

Hausdorff measure of escaping sets of certain meromorphic functions

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Abstract

Barański and Schubert showed that for a meromorphic function $f \in \mathcal{B}$ of finite order and for which ∞ is an asymptotic value, the Hausdorff dimension of its escaping set, denoted by $\dim I(f)$, is two. Assume that for $f \in \mathcal{B}$ the value ∞ is not asymptotic and that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles, except finitely many, is at most M . Bergweiler and Kotus proved that for such a function, the Hausdorff dimension of its escaping set is bounded by $2M\rho/(2 + M\rho)$, where ρ is the order of f . If f is chosen as above except that the order is infinite, then the area of $I(f)$ is zero, while an example of such a function f with $\dim I(f) = 2$ was given. In 1987 McMullen proved that the Julia set of the exponential function λe^z with $\lambda \neq 0$ has Hausdorff dimension two but, in the presence of an attracting periodic cycle, its area is zero. He further remarked that $\mu_h(J(\lambda e^z)) = \infty$ for $h(r) = r^2 \log^n(1/r)$, for arbitrary $n \in \mathbb{N}$, where μ_h is the Hausdorff measure induced by h . Peter gave in his dissertation a fairly precise description of the gauge functions h for which $\mu_h(J(\lambda e^z)) = \infty$.

We aim in this thesis to find a gauge function for which the Hausdorff measure of $I(f)$ is 0 or ∞ for certain meromorphic functions in \mathcal{B} of infinite order. We shall use the n -th order $\rho_n(f)$ as a further description of the growth rate. Set $h(t) = t^2(\log^n 1/t)^\gamma$, where $t \in (0, \delta_n]$, $\delta_n = \min\{1/\exp^n(2), 1/\exp^n(\gamma)\}$, for $n \in \mathbb{N}$ and $\gamma > 0$.

We show the following result: Let $f \in \mathcal{B}$ be a meromorphic function of finite n -th order ρ . Suppose that ∞ is not an asymptotic value of f and that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles of f , except possibly finitely many, is at most M . If h is given as above and $\gamma < 2/(M\rho)$, then we have $\mu_h(I(f)) = 0$.

The bound $2/(M\rho)$ is probably not sharp. However, it cannot be replaced by any

value greater than $8/(M\rho)$. Let $0 < \rho < \infty$ and $n, M \in \mathbb{N}$. We show that there exists a meromorphic function $f \in \mathcal{B}$ of n -th order $\rho_n(f) = \rho$ for which all poles have multiplicity M and ∞ is not an asymptotic value such that if h is given as above and $\gamma > 8/(M\rho)$, then $\mu_h(I(f)) = \infty$.

Zusammenfassung

Barański and Schubert zeigten, dass für eine meromorphe Funktion in \mathcal{B} von endlicher Ordnung, welche ∞ als asymptotischen Wert besitzt, die Hausdorffdimension der entkommenden Menge, welche wir mit $\dim(I(f))$ bezeichnen, gleich 2 ist. Es sei $f \in \mathcal{B}$ so, dass ∞ nicht asymptotischer Wert ist und dass ein $M \in \mathbb{N}$ existiert, sodass die Vielfachheit aller Pole, bis auf endliche viele, höchstens M ist. Bergweiler und Kotus haben für so eine Funktion gezeigt, dass die Hausdorffdimension der entkommenden Menge durch $2M\rho/(2+M\rho)$ beschränkt ist, wobei ρ die Ordnung von f ist. Ist f wie oben mit unendlicher Ordnung, so hat $I(f)$ Flächeninhalt 0, während ein Beispiel einer solchen Funktion f mit $\dim(I(f)) = 2$ gegeben wurde. Im Jahr 1987 hat McMullen gezeigt, dass die Juliamenge der Exponentialfunktion λe^z mit $\lambda \neq 0$ Hausdorffdimension 2 hat, jedoch in Anwesenheit eines anziehenden periodischen Zyklus die Flächeninhalt 0 ist. Weiterhin bemerkte er, dass $\mu_h(J(\lambda e^z)) = \infty$ mit $h(r) = r^2 \log^n(1/r)$ für alle $n \in \mathbb{N}$ ist, wobei μ_h das Hausdorffmaß bezüglich h beschreibt. Peter gab eine präzise Beschreibung der Eichfunktionen h an, für die $\mu_h(J(\lambda e^z)) = \infty$ gilt.

Ziel dieser Arbeit ist es eine Eichfunktion zu finden, für welche das Hausdorffmaß von $I(f)$ entweder 0 oder ∞ für gewisse meromorphe Funktionen $f \in \mathcal{B}$ unendlicher Ordnung ist. Für die Beschreibung der Wachstumsrate verwenden wir die n -te Ordnung $\rho_n(f)$. Wir setzen $h(t) = t^2(\log^n 1/t)^\gamma$, wobei $t \in (0, \delta_n]$, $\delta_n = \min\{1/\exp^n(2), 1/\exp^n(\gamma)\}$ sind, für $n \in \mathbb{N}$ und $\gamma > 0$.

Wir zeigen das folgende Resultat: Sei $f \in \mathcal{B}$ eine meromorphe Funktion endlicher n -Ordnung ρ . Weiter sei ∞ kein asymptotischer Wert von f und es existiere $M \in \mathbb{N}$, sodass die Vielfachheit aller Pole, bis auf endliche viele, höchstens M ist. Ist h wie oben mit $\gamma < 2/(M\rho)$, so gilt $\mu_h(I(f)) = 0$.

Die Schranke $2/(M\rho)$ ist vermutlich nicht optimal. Jedoch kann gezeigt werden, dass sie nicht durch einen Wert größer als $8/(M\rho)$ ersetzt werden kann. Sei $0 < \rho < \infty$ und $n, M \in \mathbb{N}$. Wir zeigen, dass es eine meromorphe Funktion $f \in \mathcal{B}$ mit n -Ordnung $\rho_n(f) = \rho$ gibt, wobei alle Pole Vielfachheit M haben und ∞ nicht asymptotischer Wert ist, so dass $\mu_n(I(f)) = \infty$ für $\gamma > 8/(M\rho)$.

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Chapter 1

Preliminaries

Complex dynamics is nowadays a very active research topic, which was initially developed by Pierre Fatou and Gaston Julia independently around 1920. Here we only mention some basic concepts and theorems needed in the sequel, concentrating on the dynamical behaviours of the iterates for one variable complex functions.

More accurately, we focus on the Hausdorff measure of escaping sets for meromorphic functions in \mathcal{B} of infinite order. Therefore in the second section, we present definitions of Hausdorff measure and Hausdorff dimension, the Eremenko-Lyubich class \mathcal{B} and the growth order of a meromorphic function.

1.1 Complex dynamics

We start with normality, see [23], the notion of which has played a central role of complex function theory.

Definition (Normal families). A family \mathcal{F} of functions meromorphic in a domain D is normal at a point $z_0 \in D$, if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains either a subsequence which converges to a limit function $f \neq \infty$ uniformly or a subsequence which converges uniformly to ∞ in some neighbourhood of z_0 .

Assume that

$$f: \mathbb{C} \rightarrow \hat{\mathbb{C}}$$

is a non-constant transcendental meromorphic function, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We consider its iterates and the whole complex plane is partitioned into two disjoint sets. On one of them f is well behaved (the Fatou set), on the other f has chaotic behaviour (the Julia set). Denote by $f^n = f(f^{n-1})$ the n -th iterate of f , for a natural number n . We give formal definitions.

Definition (Fatou set). The Fatou set of the function f is defined as the set of all points in \mathbb{C} with a neighbourhood where the iterates f^n of f are defined and form a normal family. It is denoted by $F(f)$.

Definition (Julia set). The Julia set $J(f)$ is the complement of $F(f)$, that is $J(f) = \hat{\mathbb{C}} \setminus F(f)$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

By the Arzéla-Ascoli theorem, a point $z_0 \in \mathbb{C}$ is in the Fatou set if and only if the family f^n is equicontinuous on some neighbourhood of z_0 , with respect to the chordal metric of \mathbb{C} . Evidently $F(f)$ is open on \mathbb{C} . There have been a lot of works on the properties of the two sets, see [3] for a comprehensive survey. We list the following theorems as a brief introduction.

For a complex number z_0 and f meromorphic, if $f(z) \neq z_0$ for all $z \in \mathbb{C}$, then z_0 is called an omitted value of f .

Theorem 1.1.1. *If f is rational, entire or transcendental meromorphic with exactly one pole which is an omitted value, then $F(f) = F(f^n)$ and $J(f) = J(f^n)$ for all $n \geq 2$.*

If f is transcendental meromorphic with a pole which is not omitted, then f^n is not meromorphic and so the Fatou set and Julia set of f^n are not defined.

Theorem 1.1.2. *Let $z \in \mathbb{C}$. Then $z \in F(f)$ if and only if $f(z) \in F(f)$ and, $z \in J(f)$ if and only if $f(z) \in J(f)$.*

Note that we take only $z \in \mathbb{C}$ since $f(\infty)$ is not defined if f is transcendental. The above property is called the complete invariance of $F(f)$ and $J(f)$.

Theorem 1.1.3. $F(f) \neq \mathbb{C}$ for any f meromorphic, while $J(f) = \hat{\mathbb{C}}$ is possible. Otherwise $J(f)$ has no interior.

We call a non-empty set perfect if it is closed and with no isolated points.

Theorem 1.1.4. $J(f)$ is a perfect set.

Another important set is the escaping set of f , defined as follows

$$I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty, \text{ as } n \rightarrow \infty\}.$$

In 1989 Eremenko [8] proved the following theorem.

Theorem 1.1.5. For every entire function f the set $I(f)$ is non-empty.

We will give more information about the relation between the Julia set and the escaping set in the next chapter. Here we mention another set that will be useful in our proof,

$$I_R(f) = \left\{ z \in \mathbb{C} : \liminf_{n \rightarrow \infty} |f^n(z)| \geq R \right\},$$

for $R > 0$. In fact,

$$I(f) = \bigcap_{R>0} I_R(f).$$

1.2 Hausdorff measure and Hausdorff dimension

In this section we recall some definitions and results concerning Hausdorff measure, for a good resource we refer to [22]. We want to emphasize that Hausdorff measure is in fact a much more general concept that includes s -dimensional Hausdorff measure. The choice of the gauge function could be diverse as long as it is a pre-measure.

Suppose that Ω is a metric space. We first introduce the pre-measure.

Definition (Pre-measure). A function τ defined on a class ϕ of subsets of Ω will be called a pre-measure if :

- (a) $\emptyset \in \phi$;
- (b) $0 \leq \tau(A) \leq +\infty$ for all A in ϕ ;
- (c) $\tau(\emptyset) = 0$.

The diameter is also indispensable for defining the pre-measures that lead to the Hausdorff measures.

Definition (Diameter). The diameter of a set E of a metric space Ω with metric ρ is denoted by $d(E)$ and defined by

$$d(E) = \sup_{x,y \in E} \rho(x,y),$$

with the convention $d(\emptyset) = 0$.

Now we shall introduce a method to construct a measure from a given pre-measure by the following theorem.

Theorem 1.2.1. *If τ is a pre-measure defined on a class ϕ of sets, in a metric space Ω with metric ρ , E is a subset of Ω , the set function*

$$\mu(E) = \sup_{\delta > 0} \mu_\delta(E),$$

where

$$\mu_\delta(E) = \inf_{C_j \in \phi, d(C_j) \leq \delta, \cup C_j \supset E} \sum_{j=1}^{\infty} \tau(C_j),$$

is a measure on Ω .

Alternatively we could write

$$\mu(E) = \lim_{\delta \rightarrow 0^+} \mu_\delta(E).$$

Hausdorff measure is one certain type of measure constructed as above. To give a formal definition we shall first define

$$h : [0 + \infty) \mapsto [0, +\infty],$$

which is continuous, monotonic increasing and positive for $t > 0$. The class of such functions we denote by \mathcal{H} .

Definition (Hausdorff measure). Let Ω be a metric space and $G \subset \Omega$ an open set. Let h be a function of \mathcal{H} . Define $h(G) = h(d(G))$ if $G \neq \emptyset$ and $h(\emptyset) = 0$. Then the measure constructed from the pre-measure h , defined on ϕ , by the method in Theorem 1.2.1 is called the Hausdorff measure corresponding to the function h , or simply h -measure, and is denoted by μ_h .

As a special case, we take $h(t) = t^s$, then we get s -dimensional Hausdorff measure.

Definition (s -dimensional Hausdorff measure). For A , $(A_j)_{j \in \mathbb{N}} \subset \mathbb{R}^n$, $\delta > 0$ and $s > 0$ we set

$$\mu_\delta^s(A) = \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } A_j)^s : \bigcup A_j \supseteq A, \text{diam } A_j \leq \delta \right\}$$

and call the limit

$$\mu^s(A) = \lim_{\delta \rightarrow 0} \mu_\delta^s(A)$$

the s -dimensional Hausdorff measure of the set A .

We can see that for any given set A and $\delta < 1$, $\mu_\delta^s(A)$ is non-increasing in s and μ^s is also non-increasing. In fact, there exists some value of s , where $\mu_s(A)$ jumps from ∞ to 0, which is called the Hausdorff dimension of A . Formally it is defined as follows.

Definition (Hausdorff dimension). We set

$$\dim(A) = \inf\{s : \mu^s(A) = 0\} = \sup\{s : \mu^s(A) = \infty\}.$$

and call it the Hausdorff dimension of the set A .

It is clear that $\mu^s(A) = \infty$ if $s < \dim(A)$, and $\mu^s(A) = 0$ if $s > \dim(A)$. If $s = \dim(A)$, then $\mu^s(A)$ could be zero or infinite, or may satisfy $0 < \mu^s(A) < \infty$.

1.3 Eremenko-Lyubich class and the n -th order

Assume that f is a meromorphic function on the complex plane. We first give the definitions of critical and asymptotic values.

Definition (Critical value). A point $c \in \mathbb{C}$ is called a critical point of f , if $f'(c) = 0$ or c is a multiple pole of f . Then $f(c)$ is a critical value of f .

Definition (Asymptotic value). A value $a \in \widehat{\mathbb{C}}$ is called an asymptotic value of f if there exists a curve $\gamma : [0, \infty) \rightarrow \mathbb{C}$ such that, as $t \rightarrow \infty$ we have $\gamma(t) \rightarrow \infty$ and $f(\gamma(t)) \rightarrow a$.

We denote by $\text{sing}(f^{-1})$ the set of finite critical and asymptotic values of f .

Definition (Eremenko-Lyubich class). A meromorphic function f is in the Eremenko-Lyubich class, denoted by \mathcal{B} , if the set $\text{sing}(f^{-1})$ is bounded.

In the following we recall the growth order of a function from Nevanlinna theory, see [11, 16, 29].

For a meromorphic function f we write $n(r, f)$ the number of poles of f in $|z| \leq r$, counting multiplicities. The counting function is defined by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r.$$

Set for x real and positive,

$$\begin{aligned} \log^+ x &= \log x, & \text{if } x \geq 1, \\ \log^+ x &= 0, & \text{if } x < 1. \end{aligned}$$

Then clearly $\log x = \log^+ x - \log^+ \frac{1}{x}$. The mean value given by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

is called the proximity function. The Nevanlinna characteristic function is defined by

$$T(r, f) = m(r, f) + N(r, f).$$

The order of a meromorphic function f is defined by

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

If $\rho(f) < \infty$ we say f is a function of finite order and if $\rho(f) = \infty$, then we say f is of infinite order.

In case that f is an entire function, $T(r, f)$ can be replaced by $\log M(r, f)$, where $M(r, f) = \max_{|z|=r} |f(z)|$. Hence the order can be written alternatively as,

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}.$$

Furthermore, the n -th order of a meromorphic function f is defined by

$$\rho_n(f) = \limsup_{r \rightarrow \infty} \frac{\log^{n+1} T(r, f)}{\log r}.$$

Clearly $\rho(f) = \rho_0(f)$. It is easy to see that if $\rho(f) < \infty$, then $\rho_n(f) = 0$ for all $n \geq 1$. So the n -th order $\rho_n(f)$ is of interest only if $\rho(f) = \infty$. In this case it is used as a further description of the growth of f . In particular, $\rho_1(f)$ is called the hyper-order.

Chapter 2

Main theorems

In this chapter we state the results of this dissertation. Before doing so we recall some results on the Hausdorff dimension of Julia sets and escaping sets, as the background of the problem that we studied.

2.1 Julia set and escaping set

The Julia set and the escaping set are closely related for entire or meromorphic functions.

In 1968 Baker [1] proved the following result for transcendental entire functions.

Theorem 2.1.1. *The Julia set coincides with the closure of the set of repelling periodic points.*

For any polynomial the escaping set is a domain containing ∞ . In this case we have the Julia set is the boundary of the escaping set.

In 1989 Eremenko [8] investigated the escaping set for transcendental entire functions and obtained the next theorem.

Theorem 2.1.2. *For every entire function f , $J(f) = \partial I(f)$.*

Furthermore, he proved the following result.

Theorem 2.1.3. *Let f be an transcendental entire function, then $J(f) \cap I(f) \neq \emptyset$.*

The closure of the escaping set was also studied in the same article.

Theorem 2.1.4. *The closure $\overline{I(f)}$ of the escaping set has no bounded components.*

Corresponding results for meromorphic functions were given by Domínguez [6] in 1998.

Theorem 2.1.5. *For a transcendental meromorphic function f , we have $J(f) = \partial I(f)$ and $J(f) \cap I(f) \neq \emptyset$.*

However in this case the components of the closure of $I(f)$ need not to be unbounded. Now we continue with functions in class \mathcal{B} . The following result was proved by Eremenko and Lyubich [9].

Theorem 2.1.6. *Let $f \in \mathcal{B}$ be a transcendental entire function. If $f^n(z) \rightarrow \infty$, as $n \rightarrow \infty$ then $z \in J(f)$.*

It can be deduced that for transcendental entire functions in \mathcal{B} , we have the Julia set is the closure of the escaping set.

Rippon and Stallard [15] in 1999 extended the result of Eremenko and Lyubich to meromorphic functions.

Theorem 2.1.7. *If f is a meromorphic function in \mathcal{B} , then there is no component of the Fatou set $F(f)$ in which $f^n(z) \rightarrow \infty$ as $n \rightarrow \infty$.*

We can draw the same conclusion as in the entire case that $I(f) \subset J(f)$. Together with the theorems above it can be seen that for meromorphic functions in the Eremenko-Lyubich class the Julia set is the closure of the escaping set.

2.2 Hausdorff dimension of Julia sets and escaping sets

Many mathematicians have investigated the area and Hausdorff dimension of Julia sets. In 1987 McMullen [14] considered the functions in the sine family and the exponential family and obtained the following theorems.

We say f is in the sine family if it is of the form $f(z) = \sin(\alpha z + \beta)$ where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$.

Theorem 2.2.1. *The Julia set of any member in the sine family always has positive area.*

The exponential family is the set of functions f satisfying $f(z) = \lambda e^z$ for some $\lambda \neq 0$.

Theorem 2.2.2. *The Julia set of any member in the exponential family has Hausdorff dimension 2.*

However, when the function f in the exponential family has an attracting periodic cycle, then the area of $J(f)$ is zero.

In 2008 Barański [2] generalised McMullen's result about the Hausdorff dimension of the Julia sets. The same result was proved by Schubert independently in 2007 in his dissertation [24]. Actually they showed for $R > 0$ that $\dim(I_R(f)) = 2$.

Theorem 2.2.3. *For every transcendental entire map of finite order from class \mathcal{B} , the Hausdorff dimension of the Julia set is 2.*

In the proofs, they both adopted a logarithmic change of variables that Eremenko and Lyubich [9] introduced to complex dynamics. It was pointed out by Bergweiler and Kotus [4] that the conclusion holds more generally for meromorphic functions from class \mathcal{B} of finite order for which ∞ is an asymptotic value.

However, for $f \in \mathcal{B}$ meromorphic without ∞ as its asymptotic value we have a different situation, which was shown by their results.

Theorem 2.2.4. *Let $f \in \mathcal{B}$ be a transcendental meromorphic function satisfying $\rho = \rho(f) < \infty$. Suppose that ∞ is not an asymptotic value and that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles, except possibly finitely many, is at most M . Then*

$$\dim(I(f)) \leq \frac{2M\rho}{2 + M\rho}.$$

More generally,

$$\lim_{R \rightarrow \infty} \dim(I_R(f)) \leq \frac{2M\rho}{2 + M\rho}.$$

On the other hand they found function satisfying these hypotheses for which the Hausdorff dimension of the escaping set is the value on the right side of the inequalities.

Theorem 2.2.5. *Let $0 < \rho < \infty$ and $M \in \mathbb{N}$. Then there exists a meromorphic function $f \in \mathcal{B}$ of order ρ for which all poles have multiplicity M and for which ∞ is not an asymptotic value such that*

$$\dim(I(f)) = \frac{2M\rho}{2 + M\rho}$$

and

$$\dim(I_R(f)) > \frac{2M\rho}{2 + M\rho}$$

for all $R > 0$.

Moreover, they showed without the restriction to the order, the area of the escaping set is zero.

Theorem 2.2.6. *Let $f \in \mathcal{B}$ be a transcendental meromorphic function for which ∞ is not an asymptotic value. Suppose that there exists $M \in \mathbb{N}$ such that all poles of f have multiplicity at most M . Then $\text{area}(I_R(f)) = 0$ for sufficiently large R . In particular, $\text{area}(I(f)) = 0$.*

2.3 Main results

Theorem 2.2.4 deals with the functions of finite order. For the case of infinite order, we would like to give more information by the following results.

We set

$$h(t) = t^2 \left(\log^n \frac{1}{t} \right)^\gamma, \quad (2.3.1)$$

where $t \in (0, \delta_n]$, $\delta_n = \min\{1/\exp^n(2), 1/\exp^n(\gamma)\}$ for $n \in \mathbb{N}$ and $\gamma > 0$.

Theorem 2.3.1. *Let $f \in \mathcal{B}$ be a meromorphic function with $\rho = \rho_n(f)$ satisfying $0 < \rho < \infty$. Suppose that ∞ is not an asymptotic value of f and that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles of f , except possibly finitely many, is at most M . If h is given by (2.3.1) and $\gamma < 2/(M\rho)$, then $\mu_h(I(f)) = 0$.*

The bound $2/(M\rho)$ is probably not sharp. However the following result shows that it cannot be replaced by any value greater than $8/(M\rho)$.

Theorem 2.3.2. *Let $0 < \rho < \infty$ and $n, M \in \mathbb{N}$. Then there exists a meromorphic function $f \in \mathcal{B}$ of n -th order $\rho_n(f) = \rho$ for which all poles have multiplicity M and ∞ is not an asymptotic value such that if h is as in (2.3.1) and $\gamma > 8/(M\rho)$, then $\mu_h(I(f)) = \infty$.*

Chapter 3

Estimate from above

3.1 Gauge function

We will show some properties of interest for the gauge function h in (2.3.1), which are also used in our following proofs.

Lemma 3.1.1. *Let $h(t)$ be defined as in (2.3.1). Then for all $c > 1$,*

$$h(ct) \leq c^2 h(t). \quad (3.1.1)$$

Proof. By definition we have

$$h(ct) = (ct)^2 \left(\log^n \frac{1}{ct} \right)^\gamma \leq (ct)^2 \left(\log^n \frac{1}{t} \right)^\gamma = c^2 h(t).$$

□

Lemma 3.1.2. *Let $n, l \in \mathbb{N}$ with $l \geq 1$ and t_1, t_2, \dots, t_l be real numbers. If $0 < t_j \leq 1/\exp^n 2$ for $j = 1, 2, \dots, l$, then we have*

$$\log^n \left(\frac{1}{t_1 t_2 \cdots t_l} \right) \leq \left(\log^n \frac{1}{t_1} \right) \left(\log^n \frac{1}{t_2} \right) \cdots \left(\log^n \frac{1}{t_l} \right). \quad (3.1.2)$$

Proof. We denote $t_{j_0} = \min\{t_1, t_2, \dots, t_l\}$. Consider first the case that $n = 1$ and let $u_1 = \log(1/t_{j_0})$. Then we have

$$\log \frac{1}{t_1 t_2 \cdots t_l} = \sum_{j=1}^l \log \frac{1}{t_j} \leq u_1 l. \quad (3.1.3)$$

On the other hand, noting that $1/t_j \geq \exp 2$,

$$\left(\log \frac{1}{t_1}\right) \left(\log \frac{1}{t_2}\right) \cdots \left(\log \frac{1}{t_l}\right) \geq 2^{l-1} \log \frac{1}{t_{j_0}} = u_1 2^{l-1}. \quad (3.1.4)$$

Since $l \leq 2^{l-1}$ for $l \geq 1$, we deduce from (3.1.3) and (3.1.4) that (3.1.2) holds for $n = 1$ and $t_j \leq e^{-2}$, $j = 1, 2, \dots, l$, that is,

$$\log \left(\frac{1}{t_1 t_2 \cdots t_l}\right) \leq \left(\log \frac{1}{t_1}\right) \left(\log \frac{1}{t_2}\right) \cdots \left(\log \frac{1}{t_l}\right). \quad (3.1.5)$$

We now prove the conclusion by induction. Suppose that (3.1.2) holds for $n = k$ and $t_j \leq 1/\exp^k 2$ for $j = 1, 2, \dots, l$, that is,

$$\log^k \left(\frac{1}{t_1 t_2 \cdots t_l}\right) \leq \left(\log^k \frac{1}{t_1}\right) \left(\log^k \frac{1}{t_2}\right) \cdots \left(\log^k \frac{1}{t_l}\right). \quad (3.1.6)$$

Suppose that $t_j \leq 1/\exp^{k+1} 2$ for $j = 1, 2, \dots, l$. Therefore $1/\log^k(1/t_j) \leq e^{-2}$. Then from (3.1.6) and (3.1.5) we obtain

$$\begin{aligned} \log^{k+1} \left(\frac{1}{t_1 t_2 \cdots t_l}\right) &= \log \log^k \left(\frac{1}{t_1 t_2 \cdots t_l}\right) \\ &\leq \log \left(\left(\log^k \frac{1}{t_1}\right) \left(\log^k \frac{1}{t_2}\right) \cdots \left(\log^k \frac{1}{t_l}\right) \right) \\ &\leq \left(\log \log^k \frac{1}{t_1}\right) \left(\log \log^k \frac{1}{t_2}\right) \cdots \left(\log \log^k \frac{1}{t_l}\right) \\ &= \left(\log^{k+1} \frac{1}{t_1}\right) \left(\log^{k+1} \frac{1}{t_2}\right) \cdots \left(\log^{k+1} \frac{1}{t_l}\right), \end{aligned}$$

from which we see that (3.1.2) holds for $n = k + 1$ if $t_j \leq 1/\exp^{k+1} 2$, $j = 1, 2, \dots, l$. \square

Lemma 3.1.3. *Let $n, l \in \mathbb{N}$ with $l \geq 1$. Set $h(t)$ as in (2.3.1). Suppose that $t_j \in (0, \delta_n]$, for $1 \leq j \leq l$. Then we have*

$$h(t_1 t_2 \cdots t_l) \leq \prod_{j=1}^l h(t_j). \quad (3.1.7)$$

Proof. Since $t_j \leq 1/\exp^n 2$ for $1 \leq j \leq l$ we deduce from (3.1.2) that

$$\begin{aligned} h(t_1 t_2 \cdots t_l) &= (t_1 t_2 \cdots t_l)^2 \left(\log^n \frac{1}{t_1 t_2 \cdots t_l} \right)^\gamma \\ &\leq (t_1 t_2 \cdots t_l)^2 \left(\left(\log^n \frac{1}{t_1} \right) \left(\log^n \frac{1}{t_2} \right) \cdots \left(\log^n \frac{1}{t_l} \right) \right)^\gamma \\ &= t_1^2 \left(\log^n \frac{1}{t_1} \right)^\gamma t_2^2 \left(\log^n \frac{1}{t_2} \right)^\gamma \cdots t_l^2 \left(\log^n \frac{1}{t_l} \right)^\gamma \\ &= h(t_1) h(t_2) \cdots h(t_l). \end{aligned}$$

Therefore we obtain (3.1.7). \square

Lemma 3.1.4. *Suppose that $h(t)$ is defined as in (2.3.1) for $\gamma > 0$ and $n \in \mathbb{N}$. Define the function $G(t) = h(\sqrt{t})$. Then $G(t)$ is increasing and concave on $(0, \delta_n^2]$.*

Proof. According to the definition and (2.3.1) we have

$$G(t) = t \left(\log^n \frac{1}{\sqrt{t}} \right)^\gamma.$$

Thus

$$\begin{aligned} G'(t) &= \left(\log^n \frac{1}{\sqrt{t}} \right)^\gamma + t \left(\left(\log^n \frac{1}{\sqrt{t}} \right)^\gamma \right)' \\ &= \left(\log^n \frac{1}{\sqrt{t}} \right)^\gamma - \frac{1}{2} \gamma \frac{\left(\log^n \frac{1}{\sqrt{t}} \right)^\gamma}{\left(\log^n \frac{1}{\sqrt{t}} \right) \left(\log^{n-1} \frac{1}{\sqrt{t}} \right) \cdots \left(\log \frac{1}{\sqrt{t}} \right)} \\ &= \left(\log^n \frac{1}{\sqrt{t}} \right)^\gamma \left(1 - \frac{\frac{\gamma}{2}}{\left(\log^n \frac{1}{\sqrt{t}} \right) \left(\log^{n-1} \frac{1}{\sqrt{t}} \right) \cdots \left(\log \frac{1}{\sqrt{t}} \right)} \right). \end{aligned} \quad (3.1.8)$$

If $t \leq \delta_n^2 \leq (1/\exp^n(\gamma))^2$ then

$$\left(\log^n \frac{1}{\sqrt{t}} \right) \left(\log^{n-1} \frac{1}{\sqrt{t}} \right) \cdots \log \frac{1}{\sqrt{t}} \geq \gamma \exp(\gamma) \cdots \exp^{n-1}(\gamma) \geq \gamma > \frac{\gamma}{2},$$

which yields with (3.1.8) that

$$G'(t) \geq 0.$$

Hence $G(t)$ is increasing. One can also find that $G'(t)$ is decreasing on $(0, \delta_n^2]$ since both factors on the right side of (3.1.8) are decreasing. Therefore $G(t)$ is a concave function on $(0, \delta_n^2]$. \square

3.2 Notations and lemmas

The following lemma is known as Iversen's theorem, see e.g. [16, chapter 5].

Lemma 3.2.1. *Let f be a transcendental meromorphic function for which ∞ is not an asymptotic value. Then f has infinitely many poles.*

We recall some results of distortion theory, which are usually stated only for univalent functions defined on the open unit disk, see [20, Theorem 1.6] and [7, Theorem 2.13, p.44], but the following versions follow immediately from this special case.

For $a \in \mathbb{C}$ and $r > 0$ we use the notation $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$. The next lemma is called Koebe's theorem, see also [4, Lemma 2.1].

Lemma 3.2.2. *Let $g: D(a, r) \rightarrow \mathbb{C}$ be univalent, $0 < \lambda < 1$ and $z \in D(a, \lambda r)$. Then*

$$\frac{\lambda}{(1 + \lambda)^2} \leq \frac{|g(z) - g(a)|}{|(z - a)g'(a)|} \leq \frac{\lambda}{(1 - \lambda)^2}, \quad (3.2.1)$$

$$\frac{1 - \lambda}{(1 + \lambda)^3} \leq \frac{|g'(z)|}{|g'(a)|} \leq \frac{1 + \lambda}{(1 - \lambda)^3}, \quad (3.2.2)$$

and

$$g(D(a, r)) \supset D\left(g(a), \frac{1}{4}|g'(a)|r\right). \quad (3.2.3)$$

Regarding the radius of convexity, we recall another result of univalent functions.

Lemma 3.2.3. *Let $f: D(a, r) \rightarrow \mathbb{C}$ be univalent, then $f(D(a, \rho r))$ is a convex set for $\rho \leq 2 - \sqrt{3}$.*

We put $B(R) = \{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$. Rippon and Stallard [15, Lemma 2.1] proved the following result.

Lemma 3.2.4. *Let $f \in \mathcal{B}$ be transcendental. If $R > 0$ such that $\text{sing}(f^{-1}) \subset D(0, R)$, then all components of $f^{-1}(B(R))$ are simply connected.*

Bergweiler and Kotus [4, Lemma 2.2] made a supplement. They pointed out that if ∞ is not an asymptotic value of f then all components of $f^{-1}(B(R))$ are bounded and contain exactly one pole of f .

We continue with Jensen's inequality [18, p.12], one of the crucial tools used in our proof.

Lemma 3.2.5. *Suppose that I is an interval and the function $f: I \rightarrow \mathbb{R}$ is concave. For any points $x_1, x_2, \dots, x_n \in I$ and any real non-negative numbers r_1, r_2, \dots, r_n such that $r_1 + r_2 + \dots + r_n = 1$, we have*

$$f\left(\sum_{j=1}^n r_j x_j\right) \geq \sum_{j=1}^n r_j f(x_j).$$

3.3 Proof of the first theorem

Under the assumptions that f is a meromorphic function without ∞ as its asymptotic value, by Lemma 3.2.1 f has infinitely many poles, say denoted by a_j and ordered such that $|a_j| \leq |a_{j+1}|$ for all $j \in \mathbb{N}$. Let m_j be the multiplicity of a_j . Thus for some $b_j \in \mathbb{C} \setminus \{0\}$,

$$f(z) \sim \left(\frac{b_j}{z - a_j}\right)^{m_j} \quad \text{as } z \rightarrow a_j.$$

We may assume that $|a_j| \geq 1$ for all j . Choose $R_0 > 1$ such that $\text{sing}(f^{-1}) \subset D(0, R_0)$ and $|f(0)| < R_0$.

If $R \geq R_0$, then all the components of $f^{-1}(B(R))$ are bounded, simply-connected and each component contains exactly one pole by Lemma 3.2.4. Let U_j be the component containing a_j . By the Riemann mapping theorem we may choose a conformal map

$$\phi_j: U_j \rightarrow D(0, R^{-1/m_j})$$

with $\phi_j(a_j) = 0$. We claim that $|f(z)\phi_j(z)^{m_j}| = 1$ for $z \in U_j \setminus \{a_j\}$ and that $\phi_j'(a_j) = 1/|b_j|$. In fact since ϕ_j is a conformal mapping there exists a non-zero constant c_j such that

$$\phi_j(z) \sim c_j(z - a_j), \quad z \rightarrow a_j.$$

Thus

$$\phi_j(z)^{m_j} \sim c_j^{m_j} (z - a_j)^{m_j}, \quad z \rightarrow a_j,$$

and

$$f(z)\phi_j(z)^{m_j} \sim \left(\frac{b_j}{z - a_j}\right)^{m_j} c_j^{m_j} (z - a_j)^{m_j} = b_j^{m_j} c_j^{m_j}, \quad z \rightarrow a_j \quad (3.3.1)$$

As z approaches the boundary of U_j it is noticed that

$$f(z) \rightarrow R \quad \text{and} \quad \phi_j(z) \rightarrow R^{-\frac{1}{m_j}},$$

therefore

$$|f(z)\phi_j(z)^{m_j}| \rightarrow 1.$$

By (3.3.1) we can extend ϕ_j holomorphically to the closure $\overline{U_j}$ of $U_j \setminus \{a_j\}$. Then

$$|f(z)\phi_j(z)^{m_j}| = 1 \quad \text{for } z \in \partial U_j.$$

By the maximum principle immediately we have

$$|f(z)\phi_j(z)^{m_j}| = 1 \quad \text{for all } z \in U_j.$$

Since $c_j = \phi_j'(a_j)$ with (3.3.1) we have

$$|\phi_j'(a_j)| = \frac{1}{|b_j|}.$$

Hence we may normalize ϕ_j such that $\phi_j'(a_j) = 1/b_j$.

Denote the inverse function of ϕ_j by ψ_j . Since $\psi_j(0) = a_j$ and

$$\psi_j'(0) = \frac{1}{\phi_j'(a_j)} = b_j \quad (3.3.2)$$

we can deduce from (3.2.3) that

$$U_j = \psi_j(D(0, R^{-1/m_j})) \supset D\left(a_j, \frac{1}{4}|b_j|R^{-1/m_j}\right) \supset D\left(a_j, \frac{1}{4R}|b_j|\right). \quad (3.3.3)$$

Since $|f(0)| < R$ we have $0 \notin U_j$. Then (3.3.3) implies that

$$\frac{1}{4R}|b_j| \leq |a_j|$$

for all $R \geq R_0$. Hence

$$|b_j| \leq 4R_0|a_j|. \quad (3.3.4)$$

Note that ψ_j actually extends to a map univalent in $D(0, R_0^{-1/m_j})$. Choosing $R \geq 2^M R_0$ we can apply (3.2.1) with

$$\lambda = (R/R_0)^{-1/m_j} = (R_0/R)^{1/m_j} \leq \frac{1}{2}$$

and obtain

$$\begin{aligned} |\psi_j(z) - \psi_j(0)| &\leq 2|(z - a_j)\psi_j'(a_j)| \\ &= 2|b_j||z - a_j| \\ &\leq 2|b_j| \left(\frac{R_0}{R}\right)^{\frac{1}{m_j}} \frac{1}{R_0^{m_j}} \\ &\leq 2|b_j|R^{\frac{1}{M}} \end{aligned}$$

for $z \in D(a_j, \lambda R_0^{-1/m_j})$. It follows that

$$U_j \subset D(a_j, 2|b_j|R^{-1/M}), \quad (3.3.5)$$

provided j is so large that $m_j \leq M$. With (3.3.3) and (3.3.5) we see that

$$D\left(a_j, \frac{1}{4R}|b_j|\right) \subset U_j \subset D(a_j, 2R^{-1/M}|b_j|)$$

for large j . Combining (3.3.4) and (3.3.5) and choosing $R \geq (16R_0)^M$ we have

$$U_j \subset D\left(a_j, \frac{1}{2}|a_j|\right) \subset D\left(0, \frac{3}{2}|a_j|\right) \quad (3.3.6)$$

for large j , say $j \geq j_0$ for some $j_0 \in \mathbb{N}$. Let $n(r)$ denote the number of a_j contained in the closed disc $\overline{D(0, r)}$. Since the U_j are pairwise disjoint we see from (3.3.3) that

$$\text{area} \left(\bigcup_{j=j_0}^{n(r)} U_j \right) \geq \text{area} \left(\bigcup_{j=j_0}^{n(r)} D\left(a_j, \frac{1}{4R}|b_j|\right) \right) = \sum_{j=j_0}^{n(r)} \pi \left(\frac{|b_j|}{4R} \right)^2. \quad (3.3.7)$$

On the other hand from (3.3.6) we have

$$\text{area} \left(\bigcup_{j=j_0}^{n(r)} U_j \right) \leq \text{area} \left(D\left(0, \frac{3}{2}r\right) \right) = \pi \frac{9}{4}r^2. \quad (3.3.8)$$

Combining (3.3.7) and (3.3.8) we obtain

$$\sum_{j=j_0}^{n(r)} |b_j|^2 \leq 36R^2 r^2. \quad (3.3.9)$$

Let $D \subset B(R) \setminus \{\infty\}$ be a simply connected domain. Then any branch of the inverse of f defined in a subdomain of D can be continued analytically to D . Let g_j be a branch of f^{-1} that maps D into U_j . Recall that

$$f(z)\phi_j(z)^{m_j} = 1,$$

after normalization. Thus

$$f(\psi_j(z))\phi_j(\psi_j(z))^{m_j} = 1.$$

Since ψ_j is the inverse function of ϕ_j we have

$$f(\psi_j(z^{-1})) = z^{m_j}.$$

Therefore

$$f(\psi_j(z^{-1/m_j})) = z.$$

and hence

$$g_j(z) = \psi_j\left(\frac{1}{z^{1/m_j}}\right), \quad (3.3.10)$$

for some branch of the m_j -th root. Since we assumed that $R \geq 2^M R_0$ we deduce from (3.2.1) with $\lambda = 1/2$ and (3.3.2) that

$$|g'_j(z)| = \left| \psi' \left(\frac{1}{z^{1/m_j}} \right) \right| \frac{1}{m_j |z|^{1+\frac{1}{m_j}}} \leq \frac{1 + \frac{1}{2}}{(1 - \frac{1}{2})^3} \frac{|\psi'_j(0)|}{m_j |z|^{1+\frac{1}{m_j}}} \leq \frac{12|b_j|}{m_j |z|^{1+\frac{1}{M}}} \leq \frac{12|b_j|}{|z|^{1+\frac{1}{M}}} \quad (3.3.11)$$

for $z \in D \subset B(R) \setminus \{\infty\}$, provided j is so large that $m_j \leq M$.

By Lemma 3.2.3 for R large, we have U_j is a convex set. Therefore, if $U_k \subset B(R)$ from (3.3.5) and (3.3.6) we obtain

$$\text{diam } g_j(U_k) \leq \sup_{z \in U_k} |g'_j(z)| \text{diam } U_k \leq 2^{1+\frac{1}{M}} 12 \frac{4}{R^{\frac{1}{M}}} \frac{|b_j|}{|a_k|^{1+\frac{1}{M}}} |b_k|.$$

By induction if $U_{j_1}, U_{j_2}, \dots, U_{j_l} \subset B(R)$ for R large, we have

$$\text{diam}((g_{j_1} \circ g_{j_2} \circ \dots \circ g_{j_{l-1}})(U_{j_l})) \leq \left(2^{1+\frac{1}{M}} 12\right)^{l-1} \frac{4}{R^{\frac{1}{M}}} |b_{j_1}| \frac{|b_{j_2}|}{|a_{j_2}|^{1+1/M}} \dots \frac{|b_{j_l}|}{|a_{j_l}|^{1+1/M}}. \quad (3.3.12)$$

For $z_1, z_2 \in U_j \subset D(a_j, \frac{1}{2}|a_j|)$ we estimate the spherical distance,

$$\chi(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}} \leq \frac{2|z_1 - z_2|}{1 + \frac{1}{4}|a_j|^2} \leq \frac{8|z_1 - z_2|}{|a_j|^{1+1/M}}.$$

Hence for $K \subset U_j$,

$$\text{diam}_\chi(K) \leq \frac{8}{|a_j|^{1+1/M}} \text{diam}(K).$$

Then with (3.3.5) we have

$$\text{diam}_\chi((g_{j_1} \circ g_{j_2} \circ \dots \circ g_{j_{l-1}})(U_{j_l})) \leq \left(2^{1+\frac{1}{M}} 12\right)^{l-1} \frac{32}{R^{\frac{1}{M}}} \prod_{k=1}^l \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}}. \quad (3.3.13)$$

Before proceeding we shall prove the following result. This corresponds to [4, Lemma 3.1], dealing with gauge functions of the form $h(t) = t^\alpha$. These gauge functions are estimated using Hölder's inequality. Instead, here we consider the gauge functions defined by (2.3.1) and use the results of Section 3.2 to estimate them.

Lemma 3.3.1. *Let h be defined as in (2.3.1). If $\gamma < 2/(M\rho)$ then*

$$\sum_{j=1}^{\infty} h\left(\frac{|b_j|}{|a_j|^{1+\frac{1}{M}}}\right) < \infty. \quad (3.3.14)$$

Proof. For $l \geq 0$, we put

$$P_l = \{j \in \mathbb{N} : n(2^l) \leq j < n(2^{l+1})\} = \{j \in \mathbb{N} : 2^l \leq |a_j| < 2^{l+1}\}.$$

Denote by $\text{Card } P_l$ the cardinality of P_l and put

$$c_j = \left(\frac{|b_j|}{|a_j|^L}\right)^2,$$

where $L = 1 + 1/M$. We choose l large such that $j \geq j_0$ for $j \in P_l$ and j_0 is as in (3.3.7). With (3.3.9) we obtain

$$\begin{aligned} \sum_{j \in P_l} c_j &= \sum_{j \in P_l} \frac{|b_j|^2}{|a_j|^{2L}} \\ &\leq 2^{-2lL} \sum_{j \in P_l} |b_j|^2 \\ &\leq 2^{-2lL} \sum_{j=j_0}^{n(2^{l+1})} |b_j|^2 \\ &\leq 2^{-2lL} 36R^2 2^{2(l+1)}. \end{aligned}$$

Thus

$$\sum_{j \in P_l} c_j \leq K 2^{-\frac{2l}{M}}, \quad (3.3.15)$$

where $K = 144R^2$.

Set

$$S_l := \sum_{j \in P_l} h \left(\frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} \right)$$

and

$$G(t) := h(\sqrt{t}).$$

Then

$$S_l = \sum_{j \in P_l} G(c_j). \quad (3.3.16)$$

Let $\delta_n \leq 1/\exp^n(\gamma)$ as in (2.3.1). We consider the following two cases.

Case 1. Suppose that

$$\frac{K 2^{-\frac{2l}{M}}}{\text{Card } P_l} \geq \delta_n^2.$$

Then

$$\text{Card } P_l \leq \delta_n^{-2} K 2^{-\frac{2l}{M}} < 1, \quad \text{as } l \rightarrow \infty.$$

Thus $P_l = \emptyset$ for large l . For such l we have $S_l = 0$.

Case 2. Suppose that

$$\frac{K 2^{-\frac{2l}{M}}}{\text{Card } P_l} < \delta_n^2.$$

By (3.3.15) we have

$$\frac{\sum_{j \in P_l} c_j}{\text{Card } P_l} \leq \frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l}.$$

Then from Lemma 3.1.4,

$$G\left(\frac{\sum_{j \in P_l} c_j}{\text{Card } P_l}\right) \leq G\left(\frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l}\right). \quad (3.3.17)$$

Therefore we may apply Lemma 3.2.5 to $G(t)$ with $r_j = 1/\text{Card } P_l$ and $x_j = c_j$ for $j \in P_l$. We obtain

$$G\left(\frac{\sum_{j \in P_l} c_j}{\text{Card } P_l}\right) \geq \frac{\sum_{j \in P_l} G(c_j)}{\text{Card } P_l}.$$

Thus, using (3.3.15), (3.3.16) and (3.3.17)

$$\begin{aligned} S_l &\leq (\text{Card } P_l) G\left(\frac{\sum_{j \in P_l} c_j}{\text{Card } P_l}\right) \\ &\leq (\text{Card } P_l) G\left(\frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l}\right) \\ &= (\text{Card } P_l) \frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l} \left(\log^n \frac{1}{\sqrt{\frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l}}}\right)^\gamma \\ &= K2^{-\frac{2l}{M}} \left(\log^n \frac{\sqrt{\text{Card } P_l}}{\sqrt{K2^{-\frac{2l}{M}}}}\right)^\gamma. \end{aligned} \quad (3.3.18)$$

Lemma 5.1.1 implies for $\varepsilon > 0$ that

$$\text{Card } P_l \leq n(2^{l+1}) \leq \exp^n ((2^{l+1})^{\rho+\varepsilon}), \quad (3.3.19)$$

for large l . Then (3.3.18) and (3.3.19) give,

$$\begin{aligned} S_l &\leq K2^{-\frac{2l}{M}} \left(\log^n \frac{\sqrt{\exp^n 2^{(\rho+\varepsilon)(l+1)}}}{K2^{-\frac{2l}{M}}}\right)^\gamma \\ &= K2^{-\frac{2l}{M}} \left(\log^{n-1} \left(\frac{1}{2} \exp^{n-1} 2^{(\rho+\varepsilon)(l+1)} - \log K + \frac{2l}{M} \log 2\right)\right)^\gamma \\ &\leq K2^{-\frac{2l}{M}} (\log^{n-1} (\exp^{n-1} 2^{(\rho+\varepsilon)(l+1)}))^\gamma \\ &\leq K2^{-\frac{2l}{M}} 2^{(\rho+\varepsilon)(l+1)\gamma} \\ &= K2^{(\rho+\varepsilon)\gamma} 2^{-l(\frac{2}{M} - (\rho+\varepsilon)\gamma)} \end{aligned}$$

for $n \geq 2$ and l large. If $\gamma < 2/(M(\rho + \varepsilon))$, then

$$\frac{2}{M} - (\rho + \varepsilon)\gamma > 0,$$

which implies the series $\sum_{l=0}^{\infty} S_l$ converges. The conclusion follows as $\varepsilon \rightarrow 0$. \square

We continue by denoting E_l the collections of all components V of $f^{-l}(B(R))$ for which $f^k(V) \subset U_{j_{k+1}} \subset B(R)$ for $k = 0, 1, \dots, l-1$ and certain $j_1, j_2, \dots, j_l \geq n(R)$. For such a component $V \in E_l$, we then have

$$V = (g_{j_1} \circ g_{j_2} \circ \dots \circ g_{j_{l-1}})(U_{j_l}).$$

Since there are m_{j_k} branches of f^{-1} mapping $U_{j_{k+1}}$ into U_{j_k} for $k = 1, 2, \dots, l-1$, we conclude that there are

$$\prod_{k=1}^{l-1} m_{j_k} \leq M^{l-1}$$

sets of diameters bounded as in (3.3.13) which cover all those components V such that $f^k(V) \subset U_{j_{k+1}} \subset B(R)$ for $k = 0, 1, \dots, l-1$.

From (3.3.13) we have

$$\text{diam}_{\chi}(V) \leq (2^{1+\frac{1}{M}} 12)^{l-1} \frac{32}{R^{1/M}} \prod_{k=1}^l \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}}. \quad (3.3.20)$$

It is easy to see from (3.3.4) that for R large,

$$2^{1+\frac{1}{M}} 12 \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}} \leq \frac{1}{\exp^n 2}$$

and

$$\frac{32}{R^{\frac{1}{M}}} \leq 2^{1+\frac{1}{M}} 12.$$

Therefore we may apply Lemma 3.1.3, which together with (3.3.20) gives,

$$\begin{aligned} \sum_{V \in E_l} h(\text{diam}_{\chi}(V)) &\leq M^{l-1} \sum_{j_1=n(R)}^{\infty} \dots \sum_{j_l=n(R)}^{\infty} h \left((2^{1+\frac{1}{M}} 12)^{l-1} \frac{32}{R^{\frac{1}{M}}} \prod_{k=1}^l \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}} \right) \\ &\leq M^{l-1} \sum_{j_1=n(R)}^{\infty} \dots \sum_{j_l=n(R)}^{\infty} \prod_{k=1}^l h \left(2^{1+\frac{1}{M}} 12 \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}} \right) \\ &= \frac{1}{M} \left(M \sum_{j=n(R)}^{\infty} h \left(2^{1+\frac{1}{M}} 12 \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} \right) \right)^l \end{aligned}$$

for R large enough.

We can get from (3.1.1) and Lemma 3.3.1 that if $\gamma < 2/(M\rho)$,

$$M \sum_{j=n(R)}^{\infty} h \left(2^{1+\frac{1}{M}} 12 \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} \right) \leq M \left(2^{1+\frac{1}{M}} 12 \right)^2 \sum_{j=n(R)}^{\infty} h \left(\frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} \right) < 1,$$

for R large. For such R we find that

$$\lim_{l \rightarrow \infty} \sum_{V \in E_l} h(\text{diam}_\chi(V)) = 0, \quad \text{if } \gamma < \frac{2}{M\rho}.$$

Next we note that (3.3.6) implies that if $U_j \cap B(3R) \neq \emptyset$, then $|a_j| > 2R$ and $U_j \subset B(R)$. It follows that E_l is a cover of the set

$$\{z \in B(3R) : f^k(z) \in B(3R) \text{ for } 1 \leq k \leq l-1\}.$$

Therefore

$$\mu_h(I_{3R}(f)) = 0, \quad \text{for } \gamma < \frac{2}{M\rho}.$$

The conclusion follows since $I(f) = \bigcap_{R>0} I_{3R}(f)$.

Chapter 4

Construction of the example

4.1 First step and some lemmas

Let $0 < \rho < \infty$ and $n \in \mathbb{N}$. Put

$$k_0 = \lfloor \exp^n 2 \rfloor + 1.$$

We introduce the following function

$$q: [2^{1/\rho}, \infty) \rightarrow [\exp^n 2, \infty), \quad q(r) = \exp^n(r^\rho)$$

and the inverse function

$$p: [\exp^n 2, \infty) \rightarrow [2^{1/\rho}, \infty), \quad p(t) = (\log^n t)^{1/\rho}. \quad (4.1.1)$$

For $k \geq k_0$, $k \in \mathbb{N}$ set

$$n_k = \left\lfloor \frac{p(k)}{p'(k)} \right\rfloor. \quad (4.1.2)$$

The next lemmas give some essential features of these functions, which help us to construct the function in Theorem 4.2.1.

Lemma 4.1.1.

$$\frac{d}{dr} \left(\frac{q(r)}{q'(r)} \right) \rightarrow 0, \quad \frac{q(r)}{r q'(r)} \rightarrow 0, \quad (4.1.3)$$

as $r \rightarrow \infty$.

Proof. By differentiation,

$$q'(r) = (\exp^n r^\rho)(\exp^{n-1} r^\rho) \cdots (\exp r^\rho) \rho r^{\rho-1}. \quad (4.1.4)$$

Thus for $n = 1$,

$$\begin{aligned} \frac{q(r)}{r q'(r)} &= \frac{1}{\rho r^\rho} \rightarrow 0, \\ \frac{d}{dr} \frac{q(r)}{q'(r)} &= \frac{1-\rho}{\rho r^\rho} \rightarrow 0, \end{aligned}$$

as $r \rightarrow \infty$.

For $n \geq 2$,

$$\frac{q(r)}{q'(r)} = \frac{1}{(\exp^{n-1} r^\rho)(\exp^{n-2} r^\rho)(\exp r^\rho) \rho r^{\rho-1}}, \quad (4.1.5)$$

and clearly

$$\frac{q(r)}{r q'(r)} \rightarrow 0, \quad r \rightarrow \infty.$$

Differentiating on both sides of (4.1.5) we obtain

$$\begin{aligned} \frac{d}{dr} \left(\frac{q(r)}{q'(r)} \right) &= -\frac{1}{\exp^{n-1} r^\rho} - \frac{1}{(\exp^{n-1} r^\rho)(\exp^{n-2} r^\rho)} - \cdots \\ &\quad - \frac{1}{(\exp^{n-1} r^\rho)(\exp^{n-2} r^\rho) \cdots (\exp r^\rho)} \left(1 + \frac{\rho-1}{\rho r^\rho} \right). \end{aligned}$$

It is easy to see that

$$\frac{d}{dr} \left(\frac{q(r)}{q'(r)} \right) \rightarrow 0, \quad r \rightarrow \infty.$$

□

Lemma 4.1.2.

$$p \left(t + \frac{1}{2} \right) - p(t) \sim \frac{1}{2} p'(t), \quad t \rightarrow \infty.$$

Proof. From (4.1.1) we have

$$p'(t) = \frac{1}{\rho} (\log^n t)^{\frac{1}{\rho}} \frac{1}{t(\log t)(\log^2 t) \cdots (\log^n t)}, \quad (4.1.6)$$

from which we can deduce that there exists t_0 such that $p'(t)$ is non-increasing on $(t_0, +\infty)$. Therefore

$$\int_t^{t+\frac{1}{2}} p'(s) ds \leq \frac{1}{2} \max_{t \leq s \leq t+\frac{1}{2}} p'(s) = \frac{1}{2} p'(t), \quad (4.1.7)$$

and

$$\int_t^{t+\frac{1}{2}} p'(s)ds \geq \frac{1}{2} \min_{t \leq s \leq t+\frac{1}{2}} p'(t) = \frac{1}{2} p' \left(t + \frac{1}{2} \right) \quad (4.1.8)$$

for $t \geq t_0$. Note that

$$\frac{p'(t + \frac{1}{2})}{p'(t)} = \left(\frac{\log^n(t + \frac{1}{2})}{\log^n t} \right)^{\frac{1}{\rho n}} \frac{t(\log t) \cdots (\log^n t)}{(t + \frac{1}{2}) \log(t + \frac{1}{2}) \cdots \log^n(t + \frac{1}{2})} \rightarrow 1$$

as $t \rightarrow \infty$. Together with (4.1.8) and (4.1.7) we have

$$p \left(t + \frac{1}{2} \right) - p(t) = \int_t^{t+\frac{1}{2}} p'(s)ds \sim \frac{1}{2} p'(t), \quad t \rightarrow \infty.$$

□

Lemma 4.1.3. For $l \in \mathbb{R}$ with $[l] \geq k_0 + 1$,

$$\sum_{k=k_0+1}^{[l]} n_k \sim \int_{k_0+1}^l \frac{p(t)}{p'(t)} dt$$

as $l \rightarrow \infty$.

Proof. Denote

$$P(t) = \frac{p(t)}{p'(t)} \quad \text{and} \quad I(l) = \int_{k_0+1}^l P(t)dt.$$

With (4.1.1) and (4.1.2) we get

$$n_k \sim P(k) = \rho k \log k \cdots \log^n k, \quad (4.1.9)$$

for $k \geq k_0$. Since $P(t)$ is increasing with t we have for $k_0 \leq k \leq l$,

$$P(k) = P(k) \int_k^{k+1} dt \leq \int_k^{k+1} P(t)dt.$$

Therefore

$$\sum_{k=k_0+1}^{[l]} P(k) \leq \int_{k_0+1}^{l+1} P(t)dt = I(l) + \int_l^{l+1} P(t)dt \leq I(l) + P(l+1). \quad (4.1.10)$$

Similarly we obtain

$$\sum_{k=k_0+1}^{[l]} P(k) \geq \int_{k_0}^{l-1} P(t)dt \geq I(l) + c_0 - P(l), \quad (4.1.11)$$

where $c_0 = \int_{k_0}^{k_0+1} P(t)dt$. From (4.1.9) we may take $l \geq k_1 > k_0 + 1$ so large that $P(t) \geq t$ for all $t > k_1$. Thus

$$I(l) \geq \int_{k_1}^l t dt \geq \frac{1}{2}l^2 - \frac{1}{2}k_1^2.$$

Since

$$P(l) = \rho l \log(l) \cdots \log^n(l) = o(l^2) \quad \text{as } l \rightarrow \infty,$$

we have

$$P(l) = o(I(l)), \quad P(l+1) = o(I(l))$$

as $l \rightarrow \infty$. Together with (4.1.9), (4.1.10) and (4.1.11) we have our conclusion. \square

Lemma 4.1.4. For $k_0 \leq k < l$, $k \in \mathbb{N}$ and $l \in \mathbb{R}$,

$$\left(\frac{p(l)}{p(k)} \right)^{n_k} \geq \exp(c \min\{k, l - k\}), \quad (4.1.12)$$

where $c = (\log 2)^{n+1}/2$.

Proof. From (4.1.1), (4.1.2) and (4.1.9) we have

$$\begin{aligned} \left(\frac{p(l)}{p(k)} \right)^{n_k} &= \exp \left(n_k \log \frac{p(l)}{p(k)} \right) \\ &\geq \exp \left(\frac{1}{2} \frac{p(k)}{p'(k)} \log \frac{p(l)}{p(k)} \right) \\ &= \exp \left(\frac{1}{2} \rho k (\log k) \cdots (\log^n k) \log \left(\frac{\log^n l}{\log^n k} \right)^{\frac{1}{\rho}} \right) \\ &= \exp \left(\frac{1}{2} k (\log k) \cdots (\log^n k) \log \frac{\log^n l}{\log^n k} \right), \end{aligned} \quad (4.1.13)$$

We claim that

$$k (\log k) \cdots (\log^n k) \log \frac{\log^n l}{\log^n k} \geq (\log 2)^{n+1} \min\{k, l - k\}, \quad (4.1.14)$$

which is verified as follows by induction to n . We first consider that $n = 0$.

Case 1. If $k < \frac{l}{2}$ then

$$k \log \frac{l}{k} > k \log 2. \quad (4.1.15)$$

Case 2. If $\frac{l}{2} \leq k < l$, then $(l - k)/k < 1$ and thus

$$\begin{aligned} k \log \frac{l}{k} &= k \log \left(1 + \frac{l - k}{k} \right) \\ &\geq k \frac{l - k}{k} \log 2 \\ &= (l - k) \log 2. \end{aligned} \tag{4.1.16}$$

Together (4.1.15) and (4.1.16) give

$$k \log \frac{l}{k} \geq (\log 2) \min\{k, l - k\}, \tag{4.1.17}$$

which is (4.1.14) for $n = 0$. Suppose now that (4.1.14) holds for some n .

Case 1. If $\log^{n+1} l > 2 \log^{n+1} k$ and since $k \geq k_0$ then

$$\begin{aligned} k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} l}{\log^{n+1} k} &> k(\log k) \cdots (\log^{n+1} k) \log 2 \\ &> k(\log 2)^{n+2}. \end{aligned}$$

Case 2. If $\log^{n+1} l \leq 2 \log^{n+1} k$ then

$$\begin{aligned} &k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} l}{\log^{n+1} k} \\ &= k(\log k) \cdots (\log^{n+1} k) \log \left(1 + \frac{\log^{n+1} l - \log^{n+1} k}{\log^{n+1} k} \right) \\ &\geq k(\log k) \cdots (\log^{n+1} k) \frac{\log^{n+1} l - \log^{n+1} k}{\log^{n+1} k} \log 2 \\ &= k(\log k) \cdots (\log^n k) \log \frac{\log^n l}{\log^n k} \log 2 \\ &\geq (\log 2)^{n+1} \min\{k, l - k\} \log 2 \\ &= (\log 2)^{n+2} \min\{k, l - k\}. \end{aligned}$$

Therefore

$$k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} l}{\log^{n+1} k} \geq (\log 2)^{n+2} \min\{k, l - k\}. \tag{4.1.18}$$

From (4.1.17) and (4.1.18) we see that (4.1.14) holds with n replaced by $n + 1$. Together with (4.1.13) this gives (4.1.12), by taking $c = (\log 2)^{n+1}/2$.

□

Lemma 4.1.5. For $k \in \mathbb{N}$, $l \in \mathbb{R}$ with $k > l \geq k_0$,

$$\left(\frac{p(k)}{p(l)}\right)^{n_k} \geq \exp(c(k-l)), \quad (4.1.19)$$

where $c = (\log 2)^{n+1}/2$.

Proof. We prove this result in the same way as in Lemma 4.1.4. For $k > l$, instead of (4.1.13) we have

$$\left(\frac{p(k)}{p(l)}\right)^{n_k} \geq \exp\left(\frac{1}{2}k(\log k) \cdots (\log^n k) \log \frac{\log^n k}{\log^n l}\right). \quad (4.1.20)$$

Next we show that for $n \in \mathbb{N}$.

$$k(\log k) \cdots (\log^n k) \log \frac{\log^n k}{\log^n l} > (k-l)(\log 2)^{n+1}. \quad (4.1.21)$$

We first consider that $n = 0$.

Case 1. If $k > 2l$, then

$$k \log \frac{k}{l} > k \log 2 > (k-l) \log 2.$$

Case 2. If $l < k \leq 2l$, then

$$k \log \frac{k}{l} = k \log \left(1 + \frac{k-l}{l}\right) \geq k \frac{k-l}{l} \log 2 > (k-l) \log 2.$$

Therefore we have (4.1.21) for $n = 0$. Now we suppose that (4.1.21) holds for some $n \in \mathbb{N}$.

Case 1. If $\log^{n+1} k > 2 \log^{n+1} l$, since $k \geq k_0$ then

$$\begin{aligned} k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} k}{\log^{n+1} l} &> k(\log k) \cdots (\log^{n+1} k) \log 2 \\ &> k(\log k) \cdots (\log^n k) (\log 2)^2 \\ &> (k-l)(\log 2)^{n+2} \end{aligned}$$

Case 2. If $\log^{n+1} l < \log^{n+1} k \leq 2 \log^{n+1} l$, then by the assumption,

$$\begin{aligned}
k(\log k) \cdots (\log^{n+1} k) \frac{\log^{n+1} k}{\log^{n+1} l} &= k(\log k) \cdots (\log^{n+1} k) \log \left(1 + \frac{\log^{n+1} k - \log^{n+1} l}{\log^{n+1} l} \right) \\
&\geq k(\log k) \cdots (\log^{n+1} k) \frac{\log^{n+1} k - \log^{n+1} l}{\log^{n+1} l} \log 2 \\
&> k(\log k) \cdots (\log^n k) \log \frac{\log^n k}{\log^n l} \log 2 \\
&> (\log 2)^{n+1} (k - l) \log 2 \\
&= (k - l) (\log 2)^{n+2}.
\end{aligned}$$

Hence we have (4.1.21) by induction. Together with (4.1.20) and $c = (\log 2)^{n+1}/2$ we obtain (4.1.19). \square

4.2 The function is in the Eremenko-Lyubich class

Theorem 4.2.1. *Let $p(k)$ and n_k be defined as in (4.1.1) and (4.1.2). Set the function*

$$g(z) = 2 \sum_{k=k_0+1}^{\infty} \frac{p(k)^{n_k} z^{n_k}}{z^{2n_k} - p(k)^{2n_k}}. \quad (4.2.1)$$

Then g is a meromorphic function in \mathcal{B} and ∞ is not an asymptotic value of g .

Remark. Bergweiler and Kotus [4] gave an example for the case of infinite order,

$$f(z) = 2 \sum_{k=2}^{\infty} \frac{(\log k)^{n_k} z^{n_k}}{z^{2n_k} - (\log k)^{2n_k}},$$

where $n_k = \lfloor k \log k \rfloor$. Here we take $n_k = \lfloor \frac{p(k)}{p'(k)} \rfloor$ instead. If we let $n = 1$ and $\rho = 1$, then (4.2.1) is essentially the function above.

Proof. If $|z| \leq p(k)/2$, then

$$\left| \frac{p(k)^{n_k} z^{n_k}}{z^{2n_k} - p(k)^{2n_k}} \right| \leq \frac{|z|^{n_k} p(k)^{n_k}}{p(k)^{2n_k} - |z|^{2n_k}} \leq 2 \frac{|z|^{n_k}}{p(k)^{n_k}} \leq 2^{1-n_k}.$$

From (4.1.9) we see that $n_k \geq k$ for large k . Thus the series in (4.2.1) converges locally uniformly and hence it defines a function g meromorphic in \mathbb{C} .

Note that

$$u_{k,l} = p(k) \exp\left(\frac{\pi il}{n_k}\right) \quad (4.2.2)$$

are the poles of g , where $k \in \mathbb{N}$ and $0 \leq l \leq 2n_k - 1$. With $u_{k,l}$ we rewrite g as

$$g(z) = 2 \sum_{k=k_0+1}^{\infty} \sum_{l=0}^{2n_k-1} \frac{\nu_{k,l}}{z - u_{k,l}},$$

where

$$\begin{aligned} \nu_{k,l} &= \lim_{z \rightarrow u_{k,l}} (z - u_{k,l}) 2 \frac{p(k)^{n_k} z^{n_k}}{z^{2n_k} - p(k)^{2n_k}} \\ &= 2p(k)^{n_k} u_{k,l}^{n_k} \lim_{z \rightarrow u_{k,l}} \frac{z - u_{k,l}}{z^{2n_k} - p(k)^{2n_k}} \\ &= 2p(k)^{n_k} u_{k,l}^{n_k} \lim_{z \rightarrow u_{k,l}} \frac{1}{2n_k z^{2n_k-1}} \\ &= p(k)^{n_k} \frac{1}{n_k p(k)^{n_k-1} \left(\exp\left(\frac{\pi il}{n_k}\right)\right)^{n_k-1}} \\ &= \frac{p(k)}{n_k} \exp\left(\frac{\pi il(1 - n_k)}{n_k}\right). \end{aligned} \quad (4.2.3)$$

For $m \in \mathbb{N}$ we set

$$W_1 = \bigcup_{m \geq k_0+1} \left\{ z : |z| = p\left(m + \frac{1}{2}\right) \right\}.$$

and for $\eta \in \mathbb{N}$,

$$W_2 = \bigcup_{m \geq k_0+1} \left\{ r \exp\left(\frac{i\pi(2\eta - 1)}{2n_m}\right), 1 \leq \eta \leq 2n_m : p\left(m - \frac{1}{2}\right) \leq r \leq p\left(m + \frac{1}{2}\right) \right\}.$$

Let $W = W_1 \cup W_2$. We will show that g is bounded on this 'spider's web' W . First, let $z \in W_1$ and take m such that $|z| = p(m + 1/2)$. Note that, if $0 < x < y$, then

$$\frac{xy}{y^2 - x^2} = \frac{x}{y-x} \frac{y}{y+x} \leq \frac{x}{y-x}. \quad (4.2.4)$$

Since $p(k)$ is increasing with k , from (4.2.1) and (4.2.4) we have

$$\begin{aligned}
\frac{1}{2}|g(z)| &\leq \sum_{k=k_0+1}^m \frac{p(k)^{n_k} |z|^{n_k}}{|z|^{2n_k} - p(k)^{2n_k}} + \sum_{k=m+1}^{\infty} \frac{p(k)^{n_k} |z|^{n_k}}{p(k)^{2n_k} - |z|^{2n_k}} \\
&\leq \sum_{k=k_0+1}^m \frac{p(k)^{n_k}}{|z|^{n_k} - p(k)^{n_k}} + \sum_{k=m+1}^{\infty} \frac{|z|^{n_k}}{p(k)^{n_k} - |z|^{n_k}} \\
&= \sum_{k=k_0+1}^m \frac{1}{\left(\frac{p(m+\frac{1}{2})}{p(k)}\right)^{n_k} - 1} + \sum_{k=m+1}^{\infty} \frac{1}{\left(\frac{p(k)}{p(m+\frac{1}{2})}\right)^{n_k} - 1} \\
&=: \Sigma_{1,m} + \Sigma_{2,m}.
\end{aligned}$$

From Lemma 4.1.4 with $l = m + \frac{1}{2}$ we have, for $k_0 \leq k \leq m$,

$$\left(\frac{p(m+\frac{1}{2})}{p(k)}\right)^{n_k} \geq \exp\left(c \min\left\{k, m + \frac{1}{2} - k\right\}\right),$$

where $c = (\log 2)^{n+1}/2$. Thus

$$\begin{aligned}
\Sigma_{1,m} &\leq \sum_{k=k_0+1}^m \frac{1}{\exp\left(c \min\left\{k, m + \frac{1}{2} - k\right\}\right) - 1} \\
&= \sum_{k=k_0+1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{\exp(ck) - 1} + \sum_{k=\lfloor \frac{m}{2} \rfloor + 1}^m \frac{1}{\exp\left(c\left(m + \frac{1}{2} - k\right)\right) - 1} \\
&= \sum_{k=k_0+1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{\exp(ck) - 1} + \sum_{j=0}^{m - \lfloor \frac{m}{2} \rfloor - 1} \frac{1}{\exp\left(c\left(j + \frac{1}{2}\right)\right) - 1} \\
&\leq \sum_{k=1}^{\infty} \frac{1}{\exp(ck) - 1} + \sum_{j=0}^{\infty} \frac{1}{\exp\left(c\left(j + \frac{1}{2}\right)\right) - 1} =: C.
\end{aligned}$$

Similarly, by Lemma 4.1.5 with $l = m + \frac{1}{2}$, we obtain

$$\Sigma_{2,m} \leq \sum_{k=m+1}^{\infty} \frac{1}{\exp\left(c\left(k - \left(m + \frac{1}{2}\right)\right)\right) - 1} = \sum_{j=0}^{\infty} \frac{1}{\exp\left(c\left(j + \frac{1}{2}\right)\right) - 1} \leq C.$$

Therefore

$$|g(z)| \leq 2(\Sigma_{1,m} + \Sigma_{2,m}) \leq 4C, \tag{4.2.5}$$

for $|z| = p\left(m + \frac{1}{2}\right)$, $m \in \mathbb{N}$.

Next we consider $z \in W_2$. Then $z = r \exp(i\pi(2\eta - 1)/(2n_m))$, where $p(m - \frac{1}{2}) \leq r \leq p(m + \frac{1}{2})$ and $\eta \in \mathbb{N}$, $1 \leq \eta \leq 2n_m$. Thus

$$z^{2n_m} = \left(r \exp\left(i\pi \frac{2\eta - 1}{2n_m}\right) \right)^{2n_m} = -r^{2n_m}$$

With this, (4.2.4) and since $p(k)$ is increasing with k , we have

$$\begin{aligned} \frac{1}{2}|g(z)| &\leq \sum_{k=k_0+1}^{m-1} \frac{p(k)^{n_k} r^{n_k}}{r^{2n_k} - p(k)^{2n_k}} + \frac{p(m)^{n_m} r^{n_m}}{r^{2n_m} + p(m)^{2n_m}} + \sum_{k=m+1}^{\infty} \frac{p(k)^{n_k} r^{n_k}}{p(k)^{2n_k} - r^{2n_k}} \\ &\leq \sum_{k=k_0+1}^{m-1} \frac{p(k)^{n_k}}{p(m - \frac{1}{2})^{n_k} - p(k)^{n_k}} + 2 + \sum_{k=m+1}^{\infty} \frac{p(m + \frac{1}{2})^{n_k}}{p(k)^{n_k} - p(m + \frac{1}{2})^{n_k}} \\ &\leq \Sigma_{1,m-1} + 2 + \Sigma_{2,m} \\ &\leq 2C + 2. \end{aligned}$$

Combining with (4.2.5) it follows that

$$|g(z)| \leq 4C + 4 \text{ for } z \in W. \quad (4.2.6)$$

Actually g is bounded on a larger set, as we want to show next.

From Lemma 4.1.2 we have

$$p\left(m + \frac{1}{2}\right) - p(m) = \int_m^{m+\frac{1}{2}} p'(t) dt \sim \frac{1}{2} p'(m), \quad m \rightarrow \infty. \quad (4.2.7)$$

And note that by (4.1.2)

$$|u_{m,\eta} - u_{m,\eta+1}| = p(m) \left| \exp\left(\frac{i\pi}{n_m}\right) - 1 \right| \sim p(m) \frac{\pi}{n_m} \sim \pi p'(m), \quad m \rightarrow \infty. \quad (4.2.8)$$

If $W_{m,\eta}$ denotes the component of $\mathbb{C} \setminus W$ that contains $u_{m,\eta}$, we find that there exists $\lambda > 0$ such that

$$\text{dist}(u_{m,\eta}, \partial W_{m,\eta}) \geq 2\lambda p'(m), \quad (4.2.9)$$

for m large and $\eta \in \{0, 1, \dots, 2n_m - 1\}$.

Consider the function

$$\zeta(z) = g(z) - \frac{\nu_{m,\eta}}{z - u_{m,\eta}}, \quad (4.2.10)$$

which is holomorphic in the closure of $W_{m,\eta}$. For $z \in \partial W_{m,\eta}$ by (4.2.3), (4.2.6), (4.1.2) and $n_m \geq p(m)/(2p'(m))$ we have

$$\begin{aligned} |\zeta(z)| &\leq |g(z)| + \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|} \\ &\leq 4C + 4 + \frac{p(m)}{2\lambda n_m p'(m)} \\ &\leq 4C + 4 + \frac{1}{\lambda}, \end{aligned}$$

for m large. By the maximum principle,

$$|\zeta(z)| \leq 4C + 4 + \frac{1}{\lambda} \text{ for } z \in W_{m,\eta}.$$

We put $r_m = \lambda p'(m)$ and deduce that if $z \in W_{m,\eta} \setminus D(u_{m,\eta}, r_m)$, then

$$\begin{aligned} |g(z)| &\leq |\zeta(z)| + \frac{|\nu_{m,\eta}|}{r_m} \\ &\leq 4C + 4 + \frac{1}{\lambda} + \frac{p(m)}{\lambda n_m p'(m)} \\ &\leq 4C + 4 + \frac{3}{\lambda}. \end{aligned} \tag{4.2.11}$$

This means that g is large only in small neighbourhoods of the poles.

On the other hand we will show that the set of critical values of f is bounded by verifying that there are no critical points of g in these small neighbourhoods of the poles.

Assume that $z \in \partial W_{m,\eta}$ and m', η' are such that $z \in \partial W_{m',\eta'}$. Then $|m - m'| \leq 1$ and so $r_m \leq 2r_{m'}$. Hence by (4.2.9)

$$|z - u_{m',\eta'}| \geq \text{dist}(u_{m',\eta'}, \partial W_{m',\eta'}) \geq 2\lambda p'(m') \geq \frac{1}{2}r_m + r_{m'}$$

and we have $\overline{D(z, \frac{1}{2}r_m)} \cap D(u_{m',\eta'}, r_{m'}) = \emptyset$. Together with (4.2.11) it yields that

$$|g'(z)| = \frac{1}{2\pi} \left| \int_{|\xi-z|=\frac{1}{2}r_m} \frac{g(\xi)}{(\xi-z)^2} d\xi \right| \leq \frac{r_m}{2} \max_{|\xi-z|=\frac{1}{2}r_m} \frac{|g(\xi)|}{|\xi-z|^2} \leq \frac{2}{r_m} \left(4C + 4 + \frac{3}{\lambda} \right).$$

Since $n_m > p(m)/(2p'(m))$ for m large, from (4.2.10), (4.2.3) and (4.2.9) we have

$$\begin{aligned}
|\zeta'(z)| &\leq |g'(z)| + \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|^2} \\
&\leq \frac{2}{r_m} \left(4C + 4 + \frac{3}{\lambda}\right) + \frac{p(m)}{n_m(2\lambda p'(m))^2} \\
&= \frac{2}{r_m} \left(4C + 4 + \frac{3}{\lambda}\right) + \frac{p(m)}{4\lambda n_m r_m p'(m)} \\
&\leq \frac{2}{r_m} \left(4C + 4 + \frac{3}{\lambda}\right) + \frac{1}{2\lambda r_m} \\
&= \frac{2}{r_m} \left(4C + 4 + \frac{13}{4\lambda}\right)
\end{aligned}$$

for $z \in \partial W_{m,\eta}$. It implies by maximum principle that

$$|\zeta'(z)| \leq \frac{2}{r_m} \left(4C + 4 + \frac{13}{4\lambda}\right) \quad \text{for } z \in W_{m,\eta}.$$

Choose $\delta > 0$ sufficiently small and $z \in D(u_{m,\eta}, \delta r_m)$. Since $n_m < 2p(m)/p'(m)$ for m large we have

$$\begin{aligned}
|g'(z)| &\geq \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|^2} - |\zeta'(z)| \\
&\geq \frac{p(m)}{\delta^2 n_m r_m \lambda p'(m)} - \frac{2}{r_m} \left(4C + 4 + \frac{13}{4\lambda}\right) \\
&\geq \frac{2}{r_m} \left(\frac{1}{4\delta^2 \lambda} - 4C - 4 - \frac{13}{4\lambda}\right) > 0.
\end{aligned}$$

Hence if $g'(z) = 0$ for some $z \in W_{m,\eta}$ then $|z - u_{m,\eta}| \geq \delta r_m$. Then

$$|g(z)| \leq |\zeta(z)| + \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|} \leq 4C + 4 + \frac{1}{\lambda} + \frac{p(m)}{\delta \lambda n_m p'(m)} \leq 4C + 4 + \frac{1}{\lambda} + \frac{2}{\delta \lambda} \quad (4.2.12)$$

Therefore the set of critical values is bounded by $4C + 4 + \frac{1}{\lambda} + \frac{2}{\delta \lambda}$. From (4.2.6) the same is true for the set of asymptotic values. Thus $g \in \mathcal{B}$. \square

4.3 The growth order of the function

We have shown that the functions constructed in the last two sections are meromorphic and in the Eremenko-Lyubich class. In this section we estimate the growth

order. More precisely, for $0 < \rho < \infty$ and $n \in \mathbb{N}$ as in Section 4.1, we show the n -th order of those functions is exactly ρ .

Theorem 4.3.1. *Let g be defined as in (4.2.1). Then $\rho_n(g) = \rho$.*

Proof. From Lemma 4.1.3 the number $n(r, g)$ of poles of g in $\overline{D(0, r)}$ satisfies

$$n(r, g) = \sum_{k=k_0+1}^{\lfloor q(r) \rfloor} 2n_k \sim 2 \int_{k_0+1}^{q(r)} \frac{p(t)}{p'(t)} dt.$$

Now let $t = q(s)$ and $r_0 = p(k_0 + 1)$. Then

$$\begin{aligned} \int_{k_0+1}^{q(r)} \frac{p(t)}{p'(t)} dt &= \int_{r_0}^r \frac{p(q(s))}{p'(q(s))} q'(s) ds \\ &= \int_{r_0}^r \frac{s}{p'(q(s))} q'(s) ds \\ &= \int_{r_0}^r q'(s)^2 s ds. \end{aligned}$$

Hence

$$n(r, g) \sim 2 \int_{r_0}^r q'(s)^2 s ds \tag{4.3.1}$$

for $r \rightarrow \infty$. By Lemma 4.1.1 we have

$$\frac{q(r)q'(r)}{q'(r)^2 r} = \frac{q(r)}{r q'(r)} \rightarrow 0, \quad r \rightarrow \infty. \tag{4.3.2}$$

On the other hand,

$$\begin{aligned} \frac{q''(r)q(r)}{q'(r)^2} &= 1 - \frac{q'(r)q'(r) - q(r)q''(r)}{q'(r)^2} \\ &= 1 - \frac{d}{dr} \left(\frac{q(r)}{q'(r)} \right). \end{aligned}$$

Again by lemma 4.1.1 we have

$$\frac{q(r)q''(r)}{q'(r)^2} \rightarrow 1, \quad r \rightarrow \infty, \tag{4.3.3}$$

We claim that

$$2 \int_{r_0}^r q'(s)^2 s ds \sim q(r)q'(r)r \quad \text{as } r \rightarrow \infty. \tag{4.3.4}$$

In fact, with (4.3.2) and (4.3.3) by l'Hospital's rule we have

$$\begin{aligned}
\lim_{r \rightarrow \infty} \frac{q(r)q'(r)r}{\int_{r_0}^r q'(s)^2 s ds} &= \lim_{r \rightarrow \infty} \frac{\frac{d}{dr}(q(r)q'(r)r)}{\frac{d}{dr} \int_{r_0}^r q'(s)^2 s ds} \\
&= \lim_{r \rightarrow \infty} \frac{q'(r)^2 r + q(r)q''(r)r + q(r)q'(r)}{q'(r)^2 r} \\
&= \lim_{r \rightarrow \infty} \left(1 + \frac{q(r)q''(r)}{q'(r)^2} + \frac{q(r)}{q'(r)r} \right) \\
&= 2.
\end{aligned}$$

Therefore from (4.3.1) and (4.3.4)

$$N(r, g) = \int_0^r \frac{n(t, g)}{t} dt \sim \int_0^r \frac{q(t)q'(t)t}{t} dt = \frac{1}{2}q(r)^2$$

as $r \rightarrow \infty$.

Suppose that r has the form $r = p(k + \frac{1}{2})$ for $k_0 \leq k \in \mathbb{N}$ large. From (4.2.6) we have $m(r, g) \leq 4C + 4$. Since

$$T(r, g) = N(r, g) + m(r, g),$$

we have

$$T(r, g) \sim \frac{1}{2}q(r)^2, \quad r \rightarrow \infty.$$

It yields that

$$\begin{aligned}
\log T(r, g) &= (1 + o(1))2 \log q(r) \\
&= (1 + o(1))2 \exp^{n-1}(r^\rho),
\end{aligned}$$

and thus

$$\log^{n+1} T(r, g) \sim \rho \log r \tag{4.3.5}$$

as $r \rightarrow \infty$ through r -values of the form $r = p(k + \frac{1}{2})$. It follows that (4.3.5) holds for all r since $T(r, g)$ is increasing with r . Hence

$$\rho_n(g) = \limsup_{r \rightarrow \infty} \frac{\log^{n+1} T(r, g)}{\log r} = \rho.$$

□

Chapter 5

Estimate from below

5.1 Some lemmas

The following lemma is a simple consequence of the first fundamental theorem of Nevanlinna theory, see [12, p.105]. It shows the relation between the n -th order and the number of poles for a meromorphic function.

Lemma 5.1.1. *Suppose that f is a meromorphic function and its n -th order is defined as in Section 1.3 and that $n(r)$ denotes the number of the poles contained in the closed disc $\overline{D(0, r)}$. Then we have*

$$\rho_n(f) \geq \limsup_{r \rightarrow \infty} \frac{\log^{n+1} n(r)}{\log r}.$$

For $k \in \mathbb{N}$, let N_k be a collection of disjoint compact sets in \mathbb{R}^n such that

- (a) every element of N_k contains an element of N_{k+1} ,
- (b) every element of N_{k+1} is contained in an element of N_k .

Let $\overline{N_k} = \bigcup_{A \in N_k} A$ and $N = \bigcap_{k=1}^{\infty} \overline{N_k}$.

McMullen [14] gave a lower bound for the Hausdorff dimension of a set N constructed this way. Peter [19, p.33] used McMullen's method to obtain a sufficient condition for the set N to have infinite Hausdorff measure with respect to some gauge function

h. We mention that they both worked with the Euclidean distance but the following lemma follows directly from the original one.

For measurable subsets X, Y of the plane (or sphere) we define the Euclidean and the spherical density of X in Y by

$$\text{dens}(X, Y) = \frac{\text{area}(X \cap Y)}{\text{area}(Y)} \quad \text{and} \quad \text{dens}_\chi(X, Y) = \frac{\text{area}_\chi(X \cap Y)}{\text{area}_\chi(Y)}.$$

Note that

$$\left(\frac{1+R^2}{1+S^2}\right)^2 \text{dens}(X, Y) \leq \text{dens}_\chi(X, Y) \leq \left(\frac{1+S^2}{1+R^2}\right)^2 \text{dens}(X, Y), \quad (5.1.1)$$

if Y is a subset of the annulus $\{z \in \mathbb{C} : R < |z| < S\}$.

With this terminology Peter's result takes the following form.

Lemma 5.1.2. *For $k \in \mathbb{N}$, let N_k, N be as above. Suppose that $\Delta_k > 0$, $d_k > 0$, $d_k \rightarrow 0$, such that if $B \in N_k$, then*

$$\text{dens}_\chi(\overline{N_{k+1}}, B) \geq \Delta_k \quad \text{and} \quad \text{diam}_\chi B \leq d_k.$$

Set $h(t) = t^2 g(t)$ for $t > 0$, where $g(t)$ is a decreasing continuous function such that $h(t)$ is increasing and satisfies $\lim_{t \rightarrow 0} t^2 g(t) = 0$. Then we have $\mu_h(N) = \infty$ if

$$\lim_{k \rightarrow \infty} g(d_k) \prod_{j=1}^k \Delta_j = \infty.$$

5.2 Proof of the second theorem

Let g be the function constructed in Section 4.2 and put $f(z) = g(z)^M$. Hence the multiplicity of all poles of f is M , $f \in \mathcal{B}$ without ∞ as its asymptotic value and $\rho_n(f) = \rho$.

As in Section 3.3 we denote the sequence of poles by a_j , ordered such that $|a_j| \leq |a_{j+1}|$ for all $j \in \mathbb{N}$. Choose b_j as in Section 3.3 so that

$$f(z) \sim \left(\frac{b_j}{z - a_j}\right)^M \quad \text{as } z \rightarrow a_j,$$

for each $j \in \mathbb{N}$. We thus have $a_j = u_{m,\eta}$ and $b_j = \nu_{m,\eta}$ by (4.2.3) for some $m, \eta \in \mathbb{N}$ and $0 \leq \eta \leq 2n_m - 1$.

Choose $R_0 \geq 4C + 4 + \frac{1}{\lambda} + \frac{2}{\delta\lambda}$, where λ, δ are as in (4.2.12) and $R_l = R_0 \exp(2^l)$ for $l \in \mathbb{N}$. We denote by E_l the collections of components V of $f^{-l}(B(R_l))$ which satisfy $f^k(V) \subset U_{j_{k+1}} \subset B(R_k)$ for $0 \leq k \leq l-1$ and certain $j_1, j_2, \dots, j_l \in \mathbb{N}$. Put $\overline{E}_l = \bigcup_{A \in E_l} A$. It follows that $E = \bigcap_{l=1}^{\infty} \overline{E}_l \subset I(f)$.

The estimates obtained in Section 3.3 also hold with R replaced by R_l . So we may use them for the map g_j that maps a simply connected domain $D \subset B(R_l) \setminus \{\infty\}$ to U_j^l , the component of $f^{-1}(B(R_l))$ containing a_j . From (3.3.13) we deduce that if $V \in E_l$ such that $f^k(V) \subset U_{j_{k+1}} \subset B(R_k)$ for $0 \leq k \leq l-1$, $l \in \mathbb{N}$ and certain $j_1, j_2, \dots, j_l \in \mathbb{N}$, then (3.3.20) holds.

Here a_{j_k} is the pole of f that is contained in U_{j_k} for $k = 1, 2, \dots, l$ and $l \in \mathbb{N}$. From (4.2.2) and (4.2.3) we know that $|a_{j_k}| =: r_{j_k} = p(l)$ for some $l \geq k_0 + 1$ and accordingly $|b_{j_k}| = p(l)/n_l$. With the definition of p, q and n_l we have

$$|b_{j_k}| \sim \frac{p(l)}{p(l)/p'(l)} = p'(l) = \frac{1}{q'(p(l))}.$$

Therefore

$$\frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}} \sim \frac{1}{q'(r_{j_k})r_{j_k}^{1+\frac{1}{M}}}. \quad (5.2.1)$$

Recall that $q(r) = \exp^n(r^\rho)$ is convex and thus $q'(r)$ is increasing. Moreover $r_{j_k} \geq R_{k-1}$ for $k = 1, 2, \dots, l$. It follows from (3.3.20) and (5.2.1) that

$$\text{diam}_\chi(V) \leq \prod_{k=1}^l \frac{A}{q'(R_{k-1})R_{k-1}^{1+\frac{1}{M}}} =: d_l, \quad (5.2.2)$$

where $A \neq 0$ is a constant.

With d_l we intend to apply Lemma 5.1.2. In order to do so we are estimating Δ_l . From (3.3.3) and (4.2.6) we deduce that

$$D \left(a_j, \frac{|b_j|}{4R_l^{\frac{1}{M}}} \right) \subset U_j^l = W_{m,\eta} \cap f^{-1}(B(R_l)) \quad (5.2.3)$$

for certain m and η . Then $a_j = u_{m,\eta}$ and thus $|b_j| = r_j/n_m = |\nu_{m,\eta}|$. Hence (4.2.7) and (4.2.8) imply that

$$W_{m,\eta} \subset D(u_{m,\eta}, \tau|\nu_{m,\eta}|) = D(a_j, \tau|b_j|), \quad (5.2.4)$$

where $\tau = 1/2 + \pi/2$, $a_j = u_{m,\eta}$, and m is large. For $\varepsilon > 0$ small set

$$A(S) = \{z \in \mathbb{C} : S < |z| < 2S\},$$

and

$$A_\varepsilon(S) = \{z \in \mathbb{C} : (1 + \varepsilon)S < |z| < (1 - \varepsilon)2S\}.$$

Then from (5.2.3) we have

$$\begin{aligned} \text{area} \left(\bigcup_{k \in \mathbb{N}} U_k^l \cap A(S) \right) &\geq \text{area} \left(\bigcup_{a_k \in A_\varepsilon(S)} U_k^l \right) \\ &\geq \sum_{a_k \in A_\varepsilon(S)} \pi \left(\frac{1}{4R_l^{\frac{1}{M}}} |b_k| \right)^2 \\ &= \pi \frac{1}{16R_l^{\frac{2}{M}}} \sum_{a_k \in A_\varepsilon(S)} |b_k|^2. \end{aligned}$$

On the other hand with (5.2.4) there exists a $\delta > 0$ such that

$$\begin{aligned} \text{area } A(S) &\leq (1 + \delta) \text{area } A_\varepsilon(S) \\ &\leq (1 + \delta) \text{area} \left(\bigcup_{a_k \in A_\varepsilon(S)} W_{m,\eta} \right) \\ &\leq \pi(1 + \delta)\tau^2 \sum_{a_k \in A_\varepsilon(S)} |b_k|^2. \end{aligned}$$

We conclude that

$$\text{dens} \left(\bigcup_{k \in \mathbb{N}} U_k^l, A(S) \right) \geq \frac{1}{16(1 + \delta)\tau^2 R_l^{\frac{2}{M}}}. \quad (5.2.5)$$

Now consider g_j as defined in (3.3.10), which is a branch of f^{-1} mapping

$$A'(S) = A(S) \setminus (-2S, -S)$$

into U_j^l . We apply Koebe's theorem to g_j with $\lambda = 1/2$. From (3.3.11) we obtain

$$|g_j'(z)| \leq \frac{12|b_j|}{m_j S^{1+\frac{1}{m_j}}},$$

for $z \in A'(S)$.

On the other hand from the left side of (3.2.2) we have

$$|g_j'(z)| \geq \frac{1 - \frac{1}{2}}{(1 + \frac{1}{2})^3} \frac{1}{m_j |z|^{1+\frac{1}{m_j}}} |\psi'(0)| = \frac{4|b_j|}{27m_j(2S)^{1+\frac{1}{m_j}}}$$

for $z \in A'(S)$.

Therefore

$$\sup_{u, v \in A'(S)} \left| \frac{g_j'(u)}{g_j'(v)} \right| \leq 324,$$

for S large enough. Hence by (5.2.5) and Lemma 3.2.3 it yields

$$\text{dens} \left(g_j \left(\bigcup_{k \in \mathbb{N}} U_k^l \right), g_j(A'(S)) \right) \geq \frac{1}{324^2} \text{dens} \left(\bigcup_{k \in \mathbb{N}} U_k^l, A'(S) \right) \geq \frac{\alpha}{R_l^{2/M}}, \quad (5.2.6)$$

where $\alpha = 1/(16(1 + \delta)324^2\tau^2)$.

Now we let $S = 2^k R_0$ with $k \geq 0$. Applying the above for all such S and for all branches g_j mapping $A'(S)$ to U_j^l , from (5.2.6) we deduce that

$$\text{dens} \left(f^{-1} \left(\bigcup_{k \in \mathbb{N}} U_k^l \right), U_j^l \right) \geq \frac{\alpha}{R_l^{2/M}}, \quad (5.2.7)$$

for each U_j^l in $f^{-1}(B(R_l))$.

Now suppose that $V \subset U_{j_1}$ is a component of E_l . Let j_2, \dots, j_l be such that $f^k(V) \subset U_{j_{k+1}} \subset B(R_k)$ for $0 \leq k \leq l-1$. Then $f^{l-1}(V) = U_{j_l}$, and thus

$$f^{l-1}(\overline{E_{l+1}} \cap V) = f^{-1} \left(\bigcup_{k \in \mathbb{N}} U_k^l \right) \cap U_{j_l}. \quad (5.2.8)$$

Denote by g a branch of the inverse of f^{l-1} which maps U_{j_l} to V and g_{j_l} a branch of f^{-1} that maps U_{j_l} into $U_{j_{l-1}}$. Given a constant $c \in (0, 1)$ such that $D(a_{j_l}, c|a_{j_l}|) \subset D(a_{j_l}, |a_{j_l}| - R_l)$ for l large.

Since $D(a_{j_l}, |a_{j_l}| - R_l) \cap D(0, R_l) = \emptyset$ for l large g_{j_l} extends univalently to a map from $D(a_{j_l}, c|a_{j_l}|)$ to $U_{j_{l-1}}$. It implies that g extends univalently to $D(a_{j_l}, c|a_{j_l}|)$.

Since $f(U_{j_l}) = f^l(V) = B(R_l)$, we have $U_{j_l} = U_k^l$ for some $k \in \mathbb{N}$. Noting that $U_{j_l} \subset D(a_{j_l}, \frac{1}{2}|a_{j_l}|)$ by (3.3.5), we can now take $c \in (\frac{1}{2}, 1)$ say, $c = \frac{3}{4}$ and apply Koebe's distortion theorem to g with $\lambda = \frac{2}{3}$. From Lemma 3.2.3 and by (5.2.7), (5.2.8) and (3.2.2) we obtain

$$\begin{aligned} \text{dens}(\overline{E_{l+1}}, V) &\geq \text{dens}\left(g\left(f^{-1}\left(\bigcup_{k \in \mathbb{N}} U_k^l\right), g(U_{j_l})\right)\right) \\ &\geq \left(\frac{1+\lambda}{1-\lambda}\right)^8 \text{dens}\left(f^{-1}\left(\bigcup_{k \in \mathbb{N}} U_k^l\right), U_{j_l}\right) \\ &\geq \left(\frac{1+\lambda}{1-\lambda}\right)^8 \frac{\alpha}{R_l^{2/M}}. \end{aligned}$$

Together with (5.1.1) we conclude that there exists a constant B such that

$$\text{dens}_\chi(\overline{E_{l+1}}, V) \geq \frac{B}{R_l^{2/M}} =: \Delta_l. \quad (5.2.9)$$

Next set $h(t)$ be as in (2.3.1) and $g(t) = (\log^n \frac{1}{t})^\gamma$. It is easy to see that $g(t)$ is a decreasing continuous function and $\lim_{t \rightarrow 0} t^2 g(t) = \lim_{t \rightarrow 0} h(t) = 0$. Now we shall apply Lemma 5.1.2 with $h(t)$ and $g(t)$.

From (5.2.2) we have

$$\begin{aligned} \log^n \frac{1}{d_l} &= \log^n \left(\prod_{k=1}^l \frac{1}{A} q'(R_{k-1}) R_{k-1}^{1+\frac{1}{M}} \right) \\ &= \log^{n-1} \log \left(\prod_{k=1}^l \frac{1}{A} q'(R_{k-1}) R_{k-1}^{1+\frac{1}{M}} \right) \\ &= \log^{n-1} \sum_{k=1}^l \left(\log \frac{1}{A} + \log q'(R_{k-1}) + \left(1 + \frac{1}{M}\right) \log R_{k-1} \right). \end{aligned}$$

Noting that by (4.1.4),

$$\log q'(r) = \log((\exp^n r^\rho)(\exp^{n-1} r^\rho) \cdots (\exp r^\rho) \rho r^{\rho-1}) \sim \exp^{n-1}(r^\rho)$$

as $r \rightarrow \infty$, we have

$$\log^n \frac{1}{d_l} \sim \log^{n-1} \sum_{k=1}^l \log q'(R_{k-1}) \sim \log^{n-1} \sum_{k=1}^l \exp^{n-1}(R_{k-1}^\rho) \sim R_{l-1}^\rho$$

as $l \rightarrow \infty$. Since $R_l = R_0 \exp(2^l)$ we obtain

$$g(d_l) = \left(\log^n \frac{1}{d_l} \right)^\gamma \sim R_{l-1}^{\gamma\rho} \sim R_0^{\gamma\rho} \exp(\rho\gamma 2^{l-1}), \quad l \rightarrow \infty.$$

Thus there exists a constant $K > 0$ such that

$$g(d_l) \geq K R_0^{\gamma\rho} \exp(\rho\gamma 2^{l-1}) \quad (5.2.10)$$

for l large.

On the other hand since R_l is non-decreasing it follows that

$$\begin{aligned} \prod_{k=1}^l \Delta_k &= B^l \prod_{k=1}^l R_k^{-2/M} = B^l R_0^{-2l/M} \prod_{k=1}^l \exp\left(-\frac{2}{M} 2^k\right) \\ &= B^l R_0^{-2l/M} \exp\left(-\frac{2}{M} \sum_{k=1}^l 2^k\right) \\ &= B^l R_0^{-2l/M} \exp\left(-\frac{8}{M} 2^{l-1} + \frac{4}{M}\right). \end{aligned}$$

Together with (5.2.10) we have

$$g(d_l) \prod_{k=1}^l \Delta_k \geq K B^l R_0^{\gamma\rho - 2l/M} \exp\left(\frac{4}{M}\right) \exp\left(2^{l-1} \left(\rho\gamma - \frac{8}{M}\right)\right) \rightarrow \infty \quad (5.2.11)$$

as $l \rightarrow \infty$ if $\gamma > 8/(M\rho)$.

With Lemma 5.1.2 and (5.2.11) we complete the proof.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht.

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