

# Escaping sets of entire and quasiregular maps

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*Dedicated to my grandpa Zygmunt Stankowski (†2004)*



# Abstract

A transcendental entire function is a holomorphic function on the whole complex plane which is not a polynomial. The Julia set of an entire function  $f$  denotes the set of all points in  $\mathbb{C}$  where the iterates  $f^k$  of  $f$  do not form a normal family in the sense of Montel. Many mathematicians investigated the iteration of these functions. Even the simplest example of a transcendental entire function, namely the exponential map  $\lambda e^z$  with  $\lambda \in \mathbb{C} \setminus \{0\}$ , is of rich structure. Devaney and Krych showed in 1984 that for the exponential family  $\lambda e^z$ , where  $0 < \lambda < 1/e$ , the Julia set consists of uncountably many pairwise disjoint simple curves tending to  $\infty$ , which they called hairs.

The existence of hairs is not restricted to the situation considered by Devaney and Krych. For the exponential family it was proved by Devaney, Goldberg and Hubbard in 1986 that these hairs even appear for all  $\lambda \in \mathbb{C} \setminus \{0\}$ . This result was generalized to a larger class of functions by different authors, for example by Devaney and Tangerman in 1986, by Barański in 2007 and by Rückert, Rottenfußler, Rempe and Schleicher in 2011.

Recently, the iteration of a natural higher-dimensional analogue of holomorphic maps, the so-called quasiregular maps, became of great interest. A special family of quasiregular functions, the so-called Zorich maps, was introduced by Zorich in 1969 and can be considered as a higher-dimensional counterpart of the exponential family. Bergweiler showed in 2010 that for several Zorich maps hairs also exist.

In this thesis we want to focus on the differentiability of hairs for transcendental entire functions and Zorich maps. A first result was given by Viana in 1988, who proved that the hairs of the exponential family are smooth. In the first part of this thesis, we show that his result generalizes to the Zorich maps considered by Bergweiler under some mild additional regularity assumptions.

At first glance it seems that the regularity of the function transfers to the regularity of the hairs. In the second part of this thesis we construct in contrast to the first result a transcendental entire function of the so-called Eremenko-Lyubich class, where every hair is nowhere differentiable.



# Zusammenfassung

Eine ganz transzendente Funktion ist eine auf der gesamten komplexen Zahlenebene holomorphe Funktion, welche nicht polynomiell ist. Die Juliamenge einer ganzen Funktion  $f$  bezeichnet die Menge aller Punkte in  $\mathbb{C}$ , in welchen die Iterierten  $f^k$  von  $f$  keine normale Familie im Montelschen Sinne bilden. Viele Mathematiker befassten sich mit der Iteration solcher Funktionen. Selbst der einfachste Fall einer ganz transzendenten Funktion, nämlich der Exponentialfunktion  $\lambda e^z$  mit  $\lambda \in \mathbb{C} \setminus \{0\}$ , ist von reicher Struktur. Devaney und Krych haben 1984 gezeigt, dass für die Exponentialfamilie  $\lambda e^z$  mit  $0 < \lambda < 1/e$  die Juliamenge aus überabzählbar vielen paarweise disjunkten einfachen Kurven nach  $\infty$  besteht, welche von ihnen Haare genannt werden.

Die Existenz von Haaren ist nicht auf den von Devaney und Krych betrachteten Fall beschränkt. Für die Exponentialfamilie wurde von Devaney, Goldberg und Hubbard 1986 gezeigt, dass diese Haare für alle  $\lambda \in \mathbb{C} \setminus \{0\}$  auftreten. Dieses Resultat wurde auf eine größere Funktionenklasse verallgemeinert, zum Beispiel von Devaney und Tangerman im Jahr 1986, von Barański im Jahr 2007 und von Rückert, Rottenfuß, Rempe und Schleicher im Jahr 2011.

In letzter Zeit gab es ein wachsendes Interesse an der Iteration eines natürlichen höherdimensionalen Analogons holomorpher Funktionen, den sogenannten quasiregulären Funktionen. Eine spezielle Familie quasiregulärer Funktionen, den von Zorich 1969 eingeführten und heute nach ihm benannten Zorichfunktionen, kann als höherdimensionales Analogon der Exponentialfamilie angesehen werden. Bergweiler hat 2010 gezeigt, dass für gewisse Zorichfunktionen Haare existieren.

In dieser Arbeit legen wir unser Augenmerk auf die Differenzierbarkeit von Haaren von Zorich- und ganz transzendenten Funktionen. Ein erstes Resultat wurde von Viana im Jahr 1988 gegeben, in welchem er zeigt, dass die Haare der Exponentialfamilie glatt sind. In dem ersten Teil dieser Arbeit zeigen wir, dass sein Resultat für die Zorichfunktionen, welche Bergweiler betrachtet hat, unter schwachen Zusatzvoraussetzungen an die Regularität der Funktion verallgemeinert werden kann.

Auf den ersten Blick könnte man vermuten, dass sich die Regularität der Funktion auf die Regularität der Haare überträgt. Im Gegensatz zum ersten Resultat zeigen wir im zweiten Teil dieser Arbeit, dass es eine ganz transzendente Funktion aus der sogenannten Eremenko-Lyubich-Klasse gibt, bei der jedes Haar nirgendwo differenzierbar ist.





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# Chapter 1

## Introduction

The iteration of holomorphic maps plays an important role in mathematics. The foundations of the theory were laid by the memoirs of Pierre Fatou [Fat19], [Fat20] and Gaston Julia [Ju18] in the beginning of the 20th century. Each of them considered the iteration of rational functions from a different point of view: Fatou chose a way which is nowadays the standard approach by investigating where the sequence of the iterates forms a normal family in the sense of Montel. Here we say that a family  $\mathcal{F}$  of entire or meromorphic functions is normal in  $U \subset \mathbb{C}$ , if every sequence in  $\mathcal{F}$  has a locally uniformly convergent subsequence with respect to the chordal metric. In contrast to Fatou's work, Julia investigated the closure of the set of repelling periodic points for rational maps. He showed that this set is the complement of the set considered by Fatou, and independently of Julia it was shown by Fatou that his set is the complement of the closure of the repelling periodic points. These two sets form a natural partition of the complex plane  $\mathbb{C}$  respectively the Riemann sphere  $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ , which is named after Fatou and Julia. In 1968 Baker [Bak68] has shown that these sets coincide in the situation of entire functions. The systematic study of the iteration of entire and rational functions is nowadays called complex dynamics.

Denote for a given entire or rational map  $f$  by  $\mathcal{F}(f)$  the Fatou set, which is the set of all points where the sequence  $(f^n)_{n \in \mathbb{N}}$  is normal, and by  $\mathcal{J}(f)$  the Julia set, which is its complement. Here  $f^n$  denotes the  $n$ -th iterate of the function  $f$ . By Arzelà-Ascoli's theorem, normality in a set  $U \subset \mathbb{C}$  can be seen as a locally stable behaviour of the iterates in  $U$ .

The theory of complex dynamics became of great interest in the last 40 years, and many people investigated the behaviour of special functions. In the early 1980s Sullivan [Su83, Su85] and Douady and Hubbard [DH84, DH85] introduced new techniques which provided new perspectives in complex dynamics. Besides the Fatou and Julia set, there is a third set of much relevance, the escaping set  $\mathcal{I}(f) := \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$ , which was introduced by Eremenko [Er89] in 1989. He has shown that this set is non-empty for transcendental entire functions and that  $\partial \mathcal{I}(f) = \mathcal{J}(f)$ , which justifies the relevance of this definition. There is a strong connection between the Julia set and the escaping set

for a large class of transcendental entire functions, the so-called Eremenko-Lyubich class  $\mathcal{B}$  [EL92]. This class consists of those functions where the set of singular values  $\text{sing}(f^{-1})$ , i.e. the set of critical and asymptotic values of  $f$ , is bounded. Because of this condition functions in class  $\mathcal{B}$  are said to be of bounded type. For these functions it was shown by Eremenko and Lyubich [EL92] that  $\overline{\mathcal{I}(f)} = \mathcal{J}(f)$ . Many standard functions like  $e^z$  and  $\sin(z)$  belong to the class  $\mathcal{B}$ .

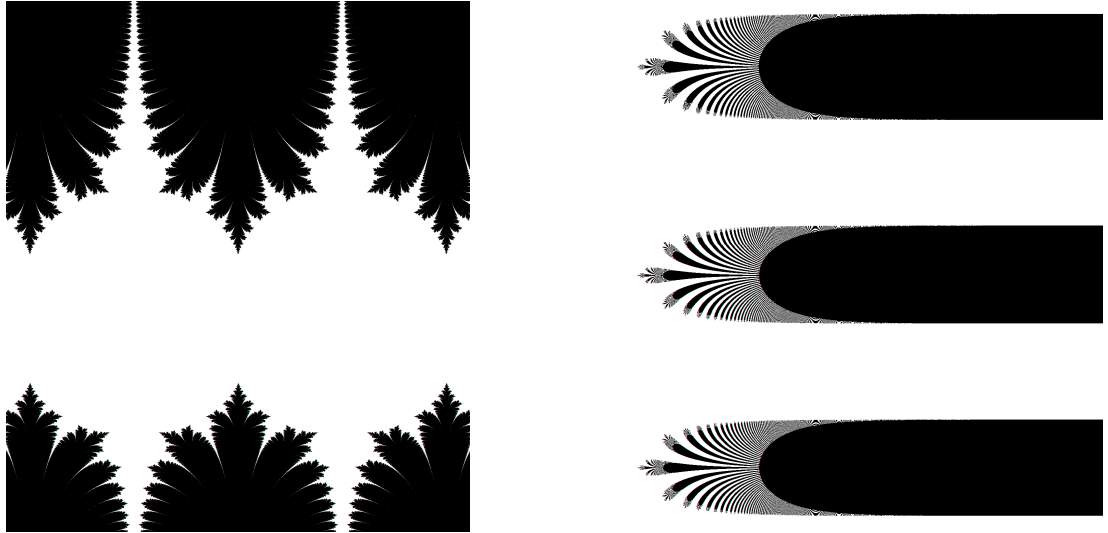


Figure 1.1. The Julia set for  $f(z) = 0.8 \sin z$  (shown on the left side) and the Julia set for  $f(z) = e^z - 1$ . Both of them are shown in black.

Although the components of the Fatou set are well understood, the Julia set is of rich structure, and the topology depends sensitively on the parameter for several parameter families. Even for the parameter family of quadratic polynomials  $f_c(z) = z^2 + c$ , where  $c \in \mathbb{C}$ , the Julia set  $\mathcal{J}(f)$  has a fractal behaviour for most  $c \neq 0$ , while the Julia set of  $f_0(z) = z^2$  is the unit circle, and for  $f_{-2}(z) = z^2 - 2$  it is the interval  $[-2, 2]$ . First results on the structure of the Julia set for non-polynomial entire maps, the so-called transcendental entire maps, were given by Devaney and Krych in 1984 [DK84]. They have shown that for the exponential family  $\lambda e^z$ , where  $0 < \lambda < 1/e$ , the Julia set consists of uncountably many pairwise disjoint simple curves tending to  $\infty$ , which they called hairs. On these hairs, except for their endpoints, the iterates tend to  $\infty$  locally uniformly. The existence of hairs is not a special feature of the case considered by Devaney and Krych. For the exponential family it was proved by Devaney, Goldberg and Hubbard in 1986 [DGH86] that these hairs even appear for all  $\lambda \in \mathbb{C} \setminus \{0\}$ . This result was generalized to a larger class of functions by different authors. In 1986 Devaney and Tangerman [DT86] considered certain transcendental entire maps, where  $\text{sing}(f^{-1})$  is finite. Examples for this class of functions are  $e^z$ ,  $\sin(z)$  and  $\cos(z)$ , see Figure 1.1. In 2007 Barański [Bar07] considered so-called hyperbolic maps, i.e. maps, where the closure of the union of all  $f^n(\text{sing}(f^{-1}))$  with  $n \in \mathbb{N}_0$  is a compact subset of  $\mathcal{F}(f)$ . In 2011 Rückert, Rottenfußer, Rempe and

Schleicher generalized this result to finite compositions of functions in class  $\mathcal{B}$  of finite order. In the same paper it was shown that the assumption on the order of  $f$  cannot be dropped. In fact they constructed a function of bounded type of infinite order such that every path-connected component of  $\mathcal{J}(f)$  is bounded, and thus this function has no hairs.

Recently, the question about a generalization of such a theory to higher real dimensions arose. To get an idea which class of functions is an appropriate higher-dimensional counterpart, one should understand the geometrical mapping behaviour of holomorphic maps. Holomorphic maps are  $C^1$  functions in a real sense which satisfy the Cauchy-Riemann differential equations. From a geometrical point of view, this means that holomorphic functions map infinitesimally small circles to infinitesimally small circles. Thus a first candidate for a higher dimensional analogue in dimension  $d \geq 3$  would be a  $C^1$  function  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  which maps infinitesimally small spheres to infinitesimally small spheres. By a theorem of Liouville [Li50] from 1850, it is known that the only functions satisfying these conditions in higher dimensions are Möbius transformations, i.e. finite compositions of reflections on spheres and hyperplanes. More precisely, Liouville has shown this result only for  $C^3$  functions, but later Hartman [Ha47, Ha58] generalized this theorem in 1958 to the case of  $C^1$  functions. In order to obtain a larger class of maps, our first candidate should satisfy a relaxed version of these conditions. It turns out that for an interesting class of functions, we should allow that  $Df(x)$  maps spheres to ellipsoids of bounded eccentricity and consider the existence of weak first derivatives. We call a continuous map satisfying the relaxed conditions quasiregular; see Section 2.4 for the precise definition.

Two-dimensional quasiregular maps were introduced by Grötsch [Gr28] in 1928, and higher-dimensional maps were first mentioned by Lavrentiev [La38] in 1938. A systematic study of these functions began in 1966 by Rešetnjak [Reš66]. For further information on the theory of quasiregular maps we refer to Rickman's book [Ri93] and for the history to a survey by Väisälä [Vä80].

Now we want to consider a higher-dimensional analogue of the exponential family, the so-called Zorich maps. These maps were introduced by Zorich [Zo67] in 1967. To get a feeling for the construction of these maps, we first consider the holomorphic case. The standard exponential map  $\exp(x + iy)$  decomposes into a product of the scaling function  $\exp(x)$  and the angular function  $\exp(iy)$ . If we restrict this function to the horizontal strip  $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \pi/2\}$ , we know that any vertical line segment of length  $\pi$  is mapped to a half circle to the right and any horizontal line segment is mapped again to a line segment in the right half plane. By Schwarz's reflection principle we can extend this map to the whole complex plane using the mapping behaviour on this initial strip. We want to define a Zorich map in  $\mathbb{R}^d$  as follows: First we consider the hypercube  $Q = [-1, 1]^{d-1}$  and map this cube to the upper hemisphere  $\mathbb{S}_+ \subset \mathbb{R}^d$  by a bi-Lipschitz map  $h: Q \rightarrow \mathbb{S}_+$  as shown in Figure 1.2. Thus  $h$  is a substitute for the angular function as in the exponential case. Now we keep the scaling function and consider the map  $F: Q \times \mathbb{R} \rightarrow \mathbb{R}^d$ ,  $F(x) = e^{x_d} h(x_1, \dots, x_{d-1})$ . As

for the exponential map, we obtain a map on  $\mathbb{R}^d$  by reflections on the faces of the square beam  $Q \times \mathbb{R}$  and call this extension a Zorich map (see Figure 1.3).

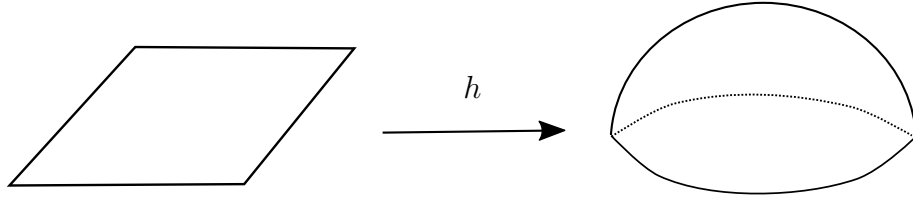


Figure 1.2. Bi-Lipschitz map  $h: Q \rightarrow \mathbb{S}_+$  of the Zorich map.

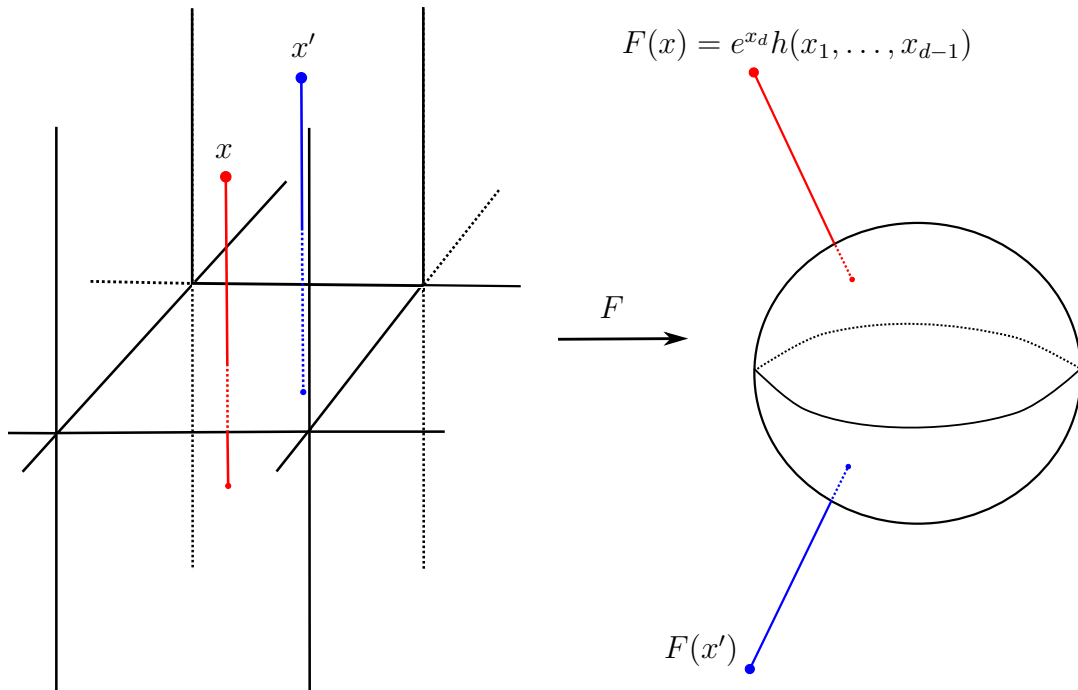


Figure 1.3. Construction of the function  $F$  via reflections.

Consider now  $f_a(x) := F(x) - (0, \dots, 0, a)$ , where  $a > 0$  is large. Bergweiler [Ber10] has shown that  $f$  behaves similarly to  $\lambda e^z$  for small  $\lambda$  and has hairs which lie in a suitable upper halfspace. Besides the Zorich maps there is a higher-dimensional analogue of the sine family. For further information see [BE11], or for several other results on a Fatou-Julia theory for quasiregular maps we refer to [Ber13, BN14].

In this thesis we want to focus on the differentiability of hairs for entire functions and Zorich maps. In 1988 Viana [Vi88] investigated the differentiability of hairs for the exponential family  $\lambda e^z$  with  $\lambda \in \mathbb{C} \setminus \{0\}$ . He has shown that the hairs of  $\lambda e^z$  are  $C^\infty$ -smooth independently of the parameter  $\lambda$ . Later Kisaka and Shishikura [KS11] generalized this result to the family  $P(z)e^{Q(z)}$ , where  $P$  and  $Q$  are non-zero polynomials.



Our first result on the differentiability of hairs for the Zorich maps considered by Bergweiler is the following:

**Theorem 1.** Let  $f_a$  be as above and assume that  $h|_{\text{int}(Q)}$  is  $C^1$  and  $Dh$  is Hölder continuous. Then the hairs of  $f_a$  are  $C^1$ -smooth.

Recall that for  $\beta > 0$  a function  $g: Q \rightarrow \mathbb{R}^d$  is  $\beta$ -Hölder continuous, if there exists a constant  $C > 0$  such that  $\|g(x) - g(y)\|_2 \leq C \|x - y\|_2^\beta$  for all  $x, y \in Q$ .

The proof of Theorem 1 will show that the conclusion of the theorem still holds if the assumptions on  $h$  and  $Dh$  are satisfied on a suitable subset of  $\text{int}(Q)$  (see Section 3.6). In the proof we use a suitable sequence of  $C^1$  curves which is given as follows: Bergweiler [Ber10] obtained the hairs as a locally uniform limit of a sequence of certain curves. We show - under the hypotheses made - that these sequence converges locally uniformly in  $C^1$ -sense and thus we obtain the  $C^1$ -smoothness of the hairs. Using this method we can show that the claim holds for every hair.

In the second part of the thesis, we investigate the differentiability of hairs of entire maps. Beginning with Weierstraß in 1895 [We95], it is well-known that there exist continuous functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  which are nowhere differentiable. Fatou [Fat06] has already shown that for  $f(z) = z^2 + c$ , the Julia set  $\mathcal{J}(f)$  is a Jordan curve for sufficiently small  $|c| \in \mathbb{R}$ , but it is nowhere differentiable. In contrast to the first result of this thesis we show the following:

**Theorem 2.** There exists a function in class  $\mathcal{B}$  of finite order, where the Julia set consists of hairs which are nowhere differentiable.

By saying that a hair  $H \subset \mathbb{C}$  is nowhere differentiable, we mean that no parametrization  $\gamma: [0, \infty) \rightarrow H$  has a non-zero derivative at any point of  $[0, \infty)$ .

To obtain this result we use the so-called Cauchy integral method to construct a transcendental entire map of finite order which grows in a prescribed tract to  $\infty$ . The existence of hairs for such a function is guaranteed by a result of Rückert, Rottenfuß, Rempe and Schleicher [RRRS11]. By choosing the geometry of the tract in a suitable way, one can obtain that the hairs cannot be differentiable at any point.



# Chapter 2

## Preliminaries

This thesis splits up into two parts which deal with quasiregular maps respectively entire functions. In this chapter on preliminaries we recall results which are needed for the main chapters 3 and 4. For the understanding of both subsequent chapters, the reader should read the first two sections and the beginning of Section 2.7. The sections 2.3 and 2.4 are required to understand the results on quasiregular maps. A solid background to understand the results in Chapter 4 is given in the sections 2.5 to 2.7.

### 2.1 Notation

In this section we want to fix the necessary notation for this thesis. As usual denote  $\mathbb{N} := \{1, 2, \dots\}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Let  $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  denote the Riemann sphere. For  $z \in \mathbb{C}$  and  $r > 0$  we write  $D(z, r) := \{w \in \mathbb{C} : |z - w| < r\}$  for the open Euclidean disc around  $z$  with radius  $r$ . With  $\text{dist}(x, y)$  we denote the Euclidean distance and with  $\text{diam}(A)$  the diameter of a set  $A \subset \mathbb{R}^d$ , i.e.

$$\text{diam}(A) := \sup_{x, y \in A} \|x - y\|_2.$$

Moreover, for  $c \in \mathbb{R}$  we define  $\mathbb{H}_{\geq c} := \{x \in \mathbb{R}^d : x_d \geq c\}$ , analogously  $\mathbb{H}_{>c}$ ,  $\mathbb{H}_{\leq c}$  and  $\mathbb{H}_{<c}$ . Only for  $d = 2$  we use this notation for the right half plane, i.e. we have  $\mathbb{H}_{>0} := \{z \in \mathbb{C} : \text{Re } z > 0\}$ .

For a set  $A \subset \mathbb{R}^d$  we use  $\text{int}(A)$  for the interior of  $A$  in  $\mathbb{R}^d$ .

### 2.2 General definitions and results from analysis

In this part we collect definitions and facts from analysis.

#### Hausdorff dimension

For a reference on Hausdorff dimensions see [Fal86].

**Definition 2.2.1** (Hausdorff measure/dimension). Let  $A \subset \mathbb{R}^d$ . For  $s, \delta > 0$  we define

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{j=0}^{\infty} (\text{diam } A_j)^s : A \subset \bigcup_{j=0}^{\infty} A_j, \text{diam } A_j < \delta \right\},$$

where  $A_j \subset \mathbb{R}^d$ . Then we call

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A)$$

the  $s$ -dimensional Hausdorff measure of  $A$  and

$$\dim A := \inf \{s > 0 : \mathcal{H}^s(A) = 0\}$$

the Hausdorff dimension of  $A$ .

*Remark 2.2.2.*

1. The  $s$ -dimensional Hausdorff measure is an outer measure, for which Borel sets are measurable.
2. The upper limit exists since the function  $\delta \mapsto \mathcal{H}_\delta^s(A)$  is non-increasing.
3. For  $s \in \mathbb{N}$  the  $s$ -dimensional Hausdorff measure and the  $s$ -dimensional Lebesgue measure on  $\mathbb{R}^s$  only differ by a constant depending on  $s$ .

## Infinite products

Now we want to collect a basic result on infinite products which will play an important role for some details in our proofs. For a reference see [FL80].

**Definition 2.2.3** (Infinite products). Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$ . We say that

$$\prod_{n=1}^{\infty} a_n$$

converges if there exists an  $N \in \mathbb{N}$  such that the limit

$$\lim_{k \rightarrow \infty} \prod_{n=N}^k a_n$$

exists and is nonzero. In this case we put

$$\prod_{n=1}^{\infty} a_n := \left( \prod_{n=1}^{N-1} a_n \right) \cdot \lim_{k \rightarrow \infty} \prod_{n=N}^k a_n.$$

**Lemma 2.2.4** (Convergence criterion for infinite products). *Let  $(b_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Then the infinite product*

$$\prod_{n=1}^{\infty} (1 + |b_n|)$$

*converges if and only if the series*

$$\sum_{n=1}^{\infty} |b_n|$$

*converges.*

## 2.3 Sobolev spaces and the space $ACL^p$

In the definition of quasiregular maps we need a weaker regularity version as the  $C^1$  condition which was discussed in the introduction. This version can be phrased in terms of Sobolev spaces which allows us to differentiate in a non-classical way. This elegant theory was given by Sergei Sobolev [So39] using  $L^p$  functions and giving them an additional structure which coincides with our structure for smooth functions. Here we refer to the standard book of Evans [Ev98].

For  $U \subset \mathbb{R}^d$  open and  $f: U \rightarrow \mathbb{R}$ , denote by

$$\text{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}}$$

the *support* of  $f$ .

**Definition 2.3.1** (Space of test functions). Let  $U \subset \mathbb{R}^d$  be open. Then denote by

$$C_0^\infty(U) := \{f \in C^\infty(U) : \text{supp}(f) \text{ is a compact subset of } U\}$$

the space of smooth functions with compact support in  $U$ . Moreover, we call each function  $f \in C_0^\infty(U)$  a *test function* and  $C_0^\infty(U)$  the *space of test functions* on  $U$ .

**Definition 2.3.2** (Weak derivative). Let  $U \subset \mathbb{R}^d$  be open and let  $f \in L^1_{loc}(U)$ . If there exists a function  $g \in L^1_{loc}(U)$  such that for all  $\varphi \in C_0^\infty(U)$  the equality

$$\int_U g_j(x) \varphi(x) dx = - \int_U f(x) \frac{\partial}{\partial x_j} \varphi(x) dx,$$

holds, then we call  $g_j$  the *first  $j$ -th weak derivative* of  $f$  and write  $\frac{\partial}{\partial x_j} f = g_j$ .

*Remark 2.3.3.*

1. For continuously differentiable functions the notion of weak derivatives coincides with the classical derivatives, because in this case the formula in the definition is the same as the rule of partial integration since the boundary terms vanish for test functions in  $I$ .
2. Weak partial derivatives are unique up to a set of measure zero.

Using the definitions from above we are ready to define Sobolev spaces.

**Definition 2.3.4** (Sobolev space). Let  $U \subset \mathbb{R}^d$  be open. For  $k \in \mathbb{N}_0$  and  $p \in [1, \infty)$  we say that a function  $f: U \rightarrow \mathbb{R}$  lies in the *Sobolev space*  $W^{k,p}(U)$  if for all  $\alpha \in \mathbb{N}_0^d$  with  $|\alpha| \leq k$ , the  $\alpha$ -th weak derivative of  $f$  exists and  $\|f\|_{W^{k,p}(U)} < \infty$ , where

$$\|f\|_{W^{k,p}(U)} = \left( \sum_{|\alpha| \leq k} \int_U \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha} \right|^p dx \right)^{\frac{1}{p}}.$$

Moreover, denote by  $W_{loc}^{k,p}(U)$  the space of all functions, such that for all  $x \in U$  there exists an open subset  $V \subset U$  containing  $x$  such that  $f|_V \in W^{k,p}(V)$ .

*Remark 2.3.5.*

1. The Sobolev space  $W^{k,p}$  equipped with the norm  $\|\cdot\|_{W^{k,p}}$  is a Banach space.
2. It is standard to define Sobolev spaces in the above way. In our situation we will deal with vector-valued functions since  $d \geq 3$ . In this case we say that each coordinate function should satisfy the conditions from above.

Another way to look at this problem is to talk about functions which are absolutely continuous on lines almost everywhere. Therefore we give the following definitions which one can find in [IM01, page 105] and [Ri93, page 5].

**Definition 2.3.6** (Absolute continuity). Let  $I \subset \mathbb{R}$  be an interval. We say that  $f: I \rightarrow \mathbb{R}$  is *absolutely continuous* if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $l \in \mathbb{N}$ ,  $k \in \{1, \dots, l\}$  and all non-overlapping intervals  $(a_k, b_k)$  with  $[a_k, b_k] \subset I$  such that

$$\sum_{k=1}^l (b_k - a_k) < \delta$$

we obtain

$$\sum_{k=1}^l |f(b_k) - f(a_k)| < \varepsilon.$$

**Definition 2.3.7** (Absolute continuity on lines). Let  $n \in \mathbb{N}$  and  $Q$  be an  $n$ -interval, i.e.  $Q := \{x \in \mathbb{R}^n : a_j \leq x_j \leq b_j \text{ for all } j \in \{1, \dots, n\}\}$ . We say that  $f: Q \rightarrow \mathbb{R}^d$  is ACL (*absolute continuous on lines*), if  $f$  is continuous and if  $f$  is absolutely continuous on almost every line segment in  $Q$  parallel to the coordinate axes.

For  $U \subset \mathbb{R}^n$  we say that  $f: U \rightarrow \mathbb{R}^d$  is ACL, if  $f|_Q$  is ACL for every closed  $n$ -interval  $Q \subset U$ . If, in addition, the partial derivatives of  $f$  are in  $L_{loc}^p(U)$  for some  $p \geq 1$ , then we say that  $f$  is  $ACL^p(U)$ .

Now we can state the lemma connecting the Sobolev spaces with the space  $ACL^p$  (see [Ri93, Proposition I.1.2]).

**Lemma 2.3.8.** *A function  $f: U \rightarrow \mathbb{R}^d$  is  $ACL^p(U)$  if and only if it is continuous and belongs to  $W_{loc}^{1,p}(U)$ . In this case the weak and the ordinary partial derivatives coincide almost everywhere.*

## 2.4 Quasiregular maps

An essential tool for this thesis is given by the theory of quasiregular maps. In this section we give a short introduction to this theory by collecting some results. Most of the time we will follow the book of Rickman [Ri93].

We want to motivate the definition of quasiregular maps by focusing on the geometrical mapping behaviour of holomorphic maps. Here we identify  $\mathbb{C}$  with  $\mathbb{R}^2$ .

For an open set  $U \subset \mathbb{R}^2$  the Cauchy-Riemann equations say that  $f: U \rightarrow \mathbb{R}^2$  is holomorphic if

1.  $f$  is  $C^1$  in the real sense,
2.  $\|Df(x)\|^2 = J_f(x)$ ,

where  $Df(x)$  denotes the derivative of  $f$  in  $x$ , by  $J_f(x)$  we denote its Jacobian, and

$$\|Df(x)\| := \sup_{\|h\|_2=1} \|Df(x)h\|_2$$

denotes the operator norm of  $Df$  in  $x$ . From a geometrical point of view this means that  $Df(x)$  maps circles to circles. Thus a first candidate for a higher-dimensional analogue of a holomorphic map would be a function  $f: U \rightarrow \mathbb{R}^d$  on an open set  $U \subset \mathbb{R}^d$  such that

1.  $f$  is  $C^1$  in the real sense,
2.  $\|Df(x)\|^d = J_f(x)$ .

This means that  $Df(x)$  maps spheres to spheres. From the following theorem of Liouville [Li50] we obtain that this choice yields a very small class of functions.

**Theorem 2.4.1.** A  $C^1$  function  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  which maps spheres to spheres is either constant or a Möbius transformation, i.e. a finite composition of reflections on spheres and hyperplanes.

*Remark 2.4.2.* The theorem from above was shown by Liouville in 1850 for  $C^3$  functions, but later Hartman proved the  $C^2$  case in 1947 [Ha47] and finally generalized his result to the  $C^1$  case in 1958 [Ha58].

It seems that we need to consider weaker conditions for an interesting class of functions. This leads to the following definition.

**Definition 2.4.3** (Quasiregular). Let  $G \subset \mathbb{R}^d$  be a domain and let  $f \in W_{loc}^{1,d}(G)$  be continuous. We say that  $f$  is *quasiregular* if there exists a constant  $K_O := K_O(f) \geq 1$  such that

$$\|Df(x)\|^d \leq K_O J_f(x) \quad \text{a.e.} \tag{2.4.1}$$

This is equivalent to the condition

$$J_f(x) \leq K_I l(Df(x))^d \quad \text{a.e.} \quad (2.4.2)$$

for some  $K_I := K_I(f) \geq 1$ , where

$$l(Df(x)) := \inf_{\|h\|_2=1} \|Df(x)h\|_2.$$

The smallest constants  $K_O$  and  $K_I$  such that (2.4.1) and (2.4.2) hold are called the *outer* and *inner dilatation* of  $f$  and  $K := \max\{K_I, K_O\}$  is called the (maximal) *dilatation* of  $f$ .

*Remark 2.4.4.* The condition in (2.4.1) says that  $Df(x)$  maps spheres to ellipsoids of bounded eccentricity, i.e. the quotient of the largest semiaxis and the smallest semiaxis is bounded.

The following result shows parallels to the holomorphic case and the proof can be found in [Ri93, Theorem I.4.1].

**Theorem 2.4.5.** Non-constant quasiregular maps are open and discrete.

Quasiregularity can be also defined for maps on a domain  $G \subset \overline{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$ . As in the case of holomorphic functions we want to give the following definitions for quasiregular maps (see [Vu88, Definition 10.6]).

**Definition 2.4.6.** Let  $G \subset \overline{\mathbb{R}^d}$  be a domain. A continuous map  $f: G \rightarrow \overline{\mathbb{R}^d}$  is *quasiregular* if either  $f(G) = \{\infty\}$ , or  $f^{-1}(\infty)$  is discrete and  $f|_{G \setminus (f^{-1}(\infty) \cup \{\infty\})}$  is quasiregular.

**Definition 2.4.7** (Polynomial/transcendental type). We say that a quasiregular map  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is of *polynomial type* if  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If  $f$  has an essential singularity at  $\infty$ , the map  $f$  is said to be of *transcendental type*.

*Remark 2.4.8.* A non-constant quasiregular map is either of polynomial type or of transcendental type.

There are some analogues for quasiregular maps to the theory of holomorphic maps. For example Picard's theorem and Montel's theorem have higher-dimensional counterparts, see e.g. [Ri80, Theorem 1.1] and [Min82, Theorem 4].

*Remark 2.4.9.* By putting  $f(\infty) = \infty$  one can extend a quasiregular map  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  of polynomial type to a quasiregular selfmap of  $\overline{\mathbb{R}^d}$ .

For our purpose it is interesting to investigate an analogue of the escaping set for quasiregular maps. It can be defined as follows.

**Definition 2.4.10.** Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be quasiregular. We call the set

$$\mathcal{I}(f) := \{x \in \mathbb{R}^d : f^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

the *escaping set* of  $f$ .



Bergweiler, Fletcher, Langley and Meyer [BFLM09, page 643] proved the following Lemma which justifies the definition of the escaping set.

**Lemma 2.4.11.** *Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be quasiregular of transcendental type. Then  $\mathcal{I}(f) \neq \emptyset$ .*

Further results on the dynamics of quasiregular maps can be found in [Ber13, BN14].

## 2.5 Complex analysis

In this section we summarize definitions and results from complex analysis which will be used in the following sections. We do not recall basic facts and focus on selected results.

**Lemma 2.5.1.** *Let  $G \subset \mathbb{C}$  be a convex domain and  $f: G \rightarrow \mathbb{C}$  holomorphic. If*

$$\operatorname{Re} f'(z) > 0 \text{ for all } z \in G,$$

*then  $f$  is univalent.*

*Proof.* Let  $z, w \in G$  such that  $f(z) = f(w)$ . Assume that  $z \neq w$ . Then

$$0 = \frac{f(z) - f(w)}{z - w} = \int_w^z \frac{f'(\zeta)}{z - w} d\zeta = \int_0^1 f'(tz + (1-t)w) dt.$$

Since  $G$  is convex we know that  $tz + (1-t)w \in G$  for all  $t \in [0, 1]$ . By our assumption this yields

$$0 = \operatorname{Re} \left( \frac{f(z) - f(w)}{z - w} \right) = \int_0^1 \operatorname{Re} (f'(tz + (1-t)w)) dt > 0,$$

a contradiction. □

**Definition 2.5.2** (Symmetric domain). Let  $G \subset \mathbb{C}$  be a domain. Then we call

$$G^* := \{z \in \mathbb{C} : \bar{z} \in G\}$$

the *complex conjugate domain* of  $G$ . Moreover, we say that  $G$  is *symmetric* if  $G = G^*$ .

**Lemma 2.5.3** (Schwarz's reflection principle). *Let  $G \subset \mathbb{C}$  be a symmetric domain and let  $f: G \cap \mathbb{H}_{\geq 0} \rightarrow \mathbb{C}$  be continuous and holomorphic on  $G \cap \mathbb{H}_{> 0}$ . If  $f(G \cap \mathbb{R}) \subset \mathbb{R}$ , then the reflected map*

$$\tilde{f}: G \rightarrow \mathbb{C}, \tilde{f}(z) = \begin{cases} f(z) & , \text{if } z \in G \cap \mathbb{H}_{\geq 0}, \\ \overline{f(\bar{z})} & , \text{if } z \notin G \cap \mathbb{H}_{\geq 0}, \end{cases}$$

*is holomorphic.*

**Definition 2.5.4** (Singular value). Let  $f$  be a transcendental entire map. We call a point  $w \in \mathbb{C}$  *critical value* of  $f$  if there exists  $z \in \mathbb{C}$  such that  $f'(z) = 0$  and  $f(z) = w$ . Moreover we say that  $a \in \mathbb{C}$  is an *asymptotic value* of  $f$  if there exists a curve  $\gamma: (0, \infty) \rightarrow \mathbb{C}$  such that  $\gamma(t) \rightarrow \infty$  and  $f(\gamma(t)) \rightarrow a$  as  $t \rightarrow \infty$ . A *singular value* of  $f$  is either a critical value or an asymptotic value of  $f$ . We denote the *set of singular values* of  $f$  by  $\operatorname{sing}(f^{-1})$ .

*Remark 2.5.5.*

1. Equivalently, the set  $\overline{\text{sing}(f^{-1})}$  is the smallest set  $S$  such that  $f: \mathbb{C} \setminus f^{-1}(S) \rightarrow \mathbb{C} \setminus S$  is a covering map.
2. For a transcendental entire map  $f$ , a point  $z \in \mathbb{C}$  is a singular value of  $f$  if and only if there exists a branch of  $f^{-1}$  which cannot be continued into  $z$ . More precisely, if a branch of  $f^{-1}$  is defined on an open set  $U \subset \mathbb{C}$  and  $\gamma$  is a curve connecting  $U$  and  $z$ , then this branch of  $f^{-1}$  can be continued along every piece of  $\gamma$  except for its endpoint  $z$ .

Standard examples are given by  $f(z) := z^2$  and  $g(z) := \exp(z)$ . Here 0 is a critical value of  $f$  and an asymptotic value of  $g$  since for  $\gamma: (0, \infty) \rightarrow \mathbb{C}$ ,  $\gamma(t) = -t$  we obtain

$$\lim_{t \rightarrow \infty} g(\gamma(t)) = \lim_{t \rightarrow \infty} \exp(-t) = 0.$$

In fact, no branch of the complex logarithm can be continued into 0.

**Definition 2.5.6** (Order of a function). We say that an entire function  $f$  is of *finite order* if there exist  $\mu, R > 0$  such that

$$|f(z)| \leq \exp(|z|^\mu) \quad \text{for } |z| \geq R.$$

The infimum over all  $\mu$  such that this holds for some  $R > 0$  is called *the order* of  $f$  and is denoted by  $\rho(f)$ , where we use the convention  $\inf \emptyset = \infty$ , if such an  $R$  does not exist for any  $\mu > 0$ . Then we say that  $f$  is of *infinite order*.

*Remark 2.5.7.* It is not difficult to show that for an entire function  $f$ , we have

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$

where  $M(r, f) := \max_{|z|=r} |f(z)|$  is the *maximum modulus* of  $f$ .

The following result [GO08] gives a connection between the number of asymptotic values and the order of an entire function.

**Lemma 2.5.8** (Denjoy-Carleman-Ahlfors theorem). *Let  $f$  be entire and  $R > 0$ . If  $N$  denotes the number of components of*

$$A_R := \{z \in \mathbb{C} : |f(z)| > R\},$$

*then  $N \leq \max\{1, 2\rho(f)\}$ .*

As a corollary [GO08], we obtain the following result.

**Corollary** (Denjoy-Carleman-Ahlfors). *Let  $f$  be entire. Then the number of asymptotic values of  $f$  is at most  $\max\{1, 2\rho(f)\}$ .*

Now we focus on the distortion of bounded sets under holomorphic maps which can be found in [Po92]. Many results of Pommerenke are stated for univalent functions  $f: \mathbb{D} \rightarrow \mathbb{C}$ , but these results can be easily generalized to the case  $f: D(z_0, r) \rightarrow \mathbb{C}$  by considering the function

$$g: \mathbb{D} \rightarrow \mathbb{C}, g(z) = \frac{f(z_0 + rz) - f(z_0)}{rf'(z_0)}$$

instead of  $f$  since  $g(0) = 0$  and  $g'(0) = 1$ .

**Lemma 2.5.9.** *Let  $f: \mathbb{D} \rightarrow \mathbb{C}$  be univalent. Then  $f(D(0, r))$  is a convex set for  $r \leq \sqrt{2} - 1$ .*

*Remark 2.5.10.* The largest  $r$  such that  $f(D(0, r))$  is convex is called the *radius of convexity*. Hengartner and Schober [HS73] and Goodman and Saff [GS79] have shown that  $f(D(0, r))$  is for  $r > \sqrt{2} - 1$  in general not convex. Goodman and Saff conjectured that  $r_0 := \sqrt{2} - 1$  is the best possible constant to preserve convexity. In 1989 Ruscheweyh and Salinas [RS89] proved that this constant is optimal.

**Definition 2.5.11** (Distortion). Let  $U \subset \mathbb{C}$  be open and  $f: U \rightarrow \mathbb{C}$  univalent. If  $D \subset U$  is bounded, then we define by

$$\mathcal{L}(f|_D) := \frac{\sup\{|f'(z)| : z \in D\}}{\inf\{|f'(z)| : z \in D\}}$$

the *distortion of  $f$  on  $D$* .

**Lemma 2.5.12** (Koebe's distortion theorem). *Let  $r > 0$ ,  $z_0 \in \mathbb{C}$  and let  $f: D(z_0, r) \rightarrow \mathbb{C}$  be univalent. Then we have for all  $z \in D(z_0, r)$  and  $s := |z - z_0|$*

$$\frac{r-s}{(r+s)^3} |f'(z_0)| \leq |f'(z)| \leq \frac{r+s}{(r-s)^3} |f'(z_0)|.$$

**Lemma 2.5.13.** *Under the assumptions of Koebe's distortion theorem we have*

$$\mathcal{L}(f|_{D(z_0, s)}) \leq \left( \frac{r+s}{r-s} \right)^4.$$

In the following denote by  $\Delta(z_1, z_2, z_3) \subset \mathbb{C}$  the triangle with vertices  $z_1, z_2$  and  $z_3$ , where these points are pairwise distinct. Moreover, we denote by  $d_{j,k} := |z_j - z_k|$  for  $j, k \in \{1, 2, 3\}$  with  $j \neq k$ .

**Definition 2.5.14** (Degenerate/ $\delta$ -degenerate). We say that for  $\delta \in (0, 1)$  the triangle  $\Delta(z_1, z_2, z_3)$  is  $\delta$ -close to degenerate, if there exists pairwise different  $j, k, l \in \{1, 2, 3\}$  such that

$$\frac{d_{j,k}}{d_{j,l} + d_{k,l}} \geq 1 - \delta. \quad (2.5.1)$$

If the quotient equals 1, we say that  $\Delta(z_1, z_2, z_3)$  is *degenerate*.

*Remark 2.5.15.* By the triangle inequality we know that the quotient in (2.5.1) is always less than or equal to 1. Thus equality means that the points  $z_1$ ,  $z_2$  and  $z_3$  lie on a line. In other words, if we have

$$\mathcal{D}(\Delta(z_1, z_2, z_3)) := \max \left\{ \frac{d_{j,k}}{d_{j,l} + d_{k,l}} : j, k, l \in \{1, 2, 3\} \text{ pairwise different} \right\} < 1 - \delta,$$

then the considered triangle is non-degenerate.

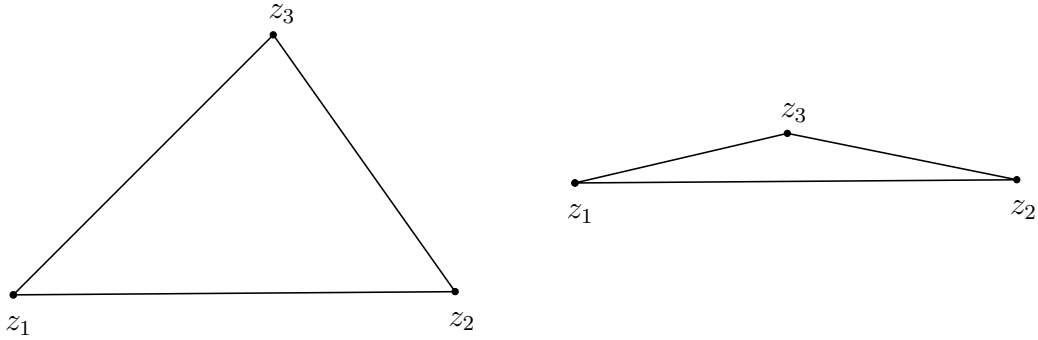


Figure 2.1. Examples of a non-degenerate triangle (left) and the  $\delta$ -close to degeneracy case.

**Lemma 2.5.16** (Distortion). *Let  $U \subset \mathbb{C}$  be an open, bounded and convex set.*

*Let  $\Delta(z_1, z_2, z_3) \subset U$  and let  $f: U \rightarrow \mathbb{C}$  be univalent. Suppose that  $f(U)$  is convex. Then*

$$\mathcal{D}(\Delta(f(z_1), f(z_2), f(z_3))) \leq \mathcal{L}(f|_U) \cdot \mathcal{D}(\Delta(z_1, z_2, z_3)).$$

*Proof.* We have for all  $j, k \in \{1, 2, 3\}$  with  $j \neq k$  that

$$\begin{aligned} |f(z_j) - f(z_k)| &= \left| \int_{z_k}^{z_j} f'(z) dz \right| \\ &\leq d_{j,k} \cdot \sup\{|f'(z)| : z \in [z_j, z_k]\} \\ &\leq d_{j,k} \cdot \sup\{|f'(z)| : z \in U\}. \end{aligned}$$

Since  $f$  is univalent in  $U$ , we have

$$\begin{aligned} d_{j,k} &= \left| \int_{f(z_k)}^{f(z_j)} (f^{-1})'(z) dz \right| \\ &\leq |f(z_j) - f(z_k)| \cdot \sup \left\{ |(f^{-1})'(z)| : z \in [f(z_j), f(z_k)] \right\} \\ &= |f(z_j) - f(z_k)| \cdot \sup \left\{ \frac{1}{|f'(f^{-1}(z))|} : z \in [f(z_j), f(z_k)] \right\} \\ &= \frac{|f(z_j) - f(z_k)|}{\inf \{|f'(f^{-1}(z))| : z \in [f(z_j), f(z_k)]\}}. \end{aligned}$$

and hence

$$\begin{aligned} |f(z_j) - f(z_k)| &\geq d_{j,k} \cdot \inf \{ |f'(f^{-1}(z))| : z \in [f(z_j), f(z_k)] \} \\ &\geq d_{j,k} \cdot \inf \{ |f'(z)| : z \in U \}. \end{aligned}$$

Thus we obtain for all pairwise different  $j, k, l \in \{1, 2, 3\}$

$$\frac{|f(z_j) - f(z_k)|}{|f(z_j) - f(z_l)| + |f(z_k) - f(z_l)|} \leq \frac{d_{j,k}}{d_{j,l} + d_{k,l}} \cdot \frac{\sup \{ |f'(z)| : z \in U \}}{\inf \{ |f'(z)| : z \in U \}}.$$

This yields

$$\mathcal{D}(\Delta(f(z_1), f(z_2), f(z_3))) \leq \mathcal{L}(f|_U) \cdot \mathcal{D}(\Delta(z_1, z_2, z_3)).$$

□

**Lemma 2.5.17.** *Let  $\Delta(z_1, z_2, z_3) \subset \mathbb{C}$  be non-degenerate in the sense of Definition 2.5.14 for some  $\delta > 0$ . Then every angle in  $\Delta(z_1, z_2, z_3)$  is bounded away from 0 and  $\pi$ , with bounds depending only on  $\delta$ .*

*Proof.* We show the contraposition of the result. Without loss of generality we assume that  $d_{1,2}$  is the longest edge of  $\Delta(z_1, z_2, z_3)$ . If  $\alpha_j$  denotes the angle at the vertex  $z_j$ , the law of sines yields that the angle  $\alpha_3$  is the largest one. We consider two cases:

**1. Both angles  $\alpha_1$  and  $\alpha_2$  are small.**

Suppose that  $\delta' > 0$  is small and that  $\alpha_1, \alpha_2 < \delta'$ . Put  $\delta := 1 - \cos \delta' \in (0, 1)$ . Then  $\delta$  is close to 0. By elementary geometry we know that

$$d_{1,2} = d_{1,3} \cos \alpha_1 + d_{2,3} \cos \alpha_2 \geq (d_{1,3} + d_{2,3}) \cos \delta'$$

and thus

$$\frac{d_{1,2}}{d_{1,3} + d_{2,3}} \geq \cos \delta' = 1 - \delta.$$

Thus  $\Delta(z_1, z_2, z_3)$  is  $\delta$ -close to degenerate if  $\alpha_3$  is close to  $\pi$ .

**2. Only one of the angles  $\alpha_1$  or  $\alpha_2$  is small.**

Suppose now that  $\delta' > 0$  is small and assume without loss of generality that  $\alpha_1 < \delta'$ . Thus we know that  $d_{2,3}$  is small compared to the length of the other edges. Using similar arguments as in the first case, we obtain the result. □

**Lemma 2.5.18.** *Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be continuous and injective, and differentiable at  $t_0 \in [a, b]$  with  $\gamma'(t_0) \neq 0$ . Then for all  $\delta \in (0, 1)$  there exists  $\varepsilon > 0$  such that the triangle  $\Delta(z_1, \gamma(t_0), z_2)$  is  $\delta$ -close to degenerate for all  $z_1, z_2 \in D(\gamma(t_0), \varepsilon) \cap \gamma([a, b])$ .*

*Proof.* Since  $\gamma$  is injective, we know that for all  $\delta > 0$  there exists  $\varepsilon > 0$  such that for all  $t \in [a, b]$  with  $|\gamma(t) - \gamma(t_0)| < \varepsilon$  we have  $|t - t_0| < \delta$ . For all  $t \in [a, b] \setminus \{t_0\}$  we have

$$\frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \gamma'(t_0) + \kappa(t)$$

for a suitable  $\kappa(t)$ . Let  $\eta > 0$  be small. Then we can choose  $\delta$  small enough such that for all  $t \in [a, b]$  such that  $|t - t_0| < \delta$  we have  $|\kappa(t)| < |\gamma'(t_0)| \sin \eta$ .

Let  $z_1, z_2 \in (D(\gamma(t_0), \varepsilon) \cap \gamma([a, b])) \setminus \{\gamma(t_0)\}$ , say  $z_1 = \gamma(t_1)$  and  $z_2 = \gamma(t_2)$ . Then  $|t_j - t_0| < \delta$  and we obtain

$$|\arg(\gamma'(t_0) + \kappa(t_j)) - \arg(\gamma'(t_0))| \leq \arcsin\left(\frac{|\gamma'(t_0)| \sin \eta}{|\gamma'(t_0)|}\right) = \eta$$

for  $j \in \{1, 2\}$ . Thus the equation

$$\gamma(t_j) - \gamma(t_0) = (\gamma'(t_0) + \kappa(t_j))(t_j - t_0)$$

yields

$$\arg(\gamma(t_j) - \gamma(t_0)) = \arg \gamma'(t_0) + r_j,$$

where  $|r_j| \leq \eta$  for  $t_j > t_0$  and  $|r_j - \pi| \leq \eta$  for  $t_j < t_0$  for a suitable choice of the argument. Hence

$$|\arg(\gamma(t_2) - \gamma(t_0)) - \arg(\gamma(t_1) - \gamma(t_0))| \leq 2\eta$$

or

$$|\arg(\gamma(t_2) - \gamma(t_0)) - \arg(\gamma(t_1) - \gamma(t_0)) - \pi| \leq 2\eta.$$

Noticing that  $\arg(\gamma(t_2) - \gamma(t_0)) - \arg(\gamma(t_1) - \gamma(t_0))$  is the angle of the triangle at  $\gamma(t_0)$ , the conclusion follows.  $\square$

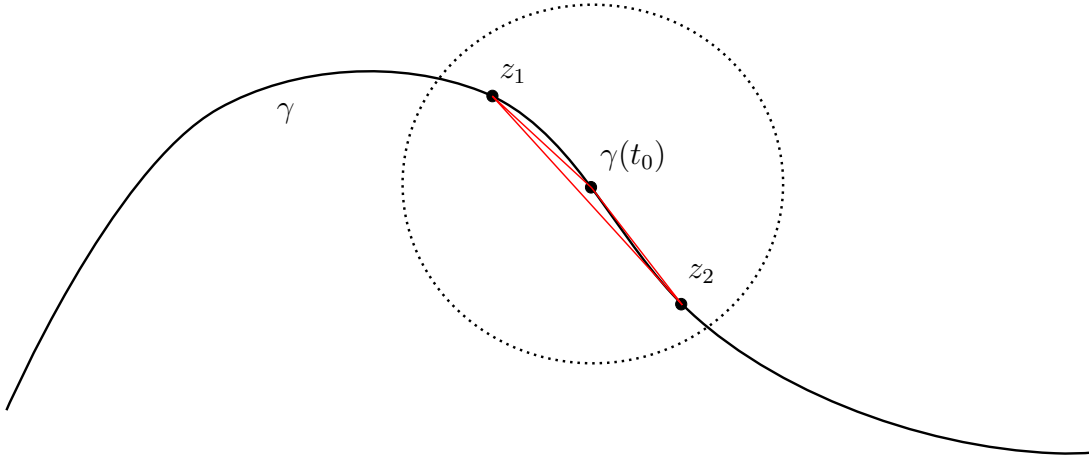


Figure 2.2. Connection between differentiability of curves and  $\delta$ -close to degeneracy for triangles.

## 2.6 Complex dynamics

In this section we want to deal with a short introduction to the theory of complex dynamics, which goes back to the beginning of the 20th century. In the sequel we collect some results of more recent work. Details can be found in [Bea91, Mil06, St93].

**Definition 2.6.1** (Normal family). Let  $U \subset \hat{\mathbb{C}}$  be open and  $\mathcal{F}$  be a family of functions from  $U$  to  $\hat{\mathbb{C}}$ . We say that  $\mathcal{F}$  is *normal* if for every sequence in  $\mathcal{F}$  there is a subsequence which converges locally uniformly with respect to the chordal metric. Moreover, we call  $\mathcal{F}$  *normal in*  $z \in U$  if there exists a neighborhood  $V$  of  $z$  such that the family  $\{f|_V : f \in \mathcal{F}\}$  is normal.

The following criterion for normality is due to Montel.

**Lemma 2.6.2.** *Let  $U \subset \hat{\mathbb{C}}$  be open,  $a_1, a_2$  and  $a_3 \in \hat{\mathbb{C}}$  distinct points and let  $\mathcal{F}$  be a family of meromorphic functions in  $U$ . If  $f(z) \neq a_j$  for all  $f \in \mathcal{F}$ ,  $z \in U$ , and  $j \in \{1, 2, 3\}$ , then  $\mathcal{F}$  is normal.*

Now we turn to the iteration of holomorphic maps. In this context normality of the family  $\{f^n : n \in \mathbb{N}\}$  in  $z \in \mathbb{C}$  means that the iterates of  $f$  behave stable in a suitable neighborhood of  $z$ . The complex plane splits up into two disjoint parts, a stable and a chaotic one, which are named after Pierre Fatou and Gaston Julia.

**Definition 2.6.3** (Fatou/Julia set). Let  $f$  be entire. We call

$$\mathcal{F}(f) := \{z \in \mathbb{C} : \{f^n : n \in \mathbb{N}\} \text{ is normal in } z\}$$

the *Fatou set* of  $f$  and

$$\mathcal{J}(f) := \mathbb{C} \setminus \mathcal{F}(f)$$

the *Julia set* of  $f$ .

In our context the next set plays a major role, which was introduced by Eremenko [Er89].

**Definition 2.6.4** (Escaping set). Let  $f$  be entire. Then we call

$$\mathcal{I}(f) := \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

the *escaping set* of  $f$ .

For polynomials it is well known that the escaping set is always a non-empty set containing  $\infty$ . The following result of Eremenko [Er89, Theorem 1] justifies the definition of the escaping set for transcendental entire maps.

**Lemma 2.6.5.** *Let  $f$  be transcendental entire. Then  $\mathcal{I}(f) \neq \emptyset$ .*

The next result is also due to Eremenko [Er89] and gives a connection between the escaping set and the Julia set in the following way.

**Lemma 2.6.6.** *Let  $f$  be entire. Then  $\partial\mathcal{I}(f) = \mathcal{J}(f)$ .*

Now we want to define a class of functions which is used in subsequent sections.

**Definition 2.6.7** (Eremenko-Lyubich class). We call

$$\mathcal{B} := \{f : f \text{ is transcendental entire and } \text{sing}(f^{-1}) \text{ is bounded}\}$$

the *Eremenko-Lyubich class*. Moreover, we say that a function  $f \in \mathcal{B}$  is of *bounded type*.

*Remark 2.6.8.* This class of functions contains many standard functions, e.g. the functions  $f_1(z) := \lambda \exp(z)$  for  $\lambda \in \mathbb{C} \setminus \{0\}$ ,  $f_2(z) := \sin(z)$  and  $f_3(z) := \sin(z)/z$  since

$$\text{sing}(f_1^{-1}) = \{0\}, \quad \text{sing}(f_2^{-1}) = \{-1, 1\} \quad \text{and} \quad \text{sing}(f_3^{-1}) \subset \overline{\mathbb{D}}.$$

Notice that  $\text{card}(\text{sing}(f_3^{-1})) = \infty$ . An example of a function which is not of bounded type is  $f(z) := z \sin(z)$ .

The following result due to Eremenko and Lyubich [EL92] is a key element in many proofs dealing with functions of bounded type.

**Lemma 2.6.9.** *Let  $f \in \mathcal{B}$ . Then  $\mathcal{I}(f) \subset \mathcal{J}(f)$ .*

*Remark 2.6.10.* Together with Lemma 2.6.6 we obtain  $\overline{\mathcal{I}(f)} = \mathcal{J}(f)$  for functions  $f$  of bounded type.

For general transcendental entire functions this is not always the case. Consider for example the so-called Fatou function [Fat26]

$$f: \mathbb{C} \rightarrow \mathbb{C}, \quad f(z) = z + 1 + e^{-z}.$$

Then we obtain  $\{x + (2k + 1)\pi i : x < 0 \text{ and } k \in \mathbb{Z}\} \subset \mathcal{I}(f) \cap \mathcal{J}(f)$ , but we also have  $\{z \in \mathbb{C} : \text{Re } z > 0\} \subset \mathcal{I}(f) \cap \mathcal{F}(f)$ .

**Definition 2.6.11** (Hyperbolic/disjoint type). Let  $f$  be transcendental entire. We say that  $f$  is *hyperbolic* if the postsingular set

$$\mathcal{P}(f) := \overline{\bigcup_{j \geq 0} f^j(\text{sing}(f^{-1}))}$$

is a compact subset of  $\mathcal{F}(f)$ . The function  $f$  is called of *disjoint type* if  $f$  is hyperbolic and furthermore  $\mathcal{F}(f)$  is connected.

The following result (see [Bar07, Lemma 3.1] or [MB12, Proposition 2.8]) provides an alternative definition of disjoint-type entire functions which we will use in the sequel.

**Lemma 2.6.12** (Characterization of disjoint-type functions). *A transcendental entire function  $f$  is of disjoint type if and only if there exists a bounded Jordan domain  $D \subset \mathbb{C}$  such that  $\text{sing}(f^{-1}) \subset D$  and  $f(\overline{D}) \subset D$ .*

*Remark 2.6.13.* Since the set  $\text{sing}(f^{-1})$  is bounded for every function  $f \in \mathcal{B}$ , one can obtain a function of disjoint type by considering the function  $f_0 := \lambda f$  for small  $\lambda \in \mathbb{R}$ .



### Logarithmic change of variable

In this part we introduce a key tool given by [EL92], the so-called logarithmic change of variable, to study functions in class  $\mathcal{B}$ . We have the following result.

**Theorem 2.6.14.** Let  $f \in \mathcal{B}$  and let  $R > |f(0)|$  be such that  $\text{sing}(f^{-1}) \subset D(0, R)$ . Let

$$W := \exp^{-1} (f^{-1} (\{z \in \mathbb{C} : |z| > R\}))$$

and

$$H := \{z \in \mathbb{C} : \text{Re } z > \log R\}.$$

Then every component of  $W$  is simply connected and there exists a holomorphic function  $F: W \rightarrow H$  such that

$$\exp F(z) = f(\exp z)$$

for all  $z \in W$ . Moreover, for every component  $U$  of  $W$  the restriction  $F|_U$  is a conformal map from  $U$  to  $H$ .

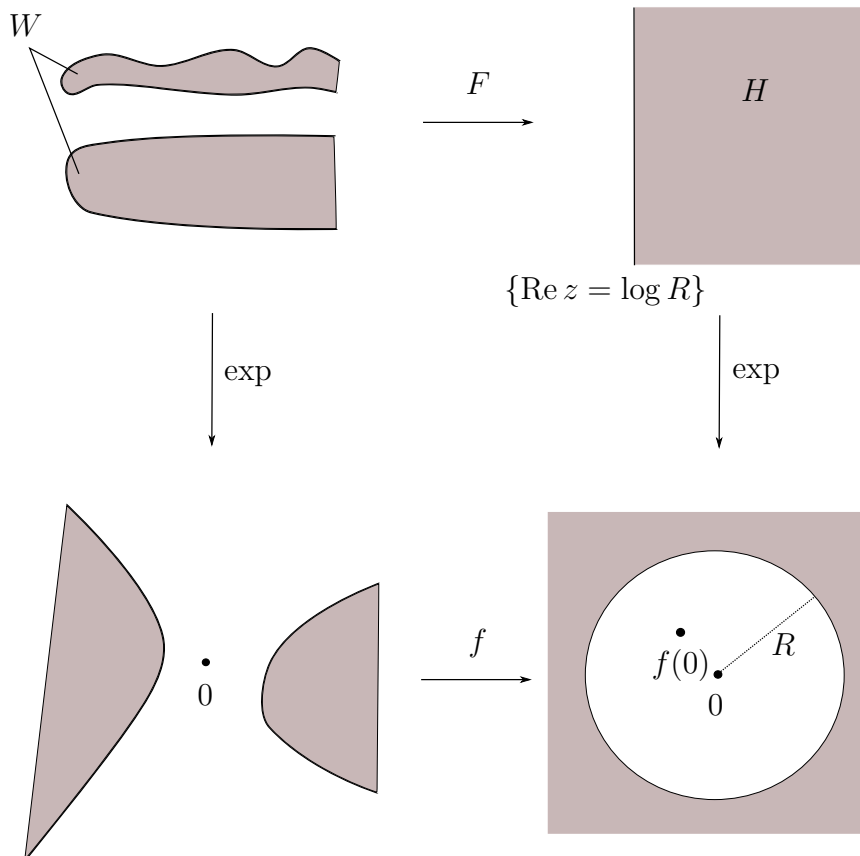


Figure 2.3. The logarithmic change of variable.

The function  $F$  obtained by the logarithmic change of variable of  $f$  is called a *logarithmic transform of  $f$* . Moreover, we call every component of  $W$  a *tract*.

Functions of bounded type satisfy the following growth condition [EL92].

**Lemma 2.6.15.** *Let  $F: U \rightarrow H$  be a logarithmic transform of the function  $f \in \mathcal{B}$ . Then there exists  $R > 0$  such that*

$$|F'(z)| \geq 2$$

for all  $z \in U$  with  $\operatorname{Re} z \geq R$ .

*Remark 2.6.16.* If in addition  $f$  is of disjoint type, then there exists a constant  $K > 1$  such that  $|F'(z)| \geq K$  for all  $z \in U$ .

Following [BJR12, Section 3], we can define disjoint type logarithmic transforms as follows.

**Definition 2.6.17** (Disjoint type for  $F$ ). Let  $F: W \rightarrow H$  be a logarithmic transform of the function  $f \in \mathcal{B}$ . Then we say that  $F$  is of *disjoint type*, if  $\overline{W} \subset H$ .

We only state the part of [BJR12, Lemma 3.1] which is relevant for us.

**Lemma 2.6.18** (Connection between disjoint type functions). *Let  $F: W \rightarrow H$  be a logarithmic transform of the function  $f \in \mathcal{B}$ . Then  $f$  is of disjoint type if and only if  $F$  is of disjoint type.*

## 2.7 Hairs of entire and quasiregular maps

Devaney and Krych [DK84] showed that for  $f(z) = \lambda e^z$ , where  $0 < \lambda < 1/e$ , there exists an attracting fixed point  $\xi \in \mathbb{R}$  such that  $\mathcal{J}(f) = \mathbb{C} \setminus A(\xi)$  and gave a detailed description of the structure of  $\mathcal{J}(f)$ . Here denote by

$$A(\xi) := \{z \in \mathbb{C} : f^n(z) \rightarrow \xi \text{ as } n \rightarrow \infty\}$$

the *basin of attraction* of the fixed point  $\xi$ . We now want to put the description of  $\mathcal{J}(f)$  in a precise form.

**Definition 2.7.1** (Hairs). We say that a subset  $H$  of  $\mathbb{C}$  (or  $\mathbb{R}^d$ ) is a *hair* if there exists a homeomorphism  $\gamma: [0, \infty) \rightarrow H$  such that  $\lim_{t \rightarrow \infty} \gamma(t) = \infty$ . We call  $\gamma(0)$  the *endpoint* of the hair.

We only state the part of the result due to Devaney and Krych which is relevant for us.

**Theorem 2.7.2.** For  $0 < \lambda < 1/e$  the set  $\mathcal{J}(\lambda e^z)$  is an uncountable union of pairwise disjoint hairs.

McMullen [McM87] investigated the Hausdorff dimension of the Julia set of the exponential family.

**Theorem 2.7.3.** Let  $\lambda \in \mathbb{C} \setminus \{0\}$ . Then  $\dim \mathcal{J}(\lambda e^z) = 2$ .

McMullen's result implies that in the situation of Theorem 2.7.2 the union of the hairs has Hausdorff dimension 2. The following result of Karpińska [Ka99] is also known as Karpińska's paradox, see e.g. [SZ03].

**Theorem 2.7.4.** Let  $0 < \lambda < 1/e$  and let  $\mathcal{C}$  be the set of endpoints of the hairs that form  $\mathcal{J}(\lambda e^z)$ . Then  $\dim \mathcal{C} = 2$  and  $\dim(\mathcal{J}(\lambda e^z) \setminus \mathcal{C}) = 1$ .

*Remark 2.7.5.* The existence of hairs is not restricted to the situation considered by Devaney and Krych. Hairs appear for  $\lambda e^z$  for all  $\lambda \in \mathbb{C} \setminus \{0\}$  as shown in [DGH86, SZ03], see Figure 2.4. The function which is shown is conjugated to  $e^{z-2}$  by  $L(z) = z - 2$ . Many authors investigated a generalization for more general classes of functions. In 1986 Devaney and Tangerman [DT86] considered certain transcendental entire maps where  $\text{sing}(f^{-1})$  is finite. Examples for this class of functions are  $\lambda e^z$  and  $\lambda \sin(z) + \mu$ , where  $\lambda \in \mathbb{C} \setminus \{0\}$ . In 2007 Barański [Bar07] proved that for functions of disjoint type and finite order the Julia set is an uncountable union of pairwise disjoint hairs. In 2011 Rückert, Rottenfußer, Rempe and Schleicher generalized this result to finite compositions of functions in class  $\mathcal{B}$  of finite order.

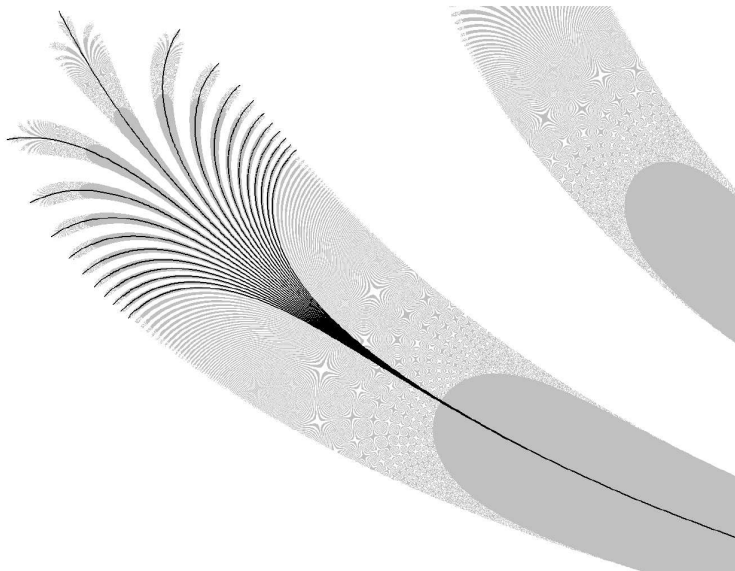


Figure 2.4. Some hairs for the function  $f(z) = e^z - 2$ , see [Re03, page 22].

We want to state a part of this result on hairs for functions of bounded type of finite order given by Rückert, Rottenfußer, Rempe and Schleicher [RRRS11, Theorem 1.2], which is relevant for us.

**Theorem 2.7.6.** Let  $f \in \mathcal{B}$  be of finite order. Then every  $z \in \mathcal{I}(f)$  can be connected to  $\infty$  by a curve  $\gamma$  such that  $f^n|_\gamma \rightarrow \infty$  uniformly as  $n \rightarrow \infty$ .

*Remark 2.7.7.*

1. In the same paper it was shown that the assumption on the order of  $f$  cannot be dropped. In fact, they constructed a function of bounded type of infinite order such that every path-connected component of  $\mathcal{J}(f)$  is bounded and thus this function has no hairs. This answered a question of Eremenko to the negative whether for every  $z \in \mathcal{I}(f)$  there exists an unbounded and connected set  $A \subset \mathbb{C}$  with  $z \in A$  such that  $f^n|_A \rightarrow \infty$  uniformly.
2. It was shown in [BJR12, Proposition 4.2, Corollary 4.4] that the Julia set  $\mathcal{J}(F)$  of a logarithmic transform  $F: W \rightarrow H$  of a disjoint type function  $f$  also consists of uncountable union of hairs which are pairwise disjoint, where

$$\mathcal{J}(F) := \{z \in \overline{W} : F^j(z) \in \overline{W} \text{ for all } j \in \mathbb{N}_0\}.$$

Note that this continuous extension of  $F|_W$  exists by Carathéodory's Theorem (see e.g. [CG93, Theorem 2.1]).

Moreover, we have  $\exp(\mathcal{J}(F)) = \mathcal{J}(f)$ . This yields that, given a parametrization  $\gamma$  of a hair  $H \subset \mathcal{J}(F)$ , the function  $\exp \circ \gamma$  is a parametrization of the hair  $\exp(H)$  in  $\mathcal{J}(f)$ . Thus  $H$  is nowhere differentiable if and only if  $\exp(H)$  is nowhere differentiable.

McMullen [McM87, Theorem 1.1] investigated the two dimensional Lebesgue measure of the Julia set of the sine family and proved the following result.

**Theorem 2.7.8.** The Julia set of  $\lambda \sin(z) + \mu$  with  $\lambda \in \mathbb{C} \setminus \{0\}$  has positive area.

*Remark 2.7.9.* McMullen proved that  $\mathcal{I}(f)$  has positive area. Since the sine family belongs to the class  $\mathcal{B}$  we have  $\mathcal{I}(f) \subset \mathcal{J}(f)$ , which yields Theorem 2.7.8.

For a subset of this family of functions, namely those functions where  $\lambda$  and  $\mu$  are chosen such that the critical values are strictly preperiodic, it is well-known that the Julia set is the whole complex plane. Schleicher [Sch07] used these functions to prove a representation of  $\mathbb{C}$  as a union of hairs, which can be seen as a strong version of Karpínska's paradox.

**Theorem 2.7.10.** There is a representation of  $\mathbb{C}$  as a union of hairs with the following properties:

1. The intersection of two hairs is either empty or consists of the common endpoint,
2. The union of the hairs without the endpoints has Hausdorff dimension 1.

For further information on dynamics of exponential functions we refer to papers by Rempe [Re06] and Schleicher [Sch03].

Viana [Vi88] investigated the differentiability of hairs for exponential maps.

**Theorem 2.7.11.** For all  $\lambda \in \mathbb{C} \setminus \{0\}$  the hairs of  $\lambda e^z$  are  $C^\infty$ -smooth.

In 2011 Kisaka and Shishikura [KS11] generalized this result to the family  $P(z)e^{Q(z)}$ , where  $P$  and  $Q$  are non-zero polynomials.

Now we turn to quasiregular maps. Motivated by Devaney and Krych's result, Bergweiler [Ber10] has shown that hairs exist for several Zorich maps, which can be seen as a higher-dimensional counterpart of the exponential family (see Section 3.1 for more details). Subsequently Bergweiler and Eremenko [BE11] have shown in 2011 that there is a quasiregular counterpart of Schleicher's representation result by introducing a quasiregular sine function.

**Theorem 2.7.12.** There is a representation of  $\mathbb{R}^d$  as a union of hairs with the following properties:

1. The intersection of two hairs is either empty or consists of the common endpoint,
2. The union of the hairs without the endpoints has Hausdorff dimension 1.

A counterpart to McMullen's result concerning the area of the Julia set of the sine family was given by Vogel in 2015 [Vo15]. Here  $\text{Sin}(x)$  denotes a quasiregular sine map.

**Theorem 2.7.13.** The measure of the escaping set of  $\lambda \text{Sin}(x)$  is positive for every  $\lambda > 0$ .



# Chapter 3

## Smooth hairs for Zorich maps

In the previous section we mentioned that hairs exist for several Zorich maps considered by Bergweiler [Ber10]. We want to put this in a precise form. In the first part of this section we are recalling mainly results from [Ber10], formulating them for functions in  $\mathbb{R}^d$  with  $d \geq 3$  instead of  $\mathbb{R}^3$ , however. Subsequent to this part we discuss suitable conditions for Zorich maps to obtain hairs which are  $C^1$ .

### 3.1 Construction of Zorich maps

Zorich introduced in the article [Zo67] a quasiregular self-map of  $\mathbb{R}^d$ , which can be considered as a higher dimensional counterpart of the exponential map. To put this in a precise form, we follow an idea which is given in [IM01, page 119]. While Iwaniec and Martin considered the 3-dimensional case, we replace the dimension 3 by  $d \geq 3$ .

Therefore define for  $d \in \mathbb{N}$  with  $d \geq 3$  the hypercube

$$Q := \{x \in \mathbb{R}^{d-1} : \|x\|_\infty \leq 1\} = [-1, 1]^{d-1}$$

and the upper hemisphere

$$\mathbb{S}_+ := \{x \in \mathbb{R}^d : \|x\|_2 = 1 \text{ and } x_d \geq 0\}.$$

For a bi-Lipschitz map  $h: Q \rightarrow \mathbb{S}_+$  we define

$$F: Q \times \mathbb{R} \rightarrow \mathbb{H}_{\geq 0}, F(x) = e^{x_d} h(x_1, \dots, x_{d-1}).$$

We want to extend  $F$  to the space  $\mathbb{R}^d$ . The square beam  $Q \times \mathbb{R}$  has  $2(d-1)$  faces. For  $j \in \{1, \dots, d-1\}$  and  $\sigma \in \{-1, 1\}$  denote by

$$\kappa_{j,\sigma}: Q \times \mathbb{R} \rightarrow \mathbb{R}^d, \kappa_{j,\sigma}(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) = (x_1, \dots, x_j, 2\sigma - x_j, x_{j+1}, \dots, x_d)$$

the reflection on the hyperplane  $\{y \in \mathbb{R}^d : y_j = \sigma\}$ . We define  $F$  for the adjacent square beams of  $Q \times \mathbb{R}$  as follows: If  $x \in Q \times \mathbb{R}$  and  $x' = \kappa_{j,\sigma}(x)$ , then

$$F(x') = (F_1(x), \dots, F_{d-1}(x), -F_d(x)),$$

i.e.  $F(x')$  is the point  $F(x)$  reflected on the hyperplane  $\{y \in \mathbb{R}^d : y_d = 0\}$ . By continuing this procedure, we get a function  $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$  which we call a *Zorich map*.

*Remark 3.1.1.*

1. One can show that  $F$  is a quasiregular map since  $h$  is bi-Lipschitz. The dilatation can be written in terms of the Lipschitz constant of  $h$ .
2. Due to the construction  $F$  is 4-periodic in each component except the last one.

If  $h$  is differentiable at  $(x_1, \dots, x_{d-1})$ , then  $DF(x_1, \dots, x_{d-1}, 0)$  exists and we have

$$DF(x_1, \dots, x_{d-1}, x_d) = e^{x_d} DF(x_1, \dots, x_{d-1}, 0) \quad (3.1.1)$$

and thus there exist  $\alpha, m, M \in \mathbb{R}$ ,  $\alpha \in (0, 1)$ ,  $m < M$ ,  $M \geq 1$  such that

$$\|DF(x)\| \leq \alpha \quad \text{a.e. for } x_d \leq m \quad (3.1.2)$$

$$l(DF(x)) \geq \frac{1}{\alpha} \quad \text{a.e. for } x_d \geq M. \quad (3.1.3)$$

## 3.2 Zorich maps with an attracting fixed point

Consider now for  $a \in \mathbb{R}$  such that

$$a \geq e^M - m \quad (3.2.1)$$

the map

$$f_a: \mathbb{R}^d \rightarrow \mathbb{R}^d, f_a(x) = F(x) - (0, \dots, 0, a).$$

In our context we call  $f_a$  a Zorich map, too. The following theorem is due to Bergweiler [Ber10].

**Theorem 3.2.1.** Let  $f_a$  be as above with  $a$  as in (3.2.1). Then there exists a unique fixed point  $\xi = (\xi_1, \dots, \xi_d)$  satisfying  $\xi_d \leq m$ , the set

$$J := \{x \in \mathbb{R}^d : f_a^k(x) \rightarrow \xi\}$$

consists of uncountably many pairwise disjoint hairs, and the set  $C$  of endpoints of these hairs has Hausdorff dimension  $d$ , while  $J \setminus C$  has Hausdorff dimension 1.

*Remark 3.2.2.* Theorem 3.2.1 generalizes the theorem on the endpoints of Karpińska 2.7.4. For our purpose we are just interested in the existence of the hairs for the Zorich case.



The main result of this chapter is the following.

**Theorem 3.2.3.** Let  $f_a$  be as above with  $a$  as in (3.2.1) and assume that  $h|_{\text{int}(Q)}$  is  $C^1$  and  $Dh$  is Hölder continuous. Then the hairs of  $f_a$  are  $C^1$ -smooth.

*Remark 3.2.4.* This theorem corresponds to Theorem 2.7.11 concerning the differentiability of hairs. As in Viana's result, we do not obtain differentiability of the hairs in the endpoints since they can spiral around a point. Rempe [Re03, 3.4.2 Theorem, page 31] gave a condition under which we obtain smoothness up to the endpoints for exponential maps. This result will be discussed in Section 3.7. The proof of Theorem 3.2.3 will show that the conclusion of the theorem still holds if the assumptions on  $h$  and  $Dh$  are satisfied on a suitable subset of  $\text{int}(Q)$  (see Section 3.6).

In the following we write for simplicity  $f = (f_1, \dots, f_d)$  instead of  $f_a$ . For  $r \in \mathbb{Z}^{d-1}$  denote by

$$\begin{aligned} P(r) &:= P(r_1, \dots, r_{d-1}) := \{x \in \mathbb{R}^{d-1} : \forall j \in \{1, \dots, d-1\} : |x_j - 2r_j| < 1\} \\ &= \text{int}(Q) + (2r_1, \dots, 2r_{d-1}) \end{aligned}$$

the shifted and open hypercube  $Q$  with centre  $2r$ . We put

$$S := \left\{ r \in \mathbb{Z}^{d-1} : \sum_{j=1}^{d-1} r_j \in 2\mathbb{Z} \right\}.$$

Then

$$F \text{ maps } P(r) \times \mathbb{R} \text{ bijectively onto } \begin{cases} \mathbb{H}_{>0}, & \text{if } r \in S \\ \mathbb{H}_{<0}, & \text{if } r \notin S \end{cases}$$

and thus

$$f \text{ maps } P(r) \times \mathbb{R} \text{ bijectively onto } \begin{cases} \mathbb{H}_{>-a}, & \text{if } r \in S, \\ \mathbb{H}_{<-a}, & \text{if } r \notin S. \end{cases} \quad (3.2.2)$$

**Definition 3.2.5** (Tract). For  $r \in S$  we call the set

$$T(r) := P(r) \times (M, \infty)$$

the *tract* above  $P(r)$ .

Now we want to understand the behaviour of our function  $f$  and collect some important facts about it. Since  $f(P(r) \times \mathbb{R}) = \mathbb{H}_{>-a}$  for  $r \in S$  and

$$f_d(x_1, \dots, x_d) = e^{x_d} h_d(x_1, \dots, x_{d-1}) - a \leq e^{x_d} - e^M + m \leq m < M \quad (3.2.3)$$

for  $x_d \leq M$  and hence  $f(P(r) \times (-\infty, M]) \subset \mathbb{H}_{<M}$ , we have  $f(T(r)) \supset \mathbb{H}_{\geq M}$ . Thus there exists a branch  $\Lambda^r : \mathbb{H}_{\geq M} \rightarrow T(r)$  of the inverse function of  $f$ . Using the notation  $\Lambda := \Lambda^{(0, \dots, 0)}$ , we have

$$\Lambda^{(r_1, \dots, r_{d-1})}(x) = \Lambda(x) + (2r_1, \dots, 2r_{d-1}, 0) \quad (3.2.4)$$

for all  $x \in \mathbb{H}_{\geq M}$  and  $r \in S$ .

In our proofs of the main results we will use the derivative of  $\Lambda^r$ . Together with (3.2.4) we obtain

$$D\Lambda^r(x) = D\Lambda(x)$$

for all  $x \in \mathbb{H}_{\geq M}$  for which the derivative exists. Because  $DF = Df$ , we deduce from (3.1.3) that

$$\|D\Lambda(x)\| \leq \alpha \quad \text{a.e.} \quad (3.2.5)$$

for  $x \in \mathbb{H}_{\geq M}$ . This implies for  $x, y \in \mathbb{H}_{\geq M}$  that

$$\|\Lambda(x) - \Lambda(y)\|_2 \leq \|x - y\|_2 \operatorname{ess\,sup}_{z \in [x, y]} \|D\Lambda(z)\| \leq \alpha \|x - y\|_2. \quad (3.2.6)$$

Since (3.2.3) and thus  $f(\mathbb{H}_{\leq M}) \subset \mathbb{H}_{\leq m}$ , we obtain using (3.1.2)

$$\|f(x) - f(y)\|_2 \leq \|x - y\|_2 \operatorname{ess\,sup}_{z \in [x, y]} \|Df(z)\| \leq \alpha \|x - y\|_2.$$

So Banach's fixed point theorem gives us the existence of a unique fixed point  $\xi \in \mathbb{H}_{\leq m}$  such that  $\lim_{n \rightarrow \infty} f^n(x) = \xi$  for all  $x \in \mathbb{H}_{\leq m}$ . Together with (3.2.2) this yields

$$J \subset \bigcup_{r \in S} T(r).$$

In this part our goal is to find additional estimates for  $D\Lambda$ . The equation (3.1.1) implies that there exist  $c_1, c_2 > 0$  such that

$$c_1 e^{x^d} \leq l(Df(x)) \leq \|Df(x)\| \leq c_2 e^{x^d} \quad \text{a.e.}$$

Thus there are constants  $c_3 > 0$  and  $c_4 \geq 1$  such that for  $x \in \mathbb{H}_{\geq M}$ .

$$\begin{aligned} l(D\Lambda(x)) &\geq \frac{c_3}{\|x\|_2} \quad \text{a.e.} \\ \|D\Lambda(x)\| &\leq \frac{c_4}{\|x\|_2} \quad \text{a.e.} \end{aligned} \quad (3.2.7)$$

Moreover, there exist  $c_5 > 0$  and  $c_6 \geq 1$  such that

$$\frac{c_5}{\|x\|_2^d} \leq J_\Lambda(x) \leq \frac{c_6}{\|x\|_2^d} \quad \text{a.e.}$$

Let us fix  $x, y \in \mathbb{H}_{\geq M}$ . Then we can connect  $x$  and  $y$  by a path  $\gamma$  in

$$\mathbb{H}_{\geq M} \cap \left\{ z \in \mathbb{R}^d : \|z\|_2 \geq \min\{\|x\|_2, \|y\|_2\} \right\}$$

with  $\text{length}(\gamma) \leq \pi \|x - y\|_2$ . Together with (3.2.7) this yields

$$\|\Lambda(x) - \Lambda(y)\|_2 \leq \pi \|x - y\|_2 \operatorname{ess\,sup}_{z \in \gamma} \|D\Lambda(z)\| \leq c_4 \pi \frac{\|x - y\|_2}{\min\{\|x\|_2, \|y\|_2\}}. \quad (3.2.8)$$

Since  $h$  is bijective, there exists a unique point  $(v_1, \dots, v_{d-1}) \in Q$  which is mapped under  $h$  to the north pole of the sphere, i.e.

$$h(v_1, \dots, v_{d-1}) = (0, \dots, 0, 1). \quad (3.2.9)$$

This implies  $F(v_1, \dots, v_{d-1}, x_d) = (0, \dots, 0, e^{x_d})$  and for  $r \in S$  and  $t \geq M$  this yields

$$\Lambda^r(0, \dots, 0, t) = (v_1 + 2r_1, \dots, v_{d-1} + 2r_{d-1}, \log(t + a)).$$

Moreover, the equation

$$\|F(x)\|_2 = e^{x_d}$$

yields

$$\|f(\Lambda(x)) + (0, \dots, 0, a)\|_2 = e^{\Lambda_d(x)}$$

and thus

$$\Lambda_d(x) = \log(\|x + (0, \dots, 0, a)\|_2). \quad (3.2.10)$$

### 3.3 Existence of hairs for these Zorich maps

To discuss the existence of hairs, it is useful, following Schleicher and Zimmer [SZ03], to define a reference function

$$E: [0, \infty) \rightarrow [0, \infty), \quad E(t) = e^t - 1.$$

This function has the following properties:

1. We have  $E(0) = 0$  and  $\lim_{k \rightarrow \infty} E^k(t) = \infty$  for  $t > 0$ .
2. For  $b > 1$  we have

$$E^k(t) < \log(E^{k+1}(t) + b) \leq E^k(t) + \log(b). \quad (3.3.1)$$

3. If  $0 < t' < t'' < \infty$ , then

$$\lim_{k \rightarrow \infty} (E^k(t'') - E^k(t')) = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{E^k(t'')}{E^k(t')} = \infty. \quad (3.3.2)$$

We need an analogous definition as in the case of exponential maps:

**Definition 3.3.1** (External address/Symbolic space). For each  $x \in J$  we call the sequence

$$\underline{s}(x) := s_0 s_1 s_2 \cdots = (s_k)_{k \geq 0} \in S^{\mathbb{N}_0}$$

such that  $f^k(x) \in T(s_k)$  for all  $k \geq 0$  the *external address* of  $x$ . Moreover, we call  $\Sigma := S^{\mathbb{N}_0}$  the *symbolic space*.

**Definition 3.3.2** (Admissibility/exponentially boundedness). We say that  $\underline{s} \in \Sigma$  is *admissible* (or *exponentially bounded*), if there exists a  $t > 0$  such that

$$\limsup_{k \rightarrow \infty} \frac{\|s_k\|_2}{E^k(t)} < \infty.$$

Moreover, denote by  $\Sigma' \subset \Sigma$  the set of all admissible points.

With these definitions we obtain the following lemmas, see [Ber10, Propositions 3.1 and 3.2].

**Lemma 3.3.3.** *Let  $x \in J$ . Then  $\underline{s}(x)$  is admissible.*

**Lemma 3.3.4.** *Let  $\underline{s} \in \Sigma'$ . Then  $\{x \in J : \underline{s}(x) = \underline{s}\}$  is a hair.*

From Lemma 3.3.3 and Lemma 3.3.4 it follows that  $J$  is the union of hairs as stated in Theorem 3.2.1.

Fixing  $\underline{s} \in \Sigma'$ , denote

$$t_{\underline{s}} := \inf \left\{ t > 0 : \limsup_{k \rightarrow \infty} \frac{\|s_k\|_2}{E^k(t)} < \infty \right\}.$$

Choosing  $t_k \in [0, \infty)$  such that  $2\|s_k\|_2 = E^k(t_k)$  and putting  $\tau_k := \sup_{j \geq k} t_j$ , we obtain

$$t_{\underline{s}} = \limsup_{k \rightarrow \infty} t_k = \lim_{k \rightarrow \infty} \tau_k.$$

Using the abbreviation

$$L_k := \Lambda^{s_k} = \Lambda^{(s_{k,1}, \dots, s_{k,d-1})}$$

we define for  $k \in \mathbb{N}_0$

$$g_k : [0, \infty) \rightarrow \mathbb{H}_{\geq M}, g_k(t) = (L_0 \circ L_1 \circ \dots \circ L_k)(0, \dots, 0, E^{k+1}(t) + M). \quad (3.3.3)$$

The three main lemmas in the proof of Lemma 3.3.4 are the following [Ber10, Lemmas 3.1, 3.2 and 3.6].

**Lemma 3.3.5.** *The sequence  $(g_k)_{k \geq 0}$  converges locally uniformly on  $(t_{\underline{s}}, \infty)$ .*

**Lemma 3.3.6.** *The sequence  $(g_k)_{k \geq 0}$  has a subsequence which converges uniformly on  $[t_{\underline{s}}, \infty)$  and thus  $g$  extends to a continuous map  $g : [t_{\underline{s}}, \infty) \rightarrow \mathbb{H}_{\geq M}$ .*

**Lemma 3.3.7.** *Let  $x \in J$ . Then there exists  $t \geq t_{\underline{s}(x)}$  with  $g_{\underline{s}(x)}(t) = x$ .*

### 3.4 $C^1$ estimates for the hairs

To have a chance for a  $C^1$  condition for our hairs, we need enough regularity of the bi-Lipschitz mapping  $h$ . In this section we want to specify this condition and want to give precise  $C^1$  estimates for the hairs.

Therefore we will introduce some new notations. For  $n, m \in \mathbb{N}$  with  $m < n$  and functions  $f_1, \dots, f_n: \mathbb{R}^d \rightarrow \mathbb{R}^d$  denote

$$\bigcirc_{j=m}^n f_j := f_m \circ \dots \circ f_n,$$

where we use the convention

$$\bigcirc_{j=m+1}^m f_j := id.$$

Then  $g_k$  defined by (3.3.3) takes the form

$$g_k(t) = \left( \bigcirc_{j=0}^k L_j \right) (0, \dots, 0, E^{k+1}(t) + M)$$

for all  $k \in \mathbb{N}_0$  and  $t \in [0, \infty)$ . If  $h$  is locally  $C^1$ , the derivative  $g'_k$  then reads as

$$\begin{aligned} g'_k(t) &= \frac{d}{dt} \left( (L_0 \circ \dots \circ L_k) (0, \dots, 0, E^{k+1}(t) + M) \right) \\ &= \prod_{l=1}^{k+1} \left( D\Lambda \left( \left( \bigcirc_{j=l}^k L_j \right) (0, \dots, 0, E^{k+1}(t) + M) \right) \right) \cdot (0, \dots, 0, (E^{k+1})'(t))^T \end{aligned} \quad (3.4.1)$$

for  $k \in \mathbb{N}$ .

To prove Theorem 3.2.3 it is enough to show the following result.

**Theorem 3.** Let  $f$  be as before and assume that  $h|_{\text{int}(Q)}$  is  $C^1$  and  $Dh$  is Hölder continuous. Then for all  $\underline{s} \in \Sigma'$  the sequence  $(g_k)_{k \geq 0}$  consists of  $C^1$ -curves which converge (in  $C^1$ -sense) locally uniformly on  $(t_{\underline{s}}, \infty)$ .

For the proof of the theorem we will compare  $g'_k$  and  $g'_{k-1}$  in a suitable way. Therefore we define as a preparation for all  $k \in \mathbb{N}_0$  the auxiliary function

$$\phi_k: [0, \infty) \rightarrow \mathbb{H}_{\geq M}, \quad \phi_k(t) = L_k(0, \dots, 0, E^{k+1}(t) + M). \quad (3.4.2)$$

Then we obtain for all  $t \geq 0$

$$\phi_k(t) = (v_1 + 2s_{k,1}, \dots, v_{d-1} + 2s_{k,d-1}, \log(E^{k+1}(t) + M + a)).$$

Thus

$$\phi'_k(t) = \left( 0, \dots, 0, \frac{1}{E^{k+1}(t) + M + a} E'(E^k(t)) \cdot (E^k)'(t) \right) \quad (3.4.3)$$

and

$$\phi'_k(t) = D\Lambda(0, \dots, 0, E^{k+1}(t) + M) \cdot (0, \dots, 0, (E^{k+1})'(t))^T. \quad (3.4.4)$$

Noticing that

$$a \geq e^M - m \geq 1 + M - m > 1 \quad (3.4.5)$$

by (3.2.1), we obtain

$$\|\phi'_k(t)\|_2 = \frac{E^{k+1}(t) + 1}{E^{k+1}(t) + M + a} \cdot (E^k)'(t) \leq (E^k)'(t). \quad (3.4.6)$$

**Lemma 3.4.1.** *For all  $k \in \mathbb{N}_{\geq 2}$  and  $l \in \{1, \dots, k-1\}$  and for all  $t \geq 0$  we have*

$$\left\| \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right\|_2 \geq \left( \bigcirc_{j=l}^{k-1} L_j \right)_d (\phi_k(t)) \geq E^l(t). \quad (3.4.7)$$

*Proof.* We prove this Lemma by induction on  $l$ . From (3.2.10), (3.3.1) and (3.4.5) we deduce that

$$\begin{aligned} \|L_{k-1}(\phi_k(t))\|_2 &\geq |L_{k-1,d}(\phi_k(t))| \\ &= |\Lambda_d(\phi_k(t))| \\ &= \log(\|\phi_k(t) + (0, \dots, 0, a)\|_2) \\ &\geq \log(\phi_{k,d}(t) + a) \\ &= \log(\log(E^{k+1}(t) + M + a) + a) \\ &\geq \log(E^k(t) + a) \\ &\geq E^{k-1}(t). \end{aligned}$$

This shows that the conclusion holds for  $l = k-1$ . Take now  $l < k-1$  such that (3.4.7) is true with  $l$  replaced by  $l+1$ . Then we obtain

$$\begin{aligned} \left\| \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right\|_2 &= \left\| L_l \circ \left( \bigcirc_{j=l+1}^{k-1} L_j \right) (\phi_k(t)) \right\|_2 \\ &\geq \left| L_{l,d} \left( \left( \bigcirc_{j=l+1}^{k-1} L_j \right) (\phi_k(t)) \right) \right| \\ &= \log \left( \left\| \left( \bigcirc_{j=l+1}^{k-1} L_j \right) (\phi_k(t)) + (0, \dots, 0, a) \right\|_2 \right) \\ &\geq \log \left( \left( \bigcirc_{j=l+1}^{k-1} L_j \right)_d (\phi_k(t)) + a \right) \\ &\geq \log(E^{l+1}(t) + a) \\ &\geq E^l(t) \end{aligned}$$

which proves the result. □

*Remark 3.4.2.* The argument shows that the conclusion of Lemma 3.4.1 also holds if  $\phi_k(t)$  is replaced by  $(0, \dots, 0, E^k(t) + M)$ .

**Lemma 3.4.3.** *For all  $k \in \mathbb{N}$  and  $t \in (0, \infty)$  we have*

$$\frac{(E^k)'(t)}{\prod_{j=1}^k E^j(t)} = \prod_{j=1}^k \left(1 + \frac{1}{E^j(t)}\right).$$

Moreover, for all  $\varepsilon > 0$  and  $t \in [\varepsilon, \infty)$  the product

$$\prod_{j=1}^{\infty} \left(1 + \frac{1}{E^j(t)}\right)$$

converges uniformly on  $[\varepsilon, \infty)$ .

*Proof.* Since

$$(E^k)'(t) = \prod_{j=1}^k (E^j(t) + 1),$$

the first part follows immediately.

If  $t \in [\varepsilon, \infty)$ , then by Lemma 2.2.4 it suffices to show that

$$\sum_{j=1}^{\infty} \frac{1}{E^j(t)}$$

converges. Since  $\lim_{j \rightarrow \infty} E^j(t) = \infty$  for all  $t > 0$  and  $E(t) > 2t$  for large  $t$  we obtain by induction  $E^k(t) > 2^k t$  for all  $k \in \mathbb{N}$  and large  $t$ . The conclusion follows.  $\square$

Since the operator norm of  $DF(x)$  is comparable to the maximum of all entries of this matrix, there exists a constant  $C > 0$  such that

$$\|DF(x)\| \leq C \max_{\substack{1 \leq j \leq d \\ 1 \leq k \leq d}} |DF_{jk}(x)|. \quad (3.4.8)$$

In the following let  $\beta \in (0, 1]$  and let  $Dh$  be  $\beta$ -Hölder continuous, i.e. there is a constant  $H_\beta > 0$  such that

$$\|Dh(x) - Dh(y)\| \leq H_\beta \|x - y\|_2^\beta$$

for all  $x, y \in Q$ . Moreover, denote by  $L_h$  the Lipschitz constant of  $h$ .

We want to use this to prove the following Lemma:

**Lemma 3.4.4.** *Let  $h$  be as in Theorem 3. Then there is a constant  $c_7 > 0$  such that for all  $x, y \in \mathbb{H}_{\geq M}$*

$$\|DF(\Lambda(x)) - DF(\Lambda(y))\| \leq c_7 \min \left\{ \|x\|_2^{1-\beta}, \|y\|_2^{1-\beta} \right\} \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}.$$

*Proof.* First of all we have for  $x \in \mathbb{H}_{\geq M}$ ,  $x = (\tilde{x}, x_d)$

$$DF(x_1, \dots, x_d) = e^{x_d} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1} h_1(\tilde{x}) & \dots & \frac{\partial}{\partial x_{d-1}} h_1(\tilde{x}) & h_1(\tilde{x}) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial}{\partial x_1} h_{d-1}(\tilde{x}) & \dots & \frac{\partial}{\partial x_{d-1}} h_{d-1}(\tilde{x}) & h_{d-1}(\tilde{x}) \\ \frac{\partial}{\partial x_1} h_d(\tilde{x}) & \dots & \frac{\partial}{\partial x_{d-1}} h_d(\tilde{x}) & h_d(\tilde{x}) \end{pmatrix}.$$

We obtain for  $x, y \in \mathbb{H}_{\geq M}$  using (3.4.8)

$$\begin{aligned} \|DF(x) - DF(y)\| &\leq C \max_{\substack{1 \leq j \leq d \\ 1 \leq k \leq d}} |DF_{jk}(x) - DF_{jk}(y)| \\ &\leq C \max_{\substack{1 \leq j \leq d \\ 1 \leq k \leq d}} |e^{x_d} DF_{jk}(\tilde{x}, 0) - e^{y_d} DF_{jk}(\tilde{y}, 0)|. \end{aligned}$$

Because  $|DF_{jk}(x) - DF_{jk}(y)|$  is symmetric in  $x$  and  $y$ , we assume without loss of generality that  $y_d \leq x_d$ . Then we obtain that

$$\begin{aligned} &|e^{x_d} DF_{jk}(\tilde{x}, 0) - e^{y_d} DF_{jk}(\tilde{y}, 0)| \\ &\leq e^{y_d} |DF_{jk}(\tilde{x}, 0) - DF_{jk}(\tilde{y}, 0)| + (e^{x_d} - e^{y_d}) \cdot \max\{|DF_{jk}(\tilde{x}, 0)|, |DF_{jk}(\tilde{y}, 0)|\}. \end{aligned}$$

Since  $h$  is as in Theorem 3 and for all  $z \in \mathbb{R}^d$  and  $j, k \in \{1, \dots, d\}$

$$DF_{jk}(\tilde{z}, 0) = \begin{cases} \frac{\partial}{\partial x_k} h_j(\tilde{z}), & \text{if } k \leq d-1, \\ h_j(\tilde{z}), & \text{if } k = d, \end{cases}$$

we have with  $\tilde{C} = \max\{L_h, H_\beta\} \geq 1$ , noting that  $\left| \frac{\partial}{\partial x_k} h_j(\tilde{z}) \right| \leq L_h$ ,

$$\begin{aligned} \|DF(x) - DF(y)\| &\leq C \left( \tilde{C} e^{y_d} \cdot \max \left\{ \|\tilde{x} - \tilde{y}\|_2, \|\tilde{x} - \tilde{y}\|_2^\beta \right\} + L_h \cdot (e^{x_d} - e^{y_d}) \right) \\ &\leq C \tilde{C} \left( e^{y_d} \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\} + (e^{x_d} - e^{y_d}) \right). \end{aligned}$$

Using the fact that  $\min\{\|x\|_2, \|y\|_2\} \geq M \geq 1$  and (3.2.8) we obtain

$$\begin{aligned} \tilde{M} &:= \max \left\{ \|\Lambda(x) - \Lambda(y)\|_2, \|\Lambda(x) - \Lambda(y)\|_2^\beta \right\} \\ &\leq \max \left\{ c_4 \pi \frac{\|x - y\|_2}{\min\{\|x\|_2, \|y\|_2\}}, c_4^\beta \pi^\beta \frac{\|x - y\|_2^\beta}{\min\{\|x\|_2^\beta, \|y\|_2^\beta\}} \right\} \\ &\leq c_4 \pi \frac{\max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}}{\min \left\{ \|x\|_2^\beta, \|y\|_2^\beta \right\}}. \end{aligned}$$



This yields together with (3.2.8), (3.2.10),  $\|x + (0, \dots, 0, a)\|_2 \leq \|x\|_2 + a$  and replacing  $x$  and  $y$  by  $\Lambda(x)$  and  $\Lambda(y)$

$$\begin{aligned}
& \|DF(\Lambda(x)) - DF(\Lambda(y))\| \\
&= C\tilde{C} \left( \exp(\min\{\Lambda_d(x), \Lambda_d(y)\}) \cdot \tilde{M} + |\exp(\Lambda_d(x)) - \exp(\Lambda_d(y))| \right) \\
&\leq C\tilde{C} \left( (\min\{\|x\|_2, \|y\|_2\} + a) \cdot \tilde{M} + \|x - y\|_2 \right) \\
&\leq C\tilde{C} \left( c_4\pi \frac{\min\{\|x\|_2, \|y\|_2\} + a}{\min\{\|x\|_2^\beta, \|y\|_2^\beta\}} + 1 \right) \cdot \max\{\|x - y\|_2, \|x - y\|_2^\beta\} \\
&\leq C\tilde{C}c_4\pi \left( \frac{\min\{\|x\|_2, \|y\|_2\} + a}{\min\{\|x\|_2^\beta, \|y\|_2^\beta\}} + 1 \right) \cdot \max\{\|x - y\|_2, \|x - y\|_2^\beta\}.
\end{aligned}$$

Since  $\min\{\|x\|_2, \|y\|_2\} \geq \min\{\|x\|_2^\beta, \|y\|_2^\beta\} \geq 1$  and

$$\begin{aligned}
\frac{\min\{\|x\|_2, \|y\|_2\} + a}{\min\{\|x\|_2^\beta, \|y\|_2^\beta\}} + 1 &= \frac{\min\{\|x\|_2, \|y\|_2\} + a + \min\{\|x\|_2^\beta, \|y\|_2^\beta\}}{\min\{\|x\|_2^\beta, \|y\|_2^\beta\}} \\
&\leq (2 + a) \cdot \min\{\|x\|_2^{1-\beta}, \|y\|_2^{1-\beta}\},
\end{aligned}$$

there exists  $c_7 > 0$  such that we have for all  $x, y \in \mathbb{H}_{\geq M}$

$$\begin{aligned}
& \|DF(\Lambda(x)) - DF(\Lambda(y))\| \\
&\leq c_7 \min\{\|x\|_2^{1-\beta}, \|y\|_2^{1-\beta}\} \cdot \max\{\|x - y\|_2, \|x - y\|_2^\beta\},
\end{aligned} \tag{3.4.9}$$

which proves the Lemma.  $\square$

**Lemma 3.4.5.** *For all  $k \in \mathbb{N}$  and  $t \in (0, \infty)$  we have*

$$\left\| \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 \leq (I_{1,k}(t) + I_{2,k}(t)) \cdot (E^k)'(t)$$

with

$$I_{1,k}(t) := \frac{(M + a)c_4}{E^{k+1}(t) \cdot E^k(t)} \tag{3.4.10}$$

and

$$I_{2,k}(t) := c_8 \cdot \frac{2\|s_k\|_2 + 1}{(E^k(t))^{1+\beta}}, \tag{3.4.11}$$

where  $c_8 := c_4^2 c_7 (d + \log(M + a) + M)$ .

*Proof.* By the triangle inequality we have

$$\begin{aligned}
& \left\| \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 \\
&= \left\| D\Lambda(\phi_k(t)) \cdot \phi'_k(t) - D\Lambda(0, \dots, 0, E^k(t) + M) \cdot (0, \dots, 0, (E^k)'(t))^T \right\|_2 \\
&= \left\| D\Lambda(\phi_k(t)) \cdot (\phi'_k(t) - (0, \dots, 0, (E^k)'(t))^T) \right. \\
&\quad \left. + (D\Lambda(\phi_k(t)) - D\Lambda(0, \dots, 0, E^k(t) + M)) \cdot (0, \dots, 0, (E^k)'(t))^T \right\|_2 \\
&\leq \|D\Lambda(\phi_k(t))\| \cdot \|\phi'_k(t) - (0, \dots, 0, (E^k)'(t))^T\|_2 \\
&\quad + \|D\Lambda(\phi_k(t)) - D\Lambda(0, \dots, 0, E^k(t) + M)\| \cdot \|(0, \dots, 0, (E^k)'(t))^T\|_2
\end{aligned} \tag{3.4.12}$$

We estimate the first part on the right hand side. Since, by (3.4.3)

$$\begin{aligned}
\|\phi'_k(t) - (0, \dots, 0, (E^k)'(t))^T\|_2 &= \left\| \left( 0, \dots, 0, \left( \frac{E'(E^k(t))}{E^{k+1}(t) + M + a} - 1 \right) \cdot (E^k)'(t) \right) \right\|_2 \\
&= \left( 1 - \frac{E^{k+1}(t) + 1}{E^{k+1}(t) + M + a} \right) \cdot (E^k)'(t) \\
&= \frac{M + a - 1}{E^{k+1}(t) + M + a} \cdot (E^k)'(t) \\
&\leq \frac{M + a}{E^{k+1}(t)} \cdot (E^k)'(t),
\end{aligned}$$

and, by (3.2.7) and (3.3.1)

$$\begin{aligned}
\|D\Lambda(\phi_k(t))\| &\leq \frac{c_4}{\|\phi_k(t)\|_2} \\
&\leq \frac{c_4}{\phi_{k,d}(t)} \\
&= \frac{c_4}{\log(E^{k+1}(t) + M + a)} \\
&\leq \frac{c_4}{E^k(t)},
\end{aligned}$$

we have

$$\|D\Lambda(\phi_k(t))\| \cdot \|\phi'_k(t) - (0, \dots, 0, (E^k)'(t))^T\|_2 \leq \frac{(M + a)c_4}{E^{k+1}(t) \cdot E^k(t)} \cdot (E^k)'(t). \tag{3.4.13}$$

Now we estimate the second part on the right hand side. Since the estimates in (3.4.9) are symmetric in  $x$  and  $y$ , we may assume without loss of generality that  $\|x\|_2 \leq \|y\|_2$ .

Using the fact that for all  $A, B \in GL(d, \mathbb{R})$

$$A^{-1} - B^{-1} = B^{-1} \cdot (B - A) \cdot A^{-1}$$

and for all  $x \in \mathbb{H}_{\geq M}$

$$D\Lambda(x) \cdot DF(\Lambda(x)) = I,$$

we obtain from Lemma 3.4.4 for all  $x, y \in \mathbb{H}_{\geq M}$  with  $\|x\|_2 \leq \|y\|_2$

$$\begin{aligned}
& \|D\Lambda(x) - D\Lambda(y)\| \\
&= \|DF(\Lambda(x))^{-1} - DF(\Lambda(y))^{-1}\| \\
&= \|DF(\Lambda(y))^{-1} \cdot (DF(\Lambda(y)) - DF(\Lambda(x))) \cdot DF(\Lambda(x))^{-1}\| \\
&\leq \|DF(\Lambda(y))^{-1}\| \cdot \|DF(\Lambda(y)) - DF(\Lambda(x))\| \cdot \|DF(\Lambda(x))^{-1}\| \\
&\leq c_7 \min \left\{ \|x\|_2^{1-\beta}, \|y\|_2^{1-\beta} \right\} \cdot \|D\Lambda(x)\| \cdot \|D\Lambda(y)\| \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\} \\
&= c_7 \|x\|_2^{1-\beta} \cdot \|D\Lambda(x)\| \cdot \|D\Lambda(y)\| \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}
\end{aligned}$$

With inequality (3.2.7) this yields for all  $x, y \in \mathbb{H}_{\geq M}$  with  $\|x\|_2 \leq \|y\|_2$

$$\begin{aligned}
\|D\Lambda(x) - D\Lambda(y)\| &\leq c_4^2 c_7 \|x\|_2^{1-\beta} \cdot \frac{\max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}}{\|x\|_2 \cdot \|y\|_2} \\
&= c_4^2 c_7 \cdot \frac{\max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}}{\|x\|_2^\beta \cdot \|y\|_2}.
\end{aligned} \tag{3.4.14}$$

Notice that we obtain with  $M \geq 1$  and (3.3.1)

$$\|\phi_k(t)\|_2 \geq |\phi_{k,d}(t)| = \log(E^{k+1}(t) + M + a) \geq E^k(t)$$

for all  $k \in \mathbb{N}$  and thus

$$\frac{1}{\|\phi_k(t)\|_2^\beta \cdot (E^k(t) + M)} \leq \frac{1}{E^k(t)^{1+\beta}}.$$

Recall that again by (3.3.1) we have  $\log(E^{k+1}(t) + M + a) \leq E^k(t) + \log(M + a)$  for all  $k \in \mathbb{N}$ . With

$$c_8 := c_4^2 c_7 (d + \log(M + a) + M)$$

and by putting  $x := \phi_k(t) = (v_1 + 2s_{k,1}, \dots, v_{d-1} + 2s_{k,d-1}, \log(E^{k+1}(t) + M + a))$  and  $y := (0, \dots, 0, E^k(t) + M)$  we obtain

$$\begin{aligned}
& \|D\Lambda(\phi_k(t)) - D\Lambda(0, \dots, 0, E^k(t) + M)\| \\
&\leq c_4^2 c_7 \cdot \frac{\max \left\{ \|(v + 2s_k, \log(M + a) + M)\|_2, \|(v + 2s_k, \log(M + a) + M)\|_2^\beta \right\}}{E^k(t)^{1+\beta}} \\
&\leq c_4^2 c_7 \cdot \frac{\max \left\{ d + 2\|s_k\|_2 + \log(M + a) + M, (d + 2\|s_k\|_2 + \log(M + a) + M)^\beta \right\}}{E^k(t)^{1+\beta}} \\
&\leq c_4^2 c_7 \cdot \frac{d + 2\|s_k\|_2 + \log(M + a) + M}{E^k(t)^{1+\beta}} \\
&\leq c_8 \cdot \frac{2\|s_k\|_2 + 1}{E^k(t)^{1+\beta}}
\end{aligned}$$

for all  $k \in \mathbb{N}$ . This yields

$$\begin{aligned} & \left\| D\Lambda(\phi_k(t)) - D\Lambda(0, \dots, 0, E^k(t) + M) \right\| \cdot \left\| (0, \dots, 0, (E^k)'(t)) \right\|_2 \\ & \leq c_8 \cdot \frac{2 \|s_k\|_2 + 1}{E^k(t)^{1+\beta}} \cdot (E^k)'(t) \\ & = I_{2,k}(t) \cdot (E^k)'(t). \end{aligned} \tag{3.4.15}$$

Finally we obtain the conclusion from (3.4.12), (3.4.13) and (3.4.15).  $\square$

From (3.4.1) we know that

$$\begin{aligned} g'_{k-1}(t) &= \prod_{l=1}^k D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (0, \dots, 0, E^k(t) + M) \right) \cdot (0, \dots, 0, (E^k)'(t))^T \\ &= \prod_{l=1}^{k-1} D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (0, \dots, 0, E^k(t) + M) \right) \cdot \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \end{aligned}$$

for all  $k \in \mathbb{N}$ . Moreover, we obtain with (3.4.2) and (3.4.4)

$$\begin{aligned} g'_k(t) &= \prod_{l=1}^{k+1} \left( D\Lambda \left( \left( \bigcirc_{j=l}^k L_j \right) (0, \dots, 0, E^{k+1}(t) + M) \right) \right) \cdot (0, \dots, 0, (E^{k+1})'(t))^T \\ &= \prod_{l=1}^{k-1} \left( D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right) \right) \cdot D\Lambda(L_k(0, \dots, 0, E^{k+1}(t) + M)) \\ &\quad \cdot D\Lambda(0, \dots, 0, E^{k+1}(t) + M) \cdot (0, \dots, 0, (E^{k+1})'(t))^T \\ &= \prod_{l=1}^{k-1} D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right) \cdot D\Lambda(\phi_k(t)) \cdot \phi'_k(t) \\ &= \prod_{l=1}^{k-1} D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right) \cdot \frac{d}{dt}(L_{k-1} \circ \phi_k)(t). \end{aligned}$$

Putting

$$\begin{aligned} A_k(t) &:= \prod_{l=1}^{k-1} D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right) \\ B_k(t) &:= \prod_{l=1}^{k-1} D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (0, \dots, 0, E^k(t) + M) \right) \end{aligned}$$

for all  $k \in \mathbb{N}$  and  $t \in [0, \infty)$ , we obtain

$$\begin{aligned} & g'_k(t) - g'_{k-1}(t) \\ &= A_k(t) \cdot \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - B_k(t) \cdot \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)). \end{aligned} \tag{3.4.16}$$

Now we need suitable estimates for  $A_k(t)$  and  $B_k(t)$ .

**Lemma 3.4.6.** *For all  $k \in \mathbb{N}_{\geq 2}$  and  $t \in [0, \infty)$  we have*

$$\begin{aligned} & \|A_k(t) - B_k(t)\| \\ & \leq \sum_{r=1}^{k-1} \left( \prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \|X_{r,k-1}(t) - Y_{r,k-1}(t)\| \cdot \prod_{s=r+1}^{k-1} \|Y_{s,k-1}(t)\| \right), \end{aligned} \quad (3.4.17)$$

where for  $l \in \{0, \dots, k-1\}$

$$\begin{aligned} X_{l,k-1}(t) & := D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right) \\ Y_{l,k-1}(t) & := D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (0, \dots, 0, E^k(t) + M) \right). \end{aligned}$$

*Proof.* By the triangle inequality and the submultiplicity of the operator norm we obtain

$$\begin{aligned} \|A_k(t) - B_k(t)\| & = \left\| \prod_{l=1}^{k-1} X_{l,k-1}(t) - \prod_{l=1}^{k-1} Y_{l,k-1}(t) \right\| \\ & = \left\| \prod_{l=1}^{k-1} X_{l,k-1}(t) - \left( \prod_{l=1}^{k-2} X_{l,k-1}(t) \right) \cdot Y_{k-1,k-1}(t) \right. \\ & \quad \left. + \left( \prod_{l=1}^{k-2} X_{l,k-1}(t) \right) \cdot Y_{k-1,k-1}(t) - \prod_{l=1}^{k-1} Y_{l,k-1}(t) \right\| \\ & \leq \prod_{l=1}^{k-2} \|X_{l,k-1}(t)\| \cdot \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| \\ & \quad + \left\| \prod_{l=1}^{k-2} X_{l,k-1}(t) - \prod_{l=1}^{k-2} Y_{l,k-1}(t) \right\| \cdot \|Y_{k-1,k-1}(t)\|. \end{aligned}$$

Assume now that we have  $j < k-1$  such that

$$\begin{aligned} \|A_k(t) - B_k(t)\| & \leq \sum_{r=j+1}^{k-1} \left( \prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \|X_{r,k-1}(t) - Y_{r,k-1}(t)\| \cdot \prod_{s=r+1}^{k-1} \|Y_{s,k-1}(t)\| \right) \\ & \quad + \left\| \prod_{l=1}^j X_{l,k-1}(t) - \prod_{l=1}^j Y_{l,k-1}(t) \right\| \cdot \prod_{s=j+1}^{k-1} \|Y_{s,k-1}(t)\|. \end{aligned}$$

Then

$$\begin{aligned}
& \left\| \prod_{l=1}^j X_{l,k-1}(t) - \prod_{l=1}^j Y_{l,k-1}(t) \right\| \cdot \prod_{s=j+1}^{k-1} \|Y_{s,k-1}(t)\| \\
& \leq \left( \prod_{l=1}^{j-1} \|X_{l,k-1}(t)\| \cdot \|X_{j,k-1}(t) - Y_{j,k-1}(t)\| \right. \\
& \quad \left. + \left\| \prod_{l=1}^{j-1} X_{l,k-1}(t) - \prod_{l=1}^{j-1} Y_{l,k-1}(t) \right\| \cdot \|Y_{j,k-1}(t)\| \right) \cdot \prod_{s=j+1}^{k-1} \|Y_{s,k-1}(t)\| \\
& = \sum_{r=j}^j \left( \prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \|X_{r,k-1}(t) - Y_{r,k-1}(t)\| \cdot \prod_{s=r+1}^{k-1} \|Y_{s,k-1}(t)\| \right) \\
& \quad + \left\| \prod_{l=1}^{j-1} X_{l,k-1}(t) - \prod_{l=1}^{j-1} Y_{l,k-1}(t) \right\| \cdot \prod_{s=j}^{k-1} \|Y_{s,k-1}(t)\|
\end{aligned}$$

and thus

$$\begin{aligned}
\|A_k(t) - B_k(t)\| & \leq \sum_{r=j}^{k-1} \left( \prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \|X_{r,k-1}(t) - Y_{r,k-1}(t)\| \cdot \prod_{s=r+1}^{k-1} \|Y_{s,k-1}(t)\| \right) \\
& \quad + \left\| \prod_{l=1}^{j-1} X_{l,k-1}(t) - \prod_{l=1}^{j-1} Y_{l,k-1}(t) \right\| \cdot \prod_{s=j}^{k-1} \|Y_{s,k-1}(t)\|.
\end{aligned}$$

This proves the Lemma. □

### 3.5 Proof of Theorem 3

*Proof of Theorem 3.* Let  $\varepsilon > 0$ . We will show that there is a constant  $C_1 > 0$  such that

$$\|g'_k(t) - g'_{k-1}(t)\|_2 \leq C_1 \alpha^{k-1}$$

for  $t \in [t_{\underline{s}} + \varepsilon, \infty)$  and large  $k \in \mathbb{N}$ , where  $\alpha$  is the constant given in (3.1.2). This implies that  $(g'_k)_k$  converges locally uniformly on  $(t_{\underline{s}}, \infty)$  and thus the hairs of  $f$  are  $C^1$ -smooth, which yields Theorem 3.

First of all we deduce from (3.4.16) that

$$\begin{aligned}
& \|g'_k(t) - g'_{k-1}(t)\|_2 \\
&= \left\| A_k(t) \cdot \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - B_k(t) \cdot \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 \\
&= \left\| A_k(t) \cdot \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - A_k(t) \cdot \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right. \\
&\quad \left. + (A_k(t) - B_k(t)) \cdot \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 \\
&\leq \|A_k(t)\| \cdot \left\| \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 \\
&\quad + \|A_k(t) - B_k(t)\| \cdot \left\| \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2.
\end{aligned}$$

With inequality (3.4.6) we obtain

$$\left\| \frac{d}{dt}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 = \|\phi'_{k-1}(t)\|_2 \leq (E^{k-1})'(t).$$

Thus using Lemma 3.4.5 we obtain that

$$\|g'_k(t) - g'_{k-1}(t)\|_2 \leq \|A_k(t)\| \cdot (I_{1,k}(t) + I_{2,k}(t)) \cdot (E^k)'(t) + \|A_k(t) - B_k(t)\| \cdot (E^{k-1})'(t).$$

For simplicity we split the last step up into three parts which we will estimate separately.

Therefore we define

$$\begin{aligned}
J_{1,k}(t) &:= \|A_k(t)\| \cdot I_{1,k}(t) \cdot (E^k)'(t) \\
J_{2,k}(t) &:= \|A_k(t)\| \cdot I_{2,k}(t) \cdot (E^k)'(t) \\
J_{3,k}(t) &:= \|A_k(t) - B_k(t)\| \cdot (E^{k-1})'(t).
\end{aligned}$$

Then the last inequality reads as

$$\|g'_k(t) - g'_{k-1}(t)\|_2 \leq J_{1,k}(t) + J_{2,k}(t) + J_{3,k}(t).$$

### 1. Estimate of $J_{1,k}(t)$ .

By the submultiplicity of the operator norm we have, using inequality (3.2.5),

$$\|A_k(t)\| = \left\| \prod_{l=1}^{k-1} D\Lambda \left( \left( \begin{smallmatrix} k-1 \\ j=l \end{smallmatrix} \right) L_j \right) (\phi_k(t)) \right\| \leq \prod_{l=1}^{k-1} \left\| D\Lambda \left( \left( \begin{smallmatrix} k-1 \\ j=l \end{smallmatrix} \right) L_j \right) (\phi_k(t)) \right\| \leq \alpha^{k-1}.$$

Hence we obtain with (3.4.10)

$$J_{1,k}(t) \leq \frac{ac_4 \alpha^{k-1}}{E^{k+1}(t) \cdot E^k(t)} \cdot (E^k)'(t).$$

Since  $t \in [t_s + \varepsilon, \infty) \subset (0, \infty)$  this implies

$$\lim_{k \rightarrow \infty} \frac{(E^k)'(t)}{E^{k+1}(t)} = \lim_{k \rightarrow \infty} \frac{\prod_{j=1}^k (E^j(t) + 1)}{E^{k+1}(t)} = 0$$

uniformly in  $t$  for  $t \in [t_s + \varepsilon, \infty)$  and thus

$$\lim_{k \rightarrow \infty} \frac{(E^k)'(t)}{E^{k+1}(t) \cdot E^k(t)} = 0$$

uniformly in  $t$  for  $t \in [t_s + \varepsilon, \infty)$ . So there exists  $c_9 > 0$  such that

$$J_{1,k}(t) \leq c_9 \alpha^{k-1}.$$

## 2. Estimate of $J_{2,k}(t)$ .

We obtain again by the submultiplicity of the operator norm, equation (3.4.7) and inequality (3.2.7)

$$\begin{aligned} \|A_k(t)\| &\leq \prod_{l=1}^{k-1} \left\| D\Lambda \left( \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right) \right\| \\ &\leq c_4^{k-1} \cdot \left( \prod_{l=1}^{k-1} \left\| \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right\|_2 \right)^{-1} \\ &\leq c_4^{k-1} \cdot \left( \prod_{l=1}^{k-1} E^l(t) \right)^{-1}. \end{aligned}$$

Thus we obtain with (3.4.11)

$$\begin{aligned} J_{2,k}(t) &\leq c_4^{k-1} \cdot \frac{(E^k)'(t)}{\prod_{l=1}^{k-1} (E^l(t))} \cdot I_{2,k}(t) \\ &\leq c_4^{k-1} \cdot \prod_{l=1}^{k-1} \left( \frac{E^l(t) + 1}{E^l(t)} \right) \cdot (E^k(t) + 1) \cdot I_{2,k}(t) \\ &\leq c_4^{k-1} c_8 \frac{2 \|s_k\|_2 + 1}{E^k(t)^\beta} \cdot \prod_{l=1}^k \left( 1 + \frac{1}{E^l(t)} \right) \\ &= c_8 \frac{c_4^{k-1}}{E^k(t)^{\beta/2}} \cdot \frac{E^k(t_k) + 1}{E^k(t)^{\beta/2}} \cdot \prod_{l=1}^k \left( 1 + \frac{1}{E^l(t)} \right). \end{aligned}$$

At this point we need to show that

$$\lim_{k \rightarrow \infty} \frac{2 \|s_k\|_2 + 1}{(E^k(t))^\delta} = 0$$

for  $t > t_s$  and  $\delta > 0$ . We already know by equation (3.3.2) that for  $0 < t' < t'' < \infty$

$$\lim_{k \rightarrow \infty} \frac{E^k(t'')}{E^k(t')} = \infty.$$

Thus for all  $C > 2 + \frac{1}{\delta}$  there exists  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$

$$E^k(t'') > C \cdot E^k(t')$$



and thus

$$\begin{aligned}
E^{k+1}(t'') &> E(C \cdot E^k(t')) \\
&> \frac{1}{2} \exp(C \cdot E^k(t')) \\
&= \frac{1}{2} \exp(E^k(t'))^C \\
&> \frac{1}{2} E^{k+1}(t')^C \\
&> E^{k+1}(t')^{C-1} \\
&> C \cdot E^{k+1}(t')^{C-2} \\
&> C \cdot E^{k+1}(t')^{1/\delta}.
\end{aligned}$$

for all large  $k$ . Therefore

$$\lim_{k \rightarrow \infty} \frac{E^k(t')}{(E^k(t''))^\delta} < \frac{1}{C^\delta}$$

for all  $t_s < t' < t''$ . Since  $C$  was arbitrary, we obtain

$$\lim_{k \rightarrow \infty} \frac{E^k(t_k)}{(E^k(t))^\delta} = 0$$

for  $t > t_s$  and the conclusion follows.

With Lemma 3.4.3 we have that

$$\prod_{l=1}^{\infty} \left(1 + \frac{1}{E^l(t)}\right) < \infty$$

for  $t > t_s$ . Moreover,

$$\frac{c_4^{k-1}}{E^k(t)^{\beta/2}} \leq \alpha^{k-1}$$

for large  $k \in \mathbb{N}$ . Since  $t \in [t_s + \varepsilon, \infty)$  there is a constant  $c_{10} > 0$  such that we have for large  $k \in \mathbb{N}$

$$J_{2,k}(t) \leq c_{10} \alpha^{k-1}.$$

### 3. Estimate of $J_{3,k}(t)$ .

We want to estimate the right hand side of (3.4.17) step by step. From (3.2.7) and (3.4.7) we obtain for all  $r \in \{1, \dots, k-1\}$

$$\prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \leq c_4^{r-1} \cdot \left( \prod_{l=1}^{r-1} \left\| \left( \bigcirc_{j=l}^{k-1} L_j \right) (\phi_k(t)) \right\|_2 \right)^{-1} \leq c_4^{r-1} \cdot \left( \prod_{l=1}^{r-1} E^l(t) \right)^{-1}.$$

Similarly Remark 3.4.2 yields

$$\prod_{l=r+1}^{k-1} \|Y_{l,k-1}(t)\| \leq c_4^{k-r-1} \cdot \left( \prod_{l=r+1}^{k-1} E^l(t) \right)^{-1}.$$

This implies that

$$\prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \prod_{l=r+1}^{k-1} \|Y_{l,k-1}(t)\| \leq c_4^{k-2} \cdot \left( \prod_{\substack{l=1 \\ l \neq r}}^{k-1} E^l(t) \right)^{-1}.$$

Recall that for

$$x = \left( \bigcirc_{j=r}^{k-1} L_j \right) (\phi_k(t)) \quad \text{and} \quad y = \left( \bigcirc_{j=r}^{k-1} L_j \right) (0, \dots, 0, E^k(t) + M)$$

the inequalities (3.4.7) and (3.4.14) ensure that

$$\|D\Lambda(x) - D\Lambda(y)\| \leq c_4^2 c_7 \frac{\max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}}{E^r(t)^{1+\beta}}.$$

Notice that there is a constant  $c_{11} \geq 1$  such that for large  $k \in \mathbb{N}$

$$\|L_{k-1}(\phi_k(t)) - L_{k-1}(0, \dots, 0, E^k(t) + M)\|_2 \leq c_{11}.$$

For  $r = k - 1$  we obtain with (3.4.7) and (3.4.14)

$$\begin{aligned} & \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| \\ &= \|D\Lambda(L_{k-1}(\phi_k(t))) - D\Lambda(L_{k-1}(0, \dots, 0, E^k(t) + M))\| \\ &\leq \frac{c_4^2 c_7 c_{11}}{E^{k-1}(t)^{1+\beta}}. \end{aligned}$$

Thus

$$\begin{aligned} & \prod_{l=1}^{k-2} \|X_{l,k-1}(t)\| \cdot \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| \cdot \prod_{l=k}^{k-1} \|Y_{l,k-1}(t)\| \cdot (E^{k-1})'(t) \\ &\leq c_7 c_{11} \frac{c_4^k}{E^{k-1}(t)^\beta} \cdot \left( \prod_{l=1}^{k-1} E^l(t) \right)^{-1} \cdot (E^{k-1})'(t) \\ &= c_7 c_{11} \frac{c_4^k}{E^{k-1}(t)^\beta} \cdot \prod_{l=1}^{k-1} \left( 1 + \frac{1}{E^l(t)} \right). \end{aligned}$$

Since

$$\frac{c_4^k}{E^{k-1}(t)^\beta} \leq \frac{\alpha^{k-1}}{k}$$

for large  $k \in \mathbb{N}$  and  $t > t_s$ , we obtain for  $r = k - 1$  from Lemma 3.4.3

$$\prod_{l=1}^{k-2} \|X_{l,k-1}(t)\| \cdot \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| \cdot (E^{k-1})'(t) \leq \frac{c_{12}}{k} \alpha^{k-1} \quad (3.5.1)$$

for a constant  $c_{12} > 0$  and large  $k \in \mathbb{N}$ .

For  $r \neq k-1$  notice that with (3.2.6) and (3.2.8) we obtain

$$\begin{aligned} & \left\| \left( \bigcirc_{j=r}^{k-1} L_j \right) (\phi_k(t)) - \left( \bigcirc_{j=r}^{k-1} L_j \right) (0, \dots, 0, E^k(t) + M) \right\|_2 \\ & \leq \alpha^{k-r-2} \left\| L_{k-2}(L_{k-1}(\phi_k(t))) - L_{k-2}(L_{k-1}(0, \dots, 0, E^k(t) + M)) \right\|_2 \\ & \leq \alpha^{k-r-2} c_4 \pi \frac{\left\| L_{k-1}(\phi_k(t)) - L_{k-1}(0, \dots, 0, E^k(t) + M) \right\|_2}{\min \{ \|L_{k-1}(\phi_k(t))\|_2, \|L_{k-1}(0, \dots, 0, E^k(t) + M)\|_2 \}} \\ & \leq \frac{c_4 c_{11} \pi}{E^{k-1}(t)}. \end{aligned}$$

Since  $E^{k-1}(t) \geq E^{k-1}(t)^\beta$  for all  $t \geq 0$  and large  $k \in \mathbb{N}$ , this implies

$$\begin{aligned} & \prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \|X_{r,k-1}(t) - Y_{r,k-1}(t)\| \cdot \prod_{s=r+1}^{k-1} \|Y_{s,k-1}(t)\| \cdot (E^{k-1})'(t) \\ & \leq c_4^2 c_7 \cdot \frac{1}{E^r(t)^{1+\beta}} \cdot \frac{c_4 c_{11} \pi}{E^{k-1}(t)^\beta} \cdot c_4^{k-2} \cdot \left( \prod_{\substack{l=1 \\ l \neq r}}^{k-1} E^l(t) \right)^{-1} \cdot (E^{k-1})'(t) \\ & = \frac{c_7 c_{11} \pi}{E^r(t)^\beta} \cdot \frac{c_4^{k+1}}{E^{k-1}(t)^\beta} \cdot \prod_{l=1}^{k-1} \left( 1 + \frac{1}{E^l(t)} \right) \\ & \leq \frac{c_{12}}{k} \alpha^{k-1}. \end{aligned} \tag{3.5.2}$$

Therefore we get again the estimate from above for large  $k \in \mathbb{N}$  which finally yields with (3.5.1) and (3.5.2)

$$J_{3,k}(t) = \|A_k(t) - B_k(t)\| \cdot (E^{k-1})'(t) \leq \sum_{r=1}^{k-1} \frac{c_{12}}{k} \alpha^{k-1} \leq c_{12} \alpha^{k-1}$$

for large  $k \in \mathbb{N}$ . Altogether we obtain with  $c_{13} := c_9 + c_{10} + c_{12}$

$$\|g'_k(t) - g'_{k-1}(t)\|_2 \leq J_{1,k}(t) + J_{2,k}(t) + J_{3,k}(t) \leq c_{13} \alpha^{k-1}. \tag{3.5.3}$$

Since for all  $k \in \mathbb{N}$

$$g'_k(t) = g'_0(t) + \sum_{l=1}^k (g'_l(t) - g'_{l-1}(t)),$$

we obtain from (3.5.3) the uniform convergence of  $(g'_k)$  on  $[t_{\underline{s}} + \varepsilon, \infty)$ . This yields for  $\varepsilon \rightarrow 0$  the locally uniform convergence of  $(g'_k)$  on  $(t_{\underline{s}}, \infty)$  which proves Theorem 3.  $\square$

### 3.6 Domain containing the hairs

A key element of Karpínska's proof of theorem 2.7.4 is that points on a hair which are not endpoints are mapped by the iterates into a comparatively small and paraboloid-like domain. In this part we want to discuss this condition. Here denote for  $\varepsilon > 0$

$$\tilde{\Omega}_\varepsilon := \{z \in \mathbb{C} : \operatorname{Re} z > 1 \text{ and } |\operatorname{Im} z| < (\operatorname{Re} z)^\varepsilon\}.$$

To prove that for  $f(z) = \lambda e^z$  the Hausdorff dimension of the union of all hairs is 2 while the Hausdorff dimension of the hairs without their endpoints is 1, Karpińska used the following lemma.

**Lemma 3.6.1.** *If  $f(z) = \lambda e^z$  with  $0 < \lambda < 1/e$  and  $z \in \mathcal{J}(f) \setminus \mathcal{C}$ , then  $f^k(z) \in \tilde{\Omega}_\varepsilon$  for all large  $k \in \mathbb{N}$ .*

Now denote by  $\tilde{x}$  the first  $d-1$  components of  $x \in \mathbb{R}^d$ , i.e.  $x = (\tilde{x}, x_d)$ . A generalization is also true for our Zorich map  $f$  using the set

$$\Omega_\varepsilon := \{x \in \mathbb{R}^d : x_d > M \text{ and } \|\tilde{x}\|_2 < x_d^\varepsilon\}.$$

Bergweiler [Ber10] even proved a stronger result by using the function

$$\psi : [1, \infty) \rightarrow [1, \infty), \quad \psi(x) = \exp\left(\sqrt{\log(x)}\right).$$

If

$$\Omega := \{x \in \mathbb{R}^d : x_d > M \text{ and } \|\tilde{x}\|_2 < \psi(x_d)\},$$

we obtain the following result.

**Lemma 3.6.2.** *If  $x \in J \setminus C$ , then  $f^k(x) \in \Omega$  for large  $k \in \mathbb{N}$ .*

We have

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x^\varepsilon} = \lim_{x \rightarrow \infty} \exp\left(-\varepsilon \log(x) + \sqrt{\log(x)}\right) = 0.$$

This implies that if  $f^k(x) \in \Omega$  for all large  $k \in \mathbb{N}$ , then, given  $\varepsilon > 0$ , also  $f^k(x) \in \Omega_\varepsilon$  for all large  $k \in \mathbb{N}$ .

Using this information it is enough to require that  $h$  satisfies the conditions of Theorem 3.2.3 on a neighborhood of the point  $v$  from (3.2.9) to obtain differentiability of the ‘tails’ of the hairs [SZ03, §3]. Moreover, it can be shown that if  $h$  is  $C^1$  on a suitable compact subset of  $\text{int}(Q)$ , we obtain the differentiability of the hairs except for the endpoints. This shows that one can relax the condition on  $Dh$  in such a way that  $Dh$  needs to be only locally Hölder continuous.

## 3.7 Further remarks

### Alternative ways of defining the bi-Lipschitz map $h$

Our bi-Lipschitz map  $h$  in Chapter 3 is a map from a hypercube to the upper or lower hemisphere. In a more general setting, a Zorich map depends on a bi-Lipschitz map which maps a hypercube to some body. A 3-dimensional example given by Iwaniec and Martin [IM01, page 121] uses a bi-Lipschitz map  $h$  which maps a square to the faces of a tetrahedron. Recently, Nicks and Sixsmith [NS17] considered a bi-Lipschitz map which

maps a square to the faces of a square based pyramid. They used these maps to obtain a quasiregular map  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of transcendental type, which has a periodic domain where all iterates of  $f$  tend locally uniformly to  $\infty$ . Thus in this thesis we restricted ourself to the ‘standard’ case.

### Differentiable endpoints for the exponential family

As in Viana’s result, we do not obtain differentiability of the hairs in the endpoints since they can spiral around a point. Rempe [Re03, 3.4.2 Theorem, page 31] gave a condition under which we obtain smoothness up to the endpoints for exponential maps. To state his result in a precise form, we first give some essential definitions [Re03, page 23-24]. Similar to the case of quasiregular maps denote by  $\Sigma := \mathbb{Z}^{\mathbb{N}_0}$  the space of all integer sequences.

**Definition 3.7.1** (Shift map). We call  $\sigma: \Sigma \rightarrow \Sigma$ ,  $\sigma(s_0s_1s_2\dots) = s_1s_2s_3\dots$  the *shift map*.

**Definition 3.7.2** (Model space). The *Model space* is defined as

$$\Xi := \Sigma \times [0, \infty) = \{(\underline{s}, t) : \underline{s} \in \Sigma, t \geq 0\},$$

where  $\Sigma$  is equipped with the order topology of the lexicographic order and  $[0, \infty)$  with its natural topology. Moreover, we define by

$$T: \Xi \rightarrow [0, \infty), T(\underline{s}, t) = t$$

the projection to the second component. Furthermore, define

$$\mathfrak{F}: \Xi \rightarrow \Sigma \times \mathbb{R}, \mathfrak{F}(\underline{s}, t) = (\sigma(\underline{s}), E(t) - 2\pi|s_1|)$$

and the sets

$$\overline{X} := \{(\underline{s}, t) \in \Xi : \forall n \geq 0 : T(\mathfrak{F}^n(\underline{s}, t)) \geq 0\}$$

and

$$X := \{(\underline{s}, t) \in \overline{X} : T(\mathfrak{F}^n(\underline{s}, t)) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

*Remark 3.7.3.* The dynamics of  $\mathfrak{F}$  on  $X$  can be seen as a model of the set of escaping points, while  $\overline{X}$  should be thought of as a model of the Julia set in the presence of an attracting fixed point.

The following theorem answers the question about the differentiability of hairs in the endpoints.

**Theorem 3.7.4.** Let  $\underline{s}$  be an exponentially bounded address such that  $(\underline{s}, t_{\underline{s}}) \in X$ . Then the curve  $g_{\underline{s}}([t_{\underline{s}}, \infty))$  is continuously differentiable in  $g_{\underline{s}}(t_{\underline{s}})$  if and only if the sum

$$\sum_{j=0}^{\infty} \frac{2\pi s_{j+1}}{T(\mathfrak{F}^j(\underline{s}, t_{\underline{s}}))}$$

converges.

When we say that the curve is continuously differentiable in  $g_{\underline{s}}(t_{\underline{s}})$  we mean that it is continuously differentiable under a suitable parametrization, e.g. by arclength and not that the function  $g_{\underline{s}}$  itself is continuously differentiable in  $t_{\underline{s}}$ . In fact, if the convergence of the sum is absolute, then Rempe showed that  $g_{\underline{s}}$  itself is differentiable in  $t_{\underline{s}}$ .

It seems reasonable that this result generalizes to the case of Zorich maps using a similar Model space.

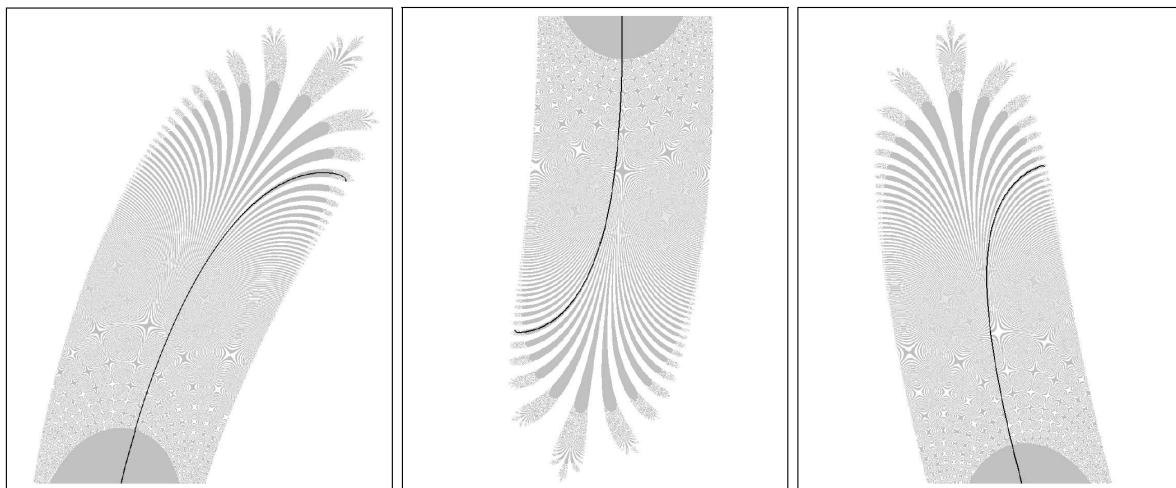


Figure 3.1. The pictures show the hair at address  $0123\dots$  for the function  $f(z) = e^z - 2$  in three different magnifications, see [Re03, page 34]. Due to Theorem 3.7.4 this hair has a non-differentiable endpoint.

# Chapter 4

## Nowhere differentiable hairs of entire functions

In the previous section we have shown that a Hölder condition for  $Dh$  implies that the hairs of our Zorich map are  $C^1$ . At first glance one might think that the regularity of the map transfers to the smoothness of its hairs. In this section we want to show that this is not the case for general functions in class  $\mathcal{B}$ . So even for analytic functions the hairs need not be smooth. More precisely we give an example of a function in class  $\mathcal{B}$  with hairs which are nowhere differentiable. For this we use so-called Cauchy integrals to obtain a function in class  $\mathcal{B}$  of finite order with prescribed tracts. We start with a suitable ‘bad’ tract and show that in this case no hair can be differentiable. More precisely, the main result of this chapter is the following.

**Theorem 4.** There exists a function in class  $\mathcal{B}$  of finite order, where the Julia set consists of hairs, which are nowhere differentiable.

### 4.1 The Cauchy integral method

In this section we use for  $\alpha \in \mathbb{R}$  and  $\beta > 0$  the notation

$$S_{\alpha,\beta} := \{z \in \mathbb{C} : \operatorname{Re}(z) > \alpha, |\operatorname{Im}(z)| < \beta\}.$$

The original Cauchy integral method was established by Pólya and Szegő [PS72, p. 115, ex. 158] using the function

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{e^{\zeta}}}{\zeta - z} d\zeta,$$

where  $\gamma$  is a curve going through the boundary of  $S_{0,\pi} = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0, |\operatorname{Im}(z)| < \pi\}$  in clockwise direction and  $z \in \mathbb{C} \setminus \overline{S_{0,\pi}}$ . This function is bounded in  $\mathbb{C} \setminus \overline{S_{0,\pi}}$ . By considering

the integral over  $\partial S_{\alpha,\pi}$  with  $\alpha > 0$  instead of  $\gamma$  and letting  $\alpha \rightarrow \infty$ , one can extend the function  $\tilde{f}$  to an entire function  $f$  in such a way that

$$f(z) = \begin{cases} \tilde{f}(z) & , \text{ if } z \in \mathbb{C} \setminus \overline{S_{0,\pi}}, \\ \tilde{f}(z) + e^{e^z} & , \text{ if } z \in S_{0,\pi}. \end{cases}$$

## 4.2 Construction of these functions

In the following we consider a tract which looks like a sine-like halfstrip. To realize this tract, we define the function

$$h: \mathbb{C} \rightarrow \mathbb{C}, \quad h(z) = 5\pi z + 2\pi i \sin z.$$

Then the function  $h|_{S_{-1,\pi/3}}$  is univalent by Lemma 2.5.1 since with  $z = x + iy$  we have

$$\begin{aligned} \operatorname{Re} h'(z) &= \operatorname{Re} (5\pi + 2\pi i \cos z) \\ &= 5\pi + 2\pi \sin x \cdot \sinh y \\ &\geq 5\pi - 2\pi \cdot \sinh \frac{\pi}{3} \\ &> 2\pi \\ &> 0. \end{aligned}$$

Moreover, we have for  $|y| < \pi/3$

$$\begin{aligned} \operatorname{Re} h(-1 + iy) &= -5\pi - 2\pi \cos(-1) \cdot \sinh y \\ &\leq -5\pi + 2\pi \sinh \frac{\pi}{3} \\ &< -2\pi \\ &< 0 \end{aligned}$$

and for  $x > -1$

$$\begin{aligned} \operatorname{Im} h\left(x + \frac{\pi}{3}i\right) &= \frac{5\pi^2}{3} + 2\pi \sin x \cdot \cosh \frac{\pi}{3} \\ &\geq \frac{5\pi^2}{3} - 2\pi \cosh \frac{\pi}{3} \\ &> \pi. \end{aligned}$$

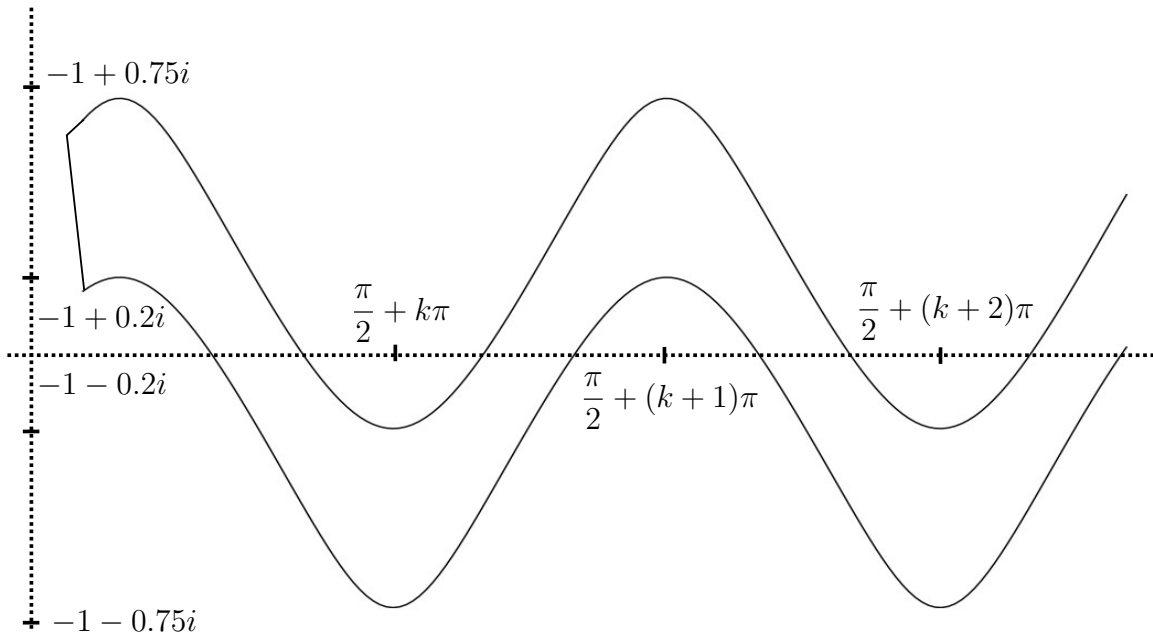
The same way one can show that

$$\operatorname{Im} h\left(x - \frac{\pi}{3}i\right) < -\pi.$$

Since  $h|_{S_{-1,\pi/3}}$  is univalent this implies

$$S_{0,\pi} \subset h(S_{-1,\pi/3}).$$



Figure 4.1. Shape of the tract  $T$ .

*Remark 4.2.1.* For our purpose it is useful to show that  $T := h^{-1}(S_{0,\pi})$  is a sine-like halfstrip and contains no straight line which is unbounded to the right as illustrated in Figure 4.1.

Since

$$\operatorname{Im} h(x + iy) = 5\pi y + 2\pi \sin x \cdot \cosh y,$$

we first consider the equation  $\operatorname{Im} h(x + iy) = \pi$  for the upper boundary of  $T$ . Thus we have for all  $k \in \mathbb{Z}$  and  $x_k = \pi/2 + 2\pi k$

$$5\pi y + 2\pi \cosh y = \pi$$

and since  $\cosh y \geq 1$ , we obtain  $y \leq -1/5$ . For the lower boundary of  $T$  we consider the equation  $\operatorname{Im} h(x + iy) = -\pi$  and with  $x_k = -\pi/2 + 2\pi k$  we obtain

$$5\pi y - 2\pi \cosh y = -\pi$$

and hence  $y \geq 1/5$ .

If  $y = 0$ , then the equation  $\operatorname{Im} h(x + iy) = \pi$  yields  $2\pi \sin x = \pi$  and thus we have  $\sin x = 1/2$ . Hence the solutions in  $[0, 2\pi]$  are  $\pi/3$  and  $2\pi/3$ . Similarly, we obtain for  $y = 0$  in  $\operatorname{Im} h(x + iy) = -\pi$  the solutions  $4\pi/3$  and  $5\pi/3$ .

The choice of  $x_k$  also shows that  $T$  is  $2\pi$ -periodic, i.e. if  $z \in T$ , then  $z + 2\pi \in T$ . In addition, it is not difficult to show that  $|\operatorname{Im}(z)| \leq 3/4$  for all  $z \in T$ .

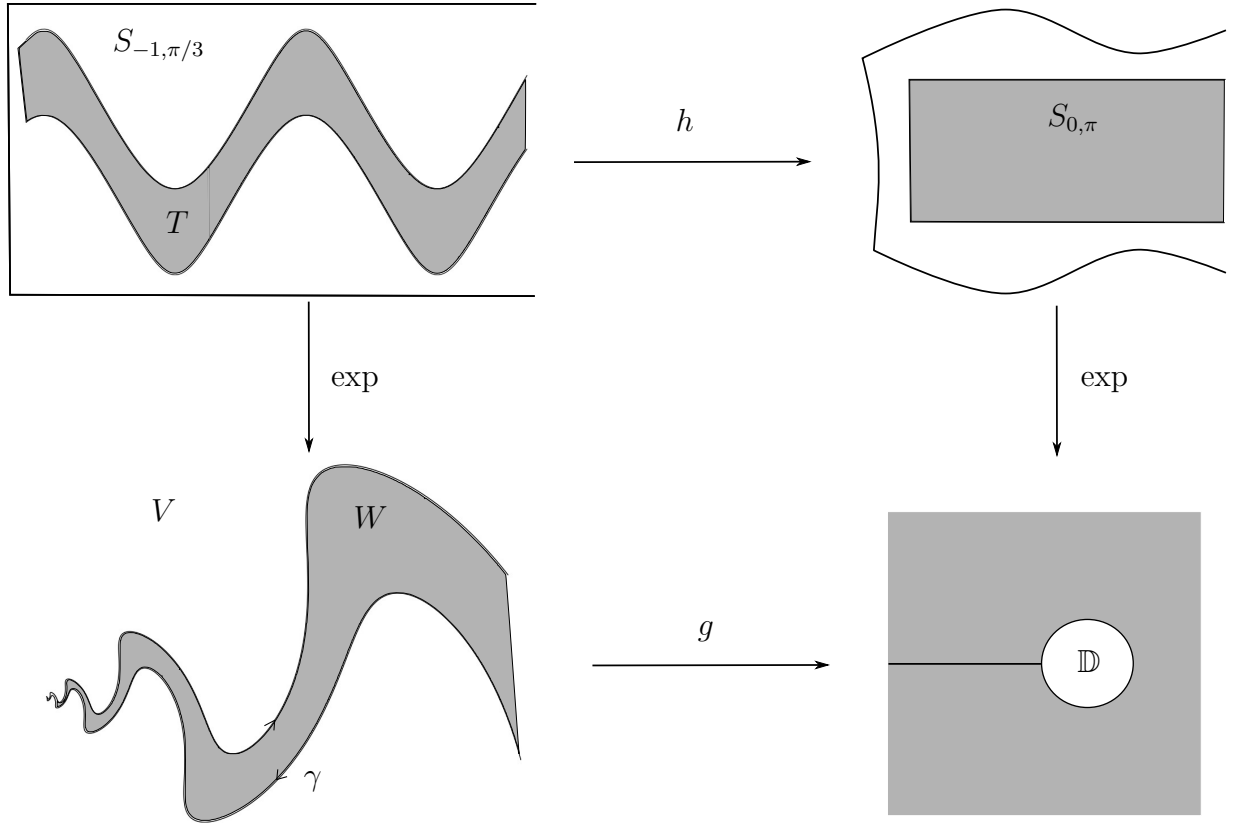


Figure 4.2. Construction of the function  $g$  using the logarithmic change of variable of the map  $h$ .

We define

$$g: \left\{ z \in \mathbb{C} : |z| > 1/e, |\arg z| < \frac{\pi}{3} \right\} \rightarrow \mathbb{C}, \quad g(z) = \exp(h(\operatorname{Log} z)),$$

where  $\operatorname{Log}$  denotes the principal branch of the complex logarithm. Then  $g$  is holomorphic and with  $W := \exp(T)$  we obtain

$$\overline{W} \subset \{z \in \mathbb{C} : |z| > 1/e, |\arg z| < \pi/3\}.$$

Now we parametrize  $\partial W$  by its radius for large  $R > 1$ , i.e. we define

$$\gamma: \mathbb{R} \rightarrow \partial W, \quad \gamma(t) = \begin{cases} |t| e^{i\varphi(t)}, & \text{if } |t| \geq R, \\ \tilde{\gamma}(t), & \text{else,} \end{cases}$$

where  $\varphi$  is an angular function and  $\tilde{\gamma}$  is a suitable piecewise  $C^1$  curve ‘gluing’ the outer parts together. Because  $\operatorname{Re} h'(z) > 0$  we have  $|\arg h'(z)| < \pi/2$  and the upper respectively the lower boundary of  $T$  has a parametrization  $t \mapsto \operatorname{Log} |t| + i\varphi(t)$ . This guarantees the parametrization for  $\partial W$  from above.

We want to show the following lemma which is similar to [RRRS11, Prop. 7.1]. Here  $V := \mathbb{C} \setminus W$ .

**Lemma 4.2.2.** *There exists an entire function  $f \in \mathcal{B}$  of finite order and a constant  $K > 0$  such that*

1.  $A := \{z \in \mathbb{C} : |f(z)| > K\} \subset W$ .
2. We have  $|f(z) - e^{g(z)}| = O(1)$  on  $A$  and  $|f(z)| = O(1)$  on  $V$ .

We split the proof up into several parts.

**Claim 1.** The map

$$\tilde{f}(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{\exp g(\zeta)}{\zeta - z} d\zeta$$

defines a holomorphic function  $\tilde{f}: \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}$ .

*Proof.* We show that the integral

$$\frac{1}{2\pi i} \int_{\gamma} \exp g(\zeta) d\zeta$$

converges absolutely for all  $z \in \mathbb{C} \setminus \gamma$ . For  $|t| \geq R$  we have

$$h(\text{Log}(\gamma(t))) = 5\pi(\text{Log } |t| + i\varphi(t)) + 2\pi i \sin(\text{Log } |t| + i\varphi(t)).$$

Since

$$\sin(\text{Log } |t| + i\varphi(t)) = \sin(\text{Log } |t|) \cdot \cosh \varphi(t) + i \cos(\text{Log } |t|) \cdot \sinh \varphi(t),$$

we obtain

$$\begin{aligned} & g(\gamma(t)) \\ &= \exp(h(\text{Log}(\gamma(t)))) \\ &= \exp(5\pi \text{Log } |t|) \cdot \exp(5\pi i\varphi(t)) \cdot \exp(2\pi i \sin(\text{Log } |t| + i\varphi(t))) \\ &= |t|^{5\pi} \cdot \exp(-2\pi \cos(\text{Log } |t|) \cdot \sinh \varphi(t)) \cdot \exp(\pi i (5\varphi(t) + 2 \sin(\text{Log } |t|) \cdot \cosh \varphi(t))). \end{aligned} \tag{4.2.1}$$

By our construction of  $h$  we know that for all  $|t| \geq R$

$$h(\text{Log } \gamma(t)) \in \mathbb{R}^+ \pm i\pi$$

and thus  $g(\gamma(t)) \in (-\infty, 0)$  for  $|t| \geq R$ . Therefore we obtain

$$5\varphi(t) + 2 \sin(\text{Log } |t|) \cdot \cosh \varphi(t) = \begin{cases} 1, & \text{if } t \geq R, \\ -1, & \text{if } t \leq -R. \end{cases}$$

Differentiating both sides with respect to  $t$  we obtain

$$5\varphi'(t) + 2 \frac{\cos(\text{Log } |t|)}{t} \cdot \cosh \varphi(t) + 2 \sin(\text{Log } |t|) \cdot \sinh \varphi(t) \cdot \varphi'(t) = 0.$$

From the construction of  $h$  it is clear that  $|\varphi(t)| \leq \pi/3$  for all  $t$ . Since

$$5 + 2 \sin(\operatorname{Log} |t|) \cdot \sinh \varphi(t) \geq 5 - 2 \sinh \frac{\pi}{3} > \frac{5}{2},$$

this yields

$$|\varphi'(t)| = \frac{1}{|t|} \cdot \frac{2 |\cos(\operatorname{Log} |t|)| \cdot \cosh \varphi(t)}{|5 + 2 \sin(\operatorname{Log} |t|) \cdot \sinh \varphi(t)|} \leq \frac{4 \cosh \frac{\pi}{3}}{5 |t|} \leq \frac{2}{|t|}.$$

With

$$\gamma'(t) = \operatorname{sgn}(t) \cdot e^{i\varphi(t)} \cdot (1 + it\varphi'(t))$$

we obtain

$$|\gamma'(t)| = |1 + it\varphi'(t)| \leq 3 \tag{4.2.2}$$

for  $|t| \geq R$ . Since  $\gamma$  is piecewise smooth, there is a constant  $C_1 > 0$  such that

$$\int_{-R}^R \exp(\operatorname{Re} g(\gamma(t))) \cdot |\gamma'(t)| dt \leq C_1. \tag{4.2.3}$$

With  $C_2 := \exp(-2\pi \sinh \frac{\pi}{3})$  we have

$$\begin{aligned} \exp \operatorname{Re} g(\gamma(t)) &= \exp(-|t|^{5\pi} \cdot \exp(-2\pi \cos(\operatorname{Log} |t|) \cdot \sinh \varphi(t))) \\ &\leq \exp\left(-|t| \cdot \exp\left(-2\pi \sinh \frac{\pi}{3}\right)\right) \\ &= \exp(-C_2 \cdot |t|). \end{aligned} \tag{4.2.4}$$

This yields with (4.2.2), (4.2.3) and (4.2.4) for  $z \in C \setminus \gamma$

$$\begin{aligned} &\left| \frac{1}{2\pi i} \int_{\gamma} \exp g(\zeta) d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{\gamma} \exp \operatorname{Re} g(\zeta) |d\zeta| \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \operatorname{Re} g(\gamma(t)) \cdot |\gamma'(t)| dt \\ &\leq \frac{1}{2\pi} \left( C_1 + \int_{-\infty}^{-R} \exp(\operatorname{Re} g(\gamma(t))) |\gamma'(t)| dt + \int_R^{\infty} \exp(\operatorname{Re} g(\gamma(t))) |\gamma'(t)| dt \right) \\ &\leq \frac{1}{2\pi} \left( C_1 + 6 \int_R^{\infty} \exp(-C_2 t) dt \right) \\ &\leq \frac{1}{2\pi} \left( C_1 + \frac{6}{C_2} \right). \end{aligned} \tag{4.2.5}$$

Thus  $\int_{\gamma} \exp(g(\zeta)) d\zeta$  converges absolutely and the claim follows.  $\square$

**Claim 2.** The function  $\tilde{f}$  is bounded.

*Proof.* We choose  $\kappa > 0$  small such that the following holds:

1. We have  $U_{\kappa}(\gamma) := \{w \in \mathbb{C} : \operatorname{dist}(\gamma, w) \leq \kappa\} \subset \{z \in \mathbb{C} : |z| > 1/e, |\arg z| < \pi/3\}$ .

2. For all  $z \in U_\kappa(\gamma)$  we have

$$g(z) \in \{3 + re^{i\theta} : \theta \in (3\pi/4, 5\pi/4)\}. \quad (4.2.6)$$

Moreover, we define

$$V_\kappa := \{z \in \mathbb{C} \setminus \overline{W} : \text{dist}(z, W) > \kappa\}$$

and

$$W_\kappa := \{z \in W : \text{dist}(z, \partial W) > \kappa\}.$$

The first step is to show that  $\tilde{f}$  is bounded in  $V_\kappa \cup W_\kappa$ . As long as  $z \in V_\kappa \cup W_\kappa$  we use the curve  $\gamma$  from above.

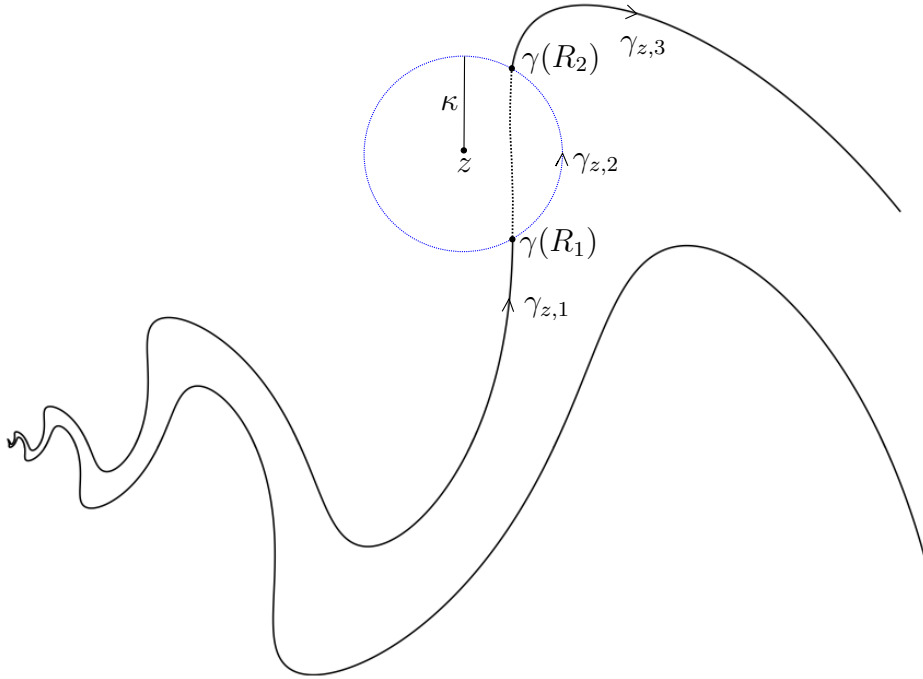


Figure 4.3. Construction of the curve  $\gamma_z$  and its decomposition into smaller parts.

If  $z \in U_\kappa(\gamma)$ , we modify the curve as follows (see Figure 4.3):

Since  $\gamma$  is unbounded, there exist  $R_1 \in \mathbb{R}$  minimal and  $R_2 \in \mathbb{R}$  maximal such that  $\text{dist}(\gamma(R_j), z) = \kappa$  for  $j \in \{1, 2\}$ . Moreover, there exist angles  $\psi_1, \psi_2 \in [0, 2\pi)$  such that  $\gamma(R_1) = R_1 e^{i\varphi(R_1)} = z + \kappa e^{i\psi_1}$  and  $\gamma(R_2) = R_2 e^{i\varphi(R_2)} = z + \kappa e^{i\psi_2}$ . Thus we define

$$\gamma_z(t) := \begin{cases} \gamma(t), & \text{if } t \in (-\infty, R_1) \cup (R_2, \infty), \\ z + \kappa \exp(i\varphi_z(t)), & \text{if } t \in [R_1, R_2], \end{cases} \quad (4.2.7)$$

with a suitable function  $\varphi_z$  satisfying  $\varphi_z(R_j) = \psi_j$  for  $j \in \{1, 2\}$ .

The next step is to show that the integral

$$\frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp g(\zeta)}{\zeta - z} d\zeta \quad (4.2.8)$$

is bounded independently of  $z$ . Notice that by Cauchy's integral theorem we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp g(\zeta)}{\zeta - z} d\zeta.$$

As illustrated in Figure 4.3, we split  $\gamma_z$  up into three parts. Therefore denote

$$\gamma_{z,1} := \gamma_z|_{(-\infty, R_1)}, \quad \gamma_{z,2} := \gamma_z|_{[R_1, R_2]} \quad \text{and} \quad \gamma_{z,3} := \gamma_z|_{(R_2, \infty)}.$$

Using this decomposition of  $\gamma_z$ , we obtain with (4.2.5)

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\gamma_z - \gamma_{z,2}} \frac{\exp g(\zeta)}{\zeta - z} d\zeta \right| &\leq \frac{1}{2\pi} \int_{\gamma_z - \gamma_{z,2}} \frac{|\exp g(\zeta)|}{|\zeta - z|} |d\zeta| \\ &\leq \frac{1}{2\pi\kappa} \int_{\gamma} \exp \operatorname{Re} g(\zeta) |d\zeta| \\ &\leq \frac{1}{2\pi\kappa} \left( C_1 + \frac{6}{C_2} \right). \end{aligned} \tag{4.2.9}$$

Moreover, we have with (4.2.6) and  $\gamma_{z,2} \subset U_{\kappa}(\gamma)$

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\gamma_{z,2}} \frac{\exp g(\zeta)}{\zeta - z} d\zeta \right| &\leq \operatorname{length}(\gamma_{z,2}) \cdot \max_{\zeta \in \gamma_{z,2}} \frac{|\exp g(\zeta)|}{|\zeta - z|} \\ &\leq \frac{2\pi\kappa}{\kappa} \cdot \max_{\zeta \in \gamma_{z,2}} \exp(\operatorname{Re} g(\zeta)) \\ &\leq 2\pi e^3. \end{aligned} \tag{4.2.10}$$

Thus the integral in (4.2.8) is bounded independently of  $z$  and hence there exists a constant  $C_3 > 0$  such that  $|\tilde{f}(z)| \leq C_3$  for all  $z \in \mathbb{C} \setminus \gamma$ .

□

**Claim 3.** The function

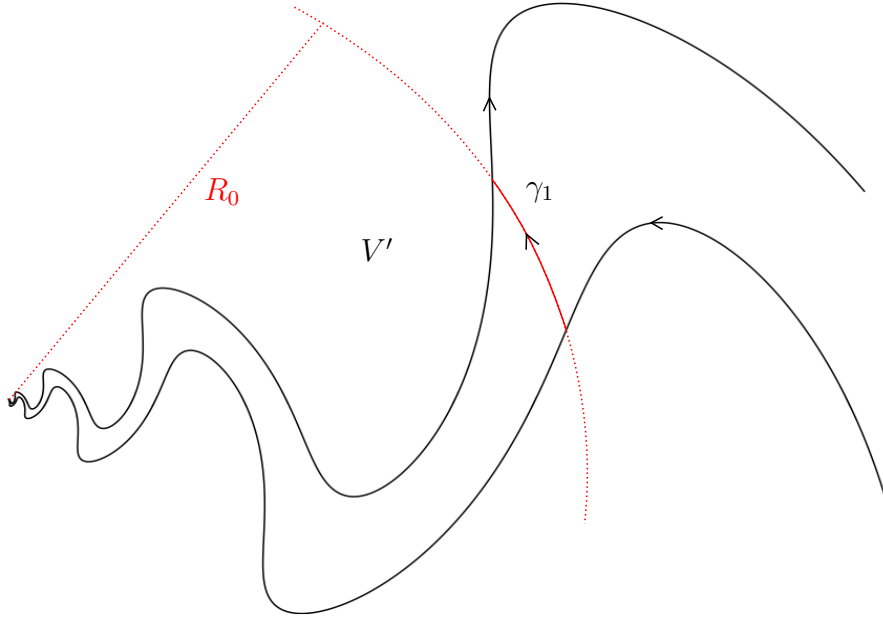
$$f(z) := \begin{cases} \tilde{f}(z), & \text{if } z \in V, \\ \tilde{f}(z) + \exp(g(z)), & \text{if } z \in W, \end{cases}$$

extends to an entire function  $f: \mathbb{C} \rightarrow \mathbb{C}$  of finite order.

*Proof.* Let  $R_0 > 0$  be large and define

$$\gamma_1 := (\gamma \cap \{z \in \mathbb{C} : |z| > R_0\}) \cup \{R_0 e^{i\theta} : \theta \in [\varphi(-R_0), \varphi(R_0)]\}$$

and let  $V'$  be the component of  $\mathbb{C} \setminus \gamma_1$  containing  $V$ .

Figure 4.4. Construction of the curve  $\gamma_1$ .

Then the function

$$f_1: V' \rightarrow \mathbb{C}, f_1(z) := \frac{1}{2\pi i} \int_{\gamma_1} \frac{\exp g(\zeta)}{\zeta - z} d\zeta$$

defines a holomorphic function on  $V'$ . By the Cauchy integral theorem we obtain for  $z \in V$

$$f(z) - f_1(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp g(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{\exp g(\zeta)}{\zeta - z} d\zeta = 0.$$

Thus  $f_1$  coincides with  $f$  on  $V$ . Moreover, we have for  $z \in V' \cap W$  by the Cauchy integral theorem that

$$f_1(z) - \tilde{f}(z) = \exp(g(z)).$$

Notice that  $f_1 = f|_{V'}$ . Since  $R_0$  was arbitrary, the function  $f$  is entire. It is left to show that  $f$  is of finite order. With equality (4.2.1) we obtain for  $z = re^{i\theta} \in W$

$$\begin{aligned} |g(re^{i\theta})| &= r^{5\pi} \cdot \exp(-2\pi \cos(\text{Log } r) \cdot \sinh \theta) \\ &\leq r^{5\pi} \cdot \exp\left(2\pi \sinh \frac{\pi}{3}\right). \end{aligned}$$

Since  $f(z) = \exp(g(z)) + O(1)$  for  $z \in W$  and  $f(z) = O(1)$  for  $z \in \mathbb{C} \setminus W$ , the function  $f$  is of order at most  $5\pi$  and thus the claim follows.  $\square$

*Proof of Lemma 4.2.2.* It is left to show that  $f \in \mathcal{B}$ . Since  $\tilde{f}$  is holomorphic on  $\mathbb{C} \setminus \gamma$ , we have

$$\tilde{f}'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp g(\zeta)}{(\zeta - z)^2} d\zeta.$$

Because the only difference between  $\tilde{f}$  and  $\tilde{f}'$  is that  $1/(\zeta - z)$  was replaced by  $1/(\zeta - z)^2$  in the integral, this yields with (4.2.5)

$$\left| \tilde{f}'(z) \right| \leq \frac{1}{2\pi\kappa^2} \left( C_1 + \frac{6}{C_2} \right)$$

for  $z \in V_\kappa \cup W_\kappa$ . For  $z \in U_\kappa(\gamma)$  and  $\gamma_z$  as in (4.2.7) we also have the same estimates in (4.2.9) and (4.2.10) with the extra factor  $1/\kappa$ . This yields that we have in total  $\left| \tilde{f}'(z) \right| \leq C_3/\kappa$  on  $\mathbb{C} \setminus \gamma$ .

Now we show that  $f \in \mathcal{B}$ , i.e. that  $\text{sing}(f^{-1})$  is bounded. Since

$$f'(z) := \begin{cases} \tilde{f}'(z), & \text{if } z \in V, \\ \tilde{f}'(z) + g'(z) \exp(g(z)), & \text{if } z \in W, \end{cases}$$

we obtain  $|f'(z) - g'(z) \exp(g(z))| \leq C_3/\kappa$  on  $W$ . Since  $f$  is bounded on  $V$ , it suffices to show that the set of critical values of  $f$  corresponding to critical points in  $W$  is bounded. That is, the set  $\{f(\zeta) : \zeta \in W, f'(\zeta) = 0\}$  is bounded. Let  $\xi \in W$  be a critical point of  $f$ . Thus

$$|f'(\xi) - g'(\xi) \exp(g(\xi))| = |g'(\xi)| \cdot \exp(\text{Re}(g(\xi))) \leq \frac{C_3}{\kappa}. \quad (4.2.11)$$

Moreover, we have

$$\begin{aligned} g'(z) &= \frac{\pi}{z} (5 + 2i \cos(\text{Log } z)) \cdot \exp(5\pi \text{Log } z + 2\pi i \sin(\text{Log } z)) \\ &= (5\pi + 2\pi i \cos(\text{Log } z)) \cdot z^{5\pi-1} \cdot \exp(2\pi i \sin(\text{Log } z)). \end{aligned}$$

In addition, we have

$$\begin{aligned} |5\pi + 2\pi i \cos(\text{Log}(x + iy))| &\geq \text{Re}(5\pi + 2\pi i \cos(\text{Log}(x + iy))) \\ &= 5\pi + 2\pi \sin\left(\text{Log}\left(\sqrt{x^2 + y^2}\right)\right) \cdot \sinh\left(\arctan\left(\frac{y}{x}\right)\right) \\ &\geq 5\pi - 2\pi \sinh\frac{\pi}{3} \\ &\geq \frac{5\pi}{2} \end{aligned}$$

and

$$\begin{aligned} |\exp(2\pi i \sin(\text{Log}(x + iy)))| &= \exp(-2\pi \text{Im}(\sin(\text{Log}(x + iy)))) \\ &= \exp\left(-2\pi \cos\left(\text{Log}\left(\sqrt{x^2 + y^2}\right)\right)\right) \cdot \sinh\left(\arctan\left(\frac{y}{x}\right)\right) \\ &\geq \exp(-3\pi). \end{aligned}$$

Together this yields for large  $z \in W$

$$|g'(z)| \geq \frac{5\pi}{2} e^{-3\pi} \cdot |z|^{5\pi-1} \geq 1.$$

With (4.2.11) we obtain

$$|f(\xi)| \leq C_3 + \exp(\text{Re } g(\xi)) \leq C_3 + \frac{C_3}{\kappa |g'(\xi)|} \leq C_3 + \frac{C_3}{\kappa}$$

Since  $f$  is of finite order, the number of asymptotic values is finite by the corollary of Lemma 2.5.8. Thus  $\text{sing}(f^{-1})$  is bounded, i.e.  $f \in \mathcal{B}$ .  $\square$



### 4.3 Proof of Theorem 4

*Proof of Theorem 4.* From Lemma 4.2.2 we have obtained a suitable function  $f$  which is close to  $e^g$  in the subset  $A$  of our desired tract. For our purpose it is an advantage to restrict to functions of disjoint type.

First choose  $R > 0$  large enough such that  $\text{sing}(f^{-1}) \subset D(0, e^R)$ . Later in the proof we will make the choice of  $R$  more precise. By Lemma 2.6.12 we obtain a disjoint type function by considering  $f_0 := \lambda f$ , where  $0 < \lambda < e^{-R}$  is small enough such that  $f_0(D(0, e^R)) \subset \mathbb{D}$  and  $\text{sing}(f_0^{-1}) \subset \mathbb{D}$ . Moreover, we have  $D(0, e^R) \subset \mathcal{F}(f_0)$  by Montel's theorem.

By Remark 2.7.5 we know that  $\mathcal{J}(f_0)$  consists of an uncountable union of pairwise disjoint hairs. We now want to show that every hair of  $f_0$  is nowhere differentiable including all endpoints. For  $\tilde{A} := f_0^{-1}(\mathbb{C} \setminus \mathbb{D})$  denote  $T(0) := \text{Log}(\tilde{A})$  and for  $k \in \mathbb{Z}$  denote  $T(k) := T(0) + 2\pi ik$ . With

$$\mathcal{T} := \bigcup_{k \in \mathbb{Z}} T(k)$$

the logarithmic transform of  $f_0$  takes the form  $F: \mathcal{T} \rightarrow \mathbb{H}_{>0}$ . By Remark 2.7.7(2) we know that  $\mathcal{J}(F)$  also consists of an uncountable union of pairwise disjoint hairs. As we did for Zorich maps, we define an admissible address  $\underline{s} = s_0 s_1 \cdots \in \mathbb{Z}^{\mathbb{N}_0}$  in such a way that  $z$  has address  $\underline{s}$  if  $F^k(z) \in T(s_k)$  for all  $k \in \mathbb{N}_0$ . From now on we fix an admissible address  $\underline{s}$  and consider the hair  $g_{\underline{s}}$  which is the set  $\{z \in \mathbb{C} : z \text{ has address } \underline{s}\}$ . Then  $\text{Re } F^n|_{g_{\underline{s}}([t_{\underline{s}}, \infty))} \rightarrow \infty$  locally uniformly as  $n \rightarrow \infty$ . It follows from the construction of the tract  $T \supset T(0)$  and the fact that  $F^n(g_{\underline{s}}) = g_{\sigma^n(\underline{s})}$  for all  $n \in \mathbb{N}$  that  $t \mapsto \text{Im}(g_{\sigma^n(\underline{s})}(t) - 2\pi i s_n)$  changes its sign infinitely often. Remark 4.2.1 yields that there exists for all  $n \in \mathbb{N}_0$  a sequence  $(z_{n,k})_{k \in \mathbb{N}} \in (g_{\sigma^n(\underline{s})})^{\mathbb{N}}$  such that the following holds:

1. We have for all  $n \in \mathbb{N}_0$

$$\text{Im}(z_{n,k} - 2\pi i s_n) \begin{cases} > 1/5, & \text{if } k \text{ is odd,} \\ < -1/5, & \text{if } k \text{ is even.} \end{cases}$$

2. For all  $n \in \mathbb{N}_0$  and  $k \in \mathbb{N}$  we have  $\text{Re } z_{n,k+1} - \text{Re } z_{n,k} = \pi$  and thus  $\lim_{k \rightarrow \infty} \text{Re } z_{n,k} = \infty$ .

Let  $w_0 \in g_{\underline{s}}([t_{\underline{s}}, \infty))$  and denote  $w_n := F^n(w_0)$  for all  $n \in \mathbb{N}$ . In this case we will take  $z_{n,k} := x_{n,k} + iy_{n,k}$ , where  $x_{n,k}$  can be chosen to be  $\pi/2 + \pi(\lceil \text{Re } g_{\sigma^n(\underline{s})}(t_{\underline{s}})/\pi \rceil + k)$  according to Remark 4.2.1. Thus the imaginary part of  $z_{n,k} - 2\pi i s_n$  is bounded uniformly away from 0.

Since  $F$  is the logarithmic transform of the function  $f_0$  of disjoint type, we have  $\overline{\mathcal{T}} \subset \mathbb{H}_{\geq 0}$  and there exists by Lemma 2.6.15 a constant  $K > 1$  such that we have  $|F'(z)| \geq K$  for all  $z \in \mathcal{T}$ . Thus every branch  $F_k^{-1}: \mathbb{H}_{>0} \rightarrow T(s_k)$  of  $F^{-1}$  is a uniform contraction on  $\mathbb{H}_{>0}$ .

Now we proceed as follows. If  $\text{Im}(w_n - 2\pi i s_n) > 0$ , there exists a minimal  $k_n \in 2\mathbb{N}$  such that

$$\text{Re } w_n < x_{n,k_n} \quad \text{and} \quad y_{n,k_n} - 2\pi s_n < 0$$

and consider the triangle  $\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})$  (see Figure 4.5). Since for all  $n \in \mathbb{N}$

$$\frac{\pi}{4} < x_{n,k_n} - \operatorname{Re} w_n < 2\pi$$

and

$$\frac{1}{5} < \operatorname{Im} w_n - y_{n,k_n} < \frac{3}{2}$$

holds by Remark 4.2.1, we obtain for all  $n \in \mathbb{N}$

$$\frac{1}{20\pi} < \left| \frac{\operatorname{Im} w_n - y_{n,k_n}}{\operatorname{Re} w_n - x_{n,k_n}} \right| < \frac{6}{\pi}.$$

Using the fact that  $x_{n,k_n+1} - x_{n,k_n} = \pi$ , we obtain

$$\frac{2}{5\pi} < \left| \frac{y_{n,k_n} - y_{n,k_n+1}}{x_{n,k_n} - x_{n,k_n+1}} \right| < \frac{3}{2\pi}.$$

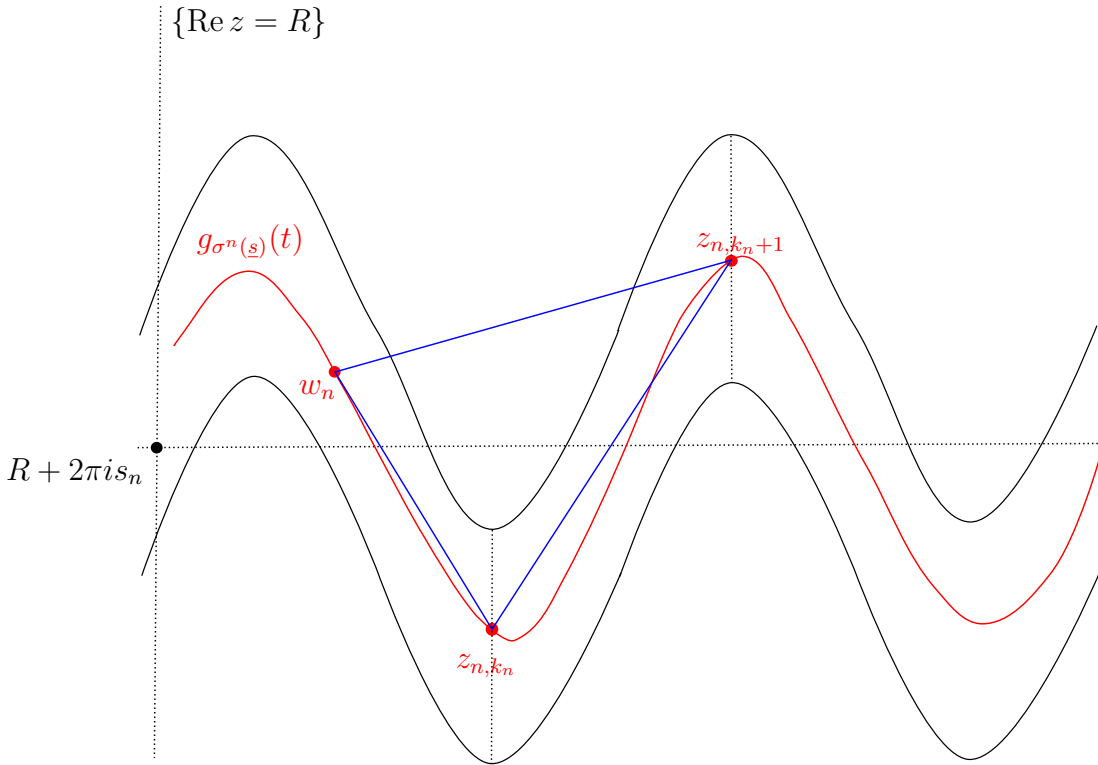


Figure 4.5. The triangle  $\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})$  stays non-degenerate under the pullback  $(\bigcirc_{j=0}^{n-1} F_j^{-1})$  for all  $n \in \mathbb{N}$ .

Hence

$$\frac{1}{20\pi} < \arctan \left( \left| \frac{\operatorname{Im} w_n - y_{n,k_n}}{\operatorname{Re} w_n - x_{n,k_n}} \right| \right) + \arctan \left( \left| \frac{y_{n,k_n} - y_{n,k_n+1}}{x_{n,k_n} - x_{n,k_n+1}} \right| \right) < \frac{\pi}{2}$$

and thus the angle at  $z_{n,k_n}$  is always greater than  $\pi/2$  and less than  $19\pi/20$ . This yields that  $\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})$  is non-degenerate, say

$$\mathcal{D}(\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})) < 1 - \delta \quad (4.3.1)$$

for a suitable  $\delta > 0$ .

Analogously, if  $\text{Im}(w_n - 2\pi i s_n) < 0$ , there exists  $k_n \in 2\mathbb{N} + 1$  with the same property. Since  $4\pi > \sqrt{(3\pi)^2 + (3/4)^2}$  and by our construction of the tract we know that the disc  $D_n := D(w_n, 4\pi)$  contains the triangle  $\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})$  for all  $n \in \mathbb{N}$ . If we choose  $R > 4\pi$  we can guarantee that  $\text{Re } w_n - 4\pi > 0$  for all  $n \in \mathbb{N}_0$  since  $\text{Re } w_n > R$ . Now we want to show that

$$\left( \bigcirc_{j=0}^{n-1} F_j^{-1} \right) \Big|_{D_n} = F_0^{-1} \circ \dots \circ F_{n-1}^{-1} \Big|_{D_n}$$

has a uniformly bounded distortion. If we put  $R_n := \text{dist}(w_n, \partial\mathbb{H}_{\geq 0})$  for all  $n \in \mathbb{N}_0$ , we obtain  $R_n = \text{Re } w_n > R$  for all  $n \in \mathbb{N}_0$ . Since  $\bigcirc_{j=0}^{n-1} F_j^{-1}$  is conformal on  $D(w_n, R_n)$ , we obtain by Lemma 2.5.13

$$\mathcal{L} \left( \left( \bigcirc_{j=0}^{n-1} F_j^{-1} \right) \Big|_{D_n} \right) \leq \left( \frac{R_n + 4\pi}{R_n - 4\pi} \right)^4 \leq \left( \frac{R + 4\pi}{R - 4\pi} \right)^4 = \left( \frac{1 + \frac{4\pi}{R}}{1 - \frac{4\pi}{R}} \right)^4 \leq 1 + \delta$$

for  $R$  large enough. To guarantee convexity of  $(\bigcirc_{j=0}^{n-1} F_j^{-1})(D_n)$ , we need by Lemma 2.5.9 to take  $R$  large enough such that  $2\pi/R < \sqrt{2} - 1$ . Together with Lemma 2.5.16 and (4.3.1) this yields

$$\begin{aligned} & \mathcal{D} \left( \Delta \left( \left( \bigcirc_{j=0}^{n-1} F_j^{-1} \right) (w_n), \left( \bigcirc_{j=0}^{n-1} F_j^{-1} \right) (z_{n,k_n}), \left( \bigcirc_{j=0}^{n-1} F_j^{-1} \right) (z_{n,k_n+1}) \right) \right) \\ & \leq (1 + \delta) \cdot \mathcal{D}(\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})) \\ & < (1 + \delta)(1 - \delta) \\ & = 1 - \delta^2. \end{aligned}$$

Since  $|F'(z)| \geq K$  for all  $z \in \mathcal{T}$  and  $(F_j^{-1})'(z) = (F_0^{-1})'(z)$  for all  $j \in \mathbb{N}_0$ , we obtain by the chain rule

$$\left| \left( \bigcirc_{j=0}^{n-1} F_j^{-1} \right)'(z) \right| = \frac{1}{|(F^n)'(F_0^{-n}(z))|} = \frac{1}{\prod_{j=0}^{n-1} |F'(F_0^{j-n}(z))|} \leq \frac{1}{K^n}.$$

Since the distortion of  $(\bigcirc_{j=0}^{n-1} F_j^{-1})$  on the triangle  $\Delta(w_n, z_{n,k_n}, z_{n,k_n+1})$  is bounded for all  $n \in \mathbb{N}_0$ , these triangles converge to the single point  $w_0$  for  $n \rightarrow \infty$ . On the other hand, these triangles are always non-degenerate. Thus  $g_{\underline{s}}$  is not differentiable in  $w_0$  by Lemma 2.5.18. Since  $w_0$  was arbitrary, the claim follows.  $\square$



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# Nomenclature

$\mathcal{H}^s(A)$	$s$ -dimensional Hausdorff measure of $A$ , page 8
$A(\xi)$	Basin of attraction of the fixed point $\xi$ , page 22
$C$	Endpoints of the hairs of $f$ , page 28
$C_0^\infty(U)$	Space of test functions in $U$ , page 9
$C_1$	Positive real constant, page 42
$E$	Reference function $E(t) = e^t - 1$ , page 31
$J$	Complement of the basin of attraction of the unique fixed point $\xi$ , page 28
$K$	(Maximal) dilatation of $f$ , page 12
$M(r, f)$	Maximum modulus of $f$ , page 14
$Q$	$d-1$ -dimensional hypercube of sidelength 2 centered at the origin, page 27
$S_{\alpha, \beta}$	Set of all points $z$ in the complex plane with $\operatorname{Re} z > \alpha$ and $ \operatorname{Im} z  < \beta$ , page 51
$T(\underline{s}, t)$	Projection in $\Xi$ to the second component, page 49
$T(r)$	Tract above $P(r)$ , page 29
$V_\kappa$	Points outside of $W$ with distance greater than $\kappa$ to $\partial W$ , page 57
$W^{k,p}(U)$	Sobolev space of functions, whose $j$ -th weak derivate, $j \leq k$ , are in $L^p(U)$ , page 9
$W_\kappa$	Points inside of $W$ with distance greater than $\kappa$ to $\partial W$ , page 57
$\mathcal{B}$	Eremenko-Lyubich class, page 20
$\Delta(z_1, z_2, z_3)$	Triangle with vertices $z_1, z_2$ and $z_3$ , page 15
$\mathcal{F}(f)$	Fatou set of $f$ , page 19

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$\mathcal{I}(f)$	Escaping set of $f$ , page 19
$\mathcal{J}(f)$	Julia set of $f$ , page 19
$\Lambda^r$	Branch of $f^{-1}$ with image in $T(r)$ , page 29
$\Omega$	Parabolic-like domain which contains large iterates of points in $J \setminus C$ , page 48
$\Sigma$	Symbolic space, page 31
$\Sigma'$	Set of admissible points, page 32
$\Xi$	Model space, page 49
$\dim A$	Hausdorff dimension of $A$ , page 8
$\left( \bigcirc_{j=0}^k L_j \right)$	Composition of $L_j$ for $j \in \{0, \dots, k\}$ , page 33
$\gamma_z(t)$	The curve $\gamma$ perturbed in a neighborhood of $z$ , page 57
$\mathbb{H}_{\geq c}$	Upper halfspace, page 7
$\mathbb{S}_+$	Upper hemisphere, page 27
$\mathcal{D}(\Delta(z_1, z_2, z_3))$	Degree of degeneracy of $\Delta(z_1, z_2, z_3)$ , page 16
$\mathcal{P}(f)$	Postsingular set of $f$ , page 20
$\ Df(x)\ $	Operator norm of $Df$ in $x$ , page 11
$\ x\ _\infty$	Maximum norm of $x \in \mathbb{R}^d$ , page 27
$\text{sing}(f^{-1})$	Set of singular values of $f$ , page 13
$\rho(f)$	Order of the function $f$ , page 14
$\sigma(\underline{s})$	Shift map of the sequence $\underline{s}$ , page 49
$\underline{s}(x)$	External address of $x$ , page 31
$c_1$	Positive real constant, page 30
$f_0$	Modified function $f$ of disjoint type, page 61
$f_a$	Shifted Zorich map by $a$ in the last coordinate, page 28
$h$	Bi-Lipschitz map which maps $Q$ onto $\mathbb{S}_+$ , page 27

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$l(Df(x))$	Operator norm of $Df$ in $x$ with inf instead of sup, page 12
$t_{\underline{s}}$	Infimum over all $t > 0$ such that $\limsup_{k \rightarrow \infty} \frac{\ s_k\ _2}{E^k(t)} < \infty$ , page 32
$ACL^p(U)$	Class of functions which are absolutely continuous on lines on $U$ and the partial derivatives are in $L^p_{loc}(U)$ , page 10



# Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Die Hauptergebnisse des dritten Kapitels der Dissertation sind in folgender Publikation enthalten:

- P. Comdühr, *On the differentiability of hairs for Zorich maps*, Preprint, <https://arxiv.org/pdf/1701.06873.pdf>. Erscheint in *Ergodic Theory and Dynamical Systems*.

Kiel, den 22.08.2017

(Patrick Comdühr)