcap_{S \in S(d)} \bigcap_{j \in S} \ker(x_{e_j}^M) = \bigcap_{i=1}^r \ker(x_{e_i}^M) = M_2.$$

Since $\langle e_j \mid j \in S \rangle_k \in \text{Gr}_{d,r}$ and $M \in \text{ESP}_d$ we get $\bigcap_{j \in S} \ker(x_{e_j}^M) = W$ and hence $M_2 = \bigcap_{S \in \mathcal{S}(d)} W = W$.

(b) Let $M \in \text{ERP}_d$, $U \in \text{Gr}_{d,r}$ and $W := \text{Rad}_U(M)$. Since M is not simple it is ([17, 5.1.1]) $\sum_{i=1}^r x_{e_i}^M = M_2$ and hence

$$W = \sum_{S \in \mathcal{S}(d)} \sum_{j \in S} x_{e_j}^M(M) = \sum_{i=1}^r x_{e_i}^M(M) = M_2.$$

□

Definition. Let $1 \leq d < r$. We define

- (a) $\text{ESP}_d := \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{d,r} : M_2 = \text{Soc}_U(M)\}$.
- (b) $\text{ERP}_d := \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{d,r} : M_2 = \text{Rad}_U(M)\}$.
- (c) $\text{CSR}_d := \{M \in \text{mod } \mathcal{K}_r \mid \exists c \in \mathbb{N}_0 \forall U \in \text{Gr}_{d,r} : \dim_k \text{Soc}_U(M) = c\}$.
- (d) $\text{CRR}_d := \{M \in \text{mod } \mathcal{K}_r \mid \exists c \in \mathbb{N}_0 \forall U \in \text{Gr}_{d,r} : \dim_k \text{Rad}_U(M) = c\}$.

Note that $\text{CRR}_1 = \text{CR} = \text{CSR}_1$, $\text{ERP}_1 = \text{EIP}$, $\text{ESP}_1 = \text{EKP}$ and for $U \in \text{Gr}_{d,r}$ with basis (u_1, \dots, u_d) we have $\text{Rad}_U(M) = \sum_{i=1}^d x_{u_i}^M(M) = \text{im } y_{(u_1, \dots, u_d)}^M$ and $\text{Soc}_U(M) = \bigcap_{i=1}^d \ker(x_{u_i}^M) = \ker x_{(u_1, \dots, u_d)}^M$. We restrict the definition to $d < r$ since $\text{Gr}_{r,r} = \{k^r\}$ and therefore every module in $\text{mod } \mathcal{K}_r$ is of constant r -socle and r -radical rank.

Lemma 3.10. *Let M be in $\text{mod } \mathcal{K}_r$ and $1 \leq d < r$.*

- (a) $M \in \text{CSR}_d$ if and only if $DM \in \text{CRR}_d$.
- (b) $M \in \text{ESP}_d$ if and only if $DM \in \text{ERP}_d$.

Proof. Note that $\text{Rad}_U(DM) = \text{im}(y_T^{DM}) = \text{im}(x_T^M)^* \cong (\text{im } x_T^M)^*$ and hence

$$\dim_k M - \dim_k \text{Soc}_U(M) = \dim_k M - \dim_k \ker x_T^M = \dim_k \text{im } x_T^M = \dim_k (\text{im } x_T^M)^* = \dim_k \text{Rad}_U(DM).$$

Hence, $M \in \text{CSR}_d$ if and only if $DM \in \text{CRR}_d$. Moreover, M in ESP_d if and only if $\text{Soc}_U(M) = M_2$ and hence $\dim_k \text{Rad}_U(DM) = \dim_k M_1 = \dim_k (DM)_2$. □

For the proof of the following proposition we use the same methods as in [48, 2.5].

Proposition 3.11. *Let $1 \leq d < r \in \mathbb{N}$. Then*

$$\begin{aligned} \text{ESP}_d &= \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{d,r} : \text{Hom}_{\mathcal{K}_r}(X_U, M) = 0\}, \\ \text{CSR}_d &= \{M \in \text{mod } \mathcal{K}_r \mid \exists c \in \mathbb{N}_0 \forall U \in \text{Gr}_{d,r} : \dim_k \text{Hom}_{\mathcal{K}_r}(X_U, M) = c\}. \end{aligned}$$

Proof. Let $U \in \text{Gr}_{d,r}$ with basis $T = (\alpha_1, \dots, \alpha_d)$. Consider the short exact sequence $0 \rightarrow (P_1)^d \xrightarrow{\bar{T}} P_2 \rightarrow X_U \rightarrow 0$. Application of $\text{Hom}_{\mathcal{K}_r}(-, M)$ yields

$$0 \rightarrow \text{Hom}_{\mathcal{K}_r}(X_U, M) \rightarrow \text{Hom}_{\mathcal{K}_r}(P_2, M) \xrightarrow{\bar{T}^*} \text{Hom}_{\mathcal{K}_r}(P_1^d, M) \rightarrow \text{Ext}_{\mathcal{K}_r}(X_U, M) \rightarrow 0.$$

Moreover, let

$$f: \text{Hom}_{\mathcal{K}_r}(P_2, M) \rightarrow M_1; g \rightarrow g(e_1)$$

and

$$g: \text{Hom}_{\mathcal{K}_r}(P_1^d, M) \rightarrow M_2^d; h \mapsto (h \circ \iota_1(e_2), \dots, h \circ \iota_d(e_2))$$

be the natural isomorphisms, where $\iota_i: P_1 \rightarrow P_1^d$ denotes the embedding into the i -th coordinate. Let $\pi_{M_2^d}: M^d \rightarrow M_2^d$ the natural projection, then $g \circ \bar{T}^* = \pi_{M_2^d} \circ x_T^M|_{M_1} \circ f$. Hence, $\dim_k \ker(\pi_{M_2^d} \circ x_T^M|_{M_1}: M_1 \rightarrow M_2^d) = \dim_k \ker(\bar{T}^*) = \dim_k \text{Hom}_{\mathcal{K}_r}(X_U, M)$. Now let $c \in \mathbb{N}_0$. We conclude

$$\begin{aligned} \dim_k \text{Hom}_{\mathcal{K}_r}(X_U, M) = c &\Leftrightarrow \dim_k \ker(\pi_{M_2^d} \circ x_T^M|_{M_1}) = c \stackrel{\text{im } x_T^M \subseteq M_2^d}{\Leftrightarrow} \dim_k \ker(x_T^M|_{M_1}) = c \\ &\stackrel{M_2 \subseteq \ker(x_T^M)}{\Leftrightarrow} \dim_k \ker(x_T^M) = c + \dim_k M_2 \\ &\stackrel{\text{Soc}_U(M) = \ker(x_T^M)}{\Leftrightarrow} \dim_k \text{Soc}_U(M) = c + \dim_k M_2. \end{aligned}$$

This finishes the proof for CSR_d . Moreover, note that $c = 0$ together with Lemma 3.9 yields

$$\begin{aligned} M \in \text{ESP}_d &\Leftrightarrow \exists W \leq M \forall U \in \text{Gr}_{d,r}: \text{Soc}_U(M) = W \\ &\Leftrightarrow \forall U \in \text{Gr}_{d,r}: \text{Soc}_U(M) = M_2 \Leftrightarrow \forall U \in \text{Gr}_{d,r}: \dim_k \text{Soc}_U(M) = 0 + \dim_k M_2 \\ &\Leftrightarrow \forall U \in \text{Gr}_{d,r}: \text{Hom}_{\mathcal{K}_r}(X_U, M) = 0. \end{aligned}$$

□

Since $\tau \circ D = D \circ \tau^{-1}$ the next result follows from the Auslander-Reiten formula and 3.10.

Proposition 3.12. *Let $1 \leq d < r \in \mathbb{N}$. Then*

$$\begin{aligned} \text{ERP}_d &= \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{d,r}: \text{Ext}_{\mathcal{K}_r}(D\tau X_U, M) = 0\} \text{ and} \\ \text{CRR}_d &= \{M \in \text{mod } \mathcal{K}_r \mid \exists c \in \mathbb{N}_0 \forall U \in \text{Gr}_{d,r}: \dim_k \text{Ext}_{\mathcal{K}_r}(D\tau X_U, M) = c\}. \end{aligned}$$

Remarks. (a) Note that for $d = 1$ we have $U = \langle \alpha \rangle_k$ with $\alpha \in k^r \setminus \{0\}$, $X_U \cong X_\alpha$ and $D\tau X_U \cong X_U$ [48, 3.1]. However, this identity holds if and only if $d = 1$.

(b) It follows immediatly from 3.11 and 3.5 that for $1 \leq d < r - 1$ and $V \in \text{Gr}_{d,r}$ the module X_V is in $\text{CSR}_{d+1} \setminus \text{CSR}_d$.

If not stated otherwise, we assume from now on that $r \geq 3$. In view of 3.4, 3.5 and the definitions of ESP_d and ERP_d we immediatly get:

Proposition 3.13. *Let $1 \leq d < r$ and \mathcal{C} be a regular component of \mathcal{K}_r .*

- (a) $\text{ESP}_1 \subseteq \text{ESP}_2 \subseteq \dots \subseteq \text{ESP}_{r-1}$ and $\text{ERP}_1 \subseteq \text{ERP}_2 \subseteq \dots \subseteq \text{ERP}_{r-1}$.
- (b) ESP_d is closed under extensions, submodules and τ^{-1} . Moreover, ESP_d contains all pre-projective modules and $\text{ESP}_d \cap \mathcal{C}$ forms a non-empty cone in \mathcal{C} .
- (c) ERP_d is closed under extensions, images and τ . Moreover ERP_d contains all preinjective modules and $\text{ERP}_d \cap \mathcal{C}$ forms a non-empty cone in \mathcal{C} .

Definition. For $1 \leq i < r$ we set $\Delta_i := \text{ESP}_i \setminus \text{ESP}_{i-1}$ and $\nabla_i := \text{ERP}_i \setminus \text{ESP}_{i-1}$, where $\text{ESP}_0 = \emptyset = \text{ERP}_0$.

The next result suggests that for each regular component \mathcal{C} and $1 < i < r$ only a small part of vertices in \mathcal{C} corresponds to modules in Δ_i . Nonetheless we will see in Section 5 that for $1 \leq i < r$ the categories Δ_i and ∇_i are of wild type.

Proposition 3.14. *Let $r \geq 3$, \mathcal{C} be a regular component and M_i, W_i ($1 \leq i < r$) in \mathcal{C} the uniquely determined quasi-simple modules such that $\text{ESP}_i \cap \mathcal{C} = (M_i \rightarrow)$ and $\text{ERP}_i \cap \mathcal{C} = (\rightarrow W_i)$.*

- (a) *There exists at most one number $1 < m(\mathcal{C}) < r$ such that $\Delta_{m(\mathcal{C})} \cap \mathcal{C}$ is non-empty. If such a number exists then $\Delta_{m(\mathcal{C})} \cap \mathcal{C} = \{M_{m(\mathcal{C})}[l] \mid l \geq 1\}$.*
- (b) *There exists at most one number $1 < w(\mathcal{C}) < r$ such that $\nabla_{w(\mathcal{C})} \cap \mathcal{C}$ is non-empty. If such a number exists then $\nabla_{w(\mathcal{C})} \cap \mathcal{C} = \{(l)W_{w(\mathcal{C})} \mid l \geq 1\}$.*

Proof. (a) We first show that $\tau(\mathcal{C} \cap \text{ESP}_1) \subseteq \mathcal{C} \cap \text{ESP}_{r-1}$. Let M be in \mathcal{C} and assume $M \notin \text{ESP}_1$. Then there exists $\alpha \in k^r \setminus \{0\}$ with $\text{Hom}_{\mathcal{K}_r}(X_U, M) \neq 0$ for $U = \langle \alpha \rangle_k$. Hence, we find a non-zero map $f: \tau X_U \rightarrow \tau M$. Consider an exact sequence $0 \rightarrow P_1^{r-2} \rightarrow X_U \rightarrow N \rightarrow 0$. Then $\underline{\dim} N = (1, 1)$ and by 3.8 N is indecomposable. By 3.7, there exists $V \in \text{Gr}_{r-1, r}$ with $X_V \cong DN$. Since $DX_U = \tau X_U$ [48, 3.1] we get a non-zero morphism $g: X_V \rightarrow \tau X_U$ and by 3.2 a non-zero morphism

$$X_V \xrightarrow{g} \tau X_U \xrightarrow{f} \tau M.$$

Therefore $\tau M \notin \text{ESP}_{r-1}$ by 3.11.

Now assume that $M_i \neq M_j$ for some i and j . Then in particular $M_1 \neq M_{r-1}$ by 3.13(a). Hence, $M_{r-1} = \tau^l M_1$ for some $l \geq 1$. By definition we have $M := \tau M_1 \notin \text{ESP}_1$ and the above considerations yield $\tau(\tau M_1) = \tau M \notin \text{ESP}_{r-1}$. Therefore $1 \leq l < 2$ since $\text{ESP}_{r-1} \cap \mathcal{C}$ is closed under τ^{-1} . Therefore $M_{r-1} = \tau M_1$.

(b) This follows by duality. \square

We state two more results that follow from 3.11 and will be needed later on. The first one is a generalization of [48, 3.5] and follows with the same arguments.

Lemma 3.15. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an almost split sequence such that two modules of the sequence are of constant d -socle rank. Then the third module also has constant d -socle rank.*

Definition. Let $r \geq 2$, $1 \leq d < r$ and $\mathfrak{X}_{d,r} := \{X_U \mid U \in \text{Gr}_{d,r}\}$. Let $\mathfrak{X}_{d,r}^\perp$ be the right orthogonal category $\mathfrak{X}_{d,r}^\perp = \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{d,r}: \text{Hom}_{\mathcal{K}_r}(X_U, M) = 0\}$ and ${}^\perp \mathfrak{X}_{d,r}$ be the left orthogonal category ${}^\perp \mathfrak{X}_{d,r} := \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{d,r}: \text{Hom}_{\mathcal{K}_r}(M, X_U) = 0\}$. Then we set $\overline{\mathfrak{X}}_{d,r} := \mathfrak{X}_{d,r}^\perp \cap {}^\perp \mathfrak{X}_{d,r}$.

Note that every module in $\overline{\mathfrak{X}}_{d,r}$ is regular since $\text{Hom}_{\mathcal{K}_r}(X_U, M) \neq 0$ for M preinjective module and $\text{Hom}_{\mathcal{K}_r}(M, X_U) \neq 0$ for N preprojective.

Lemma 3.16. *Let $r \geq 3$, $1 \leq d < r$ and M be quasi-simple regular in a component \mathcal{C} such that $M \in \overline{\mathfrak{X}}_{d,r}$. Then every module in \mathcal{C} has constant d -socle rank.*

Proof. Let $V \in \text{Gr}_{d,r}$. It follows from 3.13 that the set $\{N \mid \text{Hom}_{\mathcal{K}_r}(X_V, N) = 0\} \cap \mathcal{C}$ (resp. $\{N \mid \text{Ext}_{\mathcal{K}_r}(X_V, N) = 0\} \cap \mathcal{C}$) is closed under τ^{-1} (resp. τ). Since $0 = \dim_k \text{Hom}_{\mathcal{K}_r}(M, X_V) = \dim_k \text{Ext}_{\mathcal{K}_r}(X_V, \tau M)$ we have $\text{Ext}_{\mathcal{K}_r}(X_V, \tau^l M) = 0$ for $l \geq 1$. The Euler-Ringel form yields

$$0 = \dim_k \text{Ext}_{\mathcal{K}_r}(X_V, \tau^l M) = -\langle \underline{\dim} X_V, \underline{\dim} \tau^l M \rangle_{\Gamma_r} + \dim_k \text{Hom}_{\mathcal{K}_r}(X_V, \tau^l M).$$

Since $\langle \underline{\dim} X_V, \underline{\dim} \tau^l M \rangle_{\Gamma_r} = \langle (1, r-d), \underline{\dim} M \rangle_{\Gamma_r}$ is independent of V , $\tau^l M$ has constant d -socle rank. On the other hand $\text{Hom}_{\mathcal{K}_r}(X_V, M) = 0$ implies that $\tau^{-q} M$ has constant d -socle rank for all $q \geq 0$. It follows that each quasi-simple module in \mathcal{C} has constant d -socle rank. Now apply 3.15 and note that CSR_d is closed under direct sums since $\text{Hom}_{\mathcal{K}_r}(X_U, -)$ interchanges with finite direct sums. \square

3.4 Modules with the equal c -images and equal c -kernels property

We would like to emphasize classes EIP_c and EKP_c of modules that arise in the same fashion as ESP_d and ERP_d . These modules satisfy a stronger property than the modules in EIP respectively EKP .

Recall that a module $M \in \text{mod } \mathcal{K}_r$ is in ESP_d if and only if $\ker(x_T^M) = M_2$ for each linearly independent tuple $T = (u_1, \dots, u_d)$ with $x_T^M: M \rightarrow M^d$, i.e. $\ker(x_T^M)$ is as small as possible. Obviously each module with in EKP is in EKP_d .

In this section we study under which conditions $\text{im } x_T^M$ is as large as possible, i.e. $\text{im } x_T^M = M_2 \oplus \dots \oplus M_2$. Since $\text{im } x_T^M = \text{im } x_S^M$ and $\ker y_T^M = \ker y_S^M$ for linearly independent tuples S, T which generate the same subspace $U \subseteq k^r$, we just write x_U^M and y_U^M . Throughout this section we assume that $1 \leq c < r$.

Definition. Let $1 \leq c < r$, then we define

- (a) $\text{EIP}_c := \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{c,r} : \text{im } x_U^M = M_2 \oplus \dots \oplus M_2\}$,
- (b) $\text{EKP}_c := \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{c,r} : \ker y_U^M = M_2 \oplus \dots \oplus M_2\}$.

Remark. Observe that M is in EIP_c if and only if DM is in EKP_c , $\text{EIP}_1 = \text{EIP}$ and $\text{EKP}_1 = \text{EKP}$.

The proof of Proposition 3.11 yields the following homological characterization of EIP_c .

Proposition 3.17. $\text{EIP}_c = \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{c,r} : \text{Ext}_{\mathcal{K}_r}(X_U, M) = 0\}$.

Proof. Consider the short exact sequence and $0 \rightarrow (P_1)^c \rightarrow P_2 \rightarrow X_U \rightarrow 0$. Application of $\text{Hom}_{\mathcal{K}_r}(-, M)$ yields

$$0 \rightarrow \text{Hom}_{\mathcal{K}_r}(X_U, M) \rightarrow \text{Hom}_{\mathcal{K}_r}(P_2, M) \rightarrow \text{Hom}_{\mathcal{K}_r}(P_1, M)^c \rightarrow \text{Ext}_{\mathcal{K}_r}(X_U, M) \rightarrow 0.$$

As before we conclude that $x_U^M|_{M_1}: M_1 \rightarrow M_2 \oplus \dots \oplus M_2$ is surjective if and only if $\text{Ext}_{\mathcal{K}_r}(X_U, M) = 0$. \square

We define $\mathcal{E}_c := \{M \in \text{mod } \mathcal{K}_r \mid \forall U \in \text{Gr}_{c,r} : \text{Hom}_{\mathcal{K}_r}(D\tau X_U, M) = 0\}$.

Lemma 3.18. *Let M be a \mathcal{K}_r -module.*

- (a) *The module DM is in EIP_c if and only if M is in \mathcal{E}_c .*
- (b) $\text{EKP}_c = \mathcal{E}_c$.
- (c) $\text{EIP}_{r-1} \subseteq \text{EIP}_{r-2} \subseteq \dots \subseteq \text{EIP}_2 \subseteq \text{EIP}_1$ and $\text{EKP}_{r-1} \subseteq \text{EKP}_{r-2} \subseteq \dots \subseteq \text{EKP}_2 \subseteq \text{EKP}_1$.
- (d) $\text{EKP}_c \cap \text{EIP}_d = \{0\}$ for all $1 \leq c, d \leq r-1$.

Proof. (a) By definition we have

$$\begin{aligned} DM \in \text{EIP}_c &\Leftrightarrow \forall U \in \text{Gr}_{c,r} : \text{Ext}_{\mathcal{K}_r}(X_U, DM) = 0 \\ &\Leftrightarrow \forall U \in \text{Gr}_{c,r} : \text{Hom}_{\mathcal{K}_r}(\tau^{-1}DM, X_U) = 0 \\ &\Leftrightarrow \forall U \in \text{Gr}_{c,r} : \text{Hom}_{\mathcal{K}_r}(DM, \tau X_U) = 0 \\ &\Leftrightarrow \forall U \in \text{Gr}_{c,r} : \text{Hom}_{\mathcal{K}_r}(D\tau X_U, M) = 0 \\ &\Leftrightarrow M \in \mathcal{E}_c. \end{aligned}$$

(b) By (a) it is $\mathcal{E}_c = \{DM \mid M \in \text{EIP}_c\} = \text{EKP}_c$.

(c) follows immediatly from the definition of EIP_c and (a).

(d) By [48, 3.11] we know that $\text{EIP}_d \cap \text{EKP}_c \subseteq \text{EIP}_1 \cap \text{EKP}_1 = \{0\}$. \square

Proposition 3.19. *The class $\mathcal{T}_c := \text{EIP}_c$ is the torsion class for a torsion pair $(\mathcal{T}_c, \mathcal{F}_c)$ with*

- (a) $\text{EKP}_c \subseteq \mathcal{F}_c$.
- (b) EIP_c is closed under τ and EKP_c is closed under τ^{-1} .
- (c) EKP_c contains all preprojective modules and EIP_c contains all preinjective modules.
- (d) For each regular component \mathcal{C} there exists quasi-simple modules M and W with $\text{EKP}_c \cap \mathcal{C} = (M \rightarrow)$ and $\text{EIP}_c \cap \mathcal{C} = (\rightarrow W)$.

Proof. (a) follows from [48, 3.15.] since $\text{EIP}_c \subseteq \text{EIP}_1$ and $\text{EKP}_c \subseteq \text{EKP}_1 \subseteq \mathcal{F}_1$.

(b), (c) and (d) follow from Propositions 3.3 and 3.4. □

Lemma 3.20. *Let M be in $\text{mod } \mathcal{K}_r$.*

- (a) *If M is regular and $M \notin \text{EKP}_{r-1}$, then $\tau M \notin \text{EKP}_1$.*
- (b) *If M is regular and $M \notin \text{EIP}_{r-1}$, then $\tau^{-1}M \notin \text{EIP}_1$.*

In particular, for $1 \leq c \leq d \leq r-1$ and a regular component \mathcal{C} it is

$$\text{EKP}_c \cap \mathcal{C} = \text{EKP}_d \cap \mathcal{C},$$

or there exists a quasi-simple module M in \mathcal{C} such that

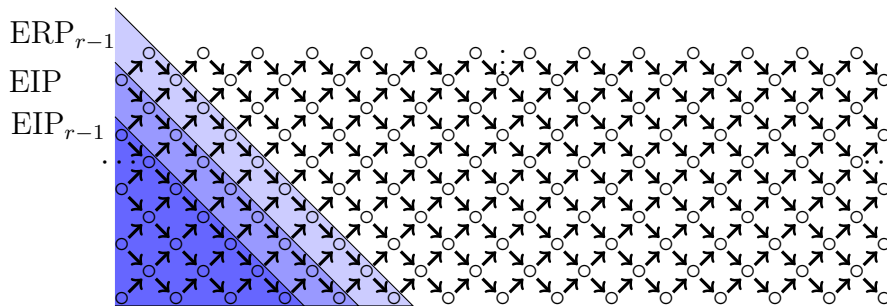
$$(\text{EKP}_c \setminus \text{EKP}_d) \cap \mathcal{C} = \{M[i] \mid i \in \mathbb{N}\}.$$

Moreover there is a most one number $1 \leq s < r-1$ such that $(\text{EKP}_s \setminus \text{EKP}_{s+1}) \cap \mathcal{C}$ is non-empty and at most one number $1 \leq t < r-1$ such that $(\text{EIP}_t \setminus \text{EIP}_{t+1}) \cap \mathcal{C}$ is non-empty.

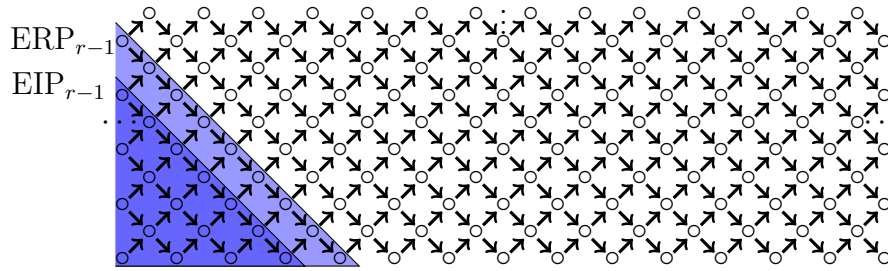
Proof. (a) Let $M \notin \text{EKP}_{r-1}$, then there exists $U \in \text{Gr}_{r-1,r}$ such that $\text{Hom}_{\mathcal{K}_r}(D\tau X_U, M) \neq 0$. Since $c = r-1$ we get $\underline{\dim} X_U = (1, r-c) = (1, 1)$. In particular, X_U is self-dual and $X_U \notin \text{EKP}_1$. We conclude $D\tau X_U = \tau^{-1}DX_U = \tau^{-1}X_U$. Hence, $0 \neq \text{Hom}_{\mathcal{K}_r}(D\tau X_U, M) \cong \text{Hom}_{\mathcal{K}_r}(X_U, \tau M)$. Since every proper factor of X_U is preinjective by 3.5, every non-zero linear map $f: X_U \rightarrow \tau M$ is injective. Since EKP_1 is closed under submodules we conclude $\tau M \notin \text{EKP}_1$. (b) follows by duality. □

Although the modules in EIP_{r-1} have more restrictive properties, the cones $\text{EIP}_{r-1} \cap \mathcal{C}$ and $\text{EIP}_1 \cap \mathcal{C}$ coincide or only differ by a ray starting in a quasi-simple module in \mathcal{C} . In Figure 3.1 we combine this result with our findings on modules in ERP_d and showcase some possible setups for \mathcal{C} .

- $\text{EIP}_{r-1} \cap \mathcal{C} \subset \text{EIP} \cap \mathcal{C} \subset \text{ERP}_{r-1} \cap \mathcal{C}$



- $\text{EIP}_{r-1} \cap \mathcal{C} \subset \text{EIP} \cap \mathcal{C} = \text{ERP}_{r-1}$



- $\text{EIP}_{r-1} \cap \mathcal{C} = \text{EIP} \cap \mathcal{C} \subset \text{ERP}_{r-1}$

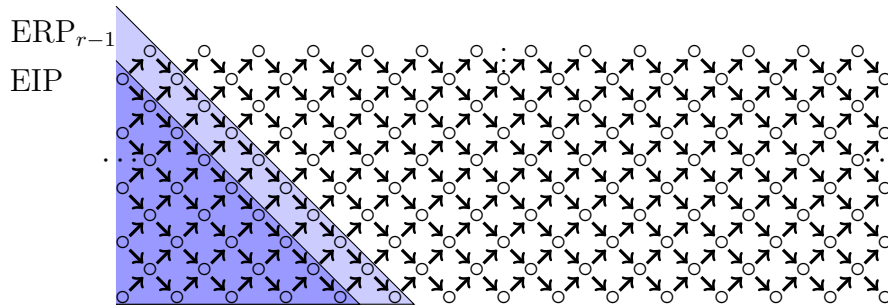


Figure 3.1: Examples of regular components.

4 Process of Simplification and Applications

4.1 Representation type

Denote by $\Lambda := kQ$ the path algebra of a connected, wild quiver Q . We stick to the notation introduced in [25]. Recall that a module M is called brick if $\text{End}_{kQ}(M) = k$ and M, N are called orthogonal if $\text{Hom}_{kQ}(M, N) = 0 = \text{Hom}_{kQ}(N, M)$.

Definition. Let \mathcal{X} be a class of pairwise orthogonal bricks in $\text{mod } \Lambda$. The full subcategory $\mathcal{E}(\mathcal{X})$ is by definition the class of all modules Y in $\text{mod } \Lambda$ with an \mathcal{X} -filtration, that is, a chain

$$0 = Y_0 \subset Y_1 \subset \dots \subset Y_{n-1} \subset Y_n = Y$$

with $Y_i/Y_{i-1} \in \mathcal{X}$ for all $1 \leq i \leq n$.

In [34, 1.] the author shows that $\mathcal{E}(\mathcal{X})$ is an exact abelian subcategory of $\text{mod } \Lambda$, closed under extensions, and \mathcal{X} is the class of all simple modules in $\mathcal{E}(\mathcal{X})$. In particular, a module M in $\mathcal{E}(\mathcal{X})$ is indecomposable if and only if it is indecomposable in $\text{mod } \mathcal{K}_r$.

Proposition 4.1. *Let $r \geq 3$ and $\mathcal{X} \subseteq \text{mod } \mathcal{K}_r$ be a class of pairwise orthogonal bricks with self-extensions (and therefore regular).*

- (a) *Every module in $\mathcal{E}(\mathcal{X})$ is regular.*
- (b) *Every regular component \mathcal{C} contains at most one module of $\mathcal{E}(\mathcal{X})$.*
- (c) *Every indecomposable module $N \in \mathcal{E}(\mathcal{X})$ is quasi-simple in $\text{mod } \mathcal{K}_r$.*
- (d) *$\mathcal{E}(\mathcal{X})$ is a wild subcategory of $\text{mod } \mathcal{K}_r$.*

Proof. (a) and (b) are proven in [25, 1.1, 1.4] for any wild quiver algebra and (c) follows by [25, 1.4] and the fact that every regular brick in $\text{mod } \mathcal{K}_r$ is quasi-simple [23, 9.2]. Let $M \in \mathcal{X}$, then $t := \dim_k \text{Ext}_{\mathcal{K}_r}(M, M) \geq 2$ by 2.19. Due to [18, 3.7] and [25, Remark 1.4] the category $\mathcal{E}(\{M\}) \subseteq \mathcal{E}(\mathcal{X})$ is equivalent to the category of finite dimensional modules over the power-series ring $k\langle\langle X_1, \dots, X_t \rangle\rangle$ in non-commuting variables X_1, \dots, X_t . Since $t \geq 2$ the category $\mathcal{E}(\{M\}) \subseteq \mathcal{E}(\mathcal{X})$ is wild by our considerations in section 2.3, and also $\mathcal{E}(\mathcal{X})$. \square

We will use the above result to prove the existence of numerous components, such that all the vertices in the components correspond to modules of constant d -socle rank. By duality, all results also follow for constant d -radical rank. As a by-product we verify the wildness of $\text{EKP} = \text{ESP}_1$ and $\text{EIP} = \text{ERP}_1$. Using the functor $\mathfrak{F}: \text{mod } \mathcal{K}_r \rightarrow \text{mod } kE_r$, we show the wildness of the corresponding full subcategories in $\text{mod}_2 kE_r$ of E_r -modules of Loewy length ≤ 2 .

4.2 Passage between \mathcal{K}_r and \mathcal{K}_s

Let $2 \leq r < s \in \mathbb{N}$. Denote by $\text{inf}_r^s: \text{mod } \mathcal{K}_r \rightarrow \text{mod } \mathcal{K}_s$ the functor that assigns to a \mathcal{K}_r -module M the module $\text{inf}_r^s(M)$ with the same underlying vector space so that the action of $e_1, e_2, \gamma_1, \dots, \gamma_r$ on $\text{inf}_r^s(M)$ stays unchanged and all other arrows act trivially on $\text{inf}_r^s(M)$. Moreover, let $\iota: \mathcal{K}_r \rightarrow \mathcal{K}_s$ be the natural k -algebra monomorphism given by $\iota(e_i) = e_i$ for $i \in \{1, 2\}$ and $\iota(\gamma_j) = \gamma_j$. Then each \mathcal{K}_s -module N becomes a \mathcal{K}_r -module N^* via pullback along ι . Denote the corresponding functor by $\text{res}_r^s: \text{mod } \mathcal{K}_s \rightarrow \text{mod } \mathcal{K}_r$. In the following r, s will be fixed, so we suppress the index and write just inf and res .

Lemma 4.2. *Let $2 \leq r < s \in \mathbb{N}$. The functor $\text{inf}: \text{mod } \mathcal{K}_r \rightarrow \text{mod } \mathcal{K}_s$ is fully faithful and exact. The essential image of inf is a subcategory of $\text{mod } \mathcal{K}_s$ closed under factors and submodules. Moreover, $\text{inf}(M)$ is indecomposable if and only if M is indecomposable in $\text{mod } \mathcal{K}_r$.*

Proof. Clearly inf is fully faithful and exact. Now let $M \in \text{mod } \mathcal{K}_r$ and $U \subseteq \text{inf}(M)$ be a submodule. Then γ_i ($i > r$) acts trivially on U and hence the pullback $\text{res}(U) =: U^p$ is a \mathcal{K}_r -module with $\text{inf}(U^p) = \text{inf} \circ \text{res}(U) = U$. Now let $f \in \text{Hom}_{\mathcal{K}_s}(\text{inf}(M), V)$ be an epimorphism. Let $v \in V$ and $m \in M$ such that $f(m) = v$. It follows $\gamma_i v = \gamma_i f(m) = f(\gamma_i m) = 0$ for all $i > r$. This shows that $\text{inf}(V^p) = \text{inf} \circ \text{res}(V) = V$.

Since inf is fully faithful, we have $\text{End}_{\mathcal{K}_s}(\text{inf}(M)) \cong \text{End}_{\mathcal{K}_r}(M)$. Hence, $\text{End}_{\mathcal{K}_s}(\text{inf}(M))$ is local if and only if $\text{End}_{\mathcal{K}_r}(M)$ is local. \square

Statement (a) of the succeeding Lemma is stated in [12, 3.1] without proof.

Lemma 4.3. *Let $2 \leq r < s$ and let M be an indecomposable \mathcal{K}_r -module that is not simple. The following statements hold.*

- (a) $\text{inf}(M)$ is regular and quasi-simple.
- (b) $\text{inf}(M) \notin \text{CSR}_m$ for all $m \in \{1, \dots, s-r\}$.

Proof. (a) Write $\underline{\dim} M = (a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$. Since M is not simple we have $ab \neq 0$ and $q_{\Gamma_r}(\underline{\dim} M) = a^2 + b^2 - rab \leq 1$. It follows $q_{\Gamma_s}(\underline{\dim} \text{inf}(M)) = a^2 + b^2 - sab = a^2 + b^2 - rab - (s-r)ab \leq 1 - (s-r)ab < 1$. Hence, $q_{\Gamma_s}(\underline{\dim} \text{inf}(M)) \leq 0$ and $\text{inf}(M)$ is regular by Theorem 2.18.

Assume that $\text{inf}(M)$ is not quasi-simple, then $\text{inf}(M) = U[i]$ for U quasi-simple with $i \geq 2$. By 4.2 we have $U[i-1] = \text{inf}(A)$ and $\tau_{\mathcal{K}_s}^{-1}U[i-1] = \text{inf}(B)$ for some A, B indecomposable in $\text{mod } \mathcal{K}_r$. Fix an irreducible monomorphism $f: \text{inf}(A) \rightarrow \text{inf}(M)$. Since inf is full, we find $g: A \rightarrow M$ with $\text{inf}(g) = f$. The faithfulness of inf implies that g is an irreducible monomorphism $g: A \rightarrow M$. By the same token there exists an irreducible epimorphism $M \rightarrow B$. As all irreducible morphisms in $\mathcal{P}(\Gamma_r)$ are injective and all irreducible morphisms in $\mathcal{I}(\Gamma_r)$ are surjective, M is located in a $\mathbb{Z}A_\infty$ component. It follows that $\tau_{\mathcal{K}_r}B = A$ in $\text{mod } \mathcal{K}_r$. Let $\underline{\dim} B = (c, d)$, then the Coxeter matrices for \mathcal{K}_r and \mathcal{K}_s yield

$$((r^2-1)c-rd, rc-d) = \underline{\dim} \tau_{\mathcal{K}_r}B = \underline{\dim} A = \underline{\dim} \text{inf}(A) = \underline{\dim} \tau_{\mathcal{K}_s} \text{inf}(B) = ((s^2-1)c-sd, sc-d).$$

This is a contradiction since $s \neq r$.

(b) Denote by $\{e_1, \dots, e_s\}$ the canonical basis of k^s . Let $1 \leq m \leq s-r$ and set $U := \langle e_{r+1}, \dots, e_{r+m} \rangle_k$. Then $\text{Soc}_U(\text{inf}(M)) = \bigcap_{i=1}^m \ker(x_{e_{r+i}}^M) = M$. Let $j \in \{1, \dots, r\}$ such that γ_j acts non-trivially on $\text{inf}(M)$. Let $V \in \text{Gr}_{m,s}$ such that $e_j \in V$. Then $\text{Soc}_V(\text{inf}(M)) \neq M$ and $\text{inf}(M)$ does not have constant m -socle rank. \square

Proposition 4.4. *Let $2 \leq r < s \in \mathbb{N}$, $1 \leq d < r$ and M be an indecomposable and non-simple \mathcal{K}_r -module. Then the following statements hold.*

- (a) *If $M \in {}^\perp \mathfrak{X}_{d,r}$ then $\inf(M) \in {}^\perp \mathfrak{X}_{d+s-r,s}$.*
- (b) *If $M \in \mathfrak{X}_{d,r}^\perp$ then $\inf(M) \in \mathfrak{X}_{d+s-r,s}^\perp$.*
- (c) *If $M \in \overline{\mathfrak{X}}_{d,r}$ then $\inf(M)$ is contained in a regular component \mathcal{C} with $\mathcal{C} \subseteq \text{CSR}_{d+s-r}$.*

Proof. By definition it is $1 \leq d + s - r < s$. Now fix $V \in \text{Gr}_{d+s-r,s}$ and note that $\underline{\dim} X_V = (1, s - (d + s - r)) = (1, r - d)$, which is the dimension vector of every \mathcal{K}_r -module X_U for $U \in \text{Gr}_{d,r}$.

(a) Assume that $\text{Hom}_{\mathcal{K}_s}(\inf(M), X_V) \neq 0$ and let $0 \neq f: \inf(M) \rightarrow X_V$. By 4.2 and 3.5 the \mathcal{K}_s -module $\inf(M)$ is regular and every proper submodule of X_V is preprojective. Hence, f is surjective onto X_V . Again 4.2 yields $Z \in \text{mod } \mathcal{K}_r$ indecomposable with $\underline{\dim} Z = (1, r - d) = \underline{\dim} X_V$ such that $X_V = \inf(Z)$. By 3.7 there exists $U \in \text{Gr}_{d,r}$ with $Z = X_U$. Since \inf is fully faithful it follows $0 = \text{Hom}_{\mathcal{K}_r}(M, X_U) \cong \text{Hom}_{\mathcal{K}_s}(\inf(M), \inf(X_U)) = \text{Hom}_{\mathcal{K}_s}(\inf(M), X_V) \neq 0$, a contradiction.

(b) Assume that $\text{Hom}_{\mathcal{K}_s}(X_V, \inf(M)) \neq 0$ and let $f: X_V \rightarrow \inf(M)$ be non-zero. Since $\inf(M)$ is regular indecomposable the module $\text{im } f \subseteq \inf(M)$ is not injective and 3.8 yields that $\text{im } f$ is indecomposable and regular. As $\text{im } f$ is a submodule of $\inf(M)$ there exists an indecomposable module $Z \in \text{mod } \mathcal{K}_r$ with $\inf(Z) = \text{im } f$. Since $\text{im } f$ is not simple we have $\underline{\dim} \text{im } f = (1, r - c)$ for $1 \leq r - c \leq r - d$. Hence, $Z = X_U$ for $U \in \text{Gr}_{c,r}$ and by 3.7(d) there exists $W \in \text{Gr}_{d,r}$ and an epimorphism $\pi: X_W \rightarrow X_U$. We conclude with $0 \neq \text{Hom}_{\mathcal{K}_s}(\text{im } f, \inf(M)) = \text{Hom}_{\mathcal{K}_s}(\inf(X_U), \inf(M)) \cong \text{Hom}_{\mathcal{K}_r}(X_U, M)$ and the surjectivity of $\pi: X_W \rightarrow X_U$ that $\text{Hom}_{\mathcal{K}_r}(X_W, M) \neq 0$, a contradiction to the assumption.

(c) By 4.3 the module $\inf(M)$ is quasi-simple in a regular component and satisfies the conditions of 3.16 for $q := d + s - r$ by (a) and (b). \square

Examples. The following two examples will be helpful later on.

- (1) Let $r = 3$. By [36, 4.1], the representation $F = (k^2, k^2, F(\gamma_1), F(\gamma_2), F(\gamma_3))$ with the linear maps $F(\gamma_1) = id_{k^2}$, $F(\gamma_2)(a, b) = (b, 0)$ and $F(\gamma_3)(a, b) = (0, a)$ is elementary. Let E be the corresponding \mathcal{K}_3 -module. Then $\underline{\dim} E = (2, 2)$ and it is easy to see that every indecomposable submodule of E has dimension vector $(0, 1)$ or $(1, 2)$. In particular, $\text{Hom}_{\mathcal{K}_3}(W, E) = 0$ for each indecomposable module with dimension vector $\underline{\dim} W = (1, 1)$. Assume now that $f: E \rightarrow W$ is non-zero, then f is surjective since every proper submodule of W is projective. Since E is elementary, $\ker f$ is a preprojective module with dimension vector $(1, 1)$, a contradiction. Hence, $E \in \overline{\mathfrak{X}}_{2,3}$.
- (2) Let \mathcal{C} be a regular component and $M_{\mathcal{C}}, W_{\mathcal{C}}$ in \mathcal{C} the quasi-simple modules with $\text{ERP}_1 \cap \mathcal{C} = (\rightarrow W_{\mathcal{C}})$ and $\text{ESP}_1 \cap \mathcal{C} = (M_{\mathcal{C}} \rightarrow)$. The width $\mathcal{W}(\mathcal{C})$ of \mathcal{C} is by definition the unique natural number satisfying $\tau^{\mathcal{W}(\mathcal{C})+1} M_{\mathcal{C}} = W_{\mathcal{C}}$. Since $X_U \cong D\tau X_U$ for $U \in \text{Gr}_{1,r}$ we conclude for an arbitrary regular component \mathcal{C} :

$$\begin{aligned} \mathcal{W}(\mathcal{C}) = 0 &\Leftrightarrow \tau M_{\mathcal{C}} = W_{\mathcal{C}} \Leftrightarrow \tau M_{\mathcal{C}} \in \text{EIP} \\ &\Leftrightarrow \text{Hom}_{\mathcal{K}_r}(X_U, M_{\mathcal{C}}) = 0 = \text{Ext}_{\mathcal{K}_r}(X_U, \tau M_{\mathcal{C}}) \text{ for all } U \in \text{Gr}_{1,r} \\ &\Leftrightarrow \text{Hom}_{\mathcal{K}_r}(X_U, M_{\mathcal{C}}) = 0 = \text{Hom}_{\mathcal{K}_r}(M_{\mathcal{C}}, X_U) \text{ for all } U \in \text{Gr}_{1,r} \Leftrightarrow M_{\mathcal{C}} \in \overline{\mathfrak{X}}_{1,r}. \end{aligned}$$

Lemma 4.5. *Let $s \geq 3$ and $2 \leq d < s$. Then there exists a regular module E_d with the following properties.*

- (a) *E_d is a (quasi-simple) brick in $\text{mod } \mathcal{K}_s$.*
- (b) *$E_d \in \overline{\mathfrak{X}}_{d,s}$.*

(c) *There exist $V, W \in \text{Gr}_{1,s}$ with $\text{Hom}_{\mathcal{K}_s}(X_V, E_d) = 0 \neq \text{Hom}_{\mathcal{K}_s}(X_W, E_d)$.*

Proof. We start by considering $s = 3$ and $d = 2$. Pick the elementary module $E_d := E$ from the preceding example. The module E is a brick and $E \in \overline{\mathfrak{X}}_{d,s}$. Set $V = \langle \alpha := (1, 0, 0) \rangle_k$, $W = \langle \beta := (0, 1, 0) \rangle_k$. By the definition of E we have

$$\dim_k \ker x_\alpha^E = 2 \neq 3 = \dim_k \ker x_\beta^E,$$

and therefore

$$\dim_k \text{Hom}_{\mathcal{K}_s}(X_V, E_d) = 0 \neq 1 = \dim_k \text{Hom}_{\mathcal{K}_s}(X_W, E_d).$$

Now let $s > 3$. If $d = s - 1$ consider $E_d := \text{inf}_3^s(E)$. In view of 4.4 we have $E_d \in \overline{\mathfrak{X}}_{2+s-3,s} = \overline{\mathfrak{X}}_{d,s}$. Moreover, $\text{inf}(E)$ is a brick in $\text{mod } \mathcal{K}_s$ and for the canonical basis vectors $e_1, e_2 \in k^s$ and $V = \langle e_1 \rangle_k$, $W := \langle e_2 \rangle_k$ we get as before

$$\dim_k \text{Hom}_{\mathcal{K}_s}(X_V, \text{inf}(E)) = 0 \neq 1 = \text{Hom}_{\mathcal{K}_s}(X_W, \text{inf}(E)).$$

Now let $1 < d < s - 1$. Set $r := 1 + s - d \geq 3$, consider a regular component for \mathcal{K}_r with $\mathcal{W}(\mathcal{C}) = 0$ such that $M_{\mathcal{C}}$ is a brick (see the example after Theorem 4.5 in [49] and [49, 3.22]) and set $M := M_{\mathcal{C}}$. Then we have $M \in \overline{\mathfrak{X}}_{1,r}$ and 4.4 yields $E_d := \text{inf}(M) \in \overline{\mathfrak{X}}_{1+s-(1+s-d),s} = \overline{\mathfrak{X}}_{d,s}$. Since M is a brick, $\text{inf}(M)$ is a brick in $\text{mod } \mathcal{K}_s$. Recall that $\text{Hom}_{\mathcal{K}_r}(X_U, M) = 0$ for all $U \in \text{Gr}_{d,r}$ implies that, viewing M as a representation, the linear map $M(\gamma_1): M_1 \rightarrow M_2$ corresponding to γ_1 is injective. Since the map is not affected by inf , $\text{inf}(M)(\gamma_1): M_1 \rightarrow M_2$ is also injective. Therefore we conclude for the first basis vector $e_1 \in k^s$ and $V := \langle e_1 \rangle_k$ that $0 = \text{Hom}_{\mathcal{K}_s}(X_V, \text{inf}(M))$. By 4.3 we find $W \in \text{Gr}_{1,s}$ with $0 \neq \text{Hom}_{\mathcal{K}_s}(X_W, \text{inf}(M))$. \square

4.3 Numerous components lying in CSR_d .

In this section we use the Simplification method to construct a family of regular components, such that every vertex in such a regular component corresponds to a module in CSR_d . By the next result it follows that $\mathcal{X} \subseteq \overline{\mathfrak{X}}_{d,r}$ implies $\mathcal{E}(\mathcal{X}) \subseteq \overline{\mathfrak{X}}_{d,r}$.

Lemma 4.6. [25, 1.9] *Let Λ be a finite dimensional k -algebra and X, Y Λ -modules with $\text{Hom}_\Lambda(X, Y)$ non-zero. If X and Y have filtrations $X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = 0$, $Y = Y_0 \supset Y_1 \supset \cdots \supset Y_s \supset Y_{s+1} = 0$, then there are i, j with $\text{Hom}_\Lambda(X_i/X_{i+1}, Y_j/Y_{j+1}) \neq 0$.*

For a regular module $M \in \mathcal{K}_r$ denote by \mathcal{C}_M the regular component that contains M .

Proposition 4.7. *Let $1 \leq d < r$ and \mathcal{X} be a family of pairwise orthogonal bricks in $\overline{\mathfrak{X}}_{d,r}$. Then*

$$\varphi: \text{ind } \mathcal{E}(\mathcal{X}) \rightarrow \mathcal{R}; M \mapsto \mathcal{C}_M$$

is an injective map such that for each component \mathcal{C} in $\text{im } \varphi$ we have $\mathcal{C} \subseteq \text{CSR}_d$. Here $\text{ind } \mathcal{E}(\mathcal{X})$ denotes the category of a chosen set of representatives of non-isomorphic indecomposable objects of $\text{mod } \mathcal{K}_r$ in \mathcal{E} .

Proof. Since each module in $\overline{\mathfrak{X}}_{d,r}$ is regular, 4.1 implies that every module $N \in \text{ind } \mathcal{E}(\mathcal{X})$ is contained in a regular component \mathcal{C}_M and is quasi-simple. By 4.6 the module N satisfies $\text{Hom}_{\mathcal{K}_r}(X_U, N) = 0 = \text{Hom}_{\mathcal{K}_r}(N, X_U)$ for all $U \in \text{Gr}_{d,r}$. But now 3.16 implies that every module in \mathcal{C}_M has constant d -socle rank. The injectivity of φ follows immediately from 4.1. \square

Corollary 4.8. *There exists an infinite set Ω of regular components such that for all $\mathcal{C} \in \Omega$*

- (a) $\mathcal{W}(\mathcal{C}) = 0$, in particular every module in \mathcal{C} has constant rank and
- (b) \mathcal{C} does not contain any bricks.

Proof. Let \mathcal{C} be a regular component that contains a brick and $\mathcal{W}(\mathcal{C}) = 0$ (see the example after Theorem 4.5 in [49] and [49, 3.22]). Let $M := M_{\mathcal{C}}$, then $M \in \overline{\mathfrak{X}}_{1,r}$ by the previous example. Apply 4.7 with $\mathcal{X} = \{M\}$ and set $\Omega := \text{im } \varphi \setminus \{\mathcal{C}_M\}$. Let $N \in \mathcal{E}(\mathcal{X}) \setminus \{M\}$ be indecomposable, then N is quasi-simple in \mathcal{C}_N by 4.1 and has a $\{M\}$ -filtration $0 = N_0 \subset \dots \subset N_l = N$ with $l \geq 2$ and $N_1 = M = N_l/N_{l-1}$. Hence,

$$N \rightarrow N_l/N_{l-1} \rightarrow N_1 \rightarrow N$$

is a non-zero homomorphism that is not injective. Therefore N is not a brick. This finishes the proof, since every regular brick in $\text{mod } \mathcal{K}_r$ is quasi-simple [23, 9.2] and $\text{End}_{\mathcal{K}_r}(\tau^l N) \cong \text{End}_{\mathcal{K}_r}(N) \neq k$ for all $l \in \mathbb{Z}$. \square

Now we apply our results on the Simplification method to modules E_d constructed in 4.5.

Definition. [49, 2.10] Denote with $\text{GL}_r(k)$ the group of invertible $r \times r$ -matrices with coefficients in k which acts on $\bigoplus_{i=1}^r k\gamma_i$ via $A \cdot \gamma_j = \sum_{i=1}^r a_{ij} \gamma_i$ for $1 \leq j \leq r$, $A \in \text{GL}_r(k)$. For $A \in \text{GL}_r(k)$, let $\varphi_A: \mathcal{K}_r \rightarrow \mathcal{K}_r$ the algebra homomorphism with $\varphi_A(e_1) = e_1$, $\varphi_A(e_2) = e_2$ and $\varphi_A(\gamma_i) = A \cdot \gamma_i$, $1 \leq i \leq r$. For a \mathcal{K}_r -module M denote the pullback of M along $\varphi_{A^{-1}}$ by $A \cdot M$. The module M is called $\text{GL}_r(k)$ -stable if $A \cdot M \cong M$ for all $A \in \text{GL}_r(k)$, in other words if $\text{GL}_r = \text{GL}_r(k)_M := \{A \in \text{GL}_r(k) \mid A \cdot M \cong M\}$.

Theorem 4.9. *Let $2 \leq d < r$, then there exists a full wild subcategory $\mathcal{E} \subseteq \text{mod } \mathcal{K}_r$ and an injection*

$$\varphi_d: \text{ind } \mathcal{E} \rightarrow \mathcal{R}; M \mapsto \mathcal{C}_M,$$

such that for each component \mathcal{C} in $\text{im } \varphi_d$ we have $\mathcal{C} \subseteq \text{CSR}_d$ and no module in \mathcal{C} is $\text{GL}_r(k)$ -stable.

Proof. Fix $2 \leq d < r$ and let E_d as in 4.5 with $V, W \in \text{Gr}_{1,r}$ and $\text{Hom}_{\mathcal{K}_r}(X_V, E_d) = 0 \neq \text{Hom}_{\mathcal{K}_r}(X_W, E_d)$. Set $\mathcal{X} := \{E_d\}$ and let $M \in \mathcal{E}(\mathcal{X})$. By 4.7 we get an injective map

$$\varphi_d: \text{ind } \mathcal{E}(\mathcal{X}) \rightarrow \mathcal{R}; M \rightarrow \mathcal{C}_M$$

such that each component \mathcal{C} in $\text{im } \varphi_d$ satisfies $\mathcal{C} \subseteq \text{CSR}_d$.

Moreover, $\mathcal{E}(\mathcal{X})$ is a wild full subcategory of $\text{mod } \mathcal{K}_r$ by 4.1. Let $M \in \mathcal{E}(\mathcal{X})$ be indecomposable. Then M has a filtration $0 = Y_0 \subset Y_1 \subset \dots \subset Y_m$ with $Y_l/Y_{l-1} = E_d$ for all $1 \leq l \leq m$. By 4.6 we have $0 = \text{Hom}_{\mathcal{K}_r}(X_U, M)$ and since $E_d = Y_1 \subseteq M$, we conclude $0 \neq \text{Hom}_{\mathcal{K}_r}(X_W, M)$. This proves that M does not have constant 1-socle rank. Therefore \mathcal{C}_M contains a module that is not of constant 1-socle rank. By [9, 3.6] the module M is not $\text{GL}_r(k)$ -stable and by [16, 2.2] no module in the component is $\text{GL}_r(k)$ -stable. \square

4.4 Components lying almost completely in CSR_d

The following definition and two lemmas are a generalization of [49, 4.7, 4.13] and [48, 3.7]. We omit the proofs.

Definition. Let M be an indecomposable \mathcal{K}_r -module, $1 \leq d < r$ and $U \in \text{Gr}_{d,r}$. The module M is called U -trivial if $\dim_k \text{Hom}_{\mathcal{K}_r}(X_U, M) = \dim_k M_1$.

Note that the sequence $0 \rightarrow P_1^{r-d} \rightarrow P_2 \rightarrow X_U \rightarrow 0$ and left-exactness of $\text{Hom}_{\mathcal{K}_r}(-, M)$ imply that $\dim_k \text{Hom}_{\mathcal{K}_r}(X_U, M) \leq \dim_k M_1$.

Lemma 4.10. *Let M be a regular U -trivial module. Then either $\text{Ext}_{\mathcal{K}_r}(X_V, \tau M) = 0 = \text{Hom}_{\mathcal{K}_r}(X_V, \tau^{-1}M)$ for all $V \in \text{Gr}_{d,r}$ or M is isomorphic to X_U or τX_U .*

Lemma 4.11. *Let M be regular quasi-simple in a regular component \mathcal{C} such that $\text{Ext}_{\mathcal{K}_r}(X_U, \tau M) = 0 = \text{Hom}_{\mathcal{K}_r}(X_U, \tau^{-1}M)$ for all $U \in \text{Gr}_{d,r}$. If M does not have constant d -socle rank, then a module X in \mathcal{C} has constant d -socle rank if and only if X is in $(\rightarrow \tau M) \cup (\tau^{-1}M \rightarrow)$.*

Corollary 4.12. *Let $3 \leq r < s$, $d := 1 + s - r$ and $1 \leq l \leq s - r$. Let M be an indecomposable \mathcal{K}_r -module in $\overline{\mathfrak{X}}_{d,r}$ that is not elementary. Denote by \mathcal{C} the regular component that contains $\text{inf}(M)$.*

- (a) *Every module in \mathcal{C} has constant d -socle rank and*
- (b) *$N \in \mathcal{C}$ has constant l -socle rank if and only if $N \in (\rightarrow \tau \text{inf}(M)) \cup (\tau^{-1} \text{inf}(M) \rightarrow)$.*

Proof. (a) is an immediate consequence of Proposition 4.4.

(b) Consider the indecomposable projective module $P_2 = \mathcal{K}_r e_1$ in $\text{mod } \mathcal{K}_r$. We get

$$\text{Hom}_{\mathcal{K}_s}(\text{inf}(P_2), \text{inf}(M)) \cong \text{Hom}_{\mathcal{K}_r}(P_2, M) = \dim_k M_1 = \dim_k \text{inf}(M)_1 =: q.$$

Since $\underline{\dim} \text{inf}(P_2) = (1, r) = (1, s - (s - r))$ we find $W \in \text{Gr}_{s-r,s}$ with $\text{inf}(P_2) = X_W$. Now let $1 \leq l \leq s - r$. By Proposition 3.7 there is $U \in \text{Gr}_{l,s}$ and an epimorphism $\pi: X_U \rightarrow X_W$. Let $\{f_1, \dots, f_q\}$ be a basis of $\text{Hom}_{\mathcal{K}_s}(\text{inf}(P_2), \text{inf}(M))$. Since π is surjective the set $\{f_1\pi, \dots, f_q\pi\} \subseteq \text{Hom}_{\mathcal{K}_s}(X_U, \text{inf}(M))$ is linearly independent. Hence $q \leq \dim_k \text{Hom}_{\mathcal{K}_s}(X_U, \text{inf}(M))$. Since $\text{Hom}_{\mathcal{K}_s}(X_U, \text{inf}(M)) \leq \dim_k \text{inf}(M)_1 = q$ the module $\text{inf}(M)$ is U -trivial.

Since M is not elementary, $\text{inf}(M)$ is not elementary and therefore not isomorphic to X_U or τX_U . Now 4.10 yields that $\text{Ext}_{\mathcal{K}_s}(X_W, \tau_{\mathcal{K}_r} \text{inf}(M)) = 0 = \text{Hom}_{\mathcal{K}_s}(X_W, \tau_{\mathcal{K}_r}^{-1} \text{inf}(M))$ for all $W \in \text{Gr}_{l,s}$. By 4.3 the module $\text{inf}(M)$ does not have the constant l -socle rank for $1 \leq l \leq s - r$. Note that M is in regular and therefore $\text{inf}(M)$ is a quasi-simple module. Now apply Lemma 4.11. \square

Example. Let $r \geq 3$ and \mathcal{C} be a regular component with $\mathcal{W}(\mathcal{C}) = 0$ such that $M_{\mathcal{C}}$ is not a brick (see 4.8) and in particular not elementary. Then $M_{\mathcal{C}} \in \overline{\mathfrak{X}}_{1,r}$ and we can apply 4.12. Figure 4.1 shows the regular component \mathcal{D} of \mathcal{K}_s containing $\text{inf}(M_{\mathcal{C}})$. Every module in \mathcal{D} has constant $d := 1 + s - r$ socle rank. But for $1 \leq q \leq r - s$, a module in this component has constant q -socle rank if and only if it lies in the shaded region.

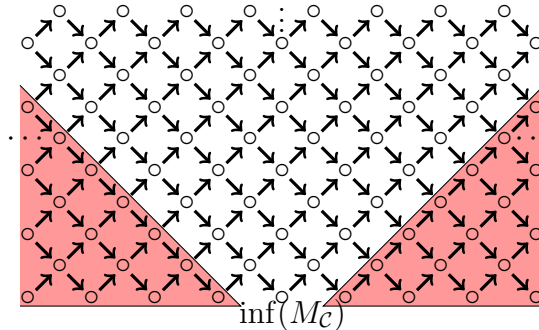


Figure 4.1: Regular component containing $\text{inf}(M_{\mathcal{C}})$.

5 Wild representation type

5.1 Wildness of strata

As another application of the Simplification method and the inflation functor $\text{inf}_r^s: \text{mod } \mathcal{K}_r \rightarrow \text{mod } \mathcal{K}_s$ we get the following result.

Theorem 5.1. *Let $s \geq 3$ and $1 \leq d \leq s - 1$. Then $\Delta_d = \text{ESP}_d \setminus \text{ESP}_{d-1} \subseteq \text{mod } \mathcal{K}_s$ is a wild subcategory, where $\text{ESP}_0 := \emptyset$.*

Proof. For $d = 1$ consider a regular component \mathcal{C} for \mathcal{K}_s that contains a brick F . By Proposition 3.13 we find a module E in the τ -orbit of F that is in ESP_1 and set $\mathcal{X} := \{E\}$. Then E is brick since $\text{End}_{\mathcal{K}_s}(E, E) \cong \text{End}_{\mathcal{K}_s}(F, F) = k$ and $\dim_k \text{Ext}_{\mathcal{K}_s}(E, E) \geq 2$ by Corollary 2.19. Therefore $\mathcal{E}(\mathcal{X})$ is wild category (see 4.1). As ESP_1 is closed under extensions, it follows $\mathcal{E}(\mathcal{X}) \subseteq \text{ESP}_1$. Note that this case does only require the application of 4.1.

Now let $d > 1$ and $r := s - d + 1 \geq 2$. Consider the projective indecomposable \mathcal{K}_r -module $P := P_2$ with $\underline{\dim} P = (1, r)$. By 4.3 and 2.19 $\text{inf}(P)$ is a regular quasi-simple module in $\text{mod } \mathcal{K}_s$ with $\dim_k \text{Ext}_{\mathcal{K}_s}(\text{inf}(P), \text{inf}(P)) \geq 2$. Since P is in ESP_1 , we have $0 = \text{Hom}_{\mathcal{K}_r}(X_U, P)$ for all $U \in \text{Gr}_{1,r}$. Hence, Proposition 4.4 implies $0 = \text{Hom}_{\mathcal{K}_s}(X_U, \text{inf}(P))$ for all $U \in \text{Gr}_{1+s-r,s} = \text{Gr}_{d,s}$, so that $\text{inf}(P)$ is in ESP_d .

Let $\mathcal{X} := \{\text{inf}(P)\}$, then $\mathcal{E}(\mathcal{X})$ is a wild category and since ESP_d is extension closed it follows $\mathcal{E}(\mathcal{X}) \subseteq \text{ESP}_d$. Since $\underline{\dim} \text{inf}(P) = (1, s - d + 1)$ we find $V \in \text{Gr}_{d-1,s}$ (see 3.7) with $\text{inf}(P) = X_V$ and $0 \neq \text{End}_{\mathcal{K}_s}(\text{inf}(P)) = \text{Hom}_{\mathcal{K}_s}(X_V, \text{inf}(P))$. That means $\text{inf}(P) \notin \text{ESP}_{d-1}$. Since ESP_{d-1} is closed under submodules we have $\mathcal{E}(\mathcal{X}) \cap \text{ESP}_{d-1} = \emptyset$. Hence, $\mathcal{E}(\mathcal{X}) \subseteq \text{ESP}_d \setminus \text{ESP}_{d-1}$. \square

Corollary 5.2. *Let $s \geq 3$ and $1 \leq d \leq s - 1$. Then $\nabla_d = \text{ERP}_d \setminus \text{ERP}_{d-1} \subseteq \text{mod } \mathcal{K}_s$ is a wild subcategory, where $\text{ERP}_0 := \emptyset$.*

Proof. Apply the duality D on the wild category Δ_d . \square

Remarks. 1. Note that all indecomposable modules in the wild category $\mathcal{E}(\mathcal{X})$ are quasi-simple in $\text{mod } \mathcal{K}_s$ and $\mathcal{E}(\mathcal{X}) \subseteq \text{ESP}_d \setminus \text{ESP}_{d-1}$.

2. From the definitions we get a chain of proper inclusions $\text{ESP}_{r-1} \supset \text{ESP}_{r-2} \supset \cdots \supset \text{ESP}_1 = \text{EKP}_1 \supset \text{EKP}_2 \supset \cdots \supset \text{EKP}_{r-1}$. By adapting the first part of the preceding proof it follows that the smallest subcategory EKP_{r-1} is wild.

Proposition 5.3. *Let $r = 2$, $B := B(3, r)$ the Beilinson algebra and $\text{EKP}(3, 2)$ the full subcategory of modules in $\text{mod } B(3, 2)$ with the equal kernels property (see [48, 2.1,3.12]). The category $\text{EKP}(3, 2)$ is of wild representation type.*

Proof. Consider the path algebra A of the extended Kronecker quiver $Q = 1 \rightarrow 2 \rightrightarrows 3$. Since the underlying graph of Q is not a Dynkin or Euclidean diagram, the algebra A is of wild

representation type. It is known that there exists a preprojective tilting module T in $\text{mod } A$ with $\text{End}_A(T) \cong B(3, 2)$, see for example [45] or [47, 4.]. We sketch the construction. The start of the preprojective component of A is illustrated in Figure 5.1 and the direct summands of T are marked with a dot.

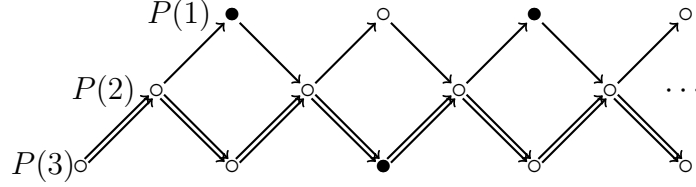


Figure 5.1: Start of the preprojective component of A .

One can check that $T = P(1) \oplus \tau_A^{-2}P(1) \oplus \tau_A^{-2}P(3)$ is a tilting module. Since preprojective components are standard [35, 2.4.11], one can show that $\text{End}_A(T)$ is given by the quiver in Figure 5.2, bound by the relation $\alpha_2\alpha_1 + \beta_2\beta_1$. Moreover, it follows from the description as a quiver with relations that $\text{End}_A(T) \cong B(3, 2)$.

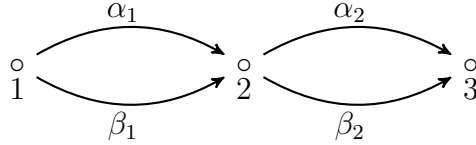


Figure 5.2: Ordinary quiver of $\text{End}_A(T)$.

Since A is hereditary, the algebra $B(3, 2)$ is a concealed algebra [1, VIII.4.6]. By [3, XVIII.5.0,5.1] the functor $\text{Hom}_A(T, -): \text{mod } A \rightarrow \text{mod } B$ induces an equivalence G between the regular categories $\text{add } \mathcal{R}(A)$ and $\text{add } \mathcal{R}(B)$ and we have an isomorphism between the two Grothendieck groups $f: K_0(A) \rightarrow K_0(B)$ with $\underline{\dim} G(M) = \underline{\dim} \text{Hom}_A(T, M) = f(\underline{\dim} M)$ for all $M \in \text{mod } A$. Now we will make use of the homological characterization [48, 2.5]

$$\text{EKP}(3, 2) = \{M \in \text{mod } B(3, 2) \mid \forall \alpha \in k^r \setminus \{0\} : \text{Hom}_B(X_\alpha^1, M) = 0\}.$$

At first note that each X_α^1 is regular; since $X_\alpha^1 \neq 0$, we have $\dim_k \text{Hom}_B(X_\alpha^1, X_\alpha^1) \neq 0$ and $0 \neq \dim_k \text{Hom}_B(X_\alpha^1, X_\alpha^1) = \dim_k \text{Ext}_B^1(X_\alpha^1, \tau_B X_\alpha^1)$ [49, 3.15], therefore X_α^1 is not in $\text{EIP}(3, 2) \cup \text{EKP}(3, 2)$ and by [48, 2.7] regular. Moreover, $\underline{\dim} X_\alpha^1$ is independent of α . Hence, we find for each $\beta \in k^r \setminus \{0\}$ a regular indecomposable module U_β in $\text{mod } A$ with $G(U_\beta) = X_\beta^1$ and $\underline{\dim} U_\beta = \underline{\dim} U_\alpha$ for all $\alpha \in k^r \setminus \{0\}$. Now let M be in $\text{mod } A$ a regular brick with $\dim_k \text{Ext}_A(M, M) \geq 2$ (see [25, 5.1]). By the dual version of [23, 4.6] we find $l \in \mathbb{N}_0$ with $\text{Hom}_A(U_\alpha, \tau_A^{-l}M) = 0$ for all $\alpha \in k^r \setminus \{0\}$. Set $N := \tau_A^{-l}M$ and $\mathcal{X} := \{N\}$. The module N is a regular brick and the AR formula yields $\dim_k \text{Ext}_A(N, N) = \dim_k \text{Ext}_A(M, M) \geq 2$ and therefore $\mathcal{E}(\mathcal{X})$ is a wild category in $\text{add } \mathcal{R}(A)$ [25, 1.4]. By Lemma 4.6 we have $\text{Hom}_A(U_\alpha, L) = 0$ for all L in $\mathcal{E}(\mathcal{X})$ and all $\alpha \in k^r \setminus \{0\}$. Hence, $0 = \text{Hom}_A(U_\alpha, L) = \text{Hom}_B(G(U_\alpha), G(L)) = \text{Hom}_B(X_\alpha^1, G(L))$ for all $\alpha \in k^r \setminus \{0\}$. This shows that the essential image of $\mathcal{E}(\mathcal{X})$ under G is a wild subcategory contained in $\text{EKP}(3, 2)$. \square

Proposition 5.4. *Let $r = 2$, $B := B(3, r)$ the Beilinson algebra and $\text{EIP}(3, 2)$ the full subcategory of modules in $\text{mod } B(3, 2)$ with the equal images property (see [48, 2.1,3.12]). The category $\text{EIP}(3, 2)$ is of wild representation type.*

Proof. The duality on $\text{mod } B(3, r)$ defined in [49] sends $\text{EIP}(3, 2)$ to $\text{EKP}(3, 2)$ (see [49, 3.15]). Hence the claim follows from Proposition 5.3. \square

5.2 $\text{ESP}_2 \cap \text{ERP}_2$ is wild

We have seen that EIP is a torsion class and EKP is contained in the corresponding torsion free class. In particular, EIP and EKP intersect trivially. In the following we show that one can not hope to extend this result to ERP_d and ESP_d .

The following example also shows that there exist numerous components of width > 0 such that every module in such a component has constant rank.

Let us mention that this example is inspired by a representation of dimension 10 for kE_3 given in [5, 5.1.4]. We consider the case $r = 3$ and the following matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let M be the 6-dimensional representation with $\underline{\dim} M = (3, 3)$ and $M(\gamma_i)(x) = A_i x$ for $1 \leq i \leq 3$. Let $\alpha = (a, b, c) \in k^3 \setminus \{0\}$ and $M^\alpha := aA_1 + bA_2 + cA_3$. We distinguish the following two cases:

- $a = 0$, then $b \neq 0$ or $c \neq 0$ and $\text{rk}(M^\alpha) = \text{rk} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{pmatrix} = 2$ and $\ker(M^\alpha) = k(c, -b, 0) = k(c, -b, a)$.
- $a \neq 0$, then

$$\begin{aligned} \text{rk}(M^\alpha) &= \text{rk} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} a & 0 & -c \\ b & c & 0 \\ 0 & a & b \end{pmatrix} = \text{rk} \begin{pmatrix} a & 0 & -c \\ 0 & c & c\frac{b}{a} \\ 0 & a & b \end{pmatrix} \\ &= \text{rk} \begin{pmatrix} a & 0 & -c \\ 0 & a & b \\ 0 & c & c\frac{b}{a} \end{pmatrix} = \text{rk} \begin{pmatrix} a & 0 & -c \\ 0 & a & b \\ 0 & 0 & c\frac{b}{a} - b\frac{c}{a} \end{pmatrix} = \text{rk} \begin{pmatrix} a & 0 & -c \\ 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} = 2 \end{aligned}$$

and $\ker(M^\alpha) = k(c, -b, a)$.

Hence, M has constant rank but is not in $\text{EKP} \cup \text{EIP}$. Moreover, we see that if $\beta = (d, e, f)$ is in k^3 with (α, β) linearly independent, then $\ker(M^\alpha) \cap \ker(M^\beta) = \{0\}$. Hence M is in ESP_2 . Let (e_1, e_2, e_3) be the canonical basis of k^3 and $\varphi: k^3 \rightarrow \text{Hom}_k(k^3, k)$ be the isomorphism given by $\varphi(e_i)(e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. We claim that $D_{\Gamma_3}M$ is isomorphic to M . Recall that $D_{\Gamma_3}M$ is the representation

$$D_{\Gamma_3}M = (\text{Hom}_k(k^3, k), \text{Hom}_k(k^3, k), M(\gamma_1)^*, M(\gamma_2)^*, M(\gamma_3)^*).$$

We show that $\Phi = (-\varphi, \varphi)$ is a morphism from M to $D_{\Gamma_3}M$. We have

$$\varphi \circ M(\gamma_1)(e_j) = \begin{cases} -\varphi(e_2), & j = 1 \\ \varphi(e_1), & j = 2 \\ 0, & j = 3 \end{cases}$$

and

$$-M(\gamma_1)^* \circ \varphi(e_j) = \begin{cases} -\varphi(e_1) \circ M(\gamma_1) = -\varphi(e_2), & j = 1 \\ -\varphi(e_2) \circ M(\gamma_1) = \varphi(e_1), & j = 2 \\ 0, & j = 3. \end{cases}$$

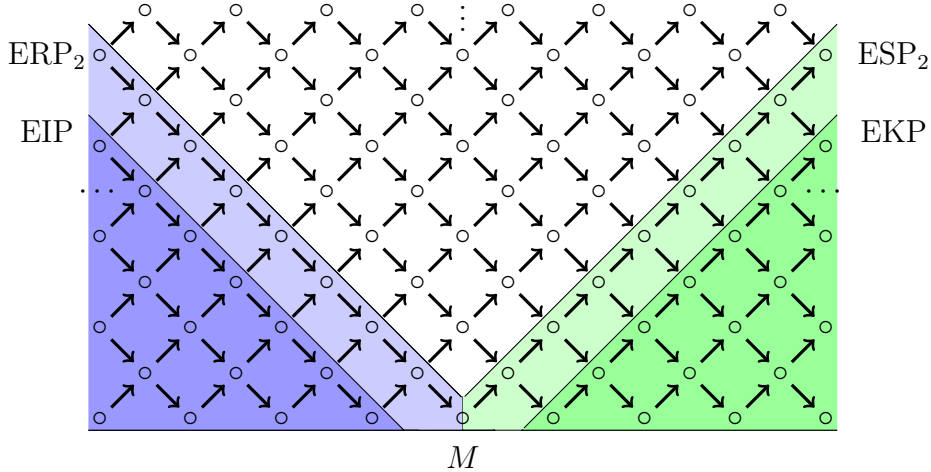
Similar computations show $\varphi \circ M(\gamma_2) = -M(\gamma_2)^* \circ \varphi$ and $\varphi \circ M(\gamma_3) = -M(\gamma_3)^* \circ \varphi$. Hence $D_{\Gamma_3}M \cong M$. Now let $f \in \text{End}_{\Gamma_3}(M)$ be non-zero and $1 \leq i \leq 3$. Since $\ker(A_i) = \langle e_{4-i} \rangle_k$, we conclude $0 = f_2 \circ M(\gamma_i)(e_{4-i}) = M(\gamma_i)(f_1(e_{4-i}))$. Hence, we find $\lambda_{4-i} \in k$ with $f_1(e_{4-i}) = \lambda_{4-i}e_{4-i}$. We conclude

$$\lambda_1 e_2 = M(\gamma_1) \circ f_1(-e_1) = f_2(e_2) = f_2(M(\gamma_3)(e_3)) = \lambda_3 M(\gamma_3)(e_3) = \lambda_3 e_2,$$

and

$$\lambda_1 e_3 = M(\gamma_2) \circ f_1(-e_1) = f_2(e_3) = f_2(M(\gamma_3)(-e_2)) = -\lambda_2 M(\gamma_3)(e_2) = \lambda_2 e_3.$$

Hence, $f_1 = \lambda_1 \cdot \text{id}_{k^3}$ and $f_2(e_1) = f_2 \circ M(\gamma_1)(e_2) = \lambda_1 M(\gamma_1)(e_2) = \lambda_1 e_1$. Moreover, we get $f_2(e_2) = f_2 \circ M(\gamma_3)(e_3) = \lambda_1 M(\gamma_3)(e_3) = \lambda_1 e_2$ and $f_2(e_3) = f_2 \circ M(\gamma_3)(-e_2) = \lambda_1 e_3$. We have shown that $f_1 = \lambda_1 \cdot \text{id}_{k^3} = f_2$. Hence, M is a brick, indecomposable and regular since $M \notin \text{EKP} \cup \text{EIP}$. Let \mathcal{C} be the regular component containing M . By [23, 9.2], every regular brick in $\text{rep}(\Gamma_3)$ is quasi-simple and $1 \leq \mathcal{W}(\mathcal{C}) \leq 2$ by Lemma 2.24. Since $D_{\Gamma_3}M \cong M$, we have $\mathcal{W}(\mathcal{C}) \neq 2$ and $\mathcal{W}(\mathcal{C}) = 1$. Moreover, M is in ERP_2 since $D_{\Gamma_3}M \cong M$.



Let us summarize our findings so far. The indecomposable representation $M \in \text{rep}(\Gamma_3)$ is a (quasi-simple) brick in $\text{ERP}_2 \cap \text{ESP}_2$. Now we set $\mathcal{X} := \{M\}$ and consider the simplified category $\mathcal{E}(\mathcal{X})$. Again, every indecomposable representation in $\mathcal{E}(\mathcal{X})$ is quasi-simple in a regular component. Moreover, $\mathcal{E}(\mathcal{X}) \subseteq \text{ERP}_2 \cap \text{ESP}_2$ since ERP_2 and ESP_2 are closed under extensions.

Corollary 5.5. *Let $r = 3$, then $\text{ERP}_2 \cap \text{ESP}_2$ has wild representation type. In particular, for each $n \in \mathbb{N}$ there is an indecomposable module N in $\text{ESP}_2 \cap \text{ERP}_2$ with $\dim_k N > n$.*

Observe that for arbitrary r and a regular component \mathcal{C} with $\text{ESP}_d \cap \text{ERP}_d \cap \mathcal{C}$ non-empty either $\text{ESP}_d \cap \mathcal{C} \neq \text{EKP} \cap \mathcal{C}$ or $\text{ERP}_d \cap \mathcal{C} \neq \text{EIP} \cap \mathcal{C}$ because EKP and EIP only intersect in the trivial module 0. Moreover, it follows from Proposition 3.14 that $\mathcal{W}(\mathcal{C}) \in \{0, 1\}$. If $\mathcal{W}(\mathcal{C}) = 1$, then $\text{ESP}_d \cap \text{ERP}_d \cap \mathcal{C}$ only contains one module (see the illustration above). If $\mathcal{W}(\mathcal{C}) = 0$, then there is also the possibility that $(\text{ESP}_d \setminus \text{EKP}) \cap \mathcal{C} \neq \emptyset \neq (\text{ERP}_d \setminus \text{EIP}) \cap \mathcal{C}$. Then $\text{ESP}_d \cap \mathcal{C}$ and $\text{ERP}_d \cap \mathcal{C}$ intersect in an Auslander-Reiten sequence with indecomposable middle term.

5.3 The module category of E_r

Throughout this section we assume that $\text{char}(k) = p > 0$ and $r \geq 2$. Moreover, let E_r be a p -elementary abelian group of rank r with generating set $\{g_1, \dots, g_r\}$. For $x_i := g_i - 1$ we get an isomorphism $kE_r \cong k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$ of k -algebras by sending X_i to x_i for all i . We recall the definition of the functor $\mathfrak{F}: \text{mod } \mathcal{K}_r \rightarrow \text{mod } kE_r$ introduced in [48]. Given a module M , $\mathfrak{F}(M)$ is by definition the vector space M and $x_i \cdot m := \gamma_i \cdot m = \gamma_i \cdot m_1 + \gamma_i \cdot m_2 = \gamma_i \cdot m_1$ where $m_i = e_i \cdot m$. Moreover, \mathfrak{F} lets the morphisms unchanged, i.e. $\mathfrak{F}(f): \mathfrak{F}(M) \rightarrow \mathfrak{F}(N); \mathfrak{F}(f)(m) = f(m)$ for all $f: M \rightarrow N$.

Definition. [9, 2.1] Let $\mathbb{V} := \langle x_1, \dots, x_r \rangle_k \subseteq \text{rad}(kE_r)$. For U in $\text{Gr}_{d, \mathbb{V}}$ with basis u_1, \dots, u_d and a kE_r -module M we set

$$\text{Rad}_U(M) := \sum_{u \in U} u \cdot M = \sum_{i=1}^d u_i \cdot M, \text{ and}$$

$$\text{Soc}_U(M) := \{m \in M \mid \forall u \in U : u \cdot m = 0\} = \bigcap_{i=1}^d \{m \in M \mid u_i \cdot m = 0\}.$$

Definition. [9, 3.1] Let $M \in \text{mod } kE_r$ and $1 \leq d < r$.

- (a) M has constant d -Rad rank (respectively, d -Soc rank) if the dimension of $\text{Rad}_U(M)$ (respectively, $\text{Soc}_U(M)$) is independent of the choice of $U \in \text{Gr}_{d, \mathbb{V}}$.
- (b) M has the equal d -Rad property (respectively, d -Soc property) if $\text{Rad}_U(M)$ (respectively, $\text{Soc}_U(M)$) is independent of the choice of $U \in \text{Gr}_{d, \mathbb{V}}$.

Proposition 5.6. *Let M be a non-simple indecomposable \mathcal{K}_r -module and $1 \leq d < r$.*

- (a) M is in CSR_d if and only if $\mathfrak{F}(M)$ has constant d -Soc rank.
- (b) M is in ESP_d if and only if $\mathfrak{F}(M)$ has the equal d -Soc property.

Proof. We fix $\mathbb{V} := \langle x_1, \dots, x_r \rangle_k \subseteq \text{rad}(kE_r)$. In the following we denote for $u \in \text{rad}(kE_r)$ with $l(u): M \rightarrow M$ the induced linear map on M . Let $U \in \text{Gr}_{d, \mathbb{V}}$ with basis $\{u_1, \dots, u_d\}$, write $u_j = \sum_{i=1}^r \alpha_j^i x_i$ for all $1 \leq j \leq d$ and set $\alpha_j = (\alpha_j^1, \dots, \alpha_j^r)$. Then $T := (\alpha_1, \dots, \alpha_d)$ is linearly independent and $\ker(l(u_j)) = \ker(\sum_{i=1}^r \alpha_j^i l(x_i)) = \ker(\sum_{i=1}^r \alpha_j^i \gamma_i) = \ker(x_{\alpha_j}^M)$. It follows

$$\text{Soc}_U(\mathfrak{F}(M)) = \bigcap_{i=1}^d \ker(l(u_i)) = \bigcap_{i=1}^d \ker(x_{\alpha_i}^M) = \text{Soc}_{\langle T \rangle}(M).$$

Hence, $M \in \text{CSR}_d$ implies that $\mathfrak{F}(M)$ has constant d -Soc rank.

Now assume that $T = (\alpha_1, \dots, \alpha_d)$ is linearly independent and set $u_j := \sum_{i=1}^r \alpha_j^i x_i$. Then $U := \langle u_1, \dots, u_d \rangle \in \text{Gr}_{d, \mathbb{V}}$ and $\text{Soc}_{\langle T \rangle}(M) = \text{Soc}_U(\mathfrak{F}(M))$. We have shown that M is in CSR_d if and only if $\mathfrak{F}(M)$ has constant d -Soc rank. The other equivalence follows in the same fashion. \square

The following result can be proven in the same fashion.

Proposition 5.7. *Let M be a non-simple indecomposable \mathcal{K}_r -module and $1 \leq d < r$.*

- (a) M is in CRR_d if and only if $\mathfrak{F}(M)$ has constant d -Rad rank.
- (b) M is in ERP_d if and only if $\mathfrak{F}(M)$ has the equal d -Rad property.

For $1 \leq d < r$ we denote with $\text{ESP}_{2,d}(E_r)$ the category of modules in $\text{mod } kE_r$ of Loewy length ≤ 2 with the equal d -Soc property and by $\text{ERP}_{2,d}(E_r)$ the category of modules of Loewy length ≤ 2 with the equal d -Rad property. As an application of Section 5.1 we get a generalization of [5, 5.6.12] and [6, 1].

Corollary 5.8. *Let $\text{char}(k) > 0$, $r \geq 3$ and $1 \leq d \leq r - 1$. Then $\text{ESP}_{2,d}(E_r) \setminus \text{ESP}_{2,d-1}(E_r)$ has wild representation type.*

Proof. Let $1 \leq c < r$. By [48, 2.1.2] and 5.6, a restriction of \mathfrak{F} to ESP_c induces a faithful exact functor

$$\mathfrak{F}_c: \text{ESP}_c \rightarrow \text{mod}_2 kE_r$$

that reflects isomorphisms and with essential image $\text{EIP}_{2,c}(E_r)$. Let $\mathcal{E} \subseteq \text{ESP}_d \setminus \text{ESP}_{d-1}$ be a wild subcategory. Since \mathfrak{F}_{d-1} and \mathfrak{F}_d reflect isomorphisms we have $\mathfrak{F}(E) \in \text{ESP}_{2,d}(E_r) \setminus \text{ESP}_{2,d-1}(E_r)$ for all $E \in \mathcal{E}$. Since \mathcal{F} preserves indecomposables, the essential image of \mathcal{E} under \mathfrak{F} is a wild category. \square

Corollary 5.9. *Assume that $\text{char}(k) = p > 2$, then the full subcategory of modules with the equal kernels property in $\text{mod } kE_2$ and Loewy length ≤ 3 is of wild representation type.*

Proof. By [48, 2.3] ($n = 3 \leq p, r = 2$) the functor $\mathfrak{F}_{\text{EKP}(3,2)}: \text{mod } B(3, 2) \rightarrow \text{mod}_3 kE_2$ is an exact, preserves indecomposables and reflects isomorphisms with essential image in $\text{EKP}(E_2)$. \square

Let us also mention the dual results which can be proven using 5.7 and 5.4.

Corollary 5.10. *Let $\text{char}(k) > 0$, $r \geq 3$ and $1 \leq d \leq r - 1$. Then $\text{ERP}_{2,d}(E_r) \setminus \text{ERP}_{2,d-1}(E_r)$ has wild representation type.*

Corollary 5.11. *Assume that $\text{char}(k) = p > 2$, then the full subcategory of modules with the equal images property in $\text{mod } kE_2$ and Loewy length ≤ 3 is of wild representation type.*

The next result follows immediately from Corollary 5.5 and the above considerations.

Corollary 5.12. *Let $\text{char}(k) > 0$ and $r = 3$, then $\text{ESP}_{2,2}(E_r) \cap \text{ERP}_{2,2}(E_r)$ has wild representation type.*

In section 3.4 we introduced for $1 \leq c < r$ the classes EKP_c and EIP_c of modules in $\text{mod } \mathcal{K}_r$ with the equal c -kernels and equal c -images property. Obviously one can define the corresponding classes in $\text{mod } kE_r$ and restrict the considerations to the modules with Loewy length ≤ 2 . If we denote these classes by $\text{EKP}_{2,c}(E_r)$ and $\text{EIP}_{2,c}(E_r)$ it follows analogously that a non-simple representation M has the equal c -kernels (resp. c -images) property if and only if $\mathfrak{F}(M)$ is in $\text{EKP}_{2,c}(E_r)$ (resp. in $\text{EIP}_{2,c}(E_r)$). We conclude:

Corollary 5.13. *Let $\text{char}(k) > 0$, $r \geq 3$ and $1 \leq c < r$, then $\text{EKP}_{2,c}(E_r)$ and $\text{EIP}_{2,c}(E_r)$ have wild representation type.*

6 The connection between $\text{rk}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C})$

We now establish the connection between the two functions $\text{rk}, \mathcal{W}: \mathcal{R}(\Gamma_r) \rightarrow \mathbb{Z}$. Let us start by recalling the definition and important properties of rk . If Q is a wild quiver and \mathcal{C} a regular component of kQ , the invariant $\text{rk}(\mathcal{C})$ gives information about morphisms in quasi-simple τ -orbit of \mathcal{C} .

The invariant was introduced by Kerner. In [21], he showed that for each wild quiver Q and each X quasi-simple in a regular component of kQ the number

$$\min\{l \in \mathbb{Z} \mid \forall m \geq l: \text{rad}_{kQ}(X, \tau_{kQ}^m X) \neq 0\}$$

is well-defined and does not depend on the choice of the quasi-simple representation in the regular component.

Definition. Let kQ be a wild quiver algebra and \mathcal{C} be a regular component with a quasi-simple module X in \mathcal{C} , then

$$\text{rk}(\mathcal{C}) := \min\{l \in \mathbb{Z} \mid \forall m \geq l: \text{rad}_{kQ}(X, \tau_{kQ}^m X) \neq 0\}$$

is called the quasi-rank of \mathcal{C} .

It was shown in [21], that $\bar{t}(kQ) := \sup\{\text{rk}(\mathcal{C}) \mid \mathcal{C} \text{ regular component}\}$ is finite. In other words, there are lots of morphisms in τ_{kQ} -direction. But also in τ_{kQ}^{-1} -direction one finds a lot of morphisms since the following result holds.

Theorem 6.1. [25, 3.1] *Let Q be a wild quiver, then*

$$\underline{t}(kQ) := \inf\{\text{rk}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{R}(Q)\} = -\infty.$$

The following lemma is very useful in the special case $Q = \Gamma_r$ since every regular \mathcal{K}_r -module has self-extensions by Corollary 2.19.

Lemma 6.2. [21, 1.7] *Let kQ be a wild hereditary algebra. Let X be a quasi-simple regular module with self-extensions in a regular component \mathcal{C} . Let s be an integer with $\text{rad}_{kQ}(X, \tau_{kQ}^s X) \neq 0$. Then we have $\text{rad}_{kQ}(X, \tau_{kQ}^r X) \neq 0$ for all $r \geq s$, that is, $\text{rk}(\mathcal{C}) \leq s$. In particular, $\text{rk}(\mathcal{C}) \leq 1$ holds in this case.*

Corollary 6.3. [21, 1.6,1.7] *For each regular component of \mathcal{K}_r we have $\text{rk}(\mathcal{C}) \leq 1$. Moreover, a regular component of \mathcal{K}_r satisfies $\text{rk}(\mathcal{C}) = 1$ if and only if each quasi-simple representation X in \mathcal{C} is a brick, i.e. $\text{End}_{\mathcal{K}_r}(X, X) \cong k$.*

In [48], the inequality $-\mathcal{W}(\mathcal{C}) \leq \text{rk}(\mathcal{C})$ is established and used in conjunction with $\underline{t}(\mathcal{K}_r) = -\infty$ to prove that $\sup\{\mathcal{W}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{R}\} = \infty$. We choose a different approach and study the number $\mathcal{W}(\mathcal{C})$ to draw conclusions for $\text{rk}(\mathcal{C})$. Therefore we prove:

Proposition 6.4. *Let \mathcal{C} be a regular component, then $-\mathcal{W}(\mathcal{C}) \leq \text{rk}(\mathcal{C}) \leq -\mathcal{W}(\mathcal{C}) + 3$.*

Proof. On the one hand let $M := \tau^{-1}W_{\mathcal{C}}$, then there is $\alpha \in k^r \setminus \{0\}$ with $0 \neq \text{Ext}_{\mathcal{K}_r}(X_{\alpha}, M)$. The Auslander-Reiten formula [23, 2.3] yields $0 \neq \text{Hom}_{\mathcal{K}_r}(\tau^{-1}M, X_{\alpha})$. On the other hand let $N := \tau M_{\mathcal{C}}$, then there is $\beta \in k^r \setminus \{0\}$ with $0 \neq \text{Hom}_{\mathcal{K}_r}(X_{\beta}, N) \cong \text{Hom}_{\mathcal{K}_r}(\tau X_{\beta}, \tau N)$. By the Euler-Ringel form we have

$$2-r = 1+(r-1)^2-r(r-1) = \langle \underline{\dim} X_{\beta}, \underline{\dim} X_{\alpha} \rangle_{\Gamma_r} = \dim_k \text{Hom}_{\mathcal{K}_r}(X_{\beta}, X_{\alpha}) - \dim_k \text{Ext}_{\mathcal{K}_r}(X_{\beta}, X_{\alpha}).$$

Hence, $0 \neq \dim_k \text{Ext}_{\mathcal{K}_r}(X_{\beta}, X_{\alpha}) = \dim_k \text{Hom}_{\mathcal{K}_r}(\tau^{-1}X_{\alpha}, X_{\beta}) = \dim_k \text{Hom}_{\mathcal{K}_r}(X_{\alpha}, \tau X_{\beta})$. Since X_{α} and τX_{β} are elementary, we get a non-zero morphism

$$\tau^{-1}M \rightarrow X_{\alpha} \rightarrow \tau X_{\beta} \rightarrow \tau N$$

by Lemma 3.2 and

$$0 \neq \text{Hom}_{\mathcal{K}_r}(M, \tau^2 N) = \text{Hom}_{\mathcal{K}_r}(\tau^{-1}W_{\mathcal{C}}, \tau^2 \circ \tau^{-\mathcal{W}(\mathcal{C})}W_{\mathcal{C}}) = \text{Hom}_{\mathcal{K}_r}(W_{\mathcal{C}}, \tau^{-\mathcal{W}(\mathcal{C})+3}W_{\mathcal{C}}).$$

Hence, $\text{rad}_{\mathcal{K}_r}(W_{\mathcal{C}}, \tau^{-\mathcal{W}(\mathcal{C})+3}W_{\mathcal{C}}) \neq 0$, since $\mathcal{W}(\mathcal{C}) = 3$ implies that $W_{\mathcal{C}}$ is not a brick (see Lemma 2.24). By 6.2 it follows that $\text{rk}(\mathcal{C}) \leq -\mathcal{W}(\mathcal{C}) + 3$. The second inequality follows from the proof of [48, 3.1.3]. \square

Examples. Let E be an indecomposable representation with $\underline{\dim} E = (1, r-1)$. Then we find $\alpha \in k^r \setminus \{0\}$ with $E \cong X_{\alpha}$. In particular, $0 \neq \text{Hom}_{\mathcal{K}_r}(X_{\alpha}, E)$ and $E \notin \text{EKP}$. The Auslander-Reiten formula yields $\dim_k \text{Ext}_{\mathcal{K}_r}(X_{\alpha}, \tau E) = \dim_k \text{Hom}_{\mathcal{K}_r}(E, X_{\alpha}) \neq 0$. Since X_{α} is elementary, it is quasi-simple in a component \mathcal{C} . Since $E \notin \text{EKP}$ and $\tau E \notin \text{EIP}$ we get that $\mathcal{W}(\mathcal{C}) \geq 2$. Since elementary representations are bricks we conclude $\mathcal{W}(\mathcal{C}) = 2$ with Lemma 2.24. Since E is a quasi-simple brick we get $\text{rk}(\mathcal{C}) = 1$ by Lemma 6.3, hence $\text{rk}(\mathcal{C}) = -\mathcal{W}(\mathcal{C}) + 3$. By [48, 3.3.1] and Corollary 4.8 there are also components $\mathcal{D}, \mathcal{E}, \mathcal{F}$ with $\text{rk}(\mathcal{D}) = 1 = \text{rk}(\mathcal{E})$, $\text{rk}(\mathcal{F}) = 0$ and $\mathcal{W}(\mathcal{D}) = 1, \mathcal{W}(\mathcal{E}) = 0, \mathcal{W}(\mathcal{F}) = 0$. Hence, the proven inequalities are as good as possible.

7 Covering Theory

From now on we consider representations over quivers instead of modules. To emphasize this we write $\text{Hom}_Q(-, -)$, $\text{Ext}_Q(-, -)$, τ_Q and D_Q .

7.1 General Theory

We follow [33] and [29] and consider the universal cover C_r of the quiver Γ_r . The underlying graph of C_r is an r -regular tree and C_r has bipartite orientation. That means each vertex $x \in (C_r)_0$ is a sink or a source and $|n(x)| = r$. In the following we recall the construction of C_r . For a quiver $Q = (Q_0, Q_1, s, t)$ with arrow set Q_1 we write $(Q_1)^{-1} := \{\alpha^{-1} \mid \alpha \in Q_1\}$ for the formal inverses of Q_1 . Moreover, we extend the functions s and t to $(Q_1)^{-1}$ by defining $s(\alpha^{-1}) := t(\alpha)$ and $t(\alpha^{-1}) := s(\alpha)$. A walk w in Q_1 is a formal sequence $w = \alpha_n^{\varepsilon_n} \cdots \alpha_1^{\varepsilon_1}$ with $\alpha_i \in Q_1$, $\varepsilon \in \{1, -1\}$ such that $s(\alpha_{i+1}^{\varepsilon_{i+1}}) = t(\alpha_i^{\varepsilon_i})$ for all $i < n$, where $\alpha^1 := \alpha$ for all $\alpha \in Q_1$. We set $t(w) := t(\alpha_n^{\varepsilon_n})$ and $s(w) := s(\alpha_1^{\varepsilon_1})$.

Let \sim be the equivalence relation on the set of walks W of Γ_r generated by

$$\gamma_i^{-1}\gamma_i \sim \epsilon_1 \text{ and } \gamma_i\gamma_i^{-1} \sim \epsilon_2.$$

Let $^{-1}: W \rightarrow W$ be the involution on W given by $(\alpha_n^{\varepsilon_n} \cdots \alpha_1^{\varepsilon_1})^{-1} := \alpha_1^{-\varepsilon_1} \cdots \alpha_n^{-\varepsilon_n}$. Now consider the fundamental group $\pi(\Gamma_r)$ of Γ_r in the point 1, i.e. the elements of $\pi(\Gamma_r)$ are the equivalence classes of unoriented paths starting and ending in 1, with multiplication given by concatenation of paths, inverse elements $[w]^{-1} := [w^{-1}]$ and identity element $[\epsilon_1]$. Note that $\pi(\Gamma_r)$ is a free group in the $r - 1$ generators $\{[\gamma_j^{-1}\gamma_1] \mid 2 \leq j \leq r\}$ and in particular torsionfree.

The quiver C_r is given by the following data:

- (a) $(C_r)_0$ is the set of equivalence classes of paths starting in 1.
- (b) There is an arrow from $[w]$ to $[w']$ whenever $w' \sim \gamma_i w$ for some $i \in \{1, \dots, r\}$.

Let $\pi: C_r \rightarrow \Gamma_r$ be the quiver morphism given by $[w] \mapsto t(w)$ and $([w] \rightarrow [\gamma_i w]) \mapsto \gamma_i$. The morphism π is a G -Galois cover for $G = \pi(\Gamma_r)$, where the action of G on C_r is given by concatenation of paths: if $g = [w] \in \pi(\Gamma_r)$ and $[v], [u] \in (C_r)_0$ with arrow $[u] \rightarrow [\gamma_i u]$ then

$$g.[v] = [vw^{-1}] \text{ and}$$

$$g.([u] \rightarrow [\gamma_i u]) = ([uw^{-1}] \rightarrow [\gamma_i uw^{-1}]).$$

We define $C_r^+ := \pi^{-1}(\{1\})$, $C_r^- := \pi^{-1}(\{2\})$ and get an induced action on $\text{rep}(C_r)$ by shifting the support of representations via $G = \pi(\Gamma_r)$: Given $M \in \text{rep}(C_r)$ and $g \in G$ we define $M^g := (((M^g)_x)_{x \in (C_r)_0}, (M^g(\alpha))_{\alpha \in (C_r)_1})$, where

$$(M^g)_x := M_{g.x} \text{ and } M^g(\alpha) := M(g.\alpha).$$

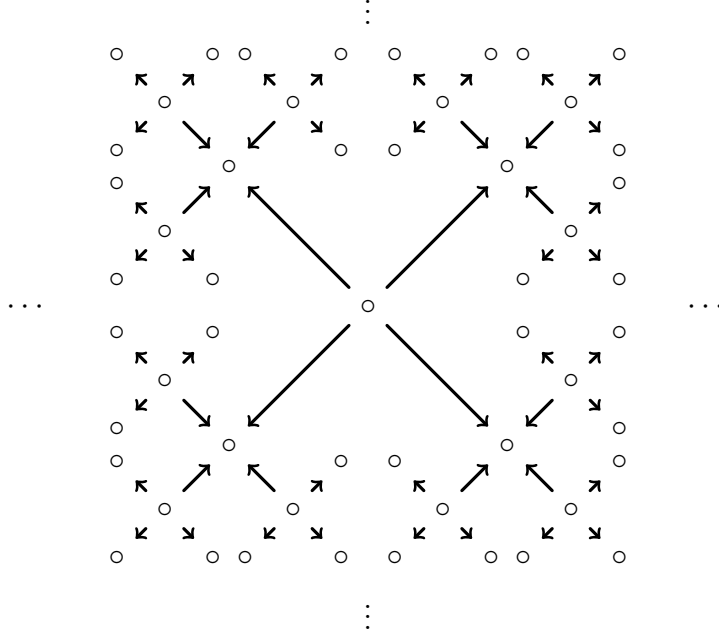


Figure 7.1: Illustration of C_r for $r = 4$.

By identifying the orbit quiver C_r/G with Γ_r we define the push-down functor $\pi_\lambda: \text{rep}(C_r) \rightarrow \text{rep}(\Gamma_r)$ on the objects via $\pi_\lambda(M) := (\pi_\lambda(M)_1, \pi_\lambda(M)_2; (\pi_\lambda(M)(\gamma_i))_{1 \leq i \leq r})$, where

$$\pi_\lambda(M)_i := \bigoplus_{\pi(y)=i} M_y \text{ and}$$

$$\pi_\lambda(M)(\gamma_i) := \bigoplus_{\pi(\beta)=\gamma_i} M(\beta): \pi_\lambda(M)_1 \rightarrow \pi_\lambda(M)_2 \text{ for } 1 \leq i \leq r.$$

If $f = (f_x)_{x \in (C_r)_0}: M \rightarrow N$ is a morphism in $\text{rep}(C_r)$ then $\pi_\lambda(f) = (g_{\pi(x)})_{x \in (C_r)_0} = (g_1, g_2)$ with

$$g_i := \bigoplus_{\pi(y)=i} f_y: \pi_\lambda(M)_i \rightarrow \pi_\lambda(N)_i.$$

By [7, 3.2] π_λ is an exact functor.

When Q is locally bounded but not finite, then kQ is not a unitary k -algebra and $\text{rep}(Q)$ is not equivalent to the module category of a finite-dimensional k -algebra. But $\text{rep}(Q)$ has almost split sequences:

We denote by $k(Q)$ the path category of Q . The category $k(Q)$ has Q_0 as set of objects and $\text{Hom}_{k(Q)}(x, y)$ is the vector space with basis given by the paths from x to y . The trivial arrow in x is denoted by ϵ_x . Since Q is locally bounded, the category $k(Q)$ is locally bounded in the sense of [7, 2.1]. A finite dimensional module over a locally bounded category A is a functor $F: A \rightarrow \text{mod } k$ such that $\sum_{x \in A} \dim_k F(x)$ is finite. The category $\text{mod } A$ has Auslander-Reiten sequences (see [7, 2.2]). Since a finite dimensional module over $k(Q)$ is the same as a representation of Q , the category $\text{rep}(Q)$ has Auslander-Reiten sequences. Since C_r is locally bounded the category $\text{rep}(C_r)$ has almost-split sequences. Moreover the following results hold:

Theorem 7.1. [19, 3.6], [29, 6.2,6.3] *The following statements hold.*

- (a) π_λ sends indecomposable representations in $\text{rep}(C_r)$ to indecomposable representations in $\text{rep}(\Gamma_r)$.
- (b) If $M \in \text{rep}(C_r)$ is indecomposable, then $\pi_\lambda(M) \cong \pi_\lambda(N)$ if and only if $M^g \cong N$ for some $g \in G$.
- (c) π_λ sends AR sequences to AR sequences and π_λ commutes with the Auslander-Reiten translates, i.e. $\tau \circ \pi_\lambda = \pi_\lambda \circ \tau_{C_r}$.
- (d) If $M \in \text{rep}(C_r)$ is indecomposable in a component \mathcal{D} with $\pi_\lambda(M)$ in a component \mathcal{C} , then π_λ induces a covering $\mathcal{D} \rightarrow \mathcal{C}$ of translation quivers.

Definition. Let $M \in \text{rep}(C_r)$ be indecomposable, then M is called regular if $\pi_\lambda(M)$ is regular. A component \mathcal{D} of $\text{rep}(C_r)$ is called regular if it contains a regular representation M . In this case we denote by $\pi_\lambda(\mathcal{D})$ the component containing $\pi_\lambda(M)$. Moreover, we let $\mathcal{R}(C_r)$ be the set of all regular components of the Auslander-Reiten quiver of $\text{rep}(C_r)$.

Corollary 7.2. *Let \mathcal{D} be regular component, then the covering $\mathcal{D} \rightarrow \pi_\lambda(\mathcal{D})$ is an isomorphism of translation quivers. In particular, \mathcal{D} is of type $\mathbb{Z}A_\infty$. Moreover, a component \mathcal{E} is regular if and only if \mathcal{E} is of type $\mathbb{Z}A_\infty$.*

Proof. By [7, 1.7], $\pi_\lambda(\mathcal{D}) \cong \mathbb{Z}A_\infty$ is a simply connected translation quiver. By [7, 1.6],[28, 1.7] the quiver morphism $\mathcal{D} \rightarrow \pi_\lambda(\mathcal{D})$ is an isomorphism. If $\pi_\lambda(\mathcal{E}) \in \{\mathcal{I}(\Gamma_r), \mathcal{P}(\Gamma_r)\}$ then there exists a vertex x with $r \geq 3$ successors (see Figure 2.5), since a covering is surjective on arrows. Hence, \mathcal{E} is not of type $\mathbb{Z}A_\infty$. \square

7.2 Duality

Recall that the duality $D_{\Gamma_r}: \text{rep}(\Gamma_r) \rightarrow \text{rep}(\Gamma_r)$ is defined by setting $(D_{\Gamma_r}M)_x := (M_{\psi(x)})^*$ and $(D_{\Gamma_r}M)(\gamma_i) := (M(\gamma_i))^*$, where $\psi: \{1, 2\} \rightarrow \{1, 2\}$ is the involution with $\psi(1) = 2$.

We define an involution $\varphi_0: (C_r)_0 \rightarrow (C_r)_0$ via $[w] \mapsto [\bar{w}\gamma_1]$, where $\bar{\epsilon}_1 := \epsilon_2$, $\bar{\epsilon}_2 := \epsilon_1$ and $\bar{\alpha}_n^{\epsilon_n} \cdots \bar{\alpha}_1^{\epsilon_1} := \alpha_n^{-\epsilon_n} \cdots \alpha_1^{-\epsilon_1}$. This induces a quiver anti-morphism $\varphi: C_r \rightarrow C_r$ in the following way.

If $[w] \rightarrow [\gamma_i w]$ is an arrow of C_r , then by definition there is a unique arrow $\varphi([w] \rightarrow [\gamma_i w])$ starting in $\varphi_0([\gamma_i w]) = [\gamma_i^{-1} \bar{w}\gamma_1]$ and ending in $\varphi_0([w]) = [\bar{w}\gamma_1]$, since $\gamma_i \gamma_i^{-1} \bar{w}\gamma_1 \sim \bar{w}\gamma_1$. Note that $\varphi(C_r^+) = C_r^-$, $\varphi(C_r^-) = C_r^+$ and $\pi(\varphi(\alpha)) = \pi(\alpha)$.

We define a duality $D_{C_r}: \text{rep}(C_r) \rightarrow \text{rep}(C_r)$ by setting $D_{C_r}M := ((D_{C_r}M)_{x \in (C_r)_0}, (D_{C_r}M(\alpha))_{\alpha \in Q_1})$ where $(D_{C_r}M)_x := (M_{\varphi(x)})^*$ and $D_{C_r}M(\alpha) := (M(\varphi(\alpha)))^*$. By construction we have $\pi_\lambda \circ D_{C_r} = D_{\Gamma_r} \circ \pi_\lambda$.

8 The categories EKP and EIP for $\text{rep}(C_r)$

In the following we give a characterization of indecomposable representations N in $\text{EKP} \cup \text{EIP}$ that are the form $N = \pi_\lambda(M)$ for some $M \in \text{rep}(C_r)$. Let $\bar{\cdot} : (C_r)_1 \rightarrow \{1, \dots, r\}$ be the unique map with $\gamma_{\bar{\beta}} = \pi(\beta)$ for all $\beta \in (C_r)_1$. Note that if $x \in (C_r)^+$ (or $x \in C_r^-$) then the restriction of $\bar{\cdot}$ to $\{\alpha \in (C_r)_1 \mid s(\alpha) = x\}$ (resp. $\{\alpha \in (C_r)_1 \mid t(\alpha) = x\}$) is a bijective map to $\{1, \dots, r\}$.

Definition. Let $\emptyset \neq X \subseteq (C_r)_0$ be a set of vertices and $T \subseteq C_r$ be a tree.

- (a) The unique minimal tree containing X is denoted by $T(X)$.
- (b) A vertex $x \in T_0$ is called a leaf of T , if $|n(x) \cap T_0| \leq 1$.

Definition. Let $x \in (C_r)_0$ and M be a representation of C_r .

- (a) The set $\text{supp}(M) := \{y \in (C_r)_0 \mid M_y \neq 0\}$ is called the support of M .
- (b) For $V \subseteq \text{supp}(M)$ we let M_V be the induced representation with $\text{supp}(M_V) = V$.
- (c) The vertex x is a leaf of M if x is a leaf of $T(M) := T(\text{supp}(M))$.
- (d) M is called balanced provided that M is indecomposable and M has leaves in C_r^+ and C_r^- .

Observe that if M is indecomposable, then we have $T(M)_0 = \text{supp}(M)$.

Definition. We define

$$\text{Inj} := \{M \in \text{rep}(C_r) \mid \forall \delta \in (C_r)_1 : M(\delta) \text{ is injective}\} \text{ and}$$

$$\text{Surj} := \{M \in \text{rep}(C_r) \mid \forall \delta \in (C_r)_1 : M(\delta) \text{ is surjective}\}.$$

Let N be an indecomposable representation of Γ_r . If N is in EKP, then the linear maps $N(\gamma_1), \dots, N(\gamma_r) : N_1 \rightarrow N_2$ are injective. But obviously the other implication does not hold in general as the example $(k, k, \text{id}_k, \text{id}_k, \text{id}_k)$ for $r = 3$ shows.

The following theorem shows that this is the case for "gradable representations", that is, for those belonging to the essential image of π_λ . So for these representations we only have to test r linear maps for injectivity instead of all non-trivial linear combinations of $N(\gamma_1), \dots, N(\gamma_r)$.

Theorem 8.1. *Let $M \in \text{rep}(C_r)$ be an indecomposable representation. The following statements are equivalent:*

- (a) $N := \pi_\lambda(M) \in \text{EKP}$.
- (b) $N(\gamma_i)$ is injective for all $i \in \{1, \dots, r\}$.
- (c) $M \in \text{Inj}$.

Proof. (a) \Rightarrow (b): Clear from the definition of EKP.

(b) \Rightarrow (c): Let $\alpha: x \rightarrow y$ and $m_x \in \ker M(\alpha)$. Let $\iota_x: M_x \rightarrow \bigoplus_{z \in C_r^+} M_z$ and $\iota_y: M_y \rightarrow \bigoplus_{z \in C_r^-} M_z$ be the natural embeddings. We conclude

$$0 = \iota_y \circ M(\alpha)(m_x) = \left[\bigoplus_{\pi(\beta)=\pi(\alpha)} M(\beta) \right] \circ \iota_x(m_x) = (N(\gamma_{\bar{\alpha}}) \circ \iota_x)(m_x).$$

Hence, $\iota_x(m_x) \in \ker N(\gamma_{\bar{\alpha}}) = \{0\}$ and $m_x = 0$.

(c) \Rightarrow (a): Let $\alpha \in k^r \setminus \{0\}$, $f := \sum_{i=1}^r \alpha_i N(\gamma_i): N_1 \rightarrow N_2$ and $m \in \ker f$. We assume w.l.o.g that $\alpha_1 \neq 0$. Write $m = (m_z)_{z \in C_r^+}$. We have to show that $m_z = 0$ for all $z \in C_r^+$.

Let $S := \{z \in C_r^+ \mid m_z \neq 0\} \subseteq C_r^+$ and suppose that $S \neq \emptyset$. Let $T(S)$ be the minimal tree that contains S . Since $T(S)_0 \subseteq \text{supp}(M)$ and $\text{supp}(M)$ is finite, there exists a leaf of $T(S)$. Moreover every leaf of $T(S)$ belongs to $C_r^+ \cap S$, by the minimality of $T(S)$. Fix a leaf x in $T(S)$ and let $\gamma: x \rightarrow y$ be the unique arrow with $\pi(\gamma) = \gamma_1$. Then

$$0 = (f(m))_y = \sum_{\beta \in (C_r)_1, t(\beta)=y} \alpha_{\bar{\beta}} M(\beta)(m_{s(\beta)}) = \alpha_1 M(\gamma)(m_x) + \sum_{\gamma \neq \beta \in (C_r)_1, t(\beta)=y} \alpha_{\bar{\beta}} M(\beta)(m_{s(\beta)}).$$

By the injectivity of $M(\gamma)$ there is $\delta: z \rightarrow y \in (C_r)_1 \setminus \{\gamma\}$ with $t(\delta) = y$ and $0 \neq \alpha_{\bar{\delta}} M(\delta)(m_z)$. It follows $m_z \neq 0$ and $z, x \in S$. Since C_r is a tree we get $y \in T(S)_0$.

Since $\bar{\delta} \neq \bar{\gamma} = 1$, we assume without loss of generality that $\bar{\delta} = 2$, so that $\alpha_2 \neq 0$. Let $\eta: x \rightarrow a$ be the unique arrow with $\bar{\eta} = 2$. Then

$$0 = (f(m))_a = \sum_{\beta \in (C_r)_1, t(\beta)=a} \alpha_{\bar{\beta}} M(\beta)(m_{s(\beta)}) = \alpha_2 M(\eta)(m_x) + \sum_{\eta \neq \beta \in (C_r)_1, t(\beta)=a} \alpha_{\bar{\beta}} M(\beta)(m_{s(\beta)}).$$

Hence, there is $\zeta: b \rightarrow a \in (C_r)_1 \setminus \{\eta\}$ with $0 \neq \alpha_{\bar{\zeta}} M(\zeta)(m_b)$ and a, b are in $T(S)_0$. We have shown that a and y are in $T(S)_0$. This is a contradiction since x is a leaf. \square

Corollary 8.2. *Let $M \in \text{rep}(C_r)$ be an indecomposable representation. The following statements are equivalent.*

- (a) $N := \pi_\lambda(M) \in \text{EIP}$.
- (b) $N(\gamma_i)$ is surjective for all $i \in \{1, \dots, r\}$.
- (c) $M \in \text{Surj}$.

Proof. (c) \Rightarrow (a): Let $M(\alpha)$ be surjective for each $\alpha \in (C_r)_1$, then $(D_{C_r} M)(\alpha)$ is injective for each $\alpha \in (C_r)_1$. By Theorem 8.1, the representation $\pi_\lambda(D_{C_r} M) \cong D_{\Gamma_r} \pi_\lambda(M)$ is in EKP. Therefore $\pi_\lambda(M) \in \text{EIP}$, since $D_{\Gamma_r}(\text{EKP}) = \text{EIP}$. \square

Corollary 8.3. *Let $\mathcal{D} \in \mathcal{R}(C_r)$ and $\mathcal{C} := \pi_\lambda(\mathcal{D})$. Then there exist uniquely determined quasi-simple representations $I_{\mathcal{D}}$ and $S_{\mathcal{D}}$ in \mathcal{D} such that*

- (a) $\pi_\lambda(I_{\mathcal{D}}) = M_{\mathcal{C}}$ and $\pi_\lambda(S_{\mathcal{D}}) = W_{\mathcal{C}}$.
- (b) $\text{Surj} \cap \mathcal{D} = (\rightarrow S_{\mathcal{D}})$ and $\text{Inj} \cap \mathcal{D} = (I_{\mathcal{D}} \rightarrow)$.
- (c) *The unique integer $\mathcal{W}_{\mathcal{C}}(\mathcal{D})$ with $\tau_{C_r}^{\mathcal{W}_{\mathcal{C}}(\mathcal{D})+1}(I_{\mathcal{D}}) = S_{\mathcal{D}}$ is given by $\mathcal{W}_{\mathcal{C}}(\mathcal{D}) = \mathcal{W}(\mathcal{C}) \in \mathbb{N}_0$.*

Proof. At first note that \mathcal{C} is of type $\mathbb{Z}A_\infty$ by Corollary 7.2. The existence of $I_{\mathcal{D}}$ and $S_{\mathcal{D}}$ is immediate. Now (b) and (c) follow from Theorem 8.1 and Corollary 8.2 since $\pi_\lambda \circ \tau_{C_r} = \tau_{\Gamma_r} \circ \pi_\lambda$. \square

Corollary 8.4. *Assume that $M \in \text{rep}(C_r)$ is balanced, then M is regular.*

Proof. If an indecomposable representation $X \in \text{rep}(C_r)$ is in Inj (respectively Surj), then all leaves of X are in C_r^+ (resp. C_r^-). Hence, $\pi_\lambda(M) \notin \text{EIP} \cup \text{EKP}$ by Theorem 8.1. Since the representations of the components $\mathcal{P}(\Gamma_r)$ and $\mathcal{I}(\Gamma_r)$ are contained in $\text{EKP} \cup \text{EIP}$ it follows that $\pi_\lambda(M)$ is regular. \square

From now on we write $x_0 := [\epsilon_1] \in (C_r)_0$ for the vertex in $(C_r)_0$ given by the trivial walk starting in the vertex 1.

Definition. For $i \in \{1, \dots, r\}$, we let $\beta_i \in (C_r)_1$ be the unique arrow with $s(\beta_i) = x_0$ and $\pi(\beta_i) = \gamma_i$. Moreover, let $z_i := t(\beta_i)$. We define an indecomposable representation X^i in $\text{rep}(C_r)$ via:

$$(X^i)_y := \begin{cases} k, & y \in \{x_0\} \cup x_0^+ \setminus \{z_i\} \\ 0, & \text{else} \end{cases}$$

and $X^i(\beta_j) := \text{id}_k$ for all $j \neq i$. By definition we have $\underline{\dim} \pi_\lambda(X^i) = (1, r-1)$ and $\pi_\lambda(X^i) \cong X_{e_i}$.

Corollary 8.5. *Let $M \in \text{rep}(C_r)$ be an indecomposable representation. The following statements are equivalent:*

- (a) M is in Inj.
- (b) $N := \pi_\lambda(M) \in \text{EKP}$.
- (c) $\text{Hom}_{\Gamma_r}(\pi_\lambda(X^i), N) = 0$ for all $i \in \{1, \dots, r\}$.
- (d) $\text{Hom}_{C_r}((X^i)^g, M) = 0$ for all $i \in \{1, \dots, r\}$ and all $g \in G$.

Proof. In view of Theorem 2.21, Theorem 7.1 and Theorem 8.1 it remains to prove (c) \Leftrightarrow (d). This follows immediately from [19, 3.6(c)]. \square

In the following we show that one also can decide in $\text{rep}(C_r)$ whether $\pi_\lambda(N)$ is in $\text{ESP}_d \cup \text{ERP}_d$ for $1 \leq d < r$ and $N \in \text{rep}(C_r)$. Hence, the results can be seen as a generalization of Theorem 8.1 and Corollary 8.2. We give a separate proof since it is technically more involved. For $x \in (C_r)_0$ we define

$$A_x := \begin{cases} \{\alpha \in (C_r)_1 \mid s(\alpha) = x\}, & x \in C_r^+ \\ \{\alpha \in (C_r)_1 \mid t(\alpha) = x\}, & x \in C_r^-. \end{cases}$$

Proposition 8.6. *Let $1 \leq d \leq r-1$ and $M \in \text{rep}(C_r)$ be an indecomposable representation. The following statements are equivalent:*

- (a) $N := \pi_\lambda(M) \in \text{ESP}_d$.
- (b) For all $I \subseteq \{1, \dots, r\}$ with $|I| = d$ we have $\{0\} = \bigcap_{i \in I} \ker N(\gamma_i)$.
- (c) For all $x \in C_r^+$ and all $S \subseteq A_x$ with $|S| = d$ we have $\{0\} = \bigcap_{s \in S} \ker M(s)$.

Proof. (a) \Rightarrow (b) Clear from the definition of ESP_d .

(b) \Rightarrow (c) Let $x \in C_r^+$, $S \subseteq A_x$ with $|S| = d$ and $m_x \in \bigcap_{s \in S} \ker M(s)$. Denote by $\iota_x: M_x \rightarrow \bigoplus_{z \in C_r^+} M_z$ the natural embedding and for $y \in C_r^-$ let $\iota_y: M_y \rightarrow \bigoplus_{z \in C_r^-} M_z$ be the natural embedding. Since $0_{kd} = (M(s)(m_x))_{s \in S}$ we conclude for $s \in S$ that

$$0 = \iota_{t(s)} \circ M(s)(m_x) = \left[\bigoplus_{\pi(\beta)=\pi(s)} M(\beta) \right] \circ \iota_x(m_x) = (N(\gamma_{\bar{s}}) \circ \iota_x)(m_x).$$

Hence, $\iota_x(m_x) \in \bigcap_{s \in S} \ker N(\gamma_{\bar{s}}) = 0$ and $m_x = 0$.

(c) \Rightarrow (a) Let $T = (\alpha_1, \dots, \alpha_d) \in (k^r)^d$ be linearly independent. For each $i \in \{1, \dots, d\}$ we write $\alpha_i = (\alpha_{1i}, \dots, \alpha_{ri})$. Moreover, we let $M(T) \in \text{Mat}_{r \times d}(k)$ be the matrix

$$M(T) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1d} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rd} \end{pmatrix}.$$

By the definition of the ESP_d we have to show that

$$\varphi_T: N_1 \rightarrow (N_2)^d, n \mapsto \left(\sum_{i=1}^r \alpha_{i1} N(\gamma_i)(n), \dots, \sum_{i=1}^r \alpha_{id} N(\gamma_i)(n) \right)$$

is injective. Let $m \in \ker \varphi_T$, write $m = (m_v)_{v \in C_r^+}$ and assume there is $u \in C_r^+$ with $m_u \neq 0$. We define

$$I := \{i \in \{1, \dots, r\} \mid \exists j \in \{1, \dots, d\} : \alpha_{ij} \neq 0\}.$$

Since T is linearly independent we have $|I| \geq d$. We let

$$\text{supp}(u, m_u) := \{s \in A_u \mid M(s)(m_u) \neq 0\}.$$

By (c) we have $|\text{supp}(u, m_u)| \geq r - d + 1 =: b$. Since $|\{\bar{s} \mid s \in \text{supp}(u, m_u)\}| \geq b$ and $d + b = r + 1$ we have $\{\bar{s} \mid s \in \text{supp}(u, m_u)\} \cap \{i \in \{1, \dots, r\} \mid \exists j \in \{1, \dots, d\} : \alpha_{ij} \neq 0\} \neq \emptyset$. Hence we find $i \in \{1, \dots, r\}$, $j \in \{1, \dots, d\}$ and $\delta \in A_u$ with $\bar{\delta} = i$ and $0 \neq \alpha_{ij} M(\delta)(m_u)$.

We now show that $|I| > d$. Assume to the contrary that $|I| = d$. Then the matrix $M(T)_I \in \text{Mat}_{d \times d}(k)$ that results from deleting the rows in $\{1, \dots, r\} \setminus I$ is invertible. In particular, the row vectors of $M(T)_I$ form a basis of k^d . Let $y := t(\delta)$ and observe that $m = (m_v)_{v \in C_r^+} \in \ker \varphi_T$ implies $0 = (Q_{\alpha_1}(y), \dots, Q_{\alpha_d}(y))$ where

$$Q_{\alpha_i}(y) := \sum_{\gamma \in (C_r)_1, t(\gamma)=y} \alpha_{\bar{\gamma}i} M(\gamma)(m_{s(\gamma)}) \in M_y.$$

Without loss of generality we assume that $M_y = k^a$ with $a \geq 1$ and that the first component $(M(\delta)(m_u))_1$ is not equal to zero. We denote for each $\gamma \in (C_r)_1$ with $t(\gamma) = y$ the scalar $\lambda_\gamma := (M(\gamma)(m_{s(\gamma)}))_1$ and get

$$0_{k^d} = ((Q_{\alpha_p}(y))_{1 \leq p \leq d}) = \sum_{\gamma \in (C_r)_1, t(\gamma)=y} (M(\gamma)(m_{s(\gamma)}))_1 (\alpha_{\bar{\gamma}p})_{1 \leq p \leq d} = \sum_{\gamma \in (C_r)_1, t(\gamma)=y} \lambda_\gamma (\alpha_{\bar{\gamma}p})_{1 \leq p \leq d}.$$

Since $\bar{\gamma} \notin I$ implies $(\alpha_{\bar{\gamma}p})_{1 \leq p \leq d} = 0_{k^d}$, we conclude

$$0_{k^d} = \sum_{\gamma \in (C_r)_1, t(\gamma)=y, \bar{\gamma} \in I} \lambda_\gamma (\alpha_{\bar{\gamma}p})_{1 \leq p \leq d}.$$

Note that for $\gamma \in (C_r)_1$ with $\bar{\gamma} \in I$ the vector $(\alpha_{\bar{\gamma}p})_{1 \leq p \leq d}$ is just a row vector of $M(T)_I$. But this is a contradiction to the fact that the rows of $M(T)_I$ form a basis of k^d , since $\bar{\delta} = i \in I$ with $\lambda_\delta = ((M(\delta)(m_u))_1) \neq 0$. Hence $|I| > d$.

Now we proceed as in the proof of Theorem 8.1. Let $S := \{z \in C_r^+ \mid m_z \neq 0\} \neq \emptyset$ and $T(S)$ be the minimal tree that contains S . Since $T(S)_0 \subseteq \text{supp}(M)$ and $\text{supp}(M)$ is finite, the tree $T(S)$ has a leaf. Moreover every leaf belongs to $S \subseteq C_r^+$, by the minimality of $T(S)$. Fix a leaf x in $T(S)$. Since $|I| \geq d + 1$ we have $|I| + b \geq d + 1 + r - d + 1 \geq 2$. Hence $|I \cap \{\bar{s} \mid s \in \text{supp}(x, m_x)\}| \geq r + 2$. Hence, we find $\beta \neq \gamma \in A_x$ and $p, q \in \{1, \dots, d\}$ with $0 \neq \alpha_{\bar{\beta}p}M(\beta)(m_x)$ and $0 \neq \alpha_{\bar{\gamma}q}M(\gamma)(m_x)$. We have $0 \neq \alpha_{\bar{\beta}p}M(\beta)(m_x)$ and conclude with

$$0 = \alpha_{\bar{\beta}p}M(\beta)(m_x) + \sum_{\eta \in (C_r)_1, t(\eta)=t(\beta), \eta \neq \beta} \alpha_{\bar{\eta}p}M(\eta)(m_{s(\eta)})$$

that there is $\beta_0 \in (C_r)_1$ such that $t(\beta_0) = t(\beta)$, $\beta_0 \neq \beta$ and $m_{s(\beta_0)} \neq 0$. By the same token, we find $\gamma_0 \in (C_r)_1$ with $t(\gamma_0) = t(\gamma)$, $\gamma_0 \neq \gamma$ and $m_{s(\gamma_0)} \neq 0$.

But that means $s(\beta_0), s(\gamma_0)$ are in S and hence $t(\beta_0) = t(\beta)$, $t(\gamma_0) = t(\gamma)$ are in $T(S)$. But this is a contradiction since x is a leaf of $T(S)$ (see Figure 8.1).

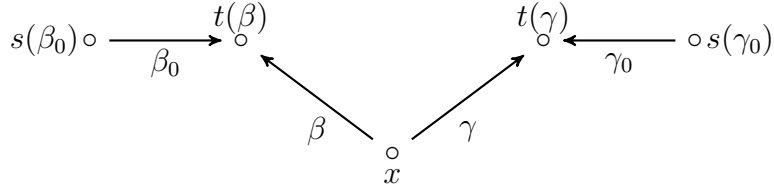


Figure 8.1: Illustration of the proof.

□

By duality we have:

Proposition 8.7. *Let $1 \leq d \leq r - 1$ and $M \in \text{rep}(C_r)$ be an indecomposable representation. The following statements are equivalent:*

- (a) $N := \pi_\lambda(M) \in \text{ERP}_d$.
- (b) For all $I \subseteq \{1, \dots, r\}$ with $|I| = d$ we have $N_2 = \sum_{i \in I} \text{im } N(\gamma_i)$.
- (c) For all $x \in C_r^-$ and all $S \subseteq A_x$ with $|S| = d$ we have $M_x = \sum_{s \in S} \text{im } M(s)$.

We end this section with a lemma that will be needed later on.

Lemma 8.8. *Let $n \in \mathbb{N}$, M be regular indecomposable, $i \in \{1, \dots, r\}$ and $g \in G = \pi(\Gamma_r)$.*

- (a) For $n \geq 2$ and $\alpha \in (C_r)_1$, the linear map $(\tau_{C_r}^n X^i)^g(\alpha)$ is surjective.
- (b) If $n \geq 1$ and $f = (f_x)_{x \in (C_r)_0} : (\tau_{C_r}^n X^i)^g \rightarrow M$ is a non-zero morphism, then each f_x is injective.
- (c) Let $n \geq 2$ and $g \in G$ be such that $\text{Hom}_{C_r}((\tau_{C_r}^n X^i)^g, M) \neq 0$. Then $\text{supp}(M) \cap n(x) = \text{supp}(M) \cap x^- = x^-$ for every $x \in \text{supp}((\tau_{C_r}^n X^i)^g) \cap C_r^-$. In particular, $|\text{supp}(M) \cap x^-| = r$.

Proof. (a) For $n \geq 2$, we have $\pi_\lambda((\tau_{C_r}^n X^i)^g) = \pi_\lambda(\tau_{C_r}^n X^i) = \tau_{\Gamma_r}^n X_{e_i} \in \text{EIP}$ (see section 6).

(b) For $n \geq 1$ each proper factor of $\tau_{\Gamma_r}^n X_{e_i}$ is preinjective by 3.5. Let $0 \neq f \in \text{Hom}_{C_r}((\tau_{C_r}^n X^i)^g, M)$, then $0 \neq \pi_\lambda(f) = (g_i)_{1 \leq i \leq 2} : \tau_{\Gamma_r}^n X_{e_1} \rightarrow \pi_\lambda(M)$ is injective, where

$$g_i = \bigoplus_{\pi(y)=i} f_y : \bigoplus_{\pi(y)=i} ((\tau_{C_r}^n X^i)^g)_y \rightarrow \bigoplus_{\pi(y)=i} M_y.$$

So each f_x is injective, since $\pi^{-1}(\{1, 2\}) = (C_r)_0$.

(c) Let $x \in \text{supp}((\tau_{C_r}^n X^i)^g) \cap C_r^-$ and $z \in n(x)$. Since x is a sink, there is $\alpha : z \rightarrow x$ and by (a), $(\tau_{C_r}^n X^i)^g(\alpha)$ is surjective. Hence, $((\tau_{C_r}^n X^i)^g)_z \neq 0$. By (b), we get $M_z \neq 0$ and $z \in \text{supp}(M) \cap x^-$. □

9 Considerations in the universal covering

Let us recall what we have shown so far. Given a component \mathcal{D} in $\mathcal{R}(C_r)$, the natural number $\mathcal{W}_C(\mathcal{D})$ is the distance between the two non-empty, non-intersecting cones $\text{Inj} \cap \mathcal{D}$ and $\text{Surj} \cap \mathcal{D}$. Moreover, we know that $\mathcal{W}(\pi_\lambda(\mathcal{D})) = \mathcal{W}_C(\mathcal{D})$.

Let now $X \in \mathcal{D}$ be indecomposable. Then there exists an integer $l \in \mathbb{Z}$ such that $\tau_{C_r}^l X \in \text{Surj}$, since $\text{Surj} \cap \mathcal{D}$ is non-empty. We also find $n \geq 1$ with $\tau_{C_r}^{-n} X \notin \text{Surj}$. Since $\text{Surj} \cap \mathcal{D}$ is closed under τ_{C_r} , we conclude $-n \leq l$. Therefore the following minima exist.

Definition. Let $X \in \text{rep}(C_r)$ be a regular indecomposable representation. We define

$$d^-(X) := \min\{l \in \mathbb{Z} \mid \tau_{C_r}^{-l} X \in \text{Inj}\} \text{ and } d^+(X) := \min\{l \in \mathbb{Z} \mid \tau_{C_r}^l X \in \text{Surj}\}.$$

Note that $|d^-(X)| \in \mathbb{N}_0$ is the distance of X to the border of the cone $\text{Inj} \cap \mathcal{D}$.

Lemma 9.1. Let $X \in \text{rep}(C_r)$ be indecomposable in a regular component \mathcal{D} . Then

$$\mathcal{W}_C(\mathcal{D}) = d^+(X) + d^-(X) - \text{ql}(X).$$

Proof. Since the equality is obvious for X quasi-simple we assume $l := \text{ql}(X) > 1$. Let Z be the unique quasi-simple representation with $X = Z[l]$. By induction we get $\mathcal{W}_C(\mathcal{D}) = d^+(Z[l-1]) + d^-(Z[l-1]) - (l-1)$. Now observe that $d^+(Z[l-1]) + 1 = d^+(Z[l])$ and $d^-(Z[l-1]) = d^-(Z[l])$. Hence $\mathcal{W}_C(\mathcal{D}) = d^+(Z[l]) - 1 + d^-(Z[l]) - (l-1) = d^+(X) + d^-(X) - \text{ql}(X)$. \square

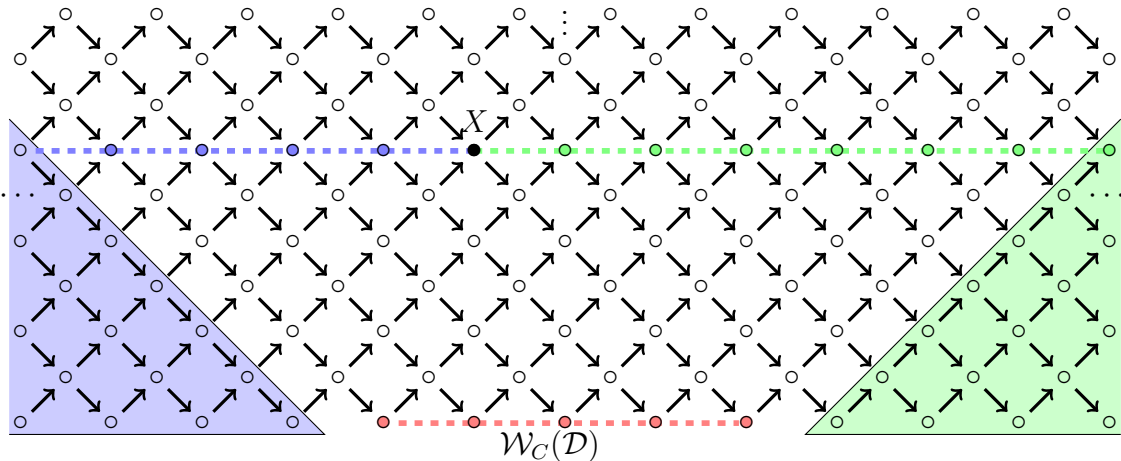


Figure 9.1: How to determine $\mathcal{W}_C(\mathcal{D})$ using an arbitrary representation in \mathcal{C} .

We show in the following how to modify a regular representation $X \rightsquigarrow X'$ such that the obtained representation X' is regular, $d^+(X) = d^+(X')$, $d^-(X) = d^-(X')$ and $\text{ql}(X') = 1$.

9.1 Indecomposable representations arising from extensions

In this section, we show how to construct non-split exact sequences with indecomposable middle term in $\text{rep}(C_r)$.

Definition. Let $M, N \in \text{rep}(C_r)$ be indecomposable. The pair (N, M) is called leaf-connected if there is $\alpha: x \rightarrow y \in (C_r)_1$ s.t.

- (a) x is a leaf of M , y is a leaf of N and
- (b) $\text{supp}(M) \cap \text{supp}(N) = \emptyset$.

Remark. Note that the assumption M and N being indecomposable together with properties (a) and (b) already implies the uniqueness of α . If (N, M) is leaf-connected, α is called the connecting arrow and (M, N) is not leaf-connected.

Definition. Let (N, M) be leaf-connected with connecting arrow $\alpha: x \rightarrow y$ and $f: M_x \rightarrow N_y$ a non-zero linear map. We define a representation $N *_f M \in \text{rep}(C_r)$ by setting

$$\text{supp}(N *_f M) := \text{supp}(N) \cup \text{supp}(M)$$

with $(N *_f M)_{\text{supp}(X)} = X$ for $X \in \{M, N\}$ and $(N *_f M)(\alpha) := f: M_x \rightarrow N_y$. Moreover, we denote by $\iota_f: N \rightarrow N *_f M$ and $\pi_f: N *_f M \rightarrow M$ the natural morphisms of quiver representations. The k -linear map along the connecting arrow is called a connecting map for (N, M) .

Remark. Note that $N *_f M$ is just an extension of M by N with corresponding exact sequence $\delta_f: 0 \rightarrow N \xrightarrow{\iota_f} N *_f M \xrightarrow{\pi_f} M \rightarrow 0$. The next lemma shows that $N *_f M$ is indecomposable. In particular, δ_f does not split. One can also show (see section 10.5) that the map

$$\Phi: \text{Hom}_k(M_x, N_y) \rightarrow \text{Ext}_{C_r}(M, N), f \mapsto [\delta_f],$$

is an isomorphism of vector spaces, where δ_0 denotes the short exact sequence $0 \rightarrow N \rightarrow N \oplus M \rightarrow M \rightarrow 0$ and $[\delta_0]$ the neutral element in the abelian group $\text{Ext}_{C_r}(M, N)$ with respect to the Baer sum.

Lemma 9.2. *Let (N, M) be leaf-connected.*

- (a) *The representation $N *_f M$ is indecomposable.*
- (b) *If M and N are regular, then $N *_f M$ is regular.*

Proof. (a) Let $U_1, U_2 \in \text{rep}(C_r)$ with $N *_f M = U_1 \oplus U_2$. Hence, we get

$$M = (N *_f M)_{\text{supp}(M)} = (U_1)_{\text{supp}(M)} \oplus (U_2)_{\text{supp}(M)}.$$

Since $M \in \text{rep}(C_r)$ is indecomposable, there is a unique $i \in \{1, 2\}$ with $(U_i)_{\text{supp}(M)} = M$. We assume $i = 1$. By the same token there is a unique j with $(U_j)_{\text{supp}(N)} = N$. Since $(N *_f M)(\alpha) = f \neq 0$, it follows that $j = i = 1$. Hence, $U_2 = 0$ and $N *_f M$ is indecomposable. (b) By construction N is a subrepresentation of $N *_f M$ and M a factor representation. Since $\pi_\lambda(N *_f M)$ is indecomposable with regular factor representation $\pi_\lambda(M)$, it is regular itself or preprojective. By the same token $\pi_\lambda(N *_f M)$ is regular or preinjective. \square

Lemma 9.3. *Let $M, N \in \text{rep}(C_r)$ be indecomposable and assume $x \in C_r^+$ is a leaf of M and $y \in C_r^-$ is a leaf of N . Then there exists $g \in G$ such that (N^g, M) is leaf-connected with connecting arrow $x \rightarrow g^{-1}.y$.*

Proof. Since x, y are leaves, we find at most one arrow $\delta_M: x \rightarrow x_1$ and at most one arrow $\delta_N: y_1 \rightarrow y$ with $x_1 \in \text{supp}(M)$ and $y_1 \in \text{supp}(N)$. Since $r \geq 3$ we find an arrow $\alpha: x \rightarrow z$ with $\pi(\delta_M) \neq \pi(\alpha) \neq \pi(\delta_N)$. In particular $\alpha \neq \delta_M$ and $x_1 \neq z$.

Now let $g \in \pi(\Gamma_r)$ be the unique element with $g.z = y$. Then $z \in n(x)$ is a leaf of N^g . By construction, we have $\pi(g.\alpha) = \pi(\alpha) \neq \pi(\delta_N)$ and conclude $x \notin \text{supp}(N^g)$. It follows that $\text{supp}(M) \cap \text{supp}(N^g) = \emptyset$. Hence, (N^g, M) is leaf-connected with connecting arrow α . \square

Definition. Let $n \geq 2$ and $M_1, \dots, M_n \in \text{rep}(C_r)$ be indecomposable. The tuple (M_n, \dots, M_1) is called leaf-connected, provided that (M_{i+1}, M_i) is leaf-connected for all $1 \leq i < n$. A tuple (f_{n-1}, \dots, f_1) of k -linear maps is called a connecting map for (M_n, \dots, M_1) if f_i is a connecting map for (M_{i+1}, M_i) for all $1 \leq i < n$.

Remark. Let (M, L) and (L, N) be leaf-connected with connecting maps f and g , then $\text{supp}(M) \cap \text{supp}(N) = \emptyset$, since C_r is a tree. Hence, $\text{supp}(M *_f L) \cap \text{supp}(N) = \emptyset$ and $(M *_f L, N)$ is leaf-connected. Therefore the following definition is meaningful.

Definition. Let $n \geq 2$, (M_n, \dots, M_1) be leaf-connected with connecting map (f_{n-1}, \dots, f_1) . Then we define inductively $M_n *_f M_{n-1} *_f \dots *_f M_i := (M_n *_f M_{n-1} *_f \dots *_f M_{i+1}) *_f M_i$ for all $2 \leq i < n$.

From now on, we assume that (M_n, \dots, M_1) is leaf-connected with connecting map (f_{n-1}, \dots, f_1) . For $1 \leq i \leq n$, we define $*_{j \geq i} M_j := M_n *_f M_{n-1} *_f \dots *_f M_i$ and $*_{j \leq i} M_j := M_i *_f M_{i-1} *_f \dots *_f M_1$. Moreover, we set $*_{j \geq n} M_j = M_n$ and $*_{j \leq 1} M_j = M_1$.

Lemma 9.4. Let $n \geq 2$, (M_n, \dots, M_1) be leaf-connected and $1 \leq i < n$.

- (a) The representation $*_{j \geq i} M_j$ is indecomposable.
- (b) There is a short exact sequence $0 \rightarrow *_{j \geq i+1} M_j \rightarrow *_{j \geq i} M_j \rightarrow M_i \rightarrow 0$.
- (c) If M_n, M_1 are regular, then $*_{j \geq i} M_j$ is regular.

Proof. Statement (a) follows from 9.2. For (c), just note that M_i is balanced for $2 \leq i \leq n-1$, hence M_i is regular. Now apply 9.2 \square

Lemma 9.5. Let $X, Y \in \text{rep}(C_r)$ be regular indecomposable. Then the following statements are equivalent:

- (a) There is $g \in G$ such that $\text{Hom}_{C_r}(X^g, Y) \neq 0$.
- (b) There is $h \in G$ such that $\text{Hom}_{C_r}(\tau_{C_r} X^h, \tau_{C_r} Y) \neq 0$.
- (c) There is $l \in G$ such that $\text{Hom}_{C_r}(\tau_{C_r}^{-1} X^l, \tau_{C_r}^{-1} Y) \neq 0$.

Proof. We only show (a) \Rightarrow (b). Let $g \in G$, such that $0 \neq \text{Hom}_{C_r}(X^g, Y)$. Then

$$0 \neq \text{Hom}_{\Gamma_r}(\pi_\lambda(X), \pi_\lambda(Y)) \cong \text{Hom}_{\Gamma_r}(\tau \circ \pi_\lambda(X), \tau \circ \pi_\lambda(Y)) \cong \text{Hom}_{\Gamma_r}(\pi_\lambda(\tau_{C_r} X), \pi_\lambda(\tau_{C_r} Y)).$$

By [19, 3.6(c)], we find $h \in G$ such that $0 \neq \text{Hom}_{C_r}((\tau_{C_r} X)^h, \tau_{C_r} Y) \cong \text{Hom}_{C_r}(\tau_{C_r} X^h, \tau_{C_r} Y)$. \square

In the following proof we repeatedly apply the above lemma for the representations X^1, \dots, X^r , which by Corollary 8.5 determine whether a representation is in Inj.

Proposition 9.6. Let $n \geq 2$, (M_n, \dots, M_1) be leaf-connected and M_1, M_n regular, then

- (a) $\max\{d^-(*_{j \geq i} M_j) \mid 1 \leq i \leq n\} \leq d^-(*_{j \geq 1} M_j) \leq \max\{d^-(M_i) \mid 1 \leq i \leq n\}$.
- (b) $\max\{d^+(*_{j \leq i} M_j) \mid 1 \leq i \leq n\} \leq d^+(*_{j \leq 1} M_j) \leq \max\{d^+(M_i) \mid 1 \leq i \leq n\}$.

Proof. Note that we have a filtration of $*_{j \geq 1} M_j$ by regular subrepresentations

$$0 \subset M_n \subset M_n * M_{n-1} \subset M_n * M_{n-1} * M_{n-2} \subset \cdots \subset *_{j \geq 2} M_j \subset *_{j \geq 1} M_j,$$

with $*_{j \geq i} M_j / *_{j \geq i+1} M_j \cong M_i$ regular for all $1 \leq i \leq n$, where $*_{j \geq n+1} M_j := 0$.

(a) Let $i \in \{1, \dots, n\}$ and $Z := *_{j \geq i} M_i$. Consider the short exact sequence

$$0 \rightarrow Z \rightarrow *_{j \geq 1} M_j \rightarrow (*_{j \geq 1} M_j) / Z \rightarrow 0.$$

Now let $l \in \mathbb{Z}$ such that $\tau_{C_r}^{-l} Z \notin \text{Inj}$. Then there exists $i \in \{1, \dots, r\}$ and $g \in G$ such that $\text{Hom}_{C_r}((X^i)^g, \tau_{C_r}^{-l} Z) \neq 0$. Hence, we find $h \in G$ with $\text{Hom}_{C_r}((\tau_{C_r}^l X^i)^h, Z) \neq 0$ for some $h \in G$. Left-exactness of $\text{Hom}_{C_r}((\tau_{C_r}^l X^i)^h, -)$ ensures that $0 \neq \text{Hom}_{C_r}((\tau_{C_r}^l X^i)^h, *_{j \geq 1} M_j)$ and therefore we find $f \in G$ with $0 \neq \text{Hom}_{C_r}((X^i)^f, \tau_{C_r}^{-l}(*_{j \geq 1} M_j))$. Hence, $\tau_{C_r}^{-l}(*_{j \geq 1} M_j) \notin \text{Inj}$ and therefore $d^-(Z) \leq d^-(*_{j \geq 1} M_j)$.

If $\text{Hom}_{C_r}((X^i)^g, \tau_{C_r}^{-l}(*_{j \geq 1} M_j)) \neq 0$ for some $g \in G$ and $i \in \{1, \dots, r\}$, then we find $h \in G$ with $0 \neq \text{Hom}_{C_r}(\tau_{C_r}^l(X^i)^h, *_{j \geq 1} M_j)$. By 4.6, we find $1 \leq p \leq n$ with $0 \neq \text{Hom}_{C_r}(\tau_{C_r}^l(X^i)^h, M_p)$. Hence, there is $u \in G$ with $0 \neq \text{Hom}_{C_r}((X^i)^u, \tau_{C_r}^{-l} M_p)$. Hence, $d^-(*_{j \geq 1} M_j) \leq d^-(M_p)$ since Inj is closed under $\tau_{C_r}^{-1}$.

(b) Note for $i \geq 2$ that $*_{j \geq 1} M_j / *_{j \geq i} M_j \cong *_{j < i} M_j$. Hence, $D_{C_r}(*_{j \geq 1} M_j)$ has a filtration

$$0 \subset D_{C_r} M_1 \subset D_{C_r} M_1 * D_{C_r} M_2 \subset \cdots \subset *_{j \leq n-1} D_{C_r} M_j \subset D_{C_r}(*_{j \geq 1} M_j),$$

here we use the fact that $D_{C_r}(X *_f Y) \cong D_{C_r} Y *_f * D_{C_r} X$ for each leaf-connected pair (X, Y) with connecting map f . Now apply (a) and note that $d^+(X) = d^-(D_{C_r} X)$ for each regular indecomposable representation X . \square

9.2 Small representations and trees

Definition. A balanced representation N is called small if $1 \leq d^-(N), d^+(N) \leq 2$.

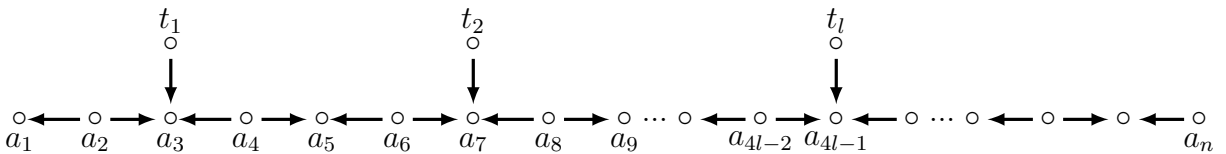
Note that N being balanced always implies $d^-(N) \geq 1$ and $d^+(N) \geq 1$.

Definition. Denote by $\overline{C_r}$ the underlying graph of C_r . Then $((C_r)_0, d)$ obtains the structure of a metric space, where $d(x, y) \in \mathbb{N}_0$ denotes the length of the unique path in $\overline{C_r}$ connecting vertices x and y .

Definition. Let $T \subseteq C_r$ be a finite subtree. T is called small if

- (a) T has leaves in C_r^+ and C_r^- ,
- (b) for all $x \in T_0$, we have $|T_0 \cap n(x)| \leq 3$,
- (c) if $|T_0 \cap n(x)| = 3 = |T_0 \cap n(y)|$ then $x = y$ or $d(x, y) \geq 3$.

Example. Let $l \in \mathbb{N}$ and $n \in 2\mathbb{N}$ with $n \geq 4l$. We denote by $\mathbb{A}_{l,n} \subseteq C_r$ a (small) subtree of the following form:



Lemma 9.7. *Assume L is an indecomposable representation with small tree $T(L)$. Then L is small.*

Proof. Since $T(L)$ is small with $T(L)_0 = \text{supp}(L)$, L is balanced and therefore regular.

We first show that $d^-(L) \leq 2$. Therefore it suffices to show that for all $l \in \mathbb{Z}$ with $\tau_{C_r}^{-l}L \notin \text{Inj}$ we have $l \leq 1$.

Assume to the contrary that $l \geq 2$. By Corollary 8.5 and Lemma 9.5 we find $i \in \{1, \dots, r\}$ and $g \in G$ with $0 \neq \text{Hom}_{C_r}((\tau_{C_r}^l X^i)^g, L)$. Fix $h: (\tau_{C_r}^l X^i)^g \rightarrow L$ non-zero.

To get a contradiction, we will now show that our assumption $l \geq 2$ yields that $\text{supp}((\tau_{C_r}^l X^i)^g)$ contains only 4 vertices. Then it will be easy to see that this can not happen.

By Lemma 8.8, h is a monomorphism and $\text{supp}((\tau_{C_r}^l X^i)^g) \subseteq \text{supp}(L)$. Let s be a sink of $(\tau_{C_r}^l X^i)^g$. By Lemma 8.8, we have $|\text{supp}(L) \cap n(s)| = r$. Since $T(L)$ is small we get $3 \leq r = |\text{supp}(L) \cap n(s)| \leq 3$. Hence,

$$(*) \quad r = 3 = |n(s) \cap \text{supp}(L)|.$$

Now let $t_1, t_2 \in \text{supp}((\tau_{C_r}^l X^i)^g)$ be sinks. Then $(*)$ yields

$$|n(t_1) \cap \text{supp}(L)| = 3 = |n(t_2) \cap \text{supp}(L)|.$$

Since $T(L)$ is small we get $t_1 = t_2$ or $d(t_1, t_2) \geq 3$. Assume that $d(t_1, t_2) \geq 3$. Since $\text{supp}((\tau_{C_r}^l X^i)^g)$ is connected we find an unoriented path from t_1 to t_2 in $\text{supp}((\tau_{C_r}^l X^i)^g)$. Let u be the unique vertex in this path with $d(t_1, u) = 2$. Since C_r has bipartite orientation we conclude that u is a sink. But now $(*)$ implies $|n(u) \cap \text{supp}(L)| = 3$, this is a contradiction because $T(L)$ is small. It follows $t_1 = t_2$.

Hence $\text{supp}((\tau_{C_r}^l X^i)^g)$ contains exactly one sink s . Write $n(s) = \{a, b, c\}$. Since $l \geq 2$, Lemma 8.8(a) implies $\text{supp}((\tau_{C_r}^l X^i)^g) = \{s, a, b, c\}$.

Hence, $Z := (\tau_{C_r}^l X^i)^g$ can be considered as a representation of the Dynkin diagram D_4 with unique sink s such that all linear maps are surjective. It follows that $Z_x = 1$ for all $x \in \{s, a, b, c\}$ (see Figure 2.2). Hence $\underline{\dim} \pi_\lambda(Z) = (3, 1)$ and $\pi_\lambda(Z)$ is indecomposable. But the only indecomposable representation $I \in \text{rep}(\Gamma_3)$ with dimension vector $(3, 1)$ is injective. This is a contradiction since Z is regular. Therefore $l = 1$ and $d^-(L) \leq 2$. For the other inequality note that $T(L)$ is small if and only if $T(D_{C_r}L)$ is small and $d^+(L) = d^-(D_{C_r}L) \leq 2$. \square

9.3 The Main Theorem

In this section we establish our main theorem, which will allow us prove the existence of regular components for each width in \mathbb{N}_0 .

Lemma 9.3.1. *Let $M \in \text{rep}(\Gamma_r)$ be an indecomposable representation with dimension vector $\underline{\dim} M = (a + 1, a)$, $a \geq 1$. Then M is a regular and quasi-simple representation.*

Proof. That M is regular follows since $(a+1, a)$ is an imaginary root of q_{Γ_r} . By [12, 3.4] it suffices to show that A_t is not a common divisor of $a + 1$ and a for all $t \geq 2$, where $\underline{\dim} P_i = (A_{i-1}, A_i)$ is the dimension vector of the preprojective indecomposable representation P_i (see Figure 2.5). But this is trivial since $\gcd(a + 1, a) = 1$. \square

Theorem 9.3.2. *Let $M \in \text{rep}(C_r)$ be balanced in the regular component \mathcal{D} and $2 \leq d^+(M), d^-(M)$. There is $n_0 \in \mathbb{N}$ and a sequence of pairwise distinct regular components $(\mathcal{D}_n)_{n \geq n_0}$ such that for each $n \geq n_0$*

$$\mathcal{W}_C(\mathcal{D}_n) = \mathcal{W}_C(\mathcal{D}) + \text{ql}(M) - 1.$$

Moreover, \mathcal{D}_n contains a balanced quasi-simple representation F_n with $\underline{\dim} \pi_\lambda(F_n) = (n+1, n)$ or $\underline{\dim} \pi_\lambda(F_n) = (n, n+1)$.

For the benefit of the reader let us give an outline of the following proof. Our strategy is to construct an indecomposable representation say M' (which depends on n) such that $d^-(M) = d^-(M')$, $d^+(M) = d^+(M')$ and $\text{ql}(M') = 1$. Then we conclude that the regular component \mathcal{D}' containing M' satisfies $\mathcal{W}_C(\mathcal{D}') = d^+(M) + d^-(M) - 1$, which will turn out to be equal to $\mathcal{W}_C(\mathcal{D}) + \text{ql}(M) + 1$.

We build the representation M' as a $*$ -product $M' = M^g * L * M$ for suitable $g \in G$ and L indecomposable with $\text{supp}(L)$ of type $\mathbb{A}_{x,y}$ (see Figure 9.2). We also choose x and y such that we can apply Lemma 9.3.1.

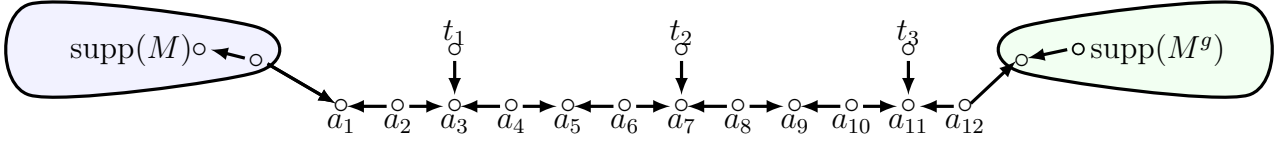


Figure 9.2: Illustration of the proof with $\text{supp}(L)$ of type $\mathbb{A}_{3,12}$.

Proof. Write $\underline{\dim} \pi_\lambda(M) = (a, b)$. After dualising M we can assume $a \leq b$. Set $l := 2(b - a) + 1 \geq 1$ and $p_0 := 4l$. Now let $p \geq p_0$ with $p \in 2\mathbb{N}$. Consider an indecomposable and thin representation L (i.e. $\dim_k L_x \leq 1$ for all $x \in (C_r)_0$) such that (L, M) is leaf-connected and $T(L) = T(\text{supp}(L))$ is of type $\mathbb{A}_{l,p}$. Let $g \in G$ be such that (M^g, L) is leaf-connected. We conclude for $n := 2b + \frac{1}{2}p \in \mathbb{N}$, $n_0 := 2b + \frac{1}{2}p_0$, and $F_n := M^g * L * M$ that

$$\begin{aligned} \underline{\dim} \pi_\lambda(F_n) &= \underline{\dim} \pi_\lambda(M^g * L * M) = 2 \underline{\dim} \pi_\lambda(M) + \underline{\dim} \pi_\lambda(L) = 2(a, b) + \left(\frac{1}{2}p + l, \frac{1}{2}p\right) \\ &= (2a + 2b - 2a + 1, 2b) + \frac{1}{2}(p, p) = (n + 1, n). \end{aligned}$$

By Lemma 9.4 F_n is a regular indecomposable representation and by Lemma 9.3.1 $\pi_\lambda(F_n)$ is quasi-simple. Therefore F_n is quasi-simple in a regular component \mathcal{D}_n . We conclude with Corollary 9.6

$d^-(M) = d^-(M^g) \leq d^-(F_n) \leq \max\{d^-(M), d^-(L)\} \stackrel{d^-(L) \leq 2, d^-(M) \geq 2}{=} \max\{d^-(M), 2\} = d^-(M)$,
i.e. $d^-(M) = d^-(F_n)$. By the same token we have $d^+(M) = d^+(F_n)$ and conclude

$$\begin{aligned} \mathcal{W}_C(\mathcal{D}_n) &= d^+(F_n) + d^-(F_n) - \text{ql}(F_n) = d^+(M) + d^-(M) - 1 \\ &= (d^+(M) + d^-(M) - \text{ql}(M)) + \text{ql}(M) - 1 = \mathcal{W}_C(\mathcal{D}) + \text{ql}(M) - 1. \end{aligned}$$

It follows immediatly from the construction that the regular components are pairwise distinct, since F_i, F_j are non-isomorphic and satisfy $\text{ql}(F_i) = \text{ql}(F_j)$ and $d^-(F_i) = d^-(F_j)$ for $i \neq j \geq n_0$. \square

Corollary 9.3.3. *Let $M \in \mathcal{D}$ be balanced and $2 \leq d^+(M), d^-(M)$. Then there exists a balanced and quasi-simple representation F in a regular component \mathcal{E} such that $\mathcal{W}_C(\mathcal{E}) = \mathcal{W}_C(\mathcal{D}) + \text{ql}(M) - 1$. Moreover, there is a leaf $x \in C_r^+$ of F with $\dim_k F_x = 1 = \dim_k F_y$ for the unique element $y \in x^+ \cap \text{supp}(F)$.*

Proof. Fix $n \geq n_0$ in the proof of the theorem and set $F := F_n = M^g * L * M$. The last claim follows since L is a thin representation of type $\mathbb{A}_{l,p}$ which has $l + 1 \geq 2$ leaves in C_r^+ (see Figure 9.2). \square

10 Applications and further considerations

10.1 Regular components for every width

The aim of this section is to construct for each $n \in \mathbb{N}$ a regular component \mathcal{D} with $\mathcal{W}_C(\mathcal{D}) = n$. Although each indecomposable representation has a leaf, it is in general not true that a regular representation has leaves in C_r^+ and C_r^- . For example if M is indecomposable in Inj , then each leaf of M is a sink by Theorem 8.1. The next results shows that self-dual representations have leaves in C_r^+ and C_r^- .

Lemma 10.1. *Let $M \in \text{rep}(C_r)$ be indecomposable such that $D_{\Gamma_r}\pi_\lambda(M) \cong \pi_\lambda(M)$, then M is balanced.*

Proof. Since $\text{supp}(M)$ is finite there exists a leaf x of M . Without loss of generality we assume that $x \in C_r^+$. We get $\pi_\lambda(M) \cong D_{\Gamma_r}\pi_\lambda(M) \cong \pi_\lambda(D_{C_r}M)$. Therefore we find $h \in G$ such that $M \cong (D_{C_r}M)^h$. Since $h^{-1}.\varphi(x) \in C_r^-$ (see 7.2) is a leaf of $(D_{C_r}M)^h$ the claim follows. \square

We denote by σC_r the quiver obtained by changing the orientation of all arrows in C_r . Note that $\sigma^2 C_r = C_r$ and $\sigma C_r \cong C_r$. We denote by Φ^+ the composition of the Bernstein-Gelfand-Ponomarev reflection functors [1, VII.5.5.] for all the sources of C_r . Φ^+ is a well-defined functor $\Phi^+ : \text{rep}(C_r) \rightarrow \text{rep}(\sigma C_r)$ (see [33, 2.3]). By the same token, we have a functor $\Phi^- : \text{rep}(\sigma C_r) \rightarrow \text{rep}(C_r)$ given by the composition of the reflection functors for all the sources of σC_r . Then $\mathcal{F} := \Phi^- \circ \Phi^+ : \text{rep}(C_r) \rightarrow \text{rep}(C_r)$ satisfies $\mathcal{F}(M) \cong \tau_{C_r}^{-1}M$ for $M \in \text{rep}(C_r)$ indecomposable and non-injective [33, 2.3],[1, VII.5.8.]. Therefore statements (a) and (b) of the next lemma follow immediately from the definition of the reflection functors.

Lemma 10.2. *Let M be in $\text{rep}(C_r)$ indecomposable and not injective.*

- (a) *For each $x \in C_r^+$ we have $\dim_k(\tau_{C_r}^{-1}M)_x = (\sum_{y \in x^+} \dim_k M_y) - \dim_k M_x$.*
- (b) *For each $y \in C_r^-$ we have $\dim_k(\tau_{C_r}^{-1}M)_y = (\sum_{x \in y^-} \dim_k(\tau_{C_r}^{-1}M)_x) - \dim_k M_y$.*
- (c) *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an almost split sequence with B indecomposable. If $a \in C_r^-$ is a leaf of A , then B has a leaf in C_r^- .*
- (d) *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an almost split sequence with B indecomposable. If $a \in C_r^+$ is a leaf of A and $b \in a^+$ satisfies $\dim_k A_b = \dim_k A_a$, then a is a leaf of B .*

Proof. (c) Consider a path $a \leftarrow b \rightarrow c$ such that b, c are not in $\text{supp}(A)$ as illustrated in Figure 10.1. Since b is in C_r^+ we get with (a) that

$$\dim_k C_b = \dim_k(\tau_{C_r}^{-1}A)_b = \left(\sum_{y \in b^+} \dim_k A_y \right) - \dim_k A_b = \dim_k A_a - \dim_k A_b = \dim_k A_a \neq 0.$$

Now let $d \in n(c) \setminus \{b\}$. Then $\dim_k C_d = (\sum_{y \in d^+} \dim_k A_y) - \dim_k A_d = 0$, since $\text{supp}(A)$ is connected and C_r is a tree. Hence, we get that $\dim_k C_c = \dim_k(\tau_{C_r}^{-1}A)_c \stackrel{(b)}{=} (\sum_{x \in c^-} \dim_k(\tau_{C_r}^{-1}A)_x) -$

$\dim_k A_c = \dim_k C_b - \dim_k A_c = \dim_k C_b \neq 0$. Hence, $c \in \text{supp}(C)$ is a leaf of C and since $\text{supp}(B) = \text{supp}(A) \cup \text{supp}(C)$. Since B is indecomposable, $\text{supp}(B)$ is connected and we conclude as before that $c \in C_r^-$ is a leaf of B .

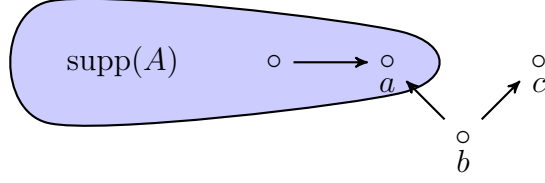


Figure 10.1: Illustration of the setup for (c).

(d) Application of (a) yields (see Figure 10.2)

$$\dim_k C_a = \dim_k(\tau_{C_r}^{-1}A)_a = \left(\sum_{z \in a^+} \dim_k A_z \right) - \dim_k A_a = \dim_k A_b - \dim_k A_a = 0.$$

Now fix $c \in a^+ \setminus \{b\}$, then $c \in C_r^-$. Let $d \in c^- \setminus \{a\}$ and $f \in d^+ \cup \{d\}$. Then $f \notin \text{supp}(A)$ since $\text{supp}(A)$ is connected and C_r is a tree. Hence $\dim_k A_f = 0$ for all $f \in d^+ \cup \{d\}$. Hence, we get

$$\dim_k C_d = \dim_k(\tau_{C_r}^{-1}A)_d \stackrel{(a)}{=} \left(\sum_{y \in d^+} \dim_k A_y \right) - \dim_k A_d = 0.$$

We conclude

$$\dim_k C_c = \dim_k(\tau_{C_r}^{-1}A)_c \stackrel{(b)}{=} \left(\sum_{z \in c^-} \dim_k(\tau_{C_r}^{-1}A)_z \right) - \dim_k A_c = \left(\sum_{z \in c^-} \dim_k C_z \right) - \dim_k A_c = 0 - 0 = 0.$$

We have shown that $(a^+ \setminus \{b\}) \cap (\text{supp}(A) \cup \text{supp}(C)) = \emptyset$. Since $\text{supp}(A) \cup \text{supp}(C) = \text{supp}(B)$ we get $|\text{supp}(B) \cap n(a)| = |(\text{supp}(A) \cup \text{supp}(C)) \cap n(a)| \leq |\{b\}| = 1$. Since $a \in \text{supp}(A) \subseteq \text{supp}(B)$ the vertex $a \in C_r^+$ is a leaf of $T(\text{supp}(B))$. Since B is indecomposable we have $T(B) = T(\text{supp}(B))$.

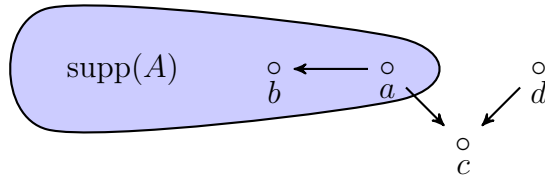


Figure 10.2: Illustration of the setup for (d).

□

Let $M_1 \in \text{rep}(C_r)$ be regular with $\dim_k M_1 = 2$. We define inductively a sequence of indecomposable representations in the regular component $\overline{\mathcal{D}}$ of M_1 . The representation M_1 is quasi-simple. Assume that M_n is already defined. If n is odd, then M_{n+1} is the unique indecomposable representation with irreducible epimorphism $M_{n+1} \rightarrow M_n$; if n is even, then M_{n+1} is the unique indecomposable representation with irreducible monomorphism $M_n \rightarrow M_{n+1}$. The component $\overline{\mathcal{D}}$ is shown in Figure 10.3. We have $\mathcal{W}_C(\overline{\mathcal{D}}) = \mathcal{W}(\pi_\lambda(\overline{\mathcal{D}})) = 1$ [48, Example 3.3].

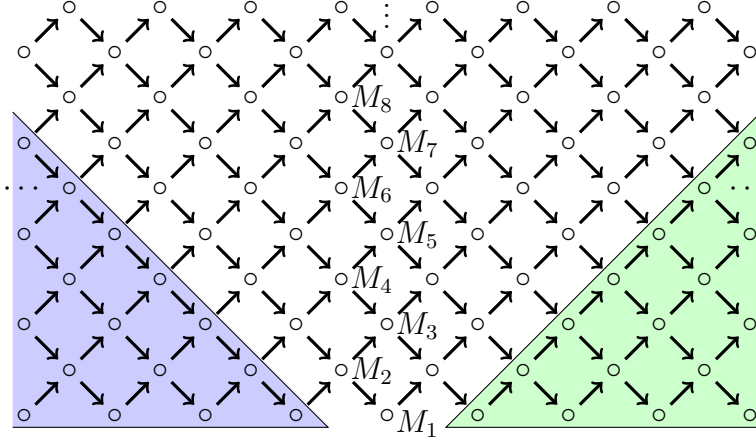


Figure 10.3: Component containing the family $(M_n)_{n \geq 1}$.

Theorem 10.3. *Let $r \geq 3$.*

- (a) *For each $m \geq 3$ there is $n_0 \geq 1$ and a family $(\mathcal{D}_n)_{n \geq n_0}$ of regular components with $\mathcal{W}_C(\mathcal{D}_n) = m$ and \mathcal{D}_n contains a quasi-simple representation E_n with $\underline{\dim} \pi_\lambda(E_n) = (n+1, n)$ or $\underline{\dim} \pi_\lambda(E_n) = (n, n+1)$.*
- (b) $\mathbb{N} \subseteq \{\mathcal{W}_C(\mathcal{E}) \mid \mathcal{E} \in \mathcal{R}(C_r)\}$.
- (c) $\{\mathcal{W}_C(\mathcal{C}) \mid \mathcal{C} \in \mathcal{R}\} = \mathbb{N}_0$.

Proof. (a) We first consider the case that $m \geq 3$ is odd. Hence, there is $l \geq 1$ with $m = 2l + 1$. Consider the representation M_{2l+1} defined above. We have $D\pi_\lambda(M_{2l+1}) \cong \pi_\lambda(M_{2l+1})$ by [48, Example 3.3]. By Lemma 10.1 M_{2l+1} is balanced. Moreover, we have $d^-(M_{2l+1}), d^+(M_{2l+1}) \geq 2$. Hence, Theorem 9.3.2 yields $n_0 \in \mathbb{N}$ and a sequence of pairwise distinct regular components $(\mathcal{D}_n)_{n \geq n_0}$ of width

$$\mathcal{W}_C(\mathcal{D}_n) = \mathcal{W}_C(\overline{\mathcal{D}}) + \text{ql}(M_{2l+1}) - 1 = 1 + 2l + 1 - 1 = 2l + 1 = m,$$

and \mathcal{D}_n contains a representation E_n with the desired properties. Hence, we have proven the statement for each $m \geq 3$ odd.

Moreover, Corollary 9.3.3 yields a balanced and quasi-simple representation A , in a regular component \mathcal{E} of width $\mathcal{W}_C(\mathcal{E}) = \mathcal{W}_C(\overline{\mathcal{D}}) + \text{ql}(M_{2l+1}) - 1 = m$, that satisfies the assumption of Lemma 10.2(d). Consider the AR sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ starting in A . Then B is balanced by Lemma 10.2(c)(d), $2 \leq d^+(B), d^-(B)$ and $\text{ql}(B) = 2$. By Theorem 9.3.2 we get an infinite sequence of pairwise distinct components of width $\mathcal{W}_C(\mathcal{E}) + \text{ql}(B) - 1 = (2l+1) + 2 - 1 = 2l + 2 = m + 1$ with the desired properties. Hence, we have also proven the statement for all even numbers ≥ 3 .

(b) Recall from the example in section 6 that $\mathcal{W}_C(\overline{\mathcal{E}}) = 2$ for the regular component $\overline{\mathcal{E}}$ containing X^1 . A regular component with width 1 is given by $\overline{\mathcal{D}}$.

(c) There exists a regular component $\mathcal{C} \in \mathcal{R}(\Gamma_r)$ with $\mathcal{W}_C(\mathcal{C}) = 0$ (see 4.8). □

Remark. Observe that one can generalize statement (a); if \mathcal{D}_n is a component with $\mathcal{W}_C(\mathcal{D}_n) = m$ and E_n the quasi-simple representation in \mathcal{D}_n with $\underline{\dim} E_n = (n+1, n)$, then $D_{C_r} E_n$ is quasi-simple in a regular component \mathcal{R}_n with $\mathcal{W}_C(\mathcal{R}_n) = m$ and $\underline{\dim} D_{C_r} E_n = (n, n+1)$.

10.2 Counting regular components of fixed width

This section is motivated by the following result by Kerner and Lukas.

Proposition. [25, 5.2] *Assume that k is uncountable and kQ is a wild quiver algebra with $|Q_0| \geq 3$. Then the number of regular component of kQ with quasi-rank -1 is uncountable. Moreover, the set of components of quasi-rank ≤ -1 for the Kronecker algebra is uncountable.*

The proof of the second statement uses the first statement for the path algebra kQ of the wild quiver $Q = 1 \rightarrow 2 \rightrightarrows 3$ with $|Q_0| = 3$ simple modules and the existence of a regular tilting module T_r in $\text{mod } kQ$ that induces a bijection

$$\varphi: \mathcal{R}(Q) \rightarrow \mathcal{R}(\Gamma_r)$$

between the set of regular components of kQ and the set of regular components of \mathcal{K}_r , with $\text{rk}(\varphi(\mathcal{C})) \leq \text{rk}(\mathcal{C})$ for all $\mathcal{C} \in \mathcal{R}(Q)$. To generalize the arguments from ≤ -1 to $\leq -p$ for $p \in \mathbb{N}$ one would need the existence of bricks of arbitrary quasi-length. Unfortunately for each wild quiver algebra there is a finite upper bound for the quasi-length of regular bricks given by the number of simple modules -1 , which is in our case $|Q_0| - 1 = 2$. We show how to circumvent this obstacle by considering the action of $\text{GL}_r(k)$ on $\text{rep}(\Gamma_r)$. We begin by considering the following examples.

- Examples.** (a) The simple representations of Γ_r are $\text{GL}_r(k)$ -stable and by [16, 2.2] every preinjective and every preprojective representation is $\text{GL}_r(k)$ -stable.
- (b) There are $\text{GL}_r(k)$ -stable representations that are regular [48, 1.2]. In this case all representations in the same component are also $\text{GL}_r(k)$ -stable.
- (c) Recall that the preinjective representation $I_3 = \tau_{\Gamma_r} I_1$ has dimension vector $(3r - 1, r)$. Since I_1 is G -gradable and π_λ interchanges with the Auslander-Reiten translates, there exists M in $\text{rep}(C_r)$ with $\pi_\lambda(M) \cong I_3$. The support of M for $r = 3$ is shown in Figure 10.4. Let $c \in T(M)_0$ be the vertex with $\dim_k M_c = 2$.

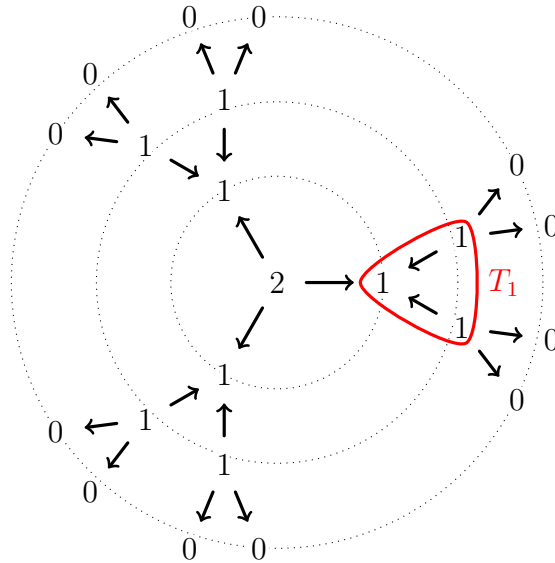


Figure 10.4: Support of M with $\pi_\lambda(M) \cong I_3$ for $r = 3$.

The underlying tree of $\text{supp}(M)$ is symmetric in the following sense. The quiver $T(M) \setminus \{c\}$ is not connected and consists of $r = 3$ isomorphic trees T_1, T_2, T_3 . Moreover, for $n \in \{1, 2\}$ the sum $\dim_k M_x$ for $x \in (T_i)_0$ with distance $d(c, x) = n$ is independent of $i \in \{1, 2, 3\}$. We will prove that this is not a coincidence. We show that every representation M such that $\pi_\lambda(M)$ is $\text{GL}_r(k)$ -stable, has a central point.

10.2.1 \mathcal{S}_r -stability

Denote by \mathcal{S}_r the symmetric group on $\{1, \dots, r\}$. Then each for each $\sigma \in \mathcal{S}_r$ there is an induced bijection on $\{1, \dots, r\} \rightarrow (\Gamma_r)_1$ given by $i \mapsto \gamma_{\sigma(i)}$ which extends in a natural way to the set of equivalence classes of walks in Γ_r . By abuse of notation we denote by $\sigma: C_r \rightarrow C_r$ the induced quiver automorphism. Let $\alpha: [w] \rightarrow [\gamma_i w]$ be an arrow in C_r , then by definition $\sigma(\alpha)$ is the unique arrow $\sigma(\alpha): \sigma([w]) \rightarrow [\gamma_{\sigma(i)} \sigma([w])]$. Note that $\pi(\alpha) = \gamma_i$ and $\pi(\sigma(\alpha)) = \gamma_{\sigma(i)}$. Now let $M \in \text{rep}(C_r)$ be an indecomposable representation. We define $\sigma(M)$ to be the indecomposable representation with

$$\sigma(M)_x := M_{\sigma(x)} \text{ and } \sigma(M)(\alpha) := M(\sigma(\alpha)).$$

We say that M is \mathcal{S}_r -stable if for each $\sigma \in \mathcal{S}_r$ there is g_σ with $M \cong \sigma(M)^{g_\sigma}$. This definition is motivated by the following obvious result:

Corollary 10.4. *Let $M \in \text{rep}(C_r)$ be an indecomposable representation. If $\pi_\lambda(M)$ is $\text{GL}_r(k)$ -stable then M is \mathcal{S}_r -stable.*

Proof. We let $I(\sigma)$ be the permutation matrix given by σ , i.e. $I(\sigma)_{ij} = 1$ if and only if $\sigma(i) = j$ and $I(\sigma)_{ij} = 0$ otherwise. Now we assume that $\pi_\lambda(M)$ is $\text{GL}_r(k)$ -stable. Then we get for each $\sigma \in \mathcal{S}_r$ that

$$\pi_\lambda(\sigma(M)) = I(\sigma) \cdot \pi_\lambda(M) \cong \pi_\lambda(M).$$

Hence, we find $g_\sigma \in G$ such that

$$M \cong \sigma(M)^{g_\sigma}.$$

□

Note that $\sigma([\epsilon_1]) = [\epsilon_1]$ and since G acts freely C_r , the element g_σ is uniquely determined. In the following we study the quiver automorphisms $\sigma \circ g_\sigma: C_r \rightarrow C_r$.

10.2.2 Automorphisms of trees

A group H is said to have property FA [42, I.6.1] if every action of H on an unoriented tree T by graph automorphisms with the property that every element $h \in H$ does not invert an edge, has a global fixed point $z \in T_0$, i.e. $hz = z$ for all $h \in H$. It is known [42, I.6.3.1] that all finitely generated torsion groups have the property FA. In particular, for every finite group acting on a quiver Q by quiver automorphisms, which underlying graph \bar{Q} is a tree, there is a global fixed point since the induced action on \bar{Q} does not invert edges.

Definition. For $x \in C_r^+$ and $1 \leq i \leq r$ denote by $T(x, i)$ the connected component of $C_r \setminus \{x\}$ containing $t(\alpha_i)$, where $\alpha_i: x \rightarrow t(\alpha_i)$ is the unique arrow with $\pi(\alpha_i) = \gamma_i$.

Let M in $\text{rep}(C_r)$ be indecomposable and $x \in \text{supp}(M)$, then we define $T(x, i, M) := T(M) \cap T(x, i)$.

Note that $\text{supp}(M) = \{x\} \cup \bigcup_{i=1}^r T(x, i, M)_0$.

Proposition 10.5. *Let $M \in \text{rep}(C_r)$ be \mathcal{S}_r -stable. Then there is $c \in \text{supp}(M)$ such that*

(a) $\sigma \circ g_\sigma(c) = c$ for all $\sigma \in \mathcal{S}_r$.

(b) For each $n \in \mathbb{N}$ the number r divides $D(n, c) := \sum_{x \in \text{supp}(M), d(x, c) = n} M_x$.

Proof. Since $M \cong \sigma(M)^{g_\sigma}$, we have

$$\text{supp}(M) = \text{supp}(\sigma(M)^{g_\sigma}) = \{g_\sigma^{-1} \cdot x \mid x \in \text{supp}(\sigma(M))\} = \{g_\sigma^{-1} \circ \sigma^{-1}(x) \mid x \in \text{supp}(M)\}.$$

We assume that $\dim_k M \neq 1$, otherwise there is nothing to show. Let $T \subseteq C_r$ be the finite subtree $T := T(M)$. Then $\sigma \circ g_\sigma: T \rightarrow T$ is a quiver automorphism of T . Since T is finite, $\text{Aut}(T)$ is finite and there exists a vertex $c \in T_0$ with $\varphi(c) = c$ for all $\varphi \in \text{Aut}(T)$. We assume that $c \in C_r^+$. For $1 \leq i \leq r$ we let $\beta_i: c \rightarrow t(\beta_i)$ be the unique arrow with $\pi(\beta_i) = \gamma_i$ and set $T_i := T(c, i, M)$. Since $\dim_k M \neq 1$ and M is indecomposable, we can assume w.l.o.g. that $e := t(\beta_1) \in \text{supp}(M)$. Since every automorphism of C_r respects the metric (see 9.2) we get

$$1 = d(c, e) = d(\sigma \circ g_\sigma(c), \sigma \circ g_\sigma(e)) = d(c, \sigma \circ g_\sigma(e)).$$

In particular, $\sigma \circ g_\sigma(\beta_1) \in \{\beta_1, \dots, \beta_r\}$. Now fix $j \in \{1, \dots, r\} \setminus \{1\}$ and $\sigma := (1 \ j) \in \mathcal{S}_r$. Since $\pi(\tau(\beta)) = \gamma_{\tau(\beta)}$ for all $\tau \in \mathcal{S}_r$ and all $\beta \in (C_r)_1$ by definition of the action we get $\pi(\sigma \circ g_\sigma(\beta_1)) = \gamma_{\sigma(\overline{\beta_1})}$. Since $\overline{g_\sigma(\beta_1)} = \overline{\beta_1} = 1$, it follows $\pi(\sigma \circ g_\sigma(\beta_1)) = \gamma_{\sigma(\overline{\beta_1})} = \gamma_{\sigma(1)} = \gamma_j$. Hence $\sigma \circ g_\sigma(\beta_1) = \beta_j$. Since $\sigma \circ g_\sigma$ induces an automorphism on $\bigcup_{i=1}^r T_i$, which permutes the connected components T_1, \dots, T_r we conclude $\sigma \circ g_\sigma(T_1) = T_j$. Hence, T_1, \dots, T_r are non-empty isomorphic quivers. For each $n \in \mathbb{N}$ and $i \in \{1, \dots, r\}$ we define $d_{n,i,c} := \{x \in (T_i)_0 \mid d(x, c) = n\}$. Let now $x \in d_{n,1,c}$, then we have $\sigma \circ g_\sigma(x) \in d_{n,j,c}$ since $\sigma \circ g_\sigma(x) \in T_j$ and

$$n = d(c, x) = d(\sigma \circ g_\sigma(c), \sigma \circ g_\sigma(x)) = d(c, \sigma \circ g_\sigma(x)).$$

Moreover, we have $M_x = (\sigma(M)^{g_\sigma})_x = M_{\sigma \circ g_\sigma(x)}$. It follows $\sum_{y \in d_{n,1,c}} \dim_k M_y = \sum_{y \in d_{n,j,c}} \dim_k M_y$ and we conclude

$$D(n, c) = \sum_{x \in \text{supp}(M), d(x, c) = n} \dim_k M_x = r \cdot \sum_{y \in d_{n,1,c}} \dim_k M_y.$$

This finishes the proof for $c \in C_r^+$. If c is in C_r^- a small adaption of arguments yields the claim; for $1 \leq i \leq r$ we define $S(c, i)$ to be the connected component of $C_r \setminus \{c\}$ containing $s(\alpha_i)$, where $\alpha_i: s(\alpha_i) \rightarrow c$ is the unique arrow with $\pi(\alpha_i) = \gamma_i$. Moreover we define $S_i := S(c, i, M) := T(M) \cap S(c, i)$. Now proceed as before. \square

Corollary 10.6. *Assume that M is \mathcal{S}_r -stable. If $\dim \pi_\lambda(M) = (a, b)$ then r divides a or b .*

Proof. Let $c \in \text{supp}(M)$ be as in Proposition 10.5. Then we have

$$\dim_k M = \dim_k M_c + \sum_{n \in 2\mathbb{N}-1} D(n, c) + \sum_{n \in 2\mathbb{N}} D(n, c).$$

If $c \in C_r^+$, then $b = \sum_{n \in 2\mathbb{N}-1} D(n, c)$ and $a = \sum_{n \in 2\mathbb{N}} D(n, c)$ otherwise. Hence, r divides b or a . \square

As an application we get the following result for components of the Kronecker quiver Γ_r .

Corollary 10.7. *Let $m \in \mathbb{N}$, then there exists a regular component \mathcal{C} with $\mathcal{W}(\mathcal{C}) = m$ and no representation in \mathcal{C} is $\text{GL}_r(k)$ -stable.*

Proof. For $m = 1$ consider the regular component $\pi_\lambda(\overline{\mathcal{D}})$ and for $m = 2$ the regular component $\pi_\lambda(\overline{\mathcal{E}})$ (see Theorem 10.3(b)). The component $\pi_\lambda(\overline{\mathcal{D}})$ (resp. $\pi_\lambda(\overline{\mathcal{E}})$) contains a representation with dimension vector $(1, 1)$ (resp. $(1, r - 1)$) which is not $\mathrm{GL}_r(k)$ -stable by Corollary 10.6. By [16, 2.2] a component contains a $\mathrm{GL}_r(k)$ -stable representation if and only if all representations in the component are $\mathrm{GL}_r(k)$ -stable.

Now let $m \geq 3$. By Theorem 9.3.2 there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there is a regular component \mathcal{D}_n with $\mathcal{W}(\mathcal{D}_n) = m$ and \mathcal{D}_n contains a quasi-simple representation $E_n = \pi_\lambda(F_n)$ with $F_n \in \mathrm{rep}(C_r)$ and $\underline{\dim} E_n = (n + 1, n)$ or $\underline{\dim} E_n = (n, n + 1)$. Since $r \geq 3$, we find $l \geq n_0$ (even infinitely many) such that r does not divide l and $l + 1$. Hence, F_l is not \mathcal{S}_r -stable and E_l not $\mathrm{GL}_r(k)$ -stable. Therefore no representation in \mathcal{D}_l is $\mathrm{GL}_r(k)$ -stable by [16, 2.2]. □

10.2.3 The number of regular components in $\mathrm{rep}(\Gamma_r)$

Definition. A locally closed set is an open subset of a closed set. A constructible set is a finite union of locally closed sets.

Lemma 10.8. *Let $M \in \mathrm{rep}(\Gamma_r)$ with $\mathrm{GL}_r(k)_M \neq \mathrm{GL}_r(k)$. There is an injection $\iota: k \rightarrow \mathrm{GL}_r(k)/\mathrm{GL}_r(k)_M$.*

Proof. By [16, 2.1] $\mathrm{GL}_r(k)_M$ is a closed subgroup of $\mathrm{GL}_r(k)$ and by [26, 5.5] $\mathrm{GL}_r(k)/\mathrm{GL}_r(k)_M$ is an algebraic variety. Hence, we find an affine variety $V \subseteq \mathrm{GL}_r(k)/\mathrm{GL}_r(k)_M$ with $d := \dim V = \dim \mathrm{GL}_r(k)/\mathrm{GL}_r(k)_M$. Since $\mathrm{GL}_r(k)$ is irreducible we have $\dim \mathrm{GL}_r(k)/\mathrm{GL}_r(k)_M = \dim \mathrm{GL}_r(k) - \dim \mathrm{GL}_r(k)_M \geq 1$. Let $k[t_1, \dots, t_d] \rightarrow k[V]$ be a Noether-normalization, $\varphi: k[t_1] \rightarrow k[t_1, \dots, t_d] \rightarrow k[V]$ be the natural injection and $\varphi^*: V \rightarrow \mathbb{A}^1$ be the comorphism. Then φ^* is dominant. By Chevalley's Theorem $f(V)$ is constructible and hence finite or cofinite. Since $f(V)$ is dense in \mathbb{A} , $f(V)$ is not finite and therefore cofinite. That means $|k \setminus f(V)|$ is finite. Since k is infinite we have $|f(V)| = |k|$. It follows $|k| = |\mathbb{A}| = |f(V)| \leq |V| \leq |\mathrm{GL}_r(k)/\mathrm{GL}_r(k)_M|$. □

Let E be a regular representation in a component \mathcal{C} . Since $\mathrm{GL}_r(k)$ acts via auto equivalences on $\mathrm{rep}(\Gamma_r)$, the representation $A.E$ is regular indecomposable for each $A \in \mathrm{GL}_r(k)$, the action interchanges with the Auslander-Reiten translation and respects the quasi-length of E .

Moreover, it follows from Proposition 3.7 and [49, 3.1.3] that $\{X_\beta \mid \beta \in k^r \setminus \{0\}\} = \mathrm{GL}_r(k).X_\alpha$ for each $\alpha \in k^r \setminus \{0\}$.

So if \mathcal{D} is the component containing $A.E$, then $M_{\mathcal{D}} = A.M_{\mathcal{C}}$ and $W_{\mathcal{D}} = A.W_{\mathcal{C}}$. It follows $\mathcal{W}(\mathcal{C}) = \mathcal{W}(\mathcal{D})$. With these considerations we conclude:

Theorem 10.9. *Let $m \in \mathbb{N}$. There is a bijection $\{\mathcal{C} \in \mathcal{R}(\Gamma_r) \mid \mathcal{W}(\mathcal{C}) = m\} \rightarrow k$.*

Proof. It is well known that $|\mathcal{C} \in \mathcal{R}(\Gamma_r)| = |k|$, see [3, XVIII.1.8]. In particular, there is an injection

$$\{\mathcal{C} \in \mathcal{R}(\Gamma_r) \mid \mathcal{W}(\mathcal{C}) = m\} \rightarrow k.$$

By Lemma 10.7 there is a regular component \mathcal{C} with $\mathcal{W}(\mathcal{C}) = m \in \mathbb{N}$ such that no representation in \mathcal{C} is $\mathrm{GL}_r(k)$ -stable. For E in \mathcal{C} the map

$$\mathrm{GL}_r(k)/\mathrm{GL}_r(k)_E \rightarrow \{Z \in \mathrm{rep}(\Gamma_r) \mid \underline{\dim} Z = \underline{\dim} E, Z \text{ indecomposable}\}; A \mathrm{GL}_r(k)_E \mapsto A.E$$

is well defined and injective. Since the number of representations in a regular component with given dimension vector (a, b) is ≤ 1 [3, XIII.1.7] and $\mathrm{GL}_r(k)$ acts via auto equivalences we get with Lemma 10.8 an injection

$$k \rightarrow \mathrm{GL}_r(k)/\mathrm{GL}_r(k)_E \rightarrow \{Z \mid \underline{\dim} Z = \underline{\dim} E, Z \text{ indecomposable}\} \rightarrow \{\mathcal{C} \in \mathcal{R} \mid \mathcal{W}(\mathcal{C}) = m\}.$$

By the Schröder-Bernstein Theorem we get the desired bijection. \square

Remark. Note that we restrict ourselves to components of width ≥ 1 , since we do not know whether components in $\mathrm{rep}(C_r)$ of width 0 exist. Also the examples [48, 3.3] of components of width 0 in $\mathrm{rep}(\Gamma_r)$ are $\mathrm{GL}_r(k)$ -stable. For the case $n = 0$ we argue as follows.

Lemma 10.10. *Let $r \geq 3$, then there exists a bijection $k \rightarrow \{\mathcal{C} \in \mathcal{R}(\Gamma_r) \mid \mathcal{W}(\mathcal{C}) = 0\}$.*

Proof. The proof of Corollary 4.8 yields an injective map $\varphi: \mathrm{ind} \mathcal{E}(\mathcal{X}) \rightarrow \{\mathcal{C} \in \mathcal{R}(\Gamma_r) \mid \mathcal{W}(\mathcal{C}) = 0\}$, where $\mathrm{ind} \mathcal{E}(\mathcal{X})$ are the indecomposable objects in a category $\mathcal{E}(\mathcal{X})$ equivalent to the category of finite dimensional modules over the power-series ring $k\langle\langle X_1, \dots, X_t \rangle\rangle$ in non-commuting variables X_1, \dots, X_t and $t \geq 2$. Now let $\lambda \in k \setminus \{0\}$ and consider the indecomposable module $M_\lambda = k^2$ given by $X_1.(a, b) = (\lambda b, 0)$, $X_2.(a, b) = (b, 0)$ and $X_i.(a, b) = 0$ for $i > 2$. Note this is well defined since $X_i X_j$ is in the annihilator of k^2 for all $i, j \in \{1, \dots, t\}$. Then $M_\lambda \not\cong M_\mu$ for $\lambda \neq \mu$ and we have an injection $k \rightarrow \mathrm{ind} \mathcal{E}(\mathcal{X})$. The claim follows as in 10.9. \square

With Proposition 6.4 we conclude:

Corollary 10.11. *Let $r \geq 3$, then for each $n \in \mathbb{N}$ there is a bijection between k and the set of regular components with quasi-rank in $[-n, -n + 3]$.*

Corollary 10.12. *Assume that k is uncountable and $p \in \mathbb{N}$. The set of components of quasi-rank $\leq -p$ in $\mathcal{R}(\Gamma_r)$ is uncountable.*

10.3 Thin representations

The proof of Theorem 10.3 relies on the fact that we find for each odd number $n \in \mathbb{N}$ a balanced representation $M \in \mathrm{rep}(C_r)$ of quasi-length n . An easy way to write down balanced representations is to consider thin representations:

Definition. An indecomposable representation $L \in \mathrm{rep}(C_r)$ is called thin if $\dim_k L_x \leq 1$ for all $x \in (C_r)_0$.

Note that for each subtree $\emptyset \neq T \subseteq C_r$ there is a (up to isomorphism) unique thin indecomposable representation with $T(L) = T$. Thus, if T has leaves in C_r^+ and C_r^- , then we find a balanced and thin representation L with $T(L) = T$.

Lemma 10.13. *Assume that Y and $\tau_{C_r} Y$ are thin representations. Then for all $a \in \mathrm{supp}(Y) \cap \mathrm{supp}(\tau_{C_r} Y)$ we have $|n(a) \cap \mathrm{supp}(Y) \cap \mathrm{supp}(\tau_{C_r} Y)| \leq 2$*

Proof. Define $X := \tau_{C_r} Y$ and assume that $a \in \cap \text{supp}(X) \cap \text{supp}(Y)$. If $a \in C_r^-$, then we have by the dual result of 10.2 that

$$1 = \dim_k X_a = \sum_{b \in n(a)} \dim_k Y_b - \dim_k Y_a = \sum_{b \in n(a)} \dim_k Y_b - 1.$$

Since $0 \leq \dim_k Y_b \leq 1$ for all $b \in n(a)$, there are exactly two elements $i, j \in n(a)$ with $\dim_k Y_i = 1 = \dim_k Y_j$. Hence $|n(a) \cap \text{supp}(Y)| \leq 2$. By the same token, $a \in C_r^+$ implies $|n(a) \cap \text{supp}(X)| \leq 2$. \square

Corollary 10.14. *Let $0 \rightarrow X \rightarrow M \oplus N \rightarrow Y \rightarrow 0$ be an almost split sequence in $\text{rep}(C_r)$ of regular representations with $\text{ql}(N) = \text{ql}(M) - 2$. If M is a thin representation, then $\text{supp}(N)$ is a line, i.e. $\text{supp}(N)$ is connected and the underlying graph $\text{supp}(N)$ is of type \mathbb{A}_m for some m .*

Proof. Since X is isomorphic to a submodule of M and Y is a factor of M , X and Y are thin representations. Since N is a factor of X , isomorphic to a submodule of Y and indecomposable we conclude with Lemma 10.13 that

$$\text{supp}(N) \subseteq \text{supp}(X) \cap \text{supp}(Y) = \text{supp}(\tau_{C_r} Y) \cap \text{supp}(Y).$$

is a line. \square

Proposition 10.15. *If $M \in \text{rep}(C_r)$ is a regular and thin representation, then $\text{ql}(M) \leq 2$.*

Proof. Assume that M is a regular and thin representation with $\text{ql}(M) \geq 3$. Consider the almost split sequence $0 \rightarrow X \rightarrow M \oplus N \rightarrow Y \rightarrow 0$ with $\text{ql}(N) = \text{ql}(M) - 2$. Since $A := \tau_{C_r}^{-1} N$ is a factor of M and $B := \tau_{C_r} N$ is a submodule of M we conclude with 10.14 that A, B and N are thin representations with $\text{supp}(N)$ being a line. Since N is regular, we obtain $|\text{supp}(N)| \geq 2$. We consider two cases:

- $|\text{supp}(N)| = 2$. Since N is a quotient of the thin representation X , N is thin. Hence $\underline{\dim} \pi_\lambda(N) = (1, 1)$ and N is quasi-simple. It follows that $\text{ql}(Y) = 2$ and that there is an almost split sequence $0 \rightarrow N \rightarrow Y \rightarrow A \rightarrow 0$. Hence $\text{supp}(Y) = \text{supp}(A) \cup \text{supp}(N)$. Let now $y \in \text{supp}(N)$ be the unique sink and $x \in \text{supp}(N)$ the unique source of N . It follows from 10.2(a) that $\dim_k A_x = 0$ and $\dim_k A_z = 1$ for all $z \in n(y) \setminus \{x\}$. Now 10.2(b) yields $\dim_k A_y = r - 1 - 1 = r - 2 \neq 0$. Hence $y \in \text{supp}(N) \cap \text{supp}(A)$, a contradiction since Y is thin.
- If $|\text{supp}(N)| \geq 3$, we fix a leaf a of N and assume that a is in C_r^+ . Since $|\text{supp}(N)| \geq 3$ and $\text{supp}(N)$ is a line we find $a \rightarrow b \leftarrow c$ in $\text{supp}(N)$ with $n(b) \cap \text{supp}(N) = \{a, c\}$. Then

$$\dim_k A_b = \dim_k (\tau_{C_r}^{-1} N)_b = \sum_{x \in n(b)} \dim_k N_x - \dim_k N_b = \dim_k N_a + \dim_k N_c - \dim_k N_b = 1$$

by 10.2 and for all $y \in n(a) \setminus \{b\}$ we get $\dim_k A_y = \dim_k N_a - 0 = 1$. We conclude

$$\dim_k A_a = \sum_{v \in n(a)} \dim_k A_v - \dim_k N_a = r \cdot 1 - 1 = r - 1 \geq 2,$$

which is a contradiction since A is a thin representation. If a is in C_r^- , then we consider the dual representation $D_{C_r} N$ and get a contradiction.

Hence, each regular and thin representation is of quasi-length ≤ 2 . □

Corollary 10.16. *Let $M \in \text{rep}(C_r)$ be a regular and thin representation with $\text{ql}(M) = 2$. Let $0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ be the almost split sequence in $\text{rep}(C_r)$. Then Y is in the G -orbit of X^i for some $i \in \{1, \dots, r\}$, i.e. $\pi_\lambda(Y) \cong X_{e_i}$.*

Proof. Since M is a thin representation, X and Y are thin representations. With $\text{supp}(X) \cup \text{supp}(Y) = \text{supp}(M)$ we get $\text{supp}(X) \cap \text{supp}(Y) = \emptyset$. Since $\dim_k X \neq 1$ we find $d \in C_r^- \cap \text{supp}(X)$. Then Lemma 10.2 yields

$$0 = \dim_k Y_d = \dim_k \tau_{C_r}^{-1} X_d = \left(\sum_{c \in n(d)} \dim_k Y_c \right) - \dim_k X_d = \left(\sum_{c \in n(d)} \dim_k Y_c \right) - 1.$$

Hence, there is exactly one $c \in n(d)$ with $\dim_k Y_c = 1$. Consider the arrow $c \rightarrow d$. Since there is exactly one arrow $\alpha: y \rightarrow x$ with $x \in \text{supp}(X)$ and $y \in \text{supp}(Y)$ and C_r is a tree, we conclude that $\{d\} = \text{supp}(X) \cap C_r^-$. Now let $a \in n(d) \setminus \{c\}$. Then $a \in C_r^+$ and $a \notin \text{supp}(Y)$ since C_r is a tree. It follows

$$0 = \dim_k Y_a = \dim_k \tau_{C_r}^{-1} X_a = \left(\sum_{b \in n(a)} \dim_k X_b \right) - \dim_k X_a = \dim_k X_d - \dim_k X_a = 1 - \dim_k X_a.$$

Hence, $1 = \dim_k X_a$ and $\text{supp}(X) = \{d\} \cup (n(d) \setminus \{c\})$. Let i be the unique number in $\{1, \dots, r\}$ with $\pi(c \rightarrow d) = \gamma_i$. Direct computation using 10.2(a), (b) shows that $\text{supp}(Y) = \{c\} \cup (n(c) \setminus \{d\})$. Since Y is a thin representation it follows $\pi_\lambda(Y) \cong X_{e_i}$. Hence, we find $g \in G$ such that $Y^g \cong X^i$. □

Remark. The proof of 10.3 uses the existence of a family of indecomposable modules $(M_n)_{n \in \mathbb{N}}$ such that $\text{ql}(M_n) = n$ for all $n \in \mathbb{N}$ and each M_n is balanced. Although one can easily write down a balanced and thin representation, our considerations show that thin representations can not replace the representations M_n , $n \in \mathbb{N}$.

10.4 Representations with different quasi-lengths

In [12] Chen studied the distribution of dimension vectors in $\text{rep}(\Gamma_r)$ and showed that the dimension of a representation does not give much information about its quasi-length:

Theorem 10.17. [12, 4.7] *Let $r \geq 3$ and $m, n \in \mathbb{N}$. Then there exist regular indecomposable representations $M, N \in \text{rep}(\Gamma_r)$ such that $\dim_k M = \dim_k N$, $\text{ql}(M) = m$ and $\text{ql}(N) = n$.*

We consider short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of regular indecomposable representations $A, B \in \text{rep}(\Gamma_r)$ with C simple and show that even in this case the number $|\text{ql}(A) - \text{ql}(B)|$ can get arbitrarily large.

Corollary 10.18. *Let $r \geq 3$ and $m \in \mathbb{N}$. Then there exists a non-split exact sequence $0 \rightarrow A \rightarrow B \rightarrow I_1 \rightarrow 0$ such that $A, B \in \text{rep}(\Gamma_r)$ are regular indecomposable with $\text{ql}(A) > m$ and $\text{ql}(B) = 1$.*

Proof. Consider the sequence $(M_n)_{n \geq 1}$ of regular representations constructed for the proof of 10.3, fix $n \geq m$ odd and set $A' := M_n$. Then $\text{ql}(A') = n \geq m$ and A' has leaves in C_r^+ and C_r^- . Hence, we can build the $*$ -product $B' := A' * C'$ with a representation C' with $C := \pi_\lambda(C') = I_1$. Since B' is indecomposable the representation $\pi_\lambda(B')$ is indecomposable and we have a non-split exact sequence $0 \rightarrow \pi_\lambda(A') \rightarrow \pi_\lambda(B') \rightarrow I_1 \rightarrow 0$. Since $\dim_k \pi_\lambda(B') = (n, n) + (1, 0) = (n+1, n)$, we have $\text{ql}(\pi_\lambda(B')) = 1$ by Lemma 9.3.1. □

Observe that application of D_{Γ_r} yields the dual result:

Corollary 10.19. *Let $m \in \mathbb{N}$, then there exists a non-split exact sequence $0 \rightarrow P_1 \rightarrow B \rightarrow A \rightarrow 0$ such that A, B are regular indecomposable with $\text{ql}(A) > m$ and $\text{ql}(B) = 1$.*

10.5 The *-construction revisited

In this section we show that for a connected pair (N, M) the *-construction yields an alternative description of $\text{Ext}_{C_r}(M, N)$.

Lemma 10.20. *Let (N, M) be connected with connecting arrow $\alpha: x \rightarrow y$ and*

$$0 \rightarrow N \xrightarrow{u} E \xrightarrow{v} M \rightarrow 0$$

*be a short exact sequence. Then there exists $g: M_x \rightarrow N_y$ k -linear, an isomorphism $\psi: E \rightarrow N *_g M$ and a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{u} & E & \xrightarrow{v} & M & \longrightarrow & 0 \\ & & \parallel & & \downarrow \psi & & \parallel & & \\ 0 & \longrightarrow & N & \xrightarrow{\iota_g} & N *_g M & \xrightarrow{\pi_g} & M & \longrightarrow & 0 \end{array}$$

Proof. If the short exact sequence splits we have $E \cong N \oplus M =: N *_f M$ for $f = 0$. Hence, we can assume that the exact sequence does not split, thus $E(\alpha) \neq 0$. Since u is injective and $\text{supp}(M) \cap \text{supp}(N) = \emptyset$ the map $u_x: N_x \rightarrow E_x$ is an isomorphism of vector spaces for each $x \in \text{supp}(N)$ and there is a k -linear map $u_x^{-1}: E_x \rightarrow N_x$ with $u_x^{-1} \circ u_x = \text{id}_{N_x}$. By the same token, there is a k -linear isomorphism $v_y^{-1}: M_y \rightarrow E_y$ with $v_y \circ v_y^{-1} = \text{id}_{M_y}$ for each $y \in \text{supp}(M)$.

Since $E|_{\text{supp}(X)}(\alpha) = 0$ for $X \in \{N, M\}$ we get isomorphisms of representations $u^{-1}: E|_{\text{supp}(N)} \rightarrow N$ and $v^{-1}: M \rightarrow E|_{\text{supp}(M)}$ by setting

$$(u^{-1})_a := \begin{cases} u_a^{-1}, a \in \text{supp}(N) \\ 0, \text{ else} \end{cases}$$

and

$$(v^{-1})_a := \begin{cases} v_a^{-1}, a \in \text{supp}(M) \\ 0, \text{ else.} \end{cases}$$

We now set $0 \neq g := u_y^{-1} \circ E(\alpha) \circ v_x^{-1}: M_x \rightarrow N_y$ and define $\psi: E \rightarrow M *_g N$ via

$$\psi_a := \begin{cases} (u^{-1})_a, a \in \text{supp}(N) \\ v_a, a \in \text{supp}(M) \\ 0, \text{ else.} \end{cases}$$

Note that this is well-defined since $\text{supp}(M) \cap \text{supp}(N) = \emptyset$. By definition of ψ each ψ_z is an isomorphism of vector spaces. Moreover, $\psi_{t(\beta)} \circ E(\beta) = (N *_g M)(\beta) \circ \psi_{s(\beta)}$ for each arrow

$\beta \in (C_r)_1 \setminus \{\alpha\}$ since u^{-1} and v are morphism of representations. It remains to consider the case $\beta = \alpha$, then

$$\psi_y \circ E(\alpha) = u_y^{-1} \circ E(\alpha) = g \circ v_x = (N *_g M)(\alpha) \circ v_x = (N *_g M)(\alpha) \circ \psi_x.$$

Hence, ψ is an isomorphism of representations. The representation $N *_g M \cong E$ is indecomposable since $g \neq 0$. Moreover, by definition of g we have $\psi \circ u = \iota_g$ and $\pi_g \circ \psi = v$. \square

Given $f \in \text{Hom}_k(M_x, N_y) \setminus \{0\}$ we now define δ_f to be the short exact sequence

$$\delta_f := 0 \rightarrow N \xrightarrow{\iota_f} N *_f M \xrightarrow{\pi_f} M \rightarrow 0.$$

and set $\delta_0 := 0 \rightarrow N \rightarrow N \oplus M \rightarrow M \rightarrow 0$. We let $\Xi: \text{Hom}_k(M_x, N_y) \rightarrow \text{Ext}_{C_r}(M, N)$ be the map that sends f to $[\delta_f]$. Recall that $\text{Ext}_{C_r}(M, N)$ is an abelian group (even a k -vector space) with addition given by the Baer sum \boxplus . The neutral element 0 is represented by the split exact sequence $\delta_0 = 0 \rightarrow N \rightarrow M \oplus N \rightarrow M \rightarrow 0$, i.e. $0 = [\delta_0]$. By 10.20 we know that Ξ is surjective. If we show that Ξ is linear, then we can conclude that Ξ is an isomorphism of vector spaces. Just note that $\Xi^{-1}(\{[\delta_0]\}) = \{0_{\text{Hom}_k(M_x, N_y)}\}$ by 9.2. Another way to see this, is to compute that

$$\dim_k \text{Ext}_{C_r}(M, N) = \langle \underline{\dim} M, \underline{\dim} N \rangle_{C_r} = \dim_k M_x \cdot \dim_k N_y = \dim_k \text{Hom}_k(M_x, N_y),$$

since $0 = \dim_k \text{Hom}_{C_r}(M, N)$.

Lemma 10.21. *The map $\Xi: \text{Hom}_k(M_x, N_y) \rightarrow \text{Ext}_{C_r}(M, N)$ is k -linear.*

Proof. Let $f, g \in \text{Hom}_k(M_x, N_y)$ and consider the two exact sequences

$$\delta_f = 0 \rightarrow N \xrightarrow{\iota_f} N *_f M \xrightarrow{\pi_f} M \rightarrow 0,$$

$$\delta_g = 0 \rightarrow N \xrightarrow{\iota_g} N *_g M \xrightarrow{\pi_g} M \rightarrow 0.$$

Let

$$p: N *_f M \oplus N *_g M \rightarrow M; (a, b) \mapsto \pi_f(a) - \pi_g(b)$$

$$q: N \rightarrow N *_f M \oplus N *_g M; a \mapsto (\iota_f(a), -\iota_g(a)).$$

By definition we have $\text{im } q \subseteq \ker p$ and a short sequence

$$0 \rightarrow N \xrightarrow{s} \ker p / \text{im } q \xrightarrow{t} M \rightarrow 0$$

with

$$s: N \rightarrow \ker p / \text{im } q, n \mapsto (\iota_f(n), 0) + \text{im } q$$

$$t: \ker p / \text{im } q \rightarrow M; (a, b) + \text{im } q \mapsto \pi_f(a).$$

The short sequence is also exact. We show that $\ker t \subseteq \text{im } s$. Let $(a, b) + \text{im } q \in \ker t$ with $(a, b) \in \ker p$, then $0 = \pi_f(a)$ and by exactness of δ_f there is $n_1 \in N$ with $a = \iota_f(n_1)$. Since (a, b) is in $\ker p$ we get $0 = \pi_f(a) - \pi_g(b) = \pi_f \circ \iota_f(n_1) - \pi_g(b) = -\pi_g(b)$. Hence, $b \in \ker \pi_g = \text{im } \iota_g$ and $b = \iota_g(n_2)$ for some $n_2 \in N$. We have shown that $(a, b) + \text{im } q = (\iota_f(n_1), \iota_g(n_2)) + \text{im } q$. Since $(\iota_f(n_2), -\iota_g(n_2))$ is in $\text{im } q$ we get

$$(a, b) + \text{im } q = (\iota_f(n_1), \iota_g(n_2)) + (\iota_f(n_2), -\iota_g(n_2)) + \text{im } q = (\iota_f(n_1 + n_2), 0) + \text{im } q = s(n_1 + n_2).$$

For the rest of the proof we just write ι and π , since $(\pi_f)_z = (\pi_g)_z$ and $(\iota_f)_z = (\iota_g)_z$ for all $z \in (C_r)_0$ on the level of k -linear maps. Set $X := \ker p / \text{im } q$ and let $v \in (C_r)_0$, then

(a) $v \in \text{supp}(N)$ implies:

$$\begin{aligned} X_v &= (\ker p / \text{im } q)_v = \{(a, b) \in (N *_f M \oplus N *_g M)_v \mid \pi_v(a - b) = 0\} / \\ &\quad \{(\iota_v(u), -\iota_v(u)) \in (N *_f M \oplus N *_g M)_v \mid u \in N_v\} \\ &= \{(a, b) \in (N *_f M \oplus N *_g M)_v \mid (a - b)_v \in N_v\} / \\ &\quad \{(a, -a) \mid a \in N_v\} \\ &= N_v \oplus N_v / \{(a, -a) \mid a \in N_v\}. \end{aligned}$$

(b) $v \in \text{supp}(M)$ implies:

$$\begin{aligned} X_v &= (\ker p / \text{im } q)_v = \{(a, b) \in (N *_f M \oplus N *_g M)_v \mid \pi_v(a - b) = 0\} / \\ &\quad \{(\iota_v(u), -\iota_v(u)) \in (N *_f M \oplus N *_g M)_v \mid u \in M_v\} \\ &= \{(a, b) \in M_v \oplus M_v \mid a - b = 0\} / \{(0, 0)\} \\ &= \{(a, a) \mid a \in M_v\}. \end{aligned}$$

(c) $v \notin \text{supp}(M) \cup \text{supp}(N)$ implies: $X_v = (\ker(p) / \text{im}(q))_v = 0$.

Now let $\beta = u \rightarrow v \in (C_r)_1$. If

(a) $u, v \in \text{supp}(M)$, then

$$X(\beta): \{(a, a) \mid a \in M_u\} \rightarrow \{(a, a) \mid a \in M_v\}; (a, a) \mapsto (M(\beta)(a), M(\beta)(a)).$$

(b) $u, v \in \text{supp}(N)$, then

$$X(\beta): N_u \oplus N_u / \{(a, -a) \mid a \in N_u\} \rightarrow N_v \oplus N_v / \{(a, -a) \mid a \in N_v\}$$

$$(r, s) + \{(a, -a) \mid a \in N_u\} \mapsto (N(\beta)(r), N(\beta)(s)) + \{(a, -a) \mid a \in N_v\}.$$

(c) $u = x, v = y$, then $\beta = \alpha$ and

$$X(\alpha): \{(a, a) \mid a \in M_x\} \rightarrow N_y \oplus N_y / \{(a, -a) \mid a \in N_y\}$$

$$(a, a) \mapsto (f(a), g(a)) + \{(a, -a) \mid a \in N_y\}.$$

(d) $X(\beta) = 0$ else.

We define $h: X \rightarrow N *_f M$ via

$$h_v := \begin{cases} 0, v \notin \text{supp}(M) \cup \text{supp}(N) \\ \{(a, a) \mid a \in M_v\} \rightarrow M_v; (a, a) \mapsto a, v \in \text{supp}(M) \\ N_v \oplus N_v / \{(a, -a) \mid a \in N_v\} \rightarrow N_v; (r, s) + \{(a, -a) \mid a \in N_v\} \mapsto r + s. \end{cases}$$

Now it is an easy exercise to check that $h = (h_v)_{v \in (C_r)_0}: X \rightarrow N *_f M$ is an isomorphism of representations. Moreover, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \xrightarrow{s} & X & \xrightarrow{t} & M \longrightarrow 0 \\
& & \parallel & & \downarrow h & & \parallel \\
0 & \longrightarrow & N & \xrightarrow{\iota_{f+g}} & N *_{f+g} M & \xrightarrow{\pi_{f+g}} & M \longrightarrow 0
\end{array},$$

Note that by construction the upper row is the Baer sum (see [1, A.5]) of δ_f and δ_g . Therefore

$$\Xi(f) \boxplus \Xi(g) = [\delta_f] \boxplus [\delta_g] = [\delta_{f+g}] = \Xi(f+g).$$

Now let $\lambda \in k \setminus \{0\}$ and δ be a short exact sequence, then $\lambda\delta$ denotes the short exact sequence obtained from δ by multiplying all maps in δ by λ . Now it follows from the definition of the scalar multiplication on $\text{Ext}_{C_r}(M, N)$ that

$$\lambda \cdot \Xi(f) = \lambda \cdot [\delta_f] = [\lambda\delta_f] = [\delta_{\lambda f}] = \Xi(\lambda f).$$

If $\lambda = 0$, then by definition $\lambda \cdot \Xi(f) = [\delta_0] = \Xi(0)$. \square

Corollary 10.22. *Let (N, M) be leaf-connected with connecting arrow $\alpha: x \rightarrow y$. The map*

$$\Xi: \text{Hom}_k(M_x, N_y) \rightarrow \text{Ext}_{C_r}(M, N); f \mapsto [\delta_f]$$

is an isomorphism of k -vectorspaces. Moreover, for a short exact sequence $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ the following statements are equivalent:

- (a) *E is indecomposable.*
- (b) *$0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ does not split.*
- (c) *$E \cong N *_f M$ for a non-zero linear map $f \in \text{Hom}_k(M_x, N_y)$.*

Observe that for $f \neq g$ we know that $[\delta_f] \neq [\delta_g]$ but it is not clear whether $N *_f M \not\cong N *_g M$.

Lemma 10.23. *Let (N, M) be leaf-connected with connecting arrow $\alpha: x \rightarrow y$. Let $f \in \text{Hom}_k(M_x, N_y)$ and $\lambda \in k \setminus \{0\}$.*

- (a) *There is an isomorphism $N *_f M \cong N *_{\lambda f} M$.*
- (b) *If $\dim_k \text{Ext}_{C_r}(M, N) = 1$, then there is $f: M_x \rightarrow N_y$ non-zero such that for each non-split exact sequence $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ we have $E \cong N *_f M$.*

Proof. (a) We define $\varphi: N *_f M \rightarrow N *_{\lambda f} M$ by setting

$$\varphi_z := \begin{cases} \text{id}_{M_z}, z \in \text{supp}(M) \\ \lambda \cdot \text{id}_{N_z}, z \in \text{supp}(N) \\ 0, \text{else.} \end{cases}$$

It is an easy exercise to check that φ is an isomorphism of representations.

(b) By Corollary 10.22 we have

$$1 = \dim_k \text{Ext}_{C_r}(M, N) = \dim_k \text{Hom}_k(M_x, N_y) = \dim_k M_x \cdot \dim_k N_y.$$

Hence $\dim_k M_x = 1 = \dim M_y$. Now let $f: M_x \rightarrow N_y$ be non-zero and $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ be a non-split exact sequence. By Corollary 10.22 we find $g: M_x \rightarrow N_y$ non-zero with $N *_g M \cong E$. Since $\dim_k \text{Hom}_k(M_x, N_y) = 1$ we find $\lambda \in k \setminus \{0\}$ with $g = \lambda f$. With (a) we conclude $E \cong N *_{\lambda f} M = N *_f M$. \square

Corollary 10.24. *Let M be a thin representation of dimension n . Then there exists a leaf-connected sequence (S_1, \dots, S_n) with S_1, \dots, S_n simple such that $M = S_1 * S_2 * \dots * S_{n-1} * S_n$.*

Proof. We use induction on n , the case $n = 1$ being trivial. Let $n > 1$ and fix a leaf x of M . Without loss of generality we assume that $x \in C_r^+$ and let y be the unique vertex in $\text{supp}(M) \cap n(x)$. We also can assume that $M_x = k = M_y$. Let S_n be the representation with $(S_n)_x = k$ and $(S_n)_z = 0$ for $z \in (C_r)_0 \setminus \{x\}$. Then $M = M_{|\text{supp}(M) \setminus \{x\}} *_{id_k} S_n$ and since $\dim_k M_{|\text{supp}(M) \setminus \{x\}} = n - 1$, the induction hypothesis yields the result. □

11 Gabriel-Roiter Theory for quivers

In this chapter Λ denotes a finite dimensional k -algebra and k is an algebraically closed field of arbitrary characteristic.

Definition. Suppose there is $n_\Lambda \in \mathbb{N}$ such that for each $M \in \text{mod } \Lambda$ indecomposable $\dim_k M \leq n_\Lambda$ holds, then Λ is of bounded representation type.

Recall the first Brauer-Thrall conjecture.

Conjecture. *If Λ is of bounded representation type, then Λ is of finite representation type.*

The conjecture was proven by Roiter in [40] and his method used in the proof was formalized and reformulated by Gabriel in [18, 5.]. This method can be seen as an induction scheme that attaches to every module M an invariant $\mu(M)$ that gives information about its dimension and submodule structure.

In [37],[32] Ringel showed that this method is also applicable if Λ is a representation infinite algebra and he revealed connections to the Auslander-Reiten theory of Λ . He called the invariant $\mu(M)$ the Gabriel-Roiter measure of M .

We show that the modules with the equal d -socle property can be characterized by the Gabriel-Roiter measure and study the behavior of $\mu(-)$ in τ -orbits of regular components. Then we extend our findings to wild quivers that have a "similar" Gabriel-Roiter theory as Γ_r .

In the following we recall the definition of $\mu(-)$ and collect important properties. Most of the presented results are true in the framework of artin algebras. Since we are mainly interested in quiver algebras, we restrict us to those. Hence, we can replace the length appearing in the definition of [37] by the dimension over k .

We consider the set $\mathcal{P}(\mathbb{N})$ of all subsets of \mathbb{N} . For $I, J \in \mathcal{P}(\mathbb{N})$ we write $I < J$ if the minimal element of the symmetric difference $(I \setminus J) \cup (J \setminus I)$ is contained in J . The relation defines a total order on $\mathcal{P}(\mathbb{N})$ which is complete, i.e. for each non-empty set $S \subseteq \mathcal{P}(\mathbb{N})$ which is bounded above (below) the supremum (infimum), $\sup S$ ($\inf S$) exists.

We write $I \ll J$ if $I \subset J$ and for all $a \in I$ and all $b \in J \setminus I$ we have $a < b$. We say that J starts with I if $I = J$ or $I \ll J$. It is an easy exercise to check that the defined order satisfies the following properties.

Lemma 11.1. *Let $I, J, K \in \mathcal{P}(\mathbb{N})$.*

- (a) *$I \subset J$ implies $I < J$.*
- (b) *If $I < J < K$ and K starts with I , then J starts with I .*

Let Q be a finite, connected and acyclic quiver. The considered modules will all be not equal to zero.

Let $M \in \text{Mod } kQ$ be not equal to zero. Assume that $M_1 \subset M_2 \subset \dots \subset M_t$ is a chain of indecomposable submodules of M . If $a \notin \{\dim_k M_1, \dots, \dim_k M_t\}$, then $\{\dim_k M_1, \dots, \dim_k M_t\} < \{1, 2, \dots, a\}$. Therefore and the fact that the order $(\mathcal{P}(\mathbb{N}), <)$ is complete the following definition is meaningful.

Definition. For M in $\text{Mod } kQ$ not equal to zero we define $\mu(M)$ to be the supremum of the all sets $\{\dim_k M_1, \dots, \dim_k M_t\} \subseteq \mathbb{N}$, where $M_1 \subset M_2 \subset \dots \subset M_t$ is a chain of indecomposable submodules of M . The set $\mu(M)$ is called the Gabriel-Roiter measure of M . An inclusion $X \subset M$ of finite dimensional, indecomposable kQ -modules is called a Gabriel-Roiter inclusion, provided that $\mu(M) = \mu(X) \cup \{\dim_k M\}$. In this case X will be called a Gabriel-Roiter submodule of M .

Definition. Let M be a kQ -module. If there exists a sequence of indecomposable finite dimensional modules

$$M_1 \subset M_2 \subset \dots \subseteq \bigcup_{i \in I} M_i = M \text{ such that } \mu(M) = \{\dim_k M_i \mid i \in I\}$$

with I finite or countable, then this chain is called a Gabriel-Roiter filtration of M .

- Remarks.** (a) If X is indecomposable, then $\mu(X) = \{1\}$ if and only if X is simple.
(b) If $Y \subseteq M$, then $\mu(Y) \leq \mu(M)$.
(c) For M in $\text{mod } kQ$, we have $\mu(M) \subseteq \{1, \dots, \dim_k M\}$ and if M is indecomposable we have $\{1, \dim_k M\} \subseteq \mu(M)$.
(d) M in $\text{mod } kQ$ has a GR filtration if and only if M is indecomposable.
(e) $X \subset M$ is a GR submodule of M if and only if $\mu(Z) \leq \mu(X)$ for all proper submodules Z of M .
(f) For any M we have $\mu(M) = \sup_{Y \subset M, Y \in \text{mod } kQ} \mu(Y)$.
(g) A chain $M_1 \subset \dots \subseteq \bigcup_{i \in I} M_i = M$ is a GR filtration if and only if $M_i \subset M_{i+1}$ is a GR inclusion for all i .

The following result is due to Gabriel and known as the Main Property (see [37]) of the Gabriel-Roiter measure.

Theorem 11.2. [37, 1.]

Let X, Y_1, \dots, Y_t be indecomposable kQ -modules of finite dimension and assume there is a monomorphism $f: X \rightarrow \bigoplus_{i=1}^t Y_i$. The following statements hold.

- (a) $\mu(X) \leq \max \mu(Y_i)$.
(b) If $\mu(X) = \max \mu(Y_i)$, then f is a split monomorphism.
(c) If $\max \mu(Y_i)$ starts with $\mu(X)$, then there is some j such that $\pi_j \circ f$ is injective, where $\pi_j: \bigoplus_{i=1}^t Y_i \rightarrow Y_j$ is the canonical projection.

Corollary 11.3. Let Y_1, \dots, Y_t in $\text{Mod } kQ$ be indecomposable. Then $\mu(\bigoplus_{i=1}^t Y_i) = \max \mu(Y_i)$.

The following results give a connection between GR and AR theory.

Lemma 11.4. [31, 3.1] Let $X \subset M$ be a GR inclusion. Then M/X is indecomposable and a factor module of $\tau_{kQ}^{-1}X$.

Proposition 11.5. [32, 1.1] Let $\iota: X \rightarrow M$ be a GR inclusion, then there exist an indecomposable module Y , an irreducible monomorphism $u: X \rightarrow Y$ and an epimorphism $\pi: Y \rightarrow M$ such that $\pi \circ u: X \rightarrow M$ is injective.

11.1 A connection to modules with the equal d -socle property

Recall that for $1 \leq d \leq r - 1$ we denote by Q^d the set of isomorphism classes $[M]$ of indecomposable modules M with dimension vector $(1, r - d)$. Moreover, recall Lemma 3.7:

Proposition 11.6. *Let M be indecomposable.*

- (a) *If $[M] \in Q^d$, then there exists $U \in \text{Gr}_{d,r}$ with $M \cong X_U$.*
- (b) *The map $\varphi: \text{Gr}_{d,r} \rightarrow Q^d; U \mapsto [X_U]$ is bijective.*
- (c) *Let $1 \leq c \leq d < r$ and $[M] \in Q^d$. There is $[N] \in Q^c$ and an epimorphism $\pi: N \rightarrow M$.*

As a consequence, we have a characterization of the family X_U , $U \in \text{Gr}_{d,r}$ in purely combinatorial terms.

Lemma 11.7. *Let $X \in \text{mod } \mathcal{K}_r$ be an indecomposable module and let $1 \leq d \leq r - 1$. The following statements are equivalent:*

- (a) $\underline{\dim} X = (1, r - d)$.
- (b) $X \cong X_U$ for some $U \in \text{Gr}_{d,r}$.
- (c) $\mu(X) = \{1, r - d + 1\}$.

Proof. Note that (a) \Leftrightarrow (b) follows from the above proposition.

Assume that $X \cong X_U$ for $U \in \text{Gr}_{d,r}$, then $\underline{\dim} X = (1, r - d)$ and X is not simple. Therefore we find a Gabriel-Roiter submodule $V \subset X$. Since every proper submodule of X_U is in $\text{add } P_1$, we conclude $V = P_1$ and $\mu(X) = \mu(V) \cup \{\dim_k X\} = \{1, r - d + 1\}$.

Now assume that $\mu(X) = \{1, r - d + 1\}$. Then $\dim_k X = r - d + 1$ and P_1 is a Gabriel-Roiter submodule of X since I_1 is not a submodule of X . By 11.5 we find an epimorphism $\pi: P_2 \rightarrow X$. In particular $\dim_k X_1 \leq \dim_k (P_2)_1 = 1$. Since X is indecomposable we conclude $\dim_k X_1 = 1$ and $\dim_k X_2 = r - d$. \square

Proposition 11.8. *Let M be an indecomposable, non-simple \mathcal{K}_r -module and $1 \leq d \leq r - 1$. The following statements are equivalent:*

- (a) $M \in \text{ESP}_{r-d}$.
- (b) $\mu(M) < \{1, d + 1\}$.

Proof. Assume first that $\mu(M) < \{1, d + 1\}$. It is known (see Proposition 12.2) that $\mu(M) < \{1, d + 1\} \leq \{1, 2\}$ implies that M not preinjective, since M is not simple. If $M \notin \text{ESP}_{r-d}$, then there is $U \in \text{Gr}_{r-d,r}$ with $\text{Hom}_{\mathcal{K}_r}(X_U, M) \neq 0$. Let $f: X_U \rightarrow M$ be a non-zero morphism. Since $\mu(X_U) = \{1, d + 1\}$, the morphism is not injective. By 3.8 the submodule $Z := \text{im } f \subset M$ is an indecomposable regular module with $\underline{\dim} Z = (1, c)$ and $1 \leq c \leq d - 1$. Application of 11.7 yields that $\{1, d + 1\} > \mu(M) \geq \mu(Z) = \{1, c + 1\}$, a contradiction since $c + 1 < d + 1$. Now assume that $\mu(M) \geq \{1, d + 1\}$ and $M \in \text{ESP}_{r-d}$. By 11.7, we know that $\mu(N) = \{1, d + 1\}$ if and only if $N \cong X_U$ for some $U \in \text{Gr}_{r-d,r}$. Hence $\mu(M) > \{1, d + 1\}$, because $X_U \notin \text{ESP}_{r-d}$. By 11.6(c), we find $1 < s \leq d$ minimal such that $\{1, s\} \subseteq \mu(M)$. Hence there is an indecomposable submodule $X \subseteq M$ with $\mu(X) = \{1, s\}$. By 11.7 we get $\underline{\dim} X = (1, s - 1)$. By 11.6(c), we know that $X \notin \text{ESP}_{r-d}$, a contradiction since ESP_{r-d} is closed under submodules. \square

Lemma 11.9. *Let M be an indecomposable module and $1 < e \leq r$. Then $\mu(M) = \{1, 2, \dots, e\}$ if and only if M is regular with $\underline{\dim} M = (e - 1, 1)$.*

Proof. Assume that $\mu(M) = \{1, 2, \dots, e\} = \{\dim_k U_1, \dim_k U_2, \dots, \dim_k U_e\}$ with GR filtration $U_1 \subset U_2 \subset \dots \subset U_e = M$. In particular, $\underline{\dim} U_{i+1} = \underline{\dim} U_i + (1, 0)$ for all $i < e$. Hence, $\underline{\dim} M = \underline{\dim} U_r = \underline{\dim} U_1 + (e-1)(1, 0) = (0, 1) + (e-1, 0) = (e-1, 1)$.

Now let N be an indecomposable module with dimension vector $\underline{\dim} N = (d, 1)$ and $1 \leq d < r$. We show by induction on d that $\mu(N) = \{1, 2, \dots, d+1\}$. The statement is obvious for $d = 0, 1$. Now let $X \subset N$ be a GR submodule with $\underline{\dim} X = (a, b)$. Since $X \not\cong I_1$, we conclude $\underline{\dim} X \neq (a, 0)$. Hence $\underline{\dim} X = (a, 1)$, $\underline{\dim} M/X = (d-a, 0)$ and $a = d-1$. It follows $\mu(N) = \mu(X) \cup \{d+1\}$, which by induction hypothesis simplifies to $\mu(N) = \{1, 2, \dots, d\} \cup \{d+1\}$. \square

Lemma 11.10. *Let M be an indecomposable module, $J \in \mathcal{P}(\mathbb{N})$, $e \in \mathbb{N}$ and $1 \leq c < r$. The following statements hold.*

- (a) $\{1, 2, \dots, e\} \leq J$ if and only if $\{1, 2, \dots, e\} \subseteq J$.
- (b) $\mu(M) = \{1, 2, \dots, c+1\}$ if and only if $M \cong DX_U$ for some $U \in \text{Gr}_{r-c,r}$.
- (c) $\{1, 2, \dots, c+1\} \leq \mu(M)$ if and only if $DX_U \subseteq M$ for some $U \in \text{Gr}_{r-c,r}$.

Proof. (a) This is straightforward.

(b) By 11.9 we know that $\mu(M) = \{1, 2, \dots, c+1\}$ if and only if $\underline{\dim} M = (c, 1)$. This is true if and only if $\underline{\dim} DM = (1, c)$ and 11.6 yields $DM = X_U$ for some $U \in \text{Gr}_{r-c,r}$. Hence $M = DX_U$.

(c) Let $\{1, 2, \dots, c+1\} \leq \mu(M)$. By (a) we have $\{1, 2, \dots, c+1\} \subseteq \mu(M)$ and find an indecomposable submodule $X \subseteq M$ such that $\mu(X) = \{1, 2, \dots, c+1\}$. By (b) we get $X \cong DX_U$ for some $U \in \text{Gr}_{r-c,r}$. On the other hand $DX_U \subseteq M$ together with (b) yields $\{1, 2, \dots, c+1\} = \mu(DX_U) \leq \mu(M)$. \square

Proposition 11.11. *Let $Y \in \text{mod } \mathcal{K}_r$ be an indecomposable module and let $1 \leq d \leq r-1$. The following statements are equivalent:*

- (a) $\underline{\dim} Y = (r-d, 1)$.
- (b) $Y \cong DX_U$ for some $U \in \text{Gr}_{d,r}$.
- (c) $\mu(Y) = \{1, 2, \dots, r-d+1\}$.

Proof. The equivalence of (a) and (b) follows immediately from 11.7 by applying the duality D . The implications (a) \Rightarrow (c) and (c) \Rightarrow (a) follow from Lemma 11.10. \square

Corollary 11.12. *Let M be indecomposable regular and $1 \leq c \leq r-1$. Then $\mu(M) < \{1, 2, \dots, c+1\}$ if and only if $\tau^{-1}M \in \text{EKP}_{r-c}$.*

Proof. First assume that $\{1, 2, \dots, c+1\} \leq \mu(M)$. By 11.10 we find $U \in \text{Gr}_{r-c,r}$ such that $DX_U \subseteq M$. It follows

$$0 \neq \text{Hom}_{\mathcal{K}_r}(DX_U, M) \cong \text{Hom}_{\mathcal{K}_r}(\tau^{-1}DX_U, \tau^{-1}M) \cong \text{Hom}_{\mathcal{K}_r}(D\tau X_U, \tau^{-1}M).$$

Hence $\tau^{-1}M \notin \text{EKP}_{r-c}$.

The assumption $\tau^{-1}M \notin \text{EKP}_{r-c}$ yields $U \in \text{Gr}_{r-c,r}$ (see 3.18) such that

$$\text{Hom}_{\mathcal{K}_r}(DX_U, M) \cong \text{Hom}(D\tau X_U, \tau^{-1}M) \neq 0.$$

Hence, there exists $f: DX_U \rightarrow M$ non-zero, which has to be a monomorphism because every proper factor module of DX_U is injective by 3.5. It follows $\{1, 2, \dots, c+1\} = \mu(DX_U) \leq \mu(M)$ and $\{1, 2, \dots, c+1\} \leq \mu(M)$. \square

Proposition 11.13. *Let M be an indecomposable non-simple \mathcal{K}_r -module not isomorphic to P_2 . Then the following statements are equivalent:*

- (a) $M \in \text{EKP}_1 = \text{ESP}_1$.
- (b) $\mu(M) < \{1, r\}$.
- (c) $\mu(\tau M) < \{1, 2, \dots, r\}$.

Proof. This follows from 11.12, 11.8 and the fact that every preprojective module P satisfies $\mu(P) < \{1, r\}$. □

12 Limiting behavior

Definition. For a finite set $I \in \mathcal{P}(\mathbb{N})$, we set

$$\mathcal{A}(I) := \{M \in \text{mod } kQ \mid \mu(M) = I, M \text{ indecomposable}\}.$$

If $\mathcal{A}(I)$ is non-empty, we call I a GR measure of kQ . A GR measure of finite type is a GR measure such that $\text{ind } \mathcal{A}(I)$ is a finite set. Here $\text{ind } \mathcal{A}(I)$ denotes a chosen set of the representatives of the indecomposable modules in $\mathcal{A}(I)$.

In [37] the following remarkable result is proven for each representation infinite algebra Λ .

Theorem 12.1. [37, 2.] *If kQ is of infinite representation type, then there are Gabriel-Roiter measures of finite type $J_t, J^t, t \in \mathbb{N}$ such that*

$$J_1 < J_2 < J_3 < \dots < J^3 < J^2 < J^1,$$

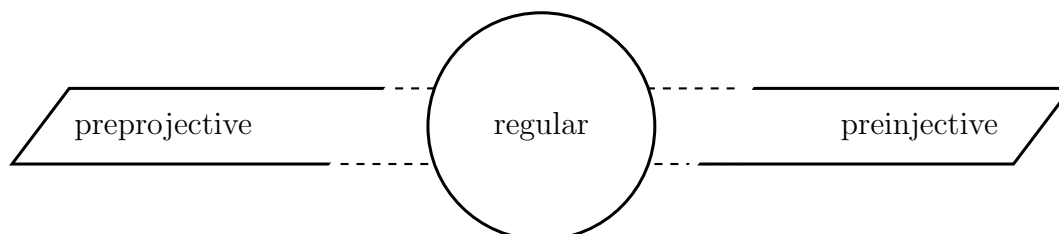
and for every other GR measure J we have $J_t < J < J^t$ for all $t \in \mathbb{N}$.

The indecomposable modules with measure J_i are called take-off modules, the indecomposable modules with measure J^i are called landing modules and all other indecomposable modules are called central modules. We denote the corresponding classes of indecomposable modules with respective measure by $\mathcal{T}(Q), \mathcal{L}(Q)$ and $\mathcal{Z}(Q)$.

The classes $\mathcal{T}(Q), \mathcal{L}(Q), \mathcal{Z}(Q)$ are called take-off, landing and central part. Hence, we obtain a partition $\mathcal{T}(Q) \cup \mathcal{Z}(Q) \cup \mathcal{L}(Q)$ of the class of finite dimensional indecomposable kQ -modules. Moreover, each indecomposable module with landing measure is preinjective (see [37, 4.]) and the ordering on the set of measures satisfies by definition

$$\mathcal{T}(Q) < \mathcal{Z}(Q) < \mathcal{L}(Q).$$

A natural question is how this partition of $\text{ind } kQ$ relates to the partition into preprojective, regular and preinjective indecomposable modules of kQ and the often used illustration



In [11] Chen shows that in the case of Q being tame, every preprojective module of kQ lies in the take-off part. For wild quivers this is no longer the case, as the example studied in [14] shows. For the wild Kronecker quiver however, the following statements hold.

Proposition 12.2. [13, 2.3] *Let $r \geq 3$.*

- (a) *The take-off part $\mathcal{T}(\Gamma_r)$ contains precisely the simple injective module and the indecomposable preprojective modules.*
- (b) *The landing part $\mathcal{L}(\Gamma_r)$ contains precisely all non-simple indecomposable preinjective modules.*
- (c) *An indecomposable module is a central module if and only if it is regular.*

Moreover, $\mu(P_i) = J_i$, $\mu(I_i) = J^i$ for $i \geq 2$, $\mu(I_1) = \mu(P_1) = J_1 = \{1\}$ and P_i is the unique GR submodule of P_{i+1} .

In this chapter we are concerned with two questions:

- Let $n \in \mathbb{N}_{\geq 2}$. Is there a wild quiver Q with $n(Q) = n$ such that $\text{ind } \mathcal{P}(Q)$ is contained in the take-off part of kQ ? Here $\text{ind } \mathcal{P}(Q)$ denotes a chosen set of the indecomposable preprojective modules of kQ .
- Whats can we say about the behavior of the Gabriel-Roiter measure in a given τ -orbit of a regular component?

A partial answer to the second question can be given using the following result.

Proposition 12.3. [23, 10.7] *Let Q be a wild quiver and X, Y be non-zero regular modules. There exists $n \in \mathbb{N}$ such that the module X is cogenerated by $\tau_{kQ}^m Y$ and generated by $\tau_{kQ}^{-m} Y$ for all $m \geq n$.*

Using the Main property of $\mu(-)$ we conclude:

Corollary 12.4. *Let Q be a wild quiver. If X, Y are non-zero regular modules, then there exists $n \in \mathbb{N}$ such that $\mu(X) < \mu(\tau_{kQ}^m Y)$ for all $m \geq n$.*

Let $r = 3$ and consider for $\alpha \in k^r \setminus \{0\}$ a short exact sequence $0 \rightarrow P_1 \rightarrow P_2 \rightarrow X_\alpha \rightarrow 0$ as well as a short exact sequence $0 \rightarrow (P_1)^2 \rightarrow X_\alpha \rightarrow I_1 \rightarrow 0$. Then $\mu(X_\alpha) = \{1, 3\} > \{1, 4\} = \mu(P_2) > \{1\} = \mu(I_1)$. These examples show that the question about the relation between X and $\tau^{-m} Y$ is way more subtle, since there is no general rule to compare $\mu(A)$ and $\mu(B)$ for an epimorphism $\pi: A \rightarrow B$.

Proposition 12.5. [11, 4.3] *Let Q be a quiver and X be an indecomposable, preprojective kQ -module, then $|\{Y \in \text{ind } \mathcal{P}(Q) \mid \mu(Y) < \mu(X)\}| < \infty$.*

Corollary 12.6. *Let Q be a wild quiver. Then $\text{ind } \mathcal{P}(Q) \cap \mathcal{T}(Q)$ is finite, or $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$.*

Proof. Assume there is P preprojective and indecomposable not contained in $\mathcal{T}(Q)$. Then $\text{ind } \mathcal{P}(Q) \cap \mathcal{T}(Q) \subseteq \text{ind } \mathcal{P}(Q) \cap \{Y \in \text{ind } \mathcal{P}(Q) \mid \mu(Y) < \mu(P)\}$, with the latter being a finite set. □

The following example shows that in the wild case it can happen that the only preprojective modules with take-off measure are simple. We basically use the same arguments as in [14] and the fact that the quiver $1 \rightrightarrows 2 \rightarrow 3$ contains the category $\text{mod } k\Gamma_2$ as a full subcategory.

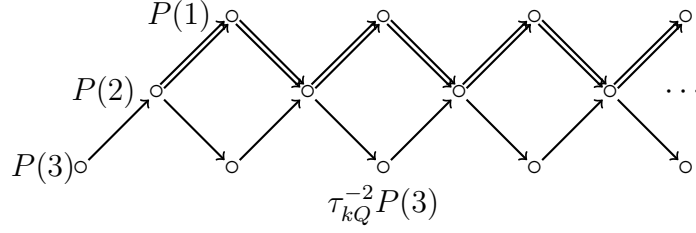


Figure 12.1: Preprojective component of kQ .

Example. We consider the wild quiver $Q = 1 \rightrightarrows 2 \rightarrow 3$ with arrows $\alpha, \beta: 1 \rightarrow 2$ and $\gamma: 2 \rightarrow 3$. We have $\underline{\dim} P(1) = (1, 2, 2)$, $\underline{\dim} P(2) = (0, 1, 1)$ and $\underline{\dim} P(3) = (0, 0, 1)$. It is $\underline{\dim} \tau_{kQ}^{-1}P(3) = (0, 1, 0)$ and hence we have that $\text{Hom}_{kQ}(P(1), \tau_{kQ}^{-1}P(3)) = 0$, since every proper submodule of $P(1)$ is in $\text{add}(P(2) \oplus P(3))$.

At first we show that $\mu(P(2)) \leq \mu(P)$ for every module $P \in \text{ind } \mathcal{P}(Q)$ not equal to $P(3)$ and $\tau_{kQ}^{-1}P(3)$. This is obvious for $\tau_{kQ}^{-j}P(i)$ with $j \geq 0$ and $i \in \{1, 2\}$ since every irreducible morphism corresponding to one of the 2-folded arrows is injective. Now let $i \in \mathbb{N}_0$. Since there is a sectional path $\tau_{kQ}^{-i}P(1) \rightarrow \tau_{kQ}^{-(i+2)}P(3)$, we have $\text{Hom}_{kQ}(\tau_{kQ}^{-i}P(1), \tau_{kQ}^{-(i+2)}P(3)) \neq 0$ (see [1, XI.2, XI.2.2]). We also have

$$\begin{aligned} \dim_k \text{Ext}_{kQ}(\tau_{kQ}^{-(i+2)}P(3), \tau_{kQ}^{-i}P(1)) &= \dim_k \text{Hom}_{kQ}(\tau_{kQ}^{-(i+1)}P(1), \tau_{kQ}^{-(i+2)}P(3)) \\ &= \dim_k \text{Hom}_{kQ}(P(1), \tau_{kQ}^{-1}P(3)) = 0. \end{aligned}$$

Hence we may apply [1, VIII.3.3] and get that every non-zero morphism $f: \tau_{kQ}^{-i}P(1) \rightarrow \tau_{kQ}^{-(i+2)}P(3)$ is injective or surjective. Now an easy induction on i shows that $\dim_k \tau_{kQ}^{-i}P(1) \leq \dim_k \tau_{kQ}^{-(i+2)}P(3)$ for all $i \in \mathbb{N}_0$. Hence, $\tau_{kQ}^{-i}P(1)$ is isomorphic to a submodule of $\tau_{kQ}^{-(i+2)}P(3)$ and

$$\mu(\tau_{kQ}^{-(i+2)}P(3)) \geq \mu(\tau_{kQ}^{-i}P(1)) \geq \mu(P(2)) = \{1, 2\}.$$

Since every quasi-simple module in a homogenous tube of the AR quiver of \mathcal{K}_2 has GR measure $\{1, 2\}$, the natural embedding $\text{mod } \mathcal{K}_2 \rightarrow \text{mod } kQ$ shows that $\mu(P(2)) = \{1, 2\}$ is of infinite type and hence not in $\mathcal{T}(Q)$. In particular, $\mathcal{T}(Q) \cap \text{ind } \mathcal{P}(Q) = \{P(3), \tau_{kQ}^{-1}P(3)\}$.

12.1 Existence of wild quivers with $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$

Lemma 12.7. [13, 2.2] *Let $X \subset M$ be a GR inclusion. If all irreducible maps to M are monomorphisms, then the GR inclusion is an irreducible map.*

Lemma 12.8. *Let A be a finite dimensional k -algebra. Let $s \in \mathbb{N}$, $s \geq 2$ and $0 \rightarrow A \xrightarrow{f^{tr}} \bigoplus_{i=1}^s E_i \xrightarrow{g} B \rightarrow 0$ be an Auslander-Reiten sequence in $\text{mod } A$ such that $f = (f_1, \dots, f_s)$, $g = (g_1, \dots, g_s)$ and $\dim_k E_i = \dim_k E_j$ for all $i, j \in \{1, \dots, s\}$. If f_1 is injective (if and only if all f_i are injective), then each g_i is injective.*

Proof. To the contrary assume that some g_i is not injective. Since g_i is irreducible, g_i is onto and $\dim_k E_i > \dim_k B$. With $s \geq 2$ we get

$$s \cdot \dim_k E_i = \dim_k \bigoplus_{i=1}^s E_i = \dim_k A + \dim_k B < \dim_k A + \dim_k E_i \leq \dim_k A + (s-1) \dim_k E_i,$$

and conclude $\dim_k E_1 = \dim_k E_i < \dim_k A$. A contradiction to the injectivity of f_1 . \square

With this lemma we can generalize the proof of [13, 2.3] and show:

Proposition 12.9. *For each $n \geq 2$ there exist infinitely many wild quivers Q with n simple modules and the following properties.*

- (a) $\mathcal{T}(Q)$ contains precisely all preprojective indecomposable modules and every simple module.
- (b) $\mathcal{L}(Q)$ contains precisely all non simple, preinjective indecomposable modules.
- (c) An indecomposable module has central measure if and only if it is regular.

Proof. Fix $r \in \mathbb{N}_{\geq 2}$, we set $S_{n,r}$ as the connected quiver with vertex set $\{1, \dots, n\}$, unique sink n and sources $1, \dots, n-1$. For each $i \in \{1, \dots, n-1\}$ there are r arrows from i to n .

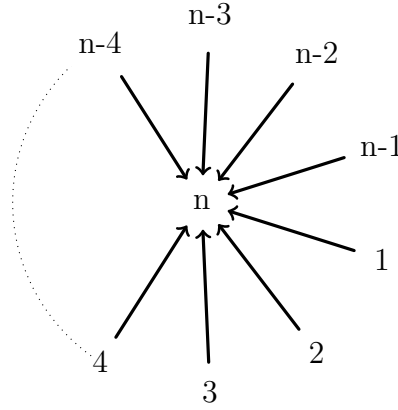


Figure 12.2: Star quiver with r -folded arrows.

Now we consider $kS_{n,r}$. Note that $\overline{S_{n,r}}$ is not a simply laced Dynkin or Euclidean diagram for $n \neq 2$ or $r \neq 2$ and hence $S_{n,r}$ is a wild quiver. By direct computation we get for the projective indecomposable modules $P(i)$ corresponding to the vertex $i \in \{1, \dots, n\}$:

$$\underline{\dim} P(n) = e_n \quad \text{and} \quad \underline{\dim} P(i) = e_i + r \cdot e_n \quad \text{for} \quad 1 \leq i \leq n-1,$$

where e_i denotes the canonical basis vector of \mathbb{R}^n . In particular $\dim_k P(i) = r+1 = \dim_k P(j)$ for all $i, j \in \{1, \dots, n-1\}$. The start of the preprojective component of the Auslander-Reiten quiver is illustrated in Figure 12.3. Note that every arrow is an r -fold arrow and every non-zero morphism starting in $P(n)$ is injective.

The Auslander-Reiten sequence

$$0 \rightarrow P(n) \rightarrow \bigoplus_{i=1}^{n-1} P(i)^r \rightarrow \tau_{kS_{n,r}}^{-1} P(n) \rightarrow 0$$

together with 12.8 yields that every irreducible morphism $P(i) \rightarrow \tau_{kS_{n,r}}^{-1} P(n)$ is injective. Now consider the AR sequence

$$0 \rightarrow P(i) \rightarrow (\tau_{kS_{n,r}}^{-1} P(n))^r \rightarrow \tau_{kS_{n,r}}^{-1} P(i) \rightarrow 0.$$

Application of 12.8 shows that each irreducible morphism $\tau_{kS_{n,r}}^{-1} P(n) \rightarrow \tau_{kS_{n,r}}^{-1} P(i)$ is injective. Also the AR sequence shows that $\dim_k \tau_{kS_{n,r}}^{-1} P(i) = \dim_k \tau_{kS_{n,r}}^{-1} P(j)$ for all $i, j \in \{1, \dots, n-1\}$.

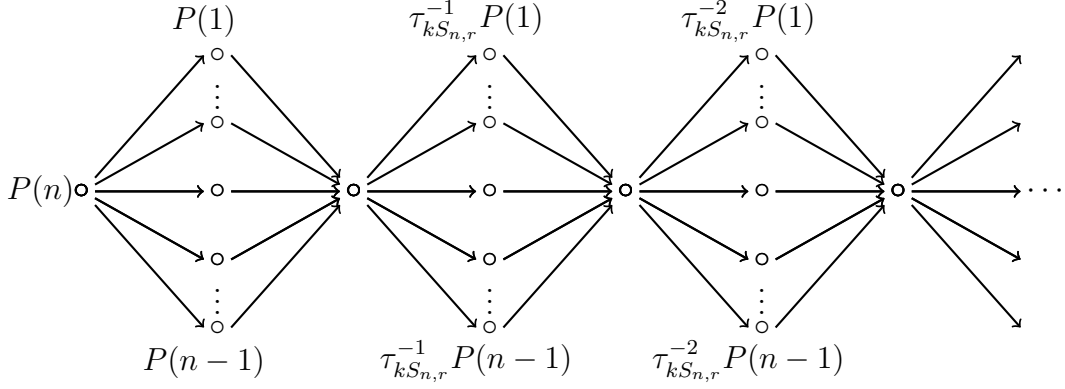


Figure 12.3: Start of the preprojective component of $kS_{n,r}$.

Inductively it follows that each irreducible morphism in $\mathcal{P}(S_{n,r})$ is injective.

We set $Q_1 := \{P(n)\}$, $Q_2 := \{P(1), \dots, P(n-1)\}$ and for $i \geq 3$

$$Q_i := \begin{cases} \{\tau_{kS_{n,r}}^{-\frac{i-2}{2}} X \mid X \in Q_2\}, & i \in 2\mathbb{N} \\ \{\tau_{kS_{n,r}}^{-\frac{i-1}{2}} P(n)\}, & i \in 2\mathbb{N} + 1. \end{cases}$$

Clearly $\mu(P(n)) = \{1\}$ and $\mu(P(i)) = \{1, r+1\}$. Now let $i \geq 3$ and $M \in Q_i$. Since each irreducible morphism to M is injective 12.7 shows that a GR submodule $X \subset M$ is in Q_{i-1} . By an easy induction on i it follows that for each $i \geq 3$

- $\mu(M) = \mu(N)$ for all $M, N \in Q_i$,
- $\dim_k X = \dim_k Y$ for all $X, Y \in Q_i$ and
- $\mu(M) = \{1, r+1, \dim_k P_3, \dim_k P_4, \dots, \dim_k P_i\}$ with $P_j \in Q_j$, $3 \leq j \leq i$.

Now the remaining proof is just a slight adaptation of [13, 2.3] to the present situation. We give the complete proof for convenience of the reader. We identify $\mathcal{A}(J_i)$ with a set of representatives of $\mathcal{A}(J_i)$ for each $i \in \mathbb{N}$. We claim that

$$\mathcal{A}(J_i) = \begin{cases} Q_1 \cup \{S(1), \dots, S(n-1)\}, & i = 1 \\ Q_i, & i > 1. \end{cases}$$

Note that the simple modules $S(1), \dots, S(n-1)$ are injective and hence do not occur as a proper submodule of an indecomposable module. Now we proceed by induction on i , the case $i = 1$ being trivial. Let $M \in \mathcal{A}(J_2)$. Then M is not simple and we find a GR submodule $X \subset M$ of M . Then $X \in \mathcal{A}(J_1)$, hence $X = P(n)$ and by 11.5 there is a surjection $\pi: P(i) \rightarrow M$ for some $i \in \{1, \dots, n-1\}$.

It follows $\{1, \dim_k M\} = \mu(M) \leq \mu(P(i)) = \{1, r+1\}$. Hence $\dim_k M \geq r+1 = \dim_k P(i)$ and π is an isomorphism. It follows $\mathcal{A}(J_2) = Q_2$.

Let $i \geq 2$, $M \in \mathcal{A}(J_{i+1})$ and $X \in \mathcal{A}(J_j)$ a GR submodule of M with $j \leq i$. By induction $X \in Q_j$ and again by 11.5 we find an epimorphism $\pi: Y \rightarrow M$ for some $Y \in Q_{j+1}$. Assume that $\dim_k Y > \dim_k M$ and fix $N \in Q_i = \mathcal{A}(J_i)$. By the above considerations and the induction hypothesis, we have $\mu(N) = \{1, r+1, \dim_k P_3, \dots, \dim_k P_i = \dim_k N\}$ for arbitrary $P_k \in Q_k$

and $\mu(X) = \mu(P_j)$. Let $P_{i+1} \in Q_{i+1}$. Since every two elements in Q_{i+1} have the same dimension we conclude with $\dim_k P_{j+1} = \dim_k Y > \dim_k M$ that

$$\begin{aligned} J_{i+1} = \mu(M) &= \mu(X) \cup \{\dim_k M\} \stackrel{\mu(P_j) = \mu(X)}{=} \mu(P_j) \cup \{\dim_k M\} \\ &\stackrel{\dim_k M < \dim_k P_{j+1}}{>} \mu(P_j) \cup \{\dim_k P_{j+1}, \dots, \dim_k P_i\} \cup \{\dim_k P_{i+1}\} = \mu(P_{i+1}) \\ &> \mu(P_j) \cup \{\dim_k P_{j+1}, \dots, \dim_k P_i\} \\ &= \mu(N) = J_i. \end{aligned}$$

This is a contradiction, because there does not exist a GR measure greater than J_i and less than J_{i+1} . Hence $\dim_k Y \leq \dim_k M$ and $M \cong Y \in Q_{j+1}$. It follows $j = i$. Statement (c) follows as in the proof of [13, 2.3]. \square

Remark. For $r = 1$ and $n \geq 6$ the quiver $S_{n,r}$ is wild but each irreducible morphisms from $\tau_{kS_{n,r}}^{-1} P(n) \rightarrow \tau_{kS_{n,r}}^{-1} P(1)$ is surjective. Hence, we can not apply 12.7 to compute the take-off part of $kS_{n,r}$.

12.2 Morphisms between regular modules

Throughout this section Q denotes a wild quiver. The proof of the next lemma may be found in [21].

Lemma 12.10. *Let $c, d \in \mathbb{N}$, then there exists*

- (a) $n(c, d) \in \mathbb{N}$ such that $\dim_k \tau_{kQ}^m M > c$ for all integers $m \in \mathbb{Z}$ with $|m| \geq n(c, d)$ and all regular indecomposable modules with $\dim_k M \leq d$.
- (b) $n(c) \in \mathbb{N}$ such that $\dim_k \tau_{kQ}^m I > c$ for each $m \geq n(c)$ and each injective indecomposable kQ -module I .
- (c) $n(d) \in \mathbb{N}$ such that $\dim_k \tau_{kQ}^{-m} P > d$ for each $m \geq n(d)$ and each projective indecomposable kQ -module P .

For $(x_i), (y_i) \in \mathbb{Z}^n$, we write $(x_i) \leq (y_i)$, if $x_i \leq y_i$ for all $i = 1, \dots, n$.

Lemma 12.11. [23, 4.2] *For any non-zero kQ -module X , there is an integer n such that all maps in $\text{Hom}_{kQ}(R, \tau_{kQ}^{-m} Y)$ have regular cokernel for all regular modules R , for all $m \geq n$ and for all regular modules Y with $\underline{\dim} Y \leq \underline{\dim} X$.*

Let $\mathcal{E} := \{\mathcal{C} \in \mathcal{R}(Q) \mid \mathcal{C} \text{ contains an elementary module}\}$. For each component $\mathcal{C} \in \mathcal{E}$ fix an indecomposable module $E_{\mathcal{C}} \in \mathcal{C}$ of minimal dimension. Observe that each $E_{\mathcal{C}}$ is elementary and quasi-simple.

Corollary 12.12. *Let E be a regular kQ -module.*

- (a) *There exists $n \in \mathbb{N}$ such that every proper submodule of $\tau_{kQ}^{-m} F$ is preprojective for each elementary module with $\underline{\dim} F \leq \underline{\dim} E$ and each $m \geq n$.*
- (b) *For each $d \in \mathbb{N}$ there exists $n(d) \in \mathbb{N}$ such that for each $m \geq n(d)$ and each elementary module F with $\dim_k F \leq d$ every proper submodule of $\tau_{kQ}^{-m} F$ is preprojective.*
- (c) *There exists an integer $n \in \mathbb{N}$ such that for each module $E \in \{E_{\mathcal{C}} \mid \mathcal{C} \in \mathcal{E}\}$, $m \geq n$ every proper submodules of $\tau_{kQ}^{-m} E$ is preprojective.*

Proof. (a) Fix $n \in \mathbb{N}$ as in 12.11 for $X = E$. Now let F be elementary with $\underline{\dim} F \leq \underline{\dim} E$. Let $m \geq n$ and $R \subseteq \tau_{kQ}^{-m} F$ be a regular indecomposable submodule. By 12.11 $(\tau_{kQ}^{-m} F)/R$ is regular and since $\tau_{kQ}^{-m} F$ is elementary, we get $R \in \{0, \tau_{kQ}^{-m} F\}$.

(b) Note that $\mathcal{X} := \{\underline{\dim} M \mid M \text{ elementary, } \dim_k M \leq d\}$ is a finite set. Now apply (a) for each element in \mathcal{X} and take the maximum integer over all obtained numbers.

(c) It is known (see [24, 2.1]) that there are only finitely many Coxeter-orbits of dimension vectors of elementary modules. In particular, there is $d \in \mathbb{N}$ such that $d \geq \dim_k E$ for all $E \in \{E_C \mid C \in \mathcal{E}\}$. Now apply (b). \square

With the above observations we are able to reformulate the result [23, 4.6] in a slightly stronger version, which will allow us to make statements about the limiting behavior of GR measures in regular components. However, the proof stays nearly unchanged.

Lemma 12.13. *Let $c, d \in \mathbb{N}$. Then there exists an integer $n(c, d)$ with $\text{Hom}_{kQ}(Z, \tau_{kQ}^{-m} X) = 0$ for all $m \geq n(c, d)$, for all regular modules X and all regular modules Z with $\dim_k X \leq d$ and $\dim_k Z \leq c$.*

Proof. By 12.12, there exists n_1 such that for each elementary module E with $\dim_k E \leq d$ the module $\tau_{kQ}^{-m} E$ has no non-trivial regular submodule, i.e. all non-zero homomorphisms $f: R \rightarrow \tau_{kQ}^{-m} E$ with R regular are surjective for $m \geq n_1$. By 12.10, we find n_2 such that $\dim_k \tau_{kQ}^{-m} U > c$ for all $m \geq n_2$ and every regular module U with $\dim_k U \leq d$. Now set $n(c, d) := \max\{n_1, n_2\}$ and let Z be regular with $\dim_k Z \leq c$. Furthermore, let $m \geq n(c, d)$, E be elementary with $\dim_k E \leq d$ and $f: Z \rightarrow \tau_{kQ}^{-m} E$. On the one hand, $m \geq n_1$ implies that f is surjective or $f = 0$. On the other hand it is $m \geq n_2$ and hence $\dim_k \tau_{kQ}^{-m} E > c \geq \dim_k Z$. We conclude that f is not surjective and hence zero.

If X is not elementary, it has a filtration $X = X_0 \supset \dots \supset X_r \supset X_{r+1} = 0$ with $X_i/X_{i+1} =: E_i$ elementary. Note that $\dim_k E_i \leq \dim_k X \leq d$ for all $i \in E_i$. We get $\text{Hom}_{kQ}(\tau_{kQ}^m Z, E_i) \cong \text{Hom}_{kQ}(Z, \tau_{kQ}^{-m} E_i) = 0$ for all $m \geq n(c, d)$ and Z regular with $\dim_k Z \leq c$. Hence 4.6 yields $0 = \text{Hom}_{kQ}(\tau_{kQ}^m Z, X) \cong \text{Hom}_{kQ}(Z, \tau_{kQ}^{-m} X)$. \square

12.3 Limiting behavior for wild algebras with $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$

Lemma 12.14. *Let I, J, K be finite, non-empty subsets of \mathbb{N} such that*

- (a) $J \ll K$,
- (b) $J < I$ and
- (c) $\max I < \max J$.

Then $J < K < I$.

Proof. By definition, we have $J < K$. Assume that $J < I \leq K$. Since $\max I < \max J$ and $J \subset K$, we get $\max I < \max K$ and hence $J < I < K$. By 11.1 we know $J \ll I$. In particular, $\max J \leq \max I$, a contradiction. \square

Definition. Let Q be a quiver, then we denote by p_Q the dimension of an indecomposable projective kQ -module of maximal dimension and by i_Q the dimension of an indecomposable injective kQ -module of maximal dimension.

Lemma 12.15. [31, 3.1] *Let X, M be indecomposable kQ -modules. If X is a GR submodule of M , then $\dim_k M \leq p_Q \cdot i_Q \cdot \dim_k X$. In other words, the quotient $\dim_k N / \dim_k Y$ is bounded by $p_Q \cdot i_Q$ for each GR inclusion $Y \rightarrow N$.*

Lemma 12.16. *Let Q be a wild quiver and $c, d \in \mathbb{N}$. Then there exists $n(c, d) \in \mathbb{N}$ such that for each $m > n(c, d)$ and each regular indecomposable module M with $\dim_k M \leq d$ there is an indecomposable preprojective module $M(m) \subset \tau_{kQ}^{-m} M$ such that*

- (a) $c < \dim_k M(m) < \dim_k \tau_{kQ}^{-m} M$,
- (b) $\mu(M(m)) \ll \mu(\tau_{kQ}^{-m} M)$.

Moreover, $M(m)$ appears in a Gabriel-Roiter filtration of $\tau_{kQ}^{-m} M$.

Proof. Set $q := p_Q \cdot i_Q > 1$. Choose b in \mathbb{N} such that $c < \frac{1}{q}b < b$.

Now 12.10 yields $n_1 \in \mathbb{N}$ such that $b < \dim_k \tau_{kQ}^{-m} M$ for all $m > n_1$ and all regular indecomposable modules M with $\dim_k M \leq d$. By 12.13, we find $n_2 \in \mathbb{N}$ such that $\text{Hom}_{kQ}(Z, \tau_{kQ}^{-m} M) = 0$ for all $m > n_2$, all regular modules M with $\dim_k M \leq d$ and all regular modules $Z \leq b$. Now define $n(c, d) := 2 \max\{n_1, n_2\}$, let $m > n(c, d)$ and fix a regular module M with $\dim_k M \leq d$. Let $B_1 \subset B_2 \subset \dots \subset B_l = \tau_{kQ}^{-m} M$ a Gabriel-Roiter filtration of $\tau_{kQ}^{-m} M$. In particular, we have $\mu(\tau_{kQ}^{-m} M) = \{\dim_k B_1, \dots, \dim_k B_l\}$ with $\dim_k B_1 = 1$ and $\dim_k B_l = \dim_k \tau_{kQ}^{-m} M$. Since $\dim_k B_l = \dim_k \tau_{kQ}^{-m} M > b \geq 1$, there exists $i \in \{1, \dots, l-1\}$ such that $\dim_k B_i \leq b \leq \dim_k B_{i+1}$. It follows that B_i is preprojective indecomposable; assume, to the contrary, that B_i is regular. Since $B_i \subseteq \tau_{kQ}^{-m} M$ we have $\text{Hom}_{kQ}(B_i, \tau_{kQ}^{-m} M) \neq 0$. This contradicts the fact that $\dim_k B_i \leq b$.

Hence, B_i is preprojective, a Gabriel-Roiter submodule of B_{i+1} and

$$\frac{qc}{\dim_k B_i} < \frac{b}{\dim_k B_i} \leq \frac{\dim_k B_{i+1}}{\dim_k B_i} \leq q.$$

The last inequality follows from 12.15. We set $M(m) := B_i$ and conclude $c < \dim_k B_i = \dim_k M(m) < \dim_k \tau_{kQ}^{-m} M$. Since $B_1 \subset B_2 \subset \dots \subset B_n = \tau_{kQ}^{-m} M$ is a GR filtration it follows that $\mu(M(m)) \ll \mu(\tau_{kQ}^{-m} M)$. \square

Proposition 12.17. *Let Q be a wild quiver such that $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$. There does not exist a minimal central measure, i.e. for each kQ -module M with central measure there exists a module N with $\mu(N)$ central and $\mu(N) < \mu(M)$.*

Proof. Assume, to the contrary, that kQ has a minimal central measure I_0 realised by an indecomposable module M_0 . Since Q is a wild quiver algebra, there exists an infinite family $(X_i)_{i \in \mathbb{N}}$ of pairwise non isomorphic indecomposable regular modules with $\underline{\dim} X_i = \underline{\dim} X_j$ for all i, j (this follows from Kac's Theorem). In particular, $d := \dim_k X_i = \dim_k X_j$ for all $i, j \in \mathbb{N}$. Now 12.16 yields an $n(c, d)$ such that for $m \geq n$ there is $X_i(m) \subset \tau_{kQ}^{-m} X_i$ preprojective indecomposable with $c := 2 \dim_k M_0 < \dim_k X_i(m)$ and $\mu(X_i(m)) \ll \mu(\tau_{kQ}^{-m} X_i)$. Since $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$, we have $\mu(X_i(m)) < \mu(M_0)$ and conclude with Lemma 12.14 that $\mu(\tau_{kQ}^{-m} X_i) < \mu(M_0) = I_0$ for all $i \in \mathbb{N}$. By minimality, each measure $\mu(\tau_{kQ}^{-m} X_i)$ has to be a take-off measure. Since $\underline{\dim} \tau_{kQ}^{-m} X_i = \underline{\dim} \tau_{kQ}^{-m} X_j$ for all i, j it is $C := \dim_k \tau_{kQ}^{-m} X_i = \dim_k \tau_{kQ}^{-m} X_j$. Hence, we get $\mu(\tau_{kQ}^{-m} X_i) \in \{1, \dots, C\}$ for all $i \in \mathbb{N}$. This is a contradiction since each take-off measure is of finite type and there are only finitely many subsets in $\{1, \dots, C\}$. \square

Let us mention that in every example known to us with $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$ we are in the situation that $\mathcal{T}(Q)$ contains precisely the preprojective indecomposable modules and the simple modules. In this case a small adaptation of the proof of [13, 3.6] would yield the same result. But then we can prove a stronger statement:

Proposition 12.18. *Let Q be a wild quiver such that $\mathcal{T}(Q)$ contains precisely the preprojective indecomposable modules and all simple modules. Then every non-simple regular indecomposable module has central measure and for each $c, d \in \mathbb{N}$ and families $(X_i)_{i \in I}$, $(Y_j)_{j \in J}$ of non simple, regular indecomposable modules with $\dim_k X_i \leq d$ and $\dim_k Y_j \leq c$ there is an $n(c, d) \in \mathbb{N}$ such that $\mu(\tau_{kQ}^{-m} X_i) < \mu(Y_j)$ for all $m \geq n(c, d)$ and all i, j .*

In particular, each τ -orbit in a regular component does not possess a minimal central measure.

Proof. Since each landing measure is preinjective [37, 4.], we conclude that every non-simple regular module has central measure. By 12.16, we find an integer $n(c, d)$ such that for each $m \geq n(c, d)$ and each X_i there exists an indecomposable preprojective module $X_i(m) \subset \tau_{kQ}^{-m} X_i$ with $\dim_k Y_j \leq c < \dim_k X_i(m)$ and $\mu(X_i(m)) \ll \mu(\tau_{kQ}^{-m} X_i)$. By 12.14, we conclude $\mu(\tau_{kQ}^{-m} X_i) < \mu(Y_j)$ for all i and j . \square

13 Central quivers

In this section we assume that Q is a wild quiver. We consider wild quivers such that the two partitions

$$\text{ind } \mathcal{P}(Q) \cup \text{ind } \mathcal{R}(Q) \cup \text{ind } \mathcal{I}(Q) = \text{ind } kQ = \text{ind } \mathcal{T}(Q) \cup \text{ind } \mathcal{Z}(Q) \cup \text{ind } \mathcal{L}(Q),$$

are as "similar" as possible. Therefore we define:

Definition. A wild quiver Q is central if every preprojective indecomposable kQ -module has take-off measure and an indecomposable module M has central measure if and only if M is regular.

Note that Q being central implies that each simple module is preprojective or preinjective and therefore $\mathcal{L}(Q)$ differs from $\mathcal{I}(Q)$ only by the simple injective modules.

We have seen in 12.1 that for each $l \geq 2$ there exist infinitely many central quivers with l vertices. From now on let Q be a central quiver. The next statement follows immediately from proposition 12.18.

Corollary 13.1. *Let X, Y be indecomposable regular kQ -modules. There exists $n \in \mathbb{N}$ such that $\mu(\tau_{kQ}^{-m} X) < \mu(Y)$ for all $m \geq n$.*

Let M be regular and indecomposable. The set $M^\mu := \{\mu(\tau_{kQ}^m M) \mid m \in \mathbb{Z}\}$ is bounded below (for example by $\{1\}$). By completeness of $(\mathcal{P}(\mathbb{N}), <)$ the infimum $\mathcal{X}(M) := \inf\{\mu(\tau_{kQ}^m M) \mid m \in \mathbb{Z}, \tau_{kQ}^m M\}$ exists.

Corollary 13.2. *If M, N are regular indecomposable, then $\mathcal{X}(M) = \mathcal{X}(N)$ and $\mathcal{X}(M) < \mu(Y)$ for each regular indecomposable module Y . In particular $\mathcal{X}(Q) := \mathcal{X}(M)$ is well-defined.*

Proof. Let $m \in \mathbb{Z}$, then there is $n \in \mathbb{N}$ such that $\mu(\tau_{kQ}^{-n} N) < \mu(\tau_{kQ}^{-m} M)$. We conclude $\mathcal{X}(N) \leq \mu(\tau_{kQ}^{-n} N) < \mu(\tau_{kQ}^{-m} M)$ for all $m \in \mathbb{Z}$ and $\mathcal{X}(N) \leq \mathcal{X}(M)$. By the same token we get $\mathcal{X}(M) \leq \mathcal{X}(N)$. By considering the case $m = 0$ and $Y = M$, the second claim follows. \square

Definition. Let $\mathcal{E}(Q)$ be the set of all preprojective indecomposable modules that occur as a GR submodule of a regular indecomposable module.

Example. It follows from [13, 3.5] that $\mathcal{E}(\Gamma_r) = \text{ind } \mathcal{P}(\Gamma_r)$, i.e. $\mathcal{E}(\Gamma_r)$ is as large as possible.

Lemma 13.3. *The following statements hold.*

- (a) *A preprojective module P is in $\mathcal{E}(Q)$ if and only if P is a GR submodule of a regular module and each GR inclusion $P \subset X$ with X regular implies that X is elementary.*
- (b) *For each $n \geq \mathbb{N}$ we find $P \in \mathcal{E}(Q)$ with $\dim_k P > n$.*

Proof. (a) Let X be an elementary module and P be an indecomposable preprojective submodule such that P is a GR submodule of X . Let $Y \subset X$ be a proper indecomposable submodule of X . Then by definition $\mu(Y) \leq \mu(P)$. The measure $\mu(P)$ is a take-off measure and therefore so is $\mu(Y)$. Since Q is central, the only indecomposable modules with take-off measure are preprojective and preinjective simple module. Since $\text{Hom}_{kQ}(\mathcal{I}(Q), \mathcal{R}(Q)) = 0$, we conclude $Y \in \mathcal{P}(Q)$. We have shown that every proper submodule of X is preprojective and X is elementary.

(b) Recall (see 12.15) that there exists $r \in \mathbb{N}$ such that $\dim_k Y / \dim_k X \leq r$ for each GR inclusion $X \subset Y$.

Now let E be an elementary module and $n \in \mathbb{N}$. Then there exists $q \in \mathbb{N}$ (3.1, 12.10) such that $\tau_{kQ}^{-q} E$ has only preprojective submodules and $\dim_k \tau_{kQ}^{-q} E \geq nr$. Let X be a GR submodule of $\tau_{kQ}^{-q} E$, then X is preprojective and $\dim_k \tau_{kQ}^{-p} E / \dim_k X \leq r$. Hence $\dim_k X \geq n$. \square

Corollary 13.4. *Let M be a regular module. Then*

$$\mathcal{X}(Q) = \bigcup_{P \in \mathcal{E}(Q)} \mu(P) = \sup\{J_t \mid t \in \mathbb{N}\}.$$

Proof. Let $P \in \mathcal{E}(Q)$ and E be elementary such that $P \subset E$ is a GR inclusion. Since P is a take-off module, we know $\mu(P) < \mu(\tau_{kQ}^{-m} M)$ for all $m \in \mathbb{Z}$. Hence

$$\mu(P) \leq \mathcal{X}(M) \stackrel{13.2}{=} \mathcal{X}(E) < \mu(E) = \mu(P) \cup \{\dim_k E\}.$$

Since each preprojective indecomposable module is in $\mathcal{T}(Q)$, there is $H \in \text{ind } \mathcal{P}(Q)$ with $\mu(P) < \mu(H)$. By the same argument as before, we have $\mu(H) \leq \mathcal{X}(M)$ and conclude

$$\mu(P) < \mu(H) \leq \mathcal{X}(M) < \mu(P) \cup \{\dim_k E\}.$$

Now we can apply 11.1 and conclude that $\mathcal{X}(M)$ starts with $\mu(P)$. In particular, $\mu(P) \subseteq \mathcal{X}(M)$. Since P was an arbitrary module in $\mathcal{E}(Q)$, we conclude $\bigcup_{P \in \mathcal{E}(Q)} \mu(P) \subseteq \mathcal{X}(M) = \mathcal{X}(Q)$.

Now let $n \in \mathcal{X}(M)$. By 13.3 there exists $P_0 \in \mathcal{E}(Q)$ with $\dim_k P_0 \geq n$. By the same argument as before, it is $n \in \mathcal{X}(M) \cap \{1, \dots, \dim_k P_0\} = \mu(P_0)$. Hence $\mathcal{X}(Q) \subseteq \bigcup_{P \in \mathcal{E}(Q)} \mu(P)$. We have proven that

$$\mathcal{X}(Q) = \bigcup_{P \in \mathcal{E}(Q)} \mu(P).$$

Since $J_i \leq \mathcal{X}(Q)$ for each take-off measure, we have

$$\sup\{J_t \mid t \in \mathbb{N}\} \leq \mathcal{X}(Q) = \bigcup_{P \in \mathcal{E}(Q)} \mu(P).$$

Now let $P \in \mathcal{E}(Q)$ and $t \in \mathbb{N}$ such that $J_t > \mu(P)$. It follows $\mu(P) < J_t < \sup\{J_t \mid t \in \mathbb{N}\} \leq \mathcal{X}(Q) < \mu(E)$ with GR inclusion $P \subset E$ and E elementary. Since E is a GR submodule of E we have $\mu(P) \ll \mu(E)$. By 11.1, we have $\mu(P) \ll \sup\{J_t \mid t \in \mathbb{N}\}$ and $\mu(P) \subset \sup\{J_t \mid t \in \mathbb{N}\}$. It follows

$$\mathcal{X}(Q) = \bigcup_{P \in \mathcal{E}(Q)} \mu(P) \subseteq \sup\{J_t \mid t \in \mathbb{N}\} \leq \mathcal{X}(Q).$$

\square

Figure 13 illustrates the limiting behavior for a central quiver.

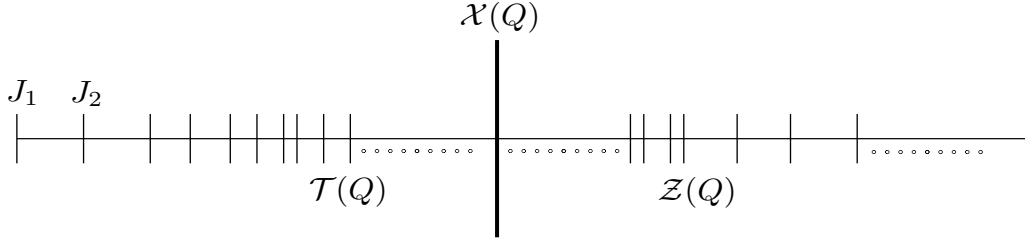


Figure 13.1: Limiting behavior of central quivers.

Example. For $r = 3$ it is $\mathcal{X}(\Gamma_r) = \{F_{2n-2} + F_{2n} \mid n \in \mathbb{N}\} = \{1, 4, 11, 29, 76, 199, 521, \dots\}$, where F_n denotes the n -th Fibonacci number.

Fahr conjectures in his PhD thesis the existence of a maximal central measure for \mathcal{K}_3 , realized by an infinite dimensional module called \mathcal{F} -GR measure. Moreover, he expects that \mathcal{F} can be realized as the limit $\lim_{i \rightarrow \infty} \mu(\tau^i M)$ for some indecomposable module M with dimension vector $(1, 1)$. The following statement is in some sense a dual result.

Corollary 13.5. *Let Q be a central quiver. Then the following statements hold:*

- (a) *The algebra kQ does not have a minimal central measure.*
- (b) *There exists an indecomposable, infinite dimensional kQ -module \mathfrak{M} with GR measure $\mathcal{X}(Q)$ such that every finite dimensional submodule of \mathfrak{M} is preprojective.*
- (c) *It is $J_t < \mathcal{X}(Q)$ for all $t \in \mathbb{N}$ and $\mathcal{X}(Q) < I$ for all other Gabriel-Roiter measures I . $\mathcal{X}(Q)$ can be obtained by every regular indecomposable module M as $\mathcal{X}(Q) = \inf\{\mu(\tau_{kQ}^m M) \mid m \in \mathbb{Z}, \tau_{kQ}^m M\}$.*

Proof. The existence of \mathfrak{M} is shown in [31, 7.3]. \mathfrak{M} is an infinite dimensional and indecomposable kQ -module with GR filtration $M_1 \subseteq M_2 \subset \dots \subseteq \bigcup_{i \in I} M_i = \mathfrak{M}$. Moreover, all M_i have take-off measure. It follows $\mu(\mathfrak{M}) = \sup\{J_t \mid t \in \mathbb{N}\} = \mathcal{X}(Q)$. \square

13.1 Wild quivers with $l \geq 3$ vertices

In this section we study the limiting behavior of wild quivers with $l \geq 3$ vertices.

Lemma 13.6. *Let Q be a wild quiver with $|Q_0| \geq 3$. The take-off part $\mathcal{T}(Q)$ does not contain a regular tilting module, i.e. if T is a regular tilting module, then there is a direct indecomposable summand S of T such that $\mu(S)$ is not a take-off measure (if and only if $\mu(T)$ is not a take-off measure).*

Proof. Since $|Q_0| \geq l$ there exists a regular tilting module $T = T_1 \oplus \dots \oplus T_l$. We consider the regular tilting module $\tau_{kQ}^{-1}T$. Since $\text{Hom}_{kQ}(\mathcal{R}(Q), \mathcal{P}(Q)) = 0$ we get $\text{ind } \mathcal{P}(Q) \subseteq \{M \in \text{mod } kQ \mid \text{Hom}_{kQ}(\tau_{kQ}^{-1}T, M) = 0\} \stackrel{2.16}{=} \text{Cogen } \tau_{kQ} \tau_{kQ}^{-1}T = \text{Cogen } T$. Hence, for each preprojective indecomposable module P there exists $d \in \mathbb{N}$ and a monomorphism $P \rightarrow T^d$. In particular, $\mu(P) < \mu(T^d) = \mu(T) = \max\{\mu(T_i)\} = \mu(T_{i_0})$ for some $1 \leq i_0 \leq l$. Hence, T_{i_0} is not a take-off module, since for each take-off module X the set $\{Y \in \text{mod } kQ \mid Y \text{ indecomposable and } \mu(Y) < \mu(X)\}$ is finite. \square

Corollary 13.7. *Let Q be a wild quiver with $l := |Q_0| \geq 3$. There exists a regular indecomposable module M such that $\{\tau_{kQ}^{-m}M \mid m \geq 0\}$ contains infinitely many modules with central measure.*

Proof. Since $l \geq 3$, there exists a regular tilting module T by 2.15. Let $m \geq 0$, then $\tau_{kQ}^{-m}T$ is a regular tilting module and we find by Lemma 13.6 $i_m \in \{1, \dots, l\}$ such that $\mu(\tau_{kQ}^{-m}T_{i_m})$ is a central measure. Hence, one of the sequences $(\tau_{kQ}^{-m}T_1)_{m \geq 0}, \dots, (\tau_{kQ}^{-m}T_l)_{m \geq 0}$ has to contain infinitely many modules with central measure. \square

We give an alternative proof for 12.17 in the case that $|Q_0| \geq 3$:

Proposition 13.8. *Let Q be a wild quiver with $\text{ind } \mathcal{P}(Q) \subseteq \mathcal{T}(Q)$ and $|Q_0| \geq 3$, then kQ does not possess a minimal central measure.*

Proof. Assume that M_0 has minimal central measure. Since $|Q_0| \geq 3$, we can apply 13.7 and find an indecomposable regular module N such that $\{\tau_{kQ}^{-m}N \mid m \geq 0\}$ contains infinitely many indecomposable modules with central measure. Now application of 12.16 shows the existence of $n \in \mathbb{N}$ such that for all $m \geq n$ the module $\mu(\tau_{kQ}^{-m}N)$ starts with $\mu(X(m))$, $\dim_k X(m) > \dim_k M_0$ and $X(m)$ is preprojective. By construction there is $m_0 \geq n$ with $\mu(\tau_{kQ}^{-m_0}N)$ being central. Hence, $\mu(M_0) \leq \mu(\tau_{kQ}^{-m_0}N)$ by the minimality of $\mu(M_0)$. Now we apply 12.14 with $I := \mu(X(m_0))$, $J := \mu(X(m_0))$ and $K := \mu(\tau_{kQ}^{-m_0}N)$. Note that the assumption (b) in 12.14 is satisfied since Q is a central quiver and $X(m_0)$ is a preprojective module. We get $\mu(\tau_{kQ}^{-m_0}N) < \mu(M_0)$. This is a contradiction to the minimality of $\mu(M_0)$. \square

Theorem 13.9. *Let Q be a central quiver with $|Q_0| \geq 3$. For every $X, Y \in \text{ind } \mathcal{R}(Q)$ with X quasi-simple, there exists $n \in \mathbb{N}$ such that for every S in $(\tau_{kQ}^{-n}X \rightarrow)$ we have $\mu(S) < \mu(Y)$.*

Proof. Let $l := |Q_0|$ and $T = T_1 \oplus \dots \oplus T_l$ be a regular tilting module. By 13.1, we find $a \in \mathbb{N}$ such that for all $b \geq a$ and all $i \in \{1, \dots, l\}$ we have $\mu(\tau_{kQ}^{-b}T_i) < \mu(Y)$. Set $\bar{T} := \tau_{kQ}^{-a}T$. Now there exists $n \in \mathbb{N}$ such that $\text{Hom}_{kQ}(\bar{T}, \tau_{kQ}^{-m}X) = 0$ for all $m \geq n$. Since $\{U \in \text{mod } kQ \mid \text{Hom}_{kQ}(\bar{T}, U) = 0\}$ is closed under submodules and extensions, we get $\text{Hom}_{kQ}(\bar{T}, S) = 0$ for all S in $(\tau_{kQ}^{-n}X \rightarrow)$. By 2.16, we find for each S in $(\tau_{kQ}^{-n}X \rightarrow)$ a $d \in \mathbb{N}$ and a monomorphism $S \rightarrow \bar{T}^d$. Hence $\mu(S) \leq \mu(\bar{T}^d) = \mu(\tau_{kQ}^{-a}T) = \max \mu(\tau_{kQ}^{-a}T_i) < \mu(Y)$. \square

Corollary 13.10. *Let Q be a central quiver, $|Q_0| \geq 3$ and $X, Y \in \text{mod } kQ$ regular indecomposable with X quasi-simple. There is $n \in \mathbb{N}$ such that $\mu(S) < \mu(Y) < \mu(T)$ for every S in $(\tau_{kQ}^{-n}X \rightarrow)$ and every T in $\{\tau_{kQ}^m X[i] \mid m \geq n, i \in \mathbb{N}\}$.*

Proof. Fix $n_1 \in \mathbb{N}$ as in Theorem 13.9. By Corollary 12.4, we find n_2 with $\mu(\tau_{kQ}^m X) > \mu(Y)$ for all $m \geq n_2$. Set $n := \max\{n_1, n_2\}$. Now the claim follows since $\mu(\tau_{kQ}^m X[i]) \geq \mu(\tau_{kQ}^m X)$ for each $m \in \mathbb{Z}$ and $i \in \mathbb{N}$. \square

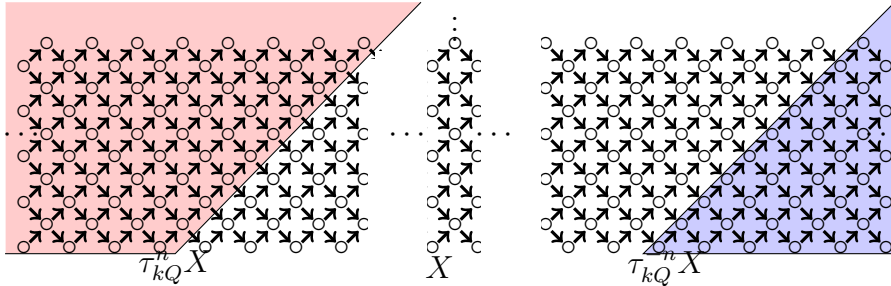


Figure 13.2: Illustration of the set $(\tau_{kQ}^{-n}X \rightarrow) \cup \{\tau_{kQ}^m X[i] \mid m \geq n, i \in \mathbb{N}\}$.

13.2 A counter example

In [14] Chen considers the extended Kronecker quiver $K = 1 \rightarrow 2 \rightrightarrows 3$ and shows that in this case $\text{ind } \mathcal{P}(K) \cap \mathcal{T}(K)$ is a finite set and that there exists a minimal central measure which is realized by the preprojective indecomposable module with dimension vector $(1, 1, 2)$.

In view of 12.17 one might hope that the existence of a minimal central measure would characterize wild quivers Q with $\text{ind } \mathcal{P}(Q) \cap \mathcal{T}(Q)$ finite.

We will construct a family $(Q_r)_{r \geq 3}$ of wild quivers such that $\text{ind } \mathcal{P}(Q_r) \cap \mathcal{T}(Q_r)$ is a finite set but there does not exist a minimal central measure. Let $r \geq 3$ and Q_r be the quiver shown in 13.3.

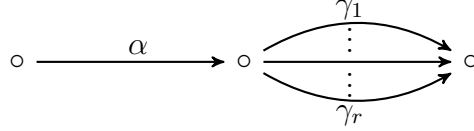


Figure 13.3: Quiver Q_r .

We have $\underline{\dim} P(1) = (1, 1, r)$, $\underline{\dim} P(2) = (0, 1, r)$ and $\underline{\dim} P(3) = (0, 0, 1)$. We denote by \overline{P}_i the indecomposable module that corresponds to the indecomposable preprojective \mathcal{K}_r -module P_i , hence $\underline{\dim} \overline{P}_i = (0, a, b)$ with $\underline{\dim} P_i = (a, b)$. Now let M be an indecomposable regular kQ_r -module with $\dim_k M_1 \neq 0$. We will show that

$$(*) \quad \mu(M) > \mu(\overline{P}_i) = \mu(P_i) \text{ for all } i \in \mathbb{N}.$$

Since $0 \neq \dim_k M_1 = \dim_k \text{Hom}_{kQ}(P(1), M)$, we find a non-zero morphism $f: P(1) \rightarrow M$. Let $Y := \text{im } f$. Since f is surjective onto Y , we get

$$1 \leq \dim_k \text{End}(Y) \leq \dim_k \text{Hom}_{kQ}(P(1), Y) = \dim_k Y_1 \leq \dim_k P(1)_1 = 1.$$

Hence, Y is a brick and therefore indecomposable. Note that every proper submodule of $P(1)$ is in $\text{add}(P(2) \oplus P(3))$. Consider the short exact sequence $0 \rightarrow \ker f \rightarrow P(1) \rightarrow Y \rightarrow 0$. Since a module with dimension vector $(1, 0, 0)$ is injective, we get $\ker f \in \text{add } P(3)$. Hence $\underline{\dim} Y = (1, 1, r - b)$ for some $0 \leq b \leq r$. Since a module with dimension vector $(1, 1, 0)$ is injective we get $0 \leq b < r$. Now we consider three different cases:

- (a) If $1 < b < r$, then $\dim_k Y \leq 1 + 1 + r - 2 = r$ and hence $\mu(M) \geq \mu(Y) \geq \{1, r\} > \mu(P_i)$ for all $i \in \mathbb{N}$.
- (b) If $b = 0$, then $Y = P(1)$ and $\mu(M) \geq \mu(P(1)) = \{1, r + 1, r + 2\} > \mu(P_i)$.
- (c) If $b = 1$, then $\underline{\dim} Y = (1, 1, r - 1)$. Assume that $\mu(Y) = \{1, r + 1\}$. By 11.5 there exists a surjection from $P(2) \rightarrow Y$, a contradiction since $\dim_k P(2)_1 = 0 \neq 1 = \dim_k Y_1$. Hence, $\mu(Y) \cap \{1, \dots, r + 1\} \supset \{1, r + 1\}$ and we get $\mu(M) \geq \mu(Y) \geq \{1, r\} > \mu(P_i)$.

Hence $\mu(M) > \mu(\overline{P}_i)$ for all $i \in \mathbb{N}$. Now let N be an indecomposable module with take-off measure. We conclude with $(*)$ that $\dim_k N_1 = 0$ and therefore every take-off module can be considered as a \mathcal{K}_r -module. Now it follows from 12.2 that $J_i = \mu(\overline{P}_i) = \mu(P_i)$ and $\mathcal{A}(J_i) = \{\overline{P}_i\}$ for all $i \geq 2$. We assume that M is an indecomposable module with minimal central measure. Then $\mu(M) > \mu(\overline{P}_i) = \mu(P_i)$ for all $i \in \mathbb{N}$.

We also find for each $i \in \mathbb{N}$ an elementary \mathcal{K}_r -module E_i with $\mu(E_i) = \mu(P_i) \cup \{\dim_k E_i\}$ (see [13, 3.5]). We consider E_i as a kQ_r -module and get $\mu(P_i) < \mu(M) \leq \mu(E_i) = \mu(P_i) \cup \{\dim_k E_i\}$

for all $i \in \mathbb{N}$. Hence $\mu(M)$ has to start with $\mu(P_i)$ for all $i \in \mathbb{N}$ by 11.1. A contradiction, since M is of finite dimension.

14 Elementary Representations for Γ_r

We have seen that elementary modules play an important role in the study of modules with constant d -socle rank and constant d -radical rank. But the main motivation for studying elementary modules is the fact that each non-zero regular module M has a filtration

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_2 \subset M_1 \subset M_0 = M$$

such that M_i/M_{i+1} is elementary for $0 \leq i < n$. Moreover, the class of elementary modules is the smallest class with this property.

Recently [36], Ringel gave a description of the τ -orbits of elementary modules for the 3-Kronecker algebra. An important tool for the description is a result which says that every module M with $\underline{\dim} M = (x, y)$ and $2 \leq y \leq x + 1$ has a submodule U with dimension vector $(1, 2)$.

We use the homological characterization of ESP_d in conjunction with Westwick's Theorem to generalize this result and other techniques used in [36] for all $r \geq 3$.

Theorem 14.1. [46, 1.] (*Westwick's Theorem*) *Let $s, m, n \in \mathbb{N}$ and $V \subseteq \text{Hom}_k(k^n, k^m)$ be a vector space of largest dimension such that each non-zero map $f \in V$ has rank s . Let $l(n, m, s)$ be the dimension of V . Then*

$$l(n, m, s) \leq m + n - 2s + 1.$$

14.1 Consequences of Westwick's Theorem

It is well known that if M is a representation of Γ_r contained in $\text{EKP} = \text{ESP}_1$, then $\dim_k M_1 < \dim_k M_2$. For completeness we provide a short proof that shows $\dim_k M_1 \leq \dim_k M_2$.

Lemma 14.2. *Let M be in $\text{rep}(\Gamma_r)$ and assume that M is in EKP , then $\dim_k M_1 \leq \dim_k M_2$.*

Proof. Just observe that $M(\gamma_1): M_1 \rightarrow M_2$ is injective if M is in EKP . Hence $\dim_k M_1 \leq \dim_k M_2$. \square

In [36] the following result is proven.

Lemma 14.3. *Each representation M of Γ_3 with dimension vector (x, y) and $2 \leq y \leq x + 1$ has a subrepresentation U with dimension vector $(1, 2)$.*

Assume that M is such a representation with subrepresentation U with dimension vector $(1, 2)$. If U is indecomposable, then U is isomorphic to X_T for some $T \in \text{Gr}_{1,3}$ by Proposition 3.7, thus $0 \neq \text{Hom}_{\Gamma_r}(X_T, U)$. If U decomposes with a direct summand isomorphic to I_1 , then $\text{Hom}_{\Gamma_r}(X_T, U) \neq 0$ for all $T \in \text{Gr}_{1,3}$. In the other case U decomposes as $U = W \oplus P_1$, where W is indecomposable with dimension vector $(1, 1)$. Hence $W \cong X_T$ for some $T \in \text{Gr}_{2,3}$. By Proposition 3.7 we find $S \in \text{Gr}_{1,3}$ and an epimorphism $X_S \rightarrow X_T$. Hence $\text{Hom}_{\Gamma_r}(X_S, U) \neq 0$ and we conclude:

Corollary 14.4. *Let M in $\text{rep}(\Gamma_3)$ be a representation with dimension vector $\underline{\dim} M = (x, y)$, $x > 0$ and $M \in \text{EKP}$, then $x + 1 < y (\Leftrightarrow y - x \geq 2)$.*

Proof. Assume to the contrary that $y \leq x + 1$. Then the above considerations yield $y \leq 1$. If $y = 0$, then M is in $\text{add } I_1$ which is not in EKP. If $y = 1$, then Lemma 14.2 and the assumption yield $x = y = 1$. But we have already seen that such representations do not have the equal kernels property. \square

In the following we use Westwick's Theorem to generalize the corollary for arbitrary $r \geq 3$ and $M \in \text{ESP}_d$ with $1 \leq d < r$. As a consequence we get a result similar to Lemma 14.3 for each pair of r and d .

Lemma 14.5. *If $1 \leq d < r$, $M \in \text{ESP}_d$ and $\dim_k M_1 \neq 0$, then $d \cdot \dim_k M_2 - \dim_k M_1 \geq r - d$.*

Proof. Since each preinjective representation is not in ESP_d , every indecomposable direct summand of M is preprojective or regular and there is at least one summand which is not isomorphic to P_1 . Therefore we can reduce our considerations to an indecomposable representation M with $\dim_k M_1 \neq 0$. We set $m := \dim_k M_2$ and $n := \dim_k M_1$. Let $l(n, dm, n)$ be the dimension of a largest subspace V in $\text{Hom}_k(M_1, (M_2)^d)$ such that every non-zero element f in V has rank n , i.e. f is injective. We get

$$l(n, dm, n) \leq dm + n - 2n + 1 = dm - n + 1.$$

By the definition of ESP_d the indecomposable and non-simple representation M is in ESP_d and only if

$$x_{(\alpha_1, \dots, \alpha_d)}: M_1 \rightarrow (M_2)^d, m \mapsto \left(\sum_{i=1}^r (\alpha_1)_i M(\gamma_i)(m), \dots, \sum_{i=1}^r (\alpha_d)_i M(\gamma_i)(m) \right)$$

is injective for each linearly independent tuple $(\alpha_1, \dots, \alpha_d) \in (k^r)^d$. Denote by σ the cycle $(1 \ 2 \ \dots \ r)$ in the symmetric group on $\{1, \dots, r\}$ and by e_1, \dots, e_r the canonical basis vectors of k^r . We set $f_0 := x_{(e_1, e_2, \dots, e_d)}$ and for $1 \leq i \leq r - d$ we define $f_i := x_{(e_{\sigma^i(1)}, \dots, e_{\sigma^i(d)})}$. Now let $a_0, \dots, a_{r-d} \in k$ with $(a_0, \dots, a_{r-d}) \neq 0$ and assume that $\sum_{i=0}^{r-d} a_i f_i(m) = 0$. Then we have

$$\begin{aligned} 0 &= \sum_{i=0}^{r-d} a_i f_i(m) = \left(\sum_{i=0}^{r-d} a_i M(\gamma_{i+1})(m), \dots, \sum_{i=0}^{r-d} a_i M(\gamma_{i+d})(m) \right) \\ &= x_{(\alpha_1, \dots, \alpha_d)}(m), \end{aligned}$$

where $\alpha_1 = (a_0, \dots, a_{r-d}, 0, \dots, 0) \in k^r$ and α_{i+1} is the right shift of α_i for $1 \leq i < d$. Since $(a_0, \dots, a_{r-d}) \neq 0$ we have

$$\text{rk} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{r-d} & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots & a_{r-d} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_0 & \cdots & \cdots & \cdots & a_{r-d-1} & a_{r-d} \end{pmatrix} = d.$$

Hence, the tuple $(\alpha_1, \dots, \alpha_d)$ is linearly independent and $x_{(\alpha_1, \dots, \alpha_d)}$ is injective. We have proven that each non-zero element in $W := \langle f_0, \dots, f_{r-d} \rangle_k$ is injective. In particular, $\{f_0, \dots, f_{r-d}\}$ is linearly independent. We conclude $r - d + 1 = \dim_k W \leq \dim_k V \leq dm - n + 1 = d \cdot \dim_k M_2 - \dim_k M_1 + 1$. \square

Remark. For $r = 3$ and $d = 1$ this inequality is sharp:

Consider an elementary representation with dimension vector $(2, 4)$ as defined in [36]. Then E is in $\text{EKP} = \text{ESP}_1$ and $\dim_k E_2 - \dim_k E_1 = 2 = 3 - 1 = r - d$.

Corollary 14.6. *Let $M \in \text{rep}(\Gamma_r)$, $1 \leq d \leq r - 1$, $\underline{\dim} M = (x, y)$, $x \neq 0$ and $dy - x < r - d$, then there exists $U \in \text{Gr}_{d,r}$ and such that $\text{Hom}_{\Gamma_r}(X_U, M) \neq 0$.*

Proof. This follows now immediatly from Proposition 3.11. □

Corollary 14.7. *Let $M \in \text{rep}(\Gamma_r)$, $1 \leq d \leq r - 1$, $\underline{\dim} M = (x, y)$, $x \neq 0$ and $dy - x < r - d \leq y$, then M has a non-preprojective subrepresentation T with $\underline{\dim} T = (1, r - d)$.*

Proof. Since $dy - x < r - d$, we find $U \in \text{Gr}_{d,r}$ with $\text{Hom}_{\Gamma_r}(X_U, M) \neq 0$. Let $f \in \text{Hom}_{\Gamma_r}(X_U, M)$ be non-zero, then $\text{im } f$ is indecomposable by Lemma 3.8 with dimension vector $(1, b)$ and $0 \leq b \leq r - d$. Since the radical of M is isomorphic to P_1^y and $b \leq r - d \leq y$, we find a representation V isomorphic to P_1^{r-d-b} inside the radical of M with $V \cap \text{im } f = \{0\}$. We set $T := V \oplus \text{im } f$, then $\underline{\dim} T = (1, r - d - b) + (1, b) = (1, r - d)$. The module T is not preprojective since $\underline{\dim} P_1 = (0, 1)$ and $\underline{\dim} P_2 = (1, r)$. □

14.2 A fundamental domain of dimension vectors

It is well known (see for example [34, 3.4]) that for $r \geq 2$ the region

$$\mathcal{F}_r = \{(x, y) \in \mathbb{N}^2 \mid \frac{1}{r-1}x \leq y \leq (r-1)x\}$$

is a fundamental domain for the action of the Coxeter transformation $C = \begin{pmatrix} r^2 - 1 & -r \\ r & -1 \end{pmatrix}$ on the set of regular dimension vectors $\underline{\dim} \mathcal{R}(\Gamma_r)$ not equal to $(0, 0)$. We will prove that

$$\mathcal{F} := \{(x, y) \in \mathbb{N}^2 \mid \frac{2}{r}x \leq y \leq x\}$$

and

$$\mathcal{G} := \{(x, y) \in \mathbb{N}^2 \mid \frac{1}{r-1}x \leq y \leq \frac{2}{r}x\}$$

are fundamental domains for the action of the group generated by σ and δ on the set of regular dimension vectors, where $\delta(x, y) = (y, x)$ and $\sigma(x, y) = (rx - y, x)$. Observe that the duality $D_{\Gamma_r}: \text{rep}(\Gamma_r) \rightarrow \text{rep}(\Gamma_r)$ satisfies $\underline{\dim} D_{\Gamma_r} M = \delta \underline{\dim} M$ for each representation M .

Let $\mathcal{H}_2: \text{rep}(\Gamma_r) \rightarrow \text{rep}(\Gamma_r)$ be the reflection functor at the vertex 2. The functor is an auto-equivalence and $\underline{\dim} \mathcal{H}_2(M) = \sigma \underline{\dim} M$ and σ^{-1} is given by the reflection functor \mathcal{H}_1 at the vertex 1. Moreover, $\mathcal{H}_1, \mathcal{H}_2$ and D_{Γ_r} take regular representations to regular representations.

We elaborate the details of the proof of the lemma in section 2 of [36].

Lemma 14.8. *The set \mathcal{F} is a fundamental domain for the action of the group generated by δ and σ on the set of regular dimension vectors. Moreover, there exists an elementary representation E with $\underline{\dim} E$ in \mathcal{F}_r if and only if there is an elementary representation F with $\underline{\dim} F \in \mathcal{F}$.*

Proof. Since $\sigma^2 = C$ we know that there is a fundamental domain contained in \mathcal{F}_r . For $(x, y) \in \mathcal{F}_r$ with $x \leq y$ we have $\delta(x, y) = (y, x) \in \mathcal{F}_r$. Hence, we can reduce our considerations to the set

$$\{(x, y) \in \mathbb{N}^2 \mid \frac{1}{r-1}x \leq y \leq x\} = \mathcal{F} \cup \mathcal{G}.$$

Now we show that (see [36, 2]) there is a bijective correspondence between the dimension vectors (x, y) in \mathcal{G} and dimension vectors (x', y') in \mathcal{F} .

At first assume that $\frac{1}{r-1}x \leq y \leq \frac{2}{r}x$, then $(x', y') := \sigma \circ \delta(x, y) = \delta \circ \sigma^{-1}(x, y) = (ry - x, y)$. We conclude $\frac{2}{r}x' = \frac{2}{r}(ry - x) = 2y - \frac{2}{r}x = y + (y - \frac{2}{r}x) \leq y = y'$ and $y' = y \leq y + (r-1)y - x = ry - x = x'$.

We assume that $\frac{2}{r}x \leq y \leq x$, then $(x', y') := \sigma \circ \delta(x, y) = (ry - x, y)$. We conclude $\frac{1}{r-1}x' = \frac{1}{r-1}(ry - x) = \frac{r}{r-1}y - \frac{1}{r-1}x = y + \frac{1}{r-1}(y - x) \leq y = y'$ and $y' = y \leq y + (y - \frac{2}{r}x) = 2y - \frac{2}{r}x = \frac{2}{r}(ry - x) = \frac{2}{r}x'$. Hence a fundamental domain is contained in \mathcal{F} . Let $d = (x, y)$ be in $\underline{\dim} \mathcal{R}$ and h be an element in the group generated by σ and δ . Since $\delta^2 = 1$ and $\delta \circ \sigma = \sigma^{-1} \circ \delta$ we find $l \in \mathbb{N}_0$ with and $\varepsilon \in \{0, 1\}$ with $h = \sigma^l \circ \delta^\varepsilon$ or $h = \delta^\varepsilon \circ \sigma^l$. Let $(x', y') := \sigma(x, y)$ and observe that

- (a) $\frac{y}{x} > \frac{y'}{x'}$ and
- (b) $\frac{y}{x} \leq 1$ implies $\frac{2}{r}x' > y'$.

We now let $(x, y) \in \mathcal{F}$ and assume that $(a, b) := h(x, y)$ is also in \mathcal{F} .

- If $\varepsilon = 0$ we get with (a) and (b) that $l = 0$ and $(x, y) = (a, b)$.
- If $\varepsilon = 1$ and $h = \delta^\varepsilon \circ \sigma^l$, we set $(c, d) := \sigma^l(x, y)$ and conclude with (a) that $\frac{d}{c} \leq \frac{y}{x} \leq 1$. Hence $d \leq c$ and $a \leq b$. Since $(a, b) \in \mathcal{F}$ we get $a = b$, $c = d$ and $l = 0$ by (a), hence $(a, b) = (x, y)$.
- If $\varepsilon = 1$ and $h = \sigma^l \circ \delta^\varepsilon$ set $(x', y') := \delta(x, y) = (y, x)$ and consider $(c, d) := \sigma(x', y') = (rx' - y', x') = (ry - x, y)$. Then $\frac{d}{c} \leq 1$ since $d \leq c \Leftrightarrow y \leq ry - x \Leftrightarrow \frac{x}{y} \leq r - 1$, which is the case since $\frac{x}{y} \leq \frac{r}{2}$ and $\frac{r}{2} \leq r - 1$. By (a) and (b) we conclude that $l \leq 1$ since $\sigma^{l-1}(c, d) = (a, b) \in \mathcal{F}$. For $l = 0$ we get $(a, b) = (y, x)$, $x = y$ i.e. $(a, b) = (x, y)$. For $l = 1$ we conclude $(ry - x, y) \in \mathcal{F}$, hence $\frac{2}{r}(ry - x) \leq y \Leftrightarrow y \leq \frac{2}{r}x$. Since $(x, y) \in \mathcal{F}$ we conclude $y = \frac{2}{r}x$ and $(a, b) = \sigma(y, x) = \sigma(\frac{2}{r}x, x) = (x, y)$.

It remains to prove the second statement, but this is trivial since D_{Γ_r} , \mathcal{H}_1 and \mathcal{H}_2 take elementary representations to elementary representations. \square

Lemma 14.9. (a) Let $n \in \mathbb{N}$, and for each $1 \leq i \leq n$ let $(a_i, b_i) \in \mathbb{N} \times \mathbb{N}$, then

$$\min\left\{\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\right\} \leq \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}.$$

- (b) Let I be a preinjective representation that is not semisimple with $\underline{\dim} I = (a, b)$, then $\frac{a}{b} \geq \frac{r+\sqrt{r^2-4}}{2}$.
- (c) If $r \geq 3$, then $(r-1) < \frac{r+\sqrt{r^2-4}}{2}$.

Proof. (a) For each $1 \leq i \leq n$ let $c_i := \frac{a_i}{b_i}$, then obviously $\sum_{i=1}^n b_i c_i \geq (\sum_{i=1}^n b_i) \min\{c_1, \dots, c_n\}$. Hence $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i c_i$ implies $\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \geq \min\{c_1, \dots, c_n\}$.

(b) Since I is not semisimple we have $b \neq 0$. Write $I = S \oplus T$ such that S is semisimple and I_1

does not divide T . Then $T \neq 0$ and $\dim S = (x, 0)$ for some $x \in \mathbb{N}_0$. Hence, we can assume that $S = 0$, i.e. I is the direct sum of non-simple indecomposable preinjective representations. In conjunction with (a) it suffices to prove the inequality for each indecomposable representation I_i with $i \geq 2$. Let $i \geq 2$ and $\underline{\dim} I_i = (a, b)$. Then

$$\begin{aligned} \frac{a}{b} - \frac{r + \sqrt{r^2 - 4}}{2} > 0 &\Leftrightarrow \left(\frac{a}{b} - \frac{r}{2}\right)^2 > \frac{1}{4}(r^2 - 4) \Leftrightarrow \frac{a^2}{b^2} - \frac{ar}{b} + 1 > 0 \\ &\Leftrightarrow a^2 + b^2 - rab > 0 \Leftrightarrow q_{\Gamma_r}(\underline{\dim} I_i) > 0. \end{aligned}$$

The Euler-Ringel form yields $q_{\Gamma_r}(\underline{\dim} I_i) = \dim_k \text{End}_{\Gamma_r}(I_i) - \dim_k \text{Ext}_{\Gamma_r}(I_i, I_i) = 1 - 0 = 1$, since preinjective indecomposable representations are bricks and do not have self-extensions.

(c) We have

$$r - 1 < \frac{r + \sqrt{r^2 - 4}}{2} \Leftrightarrow \frac{r - 2}{2} < \frac{\sqrt{r^2 - 4}}{2} \Leftrightarrow (r - 2)^2 < r^2 - 4 \Leftrightarrow 2 < r.$$

□

14.3 Restricting x and y

Lemma 14.10. *Let E be an elementary representation with $\underline{\dim} E = (x, y) \in \mathcal{F}$ and $r - 1 < y$, then $\sqrt{r^2 - 4} \leq \frac{r^2 - r - 2}{y - r + 1}$.*

Proof. We deal with the case $\frac{2}{r}x \leq y \leq x$. Thus $y - x < r - 1 \leq y$ and by Corollary 14.7 we find a subrepresentation $U \subseteq E$ with $\underline{\dim} U = (1, r - 1)$. By Lemma 3.1, the dimension vector $(x - 1, y - r + 1)$ is preinjective. In particular,

$$\frac{1}{2}r + \frac{\sqrt{r^2 - 4}}{2} = \frac{r + \sqrt{r^2 - 4}}{2} \leq \frac{x - 1}{y - (r - 1)}.$$

Since

$$\frac{x - 1}{y - (r - 1)} \leq \frac{\frac{r}{2}y - 1}{y - r + 1} = \frac{1}{2} \frac{ry - 2}{y - r + 1} = \frac{1}{2} \left(\frac{ry - r^2 + r}{y - r + 1} + \frac{r^2 - r - 2}{y - r + 1} \right) = \frac{1}{2}r + \frac{1}{2} \cdot \frac{r^2 - r - 2}{y - r + 1},$$

we conclude

$$\sqrt{r^2 - 4} \leq \frac{r^2 - r - 2}{y - r + 1}.$$

□

Lemma 14.11. *Let E be an elementary representation with $\underline{\dim} E = (x, y)$ and $y \leq x \leq y + r - 2$, then $x < r$.*

Proof. Assume to the contrary that $x \geq r$. Set $F := D_{\Gamma_r}E$, then $\underline{\dim} F = (x', y') = (y, x)$ with $y' - x' \leq r - 2 < r - 1 \leq y'$. Now Corollary 14.7 yields a subrepresentation U with dimension vector $(1, r - 1)$. Since F is elementary and U is not preprojective the factor F/U with dimension vector $(y - 1, x - r + 1)$ is a preinjective. By Lemma 14.9, we have

$$r - 1 < \frac{y - 1}{x - r + 1} \stackrel{y \leq x}{\leq} \frac{x - 1}{x - r + 1}.$$

Since $x - r + 1 > 0$ we conclude

$$(x - r + 1)(r - 1) < x - 1 \Leftrightarrow x(r - 2) < r(r - 2) \Leftrightarrow x < r,$$

a contradiction.

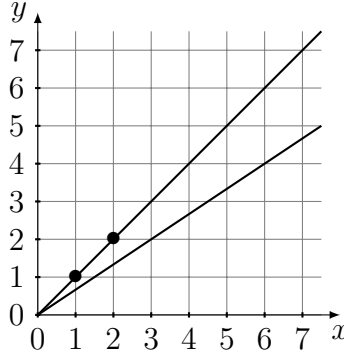
□

14.4 Elementary representations for Γ_3

Let $r = 3$, E be elementary with $\underline{\dim} E = (x, y) \in \mathcal{F}$ and assume that $y > 2(r-1) = 4 > r-1$. Then Lemma 14.10 yields $\sqrt{5} \leq \frac{9-3-2}{y-3+1} \leq \frac{5}{3}$, a contradiction. Hence $y \leq 4$.

- If $y = 1$ we get $(x, y) = (1, 1)$.
- If $y = 2$ it follows $2 \leq x \leq \frac{3}{2}2 = 3$ and $(x, y) \in \{(2, 2), (3, 2)\}$. Since $3 \leq y + r - 2 = 3$ we conclude with Lemma 14.11 that $x = 2$.
- Assume that $y = 3$, then $x \geq y = 3$ and 14.11 yields $x > y + 1 = 4$. But $3 \leq x \leq \frac{3}{2}3$ implies $x \in \{3, 4\}$, a contradiction.
- Assume that $y = 4$, then $x \geq y = 4$ and $x \in \{4, 5, 6\}$. Hence, $x \geq r$ and Lemma 14.11 yields $x > y + r - 2 = 5$, hence $x = 6$. By 14.7 we find a non-projective subrepresentation $T \subseteq E$ with dimension vector $(1, 2)$. It follows $\frac{5}{2} = \frac{x-1}{y-2} \stackrel{14.9}{\geq} 1.5 + \frac{\sqrt{5}}{2} \approx 2.61$, a contradiction.

Note that this was already proven in [36].



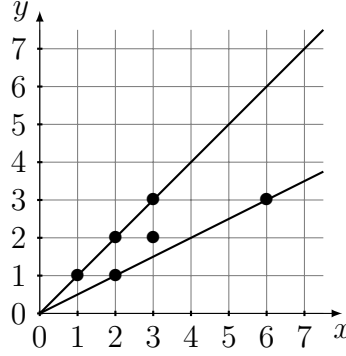
14.5 Elementary representations for Γ_4

We consider $r = 4$. Let E be elementary with $\underline{\dim} E = (x, y) \in \mathcal{F}$ and assume that $y \geq 2(r-1) = 6 > r-1$. Then Lemma 14.10 yields $\sqrt{12} \leq \frac{16-4-2}{y-4+1} \leq \frac{10}{3}$, a contradiction. Hence $y \leq 5$. Since $y \leq x \leq 2y$, we get for

- $y = 1$: $(x, y) \in \{(1, 1), (2, 1)\}$.
- $y = 2$: $x \in \{2, 3, 4\}$. Since $y + r - 2 = 4$, Lemma 14.11 implies $x \in \{2, 3\}$.
- $y = 3$: $x \in \{3, 4, 5, 6\}$. By 14.11 we get $x \in \{3, 6\}$.
- $y = 4$: $x \in \{4, 5, 6, 7, 8\}$. By 14.11 we get $x \in \{7, 8\}$. We set $d = 2$ and get $dy - x \leq 8 - 7 = 1 < r - d$. Hence, Corollary 14.7 yields a non-projective subrepresentation $T \subseteq E$ with $\underline{\dim} T = (1, 2)$. Since E is elementary, E/T is preinjective and $\frac{x-1}{2} = \frac{x-1}{y-2} \stackrel{14.9}{\geq} 2 + \frac{1}{2}\sqrt{12} \approx 3.73$, a contradiction.

- $y = 5, x \in \{5, 6, 7, 8, 9, 10\}$. By 14.11 we get $x \in \{8, 9, 10\}$. We set $d = 2$ and get for $x \in \{9, 10\}$ that $dy - x \leq 10 - 9 = 1 < r - d$. Hence, Corollary 14.7 yields a non-projective subrepresentation $T \subseteq E$ with $\underline{\dim} T = (1, 2)$. Since E is elementary, E/T is preinjective and $\frac{x-1}{3} = \frac{x-1}{y-2} \geq 2 + \frac{1}{2}\sqrt{12} \approx 3.73$, a contradiction. It remains to consider the case $x = 8$. Then we find a non-projective subrepresentation $S \subseteq E$ with $\underline{\dim} S = (1, 3)$. Since E is elementary E/S is preinjective and $\frac{7}{2} = \frac{8-1}{5-3} \geq 2 + \frac{1}{2}\sqrt{12} \approx 3.73$, a contradiction.

We get $(x, y) \in \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (6, 3)\} =: \mathcal{A}$.



In the following we prove the existence of an elementary representation E with $\underline{\dim} E = (x, y)$ for each $(x, y) \in \mathcal{A} \setminus \{(6, 3)\}$.

1. Let $\alpha = (a, b, c) \in k^3 \setminus \{0\}$. We define the Matrix $A_\alpha = \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \end{pmatrix}$. Note that for each $\alpha \neq 0$ we have $\text{rk}(A_\alpha) = 2$. We define the representation $E \in \text{rep}(\Gamma_4)$ by setting $E_1 = k^3$, $E_2 = k^2$, $M(\gamma_1)(x, y, z) = (0, x)$, $M(\gamma_2)(x, y, z) = (x, y)$, $M(\gamma_3)(x, y, z) = (y, z)$ and $M(\gamma_4)(x, y, z) = (z, 0)$. Now let $e \in E_1 \setminus \{0\}$. Since $\text{rk}(A_\alpha) = 2$ the element e generates a subrepresentation U with dimension vector $(1, 2)$, thus $\underline{\dim} E/U = (2, 0)$. We have proven, that every proper factor of E is preinjective. Moreover, E is indecomposable because α was arbitrary; if $E = U \oplus V$, then we can assume that $U_1 \neq 0$. The above considerations yield $\dim_k U_2 = 2$. Since $\dim_k M_2 = 2 = \dim_k V_2$ we get $V_2 = 0$. Hence $\dim_k V_1 = 0$, since otherwise we would get $\dim_k V_2 = 2$ by the same argument. We have proven that E is an elementary representation with dimension vector $(3, 2)$.

2. Let $\alpha = (a, b, c) \in k^3 \setminus \{0\}$. We define the Matrix $\begin{pmatrix} a & b & -c & 0 \\ 0 & a & b & c \\ c & 0 & a & b \end{pmatrix}$ We consider $\text{rk}(A_\alpha)$,

therefore we distinguish the following cases:

- $\alpha \in ke_i$ for some $i \in \{1, 2, 3\}$, then $\text{rk}(A_\alpha) = 3$.
- $a = 0, b, c \neq 0$. Then

$$\text{rk}(A_\alpha) = \text{rk} \begin{pmatrix} 0 & b & -c & 0 \\ 0 & 0 & b & c \\ c & 0 & 0 & b \end{pmatrix} = \text{rk} \begin{pmatrix} c & 0 & 0 & b \\ 0 & b & -c & 0 \\ 0 & 0 & b & c \end{pmatrix} = 3.$$

- $b = 0, a, c \neq 0$. Then

$$\text{rk}(A_\alpha) = \text{rk} \begin{pmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & c \\ c & 0 & a & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & c \\ 0 & 0 & a + \frac{c^2}{a} & 0 \end{pmatrix} = 3.$$

Observe that $a + \frac{c^2}{a} \neq 0$ since $a^2 + c^2 > 0$.

- $c = 0, a, b \neq 0$. Then

$$\text{rk}(A_\alpha) = \text{rk} \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix} = 3.$$

- $a, b, c \neq 0$. Then

$$\begin{aligned} \text{rk}(A_\alpha) &= \text{rk} \begin{pmatrix} a & b & -c & 0 \\ 0 & a & b & c \\ c & 0 & a & b \end{pmatrix} = \text{rk} \begin{pmatrix} a & b & -c & 0 \\ 0 & a & b & c \\ 0 & -\frac{c}{a}b & a - \frac{c}{a}(-c) & b \end{pmatrix} \\ &= \text{rk} \begin{pmatrix} a & b & -c & 0 \\ 0 & a & b & c \\ 0 & -\frac{c}{a}b & a + \frac{c^2}{a} & b \end{pmatrix} = \text{rk} \begin{pmatrix} a & b & -c & 0 \\ 0 & a & b & c \\ 0 & 0 & a + \frac{c^2}{a} + \frac{cb}{a^2}b & b + \frac{cb}{a^2}c \end{pmatrix} = 3, \end{aligned}$$

because $b(a^2 + c^2) \neq 0$ implies $b + \frac{cb}{a^2}c \neq 0$.

Consider the representation $(M_1, M_2, M(\gamma_1), M(\gamma_2), M(\gamma_3), M(\gamma_4))$ with $M_1 = k^3 = M_2$ and $M(\gamma_1)(x, y, z) = (x, 0, z)$, $M(\gamma_2)(x, y, z) = (y, x, 0)$, $M(\gamma_3)(x, y, z) = (-z, y, x)$ and $M(\gamma_4)(x, y, z) = (0, z, y)$. Now let $(a, b, c) \in M_1 \setminus \{0\}$. Since $\text{rk}(A_\alpha) = 3$, the representation generated by (a, b, c) has dimension vector $(1, 3)$. So if N is proper subrepresentation of M with $\dim_k N_1 \neq 0$, then M/N is semisimple and therefore preinjective. Moreover, M is indecomposable because α was arbitrary (see the argument for the dimension vector $(3, 2)$). Hence M is elementary.

3. For $(x, y) = (2, 2)$ consider an elementary representation of Γ_3 , defined in [36] and note that it is also elementary considered as a representation of Γ_4 .

Corollary 14.12. *Let $(x, y) \in \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2)\}$. Then there exists an elementary representation E with $\underline{\dim} E = (x, y)$.*

The question that remains is whether there exists a regular representation with dimension vector $(6, 3)$.

Bibliography

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, I: Techniques of Representation Theory*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2006.
- [2] D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, II: Tubes and Concealed Algebras of Euclidean type*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2007.
- [3] ———, *Elements of the Representation Theory of Associative Algebras, III: Representation-Infinite Tilted Algebras*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2007.
- [4] D. Benson, *Representations and cohomology I*, Cambridge studies in advanced mathematics, vol. 30, Cambridge University Press, 1995.
- [5] ———, *Representations of elementary abelian p -groups and vector bundles*, Cambridge Tracts in Mathematics, vol. 208, Cambridge University Press, 2016.
- [6] V. M. Bondarenko and I. V. Lytvynchuk, *The representation type of elementary abelian p -groups with respect to the modules of constant Jordan type*, Algebra and Discrete Mathematics **14** (2012), no. 1, 29-36.
- [7] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Inventiones mathematicae **65** (1981/82), 331-378.
- [8] N. Bourbaki, *Algebra II: Chapters 4-7*, Elements of Mathematics, Springer-Verlag, 2003.
- [9] J. F. Carlson, E. M. Friedlander, and J. Pevtsova, *Representations of elementary abelian p -groups and bundles of Grassmannians*, Advances in Mathematics **229** (2012), 2985-3051.
- [10] J. F. Carlson, E. M. Friedlander, and A. Suslin, *Modules for $\mathbb{Z}_p \times \mathbb{Z}_p$* , Commentarii Math. Helv. **86** (2011), 609-657.
- [11] B. Chen, *Comparison of Auslander-Reiten theory and Gabriel-Roiter measure approach to the module categories of tame hereditary algebras*, Comm. Algebra **36** (2008), 4186-4200.
- [12] ———, *Dimension vectors in regular components over wild Kronecker quivers*, Bulletin des Sciences Mathématiques **137** (2013), 730-745.
- [13] ———, *Regular modules with preprojective Gabriel-Roiter submodules over n -Kronecker quivers*, J. Pure Appl. Algebra **215** (2011), 3025-3037.
- [14] ———, *The number of the Gabriel-Roiter measures admitting no direct predecessors over a wild quiver*, Pure Appl. Algebra **215** (2011), 2341-2351.
- [15] P. Donovan and M. R. Freislich, *The representation theory of finite graphs and associated algebras*, Carleton Math. Lecture Notes, vol. 5, Carleton Univ., Ottawa, 1973.
- [16] R. Farnsteiner, *Categories of modules given by varieties of p -nilpotent operators*, 2011, Preprint, arXiv:1110.2706v1.
- [17] ———, *Lectures Notes: Nilpotent Operators, Categories of Modules, and Auslander-Reiten Theory*, 2012. <http://www.math.uni-kiel.de/algebra/de/farnsteiner/material/Shanghai-2012-Lectures.pdf>.
- [18] P. Gabriel, *Indecomposable Representations II*, Symposia Mathematica. Inst. Naz. Alta Mat. **XI** (1973), 81-104.

- [19] ———, *The universal cover of a representation finite algebra*. *Representations of algebras*, Lecture Notes in Mathematics **903** (1981), 68-105.
- [20] V.G. Kac, *Root systems, representations of quivers and invariant theory*, in: *Invariant Theory*, Montecatini, Springer Lecture Notes in Mathematics **996** (1982), 74-108.
- [21] O. Kerner, *Exceptional Components of Wild Hereditary Algebras*, *Journal of Algebra* **152** (1992), no. 1, 184-206.
- [22] ———, *More Representations of Wild Quivers*, *Contemporary Mathematics* **607** (2014), 35-66.
- [23] ———, *Representations of Wild Quivers*, *Representation theory of algebras and related topics*, CMS Conf. Proc. **19** (1996), 65-107.
- [24] O. Kerner and F. Lukas, *Elementary modules*, *Math. Z.* **223** (1996), 421-434.
- [25] ———, *Regular modules over wild hereditary algebras*, in *Proc. Conf. ICRA '90*, CMS Conf. Proc. **11** (1991), 191-208.
- [26] G. Malle and D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*, *Cambridge Studies in Advanced Mathematics*, 133, Cambridge University Press, 2011.
- [27] M. Prest, *Module Theory and Modules*, L.M.S. Lecture Note Series, vol. 130, Cambridge University Press, 1988.
- [28] C. Riedtmann, *Algebren, Darstellungsköcher, Ueberlagerungen und zurück*, *Commentarii Math. Helv.* **55** (1980), 199-224.
- [29] C.M. Ringel, *Covering Theory*, <https://www.math.uni-bielefeld.de/~ringel/lectures/izmir/izmir-6.pdf>.
- [30] ———, *Finite-dimensional hereditary algebras of wild representation type*, *Math. Z.* **161** (1978), 235-255.
- [31] ———, *Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure*. In: *Trends in Representation Theory of Algebras and Related Topics*, *Contemporary Math. Amer.Math.Soc.* **406** (2006), 105-135.
- [32] ———, *Gabriel-Roiter inclusions and Auslander-Reiten theory*, *Journal of Algebra* **324** (2010), 3579-3590.
- [33] ———, *Indecomposable representations of the Kronecker quivers*, *Proc. Amer. Math. Soc.* **141** (2013), no. 1, 115-121.
- [34] ———, *Representations of K -species and bimodules*, *Journal of Algebra* **41** (1976), no. 2, 269-302.
- [35] ———, *Tame Algebras and Integral Quadratic Forms*, *Lecture Notes in Mathematics*, vol. 1099, Springer Verlag, 1984.
- [36] ———, *The elementary 3-Kronecker modules* (2016), Preprint, arXiv:1612.09141.
- [37] ———, *The Gabriel-Roiter measure.*, *Bull. Sci. math.* **129** (2005), 726-748.
- [38] ———, *The regular components of the Auslander-Reiten quiver of a tilted algebra*, *Chinese Ann. Math.* **9B** (1988), 1-18.
- [39] ———, *The representation type of local algebras*, In: *Representations of Algebras*, *Lecture Notes in Mathematics* **488** (1975), 282-305.
- [40] A.V. Roiter, *Unboundedness of the dimensions of the indecomposable representations of an algebra which has infinitely many indecomposable representations*, *Izd. Akad. Nauk SSSR. Ser. Mat.* **32** (1986), 1275-1282.
- [41] J. J. Rotman, *An Introduction to Homological Algebra: Second Edition*, Springer New York, 2009.
- [42] J-P. Serre, *Trees*, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 1980.
- [43] D. Simson, *On large indecomposable modules, endo-wild representation type and right pure semisimple rings*, *Algebra and Discrete Mathematics* **2** (2003), 93-118.
- [44] ———, *Representation Types of Module Subcategories and Orders*, *Bulletin of the Polish Academy of Sciences Mathematics* **41** (1993), 77-93.

- [45] L. Unger, *The concealed algebras of the minimal wild, hereditary algebra*, Bayreuther Mathematische Schriften **31** (1990), 145–154.
- [46] R. Westwick, *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra **20** (1987), 171-174.
- [47] J. Worch, *AR-components for generalized Beilinson algebras*, Proceedings of the AMS **143** (2015), 4271–4281.
- [48] ———, *Categories of modules for elementary abelian p -groups and generalized Beilinson algebras*, J. London Math. Soc. **88** (2013), 649-668.
- [49] ———, *Module categories and Auslander-Reiten theory for generalized Beilinson algebras. PhD-Thesis* (2013), http://macau.uni-kiel.de/receive/dissertation_diss_00013419.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Einige der Hauptergebnisse der Dissertation sind in folgenden Publikationen enthalten:

- Bissinger, Daniel. *Representations of constant socle rank for the Kronecker algebra*. Preprint, <https://arxiv.org/pdf/1610.01377v1.pdf>. Zur Publikation eingereicht.
- Bissinger, Daniel. *Representations of regular trees and invariants of AR-components for generalized Kronecker quivers*, *Algebras and Representation Theory*, 2017, doi: 10.1007/s10468-017-9716-x.

Kiel, den 12. November 2017

(Daniel Bissinger)