

Applications of foldings of \mathbb{A} -graphs

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Abstract

This thesis gives some applications of the theory of folding of \mathbb{A} -graphs. In the first application, we investigate the relationship between the bridge number of a knot \mathfrak{k} , a geometrically defined invariant of the knot, and the meridional rank of the knot group $G(\mathfrak{k})$. We give a new proof that these two invariants coincide if \mathfrak{k} is a torus knot. With similar techniques we show that the same holds if \mathfrak{k} is a knot whose complement is a graph manifold. Next, we study killers of knot groups. We prove that the group $G(\mathfrak{k})$ of any cable knot \mathfrak{k} contains infinitely many killers, none of which is the automorphic image of another. As a corollary we show that if \mathfrak{k} is a knot whose complement is a graph manifold, then $G(\mathfrak{k})$ has the same property. Finally, we study generating sets of planar discontinuous groups. We show for a large class of planar discontinuous groups that there are only finitely many Nielsen equivalence classes of generating tuples.

Zusammenfassung

Diese Thesis beinhaltet einige Anwendungen der Faltungstheorie von Graphen von Gruppen. Dabei beschäftigen wir uns in der ersten Anwendung mit der Beziehung der sogenannten Brückenzahl, einer geometrischen Invarianten eines Knoten \mathfrak{k} , und dem sogenannten meridialen Rang der assoziierten Knotengruppe $G(\mathfrak{k})$. Genauer gesagt, werden wir in diesem Abschnitt einen neuen Beweis dafür liefern, dass diese beiden Invarianten übereinstimmen, falls \mathfrak{k} ein sogenannter Torusknoten ist. Mit ähnlichen Methoden beweisen wir in der ersten Veröffentlichung aus dieser Thesis, dass die beiden erwähnten Invarianten ebenfalls gleich sind, falls \mathfrak{k} ein Knoten ist, dessen Komplement die Struktur einer Graph Mannigfaltigkeit hat. Danach widmen wir uns den sogenannten "Killern" von Knotengruppen. In diesem Kontext werden wir nachweisen, dass die Knotengruppe $G(\mathfrak{k})$ eines jeden "Kabelknoten" \mathfrak{k} unendlich viele Äquivalenzklassen von Killern enthält, wobei wir zwei Killer x und y als äquivalent bezeichnen, falls ein Automorphismus φ der Gruppe $G(\mathfrak{k})$ existiert, s.d. $\varphi(x) = y$.

Als eine direkte Konsequenz dieses Satzes, werden wir einsehen, dass für jeden Knoten \mathfrak{k} , dessen Komplement eine Graph Mannigfaltigkeit ist, die Gruppe $G(\mathfrak{k})$ ebenfalls unendlich viele Äquivalenzklassen von Killern enthält.

Am Ende dieser Arbeit werden wir uns dann noch Erzeugendensystemen planarer diskontinuierlicher Gruppen zuwenden. In diesem Abschnitt werden wir beweisen, dass es für eine große Klasse dieser Gruppen nur endlich viele Nielsen-äquivalente Tupel von Erzeugern gibt.

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The idea of using foldings to study group actions on trees was introduced by J. Stallings in a seminal paper [43], where he applied foldings to investigate free groups. Free groups are exactly those groups that admit free actions on simplicial trees. Later Stallings [44] offered a way to extend these ideas to non-free actions of groups on graphs and trees. Bestvina and Feighn [5] gave a systematic treatment of Stallings' approach in the context of graph of groups and applied this theory to prove a generalization of Dunwoody's accessibility result.

Later Dunwoody [12] refined the theory by introducing the notion of vertex morphisms. Dunwoody [13] used foldings to construct a small unstable action on a real tree. Some other applications of foldings in the graph of groups setting can be found in [31, 54, 55].

The notions of \mathbb{A} -graphs and of foldings of \mathbb{A} -graphs were introduced by Kapovich, Myasnikov and Weidmann [18] who developed a combinatorial treatment of foldings geared towards more computational questions. Because of their algorithmic goals, they developed the theory at the level of the quotient graph of groups rather than at the level of the Bass-Serre covering tree, as had been done before in the Stallings-Bestvina-Feighn-Dunwoody treatment of foldings. As an application of their new methods they proved that the membership problem is solvable for the fundamental groups of a large class of graph of groups.

The theory of \mathbb{A} -graphs along with the notion of foldings proved to be a powerful tool to approach problems in low dimensional topology as well as in group theory, see for example [49, 52, 51]. In this thesis we give more application of the theory of \mathbb{A} -graphs and foldings to knot theory and to the theory of Fuchsian groups.

Meridional rank vs. bridge number of knots

Let \mathfrak{k} be a knot in S^3 . It is well-known that the knot group $\pi_1(S^3 - \mathfrak{k})$ of \mathfrak{k} can be generated by $b(\mathfrak{k})$ conjugates of the meridian, where $b(\mathfrak{k})$ is the bridge number of \mathfrak{k} . The meridional rank $w(\mathfrak{k})$ of \mathfrak{k} is the smallest number of conjugates of the meridian that generate its group. Thus we always have $w(\mathfrak{k}) \leq b(\mathfrak{k})$. It was asked by S. Cappell and J. Shaneson [19, pb 1.11], as well as by K. Murasugi, whether the opposite inequality always holds, i.e. whether $b(\mathfrak{k}) = w(\mathfrak{k})$ for any knot \mathfrak{k} . To this day no counterexamples are known but the equality has been verified in a number of cases:

1. For generalized Montesinos knots this is due to Boileau and Zieschang [7].
2. For torus knots this is a result of Rost and Zieschang [32], see Chapter 2 for a new proof of this fact using foldings of \mathbb{A} -graphs.
3. The case of knots of meridional rank 2 (and therefore also knots with bridge number 2) is due to Boileau and Zimmermann [8].
4. For a class of knots also referred to as generalized Montesinos knots, the equality is due to Lustig and Moriah [23].
5. For some iterated cable knots this is due to Cornwell and Hemminger [11].
6. For knots of meridional rank 3 whose double branched cover is a graph manifold the equality can be found in [6].

A knot \mathfrak{k} is called a *torus knot* if \mathfrak{k} can be isotoped into the surface of a standardly embedded torus in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. We give a new proof of the following theorem from Rost and Zieschang [32].

Theorem (Rost and Zieschang). *If \mathfrak{k} is a torus knot, then $w(\mathfrak{k}) = b(\mathfrak{k})$.*

In an unpublished 2013 manuscript M. Boileau, Y. Jang and R. Weidmann proved that the meridional rank coincides with the bridge number for knots obtained from torus knots by satellite operations with braid patterns. In this thesis we generalize their result to the case that the complement of the knot contains JSJ-components homoeomorphic to a composing space. More precisely, we consider the class of *graph knots*, i.e the smallest class of knots that contains all torus knots and is closed under connected sums and cabling.

Theorem. *If \mathfrak{k} is a graph knot, then $w(\mathfrak{k}) = b(\mathfrak{k})$.*

Killers of knot groups

We say that an element of a group is a killer if the element normally generates the group. If the group in question is the group of a knot then the meridian is clearly a killer of the knot group. It is natural to ask whether there are other killers other than the meridian of the knot. Since the automorphic image of a killer is again a killer, it is natural to say that two killers are equivalent if there exists an automorphism sending one to the other.

From either a Wirtinger presentation of the group of a knot or a bridge presentation of the knot, we see that the meridian of a knot is a killer of its group. The first example of a killer of a knot group, that is not the automorphic image of the meridian, was given in [45, Theorem 3.11]. Recently, Silver–Whitten–Williams [41, Corollary 1.3] showed that if the knot is hyperbolic with 2-bridges or a torus knot or a hyperbolic knot with unknotting number one, then its group contains infinitely many pairwise inequivalent killers.

In [41, Conjecture 3.3] it is conjectured that the group of any nontrivial knot has infinitely many inequivalent killers, see also [2, Question 9.26].

In this thesis we show that this conjecture holds for a large class of knots. The following is our main result on this question.

Theorem. *The group of a cable knot contains infinitely many pairwise inequivalent killers.*

We then observe the following property of knots with regard to killers.

Theorem. *If one of the prime factors of a knot has infinitely many inequivalent killers then the knot itself has infinitely many killers.*

As a corollary we obtain the following result.

Corollary. *The group of a knot whose complement is a graph manifold contains infinitely many killers.*

Nielsen equivalence in planar discontinuous groups

Let G be a group, and $\mathcal{T} = (g_1, \dots, g_n)$ and $\mathcal{T}' = (g'_1, \dots, g'_n)$ be two n -tuples of elements of G . Recall that \mathcal{T} and \mathcal{T}' are called *elementary equivalent* if one of the following conditions holds:

1. There exists some permutation $\sigma \in S_n$ such that $g'_i = g_{\sigma(i)}$ for $1 \leq i \leq n$.
2. $g'_i = g_i^{-1}$ for some $i \in \{1, \dots, n\}$ and $g'_j = g_j$ for $j \neq i$.
3. $g'_i = g_i g_j^\varepsilon$ for some $i \neq j$ and $\varepsilon \in \{-1, 1\}$. Furthermore, $g'_k = g_k$ for $k \neq i$.

We say that \mathcal{T} and \mathcal{T}' are *Nielsen equivalent*, and write $\mathcal{T} \sim \mathcal{T}'$, if there exists a finite sequence of n -tuples

$$\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k = \mathcal{T}'$$

such that \mathcal{T}_{i-1} and \mathcal{T}_i are elementary equivalent for $1 \leq i \leq k$.

Planar discontinuous groups are the groups given by the presentation

$$\langle a_1, \dots, a_p, s_1, \dots, s_k \mid s_1^{h_1}, \dots, s_k^{h_k}, s_1 \cdots s_k R(a_1, \dots, a_p) \rangle.$$

where $k \geq 0$, $2 \leq h_i < \infty$ and

- (a) $p = 2g$ and $R(a_1, \dots, a_p) = [a_1, a_2][a_3, a_4] \cdots [a_{p-1}, a_p]$ if $\varepsilon = +1$.
- (b) $p = g$ and $R(a_1, \dots, a_p) = a_1^2 \cdots a_g^2$ if $\varepsilon = -1$.

The signature of Γ is defined as the tuple $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$. The fundamental group $\pi_1(\Sigma)$ of a closed surface Σ and more generally, Fuchsian groups, are planar discontinuous group.

Definition. A *standard generating tuple* of the planar group Γ with $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$ is a tuple of the form

$$\mathcal{T} = (a_1, \dots, a_p, s_{i_1}^{\nu_1}, \dots, s_{i_{k-1}}^{\nu_{k-1}})$$

where $1 \leq i_1 < \dots < i_{k-1} \leq k$ and $\nu_1, \dots, \nu_{k-1} \in \mathbb{N}$ such that $\gcd(\nu_j, h_j) = 1$ for all $1 \leq j \leq k-1$.

In [56] H. Zieschang proves that any minimal generating tuple of a surface group is Nielsen equivalent to the standard generating tuple and proves a similar result for Fuchsian groups that lead to the solution of the rank problem [30]. Nielsen equivalence in Fuchsian groups has been studied by many authors and has some important applications in 3-manifold topology, see [21, 22, 33, 34] for example.

Recently L. Louder [20] has generalized Zieschang's result to arbitrary generating tuples of a surface group proving that any generating tuple is Nielsen equivalent to a standard generating tuple. We generalize Louder's ideas for the case of planar groups and prove the following result.

Theorem. *Let Γ be a planar group with signature $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$. Suppose that the following hold:*

1. $2 \leq h_i < \infty$ for all $1 \leq i \leq k$.
2. $g \geq 1$ if $\varepsilon = 1$ and $g \geq 2$ if $\varepsilon = -1$.

Then any irreducible generating tuple of Γ is Nielsen equivalent to a standard generating tuple.

In this thesis we need some facts from Bass-Serre theory, \mathbb{A} -graphs, foldings of \mathbb{A} -graphs and basic notions from 3-manifolds topology and knot theory. This chapter will provide some preliminaries from those theories.

1.1 Graph of groups

In this section we recall some basic notions about graphs of groups and morphisms of graphs of groups. For a detailed account of Bass-Serre theory we refer to the article of H. Bass [3], the book of J.-P. Serre [40] or to [18] for slightly more similar notation.

A *graph* A is understood to consist of a vertex set VA , a set of oriented edges EA , a fixed point free involution $^{-1} : EA \rightarrow EA$ and a map $\alpha : EA \rightarrow VA$ which assigns to each edge e its initial vertex $\alpha(e)$. Moreover, we will denote $\alpha(e^{-1})$ by $\omega(e)$ and call $\omega(e)$ the terminal vertex of the edge e .

A finite sequence $\gamma = e_1, \dots, e_k$ of edges of a graph A is called a *path* of length k in A if $\omega(e_i) = \alpha(e_{i+1})$ for all $1 \leq i < k$. The vertex $\alpha(e_1)$ is called the initial vertex of γ , denoted $\alpha(\gamma)$, and $\omega(e_k)$ is called the terminal vertex of γ , denoted $\omega(\gamma)$. We assume that any vertex v of A is a (*degenerate*) path of length 0 with initial vertex and terminal vertex v . A path γ is said to be *reduced* if $e_{i+1} \neq e_i^{-1}$ for all $1 \leq i < k$. A path γ is *closed* if its initial vertex $\alpha(\gamma)$ and terminal vertex $\omega(\gamma)$ coincide.

A graph A is *connected* if for any two of its vertices u and v , there exists a path in A from u to v . A connected graph A is called a *tree* if there exists no non-degenerate reduced closed path in A . Equivalently, a tree is a graph in which any two vertices are connected by exactly

one reduced path.

A *morphism* $\varphi : B \rightarrow A$ from the graph B to the graph A is a pair (φ_V, φ_E) , where $\varphi_V : VB \rightarrow VA$ and $\varphi_E : EB \rightarrow EA$ are maps such that $\alpha(\varphi_E(f)) = \varphi_V(\alpha(f))$ and $\varphi_E(f^{-1}) = \varphi_E(f)^{-1}$ for all $f \in EB$. We will often denote the maps φ_V and φ_E simply by φ . If $\gamma = f_1, \dots, f_k$ is a path in B from $u \in VB$ to $u' \in VB$, then $\varphi(\gamma) := \varphi(f_1), \dots, \varphi(f_k)$ is a path in A from $\varphi(u) \in VA$ to $\varphi(u') \in VA$.

A *graph of groups* \mathbb{A} consists of an underlying graph A (which will always be tacitly assumed to be connected) together with the following data. For each vertex $v \in VA$ there is an associated *vertex group* A_v and for each edge $e \in EA$ there is an associated *edge group* A_e , with $A_{e^{-1}} = A_e$ for all $e \in EA$. Every edge $e \in EA$ comes equipped with two *boundary monomorphisms* $\alpha_e : A_e \rightarrow A_{\alpha(e)}$ and $\omega_e : A_e \rightarrow A_{\omega(e)}$, with $\alpha_{e^{-1}} = \omega_e$ for all $e \in EA$.

An \mathbb{A} -*path* of length $k \geq 0$ from $v \in VA$ to $v' \in VA$ is a sequence

$$p = a_0, e_1, a_1, \dots, a_{k-1}, e_k, a_k$$

where e_1, \dots, e_k is a path in the underlying graph A from v to v' , $a_0 \in A_v$ and $a_i \in A_{\omega(e_i)}$ for all $1 \leq i \leq k$. We will call k the length of p and denote it by $|p|$. Note that we allow $k = |p|$ to be equal to zero, in which case $v = v'$ and $p = a_0$ with $a_0 \in A_v$. For an \mathbb{A} -path $p = a_0, e_1, a_1, \dots, a_{k-1}, e_k, a_k$ and $1 \leq i \leq j \leq k$ we call the \mathbb{A} -path $a_{i-1}, e_i, a_i, \dots, a_{j-1}, e_j, a_j$ an \mathbb{A} -subpath of p .

If $p = a_0, e_1, a_1, \dots, a_{k-1}, e_k, a_k$ is an \mathbb{A} -path from $v \in VA$ to $v' \in VA$ and $p' = a'_0, e'_1, a'_1, \dots, a'_{l-1}, e'_l, a'_l$ is an \mathbb{A} -path from $v' \in VA$ to $v'' \in VA$, then the *concatenation* pp' of p and p' is defined as

$$pp' = a_0, e_1, a_1, \dots, a_{k-1}, e_k, a_k a'_0, e'_1, a'_1, \dots, a'_{l-1}, e'_l, a'_l.$$

Note that pp' is an \mathbb{A} -path from v to v'' and that $|pp'| = |p| + |p'|$.

Let \mathbb{A} be a graph of groups. We say that two \mathbb{A} -paths p and p' are *elementary homotopic* if p' can be obtained from p by one of the following two operations or their inverses:

1. (Elementary reduction) Replace an \mathbb{A} -subpath of type $a, e, \omega_e(c), e^{-1}, a'$ by $a\alpha_e(c)a'$ where $c \in A_e$ and $a, a' \in A_{\alpha(e)}$.
2. Replace an \mathbb{A} -subpath a, e, a' by $a\alpha_e(c), e, \omega(c^{-1})a'$ for some $c \in A_e$.

We say that two \mathbb{A} -paths p and p' are *homotopic* and write $p \sim p'$ if there exists a finite sequence $p = p_0, p_1, \dots, p_r = p'$ of \mathbb{A} -paths such that p_i and p_{i+1} are elementary homotopic for $1 \leq i \leq r - 1$. This clearly defines an equivalence relation on the set of all \mathbb{A} -paths.

If p is an \mathbb{A} -path, we will denote the \sim -equivalence class of p by $[p]$. Note $p \sim p'$ implies that p and p' have the same initial vertex and the same terminal vertex. Moreover, if q and q'

are \mathbb{A} -paths such that $q \sim q'$ and that the initial vertex of q coincides with the terminal vertex of p , then $pq \sim p'q'$.

Definition 1.1.1 (Fundamental group of a graph of groups). Let v_0 be a vertex of A . We define the fundamental group $\pi_1(\mathbb{A}, v_0)$ of the graph of groups \mathbb{A} as the set of \sim -equivalence classes of \mathbb{A} -paths from v_0 to v_0 endowed with multiplication $[p][q] := [pq]$.

We say that an \mathbb{A} -path p is *reduced* if p has no \mathbb{A} -subpath of the form $a, e, \omega_e(c), e^{-1}, a'$ for $a, a' \in A_{\alpha(e)}$ and $c \in A_e$. Any \mathbb{A} -path is homotopic to a reduced \mathbb{A} -path, and such a reduced \mathbb{A} -path can be obtained by applying elementary reductions as long as possible. The following proposition implies that the reduced \mathbb{A} -path obtained in this way is almost unique. For a proof see [40, Theorem 11].

Proposition 1.1.2. (*Reduced form theorem*) Let \mathbb{A} be a graph of groups and

$$p = a_0, e_1, a_1, \dots, a_{k-1}, e_k, a_k \quad \text{and} \quad q = b_0, f_1, b_1, \dots, b_{l-1}, f_l, b_l$$

be reduced \mathbb{A} -paths.

Then p is homotopic to q if, and only if, $k = l$, $e_i = f_i$ for $1 \leq i \leq k$ and there exist elements $c_i \in A_{e_i}$ ($1 \leq i \leq k$) such that the following hold:

1. $b_0 = a_0 \alpha_{e_1}(c_1)$.
2. $b_i = \omega_{e_i}(c_i) a_i \alpha_{e_{i+1}}(c_{i+1})$ for $1 \leq i \leq k-1$.
3. $b_k = \omega_{e_k}(c_k) a_k$.

A *morphism* from the graph of groups \mathbb{B} to the graph of groups \mathbb{A} is a tuple

$$\varphi = (\varphi, \{\varphi_u \mid u \in VB\}, \{\varphi_f \mid f \in EB\}, \{o_f \mid f \in EB\})$$

where $\varphi : B \rightarrow A$ is a morphism from the graph B to the graph A , $\varphi_u : B_u \rightarrow A_{\varphi(u)}$ is a homomorphism for all $u \in VB$, $\varphi_f : B_f \rightarrow A_{\varphi(f)}$ is a homomorphism for all $f \in EB$ with $\varphi_f = \varphi_{f^{-1}}$, and $o_f \in A_{\alpha(\varphi(f))}$ for all $f \in EB$ such that

$$\varphi_{\alpha(f)} \circ \alpha_f(c) = o_f \cdot (\alpha_{\varphi(f)} \circ \varphi_f(c)) \cdot o_f^{-1}$$

for all $c \in B_f$. For brevity we write $\varphi : \mathbb{B} \rightarrow \mathbb{A}$.

We will denote the element $o_{f^{-1}} \in A_{\alpha(\varphi(f^{-1}))} = A_{\omega(\varphi(f))}$ alternatively by t_f^{-1} . We will write o_f^φ or t_f^φ instead of o_f or t_f if we want to make explicit that the elements come from the morphism φ .

Let now $\varphi : \mathbb{B} \rightarrow \mathbb{A}$ be a graph of groups morphism. To any \mathbb{B} -path $p = b_0, f_1, b_1, \dots, f_k, b_k$ we associate the \mathbb{A} -path

$$\mu(p) = a_0, \varphi(f_1), a_1, \dots, \varphi(f_k), a_k$$

where $a_0 = \varphi_{\alpha(f_1)}(b_0) \cdot o_{f_1}$, $a_k = t_{f_k} \cdot \varphi_{\omega(f_k)}(b_k)$ and $a_i = t_{f_i} \cdot \varphi_{\omega(f_i)}(b_i) \cdot o_{f_{i+1}}$ for all $1 \leq i \leq k-1$. Notice that μ preserves the length of \mathbb{B} -paths, i.e. $|\mu(p)| = |p|$ for all \mathbb{B} -paths p . A simple calculation further shows that if p and p' are homotopic \mathbb{B} -paths, then $\mu(p)$ and $\mu(p')$ are homotopic \mathbb{A} -paths.

Definition 1.1.3 (The induced homomorphism). Let $\varphi : \mathbb{B} \rightarrow \mathbb{A}$ be a graph of groups morphism, $u_0 \in VB$ and $v_0 = \varphi(u_0) \in VA$. The induced homomorphism $\varphi_* : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$ is defined by $\varphi_*[p] = [\mu(p)]$.

In the following we will need to modify a graph of groups morphism in an inessential way as follows. Let $\varphi : \mathbb{B} \rightarrow \mathbb{A}$ be a graph of groups morphism, $f \in EB$ and $b \in B_u$, where $u := \alpha(f) \in VB$. Denote by \mathbb{B}' the graph of groups obtained from \mathbb{B} by replacing the boundary monomorphism $\alpha_f : B_f \rightarrow B_u$ by the monomorphism

$$\alpha'_f : B_f \rightarrow B_u, c \mapsto b \cdot \alpha_f(c) \cdot b^{-1}.$$

There is an obvious morphism $\varphi' : \mathbb{B}' \rightarrow \mathbb{A}$ obtained from φ by replacing $o_f \in A_{\varphi(u)}$ by the element $o'_f := \varphi_u(b) \cdot o_f \in A_{\varphi(u)}$. We have the following proposition which relates the homomorphisms φ_* and φ'_* .

Proposition 1.1.4. *There exists a morphism $\sigma : \mathbb{B} \rightarrow \mathbb{B}'$ such that the induced homomorphism σ_* is an isomorphism and that the diagram*

$$\begin{array}{ccc} \pi_1(\mathbb{B}, u_0) & & \\ \sigma_* \downarrow \cong & \searrow \varphi_* & \\ \pi_1(\mathbb{B}', u_0) & \xrightarrow{\varphi'_*} & \pi_1(\mathbb{A}, v_0) \end{array}$$

commutes.

Proof. Indeed, define the morphism $\sigma : \mathbb{B} \rightarrow \mathbb{B}'$ by

$$\sigma = (\sigma, \{\sigma_u \mid u \in VB\}, \{\sigma_f \mid f \in EB\}, \{o_f^\sigma \mid f \in EB\})$$

where $\sigma = id : B \rightarrow B$, $\sigma_x = id : B_x \rightarrow B_x$ for all $x \in VB \cup EB$, $o_g^\sigma = 1$ for all $g \neq f$ and $o_f^\sigma = b^{-1}$. It is easy to see that σ_* is an isomorphism and that $\varphi'_* \circ \sigma_* = \varphi_*$. \square

1.2 \mathbb{A} -graphs

In this section we discuss \mathbb{A} -graphs. The concept of \mathbb{A} -graph was first introduced in [18]. In this thesis we will follow the exposition of [49], which is a variation of the content contained in [18]. Essentially, \mathbb{A} -graphs encode a morphism $\mathbb{B} \rightarrow \mathbb{A}$ which is the inclusion map on the vertex and edges groups.

Definition 1.2.1 (\mathbb{A} -graphs). Let \mathbb{A} be a graph of groups. An \mathbb{A} -graph \mathcal{B} consists of an underlying connected graph B with the following additional data:

- (1) a graph-morphism $[\cdot] : B \rightarrow A$;
- (2) for each vertex $u \in VB$ there is a group B_u with $B_u \leq A_{[u]}$;
- (3) to each edge $f \in EB$ there are two associated group elements $f_\alpha \in A_{\alpha([f])}$ and $f_\omega \in A_{\omega([f])}$ such that $(f_\omega)^{-1} = (f^{-1})_\alpha$ for all $f \in EB$;
- (4) for each edge $f \in EB$ there is a group $B_f \leq A_{[f]}$ such that $B_f = B_{f^{-1}}$ and that $f_\alpha \cdot \alpha_{[f]}(B_f) \cdot f_\alpha^{-1} \leq B_{\alpha(f)}$.

If $f \in EB$ and $u \in VB$, we will refer to $e = [f] \in EA$ and $v = [u] \in VA$ as the type of f and u accordingly. When representing \mathbb{A} -graphs by pictures, we will say that a vertex $u \in VB$ has label $(B_u, [u])$. Similarly, we will say that an edge $f \in EB$ has label $l(f) = (f_\alpha, [f], B_f, f_\omega)$ or simply $l(f) = (f_\alpha, [f], f_\omega)$ when the associated edge group B_f is clear from the context or there is no need to specify B_f .

To any \mathbb{A} -graph we can associate in a natural way a graph of groups and a morphism.

Definition 1.2.2 (Graph of groups and morphism defined by an \mathbb{A} -graph). Let \mathcal{B} be an \mathbb{A} -graph. The associated graph of groups $\mathbb{B} = \mathbb{B}(\mathcal{B})$ is defined as follows:

- (1) the underlying graph of \mathbb{B} is the graph B ;
- (2) the vertex and edge groups are the groups B_u for $u \in VB$ and B_f for $f \in EB$;
- (3) for each $f \in EB$ we define the boundary monomorphism $\alpha_f : B_f \rightarrow B_{\alpha(f)}$ as $\alpha_f(x) := f_\alpha \cdot \alpha_{[f]}(x) \cdot f_\alpha^{-1}$.

The associated morphism $\varphi_{\mathcal{B}} = \varphi : \mathbb{B}(\mathcal{B}) \rightarrow \mathbb{A}$ is defined as follows:

- (1) the graph morphism $\varphi : B \rightarrow A$ coincides with the morphism $[\cdot]$;
- (2) for each $x \in B$, φ_x is the inclusion map $B_x \hookrightarrow A_{[x]}$;
- (3) for each $f \in EB$, φ_f is equal to $f_\alpha \in A_{[\alpha(f)]}$.

We also say that $\varphi = \varphi_{\mathcal{B}}$ is induced by the \mathbb{A} -graph \mathcal{B} .

Convention 1.2.3. Let \mathcal{B} be an \mathbb{A} -graph and $u_0 \in VB$ of type $v_0 = [u_0] \in VA$. The homomorphism $\varphi_* : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$ induced by the morphism $\varphi = \varphi_{\mathcal{B}} : \mathbb{B}(\mathcal{B}) \rightarrow \mathbb{A}$ will be denoted by ϕ .

We say that the \mathbb{A} -graph \mathcal{B} represents the subgroup $U = \phi(\pi_1(\mathbb{B}, u_0)) \leq \pi_1(\mathbb{A}, v_0)$. Any subgroup U of $G = \pi_1(\mathbb{A}, v_0)$ arises in this way, i.e. there exists a (non-unique) \mathbb{A} -graph \mathcal{B} such that $U = \phi(\pi_1(\mathbb{B}, u_0))$, for details see [49, Section 1.4].

Definition 1.2.4 (\mathbb{A} -subgraph). Let \mathcal{B} be an \mathbb{A} -graph. An \mathbb{A} -subgraph of \mathcal{B} is an \mathbb{A} -graph \mathcal{B}' such that:

1. $B' \subset B$.
2. $[\cdot]' = [\cdot]|_{B'} : B' \rightarrow A$.
3. $B'_x = B_x$ for all $x \in VB' \cup EB'$.
4. $l_{B'}(f') = l_B(f')$ for all $f' \in EB'$.

Definition 1.2.5. (Folded \mathbb{A} -graphs) Let \mathcal{B} be an \mathbb{A} -graph. We say that \mathcal{B} is *folded* if the following hold:

- (F1)** For any two distinct edges $f_1, f_2 \in EB$ with $\alpha(f_1) = \alpha(f_2) = u$ and labels $(a_1, e, b_1), (a_2, e, b_2)$ we have $a_2 \neq ba_1\alpha_e(c)$ for all $b \in B_u$ and all $c \in A_e$.
- (F2)** If $f \in EB$ is an edge with label (a, e, b) then $B_f = \alpha_e^{-1}(a^{-1}B_{\alpha(f)}a) = \omega_e^{-1}(bB_{\omega(f)}b^{-1})$.

Proposition 1.2.6. *Let \mathcal{B} be an \mathbb{A} -graph defining the graph of groups \mathbb{B} . Let $u_0 \in VB$ be of type $v_0 \in VA$. If \mathcal{B} is folded, then the canonical homomorphism*

$$\phi : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$$

induced by the morphism $\varphi_{\mathcal{B}}$, is injective.

Proof. This follows from the fact that any reduced \mathbb{B} -path translates into a reduced \mathbb{A} -path. For details we refer to [18, Lemma 4.16]. \square

1.3 Foldings of \mathbb{A} -graphs

We will first introduce three moves that can be applied to any \mathbb{A} -graph without substantially changing their structure.

Conjugation move A0. Let \mathcal{B} be an \mathbb{A} -graph. Suppose that u is a vertex of \mathcal{B} labeled (B_u, v) and that $g \in A_v$.

Let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B} as follows:

(1) replace $B_u \leq A_v$ by $gB_u g^{-1} \leq A_v$;

(2) for each edge $f \in EB$ with $\alpha(f) = u$ replace $f_\alpha \in A_v$ with $gf_\alpha \in A_v$.

We will say that \mathcal{B}' is obtained from \mathcal{B} by an auxiliary move of type A0. See Figure 1.1.

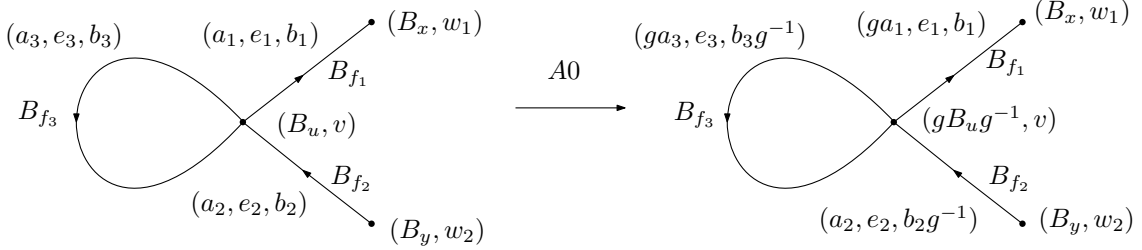


Figure 1.1: An auxiliary move of type A0 with $g \in A_v$.

If $u' \in VB$ with $u \neq u'$ we will say that this A0-move is admissible with respect to u' .

Remark 1.3.1. Note that for each edge f of \mathcal{B} with $\omega(f) = u$ the element $f_\omega \in A_v$ is replaced with $f_\omega g^{-1} \in A_v$.

Bass-Serre move A1. Let \mathcal{B} be an \mathbb{A} -graph. Suppose that f is an edge of \mathcal{B} , that $e = [f] \in EA$ and that $c \in A_e$.

Let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B} by replacing the edge group $B_f \leq A_e$ with $cB_f c^{-1} \leq A_e$, the element f_α with $f_\alpha \alpha_e(c)^{-1}$ and the element f_ω with $\omega_e(c) f_\omega$.

We say that \mathcal{B}' is obtained from \mathcal{B} by an auxiliary move of type A1. See Figure 1.2.

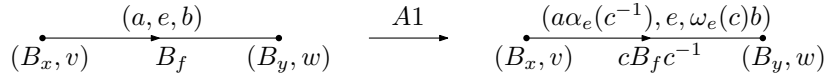


Figure 1.2: An auxiliary move of type A1 with $c \in A_e$.

Simple adjustment A2. Let \mathcal{B} be an \mathbb{A} -graph. Suppose that f is an edge of \mathcal{B} and that $b' \in B_{\alpha(f)}$.

Let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B} by replacing the element f_α with $b' f_\alpha$. We say that \mathcal{B}' is obtained from \mathcal{B} by an auxiliary move of type A2. See Figure 1.3.

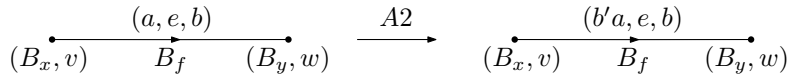


Figure 1.3: An auxiliary move of type A2 with $b' \in B_{\alpha(f)}$.

The next proposition implies that the \mathbb{A} -graphs \mathcal{B} and \mathcal{B}' represent the same subgroup of $\pi_1(\mathbb{A}, v_0)$ unless the base vertex is affected by an auxiliary move of type A0. For a proof we refer to [18, Proposition 4.15].

Proposition 1.3.2. *Let \mathbb{A} be a graph of groups with base vertex v_0 and \mathcal{B} be an \mathbb{A} -graph with base vertex u_0 of type v_0 . Suppose that \mathcal{B}' is obtained from \mathcal{B} by an auxiliary move of type A1, A2 or an auxiliary move of type A0 that is admissible with respect to u_0 . Then there exists a canonical morphism $\sigma : \mathbb{B} \rightarrow \mathbb{B}'$ such that $\sigma_* : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u_0)$ is an isomorphism and that the diagram*

$$\begin{array}{ccc} \pi_1(\mathbb{B}, u_0) & & \\ \sigma_* \downarrow \cong & \searrow \phi & \\ \pi_1(\mathbb{B}', u_0) & \xrightarrow{\phi'} & \pi_1(\mathbb{A}, v_0) \end{array}$$

commutes. If the move is of type A0 and is not admissible with respect to u_0 , then there exists $g \in A_{v_0}$ such that $\phi(h) = g \cdot \phi' \circ \sigma_(h) \cdot g^{-1}$ for all $h \in \pi_1(\mathbb{B}, u_0)$.*

Now we introduce folding moves for \mathbb{A} -graphs. They can be applied to an \mathbb{A} -graph \mathcal{B} if and only if \mathcal{B} is not folded. These moves are combinatorial versions of Stallings folds discussed by M. Bestvina and M. Feighn [5] combined with vertex morphisms as introduced by M. Dunwoody in [12, 13]. In the context of graphs rather than graph of groups Stallings folds were first introduced in [43].

For the remainder of this section let (\mathcal{B}, u_0) be an \mathbb{A} -graph with base vertex u_0 . Suppose that \mathcal{B} is not folded. We distinguish the case that (F1) is not satisfied and the case that (F2) is not satisfied.

If (F1) is not satisfied then there exist distinct edges f_1 and f_2 with same initial vertex $u := \alpha(f_1) = \alpha(f_2) \in VB$ and labels (a_1, e, b_1) and (a_2, e, b_2) such that $a_2 = a' a_1 \alpha_e(c)$ for some $a' \in B_u$ and some $c \in A_e$. Suppose further that $x = \omega(f_1)$ and $y = \omega(f_2)$. Clearly x and y are of same type $w := \omega(e) \in VA$. We denote the type of u by $v \in VA$. By applying an auxiliary

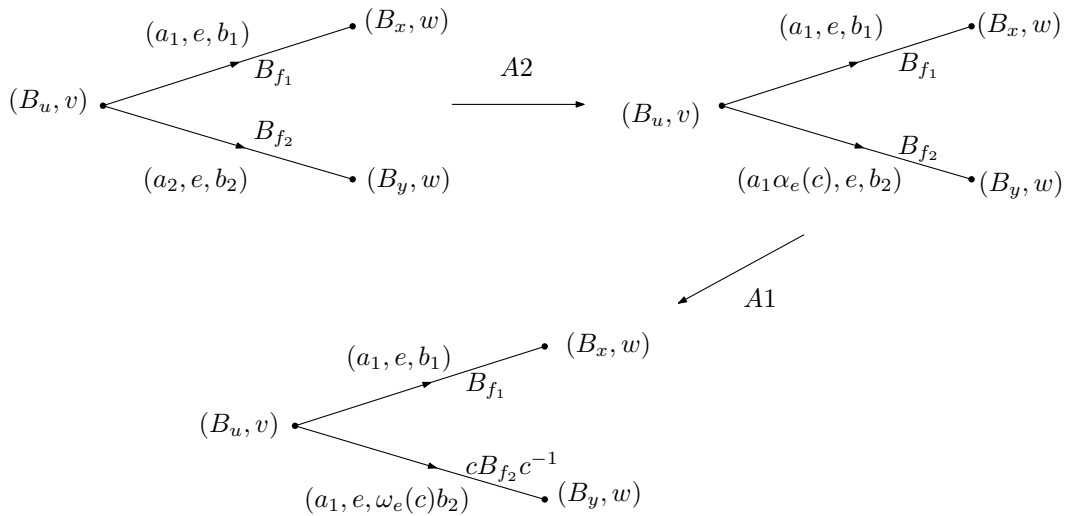


Figure 1.4: The intermediate \mathbb{A} -graph \mathcal{B}' .

move of type A2 to the edge f_2 we can change the label of f_2 to

$$(a'^{-1}a_2, e, b_2) = (a'^{-1}a'a_1\alpha_e(c), e, b_2) = (a_1\alpha_e(c), e, b_2).$$

An auxiliary move of type A1 applied to f_2 then yields the label $(a_1, e, \omega_e(c)b_2)$ on f_2 and replaces the edge group $B_{f_2} \leq A_e$ with $cB_{f_2}c^{-1} \leq A_e$. We denote the resulting \mathbb{A} -graph by \mathcal{B}' . See Figure 1.4.

The \mathbb{A} -graph \mathcal{B}' will be used as an intermediate object before we define the main folding moves on \mathcal{B} . Note that \mathcal{B}' is obtained from \mathcal{B} by moves that only change labels of edges and one edge group, no changes are performed on vertex groups.

We now introduce four different types of folds. They are distinguished by the topological type of the subgraph $f_1 \cup f_2$ in B . Each of these moves will be defined as a sequence of several transformations, exactly one of which will correspond to performing a Stallings fold identifying the edges f_1 and f_2 in B . That particular portion of a move will be called an elementary move of the respective type.

Fold of type IA. Suppose that f_1 and f_2 are two distinct non-loop edges and that $x = \omega(f_1) \neq \omega(f_2) = y$. Possibly after exchanging f_1 and f_2 we can assume that y is not the base vertex u_0 of \mathcal{B} .

We first perform an auxiliary move of type A0 on \mathcal{B}' at the vertex y making the label of f_2 to be (a_1, e, b_1) and the label of y to be $(b_1^{-1}\omega_e(c)b_2 \cdot B_y \cdot b_2^{-1}\omega_e(c)^{-1}b_1, w)$. Now both edges f_1 and f_2 have label (a_1, e, b_1) .

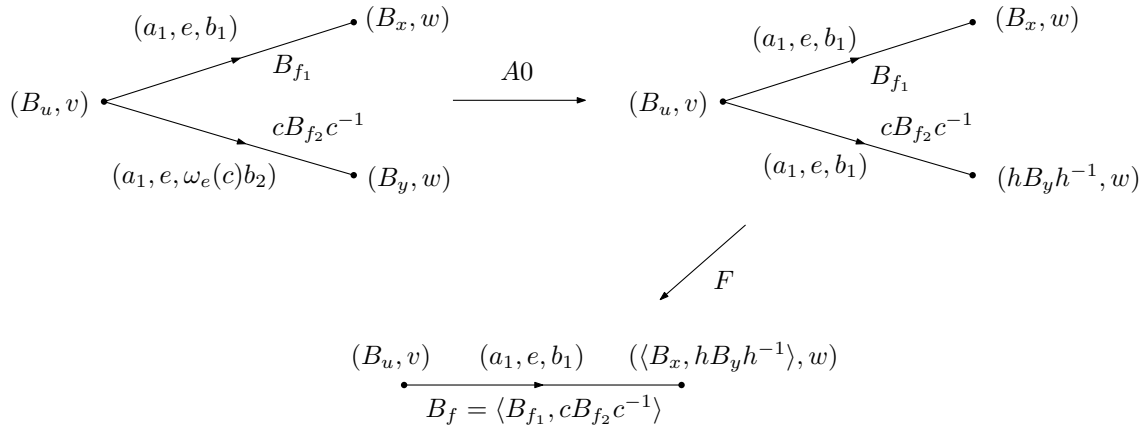


Figure 1.5: A fold of type IA, where $h = b_1^{-1}\omega_e(c)b_2 \in A_w$.

Next we identify f_1 and f_2 into a single edge f with edge group $B_f = \langle B_{f_1}, cB_{f_2}c^{-1} \rangle \leq A_e$ and label $l(f) = (a_1, e, b_1)$. The label of the vertex $\omega(f)$ is equal to

$$(\langle B_x, b_1^{-1}\omega_e(c)b_2 \cdot B_y \cdot b_2^{-1}\omega_e(c)^{-1}b_1 \rangle, w).$$

The remaining labels do not change. See figure 1.5.

We call this last operation, labeled F in Figure 1.5, an elementary move of type IA and say that the resulting \mathbb{A} -graph is obtained from the original \mathbb{A} -graph \mathcal{B} by a folding move of type IA.

Fold of type IB. Suppose now that f_1 is a loop edge and that f_2 is a non-loop edge. This implies that $e = [f_1] = [f_2] \in EA$ is a loop edge based at the vertex $v = w \in VA$.

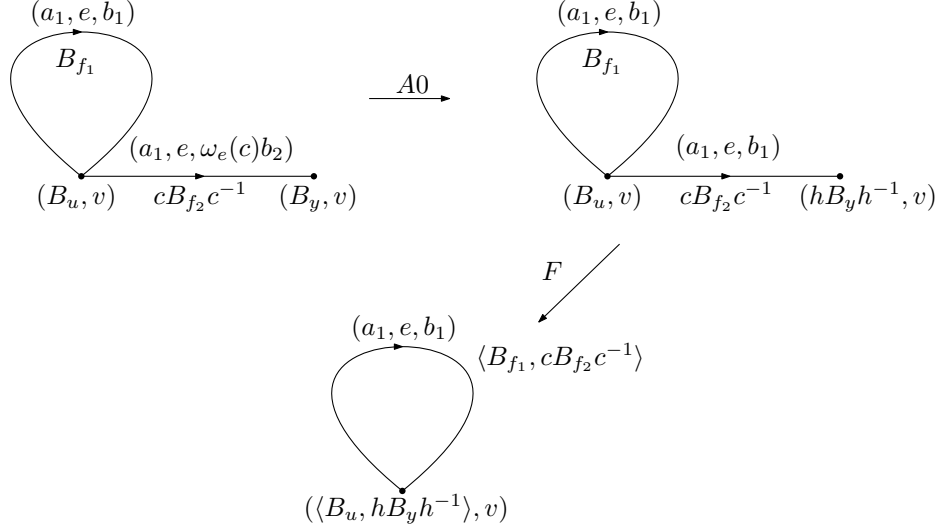


Figure 1.6: A move of type IB, where $h := b_1^{-1}\omega_e(c)b_2 \in A_w = A_v$.

We first perform an auxiliary move A0 on \mathcal{B}' making the label of f_2 to be (a_1, e, b_1) . Next we fold f_1 and f_2 into a single loop-edge f with edge group $\langle B_{f_1}, cB_{f_2}c^{-1} \rangle \leq A_e$ and label (a_1, e, b_1) . The label of $\alpha(f) = \omega(f)$ is equal to

$$\langle \langle B_x, b_1^{-1}\omega_e(c)b_2B_yb_2^{-1}\omega_e(c)^{-1}b_1 \rangle, v \rangle.$$

We call this operation, labeled F on Figure 1.6, an elementary move of type IB.

If $y = \omega(f_2) = u_0$ (the base vertex of \mathcal{B}), then we perform the auxiliary move A0 corresponding to the element $b_2^{-1}\omega_e(c)^{-1}b_1 \in A_{v_0}$.

We will say that the resulting \mathbb{A} -graph is obtained from \mathcal{B} by a folding move of type IB.

Fold of type IIIA. Suppose that f_1 and f_2 are both non-loop edges that have same terminal vertex $x = \omega(f_1) = \omega(f_2) = y$.

We identify the edges f_1 and f_2 of \mathcal{B}' into a single edge f with edge group $\langle B_{f_1}, cB_{f_2}c^{-1} \rangle$ and label (a_1, e, b_1) . The label of $\omega(f)$ is defined as $\langle \langle B_x, b_1^{-1}\omega_e(c)b_2 \rangle, w \rangle$.

We call this operation, labeled F in Figure 1.7, an elementary move of type IIIA and say that the resulting \mathbb{A} -graph is obtained from \mathcal{B} by a folding move of type IIIA.

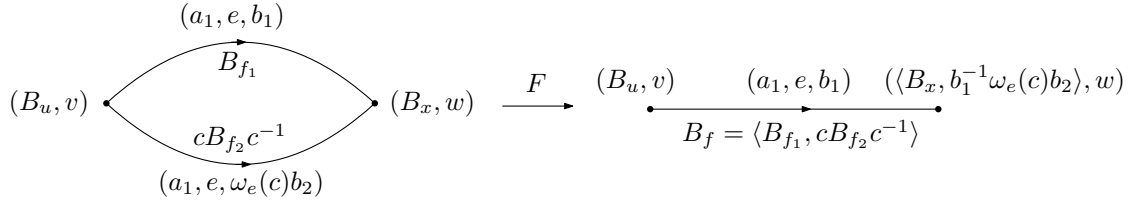


Figure 1.7: An elementary move of type IIIA.

Fold of type IIIB. Suppose that f_1 and f_2 are distinct loop edges. Thus $u = x = y \in VB$ and $e \in EA$ is a loop-edge at $v = w \in VA$.

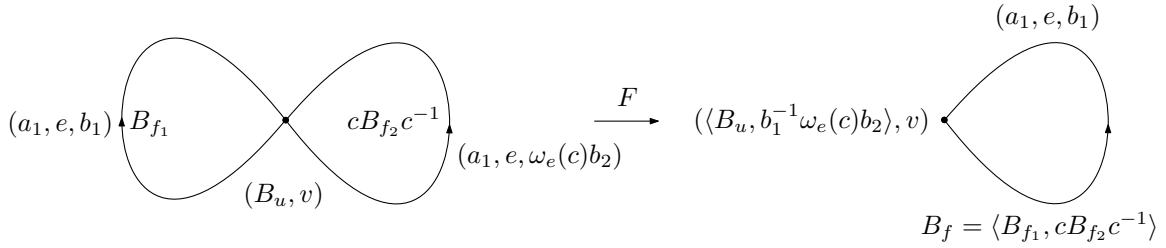


Figure 1.8: An elementary move of type IIIB.

We identify the edges f_1 and f_2 in \mathcal{B}' into a single loop-edge f with edge group $B_f = \langle B_{f_1}, cB_{f_2}c^{-1} \rangle \leq A_e$ and label (a_1, e, b_1) . The label of u is set to be $(\langle B_u, b_1^{-1}\omega_e(c)b_2 \rangle, v)$.

We will call this operation, labeled F in Figure 1.8, an elementary move of type IIIB and say that the resulting \mathbb{A} -graph is obtained from \mathcal{B} by a folding move of type IIIB.

Suppose now that the \mathbb{A} -graph \mathcal{B} is not folded because condition (F2) is violated. We introduce two folding moves that can be applied to such an \mathbb{A} -graph. For these folds there is no distinction between folds and elementary folds.

It follows that there exists an edge f of \mathcal{B} with label (a, e, b) , initial vertex $x = \alpha(f)$ labeled (B_x, v) , and terminal vertex $y = \omega(f)$ labeled (B_y, w) such that $B_f \not\subseteq \alpha_e(a^{-1}B_x a)$ or $B_f \not\subseteq \omega_e^{-1}(bB_y b^{-1})$. Possibly after replacing f with f^{-1} , we can assume that there exists some element $c \in \alpha_e^{-1}(a^{-1}B_x a)$ such that $c \notin B_f$. We distinguish the cases $x = y$ and $x \neq y$.

Fold of type IIA. Suppose that $x \neq y$. Let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B} by replacing the edge group $B_f \leq A_e$ with $\langle B_f, c \rangle \leq A_e$ and the vertex group B_y by $\langle B_y, b^{-1}\omega_e(c)b \rangle \leq A_w$. In this case we say that \mathcal{B}' is obtained from \mathcal{B} by a move of type IIA, see Figure 1.9.

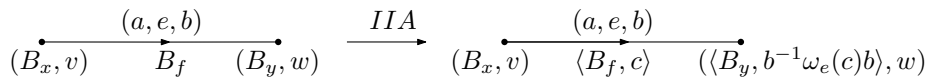


Figure 1.9: A move of type IIA.

Fold of type IIB. Suppose that $x = y$. Let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B} by replacing the edge group $B_f \leq A_e$ with $\langle B_f, c \rangle \leq A_e$ and the vertex group $B_x \leq A_v$ with the group $\langle B_x, b^{-1}\omega_e(c)b \rangle \leq A_v$. We will say that \mathcal{B}' is obtained from \mathcal{B} by a move of type IIB, see Figure 1.10.

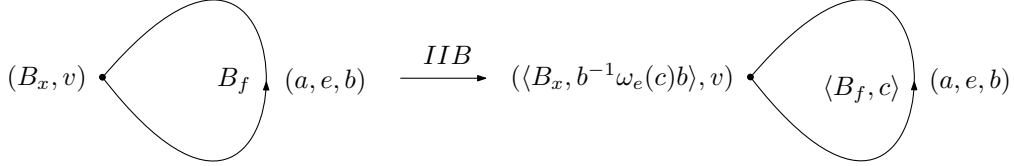


Figure 1.10: A move of type IIB.

The following important proposition states that folding moves preserve the subgroup defined by an \mathbb{A} -graph. For a proof see [18, Proposition 4.15].

Proposition 1.3.3. *Let \mathbb{A} be a graph of groups with base vertex v_0 and \mathbb{B}' be an \mathbb{A} -graph obtained from the \mathbb{A} -graph \mathbb{B} by one of the folding moves IA, IIA, IIIA, IB, IIB or IIIB. Let u_0 be the base vertex of \mathbb{B} and let u'_0 be the image of u_0 in \mathbb{B}' . Suppose that the type of the vertices u_0 and u'_0 is $v_0 \in VA$. Then there exists a canonical morphism $\sigma : \mathbb{B} \rightarrow \mathbb{B}'$ which induces an epimorphism $\sigma_* : \pi_1(\mathbb{B}, u_0) \twoheadrightarrow \pi_1(\mathbb{B}', u'_0)$ such that the diagram*

$$\begin{array}{ccc}
 \pi_1(\mathbb{B}, u_0) & & \\
 \sigma_* \downarrow & \searrow \phi & \\
 \pi_1(\mathbb{B}', u'_0) & \xrightarrow{\phi'} & \pi_1(\mathbb{A}, v_0)
 \end{array}$$

commutes.

1.4 JSJ-decomposition of 3-manifolds

In this section we will give an elementary descriptions of Seifert Fibered manifolds and explain the JSJ-decomposition of compact 3-manifolds with empty or toroidal boundary.

A 3-manifold M is called *irreducible* if every embedded 2-sphere $S \hookrightarrow M$ bounds a 3-cell in M .

We say that a 3-manifold M is *atoroidal* if any map $T \rightarrow M$ from a torus T to M which induces an injective homomorphism $\pi_1(T) \rightarrow \pi_1(M)$ can be homotoped into the boundary of M .

A *Seifert fibered manifold* is a 3-manifold M together with a decomposition into disjoint simple closed curves (called Seifert fibers) such that each Seifert fiber has a regular neighborhood that forms a standard fibered torus. The *standard fibered torus* corresponding to a pair of

coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$, equipped with the natural fibering by circles. If $a > 1$, then the middle Seifert fiber is called *singular*. A compact Seifert fibered manifold has only finitely many singular fibers. We refer to [28, 37, 39] for further information and for the classification of Seifert fibered manifolds.

The following theorem was first announced by Waldhausen [46] and was proved independently by Jaco–Shalen [17, p. 157] and Johannson [16]. In the case of knot complements the JSJ decomposition theorem was foreshadowed by the work of Schubert.

Theorem 1.4.1 (JSJ Decomposition Theorem). *Let M be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori*

$$T_1, \dots, T_k \subset \text{int}(M)$$

such that each component of $M - \text{int}(V(T_1 \cup \dots \cup T_k))$ is atoroidal or Seifert fibered. Furthermore any such collection of tori with a minimal number of components is unique up to isotopy.

The tori $T_1 \cup \dots \cup T_k$ are called the JSJ tori and the components of $M - \text{int}(V(T_1 \cup \dots \cup T_k))$ are called the JSJ components of M . We can view $\pi_1(M)$ as the fundamental group of a graph of groups with vertex groups the fundamental groups of the JSJ components and with edge groups the fundamental groups of the JSJ tori. Indeed, let N be a JSJ component of M . After picking base points for M and N and a path connecting these base points, the inclusion $N \subseteq M$ induces a map on the level of fundamental groups. This map is injective since the tori we cut along are incompressible. We can thus view $\pi_1(M)$ as a subgroup of $\pi_1(N)$, which is well defined up to the above choices.

Definition 1.4.2 (Graph manifold). A compact, orientable, irreducible 3-manifold with empty or toroidal boundary is called a graph manifold if every JSJ component of M is a Seifert fibered space.

Graph manifolds were invented and classified by F. Waldhausen [47, 48]. He originally defined a graph manifold as a 3-manifold that is obtained by gluing some circle bundles and can be seen as the first breakthrough towards the discovery of JSJ decomposition.

1.5 Basic knot theory

This section contains elementary definitions from knot theory such as the knot complement and the knot group. We will also recall the definition of the bridge number of a knot.

A knot \mathfrak{k} in S^3 is a piecewise linear (P.L.) homeomorphic image of the circle S^1 in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Two knots are said to be equivalent if they are ambient isotopic. There will

be a certain abuse of language in this thesis to avoid complicating the notation. A knot \mathfrak{k} will be the image $i(S^1) = \mathfrak{k}$ of an embedding $i : S^1 \hookrightarrow S^3$ or the isotopy class of such curves.

Definition 1.5.1. Let \mathfrak{k} be a knot in S^3 and $V(\mathfrak{k})$ a regular neighborhood of \mathfrak{k} . The 3-manifold $\overline{S^3 - V(\mathfrak{k})}$ is called the complement of the knot \mathfrak{k} and denoted by $C(\mathfrak{k})$.

There are, up to isotopy, unique simple closed curves μ and λ on the boundary $\partial C(\mathfrak{k}) = \partial V(\mathfrak{k})$ of $C(\mathfrak{k})$ such that μ and λ intersect in one point, μ bounds a disk in $V(\mathfrak{k})$ but not in $\partial V(\mathfrak{k})$, and λ is not homologous to 0 on $\partial C(\mathfrak{k})$ but homologous to 0 in $C(\mathfrak{k})$. μ is called a *meridian* of \mathfrak{k} and λ is called a *longitude* of \mathfrak{k} . A choice of the pair (μ, λ) , with any fixed orientation, is called a preferred meridian and longitude pair for \mathfrak{k} , $C(\mathfrak{k})$, and $V(\mathfrak{k})$.

Definition 1.5.2. Let \mathfrak{k} be a knot in S^3 and $C(\mathfrak{k})$ its complement. The group of the knot \mathfrak{k} , denoted by $G(\mathfrak{k})$, is defined as the fundamental group of $C(\mathfrak{k})$.

The element of $G(\mathfrak{k})$ represented by a meridian $\mu \subset \partial V(\mathfrak{k})$ of \mathfrak{k} will also be referred to as a *meridian* of \mathfrak{k} , denoted by m . In the same way, the element l of $G(\mathfrak{k})$ represented by a longitude $\lambda \subset \partial V(\mathfrak{k})$ will be referred to as a *longitude* of \mathfrak{k} . It follows from [9, Proposition 3.15] that m and l are well defined up to a common conjugating element in $G(\mathfrak{k})$. In the sequel we refer to any conjugate $gmg^{-1} \in G(\mathfrak{k})$ ($g \in G(\mathfrak{k})$) of m as a meridian of \mathfrak{k} .

We further denote by $P(\mathfrak{k}) \leq G(\mathfrak{k})$ the peripheral subgroup of \mathfrak{k} , that is, $P(\mathfrak{k}) = \pi_1 \partial C(\mathfrak{k})$. Note that $P(\mathfrak{k}) = \langle m, l \rangle$ and for a non-trivial knot \mathfrak{k} we further have $P(\mathfrak{k}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Next we recall the definition of a numerical invariant of a knot called the bridge number or bridge index. Bridge number was first studied in the 1950s by Horst Schubert [35]. His investigation was motivated by the question as to whether a knot can have only finitely many companions. Together with the fact that the only bridge number one knot is the unknot, his result showed that the answer to this question is yes.

An *n-bridge sphere* of a knot \mathfrak{k} in the 3-sphere S^3 is a 2-sphere S dividing S^3 into two 3-cells B_1 and B_2 :

$$S^3 = B_1 \cup B_2, \quad B_1 \cap B_2 = \partial B_1 = \partial B_2 = S$$

and dividing \mathfrak{k} into $2n$ arcs $\kappa_{i,j}$ ($i \in \{1, 2\}$, $j \in \{1, \dots, n\}$) such that

$$\kappa_{i,j} \subset B_i, \quad \partial \kappa_{i,j} \subset \partial B_i = S$$

and such that there are 2-cells $\delta_{i,j}$ ($i \in \{1, 2\}$, $j \in \{1, \dots, n\}$) with the properties

$$\delta_{i,j} \cap \delta_{i,j'} = \emptyset \text{ for } 1 \leq j \neq j' \leq n \text{ and } \partial \delta_{i,j} = \kappa_{i,j} \cup (\delta_{i,j} \cap S).$$

In each ball B_i the picture is as in Figure 1.11, and any two such balls with embedded arcs are of the same topological type. The notion of an *n-bridge sphere* becomes clear with

the following description. Identify S with the plane $z = 0$ plus a point at infinity ∞ and B_1 with the upper half-space. Assume that the projection parallel to the z -axis maps the n -arcs $\mathfrak{k} \cap B_1 = \kappa_{1,1} \cup \dots \cup \kappa_{1,n}$ to segments β_1, \dots, β_n of the plane $z = 0$. Then $\kappa_{1,1}, \dots, \kappa_{1,n}$ are called *bridges*. From this viewpoint it is clear that any knot admits at least one bridge sphere, namely, the plane of a regular projection of \mathfrak{k} gives a bridge sphere for \mathfrak{k} .

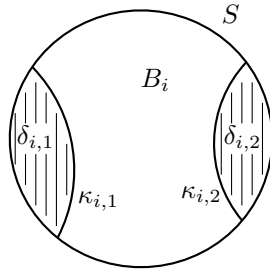


Figure 1.11: 2-bridge sphere of \mathfrak{k} .

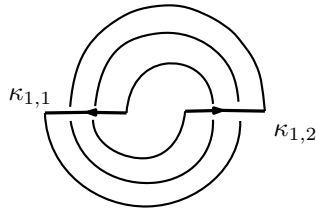


Figure 1.12: A 2-bridge knot. The arcs $\kappa_{1,1}$ and $\kappa_{1,2}$ depicted in fat lines are the bridges.

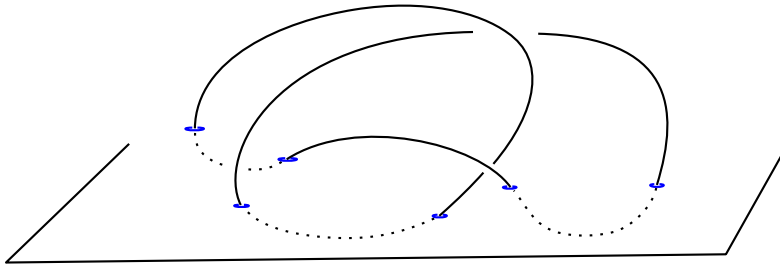


Figure 1.13: A 3-bridge sphere for the trefoil knot.

Let \mathfrak{k} be a knot in S^3 . Suppose that $S \subset S^3$ is an n -bridge sphere for \mathfrak{k} . It is easy to see that

$$C(\mathfrak{k}) = X_1 \sqcup_{X_0} X_2$$

where $X_i = C(\mathfrak{k}) \cap B_i$ is homeomorphic with a cube with n -handles and $X_0 = X_1 \cap X_2 = S \cap C(\mathfrak{k})$ is homeomorphic with a $2n$ -punctures sphere. Moreover, $\pi_1(X_i)$ is generated by $m_{i,1}, \dots, m_{i,k}$ ($0 \leq i \leq 2$, $k = n$ if $i = 1, 2$ and $k = 2n$ if $k = 0$) where each $m_{i,j}$ is a meridian of \mathfrak{k} . The

inclusion map $X_0 \hookrightarrow X_i$ induces a surjective homomorphism $\pi_1(X_0) \rightarrow \pi_1(X_i)$ which maps $m_{0,j}$ onto some m_{i,s_j} and $\{s_1, \dots, s_{2n}\} = \{1, \dots, n\}$. From the Theorem of Seifert and van-Kampen it follows that the group $G(\mathfrak{k})$ is generated by n meridians.

Definition 1.5.3 (Bridge number). A knot \mathfrak{k} is called an n -bridge knot if \mathfrak{k} admits an n -bridge sphere and does not admit an $(n - 1)$ -bridge sphere. We call n the bridge number of \mathfrak{k} and denote it by $b(\mathfrak{k})$.

We now recall the definition of satellite knots. Let $V = \mathbb{D}^2 \times S^1$ be a standardly embedded solid torus in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ and \mathfrak{k}' a knot contained in the interior of V such that the inclusion map $\partial V \hookrightarrow V - \mathfrak{k}'$ induces a monomorphism $\pi_1(\partial V) \rightarrow \pi_1(V - \mathfrak{k}')$, equivalently, \mathfrak{k}' is not contained in a 3-cell of V . A homeomorphism $h : V \rightarrow V(\mathfrak{k}'')$ onto a tubular neighborhood of a non-trivial knot \mathfrak{k}'' which maps a meridian of $\overline{S^3 - V}$ onto a longitude of \mathfrak{k}'' maps \mathfrak{k}' onto a knot $\mathfrak{k} = h(\mathfrak{k}') \subset S^3$. The knot \mathfrak{k} is called a *satellite* of \mathfrak{k}'' , and \mathfrak{k}'' is its *companion*. The pair (V, \mathfrak{k}') is called the *pattern* of \mathfrak{k} .

Example 1.5.4 (Connected sum of knots). If there exists a meridional disk of V which intersects \mathfrak{k}' in exactly one point, see Figure 1.14, then we call the satellite knot $\mathfrak{k} = h(\mathfrak{k}')$ the *connected sum* of \mathfrak{k}' and \mathfrak{k}'' . We denote \mathfrak{k} by $\mathfrak{k}' \# \mathfrak{k}''$.

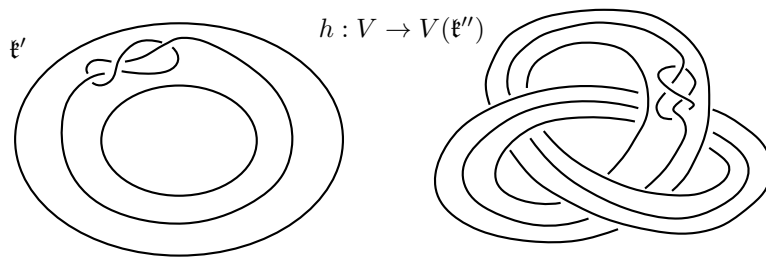


Figure 1.14: Connected sum $\mathfrak{k} = h(\mathfrak{k}')$ of \mathfrak{k}' and \mathfrak{k}'' .

Example 1.5.5 (Satellite knots with braid pattern). Let β an n -braid with $n \geq 2$ such that the associated permutation $\sigma \in S_n$ is a cycle of length n . Assume that the closed braid $\mathfrak{k}' = \widehat{\beta}$ is standardly embedded in the interior of V , see Figure 1.15. In this case the satellite knot $\mathfrak{k} = h(\mathfrak{k}')$ is called a β -*satellite* of \mathfrak{k}'' and it is denoted by $\beta(\mathfrak{k}'')$. We also say that \mathfrak{k} is obtained from \mathfrak{k}'' by a *satellite operation with braid pattern* β .

Example 1.5.6 (Cable knots). If $\mathfrak{k}' \subset \text{int}(V)$ is ambient isotopic (in V) to a simple closed curve lying on a torus $T \subset \text{int}(V)$ which is parallel to ∂V , or equivalently, if $V - \mathfrak{k}'$ has the structure of a Seifert fibered space, then $\mathfrak{k} = h(\mathfrak{k}')$ is called a *cable knot on* \mathfrak{k}'' . Note that a cable knot is a special instance of a satellite knot with braid pattern.

Later on we will see the following result of Schubert [35], see also [36] for a more modern proof. Note that the result of Schubert is actually stronger than what we state.

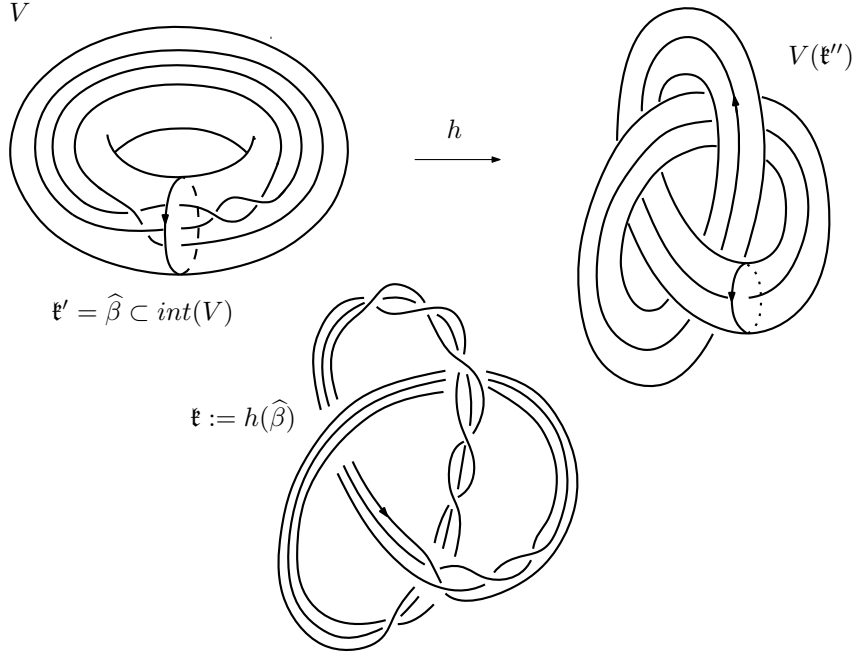


Figure 1.15: β is the 3-braid $\sigma_2\sigma_1^{-1}\sigma_2^2$ and \mathfrak{k}_1 is the trefoil knot.

Theorem 1.5.7 (Schubert). *Let $\mathfrak{k}_1, \dots, \mathfrak{k}_d$ be knots in S^3 and β be an n -braid such that the closed braid $\widehat{\beta}$ is a knot. Then the following hold:*

(i) $b(\mathfrak{k}_1 \# \dots \# \mathfrak{k}_d) = b(\mathfrak{k}_1) + \dots + b(\mathfrak{k}_d) - (d - 1)$.

(ii) $b(\beta(\mathfrak{k}_1)) = n \cdot b(\mathfrak{k}_1)$.

Definition 1.5.8. The meridional rank of the knot \mathfrak{k} , denoted by $w(\mathfrak{k})$, is defined as the minimal number of meridians needed to generate $G(\mathfrak{k})$.

From the discussion above we see that the bridge number is an upper bound for the meridional rank of \mathfrak{k} , that is, we always have $w(\mathfrak{k}) \leq b(\mathfrak{k})$ for any knot \mathfrak{k} .

We call a subgroup $U \leq G(\mathfrak{k})$ *meridional* if U is generated by finitely many meridians of \mathfrak{k} . The minimal number of meridians needed to generate U , denoted by $w(U)$, is called the *meridional rank* of U . Observe that the knot group $G(\mathfrak{k})$ is meridional and its meridional rank is equal to $w(\mathfrak{k})$.

Definition 1.5.9. A meridional subgroup U of meridional rank $w(U) = l$ is called *tame* if for any $g \in G(\mathfrak{k})$ one of the following holds:

1. $gP(\mathfrak{k})g^{-1} \cap U = 1$.
2. $gP(\mathfrak{k})g^{-1} \cap U = g\langle m \rangle g^{-1}$ and there exist meridians m'_2, \dots, m'_l such that U is generated by $\{gmg^{-1}, m'_2, \dots, m'_l\}$.

Definition 1.5.10 (Meridionally tame knot). A non-trivial knot \mathfrak{k} in S^3 is called meridionally tame if any meridional subgroup $U \leq G(\mathfrak{k})$ generated by less than $b(\mathfrak{k})$ meridians is tame.

The class of meridionally tame knots trivially contains the class of 2-bridge knots as any cyclic meridional subgroup is obviously tame. In Lemma 4.1.1 below we show that the meridional tameness of torus knots is implicit in [32] and in Proposition 4.2.1 we show that prime 3-bridge knots are meridionally tame. However it follows from [1] and the discussion at the end of Section 4.2 that satellite knots are in general not meridionally tame. For examples Whitehead doubles of non-trivial knots are never meridionally tame. These are prime satellite knots, whose satellite patterns have winding number zero, which is opposite to the braid patterns considered in this thesis. Moreover Whitehead doubles of 2-bridge knots are prime 4-bridge knots for which the question of Cappell and Shaneson has a positive answer by [6, Corollary 1.6]. In contrast, it should be noted that we do not know any hyperbolic knots that are not meridionally tame, but it is likely that such knots exist. The following Lemma shows that the question of Cappell and Shaneson [19, Pb 1.11] has positive answer .

Lemma 1.5.11. *If \mathfrak{k} is meridionally tame, then $w(\mathfrak{k}) = b(\mathfrak{k})$.*

Proof. Indeed, as \mathfrak{k} is meridionally tame it follows that any meridional subgroup U of $G(\mathfrak{k})$ generated by less than $b(\mathfrak{k})$ is tame. Thus $U \cap P(\mathfrak{k})$ is a proper subgroup of $P(\mathfrak{k})$. Consequently $U \subsetneq G(\mathfrak{k})$ since we clearly have $G(\mathfrak{k}) \cap P(\mathfrak{k}) = P(\mathfrak{k})$. Therefore $w(\mathfrak{k}) \geq b(\mathfrak{k})$ which implies that $b(\mathfrak{k}) = w(\mathfrak{k})$. □

Meridional rank of torus knots

In this chapter we provide a new calculation of the meridional rank of a torus knot. In [32] M. Rost and H. Zieschang compute the meridional rank using a long combinatorial argument which relies on a linear ordering of the free product of two finite cyclic groups. Our proof relies only on foldings of \mathbb{A} -graphs and is rather elementary. Thus the aim of this chapter is to reprove the following result:

Theorem 2.0.12 (Rost and Zieschang [32]). *If \mathfrak{k} is a torus knot of type (p, q) , then $b(\mathfrak{k}) = w(\mathfrak{k}) = \min(p, q)$.*

2.1 Definition of torus knots

Let $\mathfrak{k}_0 \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the trivial knot standardly embedded in \mathbb{R}^3 and $V(\mathfrak{k}_0)$ a regular neighborhood of \mathfrak{k}_0 . Let further μ_0 and λ_0 be a meridian and a longitude of \mathfrak{k}_0 , see Figure 2.1.

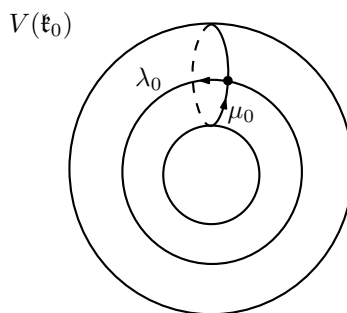


Figure 2.1: Regular neighborhood of the trivial knot.

Any closed curve c on $F := \partial V(\mathfrak{k})$ is homotopic to a curve $\mu_0^p \cdot \lambda_0^q$ where $p, q \in \mathbb{Z}$. The homotopy class of c contains a simple closed curve if and only if p and q are relatively prime.

Definition 2.1.1 (Torus knots). Let $V(\mathfrak{k}_0)$ be the regular neighborhood of the trivial knot \mathfrak{k}_0 as described above. If \mathfrak{k} is a simple closed curve lying on F represented by the curve $\mu_0^p \cdot \lambda_0^q$ and if $|p|, |q| \geq 2$ then \mathfrak{k} is called a torus knot of type (p, q) , denoted $\mathfrak{k}(p, q)$.

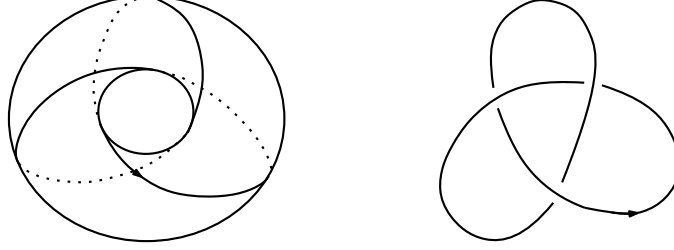


Figure 2.2: The torus knot $\mathfrak{k} = \mathfrak{k}(3, 2)$.

Let $\mathfrak{k} = \mathfrak{k}(p, q)$ be a torus knot of type (p, q) . From [9, Proposition 3.38] it follows that the group $G(\mathfrak{k})$ of \mathfrak{k} is isomorphic to the fundamental group of the graph of groups \mathbb{A} defined as follows:

1. Its underlying graph A consists of the vertex set $VA = \{v_0, v_1\}$ and the edge set $EA = \{e, e^{-1}\}$ where $\alpha(e) = v_0$ and $\omega(e) = v_1$.
2. The vertex groups are $A_{v_0} = \langle u | - \rangle$ and $A_{v_1} = \langle v | - \rangle$. The single edge group is $A_e = A_{e^{-1}} = \mathbb{Z}$ and the boundary monomorphisms are given by $\alpha_e(z) = u^{pz}$ and $\omega_e(z) = v^{qz}$ for $z \in A_e$.

$$\begin{array}{ccccc}
 & & u^{pz} \longleftarrow z \longrightarrow v^{qz} & & \\
 A_{v_0} = \langle u | - \rangle & & A_e & & A_{v_1} = \langle v | - \rangle \\
 \bullet & \xrightarrow{e} & \bullet & & \bullet \\
 v_0 & & e & & v_1
 \end{array}$$

The meridian m of \mathfrak{k} is represented by the reduced \mathbb{A} -path $u^c, e, v^{-d}, e^{-1}, 1$ where $pd - qc = 1$. The longitude l of \mathfrak{k} is equal to $u^p m^{-pq}$.

2.2 Proof of Theorem 2.0.12

Since we obviously have the isomorphisms $G(\mathfrak{k}(p, q)) \cong G(\mathfrak{k}(q, p)) \cong G(\mathfrak{k}(|p|, |q|))$, we can assume that $2 \leq p < q$.

Before we start with the proof we need an auxiliary construction. Let T be a finite tree. For each function $l : ET \rightarrow \mathbb{N}$ with the property that $l(f) = l(f)^{-1}$ for all $f \in ET$, we construct a

graph $T(l)$ by replacing each edge $f \in ET$ by $l(f)$ edges with initial vertex $\alpha(f)$ and terminal vertex $\omega(f)$, see Figure 2.3. More precisely, $T(l)$ is given by:

1. $VT(l) = VT$
2. $ET(l) = \bigcup_{f \in ET} \{e_{f,i}^\varepsilon \mid 1 \leq i \leq l(f), \varepsilon \in \{-1, +1\} \text{ and } e_{f^{-1},i} = e_{f,i}^{-1}\}$.
3. $\alpha(e_{f,i}) := \alpha(f)$ and $\omega(e_{f,i}) := \omega(f)$ for $f \in ET$ and $1 \leq i \leq l(f)$.

We will say that the edge $e' \in ET(l)$ corresponds to the edge $f \in ET$ if $e' = e_{f,i}$ for some $1 \leq i \leq l(f)$.

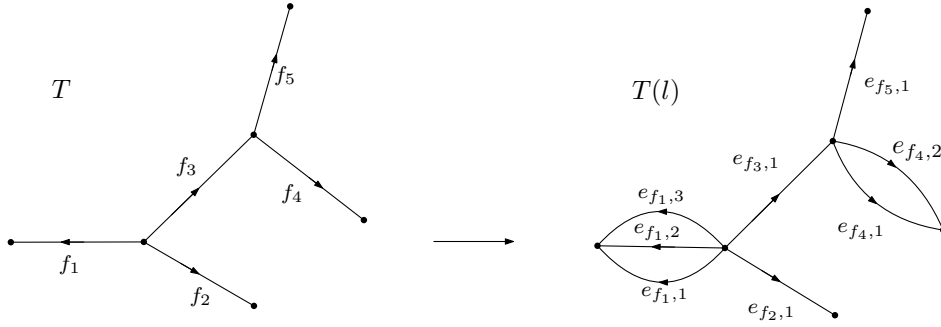


Figure 2.3: The graph $T(l)$ with $l(f_1) = 3$, $l(f_4) = 2$ and $l(f_i) = 1$ for $i = 2, 3, 5$.

For any morphism $[\cdot] : T \rightarrow A$ we can naturally define a morphism $[\cdot] : T(l) \rightarrow A$ by putting $[e_{f,i}^\varepsilon] := [f^\varepsilon] = e^\varepsilon$. Conversely, given a morphism $[\cdot] : T(l) \rightarrow A$ we define $[\cdot] : T \rightarrow A$ by $[f] = [e']$ where $e' \in ET(l)$ corresponds to f . Note that $[\cdot] : T \rightarrow A$ is well defined since for any two edges e' and e'' corresponding to the same edge $f \in ET$ we have $[e'] = [e'']$.

Definition 2.2.1. We say that an \mathbb{A} -graph \mathcal{B} is *tame* if the following hold:

- (T1) There exist a finite tree T and a function $l : ET \rightarrow \mathbb{N}$ such that the underlying graph B of \mathcal{B} is equal to $T(l)$.
- (T2) Vertex and edge groups are trivial.
- (T3) For each edge $f \in ET_+ := \{f \in ET \mid [f] = e\}$ there is a surjective map $\sigma_f : \{1, \dots, l(f)\} \rightarrow \{-n_f, \dots, m_f\}$, where $n_f, m_f \geq 0$ depends only on f , such that for each $1 \leq i \leq l(f)$ the corresponding edge $e_{f,i}$ has label

$$(u^{\alpha_f} u^{(\sigma_f(i)-1)c} u^{pz_{f,i}}, e, v^{-qz_{f,i}} v^{-(\sigma_f(i)-2)d} v^{-\beta_f})$$

where $\alpha_f, \beta_f, z_{f,i} \in \mathbb{Z}$ for all $1 \leq i \leq l(f)$.

Note that for $f \in ET_+$ with $l(f) = 1$ there is no restriction on the label of the corresponding edge $e_{f,1} \in EB$. Indeed, given $u^h \in A_{v_0}$ and $v^k \in A_{v_1}$ we can write $u^h = u^{\alpha_f} u^{(n-1)c} u^{pz_{f,1}}$ and $v^k = v^{-qz_{f,1}} v^{-(n-2)d} v^{-\beta_f}$, where $z_{f,1} = 0$, $\alpha_f = h - (n-1)c$ and $\beta_f = (n-2)d - k$.

The fundamental group $\pi_1(\mathbb{B}, u_0)$ of the graph of groups induced by a tame \mathbb{A} -graph \mathcal{B} is free of rank $\sum_{f \in ET_+} (l(f) - 1)$. Moreover, the image of $\pi_1(\mathbb{B}, u_0)$ under the canonical homomorphism $\phi : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$ is a meridional subgroup of $G(\mathfrak{k})$.

Definition 2.2.2. Let \mathcal{B} be a tame \mathbb{A} -graph. We define the c -complexity of \mathcal{B} as

$$c(\mathcal{B}) = (c_1(\mathcal{B}), c_2(\mathcal{B})) \in \mathbb{N} \times \mathbb{N}$$

where $c_1(\mathcal{B}) = rk(\pi_1(\mathbb{B}, u_0))$ and $c_2(\mathcal{B}) = \frac{1}{2}|EB|$.

Since we want to compare complexities we endow the set $\mathbb{N} \times \mathbb{N}$ with the lexicographic order, i.e. $(n, m) < (n', m')$ if $n < n'$ or $n = n'$ and $m < m'$.

Lemma 2.2.3. Any meridional subgroup of $G(\mathfrak{k})$ generated by $r < p = \min(p, q)$ meridians is freely generated by $s \leq r$ meridians.

Proof. Let U be a meridional subgroup of $G(\mathfrak{k})$ generated by $g_1 m g_1^{-1}, \dots, g_r m g_r^{-1}$ where $r < p = \min(p, q)$ and $g_i \in G(\mathfrak{k})$ for all $1 \leq i \leq r$. We will construct a tame \mathbb{A} -graph \mathcal{B}_0 such that $\phi_0(\pi_1(\mathbb{B}_0, u_0)) = U$ as follows. For $1 \leq i \leq r$ the element $g_i m g_i^{-1} \in G(\mathfrak{k}) = \pi_1(\mathbb{A}, v_0)$ can be written as

$$g_i m g_i^{-1} = [p_i \cdot (u^c, e, v^{-d}, e^{-1}, 1) \cdot p_i^{-1}]$$

where p_i is a reduced \mathbb{A} -path from v_0 to v_0 of the form

$$a_{i,0}, e, a_{i,1}, e^{-1}, a_{i,2}, \dots, e^{-1}, a_{i,2n_i}$$

for some $n_i \geq 0$. We construct the \mathbb{A} -graph \mathcal{B}_0 as follows, see Figure 2.4:

1. The underlying graph B_0 is given by:

- (a) $VB = \{u_0\} \cup \{u_{i,1}, \dots, u_{i,2n_i}, u_{i,2n_i+1}\}$
- (b) $EB = \{f_{i,j}^\varepsilon, e_{i,1}^\varepsilon, e_{i,2}^\varepsilon \mid 1 \leq i \leq r, 1 \leq j \leq 2n_i, \varepsilon \in \{-1, +1\}\}$
- (c) $\omega(f_{i,j}) = u_{i,j}$ and $\omega(e_{i,j}) = u_{i,2n_i+1}$ for all $1 \leq i \leq r, 1 \leq j \leq 2n_i$.
- (d) $\alpha(f_{i,1}) = u_0$, $\alpha(f_{i,j}) = u_{i,j-1}$, and $\alpha(e_{i,1}) = \alpha(e_{i,2}) = u_{i,2n_i}$ for all $1 \leq i \leq r$ and all $1 \leq j \leq 2n_i$.

2. The graph morphism $[\cdot] : B_0 \rightarrow A$ is defined by $[f_{i,j}] = e^{(-1)^j}$ and $[e_{i,1}] = [e_{i,2}] = e$ for $1 \leq i \leq r$ and $1 \leq j \leq 2n_i$.

3. $B_x = 1$ for all $x \in VB_0 \cup EB_0$.
4. $(f_{i,j})_\alpha = a_{i,j-1}$, $(e_{i,1})_\alpha = u^c$ and $(e_{i,2})_\alpha = 1$ for $1 \leq i \leq r$ and $1 \leq j \leq 2n_i$.
5. $(f_{i,j})_\omega = 1$ for all $j < 2n_i$, $(f_{i,2n_i})_\omega = a_{i,2n_i}$, $(e_{i,1})_\omega = 1$, and $(e_{i,2})_\omega = v^d$ for $1 \leq i \leq r$.

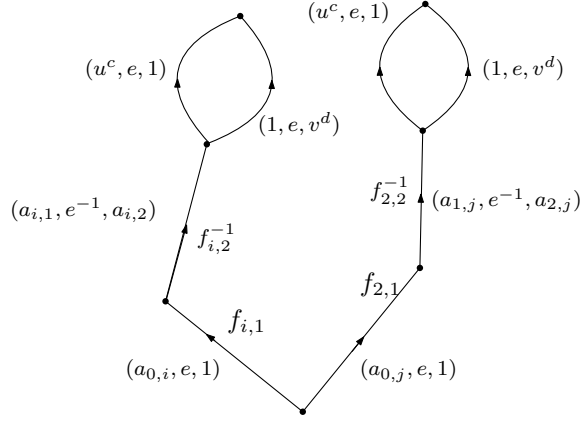


Figure 2.4: The \mathbb{A} -graph \mathcal{B}_0 with $n_i = n_j = 2$.

Note that the graph B_0 is equal to $T_0(l_0)$, where T_0 is the tree obtained by gluing segments $f_{i,1}, f_{i,2}, \dots, f_{i,2n_i}, f_{i,2n_i+1}$ of length $2n_i + 1$ to u_0 and the map $l_0 : ET_0 \rightarrow \mathbb{N}$ is defined by $l_0(f_{i,j}) = 1$ for $j \leq 2n_i$ and $l_0(f_{i,2n_i+1}) = 2$ for all $1 \leq i \leq r$. Conditions (T2) and (T3) are easily verified. Thus \mathcal{B}_0 is tame. The image of the induced homomorphism $\phi_0 : \pi_1(\mathbb{B}_0, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$ is equal to U since

$$\phi([1, f_{i,1}, 1, \dots, f_{i,2n_i}, 1, e_{i,1}, 1, e_{i,2}^{-1}, 1, f_{i,2n_i}^{-1}, 1, \dots, 1, f_{i,1}^{-1}, 1]) = g_i m g_i^{-1}$$

for all $1 \leq i \leq r$. Moreover $c_1(\mathcal{B}_0) = r < p$ and $c_2(\mathcal{B}_0) = 2r + \sum_{i=1}^r 2n_i$.

Now choose a tame \mathbb{A} -graph with the following properties:

1. \mathcal{B} represents the meridional subgroup $U = \langle g_1 m g_1^{-1}, \dots, g_r m g_r^{-1} \rangle \leq G(\mathfrak{k})$.
2. If \mathcal{B}' is a tame \mathbb{A} -graph that represents U , then $c(\mathcal{B}') \geq c(\mathcal{B})$.

It follows immediately from the choice of \mathcal{B} that $rk(\pi_1(\mathbb{B}, u_0)) = c_1(\mathcal{B}) \leq c_1(\mathcal{B}_0) = r < p$ since \mathcal{B}_0 is tame and represents U .

Claim: The tame \mathbb{A} -graph \mathcal{B} is folded.

Proof. The proof of the claim is by contradiction. Thus we suppose that \mathcal{B} is not folded. We will derive a contraction by showing the existence of a tame \mathbb{A} -graph with strictly smaller c -complexity than that of \mathcal{B} .

As \mathcal{B} is not folded a fold is applicable to \mathcal{B} . Since $B_x = 1$ for all $x \in VB$ it follows that condition (F2) from the definition of folded \mathbb{A} -graphs is verified. Thus we can only apply a fold of type IA or a fold of type IIIA to \mathcal{B} . We consider each case separately.

Case 1: Suppose that a fold of type IIIA is applicable to \mathcal{B} . This implies that there exists an edge $f \in ET_+$ with $l(f) \geq 2$ and $1 \leq i \neq j \leq l(f)$ such that one of the following holds:

1. $u^{\alpha_f} u^{(\sigma_f(i)-1)c} u^{pz_{f,i}} = u^{\alpha_f} u^{(\sigma_f(j)-1)c} u^{pz_{f,j}} u^{zp}$.
2. $v^{\beta_f} v^{(\sigma_f(i)-2)d} v^{qz_{f,i}} = v^{\beta_f} v^{(\sigma_f(j)-2)d} v^{qz_{f,j}} v^{zq}$.

for some $z \in A_e = \mathbb{Z}$. Assume that the first occurs, the second case is entirely analogous. If $\sigma_f(i) \neq \sigma_f(j)$ then

$$u^{(\sigma_f(j)-\sigma_f(i))c} = u^{(\sigma_f(j)-1)c} u^{-(\sigma_f(i)-1)c} = u^{p(z_{f,i}-z_{f,j}-z)}.$$

Since p and c are relatively prime it follows that $\sigma_f(j) - \sigma_f(i) \in p\mathbb{Z}$ which is a contradiction as we have

$$0 < |\sigma_f(j) - \sigma_f(i)| \leq l(f) - 1 \leq \sum_{f' \in ET_+} (l(f') - 1) = c_1(\mathcal{B}) \leq c_1(\mathcal{B}_0) = r < p.$$

Thus $\sigma_f(i) = \sigma_f(j) := k \in \{1, \dots, n_f\}$ which implies that $z_{f,i} = z_{f,j} + z \in A_e$. By applying an auxiliary move of type A1 to the edge $e_{f,j}$ with $c = z \in A_e$ we make the labels of $e_{f,j}$ and $e_{f,i}$ coincide. Therefore the \mathbb{A} -graph \mathcal{B}' obtained from \mathcal{B} by folding the edges $e_{f,i}$ and $e_{f,j}$ into a single edge is an \mathbb{A} -subgraph of \mathcal{B} with underlying graph $B' = B \setminus \{e_{f,i}, e_{f,i}^{-1}\} = T(l')$, where $l' : ET \rightarrow \mathbb{N}$ is given by $l'(f) = l(f) - 1$ and $l'(g) = l(g)$ for all $g \in ET \setminus \{f, f^{-1}\}$. Thus \mathcal{B}' is tame. From Proposition 1.3.2 and Proposition 1.3.3 it follows that \mathcal{B}' represents U .

Since the rank of $\pi_1(\mathbb{B}', u_0)$ is clearly smaller than the rank of $\pi_1(\mathbb{B}, u_0)$ we conclude that $c(\mathcal{B}') < c(\mathcal{B})$, which is a contradiction to the fact that \mathcal{B} has the smallest c -complexity.

Case 2: Suppose now that a fold of type IA is applicable to \mathcal{B} . Without loss of generality we can assume that there exist two distinct edges $f, g \in ET_+$ with $\alpha(f) = \alpha(g)$, $\omega(g) \neq u_0$ and corresponding edges $e_{f,i}, e_{g,j} \in EB$ with $1 \leq i \leq l(f)$ and $1 \leq j \leq l(g)$ such that

$$u^{\alpha_g} u^{(\sigma_g(j)-1)c} u^{pz_{g,j}} = u^{\alpha_f} u^{(\sigma_f(i)-1)c} u^{pz_{f,i}} u^{pz}$$

for some $z \in A_e = \mathbb{Z}$. Thus $u^{\alpha_g} = u^{\alpha_f} u^{(\sigma_f(i)-\sigma_g(j))c} u^{p(z_{f,i}+z-z_{g,j})} \in A_{v_0}$.

We first apply an auxiliary move of type A1 to $e_{g,j}$ which replaces its label by

$$\underbrace{(u^{\alpha_g} u^{(\sigma_g(j)-1)c} u^{p(z_{g,j}-z)})}_{=u^{\alpha_f} u^{(\sigma_f(i)-1)c} u^{pz_{f,i}}}, e, v^{-q(z_{g,j}-z)} v^{-(\sigma_g(j)-2)d} v^{-\beta_g}$$

We further apply an auxiliary move of type A0 to $\omega(e_{g,j})$ with

$$b = v^{\beta_f} v^{(\sigma_f(i)-2)d} v^{qz_{f,i}} v^{-q(z_{g,j}-z)} v^{-(\sigma_g(j)-2)d} v^{-\beta_g} \in A_{v_1}$$

which makes the labels of $e_{g,j}$ and $e_{f,i}$ coincide. Note that for each $k \in \{1, \dots, l(g)\} \setminus \{j\}$ the label of $e_{g,k}$ is equal to

$$(u^{\alpha_f} u^{(\sigma_g(k)+\sigma_f(i)-\sigma_g(j)-1)c} u^{pw_k}, e, v^{-qw_k} v^{-(\sigma_g(k)+\sigma_f(i)-\sigma_g(j)-1)d} v^{-\beta_f}).$$

where $w_k = z_{g,k} - z_{g,j} + z_{f,i} + z \in A_e$.

Denote by \mathcal{B}' the \mathbb{A} -graph obtained from \mathcal{B} by folding the edges $e_{f,i}$ and $e_{f,j}$ into a single edge. The underlying graph B' of \mathcal{B}' is equal to $T'(l')$, where T' is the tree obtained from T by folding f and g into a single edge h and $l' : ET' \rightarrow \mathbb{N}$ is the function given by $l'(h) = l'(h^{-1}) = l(f) + l(g) - 1$ and $l'(h') = l(h')$ for all $h' \in ET'(l') \setminus \{h, h^{-1}\} = ET \setminus \{f^{\pm 1}, g^{\pm 1}\}$. Hence condition (T1) is verified.

Condition (T2) is trivially satisfied. Thus it remains to verify condition (T3). Let $\iota : \{1, \dots, l(g)\} \setminus \{j\} \rightarrow \{l(f) + 1, \dots, l(f) + l(g) - 1\}$ be a bijection such that $\iota(k) \geq l(f) + 1$ and $\iota(k) < \iota(k + 1)$ for all k . We define σ_h by putting $\sigma_h(k) = \sigma_f(k)$ for $1 \leq i \leq l(f)$ and $\sigma_h(k) = \sigma_g(\iota^{-1}(k)) + \sigma_f(i) - \sigma_g(j)$ for $l(f) < k \leq l(f) + l(g) - 1$. Thus the edge $e_{h,k}$ has label

$$(u^{\alpha_f} u^{(\sigma_h(k)-1)c} u^{pz_{h,k}}, e, v^{-qz_{h,k}} v^{-(\sigma_h(k)-2)d} v^{-qz_{h,k}})$$

where $z_{h,k} = z_{f,k}$ for $k \leq l(f)$ and $z_{h,k} = z_{g,k} + z_{f,i} - z_{g,j}$ for $z > l(f)$. Thus \mathcal{B} is tame. Since the number of edges decrease we conclude that $c(\mathcal{B}') < c(\mathcal{B})$, contradiction. \square

Therefore \mathcal{B} is folded. This implies, in particular, that for each edge $f \in ET$ the map $\sigma_f : \{1, \dots, l(f)\} \rightarrow \{-n_f, \dots, m_f\}$ is a bijection. The fundamental group $\pi_1(\mathbb{B}, u_0)$ of the graph of groups induced by \mathcal{B} is freely generated by the elements $x_{f,i} := [p_{f,i} \cdot 1, e_{f,i+1}, 1, e_{f,i}^{-1}, 1, p_{f,i}^{-1}]$ for $f \in ET_+$ with $l(f) \geq 2$ and $1 \leq i \leq l(f) - 1$, where p_i is an \mathbb{B} -path from u_0 to $\alpha(e_{f,i})$. Since $\phi : \pi_1(\mathbb{B}, u_0) \rightarrow U$ is an isomorphism and ϕ maps each element $x_{f,i}$ onto a meridian of \mathfrak{k} , we conclude that U is free generated by $c_1(\mathcal{B}) \leq r$ meridians. \square

Theorem 2.2.4 (Rost, Zieschang). *Let \mathfrak{k} be a torus knot of type (p, q) . Then*

$$w(\mathfrak{k}) = b(\mathfrak{k}) = \min(p, q).$$

Proof. From Lemma 2.2.3 it follows that any meridional subgroup of $G(\mathfrak{k})$ generated by less than $p = \min(p, q)$ meridians is free. Since $\mathbb{Z} \times \mathbb{Z} \cong \langle m, l \rangle \leq G(\mathfrak{k})$ we conclude that $G(\mathfrak{k})$ is not free. Therefore $w(\mathfrak{k}) \geq p$. On the other hand we have

$$G(\mathfrak{k}) = \langle m, v^d m v^{-d}, \dots, v^{(p-1)d} m v^{-(p-1)d} \rangle$$

which implies that $w(\mathfrak{k}) \leq p$. Thus $w(\mathfrak{k}) = p$.

It is a result of H. Schubert [35] that the bridge number of a torus knot of type (p, q) is equal to $\min(p, q)$. Therefore $w(\mathfrak{k}) = b(\mathfrak{k}) = \min(p, q)$. \square

Meridional rank of knots whose complement is a graph manifold

In this chapter we consider the class of knots \mathcal{K} that is the smallest class of knots that contains all meridionally tame knots and is closed under connected sums and satellites with braid pattern, see Section 3.1 for details. The following result is the main theorem of this chapter:

Theorem 3.0.5. *Let \mathfrak{k} be a knot from \mathcal{K} . Then $w(\mathfrak{k}) = b(\mathfrak{k})$.*

As the only Seifert-fibred manifolds that can be embedded into a knot manifold with incompressible boundary are torus knot complements, composing spaces and cable spaces (see Lemma VI.3.4 of [17]) and as cable spaces are special instances of braid patterns we immediately obtain the following consequence of Theorem 3.0.5.

Corollary 3.0.6. *Let \mathfrak{k} be a knot such that its complement $C(\mathfrak{k})$ is a graph manifold. Then $w(\mathfrak{k}) = b(\mathfrak{k})$.*

3.1 Description of the class \mathcal{K} .

In this section we introduce the appropriate formalism to study knots that lie in the class \mathcal{K} . Recall that these knots are obtained from meridionally tame knots by repeatedly taking connected sums and satellites with braid pattern.

In order to define the class \mathcal{K} we need some terminology. Let A be a graph. For any vertex $v \in VA$, the *star* of v is defined as

$$St(v, A) = \{e \in EA \mid \alpha(e) = v\}.$$

The cardinality $|St(v, A)|$ of $St(v, A)$ is called the *valence* of v , written $val(v, A)$.

An *orientation* of a graph A consists of a subset $E \subset EA$ such that $EA = E \cup E^{-1}$ and $E \cap E^{-1} = \emptyset$. If A is a graph with some orientation $E \subset EA$ we say that A is oriented.

Let A be an oriented graph with orientation $E \subset EA$. We define the *positive star* of v as

$$St_+(v, A) = St(v, A) \cap E$$

and the *positive valence* of v as $val_+(v, A) = |St_+(v, A)|$. For $i \in \{0, 1\}$ we further define

$$V_i := \{v \in VA \mid val_+(v, A) = i\}$$

and

$$V_2 := \{v \in VA \mid val_+(v, A) \geq 2\}.$$

We say that a path e_1, \dots, e_k in A is *oriented* if $e_i \in E$ for all $1 \leq i \leq k$. In this case we say that the vertex $\alpha(e_1)$ *projects* onto the vertex $\omega(e_k)$. We record the following simple lemma for future use.

Lemma 3.1.1. *Let A be a finite oriented tree. Then for any $v \in VA$ we can choose a set $P(v) = \{w_{v,1}, \dots, w_{v,r_v}\} \subseteq V_0$ such that the following hold:*

1. $r_v \geq val_+(v, A) - 1$.
2. v projects onto $w_{v,i}$ for $1 \leq i \leq r_v$.
3. If $v \neq v'$ then $P(v) \cap P(v') = \emptyset$.

Proof. The proof is by induction on the number of oriented edges of A . If the tree consists of a single vertex v then the lemma holds trivially by choosing $P(v) = \emptyset$.

Suppose that the lemma holds for any finite oriented tree with at most n oriented edges.

Let A be a finite oriented tree with orientation $EA_+ \subset EA$ and $|EA_+| = n + 1$. Since A is finite there exists $v_0 \in VA$ with $val(v_0, A) = 1$, i.e. that $St(v_0, A) = \{e\}$ for some $e \in EA$. Let B be the subtree of A spanned by $VB = VA \setminus \{v_0\}$, we clearly have $EB = EA \setminus \{e, e^{-1}\}$. Consider B with the orientation $EB_+ := EA_+ \cap EB$. Thus B is a finite oriented tree with n oriented edges. It follows from the induction hypothesis that for each $v \in VB$ we can choose $P'(v) \subseteq \{v \in VB \mid val_+(v, B) = 0\}$ satisfying (1)-(3).

We distinguish two cases according to whether $e \in EA_+$ or $e^{-1} \in EA_+$.

Suppose first that $e \in EA_+$. Note that $val_+(v_0, A) = 1$ and $val_+(v, A) = val_+(v, B)$ for all $v \in VB$. We set $P(v_0) = \emptyset$ and $P(v') = P'(v')$ for all $v' \in VB$.

Now suppose that $e^{-1} \in EA_+$. Note that $val_+(v_0, A) = 0$, $val_+(\omega(e), A) = val_+(\omega(e), B) + 1$ and $val_+(v, B) = val_+(v, A)$ for all $v \in VA \setminus \{v_0, \omega(e)\}$. Moreover, $\omega(e)$ projects onto v_0 . If

$val_+(\omega(e), B) = 0$ and $\omega(e) \in P'(v)$ for some $v \in VB$, we set $P(v) := (P'(v) \setminus \{\omega(e)\}) \cup \{v_0\}$, $P(v_0) = \emptyset$ and $P(u) = P'(u)$ for all $u \in VA \setminus \{v_0, v\}$. Otherwise we set $P(\omega(e)) := P'(\omega(e)) \cup \{v_0\}$, $P(v_0) = \emptyset$ and $P(u) = P'(u)$ for all $u \in VA \setminus \{v_0, \omega(e)\}$.

In all cases it is immediate that the defined sets satisfy (1)-(3). \square

Let A be a finite oriented tree. For any vertex $w \in VA$ we define $A(w)$ as the sub-tree of A spanned by the set

$$VA(w) = \{\omega(\gamma) \mid \gamma \text{ is an oriented path in } A \text{ and } \alpha(\gamma) = w\}.$$

For a finite rooted tree (A, v_0) we will define a natural orientation determined by the root $v_0 \in VA$ in the following way: for each vertex $v \in VA$ there exists a unique reduced path $\gamma_v := e_{v,1}, \dots, e_{v,r_v}$ in A from v_0 to v . We define $E(A, v_0) \subset EA$ as the unique orientation of A such that all paths γ_v are oriented, that is,

$$E(A, v_0) = \{e_{v,i} \mid v \in VA, 1 \leq i \leq r_v\}.$$

Throughout this paper we assume that any rooted tree (A, v_0) is endowed with this orientation.

Note that the canonical orientation of the rooted tree $(A(w), w)$ coincides with the orientation that $A(w)$ inherits as a sub-tree of (A, v_0) , i.e., $E(A(w), w) = E(A, v_0) \cap EA(w)$, see Figure 3.1.

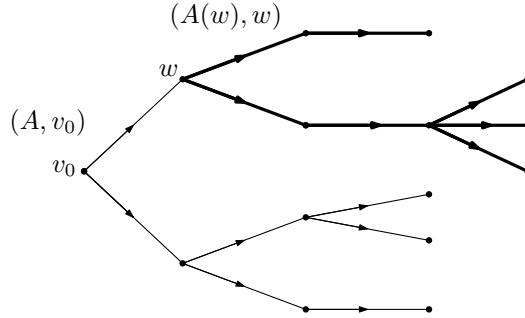


Figure 3.1: A rooted tree (A, v_0) and a rooted subtree $(A(w), w)$ (fat lines) with their canonical orientations.

We now define labelings of rooted trees. A labeled rooted tree is a tuple

$$\mathcal{A} = ((A, v_0), \{\mathfrak{k}_v \mid v \in V_0\}, \{\beta_v \mid v \in V_1\})$$

such that the following hold:

1. (A, v_0) is a finite rooted tree.

2. For any $v \in V_0$, \mathfrak{k}_v is a non-trivial knot.
3. For any $v \in V_1$, β_v is an n_v -braid having $n_v \geq 2$ strands such that the closed braid $\widehat{\beta}_v$ is a knot.

For $w \in VA$ we define the labeled rooted tree \mathcal{A}_w as

$$\mathcal{A}_w := ((A(w), w), \{\mathfrak{k}_v \mid v \in V_0 \cap VA(w)\}, \{\beta_v \mid v \in V_1 \cap VA(w)\}).$$

We now associate to any labeled rooted tree \mathcal{A} a knot $\mathfrak{k} = \mathfrak{k}_{\mathcal{A}} \subset S^3$. We define $l(\mathcal{A}) := \max\{d(v, v_0) \mid v \in VA\}$. We recursively define $\mathfrak{k}_{\mathcal{A}}$ in the following way:

1. If $l(\mathcal{A}) = 0$, then we define $\mathfrak{k}_{\mathcal{A}} = \mathfrak{k}_{v_0}$.
2. If $l(\mathcal{A}) > 0$ and $\text{val}(v_0, A) = 1$, then we define $\mathfrak{k}_{\mathcal{A}} := \beta_{v_0}(\mathfrak{k}_{\mathcal{A}_{v_1}})$ where $v_1 \in VA$ is the unique vertex of A such that $d(v_1, v_0) = 1$.
3. If $l(\mathcal{A}) > 0$ and $d := \text{val}(v_0, A) \geq 2$ then we define $\mathfrak{k}_{\mathcal{A}} := \#_{i=1}^d \mathfrak{k}_{\mathcal{A}_{v_i}}$ where $v_1, \dots, v_d \in VA$ are those vertices such that $d(v_i, v_0) = 1$.

Note that this is a (recursive) definition indeed as in situation (2) and (3) we have $l(\mathcal{A}_{v_i}) < l(\mathcal{A})$ for all occurring i . Note moreover that a knot \mathfrak{k} lies in \mathcal{K} if and only if there exists a labeled rooted tree \mathcal{A} such that $\mathfrak{k}_{\mathcal{A}} = \mathfrak{k}$ and such that \mathfrak{k}_v is meridionally tame when $v \in V_0 \subset VA$. Thus we can rephrase the main theorem in the following way:

Theorem 3.1.2. *Let \mathcal{A} be a labeled rooted tree and $\mathfrak{k} = \mathfrak{k}_{\mathcal{A}}$. Suppose that \mathfrak{k}_v is meridionally tame for all $v \in V_0$. Then*

$$b(\mathfrak{k}) = w(\mathfrak{k}).$$

The proof relies on computing both the bridge number and the meridional rank. We conclude this section with the computation of the bridge number which is an easy consequence of the work of Schubert.

Remember that any vertex $v \in V_1$ is labeled β_v , where β_v is a braid with n_v strands. We define the function $n : VA \rightarrow \mathbb{N}$ by

$$n(v) = \begin{cases} n_v & \text{if } v \in V_1 \\ 1 & \text{if } v \in V_0 \cup V_2 \end{cases}$$

Recall that, for any $v \in VA$, $\gamma_v = e_{v,1}, \dots, e_{v,r_v}$ is the unique reduced path in A from v_0 to v . We define the *height* of a vertex $v \in VA$ as

$$h(v) := \prod_{i=1}^{r_v} n(\alpha(e_{v,i}))$$

for $v \neq v_0$ and $h(v_0) := 1$.

Lemma 3.1.3. *Let $\mathfrak{k} = \mathfrak{k}_A$ be the knot defined by the labeled rooted tree A . Then the bridge number of \mathfrak{k} is given by*

$$b(\mathfrak{k}) = \left[\sum_{v \in V_0} h(v) \cdot b(\mathfrak{k}_v) \right] - \left[\sum_{v \in V_2} h(v) \cdot (\text{val}_+(v, A) - 1) \right]$$

Proof. The proof is by induction on $l(A) := \max\{d(v, v_0) \mid v \in VA\}$. If $l(A) = 0$, then $\mathfrak{k} = \mathfrak{k}_{v_0}$. Hence we have

$$b(\mathfrak{k}) = b(\mathfrak{k}_{v_0}) = \left[\sum_{v \in V_0 = \{v_0\}} h(v) \cdot b(\mathfrak{k}_v) \right] - \left[\sum_{v \in V_2 = \emptyset} h(v) \cdot (\text{val}_+(v, A) - 1) \right]$$

as $h(v_0) = 1$.

Suppose that $l(A) > 0$ and $\text{val}_+(v_0, A) = 1$. Let $v_1 \in VA$ be the unique vertex such that $d(v_0, v_1) = 1$. In this case $V_0 \cup V_2 \subset VA(v_1)$. If h_{v_1} denotes the height of a vertex in the rooted tree $(A(v_1), v_1)$, then it is easy to see that $h(v) = n(v_0)h_{v_1}(v)$ for all $v \in VA(v_1)$. Moreover, $\text{val}_+(v, A(v_1)) = \text{val}_+(v, A)$ for all $v \in VA(v_1)$. By Theorem 1.5.7(ii) and the induction hypothesis we obtain:

$$\begin{aligned} b(\mathfrak{k}) &\stackrel{\text{Thm. 1.5.7(ii)}}{=} n_{v_0} \cdot b(\mathfrak{k}_{A_{v_1}}) \\ &\stackrel{\text{i.h.}}{=} n_{v_0} \cdot \left[\sum_{v \in V_0 \cap VA(v_1)} h_{v_1}(v) \cdot b(\mathfrak{k}_v) - \right. \\ &\quad \left. - \sum_{v \in V_2 \cap VA(v_1)} h_{v_1}(v) \cdot (\text{val}_+(v, A(v_1)) - 1) \right] \\ &= \sum_{v \in V_0 \cap VA(v_1)} n_{v_0} h_{v_1}(v) \cdot b(\mathfrak{k}_v) - \sum_{v \in V_2 \cap VA(v_1)} n_{v_0} h_{v_1}(v) \cdot (\text{val}_+(v, A) - 1) \\ &= \left[\sum_{v \in V_0} h(v) \cdot b(\mathfrak{k}_v) \right] - \left[\sum_{v \in V_2} h(v) \cdot (\text{val}_+(v, A) - 1) \right] \end{aligned}$$

Suppose now that $l(A) > 0$ and $d := \text{val}_+(v_0, A) \geq 2$. Let $v_1, \dots, v_d \in VA$ such that $d(v_i, v_0) = 1$. By definition, \mathfrak{k} is equal to the connected sum $\#_{i=1}^d \mathfrak{k}_{A_{v_i}}$. Observe that if h_{v_i} denotes the height of a vertex in the rooted tree $(A(v_i), v_i)$, then $h_{v_i}(v) = h(v)$ for any $v \in VA(v_i)$. By

Theorem 1.5.7(i) and the induction hypothesis we obtain:

$$\begin{aligned}
 b(\mathfrak{k}) &\stackrel{\text{Thm. 1.5.7(i)}}{=} \left[\sum_{i=1}^d b(\mathfrak{k}_{\mathcal{A}_{v_i}}) \right] - (d-1) = \left[\sum_{i=1}^d b(\mathfrak{k}(\mathcal{A}_{v_i})) \right] - (\text{val}_+(v_0, A) - 1) = \\
 &\stackrel{\text{i.h.}}{=} \left[\sum_{i=1}^d \left(\sum_{v \in V_0 \cap VA(v_i)} h_{v_i}(v) b(\mathfrak{k}_{\mathcal{A}_{v_i}}) - \sum_{v \in V_2 \cap VA(v_i)} h_{v_i}(v) (\text{val}_+(v, A(v_i)) - 1) \right) \right] - \\
 &\quad - (\text{val}_+(v_0, A) - 1) \\
 &= \left[\sum_{v \in V_0} h(v) \cdot b(\mathfrak{k}_v) \right] - \left[\sum_{v \in V_2} h(v) \cdot (\text{val}_+(v, A) - 1) \right]
 \end{aligned}$$

since $V_0 = \cup_i (V_0 \cap VA(v_i))$ and $V_2 = \{v_0\} \cup (\cup_i (V_2 \cup VA(v_i)))$. \square

3.2 Description of $G(\mathfrak{k}_{\mathcal{A}})$.

In the previous section we have constructed a knot $\mathfrak{k}_{\mathcal{A}}$ from a labeled tree \mathcal{A} . The construction implies that the knot complement $C(\mathfrak{k}_{\mathcal{A}})$ contains a collection \mathcal{T} of incompressible tori corresponding to the edges of A such that to each vertex $v \in VA$ there corresponds a component of the complement of \mathcal{T} such that the following hold:

1. The vertex space associated to each vertex $v \in V_0$ is $C(\mathfrak{k}_v)$.
2. The vertex space associated to a vertex $v \in V_1$ is the braid space $CS(\beta_v)$, see below for details.
3. The vertex space associated to a vertex $v \in V_2$ is an r -fold composing space where $r = \text{val}_+(v, A)$.

Thus $C(\mathfrak{k}_{\mathcal{A}})$ can be thought of as a tree of spaces. It follows from the theorem of Seifert and van Kampen that corresponding to the tree of spaces there exists a tree of groups decomposition \mathbb{A} of $G(\mathfrak{k}_{\mathcal{A}})$ such that all edge groups are free Abelian of rank 2. It is the aim of this section to describe this tree of groups. We will first describe the vertex groups that occur in this splitting and then conclude by describing the boundary monomorphisms of the tree of groups.

(1) If $v \in V_0$ then the complementary component of \mathcal{T} corresponding to v is the knot space of \mathfrak{k}_v . Thus we put $A_v := G(\mathfrak{k}_v)$. Denote by $m_v \in P(\mathfrak{k}_v)$ the meridian and by $l_v \in P(\mathfrak{k}_v)$ the longitude of \mathfrak{k}_v .

(2) We now describe the vertex group A_v for $v \in V_1$. Let $n := n_v$ be the number of strands of the associated braid β_v . By definition the associated permutation $\tau \in S_n$ of β_v is an n -cycle (equivalently the closed braid $\widehat{\beta}_v$ is a knot) and $\widehat{\beta}_v$ is standardly embedded in the interior of an

unknotted solid torus $V_0 \subset \mathbb{R}^3$, see Figure 1.15. The *braid space* of β_v is defined as

$$CS(\beta_v) := \overline{V_0 - V(\beta_v)}$$

where $V(\beta_v)$ is a regular neighborhood of $\widehat{\beta}_v$ contained in the interior of V_0 , see Figure 3.2. The complementary component of \mathcal{T} corresponding to v is by construction homeomorphic to $CS(\beta_v)$.

There is an obvious fibration $CS(\beta_v) \rightarrow S^1$ of $CS(\beta_v)$ onto S^1 induced by the projection of $V_0 = \mathbb{D}^2 \times S^1$ onto the second factor. The fiber is clearly the space

$$X := \mathbb{D}^2 - Q_n,$$

where Q_n is the union of the interior of n disjoint disks contained in the interior of the unit disk.

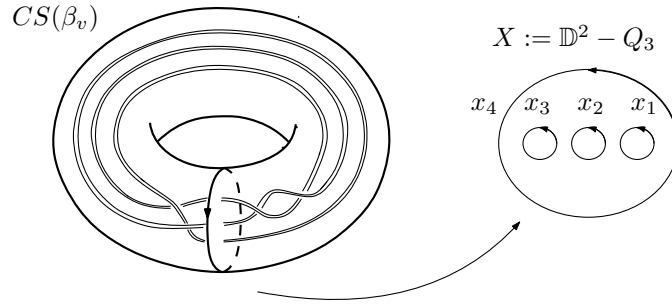


Figure 3.2: The 3-manifold $CS(\beta)$.

We will denote the free generators of $\pi_1(X)$ corresponding to the boundary paths of the removed disks by x_1, \dots, x_n . This gives a natural identification of $\pi_1(X)$ with F_n . We obtain the short exact sequence

$$1 \rightarrow F_n = \pi_1(X) \rightarrow \pi_1(CS(\beta_v)) \rightarrow \pi_1(S^1) = \mathbb{Z} \rightarrow 1.$$

Let $t \in A_v := \pi_1 CS(\beta_v)$ be the element represented by the loop in $CS(\beta_v)$ defined by $q_0 \times S^1$, for a point $q_0 \in \partial \mathbb{D}^2$. Thus t represents a longitude of V_0 in the above sense. We can write A_v as the semidirect product $F_n \rtimes \mathbb{Z}$ where the action of $\mathbb{Z} = \langle t \rangle$ on the fundamental group of the fiber is given by

$$tx_i t^{-1} = A_i x_{\tau(i)} A_i^{-1} \quad 1 \leq i \leq n \quad (3.1)$$

and the words $A_1, \dots, A_n \in F_n$ satisfy the identity

$$A_1 x_{\tau(1)} A_1^{-1} \cdot \dots \cdot A_n x_{\tau(n)} A_n^{-1} = x_1 \cdot \dots \cdot x_n \quad (3.2)$$

in the free group F_n .

Since $A_v = F_n \rtimes \mathbb{Z}$ it follows that any element of A_v can be uniquely written in the form $w \cdot t^r$ with $w \in F_n$ and $r \in \mathbb{Z}$. Moreover

$$C_v := \pi_1(\partial V_0) = \langle x_{n+1}, t \rangle$$

where $x_{n+1} := x_1 \cdot \dots \cdot x_n$. Note that in the case of a satellite knot with braid pattern β_v the curve corresponding to x_{n+1} is identified with the meridian of the companion knot and the curve corresponding to t is identified with the longitude of the companion knot. Finally

$$P_v := \pi_1(\partial V(\beta_v)) = \langle m_v, l_v \rangle$$

where $m_v := x_1$ and $l_v := u \cdot t^n$ for some $u \in F_n$. Note that for a satellite with braid pattern β_v (in particular for the knot \mathfrak{k}_{A_v}) m_v and l_v represent the meridian and the longitude of the satellite knot.

Note further that $F_n = \langle\langle m_v \rangle\rangle_{A_v}$ (normal closure in A_v) as any two elements of $\{x_1, \dots, x_n\}$ are conjugate in A_v . It follows in particular that $C_v \cap \langle\langle m_v \rangle\rangle_{A_v} = \langle x_{n+1} \rangle$.

(3) We now describe the vertex group A_v if $v \in V_2$. Let $n = \text{val}_+(v, A)$. By construction the complementary component of \mathcal{T} corresponding to the vertex v is homeomorphic to an n -fold composing space $W_n := X \times S^1$, where $X = \mathbb{D}^2 - Q_n$ is as before, see [17]. Thus

$$A_v := \pi_1(W_n) = \pi_1(X) \times \pi_1(S^1) = \langle x_1, \dots, x_n, t \mid [x_i, t] = 1 \rangle.$$

Consequently any element of A_v can be uniquely written as $w \cdot t^z$ with $w \in F_n$ and $z \in \mathbb{Z}$. Clearly t generates the center of A_v .

If the homeomorphism is chosen appropriately then we get the following with the above notation:

1. $P_v := \langle l_v, m_v \rangle$ corresponds to the peripheral subgroup of \mathfrak{k}_{A_v} where $l_v := x_{n+1}$ is the longitude and $m_v := t$ is the meridian.
2. There exists a bijection $j : St_+(v, A) \rightarrow \{1, \dots, n\}$ such that for any $e \in St_+(v, A)$ the subgroup $C_e := \langle l_e, m_v \rangle$ corresponds to the peripheral subgroup of $\mathfrak{k}_{A_{\omega(e)}}$ with $l_e := x_{j(e)}$ the longitude and $m_v := t$ the meridian.

With the notation introduced we have

$$A_v = F(\{l_e \mid e \in St_+(v, A)\}) \times \langle m_v \rangle.$$

We further denote $F(\{l_e \mid e \in St_+(v, A)\})$ by F_v .

(4) For any edge $e \in E(A, v_0)$ the associated edge group A_e is free Abelian generated by $\{m_e, l_e\}$.

We now describe the boundary monomorphisms. For any $e \in E(A, v_0)$ with $v := \alpha(e)$ and $w := \omega(e)$ the boundary monomorphism $\alpha_e : A_e \rightarrow A_v$ is given by:

$$\alpha_e(m_e^{z_1} \cdot l_e^{z_2}) = \begin{cases} x_{n+1}^{z_1} \cdot t^{z_2} & \text{if } v \in V_1. \\ m_v^{z_1} \cdot l_e^{z_2} & \text{if } v \in V_2. \end{cases}$$

while $\omega_e : A_e \rightarrow A_w$ is given by $\omega_e(m_e^{z_1} \cdot l_e^{z_2}) = m_w^{z_1} \cdot l_w^{z_2}$.

3.3 Vertex Groups

In this section we will study subgroups of the vertex groups A_v of \mathbb{A} for $v \in V_1 \cup V_2$. As these vertex groups are semidirect products of a finitely generated free group and \mathbb{Z} we start by considering certain subgroups of the free group F_n .

We think of $F_n = F(x_1, \dots, x_n)$ as the group given by the presentation

$$\langle x_1, \dots, x_{n+1} \mid x_1 \cdot \dots \cdot x_n = x_{n+1} \rangle$$

and identify F_n with $\pi_1(X)$ where X is the $(n+1)$ -punctured sphere as before.

We will study subgroups of F_n that are generated by finitely many conjugates of the x_i . We call an element of F_n *peripheral* if it is conjugate to x_i^z for some $i \in \{1, \dots, n+1\}$ and $z \in \mathbb{Z}$.

Lemma 3.3.1. *Let $S = \{g_i x_{j_i} g_i^{-1} \mid 1 \leq i \leq k\}$ with $k \geq 0$, $g_i \in F_n$, and $j_i \in \{1, \dots, n+1\}$ for $1 \leq i \leq k$. Suppose that $U := \langle S \rangle \neq F_n$.*

Then there exists $T = \{h_l x_{p_l} h_l^{-1} \mid 1 \leq l \leq m\}$ with $m \leq k$ and $h_l \in F_n$ and $p_l \in \{1, \dots, n+1\}$ for $1 \leq l \leq m$ such that the following hold:

1. U is freely generated by T .
2. Any $h_l x_{p_l} h_l^{-1}$ is conjugate in U to some $g_i x_{j_i} g_i^{-1}$.
3. Any peripheral element of U is conjugate in U to an element of $\langle h_l x_{p_l} h_l^{-1} \rangle$ for some $l \in \{1, \dots, m\}$.

In particular $\{j_1, \dots, j_k\} = \{p_1, \dots, p_m\}$.

Remark 3.3.2. Note that the conclusion of Lemma 5.1.2 does not need to hold if $U = F_n$. Indeed if $S = \{x_1, \dots, x_{n+1}\} \setminus \{x_i\}$ for some $i \in \{1, \dots, n+1\}$ then $\langle S \rangle = U = F_n$ and U has $n+1$ conjugacy classes of peripheral subgroups, it follows that conclusion (3) cannot hold.

Proof. Let \tilde{X} be the cover of X corresponding to $U \leq F_n = \pi_1(X)$. Note that the U -conjugacy classes of maximal peripheral subgroups of U correspond to compact boundary components of \tilde{X} that finitely cover boundary components of X . Note that for any i the U -conjugacy class of $\langle g_i x_{j_i} g_i^{-1} \rangle$ must correspond to a compact boundary component of \tilde{X} for which the covering is of degree 1.

As U is generated by peripheral elements it follows that the interior of \tilde{X} is a punctured sphere. Note further that all punctures but one must correspond to compact boundary components of \tilde{X} that cover components of ∂X with degree 1 and correspond to some $\langle g_i x_{j_i} g_i^{-1} \rangle$. Otherwise U could not be generated by S by the above remark.

We now show that \tilde{X} is an infinite sheeted cover of X , i.e. that the last puncture does not correspond to a compact boundary component of \tilde{X} . Suppose that \tilde{X} is a q -sheeted cover of X . As $\chi(X) = 2 - (n + 1) = 1 - n$ it follows that $\chi(\tilde{X}) = q(1 - n) = q - qn$. Clearly \tilde{X} has at least $qn + 1$ boundary components as n boundary components of X must have q lifts each in \tilde{X} . It follows that

$$\chi(\tilde{X}) \leq 2 - (qn + 1) = 1 - qn.$$

Thus $q = 1$ and therefore $U = F_n$, a contradiction.

Thus \tilde{X} is an infinite sheeted cover of X whose interior is homeomorphic to a punctured sphere such that all but one punctures correspond to compact boundary components of \tilde{X} and such that the conjugacy classes of peripheral subgroups of U corresponding to these boundary are represented by some $\langle g_i x_{j_i} g_i^{-1} \rangle$. This implies that there exists T satisfying (1) and (2), indeed we take T to be the tuple of elements corresponding to the compact boundary components \tilde{X} . Item (3) is obvious.

Finally by combining (2) and (3) we conclude that $\{j_1, \dots, j_k\} = \{p_1, \dots, p_m\}$. □

We now use Lemma 5.1.2 to describe subgroups of $A := \pi_1 CS(\beta)$ that are generated by finitely many meridians, i.e. conjugates of x_1 . Here β is an n -strand braid such that its associated permutation $\tau \in S_n$ of β is an n -cycle. Recall that A is generated by $\{x_1, \dots, x_n, t\}$, that $\langle\langle x_1 \rangle\rangle = \langle x_1, \dots, x_n \rangle$ is free in $\{x_1, \dots, x_n\}$ and that in A the element x_1 is conjugate to x_i for $1 \leq i \leq n$. As in the previous section we denote the peripheral subgroups of A by P and C .

Corollary 3.3.3. *Let $S = \{g_i x_1 g_i^{-1} | 1 \leq i \leq k\}$ with $k \geq 0$ and $g_i \in A$ for $1 \leq i \leq k$. Suppose that $U := \langle S \rangle \leq A$.*

Then either $U = \langle\langle x_1 \rangle\rangle$ or there exists

$$T = \{g'_l x_1 g'^{-1}_l | 1 \leq l \leq m\}$$

with $m \leq k$ and $g'_l \in A$ for $1 \leq l \leq m$ such that the following hold:

1. U is freely generated by T .

2. For any $g \in A$ one of the following holds:

(a) $gPg^{-1} \cap U = \{1\}$.

(b) $gPg^{-1} \cap U = g\langle x_1 \rangle g^{-1}$ and gx_1g^{-1} is conjugate in U to $g'_l x_1 g'^{-1}_l$ for some $l \in \{1, \dots, m\}$.

3. For any $g \in A$ we have $gCg^{-1} \cap U = \{1\}$.

Proof. Suppose that $U \neq F_n = \langle\langle x_1 \rangle\rangle$. We will show the existence of T as above such that (1)-(3) are satisfied.

(1) For any $g = u \cdot t^r \in A$ we have $gx_1g^{-1} = ut^r x_1 t^{-r} u^{-1} = wx_i w^{-1}$ where $i = \tau^r(1) \in \{1, \dots, n\}$ and $w \in F_n$. Hence S can be rewritten in the form

$$S = \{w_1 x_{j_1} w_1^{-1}, \dots, w_k x_{j_k} w_k^{-1}\}$$

with $w_i \in F_n$, $j_i \in \{1, \dots, n\}$ for $1 \leq i \leq k$ and so U is a subgroup of F_n generated by conjugates of the x_i . By Lemma 5.1.2 there exists

$$T = \{w'_1 x_{p_1} w'^{-1}_1, \dots, w'_m x_{p_m} w'^{-1}_m\}$$

with $m \leq k$ and $w'_l \in F_n \leq A$ and $p_l \in \{1, \dots, n\}$ for $1 \leq l \leq m$ such that U is freely generated by T and such that the other conclusions of Lemma 5.1.2 are satisfied. Now each $w'_l x_{p_l} w'^{-1}_l$ can be rewritten as $g'_l x_1 g'^{-1}_l$ for some $g'_l \in A$ and so item (1) follows.

(2) Note that for any $g \in A$ we have

$$gPg^{-1} \cap U \leq gPg^{-1} \cap F_n = gPg^{-1} \cap gF_n g^{-1} = g(P \cap F_n)g^{-1} = g\langle x_1 \rangle g^{-1}.$$

Suppose that $gPg^{-1} \cap U \neq 1$, i.e. $gx_1^z g^{-1} \in U$ for some integer $z \neq 0$. It follows from Lemma 5.1.2(3) and the fact that any element in a free group is contained in a unique maximal cyclic subgroup that $gx_1g^{-1} \in U$ and that gx_1g^{-1} is conjugate in U to $g'_l x_1 g'^{-1}_l$ for some $l \in \{1, \dots, m\}$, thus we have shown item (2).

(3) For any $g = w \cdot t^r \in A$ we have

$$gCg^{-1} \cap U \leq gCg^{-1} \cap F_n = gCg^{-1} \cap gF_n g^{-1} = g(C \cap F_n)g^{-1} = g\langle x_{n+1} \rangle g^{-1}.$$

It follows from the discussion of the algebraic structure of $\pi_1(CS(\beta))$ in Section 3.2 that $tx_{n+1}t^{-1} = x_{n+1}$ and therefore $gx_{n+1}^z g^{-1} = wx_{n+1}^z w^{-1}$ for all $z \in \mathbb{Z}$. If $gx_{n+1}^z g^{-1} = wx_{n+1}^z w^{-1} \in U$ for some integer $z \neq 0$, then Lemma 5.1.2 implies that $wx_{n+1}^z w^{-1}$ is conjugate in U to $w'_l x_{p_l}^z w'^{-1}_l$ for some l . But this is a contradiction since in F_n the element x_i^z is not conjugate to x_j^z for $i \neq j \in \{1, \dots, n+1\}$. \square

We now use Lemma 5.1.2 to describe subgroups of

$$B = \pi_1 W_n = \langle x_1, \dots, x_n, t \mid [x_i, t] = 1 \rangle$$

that are generated by t and conjugates of the x_i . Here $P = \langle x_{n+1}, t \rangle$ and $C_i = \langle x_i, t \rangle$ for $1 \leq i \leq n$.

Corollary 3.3.4. *Let $S = \{g_i x_{j_i} g_i^{-1} \mid 1 \leq i \leq k\}$ with $g_i \in B$ and $j_i \in \{1, \dots, n\}$ for $1 \leq i \leq k$. Let $U := \langle S \rangle \times \langle t \rangle \leq B$.*

Then either $U = B$ or there exists

$$T = \{h_l x_{p_l} h_l^{-1} \mid 1 \leq l \leq m\}$$

with $m \leq k$, $h_l \in F_n$ and $p_l \in \{1, \dots, n\}$ for $1 \leq l \leq m$ such that the following hold:

1. $\{j_1, \dots, j_k\} = \{p_1, \dots, p_m\}$.
2. $\langle S \rangle$ is freely generated by T .
3. For any $g \in B$ and any $i \in \{1, \dots, n\}$ one of the following holds:
 - (a) $gC_i g^{-1} \cap U = \langle t \rangle$.
 - (b) $gC_i g^{-1} \cap U = gC_i g^{-1} = g\langle x_i \rangle g^{-1} \times \langle t \rangle$ and $g x_i g^{-1}$ is conjugate in U to $h_l x_{p_l} h_l^{-1}$ for some $l \in \{1, \dots, m\}$. In particular $i = p_l$.
4. For any $g \in B$ we have $gP g^{-1} \cap U = \langle t \rangle$.

Proof. Assume that $U \neq B$. We will show the existence of T as above such that (1)-(4) are satisfied.

(1)-(2) For any $g = w \cdot t^z \in B$ and $1 \leq i \leq n+1$ we have

$$g x_i g^{-1} = w x_i w^{-1}.$$

Hence S can be rewritten in the form

$$S = \{w_1 x_{j_1} w_1^{-1}, \dots, w_k x_{j_k} w_k^{-1}\}$$

with $w_i \in F_n$ for $1 \leq i \leq k$ and so $\langle S \rangle \leq F_n = \langle x_1, \dots, x_n \rangle$. Since $U \neq B$ it follows that $\langle S \rangle$ is a proper subgroup of F_n . By Lemma 5.1.2 there exists

$$T = \{h_1 x_{p_1} h_1^{-1}, \dots, h_m x_{p_m} h_m^{-1}\}$$

with $m \leq k$, $h_l \in F_n$ and $p_l \in \{1, \dots, n\}$ for $1 \leq l \leq m$ such that $\langle S \rangle$ is freely generated by T and that (1) and (2) are satisfied.

(3) Let $g \in B$ and $i \in \{1, \dots, n\}$. Clearly we have

$$\langle t \rangle \subseteq gC_i g^{-1} \cap U.$$

If $\langle t \rangle = gC_i g^{-1} \cap U$ then there is nothing to show. Thus we may assume that $\langle t \rangle \subsetneq gC_i g^{-1} \cap U$.

It follows that $gx_i^z g^{-1} \in U$ for some $z \neq 0$. This clearly implies that $gx_i^z g^{-1} \in \langle S \rangle$. It follows from Lemma 5.1.2(3) that $gx_i g^{-1} \in \langle S \rangle$ and that $gx_i g^{-1}$ is conjugate in $\langle S \rangle$ to $h_l x_{p_l} h_l^{-1}$ for some $l \in \{1, \dots, m\}$, thus we have shown item (3).

(4) For any $g = w \cdot t^z \in B$ we have

$$\langle t \rangle \subseteq gPg^{-1} \cap U.$$

If $\langle t \rangle \subsetneq gPg^{-1} \cap U$ then $gx_{n+1}^z g^{-1} = wx_{n+1}^z w^{-1} \in U$ for some integer $z \neq 0$. It follows that $gx_{n+1}^z g^{-1} \in \langle S \rangle \leq U$. Lemma 5.1.2 implies that $wx_{n+1}^z w^{-1}$ is conjugate in $\langle S \rangle$ to $h_l x_{p_l}^z h_l^{-1}$ for some l . But this is impossible as in F_n the element x_i^z is not conjugate to x_j^z for $i \neq j \in \{1, \dots, n+1\}$. \square

3.4 Proof of the main theorem

In this section we give the proof of Theorem 3.0.5 or equivalently of Theorem 3.1.2. We will show that for any knot \mathfrak{k} from \mathcal{K} the meridional rank $w(\mathfrak{k})$ is bounded from below by the bridge number that is given by Lemma 3.1.3.

The general idea of the proof is similar to the proof of Grushko's theorem. Clearly there is an epimorphism from the free group of rank $w(\mathfrak{k})$ to $G(\mathfrak{k})$ that maps any basis element (of some fixed basis) to a conjugate of the meridian. This epimorphism can be realized by a morphism of graphs of groups. Such a morphism can be written as a product of folds and the main difficulty of a proof is to define a complexity that does not increase in the folding sequence.

The proof is now structured as follows:

1. We assume that the knot group $G(\mathfrak{k})$ of \mathfrak{k} is generated by $l < b(\mathfrak{k})$ meridians, we prove the theorem by obtaining a contradiction to this assumption.
2. We construct an \mathbb{A} -graph \mathcal{B}_0 whose fundamental group is free of rank l that represents $G(\mathfrak{k}) = \pi_1(\mathbb{A})$.

3. We define the class of tame \mathbb{A} graphs endowed with a lexicographically ordered complexity function $c = (c_1, c_2)$ and observe that \mathcal{B}_0 is tame with $c_1(\mathcal{B}_0) = (l, c_2(\mathcal{B}_0))$. In particular there exists a tame \mathbb{A} -graph representing $G(\mathfrak{k})$ with $c_1(\mathfrak{k}) < b(\mathfrak{k})$.
4. We now choose a tame \mathbb{A} -graph \mathcal{B} representing $G(\mathfrak{k})$ that is of minimal complexity. Thus in particular $c_1(\mathcal{B}) \leq l < b(\mathfrak{k})$.
5. We show that \mathcal{B} is not folded as otherwise $c_1(\mathcal{B}) = b(\mathfrak{k})$. Thus a fold can be applied to \mathcal{B} .
6. By applying a fold we construct a tame \mathbb{A} -graph of smaller complexity than \mathcal{B} also representing $G(\mathfrak{k})$ obtaining a contradiction to the minimality assumption made in (4). Thus $l \geq b(\mathfrak{k})$ and the theorem is proven.

Assume that the knot group $G(\mathfrak{k})$ of \mathfrak{k} is generated by $l < b(\mathfrak{k})$ meridians, namely by $g_1 m g_1^{-1}, \dots, g_l m g_l^{-1}$ where $g_i \in G(\mathfrak{k})$ for $i = 1, \dots, l$. Since $G(\mathfrak{k})$ splits as $\pi_1(\mathbb{A}, v_0)$ where \mathbb{A} is the tree of groups described in Section 3.2, we see that for $1 \leq i \leq l$ the element g_i can be written as $g_i = [\gamma_i]$ where γ_i is an \mathbb{A} -path from v_0 to v_0 of the form

$$\gamma_i = a_{i,0}, e_{i,1}, a_{i,1}, e_{i,2}, \dots, a_{i,q_i-1}, e_{i,q_i}, a_{i,q_i}$$

for some $q_i \geq 1$. Observe that we do not require γ_i to be reduced, otherwise we could possibly not choose γ_i such that $q_i \geq 1$.

We now define the \mathbb{A} -graph \mathcal{B}_0 as follows:

1. The underlying graph B_0 is a finite tree given by:
 - (a) $EB_0 := \{f_{i,j}^\varepsilon \mid 1 \leq i \leq l, 1 \leq j \leq q_i, \varepsilon \in \{-1, +1\}\}$.
 - (b) $VB_0 := \{u_0\} \cup \{u_{i,j} \mid 1 \leq i \leq l, 1 \leq j \leq q_i\}$.
 - (c) For $1 \leq i \leq l$ the initial vertex of $f_{i,j}$ is

$$\alpha(f_{i,j}) = \begin{cases} u_0 & \text{if } j = 1. \\ u_{i,j-1} & \text{if } j > 1. \end{cases}$$

while the terminal vertex of $f_{i,j}$ is $\omega(f_{i,j}) = u_{i,j}$ for $1 \leq j \leq q_i$.

2. The graph morphism $[\cdot] : B_0 \rightarrow A$ is given by $[f_{i,j}^\varepsilon] = e_{i,j}^\varepsilon$.
3. For each $x \in VB_0 \cup EB_0$ the associated group is

$$B_x = \begin{cases} \langle m_{v_0} \rangle & \text{if } x = u_{i,q_i} \text{ for some } 1 \leq i \leq l. \\ 1 & \text{otherwise.} \end{cases}$$

4. For $1 \leq i \leq l$, $(f_{i,j})_\alpha = a_{i,j-1}$ for all $1 \leq j \leq q_i$ while

$$(f_{i,j})_\omega = \begin{cases} 1 & \text{if } j < q_i. \\ a_{i,q_i} & \text{if } j = q_i. \end{cases}$$

Observe that the fundamental group of the associated graph of groups \mathbb{B}_0 is freely generated by the elements

$$y_i := [1, f_{i,1}, 1, \dots, f_{i,q_i}, m_{v_0}, f_{i,q_i}^{-1}, \dots, 1, f_{i,1}^{-1}, 1]$$

for $1 \leq i \leq l$. Additionally, the induced homomorphism $\phi : \pi_1(\mathbb{B}_0, u_0) \rightarrow \pi(\mathbb{A}, v_0)$ is surjective as $g_i m g_i^{-1} = \phi(y_i)$ by our construction of \mathcal{B}_0 .

Recall that an \mathbb{A} -subgraph of \mathcal{B} is an \mathbb{A} -graph \mathcal{B}' such that:

1. $B' \subset B$.
2. $[\cdot]' = [\cdot]|_{B'} : B' \rightarrow A$.
3. $B'_x = B_x$ for all $x \in VB' \cup EB'$.
4. $l_{\mathcal{B}'}(f') = l_{\mathcal{B}}(f')$ for all $f' \in EB'$.

Let now \mathcal{B} be an arbitrary \mathbb{A} -graph whose underlying graph B is a finite tree. Throughout the proof we assume that B has the orientation $EB_+ := \{f \in EB \mid [f] \in E(A, v_0)\}$.

We say that a vertex $u \in VB$ is *isolated* if $B_f = 1$ for any $f \in St_+(u, B)$.

We further say that a vertex $u \in VB$ is *full* if there exists a sub-tree B' of $B(u)$ such that the following hold:

1. $u \in VB'$.
2. $[\cdot] : B \rightarrow A$ maps B' isomorphically onto $A([u])$.
3. $B_x = A_{[x]}$ for all $x \in VB' \cup EB'$.

The following lemma follows immediately from the definition of fullness:

Lemma 3.4.1. *Let \mathcal{B} be an \mathbb{A} -graph whose underlying graph is a finite tree and $u \in VB$. Then u is full if and only if $B_u = A_{[u]}$ and there exists $S_+ \subseteq St_+(u, B)$ such that the following hold:*

1. $[\cdot]|_{S_+} : S_+ \rightarrow St_+([u], A)$ is a bijection.
2. $B_e = A_{[e]}$ for all $e \in S_+$.
3. $\omega(e)$ is full for all $e \in S_+$.

Lemma 3.4.2. *Let \mathcal{B} be an \mathbb{A} -graph whose underlying graph is a finite tree. Assume that \mathcal{B}_1 is obtained from \mathcal{B} by a fold. Then the image of any full vertex in \mathcal{B} is full in \mathcal{B}_1 .*

Proof. It follows immediately from the definition of foldings of \mathbb{A} -graphs (see section 1.5 of [49]) that if \mathcal{B}' is a folded \mathbb{A} -subgraph of \mathcal{B} , then the image of \mathcal{B}' in \mathcal{B}_1 is an \mathbb{A} -subgraph \mathcal{B}'_1 of \mathcal{B}_1 such that the following hold:

- (i) B'_1 is isomorphic to B' .
- (ii) $B'_x \leq (B'_1)_x$ for all $x \in VB' \cup EB'$.

Assume that $u \in VB$ is a full vertex of \mathcal{B} . By definition there exists a sub-tree $B' \subseteq B(u)$ such that:

- 1. $u \in B'$.
- 2. $[\cdot]_{B'} : B' \rightarrow A([u])$ is an isomorphism.
- 3. $B_x = A_{[x]}$ for all $x \in VB' \cup EB'$.

Items (2) and (3) imply that the \mathbb{A} -subgraph \mathcal{B}' of \mathcal{B} having B' as its underlying graph is folded. Thus the image of \mathcal{B}' in \mathcal{B}_1 is an \mathbb{A} -subgraph \mathcal{B}'_1 of \mathcal{B}_1 such that (i) and (ii) hold. From (i) it follows that B'_1 is isomorphic to $A([u])$. Item (ii) implies that $(B_1)_x = A_{[x]}$ for all $x \in VB' \cup EB'$ as $A_{[x]} = B_x$. Therefore the image of u in \mathcal{B}' is full. \square

Definition 3.4.3. We call an \mathbb{A} -graph \mathcal{B} with associated graph of groups \mathbb{B} *tame* if the graph B underlying \mathcal{B} is a finite tree and the following conditions hold:

- (1) For each $f \in EB_+$ with $e := [f]$ one of the following holds:
 - (a) $B_f = 1$.
 - (b) $B_f = \langle m_e \rangle$.
 - (c) $B_f = A_e = \langle m_e, l_e \rangle$ and $\omega(f) \in VB$ is full.
- (2) For every vertex $u \in VB$ with $v := [u] \in V_0$ one of the following holds:
 - (a) B_u is generated by $r_u < b(\mathfrak{k}_v)$ meridians of \mathfrak{k}_v .
 - (b) $B_u = A_v = G(\mathfrak{k}_v)$.
- (3) For every vertex $u \in VB$ with $v := [u] \in V_1$ one of the following holds:
 - (a) B_u is freely generated by finitely many conjugates of m_v .
 - (b) $B_u = A_v$ and u is full.

(4) For every vertex $u \in VB$ with $v := [u] \in V_2$ one of the following holds:

(a) $B_u = 1$.

(b) There exists $S_u \subseteq \{f \in St_+(u, B) \mid B_f = A_{[f]}\}$ such that $B_u = F_u \times \langle m_v \rangle$, with F_u is freely generated by

$$\{g_f l_{[f]} g_f^{-1} \mid f \in S_u\}$$

where $g_f \in A_v$ and $f_\alpha l_{[f]} f_\alpha^{-1}$ is conjugate in B_u to $g_f l_{[f]} g_f^{-1}$ for all $f \in S_u$. Moreover, if $B_u = A_v$ then u is full.

Put $[S_u] := \{[f] \mid f \in S_u\} \subseteq St_+(v, A)$.

Observe that the \mathbb{A} -graph \mathcal{B}_0 defined above is tame as $B_x = \{1\}$ for all $x \in VB_0 \cup EB_0 \setminus \{u_{i,q_i} \mid 1 \leq i \leq l\}$ and $B_{i,q_i} \leq A_{v_0}$ is infinite cyclic, generated by the meridian m of \mathfrak{k} .

Next we will define the complexity of a tame \mathbb{A} -graph \mathcal{B} . First we need to introduce the following notion. We define the *positive height* of a vertex $v \in VA$ as

$$h_+(v) := h(v)n(v)$$

where $h(v)$ denotes the height of a vertex in A defined in Section 3.1. Note that for any edge $e \in E(A, v_0)$ we have $h_+(\alpha(e)) = h(\omega(e))$. In particular $h_+(v) = h(v)$ if $v \in V_2$ and $h_+(v) = h(v) \cdot n_v$ if $v \in V_1$.

Definition 3.4.4. Let \mathcal{B} be a tame \mathbb{A} -graph.

We define *the c_1 -complexity* of \mathcal{B} as

$$c_1(\mathcal{B}) := \sum_{\substack{u \in VB \\ u \text{ isol.}}} h([u]) \cdot w(B_u) - \sum_{\substack{u \in VB \\ u \text{ isn't isol.}}} h_+([u]) \cdot (val_+^1(u, \mathcal{B}) - 1)$$

where $val_+^1(u, \mathcal{B}) := |\{f \in St_+(u, B) \mid B_f \neq 1\}|$.

We further define the c_2 -complexity of \mathcal{B} as

$$c_2(\mathcal{B}) := 2|EB| - |E_2B| - \frac{1}{2}|E_1B|$$

where $E_iB \subset EB$ denotes the set of edges whose edge group is isomorphic to \mathbb{Z}^i ($i = 1, 2$).

Remark 3.4.5. Note the similarity of the definition of c_1 with the formula for the bridge number from Lemma 3.1.3. This implies that the c_1 complexity of a folded \mathbb{A} -graph representing $G(\mathfrak{k})$ coincides with $b(\mathfrak{k})$, see the proof of Lemma 3.4.7 for details.

Lemma 3.4.6. *Let \mathcal{B} be a tame \mathbb{A} -graph. Then $c_1(\mathcal{B}) \geq 0$.*

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Proof. Let B' be the subgraph of B with $VB' := \{v \in VB \mid B_v \neq 1\}$ and $EB' := \{e \in EB \mid B_e \neq 1\}$. For any component C of B' let \mathcal{C} be the corresponding \mathbb{A} -subgraph of \mathcal{B} . Clearly any such \mathcal{C} is tame and we have

$$c_1(\mathcal{B}) = \sum_{C \text{ comp. of } B'} c_1(\mathcal{C}).$$

Thus it suffices to show that $c_1(\mathcal{C}) \geq 0$ for any component C of B' .

As C is a finite oriented tree it follows from Lemma 3.1.1 that for each vertex $u \in VC$ we can choose a set $P(u) = \{w_{u,1}, \dots, w_{u,r_u}\} \subseteq \{u' \in VC \mid \text{val}_+(u', C) = 0\}$ such that the following hold:

1. $r_u \geq \text{val}_+(u, C) - 1$.
2. u projects onto $w_{u,i}$ for $1 \leq i \leq r_u$.
3. If $u \neq u'$ then $P(u) \cap P(u') = \emptyset$.

Since each edge of C has non-trivial edge group we have $\text{val}_+(u, C) = \text{val}_+^1(u, \mathcal{C})$ for all $u \in VC$. In particular any $w_{u,i}$ is an isolated vertex.

The vertex group $C_{w_{u,i}}$ is non-trivial as there is an incident edge at $w_{u,i}$ with non-trivial group. Consequently $w(C_{w_{u,i}}) \geq 1$. As we further have $h_+([u]) \leq h([w_{u,i}])$ we conclude that

$$\begin{aligned} c_1(\mathcal{C}) &= \sum_{\substack{u \in VC \\ u \text{ isol.}}} h([u]) \cdot w(C_u) - \sum_{\substack{u \in VC \\ u \text{ isn't isol.}}} h_+([u]) \cdot (\text{val}_+^1(u, \mathcal{C}) - 1) \\ &\geq \sum_{\substack{u \in VC \\ u \text{ isn't isol.}}} \left[\left(\sum_{i=1}^{r_u} h([w_{u,i}]) \cdot w(C_{w_{u,i}}) \right) - h_+([u]) \cdot (\text{val}_+^1(u, \mathcal{C}) - 1) \right] \\ &\geq \sum_{\substack{u \in VC \\ u \text{ isn't isol.}}} \left[\left(\sum_{i=1}^{r_u} h_+([u]) \cdot 1 \right) - h_+([u]) (\text{val}_+^1(u, \mathcal{C}) - 1) \right] \\ &= \sum_{\substack{u \in VC \\ u \text{ isn't isol.}}} \left[h_+([u]) \cdot r_u - h_+([u]) \cdot (\text{val}_+^1(u, \mathcal{C}) - 1) \right] \\ &\geq \sum_{\substack{u \in VC \\ u \text{ isn't isol.}}} \left[h_+([u]) \cdot (\text{val}_+^1(u, \mathcal{C}) - 1) - h_+([u]) \cdot (\text{val}_+^1(u, \mathcal{C}) - 1) \right] \\ &= 0 \end{aligned}$$

□

The c -complexity of \mathcal{B} is defined as

$$c(\mathcal{B}) = (c_1(\mathcal{B}), c_2(\mathcal{B})) \in \mathbb{N} \times \mathbb{N}.$$

As we want to compare complexities we endow the set $\mathbb{N} \times \mathbb{N}$ with the lexicographic order, i.e. we write $(n_1, m_1) < (n_2, m_2)$ if one of the following occurs:

1. $n_1 < n_2$.
2. $n_1 = n_2$ and $m_1 < m_2$.

For the tame \mathbb{A} -graph \mathcal{B}_0 we have $c_1(\mathcal{B}_0) = l < b(\mathfrak{k})$. Thus our assumption (which will yield a contradiction) implies that there exists a tame \mathbb{A} -graph \mathcal{B} with c_1 -complexity strictly smaller than the bridge number of $\mathfrak{k} = \mathfrak{k}_{\mathcal{A}}$ such that the induced homomorphism

$$\phi : \pi_1(\mathbb{B}_0, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$$

is an epimorphism.

Let now \mathcal{B} be a tame \mathbb{A} -graph such that there exists a vertex $u_0 \in VB$ of type v_0 such that $\phi : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$ is surjective and such that among all such \mathcal{B} the c -complexity is minimal. Note that $c_1(\mathcal{B}) \leq l < b(\mathfrak{k})$ as the \mathbb{A} -graph \mathcal{B}_0 constructed above is tame and the map ϕ is surjective. We will prove the theorem by deriving a contradiction to this minimality assumption.

Lemma 3.4.7. *\mathcal{B} is not folded.*

Proof. Assume that \mathcal{B} is folded. Proposition 4.3 of [18] and the fact that the map

$$\phi : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$$

is surjective imply that \mathbb{B} is isomorphic to \mathbb{A} , i.e. the graph morphism is bijective and that all vertex and edge groups are mapped bijectively. Observe that this implies the following:

1. The morphism $[\cdot]$ maps the isolated vertices of VB bijectively to $V_0 \subset VA$. It follows in particular that for any isolated vertex $u \in VB$ we have $w(B_u) = w(A_{[u]}) = w(G(\mathfrak{k}_{[u]})) = b(\mathfrak{k}_{[u]})$ as $\mathfrak{k}_{[u]}$ is meridionally tame by assumption.
2. As all edge groups are non-trivial we have $val_+^1(u, \mathcal{B}) = val_+(u, B) = val_+([u], A)$ for all $u \in VB$.
3. If $u \in VB$ is not isolated, then either $[u] \in V_1$ and therefore $val_+^1(u, \mathcal{B}) - 1 = 0$ or $[u] \in V_2$ and therefore $h_+([u]) = h([u])$.

Using Lemma 3.1.3 this implies that

$$\begin{aligned}
 b(\mathfrak{k}) &= \left[\sum_{v \in V_0} h(v) \cdot b(\mathfrak{k}_v) \right] - \left[\sum_{v \in V_2} h(v) \cdot (\text{val}_+(v, A) - 1) \right] \\
 &= \sum_{\substack{u \in VB \\ u \text{ isol.}}} h([u]) \cdot w(B_u) - \sum_{\substack{u \in VB \\ u \text{ isn't isol.}}} h_+([u]) \cdot (\text{val}_+^1(u, \mathcal{B}) - 1) \\
 &= c_1(\mathcal{B})
 \end{aligned}$$

contradicting the assumption that $c_1(\mathcal{B}) < b(\mathfrak{k})$.

Thus \mathcal{B} is not folded. □

As the \mathbb{A} -graph \mathcal{B} is not folded, a fold can be applied to \mathcal{B} ; this follows immediately from the definition of the various folds and the definition of a folded \mathbb{A} -graph, see [49, p. 615].

Any fold is a composition of finitely many auxiliary moves and by fold described in a particularly easy way, namely an elementary move. Thus for us auxiliary moves are merely used as preprocessing tools. They can be interpreted as isomorphisms of graphs of groups, i.e. they don't essentially affect the \mathbb{A} -graph. The following lemma implies that we can restrict attention to elementary moves as auxiliary moves affect neither tameness nor complexity. The proof of Lemma 3.4.8 is a trivial consequence of the definition of the auxiliary moves. For a detailed description of auxiliary moves and folds we refer to [49, Section 1.5], cf. also Figures 3.3 and 3.4 below.

Lemma 3.4.8. *Let \mathcal{B} be a tame \mathbb{A} -graph with base vertex u_0 of type v_0 . Suppose that \mathcal{B}' is obtained from \mathcal{B} by an auxiliary move. Then \mathcal{B}' is tame and $c(\mathcal{B}') = c(\mathcal{B})$.*

As the graph B underlying \mathcal{B} is a tree it follows that only folds of type IA and IIA can occur. Because of the discussion preceding Lemma 3.4.8 we can assume that one of the following holds after first performing some auxiliary moves:

1. An elementary move of type IA can be applied to \mathcal{B} .
2. No fold of type IA can be applied to \mathcal{B} but an elementary move of type IIA can be applied to \mathcal{B} .

In both cases we will derive the desired contradiction to the minimality of the complexity of \mathcal{B} by producing a tame \mathbb{A} -graph \mathcal{B}'' that is π_1 -surjective and such that $c(\mathcal{B}'') < c(\mathcal{B})$.

The following lemma will be useful when considering folds. It implies that certain type IIA folds are only possible if also a fold of type IA is possible. Because of our above choice we will therefore not need to consider such folds of type IIA.

Lemma 3.4.9. *Let \mathcal{B} be a tame \mathbb{A} -graph, $u \in VB$ and $v := [u]$. Suppose that one of the following holds:*

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1. $v \in V_2$ and there exists $f \in St_+(u, B)$ labeled (a, e, b) such that $B_f \neq A_e$ and $a\alpha_e(l_e)a^{-1} = al_ea^{-1} \in B_u$.
2. $v \in V_1$, $\langle\langle m_v \rangle\rangle \leq B_u$ and there exist distinct edges $f_1, f_2 \in St_+(u, B)$.

Then we can apply a fold of type IA to \mathcal{B} .

Proof. (1) Note first that an element

$$g \in A_v = F_v \times \langle m_v \rangle = F(\{l_e \mid e \in St_+(v, A)\}) \times \langle m_v \rangle$$

commutes with l_e if and only if $g = m_v^{z_1} \cdot l_e^{z_2} = \alpha_e(m_e^{z_1} \cdot l_e^{z_2})$ for $z_1, z_2 \in \mathbb{Z}$. This follows immediately from the fact that any maximal cyclic subgroup of F_v and therefore also $\langle l_e \rangle$ is self-normalizing in F_v .

It follows from Corollary 3.3.4(3.b) and the tameness of \mathcal{B} that there exists an edge $f_0 \in St_+(u, B)$ labeled (a_0, e, b_0) such that $B_{f_0} = A_e = \langle m_e, l_e \rangle$ and

$$al_ea^{-1} = ga_0l_ea_0^{-1}g^{-1}$$

for some $g \in B_u$. Thus, $a_0^{-1}g^{-1}a$ commutes with l_e which implies that

$$a_0^{-1}g^{-1}a = m_v^{z_1} \cdot l_e^{z_2} = \alpha_e(m_e^{z_1} \cdot l_e^{z_2})$$

for $z_1, z_2 \in \mathbb{Z}$. Hence $a = ga_0\alpha_e(m_e^{z_1}l_e^{z_2})$. Since we further have $f_0 \neq f$ and $[f_0] = [f] = e$ it follows that \mathcal{B} is not folded because condition (F1) is not satisfied, see p.615 from [49]. Thus we can apply a fold of type I or type III to \mathcal{B} . Since the underlying graph B of \mathcal{B} is a tree it follows that we can apply a fold of type IA to \mathcal{B} .

(2) Since $v \in V_1$ it follows that $St_+(v, A) = \{e\}$ for some $e \in EA$ and so f_1 and f_2 are of same type. For $i \in \{1, 2\}$ let $(f_i)_\alpha = w_i \cdot t^{z_i}$ where $w_i \in \langle\langle m_v \rangle\rangle$ and $z_i \in \mathbb{Z}$. We can write $(f_2)_\alpha \in A_v = F_{n_v} \rtimes \langle t \rangle$ as

$$(f_2)_\alpha = w_2w_1^{-1} \cdot (f_1)_\alpha \cdot t^{z_2-z_1}.$$

Note now that $w_2w_1^{-1} \in \langle\langle m_v \rangle\rangle \leq B_u$ and $t^{z_2-z_1} = \alpha_e(l_e^{z_2-z_1})$. Thus \mathcal{B} is not folded because condition (F1) is not satisfied which implies in our context we can apply a fold of type IA to \mathcal{B} . □

The following Lemma implies that if we apply an elementary move of type IIA in the direction of an oriented edge, we only ever add a meridian to the edge group.

Lemma 3.4.10. *Let \mathcal{B} be a tame \mathbb{A} -graph and $f \in EB_+$ labeled (a, e, b) with $x := \alpha(f)$. Assume that no fold of type IA is applicable to \mathcal{B} (i.e. condition (F1) is satisfied, see [49, p.615]) and that*

$$B_f \not\subseteq \alpha_e^{-1}(a^{-1}B_xa).$$

Then $\alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle$. In particular, $B_f = 1$.

Proof. We first assume that $v := \alpha(e) \in V_1$. By tameness of \mathcal{B} we know that $B_x = A_v$ or $B_x \leq \langle\langle m_v \rangle\rangle$ is freely generated by conjugates of m_v .

If $B_x = A_v$, then tameness implies that x is full. Hence there exists an edge $f_0 \in St_+(x, B)$ such that $B_{f_0} = A_e$. Since no folds of type IA are applicable to \mathcal{B} , it follows from Lemma 6.1.6 that $f = f_0$. Consequently

$$\alpha_e^{-1}(a^{-1}B_x a) = \alpha_e^{-1}(A_v) = A_e = B_f$$

contradicting the fact that $B_f \subsetneq \alpha_e^{-1}(a^{-1}B_x a)$.

Thus $B_x \leq \langle\langle m_v \rangle\rangle$ is freely generated by conjugates of m_v . As by hypothesis $B_f \subsetneq \alpha_e^{-1}(a^{-1}B_x a)$ it implies that $\alpha_e^{-1}(a^{-1}B_x a)$ is non-trivial. It is a consequence of Corollary 3.3.3(3) that in this case $B_x = \langle\langle m_v \rangle\rangle$ and so $\alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle$.

Assume now that $v \in V_2$. We know from Corollary 3.3.4 that one of the following holds true:

1. $\alpha_e^{-1}(a^{-1}B_x a) = 1$
2. $\alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle$
3. $\alpha_e^{-1}(a^{-1}B_x a) = A_e = \langle m_e, l_e \rangle$

If $\alpha_e^{-1}(a^{-1}B_x a) = A_e$, then $al_e a^{-1} \in F_x$. As by assumption $B_f \subsetneq A_e$ we conclude from Lemma 6.1.6(1) that a fold of type IA can be applied on \mathcal{B} , a contradiction. Thus (1) or (2) occurs. As $1 \subseteq B_f \subsetneq \alpha_e^{-1}(a^{-1}B_x a)$ we conclude that (2) must occur. \square

We will now deal with the two cases mentioned above. In both cases we let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B} by a fold of type IA or IIA. From Proposition 8 of [49] it follows that \mathcal{B}' is π_1 -surjective. After possibly applying auxiliary moves first we can assume that the folds are elementary.

Case 1: \mathcal{B}' is obtained from \mathcal{B} by an elementary move of type IA: By definition, there exist distinct edges $f_1, f_2 \in EB$ with $\alpha(f_i) = u \in VB$ and $l(f_i) = (a, e, b)$ for $i \in \{1, 2\}$ such that \mathcal{B}' is obtained from \mathcal{B} by identifying f_1 and f_2 into a single edge f . Put $x := \omega(f_1)$, $y := \omega(f_2)$, $v := \alpha(e)$ and $w := \omega(e)$ and $z := \omega(f)$. In particular we have $B'_f = \langle B_{f_1}, B_{f_2} \rangle$ and $B'_z = \langle B_x, B_y \rangle$, see Figure 3.3.

Let now \mathcal{B}'' be the \mathbb{A} -graph obtained from \mathcal{B}' by replacing the vertex group B'_z by the group B''_z defined as follows:

$$B''_z := \begin{cases} B'_z = \langle B_x, B_y \rangle & \text{if } w \in V_1 \cup V_2 \text{ or } w \in V_0 \text{ and } w(B_x) + w(B_y) < b(\mathfrak{k}_w). \\ A_w = G(\mathfrak{k}_w) & \text{if } w \in V_0 \text{ and } w(B_x) + w(B_y) \geq b(\mathfrak{k}_w). \end{cases}$$

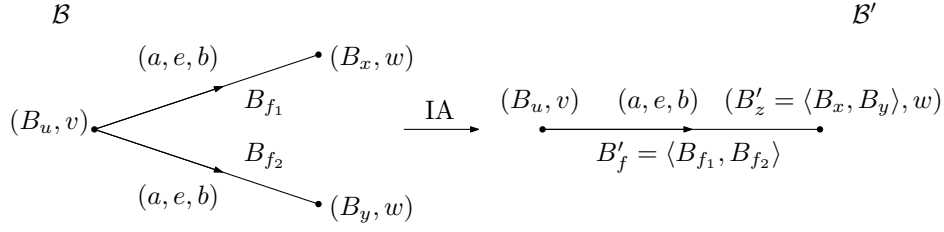


Figure 3.3: An elementary move of type IA.

The \mathbb{A} -graph \mathcal{B}'' is π_1 -surjective since it is defined from \mathcal{B}' by possibly enlarging the vertex group at z . Note that the underlying graph B'' of \mathcal{B}'' is equal to B' . Moreover, $B''_{u'} = B_{u'}$ for all $u' \in VB'' \setminus \{z\} \cup EB'' \setminus \{f, f^{-1}\}$ while $B''_f = B'_f = \langle B_{f_1}, B_{f_2} \rangle$.

Lemma 3.4.11. *The \mathbb{A} -graph \mathcal{B}'' is tame.*

Proof. We first show that edge groups are as in Definition 3.4.3. From the tameness of \mathcal{B} we conclude that one of the following occurs:

1. $B''_f = 1$.
2. $B''_f = \langle m_e \rangle$.
3. $B''_f = A_e$. This occurs if and only if $B_{f_i} = A_e$ for some $i \in \{1, 2\}$. In this case $\omega(f_i)$ is full by tameness and therefore z is also full by Lemma 3.4.2.

Since the remaining edge groups do not change we conclude that edge groups are as in Definition 3.4.3.

The \mathbb{A} -graph \mathcal{B}'' is tame if $w \in V_0$. Indeed, by definition $B''_z = G(\mathfrak{k}_w)$ or $B''_z = \langle B_x, B_y \rangle$ is generated by at most $w(B_x) + w(B_y) < b(\mathfrak{k}_w)$ meridians of \mathfrak{k}_w . Hence condition (2) of Definition 3.4.3 is fulfilled.

We next show that \mathcal{B}'' is tame if $w \in V_1$. Note first that if $B_q = A_w$ for some $q \in \{x, y\}$, then obviously $B''_z = B'_z = A_w$. Tameness of \mathcal{B} implies that $q \in VB$ is full and so z is full by Lemma 3.4.2. Thus we may assume that both B_x and B_y are freely generated by conjugates of m_w . It follows from Corollary 3.3.3 that $B''_z = \langle B_x, B_y \rangle$ is freely generated by at most $w(B_x) + w(B_y)$ conjugates of m_w . Since any subgroup of A_w that is generated by conjugates of m_w is contained in $\langle\langle m_w \rangle\rangle$, which is a proper subgroup of A_w , we conclude that $B''_z = A_w$ if and only if $B_q = A_w$ for some $q \in \{x, y\}$. Therefore \mathcal{B}'' is tame.

Suppose now that $w \in V_2$. If $B_x = 1$ or $B_y = 1$, say $B_x = 1$, then clearly $B''_z = B_y$ and there is nothing to show. Thus we may assume that for $q \in \{x, y\}$ we have $B_q = F_q \times \langle m_w \rangle$ with

$$F_q = F(\{\{g_{f'} l_{[f']} g_{f'}^{-1} \mid f' \in S_q\}\})$$

where $S_q \subseteq \{f' \in St_+(q, B) \mid B_{f'} = A_{[f']}\}$ and $g_{f'} \in A_w$ for all $f' \in S_q$. Thus $B_z'' = F_z'' \times \langle m_w \rangle$ where

$$F_z'' := \langle \{g_{f'} l_{[f']} g_{f'}^{-1} \mid f' \in S_x \cup S_y\} \rangle \leq F_w.$$

From Corollary 3.3.4 we conclude that there exists a subset $S_z'' \subseteq S_x \dot{\cup} S_y$ such that

$$F_z'' = F(\{h_{f'} l_{[f']} h_{f'}^{-1} \mid f' \in S_z''\})$$

and $h_{f'} l_{[f']} h_{f'}^{-1}$ is conjugate in B_z to $g_{f'} l_{[f']} g_{f'}^{-1}$ for all $f' \in S_z''$. From tameness we know that $(f')_\alpha l_{[f']} (f')_\alpha^{-1}$ is conjugate to $g_{f'} l_{[f']} g_{f'}^{-1}$ and so it is also conjugate to $h_{f'} l_{[f']} h_{f'}^{-1}$.

We next show that S_z'' is preserved by the fold, i.e. that S_z'' is mapped injectively into $St_+(z, B'')$. This is trivial if $f_1, f_2 \in EB_+$ since

$$St_+(z, B'') = St_+(x, B) \dot{\cup} St_+(y, B).$$

If $f_1, f_2 \in EB \setminus EB_+$, then it is clear that

$$St_+(z, B'') = (St_+(x, B) \setminus \{f_1^{-1}\}) \cup (St_+(y, B) \setminus \{f_2^{-1}\}) \cup \{f^{-1}\}.$$

As F_z'' is free on the set $\{h_{f'} l_{[f']} h_{f'}^{-1} \mid f' \in S_z''\}$ and the label of f_i^{-1} is (b^{-1}, e^{-1}, a^{-1}) for $i = 1, 2$, we conclude that S_z'' contains at most one element of $\{f_1^{-1}, f_2^{-1}\}$. Thus S_z'' is mapped injectively into $St_+(z, B'')$.

Finally we need to show that z is full if $B_z'' = A_w$. Observe that if $F_z'' = F_w$ then $[S_z''] = St_+(w, A)$ as F_w is freely generated by $\{l_e \mid e \in St_+(w, A)\}$. Thus there exists

$$S_+'' \subseteq S_z'' \subseteq \{f' \in St_+(z, B'') \mid B_{f'} = A_{[f']}\}$$

such that $[\cdot] : B'' \rightarrow A$ maps S_+'' bijectively onto $St_+(w, A)$. From Lemma 3.4.2 we conclude that for all $f' \in S_+''$ the vertex $\omega(f')$ is full in \mathcal{B}'' as it is full in \mathcal{B} . By Lemma 3.4.1, z is full in \mathcal{B}'' . \square

Lemma 3.4.12. $c(\mathcal{B}'') < c(\mathcal{B})$.

Proof. Case i(a): Assume that $f_1, f_2 \in EB_+$ and $B_{f_1} = 1$ or $B_{f_2} = 1$.

If both x and y are isolated in \mathcal{B} , then z is isolated in \mathcal{B}'' since $St_+(z, B') = St_+(x, B) \cup St_+(y, B)$ and $B'_{f'} = B_{f'}$ for $f' \in St_+(z, B)$. From the definition of B_z'' we see that $w(B_z'') \leq w(B_x) + w(B_y)$. Thus

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h(w)w(B_x) - h(w)w(B_y) + h(w)w(B_z'') \leq c_1(\mathcal{B}).$$

If one of the vertices x or y , say x , is not isolated in \mathcal{B} , then z is not isolated in \mathcal{B}'' . In this

case we have $val_+^1(z, \mathcal{B}'') = val_+^1(x, \mathcal{B}) + val_+^1(y, \mathcal{B})$. Consequently

$$c_1(\mathcal{B}'') = \begin{cases} c_1(\mathcal{B}) - h(w)w(B_y) & \text{if } y \text{ is isolated in } \mathcal{B}. \\ c_1(\mathcal{B}) - h_+(w) & \text{if } y \text{ isn't isolated in } \mathcal{B}. \end{cases}$$

Since $w(B_y) \geq 0$ and $h_+(w) \geq h(w) \geq 1$ it follows that $c_1(\mathcal{B}'') \leq c_1(\mathcal{B})$.

As at least one of the edges involved on the fold has trivial group we obtain

$$c_2(\mathcal{B}'') = c_2(\mathcal{B}) - 4.$$

Therefore $c(\mathcal{B}'') < c(\mathcal{B})$.

Case i(b): Assume that $f_1, f_2 \in EB_+$ and $B_{f_1} \neq 1 \neq B_{f_2}$. Thus

$$val_+^1(u, \mathcal{B}'') = val_+^1(u, \mathcal{B}) - 1.$$

If both x and y are isolated in \mathcal{B} , then by the same reasons as before we conclude that z is isolated in \mathcal{B}'' . Moreover, $w(B_z'') \leq w(B_x) + w(B_y) - 1$. Indeed, for $w \in V_0$ the inequality follows from the fact that \mathfrak{k}_w is meridionally tame and $B_{f_i} \neq 1$ for $i \in \{1, 2\}$. For $w \in V_1 \cup V_2$ we know from the tameness of \mathcal{B} that B_q is generated by conjugates of m_w since q is isolated. Thus $B_{f_i} = \langle m_e \rangle$ since otherwise the vertex $\omega(f_i)$ is full. Hence for $w \in V_1$ the inequality follows from Corollary 3.3.3(2.b) and for $w \in V_2$ this is trivial since $B_x = B_y = \langle m_w \rangle$. As $h_+(v) = h(w)$ we obtain

$$\begin{aligned} c_1(\mathcal{B}'') &= c_1(\mathcal{B}) + h(w)(w(B_z) - w(B_x) - w(B_y)) \\ &\quad + h_+(v)(val_+^1(u, \mathcal{B}) - val_+^1(u, \mathcal{B}'')) \\ &\leq c_1(\mathcal{B}) - h(w) + h_+(v) \\ &= c_1(\mathcal{B}). \end{aligned}$$

Now suppose that x is not isolated in \mathcal{B} , the case that y is not isolated in \mathcal{B} is analogous. Hence z is not isolated in \mathcal{B}'' . Note that

$$val_+^1(z, \mathcal{B}'') = val_+^1(x, \mathcal{B}) + val_+^1(y, \mathcal{B})$$

and $h_+(v) = h(w) \leq h_+(w)$. Since B_{f_2} is non-trivial it follows that $w(B_y) \geq 1$. If y is isolated in \mathcal{B} we obtain

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h(w)w(B_y) + h_+(v) = c_1(\mathcal{B}) + h(w)(1 - w(B_y)) \leq c_1(\mathcal{B}).$$

If y is not isolated in \mathcal{B} then

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h_+(w) + h_+(v) \leq c_1(\mathcal{B}).$$

Since $c_2(\mathcal{B}'') \leq c_2(\mathcal{B}) - 2$ we conclude that the c -complexity decreases.

Case ii: Suppose that $f_1, f_2 \in EB \setminus EB_+$.

As before, if x and y are isolated in \mathcal{B} then z is isolated in \mathcal{B}'' . We clearly have $w(B_z'') \leq w(B_x) + w(B_y)$. Hence

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h(w)(w(B_x) + w(B_y) - w(B_z)) \leq c_1(\mathcal{B}).$$

If x or y , say x , is not isolated in \mathcal{B} then z is not isolated in \mathcal{B}'' . Furthermore, we have

$$val_+^1(z, \mathcal{B}'') = \begin{cases} val_+^1(x, \mathcal{B}) + val_+^1(y, \mathcal{B}) & \text{if } B_{f_1} = 1 \text{ or } B_{f_2} = 1. \\ val_+^1(x, \mathcal{B}) + val_+^1(y, \mathcal{B}) - 1 & \text{if } B_{f_1} \neq 1 \text{ and } B_{f_2} \neq 1. \end{cases}$$

If y is isolated in \mathcal{B} , then a simple calculation shows that

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h(w)w(B_y).$$

If y is not isolated in \mathcal{B} then we see that

$$c_1(\mathcal{B}'') = \begin{cases} c_1(\mathcal{B}) - h_+(w) & \text{if } B_{f_1} = 1 \text{ or } B_{f_2} = 1. \\ c_1(\mathcal{B}) & \text{if } B_{f_1} \neq 1 \text{ and } B_{f_2} \neq 1. \end{cases}$$

Since $h_+(w) \geq h(w) \geq 1$ and $w(B_y) \geq 0$ it follows that $c_1(\mathcal{B}'') \leq c_1(\mathcal{B})$. In any case the c_2 -complexity decreases by at least two which implies that the c -complexity decreases. \square

Case 2: No fold of type IA can be applied to \mathcal{B} but an elementary move of type IIA can be applied to \mathcal{B} : Hence there is an edge $f \in EB$ with label (a, e, b) , initial vertex $x = \alpha(f)$ labeled (B_x, v) and terminal vertex $y = \omega(f)$ labeled (B_y, w) such that

$$B_f \neq \alpha_e^{-1}(a^{-1}B_x a).$$

We will distinguish two cases depending on whether $f \in EB_+$ or $f \in EB \setminus EB_+$.

Case i: Assume that $f \in EB_+$.

It follows from Lemma 3.4.10 that $B_f = 1$ and $\alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle$. Note that \mathcal{B}' is obtained from \mathcal{B} by replacing $B_f = 1$ by $B_f' := \langle m_e \rangle$ and B_y by $B_y' := \langle B_y, b^{-1}\omega_e(m_e)b \rangle = \langle B_y, b^{-1}m_w b \rangle$ (Note that $\omega_e(m_e) = m_w$ since $e = [f] \in E(A, v_0)$).

Let now \mathcal{B}'' be the \mathbb{A} -graph obtained from \mathcal{B}' by replacing the vertex group B_y' by the group

$$\begin{array}{ccc}
 \mathcal{B} & & \mathcal{B}' \\
 \begin{array}{c} (B_x, v) \xrightarrow{(a, e, b)} (B_y, w) \\ \xrightarrow{B_f = 1} \end{array} & \xrightarrow{\text{IIA}} & \begin{array}{c} (B_x, v) \xrightarrow{(a, e, b)} (B'_y = \langle B_y, b^{-1}m_w b \rangle, w) \\ \xrightarrow{B'_f = \langle m_e \rangle} \end{array}
 \end{array}$$

Figure 3.4: An elementary move of type IIA along an oriented edge.

B''_y defined as follows:

$$B''_y := \begin{cases} B'_y & \text{if } w \in V_1 \cup V_2 \text{ or } w \in V_0 \text{ and } w(B_y) + 1 < b(\mathfrak{k}_w). \\ G(\mathfrak{k}_w) & \text{if } w \in V_0 \text{ and } w(B_y) + 1 \geq b(\mathfrak{k}_w). \end{cases}$$

Observe that the underlying graphs of \mathcal{B}' and \mathcal{B}'' are both equal to the underlying graph B of \mathcal{B} . Moreover, $B''_u = B_u$ for all $u \in VB \setminus \{y\} \cup EB \setminus \{f, f^{-1}\}$ while $B''_f = B'_f = \langle m_e \rangle$. By the same reasons as before we see that \mathcal{B}'' is π_1 -surjective.

Lemma 3.4.13. *The \mathbb{A} -graph \mathcal{B}'' is tame.*

Proof. Edge groups are as in Definition 3.4.3 as $B''_f = \langle m_e \rangle$ and the remaining edge groups do not change.

It is trivial to see that the vertex group B_x is as in Definition 3.4.3 since $B''_x = B_x$ and $B''_f = \langle m_e \rangle$.

It remains to check that the tameness condition is satisfied for B''_y . If $[y] = w \in V_0$ then this is immediate as $B''_y = G(\mathfrak{k}_w)$ or B''_y is generated by at most $w(B_y) + 1 < b(\mathfrak{k}_w)$ meridians.

We now show that \mathcal{B}'' is tame if $w \in V_1$. Note first that $B_y = A_w$ clearly implies $B''_y = \langle B_y, b^{-1}m_w b \rangle = A_w$. From tameness it follows that y is full. By Lemma 3.4.2, y is full in \mathcal{B}'' . Assume now that B_y is freely generated by $w(B_y)$ conjugates of m_w . It follows from Corollary 3.3.3 that $B''_y = \langle B_y, b^{-1}m_w b \rangle$ is freely generated by at most $w(B_y) + 1$ conjugates of m_w and so is a proper subgroup of A_w . Thus we have shown that \mathcal{B}'' is tame if $w \in V_1$.

Finally we show that \mathcal{B}'' is tame if $w \in V_2$. This case is trivial since $B''_{f'} = B_{f'}$ for all $f' \in St_+(y, B)$ and

$$B''_y := \begin{cases} \langle m_w \rangle & \text{if } B_y = 1. \\ B_y & \text{if } B_y \neq 1. \end{cases}$$

□

Lemma 3.4.14. $c(\mathcal{B}'') < c(\mathcal{B})$.

Proof. In order to compute the complexity recall that $B_f = 1$ is replaced by $B''_f = \langle m_e \rangle$ which implies that $val_+^1(x, \mathcal{B}'') = val_+^1(x, \mathcal{B}) + 1$.

If y is not isolated in \mathcal{B} , then y is not isolated in \mathcal{B}'' . A straightforward calculation shows

that

$$c_1(\mathcal{B}'') = \begin{cases} c_1(\mathcal{B}) - h(v)w(B_x) & \text{if } x \text{ is isolated in } \mathcal{B}. \\ c_1(\mathcal{B}) - h_+(v) & \text{if } x \text{ isn't isolated in } \mathcal{B}. \end{cases}$$

As $w(B_x) \geq 1$ and $h_+(v) \geq h(v) \geq 1$ we obtain $c_1(\mathcal{B}'') < c_1(\mathcal{B})$.

Assume now that y is isolated in \mathcal{B} which implies that y is isolated in \mathcal{B}'' . For the case in which $v \in V_1$ it was shown in the proof of Lemma 3.4.10 that $B_x = \langle\langle m_v \rangle\rangle$. Since no fold of type IA is applicable to \mathcal{B} , Lemma 6.1.6 implies that $St_+(x, B) = \{f\}$. Hence the vertex x is isolated in \mathcal{B} and its contribution to the c_1 -complexity is $h([x])w(B_x) = h(v)n_v = h_+(v) = h(w)$. It is now easy to see that

$$\begin{aligned} c_1(\mathcal{B}'') &= c_1(\mathcal{B}) - h(v)n_v - h(w)w(B_y) + h(w)w(B_y'') \\ &= c_1(\mathcal{B}) - h(w)(1 + w(B_y) - w(B_y'')) \\ &\leq c_1(\mathcal{B}) \end{aligned}$$

as $w(B_y'') = w(\langle\langle B_y, b^{-1}m_w b \rangle\rangle) \leq w(B_y) + 1$.

Let us now consider that case in which $v \in V_2$. As $B_f \not\subseteq \alpha_e^{-1}(a^{-1}B_x a)$ it implies that B_x is non-trivial. If x is isolated in \mathcal{B} , then the condition 4(b) in the definition of a tame \mathbb{A} -graph, implies that $B_x = \langle m_v \rangle$. Thus the contribution of the vertex x to the c_1 -complexity of \mathcal{B} is equal to $h(v)$. A simple calculation shows that

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h(v) - h(w)w(B_y) + h(w)w(\langle\langle B_y, b^{-1}m_w b \rangle\rangle).$$

Since $h(w) = h(v)$ we conclude that $c_1(\mathcal{B}'') \leq c_1(\mathcal{B})$.

If x is not isolated in \mathcal{B} , then $c_1(\mathcal{B}'') \leq c_1(\mathcal{B})$ as $h_+(v) = h(w)$.

Finally, as $c_2(\mathcal{B}'') = c_2(\mathcal{B}) - 1$ it implies that $c(\mathcal{B}'') < c(\mathcal{B})$. □

Case ii: Assume now that $f \in EB \setminus EB_+$. Note that in this case $[y] = w \notin V_0$. Moreover as \mathcal{B} is tame the group $B_x \leq B_{[x]} = A_v$ is as described in Definition 3.4.3(2) if $v \in V_0$, as described in Definition 3.4.3(3) if $v \in V_1$ and as described in Definition 3.4.3(4) if $v \in V_2$.

Now the group $\alpha_e^{-1}(a^{-1}B_x a)$ is the image under α_e^{-1} of $a^{-1}B_x a \cap \alpha_e(A_e)$. If B_x is full we clearly have $a^{-1}B_x a \cap \alpha_e(A_e) = \alpha_e(A_e)$. In the remaining case it follows from the meridional tameness of \mathfrak{k}_v if $v \in V_0$, Corollary 3.3.3 if $v \in V_1$ and Corollary 3.3.4 if $v \in V_2$ that $a^{-1}B_x a \cap \alpha_e(A_e)$ is trivial or cyclic and generated by the meridian $\alpha_e(m_e)$.

As $a^{-1}B_x a \cap \alpha_e(A_e)$ cannot be trivial if a fold of type IIA is applicable this implies that one of the following occurs:

1. $\alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle$.
2. $\alpha_e^{-1}(a^{-1}B_x a) = A_e = \langle m_e, l_e \rangle$. This occurs iff $B_x = A_v$.

Hence the \mathbb{A} -graph \mathcal{B}' is obtained from \mathcal{B} by replacing B_f by B'_f defined as

$$B'_f := \begin{cases} \langle m_e \rangle & \text{if } \alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle. \\ A_e & \text{if } \alpha_e^{-1}(a^{-1}B_x a) = A_e. \end{cases}$$

and the vertex group B_y by B'_y defined as follows:

$$B'_y := \begin{cases} \langle B_y, b^{-1}\omega_e(m_e)b \rangle & \text{if } \alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle. \\ \langle B_y, b^{-1}\omega_e(m_e)b, b^{-1}\omega_e(l_e)b \rangle & \text{if } \alpha_e^{-1}(a^{-1}B_x a) = A_e. \end{cases}$$

Let \mathcal{B}'' be the \mathbb{A} -graph obtained from \mathcal{B}' by replacing the vertex group B'_y by

$$B''_y := \begin{cases} B'_y & \text{if } w \in V_2. \\ \langle \langle m_w \rangle \rangle & \text{if } \alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle \text{ and } w \in V_1. \\ A_w & \text{if } \alpha_e^{-1}(a^{-1}B_x a) = A_e \text{ and } w \in V_1. \end{cases}$$

Note that $w \notin V_0$ since $e^{-1} = [f^{-1}] \notin E(A, v_0)$ and so $\text{val}_+(w, A) \geq 1$.

We remark that \mathcal{B}'' is π_1 -surjective as it is defined from the π_1 -surjective \mathbb{A} -graph \mathcal{B}' by possibly enlarging the vertex group at y .

Lemma 3.4.15. *The \mathbb{A} -graph \mathcal{B}'' is tame.*

Proof. Note that $B''_f = B'_f = \langle m_e \rangle$ or $B''_f = B'_f = A_e = \langle m_e, l_e \rangle$. The latter case occurs only when $B_x = A_v$. By tameness it follows that x is full in \mathcal{B} and hence by Lemma 3.4.2 x is full in \mathcal{B}'' .

The \mathbb{A} -graph \mathcal{B}'' is obviously tame if $w \in V_1$ since by definition $B''_y = \langle \langle m_w \rangle \rangle$ or $B''_y = A_w$. The latter case occurs only when $B_x = A_v$ and so $B''_f = A_e$. As $f^{-1} \in \text{St}_+(y, B)$ and $\text{St}_+(w, A) = \{e^{-1}\}$ we conclude from Lemma 3.4.1 that y is full in \mathcal{B}'' .

Finally we show that \mathcal{B}'' is tame if $w \in V_2$. Assume first that $\alpha_e^{-1}(a^{-1}B_x a) = \langle m_e \rangle$. Tameness of \mathcal{B} implies that either $B_y = 1$ or $B_y = F_y \times \langle m_w \rangle$ for some subgroup F_y of F_w . By definition $B''_y = B'_y = \langle B_y, b^{-1}\omega(m_e)b \rangle$. Since $\omega(m_e) = m_w$ lies in the center of A_w we conclude that

$$B''_y = \begin{cases} \langle m_w \rangle & \text{if } B_y = 1. \\ B_y & \text{if } B_y \neq 1. \end{cases}$$

It is also trivial to see that \mathcal{B}'' is tame if $\alpha_e^{-1}(a^{-1}B_x a) = A_e$ and $B_y \leq \langle m_w \rangle$ since in this case $B''_y = \langle b^{-1}l_{e^{-1}}b \rangle \times \langle m_w \rangle \subsetneq A_w$. So assume that $\alpha_e^{-1}(a^{-1}B_x a) = A_e$ and $B_y = F_y \times \langle m_w \rangle$ where F_y is freely generated by

$$\{g_{f'}l_{[f']}g_{f'}^{-1} \mid f' \in S_y\}.$$

Thus

$$B''_y = F''_y \times \langle m_w \rangle$$

where

$$F_y'' := \langle \{b^{-1}l_{e^{-1}}b\} \cup \{g_{f'}l_{[f']}g_{f'}^{-1} \mid f' \in S_y\} \rangle.$$

Corollary 3.3.4 implies that there exists a subset $S_y'' \subseteq S_y \cup \{f^{-1}\}$ such that F_y'' is freely generated by $\{h_{f'}l_{[f']}h_{f'}^{-1} \mid f' \in S_y''\}$ and that $h_{f'}l_{[f']}h_{f'}^{-1}$ is conjugate in B_y'' to $g_{f'}l_{[f']}g_{f'}^{-1}$ or to $b^{-1}l_{e^{-1}}b$. Note that if $B_y'' = A_w$ then clearly $F_y'' = F_w$ since $B_y'' = F_y'' \times \langle m_w \rangle$, $A_w = F_w \times \langle m_w \rangle$ and $F_y'' \leq F_w$. Hence $[S_y''] = \{[f] \mid f \in S_y''\}$ is equal to $St_+(w, A)$. Combining Lemma 3.4.1, Lemma 3.4.2 and the tameness of \mathcal{B} we conclude that y is full in \mathcal{B}'' . Therefore \mathcal{B}'' is tame if $w \in V_2$. \square

Lemma 3.4.16. $c(\mathcal{B}'') < c(\mathcal{B})$.

Proof. Note that if y is isolated in \mathcal{B} , then we obtain

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) - h(w)w(B_y) \leq c_1(\mathcal{B})$$

since y is not isolated in \mathcal{B}'' and $val_+^1(y, \mathcal{B}'') = 1$.

If y is not isolated in \mathcal{B} , then $val_+^1(y, \mathcal{B}'') \geq val_+^1(y, \mathcal{B})$. Thus

$$c_1(\mathcal{B}'') = c_1(\mathcal{B}) + h_+(w)(val_+^1(y, \mathcal{B}) - 1) - h_+(w)(val_+^1(y, \mathcal{B}'') - 1) \leq c_1(\mathcal{B}).$$

In both cases a simple calculation shows that $c_2(\mathcal{B}'') \leq c_2(\mathcal{B}) - 1$ which implies that the c -complexity decreases. \square

In this chapter we prove that torus knots and prime 3-bridge knots are meridionally tame. As observed before, 2-bridge knots are meridionally tame.

4.1 Meridional tameness of torus knots

In this section we show that torus knots are meridionally tame. This fact is implicit in [32]. Note that this implies Corollary 3.0.6 since any knot whose complement is a graph manifold is equal to $\mathfrak{k}(\mathcal{A})$ for some labeled tree \mathcal{A} , where \mathfrak{k}_v is a torus knot for any $v \in V_0$.

Lemma 4.1.1. *Torus knots are meridionally tame.*

Proof. Let \mathfrak{k} be a (p, q) -torus knot. Without loss of generality we may assume that $p > q \geq 2$. Suppose that $U = \langle m_1, \dots, m_r \rangle$ is a meridional subgroup of \mathfrak{k} with $w(U) = r < b(\mathfrak{k}) = \min(p, q) = q$.

It follows from [32, Theorem 1.4(β)], see Lemma 2.2.3, that U is freely generated by $\{m_1, \dots, m_r\}$ since by assumption U cannot be generated by less than r meridians.

We will show that for any $g \in G(\mathfrak{k})$ one of the following holds:

1. $gP(\mathfrak{k})g^{-1} \cap U = \{1\}$.
2. $gP(\mathfrak{k})g^{-1} \cap U = g\langle m \rangle g^{-1}$ and gmg^{-1} is conjugate in U to m_j for some $j \in \{1, \dots, r\}$.

Note that this claim is implicit in the argument in [32], however, in order to avoid getting involved with the combinatorial details, we give an alternative argument.

Observe that it suffices to show that any conjugate of a peripheral element that lies in U is conjugate in U to some element of $\langle m_i \rangle$ for some $i \in \{1, \dots, r\}$ since the meridional subgroup $\langle m \rangle$ is prenormal in $G(\mathfrak{k})$ with respect to the peripheral subgroup $P(\mathfrak{k})$, that is, $g\langle m \rangle g^{-1} \cap P(\mathfrak{k}) \neq \{1\}$ implies $g \in P(\mathfrak{k})$, see Lemma 3.1 of [50].

Let $\pi : G(\mathfrak{k}) \rightarrow \mathbb{Z}_p * \mathbb{Z}_q$ be the map that quotients out the center. Note that we can think of $\mathbb{Z}_p * \mathbb{Z}_q$ as the fundamental group of the hyperbolic 2-orbifold $\mathcal{O} = S^2(\infty, p, q)$ of finite volume. Thus the boundary corresponds to a parabolic element. As U is free and has therefore trivial center it follows that $\pi|_U$ is injective. Thus $\pi(U)$ is free in the parabolic elements $\pi(m_1), \dots, \pi(m_r)$.

Now consider the covering $\tilde{\mathcal{O}}$ of \mathcal{O} corresponding to $\pi(U)$. As $\pi_1(\tilde{\mathcal{O}})$ is freely generated by r parabolic elements it follows that $\tilde{\mathcal{O}}$ is an $(r+1)$ -punctured sphere with at least r parabolic boundary components. If the last boundary component corresponds to a hyperbolic element then the claim of the lemma holds. Thus we may assume that all boundary components correspond to parabolic elements, i.e., that $\tilde{\mathcal{O}}$ is a finite sheeted cover of \mathcal{O} with

$$\chi(\tilde{\mathcal{O}}) = 2 - (r+1) = 1 - r > 1 - \min(p, q) = 1 - q. \quad (*)$$

Note that $\chi(\mathcal{O}) = 1 - (1 - \frac{1}{p}) - (1 - \frac{1}{q}) = -1 + \frac{1}{p} + \frac{1}{q}$. As $\pi(U)$ is a torsion free subgroup of $\mathbb{Z}_p * \mathbb{Z}_q$ and as p and q are coprime it follows that $|\mathbb{Z}_p * \mathbb{Z}_q : \pi(U)| \geq p \cdot q$. Thus, as $\chi(\mathcal{O}) < 0$ we obtain

$$\chi(\tilde{\mathcal{O}}) \leq p \cdot q \cdot \chi(\mathcal{O}) = p \cdot q \cdot \left(-1 + \frac{1}{p} + \frac{1}{q}\right) = -p \cdot q + p + q.$$

As $2 \leq q < p$ and therefore $p \geq 3$ it follows that

$$\chi(\tilde{\mathcal{O}}) \leq -p \cdot q + p + q = p(1 - q) + q \leq 3(1 - q) + q = 3 - 2q \leq 1 - q = 1 - \min(p, q),$$

a contradiction to $(*)$. Thus this case cannot occur. \square

4.2 Meridional tameness of 3-bridge knots

Proposition 4.2.1. *A prime 3-bridge knot is meridionally tame.*

Proof. From [6, Corollary 1.6] it follows that a prime 3-bridge knot $\mathfrak{k} \subset S^3$ is either hyperbolic or a torus knot.

In the case of a torus knot the meridional tameness follows from Lemma 4.1.1. Thus we can assume that \mathfrak{k} is hyperbolic.

We need to check that any subgroup $U \leq G(\mathfrak{k})$ that is generated by at most two meridians is tame. It follows from [6, Prop. 4.2] that any such subgroup is either cyclic or the fundamental group of a 2-bridge knot summand of \mathfrak{k} or free of rank 2. In the first case the tameness of U is

trivial and the second case cannot occur as \mathfrak{k} is a prime 3-bridge knot.

Thus we are left with the case where the knot \mathfrak{k} is hyperbolic and the meridional subgroup U is a free group generated by two parabolic elements which are meridians. In [24] a Kleinian free group U generated by two parabolics is shown to be geometrically finite. In the course of the proof it is proved that U is of Schottky type and obtained from a handlebody of genus 2 by pinching at most 3 disjoint and non parallel simple curves to a point corresponding to cusps. Each curve generates a maximal parabolic subgroup of U and there are at most 3 conjugacy classes of such subgroups.

If there are only two conjugacy classes of maximal parabolic subgroups, they correspond to the two meridian generators and so the group U is tame. This is the 4-times punctured sphere case in [24, case 4].

The group U may also have 3 conjugacy classes of maximal parabolic subgroups, corresponding to two 3-times punctured spheres on the boundary of the core of the associated Kleinian manifold, see [24, case 5]. They give π_1 -injective properly immersed pants in the hyperbolic knot complement $C(\mathfrak{k})$. Then it follows from Agol's result [1] that such a π_1 -injective properly immersed pant is either embedded or $C(\mathfrak{k})$ can be obtained by Dehn filling one boundary component of the Whitehead link complement. A knot complement in S^3 cannot contain a properly embedded pant. Therefore one may assume that $C(\mathfrak{k})$ is obtained by a Dehn filling of slope p/q along one boundary component of the Whitehead link complement. Then a homological computation shows that $H_1(C(\mathfrak{k}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/|p|\mathbb{Z}$, so $|p| = 1$. Thus \mathfrak{k} must be a twist knot, which is a 2-bridge knot. Hence this case is also impossible. \square

Agol's construction [1] of a π_1 -injective properly immersed pant in the complement of the whitehead link shows that meridional tameness never holds for Whitehead doubles of non-trivial knots. The complement of such a knot is obtained by gluing the complement of a non-trivial knot to one boundary component of the Whitehead link complement. Thus by van Kampen's theorem the π_1 -injectivity of the properly immersed pant in the complement of the whitehead link is preserved. The image is a free subgroup generated by two meridians that is not tame as the third boundary component gives rise to another conjugacy class of peripheral elements which is generated by the square of some meridian.

Note that for non-prime knots meridionally tameness does not hold in general. This is suggested by the above argument. If one summand is a 2-bridge knot (hyperbolic but also the trefoil can occur) then there might exist π_1 -injective properly immersed pants with two boundary components corresponding to a meridian. The image is a free subgroup generated by two meridians that is not tame as the third boundary component gives rise to another conjugacy class of peripheral elements which is generated by the square of some meridian.

Let G be an arbitrary group and $S \subseteq G$. We define the normal closure $\langle\langle S \rangle\rangle_G$ of S as the smallest normal subgroup of G containing S , equivalently

$$\langle\langle S \rangle\rangle_G = \left\{ \prod_{i=1}^k u_i s_i^{\varepsilon_i} u_i^{-1} \mid u_i \in G, \varepsilon_i = \pm 1, s_i \in S, k \in \mathbb{N} \right\}.$$

The following definition is taken from [42].

Definition 5.0.2. A *killer* is an element $g \in G$ such that $\langle\langle g \rangle\rangle_G = G$. We say that two killers $g_1, g_2 \in G$ are *equivalent* if there exists an automorphism $\phi : G \rightarrow G$ such that $\phi(g_1) = g_2$.

From a Wirtinger presentation of $G(\mathfrak{k})$, or a bridge presentation of \mathfrak{k} , we see that the meridian of a knot \mathfrak{k} is a killer of $G(\mathfrak{k})$. In [45, Theorem 3.11] the author exhibit a knot for which there exists a killer that is not equivalent to the meridian. Silver–Whitten–Williams [41, Corollary 1.3] showed that if \mathfrak{k} is a hyperbolic 2-bridge knot or a torus knot or a hyperbolic knot with unknotting number one, then its group contains infinitely many pairwise inequivalent killers.

In [41, Conjecture 3.3] it is conjectured that the group of any nontrivial knot has infinitely many inequivalent killers, see also [2, Question 9.26]. In this chapter we show the following.

Theorem 5.0.3. *Let \mathfrak{k} be a cable knot about a nontrivial knot \mathfrak{k}_1 . Then its group contains infinitely many pairwise inequivalent killers.*

Moreover, we show that having infinitely many inequivalent killers is preserved under connected sums. As a Corollary we show that the group of any nontrivial knot whose complement is a graph manifold contains infinitely many inequivalent killers.

5.1 Proof of Theorem 1

In this section we give a precise description of cable spaces and its fundamental group as well as the constriction of cable knots since we will need precise language later on.

Let m, n be coprime integers with $n \geq 2$. The *cable space* $CS(m, n)$ is defined as follows: let $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$ and $\rho : D^2 \rightarrow D^2$ a rotation through an angle of $2\pi(m/n)$ about the origin. Choose a disk $\delta \subset \text{Int}(D^2)$ such that $\rho^i(\delta) \cap \rho^j(\delta) = \emptyset$ for $1 \leq i \neq j \leq n$ and denote by D_n^2 the space

$$D^2 - \text{Int}\left(\bigcup_{i=1}^n \rho^i(\delta)\right).$$

ρ induces a homeomorphism $\rho_0 := \rho|_{D_n^2} : D_n^2 \rightarrow D_n^2$. We define $CS(m, n)$ as the mapping torus of ρ_0 , i.e.

$$CS(m, n) := D_n^2 \times I / (z, 0) \sim (\rho_0(z), 1).$$

Note that $CS(m, n)$ has the structure of a Seifert fibered space. Each fiber is the image of $\{\rho^i(z) \mid 1 \leq i \leq n\} \times I$ under the quotient map, where $z \in D_n^2$. There is exactly one exceptional fiber, namely the image C_0 of the arc $0 \times I$.

In order to compute the fundamental group A of $CS(m, n)$, denote the free generators of $\pi_1(D_n^2)$ corresponding to the boundary paths of the removed disks $\rho_0(\delta), \dots, \rho_0^{n-1}(\delta)$ by x_1, \dots, x_n respectively. From the definition of $CS(m, n)$ we see that we can write A as the semi-direct product $F(x_1, \dots, x_n) \rtimes \mathbb{Z}$, where the action of $\mathbb{Z} = \langle t \rangle$ on $\pi_1(D_n^2) = F(x_1, \dots, x_n)$ is given by

$$tx_i t^{-1} = x_{\sigma(i)} \text{ for } 1 \leq i \leq n.$$

The element t is represented by the exceptional fiber of $CS(m, n)$ and the permutation $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is given by $i \mapsto i + m \pmod n$. Thus

$$A = \langle x_1, \dots, x_n, t \mid tx_i t^{-1} = x_{\sigma(i)} \text{ for } 1 \leq i \leq n \rangle.$$

We finally remark that any element $a \in A$ is uniquely written as $w \cdot t^z$ for some $w \in F(x_1, \dots, x_n)$ and $z \in \mathbb{Z}$.

We next give more precise definition of cable knots. Let V_0 be the solid torus $D^2 \times I / (z, 0) \sim (\rho(z), 1)$ and for some $z_0 \in \text{Int}(D^2) - 0$ let \mathfrak{k}_0 be the image of $\{\rho^i(z_0) \mid 1 \leq i \leq n\} \times I$ under the quotient map. Note that \mathfrak{k}_0 is a simple closed curve contained in the interior of V_0 . Let \mathfrak{k}_1 be a nontrivial knot in S^3 and $V(\mathfrak{k}_1)$ a regular neighborhood of \mathfrak{k}_1 in S^3 . Let further $h : V_0 \rightarrow V(\mathfrak{k}_1)$ be a homeomorphism which maps the meridian $\partial D^2 \times 1$ of V_0 to a meridian of \mathfrak{k}_1 . The knot $\mathfrak{k} := h(\mathfrak{k}_0)$ is called a (m, n) -cable knot about \mathfrak{k}_1 .

Thus the knot complement $C(\mathfrak{k})$ of a (m, n) -cable knot \mathfrak{k} decomposes as

$$C(\mathfrak{k}) = CS(m, n) \cup C(\mathfrak{k}_1)$$

with $\partial C(\mathfrak{k}_1) = CS(m, n) \cap C(\mathfrak{k}_1)$ an incompressible torus in $C(\mathfrak{k})$. It follows from the Theorem of Seifert and van-Kampen that

$$G(\mathfrak{k}) = A *_C B$$

where $B = G(\mathfrak{k}_1)$ and $C = \pi_1(\partial C(\mathfrak{k}_1))$. Denote by m_1 the meridian of \mathfrak{k}_1 and note that in A we have $m_1 = x_1 \cdot \dots \cdot x_n$. In turn, the meridian $m \in G(\mathfrak{k})$ of \mathfrak{k} is written as $m = x_1 \in A$.

The proof of Theorem 5.0.3 is divided in two steps. In Lemma 5.1.1 we exhibit elements that normally generate the group of the cable knot and next, in Lemma 5.1.2, we prove that these killers are indeed inequivalent.

Choose $s \in \{1, \dots, n-1\}$ such that $\sigma^s(1) = 2$. Since $\sigma^s(i) = i + sm \pmod n$ it follows that $\sigma^s = (1 \ 2 \ 3 \ \dots \ n-1 \ n)$.

Lemma 5.1.1. *Let \mathfrak{k} be a (m, n) -cable knot about a nontrivial knot \mathfrak{k}_1 . Then for each $l \geq 1$ the element*

$$g_l := x_1^l x_2^{-(l-1)} = x_1^l \cdot (t^s x_1 t^{-s})^{-(l-1)}$$

normally generates the group of \mathfrak{k} .

Proof. The first step of the proof is to show that the group of the companion knot is contained in $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$.

Claim 1: The meridian $m_1 = x_1 \cdot \dots \cdot x_n$ of \mathfrak{k}_1 belongs to $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$. Consequently $B = \langle\langle m_1 \rangle\rangle_B = \langle\langle m_1 \rangle\rangle_{G(\mathfrak{k})} \cap B \subseteq \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$.

Note that for $0 \leq i \leq n-1$ we have $t^{is} g_l t^{-is} = x_{i+1}^l x_{i+2}^{-(l-1)}$, where indices are taken mod n . Thus

$$\begin{aligned} x_1 \cdot \dots \cdot x_n &= x_1^{-(l-1)} (x_1^l x_2 \cdot \dots \cdot x_n x_1^{-(l-1)}) x_1^{l-1} \\ &= x_1^{-(l-1)} \left(\prod_{i=0}^{n-1} x_{i+1}^l x_{i+2}^{-(l-1)} \right) x_1^{l-1} \\ &= x_1^{-(l-1)} \left(\prod_{i=0}^{n-1} t^{is} \cdot g_l \cdot t^{-is} \right) x_1^{l-1} \\ &= \prod_{i=0}^{n-1} \left(x_n^{-(l-1)} t^{is} \cdot g_l \cdot t^{-is} x_1^{l-1} \right) \\ &= \prod_{i=0}^{n-1} \left((x_1^{-(l-1)} t^{is}) \cdot g_l \cdot (x_1^{-(l-1)} t^{is})^{-1} \right) \end{aligned}$$

which implies that $m_1 \in \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$. Thus Claim 1 is proved.

From Claim 1 it follows that the peripheral subgroup $C = \pi_1(\partial C(\mathfrak{k}_1))$ of \mathfrak{k}_1 is contained in $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$ since $C \subseteq B$ and consequently we have

$$G(\mathfrak{k})/\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})} = A *_C B/\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})} \cong A/\langle\langle g_l, C \rangle\rangle_A.$$

Thus, we need to show that $A/\langle\langle g_l, C \rangle\rangle_A = 1$. It is easy to see that $A/\langle\langle C \rangle\rangle_A$ is cyclically generated by $\pi(x_1)$, where $\pi : A \rightarrow A/\langle\langle C \rangle\rangle_A$ is the canonical projection. The result now follows from the fact that $\pi(g_l) = \pi(x_1^l \cdot t^s x_1^{(l-1)} t^{-s}) = \pi(x_1)$. \square

Lemma 5.1.2. *If $k \neq l$, then g_k is not equivalent to g_l .*

Proof. Assume that $\phi : G(\mathfrak{k}) \rightarrow G(\mathfrak{k})$ is an automorphism such that $\phi(g_l) = g_k$ and let $f : C(\mathfrak{k}) \rightarrow C(\mathfrak{k})$ be a homotopy equivalence inducing ϕ . From [16, Theorem 14.6] it follows that f can be deformed into $\hat{f} : C(\mathfrak{k}) \rightarrow C(\mathfrak{k})$ so that \hat{f} sends $C(\mathfrak{k}_1)$ homeomorphically onto $C(\mathfrak{k}_1)$ and $\hat{f}|_{CS(m,n)} : CS(m,n) \rightarrow CS(m,n)$ is a homotopy equivalence. Thus $\phi(A)$ is conjugate to A , that is, $\phi(A) = gAg^{-1}$ for some $g \in G(\mathfrak{k})$. Since $\phi(g_l) = g_k$ it implies that $g_k \in gAg^{-1}$. As g_k is not conjugate (in A) to an element of C , it implies that $g \in A$. Thus $\phi(A) = A$. By [16, Proposition 28.4], we may assume that $\hat{f}|_{CS(m,n)}$ is fiber preserving. Since $CS(m,n)$ has exactly one exceptional fiber, which represents t , we must have $\phi(t) = at^\eta a^{-1}$ for some $a = v \cdot t^{z_1} \in A$ and some $\eta \in \{\pm 1\}$.

The automorphism $\phi|_A : A \rightarrow A$ induces an automorphism ϕ_* on the factor group $A/\langle t^n \rangle = \langle x_1, t \mid t^n = 1 \rangle = \mathbb{Z} * \mathbb{Z}_n$ such that $\phi_*(t) = at^\eta a^{-1}$. It is a standard fact about automorphisms of free products that we must have $\phi_*(x_1) = at^{e_0} x_1^\varepsilon t^{e_1} a^{-1}$ for $e_0, e_1 \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$. Thus,

$$\phi(x_1) = at^{e_0} x_1^\varepsilon t^{e_1} a^{-1} t^{dn} = at^{e_0} \cdot x_1 t^{e_0+e_1+dn} \cdot t^{-e_0} a^{-1}$$

for some $d \in \mathbb{Z}$. Since t has non-zero homology in $H_1(C(\mathfrak{k}))$ it follows that $e_0 + e_1 + dn = 0$. Consequently, $\phi(x_1) = b \cdot x_1^\varepsilon \cdot b^{-1}$, where $b = at^{e_0} = v \cdot t^{z_2} \in A$ and $z_2 = z_1 + e_0$.

Hence we obtain

$$\begin{aligned} \phi(g_l) &= \phi(x_1^l x_2^{-(l-1)}) \\ &= \phi(x_1^l \cdot t^s x_1^{-(l-1)} t^{-s}) \\ &= b x_1^{\varepsilon l} b^{-1} \cdot at^{\eta s} a^{-1} \cdot b x_1^{-\varepsilon(l-1)} b^{-1} \cdot at^{-\eta s} a^{-1} \\ &= v t^{z_2} x_1^{\varepsilon l} t^{-z_2} v^{-1} \cdot v t^{z_1} t^{\eta s} t^{-z_1} v^{-1} \cdot v t^{z_2} x_1^{-\varepsilon(l-1)} t^{-z_2} v^{-1} \cdot v t^{z_1} t^{-\eta s} t^{-z_1} v^{-1} \\ &= v x_i^{\varepsilon l} x_j^{-\varepsilon(l-1)} v^{-1} \end{aligned}$$

where $i = \sigma^{z_2}(1)$ and $j = \sigma^{z_2+\eta s}(1)$. Note that $i \neq j$ since $\sigma^s(1) = 2$ and $\sigma^{-s}(1) = n$. Hence, $\phi(g_l) = g_k$ implies that

$$v(x_i^{\varepsilon l} \cdot x_j^{-\varepsilon(l-1)})v^{-1} = x_1^k \cdot x_2^{-(k-1)}$$

in $F(x_1, \dots, x_n)$. Thus, in the abelinization of $F(x_1, \dots, x_n)$ we have

$$\varepsilon[lx_i + (1-l)x_j] = kx_1 + (1-k)x_2$$

which implies that $\{i, j\} = \{1, 2\}$. If $(i, j) = (1, 2)$, then $\varepsilon l = k$ and so $k = |k| = |\varepsilon l| = l$. If $(i, j) = (2, 1)$, then $\varepsilon l = k - 1$ and $\varepsilon(1-l) = k$. Consequently, $\varepsilon = 1$ and $l + k = 1$ which is impossible since $k, l \geq 1$. \square

5.2 Connected sums and killers

In this section we show that having infinitely many inequivalent killers is preserved under connected sums of knots. This fact, Theorem 5.0.3, and Corollary 1.3 of [41] imply that the group of knots whose complement is a graph manifold have infinitely many inequivalent killers.

Let \mathfrak{k} be a knot and $\mathfrak{k}_1, \dots, \mathfrak{k}_n$ its prime factors, that is, $\mathfrak{k} = \mathfrak{k}_1 \# \dots \# \mathfrak{k}_n$ and each \mathfrak{k}_i is a nontrivial prime knot. Assume that $x \in G(\mathfrak{k}_i)$ is a killer of $G(\mathfrak{k}_i)$. It is well-known that $G(\mathfrak{k}_i) \leq G(\mathfrak{k})$ and $\langle m \rangle \leq G(\mathfrak{k}_i)$ for all i , where m denotes the meridian of \mathfrak{k} . From this we immediately see that $m \in \langle\langle x \rangle\rangle_{G(\mathfrak{k}_i)} \subseteq \langle\langle x \rangle\rangle_{G(\mathfrak{k})}$ which implies that $G(\mathfrak{k}) = \langle\langle m \rangle\rangle_{G(\mathfrak{k})} \subseteq \langle\langle x \rangle\rangle_{G(\mathfrak{k})}$, i.e., x is a killer of $G(\mathfrak{k})$.

Now suppose that $x, y \in G(\mathfrak{k}_i)$ are killers of $G(\mathfrak{k}_i)$ and that there exists an automorphism ϕ of $G(\mathfrak{k})$ such that $\phi(x) = y$. ϕ is induced by a homotopy equivalence $f : X(\mathfrak{k}) \rightarrow X(\mathfrak{k})$. From [16, Theorem 14.6] it follows that f can be deformed into $\hat{f} : C(\mathfrak{k}) \rightarrow C(\mathfrak{k})$ so that:

1. $\hat{f}|_V : V \rightarrow V$ is a homotopy equivalence, where $V = S^1 \times (n - \text{punctured disk})$ is the peripheral component of the characteristic submanifold of $C(\mathfrak{k})$.
2. $\hat{f}|_{\overline{C(\mathfrak{k}) - V}} : \overline{C(\mathfrak{k}) - V} \rightarrow \overline{C(\mathfrak{k}) - V}$ is a homeomorphism.

Note that $\overline{C(\mathfrak{k}) - V} = C(\mathfrak{k}_1) \dot{\cup} \dots \dot{\cup} C(\mathfrak{k}_n)$. Since $\hat{f}|_{\overline{C(\mathfrak{k}) - V}}$ is a homeomorphism it follows that \hat{f} sends $C(\mathfrak{k}_i)$ homeomorphically onto $C(\mathfrak{k}_{\tau(i)})$ for some permutation τ of $\{1, \dots, n\}$. Consequently, there exists $g' \in G(\mathfrak{k})$ such that

$$\phi(G(\mathfrak{k}_i)) = g'G(\mathfrak{k}_{\tau(i)})g'^{-1}.$$

If $\tau(i) = i$ and $g' \in G(\mathfrak{k}_i)$, then ϕ induces an automorphism $\psi := \phi|_{G(\mathfrak{k}_i)}$ of $G(\mathfrak{k}_i)$ such that $\psi(x) = y$, i.e., x and y are equivalent in $G(\mathfrak{k}_i)$. If $\tau(i) \neq i$ or $g' \notin G(\mathfrak{k}_{\tau(i)})$, then it is not hard to see that y is conjugate (in $G(\mathfrak{k}_i)$) to an element of $\langle m \rangle$ since $y = \phi(x) \in G(\mathfrak{k}_i) \cap g'G(\mathfrak{k}_{\tau(i)})g'^{-1}$. As $\langle\langle m^k \rangle\rangle \neq G(\mathfrak{k})$ for $|k| \geq 2$ and y normally generates $G(\mathfrak{k})$, we conclude that y is conjugate (in $G(\mathfrak{k}_i)$) to $m^{\pm 1}$. The same argument applied to ϕ^{-1} shows that x is conjugate (in $G(\mathfrak{k}_i)$) to $m^{\pm 1}$.

Therefore, if the group of one of the prime factors of \mathfrak{k} has infinitely many inequivalent killers, then so does the group of \mathfrak{k} . As a Corollary of Theorem 5.0.3 and the remark made above we obtain the following result.

Corollary 5.2.1. *If \mathfrak{k} is a knot such that its complement $C(\mathfrak{k})$ is a graph manifold, then $G(\mathfrak{k})$ contains infinitely many pairwise inequivalent killers.*

Proof. From [17] it follows that the only Seifert-fibered manifolds that can be embedded into a knot manifold with incompressible boundary are torus knot complements, composing spaces and cable spaces. Thus, if $C(\mathfrak{k})$ is a graph manifold, then one of the following holds:

1. \mathfrak{k} is a torus knot.
2. \mathfrak{k} is a cable knot.
3. $\mathfrak{k} = \mathfrak{k}_1 \# \dots \# \mathfrak{k}_n$ where each \mathfrak{k}_i is either a torus knot or a cable knot.

Now the result follows from Theorem 5.0.3 and [41, Corollary 1.3]. □

Nielsen equivalence in planar discontinuous groups

The notion of Nielsen equivalence goes back to the origins of geometric group theory and the work of J. Nielsen [25, 26, 27]. Let G be a group and $\mathcal{T} = (g_1, \dots, g_n)$ and $\mathcal{T}' = (g'_1, \dots, g'_n)$ be two ordered n -tuples of elements of G . Recall that \mathcal{T} and \mathcal{T}' are called *elementary equivalent* if one of the following conditions holds:

1. There exists some permutation $\sigma \in S_n$ such that $g'_i = g_{\sigma(i)}$ for $1 \leq i \leq n$.
2. $g'_i = g_i^{-1}$ for some $i \in \{1, \dots, n\}$ and $g'_j = g_j$ for $j \neq i$.
3. $g'_i = g_i g_j^\varepsilon$ for some $i \neq j$ and $\varepsilon \in \{-1, 1\}$. Furthermore, $g'_k = g_k$ for $k \neq i$.

We say that \mathcal{T} and \mathcal{T}' are *Nielsen equivalent*, and write $\mathcal{T} \sim \mathcal{T}'$, if there exists a finite sequence of n -tuples

$$\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k = \mathcal{T}'$$

such that \mathcal{T}_{i-1} and \mathcal{T}_i are elementary equivalent for $1 \leq i \leq k$. Nielsen equivalence clearly defines an equivalence relation for tuples of elements of G and equivalent tuples generate the same subgroup of G .

Recall that for a finitely generated group G the *rank* of G , denoted $rk(G)$, is the smallest cardinality of a generating set of G . If $n \geq rk(G)$, an n -tuple $\mathcal{T} = (g_1, \dots, g_n) \in G^n$ is called a *generating tuple* if $G = \langle \{g_1, \dots, g_n\} \rangle$.

We say that a tuple $\mathcal{T} = (g_1, \dots, g_n)$ of elements of G is *reducible* if \mathcal{T} is Nielsen equivalent to a tuple of the type $(g'_1, \dots, g'_{n-1}, 1)$. If \mathcal{T} is not reducible we say that \mathcal{T} is *irreducible*. Note that any generating tuple of minimal cardinality is necessarily irreducible.

In [27] J. Nielsen introduced the notion of Nielsen equivalence to study subgroups and also the automorphism group of finitely generated free groups. The following result implies in particular that in a finitely generated free group all generating tuples of same size are Nielsen equivalent.

Theorem 6.0.2. *(Nielsen) Let F_n be a free group of rank $n \geq 1$ with a distinguished free basis (x_1, \dots, x_n) . Then any generating tuple (y_1, \dots, y_m) of F_n is Nielsen equivalent to $(x_1, \dots, x_n, 1, \dots, 1)$.*

Nielsen's theorem implies that an n -tuple (y_1, \dots, y_n) of elements of F_n is a free basis of F_n if and only if $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ in F_n , it further implies that the automorphism group is finitely generated by the so-called Nielsen automorphisms. It follows that if G is a group and $\mathcal{T} = (g_1, \dots, g_n), \mathcal{T}' = (g'_1, \dots, g'_n) \in G^n$, then $\mathcal{T} \sim \mathcal{T}'$ if and only if there exists an automorphism $\varphi : F_n \rightarrow F_n$ with $\varphi(x_i) = w_i(x_1, \dots, x_n)$ such that $g'_i = w_i(g_1, \dots, g_n)$ for $i = 1, \dots, n$.

One can also interpret this fact in terms of the action of $Aut(F_n)$ on $Hom(F_n, G)$. After fixing a free basis (x_1, \dots, x_n) of F_n we identify $Hom(F_n, G)$ with G^n by identifying a homomorphism $f : F_n \rightarrow G$ with the n -tuple $\mathcal{T}_f = (f(x_1), \dots, f(x_n)) \in G^n$. There is a natural action of $Aut(F_n)$ on $Hom(F_n, G)$ by pre-composition. Then two homomorphisms $f_1, f_2 \in Hom(F_n, G)$ belong to the same $Aut(F_n)$ -orbit if and only if the corresponding n -tuples \mathcal{T}_{f_1} and \mathcal{T}_{f_2} are Nielsen equivalent in G .

Under the identification between G^n and $Hom(F_n, G)$ generating n -tuples of G correspond to epimorphisms $F_n \twoheadrightarrow G$. The set $Epi(F_n, G) \subseteq Hom(F_n, G)$ of all epimorphisms from F_n to G is $Aut(F_n)$ -invariant, and, as follows from the previous discussion, two elements $f_1, f_2 \in Epi(F_n, G)$ are in the same $Aut(F_n)$ -orbit if and only if the corresponding generating n -tuples are Nielsen equivalent.

Nielsen's result was later generalized to free products by I. A. Grushko [14]. Note that Nielsen's result can be easily derived from Grushko's theorem and the Euclidean algorithm, which is nothing but Nielsen's method in the case of a free group of rank one.

Theorem 6.0.3. *(Grushko) Let $G = H * K$ be a free product. Then any generating tuple $\mathcal{T} = (g_1, \dots, g_n)$ of G is Nielsen equivalent to a tuple of the form*

$$(h_1, \dots, h_s, k_{s+1}, \dots, k_n)$$

where $h_i \in H$ for $1 \leq i \leq s$ and $k_j \in K$ for $s + 1 \leq j \leq n$.

Planar discontinuous groups are the groups that act properly discontinuously by isometries on \mathbb{S}^2 , \mathbb{E}^2 or \mathbb{H}^2 and are classified by their orbifold, that is, by the quotient space of the respective plane modulo the group action where one needs to recover rotation orders of images of fixed points and reflections. The group can then be recovered as the fundamental group of

the orbifold; for details see [37, 57]. In this work we will be concerned with planar groups that contain no reflections. Every such group Γ is characterized by a signature

$$\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$$

where $k, g \in \mathbb{N} \cup \{0\}$ with $k \neq 1$ if $g = 0$, $\varepsilon \in \{-1, 1\}$, $h_i \in \mathbb{N} \cup \{\infty\}$ with $h_i \geq 2$ for $1 \leq i \leq k$ and $h_1 = h_2$ if $g = 0$ and $k = 2$; see [57] for details. This signature is determined up to a permutation of the h_i . The signature of a planar discontinuous group is nothing but the signature of the orbifold (marked surface) $\Gamma \backslash X$ where $X \in \{\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2\}$.

A planar discontinuous group Γ with signature $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$ has presentation

$$\langle a_1, \dots, a_p, s_1, \dots, s_k \mid s_1^{h_1}, \dots, s_k^{h_k}, s_1 \cdot \dots \cdot s_k R(a_1, \dots, a_p) \rangle.$$

where

(a) $p = 2g$ and $R(a_1, \dots, a_p) = [a_1, a_2][a_3, a_4] \cdot \dots \cdot [a_{p-1}, a_p]$ if $\varepsilon = +1$.

(b) $p = g$ and $R(a_1, \dots, a_p) = a_1^2 \cdot \dots \cdot a_p^2$ if $\varepsilon = -1$.

Definition 6.0.4. A *standard generating tuple* of the planar group Γ with $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$ is a tuple of the form

$$\mathcal{T} = (a_1, \dots, a_p, s_{i_1}^{\nu_1}, \dots, s_{i_{k-1}}^{\nu_{k-1}})$$

where $1 \leq i_1 < \dots < i_{k-1} \leq k$ and $\nu_1, \dots, \nu_{k-1} \in \mathbb{N}$ such that $\gcd(\nu_j, h_j) = 1$ for all $1 \leq j \leq k-1$.

We can now state the main result of this chapter which asserts that for a large class of planar groups these are essentially all the ways we can generate them.

Theorem 6.0.5. *Let Γ be a planar group with signature $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$ such that $g \geq 1$ if $\varepsilon = 1$ and $g \geq 2$ if $\varepsilon = -1$. Then any irreducible generating tuple of Γ is Nielsen equivalent to a standard generating tuple.*

Remark 6.0.6. Note that if $h_i = \infty$ for some $i \in \{1, \dots, k\}$ then the result follows from Grushko's theorem since in this case Γ is a free product of a finitely generated free group and finitely many cyclic groups.

6.1 Structure of a planar discontinuous group

Planar discontinuous groups can be decomposed as amalgamated products over infinite cyclic in a variety of ways where the amalgam corresponds to a separating simple closed curve on the associated orbifold $\Gamma \backslash X$. This is used in [33, 34, 53] to study minimal generating tuples of

planar discontinuous groups. We will, however, decompose a planar group as the fundamental group of a graph of groups where the edge groups correspond to non-separating simple closed curves on the associated orbifold. We will need the following fact from [20].

Lemma 6.1.1 (Louder). *Let Σ be a closed surface with $\chi(\Sigma) \leq 0$. Then there exist a non-separating simple closed curve $c \subset \Sigma$, a graph $A^0 \subset \Sigma' := \Sigma - \text{int}(V(c))$, where $V(c)$ is a regular neighborhood of c , and two immersions $\alpha, \omega : c \looparrowright A^0$, c has a graph structure which depends on the Euler characteristic of Σ , such that the following hold:*

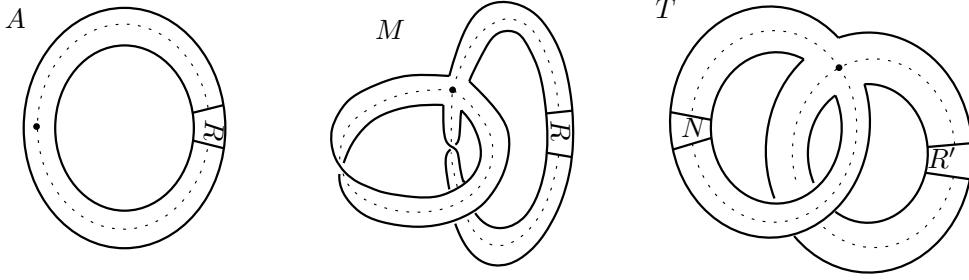
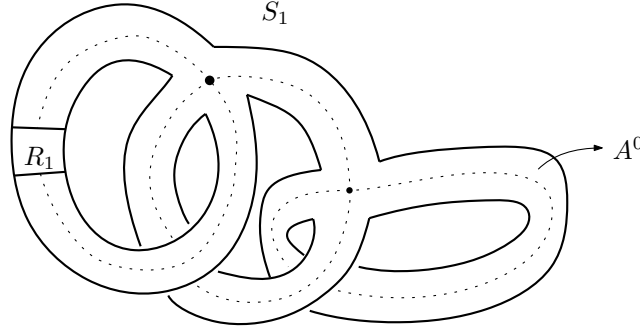
- (1) Σ' is a regular neighborhood (in Σ) of A^0 .
- (2) The paths $\alpha(c)$ and $\omega(c)$ have the same length in the graph A^0 .
- (3) The image of the induced homomorphisms $\alpha_*, \omega_* : \pi_1(c) \cong \mathbb{Z} \rightarrow \pi_1(A^0)$ correspond to the peripheral subgroups of $\pi_1(\Sigma')$.

Proof. Louder's construction goes inductively on $\chi(\Sigma)$ as follows. Let S_0 be either A ($\chi(\Sigma)$ even) or M ($\chi(\Sigma)$ odd), and let R_0 be the rectangle labeled R , see Figure 6.1. For each $i \geq 1$ we define $S_i := (S_{i-1} \sqcup T)/R_{i-1} = R'$, where T is the surface from figure 6.1. Finally, set $R_i := N$. Denote by c_1^i and c_2^i the boundary components of S_i . There is an obvious retraction $r : S_i \rightarrow A^0$ of S_i onto the core graph A^0 such that, after subdividing c_1^i and c_2^i into $-2\chi(S_i)$ segments, the maps $r|_{c_j^i} : c_j^i \rightarrow A^0$ is an immersion and the length of the paths $r(c_1^i)$ and $r(c_2^i)$ are equal. Moreover, each edge of A^0 is traversed twice by the morphisms $r|_{c_1^i} : c_1^i \rightarrow A^0$ and $r|_{c_2^i} : c_2^i \rightarrow A^0$.

The surface Σ is constructed from S_i as follows. Let $S'_i := S_i \sqcup A$ be the disjoint union of the surface S_i and the annulus A , c_1 and c_2 the boundary components of A and c the core of A . If $\chi(S_i)$ is odd, then $\Sigma := S'_i / \{c_1 = c_1^i, c_2 = c_2^i\}$. If $\chi(S_i)$ is even, then S_i is orientable. Fix an orientation for S_i and for A and give c_j^i (resp. c_i) the induced orientation. We define the surface Σ as $S'_i / \{c_1 = c_1^i, c_2 = c_2^i\}$ if Σ is orientable and $\Sigma = S'_i / \{c_1 = c_1^i, \bar{c}_2 = c_2^i\}$ if Σ is non-orientable, where \bar{c}_2 denotes the boundary component c_2 with its reverse orientation. Thus the core c of A is a non-separating simple closed curve in Σ and A is a regular neighborhood of c . \square

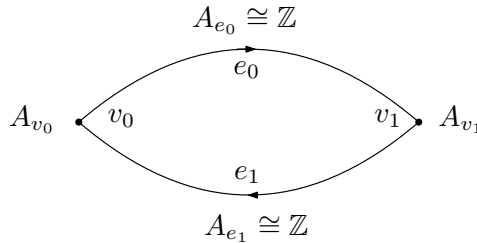
Convention 6.1.2. Any graph B will naturally be considered as a graph of groups \mathbb{B} having trivial vertex and edge groups. In this case we clearly have $\pi_1(\mathbb{B}, v_0) \cong \pi_1(B, v_0)$.

Convention 6.1.3. For each $k \geq 1$ we define the graph C_k as follows: C_k has vertex set $VC_k = \{v_0, \dots, v_{k-1}\}$ and edge-set $EC_k = \{e_1, \dots, e_k\}^{\pm 1}$. The boundary maps $\alpha, \omega : EC_k \rightarrow VC_k$ are given by $\alpha(e_i) = v_{i-1}$ and $\omega(e_i) = v_i$, where each index is taken mod k , see Figure 6.8. We will always consider v_0 as the base vertex of C_k . The fundamental group $\pi_1(C_k, v_0)$ of C_k is infinite cyclic generated by the homotopy class of the path $c_k := e_1, \dots, e_k$.


 Figure 6.1: The building blocks for the surface Σ .

 Figure 6.2: $\Sigma' = S_1$ is a 2-punctured torus. c is subdivided in 4 segments.

Let Γ be a planar discontinuous group with signature $\sigma(\Gamma) = (g, \varepsilon, k, h_1, \dots, h_k)$ where $g \geq 1$ if $\varepsilon = +1$ and $g \geq 2$ if $\varepsilon = -1$ and $2 \leq h_i < \infty$ for $i = 1, \dots, k$. Denote by Σ_Γ the underlying surface of the orbifold $\Gamma \backslash X$ ($X \in \{\mathbb{E}^2, \mathbb{H}^2\}$). From the hypothesis on Γ it follows that $\chi(\Sigma_\Gamma) \leq 0$. From 6.1.1 we obtain a non-separating simple closed curve $c \subset \Sigma_\Gamma$, a graph $A^0 \subset \Sigma'_\Gamma := \Sigma_\Gamma - \text{int}(V(c))$ and two immersions $\alpha, \omega : c \looparrowright A^0$ such that (1)-(3) hold.

We define the graph of groups \mathbb{A} as follows. The underlying graph A has vertex set $VA = \{v_0, v_1\}$, edge set $EA = \{e_0, e_1\}^{\pm 1}$ and boundary maps $\alpha, \omega : EA \rightarrow VA$ defined as $\alpha(e_0) = \omega(e_1) = v_0$ and $\omega(e_0) = \alpha(e_1) = v_1$.


 Figure 6.3: The splitting of Γ as $\pi_1(\mathbb{A}, v_0)$.

The vertex group A_{v_0} . The vertex group A_{v_0} is isomorphic to the fundamental group of the graph of trivial groups $\mathbb{A}^{v_0} := A^0$, i.e. A_{v_0} is isomorphic to the fundamental group of a compact surface with two boundary components and genus $g - 1$ if $\varepsilon = 1$ and genus $g - 2$ if $\varepsilon = -1$.

The boundary monomorphisms $\alpha_{e_0} : A_{e_0} \rightarrow A_{v_0}$ and $\alpha_{e_1} : A_{e_1} \rightarrow A_{v_0}$ are defined by

$$\alpha_{e_0}(z) := [\alpha(c)]^z \quad \text{and} \quad \omega_{e_1}(z) := [\omega(c)]^z$$

for $z \in A_{e_i} \cong \mathbb{Z}$ and $0 \leq i \leq 1$, where $\alpha, \omega : c \looparrowright A^0$ are the immersions given by Lemma 6.1.1. Denote by 1 the generator of $\mathbb{Z} = A_{e_0} = A_{e_1}$. We will denote the path $\alpha(c)$ (resp. $\omega(c)$) and the element $\alpha_{e_0}(1)$ (resp. $\omega_{e_1}(1)$) by α_{v_0} (resp. ω_{v_0}).

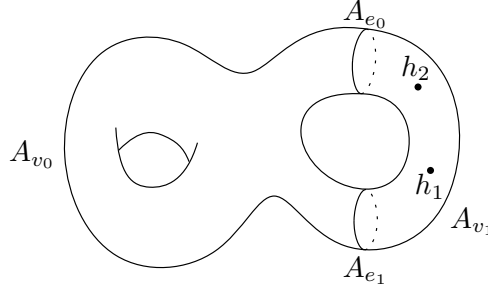


Figure 6.4: The geometric picture of the splitting of Γ .

The vertex group A_{v_1} . The vertex group A_{v_1} is isomorphic to the fundamental group of the finite graph of finite cyclic groups

$$\mathbb{A}^{v_1} := (A^1, \{A_x^1 = \langle s_x | s_x^{h_x} \rangle \mid x \in VA^1\}, \{A_e^1 = 1 \mid e \in EA^1\})$$

where $A^1 = C_l$ for some $l \geq 1$, see figure 6.8. Note that A_{v_1} is isomorphic to

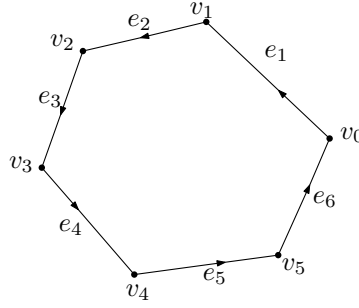
$$\pi_1(A^1, v_0) * A_{v_{i_1}}^1 * \dots * A_{v_{i_k}}^1 \cong \mathbb{Z} * \mathbb{Z}_{h_{v_{i_1}}} * \dots * \mathbb{Z}_{h_{v_{i_k}}}$$

where $v_{i_1}, \dots, v_{i_k} \in VA^1$ such that $A_{v_{i_j}} \neq 1$. We will assume that the group at the vertex v_0 , the base vertex of the graph of groups \mathbb{A}^{v_1} , is trivial.

Note that the group A_{v_1} can also be identified with the fundamental group of an orbifold having underlying surface an annulus and k cone points of order $h_{v_{i_1}}, \dots, h_{v_{i_k}}$, respectively. Let ω_{v_1} be the \mathbb{A}^{v_1} -path $1, e_1, 1, \dots, 1, e_l, 1$ and α_{v_1} the \mathbb{A}^{v_1} -path $1, e_1, s_{v_1}, \dots, s_{v_{l-1}}, e_l, 1$. Denote by ω_{v_1} (resp. α_{v_1}) the element of $A_{v_1} = \pi_1(\mathbb{A}^{v_1}, v_0)$ represented by the \mathbb{A}^{v_1} -path ω_{v_1} (resp. α_{v_1}). The boundary monomorphisms $\omega_{e_0} : A_{e_0} \rightarrow A_{v_1}$ and $\alpha_{e_1} : A_{e_1} \rightarrow A_{v_1}$ are defined by $\omega_{e_0}(z) = \omega_{v_1}^z$ and $\alpha_{e_1}(z) = \alpha_{v_1}^z$ for $z \in \mathbb{Z} = A_{e_0} = A_{e_1}$.

Thus we have completely defined the graph of groups \mathbb{A} . By looking at the standard presentation of $\pi_1(\mathbb{A}, v_0)$ we see that $\pi_1(\mathbb{A}, v_0)$ is isomorphic to the planar group Γ .

From now on in this section let $v \in VA = \{v_0, v_1\}$ be a vertex of A and \mathcal{D} be an \mathbb{A}^v -graph defining a graph of groups \mathbb{D} and the morphism $\phi_{\mathcal{D}} : \mathbb{D} \rightarrow \mathbb{A}^v$.


 Figure 6.5: The underlying graph of \mathbb{A}^1 with $l = 6$.

An element $a \in A_v = \pi_1(\mathbb{A}^v)$ is called *peripheral* if a is conjugate to γ^z for some $\gamma \in \{\alpha_v, \omega_v\}$ and some $z \in \mathbb{Z} \setminus \{0\}$. Furthermore, we call a subgroup $P \leq A_v$ *peripheral* if P is generated by a peripheral element.

We call a morphism $f : C_k \rightarrow \mathbb{D}$ (remember that we consider the circle C_k as a graph of trivial groups) an *immersion* if the morphism $f : C_k \rightarrow D$ on the graph level is an immersion. This means that the underlying path of the \mathbb{D} -path $f(c_k)$ is cyclically reduced, where $c_k = e_1, \dots, e_k$.

Let $\gamma \in \{\alpha_v, \omega_v\}$. There exists a unique, up to auxiliary moves of type A0, morphism $\gamma : C_{|\gamma|} \rightarrow \mathbb{A}^v$ such that $\gamma(c_{|\gamma|}) = \gamma$, where $|\gamma|$ denotes the length of the \mathbb{A}^v -path γ . Note that $\gamma : C_{|\gamma|} \rightarrow \mathbb{A}^v$ is an immersion.

Definition 6.1.4. Let $\mathcal{E} = \{f_i : C_{k_i} \rightarrow \mathbb{D} \mid 1 \leq i \leq n\}$ be a collection of immersions with the property that each edge of D is crossed at most twice by the f_i 's. We say that \mathcal{E} is *collapsible* if for every nonempty subset $J \subseteq \{1, \dots, n\}$ there exists at least one edge of D crossed exactly once by the collection $\mathcal{E}_J = \{f_j : C_{k_j} \rightarrow \mathbb{D} \mid j \in J\}$.

Note that, after reordering the immersions $f_i : C_{k_i} \rightarrow \mathbb{D}$, there exist edges $\epsilon_1, \dots, \epsilon_n \in ED$ with the following property: for each i there is $\tau(i) \in \{1, \dots, k_i\}$ such that $f_i(e_{\tau(i)}) = \epsilon_i$ and $f_i(e_s) \in D_i := D \setminus \{\epsilon_1, \dots, \epsilon_i\}^{\pm 1}$ for all $s \in \{1, \dots, k_i\}$ with $s \neq \tau(i)$. The graph $D_n = D \setminus \{\epsilon_1, \dots, \epsilon_n\}^{\pm 1}$ obtained from D by removing the edges $\{\epsilon_1, \dots, \epsilon_n\}^{\pm 1}$ is connected and $VD_n = VD$. Thus any maximal subtree $T \subseteq D_n$ is a maximal subtree of D .

Let $j \in \{1, \dots, n\}$. Define \mathcal{D}' as the \mathbb{A}^v -subgraph of \mathcal{D} having underlying graph $D' = D \setminus \{\epsilon_1, \dots, \epsilon_j\}^{\pm 1}$. We say that \mathcal{D}' is *obtained from \mathcal{D} by collapsing f_1, \dots, f_j* . Note that for each $s > j$, f_s has its image in \mathbb{D}' . We naturally put $\mathcal{E}' = \{f_s : C_{k_s} \rightarrow \mathbb{D}' \mid j < s \leq n\}$.

Lemma 6.1.5. *Let \mathcal{D} be an \mathbb{A}^v -graph and $\mathcal{E} = \{f_i : C_{k_i} \rightarrow \mathbb{D} \mid 1 \leq i \leq n\}$ be a collection of immersions with the property that each edge of D is crossed at most twice by \mathcal{E} . Suppose that the \mathbb{A}^v -graph \mathcal{D}' is obtained from \mathcal{D} by a fold of type IA or of type IB and let $\mathcal{E}' = \{f'_i = \sigma \circ f_i : C_{k_i} \rightarrow \mathbb{D}' \mid 1 \leq i \leq n\}$, where $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$ is induced by the fold. Assume that each f'_i is an immersion and that every edge of D' is traversed at most twice by \mathcal{E}' . If \mathcal{E} is collapsible, then \mathcal{E}' is collapsible.*

Proof. Denote by π the morphism from the graph of groups \mathbb{D} to its underlying graph D that “forgets” vertex and edge groups and by \mathcal{E}_π the collection $\{\pi \circ f_i : C_{k_i} \rightarrow D \mid 1 \leq i \leq n\}$. It follows from the definition that \mathcal{E} is collapsible if and only if \mathcal{E}_π is collapsible. Moreover, it is not hard to see that \mathcal{E}_π is collapsible if and only if the set $\{x_1, \dots, x_n\}$ is part of a minimal generating set of $\pi_1(D)$, where x_i is the element of $\pi_1(D)$ determined by the path $\pi \circ f_i(c_{k_i})$ for $i = 1, \dots, n$.

Denote by $\sigma_\pi : D \rightarrow D'$ the corresponding fold on the graph level. Note that $\sigma_\pi \circ \pi = \pi' \circ \sigma$, where $\pi' : \mathbb{D}' \rightarrow D'$. Since \mathcal{D}' is obtained from \mathcal{D} by a fold of type IA or of type IB, it follows that σ_π is a homotopy equivalence. Thus the set $\{y_1, \dots, y_n\}$ is part of a minimal generating set of $\pi_1(D')$, where y_i is the element of $\pi_1(D')$ determined by the path $\sigma_\pi \circ \pi(f_i(c_i)) = \phi' \circ \sigma(f_i(c_i)) = \pi'(f'_i(c_i))$ for $i = 1, \dots, n$. Therefore $\mathcal{E}'_{\pi'}$ is collapsible. Consequently \mathcal{E}' is collapsible. \square

Now we will consider triples $(\mathcal{D}, \mathcal{E}, Q)$ consisting of an \mathbb{A}^v -graph \mathcal{D} , a collection of immersions $\mathcal{E} = \{f_i : C_{k_i} \rightarrow \mathbb{D} \mid 1 \leq i \leq n\}$ with the property that each edge of \mathcal{D} is traversed at most twice by the f_i 's, and a collection of finite covers

$$Q = \{q_i : C_{k_i} \rightarrow C_{|\gamma_i|} \mid 1 \leq i \leq n\}$$

($\gamma_i \in \{\alpha_v, \omega_v\}$) satisfying $\phi_{\mathcal{D}} \circ f_i(c_{k_i}) = \gamma_i \circ q_i(c_{k_i})$ for $1 \leq i \leq n$. Following Louder [20], we say that $(\mathcal{D}, \mathcal{E}, Q)$ *folds squares* if one of the following holds:

- (a) There exist $i \neq j \in \{1, \dots, n\}$, $e \in EC_{k_i}$, and $e' \in EC_{k_j}$ such that $\gamma_i = \gamma_j$, $f_i(e) = f_j(e')$ and $q_i(e) = q_j(e')$.
- (b) There exists $i \in \{1, \dots, n\}$ and $e \neq e' \in EC_{k_i}$ such that $f_i(e) = f_i(e')$ and $q_i(e) = q_i(e')$.

For a finitely generated group G we denote the number of conjugacy classes of maximal finite subgroups of G by $\text{tor}(G)$. For example, if $G = \mathbb{Z}_p * \mathbb{Z}_q$ then $\text{tor}(G) = 2$.

Lemma 6.1.6. *Let Γ_0 be a finitely generated subgroup of A_v , $a_1, \dots, a_l \in A_v$ and z_1, \dots, z_l integers with $z_i \geq 1$ for $i = 1, \dots, l$. For each $i \in \{1, \dots, l\}$ denote by $\Gamma_i \leq A_v$ the peripheral subgroup generated by $a_i \gamma_i^{z_i} a_i^{-1}$, where $\gamma_i \in \{\alpha_v, \omega_v\}$. Define the group*

$$\Gamma := \Gamma_0 * \Gamma_1 * \dots * \Gamma_l.$$

Let $\eta : \Gamma \rightarrow A_v$ be the homomorphism induced by the inclusion maps $\Gamma_i \hookrightarrow A_v$ ($1 \leq i \leq l$). If $\text{rk}(\eta(\Gamma)) < \text{rk}(\Gamma)$ then

$$\Gamma = \Gamma'_0 * \Gamma'_1 * \dots * \Gamma'_l$$

where $\Gamma'_0 \leq \Gamma$, $\Gamma'_i = g_i \Gamma_i g_i^{-1}$ for some $g_i \in \Gamma$, and one of the following holds:

- (a) *There exist $i \neq j \in \{1, \dots, l\}$ such that $\eta(\Gamma'_i) \cap \eta(\Gamma'_j) \neq 1$.*

(b) There exists $i \in \{1, \dots, l\}$ such that $\eta(\Gamma'_i) \cap \eta(\Gamma') \neq 1$, where

$$\Gamma' := \Gamma'_0 * \Gamma'_{i_1} \dots * \Gamma'_{i_s} \leq \Gamma$$

and $i_1, \dots, i_s \in \{1, \dots, l\} \setminus \{i\}$.

(c) There exists an epimorphism $\sigma : \Gamma \twoheadrightarrow \bar{\Gamma}$ and a homomorphism $\bar{\eta} : \bar{\Gamma} \rightarrow A_v$ such that $\bar{\eta} \circ \sigma = \eta$. Moreover,

$$\bar{\Gamma} = \bar{\Gamma}_0 * \bar{\Gamma}_1 * \dots * \bar{\Gamma}_l$$

where $\bar{\Gamma}_i$ is conjugate to $\sigma(\Gamma_i)$ for $i = 1, \dots, l$ and either $\text{rk}(\bar{\Gamma}) < \text{rk}(\Gamma)$ or $\text{tor}(\Gamma) < \text{tor}(\bar{\Gamma})$.

(d) $\Gamma'_0 = \Gamma''_0 * \langle as_w^z a^{-1} \rangle$ for some $w \in VA^v$, $z \in \mathbb{Z} - \{0\}$ and $a \in A_v$ and $as_w^{z'} a^{-1} \in \eta(\Gamma'')$ for some $z' \in \mathbb{Z}$ such that $|\langle s_w \rangle : \langle s_w^{z'} \rangle| < |\langle s_w \rangle : \langle s_w^z \rangle|$, where

$$\Gamma'' := \Gamma''_0 * \Gamma'_1 * \dots * \Gamma'_l \leq \Gamma.$$

Proof. The group A_v is isomorphic to the fundamental group of the graph of cyclic groups \mathbb{A}^v . Choose a finite core \mathbb{A}^v -graph \mathcal{D}_{Γ_0} that represents the subgroup Γ_0 . Such an \mathbb{A}^v -graph exists since $\Gamma_0 \leq A_v$ is finitely generated. For each $i \in \{1, \dots, l\}$ represent the element $a_i \in A_v$ by an \mathbb{A}^v -path p_i of positive length and define the \mathbb{A}^v -graph \mathcal{D}_{Γ_i} as follows. The underlying graph of \mathcal{D}_{Γ_i} is obtained from the circle C_{k_i} by gluing a segment $f_1^i, \dots, f_{|p_i|}^i$ of length $|p_i|$ to the base vertex v_0 of C_{k_i} , where $k_i = z_i |\gamma_i| > 0$. Vertex and edge groups of \mathcal{D}_{Γ_i} are trivial. Edges are label so that the path

$$1, f_1^i, 1, \dots, f_{|p_i|}^i, 1, e_1, 1, \dots, e_{k_i}, 1, (f_{|p_i|}^i)^{-1}, 1, \dots, 1, (f_1^i)^{-1}, 1$$

is mapped to the \mathbb{A}^v -path $p_i \gamma_i^{z_i} p_i^{-1}$. The base vertex of \mathcal{D}_{Γ_i} is set to be $\alpha(f_1^i)$. Thus \mathcal{D}_{Γ_i} is well-defined up to auxiliary moves of type A0.

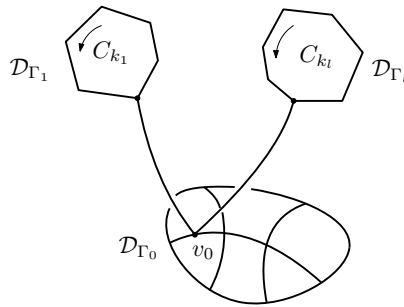


Figure 6.6: The \mathbb{A}^v -graph \mathcal{D}^0 .

Now we construct an \mathbb{A}^v -graph whose fundamental group is canonically isomorphic to Γ . Let ν_0 be the base vertex of \mathcal{D}_{Γ_0} . The \mathbb{A}^v -graph \mathcal{D}^0 is defined as the wedge of $\mathcal{D}_{\Gamma_0}, \dots, \mathcal{D}_{\Gamma_l}$, i.e. we glue the \mathbb{A}^v -graphs $\mathcal{D}_{\Gamma_0}, \dots, \mathcal{D}_{\Gamma_l}$ along their base vertices, see Figure 6.6. Note that each \mathbb{A}^v -graph \mathcal{D}_{Γ_i} is an \mathbb{A}^v -subgraph of \mathcal{D}^0 . There is a canonical isomorphism between Γ and $\pi_1(\mathbb{D}^0, \nu_0)$ that maps $\pi_1(\mathbb{D}_{\Gamma_i}, \nu_0) \leq \pi_1(\mathbb{D}^0, \nu_0)$ isomorphically onto $\Gamma_i \leq \Gamma$ such that the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{D}^0, \nu_0) & \xrightarrow{\cong} & \Gamma \\ & \searrow \phi^0 & \downarrow \eta \\ & & A_v \end{array}$$

commutes. Thus we identify Γ with $\pi_1(\mathbb{D}^0, \nu_0)$ and η with ϕ^0 . For each $i = 1, \dots, l$ there is an obvious immersion $f_i : C_{k_i} \rightarrow \mathbb{D}^0$ and a finite covering $q_i : C_{k_i} \rightarrow C_{|\gamma_i|}$ such that $\phi_{\mathcal{D}} \circ f_i(c_{k_i}) = \gamma_i \circ q_i(c_{k_i})$ for $1 \leq i \leq l$. By hypothesis $rk(\eta(\Gamma)) < rk(\Gamma)$. Thus \mathcal{D}^0 is not folded. Choose a folding sequence

$$\mathcal{D}^0, \mathcal{D}^1, \dots, \mathcal{D}^m$$

with the following property: \mathcal{D}^m is folded and either \mathcal{D}^{i+1} is obtained from \mathcal{D}^i by a fold of type I or no fold of type I is applicable to \mathcal{D}^i and \mathcal{D}^{i+1} is obtained from \mathcal{D}^i by a fold of type III.

For each $1 \leq r \leq m$, define $\mathcal{E}^r = \{f_i^r := \sigma_r \circ f_i : C_{k_i} \rightarrow \mathbb{D}^r \mid 1 \leq i \leq l\}$, where the morphism $\sigma_r : \mathbb{D}^0 \rightarrow \mathbb{D}^r$ is induced by the sequence of folds $\mathcal{D}^0 \rightarrow \dots \rightarrow \mathcal{D}^r$. Note that $f_s^r : C_{k_s} \rightarrow \mathbb{D}^i$ is an immersion for all $1 \leq r \leq l$ and all $1 \leq s \leq n$. If not, then there exist edges $e, e' \in EC_{k_s}$ with $\alpha(e) = \alpha(e')$ such that $f_s^r(e) = f_s^r(e')$. Since $\phi_{\mathcal{D}^r} \circ f_s^r(c_{k_s}) = \gamma_s \circ q_s(c_{k_s})$ we have $\gamma_s(q_s(e)) = \gamma_s(q_s(e'))$ which contradicts the fact that γ_s is an immersion.

Now let $1 \leq k \leq m - 1$ be the first index such that one of the following holds:

- (α) There exist $i \neq j \in \{1, \dots, l\}$, $e \in EC_{k_i}$ and $e' \in EC_{k_j}$ such that $q_i(e) = q_j(e')$, $f_i^k(\alpha(e)) = f_j^k(\alpha(e'))$ and $f_i^{k+1}(e) = f_j^{k+1}(e')$, or (α) does not hold and
- (β) There exists $i \in \{1, \dots, l\}$ and $e \neq e' \in EC_{k_i}$ such that $q_i(e) = q_i(e')$, $f_i^k(\alpha(e)) = f_i^k(\alpha(e'))$ and $f_i^{k+1}(e) = f_i^{k+1}(e')$, or (α) and (β) do not hold and
- (γ) $\sigma : \pi_1(\mathbb{D}^0, \nu_0) \rightarrow \pi_1(\mathbb{D}^{k+1}, \nu_{k+1})$ is not injective.

For each $s \in \{1, \dots, k\}$, the collection of immersions \mathcal{E}^s has the property that each edge of \mathcal{D}^s is traversed at most two times by the f_i^s 's since otherwise \mathcal{D}_{s-1} satisfies either (α) or (β). By the same reason we see that $(\mathcal{D}^s, \mathcal{E}^s, Q)$ does not fold squares.

If we are in situation (α) or (β), then clearly \mathcal{D}^k is obtained from \mathcal{D}^0 by folds of type I only. Lemma 6.1.5 implies that \mathcal{E}^k is collapsible.

Let \mathcal{D}_0^k the \mathbb{A}^v -subgraph obtained from \mathcal{D}^k by collapsing all the immersions $f_i^k \in \mathcal{E}^k$ and denote by Γ_0^k the group $\pi_1(\mathbb{D}_0^k, \nu_k) \leq \pi_1(\mathbb{D}^k, \nu_k)$. Choose a maximal subtree T of \mathcal{D}_0^k , which is a

maximal subtree of D^k since the vertex set of D_0^k coincides with the vertex set of D^k . For each $i = 1, \dots, l$ denote by q_i the \mathbb{D}_0^k -path $1, e_1', 1, \dots, 1, e_{d_\nu}'$ from $\nu_k \in VD^k$ to $\nu := f_i^k(v_0) \in VD^k$, where e_1', \dots, e_{d_ν}' is the unique reduced path in T from ν_k to $\nu = f_i^k(v_0)$. For each $i \in \{1, \dots, l\}$ let $\Gamma_i' \leq \pi_1(\mathbb{D}^k, \nu_k)$ be the subgroup generated by the element $[q_i \cdot f_i^k(c_{k_i}) \cdot q_i^{-1}]$. Therefore we have

$$\Gamma \cong \pi_1(\mathbb{D}^k, \nu_k) = \Gamma_0' * \Gamma_1' * \dots * \Gamma_l'$$

and clearly Γ_i' is conjugate to Γ_i for all $i = 1, \dots, l$.

Suppose that (α) occurs. In this case we can choose the maximal subtree $T \subseteq D^k$ in such a way that $q_j = q_i \cdot q$ with q contained in the \mathbb{A}^v -subgraph $f_i^k(C_{k_i}) \cup f_j^k(C_{k_j})$. Therefore we obtain (a).

Suppose now that (β) occurs. Since \mathcal{E}^k is collapsible, we collapse the immersions $f_{j_1}^k, \dots, f_{j_t}^k \in \mathcal{E}^k$ from \mathcal{D}^k where j_t is the first index equal to j . Denote by \mathcal{D}_1^k the resulting \mathbb{A}^v -graph. Then

$$\pi_1(\mathbb{D}_1^k, \nu_k) \cong \Gamma_0' * \Gamma_{i_1}' * \dots * \Gamma_{i_s}'$$

where $\{i_1, \dots, i_s\} = \{1, \dots, l\} \setminus \{j_1, \dots, j_t\}$. Clearly we have

$$\eta(\Gamma_i) \cap \eta(\Gamma') \neq 1$$

with $\Gamma' = \Gamma_0' * \Gamma_{i_1}' * \dots * \Gamma_{i_s}' \leq \Gamma$.

Suppose now that (γ) occurs. We distinguish two cases according to whether \mathcal{E}^{k+1} is collapsible or not. Suppose that \mathcal{E}^{k+1} is collapsible. As before, we obtain

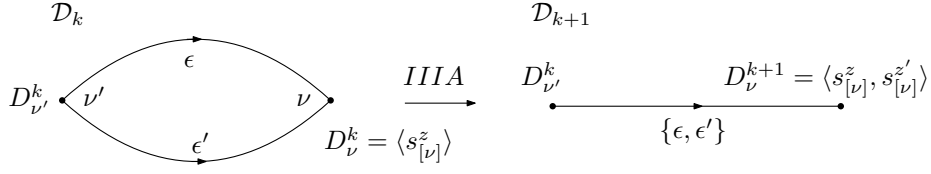
$$\pi_1(\mathbb{D}^{k+1}, \nu_{k+1}) = \bar{\Gamma}_0 * \bar{\Gamma}_1 * \dots * \bar{\Gamma}_l$$

and $\bar{\Gamma}_i$ is conjugate to $\sigma(\Gamma_i')$ for $i = 1, \dots, l$, where $\sigma : \Gamma \cong \pi_1(\mathbb{D}^k, \nu_k) \rightarrow \pi_1(\mathbb{D}^{k+1}, \nu_{k+1})$ is induced by the fold $\mathcal{D}^k \rightarrow \mathcal{D}^{k+1}$. It is easily verified that either $rk(\Gamma) > rk(\bar{\Gamma})$ or $tor(\Gamma) < tor(\bar{\Gamma})$. Therefore we obtain (b).

Suppose now that (γ) occurs and that \mathcal{E}^{k+1} is not collapsible. Thus, from Lemma 6.1.5 it follows that \mathcal{D}^{k+1} is obtained from \mathcal{D}^k by a fold of type III. By our choice of the folding sequence it follows that no fold of type I is applicable to \mathcal{D}^k .

Denote by $\epsilon, \epsilon' \in ED^k$ the edges involved in the fold and by $\nu \in VD^k$ the vertex $\omega(\epsilon) = \omega(\epsilon')$. The vertex group $D_\nu^k := \langle s_{[\nu]}^z \rangle \leq A_{[\nu]}^v$ is replaced by $D_\nu^{k+1} = \langle s_{[\nu]}^z, s_{[\nu]}^{z'} \rangle$, where $s_{[\nu]}^{z'} = (\epsilon)_\omega^{-1}(\epsilon')_\omega \in A_{[\nu]}^v = \langle s_{[\nu]} \mid s_{[\nu]}^{h_{[\nu]}} \rangle$, see Figure 6.7.

We first observe that ϵ and ϵ' are crossed at least once by the collection \mathcal{E}^k . In fact, suppose that ϵ is not crossed by any immersion. Then it follows that \mathcal{E}^k is not collapsible since the graph $D^k \setminus \{\epsilon, \epsilon^{-1}\}$ is mapped isomorphically onto the underlying graph of \mathcal{D}^{k+1} . Denote by f_i^k the immersion which traverses ϵ and by f_j^k the immersion that traverses ϵ' .


 Figure 6.7: \mathcal{D}_k folds onto \mathcal{D}_{k+1} .

Secondly, ϵ and ϵ' are crossed exactly once. For if ϵ is crossed by f_h^k and f_i^k , then \mathcal{D}^{k-1} would satisfy either (α) or (β) as each edge of \mathbb{A}^v is crossed twice by α_v and ω_v .

If $D_\nu^{k+1} = D_\nu^k$ then an auxiliary move of type A2 applied at $\nu = \alpha(\epsilon'^{-1})$ gives us $(\epsilon')_\omega = (\epsilon)_\omega$. Thus we obtain (b).

Assume now that $D_\nu^{k+1} \not\supseteq D_\nu^k$. Let $St(\nu) = \{\epsilon_1, \dots, \epsilon_s\}$ be the star of the vertex ν . Suppose that $\epsilon_1^{-1} = \epsilon$ and $\epsilon_s^{-1} = \epsilon'$ and that $(\epsilon_i)_\alpha = s_w^{a_i}$ for $1 \leq i \leq s$, where $w = [\nu] \in VA^v$ is the type of ν . After reordering the edges $\epsilon_1, \dots, \epsilon_s$ we may assume that $\epsilon_i^{-1}, \epsilon_{i+1}$ is crossed by some immersion f_d^k for $1 \leq i \leq s-1$. For if this is not the case then the collection \mathcal{E}^{k+1} is collapsible contradicting our assumption. Therefore the edges $\epsilon_2, \dots, \epsilon_{s-1}$ are crossed exactly two times by the collection \mathcal{E}^k . Since each $f_i^k \in \mathcal{E}^k$ covers the \mathbb{A}^v -path $\gamma_i \in \{\alpha_v, \omega_v\}$, we have

$$s_w^{-a_i} s_w^{\alpha_i z} s_w^{a_{i+1}} = s_w^{\varepsilon_i}$$

for some $\varepsilon_i \in \{0, 1\}$. This implies that for $i = 1, \dots, s$ we have

$$s_w^{\alpha_i z} s_w^{a_{i+1}} = s_w^{a_i} s_w^{\varepsilon_i}.$$

Thus, by applying an auxiliary move of type A2 with $s_w^{(\alpha_1 + \dots + \alpha_{i-1})z} \in D_\nu^k$ to the edge $\epsilon_i \in ED^k$ ($i \geq 2$) we obtain an \mathbb{A}^v -graph \mathcal{D}_1^k such that all paths $f_i^k(c_{k_i})$ are locally equal to $\epsilon_i^{-1}, 1, \epsilon_{i+1}$. Furthermore, the label of ϵ_s is equal to $s_w^{a_1} s_w^{\varepsilon_1 + \dots + \varepsilon_s} \in A_w^v$. Note that $|\langle s_w \rangle : \langle s_w^z \rangle| > s \geq \varepsilon_1 + \dots + \varepsilon_s$ since otherwise we would have $D_\nu^k = D_\nu^{k+1}$ or we could fold ϵ_i with some ϵ_j and so be in situation (α) or (β) .

Define the \mathbb{A}^v -graph \mathcal{D}_2^k from \mathcal{D}_1^k by replacing the vertex group $D_\nu^k = \langle s_{[\nu]}^z \rangle$ by the trivial group. By the previous discussion we see that each immersion $f_i^k \in \mathcal{E}^k$ has image in \mathcal{D}_2^k . Let further \mathcal{D}_3^k be the \mathbb{A}^v -graph obtained from \mathcal{D}_2^k by collapsing all immersions $f_i^k \in \mathcal{E}^k$. Therefore, from the discussion in the previous paragraph we have

$$\Gamma'_0 = \Gamma''_0 * D_\nu^k$$

where $\Gamma''_0 = \pi_1(\mathbb{D}_3^k, \nu_k)$ and $a \cdot s_w^{\varepsilon_1 + \dots + \varepsilon_s} \cdot a^{-1} \in \eta(\Gamma''_0)$ where

$$\Gamma''_0 = \pi_1(\mathcal{D}_2^k, \nu_k) = \Gamma''_0 * \Gamma'_1 * \dots * \Gamma_l \leq \Gamma.$$

Therefore we are in case (c). \square

6.2 Tame \mathbb{A} -graphs

In this section we introduce the class of tame \mathbb{A} -graphs. The idea of the proof of Theorem 6.0.5 is to encode the Nielsen class of a generating tuple \mathcal{T} of the planar discontinuous group $\Gamma = \pi_1(\mathbb{A}, v_0)$ by pairs $(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$, where \mathcal{B} is a tame \mathbb{A} -graph and $\mathcal{T}_{\mathcal{B}}$ is a minimal generating tuple of $\pi_1(\mathbb{B}, u_0)$.

Let \mathcal{B} be an \mathbb{A} -graph. We denote the set of edges $f \in EB$ with non-trivial group by E_1B . We say that \mathcal{B} is *collapsed* if E_1B is empty. In other words, \mathcal{B} is collapsed if all edges of \mathcal{B} have trivial group. We further denote the rank of $\pi_1(\mathbb{B}, u_0)$ by $rk(\mathcal{B})$ and the number of conjugacy classes of finite subgroups of $\pi_1(\mathbb{B}, u_0)$ by $tor(\mathcal{B})$.

Definition 6.2.1 (Weakly tame \mathbb{A} -graphs). We say that an \mathbb{A} -graph \mathcal{B} is *weakly tame* if the following hold:

- (T1) For each $u \in VB$, the associated group B_u is finitely generated.
- (T2) For each $f \in E_1B$, $f_{\alpha} \cdot \alpha_{[f]}(B_f) \cdot f_{\alpha}^{-1}$ is a maximal peripheral subgroup of $B_{\alpha(f)}$.
- (T3) For each pair of distinct edges $f, g \in E_1B$ with $\alpha(f) = \alpha(g)$, we have

$$f_{\alpha} \cdot \alpha_{[f]}(B_f) \cdot f_{\alpha}^{-1} \cap b \cdot (g_{\alpha} \cdot \alpha_{[g]}(B_g) \cdot g_{\alpha}^{-1}) \cdot b^{-1} = 1$$

for all $b \in B_{\alpha(f)} = B_{\alpha(g)}$, i.e the subgroups $f_{\alpha} \cdot \alpha_{[f]}(B_f) \cdot f_{\alpha}^{-1}$ and $g_{\alpha} \cdot \alpha_{[g]}(B_g) \cdot g_{\alpha}^{-1}$ are conjugacy separated in $B_{\alpha(f)}$.

Lemma 6.2.2. *Let \mathcal{B} be a weakly tame \mathbb{A} -graph. If \mathcal{B} is not folded then one of the following holds:*

1. *There exist distinct edges $f, g \in EB$ incident at $x \in VB$ with $l(f) := (a, e, b)$ and $l(g) := (a', e, b')$ such that $B_f = 1$ or $B_g = 1$ and that*

$$a' = b_x \cdot a \cdot \alpha_e(c)$$

for some $c \in A_e$ and some $b_x \in B_x$.

2. *There exists an edge $f \in EB$ labeled (a, e, b) with $B_f = 1$ such that*

$$\langle c_f \rangle := \alpha_e^{-1}(a^{-1} \cdot B_{\alpha(f)} \cdot a) \neq 1.$$

Proof. Since \mathcal{B} is not folded it follows from the definition of folded \mathbb{A} -graphs that at least one of the following holds:

1. There are two distinct edges $f, g \in EB$ with $\alpha(f) = \alpha(g) = x$ and labels (a, e, b) and (a', e, b') accordingly, such that $a' = b_x \cdot a \cdot \alpha_e(c)$ for some $b_x \in B_x$ and some $c \in A_e$.
2. There is an edge $f \in EB$ with label (a, e, b) such that $B_f \subsetneq \alpha_e^{-1}(a^{-1} \cdot B_{\alpha(f)} \cdot a)$.

Suppose that (1) occurs and that $B_f \neq 1 \neq B_g$. Since $a' = b_x \cdot a \cdot \alpha_e(c)$ we obtain $a' \cdot \alpha_e(B_g) \cdot a'^{-1} = b_x(a \cdot \alpha_e(B_g) \cdot a^{-1})b_x^{-1}$. Hence $b_x^{-1}(a' \cdot \alpha_e(B_g)a'^{-1})b_x \cap a \cdot \alpha_e(B_f) \cdot a^{-1} \neq 1$, contradicting condition (T3). Thus either $B_f = 1$ or $B_g = 1$.

Suppose now that (2) occurs. We will show that B_f is trivial. For if $B_f \neq 1$, then condition (T2) implies that $a \cdot \alpha_e(B_f) \cdot a^{-1}$ is a maximal peripheral subgroup of $B_{\alpha(f)}$. Thus we must have $B_f = \alpha_e^{-1}(a^{-1} \cdot B_{\alpha(f)} \cdot a)$. Therefore, $B_f \subsetneq \alpha_e^{-1}(a^{-1} \cdot B_{\alpha(f)} \cdot a)$ implies $B_f = 1$. \square

Definition 6.2.3. Let \mathcal{B} be a weakly tame \mathbb{A} -graph. We say that a vertex $u \in VB$ is *full* if $B_u \leq A_{[x]}$ is of finite index and for each maximal peripheral subgroup $P \leq B_u$ there exists $f \in E_1B$ with $\alpha(f) = u$ such that P is conjugate (in B_u) to $f_\alpha \cdot \alpha_{[f]}(B_f) \cdot f_\alpha^{-1}$.

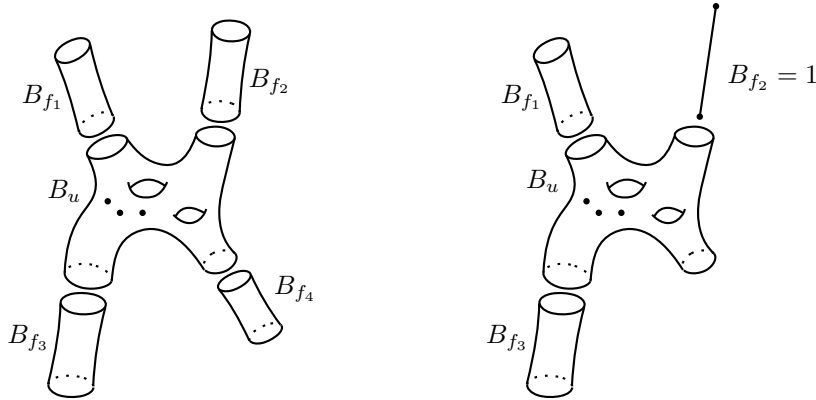


Figure 6.8: The local picture at a full vertex (left) and at a collapsible vertex (right).

Definition 6.2.4. Let \mathcal{B} be a weakly tame \mathbb{A} -graph. A vertex $u \in VB$ is called *collapsible* if u is not full there exists at least one edge $f \in E_1B$ with $\alpha(f) = u$

Suppose now that \mathcal{B} is weakly tame and that u is a collapsible vertex. Let $f_1, \dots, f_r \in E_1B$ be the set of edges incident at u with label $(a_1, e_1, b_1), \dots, (a_r, e_r, b_r)$. Since u is collapsible it follows that, after auxiliary moves of type A2 applied to the edges f_1, \dots, f_r , we have

$$B_u = U * \left(\underset{i \in \{1, \dots, r\}}{*} a_i \cdot \alpha_{e_i}(B_{f_i}) \cdot a_i^{-1} \right)$$

where U is a finitely generated subgroup of B_u . Let $i_0 \in \{1, \dots, r\}$. We define the \mathbb{A} -graph \mathcal{B}' from \mathcal{B} by replacing the edge group $B_{f_{i_0}}$ by the trivial group, and the vertex group B_u by the group

$$B'_u := U * \left(\underset{i \in \{1, \dots, r\} \setminus \{i_0\}}{*} a_i \cdot \alpha_{e_i}(B_{f_i}) \cdot a_i^{-1} \right).$$

We can recover \mathcal{B} from \mathcal{B}' by a single fold of type IIA along the edge f_{i_0} . The fundamental group of the graph of groups \mathbb{B}' , induced by \mathcal{B}' , is isomorphic to $\pi_1(\mathbb{B}, u_0)$. We say that \mathcal{B}' is obtained from \mathcal{B} by collapsing the edge f_{i_0} from the vertex u , or simply \mathcal{B}' is obtained from \mathcal{B} by collapsing. It is easily verified that \mathcal{B}' is weakly tame.

Definition 6.2.5 (Tame \mathbb{A} -graphs). We call a weakly tame \mathbb{A} -graph \mathcal{B} *tame* if there is a finite sequence of \mathbb{A} -graphs $\mathcal{B}_0, \dots, \mathcal{B}_s$ such that \mathcal{B}_s is collapsed and that for each $1 \leq i \leq s-1$, \mathcal{B}_i has a collapsible vertex and \mathcal{B}_{i+1} is obtained from \mathcal{B}_i by collapsing.

The fundamental group $\pi_1(\mathbb{B}, u_0)$ of the induced graph of groups of a tame \mathbb{A} -graph \mathcal{B} is isomorphic to $\left(\underset{x \in VB}{*} U_x \right) * \pi_1(B, u_0)$, where for each $x \in VB$, U_x is a finitely generated subgroup of B_x and B is the underlying graph of \mathcal{B} . Since $U_x \leq B_x \leq A_{[x]}$ it follows that U_x is isomorphic to a free product of a free group and finitely many finite cyclic groups. Therefore $\pi_1(\mathbb{B}, u_0)$ is isomorphic to $F(\mathcal{B}) * C_1 * \dots * C_s$, where $F(\mathcal{B})$ is a finitely generated free group, C_1, \dots, C_s are finite cyclic groups and $s = \text{tor}(\pi_1(\mathbb{B}, u_0))$. From Grushko's Theorem 6.0.3 it follows that an irreducible generating tuple of $\pi_1(\mathbb{B}, u_0)$ is minimal.

Definition 6.2.6 (Standard \mathbb{A} -graph). We say that an \mathbb{A} -graph \mathcal{B} is *standard* if \mathcal{B} is given by the following data:

- (S1) The underlying graph B of \mathcal{B} has vertex-set $VB = \{u_0, u_1\}$ and edge-set $EB = \{f_0, f_1\}^{\pm 1}$. The boundary maps $\alpha, \omega : EB \rightarrow VB$ are given by $\alpha(f_0) = \omega(f_1) = u_0$ and $\omega(f_0) = \alpha(f_1) = u_1$.
- (S2) The morphism $[\cdot] : B \rightarrow A$ is defined by $[u_i] = v_i$ and $[f_i] = e_i$ for $i \in \{1, 2\}$.
- (S3) $B_{u_0} = A_{v_0}$ and $B_{u_1} = \langle s_{i_1}, \dots, s_{i_{k-1}} \rangle \leq A_{v_1}$, where $1 \leq i_1 < \dots < i_{k-1} \leq k$
- (S4) $B_{f_0} = B_{f_1} = 1$.

6.2.1 Decreasing the number of edges

Lemma 6.2.7. *Let \mathcal{B} be a tame \mathbb{A} -graph with base vertex u_0 of type v_0 . Suppose that there exist distinct edges $f, g \in EB$ with $\alpha(f) = \alpha(g) = x$ labeled (a, e, b) and (a', e, b') accordingly, such that $a' = b_x \cdot a \cdot \alpha_e(c)$ for some $b_x \in B_x$ and some $c \in A_e$.*

Then there exists a tame \mathbb{A} -graph \mathcal{B}' with base vertex u'_0 of type v_0 , having trivial edge groups, and an epimorphism $\sigma : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$ such that the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{B}, u_0) & & \\ \sigma \downarrow & \searrow \phi & \\ \pi_1(\mathbb{B}', u'_0) & \xrightarrow{\phi'} & \pi_1(\mathbb{A}, v_0) \end{array}$$

commutes. Moreover, $|EB'| = |EB| - 2$.

Proof. It follows from Lemma 6.2.2 that either $B_f = 1$ or $B_g = 1$. Suppose that the edge group B_g is trivial. Let \mathcal{B}_1 be the \mathbb{A} -graph obtained from \mathcal{B} by an auxiliary move of type A2 and an auxiliary move of type A1 so that the label of g is equal to $(a, e, \omega_e(c)b')$. Next let \mathcal{B}_2 be the \mathbb{A} -graph obtained from \mathcal{B}_1 by collapsing all edges with non-trivial group. In the collapsing process we might need to change the label of the edge f by an auxiliary move of type A2 in order to make the group $\alpha_f(B_f)$ a free factor of B_x , but in this situation we apply the same move to the edge g so that $f_\alpha = g_\alpha$. Note that there is an isomorphism $\sigma' : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}_2, u_0)$ such that $\phi = \phi_2 \circ \sigma'$, where $\phi_2 : \pi_1(\mathbb{B}_2, u_0) \rightarrow \pi_1(\mathbb{A}, v_0)$ is the canonical homomorphism induced by \mathcal{B}_2 .

Next apply an auxiliary move of type A0 to $\omega(f)$ or $\omega(g)$, depending on whether $\omega(f) \neq u_0$ or $\omega(g) \neq u_0$, that makes the labels of f and g coincide. Finally let \mathcal{B}' be the \mathbb{A} -graph obtained from \mathcal{B}_2 by folding the edges f and g into a single edge h . By Proposition 1.3.3 there is an epimorphism $\sigma'' : \pi_1(\mathbb{B}_2, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$ such that $\phi_2 = \phi' \circ \sigma''$, where u'_0 is the image of the vertex u_0 under the fold. Define $\sigma := \sigma'' \circ \sigma'$. We have $\phi' \circ \sigma = \phi$. All vertex groups of \mathcal{B}' are finitely generated. Moreover, conditions (T2) and (T3) are trivially verified since all edge groups in \mathcal{B}' are trivial. Hence \mathcal{B}' is tame. Clearly we have $|EB'| = |EB| - 2$. \square

6.2.2 Increasing the number of non-trivial edge groups

Let \mathcal{B} be a tame \mathbb{A} -graph. Assume that \mathcal{B} is not folded but condition (F1) from Definition 1.2.5 is verified. It follows from Lemma 6.2.2 that there exists an edge $f \in EB$ with initial vertex $x := \alpha(f)$, terminal vertex $y := \omega(f)$ and label $l(f) := (a, e, b)$ such that $B_f = 1$ and that $\langle c_f \rangle := \alpha_e^{-1}(a^{-1} \cdot B_x \cdot a) \neq 1$.

Denote by \mathcal{B}' the \mathbb{A} -graph obtained from \mathcal{B} by replacing the edge group $B_f = 1$ by $B'_f := \langle c_f \rangle \leq A_e$ and the vertex group B_y by $B'_y := \langle B_y, b^{-1} \cdot \omega_e(c_f) \cdot b \rangle \leq A_{[y]}$.

Lemma 6.2.8. *If $rk(B'_y) = rk(B_y) + 1$, then the \mathbb{A} -graph \mathcal{B}' is weakly tame.*

Proof. We need to show that the \mathbb{A} -graph \mathcal{B}' satisfies the three conditions from Definition 6.2.1. Condition (T1) is trivially verified. Note that we need to look at both vertices involved in the fold. Let us first look at the vertex $x = \alpha(f)$. Since $B'_f = \alpha_e^{-1}(a^{-1} B_x \cdot a) \cong (a \cdot \alpha_e(A_e) \cdot a^{-1}) \cap B_x$,

it follows that $a \cdot \alpha_e(B'_f) \cdot a^{-1}$ is a maximal peripheral subgroup of B_x , that is, condition (T2) is verified. If there exists an edge $g \in EB$ incident at x with $B_g \neq 1$ such that

$$b(g_\alpha \cdot \alpha_{[g]}(B_g) \cdot g_\alpha^{-1})b^{-1} \cap (a \cdot \alpha_{[f]}(B'_f) \cdot a^{-1}) \neq 1$$

for some $b \in B'_x = B_x$, then $[f] = [g] := e \in EA$ and $bg_\alpha \alpha_e(d)g_\alpha^{-1}b^{-1} = a\alpha_e(d)a^{-1}$ for some $d \in A_e$. The peripheral subgroup $\alpha_e(A_e) \leq A_{[x]}$ is malnormal, hence $bg_\alpha = a\alpha_e(z)$ for some $z \in A_e$ since the peripheral subgroup $\alpha_e(A_e)$ is malnormal in $A_{[x]}$. But this implies that \mathcal{B} violates condition (F1) from Definition 1.2.5. Therefore condition (T3) holds at the vertex x .

It remains to verify tameness at the vertex y . Let $f_1, \dots, f_r \in E_1B$ be the set of edges incident at x labeled (a_i, e_i, b_i) with $f_r := f^{-1}$. Denote by c_{f_i} the generator of B_{f_i} . Note that B_y is of infinite index in $A_{[y]}$ since otherwise we would have

$$(b^{-1} \cdot \omega_e(A_e) \cdot b) \cap B_y \neq 1$$

which implies that $rk(B'_y) < rk(B_y) + 1$. Thus after auxiliary moves of type A2 we can assume that $\{a_1\alpha_{e_1}(c_1)a_1^{-1}, \dots, a_{r-1}\alpha_{e_{r-1}}(c_{r-1})a_{r-1}^{-1}\}$ is part of a minimal generating set of B_y . Thus the set

$$\{a_1\alpha_{e_1}(c_1)a_1^{-1}, \dots, a_{r-1}\alpha_{e_{r-1}}(c_{r-1})a_{r-1}^{-1}, a_r\alpha_e(c_f) \cdot a_r^{-1}\}$$

is part of a minimal generating set of B'_y since $rk(B'_y) = rk(B_y) + 1$. From this observation we conclude that conditions (T2) and (T3) are verified at the vertex y . Therefore \mathcal{B}' is weakly tame. \square

Corollary 6.2.9. *Let \mathcal{B} be a π_1 -surjective tame \mathbb{A} -graph with base vertex u_0 of type v_0 . Then there exist a tame \mathbb{A} -graph \mathcal{B}' and an epimorphism $\sigma : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$ such that $\phi = \phi' \circ \sigma$ satisfying one of the following conditions:*

(i) \mathcal{B}' is collapsed and $|EB'| < |EB|$.

(ii) σ is an isomorphism and there exists an edge $f \in EB'$ with label $l(f) := (a, e, b)$ and $B_f = 1$ such that

$$\alpha_e^{-1}(a^{-1} \cdot B_{\alpha(f)} \cdot a) = \langle c_f \rangle \neq 1$$

and $rk(\langle B_{\omega(f)}, b^{-1}\omega_e(c_f)b \rangle) < rk(B_{\omega(f)}) + 1$.

(iii) σ is an isomorphism and \mathcal{B}' is standard with $|EB'| = |EB|$.

Proof. Put $\mathcal{B}_0 := \mathcal{B}$. Suppose that the tame \mathbb{A} -graph \mathcal{B}_i , and the epimorphism $\sigma_i : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}_i, u_i)$ such that $\phi = \phi_i \circ \sigma_i$ has been defined. We define \mathcal{B}_{i+1} as follows. If \mathcal{B}_i does not satisfy condition (F1) from the definition of folded \mathbb{A} -graphs, then we apply Lemma 6.2.7 to obtain a tame \mathbb{A} -graph \mathcal{B}_{i+1} having trivial edge groups and strictly smaller complexity and an

epimorphism $\sigma_i : \pi_1(\mathbb{B}_i, u_i) \rightarrow \pi_1(\mathbb{B}_{i+1}, u_{i+1})$ such that $\phi_i = \phi_{i+1} \circ \sigma_i$. If it is not possible to reduce the number of edges of \mathcal{B}_i , then Lemma 6.2.2 implies that there exists an edge f with $B_f = 1$ such that

$$\langle c_f \rangle = \alpha_{[f]}^{-1}(f_\alpha^{-1} B_{\alpha(f)} f_\alpha) \neq 1.$$

Let $\overline{\mathcal{B}}$ be the \mathbb{A} -graph obtained from \mathcal{B}_i by replacing the edge group $B_f^i = B_f = 1$ by the group $\overline{B}_f := \langle c_f \rangle$ and the vertex group $B_{\omega(f)}^i$ by the group

$$\overline{B}_{\omega(f)} := \langle B_{\omega(f)}^i, f_\omega^{-1} \omega_{[f]}(c_f) f_\omega \rangle.$$

If $rk(\overline{B}_{\omega(f)}) < rk(B_{\omega(f)}^i) + 1$ then we put $\mathcal{B}' = \mathcal{B}_i$. If $rk(\overline{B}_{\omega(f)}) = rk(B_{\omega(f)}^i) + 1$ and $\overline{\mathcal{B}}$ is not tame (it is weakly tame) then we see that $\overline{\mathcal{B}}$ is a finite cover of \mathbb{A} . As we further know that \mathcal{B} , and thus $\overline{\mathcal{B}}$ is π_1 -surjective, it implies that the graph of groups $\overline{\mathbb{B}}$ determined by $\overline{\mathcal{B}}$ is equivalent to \mathbb{A} . Thus the \mathbb{A} -graph \mathcal{B}' obtained from \mathcal{B}_i by collapsing the edge with non-trivial group is standard. If $\overline{\mathcal{B}}$ is collapsible, then we put $\mathcal{B}_{i+1} := \overline{\mathcal{B}}$. Since $|E_1 \mathcal{B}_i| < |E_1 \mathcal{B}_{i+1}| \leq |EB|$ it follows that this process terminates at some point. \square

6.2.3 The singular case

Definition 6.2.10. Following Louder [20], we say that a tame \mathbb{A} -graph \mathcal{B} has an *obvious relation* if there exists an edge $f \in EB$ with $B_f = 1$ such that $\alpha_{[f]}^{-1}(f_\alpha^{-1} B_{\alpha(f)} f_\alpha) \neq 1 \neq \omega_{[f]}^{-1}(f_\omega B_{\omega(f)} f_\omega^{-1})$.

Lemma 6.2.11. *Let \mathcal{B} be a π_1 -surjective tame \mathbb{A} -graph with base vertex u_0 of type v_0 . Suppose that there exists an edge $f \in EB$ with label $l(f) := (a, e, b)$, initial vertex $\alpha(f) := x$, terminal vertex $\omega(f) := y$ and $B_f = 1$ such that the following hold:*

1. $\langle c_f \rangle := \alpha_e^{-1}(a^{-1} \cdot B_x \cdot a) \neq 1$.
2. $rk(\langle B_y, b^{-1} \cdot \omega_e(c_f) \cdot b \rangle) < rk(B_y) + 1$.

Then there exists a tame \mathbb{A} -graph \mathcal{B}' with base vertex u'_0 of type v_0 and an epimorphism $\sigma : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$ such that $\phi = \phi' \circ \sigma$. Furthermore, one of the following holds:

- (i) *All edge groups of \mathcal{B}' are trivial and $|EB'| < |EB|$.*
- (ii) *\mathcal{B}' has an obvious relation and $|EB'| = |EB|$.*
- (iii) *Either $rk(\mathcal{B}') < rk(\mathcal{B})$ or $rk(\mathcal{B}') = rk(\mathcal{B})$ and $tor(\mathcal{B}') > tor(\mathcal{B})$.*
- (iv) *σ is an isomorphism and $\pi_1(\mathbb{B}', u'_0) = H_0 * H_1$ for $H_0, H_1 \leq \pi_1(\mathbb{B}', u'_0)$ such that H_1 is finite cyclic, $\phi'(H_1) = \langle g s_i^z g^{-1} \rangle$ ($g \in \pi_1(\mathbb{A}, v_0)$, $i \in \{1, \dots, k\}$, $z \in \mathbb{Z} \setminus \{0\}$) and $g s_i^{z'} g^{-1} \in \phi'(H_0)$ for some $z' \in \mathbb{Z}$ such that $|\langle s_i \rangle : \langle s_i^{z'} \rangle| < |\langle s_i \rangle : \langle s_i^z \rangle|$.*

Proof. We can assume that B_y has infinite index in $A_{[y]}$ since otherwise \mathcal{B} has an obvious relation. Let $f_1, \dots, f_{l-1} \in E_1 B$ labeled $(a_1, e_1, b_1), \dots, (a_l, e_l, b_l)$ such that $\omega(f_i) = y$. After auxiliary moves of type A2 appropriately applied to the edges f_1, \dots, f_l , we can assume that

$$B_y = \Gamma_0 * \Gamma_1 * \dots * \Gamma_{l-1}$$

where $\Gamma_0 \leq B_y$ and $\Gamma_i = \alpha_{f_i}(B_{f_i}) = b_i^{-1} \omega_{e_i}(B_{f_i}) b_i \leq B_y$ for all $1 \leq i \leq l-1$. Denote the peripheral subgroup $\langle b^{-1} \omega_e(c_f) b \rangle$ by Γ_l .

Now we will construct a graph of groups which is not induced by an \mathbb{A} -graph. This graph of groups will serve as an intermediate object to obtain the \mathbb{A} -graph \mathcal{B}' . Denote by \mathbb{B}_1 the graph of groups obtained from $\mathbb{B} = \mathbb{B}(\mathcal{B})$ by replacing the edge group $B_f = 1$ by the group $\langle c_f \rangle \leq A_e$ and the vertex group B_y by the group

$$\Gamma := B_y * \langle \langle b^{-1} \omega_e(c_f) b \rangle \rangle = \Gamma_0 * \Gamma_1 * \dots * \Gamma_l.$$

There are obvious morphisms $\sigma_1 : \mathbb{B} \rightarrow \mathbb{B}_1$ and $\phi_1 : \mathbb{B}_1 \rightarrow \mathbb{A}$ such that the induced homomorphism $\sigma_1 : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}_1, u_0)$ is an isomorphism and that the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{B}, u_0) & & \\ \sigma_1 \downarrow \cong & \searrow \phi & \\ \pi_1(\mathbb{B}_1, u_0) & \xrightarrow{\phi_1} & \pi_1(\mathbb{A}, v_0) \end{array}$$

commutes. Denote by $\eta : \Gamma \rightarrow A_{[y]}$ the homomorphism induced by the inclusion maps $\Gamma_i \hookrightarrow A_{[y]}$. Since $rk(\langle \langle B_y, b^{-1} \omega_e(c_f) b \rangle \rangle) < rk(B_y) + 1$ it follows that $rk(\eta(\Gamma)) < rk(\Gamma)$. Lemma 6.1.6 implies that

$$\Gamma = \Gamma'_0 * \Gamma'_1 * \dots * \Gamma'_l$$

where $\Gamma'_0 \leq \Gamma$ and $\Gamma'_i = g_i^{-1} \Gamma_i g_i$ ($g_i \in \Gamma$, $1 \leq i \leq l$). Note that the image of Γ'_i under the homomorphism $\eta : \Gamma \rightarrow A_v$ is equal to the peripheral subgroup $\langle \eta(g_i)^{-1} b_i^{-1} \omega_{e_i}(c_{f_i}) b_i \eta(g_i) \rangle \leq A_{[y]}$. Moreover, one of the following holds:

- (a) There exist $i \neq j \in \{1, \dots, l\}$ such that $\eta(\Gamma'_i) \cap \eta(\Gamma'_j) \neq 1$.
- (b) There exists $i \in \{1, \dots, l\}$ such that $\eta(\Gamma'_i) \cap \eta(\Gamma') \neq 1$, where

$$\Gamma' := \Gamma'_0 * \Gamma'_1 \dots * \Gamma'_{i-1} * \Gamma'_{i+1} * \dots * \Gamma'_l \leq \Gamma.$$

- (c) There exists an epimorphism $\tau : \Gamma \twoheadrightarrow \bar{\Gamma}$ and a homomorphism $\bar{\eta} : \bar{\Gamma} \rightarrow A_v$ such that $\bar{\eta} \circ \tau = \eta$. Moreover,

$$\bar{\Gamma} = \bar{\Gamma}_0 * \bar{\Gamma}_1 * \dots * \bar{\Gamma}_l$$

where $\bar{\Gamma}_i$ is conjugate to $\sigma(\Gamma_i)$ for $i = 1, \dots, l$ and either $rk(\bar{\Gamma}) < rk(\Gamma)$ or $tor(\Gamma) < tor(\bar{\Gamma})$.

(d) $\Gamma'_0 = \Gamma''_0 * \langle as_w^z a^{-1} \rangle$ for some $w \in VA^v$, $z \in \mathbb{Z} - \{0\}$ and $a \in A_v$ and $as_w^{z'} a^{-1} \in \eta(\Gamma'')$ for some $z' \in \mathbb{Z}$ such that $|\langle s_w \rangle : \langle s_w^{z'} \rangle| < |\langle s_w \rangle : \langle s_w^z \rangle|$, where

$$\Gamma'' := \Gamma''_0 * \Gamma'_1 * \dots * \Gamma'_l \leq \Gamma.$$

For each $1 \leq i \leq l$ denote by $\alpha'_{f_i} : B_{f_i} \rightarrow \Gamma$ the monomorphism $x \mapsto g_i \alpha_{f_i}(x) g_i^{-1}$. Let further \mathbb{B}_2 be the graph of groups obtained from \mathbb{B}_1 by replacing the boundary monomorphisms $\alpha_{f_1}, \dots, \alpha_{f_r}$ by the monomorphisms $\alpha'_{f_1}, \dots, \alpha'_{f_l}$. There are obvious morphisms $\sigma_2 : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ and $\phi_2 : \mathbb{B}_2 \rightarrow \mathbb{A}$ such that σ_2 induces an isomorphism $\pi_1(\mathbb{B}_1, u_0) \rightarrow \pi_1(\mathbb{B}_2, u_0)$ and $\phi_2 \circ \sigma_2 = \phi_1$.

Case (1): Suppose that (a) holds. Since $\Gamma'_i \cap \Gamma'_j \neq 1$ it follows from the malnormality of the peripheral subgroups of $A_{[y]}$ that $e_i = e_j$ and that $\eta(g_j) b_j^{-1} = \eta(g_i) b_i^{-1} \omega_e(c)$ for some $c \in A_e$. Now denote by \mathbb{B}_3 the graph of groups obtained from \mathbb{B}_2 by collapsing all maps $\alpha'_{f_1}, \dots, \alpha'_{f_l}$, i.e. we replace the vertex group Γ by the group Γ'_0 and each edge group B_{f_i} by the trivial group. The graph of groups \mathbb{B}_3 is induced by a tame \mathbb{A} -graph \mathcal{B}_3 such that $(f_i)_\omega = \eta(g_i) b_i^{-1}$ for all $i = 1, \dots, l$. Since $(f_i)_\omega = (f_i)_\omega \omega_e(c)$ it follows from Lemma 6.2.7 that we obtain a collapsed tame \mathbb{A} -graph \mathcal{B}' with $|EB'| < |EB|$.

Case (b): Suppose now that (b) occurs. Let \mathcal{B}_2 be the graph of groups obtained from \mathbb{B}_1 by collapsing the boundary monomorphism $\alpha'_{f_i} : B_{f_i} \rightarrow \Gamma$. Denote by η' the homomorphism $\eta|_{\Gamma'} : \Gamma' \rightarrow A_{[y]}$. If $rk(\eta'(\Gamma')) = \eta'(\Gamma')$ then we define the \mathbb{A} -graph \mathcal{B}' by replacing the vertex group Γ' of \mathbb{B}_2 by the group $\eta'(\Gamma') \leq A_{[y]}$. Thus \mathcal{B}' has an obvious relation. Suppose that $rk(\eta'(\Gamma')) < rk(\eta')$. Let \mathcal{B}' be the \mathbb{A} -graph obtained from the graph of groups \mathbb{B}_2 by collapsing all edge groups so that the vertex group of \mathcal{B}' at y is equal to $\eta'(\Gamma')$. Thus $rk(\mathcal{B}') < rk(\mathcal{B})$ and so we obtain (iii).

Case (c): Let \mathbb{B}_2 be the graph of groups obtained from \mathbb{B}_1 by replacing the vertex group Γ by the group $\bar{\Gamma}$. The boundary monomorphisms are defined as $\tau \circ \alpha'_{f_i} : B_{f_i} \rightarrow \bar{\Gamma}$. Since each free factor $\bar{\Gamma}_i$ ($1 \leq i \leq l$) is conjugate to $\tau(\Gamma'_i) = \tau(\alpha'_{f_i}(B_{f_i}))$ it follows that, after conjugating each map $\tau \circ \alpha'_{f_i}$ appropriately, we can assume that $\tau \circ \alpha'_{f_i}(B_{f_i}) = \bar{\Gamma}_i$. Since Γ'_i and $\bar{\Gamma}_i$ are infinite cyclic for $1 \leq i \leq l$ it follows that $tor(\Gamma) = tor(\Gamma'_0)$ and $tor(\bar{\Gamma}) = tor(\bar{\Gamma}_0)$. Let now \mathbb{B}' be the graph of groups obtained from \mathbb{B}_2 by replacing the edge groups B_{f_1}, \dots, B_{f_r} by the trivial group and the vertex group $\bar{\Gamma}$ by the group $\bar{\Gamma}_0$. Then it is easily verified that \mathbb{B}' is induced by an \mathbb{A} -graph \mathcal{B}' . Therefore we have either $rk(\mathcal{B}') < rk(\mathcal{B})$ if $rk(\bar{\Gamma}) < rk(\Gamma)$ or $tor(\mathcal{B}') > tor(\mathcal{B})$ if $tor(\bar{\Gamma}) > tor(\Gamma)$.

Case (d): In this case define the \mathbb{A} -graph \mathcal{B}' from the graph of groups \mathbb{B}_1 by replacing the edge groups B_{f_1}, \dots, B_{f_l} by the trivial group and the vertex group Γ by the group $\Gamma'_0 = \Gamma''_0 * \langle as_w a^{-1} \rangle$. Thus $\pi_1(\mathbb{B}', u_0) = H_0 * H_1$ where H_1 is isomorphic to $\langle as_w a^{-1} \rangle$ and H_0 is the fundamental group of the graph of groups induced by the \mathbb{A} -graph \mathcal{B}'' obtained from \mathcal{B}' by replacing the vertex

group Γ'_0 by the group $\Gamma''_0 \leq \Gamma'_0$. Note that $\phi(H_1) = gs_i^z g^{-1}$ and (d) implies that $gs_i^{z'} g^{-1} \in \phi(H_0)$ for some z' such that $|\langle s_i \rangle : \langle s_i^{z'} \rangle| < |\langle s_i \rangle : \langle s_i^z \rangle|$. \square

6.3 Proof of Theorem 6.0.5

Let \mathcal{T} be an irreducible m -tuple of elements of Γ . We define δ_1 -complexity of \mathcal{T} as

$$\delta_1(\mathcal{T}) := \text{size}(\mathcal{T}) - \text{size}(\mathcal{T}_{\text{tor}})$$

where \mathcal{T}_{tor} is the subtuple of \mathcal{T} consisting of all elements $t \in \mathcal{T}$ of finite order. We further define the δ_2 -complexity of an irreducible tuple \mathcal{T} by

$$\delta_2(\mathcal{T}) := \sum_{t \in \mathcal{T}_{\text{tor}}} |\Gamma(t) : \langle t \rangle|$$

where $\Gamma(t) \leq \Gamma$ is the unique maximal finite subgroup of Γ containing the element $t \in \mathcal{T}$ and $|\Gamma(t) : \langle t \rangle|$ is the index of $\langle t \rangle$ in $\Gamma(t)$. The δ -complexity of \mathcal{T} is defined as the lexicographically ordered pair $\delta(\mathcal{T}) := (\delta_1(\mathcal{T}), \delta_2(\mathcal{T}))$. For example, if $\mathcal{T} = (a_1, \dots, a_p, s_{i_1}^{\nu_1}, \dots, s_{i_{k-1}}^{\nu_{k-1}})$ is a standard generating tuple then we have $d(\mathcal{T}) = (p, k - 1)$.

Now let \mathcal{T} be an irreducible generating tuple of the planar group Γ . Let $\Omega(\mathcal{T})$ be the set of all pairs $(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$, where \mathcal{B} is a tame \mathbb{A} -graph and $\mathcal{T}_{\mathcal{B}}$ is an irreducible, and therefore minimal, generating tuple of $\pi_1(\mathbb{B}, u_0)$ such that the image $\phi(\mathcal{T}_{\mathcal{B}})$ of the tuple $\mathcal{T}_{\mathcal{B}}$ under the canonical homomorphism $\phi : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{A}, v_0) = \Gamma$ is Nielsen equivalent to \mathcal{T} .

Lemma 6.3.1. *The set $\Omega(\mathcal{T})$ is non-empty.*

Proof. Suppose that $\mathcal{T} = (g_1, \dots, g_m)$. We will construct a tame \mathbb{A} -graph \mathcal{B}_0 such that the fundamental group of the induced graph of groups $\mathbb{B}_0 = \mathbb{B}(\mathcal{B}_0)$ is free of rank $m = \text{size}(\mathcal{T})$ with free generating set $\{x_1, \dots, x_m\}$ such that $\phi_0(x_i) = g_i$ for all $1 \leq i \leq m$. For each $1 \leq i \leq m$ the element $g_i \in \pi_1(\mathbb{A}, v_0)$ is represented by an \mathbb{A} -path

$$p_i = a_0^i, e_1^i, a_1^i, \dots, a_{k_i-1}^i, e_{k_i}^i, a_{k_i}^i$$

from v_0 to v_0 of length $|p_i| = k_i \geq 1$, note that we do not require p_i to be reduced.

We define the \mathbb{A} -graph \mathcal{B}_0 as follows. The underlying graph B_0 has base vertex called u_0 of type v_0 . For each $1 \leq i \leq m$ attach at the vertex u_0 a circuit $f_1^i, \dots, f_{k_i}^i$ of length k_i . We give the first $k_i - 1$ edges $f_1^i, \dots, f_{k_i}^i$ the label

$$(a_0^i, e_1^i, 1), \dots, (a_{k_i-2}^i, e_{k_i-1}^i, 1)$$

accordingly. We label the last edge of the circle by $(a_{k_i-1}^i, e_{k_i}^i, a_{k_i}^i)$. For each $x \in VB_0 \cup EB_0$ the associated group is trivial. Thus we have completely described the \mathbb{A} -graph \mathcal{B}_0 . Tameness of \mathcal{B}_0 is trivially verified since all vertex and edge groups are trivial.

For each $1 \leq i \leq m$ let q_i be the \mathbb{B}_0 -path $1, f_1^i, 1, \dots, f_{k_i}^i, 1$. and x_i the element of $\pi_1(\mathbb{B}_0, u_0)$ represented by q_i . Thus $\pi_1(\mathbb{B}_0, u_0)$ is freely generated by the set $\{x_1, \dots, x_m\}$ and the image of each free generator x_i under the canonical homomorphism

$$\phi_0 : \pi_1(\mathbb{B}_0, u_0) \rightarrow \pi_1(\mathbb{A}, v)$$

is equal to g_i since $\mu(q_i) = \mu(1, f_1^i, 1, \dots, f_{k_i}^i, 1) = a_0^i, e_1^i, a_1^i, \dots, e_{k_i}^i, a_{k_i}^i$. Thus $\mathcal{T}_{\mathcal{B}_0} = (x_1, \dots, x_m)$ is a minimal generating tuple of $\pi_1(\mathbb{B}_0, u_0)$ and $\phi_0(\mathcal{T}_{\mathcal{B}_0})$ is equal to \mathcal{T} . Consequently $(\mathcal{B}_0, \mathcal{T}_{\mathcal{B}_0})$ lies in the set $\Omega(\mathcal{T})$. \square

Definition 6.3.2. We define the c -complexity of $(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) \in \Omega(\mathcal{T})$ as the lexicographically ordered pair

$$c(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) := (|EB|, \delta(\phi(\mathcal{T}_{\mathcal{B}}))) \in \mathbb{N} \times \mathbb{N}$$

where $\delta(\phi(\mathcal{T}_{\mathcal{B}}))$ is the complexity of the irreducible tuple $\phi(\mathcal{T}_{\mathcal{B}})$.

Now choose $(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) \in \Omega(\mathcal{T})$ of smaller c -complexity, i.e if $(\mathcal{B}', \mathcal{T}_{\mathcal{B}'}) \in \Omega(\mathcal{T})$ then wither $|EB'| > |EB|$ or $|EB'| = |EB|$ and $\delta(\phi(\mathcal{T}_{\mathcal{B}'})) > \delta(\phi(\mathcal{T}_{\mathcal{B}}))$. From Corollary 6.2.9 and Lemma 6.2.11 it follows that there exists a tame \mathbb{A} -graph \mathcal{B}' and an epimorphism $\sigma : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$ such that $\phi' \circ \sigma = \phi$ and one of the following holds:

- (1) σ is an isomorphism and \mathcal{B}' is standard.
- (2) \mathcal{B}' has an obvious relation.
- (3) $\text{tor}(\mathcal{B}') > \text{tor}(\mathcal{B})$.
- (4) σ is an isomorphism and $\pi_1(\mathbb{B}', u'_0) = H_0 * H_1$ for $H_0, H_1 \leq \pi_1(\mathbb{B}', u'_0)$ such that H_1 is finite cyclic, $\phi'(H_1) = \langle gs_i^z g^{-1} \rangle$ ($g \in \pi_1(\mathbb{A}, v_0)$, $i \in \{1, \dots, k\}$, $z \in \mathbb{Z} \setminus \{0\}$) and that $gs_i^{z'} g^{-1} \in \phi'(H_0)$ for some z' such that $|\langle s_i \rangle : \langle s_i^{z'} \rangle| < |\langle s_i \rangle : \langle s_i^z \rangle|$.

Suppose that (1) holds. By Grushko's theorem 6.0.3 the tuple $\sigma(\mathcal{T}_{\mathcal{B}})$ is Nielsen equivalent to the tuple $\mathcal{T}_{\mathcal{B}'} = \mathcal{T}_{u_0} \oplus (s_{i_1}^{v_i}, \dots, s_{i_{k-1}}^{v_{k-1}}) \oplus (t)^1$, where $t \in \pi_1(\mathbb{B}', u_0)$ is represented by the \mathbb{B}' -path $1, f_1, 1, f_2^{-1}, 1$ and \mathcal{T}_{u_0} is a generating tuple of the group $B_{u_0} = A_{v_0}$. Therefore $\phi'(\mathcal{T}_{\mathcal{B}'})$ is equivalent to a standard generating tuple and by transitivity we conclude that \mathcal{T} is Nielsen equivalent to a standard generating tuple.

We will show that (3) and (4) cannot occur and that if (2) occurs then $k = 0$ (that is, Γ is the fundamental group of a closed surface) and consequently, \mathcal{B}' is standard.

¹Recall that the sum of the tuples $\mathcal{T} = (t_1, \dots, t_k)$ and $\mathcal{T}' = (t'_1, \dots, t'_l)$ is the tuple $(t_1, \dots, t_k, t'_1, \dots, t'_l)$

Suppose that (3) occurs. Then $(\mathcal{B}', \sigma(\mathcal{T}_{\mathcal{B}}))$ lies on $\Omega(\mathcal{T})$ since $\phi'(\sigma(\mathcal{T}_{\mathcal{B}})) = \phi(\mathcal{T}_{\mathcal{B}})$ is Nielsen equivalent to \mathcal{T} . Since $\text{tor}(\mathcal{B}') > \text{tor}(\mathcal{B})$ it follows that $\sigma(\mathcal{T}_{\mathcal{B}})$ is Nielsen equivalent to a generating tuple $\mathcal{T}_{\mathcal{B}'}$ of $\pi_1(\mathbb{B}', u_0)$ having one element of finite order more than \mathcal{T} . Thus $\delta(\mathcal{T}_{\mathcal{B}}) > \delta(\phi(\mathcal{T}_{\mathcal{B}'}))$ a contradiction to the minimality assumption of $(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$.

Suppose now that (4) occurs. Thus $\sigma(\mathcal{T}_{\mathcal{B}}) = \mathcal{T}_0 \oplus \mathcal{T}_1$ with $\langle \mathcal{T}_i \rangle = H_i$ for $i = 1, 2$. Since $gs_i^{z'}g^{-1} \in \phi'(H_1) = \langle gs_i^z g^{-1} \rangle$ and the index of the subgroup $\langle s_i^{z'} \rangle$ in $\langle s_i \rangle$ is smaller than the index of $\langle s_i^z \rangle$ in $\langle s_i \rangle$, it follows that $\sigma(\mathcal{T}_{\mathcal{B}})$ is Nielsen equivalent to a generating tuple $\mathcal{T}_{\mathcal{B}'}$ with strictly smaller δ_2 -complexity which implies that $c(\mathcal{B}', \mathcal{T}_{\mathcal{B}'}) < c(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$.

Now assume that (2) occurs. Thus there exists an edge $f \in EB'$ with $x = \alpha(f)$ of type $v \in VA$ and $y = \omega(f)$ of type $w \in VA$ labeled (a, e, b) such that $\langle c_f \rangle = \alpha_e^{-1}(a^{-1}B_x a) \neq 1$ and $\langle d_f \rangle = \omega_e^{-1}(bB_y b^{-1}) \neq 1$.

Lemma 6.3.3. *The group B_x has finite index in A_v . Moreover, for each maximal peripheral subgroup P of B_x that is not conjugate (in B_x) to $\langle a\alpha_e(c_f)a^{-1} \rangle$ there exists an edge $g \in E_1B'$ with $\alpha(g) = x$ such that $\alpha_g(B_g)$ is conjugate to P .*

Proof. In fact, if these does not hold then $B_x = B_x * \langle a\alpha_e(c_f)a^{-1} \rangle$. Then we define the tame \mathbb{A} -graph \mathcal{B}'' from \mathcal{B}' by replacing the vertex group B_x by B'_x and B_y by $\langle B_y, b^{-1}\omega_e(c_f)b \rangle \leq A_w$. There is an epimorphism $\sigma' : \pi_1(\mathbb{B}', u_0) \rightarrow \pi_1(\mathbb{B}'', u_0)$ such that $\phi'' \circ \sigma' = \phi'$. Since $rk(\mathcal{B}'') < rk(\mathcal{B}')$ it follows that the tuple $\sigma(\mathcal{T}_{\mathcal{B}'})$ is reducible which implies that \mathcal{T} is reducible, a contradiction. \square

With an entirely analogous argument we conclude that B_y has finite index in A_w and that for each maximal peripheral subgroup Q of B_y there exists an edge $g \in E_1B'$ with $\omega_e(g) = y$ such that $\omega_e(B_g)$ is conjugate (in B_y) to Q .

Lemma 6.3.4. *If $z \in VB' \setminus \{x, y\}$ then z is full.*

Proof. Let $z \in VB' \setminus \{x, y\}$ such that there exists a path $p = f_1, \dots, f_k$ with initial vertex $\alpha(p) \in \{x, y\}$ and $B_{f_i} \neq 1$ for all $i = 1, \dots, k$. Then the vertex z is full. For if z is not full, then z is collapsible since $\omega(f_k) = z$ and $B_{f_k} \neq 1$. By collapsing along the path $p = f_1, \dots, f_k$ we obtain a tame \mathbb{A} -graph \mathcal{B}'' with an obvious relation such that $\langle a\alpha_e(c_f)a^{-1} \rangle$ (or $\langle b^{-1}\omega_e(d_f)b \rangle$ if $\alpha(p) = y$) is a free factor of B_x (B_y). But, as seen previously, this implies that the tuple \mathcal{T} is reducible. Since B_x and B_y are of finite index in A_v and A_w respectively and all edges $g \in EB' \setminus \{f, f^{-1}\}$ with $\alpha(g) = x$ or $\alpha(g) = y$ have nontrivial group it follows that any vertex $z \neq x, y$ is connected to x or y by such a path. \square

Lemma 6.3.5. *We have $c_f = d_f$.*

Proof. Suppose that $c_f < d_f$. Let $g \in E_1B$ be an edge incident at y distinct from f . Note that such an edge exists since otherwise the underlying graph A of \mathbb{A} would have a separating edge.

The vertex y is collapsible. Let \mathcal{B}'' be the \mathbb{A} -graph obtained from \mathcal{B}' by collapsing the edge g from y . Thus \mathcal{B}'' is a tame \mathbb{A} -graph with fundamental group isomorphic to $\pi_1(\mathbb{B}', u)$. Now it is not hard to see that $rk(\langle B'_y, f_\omega^{-1} \cdot \omega_{[f]}(c_f) \cdot f_\omega \rangle) < rk(B_y)$, where B'_y is the vertex group of \mathcal{B}' at y . Therefore, after collapsing we find a tame \mathbb{A} -graph \mathcal{B}''' with smaller rank which implies that \mathcal{T} is reducible. \square

Denote by \mathcal{B}'' the \mathbb{A} -graph obtained from \mathcal{B} by a fold of type IIA. Hence the edge group $B_f = 1$ is replaced by $\langle c_f \rangle$ and the vertex group B_y is replaced by $B'_y = \langle B_y, f_\omega^{-1} \cdot \omega_{[f]}(c_f) \cdot f_\omega \rangle = B_y$. The \mathbb{A} -graph \mathcal{B}'' is weakly tame. Since all its vertices are full we conclude that \mathcal{B}'' is a covering of \mathbb{A} . Since \mathcal{B}'' is π_1 -surjective it follows \mathbb{B}'' is equivalent to \mathbb{A} .

Thus the underlying graph of \mathcal{B}' is isomorphic to A . Suppose that $y \in VB$ has type $v_1 \in VA$ and $g \in E_1B$ with $\omega(g) = y$. From our previous discussion we see that $B_y = A_{v_1}$. Now, after collapsing the edge g from y we obtain a tame \mathbb{A} -graph \mathcal{B}'' having vertex group $B_x = A_{v_0}$ and $B'_y = \langle s_1, \dots, s_k \rangle$. If $k > 0$ then it is easy to see that the tuple $\phi''(\mathcal{T}_{\mathcal{B}''})$ is reducible. Hence $k = 0$ and \mathcal{B}'' is standard which implies that \mathcal{T} is Nielsen equivalent to a standard generating tuple.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Einige der Hauptergebnisse der Dissertation sind in folgenden Publikationen enthalten:

- M. Boileau, E. Dutra, Y. Jang, R. Weidmann, *Meridional rank of knots whose exterior is a graph manifold*, *Topology and its Applications* **228** (2017)458-485.
- E. Dutra, *On killers of cable knot groups*, *Bull. of the Aust. Math. Soc.* **96** (2017), 171-176.
- E. Dutra, *Nielsen equivalence in Planar discontinuous groups*, in preparation.

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(Ederson Ricardo Fruhling Dutra)