

TEST SEQUENCES AND FORMAL SOLUTIONS OVER HYPERBOLIC GROUPS

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von
SIMON HEIL

Kiel, 2018

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Tag der mündlichen Prüfung: 6.7.2018

Zusammenfassung

Sei G eine Gruppe und σ ein Satz in der Sprache der Gruppen. Dann sagen wir, dass σ wahr in G ist (bezeichnet mit $G \models \sigma$), falls σ wahr ist, wenn man ihn in G interpretiert. Die elementare Theorie einer Gruppe G , bezeichnet mit $\text{Th}(G)$, ist die Menge aller Sätze, die in G wahr sind.

Die Gruppen, für deren elementare Theorie wir uns hauptsächlich interessieren, sind freie und hyperbolische Gruppen. Das Interesse von Modelltheoretikern an der elementaren Theorie von freien Gruppen geht zurück bis in die 1940er Jahre, als Alfred Tarski seine berühmten Fragen gestellt hat, die heute als die Tarski Probleme bekannt sind. Diese fragen ob alle nicht-abelschen freien Gruppen die gleiche elementare Theorie haben und ob diese Theorie entscheidbar ist.

2006 haben Z. Sela und unabhängig davon O. Kharlampovich und A. Myasnikov gezeigt, dass $\text{Th}(F_n) = \text{Th}(F_m)$ für alle $m, n \geq 2$ und damit Tarski's Fragen positiv beantwortet. Im Folgenden gelang es Z. Sela seine im Beweis der Tarski Probleme verwendeten Techniken zu verallgemeinern und dadurch alle endlich erzeugten Gruppen zu klassifizieren, die elementar äquivalent zu torsionsfreien hyperbolischen Gruppen sind. Ein wichtiger Schritt in seiner Analyse der elementaren Theorie von freien und torsionsfreien hyperbolischen Gruppen ist eine Verallgemeinerung des folgenden Satzes von Merzlyakov. Sei $F_k = F(a)$ eine nicht-abelsche freie Gruppe. Angenommen es gilt

$$F_k \models \forall y \exists x \Sigma(x, y, a) = 1 \wedge \Psi(x, y, a) \neq 1.$$

Dann existiert eine Retraktion $\pi : \langle x, y, a \mid \Sigma(x, y, a) = 1 \rangle \rightarrow F(y) * F_k$ und

$$F_k \models \exists y \Psi(\pi(x), y, a) \neq 1.$$

Sela's verallgemeinerter Satz von Merzlyakov betrachtet den Fall, dass die universellen Variablen y in einer Varietät liegen. In unserer Arbeit beweisen wir eine Version des verallgemeinerten Satzes von Merzlyakov über hyperbolischen Gruppen (mit Torsion). Dafür zeigen wir, dass, gegeben eine hyperbolische Gruppe Γ und eine Γ -Limes Gruppe L , eine größere Gruppe $\text{Comp}(L)$ existiert, in welche man L einbetten kann. Zusätzlich existiert eine Folge von Punkten (λ_n) in der Varietät $\text{Hom}(\text{Comp}(L), \Gamma)$ aus der man die Struktur der Gruppe $\text{Comp}(L)$ rekonstruieren kann. Mit Hilfe einer solchen Testsequenz (λ_n) können wir schließlich eine Version des verallgemeinerten Satzes von Merzlyakov über allen hyperbolischen Gruppen beweisen.

Weiterhin geben wir ein Beispiel eines Makanin-Razborov Diagramms einer Gruppe mit nicht-trivialer JSJ Zerlegung an und nutzen dieses, um zu zeigen, dass Limes Gruppen im Allgemeinen nicht frei Untergruppen-separabel sind. Zudem klassifizieren wir alle JSJ Zerlegungen von Doppeln von freien Gruppen von Rang zwei.

Abstract

Given a group G and a sentence σ in the language of groups, we say that σ is a true sentence in G (denoted by $G \models \sigma$) if σ is true if interpreted in G . The first-order or elementary theory of a group G , denoted by $\text{Th}(G)$, is the set of all sentences true in G . The groups whose first-order theory we are mainly interested in, are free and hyperbolic groups. The interest of model theorist in the first-order theory of free groups dates back to the 1940s when Alfred Tarski posed his famous questions, which are now widely known as the Tarski-problems, namely if all non-abelian free groups have the same elementary theory and if this theory is decidable.

In 2006 Z. Sela and independently O. Kharlampovich and A. Myasnikov showed that $\text{Th}(F_n) = \text{Th}(F_m)$ for all $m, n \geq 2$, answering Tarski's question to the positive. Subsequently Z. Sela generalized his techniques used in his proof of the Tarski problems to classify all finitely generated groups elementary equivalent to a given torsion-free hyperbolic group. One important step in his analysis of the elementary theory of free and torsion-free hyperbolic groups is a generalization of the following theorem of Merzlyakov. Let $F_k = F(a)$ be a non-abelian free group and suppose that

$$F_k \models \forall y \exists x \Sigma(x, y, a) = 1 \wedge \Psi(x, y, a) \neq 1.$$

Then there exists a retraction $\pi : \langle x, y, a \mid \Sigma(x, y, a) = 1 \rangle \rightarrow F(y) * F_k$ and

$$F_k \models \exists y \Psi(\pi(x), y, a) \neq 1.$$

Sela's generalized Merzlyakov's Theorem covers the case that the universal variables y are restricted to some variety. In our work we prove a version of the Generalized Merzlyakov's Theorem over hyperbolic groups (possibly with torsion). To do this, we show that given a hyperbolic group Γ and Γ -limit group L , there exists a larger group $\text{Comp}(L)$ into which L embeds, and a sequence of points (λ_n) in the variety $\text{Hom}(\text{Comp}(L), \Gamma)$ from which one can recover the structure of the group $\text{Comp}(L)$. Using such a test sequence (λ_n) we are finally able to prove a version of the Generalized Merzlyakov's Theorem over all hyperbolic groups.

We moreover give an example of a Makanin-Razborov diagram for a group with non-trivial JSJ decomposition and conclude that limit groups are in general not freely subgroup separable. We then classify all JSJ decomposition of doubles of free groups of rank two.

Danksagung

Mein herzlicher Dank gilt Richard Weidmann für seine gute Betreuung während meiner Promotion. Er hatte stets ein offenes Ohr für Fragen und Probleme und durch die vielen Diskussionen mit ihm wurde diese Arbeit überhaupt erst ermöglicht.

Besonders danken möchte ich auch Zlil Sela für seine Geduld mit mir und meinen Fragen zu seiner Arbeit über die Tarski Probleme. Er hat sich stets die Zeit genommen ausführlich auf jede meiner Emails zu antworten.

Desweiteren danke ich meinen Mitdoktoranten Daniel Bissinger, Giso Jahncke und Eric Henack für die angenehme Arbeitsatmosphäre.

Schlussendlich danke ich Marie für ihre durchgehende Unterstützung während meiner Promotion.

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1 Introduction

Model theory is a branch of mathematical logic where mathematical structures are studied by considering the first-order sentences true in those structures and the sets definable by first-order formulas. Intuitively, a structure is a set equipped with distinguished functions, relations and elements. We then use a language where we can talk about this structure. For example the first-order language L_0 of groups uses the following symbols:

- The binary function group multiplication " \cdot ", the unary function inverse " $^{-1}$ ", the constant " 1 " and the equality relation " $=$ ".
- Variables " x, y, z " or tuples of variables " $\underline{x} = (x_1, \dots, x_n)$ " (which will have to be interpreted as individual group elements).
- Logical connectives " \wedge " ("and"), " \vee " ("or"), " \neg " ("not"), the quantifiers " \forall " ("for all") and " \exists " ("there exists") and parentheses "(" and ")".

A *formula* in the language of groups is a logical expression using the above symbols. A variable in a formula is called *bound* if it is bound to a quantifier and *free* otherwise. A *sentence* is a formula in L_0 where all variables are bound.

Given a group G and a sentence σ we will say that σ is a *true sentence in G* (denoted by $G \models \sigma$) if σ is true if interpreted in G (i.e. variables interpreted as group elements in G). The *first-order* or *elementary theory* of a group G , denoted by $\text{Th}(G)$, is the set of all sentences of L_0 true in G .

The groups whose first-order theory we are mainly interested in, are free and hyperbolic groups. The interest of model theorist in the first-order theory of free groups dates back to the 1940s when Alfred Tarski posed his famous questions, which are now widely known as the Tarski-problems.

Tarski-problems. (1) $\text{Th}(F_n) = \text{Th}(F_m)$ for all $m, n \geq 2$?

(2) Is the first-order theory of a non-abelian free group F_n decidable?

There was some early progress on the subject by Vaught who proved that two non-abelian free groups of infinite rank have the same elementary theory. Meanwhile Merzlyakov was able to show that all non-abelian free groups have the same positive theory, that is all sentences not containing negations.

In the 1980s Makanin then described an algorithm that could decide if a given system of equations (with constants) has a solution in a free group. Later Razborov refined this process to give a complete description of the set of solutions for a system of equations in a free group. This is now called the Makanin-Razborov algorithm.

Rips then recognized that one could use this algorithm to study group actions on real trees. This culminated in a structure theorem for the actions of finitely presented groups on real trees, which is usually referred to as the Rips machine. The original version of the

Rips machine has later been generalized by Sela to actions of finitely generated groups and further refined by Guirardel [Gu].

Still the original questions of Tarski remained unsolved and became known for their notorious difficulty. It took well over 60 years until finally in 2006 Zlil Sela and independently O. Kharlampovich and A. Myasnikov published their astonishing solutions to Tarski's problems ([Se1]-[Se6] and [KhMy]). Sela only considered the elementary theory of free groups, while Kharlampovich/Myasnikov also answered Tarski's second question to the positive. Despite this now being more than ten years ago, up to the present day the mathematical community has not yet fully absorbed the sophisticated ideas underlying Sela's voluminous work.

His proof splits up into seven papers. In [Se1] he uses a geometric interpretation of ideas going back to Makanin and Razborov's algorithm for solving systems of equations over free groups to give a parametrization of $\text{Hom}(G, F_k)$ for a fixed non-abelian free group F_k and an arbitrary finitely generated group G . In [Se2] he proves an implicit function-type theorem over free groups, which he calls Generalized Merzlyakov's Theorem and shows that given an $\forall\exists$ -sentence true in F_k , there exist so-called formal solutions witnessing the truthfulness of this sentence.

In his third paper ([Se3]) he shows that there exists a uniform bound on exceptional solutions to systems of equations with parameters. In [Se4] he uses the results from the first three papers to show that all non-abelian free groups have the same $\forall\exists$ -theory. This is the induction start for the general proof of the elementary equivalence of non-abelian free groups. Namely he proves quantifier elimination over free groups in [Se5.1] and [Se5.2], that is he shows that an arbitrary first-order sentence over a given non-abelian free group is logically equivalent to a boolean combination of $\forall\exists$ -sentences, hence using the results from [Se4] mentioned before he finally was able to prove that all non-abelian free groups have the same elementary theory in [Se6].

Since then many of these results have been generalized and led to other important discoveries. The following list covers some of the developments following Sela's solution to the Tarski problems but is by no means complete.

- Solutions to systems of equations over
 - torsion-free hyperbolic groups by Sela ([Se7]),
 - all hyperbolic groups by Reinfeldt-Weidmann ([ReWe]),
 - torsion-free toral relatively hyperbolic groups by Groves ([Gr]),
 - acylindrical hyperbolic groups by Groves-Hull ([GrHu])
 - free products of groups by Jaligot-Sela ([JaSe])
 - free semigroups by Sela. ([Se11])
- The elementary theory of torsion-free hyperbolic groups [Se7].

- The elementary theory of free products of groups [Se10].

On the one hand the first paper of Sela on the elementary theory of free groups is now fairly well-understood and has been generalized in several ways, on the other hand for none of the remaining papers there is a generalization from someone else besides Sela himself. The only work we are aware of, dealing with one of the later papers of Sela's solution is the recent PhD thesis of J. González, who proves a Generalized Merzlyakov's theorem over, what he calls, π -free groups [Go]. Some parts of the original results from Sela's second paper are also reproved in [BySk].

We are mainly interested in the first step in Sela's proof of the Tarski problems, that is to show that $\text{Th}_{\forall\exists}(F_m) = \text{Th}_{\forall\exists}(F_n)$ for given $m, n \geq 2$, where $\text{Th}_{\forall\exists}(F_m)$ consists of all sentences with two quantifiers, true in the free group F_m . These sentences are called $\forall\exists$ - or *AE-sentences*. An $\forall\exists$ -sentence σ is of the form

$$\forall y \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1,$$

where x and y are tuples of variables, $\Sigma(x, y) = 1$ is a system of equations in the variables x, y and $\Psi(x, y) \neq 1$ is a system of inequalities.

Now fix a non-abelian free group F_k and let σ be an $\forall\exists$ -sentence, true in F_k , that does not contain inequalities. Then $F_k \models \sigma$ if and only if for every homomorphism $\varphi : F(y) \rightarrow F_k$ there exists an extension $\bar{\varphi} : G_\Sigma \rightarrow F_k$ to the group $G_\Sigma := \langle x, y \mid \Sigma(x, y) = 1 \rangle$ making the following diagram commute.

$$\begin{array}{ccc} G_\Sigma & & \\ \uparrow & \searrow \bar{\varphi} & \\ F(y) & \xrightarrow{\varphi} & F_k \end{array}$$

Here $\nu : F(y) \rightarrow G_\Sigma$ is the canonical embedding. Clearly if there exists a retraction $\pi : G_\Sigma \rightarrow F(y)$ such an extension always exists by just setting $\bar{\varphi} = \varphi \circ \pi$.

It turns out that the other direction is also true, as was shown by Yu. I. Merzlyakov.

Merzlyakov's Theorem ([Me]). *Let F_k be a non-abelian free group and suppose that*

$$F_k \models \forall y \exists x \Sigma(x, y) = 1.$$

Then there exists a retraction $\pi : G_\Sigma \rightarrow F(y)$.

The retraction π , respectively $\pi(x)$, is called a *formal solution*. The existence of formal solutions does not depend on the rank of the free group F_k and therefore it follows immediately that $\text{Th}_{\forall\exists}^+(F_m) = \text{Th}_{\forall\exists}^+(F_n)$, i.e. F_m and F_n have the same positive

theory.

Unfortunately the picture becomes a lot more complicated as soon as inequalities are involved. A general $\forall\exists$ -sentence

$$\forall y \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1$$

is true in F_k if and only if for every homomorphism $\varphi : F(y) \rightarrow F_k$ there exists an extension $\bar{\varphi} : G_\Sigma \rightarrow F_k$ such that $\varphi = \bar{\varphi} \circ \nu$ and with the additional property that $\bar{\varphi}(v(x, y)) \neq 1$ for all $v(x, y) \in \Psi(x, y)$. By Merzlyakov's Theorem there still exists a retraction $\pi : G_\Sigma \rightarrow F(y)$ but the formal solution $\pi(x)$ fails to verify the sentence for all homomorphisms $\varphi : F(y) \rightarrow F_k$ with the property that $\varphi \circ \pi(v(x, y)) = 1$ for some $v(x, y) \in \Psi(x, y)$. Let

$$v'_1(y) := v_1(\pi(x), y) \neq 1, \dots, v'_r(y) := v_r(\pi(x), y) \neq 1$$

be the inequalities of $\Psi(x, y)$ after replacing the variables x by $\pi(x) \in F(y)$. Fix $i \in \{1, \dots, r\}$. We assume for simplicity that $R_i(y) := \langle y \mid v'_i(y) = 1 \rangle$ is a limit group (in general one first has to pass to finitely many limit quotients of $R_i(y)$). Clearly $R_i(y)$ is a proper quotient of $F(y)$ and we define the basic definable set corresponding to the limit group $R_i(y)$ as

$$B_i(y) = \{y_0 \in F_k^l \mid v'_i(y_0) = 1\}.$$

If

$$y_0 \notin V_B(y) := B_1(y) \cup \dots \cup B_r(y)$$

then $v'_j(y_0) \neq 1$ for every $j \in \{1, \dots, r\}$. The formal solution $\pi(x)$ given by Merzlyakov's Theorem proves the validity of our sentence on the set $F_k^l \setminus V_B(y)$ and hence it only remains to find a witness for the truth of the sentence

$$\forall y \in V_B(y) \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1.$$

So the question becomes: Does Merzlyakov's Theorem still hold when the universal variables y are restricted to some variety?

Unfortunately, a naive generalization of Merzlyakov's Theorem cannot be true as the following example demonstrates.

Example. Let $L = \langle \underline{y} \mid \Theta(\underline{y}) \rangle$ be a limit group with JSJ decomposition

$$L = R_1 *_C R_2,$$

where $C = \langle c \rangle \cong \mathbb{Z}$ and $R_1 = \langle \underline{y}_1 \rangle, R_2 = \langle \underline{y}_2 \rangle$ are rigid. Let

$$S = \langle \underline{s} \mid \prod_{j=1}^g [s_{2j-1}, s_{2j}] = 1 \rangle$$

be the fundamental group of a closed orientable surface of genus $g \geq 2$ and suppose that S is the unique maximal shortening quotient of L with corresponding quotient map $\eta : L \rightarrow S$. This means that for every homomorphism $\varphi \in \text{Hom}(L, F_k)$ there exists a Dehn-twist $\alpha \in \text{Aut}(L)$ along the single edge of the JSJ decomposition of L and $\bar{\varphi} \in \text{Hom}(S, F_k)$ such that $\varphi = \bar{\varphi} \circ \eta \circ \alpha$. Let $V_\Theta = \{\underline{y} \in F_k^l \mid \Theta(\underline{y}) = 1\}$ and $s_i(\underline{y}_1, t\underline{y}_2t^{-1})$ an element of $(\eta \circ \alpha)^{-1}(s_i)$. Then

$$F_k \models \forall \underline{y} \in V_\Theta \exists t, \underline{s} \underbrace{([t, c] = 1) \wedge (\underline{s} = \underline{s}(\underline{y}_1, t\underline{y}_2t^{-1}) \wedge (\prod_{j=1}^g [s_{2j-1}, s_{2j}] = 1))}_{\Sigma(t, \underline{s}, \underline{y})=1},$$

i. e. the sentence is true in F_k (here we implicitly view c as a word in \underline{y}). Suppose that there exists a retraction

$$\pi : G_\Sigma = \langle t, \underline{s}, \underline{y} \mid V_\Theta, \Sigma(t, \underline{s}, \underline{y}) \rangle \rightarrow L = \langle \underline{y} \mid V_\Theta \rangle.$$

Then in particular

$$\pi\left(\prod_{j=1}^g [s_{2j-1}, s_{2j}]\right) = 1$$

and hence π induces a homomorphism $\iota : S \rightarrow L$ such that $\eta \circ \iota = \text{id}$ and therefore ι is injective.

This example shows that if one wants to prove a version of Merzlyakov's Theorem in the case that the universal variables are restricted to some variety, then one has to guarantee that the maximal shortening quotients of the corresponding limit group L are already contained as subgroups in L .

Obviously this is not true in general but nonetheless one can show the following. Let $L := \langle y \mid B(y) = 1 \rangle$ be a limit group and $G_\Sigma := \langle x, y \mid B(y), \Sigma(x, y) \rangle$ then one can embed L into a new group $\text{Comp}(L) := \langle y, z \rangle$, which is called its completion, such that every homomorphism $\varphi : L \rightarrow F_k$ can be extended to a homomorphism $\hat{\varphi} : \text{Comp}(L) \rightarrow F_k$ (in general there exist finitely many completions, but for now we suppose that it is only a single one). Moreover, for the group $\text{Comp}(L)$ a generalized version of Merzlyakov's Theorem holds.

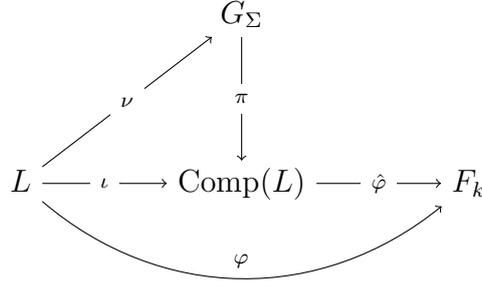
Generalized Merzlyakov's Theorem. *Let F_k be a non-abelian free group,*

$$L = \langle y \mid B(y) = 1 \rangle$$

a limit group, $V_B = \{y \in F_k^l \mid B(y) = 1\}$ and suppose that

$$F_k \models \forall y \in V_B \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1.$$

Then L can be embedded into its completion $\text{Comp}(L)$ via a homomorphism ι and there exists a homomorphism $\pi : G_\Sigma \rightarrow \text{Comp}(L)$ such that $\pi \circ \nu = \iota$, where $\nu : L \rightarrow G_\Sigma$ is the canonical embedding.



It should be noted though that this theorem is not entirely true as it stands, due to the difficulties involved when L contains non-cyclic abelian subgroups. For a precise formulation of the Generalized Merzlyakov's Theorem see [Se2] (or Theorem 10.1 for the corresponding theorem over hyperbolic groups).

Note that in contrast to the ordinary Merzlyakov's Theorem, the map π is no longer a retraction. Still this gives us a formal solution $\pi(x) = u(y, z) \in \text{Comp}(L) = \langle y, z \rangle$. But the price for finding this formal solution was that we had to introduce new variables z . Now for all $i \in \{1, \dots, r\}$ let M_i be the maximal limit quotient of $\text{Comp}(L)$ corresponding to all homomorphisms from $\text{Comp}(L)$ to F_k which map $v_i(u(y, z), y)$ to the identity (again we ignore the fact that in general there are finitely many quotients). This is a proper quotient of $\text{Comp}(L)$. Let

$$D_i(y) = \{y \in B(y) \mid \exists z \in F_k^l : v_i(u(y, z), y) = 1\}$$

be the (Diophantine) definable set corresponding to M_i . Set

$$D(y) := D_1(y) \cup \dots \cup D_r(y).$$

Then $D(y) \subsetneq B(y)$ and the formal solutions constructed by Merzlyakov's Theorem and the Generalized Merzlyakov's Theorem prove the validity of our sentence on the co-Diophantine set

$$F_k^l \setminus D(y).$$

Hence it remains to find a witness for the truth of the sentence

$$\forall y \in D(y) \exists x \Sigma(x, y) = 1 \wedge \Psi(x, a) \neq 1.$$

Repeatedly applying the Generalized Merzlyakov's Theorem yields smaller and smaller sets on which the sentence has to be analyzed. But a priori there is no reason why this process should eventually stop. Still Sela in [Se4] is able to show that the completion was constructed carefully enough such that this iterative procedure indeed terminates. Since all of this does not depend on the rank of the free group F_k it follows that $\text{Th}_{\forall\exists}(F_m) = \text{Th}_{\forall\exists}(F_n)$, i.e. F_m and F_n have the same $\forall\exists$ -theory.

In our work we build on the results of Reinfeldt-Weidmann to generalize Sela's results to hyperbolic groups with torsion. More precisely we prove a version of the Generalized Merzlyakov's Theorem for all hyperbolic groups. In Sela's paper on the elementary theory of torsion-free hyperbolic groups [Se7] a version of this theorem for torsion-free hyperbolic groups is announced, with the remark that the proof is fairly similar to the one in the free group case. Over all hyperbolic groups we have to address several problems caused by the presence of torsion, which makes some easy arguments from the torsion-free case a lot more technically involved. We therefore intend to give a short explanation here and there of the corresponding proof in the torsion-free case, supplementing Sela's result in [Se7].

We start in chapter 2 by recalling the fundamental ideas of graphs of groups and Rips theory, namely how one can deduce a graph of groups decomposition of a group G from its (appropriate) action on a real tree. In chapter 3 we collect basic properties of Γ -limit groups, describe their Dunwoody- and JSJ decomposition and their actions on real trees. Most of these results are borrowed from [ReWe].

In chapter 4 we introduce some model theory and state the Tarski problems. Then in chapter 5 we review the main results from [ReWe], that is the shortening argument and the existence of Makanin-Razborov diagrams for finitely generated groups over a hyperbolic group. We end this chapter with a description of special resolutions appearing in such a diagram, so-called strict and well-structured resolutions.

In chapter 6 we give an application of Makanin-Razborov diagrams. More precisely we construct a Makanin-Razborov diagram of a particular double of a non-abelian free group of rank two and use this diagram to show that limit groups are in general not freely subgroup separable. Unfortunately we were not able to classify all Makanin-Razborov diagrams of doubles of non-abelian-free groups of rank two, but at least we give a complete description of all possible JSJ decompositions of such doubles in chapter 7.

In chapter 8 we construct the completion of a well-structured resolution $\text{Res}(L)$, that is a Γ -limit group $\text{Comp}(L)$ into which L can be embedded. The main significance of the completion is that it admits so-called test sequences, i.e. sequences of generic points in the variety $\text{Hom}(\text{Comp}(L), \Gamma)$ which allow to recover the group $\text{Comp}(L)$ from it. Constructing these test sequences is the core of our proof of the Generalized Merzlyakov's Theorem and will take up all of chapter 9.

In chapter 10 we are then finally able to prove a version of the Generalized Merzlyakov's Theorem over all hyperbolic groups. It should be noted though that we were not able to prove a full generalization of the Generalized Merzlyakov's Theorem as stated in [Se7] to the case of hyperbolic groups with torsion. Namely in order to prove that the collection of closures which appears in the theorem forms a covering closure, we have to restrict ourselves to the torsion-free case (see Theorem 10.1). We expect that this restriction can be removed, so that a full generalization of the Generalized Merzlyakov's Theorem over all hyperbolic groups in fact holds.

2 Groups acting on trees and Rips theory

The structure of groups acting on simplicial trees is well-understood, thanks to Bass-Serre theory. The picture becomes a lot more complicated when actions of groups on real trees are considered. Still under some mild stability assumptions, Rips theory provides a description of these actions. This theory was developed by Rips (unpublished) who applied ideas from the Makanin-Razborov rewriting process for system of equations over free groups to describe free actions of finitely presented groups on real trees (see [GLP]). His ideas have then been generalized by Bestvina and Feighn [BeFe] to stable actions of finitely presented groups and by Sela [Se9] to super-stable actions of finitely-generated torsion-free groups. Guirardel [Gu] then generalized Sela's version by weakening the stability assumptions and allowing the presence of torsion.

In this chapter we fix notations for graphs of groups and the corresponding Bass-Serre tree and we recall the definition of a graph of actions. We then review some material and state the main theorem of [Gu] and explain certain simple actions on real trees which appear as the building blocks for general actions of groups on real trees. We expect the reader to be familiar with the ideas of Bass-Serre theory and only fix the notation here, which will be used later on. For more details on the subject see [Ser] or [KMW].

2.1 Graphs of groups

A *graph* A is a tuple $A = (VA, EA, \alpha, \omega, {}^{-1})$ consisting of a vertex set VA , a set of oriented edges EA , a fixed point free involution ${}^{-1}$ and maps $\alpha, \omega : EA \rightarrow VA$, which assign to every edge $e \in EA$ its initial vertex $\alpha(e)$ and terminal vertex $\omega(e)$, such that $\alpha(e^{-1}) = \omega(e)$.

A *simplicial tree* is a connected graph which contains no non-trivial closed paths.

Definition 2.1. A graph of groups \mathbb{A} is a tuple

$$(A, (A_v)_{v \in VA}, (A_e)_{e \in EA}, (\alpha_e)_{e \in EA}, (\omega_e)_{e \in EA}),$$

where for all $v \in VA$, $e \in EA$

- A is a connected graph,
- A_v and A_e are groups, called vertex and edge groups and
- $\alpha_e : A_e \rightarrow A_{\alpha(e)}$, $\omega_e = \alpha_{e^{-1}} : A_e \rightarrow A_{\omega(e)} = A_{\alpha(e^{-1})}$ are monomorphisms.

We call the maps α_e and ω_e boundary monomorphisms of the edge e .

To every graph of groups \mathbb{A} we can associate a group, namely its fundamental group $\pi_1(\mathbb{A})$. Moreover we can associate a simplicial tree to \mathbb{A} on which its fundamental group acts, its *Bass-Serre tree* T_A .

Definition 2.2. Let \mathbb{A} be a graph of groups and suppose that $A_v = \langle X_v \mid R_v \rangle$ for all $v \in VA$ and $A_e = \langle X_e \rangle$ for all $e \in EA$. Let T be a maximal subtree of A . We then define the fundamental group $\pi_1(\mathbb{A}, T)$ to be given by

$$\pi_1(\mathbb{A}, T) = \langle \bigcup_{v \in VA} X_v, \{s_e \mid e \in EA\} \mid \bigcup_{v \in VA} R_v, N \rangle,$$

where

$$N = \{s_e \mid e \in ET\} \cup \{s_e s_{e^{-1}} \mid e \in EA\} \cup \{s_e \omega_e(x) s_e^{-1} = \alpha_e(x) \mid e \in EA, x \in X_e\}$$

Remark 2.3. Up to isomorphism $\pi_1(\mathbb{A}, T)$ does not depend on the choice of the maximal subtree T and hence we denote the fundamental group of \mathbb{A} by $\pi_1(\mathbb{A})$.

Let \mathbb{A} be a graph of groups. An \mathbb{A} -path from $v \in VA$ to $w \in VA$ is a sequence

$$(a_0, e_1, a_1, \dots, e_k, a_k),$$

where e_1, \dots, e_k is an edge-path from v to w in the underlying graph A , $a_0 \in A_v$ and $a_i \in A_{\omega(e_i)}$ for $i \in \{1, \dots, k\}$. Let \sim be the equivalence relation on the set of \mathbb{A} -paths generated by the elementary equivalences

$$(a_0, e, a_1) \sim (a_0 \alpha_e(c), e, \omega_e(c^{-1}) a_1) \text{ and} \\ (a_1, e, 1, e^{-1}, a_2) \sim (a_1 a_2).$$

We denote the equivalence class of an \mathbb{A} -path p by $[p]$. For every base vertex $v_0 \in VA$, the fundamental group of \mathbb{A} with respect to v_0 is defined as

$$\pi_1(\mathbb{A}, v_0) = \{[p] \mid p \text{ is an } \mathbb{A}\text{-path from } v_0 \text{ to } v_0\}.$$

It is easy to see that this definition does not depend (up to isomorphism) on the choice of the base vertex and $\pi_1(\mathbb{A}, v_0) \cong \pi_1(\mathbb{A}, T)$.

For all $e \in EA$ let C_e be a set of left coset representatives of $\alpha_e(A_e)$ in $A_{\alpha(e)}$. Then each \mathbb{A} -path p is equivalent to a reduced \mathbb{A} -path $p' = (a_0, e_1, \dots, e_k, a_k)$ such that $a_{i-1} \in C_{e_i}$ for all $i \in \{1, \dots, k\}$. We say that p' is in *normal form* (with respect to the set $\{C_e \mid e \in EA\}$).

Definition 2.4. Let \mathbb{A} be a graph of groups, $e \in EA$ and $g \in Z(A_e)$ an element of the center of A_e . The Dehn-twist along e by g is the automorphism

$$\varphi : \pi_1(\mathbb{A}) \rightarrow \pi_1(\mathbb{A}), [a_0, e_1, \dots, e_k, a_k] \rightarrow [\bar{a}_0, e_1, \dots, e_k, \bar{a}_k],$$

where

$$\bar{a}_i = \begin{cases} \omega_e(g) a_i, & \text{if } e_i = e \\ \alpha_e(g^{-1}) a_i, & \text{if } e_i = e^{-1} \\ a_i, & \text{otherwise} \end{cases}$$

In the case of a one-edge splitting \mathbb{A} , this just recovers the usual definition of a Dehn-twist.

2.2 Rips theory

We now turn our attention from simplicial trees to \mathbb{R} -trees. An \mathbb{R} -tree (or *real tree*) is a 0-hyperbolic metric space. Suppose a group G acts on an \mathbb{R} -tree T by isometries. We call the action *minimal*, if T has no proper G -invariant subtree. We call a subtree of T non-degenerate if it contains an arc, where an arc is a set homeomorphic to the interval $[0, 1]$. We now collect the relevant definitions and results from [Gu] which we are using in this work. For more details on \mathbb{R} -trees see [Ch] or [Gu].

Definition 2.5. *Let G be a group acting on a real tree T by isometries.*

- (a) *T satisfies the ascending chain condition if for any sequence of arcs $I_1 \supset I_2 \supset \dots$ in T whose length converges to 0, the sequence of stabilizers of the sequence is eventually constant.*
- (b) *A non-degenerate subtree $S \subset T$ is called stable if for every arc $I \subset S$, $\text{Stab}_G(I) = \text{Stab}_G(S)$, where Stab_G denotes the stabilizer in G . Otherwise S is called unstable.*
- (c) *T is super-stable if any arc with non-trivial stabilizer is stable*
- (d) *A non-degenerate subtree $S \subset T$ is called indecomposable if for every pair of arcs $I, J \subset S$, there is a finite sequence g_1I, \dots, g_nI which covers J such that $g_i \in G$ for all $i \in \{1, \dots, n\}$ and $g_iI \cap g_{i+1}I$ is non-degenerate.*

Remark 2.6. (a) *By stabilizer we always mean pointwise stabilizer.*

(b) *Clearly super-stability implies the ascending chain condition.*

A graph of actions is a way of decomposing the action of a group on a real tree into pieces. For more information see [Le].

Definition 2.7. *A graph of actions is a tuple*

$$\mathcal{G} = \mathcal{G}(\mathbb{A}) = (\mathbb{A}, (T_v)_{v \in VA}, (p_e^\alpha)_{e \in EA}, l),$$

where

- \mathbb{A} is a graph of groups with underlying graph A ,
- for each $v \in VA$, $T_v = (T_v, d_v)$ is a real A_v -tree,
- for each $e \in EA$, $p_e^\alpha \in T_{\alpha(e)}$ is a point fixed by $\alpha_e(A_e)$,
- $l : EA \rightarrow \mathbb{R}_{\geq 0}$ is a function satisfying $l(e) = l(e^{-1})$ for all $e \in EA$.

We call $l(e)$ the length of e . If $l = 0$ then we omit l , i.e. we write

$$\mathcal{G} = \mathcal{G}(\mathbb{A}) = (\mathbb{A}, (T_v)_{v \in VA}, (p_e^\alpha)_{e \in EA}).$$

Given a graph of actions one can canonically construct an associated real tree T by replacing the vertices of the Bass-Serre tree T_A of \mathbb{A} by copies of the trees T_v and any edge e by a segment of length $l(e)$. Clearly the action of $\pi_1(\mathbb{A})$ on T_A extends naturally to an action of $\pi_1(\mathbb{A})$ on T .

We are now ready to state the main theorem of [Gu]. This result and its relatives are usually referred to as the Rips machine.

Theorem 2.8 ([Gu]). *Consider a non-trivial action of a finitely generated group G on a real tree T by isometries. Assume that*

1. T satisfies the ascending chain condition,
2. for any unstable arc $J \subset T$,
 - a) $\text{Stab}(J)$ is finitely generated
 - b) $\text{Stab}(J)$ is not a proper subgroup of any conjugate of itself, i.e. for all $g \in G$ $g \text{Stab}(J)g^{-1} \subset \text{Stab}(J) \implies g \text{Stab}(J)g^{-1} = \text{Stab}(J)$.

Then either G splits over the stabilizer of an unstable arc or over the stabilizer of an infinite tripod, or T splits as a graph of actions

$$\mathcal{G} = (\mathbb{A}, (T_v)_{v \in VA}, (p_e^\alpha)_{e \in EA}, l),$$

where each vertex action of A_v on the vertex tree T_v is either

1. *simplicial*: a simplicial action on a simplicial tree,
2. *of orbifold type*: the action of A_v has kernel N_v and the faithful action of A_v/N_v is dual to an arational measured foliation on a compact 2-orbifold with boundary, or
3. *axial*: T_v is a line and the image of A_v in $\text{Isom}(T_v)$ is a finitely generated group acting with dense orbits on T_v

Remark 2.9. (a) For a detailed definition of the different types of actions on the vertex trees see [Gu] or [ReWe].

(b) The actions of axial and orbifold type have the indecomposable property.

Later we will also need a relative version of the Rips machine for pairs of groups:

Definition 2.10. A pair of groups (G, \mathcal{H}) consists of a group G together with a finite set of subgroups $\mathcal{H} = \{H_1, \dots, H_k\}$ of G . A pair (G, \mathcal{H}) is finitely generated if there exists a finite subset $X \subset G$ such that G is generated by $X \cup H_1 \cup \dots \cup H_k$.

An action of a pair (G, \mathcal{H}) on a (real) tree T is an action of G on T such that each subgroup H_i acts elliptically. A splitting of (G, \mathcal{H}) is a graph of groups \mathbb{A} with $\pi_1(\mathbb{A}) = G$ such that every H_i is contained in a conjugate of a vertex group of \mathbb{A} .

The following Theorem is Theorem 5.1 of [Gu].

Theorem 2.11. *(Relative Version of the Rips machine) Consider a non-trivial action of a finitely generated pair (G, \mathcal{H}) on a real tree T by isometries. Assume that*

1. T satisfies the ascending chain condition,
2. there exists a finite family of arcs I_1, \dots, I_p such that $I_1 \cup \dots \cup I_p$ spans T and such that for any unstable arc J contained in some I_i ,
 - a) $\text{Stab}(J)$ is finitely generated
 - b) $\text{Stab}(J)$ is not a proper subgroup of any conjugate of itself, i.e. for all $g \in G$ $g \text{Stab}(J) g^{-1} \subset \text{Stab}(J) \implies g \text{Stab}(J) g^{-1} = \text{Stab}(J)$

Then either (G, \mathcal{H}) splits over the stabilizer of an unstable arc or over the stabilizer of an infinite tripod (whose normalizer contains a non-abelian free group generated by two elements having disjoint axes), or T splits as a graph of actions

$$\mathcal{G} = (\mathbb{A}, (T_v)_{v \in VA}, (p_e^\alpha)_{e \in EA}, l)$$

where each vertex action of A_v on the vertex tree T_v is either

1. simplicial: a simplicial action on a simplicial tree,
2. of orbifold type: the action of A_v has kernel N_v and the faithful action of A_v/N_v is dual to an arational measured foliation on a compact 2-orbifold with boundary, or
3. axial: T_v is a line and the image of A_v in $\text{Isom}(T_v)$ is a finitely generated group acting with dense orbits on T_v

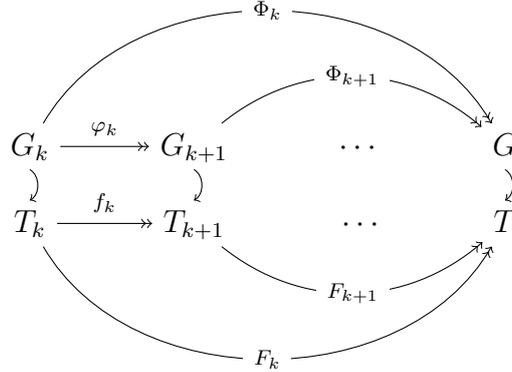
2.3 Strong convergence of group actions

In this section we describe a way how a minimal action of a finitely generated pair on a real tree can be approximated by a sequence of better-understandable actions of pairs on real trees. This is used by Guirardel in his proof of the Rips machine. In the end we state a relative version of Guirardel's Extended Scott's Lemma (Theorem 2.4 in [Gu]). The non-relative versions of the following definitions and results can be found in [LePa] while the relative versions are due to Guirardel ([Gu]). Given two actions $G_1 \curvearrowright T_1$, $G_2 \curvearrowright T_2$ of groups G_1, G_2 on \mathbb{R} -trees T_1, T_2 we call

$$(\varphi, f) : G_1 \curvearrowright T_1 \rightarrow G_2 \curvearrowright T_2$$

a morphism of group actions if $\varphi : G_1 \rightarrow G_2$ is a group homomorphism and $f : T_1 \rightarrow T_2$ is a φ -equivariant morphism of \mathbb{R} -trees. We say that (φ, f) is surjective if both φ and f are.

Definition 2.12. A direct system of actions on \mathbb{R} -trees is a sequence of actions of finitely generated groups $G_k \curvearrowright T_k$ and an action $G \curvearrowright T$ with surjective morphisms of group actions $(\varphi_k, f_k) : G_k \curvearrowright T_k \rightarrow G_{k+1} \curvearrowright T_{k+1}$ and $(\Phi, F_k) : G_k \curvearrowright T_k \rightarrow G \curvearrowright T$ such that the following diagram commutes



Following Guirardel, we will use the notation $f_{kk'} = f_{k'-1} \circ \dots \circ f_k : T_k \rightarrow T_{k'}$ and $\varphi_{kk'} = \varphi_{k'-1} \circ \dots \circ \varphi_k : G_k \rightarrow G_{k'}$ for $k' \geq k$.

Definition 2.13. A direct system of minimal actions of finitely generated groups on \mathbb{R} -trees $G_k \curvearrowright T_k$ converges strongly to $G \curvearrowright T$ if

- G is the direct limit of the groups G_k ,
- for every finite tree $K \subset T_k$, there exists $k' \geq k$ such that $F_{k'}$ restricts to an isometry on $f_{kk'}(K)$.

The action $G \curvearrowright T$ is called the strong limit of this direct system.

Strong convergence of direct systems allows us to make precise what it means to approximate a minimal action of a finitely generated group on a real tree by so-called geometric actions of groups on real trees. Since the definition of a geometric action is rather long, we refer the reader to the discussion before Definition 1.24 in [Gu] or [LePa].

Theorem 2.14 (Theorem 3.7 in [LePa]). Consider a minimal action of a finitely generated group G on a real tree T . Then $G \curvearrowright T$ is a strong limit of a direct system of geometric actions $\{(\Phi_k, F_k) : G_k \curvearrowright T_k \rightarrow G \curvearrowright T\}$ such that

- Φ_k is one-to-one in restriction to each arc stabilizer of T_k ,
- T_k is dual to a 2-complex X whose fundamental group is generated by free homotopy classes of curves contained in leaves.

We will later need a relative version of the above theorem. Note that the last claim in Theorem 2.15 is equivalent to the last claim in Theorem 2.14.

Theorem 2.15 (Proposition 1.31 in [Gu]). *Consider a minimal action $(G, \mathcal{H}) \curvearrowright T$ of a finitely generated pair on an \mathbb{R} -tree. Then there exists a direct system of minimal actions $(G_k, \mathcal{H}_k) \curvearrowright T_k$ converging strongly to $G \curvearrowright T$ and such that*

- φ_k and Φ_k are one-to-one in restriction to arc stabilizers of T_k .
- φ_k (resp. Φ_k) restricts to an isomorphism between \mathcal{H}_k and \mathcal{H}_{k+1} (resp. \mathcal{H}).
- T_k splits as a graph of actions where each non-degenerate vertex action is either axial, thin or of orbifold type and therefore indecomposable, or is an arc containing no branch point except at its endpoints.

Finally we will need the following result, which is Theorem 2.4 in [Gu].

Theorem 2.16 (Relative version of the Extended Scott's Lemma). *Let*

$$(G_k, \{H_1^k, \dots, H_p^k\}) \curvearrowright S_k$$

be a sequence of non-trivial actions of finitely generated pairs on simplicial trees, and

$$(\varphi_k, f_k) : G_k \curvearrowright S_k \rightarrow G_{k+1} \curvearrowright S_{k+1}$$

be epimorphisms mapping H_i^k onto H_i^{k+1} . Consider $G = \varinjlim G_k$ the inductive limit and $\Phi_k : G_k \rightarrow G$ the natural map. Assume that (φ_k, f_k) does not increase edge stabilizers in the following sense:

- $\forall e \in E(S_k), \forall e' \in E(S_{k+1}), e' \subset f_k(e) \Rightarrow G_{k+1}(e') = \varphi_k(G_k(e))$.
- $\forall e \in E(S_k), \forall i \in \{1, \dots, p\}, \Phi_k(H_i^k) \not\subseteq \Phi_k(e)$.

Here $G_{k+1}(e')$ denotes the pointwise edge stabilizer of the edge e' in G_{k+1} . Then the pair (G, \mathcal{H}) has a non-trivial splitting over the image of an edge stabilizer of some S_k . Moreover, any subgroup $H \subset G$ fixing a point in some S_k fixes a point in the obtained splitting of G .

3 Γ -limit groups

3.1 Γ -limit groups and their actions on real trees

From now on let Γ be a finitely generated non-elementary hyperbolic group (possibly with torsion). In this section we will define the notion of a Γ -limit group and investigate which properties these groups have. In particular we will show how one can construct faithful actions of Γ -limit groups on real trees. Most of the results are borrowed from [ReWe] and we will only sketch some of the ideas found there. For a complete proof of all the results we refer the reader to [ReWe].

Definition 3.1. Let G be a group and $(\varphi_i)_{i \in \mathbb{N}} \subset \text{Hom}(G, \Gamma)$.

- (1) The sequence (φ_i) is stable if for any $g \in G$ either $\varphi_i(g) = 1$ for almost all i or $\varphi_i(g) \neq 1$ for almost all i . If (φ_i) is stable then the stable kernel of the sequence, denoted by $\underline{\ker}(\varphi_i)$, is defined as

$$\underline{\ker}(\varphi_i) := \{g \in G \mid \varphi_i(g) = 1 \text{ for almost all } i\}.$$

We call the sequence (φ_i) stably injective if (φ_i) is stable and $\underline{\ker}(\varphi_i) = 1$.

- (2) Assume that (φ_i) is stable. We then call the quotient $G/\underline{\ker}(\varphi_i)$ the Γ -limit group associated to (φ_i) and the projection $\pi : G \rightarrow G/\underline{\ker}(\varphi_i)$ the Γ -limit map associated to (φ_i) .
- (3) We call a group L a Γ -limit group if there exists some group H and a stable sequence $(\varphi_i) \subset \text{Hom}(H, \Gamma)$ such that $L = H/\underline{\ker}(\varphi_i)$.

Remark 3.2. (1) Let L be a Γ -limit group. Then there exists a stably injective sequence $(\varphi_i) \subset \text{Hom}(L, \Gamma)$ such that $L = L/\underline{\ker}(\varphi_i)$.

- (2) The definition of a Γ -limit group also works for arbitrary groups Γ , but here we are only interested in the case that Γ is hyperbolic.

Definition 3.3. A group G is called

- residually Γ if for every non-trivial element $g \in G$ there exists a homomorphism $\varphi : G \rightarrow \Gamma$ such that $\varphi(g) \neq 1$.
- fully residually Γ if for every finite subset $S \subset G$ there exists a homomorphism $\varphi : G \rightarrow \Gamma$ such that $\varphi|_S$ is injective.

The following theorem shows that any finitely generated Γ -limit group is fully residually Γ and the converse also holds.

Theorem 3.4 ([ReWe] Lemma 1.3 and Corollary 5.4). *Let L be a finitely generated group. Then the following are equivalent*

- (1) L is a Γ -limit group.
- (2) L is fully residually Γ

Let G be a group. We now explain how a sequence of suitable group actions of G on δ -hyperbolic spaces can be used to construct an action of G on a real tree. This is usually referred to as the Bestvina-Paulin method ([Pa]).

Let G be a finitely generated group. A pseudo-metric d on G is called G -invariant if $d(h_1, h_2) = d(gh_1, gh_2)$ for all $g, h_1, h_2 \in G$. We denote by $\mathcal{ED}(G)$ the space of all G -invariant pseudo-metrics on G , with the compact-open topology (where G is given the discrete topology). Thus a sequence (d_n) of G -invariant pseudo-metrics on G converges in $\mathcal{ED}(G)$ against a pseudo-metric d if and only if the sequence $(d_n(1, g))$ converges in \mathbb{R} to $d(1, g)$ for all $g \in G$.

Definition 3.5. (1) A (based) G -space is a tuple (X, x_0, ρ) of a metric space X , a base point x_0 and an action ρ of a group G on X . If $g \in G$ and $x \in X$, we will denote the element $\rho(g)(x) \in X$ simply by gx if the action ρ is understood. Let $X = (X, x, \rho)$ be a based G -space. Then the G -action ρ on X induces a pseudo-metric

$$d_\rho^x : G \times G \rightarrow \mathbb{R}_{\geq 0}$$

on G given by

$$d_\rho^x(g, h) = d_X(\rho(g)(x), \rho(h)(x)).$$

(2) Let G be a group with finite generating set S_G and $X = (X, x, \rho)$ a based G -space. The norm of the action ρ with respect to the base point x (and the generating set S_G), denoted by $|\rho|_x$, is defined as

$$|\rho|_x := \sum_{s \in S_G} d_X(x, \rho(s)x).$$

Theorem 3.6 (Theorem 1.11 in [ReWe]). Let G be a group with finite generating set S_G . For $i \in \mathbb{N}$ let $\delta_i \geq 0$ and $X_i = (X_i, x_i, \rho_i)$ a based δ_i -hyperbolic G -space. Assume that the sequence $(d_{\rho_i}^{x_i})$ converges in $\mathcal{ED}(G)$ to a limit sequence d_∞ and

$$\lim_{i \rightarrow \infty} \delta_i = 0.$$

Then there exists a based real G -tree (T, x, ρ) such that T is spanned by $\rho(G)x$ and $d_\rho^x = d_\infty$. If moreover for any i and any $y \in X_i$,

$$|\rho_i|_y \geq |\rho_i|_{x_i},$$

then the limit action of G on T is minimal.

We explain now how the previous statement implies that Γ -limit groups act on real trees.

Let S_Γ be a generating set of the hyperbolic group Γ and $X := \text{Cay}(\Gamma, S_\Gamma)$ the Cayley graph of Γ . Let G be a group with finite generating set S_G . Let δ be the hyperbolicity constant of X . Note that any $\varphi \in \text{Hom}(G, \Gamma)$ defines an action of G on X , namely

$$G \times X \rightarrow X, (g, x) \mapsto \varphi(g)x.$$

It follows that $(X, 1, \varphi)$ is a based G -space for any $\varphi \in \text{Hom}(G, \Gamma)$. This action induces the δ -hyperbolic pseudo metric $d_\varphi := d_\varphi^1$ on G given by

$$d_\varphi(g, h) = d_X(\varphi(g), \varphi(h)) = |\varphi(g^{-1}h)|_{S_\Gamma}.$$

Then $|\varphi| := |\varphi|_1 = \sum_{s \in S_G} d_X(1, \varphi(s))$ denotes the norm of φ .

For any $g \in \Gamma$ let $c_g : \Gamma \rightarrow \Gamma$ be the inner automorphism given by $c_g(h) := g^{-1}hg$ for all $h \in \Gamma$.

Definition 3.7. *We call a homomorphism $\varphi \in \text{Hom}(G, \Gamma)$ conjugacy-short if $|\varphi| \leq |c_g \circ \varphi|$ for all $g \in \Gamma$.*

Clearly a homomorphism $\varphi \in \text{Hom}(G, \Gamma)$ is conjugacy-short if and only if $|\varphi| \leq |\varphi|_g$ for all $g \in G$, since $d_{c_g \circ \varphi} = d_\varphi^g$ as

$$d_{c_g \circ \varphi}(h, k) = d_X(g^{-1}\varphi(h)g, g^{-1}\varphi(k)g) = d_X(\varphi(h)g, \varphi(k)g) = d_\varphi^g(h, k)$$

for all $h, k \in G$ and therefore in particular $|c_g \circ \varphi| = |\varphi|_g$.

We now consider a sequence of homomorphisms $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ such that the following hold:

1. φ_i is conjugacy-short for all $i \in \mathbb{N}$.
2. (φ_i) does not contain a subsequence (φ_{j_i}) such that $\ker \varphi_{j_i} = \ker \varphi_{j_{i'}}$ for all $i, i' \in \mathbb{N}$.

The second condition implies that we may assume that the φ_i are pairwise distinct after passing to a subsequence. It follows in particular that $\lim_{i \rightarrow \infty} |\varphi_i| = \infty$ as for any K there are only finitely many homomorphisms of norm at most $|K|$.

Let now $X_i = (X_i, d_{X_i})$ be the metric space obtained from X by scaling with the factor $\frac{1}{|\varphi_i|}$, thus $X_i = X$ (the underlying sets) and $d_{X_i} = \frac{1}{|\varphi_i|} d_X$. Clearly G also acts on X_i by isometries where the action on the underlying sets $X = X_i$ coincide, hence we obtain a based G -space $(X_i, 1, \rho_i)$ where the action $\rho_i : G \times X_i \rightarrow X_i$ is given by

$$\rho_i(g)x = \varphi_i(g)x \text{ for all } x \in X_i = X.$$

It is immediate that $d_{\rho_i} = \frac{1}{|\varphi_i|} d_{\varphi_i}$. Moreover d_{ρ_i} is δ_i -hyperbolic with $\delta_i := \frac{\delta}{|\varphi_i|}$ and $\lim_{i \rightarrow \infty} \delta_i = 0$. Applying Theorem 3.6 to the sequence of based G -spaces (X_i, x_i, ρ_i) yields the following Theorem.

Theorem 3.8 (Theorem 1.12 in [ReWe]). *Let G be a finitely generated group, Γ a hyperbolic group and $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ a sequence of conjugacy-short homomorphisms. Then one of the following holds:*

- (1) (φ_i) contains a constant subsequence

(2) A subsequence of $(\frac{1}{|\varphi_i|}d_{\varphi_i})$ converges to d_ρ^x for some non-trivial, minimal based real G -tree (T, x, ρ) .

Remark 3.9. (1) Suppose we are in case (2) of Theorem 3.8 and moreover that the sequence (φ_i) is stable. Then the action of G on the limit real tree T induces an action of the limit group $L = G/\underline{\ker}(\varphi_i)$ on T without a global fixed point.

(2) Let $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ be a stable sequence of conjugacy-short homomorphisms and suppose (φ_i) contains a constant subsequence. Then after passing to this subsequence (still denoted (φ_i)) we get that $L = G/\underline{\ker}(\varphi_i) = G/\ker(\varphi_i) \cong H$ for some subgroup H of Γ .

Hence if $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ is a stable sequence of conjugacy-short homomorphisms either the limit group $G/\underline{\ker}(\varphi_i)$ is isomorphic to a subgroup of Γ or it acts on a minimal limit real tree without a global fixed point.

3.2 Restricted Γ -limit groups

We will often not be interested in arbitrary Γ -limit groups (Γ still a hyperbolic group) but in Γ -limit groups which contain an isomorphic copy of Γ as a distinguished subgroup. This leads to the following definition.

Definition 3.10. Let G be a group and $\iota : \Gamma \hookrightarrow G$ an embedding. A homomorphism $\varphi : G \rightarrow \Gamma$ is called restricted if $\varphi \circ \iota = \text{id}$. Let

$$\text{Hom}_\iota(G, \Gamma) = \{\varphi \in \text{Hom}(G, \Gamma) \mid \varphi \circ \iota = \text{id}\}.$$

A group L is called a restricted Γ -limit group if there exists a group G , an embedding $\iota : \Gamma \rightarrow G$ and a stable sequence $(\varphi_n) \subset \text{Hom}_\iota(G, \Gamma)$ such that $L = G/\underline{\ker}(\varphi_n)$.

From now on we ignore the embedding $\iota : \Gamma \rightarrow G$ and consider Γ as a subgroup of G . Clearly every restricted homomorphism is injective on Γ and hence Γ is contained in every Γ -limit group which arises as a quotient of G by a stable kernel of some sequence of restricted homomorphisms.

Now let G be a group and $(\varphi_n) \subset \text{Hom}(G, \Gamma)$ a stable sequence of pairwise distinct restricted homomorphisms. Denote by L the associated restricted Γ -limit group. Theorem 3.6 again provides us with a limit based real tree (T, x, ρ) onto which G (and also L) acts. But this tree T may a priori not be minimal or the action of G on T may contain a global fixed point since we are not allowed to postcompose the homomorphisms φ_n by arbitrary inner automorphisms of Γ . The following Theorem shows that we can still guarantee that the limit tree is minimal and the action of G on it is without a global fixed point.

Theorem 3.11 (Lemma 8.2 in [Re]). Let G be a finitely generated group, Γ a hyperbolic group and $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ a sequence of pairwise distinct restricted homomorphisms.

Then a subsequence of $(\frac{1}{|\varphi_i|}d_{\varphi_i})$ converges to d_ρ^x for some non-trivial, minimal based real G -tree (T, x, ρ) .

Remark 3.12. *Suppose that the sequence (φ_i) is stable and let L be the corresponding restricted Γ -limit group. Then as before the action of G on the limit tree T induces an action of L on T and moreover Γ fixes the base point of T .*

3.3 Properties of Γ -limit groups

In this section we collect some results mainly from [ReWe], which will be used repeatedly in the sequel. It is well-known that in a hyperbolic group Γ there exists a uniform bound on the cardinality of torsion subgroups (in particular torsion subgroups are finite). It turns out that the same holds for Γ -limit groups.

Proposition 3.13 (Proposition 1.17 and Lemma 1.18 in [ReWe]). *Let Γ be a hyperbolic group and L a Γ -limit group. Then there exists a constant $N(\Gamma)$ such that the following hold:*

- (1) *Every torsion subgroup of Γ has at most $N(\Gamma)$ elements.*
- (2) *Every torsion subgroup of L has at most $N(\Gamma)$ elements.*

Definition 3.14. (1) *A group is called K -virtually abelian if it contains an abelian subgroup of index $K \in \mathbb{N}$ and virtually abelian if it is K -virtually abelian for some $K \in \mathbb{N}$.*

- (2) *A group is called finite-by-abelian if it contains a finite normal subgroup N such that G/N is abelian.*

The following Theorem allows us to use the Rips machine (Theorem 2.8) to analyze the action of a Γ -limit group on its associated limit real tree as constructed in Theorem 3.8.

Theorem 3.15 (Theorem 1.16 in [ReWe]). *Let G be a finitely generated group, Γ a hyperbolic group, $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ a convergent sequence of pairwise distinct conjugacy-short homomorphisms with corresponding Γ -limit group L and T be the associated limit real tree. Then the following hold for the action of L on T :*

- (1) *The stabilizer of any non-degenerate tripod is finite.*
- (2) *The stabilizer of any unstable arc is finite.*
- (3) *The stabilizer of any non-degenerate arc is finite-by-abelian.*
- (4) *Every subgroup of L which leaves a line in T invariant and fixes its ends is finite-by-abelian.*

Note that beside having stated the above Theorem for unrestricted Γ -limit groups, the same holds for restricted Γ -limit groups. And since the subgroup Γ of a restricted Γ -limit group L fixes the base point of the associated limit real tree T , this allows us to analyze the action of L on T by applying the relative version of the Rips machine (Theorem 2.11).

Now we turn our attention to virtually abelian subgroups of Γ -limit groups.

Lemma 3.16 (Lemma 1.21 in [ReWe]). *Let A be an infinite Γ -limit group. Then the following hold:*

- (1) *If A is finite-by-abelian with center $Z(A)$ then $|A : Z(A)| < \infty$, i.e. A is virtually abelian.*
- (2) *A is virtually abelian if and only if A is either finite-by-abelian or contains a unique finite-by-abelian subgroup of index 2.*
- (3) *A is virtually abelian if and only if all finitely generated subgroups are virtually abelian.*

Often we will have to deal with maximal virtually abelian subgroups of a Γ -limit group L and it turns out that every element $a \in L$ of infinite order is contained in a unique maximal virtually abelian subgroup $M(a)$ and $M(a)$ is almost malnormal in L .

Proposition 3.17 (Corollary 1.34 and 1.24 in [ReWe]). *Let L be a Γ -limit group. Then the following hold:*

- (1) *If $a \in L$ is an element of infinite order, then*

$$M(a) := \langle \{a' \in L \mid \langle a, a' \rangle \text{ is virtually abelian} \} \rangle$$

is the unique maximal virtually abelian subgroup of L containing a .

- (2) *Any infinite virtually abelian subgroup H is contained in a unique maximal virtually abelian subgroup $M(H)$. In addition the group $M(H)$ is almost malnormal, i.e. if $M(H) \cap gM(H)g^{-1}$ is infinite, then $g \in M(H)$.*

Definition 3.18. *Let $K > 0$ be some integer. A group G is called K -CSA if:*

- (a) *Any finite subgroup has cardinality at most K .*
- (b) *Any element $g \in G$ of infinite order is contained in a unique maximal virtually abelian subgroup $M(g)$ and $M(g)$ is K -virtually abelian.*
- (c) *$M(g)$ is its own normalizer.*

Combining Proposition 3.13, Lemma 3.16 and Proposition 3.17 immediately yields the first assertion of the following result. The second one is Corollary 9.10 in [GuLe].

Corollary 3.19. *Γ -limit groups are $N(\Gamma)$ -CSA. Moreover any subgroup of a Γ -limit group G is either $N(\Gamma)$ -virtually abelian or contains a non-abelian free subgroup.*

3.4 Dunwoody decompositions of Γ -limit groups

Let Γ again be a non-elementary hyperbolic group and L a restricted Γ -limit group. A splitting of L is a graph of groups \mathbb{A} with $\pi_1(\mathbb{A}) \cong L$ such that Γ is contained in a vertex group. In this section we are interested in all splittings of L along finite subgroups, while the next section covers the (more difficult) case of all splittings along virtually abelian subgroups. We only state the results for unrestricted Γ -limit groups, but it is obvious that they also hold in the restricted case.

Definition 3.20. *A group G is called accessible if there exists a reduced graph of groups \mathbb{A} such that*

- (a) $\pi_1(\mathbb{A}) \cong G$,
- (b) any edge group of \mathbb{A} is finite and
- (c) any vertex group is one-ended or finite.

We call any such graph of groups a Dunwoody decomposition of G .

Remark 3.21. *The maximal vertex groups of a Dunwoody decomposition are unique up to conjugacy. Indeed they are precisely the maximal one-ended subgroups of G .*

Theorem 3.22. *Let Γ be a hyperbolic group and L a Γ -limit group. Then L is accessible.*

This follows from the following theorem of P. Linnell and the fact that the order of finite subgroups of L is bounded by a constant $N(\Gamma) > 0$ (Proposition 3.13).

Theorem 3.23. *[Li] Let G be a finitely generated group. Suppose that there exists some constant $C > 0$ such that any finite subgroup H of G is of order at most C . Then G is accessible.*

We end the section with the following observation:

Lemma 3.24. *Let Γ be hyperbolic, L a one-ended Γ -limit group. Then either L is virtually abelian, or a finite extension of the fundamental group of a closed orbifold, or has a non-trivial virtually abelian JSJ decomposition.*

Proof. Assume that L is neither virtually abelian nor a finite extension of the fundamental group of a closed orbifold. By Theorem 2.8 L splits as a graph of actions. Since the action of L on the corresponding limit tree is faithful, L does not act with a global fixpoint and hence the splitting is non-trivial. \square

3.5 The virtually abelian JSJ decomposition of a Γ -limit group

Let G be a finitely generated group and \mathcal{H} a class of groups (finite, abelian, free, etc.). One could ask, if there exists a splitting of G as a graph of groups \mathbb{A} such that one can read off (in an appropriate sense) all splittings of G along groups contained in \mathcal{H} from \mathbb{A} . JSJ theory deals with this question. This theory goes back to the work of Jaco-Shalen and Johannsen on 3-dimensional topology. For groups the first results were due to Rips and Sela for splittings of finitely presented freely indecomposable groups along infinite cyclic subgroups ([RiSe2]). Since then much has been done for splittings of groups along various classes of subgroups \mathcal{Z} . A summary of the historical development together with the most recent results in this area can be found in the excellent work of V. Guirardel and G. Levitt on JSJ decompositions of groups ([GuLe]).

Let \mathcal{Z} be a class of subgroups of a group L . Given two \mathcal{Z} -trees T_1 and T_2 , i.e. simplicial trees on which L acts with edge stabilizers in \mathcal{Z} , we say that T_1 *dominates* T_2 if any group which is elliptic in T_1 is also elliptic in T_2 . A \mathcal{Z} -tree is *universally elliptic* if its edge stabilizers are elliptic in every \mathcal{Z} -tree.

Definition 3.25. *Let L be a finitely generated group and T a \mathcal{Z} -tree such that*

- (a) *T is universally elliptic and*
- (b) *T dominates any other universally elliptic \mathcal{Z} -tree T' .*

We call T a JSJ tree and the quotient graph of groups $\mathbb{A} = T/L$ a JSJ decomposition of L (with respect to \mathcal{Z}).

For arbitrary finitely generated groups JSJ decompositions do not always exist, but since we only need the case that L is a one-ended Γ -limit group (Γ hyperbolic) and \mathcal{Z} is the class of all virtually abelian subgroups, we will show that under these assumptions a JSJ decomposition of L does exist.

Unfortunately JSJ decompositions are not unique in general, but rather form a deformation space, denoted by \mathcal{D}_{JSJ} , which consists of all JSJ trees of L . Hence two JSJ trees T_1, T_2 are in \mathcal{D}_{JSJ} if and only if they have the same elliptic subgroups.

Let now \mathbb{A} be a JSJ decomposition of a finitely generated group L . A vertex group A_v of \mathbb{A} is called *rigid* if it is elliptic in every splitting of L along a subgroup from \mathcal{Z} and *flexible* otherwise. The description of flexible vertices of JSJ decompositions (along a given class of groups) is one of the major difficulties in JSJ theory.

Let from now on Γ be a hyperbolic group and L a Γ -limit group. As seen in the previous section there exists a Dunwoody decomposition for L , which is a JSJ decomposition of L along the class of finite subgroups. Assume now that L is one-ended. For the study of equations over hyperbolic groups and the proof of a generalization of Merzlyakov's theorem we need to understand splittings of L over virtually abelian subgroups, namely

the JSJ decomposition along the class of virtually abelian subgroups. The construction of this JSJ decomposition and its properties can be found in [ReWe]. For convenience we recall the important definitions and results from there.

Definition 3.26. (1) Let \mathbb{A} be a graph of groups. Let $e \in EA$ be an edge in \mathbb{A} and $\alpha_e(A_e) \leq C \leq A_{\alpha(e)}$. A folding along the edge e is the replacement of $A_{\omega(e)}$ by $C *_{A_e} A_{\omega(e)}$, A_e by C and the corresponding replacement of the boundary monomorphisms α_e and ω_e . The inverse of a folding is called an unfolding.

(2) Let \mathbb{A} be a splitting of a finitely generated group G . We say that \mathbb{A} is a compatible splitting if all one-ended virtually abelian subgroups are elliptic in it.

(3) A vertex group Q of a graph of groups \mathbb{A} is called a QH vertex group (quadratically hanging vertex group) if the following hold:

(a) Q is finite-by-orbifold, i.e. there exists a cone-type orbifold \mathcal{O} and a finite group E such that Q fits into the short exact sequence

$$1 \rightarrow E \rightarrow Q \xrightarrow{\pi} \pi_1(\mathcal{O}) \rightarrow 1.$$

(b) For any edge $e \in EA$ such that Q is the vertex group of $\alpha(e)$ there exists a peripheral subgroup P_e of $\pi_1(\mathcal{O})$ such that $\alpha(A_e)$ is in Q conjugate to a finite index subgroup of $\pi^{-1}(P_e)$.

(4) Let Q be a QH subgroup in a graph of groups \mathbb{A} with underlying orbifold \mathcal{O} . Any essential simple closed curve on \mathcal{O} induces a splitting of $\pi_1(\mathbb{A})$ along a 2-ended subgroup. we call such a splitting geometric with respect to Q .

(5) A one-edge splitting \mathbb{A} of a group G , i.e. either a splitting as an amalgamated product or an HNN-extension, is called hyperbolic-hyperbolic with respect to another splitting \mathbb{B} of G if the edge group of \mathbb{A} is not conjugate into a vertex group of \mathbb{B} and vice versa.

Before we proceed we introduce some notation concerning virtually abelian vertex groups of splittings of a group L .

Definition 3.27. Let \mathbb{A} be a graph of groups decomposition of a one-ended Γ -limit group L . For every virtually abelian subgroup A of L we denote by A^+ the unique maximal finite-by-abelian subgroup of index at most 2.

Let now A_v be a virtually abelian vertex group of \mathbb{A} . Let Δ be the set of homomorphisms $\eta : A_v^+ \rightarrow \mathbb{Z}$ such that $\eta(\alpha_e(A_e^+)) = 0$ for all $e \in EA$ with $\alpha(e) = v$. We then define

$$P_v^+ = \{g \in A_v^+ \mid \eta(g) = 0 \text{ for all } \eta \in \Delta\}$$

to be the peripheral subgroup of A_v^+ . We call every subgroup $U \leq A_v$ with $U^+ = P(A_v^+)$ a peripheral subgroup of A_v . Note that the peripheral subgroup of A_v^+ is uniquely defined, while the peripheral subgroup of A_v is not. For every edge $e \in EA$ with $\alpha(e) = v$ we denote the peripheral subgroup of A_v containing A_e by $P(A_v, A_e)$.

Remark 3.28. A_v^+/P_v^+ is a finitely generated free abelian group.

Now we are ready to define the canonical virtually abelian JSJ decomposition of a one-ended Γ -limit group.

Definition 3.29. Let L be a one-ended Γ -limit group and \mathbb{A} be a virtually abelian compatible splitting of L . Then \mathbb{A} is called a virtually abelian JSJ decomposition of L if the following hold.

- (1) Every splitting over a 2-ended subgroup that is hyperbolic-hyperbolic with respect to another splitting over a 2-ended subgroup is geometric with respect to a QH subgroup of \mathbb{A} .
- (2) Any edge of \mathbb{A} that can be unfolded to be finite-by-abelian is finite-by-abelian.
- (3) For any virtually abelian vertex group A_v , the rank of A_v^+/P_v^+ cannot be increased by unfoldings.
- (4) \mathbb{A} is in normal form and of maximal complexity among all virtually abelian compatible splittings of L that satisfy (1)-(3).

Remark 3.30. (1) The virtually abelian JSJ decomposition is a JSJ decomposition with respect to the class of virtually abelian subgroups in the sense of Definition 3.25.

- (2) Vertex groups in a virtually abelian JSJ decomposition which are neither QH subgroups nor virtually abelian, are rigid.
- (3) The analogue for restricted limit groups of the graph of groups \mathbb{A} in the above definition is called a restricted JSJ decomposition.

Theorem 3.31. Let L be a one-ended (restricted) Γ -limit group. Then there exists a (restricted) JSJ decomposition \mathbb{A} of L .

Proof. This follows from Theorem 3.9 in [ReWe] after applying finitely many refinements of non-QH subgroups, unfoldings and the normalization process defined in [ReWe] to the graph of groups yielded by this theorem. \square

The following theorem explains in which sense one can read off any splitting of a one-ended group along a virtually abelian subgroup from its JSJ decomposition. For a definition of an edge slide and the normal form of a splitting we refer the reader to chapter 3 in [ReWe].

Theorem 3.32 (Theorem 3.12 in [ReWe]). Let L be a one-ended Γ -limit group and let \mathbb{A} be a virtually abelian JSJ decomposition of L . Then the following hold.

- (1) Let \mathbb{B} be a virtually abelian compatible splitting of L such that all maximal QH subgroups are elliptic. Assume further that \mathbb{B} is either in normal form or a one-edge splitting. Then \mathbb{B} is visible in a graph of groups obtained from \mathbb{A} by unfoldings followed by foldings and edge slides.
- (2) Any other JSJ decomposition \mathbb{B} of L can be obtained from \mathbb{A} by a sequence of foldings and unfoldings.

One of the reasons the virtually abelian JSJ decomposition is so useful, is that one can read off special classes of automorphisms of the group from it.

Definition 3.33. Let G be a one-ended group with virtually abelian JSJ decomposition \mathbb{A} . Then $\text{Mod}(G) \leq \text{Aut}(G)$, the modular group of G , is the group generated by the following automorphisms:

- (1) Inner automorphisms of G .
- (2) If $e \in EA$ with A_e finite-by-abelian and \mathbb{A}^e is the one-edge splitting obtained from \mathbb{A} by collapsing all edges but e : A Dehn-twist along $e \in \mathbb{A}^e$ by an elliptic (with respect to \mathbb{A}) element $g \in Z(M^+)$ where M is the maximal virtually abelian subgroup containing A_e .
- (3) Natural extensions of automorphisms of QH-subgroups.
- (4) Natural extensions of automorphisms of maximal virtually abelian vertex groups A_v which restrict to the identity on the peripheral subgroup P_v^+ and to conjugation on each virtually abelian subgroup $U \leq A_v$ with $U^+ = P_v^+$.

Remark 3.34. (a) By Corollary 3.18 in [ReWe] the modular group of G does not depend on the chosen virtually abelian JSJ decomposition, hence is well-defined.

- (b) Once again for restricted Γ -limit groups one can also define the restricted modular group. This group consists of all modular automorphisms of the restricted virtually abelian JSJ decomposition which fix elementwise the vertex group containing Γ .

Definition 3.35. Let Γ be a hyperbolic group, L a Γ -limit group and \mathbb{D} a Dunwoody decomposition of L . Then we call the subgroup $\text{Mod}(G) \leq \text{Aut}(G)$, consisting of those automorphisms that restrict to modular automorphisms of the vertex groups of \mathbb{D} , the modular group of L .

Note that modular automorphisms of vertex groups of a Dunwoody decomposition \mathbb{D} of L , i.e. of one-ended subgroups of L , restrict to conjugation on finite subgroups and hence are naturally extendable to L .

4 Some model theory

4.1 Introduction

The first-order language L_0 of groups uses the following symbols:

- The binary function group multiplication " \cdot ", the unary function inverse " $^{-1}$ ", the constant " 1 " and the equality relation " $=$ ".
- Variables " x, y, z " or tuples (strings) of variables " $\underline{x} = (x_1, \dots, x_n)$ " (which will have to be interpreted as individual group elements).
- Logical connectives " \wedge " ("and"), " \vee " ("or"), " \neg " ("not"), the quantifiers " \forall " ("for all") and " \exists " ("there exists") and parentheses "(" and ")".

A *formula* in the language of groups is a logical expression using the above symbols. A variable in a formula is called *bound* if it is bound to a quantifier and *free* otherwise. A *sentence* is a formula in L_0 where all variables are bound.

Example 4.1. 1. $\forall x_1((x_1x_3 = 1) \vee (\exists x_2 x_1x_2 = x_3))$ is a formula in L_0 but not a sentence, since x_3 is a free variable.

2. $\forall x, y [x, y] = 1$ is a sentence in L_0 since both x and y are bound.

3. The statement $\forall x \exists k \in \mathbb{N} x^k = 1$ is not allowed in the first-order language of groups, because it quantifies over an integer, and not a group element; similarly one cannot quantify over subsets, subgroups or morphisms. In fact the quantifier does not mention to which group the variables belong, it is the interpretation which specifies the group.

Given a group G and a sentence σ we will say that G *satisfies* σ or σ *is a true sentence in* G if σ is true if interpreted in G (i.e. variables interpreted as group elements in G). We denote by $G \models \sigma$ the fact that σ is a true sentence in G .

Example 4.2. $G \models \forall x, y [x, y] = 1$ if and only if G is abelian.

Definition 4.3. A universal sentence (of L_0) is a sentence of the form

$$\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n),$$

where $\varphi(x_1, \dots, x_n)$ is a quantifier free formula which contains at most the variables x_1, \dots, x_n . Similarly an existential sentence is one of the form $\exists \underline{x} \varphi(\underline{x})$, where $\underline{x} = (x_1, \dots, x_n)$ and $\varphi(\underline{x})$ is as above. An universal-existential- or $\forall\exists$ -sentence is one of the form

$$\forall \underline{y} \exists \underline{x} \varphi(\underline{x}, \underline{y}).$$

A positive sentence is a sentence containing no negations \neg .

The universal theory of a group G , which we denote by $\text{Th}_\forall(G)$, is the set of all universal sentences of L_0 true in G . Similarly the existential theory $\text{Th}_\exists(G)$ of a group G is the set of all existential sentences true in G .

The first-order or elementary theory of a group G , denoted by $\text{Th}(G)$ is the set of all sentences of L_0 true in G .

Remark 4.4. (1) It is well-known that every sentence of L_0 is logically equivalent to one of the form $Q_1x_1, \dots, Q_nx_n \varphi(\underline{x})$, where $\underline{x} = (x_1, \dots, x_n)$ is a tuple of distinct variables, each Q_i for $1, \dots, n$ is a quantifier, i.e. either \forall or \exists , and $\varphi(\underline{x})$ is a formula of L_0 containing no quantifiers and at most the variables of \underline{x} .

(2) Since any existential sentence is logically equivalent to the negation of an universal sentence, it follows that if two groups have the same universal theory, then they also have the same existential theory.

(3) If $H \leq G$, a quantifier free formula which is true for every tuple of elements of G is clearly also true for every tuple of elements of H , hence $\text{Th}_\forall(G) \subset \text{Th}_\forall(H)$. We immediately conclude that $\text{Th}_\forall(F_2) = \text{Th}_\forall(F_n)$ for all $n \geq 3$.

Two groups G and H are said to be *elementary equivalent* if they have the same elementary theory, i.e. $\text{Th}(G) = \text{Th}(H)$. The aim is to encode properties of a group in first-order sentences and then conclude that all groups elementary equivalent to this group possess the same properties.

Example 4.5. (1) The sentence $\forall x (x = 1 \vee x^k \neq 1)$ says that the group has no k -torsion. Thus, the property of being torsion-free can be encoded by infinitely many elementary sentences. Hence any group elementary equivalent to a torsion-free group is torsion-free.

(2) \mathbb{Z} and \mathbb{Z}^2 don't have the same elementary theory. Indeed one can encode in a sentence the fact that there are at most 2 elements modulo the doubles, i.e. the sentence

$$\forall x_1, x_2, x_3 \exists x_4 (x_1 = x_2 + 2x_4) \vee (x_1 = x_3 + 2x_4) \vee (x_2 = x_3 + 2x_4)$$

holds in \mathbb{Z} and not in \mathbb{Z}^2 since $\mathbb{Z}/2\mathbb{Z}$ has two elements and $\mathbb{Z}^2/2\mathbb{Z}^2$ has four elements.

If H and G are groups and $f : H \rightarrow G$ is a monomorphism, then f is an *elementary embedding* provided whenever $\varphi(x_1, \dots, x_n)$ is a formula of L_0 containing at most the distinct variables x_1, \dots, x_n , and $(h_1, \dots, h_n) \in H^n$, then $\varphi(h_1, \dots, h_n)$ is true in H if and only if $\varphi(f(h_1), \dots, f(h_n))$ is true in G .

Note that the existence of an elementary embedding $f : H \rightarrow G$ is a sufficient but not necessary condition for groups G and H to be elementary equivalent.

4.2 The Tarski problems

The following theorem, formerly known as the Tarski conjecture was proved independently by Z. Sela ([Se1]-[Se6]) and Kharlampovich/Myasnikov ([KhMy]).

Theorem 4.6 (Tarski Problem). *Finitely generated non-abelian free groups have the same elementary theory.*

Definition 4.7. *Let G be a group and Q a formula in the first-order language of the group G (i.e. possibly with coefficients). Let $\underline{p} = (p_1, \dots, p_n)$ be the tuple of free variables appearing in Q . Then the definable set associated to $Q(\underline{p})$ is the set of values \underline{p}_0 in G^n such that $Q(\underline{p}_0)$ is true in G .*

Example 4.8. The following formulas give definable sets:

- (a) $Q(p) \equiv \exists x, y (p = [x, y])$ yields the definable set $\{p_0 \in G \mid \exists x, y \in G : p_0 = [x, y]\}$.
- (b) $Q(p) \equiv (\exists x_1, x_2, x_3, x_4 (p = [x_1, x_2][x_3, x_4])) \wedge (\forall y_1, y_2 (p \neq [y_1, y_2]))$.

Arbitrary definable sets (as in the above example) are Diophantine sets, i.e. they may depend on additional variables aside from \underline{p} . The easiest definable sets are definable sets which are given by sentences without additional variables, thus are varieties.

Definition 4.9. *A basic definable set is a definable set which corresponds to a formula which is a system of equations, i.e. is of the form*

$$Q(\underline{p}) \equiv (\Sigma(\underline{p}, \underline{a}) = 1),$$

where \underline{a} is a tuple of constants from G . Hence a basic definable set is a variety over G .

Example 4.10. Let $F_k = \langle a \rangle$ be a non-abelian free group and

$$L = \langle y, a \mid V(y, a) \rangle$$

a restricted F_k -limit group. Then the basic definable set (respectively the variety) associated to L is

$$\{y_0 \in F_k^n \mid V(y_0, a) = 1\},$$

i.e. the definable set corresponding to the formula

$$Q(y) \equiv (V(y, a) = 1).$$

5 Makanin-Razborov resolutions and the shortening argument

5.1 Systems of equations over hyperbolic groups

One key part in Sela's solution to the Tarski problems and his subsequent study of the elementary theory of torsion-free hyperbolic groups is the understanding of the structure of definable sets over free and hyperbolic groups. As seen in Definition 4.9 the easiest of such definable sets are varieties over the given free or hyperbolic group. Let Γ be a hyperbolic group. Recall that a variety over Γ is a set

$$\{\underline{g} = (g_1, \dots, g_n) \in G^n \mid \Sigma(\underline{g}, \underline{a}) = 1\},$$

where $\Sigma(\underline{x}, \underline{a}) = 1$ is a system of equations with variables $\underline{x} = (x_1, \dots, x_n)$ and coefficients \underline{a} from Γ . In the following we will just write x instead of \underline{x} . In order to be able to understand definable sets over Γ , one first of all has to understand solution to systems of equations over Γ . If Γ is a free group this was done by the work of Makanin and Razborov, who developed an algorithm to determine if a given system of equations over a free group has a solution (Makanin) and described the set of solutions to this system of equations (Razborov). This algorithm is commonly known as the Makanin-Razborov algorithm. O. Kharlampovich and A. Myasnikov used a refined version of this algorithm in their solution to the Tarski problems, while Sela used a more geometric approach to get a description of the solution set of a system of equations by heavily exploiting the Rips machine and bypassing much of the combinatorics in the process. The advantage of Sela's approach is, that it can fairly easy be generalized to understand system of equations over other classes of groups. The most important generalizations are for systems of equations over

- torsion-free hyperbolic groups by Sela ([Se7]),
- all hyperbolic groups by Reinfeldt-Weidmann ([ReWe])
- acylindrical hyperbolic groups by Groves-Hull ([GrHu])
- free products of groups by Jaligot-Sela ([JaSe])
- free semigroups by Sela. ([Se11])

Note that this list is by no means complete.

It turns out that understanding all solutions to a system of equations over a given group Γ is equivalent to understanding $\text{Hom}(G, \Gamma)$, i.e. all homomorphisms from a finitely generated group G to Γ . This can be seen by considering a system of equations $\Sigma(x, a)$ with variables $x = (x_1, \dots, x_n)$ and coefficients $a = (a_1, \dots, a_k)$ from Γ , given

by a finite set of words $w_1(x, a), \dots, w_r(x, a) \in F(x) * \Gamma = \langle x, a \rangle$. Then clearly every solution of $\Sigma(x, a) = 1$ corresponds to a homomorphism

$$\varphi : G_\Sigma := \langle x, \Gamma \mid w_1(x, a) = 1, \dots, w_r(x, a) = 1 \rangle \rightarrow \Gamma$$

and vice versa. In the following we will describe Sela's approach to the solution of systems of equation over hyperbolic groups. The aim is to get a parametrization of $\text{Hom}(G, \Gamma)$, where G is a finitely generated group. Since we need the result for all hyperbolic groups, instead of only torsion-free hyperbolic ones, we are going to present the generalization of Sela's techniques which appear in [ReWe].

5.2 The shortening argument

In this section we will give a short account of Z. Sela's shortening argument. For the original shortening argument in the torsion free case see [RiSe2]. Since we have to deal with torsion, the version of the shortening argument presented here is borrowed from [ReWe]. For a detailed discussion on the subject and proofs of the results stated in this chapter we therefore refer the reader to chapter 4.2 in the aforementioned paper. Throughout the whole chapter let Γ be a hyperbolic group.

Definition 5.1. *Let G be a finitely generated Γ -limit group with a fixed generating set S , $H \leq G$ a subgroup, Γ a hyperbolic group with corresponding Cayley graph (X, d_X) and $\varphi : G \rightarrow \Gamma$ a homomorphism. The length of φ is defined as*

$$l(\varphi) = \sum_{s \in S} d_X(1, \varphi(s)).$$

- (1) We call φ short if $l(\varphi) \leq l(c_g \circ \varphi \circ \alpha)$ for any $g \in \Gamma$ and any $\alpha \in \text{Mod}(G)$, where c_g denotes conjugation by g .
- (2) φ is called short with respect to H if $l(\varphi) \leq l(c_g \circ \varphi \circ \alpha)$ for any $g \in C_\Gamma(\varphi(H))$ and any $\alpha \in \text{Mod}(G, H)$, i.e. modular automorphisms of G fixing H . In particular if G is a restricted Γ -limit group, φ is short with respect to Γ if $l(\varphi) \leq l(\varphi \circ \alpha)$ for any $\alpha \in \text{Mod}(G, \Gamma)$.

Theorem 5.2 (Proposition 4.6 in [ReWe]). *Let L be a one-ended (restricted) Γ -limit group and $(\varphi_n) \subset \text{Hom}(L, \Gamma)$ a stably injective sequence of pairwise distinct non-injective homomorphisms. Then the φ_n are eventually not short in the unrestricted case and not short relative Γ in the restricted case.*

Proof. For the sake of contradiction assume that the homomorphisms φ_n are short. By Theorem 3.8 the sequence (φ_n) converges into a minimal action of L on some limit real tree (T, t_0) . By Theorem 3.15 the action satisfies the stability assertions of Theorem 2.8.

Since L is one-ended and stabilizers of tripods and unstable arcs are finite, the action of L on T splits as a non-trivial graph of actions \mathbb{A} , such that every vertex action is either simplicial, of axial or orbifold type. Now we can use this decomposition as a graph of actions to construct modular automorphisms $\alpha_n \in \text{Mod}(L)$, such that $l(\varphi_n \circ \alpha) < l(\varphi_n)$ for large n , i.e. show that for large n the φ_n are not short. Denote by (X, d_X) the Cayley graph of Γ with respect to some fixed generating set.

Lemma 5.3 (Theorem 4.8 in [ReWe]). *Let $v \in VA$ be an axial vertex with corresponding vertex tree T_v . For any finite subset $S \subset L$ there exists $\alpha \in \text{Mod}(L)$ such that for any $s \in S$ the following hold.*

(a) *If $[t_0, st_0] \subset T$ has a non-degenerate intersection with a translate of T_v in T , then*

$$d_T(t_0, \alpha(s)t_0) < d_T(t_0, st_0),$$

(b) *otherwise,*

$$d_T(t_0, \alpha(s)t_0) = d_T(t_0, st_0).$$

Lemma 5.4 (Theorem 4.15 in [ReWe]). *Let $v \in VA$ be an orbifold vertex with corresponding vertex tree T_v . For any finite subset $S \subset L$ there exists $\alpha \in \text{Mod}(L)$ such that for any $s \in S$ the following hold.*

(a) *If $[t_0, st_0] \subset T$ has a non-degenerate intersection with a translate of T_v in T , then*

$$d_T(t_0, \alpha(s)t_0) < d_T(t_0, st_0),$$

(b) *otherwise,*

$$d_T(t_0, \alpha(s)t_0) = d_T(t_0, st_0).$$

Suppose we are either in the case of Lemma 5.3 or Lemma 5.4. Denote by d_n the scaled metric on the Cayleygraph X . In the unrestricted case after postcomposing φ by an inner automorphism of Γ we can assume that (1) is an approximating sequence of t_0 . In the restricted case this is given by the construction of the limit tree. Therefore

$$\lim_{n \rightarrow \infty} d_n(1, \varphi_n \circ \alpha(s)) = d_T(t_0, \alpha(s)t_0) < d_T(t_0, st_0) = \lim_{n \rightarrow \infty} d_n(1, \varphi_n(s))$$

and hence for large n , $d_X(1, \varphi_n \circ \alpha(s)) < d_X(1, \varphi_n(s))$. This implies that for large n , φ_n is not short in the unrestricted case and not short with respect to Γ in the restricted case.

It remains to consider the case of a simplicial vertex tree. In this case it is not possible to directly shorten the action on the limit tree, but we are still able to shorten the action of L on X for large n .

If there exists a simplicial vertex tree in the graph of actions \mathbb{A} we can refine the graph of actions such that every simplicial vertex tree is a point. Then in particular there exists an edge $e \in EA$ with non-zero length. By Theorem 3.15 the stabilizer A_e of this edge is finite-by-abelian and hence has infinite center. Let α be a Dehn-twist along the edge e by an element of infinite order from $Z(A_e)$.

Lemma 5.5 (Corollary 4.20 in [ReWe]). *For sufficiently large n , there exists $m_n \in \mathbb{N}$ such that*

$$l(\varphi_n \circ \alpha^{m_n}) < l(\varphi_n).$$

This completes the proof of Theorem 5.2. \square

A combination of Lemma 5.3, Lemma 5.4 and Lemma 5.5 is usually referred to as the *shortening argument*.

We draw one corollary which allows us later to shorten relative a finite set of inequalities.

Corollary 5.6. *Let L be a (restricted) one-ended Γ -limit group, $S \subset L \setminus \{1\}$ a finite set and (φ_n) a stable sequence of pairwise distinct non-injective homomorphisms, such that the φ_n are short among all homomorphisms which map all $s \in S$ to a non-trivial image. Then $\underline{\ker}(\varphi_n) \neq 1$. In particular $L/\underline{\ker}(\varphi_n)$ is a proper quotient of L .*

Proof. Suppose that $\underline{\ker}(\varphi_n) = 1$, i.e. (φ_n) is stably injective. It then follows from the shortening argument that there exist modular automorphisms $\alpha_n \in \text{Mod}(L)$ (respectively $\alpha_n \in \text{Mod}(L, \Gamma)$ in the restricted case) and $g_n \in \Gamma$ such that $l(c_{g_n} \circ \varphi_n \circ \alpha_n) < l(\varphi_n)$. Note that $g_n = 1$ in the restricted case. It is implicitly contained in the proof of the shortening argument in [ReWe] that for large n the α_n can be chosen such that $\varphi_n \circ \alpha_n(s) \neq 1$ for all $s \in S$. This yields a contradiction to the assumption that the φ_n are short among all homomorphisms with this property. \square

5.3 Shortening quotients

Equipped with the shortening argument we are now ready to draw some consequences from Theorem 5.2. Let Γ again be a hyperbolic group.

Definition 5.7. *Let L be a (restricted) one-ended Γ -limit group and $(\varphi_n) \subset \text{Hom}(L, \Gamma)$ a stable sequence of short homomorphisms. Then $Q := L/\underline{\ker}(\varphi_n)$ is called a shortening quotient of L .*

We make the following observation.

Lemma 5.8. *Let L be a (restricted) one-ended Γ -limit group. Then either there exists a proper shortening quotient Q of L , or L is isomorphic to a subgroup of Γ .*

Proof. Let $(\varphi_n) \subset \text{Hom}(L, \Gamma)$ be a stable sequence of homomorphisms from L to Γ . Clearly either (φ_n) contains a constant subsequence or we can assume after passing to a subsequence that the φ_n are pairwise distinct. Suppose the second case holds. For all $n \in \mathbb{N}$ let $\alpha_n \in \text{Mod}(L)$ be a modular automorphism such that $(\varphi_n \circ \alpha_n)$ is a stable sequence of short homomorphisms. Hence it follows from Corollary 5.6 that

$$Q := L/\underline{\ker}(\varphi_n \circ \alpha_n)$$

is a proper quotient of L .

Now assume that every stable sequence of homomorphisms from L to Γ has a constant subsequence. Since L is a Γ -limit group, there exists a stably injective sequence $(\varphi_n) \subset \text{Hom}(L, \Gamma)$. After passing to the constant subsequence of (φ_n) we have $\varphi_n = \varphi$ for all $n \in \mathbb{N}$ and some fixed injective homomorphism $\varphi : L \rightarrow \Gamma$. In particular

$$L = L/\underline{\ker}(\varphi_n) = L/\ker \varphi \cong \text{Im}(\varphi) \leq \Gamma.$$

□

In the next step we will define shortening quotients not only for one-ended Γ -limit groups but for arbitrary Γ -limit groups.

Definition 5.9. *Let L be a (restricted) Γ -limit group and \mathbb{D} a (restricted) Dunwoody decomposition of L , i.e. every vertex group is either finite or one-ended (relative Γ in the restricted case). Let $(\varphi_n) \subset \text{Hom}(L, \Gamma)$ be a sequence of homomorphisms such that $\varphi_n|_{D_v}$ is short for every vertex group D_v of \mathbb{D} for all $n \in \mathbb{N}$. Then $Q := L/\underline{\ker}(\varphi_n)$ is called a shortening quotient of L .*

We immediately get an equivalent result to Lemma 5.8.

Lemma 5.10. *Let L be a (restricted) Γ -limit group with Dunwoody decomposition \mathbb{D} . Then either there exists a proper shortening quotient Q of L , or all vertex groups of \mathbb{D} are isomorphic to subgroups of Γ .*

We now want to show that there exist only finitely many maximal shortening quotients of a given (restricted) Γ -limit group L with respect to the following partial order defined on the set of all Γ -limit quotients of L .

Definition 5.11. *Let L be a (restricted) Γ -limit group and Q_1, Q_2 Γ -limit quotients of L with corresponding quotient maps $\pi_i : L \rightarrow Q_i$ for $i = 1, 2$. We say that $Q_1 \geq Q_2$ if there exists an epimorphism $\eta : Q_1 \rightarrow Q_2$ such that $\pi_2 = \eta \circ \pi_1$. We further say that $Q_1 > Q_2$ if $Q_1 \geq Q_2$ and $Q_2 \not\geq Q_1$.*

Theorem 5.12. *Let L be a (restricted) Γ -limit group.*

- (1) *Let $Q_1 < Q_2 < Q_3 < \dots$ be an infinite ascending sequence of Γ -limit quotients of L . Then there exists a maximal Γ -limit quotient Q of L such that $Q_i < Q$ for all $i \geq 1$.*
- (2) *There exist only finitely maximal Γ -limit quotients of L with respect to " $<$ ".*

Proof. This is a combination of Theorem 6.10 and Theorem 6.11 in [ReWe]. □

Remark 5.13. *In particular there exist only finitely many maximal shortening quotients of a Γ -limit group L .*

The following result is commonly referred to as the descending chain condition for Γ -limit groups. It states that there cannot be an infinite descending sequence of Γ -limit quotients of a given Γ -limit group. This will be crucial in the sequel, to show that certain iterative procedures terminate after finitely many steps.

Proposition 5.14 (Descending chain condition). *Let L be a (restricted) Γ -limit group. Then there exists no infinite descending sequence $Q_1 > Q_2 > Q_3 > \dots$ of Γ -limit quotients of L .*

Proof. This follows immediately from Lemma 6.8 in [ReWe]. □

Later on we will need more information about the algebraic structure of special types of shortening quotients, the so-called strict shortening quotients.

Definition 5.15. *Let Γ be a hyperbolic group and L be a one-ended (restricted) Γ -limit group. Let \mathbb{A} be a virtually abelian JSJ decomposition of L . We say that a proper shortening quotient Q of L with corresponding quotient map $\eta : L \rightarrow Q$ is a strict shortening quotient if the following hold:*

- (1) η maps each QH subgroup A_v of L to a non-virtually abelian image. Moreover if $1 \rightarrow E \rightarrow A_v \xrightarrow{\pi} O \rightarrow 1$ is the short exact sequence corresponding to A_v , then for every peripheral subgroup O_e of O , $\pi^{-1}(O_e)$ is mapped isomorphically into Q by η .
- (2) If A_v is a virtually abelian vertex group, then η maps every subgroup $U \leq A_v$ with $U^+ = P_v^+$ injectively into Q . (For a definition of P_v^+ see Definition 3.27).
- (3) Every virtually abelian edge group A_e of \mathbb{A} gets embedded into Q by η .
- (4) Let R be a rigid vertex group in \mathbb{A} . The envelope $E(R)$ of R is the fundamental group of a graph of groups \mathbb{B} whose underlying graph is a star with one vertex v with vertex group R connected to all other vertices via an edge of the form (v, u) for some $u \in VB \setminus \{v\}$. Edges in \mathbb{B} are in 1-to-1 correspondence to edges adjacent to R in \mathbb{A} and the edge groups are isomorphic. Let now $e = (v, u)$ be an edge in \mathbb{B} and $e' = (v', u')$ the corresponding edge in \mathbb{A} adjacent to R (where R is the rigid vertex group of v'). If $A_{u'}$ is rigid or a QH vertex group, then B_u is the maximal virtually abelian subgroup of $A_{u'}$ containing $A_{e'}$. If $A_{u'}$ is a virtually abelian vertex group, then B_u is the peripheral subgroup of $A_{u'}$ containing $A_{e'}$. Then η is injective when restricted to the envelope $E(R)$ of R . In particular every rigid vertex group in \mathbb{A} gets embedded into Q .

As before when we generalized shortening quotients from one-ended groups to the case of arbitrary Γ -limit groups, we can do the same for strict shortening quotients.

Definition 5.16. *Let L be a (restricted) Γ -limit group with (restricted) Dunwoody decomposition \mathbb{D}_L . Let Q be a proper shortening quotient of L with (restricted) Dunwoody decomposition \mathbb{D}_Q and denote by $\eta : L \rightarrow Q$ the corresponding quotient map. We call Q a strict shortening quotient of L if the following hold.*

1. The underlying graph D_Q of \mathbb{D}_Q is a refinement of the underlying graph D_L of \mathbb{D}_L .
2. Let $v \in VD_L$. Then either $Q_v := \eta(D_v)$ is a strict shortening quotient of D_v or D_v is isomorphic to a subgroup of Γ and there exists a vertex group Q_v in \mathbb{D}_Q such that η maps D_v isomorphically onto Q_v . Note that we do not claim that Q_v is one-ended in the first alternative (hence in particular Q_v may not be isomorphic to a vertex group of \mathbb{D}_Q).

The significance of strict shortening quotients will become apparent in the following chapters.

5.4 Makanin-Razborov diagrams

In this chapter we will give a short account of the construction of a Makanin-Razborov diagram for a finitely generated group G over a hyperbolic group Γ . Essentially everything from this chapter is just a reformulation of the corresponding results in [ReWe] to better fit our needs.

Definition 5.17. *Let G be a finitely generated accessible group, Γ a group. Then we call a homomorphism $\Psi : G \rightarrow \Gamma$ locally injective if Ψ is injective when restricted to the vertex groups of a Dunwoody decomposition of G .*

The following is the main theorem in [ReWe].

Theorem 5.18 (Theorem 5.7 in [ReWe]). *Let Γ be a hyperbolic group and G be a finitely generated group. Then there exists a finite directed rooted tree T with root v_0 satisfying:*

- (a) The vertex v_0 is labeled by G .
- (b) Any vertex $v \in VT$, $v \neq v_0$, is labeled by a Γ -limit group G_v .
- (c) Any edge $e \in ET$ is labeled by an epimorphism $\pi_e : G_{\alpha(e)} \rightarrow G_{\omega(e)}$ such that for any homomorphism $\varphi : G \rightarrow \Gamma$ there exists a directed path e_1, \dots, e_k from v_0 to some vertex $\omega(e_k)$ such that

$$\varphi = \Psi \circ \pi_{e_k} \circ \alpha_{k-1} \circ \pi_{e_{k-1}} \circ \dots \circ \alpha_1 \circ \pi_{e_1}$$

where $\alpha_i \in \text{Mod } G_{\omega(e_i)}$ for $1 \leq i \leq k$ and Ψ is locally injective.

We call T a Makanin-Razborov diagram of G , and a directed path as in (c), a Makanin-Razborov resolution of G .

Despite the growing interest in these diagrams since Sela's solution to the Tarski conjectures, very few actual examples of MR diagrams have been computed. If Γ is a non-abelian free group and G is either a surface group, a free abelian group, or a non-abelian free group, Makanin-Razborov diagrams for G are well-known (see for example

[ChGu]). All these examples have in common that the JSJ decomposition of the limit group is trivial, namely consists of a single vertex. In Theorem 6.10 we construct a Makanin-Razborov diagram of a F_k -limit group L which is a double $F_2 *_{\langle w \rangle} F_2$ along a special word $w \in F_2$. In particular the JSJ decomposition of L consists of two rigid vertices connected by an edge with cyclic edge group, hence is non-trivial. To our knowledge this is the first example of an explicit Makanin-Razborov diagram of a group with non-trivial JSJ decomposition.

In our work we will only be interested in Makanin-Razborov diagrams of Γ -limit groups, that is if the group G which labels the root v_0 is a finitely generated Γ -limit group. Therefore we only sketch the proof of Theorem 5.18 in this case. It should be noted though that this case already covers essentially all the difficulties of the general case. As mentioned before, a complete proof of the above Theorem for an arbitrary finitely generated group G can be found in [ReWe].

Proof of Theorem 5.18. Since by Proposition 5.14 every descending chain of Γ -limit groups eventually stabilizes, it clearly suffices to show that any Γ -limit group L admits a finite set $\{q_i : L \rightarrow Q_i \mid 1 \leq i \leq n\}$ of proper quotient maps such that any homomorphism $\varphi : L \rightarrow \Gamma$ which is not locally injective, factors through some q_i after precomposition with an element of $\text{Mod}(L)$. Such a set of quotient maps is called a Γ -factor set of L relative $\text{Mod}(L)$.

With the construction of the set of maximal shortening quotients of L which exists by Theorem 5.12 we have already done the main part of the work.

So let \mathbb{D} be a Dunwoody decomposition of L . By Theorem 5.12 there exist finitely many maximal shortening quotients Q_1, \dots, Q_m of L . Denote the corresponding quotient maps by $\pi_i : L \rightarrow Q_i$ for all $i \in \{1, \dots, m\}$.

We claim that for every non-locally injective homomorphism $\varphi : L \rightarrow \Gamma$ there exists a modular automorphism $\alpha \in \text{Mod}(L)$ and (in the unrestricted case) an inner automorphism c_g of Γ such that $c_g \circ \varphi \circ \alpha$ factors through one of the π_i 's.

So let $\varphi : L \rightarrow \Gamma$ be a homomorphism and suppose that φ is not locally injective. Then there exists a modular automorphism $\alpha \in \text{Mod}(L)$ and (in the unrestricted case) an inner automorphism c_g of Γ such that $\hat{\varphi} = c_g \circ \varphi \circ \alpha$ is short.

The constant sequence $(\hat{\varphi})$ converges into a Γ -limit group M with corresponding Γ -limit map $\hat{\pi} : L \rightarrow M$. Since by assumption φ is not injective when restricted to at least one vertex group \mathbb{D}_v , M is a proper quotient of L . Since $\hat{\varphi}$ was chosen to be short when restricted to every vertex group of \mathbb{D} , M is a proper shortening quotient of L . Hence by Theorem 5.12 either M is itself a maximal shortening quotient of L or there exists a maximal shortening quotient Q of L such that $M < Q$. In both cases $\hat{\varphi}$ splits through a maximal shortening quotient of L . As mentioned before the descending chain condition for Γ -limit groups (Proposition 5.14) seals the deal. \square

5.5 Strict and well-structured resolutions

Definition 5.19. Let Γ be a hyperbolic group, G a Γ -limit group and let

$$G = G_0 \xrightarrow{\eta_0} G_1 \xrightarrow{\eta_1} G_2 \rightarrow \cdots \xrightarrow{\eta_{m-1}} G_m = \pi_1(\mathbb{D})$$

be a MR resolution of Γ , such that \mathbb{D} is a Dunwoody decomposition of G_m and every vertex group of \mathbb{D} can be embedded into Γ . We say that the given resolution is a strict MR resolution if G_i is a strict shortening quotient for all $i \in \{1, \dots, m\}$.

Proposition 5.20. Let L be a Γ -limit group. Then the Makanin-Razborov diagram of L contains a strict Makanin-Razborov resolution.

Proof. Using the results from [ReWe] this is identical to the proof of Proposition 1.29 in [Se7]. Note that in the case that every vertex group of a Dunwoody decomposition of L is isomorphic to a subgroup of Γ , the empty resolution is considered a strict resolution. \square

We are now going to replace the canonical collection of MR resolutions associated with a restricted Γ -limit group, with a canonical collection of strict MR resolutions.

Proposition 5.21. Let L be a restricted Γ -limit group. There exists a canonical finite collection of strict MR resolutions $\text{Res}_1(L), \dots, \text{Res}_s(L)$, such that the Γ -limit groups associated with the resolutions $\text{Res}_i(L)$ (i.e. the Γ -limit groups where the resolution starts) are either L itself or a quotient of it, and every homomorphism from L to Γ , factors through (at least) one of these resolutions.

Proof. Let U be the collection of all MR resolutions that appear in the canonical MR diagram of the restricted Γ -limit group L . Each resolution in U , which is a strict resolution is taken to be one of the resolutions in our new collection. By Proposition 5.20, U contains at least one strict resolution. Each non-strict resolution in U will be replaced by a finite collection of strict resolutions either of L or of a quotient of it in the following way: Let

$$\text{Res}(L) : L = G_0 \xrightarrow{\eta_0} G_1 \xrightarrow{\eta_1} G_2 \rightarrow \cdots \xrightarrow{\eta_{m-1}} G_m = \pi_1(\mathbb{D})$$

be a resolution in the MR diagram of L , which is not a strict resolution. For each level $j \in \{1, \dots, m-1\}$ let R_j be the quotient group of G_j obtained from the collection of homomorphisms $G_j \rightarrow \Gamma$, that factor through the part $G_j \xrightarrow{\eta_j} G_{j+1} \rightarrow \cdots \xrightarrow{\eta_{m-1}} G_m$ of $\text{Res}(L)$, i.e. the quotient of G_j by the subgroup generated by all elements which lie in the kernel of all such homomorphisms.

Since $\text{Res}(L)$ is not a strict resolution at least for some level j , R_j is a proper quotient of G_j with corresponding quotient map τ_j . Denote by R_j^1, \dots, R_j^m the finitely many maximal Γ -limit quotients of R_j that appear on the level below R_j in the MR diagram of R_j . We assume without loss of generality that R_j is already a Γ -limit group (if this is not the case we proceed in the following with the finitely many Γ -limit quotients

R_j^1, \dots, R_j^m). Let j_0 be the highest integer j (i.e. the lowest level in figure 1) for which R_{j_0} is a proper quotient of G_{j_0} . We replace the resolution $\text{Res}(L)$ by finitely many resolutions obtained in the following way:

- (1) The top part of the new resolutions is identical with the top part of the resolution $\text{Res}(L)$ starting at $j_0 - 1$ and above.
- (2) The Γ -limit group at level j_0 in all the new resolutions is equal to R_{j_0} and the corresponding (new) epimorphism $\eta'_{j_0-1} : G_{j_0-1} \rightarrow R_{j_0}$ is equal to $\tau_{j_0} \circ \eta_{j_0-1}$.
- (3) Each of the new resolutions continues after the level j_0 along one of the resolutions that appear in the MR diagram of the restricted Γ -limit group R_{j_0} .

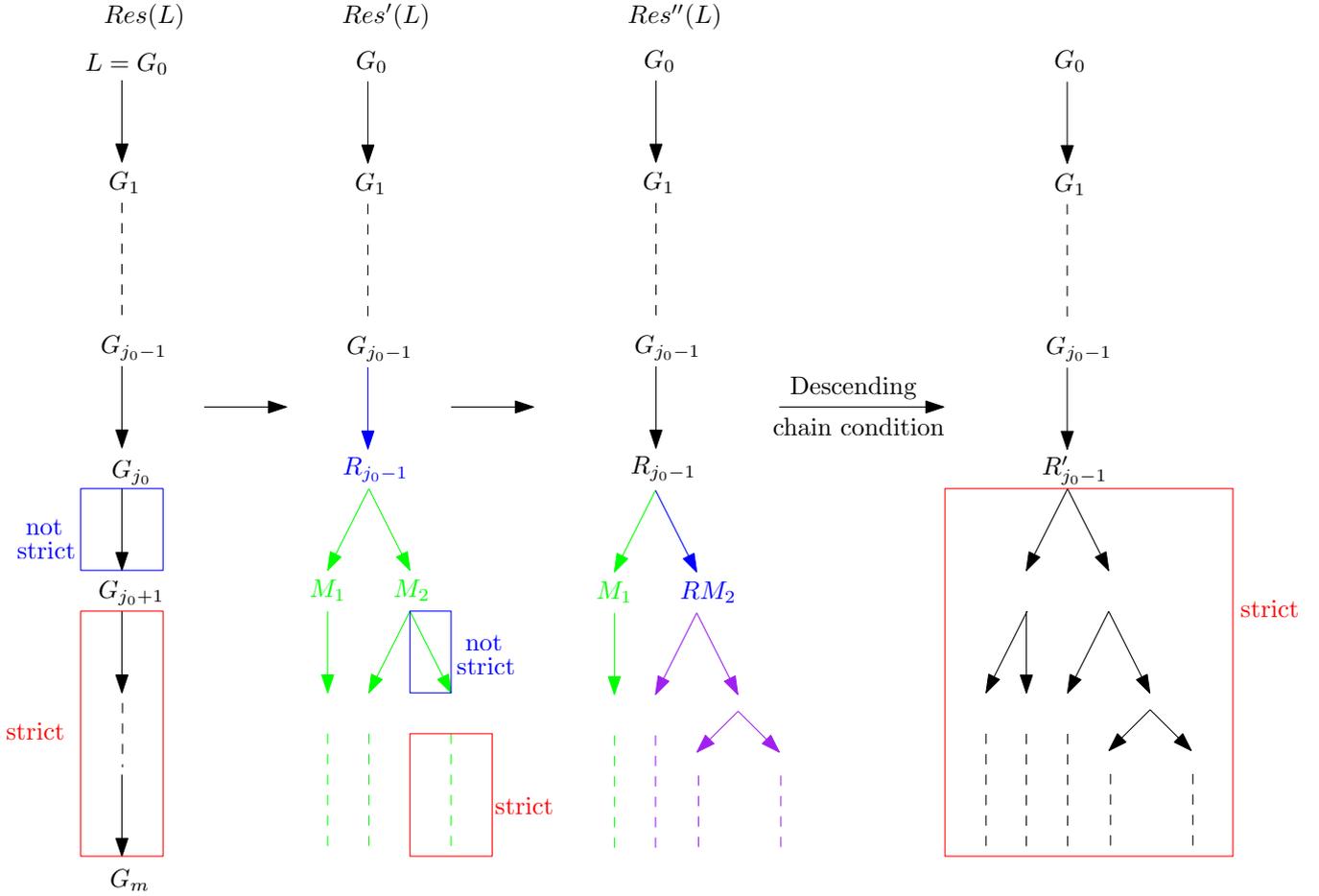


Figure 1: Constructing strict resolutions

Hence every new resolution is of the form

$$L = G_0 \xrightarrow{\eta_0} \dots \xrightarrow{\eta_{j_0-2}} G_{j_0-1} \xrightarrow{\tau_{j_0} \circ \eta_{j_0-1}} R_{j_0} \xrightarrow{\pi_0} M_1 \xrightarrow{\pi_1} M_2 \rightarrow \dots \xrightarrow{\pi_{l-1}} M_l = \pi_1(\mathbb{D}')$$

where

$$R_{j_0} \xrightarrow{\pi_0} M_1 \xrightarrow{\pi_1} M_2 \rightarrow \cdots \xrightarrow{\pi_{l-1}} M_l$$

is a MR resolution in the MR diagram of R_{j_0} .

Clearly every homomorphism from L to Γ that factors through $\text{Res}(L)$, factors through (at least) one of the new resolutions. Any strict resolution from the obtained collection of new resolutions is taken to be a resolution in our collection of strict resolutions. If one of the new obtained resolutions is not strict we replace it by finitely many new resolutions obtained by the same procedure as before.

Whenever we replace a Γ -limit group which lies on a lower level than j_0 (i.e. on some level $j > j_0$) in the new resolutions with its proper quotient R_j , then R_j is clearly also a proper quotient of G_{j_0} . Hence by the descending chain condition (Proposition 5.14), this can happen only finitely many times. Therefore after finitely many steps all of the new resolutions are strict resolutions below level $j_0 - 1$ (see figure 1) and hence the procedure described above terminates after finitely many steps. By construction after the termination of the procedure every obtained resolution is a strict MR resolution and every homomorphism from L to Γ factors through at least one of these strict resolutions. \square

Definition 5.22. We call the collection of strict MR resolutions constructed in Proposition 5.21 the strict Makanin-Razborov diagram of the restricted Γ -limit group L .

We now introduce some notation:

Let $\Gamma = \langle a_1, \dots, a_k \rangle$ be a hyperbolic group. Let L be a restricted Γ -limit group and let

$$\text{Res}(L) : L = L_0 \xrightarrow{\eta_0} L_1 \xrightarrow{\eta_1} L_2 \rightarrow \cdots \xrightarrow{\eta_{l-1}} L_l = \pi_1(\mathbb{D})$$

be a strict MR resolution of L , where $\pi_1(\mathbb{D})$ is a Dunwoody decomposition of L_l , such that every vertex group is isomorphic to a subgroup of Γ . Each restricted Γ -limit group L_i that appears along the resolution $\text{Res}(L)$, admits an associated restricted Dunwoody decomposition $\mathbb{U}^{(i)}$ with underlying graph $U^{(i)}$.

Denote by $R_1^0, \dots, R_{k_0}^0$ the vertex groups of $\mathbb{U}^{(0)}$. Let $Q_1^0, \dots, Q_{s(0)}^0$ be the QH vertex groups in the various virtually abelian JSJ decompositions \mathbb{A}_j^0 of the vertex groups R_j^0 . If Q_t^0 is contained in R_j^0 , let \mathbb{B}_t^0 be the graph of groups obtained from \mathbb{A}_j^0 by collapsing all edges which are not connected to the QH vertex group Q_t^0 . In particular all edge groups in \mathbb{B}_t^0 are virtually cyclic. We say that the *circumference* of the QH subgroup Q_t^0 , denoted by $\text{CF}(Q_t^0)$, is the subgroup of R_j^0 generated by Q_t^0 and the Bass-Serre generators in $\pi_1(B_t^0)$ (i.e. the generators of the fundamental group of the underlying graph).

We set \mathbb{H}_0 to be the empty subgraph of groups of $\mathbb{U}^{(0)}$ and for all $i \in \{1, \dots, l\}$ we define the following (possibly trivial, possibly disconnected) subgraphs $\mathbb{H}_i \subset \mathbb{U}^{(i)}$ inductively as follows:

For all $i \in \{1, \dots, l\}$ denote by $A_1^{i-1}, \dots, A_{k(i-1)}^{i-1}$ the vertex groups of $\mathbb{U}^{(i-1)}$ which are

either infinite virtually abelian or finite and by $O_1^{i-1}, \dots, O_{u(i-1)}^{i-1}$ the vertex groups of $\mathbb{U}^{(i-1)}$ which are finite-by-(closed) orbifold groups. We call such vertex groups *isolated virtually abelian* (respectively *isolated QH*) vertex groups.

For $i = 1$:

For every QH vertex group Q_t^0 defined above, let \mathbb{M}_t be the decomposition along finite subgroups of $\eta_0(CF(Q_t^0))$ inherited from the restricted Dunwoody decomposition $\mathbb{U}^{(1)}$ of L_1 . Then there exist vertex groups of \mathbb{M}_t , denoted by $H_{t_1}^0, \dots, H_{t_b}^0$, which do not contain any conjugate of the image $\eta_0(P)$ of a peripheral subgroup P of Q_t^0 (recall that peripheral subgroups correspond to boundary components of the underlying orbifold), while all other vertex groups, say V_1, \dots, V_s , of \mathbb{M}_t do contain a conjugate of the image of a peripheral subgroup of Q_t^0 . Moreover the image of every peripheral subgroup of Q_t^0 can be conjugated into one of the vertex groups V_1, \dots, V_s . To simplify notation we assume that there exists only one vertex group in \mathbb{M}_t , say H_t^0 , which does not contain any conjugate of the image $\eta_0(P)$ of a peripheral subgroup P of Q_t^0 .

Let now $\mathbb{D}^{(1)}$ be a Dunwoody decomposition of L_1 relative

$$\{\eta_0(A_1^0), \dots, \eta_0(A_{k(0)}^0), \eta_0(O_1^0), \dots, \eta_0(O_{u(0)}^0), H_1^0, \dots, H_{s(0)}^0\}$$

and denote the subgraph of $\mathbb{D}^{(1)}$ consisting of the collection of vertices with vertex groups which contain (at least) one of the groups

$$\eta_0(A_1^0), \dots, \eta_0(A_{k(0)}^0), \eta_0(O_1^0), \dots, \eta_0(O_{u(0)}^0), H_1^0, \dots, H_{s(0)}^0,$$

together with the edges connecting two such vertices, by \mathbb{H}_1 . Denote the other vertex groups of $\mathbb{D}^{(1)}$ by $R_1^1, \dots, R_{k_1}^1$.

For $i \in \{2, \dots, l\}$:

Let $Q_1^{i-1}, \dots, Q_{s(i-1)}^{i-1}$ be the QH vertex groups in the various virtually abelian JSJ decompositions \mathbb{A}_j^{i-1} of the vertex groups R_j^{i-1} . If Q_t^{i-1} is contained in R_j^{i-1} , let \mathbb{B}_t^{i-1} be the graph of groups obtained from \mathbb{A}_j^{i-1} by collapsing all edges which are not connected to the QH vertex group Q_t^{i-1} . In particular all edge groups in \mathbb{B}_t^{i-1} are virtually cyclic. We say that the *circumference* of the QH subgroup Q_t^{i-1} , denoted by $CF(Q_t^{i-1})$, is the subgroup of R_j^{i-1} generated by Q_t^{i-1} and the Bass-Serre generators in $\pi_1(B_t^{i-1})$ (i.e. the generators of the fundamental group of the underlying graph).

For every QH vertex group Q_t^{i-1} defined above, let \mathbb{M}_t be the decomposition along finite subgroups of $\eta_{i-1}(CF(Q_t^{i-1}))$ inherited from the restricted Dunwoody decomposition $\mathbb{U}^{(i)}$ of L_i . Then there exist vertex groups in \mathbb{M}_t , denoted by $H_{t_1}^{i-1}, \dots, H_{t_b}^{i-1}$, which do not contain any conjugate of the image $\eta_{i-1}(P)$ of a peripheral subgroup P of Q_t^{i-1} , while all other vertex groups, say V_1, \dots, V_s , of \mathbb{M}_t do contain a conjugate of the image of a peripheral subgroup of Q_t^{i-1} . Moreover the image of every peripheral subgroup of Q_t^{i-1} can be conjugated into one of the vertex groups V_1, \dots, V_s . Again to simplify notation we assume that there exists only one vertex group in \mathbb{M}_t , say H_t^{i-1} , which does not contain any conjugate of the image $\eta_{i-1}(P)$ of a peripheral subgroup P of Q_t^{i-1} .

Denote the vertex groups of \mathbb{H}_{i-1} by $K_1^{i-1}, \dots, K_{q(i-1)}^{i-1}$ and let now $\mathbb{D}^{(i)}$ be a Dunwoody decomposition of L_i relative

$$\{\eta_{i-1}(A_1^{i-1}), \dots, \eta_{i-1}(A_{k(i-1)}^{i-1}), \eta_{i-1}(O_1^{i-1}), \dots, \eta_{i-1}(O_{u(i-1)}^{i-1}), \\ H_1^0, \dots, H_{s(i-1)}^0, \eta_{i-1}(K_1^{i-1}), \dots, \eta_{i-1}(K_{q(i-1)}^{i-1})\}$$

and denote the subgraph of $\mathbb{D}^{(i)}$ consisting of the collection of vertices with vertex groups which contain (at least) one of the groups

$$\eta_{i-1}(A_1^{i-1}), \dots, \eta_{i-1}(A_{k(i-1)}^{i-1}), \eta_{i-1}(O_1^{i-1}), \dots, \eta_{i-1}(O_{u(i-1)}^{i-1}), \\ H_1^0, \dots, H_{s(i-1)}^0, \eta_{i-1}(K_1^{i-1}), \dots, \eta_{i-1}(K_{q(i-1)}^{i-1}),$$

together with edges connecting two such vertices, by \mathbb{H}_i . Denote the other vertex groups of $\mathbb{D}^{(i)}$ by $R_1^i, \dots, R_{k_i}^i$.

So let us briefly recall the setting we are in: $\Gamma = \langle a_1, \dots, a_k \rangle$ is as always a hyperbolic group. Let L be a restricted Γ -limit group and let

$$\text{Res}(L) : L = L_0 \xrightarrow{\eta_0} L_1 \xrightarrow{\eta_1} L_2 \rightarrow \dots \xrightarrow{\eta_{l-1}} L_l = \pi_1(\mathbb{D})$$

be a strict MR resolution of L , where $\pi_1(\mathbb{D})$ is a Dunwoody decomposition of L_l , such that every vertex group is isomorphic to a subgroup of Γ .

With each restricted Γ -limit group L_i that appears along the resolution $\text{Res}(L)$, we have constructed an associated (restricted) decomposition $\mathbb{D}^{(i)}$ with finite edge groups, such that every vertex group V of $\mathbb{D}^{(i)}$ either belongs to $\{R_1^i, \dots, R_{k_i}^i\}$ or to \mathbb{H}_i and these two sets are disjoint as (not necessarily connected) subgraphs of $\mathbb{D}^{(i)}$. Moreover $L_i = \pi_1(\mathbb{D}^{(i)})$ and the coefficient group Γ is contained in one of the vertex groups R_j^i . With each factor R_j^i there is an associated restricted virtually abelian JSJ decomposition \mathbb{A}_j^i . While reading the following definition we encourage the reader to look at Figure 2.

Definition 5.23. *Let $\text{Res}(L)$ be a strict MR resolution of a Γ -limit group L as above. We call $\text{Res}(L)$ a well-structured resolution if the following conditions hold for all i :*

- (1) $\mathbb{D}^{(i+1)}$ is a refinement of $\mathbb{D}^{(i)}$. In particular the fundamental group $\pi_1(\mathbb{D}^{(i)})$ of the underlying graph of $\mathbb{D}^{(i)}$ gets mapped injectively into $\pi_1(\mathbb{D}^{(i+1)})$ and the respective edge groups of $\mathbb{D}^{(i)}$ and their images in $\mathbb{D}^{(i+1)}$ are isomorphic.
- (2) The subgraph \mathbb{H}_i of $\mathbb{D}^{(i)}$ gets mapped isomorphically to a subgraph of \mathbb{H}_{i+1} in $\mathbb{D}^{(i+1)}$.
- (3) If the factor R_j^i is neither a finite-by-(closed) orbifold group nor a (infinite or finite) virtually abelian group, then $\eta_i(R_j^i)$ has a Dunwoody decomposition \mathbb{B} , such that \mathbb{B} is a connected subgraph of groups of $\mathbb{D}^{(i+1)}$, i.e. the vertex and edge groups coincide. This means the vertex groups of \mathbb{B} are isomorphic to (possibly) some of the $R_j^{(i+1)}$ and some of the vertex groups of \mathbb{H}_{i+1} .

- (4) Let $j \neq j'$. Then R_j^i and $R_{j'}^i$ get mapped onto fundamental groups of distinct (and disjoint) subgraphs of groups of $\mathbb{D}^{(i+1)}$ and these fundamental groups have finite intersection with the image of the vertex groups of \mathbb{H}_i under η_i . Moreover these subgraphs do not contain any edge of $ED^{(i)}$, where we view $ED^{(i)}$ as a subset of $ED^{(i+1)}$ (since $\mathbb{D}^{(i+1)}$ is a refinement of $\mathbb{D}^{(i)}$).
- (5) Since by assumption $\text{Res}(L)$ is a strict MR resolution, η_i maps every rigid vertex group and every edge group in \mathbb{A}_j^i monomorphically into L_{i+1} for all j . Moreover the image of a QH vertex group under η_i is non-virtually abelian.
- (6) Let $A_1^i, \dots, A_{q(i)}^i$ be the (infinite or finite) isolated virtually abelian vertex groups among the R_j^i 's in the Dunwoody decomposition of L_i . Then $C_u^i := \eta_i(A_u^i)$ is an (infinite or finite) virtually cyclic vertex group of \mathbb{H}_{i+1} and $C_1^i, \dots, C_{q(i)}^i$ have pairwise finite intersection with each other and with the image of every vertex group of \mathbb{H}_i .
- (7) Let $S_1^i, \dots, S_{d(i)}^i$ be the isolated (i.e. finite-by-closed) orbifold groups among the R_j^i 's. Then for all $k \in \{1, \dots, d(i)\}$, $K_k^i := \eta_i(S_k^i)$ has a decomposition as a subgraph of groups of \mathbb{H}_{i+1} and $K_1^i, \dots, K_{d(i)}^i$ have pairwise finite intersection with each other, with $C_1^i, \dots, C_{q(i)}^i$ and with the image of every vertex group of \mathbb{H}_i .
- (8) Let $Q_1^i, \dots, Q_{s(i)}^i$ be the QH vertex groups in the various abelian JSJ-decompositions \mathbb{A}_j^i of the vertex groups R_j^i . For every QH vertex group Q_t^i let \mathbb{M}_t be the decomposition along finite subgroups of $\eta_i(CF(Q_t^i))$ inherited from $\mathbb{D}^{(i+1)}$ such that several vertex groups of \mathbb{M}_t , denoted by $H_{t_1}^i, \dots, H_{t_b}^i$, do not contain any conjugate of the image $\eta_i(P)$ of a peripheral subgroup P of Q_t^i , while all other vertex groups, say $V_1, \dots, V_{m(t)}$, of \mathbb{M}_t do contain a conjugate of the image of a peripheral subgroup of Q_t^i . Moreover the image of every peripheral subgroup of Q_t^i can be conjugated into one of the vertex groups $V_1, \dots, V_{m(t)}$. Again to simplify notation we assume that there exists only one vertex group in \mathbb{M}_t , say H_t^i , which does not contain any conjugate of the image $\eta_i(P)$ of a peripheral subgroup P of Q_t^i .
Then
- (a) $H_1^i, \dots, H_{s(i)}^i$ are vertex groups of \mathbb{H}_{i+1} .
- (b) $H_1^i, \dots, H_{s(i)}^i$ have pairwise finite intersection with each other, with $K_1^i, \dots, K_{d(i)}^i$, $C_1^i, \dots, C_{q(i)}^i$ and with the image of every vertex group of \mathbb{H}_i .
- (c) $V_1, \dots, V_{m(t)} \leq \langle R_1^{i+1}, \dots, R_{q(i+1)}^{i+1}, \pi_1(D_{i+1}) \rangle$.
- (9) Denote by $U_1^i, \dots, U_{f(i)}^i$ the images of vertex groups of \mathbb{H}_i under η_i . Then a group X is a vertex group of \mathbb{H}_{i+1} if and only if

$$X \in \{H_1^i, \dots, H_{s(i)}^i, K_1^i, \dots, K_{d(i)}^i, C_1^i, \dots, C_{q(i)}^i, U_1^i, \dots, U_{f(i)}^i\}.$$

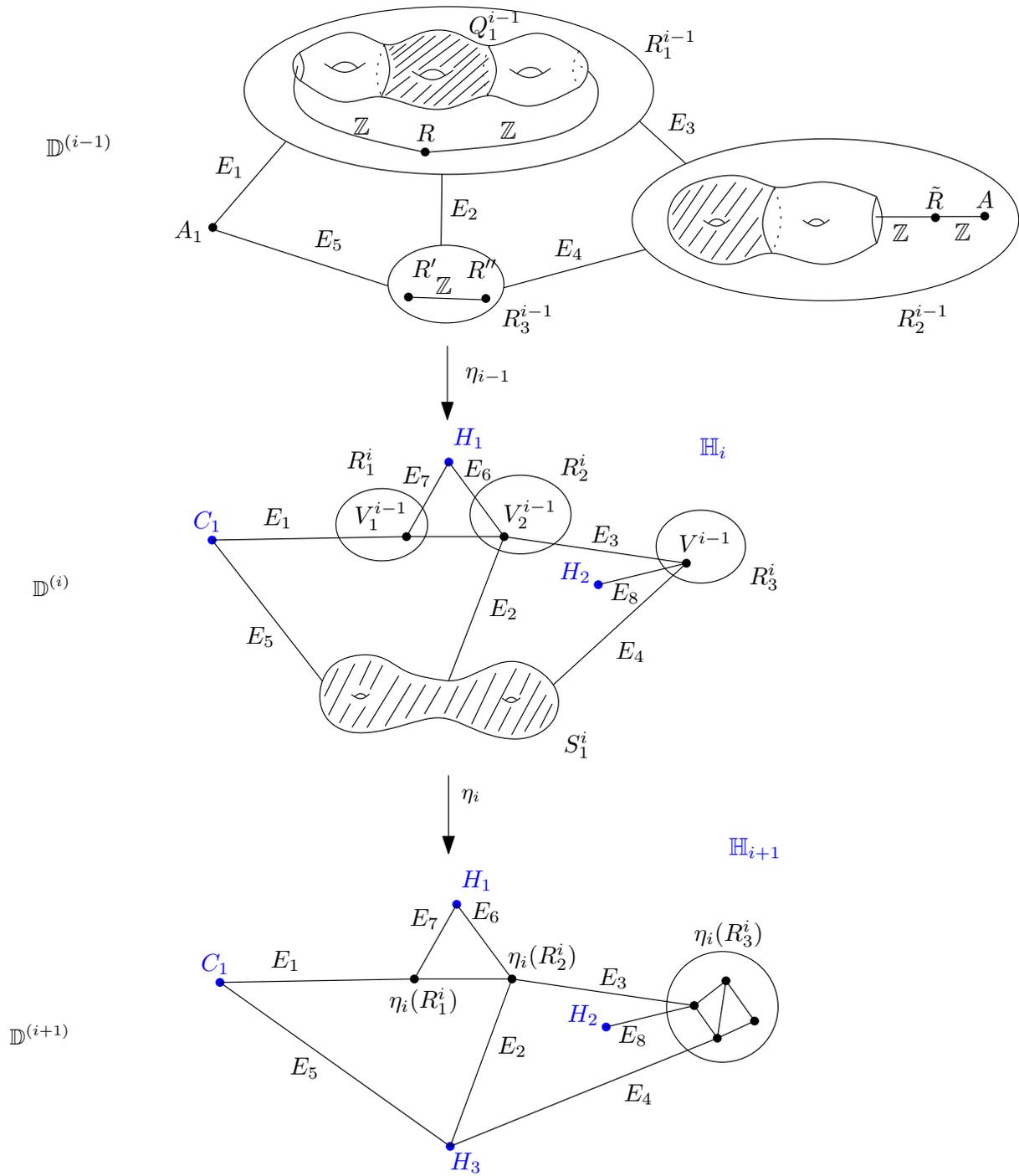


Figure 2: A well-structured resolution of a Γ -limit group

Remark 5.24. *It follows from [ReWe] that every resolution of a Γ -limit group which is constructed from a descending chain of maximal shortening quotients is a well-structured resolution. In fact the well-structured resolution is defined in order to capture the properties of a descending chain of shortening quotients. It follows that every resolution in the strict MR diagram constructed in Proposition 5.21 is well-structured.*

Later we will need an additional property to hold for our well-structured resolution.

Definition 5.25. *We call a well-structured MR resolution*

$$\text{Res}(L) : L = L_0 \xrightarrow{n_0} L_1 \xrightarrow{n_1} L_2 \rightarrow \cdots \xrightarrow{n_{l-1}} L_l = \pi_1(\mathbb{D})$$

of a restricted Γ -limit group L good if there exists a homomorphism $\varphi : L \rightarrow \Gamma$ that factors through $\text{Res}(L)$, such that the centralizer in Γ of the image of every edge group of a Dunwoody decomposition \mathbb{D} of L_l contains a quasi-convex non-abelian free group.

5.6 Γ -limit tower

Recall that a subgroup of a group G is called a QH subgroup if it is conjugate to a vertex group A_v of some splitting \mathbb{A} of G such that the following hold:

- (a) A_v is finite-by-orbifold, i.e. there exists an orbifold \mathcal{O} , some finite group E and a short exact sequence

$$1 \rightarrow E \rightarrow A_v \xrightarrow{\pi} \pi_1(\mathcal{O}) \rightarrow 1.$$

- (b) For any edge $e \in EA$, such that $\alpha(e) = v$ there exists a peripheral subgroup O_e of $\pi_1(\mathcal{O})$ such that $\alpha_e(A_e)$ is in A_v conjugate to a finite index subgroup of $\pi^{-1}(O_e)$.

Definition 5.26. *Let G be a group and H be a subgroup of G .*

- (a) *G has the structure of a virtually abelian flat over H , if G is the amalgamated product $H *_{A'} A$, where A is virtually abelian and A' is either an infinite maximal virtually abelian subgroup of H or finite, and if A' is infinite then there exists a retraction $\pi : G \rightarrow H$.*

- (b) *G has the structure of an orbifold flat over H if G admits a decomposition as a graph of groups \mathbb{A} such that*

- *There exists a vertex $v \in VA$ with a QH subgroup $Q = A_v$ as vertex group, which fits into a short exact sequence*

$$1 \rightarrow E \rightarrow Q \xrightarrow{\pi} \pi_1(\mathcal{O}) \rightarrow 1.$$

- *Every edge $e \in EA$ is of the form $e = (v, u)$ for some $u \in V \setminus \{v\}$ and for every edge e with $\alpha(e) = v$, either there exists a peripheral subgroup O_e of $\pi_1(\mathcal{O})$ such that $\alpha_e(A_e)$ is in A_v conjugate to a finite index subgroup of $\pi^{-1}(O_e)$, or A_e is finite. Moreover every boundary component of \mathcal{O} corresponds to an edge in such a way.*

- H admits a decomposition as a graph of groups along finite subgroups, such that the vertex groups of this decomposition are precisely the vertex groups of the vertices in $VA \setminus \{v\}$.

Definition 5.27. A group G has the structure of a tower of height $n \in \mathbb{N}$ over a subgroup H if there exists a sequence

$$G = G^n \geq G^{n-1} \geq \dots \geq G^0 = H,$$

such that for each $0 \leq i \leq n - 1$, one of the following holds:

- G^{i+1} is an HNN-extension of G^i along some finite group.
- G^{i+1} is the amalgamated product along a finite group of G^i with a group K which is isomorphic to a subgroup of H .
- G^{i+1} has the structure of an orbifold flat over G^i and H is contained in one of the vertex groups of the corresponding graph of groups decomposition of G^i .
- G^{i+1} has the structure of a virtually abelian flat $G^i *_{A_1} A$ over G^i and if A_1 is infinite, it cannot be conjugated into a virtually abelian group which corresponds to a virtually abelian flat which is lower in the tower hierarchy.

Remark 5.28. (1) The case that G^{i+1} is a product along a finite group with either a virtually abelian group or with an extension of a closed orbifold group is covered by the virtually abelian/orbifold flat cases.

(2) Our definition of a tower is solely designed to describe the completion of a well-structured resolution defined later. We chose this name because in the case of limit groups over free and torsion free hyperbolic groups, the completion of a resolution is always a tower in the classical sense. We refer the reader to [BySk] for a detailed discussion on towers over free groups.

(3) In the torsion free case there always exists a retraction from G^{i+1} onto G^i , since for the orbifold flat there exists one by definition and in the other cases there always exists a canonical one. In the presence of torsion we do not claim that such a retraction always exist.

Definition 5.29. Suppose G has the structure of a tower $G = G^n \geq G^{n-1} \geq \dots \geq G^0 = \Gamma$ over Γ . Then we say that G has the structure of a Γ -limit tower if Γ is a hyperbolic group and G_i is a Γ -limit group for all $i \in \{1, \dots, n\}$.

6 Separability Properties of Groups

There are a few well-known residual properties for groups. We only state the definition here and refer the reader to [BoGr] for a detailed discussion on the subject.

Definition 6.1. *Let G be a group.*

1. G is called *separable* or *residually finite*, if for any $g \in G \setminus \{1_G\}$ there exists a finite group E and a homomorphism $\varphi : G \rightarrow E$ such that $\varphi(g) \neq 1$.
2. G is called *conjugacy separable* if whenever x and y are non-conjugate elements of G , there exists a homomorphism $\varphi : G \rightarrow E$ to a finite group E such that $\varphi(x)$ and $\varphi(y)$ are non-conjugate.
3. G is called *subgroup separable* or *has the locally extendable residually finiteness (LERF) property* if for every finitely generated subgroup $H \leq G$ and $g \in G \setminus H$ there exists a homomorphism $\varphi : G \rightarrow E$ to a finite group E such that $\varphi(H) = 1$ and $\varphi(g) \neq 1$.
4. G is called *subgroup conjugacy separable* if for any two finitely generated non-conjugate subgroups $H_1, H_2 \leq G$ there exists a homomorphism ϕ from G to a finite group E such that $\phi(H_1)$ is not conjugate to $\phi(H_2)$ in E .

In this work we are mainly interested in the subgroup separability of a given group G . Subgroup separability was shown for polycyclic groups by Mal'cev [Ma], for free groups by M. Hall [Ha], for surface groups by P. Scott [Sc] and more recently for limit groups by H. Wilton [Wi2].

Now one could wonder if it is possible to achieve Wilton's result by passing to a free quotient of the limit group and then use the result of M. Hall. This was indeed phrased as a question on mathoverflow.com [Mo] by K. Bou-Rabee, namely whether or not a limit group is freely subgroup separable. A group L is called *freely subgroup separable* if for every finitely generated subgroup $H \leq L$ and every element $g \in L \setminus H$ there exists a homomorphism φ from L to a free group, such that $\varphi(g) \notin \varphi(H)$.

If limit groups would have this property, then as mentioned before, together with the result of M. Hall, one could deduce the theorem of Henry Wilton that limit groups are subgroup separable. First I. Agol notes in the discussion on matheoverflow.com that in order to have a chance of achieving a positive result, one has to assume that the subgroup H is of infinite index in L .

Henry Wilton then conjectured that one can show that limit groups are not freely subgroup separable by computing the Makanin-Razborov diagram of a particular double of a free group (which is a limit group).

We are following this approach and try to describe Makanin-Razborov diagrams of doubles of a free group in two generators. First classes of examples of Makanin-Razborov diagrams of such doubles, respectively the limit groups appearing in these diagrams were

computed by Nicholas Touikan in [To2], [To3]. We refer the reader to his papers for a broader background on the subject.

We compute the Makanin-Razborov diagram of a particular double of a free group of rank two, not covered by the work of Touikan and use this diagram to show that limit groups are in general not freely subgroup separable (even if one assumes that the subgroup H is of infinite index, as mentioned by Agol). After the first version of our paper this double and its MR diagram were independently constructed and used by Louder and Touikan to show that limit groups are also not freely conjugacy separable ([LoTo]).

The first step in constructing a MR diagram of a given finitely generated (one-ended) group L , is to understand all splittings of L along cyclic subgroups, that is to find a splitting of L as a graph of groups which encodes in some sense all of these splittings. Recall that such a graph of groups is called a JSJ decomposition of L with respect to the class of infinite cyclic subgroups \mathcal{Z} (or for short a *cyclic JSJ decomposition*).

As noted before for arbitrary finitely generated groups JSJ decompositions do not always exist, but since in our case L is a one-ended limit group and therefore in particular finitely presented, a JSJ decomposition of L does exist. The existence and much more about JSJ decompositions (not only along cyclic edge groups) can be found in [Gu]. Unfortunately JSJ decompositions are not unique in general, but rather form a deformation space, denoted by \mathcal{D}_{JSJ} , which consists of all cyclic JSJ trees of L . Hence two JSJ trees T_1, T_2 are in \mathcal{D}_{JSJ} if and only if they have the same elliptic subgroups. Still in some cases there exists a canonical cyclic JSJ decomposition, i.e. a decomposition that is invariant under automorphisms. In particular when L is a one-ended limit group one can apply the *tree of cylinder construction* of Guirardel and Levitt (see section 7 and Theorem 9.5 in [Gu]) to get a canonical cyclic JSJ decomposition.

Let now \mathbb{A} be a cyclic JSJ decomposition of a finitely generated group L . Recall that a vertex group A_v of \mathbb{A} is called *rigid* if it is elliptic in every splitting of L along a cyclic subgroup and *flexible* otherwise. In the case that L is a one-ended limit group the structure of flexible vertex groups of a cyclic JSJ decomposition \mathbb{A} is well-understood, namely these are either free abelian or *quadratically hanging (QH) vertex groups*. A vertex group A_v of \mathbb{A} is QH if A_v is isomorphic to the fundamental group of a compact surface Σ with boundary in such a way that any incident edge group can be conjugated into a boundary subgroup of $\pi_1(\Sigma)$.

In the second section we introduce particular words in free groups, so-called C -test words, and compute the Makanin-Razborov diagram of a double of a free group of rank two along such a C -test word. We then use this diagram to show that limit groups are in general not freely subgroup separable. Unfortunately we were not able to give a description of all possible Makanin-Razborov diagrams of general doubles of free groups but at least we are able to classify all possible JSJ decompositions of these doubles in

the third section. This is the first step towards the construction of more examples of MR diagrams or even to the possible classification of all MR diagrams of doubles of free groups of rank two in the future.

To compute the JSJ decompositions of a double of F_2 along a given word w it is necessary to understand relative cyclic JSJ decompositions of the non-abelian free group F_2 relative to w . In [Ca] C. Cashen has shown that for an arbitrary non-abelian free group F there exists a canonical cyclic JSJ decomposition relative to finitely many maximal cyclic subgroups of F . Moreover given finitely many such subgroups Cashen and Manning have written an algorithm that computes this relative JSJ decomposition ([CaMa]).

6.1 Limit groups are not freely subgroup separable

We first fix some notation. Let $F_2 = \langle x_1, x_2 \rangle$ be a non-abelian free group, $w \in F_2$ and G_w the double of F_2 along \mathbb{Z} given by the embedding $\iota : \mathbb{Z} \rightarrow F_2, 1 \mapsto w$. In the following we denote such a double decomposition of G_w along w by $G_w := A *_C B$ where $A = F_2(a_1, a_2)$, $B = F_2(b_1, b_2)$, $C = \langle c \rangle \cong \mathbb{Z}$ with the embeddings $\iota_A : C \rightarrow A : c \mapsto w_A$, $\iota_B : C \rightarrow B : c \mapsto w_B$, where w_A (w_B) is the image of w under the canonical isomorphisms from $F(x_1, x_2)$ to A (B) given by $x_i \mapsto a_i$ ($x_i \mapsto b_i$), $i \in \{1, 2\}$.

We are now going to describe the Makanin-Razborov diagram of G_w for a very specific word, a so-called C-test word:

Definition 6.2. [Iv] Let F_n be a non-abelian free group in n generators. A non-trivial word $w \in F_n$ is a C-test word in n letters if for any two n -tuples (A_1, \dots, A_n) , (B_1, \dots, B_n) of elements of a non-abelian free group F the equality

$$w(A_1, \dots, A_n) = w(B_1, \dots, B_n) \neq 1$$

implies the existence of an element $S \in F$ such that $SA_iS^{-1} = B_i$ for all $i \in \{1, \dots, n\}$.

Remark 6.3. Let F_n be a free group of rank $n \geq 2$.

1. The definition of a C-test word does not depend on the rank of the free group F . This follows from Sela's solution of the Tarski problems, since being a C-test word in n letters can be expressed by the following first-order sentence:

$$\begin{aligned} \forall a_1, \dots, a_n \forall b_1, \dots, b_n \exists s : (w(a_1, \dots, a_n) = w(b_1, \dots, b_n) \neq 1) \\ \Rightarrow (a_1 = sb_1s^{-1} \wedge \dots \wedge a_n = sb_ns^{-1}) \end{aligned}$$

2. [Tu] A C-test word $w \in F_n$ is not contained in a proper retract of F_n .
3. [Iv] Every C-test word $w \in F_n$ is contained in the commutator subgroup of F_n .
4. [Iv] If a C-test word $w \in F_n$ is not a proper power, then the stabilizer of w in $\text{Aut}(F_n)$ is $\langle c_w \rangle$, where c_w denotes conjugation by w .

The following result is due to Ivanov.

Theorem 6.4. [Iv] For arbitrary $n \geq 2$ there exists a C-test word $w_n \in F_n$. In addition w_n is not a proper power.

D. Lee in [Lee] generalized this result to the following:

Theorem 6.5. [Lee] For arbitrary $n \geq 2$ there exists a word $w_n \in F_n$ that is a C-test word in n letters such that w_n is not a proper power and with the additional property that for elements A_1, \dots, A_n in a free group F the following is equivalent:

- (a) $w_n(A_1, \dots, A_n) = 1$.
- (b) The subgroup $\langle A_1, \dots, A_n \rangle$ of F is cyclic.

Hence we get the following corollary.

Corollary 6.6. [Lee] There exists an element $w \in F_n$ such that if φ is an endomorphism of F_n , Ψ is an endomorphism of F_n with non-cyclic image, and $\varphi(w) = \Psi(w)$, then $\varphi = c_S \circ \Psi$ for some $S \in F_n$ such that $\langle S, \Psi(w) \rangle \cong \mathbb{Z}$. If $n = 2$, then $S \in \langle \Psi(w) \rangle$.

Definition 6.7. We call a C-test word which satisfies the assumptions of Theorem 6.5 and Corollary 6.6 an Ivanov word.

The following corollary follows immediately from the proof of the above theorem in [Lee].

Corollary 6.8. Let $F_2 = \langle x_1, x_2 \rangle$. Then

$$w = [x_1^8, x_2^8]^{100} x_1 [x_1^8, x_2^8]^{200} x_1 [x_1^8, x_2^8]^{300} x_1^{-1} [x_1^8, x_2^8]^{400} x_1^{-1} \\ \cdot [x_1^8, x_2^8]^{500} x_2 [x_1^8, x_2^8]^{600} x_2 [x_1^8, x_2^8]^{700} x_2^{-1} [x_1^8, x_2^8]^{800} x_2^{-1}$$

is an Ivanov word.

Before we start the construction of a Makanin-Razborov diagram of a double G_w along an Ivanov word w we make the following observation.

Lemma 6.9. Let F_n be a non-abelian free group, $w \in F_n$ and G_w the double of F_n along w . G_w is one-ended if and only if w is not contained in a free factor of F_n .

Proof. By Corollary 1.5 in [To1] G_w is one-ended if and only if A is one-ended relative to $\iota_A(C)$, i.e. if and only if there does not exist a free splitting $A_1 * A_2$ of A such that $\iota_A(C)$ is contained in A_1 . □

The following theorem amounts to a computation of the Makanin-Razborov diagram of a double of F_2 along an Ivanov word.

Theorem 6.10. *Let $w \in F_2$ be an Ivanov word and $G_w = A *_{\langle w \rangle} B$ be the double of F_2 along w . Then after precomposition with a Dehn-twist, every homomorphism from G_w to a non-abelian free group factors through either the canonical retraction $\pi : G_w \rightarrow A$ or the projection $\eta : G_w \rightarrow \mathbb{Z}^2 * \mathbb{Z}^2$.*

Proof. First note that by Remark 6.3 w is not contained in a proper retract of F_2 . In particular w is not contained in a free factor of F_2 and hence G_w is one-ended by Lemma 6.9. Moreover G_w is clearly a F_2 -limit group. This follows from the fact that w is not a proper power and therefore we can embed G_w into the extension of centralizer along $\langle w \rangle$ (see for example [B.Ba]).

Let now $\varphi : G_w \rightarrow F_2$ be a homomorphism. We distinguish two cases:

(1) Assume that $w \in \ker \varphi$. Then $\varphi(A)$ and $\varphi(B)$ are cyclic and hence φ factors through η .

(2) Assume now that $w \notin \ker \varphi$. Since by Remark 6.3, w is contained in $[F_2, F_2]$, it follows that $\varphi(A) \cong F_2 \cong \varphi(B)$. Now Corollary 6.6 yields that there exists $k \in \mathbb{N}$ such that $\varphi|_B = \varphi|_A \circ c_{w_A^k}$. Hence after precomposing with a Dehn-twist α along $\langle w \rangle$, $\varphi \circ \alpha$ factors through $\pi : A *_{\langle w \rangle} B \rightarrow A$. To be more precise $\alpha : G_w \rightarrow G_w$ is given by $\alpha|_A = \text{id}$ and $\alpha|_B = c_{w_B^k}$ and we have then $\varphi = \varphi|_A \circ \pi \circ \alpha$. \square

Theorem 6.10 implies that the Makanin-Razborov diagram of G_w is as in Figure 3.

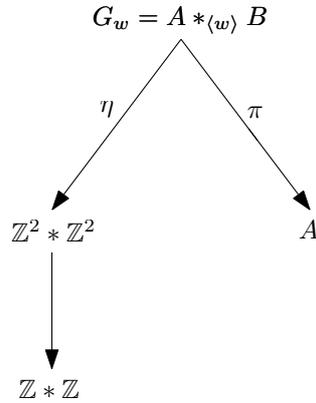


Figure 3: The MR diagram for G_w

Corollary 6.11. *Let $w \in F_2$ be an Ivanov word and G_w be the double of F_2 along w . Then G_w is not freely subgroup separable.*

Proof. Let $g = [b_1, b_2] \in G_w$. Then clearly $g \notin A$ and we claim that g cannot be separated from A in any free quotient. Indeed assume there exists a homomorphism $\varphi : G_w \rightarrow F$, where F is a non-abelian free group, such that $\varphi(g) \notin \varphi(A)$. By Theorem 6.10, after possibly precomposing with a Dehn-twist along $\langle w \rangle$, φ factors through (at least) one of the homomorphisms π or η . Since $\eta(B) \cong \mathbb{Z}^2$ it follows that $\varphi(g) = 1 \in \varphi(A)$ if φ factors

through η . Now assume that φ factors through π . But then there exists a Dehn-twist α and a monomorphism $\Psi : A \rightarrow F$ such that $\varphi(g) = \Psi \circ \pi \circ \alpha(g) \in \Psi \circ \pi(B) = \Psi \circ \pi(A) = \varphi(A)$. \square

7 JSJ decomposition of doubles of non-abelian free groups

Unfortunately we are not able to give a description of all possible Makanin-Razborov diagrams of general doubles of free groups, but at least we are able to classify all possible JSJ decompositions of these doubles. This is the first step towards the construction of more examples of MR diagrams or even to the possible classification of all MR diagrams of doubles of free groups of rank two in the future.

In this section we will give a description of the possible JSJ decompositions of (one-ended) doubles of a free group in two generators (see Theorem 7.4).

Recall that G_w is the double of $F_2 = \langle x_1, x_2 \rangle$ along \mathbb{Z} given by the embedding $\iota : \mathbb{Z} \rightarrow F_2, 1 \mapsto w$, for some $w \in F_2$. By Lemma 6.9 G_w is one-ended if and only if w is not contained in a free factor of F_2 . Therefore if G_w is not one-ended then G_w has a free decomposition of the form

$$G_w \cong \langle y \rangle * \langle x \rangle *_{\langle w=x^n \rangle} \langle x \rangle * \langle y \rangle$$

for some basis $\{x, y\}$ of F_2 . Hence from now on we will only be interested in one-ended doubles G_w such that in addition w is not a proper power (and therefore G_w is a limit group).

First we give some necessary and sufficient conditions on w such that a JSJ decomposition of G_w is as simple as possible, i.e. it is just the double decomposition $A *_C B$.

Later we describe JSJ decompositions of G_w in the case that at least one of these conditions is violated. In the following Proposition we collect some results which will be of use later.

Proposition 7.1. *Let $G \cong F_2$ be a non-abelian free group on two generators.*

- (1) [KaSo] *G does not split as a proper amalgamated product along a non-trivial malnormal subgroup C .*
- (2) [KaWe1] (Proposition 3.7) *Suppose G splits as a non-trivial amalgamated product $A *_C B$ along $C \cong \mathbb{Z}$. Then there exist $x, y \in G$ and $n > 1$, such that $G = \langle x, y \rangle$, $A = \langle x \rangle \cong \mathbb{Z}$, $B = \langle x^n, y \rangle \cong F_2$ and $A *_C B = \langle x \rangle *_{\langle x^n \rangle} \langle x^n, y \rangle$. In particular C is malnormal in B .*
- (3) [KaWe1] (Lemma 3.6, Proposition 3.8) *Suppose G splits as an HNN-extension $H *_C$ where $C \cong \mathbb{Z}$. Then $G = \langle H, t \mid ta^nt^{-1} = b \rangle$ for some $n \geq 1$ and elements $a, b \in G$*

with no roots. Moreover $\langle a \rangle$ and $\langle b \rangle$ are malnormal in H and there exist $h \in H$ such that $G = \langle ht, a \rangle$ and $H = \langle a, h b h^{-1} \rangle = \langle a, (ht)a^n(ht)^{-1} \rangle$.

We say that a group G is obtained by *adjoining a root to x* (or *pulling out a root of x*) if G has a decomposition as a graph of groups of the form $G = \langle x \rangle *_{\langle x^n \rangle} H$.

Proposition 7.2. *Let $F_2 = \langle x_1, x_2 \rangle$ be a free group, $w \in F_2$ an element which is not a proper power and G_w the double of F_2 along w . Then the graph of groups given by this decomposition is a JSJ decomposition of G if none of the following holds:*

- (1) w is contained in a subgroup generated by $\{xyx^{-1}, y\}$ for some basis $\{x, y\}$ of F_2
- (2) w is contained in a subgroup generated by $\{x^n, y\}$ for some basis $\{x, y\}$ of F_2 and some $n \geq 2$.

Proof. Assume that w does neither satisfy property (1) nor property (2). Property (2) implies in particular that w is not a power of a basis element in F_2 . Hence it follows from Lemma 6.9 that G_w is one-ended.

JSJ-theory implies that either $A *_{\langle w \rangle} B$ is a JSJ decomposition of G_w or there exists a \mathbb{Z} -splitting of A (respectively B) relative w . Therefore it suffices to show that A and B do not admit any further splittings along \mathbb{Z} relative to $\langle w \rangle$.

First suppose that $A = R_1 *_D R_2$ is a non-trivial relative splitting, where $D \cong \mathbb{Z}$. From Proposition 7.1 (2) follows the existence of $r, h \in A$ and $n > 1$, such that

$$F_2 \cong A = R_1 *_D R_2 = \langle r \rangle *_{\{r^n=b\}} \langle b, h \rangle = \langle r \rangle *_{\{r^n=r^n\}} \langle r^n, h \rangle,$$

in particular $A = \langle r, h \rangle$.

Since w is not contained in any subgroup generated by $\{x^n, y\}$ for some basis $\{x, y\}$ of A and some $n > 1$ this implies that $w \notin R_1, R_2$, a contradiction.

Now suppose that A splits as an HNN-extension $R *_Z$ relative w . From Proposition 7.1 (3) follows that $A = \langle R, t \mid ta^nt^{-1} = b \rangle$ for some $n \geq 1$ and elements $a, b \in A$. Moreover there exist $h \in R$ such that $A = \langle ht, a \rangle$ and $R = \langle a, (ht)a^n(ht)^{-1} \rangle$. Since by assumption $w \notin \langle x, yxy^{-1} \rangle$ for every basis $\{x, y\}$ of A , this implies that $w \notin R$. \square

We now describe when G_w is a surface group, in which case the JSJ decomposition of G_w consists of a single vertex with QH vertex group G_w .

Proposition 7.3. *Let G_w be the double of $F_2 = \langle x_1, x_2 \rangle$ along w . Then G_w is a surface group if and only if one of the following holds:*

- (1) w is conjugate to $[x_1, x_2]^{\pm 1}$ in which case G_w is the fundamental group of an orientable surface of genus 2.
- (2) w is either conjugate to $x_1^{2\epsilon_1} x_2^{2\epsilon_2}$, where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, to $(x_1 x_2 x_1^{-1} x_2)^{\pm 1}$, or to $(x_1 x_2 x_1 x_2^{-1})^{\pm 1}$ in which case G_w is the fundamental group of a non-orientable surface of genus 4.

Proof. If G_w is the fundamental group of a closed surface Σ , then either Σ is orientable of genus 2 or non-orientable of genus 4. In both cases there exists a basis of F_2 such that the curve corresponding to w is conjugate to one of the specified words. The claim now follows immediately from the fact that the sets

$$\{[x_1, x_2]^{\pm 1}\}$$

and

$$\{x_1^{2\epsilon_1} x_2^{2\epsilon_2}, (x_1 x_2 x_1^{-1} x_2)^{\pm 1}, (x_1 x_2 x_1 x_2^{-1})^{\pm 1} \mid \epsilon_1, \epsilon_2 \in \{\pm 1\}\}$$

are up to conjugation invariant under Nielsen transformations of F_2 . \square

It remains to consider the case that G_w is neither a surface group, nor that the double decomposition of G_w is already a JSJ decomposition.

Theorem 7.4. *Let G_w be the double of $F_2 = \langle x_1, x_2 \rangle$ along w and assume that w is not a proper power and not contained in a free factor of F_2 . Suppose moreover that G_w is not a surface group.*

(1) *If w satisfies property (2) from Proposition 7.2, but not property (1), then a JSJ decomposition \mathbb{A} of G_w has one of the following forms:*

- (a) \mathbb{A} is as in Figure 6.
- (b) \mathbb{A} is as in Figure 6 but the vertices stabilized by $\langle x \rangle$ together with their adjacent edges are replaced by Möbius strips which are glued along their boundaries to the 4-punctured sphere. Moreover $m > 2$.
- (c) \mathbb{A} has only rigid vertices and is one of the graphs of groups which we get by refining the vertices A and B in $A *_C B$ by one of the following two graphs of groups:
 - $\langle x^n, y \rangle *_{\langle x^n \rangle} \langle x \rangle$
 - $\langle y \rangle *_{\langle y^m \rangle} \langle x^n, y^m \rangle *_{\langle x^n \rangle} \langle x \rangle$.

(2) *If w satisfies property (1) from Proposition 7.2, but not property (2), then a JSJ decomposition of G_w is the graph of groups which we get by substituting either $\langle xyx^{-1}, y \rangle *_{\langle y \rangle}$ or one of the three graphs of groups in Figure 7 for A and B , or the graph of groups in Figure 9.*

(3) *If w satisfies (1) and (2) from Proposition 7.2 then a JSJ decomposition of G_w is one of the graphs of groups which we get by refining the vertices A and B in $A *_C B$ by one of the graphs of groups in Figure 11, or the graph of groups in Figure 9 with $\gcd(m, n) > 1$, or the graph of groups in Figure 12.*

We split the proof of the theorem in several lemmas. From now on we assume that w is not a proper power and not contained in a free factor of F_2 . In particular G_w is one-ended by Lemma 6.9. Moreover we assume that G_w is not isomorphic to the fundamental group of a closed surface.

Lemma 7.5. *If $w \in \langle x^n, y \rangle$ for some $n > 1$ and some basis $\{x, y\}$ of F_2 and $w \notin \langle bab^{-1}, a \rangle$ for any basis $\{a, b\}$ of F_2 , then a JSJ decomposition \mathbb{A} of G_w has one of the following forms:*

- (a) \mathbb{A} is as in Figure 6.
- (b) \mathbb{A} is as in Figure 6 but the vertices stabilized by $\langle x \rangle$ together with their adjacent edges are replaced by Möbius strips which are glued along their boundaries to the 4-punctured sphere. Moreover $m > 2$.
- (c) \mathbb{A} has only rigid vertices and is one of the graphs of groups which we get by refining the vertices A and B in $A *_C B$ by one of the following two graphs of groups:
 - $\langle x^n, y \rangle *_{\langle x^n \rangle} \langle x \rangle$
 - $\langle y \rangle *_{\langle y^m \rangle} \langle x^n, y^m \rangle *_{\langle x^n \rangle} \langle x \rangle$.

Proof. As in the proof of Proposition 7.2 we conclude that there exists no splitting of F_2 as an HNN-extension relative to w , but there exists (at least) one splitting of F_2 as an amalgamated product relative to w . We assume without loss of generality that w is cyclically reduced.

Let $n > 1$ be maximal such that $\langle x^n, y \rangle *_{\langle x^n \rangle} \langle x \rangle$ is such a splitting of F_2 relative to w which does not correspond to a splitting along a non-boundary parallel, closed curve on a QH subgroup of G_w (such a splitting exists since by assumption G_w is not a surface group).

Case 1: Assume that $w \notin \langle x^n, z^m \rangle$ for any $m > 1$ and any $z \in F_2$ such that $\{x, z\}$ is a basis of F_2 . Suppose that there exists a splitting as an amalgamated product of $\langle x^n, y \rangle$ relative to w and $\langle x^n \rangle$. Hence by using Proposition 7.1 (2) we can refine $\langle x^n, y \rangle *_{\langle x^n \rangle} \langle x \rangle$ to say

$$\langle e \rangle *_{\langle e^k \rangle} H *_{\langle x^n \rangle} \langle x \rangle$$

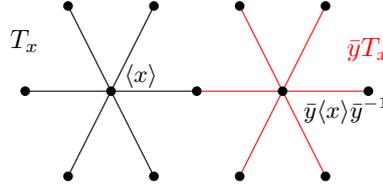
for some $H \cong F_2$ and $k > 1$. We denote this graph of groups by \mathbb{A} and the corresponding Bass-Serre tree by T_A . We moreover identify $F(x, y)$ with $\pi_1(\mathbb{A})$.

By (the proof of) Theorem 1A in [KaWe2] $\{x, y\}$ is Nielsen equivalent to $\{x, \bar{y}\}$ such that either

- (a) $T_x \cap T_{\bar{y}} \neq \emptyset$ and x and \bar{y} act elliptically on T_A or
- (b) $T_x \cap \bar{y}T_x \neq \emptyset$ and \bar{y} acts hyperbolically and x acts elliptically on T_A ,

where $T_x := \{v \in T_A \mid x^z v = v \text{ for some } z \in \mathbb{Z} \setminus \{0\}\}$ is the subtree of T_A consisting of all points fixed by a non-trivial power of x .

Suppose we are in case (b), i.e. \bar{y} acts hyperbolically and x acts elliptically on T_A and $\langle x^n \rangle$ is by Proposition 7.1 (2) malnormal in H . Moreover $\langle x^n \rangle$ is conjugacy separated from $\langle e^k \rangle$, i.e. $g\langle e^k \rangle g^{-1} \cap \langle x^n \rangle = 1$ for all $g \in F(x, y)$. Therefore T_x is the star of the vertex stabilized by $\langle x \rangle$ and all other vertices of T_x are stabilized by conjugates of H ,


 Figure 4: $T_x \cup \bar{y}T_x$

while the edges are stabilized by $\langle x^n \rangle$. Clearly $\bar{y}T_x$ is of the same form. Since we are in case (b) we have that $T_x \cap \bar{y}T_x \neq \emptyset$ (see Figure 4). But then $T := \langle x, \bar{y} \rangle T_x \subset T_A$ is a $F(x, y)$ -invariant subtree, a contradiction to the minimality of \mathbb{A} . Therefore \bar{y} cannot act hyperbolically on T_A and we are in case (a), i.e. $T_x \cap T_{\bar{y}} \neq \emptyset$ and \bar{y} acts elliptically on T_A .

Hence either \bar{y} is conjugate into $\langle e \rangle$ or into H (since \bar{y} is not conjugate into $\langle x \rangle$). So suppose that \bar{y} is conjugate to an element of H . But then mapping $H, \langle x \rangle$ to 0 and e to 1 defines a non-trivial homomorphism $\varphi : F(x, y) \rightarrow \mathbb{Z}_k$, such that $F(x, y) = \langle x, \bar{y} \rangle \subset \ker \varphi$, a contradiction.

Therefore \bar{y} is conjugate to an element of $\langle e \rangle$, hence since \bar{y} is a basis element $\bar{y} = geg^{-1}$ for some $g \in F(x, y)$. Since $T_{\bar{y}} \cap T_x \neq \emptyset$ there exists an isomorphism of graphs of groups from

$$\langle \bar{y} \rangle *_{\langle \bar{y}^k \rangle} \langle x^n, \bar{y}^k \rangle *_{\langle x^n \rangle} \langle x \rangle \text{ to } \langle e \rangle *_{\langle e^k \rangle} H *_{\langle x^n \rangle} \langle x \rangle.$$

In particular there exists an automorphism $\alpha \in \text{Aut}(F_2)$ mapping $\langle \bar{y}^k, x^n \rangle$ onto H . But since the images of e and $y = geg^{-1}$ in the abelianization of $F(x, y)$ are equal, this immediately implies that α is an inner automorphism and hence since it fixes x we conclude that $g = x^m$ for some $m \in \mathbb{N}$. Therefore $H = \langle \bar{y}^k, x^n \rangle$ and $\langle \bar{y} \rangle *_{\langle \bar{y}^k \rangle} \langle x^n, \bar{y}^k \rangle *_{\langle x^n \rangle} \langle x \rangle$ is a splitting of $F(x, y)$ relative w , a contradiction to the assumption that $w \notin \langle x^n, \bar{y}^m \rangle$ for any $m > 1$.

Therefore a JSJ decomposition of G_w is a graph of groups which we get by refining both A and B in $A *_C B$ by $\langle x^n, y \rangle *_{\langle x^n \rangle} \langle x \rangle$ and all vertex groups are rigid. This is true since $\langle x^n, y \rangle *_{\langle w \rangle} \langle x^n, y \rangle$ cannot be a QH subgroup, because then cutting the underlying surface along the curve corresponding to w , would yield a surface S with 2 boundary components and fundamental group F_2 . Hence by the classification of surfaces S is a punctured Möbius strip (respectively a twice punctured projective plane, see Figure 5). But this implies that $w \in \langle x^n, y^2 \rangle$ for some $n > 1$, which has been excluded by assumption. Therefore all vertices in the JSJ decomposition are indeed rigid.

Case 2: Now we assume that there exists some (maximal) $m > 1$ such that $w \in \langle x^n, y^m \rangle$. Then clearly

$$\langle y \rangle *_{\langle y^m \rangle} \langle x^n, y^m \rangle *_{\langle x^n \rangle} \langle x \rangle$$

is a splitting of $F(x, y)$ relative to w and we cannot refine $\langle x^n, y^m \rangle$ further as an amalgamated product relative to $\langle x^n \rangle, \langle y^m \rangle$ and w .

Now assume that w is conjugate to $(x^{n\epsilon_1} y^{m\epsilon_2})^{\pm 1}$ in $\langle x^n, y^m \rangle$, where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. We

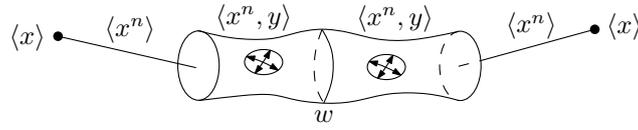


Figure 5: G_w does not contain a QH subgroup

further distinguish between the following cases depending on n and m .

First assume that $n, m > 2$. Then $\langle x^n, y^m \rangle *_{\langle w \rangle} \langle x^n, y^m \rangle$ is the fundamental group of a surface with genus 0 and 4 boundary components, two corresponding to $\langle x^n \rangle$ and two corresponding to $\langle y^m \rangle$. And since elements corresponding to essential simple closed curves on surfaces have power at most 2 in the fundamental group of the surface, the edges in $\langle y \rangle *_{\langle y^m \rangle} \langle x^n, y^m \rangle *_{\langle x^n \rangle} \langle x \rangle$ do not correspond to simple closed curves on a QH subgroup. Hence the JSJ decomposition of G_w is a graph of groups with one QH vertex group, which is the fundamental group of the 4-punctured sphere, and four rigid vertices with vertex group isomorphic to \mathbb{Z} connected to the four boundary components (see figure 6).

Now assume that $n > 2$ and $m = 2$. In this case we can replace the vertices stabilized

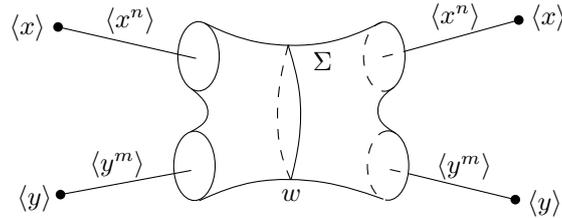


Figure 6: The JSJ decomposition of G_w if w is conjugate to $(x^n y^m)^{\pm 1}$

by $\langle x \rangle$ together with their adjacent edges in Figure 6 by Möbius strips which are glued along their boundaries to the 4-punctured sphere. Hence a JSJ decomposition of G_w is a graph of groups with one QH vertex group, which is the fundamental group of a twice-punctured Klein bottle Σ , and two rigid vertices with vertex group isomorphic to \mathbb{Z} connected to the two boundary components of Σ . The same arguments holds if $n = 2$ and $m > 2$.

So for the last case assume that $n = 2 = m$. Then we can replace all four vertices with cyclic vertex groups (and their adjacent edges) in figure 6 by Möbius strips which are glued along their boundaries to the boundary components of the 4-punctured sphere. But in this case clearly G_w is the fundamental group of a non-orientable surface of genus 4 and this was excluded by the assumption of the theorem.

Now assume that w is not conjugate to $(x^{n\epsilon_1} y^{m\epsilon_2})^{\pm 1}$, $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Then refining A and B by

$$\langle y \rangle *_{\langle y^m \rangle} \langle x^n, y^m \rangle *_{\langle x^n \rangle} \langle x \rangle$$

yields a JSJ decomposition of G_w . In particular all vertex groups are rigid. This holds since contrary to the case above, $\langle x^n, y^m \rangle *_{\langle w \rangle} \langle x^n, y^m \rangle$ cannot be a QH subgroup, because

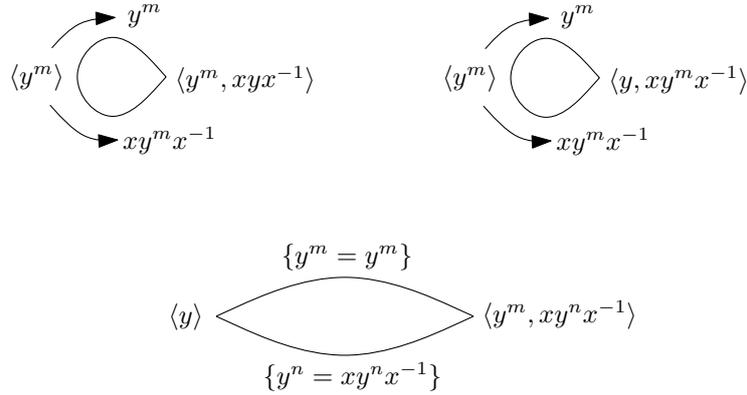


Figure 7: The possible relative JSJ decompositions of F_2 in the HNN-splitting case

otherwise cutting the underlying surface Σ along the curve corresponding to w , would yield two copies of a surface S with 3 boundary components and as $\pi_1(S)$ is generated by the elements x^n and y^m corresponding to the boundary components of Σ , S has genus 0 (see Figure 6), and the third boundary component is generated by a conjugate of $(x^{n\epsilon_1}y^{m\epsilon_2})^{\pm 1}$ and therefore w is conjugate to $(x^{n\epsilon_1}y^{m\epsilon_2})^{\pm 1}$, a contradiction. \square

Lemma 7.6. *If $w \in \langle xyx^{-1}, y \rangle$ for some basis $\{x, y\}$ of F_2 and $w \notin \langle a^n, b \rangle$ for any basis $\{a, b\}$ of F_2 and any $n > 1$, then a JSJ decomposition of G_w has either only rigid vertices and is one of the graphs of groups which we get by substituting either $\langle xyx^{-1}, y \rangle *_{\langle y \rangle}$ or one of the three graphs of groups in Figure 7 for A and B , or has one QH vertex and is the graph of groups in Figure 9.*

Proof. As in the proof of Proposition 7.2 we conclude that there exists no splitting of F_2 as an amalgamated product relative to w . But since $w \in \langle xyx^{-1}, y \rangle$ for some basis $\{x, y\}$ of F_2 there exists (at least) one relative splitting of F_2 as an HNN-extension of the form $\langle xyx^{-1}, y \rangle *_{\langle y \rangle}$ with the embeddings $y \mapsto xyx^{-1}$ and $y \mapsto y$, which does moreover not correspond to a non-boundary parallel closed curve on a surface corresponding to a QH subgroup of G_w . The moreover part holds, since by assumption G_w is not a surface group.

Since F_2 projects onto the fundamental group of the underlying graph of any refinement of A , it follows that the first Betti number of any refinement of A is at most one. Hence a (relative) refinement of $\langle xyx^{-1}, y \rangle$ has to be an amalgamated product, say $X *_E Y$. By Proposition 7.1 we can assume that $X \cong \mathbb{Z}$. Since by assumption there does not exist a splitting of $F(x, y)$ as an amalgamated product relative w , we conclude that the refined graph of groups has no separating edge and hence we can assume that either $X = \langle y \rangle$ or $X = \langle xyx^{-1} \rangle$ (see Figure 8).

Moreover we conclude that either $E = \langle y^m \rangle$ or $E = \langle xy^m x^{-1} \rangle$ for some $m > 1$. Hence the resulting graph of groups (after plugging in $X *_E Y$ for $\langle xyx^{-1}, y \rangle$) is not reduced

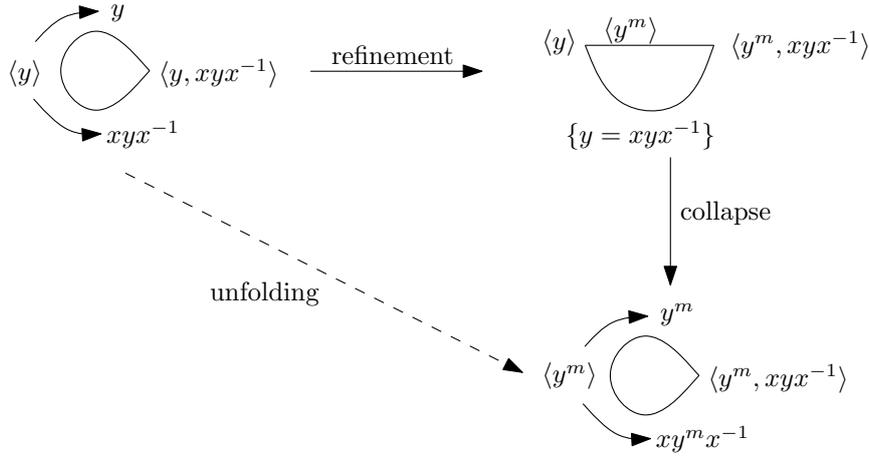


Figure 8: Unfolding the graph of groups in the HNN case

and after collapsing the original edge, we get one of the following graphs of groups:

$$\langle xyx^{-1}, y^m \rangle *_{\langle y^m \rangle} \text{ or } \langle xy^m x^{-1}, y \rangle *_{\langle xy^m x^{-1} \rangle}$$

and therefore this new graph of groups is just an unfolding of the original one (see Figure 8).

So if we choose $m, n \geq 1$ maximal such that $w \in \langle y^m, xy^n x^{-1} \rangle$ and w is not conjugate to $(y^{\pm m} xy^{\pm n} x^{-1})^{\pm 1}$, it follows that (depending on w) the only possible relative JSJ decompositions of F_2 relative w (up to folding) are the three decompositions shown in Figure 7. Note that the top two graphs of groups are in fact isomorphic, since both are graphs of groups of the form

$$F(a, b) *_{\mathbb{Z}}$$

with the embeddings $\alpha : \mathbb{Z} \rightarrow \langle a^m \rangle$ and $\omega : \mathbb{Z} \rightarrow \langle b^n \rangle$, and $w \in F(a, b)$. It follows as in the amalgamated product case that all vertex groups are rigid.

Note that in the case that $m, n > 1$ (and hence the splitting is the bottom graph of groups of Figure 7), we have that $k := \gcd(m, n) = 1$ since otherwise we could pull out a root of y , i.e. refine the graph of groups by replacing the vertex with vertex group $\langle y \rangle$ by the graph of groups

$$\langle y \rangle *_{\langle y^k \rangle} \langle y^k \rangle$$

and get a non-trivial amalgamated product, a contradiction.

So assume now that $m \neq n \geq 1$ are maximal such that w is conjugate to $(y^{\pm m} xy^{\pm n} x^{-1})^{\pm 1}$. Note that again $\gcd(m, n) = 1$. Then by the same argument as in the lemma before, $\langle y^m, xy^n x^{-1} \rangle *_{\langle w \rangle} \langle y^m, xy^n x^{-1} \rangle$ is the fundamental group of the 4-punctured sphere, i.e. the surface with genus 0 and 4 boundary components, two corresponding to $\langle y^m \rangle$ and two corresponding to $\langle xy^n x^{-1} \rangle$, and hence a JSJ decomposition of G_w is a graph of groups with one QH subgroup, which is the fundamental group of the 4-punctured sphere, and

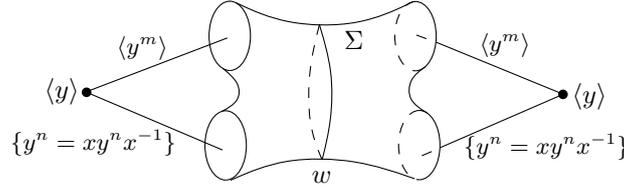


Figure 9: The JSJ decomposition of G_w if w is conjugate to $(y^{\pm m}xy^{\pm n}x^{-1})^{\pm 1}$

two rigid vertices with vertex group isomorphic to \mathbb{Z} each connected to two boundary components (see Figure 9). \square

Lemma 7.7. *If $w \in \langle a^n, b \rangle$ for some $n > 1$ and some basis $\{a, b\}$ of F_2 and moreover $w \in \langle xyx^{-1}, y \rangle$ for some basis $\{x, y\}$ of F_2 , then a JSJ decomposition of G_w is one of the graph of groups which we get by refining the vertices A and B in $A *_C B$ by one of the graphs of groups in Figure 11 (in which case all vertices are rigid), the graph of groups in Figure 9 with $\gcd(m, n) > 1$, or the graph of groups in Figure 12.*

Proof. Again it suffices to consider a JSJ decomposition of F_2 relative w . Since $w \in \langle xyx^{-1}, y \rangle$ for some basis $\{x, y\}$ of F_2 , F_2 admits in particular a relative splitting which is one of the three types in Figure 7 with neither the restriction that $\gcd(m, n) = 1$ as in Lemma 7.6 nor that necessarily $m > 1$ or $n > 1$.

Case 1: Assume that F_2 admits the third type of splitting (the one with two edges). Since the vertex group $\langle y^m, xy^n x^{-1} \rangle$ does not admit any further refinement as an amalgamated product relative $\langle y^m \rangle$, $\langle xy^n x^{-1} \rangle$ and $\langle w \rangle$, it follows that we can pull a root of order $k := \gcd(m, n) > 1$ out of the vertex group $\langle y \rangle$ to refine the graph of groups. The refined graph of groups is the bottom one in Figure 11 and since there exists no further refinement, this is a JSJ decomposition of F_2 relative w .

Case 2: So now assume that F_2 admits a splitting relative w of the form:

$$\langle y^n, xyx^{-1} \rangle *_{\{ty^{nt-1} = xy^n x^{-1}\}}$$

or

$$\langle y, xy^m x^{-1} \rangle *_{\{ty^{mt-1} = xy^m x^{-1}\}}$$

for some maximal $m, n \geq 1$, i.e. we are in one of the two top cases of Figure 7. By assumption of the lemma there exists a further refinement of the vertex group as an amalgamated product. So let $k > 1$ maximal such that $\langle e \rangle *_{\langle e^k \rangle} H$ is a splitting of $\langle y, xy^m x^{-1} \rangle$ as an amalgamated product relative to $\langle y^m \rangle$ and $\langle xy^m x^{-1} \rangle$ (we only consider the second case, the first one is similar). We want to show that $e = y$, so let \mathbb{A} be the graph of groups which we get by refining the vertex group in

$$\langle y, xy^m x^{-1} \rangle *_{\{ty^{mt-1} = xy^m x^{-1}\}}$$

by $\langle e \rangle *_{\langle e^k \rangle} H$ and denote by T_A the corresponding Bass-Serre tree. First we note that x acts hyperbolically on T_A , since otherwise

$$x \in \langle e \rangle *_{\langle e^k \rangle} H = \langle y, xy^m x^{-1} \rangle$$

and hence $\langle y, xy^m x^{-1} \rangle = F(x, y)$, a contradiction. Moreover y acts elliptically on T_A , as y^m does. As in the proof of Lemma 7.5 we can deduce from (the proof of) Theorem 1A in [KaWe2] that $\{x, y\}$ is Nielsen equivalent to $\{\bar{x}, y\}$ such that either

- (a) $T_{\bar{x}} \cap T_y \neq \emptyset$ and \bar{x}, y act elliptically on T_A or
- (b) $T_y \cap \bar{x}T_y \neq \emptyset$ and \bar{x} acts hyperbolically, y elliptically on T_A .

If \bar{x} acts elliptically on T_A this yields an immediate contradiction to the minimality of \mathbb{A} , hence we are in case (b). Since y acts elliptically on T_A , it is either conjugate to an element of H or conjugate to e .

So suppose that y is not conjugate to e , i.e. y is conjugate into H . Then $T_y \subset T_A$ is the star of the vertex stabilized by H and all other vertices of T_y are stabilized by conjugates of H , while the edges are stabilized by $\langle y^m \rangle$. Clearly $\bar{x}T_y$ is of the same form. Since $T_y \cap \bar{x}T_y \neq \emptyset$ we conclude that $\langle \bar{x}, y \rangle T_y \subset T_A$ is an $F(x, y)$ -invariant subtree, a contradiction to the minimality of \mathbb{A} . Hence y is conjugate to e and by the same arguments as in the amalgamated product case (Lemma 7.5) it follows without loss of generality that $e = y$ and k divides n . Hence F_2 has a relative splitting \mathbb{A} which is of one of the two types:

$$\langle xyx^{-1} \rangle *_{\langle xy^k x^{-1} \rangle} \langle y^n, xy^k x^{-1} \rangle *_{\{ty^n t^{-1} = xy^n x^{-1}\}}$$

where k divides n , or

$$\langle y \rangle *_{\langle y^k \rangle} \langle y^k, xy^m x^{-1} \rangle *_{\{ty^m t^{-1} = xy^m x^{-1}\}}$$

where k divides m . Since we have chosen k maximal, by the same arguments as in the proof of Lemma 7.5 \mathbb{A} has no further refinement, i.e. \mathbb{A} cannot have one loop edge and two non-loop edges as in Figure 10. In addition as in the proof of Lemma 7.6 these two graphs of groups are isomorphic.

Therefore under the assumptions of Lemma 7.7 a relative JSJ decomposition of F_2

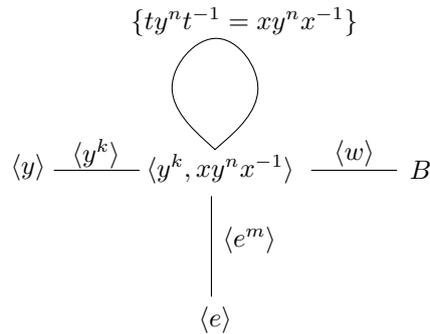


Figure 10: \mathbb{A} cannot have one loop edge and two non-loop edges originating from the same vertex

is of one of the two types pictured in Figure 11. To complete the proof that a JSJ

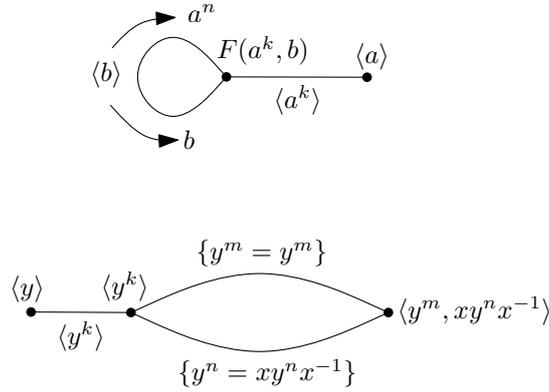


Figure 11: The possible relative JSJ decompositions of F_2 . In the top graph of groups k divides n , in the bottom one is $k = \gcd(m, n)$

decomposition of G_w has the desired form, it remains to consider QH vertices. We first consider the case that the relative JSJ decomposition of F_2 has the form

$$\langle y \rangle *_{\langle y^k \rangle} \langle y^k, xy^m x^{-1} \rangle *_{\{ty^{mt-1} = xy^m x^{-1}\}}$$

Note that there exist $l \in \mathbb{Z}$ such that $lk = m$ and therefore by sliding the edge corresponding to the loop edge over the non-loop edge we get a graph of groups which has two vertices and two non-loop edges (corresponding to HNN-extensions) connecting these vertices. Hence by the same arguments as in the HNN case (Lemma 7.6), we conclude that the JSJ decomposition is as claimed, i.e. with only rigid vertices, if w is not conjugate to $(y^{\pm m}xy^{\pm n}x^{-1})^{\pm 1}$ and as in Figure 9 otherwise. In particular in this case there exists a QH vertex group.

Now we consider the case that the relative JSJ decomposition is the bottom graph of groups in Figure 11. Again by the same arguments as before, we conclude that a JSJ decomposition of G_w is as claimed in the lemma, with only rigid vertices if w is not conjugate to $(y^{\pm m}xy^{\pm n}x^{-1})^{\pm 1}$ and as in Figure 12 otherwise. Again in this case there exists a QH vertex group. \square

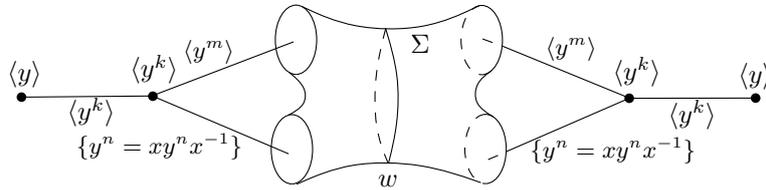


Figure 12: The JSJ decomposition of G_w if w is conjugate to $(y^{\pm m}xy^{\pm n}x^{-1})^{\pm 1}$

8 Completions of Resolutions

The aim of this section is to show, that every restricted Γ -limit group L can be embedded into a larger group, called its completion, which is a Γ -limit tower. The original paper of Sela ([Se2]) contains the construction of the completion over free groups but is lacking a proof of the embeddability. In [BySk] a slightly different construction together with a complete proof of the embeddability appears, but it only covers the case of minimal-rank well-structured resolutions over free groups (this means that all limit groups which appear along the resolution are one-ended relative to the coefficient free group).

We follow the approach of the authors of [BySk] but have to generalize their ideas in several ways in order to deal with general resolutions over hyperbolic groups and in particular with the various problems caused by the presence of torsion.

So let us briefly recall the setting we are in: $\Gamma = \langle a_1, \dots, a_k \rangle$ is as always a hyperbolic group. Let L be a restricted Γ -limit group and let

$$\text{Res}(L) : L = L_0 \xrightarrow{\eta_0} L_1 \xrightarrow{\eta_1} L_2 \rightarrow \dots \xrightarrow{\eta_{l-1}} L_l = \pi_1(\mathbb{D})$$

be a well-structured MR resolution of L , where $\pi_1(\mathbb{D})$ is a Dunwoody decomposition of L_l , such that every vertex group is isomorphic to a subgroup of Γ .

With each restricted Γ -limit group L_i that appears along the resolution $\text{Res}(L)$, we have constructed (see Definition 5.23) an associated (restricted) decomposition $\mathbb{D}^{(i)}$ along finite edge groups, such that every vertex group V of $\mathbb{D}^{(i)}$ either belongs to $\{R_1^i, \dots, R_{k_i}^i\}$ or to \mathbb{H}_i and these two sets are disjoint as (not necessarily connected) subgraphs of $\mathbb{D}^{(i)}$. Moreover $L_i = \pi_1(\mathbb{D}^{(i)})$ and the coefficient group Γ is contained in one of the vertex groups R_j^i . With each factor R_j^i there is an associated restricted (possibly trivial) virtually abelian JSJ decomposition \mathbb{A}_j^i .

8.1 The construction

We begin with a simple observation how the virtually abelian JSJ decomposition of a one-ended Γ -limit group can be modified, such that the construction of its completion will be simplified later.

We first recall some definitions. Let \mathbb{A} be the virtually abelian JSJ decomposition of a one-ended Γ -limit group L . Let A_v be a virtually abelian vertex group in \mathbb{A} and let Δ be the set of homomorphisms $\eta : A_v^+ \rightarrow \mathbb{Z}$ such that $\eta(\alpha_e(A_e^+)) = 0$ for all $e \in EA$ with $\alpha(e) = v$. We then define the *peripheral subgroup* of A_v^+ as

$$P(A_v^+) = \{g \in A_v^+ \mid \eta(g) = 0 \text{ for all } \eta \in \Delta\}.$$

We call every subgroup $U \leq A_v$ with $U^+ = P(A_v^+)$ a *peripheral subgroup* of A_v . Note that the peripheral subgroup of A_v^+ is uniquely defined, while the peripheral subgroup of A_v is not. For every edge $e \in EA$ with $\alpha(e) = v$ we denote a peripheral subgroup of

A_v containing A_e by $P(A_v, A_e)$.

Recall that the envelope $E(R)$ of a rigid vertex group R is the fundamental group of a graph of groups \mathbb{B} whose underlying graph is a star with one vertex v with vertex group R connected to all other vertices via an edge of the form (v, u) for some $u \in VB \setminus \{v\}$. Edges in \mathbb{B} are in 1-to-1 correspondence to edges adjacent to R in \mathbb{A} and the edge groups are isomorphic. Let now $e = (v, u)$ be an edge in \mathbb{B} and $e' = (v', u')$ the corresponding edge in \mathbb{A} adjacent to R (where R is the rigid vertex group of v'). If $A_{u'}$ is rigid or a QH vertex group, then B_u is the maximal virtually abelian subgroup of $A_{u'}$ containing $A_{e'}$. If $A_{u'}$ is a virtually abelian vertex group, then B_u is the peripheral subgroup of $A_{u'}$ containing $A_{e'}$.

Note that in a strict resolution the envelopes of rigid vertex groups and the peripheral subgroups of virtually abelian vertex groups get mapped injectively by the canonical quotient maps.

Lemma 8.1. *Let L be a one-ended Γ -limit group and \mathbb{A} be a virtually abelian JSJ decomposition of L . Then we can enlarge every rigid vertex group R to (a subgroup of) its envelope, such that moreover all adjacent edge groups, which connect R to another rigid vertex are maximal finite-by-abelian in it. In addition all edge groups connecting a rigid vertex to a virtually abelian vertex are maximal virtually abelian in the rigid vertex group.*

Proof. First note that it is shown in [ReWe] that every edge group adjacent to a rigid vertex group is contained in a maximal virtually abelian subgroup and this subgroup is elliptic in \mathbb{A} , i.e. it is contained in at least one of the adjacent vertex groups. Let e be an edge with edge group E adjacent to a rigid vertex group R and a virtually abelian vertex group A . We then replace R by $R *_E P(A, E)$ and E by $P(A, E)$, i.e. the subgraph of groups $R *_E A$ is replaced by $(R *_E P(A, E)) *_P(A, E) A$. Since $R *_E P(A, E) \leq E(R)$, the new vertex group is still rigid. We perform these modifications for all edges connecting a rigid and a virtually abelian vertex in parallel and denote the resulting graph of groups again by \mathbb{A} .

We will now explain how we can change the graph of groups \mathbb{A} such that every edge group connecting two rigid vertices is maximal finite-by-abelian in both adjacent vertex groups. Assume that $R_1 *_Z R_2$ is such an edge connecting two rigid vertex groups and suppose that Z is contained in a maximal finite-by-abelian subgroup C which is contained in R_2 (the HNN case is similar). Note that

$$\langle R_1, C \rangle = R_1 *_Z C \subset E(R_1).$$

We change the splitting $R_1 *_Z R_2$ to $R_1 *_Z C *_C R_2$ and collapse the edge with edge group Z , i.e. we get the splitting $(R_1 *_Z C) *_C R_2$. Note that $(R_1 *_Z C)$ is still rigid and C is maximal finite-by-abelian in both adjacent vertex groups. After repeating this step for all edges adjacent to two rigid vertex groups we get a graph of groups which satisfies the claim of the Lemma. \square

We will use these modifications either explicitly or implicitly in the following constructions whenever a JSJ decomposition of a one-ended Γ -limit group appears.

Theorem 8.2. *Let Γ be a hyperbolic group and let $\text{Res}(L)$ be a well-structured resolution of a restricted Γ -limit group L . Then L embeds into a (canonical) Γ -limit tower. We call this tower the completion of L and denote it by $\text{Comp}(L)$.*

Proof. We construct the desired Γ -limit tower iteratively from bottom to top. Assume that the well-structured resolution of L is given by the following decreasing sequence of restricted Γ -limit groups:

$$\text{Res}(L) : L = L_0 \xrightarrow{\eta_0} L_1 \xrightarrow{\eta_1} L_2 \rightarrow \cdots \xrightarrow{\eta_{l-1}} L_l = \pi_1(\mathbb{D}^{(l)}),$$

where $\eta_i : L_i \rightarrow L_{i+1}$ is the canonical quotient map for all $i \in \{0, \dots, l-1\}$.

For all $i \in \{1, \dots, l\}$ we construct a group $\text{Comp}(L_i)$ which we call the completion of L_i , together with an embedding $\iota_i : L_i \rightarrow \text{Comp}(L_i)$ and a decomposition \mathbb{B}_i of $\text{Comp}(L_i)$, such that every vertex group of \mathbb{H}_i is a vertex group in \mathbb{B}_i . In fact the image of $\pi_1(\mathbb{U}_i)$ for every connected subgraph of groups \mathbb{U}_i of \mathbb{H}_i under ι_i admits a decomposition as a subgraph of groups of \mathbb{B}_i which is isomorphic to \mathbb{U}_i . In particular $\iota_i|_{\pi_1(\mathbb{U}_i)}$ is an isomorphism.

We set $\text{Comp}(L_l) = L_l$ to be the completion of L_l and $\mathbb{B}_l := \mathbb{D}^{(l)}$. Clearly $\iota_l := \text{id}_{L_l}$ embeds L_l into its completion and \mathbb{H}_l is a subgraph of \mathbb{B}_l . Since L is a restricted Γ -limit group (and so is L_l), Γ is contained in a vertex group of $\mathbb{D}^{(l)}$. But since every vertex group of $\mathbb{D}^{(l)}$ is isomorphic to a subgroup of Γ and one-ended hyperbolic groups are cohopfian, this implies that Γ is the whole vertex group (not only a subgroup). Hence Γ is a vertex group of $\mathbb{D}^{(l)}$ and therefore it follows immediately that $\text{Comp}(L_l)$ is a Γ -limit tower. We say that $\mathbb{B}_l = \mathbb{D}^{(l)}$ is the *completed decomposition* of $\text{Comp}(L_l)$.

We continue the construction of the completion inductively. Let $i \in \{0, \dots, l-1\}$ and assume we have already constructed the completed Γ -limit group $\text{Comp}(L_{i+1})$ on the $(i+1)$ -th level together with its completed decomposition \mathbb{B}_{i+1} and an embedding $\iota_{i+1} : L_{i+1} \rightarrow \text{Comp}(L_{i+1})$. We now collapse a minimal number of edges in \mathbb{B}_{i+1} necessary to guarantee that all images under $\iota_{i+1} \circ \eta_i$ of non-QH vertex groups in all the virtually abelian JSJ decompositions A_j^i are elliptic in this new graph of groups. By abuse of notation we still call this new graph of groups \mathbb{B}_{i+1} .

We will construct (in several steps) a new graph of groups \mathbb{B}_i and call its fundamental group the completion of L_i . Throughout the construction we will heavily use the properties of a well-structured resolution (either explicitly or implicitly).

Let \mathbb{A} be a graph of groups which we get by plugging in all the virtually abelian JSJ decompositions for the one-ended vertex groups in $\mathbb{D}^{(i)}$. Note that it follows from Lemma 8.1 that after performing finitely many edge slides, we can assume that all edges adjacent to non-isolated virtually abelian vertices have infinite edge groups.

If \mathbb{A} does not contain a rigid vertex group, then every vertex group is either a finite-by-closed orbifold, a virtually abelian group or isomorphic to a subgroup of Γ . Let V be

such a vertex group. Then $\iota_{i+1} \circ \eta_i(V)$ has a decomposition as a connected subgraph of groups, say \mathbb{V} , of \mathbb{B}_{i+1} (since \mathbb{V} is contained in \mathbb{H}_{i+1}) and we replace \mathbb{V} in \mathbb{B}_{i+1} by a vertex with vertex group V . We moreover adjust the boundary monomorphisms of adjacent edges with finite edge groups in the obvious way. This is well-defined since $\text{Res}(L)$ is a well-structured resolution. Repeating this for all vertex groups V of \mathbb{A} , yields a graph of groups \mathbb{B}_i . We set $\text{Comp}(L_i) := \pi_1(\mathbb{B}_i)$ to be the completion of L_i and call \mathbb{B}_i the completed decomposition of $\text{Comp}(L_i)$. Clearly $\text{Comp}(L_i)$ is a Γ -limit tower.

Hence from now on (by introducing redundant rigid vertices if necessary) we can assume that there exists at least one rigid vertex, say v , in \mathbb{A} . Let e_1, \dots, e_m be an enumeration of the edges of A in such a way that $\alpha(e_1) = v$ and for all $t \in \{0, \dots, m\}$ there exists a graph of groups \mathbb{A}_t with the following properties:

- \mathbb{A}_t is a connected subgraph of groups of \mathbb{A} .
- \mathbb{A}_0 consists of the single vertex v .
- $\mathbb{A}_t \subset \mathbb{A}_{t+1}$ is a proper subgraph of groups and \mathbb{A}_{t+1} arises from \mathbb{A}_t by adding the edge e_{t+1} together with possibly one of the vertices $\alpha(e_{t+1})$ or $\omega(e_{t+1})$ (if both adjacent vertices don't already belong to \mathbb{A}_t).

We start the construction of the completion $\text{Comp}(L_i)$ and its completed decomposition \mathbb{B}_i with the graph of groups \mathbb{B}_{i+1} . Let V be a finite-by-closed orbifold or a virtually abelian vertex group of \mathbb{A} adjacent only to edges with finite edge group. Recall that we call such vertex groups isolated QH or virtually abelian vertex groups. Then $\iota_{i+1} \circ \eta_i(V)$ has a decomposition as a connected subgraph of groups, say \mathbb{V} , of \mathbb{B}_{i+1} (since \mathbb{V} is contained in \mathbb{H}_{i+1}) and we replace \mathbb{V} in \mathbb{B}_{i+1} by a vertex with vertex group V . We also adjust the boundary monomorphisms of edges adjacent to the new vertex with vertex group V in the obvious way. Again this is well-defined since $\text{Res}(L)$ is a well-structured resolution. Repeating this for all vertex groups V of \mathbb{A} of this type, yields a graph of groups \mathbb{B}'_0 . We set $W_0 := \pi_1(\mathbb{B}'_0)$.

We then proceed inductively through the sequence of graphs of groups $\mathbb{A}_0 \subset \dots \subset \mathbb{A}_m = \mathbb{A}$ in such a way, that we construct a graph of groups \mathbb{B}'_t which contains \mathbb{B}'_0 (which is constructed out of \mathbb{B}_{i+1}) as a subgraph, such that $\pi_1(\mathbb{A}_t)$ embeds into $W_t := \pi_1(\mathbb{B}'_t)$ via a monomorphism ν_t for all $t \in \{0, \dots, m\}$, where ν_t coincides on all rigid vertices of \mathbb{A}_t with $\iota_{i+1} \circ \eta_i$ up to conjugation by an element which is either trivial or contained in $W_t \setminus \text{Comp}(L_{i+1})$. Moreover $\nu_t|_{\pi_1(\mathbb{A}_{t-1})} = \nu_{t-1}$ and $W_t = \pi_1(\mathbb{B}'_t)$ is a Γ -limit tower.

For $t = 0$ the graph of groups \mathbb{A}_0 consists by definition of a single vertex v with rigid vertex group A_v . Since $\text{Res}(L)$ is well-structured, $\iota_{i+1} \circ \eta_i$ is injective on A_v . Hence $\nu_0 := \iota_{i+1} \circ \eta_i|_{A_v}$ yields the desired embedding of $\pi_1(\mathbb{A}_0)$ into $W_0 = \pi_1(\mathbb{B}'_0)$. (This holds since the image of a rigid vertex group under $\iota_{i+1} \circ \eta_i$ has at most finite intersection with the fundamental group of every connected subgraph of groups of \mathbb{H}_{i+1} and all the subgraphs of groups \mathbb{V} which have been replaced in the step before are contained in \mathbb{H}_{i+1}). Clearly W_0 is a Γ -limit tower, since $\text{Comp}(L_{i+1})$ is one by induction hypothesis.

So let's assume we already have constructed a graph of groups \mathbb{B}'_t out of \mathbb{B}_{i+1} together with an embedding

$$\nu_t : \pi_1(\mathbb{A}_t) \hookrightarrow W_t = \pi_1(\mathbb{B}'_t),$$

which coincides on rigid vertex groups with $\iota_{i+1} \circ \eta_i$ up to conjugation by an element which is either trivial or contained in $W_t \setminus \text{Comp}(L_{i+1})$. Assume without loss of generality that $\alpha(e_{t+1}) \in A_t$. We distinguish several cases, depending on the edge e_{t+1} . Denote by A_e the edge group of e_{t+1} and by $\alpha : A_e \hookrightarrow A_{\alpha(e_{t+1})}$, $\omega : A_e \hookrightarrow A_{\omega(e_{t+1})}$ the canonical embeddings into the adjacent vertex groups.

(1) Assume that A_e is infinite.

(a) Assume that e_{t+1} connects two rigid vertices in \mathbb{A} and hence A_e is finite-by-abelian (see the construction of the virtually abelian JSJ decomposition in [ReWe]).

Let f_1, \dots, f_m be the edges in the various JSJ decompositions \mathbb{A}_j^i connecting two rigid vertices. Let $I_1 \cup \dots \cup I_k$ be a partition of $\{1, \dots, m\}$ with the following property:

For any $l \in \{1, \dots, k\}$ there exists a maximal finite-by-abelian subgroup M_l^+ of $\text{Comp}(L_{i+1})$ such that for all $j \in I_l$ the image $\iota_{i+1} \circ \eta_i(A_{f_j})$ of the corresponding edge group can be conjugated into M_l^+ . Moreover I_l is maximal with this property for all $1 \leq l \leq k$.

Let $l \in \{1, \dots, k\}$ such that $\iota_{i+1} \circ \eta_i(A_e)$ can be conjugated into M_l^+ . If the group $M_l^+ \oplus \mathbb{Z}^{|I_l|}$ has been amalgamated along M_l^+ to $\text{Comp}(L_{i+1}) \leq W_h$ for some $h < t$ earlier in the progress then we set $\mathbb{B}'_{t+1} := \mathbb{B}'_t$ and $W_{t+1} := W_t$. Otherwise we amalgamate the group $M_l^+ \oplus \mathbb{Z}^{|I_l|}$ along M_l^+ to $\text{Comp}(L_{i+1}) \leq W_t$. (Note that M_l^+ is still contained in W_t since $\text{Res}(L)$ is well-structured, i.e. the maximal finite-by-abelian subgroups containing the images of edge groups connecting two rigid vertices have at most finite intersection with $\pi_1(\mathbb{H}_{i+1})$ and therefore were not replaced in the previous modifications). Hence \mathbb{B}'_{t+1} is the graph of groups \mathbb{B}'_t together with a new vertex with vertex group $M_l^+ \oplus \mathbb{Z}^{|I_l|}$ and an edge connecting both with edge group M_l^+ (see Figure 13). Therefore W_{t+1} has the structure of a virtually abelian (in fact even finite-by-abelian) flat over W_t and hence is a Γ -limit tower.

(b) Assume that e_{t+1} connects a rigid vertex r with vertex group R and a QH vertex q with vertex group Q in \mathbb{A} .

If $q = \omega(e_{t+1})$ and $q \notin A_t$, then we add a new vertex v to \mathbb{B}'_t with vertex group Q , together with an edge f connecting v to the vertex w with vertex group B_w in \mathbb{B}'_t which contains the image $\iota_{i+1} \circ \eta_i \circ \alpha(A_e)$ of $\alpha(A_e) \leq R$ under $\iota_{i+1} \circ \eta_i$. Note that by the previous modifications applied to \mathbb{B}_{i+1} , this image is in fact elliptic in $\mathbb{B}_{i+1} \subset \mathbb{B}'_t$. The edge group of f is A_e . The embeddings are the obvious ones, namely

$$\omega : A_e \rightarrow Q \text{ and } \iota_{i+1} \circ \eta_i \circ \alpha : A_e \rightarrow B_w.$$

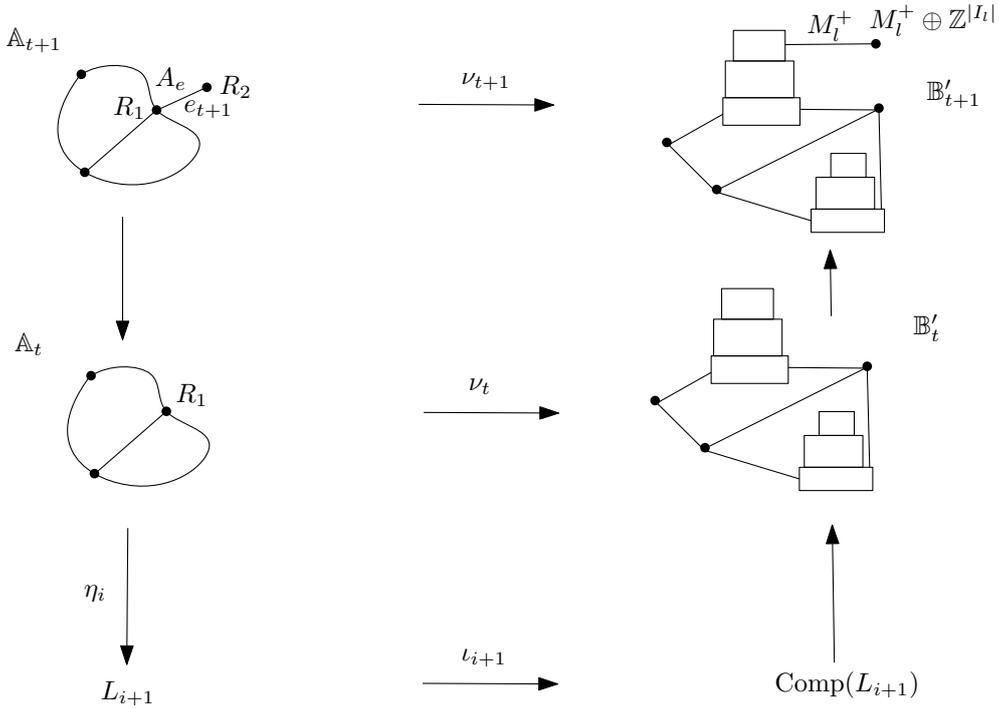


Figure 13: Case (1)(a)

If $q \in A_t$ then a vertex v with vertex group Q has already been added to \mathbb{B}'_t earlier in our iterative process and we only add a new edge f to \mathbb{B}'_t connecting v to the vertex w with vertex group B_w in \mathbb{B}'_t which contains the image $\iota_{i+1} \circ \eta_i \circ \alpha(A_e)$. The edge group of f is again A_e and the embeddings are also the same as the ones in the previous case.

If $q = \alpha(e_{t+1})$ then again a vertex v with vertex group Q has already been added to \mathbb{B}'_t earlier in our iterative process and we only add a new edge f to \mathbb{B}'_t connecting v to the vertex w with vertex group B_w in \mathbb{B}'_t which contains the image $\iota_{i+1} \circ \eta_i \circ \omega(A_e)$. The edge group of f is again A_e and the embeddings are also the same as the ones in the previous case.

- (c) Assume that e_{t+1} connects a virtually abelian vertex a with vertex group A' and a rigid vertex r with vertex group R in \mathbb{A} . Note that by Lemma 8.1 $A_e^+ = P(A'^+)$, i.e. A_e is a peripheral subgroup of A' . For simplicity we set $P(A') := A_e$. This case is by far the most difficult one, mainly because the simple strategy of the torsion-free case can no longer be applied. Assume that $a = \omega(e_{t+1}) \notin A_t$. Let $G := \pi_1(\mathbb{A}_t)$. Hence if we collapse in \mathbb{A}_{t+1} all edges but e_{t+1} we get the amalgamated free product $G *_P(A') A'$. Here we view $P(A')$ as a subgroup of G and A' (to simplify notation), hence both embeddings into the vertex groups are the identity map. Since $\eta_i : L_i \rightarrow L_{i+1}$ is strict, it is in particular injective

on $P(A')$. We set

$$M := M_{W_t}(\iota_{i+1} \circ \eta_i(P(A')))$$

to be the maximal virtually abelian subgroup of W_t which contains the (isomorphic) image of $P(A')$. Here we view $\text{Comp}(L_{i+1})$ as a subgroup of W_t . Since W_t is a Γ -limit group there exists a stably injective sequence $\varphi_n : W_t \rightarrow \Gamma$ such that

$$W_t = W_t / \varinjlim \varphi_n.$$

By the construction of W_t and since η_i is a strict quotient map, there exists a sequence of modular automorphisms $\alpha_n \in \text{Mod}(L_i)$, such that

$$\mu_n := (\varphi_n \circ \iota_{i+1} \circ \eta_i \circ \alpha_n)|_{A'}$$

is stably injective. Let

$$U := M *__{P(A')} A'$$

be the amalgamated product given by the embeddings

$$\alpha = \iota_{i+1} \circ \eta_i : P(A') \rightarrow M$$

and

$$\omega = \text{id} : P(A') \rightarrow A'.$$

We define a sequence of homomorphisms

$$\Psi_n : W_t *_M U \rightarrow \Gamma$$

($W_t *_M U$ with the obvious embeddings) via

- $\Psi_n|_{W_t} = \varphi_n$,
- $\Psi_n|_M = \varphi_n$ and
- $\Psi_n|_{A'} = \mu_n$.

We first check that this is a well-defined homomorphisms on $W_t *_M U$. First note that since $\alpha_n \in \text{Mod}(L_i)$, it fixes $P(A')$. Now by definition

$$\begin{aligned} \Psi_n(\omega(P(A'))) &= \Psi_n(\underbrace{\text{id}(P(A'))}_{\leq A'}) \\ &= \mu_n(P(A')) \\ &= \varphi_n \circ \iota_{i+1} \circ \eta_i \circ \alpha_n(P(A')) \\ &= \varphi_n \circ \iota_{i+1} \circ \eta_i(P(A')) \\ &= \Psi_n(\underbrace{\iota_{i+1} \circ \eta_i(P(A'))}_{\leq M \subset W_t}) \\ &= \Psi_n(\alpha(P(A'))) \end{aligned}$$

$$\begin{array}{ccccc}
 & \begin{array}{c} \alpha_n \\ \curvearrowright \\ \downarrow \end{array} & & & \\
 L_i \geq G *_{P(A')} A' & \longrightarrow & W_t *_{M} (M *_{P(A')} A') & \xrightarrow{\Psi_n} & \Gamma \\
 \downarrow \eta_i & & & & \\
 L_{i+1} & \xrightarrow{\iota_{i+1}} & \text{Comp}(L_{i+1}) \leq W_t & \xrightarrow{\varphi_n} & \Gamma
 \end{array}$$

Figure 14: Constructing the sequence (Ψ_n)

and hence Ψ_n is a well-defined homomorphism. Note however that (after passing to a subsequence) the sequence $(\Psi_n) \subset \text{Hom}(W_t *_{M} U, \Gamma)$ is stable but not necessarily stably injective.

We set

$$A := U / \underline{\ker} \Psi_n = (M *_{P(A')} A') / \underline{\ker} \Psi_n.$$

Since $\Psi_n|_M = \varphi_n$ is stably injective, we have that $M \leq A$. We now define $W_{t+1} := W_t *_{M} A$. Since $\text{Res}(L)$ is a well-structured resolution, W_{t+1} admits a decomposition as a graph of groups \mathbb{B}'_{t+1} which is the graph of groups \mathbb{B}'_t together with a new edge with edge group M connecting a new vertex v with vertex group A to the vertex in \mathbb{B}'_t containing M .

Clearly W_{t+1} is a Γ -limit group. It remains to show that $W_{t+1} = W_t *_{M} A$ is a

$$\begin{array}{ccc}
 \pi_1(\mathbb{A}_t) *_{P(A')} A' & \xrightarrow{\nu_{t+1}} & W_{t+1} = W_t *_{M} A \\
 \downarrow & & \downarrow \\
 \pi_1(\mathbb{A}_t) & \xrightarrow{\nu_t} & W_t
 \end{array}$$

Figure 15: Adding a virtually abelian flat to W_t

virtually abelian flat over W_t . Hence we have to show that A is in fact virtually abelian and moreover that there exists a retraction from W_{t+1} to W_t .

We start by showing that A is virtually abelian. Since $P(A')$ is infinite virtually abelian and $\Psi_n|_{P(A')}$ is by construction stably injective, it holds that for large n , $\Psi_n(P(A'))$ is an infinite subgroup of a maximal virtually cyclic subgroup C_n of Γ . Hence $\Psi_n(A')$ and $\Psi_n(M)$ are also contained in C_n . This implies that $\Psi_n(M *_{P(A')} A') \leq C_n$.

Hence after passing to a subsequence of (Ψ_n) we can assume that $C_n = C$ for all n for some maximal virtually cyclic subgroup C of Γ . Hence

$$A = M *_{P(A')} A' / \varinjlim \Psi_n$$

is a C -limit group and therefore virtually abelian (since otherwise there exists a commutator in A which cannot be mapped to a non-trivial element in C for any n). It only remains to show that there exists a retraction from $W_{t+1} = W_t *_{M'} A$ onto W_t .

It clearly suffices to show that there exists a retraction from A onto M . Recall that A'^+ denotes the unique finite-by-abelian subgroup of A' of index ≤ 2 . It follows from the discussion in [ReWe] after Definition 3.10 that $A'^+/P(A')^+ \cong \mathbb{Z}^k$ is free abelian of rank k for some $k \in \mathbb{N}$. Let $\{x_1, \dots, x_k\}$ be a \mathbb{Z} -basis of $A'^+/P(A')^+$. By abuse of notation we denote a lift of this basis to A' again by $\{x_1, \dots, x_k\}$. It follows from section 4.2.1 in [ReWe] that for large enough n we can choose the sequence of modular automorphisms (α_n) earlier in the construction such that α_n fixes $P(A')$ and such that the tuple of word lengths

$$(|\varphi_n \circ \iota_{i+1} \circ \eta_i \circ \alpha_n(x_i)|_\Gamma) = (\lambda_1, \dots, \lambda_k)$$

is linearly independent over \mathbb{Q} , where $|\cdot|_\Gamma$ denotes the word length with respect to a fixed finite generating set of Γ .

Now the sequence $\Psi_n|_{M^+ *_{P(A')^+} A'^+}$ converges to the action of A^+ on a real tree T isometric to a line. The kernel of this action is M^+ , while x_1, \dots, x_k act hyperbolically by translations on T . Since $(\lambda_1, \dots, \lambda_k)$ is linearly independent over \mathbb{Q} , $\{x_1, \dots, x_k\}$ generate a free abelian subgroup of rank k of A^+ . Since M^+ is the kernel of the action, it is a normal subgroup of A^+ , i.e. by definition of the sequence, $A^+/M^+ \cong \mathbb{Z}^k$. Hence A^+ fits into the short exact sequence

$$1 \rightarrow M^+ \rightarrow A^+ \rightarrow \mathbb{Z}^k \rightarrow 1.$$

Therefore for all $i, j \in \{1, \dots, k\}$, $p \in \{1, \dots, l\}$ there exist elements e_{ij}, \bar{e}_{ip} of finite order in M^+ such that A^+ has the presentation

$$A^+ = \langle M^+, x_1, \dots, x_k \mid [x_i, x_j] = e_{ij}, [x_i, y_p] = \bar{e}_{ip} \\ \text{for all } i, j \in \{1, \dots, k\}, p \in \{1, \dots, l\} \rangle,$$

where $\{y_1, \dots, y_l\}$ is a generating set for M^+ . Note that there exists an element s such that M is generated by $\{s, y_1, \dots, y_l\}$ and either $s = 1$ or the image of s in the quotient of A' by its torsion subgroup has order 2. Using Lemma 4.9 in [ReWe], we conclude that A has the presentation

$$A = \langle M, x_1, \dots, x_k \mid [x_i, x_j] = e_{ij}, [x_i, y_p] = \bar{e}_{ip}, s x_i s^{-1} x_i = \tilde{e}_i, \\ \text{for all } i, j \in \{1, \dots, k\}, p \in \{1, \dots, l\} \rangle,$$

where \tilde{e}_i is an element of finite order in M^+ for all $i \in \{1, \dots, k\}$.

For all $i \in \{1, \dots, k\}$, $n \in \mathbb{N}$ we set $m_n^i := \iota_{i+1} \circ \eta_i \circ \alpha_n(x_i)$. Let E be the torsion subgroup of M^+ . Since A' is virtually abelian it holds for all $i, j \in \{1, \dots, k\}$ that there exist some $e_{ij}^{(n)}, \tilde{e}_{ij}^{(n)} \in E$ such that

$$[m_n^i, m_n^j] = e_{ij}^{(n)} \text{ and } sm_n^i s^{-1} m_n^i = \tilde{e}_{ij}^{(n)}.$$

Moreover for all $n \in \mathbb{N}$, $\iota_{i+1} \circ \eta_i \circ \alpha_n$ defines an action of A' on M and since M is virtually abelian, there exists elements $\bar{e}_{ij}^{(n)} \in E$ for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, l\}$ such that

$$[m_n^i, y_j] = \bar{e}_{ij}^{(n)}.$$

Since $|E|$ is finite, after passing to a subsequence of (α_n) (still denoted (α_n)) we can assume that for all $i, j \in \{1, \dots, k\}$, $p \in \{1, \dots, l\}$:

$$\begin{aligned} [m_n^i, m_n^j] &= e_{ij}, \\ sm_n^i s^{-1} m_n^i &= \tilde{e}_{ij} \text{ and} \\ [m_n^i, y_p] &= \bar{e}_{ij} \end{aligned}$$

holds. Therefore there exists some fixed $n \in \mathbb{N}$ such that the map

$$\pi_0 : A = (M *_{P(A')} A') / \underbrace{\ker \Psi_n}_{\rightarrow} \rightarrow M$$

defined via

$$\pi_0|_M = \text{id} \text{ and } \pi_0(x_i) = m_n^i$$

extends to a retraction $\pi : A \rightarrow M$. Hence W_{t+1} has the structure of a virtually abelian flat over W_t .

Now assume that the virtually abelian vertex a is already contained in A_t . In this case we set $B'_{t+1} := B'_t$ and $W_{t+1} = W_t$.

(2) Assume that A_e is finite. By the construction of $\text{Comp}(L_{i+1})$ and \mathbb{B}_{i+1} there exists precisely one distinguished edge, say f , with edge group A_f (isomorphic to) A_e in \mathbb{B}_{i+1} (which corresponds to e_{t+1}). We set $A_w := A_{\alpha(e_{t+1})}$, $A_u := A_{\omega(e_{t+1})}$ and again have to distinguish several cases. Recall that by assumption $\alpha(e_{t+1}) \in A_t$.

(a) First suppose that A_u and A_w are either isolated QH vertex groups or isolated virtually abelian vertex groups in \mathbb{A} . In this case we set $\mathbb{B}'_{t+1} := \mathbb{B}'_t$ and $W_{t+1} := W_t$.

(b) Suppose that $A_u \in \mathbb{A}_t$.

- If A_w and A_u are rigid we replace f by a new edge f' (with edge group A_e) connecting the vertices u' and w' of \mathbb{B}'_t containing $\iota_{i+1} \circ \eta_i(A_u)$ and $\iota_{i+1} \circ \eta_i(A_w)$ respectively. The boundary monomorphisms are given by

$$\alpha_{f'} = \iota_{i+1} \circ \eta_i \circ \alpha_{e_{t+1}} : A_e \rightarrow B_{v'}$$

and

$$\omega_{f'} = \iota_{i+1} \circ \eta_i \circ \omega_{e+1} : A_e \rightarrow B_{w'}.$$

- If A_w and A_u are QH vertices of one of the JSJ decompositions \mathbb{A}_t^j , i.e. in particular not extensions of closed orbifold groups, we replace f by a new edge f' (with edge group A_e) connecting the vertices w' and u' of \mathbb{B}'_t whose vertex groups are the isomorphic copies of A_w and A_u respectively, which have been added earlier in the process. Denote by $\tau_1 : A_w \rightarrow B_{w'}$ and $\tau_2 : A_u \rightarrow B_{u'}$ the corresponding canonical isomorphisms. Then the boundary monomorphisms of f' are given by $\alpha_{f'} = \tau_1 \circ \alpha_{e+1} : A_e \rightarrow B_{w'}$ and $\omega_{f'} = \tau_2 \circ \omega_{e+1} : A_e \rightarrow B_{u'}$.
- Suppose that either A_u is rigid and A_w is a QH vertex or vice versa. Without loss of generality we assume that A_w is rigid. Then again we replace f by a new edge f' (with edge group A_e) connecting the vertex w' of \mathbb{B}'_t whose vertex group contains $\iota_{i+1} \circ \eta_i(A_w)$ to the vertex whose vertex group is the isomorphic copy of A_u , which has been added earlier in the process. Denote by $\tau : A_u \rightarrow B_{u'}$ the corresponding canonical isomorphism. Then the boundary monomorphisms of f' are given by $\alpha_{f'} = \iota_{i+1} \circ \eta_i \circ \alpha_{e+1} : A_e \rightarrow B_{w'}$ and $\omega_{f'} = \tau \circ \omega_{e+1} : A_e \rightarrow B_{u'}$.
- Suppose that either A_w is rigid or a QH vertex of one of the non-trivial JSJ decompositions \mathbb{A}_t^j and A_u is an isolated QH or virtually abelian vertex group or vice versa. Without loss of generality we assume that A_w is rigid or a non-isolated QH vertex. Then again we replace f by a new edge f' (with edge group A_e) connecting the vertex w' of \mathbb{B}'_t whose vertex group contains $\iota_{i+1} \circ \eta_i(A_w)$ (or the isomorphic copy of A_w) with the vertex u' with vertex group $B_{u'} = A_u$. If A_w is a QH vertex group, denote by $\tau : A_w \rightarrow B_{w'}$ the corresponding canonical isomorphism. Then the boundary monomorphisms of f' are given by

$$\alpha_{f'} = \iota_{i+1} \circ \eta_i \circ \alpha_{e+1} : A_e \rightarrow B_{w'}$$

(or $\alpha_{f'} = \tau \circ \alpha_{e+1} : A_e \rightarrow B_{w'}$) and

$$\omega_{f'} = \omega_{e+1} : A_e \rightarrow A_u = B_{u'}.$$

(c) Suppose that $A_u \notin \mathbb{A}_t$.

- If A_u is a rigid vertex, there exists a vertex u' in \mathbb{B}'_t which contains $\iota_{i+1} \circ \eta_i(A_u)$ and we proceed as in the respective subcases (1),(3) and (4) of (b)
- If A_u is an isolated QH or virtually abelian vertex group, we either already have dealt with this edge in case (a) or we proceed as in the subcase (4) of (b).

- If A_u corresponds to a QH vertex in one of the non-trivial JSJ decompositions we replace f by a new edge f' together with a new vertex u' , where f' connects u' to the vertex v' which either contains $\iota_{i+1} \circ \eta_i(A_w)$ (in case that A_w is rigid), or the isomorphic copy of A_w (in the QH case), or the group A_w (in the isolated QH/Virtually abelian vertex case). The edge group $A_{f'}$ is isomorphic to A_e and the embeddings are constructed in the same way as in the various subcases of (b).

It is possible that this new graph of groups is not connected. In this case we add the edge, say f , which we have replaced before again, making the graph connected again. We then proceed with the iterative procedure until removing the edge f does no longer disconnect the graph and then remove f .

By now we have constructed a new graph of groups \mathbb{B}'_{t+1} whose fundamental group $W_{t+1} = \pi_1(\mathbb{B}'_{t+1})$ has the structure of a tower over $\text{Comp}(L_{i+1})$. It remains to construct the embedding $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$. To do this, we again distinguish several cases, depending on the edge e_{t+1} .

Recall that we started with a graph of groups \mathbb{B}'_t together with an embedding

$$\nu_t : \pi_1(\mathbb{A}_t) \hookrightarrow W_t = \pi_1(\mathbb{B}'_t),$$

which coincides on rigid vertex groups with $\iota_{i+1} \circ \eta_i$ up to conjugation by an element, which is either trivial or contained in $W_t \setminus \text{Comp}(L_{i+1})$. Moreover $w := \alpha(e_{t+1}) \in \mathbb{A}_t$ and A_e denotes the edge group of e_{t+1} with the corresponding embeddings into the adjacent vertex groups $\alpha : A_e \hookrightarrow A_w := A_{\alpha(e_{t+1})}$ and $\omega : A_e \hookrightarrow A_u := A_{\omega(e_{t+1})}$.

(1) Assume that A_e is infinite.

- Assume that e_{t+1} connects two rigid vertices in \mathbb{A} . Then there exists a maximal finite-by-abelian subgroup M_l^+ of $\text{Comp}(L_{i+1})$ into which $\iota_{i+1} \circ \eta_i(A_e)$ can be conjugated by an element $g \in \text{Comp}(L_{i+1})$. Suppose without loss of generality that we are in the case that W_{t+1} is the amalgamated product

$$W_{t+1} = W_t *_{M_l^+} (M_l^+ \oplus \mathbb{Z}^{|I_l|}).$$

Let $\{z_1, \dots, z_{|I_l|}\}$ be a basis of $\mathbb{Z}^{|I_l|}$. Moreover we assume without loss of generality that the edge e_{t+1} corresponds to the first element of the ordered set I_l .

- We first consider the case that $w = \alpha(e_{t+1}) \in \mathbb{A}_t$ and $u = \omega(e_{t+1}) \notin \mathbb{A}_t$. By induction hypothesis there exists $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_w} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_w}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = c_h \circ c_{g^{-1}z_1g} \circ \iota_{i+1} \circ \eta_i.$$

We first show that this is in fact a homomorphism. Let $a \in A_e$. Then for $\omega(a) \in A_u$ holds that $g(\iota_{i+1}(\eta_i(\omega(a))))g^{-1} \in M_l^+$ and hence commutes with z_1 . It follows that

$$\begin{aligned} c_h \circ c_{g^{-1}z_1g} \circ \iota_{i+1} \circ \eta_i(\omega(a)) &= c_h c_{g^{-1}g} \circ \iota_{i+1} \circ \eta_i(\omega(a)) \\ &= c_h \circ \iota_{i+1} \circ \eta_i(\alpha(a)) \\ &= \nu_t(\alpha(a)) \end{aligned}$$

and therefore ν_{t+1} is a well-defined homomorphism. Moreover $hg^{-1}z_1g \notin \text{Comp}(L_{i+1})$. It remains to show the injectivity of ν_{t+1} . Let

$$y = a_1 b_1 \cdots a_k b_k \in \pi_1(\mathbb{A}_{t+1}) \setminus \{1\}$$

be reduced with respect to the amalgamated product decomposition

$$\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e} A_u.$$

We show that

$$\begin{aligned} \nu_{t+1}(y) &= \nu_t(a_1) h g^{-1} z_1 g (\iota_{i+1} \circ \eta_i(b_1)) g^{-1} z_1^{-1} g h^{-1} \nu_t(a_2) \cdots \\ &\quad \cdots \nu_t(a_k) h g^{-1} z_1 g (\iota_{i+1} \circ \eta_i(b_k)) g^{-1} z_1^{-1} g h^{-1} \end{aligned}$$

is reduced with respect to the decomposition

$$W_{t+1} = W_t *_{M_l^+} (M_l^+ \oplus \mathbb{Z}^{|I_l|}).$$

Suppose not. Then either there exists $j \in \{1, \dots, k\}$ such that

$$g(\iota_{i+1} \circ \eta_i(b_j))g^{-1} \in M_l^+$$

or there exists $j \in \{2, \dots, k\}$ such that

$$gh^{-1}\nu_t(a_j)hg^{-1} \in M_l^+.$$

Suppose we are in the first case. There exists a non-trivial element $x \in \omega(A_e) \leq A_u$ such that $g(\iota_{i+1} \circ \eta_i(x))g^{-1} \in M_l^+$. Hence $\iota_{i+1} \circ \eta_i(x)$ and $\iota_{i+1} \circ \eta_i(b_j)$ lie in the same maximal finite-by-abelian subgroup $g^{-1}M_l^+g$ of $\text{Comp}(L_{i+1})$. But since $\iota_{i+1} \circ \eta_i$ is injective on A_u this implies that b_j and x lie in the same maximal finite-by-abelian subgroup of A_u . By Lemma 8.1 $\omega(A_e)$ is maximal finite-by-abelian in A_u and therefore $b_j \in \omega(A_e)$, a contradiction to the reducedness of y .

Now suppose we are in the second case. There exists a non-trivial element

$x \in \alpha(A_e) \leq A_w$ such that $g(\iota_{i+1} \circ \eta_i(x))g^{-1} \in M_t^+$. Hence $h^{-1}\nu_t(a_j)h$ and $\iota_{i+1} \circ \eta_i(x)$ lie in the same maximal finite-by-abelian subgroup of W_t . It follows that $h(\iota_{i+1} \circ \eta_i(x))h^{-1}$ and $\nu_t(a_j)$ lie in the same maximal finite-by-abelian subgroup of W_t . But since by the induction hypothesis

$$\nu_t(a_j) = h(\iota_{i+1} \circ \eta_i(a_j))h^{-1}$$

and $\iota_{i+1} \circ \eta_i$ is injective on A_w we have that a_j and x lie in the same maximal finite-by-abelian subgroup of A_w . Again Lemma 8.1 yields a contradiction to the reducedness of y .

- Now consider the case that $\alpha(e_{t+1}), \omega(e_{t+1}) \in \mathbb{A}_t$, i.e. $\pi_1(\mathbb{A}_{t+1})$ splits as an HNN-extension $\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e}$. By the induction hypothesis there exist elements $h_1, h_2 \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_w} = c_{h_1} \circ \iota_{i+1} \circ \eta_i|_{A_w}$$

and

$$\nu_t|_{A_u} = c_{h_2} \circ \iota_{i+1} \circ \eta_i|_{A_u}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(p) = h_2(\iota_{i+1} \circ \eta_i(p))g^{-1}z_1gh_1^{-1},$$

where p denotes the stable letter of the HNN-extension $\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e}$. It follows from similar arguments as in the amalgamated product case that ν_{t+1} is an injective homomorphism.

- (b) Assume that e_{t+1} connects a rigid and a QH vertex in \mathbb{A} . First assume that $\omega(e_{t+1}) = u \notin \mathbb{A}_t$ is the QH vertex. In this case

$$\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e} A_u$$

and

$$W_{t+1} = W_t *_{A_e} A_u.$$

By the induction hypothesis there exists $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_w} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_w}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = c_h.$$

It is a straightforward calculation that this defines in fact an injective homomorphism.

Now assume that the QH vertex is $\alpha(e_{t+1}) = w \in \mathbb{A}_t$. Then either

$$\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e} A_u$$

for some rigid vertex group A_u or

$$\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e} .$$

In both cases $W_{t+1} = W_t *_{A_e}$. Assume we are in the first case.

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = c_s \circ \iota_{i+1} \circ \eta_i|_{A_u},$$

where s is the stable letter of the HNN-extension $W_{t+1} = W_t *_{A_e}$. Clearly $s \notin \text{Comp}(L_{i+1})$. Once again it is straightforward to show that ν_{t+1} is an injective homomorphism.

Now assume that we are in the second case, i.e. $\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e}$. Denote by p the stable letter of this HNN-extension and by s the stable letter of the HNN-extension $W_{t+1} = W_t *_{A_e}$. By the induction hypothesis there exists $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_u} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_u}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(p) = hs.$$

As in the cases before one can easily check that this is an injective homomorphism.

- (c) Assume that e_{t+1} connects a rigid and a virtually abelian vertex in \mathbb{A} . First assume that $\omega(e_{t+1}) = u \notin \pi_1(\mathbb{A}_t)$. In this case

$$\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e} A_u$$

and $A_e^+ = P(A_u^+)$. Let M be the maximal virtually abelian subgroup of W_t containing $\nu_t(A_e)$. Then by construction

$$W_{t+1} = W_t *_M A,$$

where A is virtually abelian and contains A_u . Denote by $\tau : A_u \hookrightarrow A$ the corresponding embedding. Note that by construction

$$\tau(\omega(A_e)) = \iota_{i+1} \circ \eta_i(\omega(A_e)) \leq M \leq A.$$

By the induction hypothesis there exists $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_w} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_w}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = c_h \circ \tau.$$

One can immediately check that this is a homomorphism since

$$\begin{aligned} \nu_{t+1}(\alpha(A_e)) &= c_h \circ \iota_{i+1} \circ \eta_i(\alpha(A_e)) \\ &= c_h \circ \iota_{i+1} \circ \eta_i(\omega(A_e)) \\ &= c_h \circ \tau(\omega(A_e)) \\ &= \nu_{t+1}(\omega(A_e)). \end{aligned}$$

For the injectivity let $y = a_1 b_1 \cdots a_k b_k \in \pi_1(\mathbb{A}_{t+1}) \setminus \{1\}$ be reduced with respect to the amalgamated product decomposition

$$\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *__{A_e} A_u.$$

Suppose that

$$\nu_{t+1}(y) = \nu_t(a_1) h \tau(b_1) h^{-1} \nu_t(a_2) \cdots \nu_t(a_k) h \tau(b_k) h^{-1}$$

is not reduced with respect to the splitting $W_{t+1} = W_t *_M A$. Then either $h^{-1} \nu_t(a_j) h \in M$ for some $j \in \{2, \dots, k\}$ or $\tau(b_j) \in M$ for some $j \in \{1, \dots, k\}$. First suppose that $\tau(b_j) \in M$. By construction

$$A = (M *__{A_e} A_u) / \underline{\ker} \Psi_n,$$

where $(\Psi_n) \subset \text{Hom}(M *__{A_e} A_u, \Gamma)$ is a stable sequence, and hence since $b_j \notin A_e$ we have that $\tau(b_j) b_j^{-1} \in \underline{\ker} \Psi_n$. Once again by the construction of the stable

sequence (Ψ_n) this would imply that $b_j \in A_e$, a contradiction to the reducedness of y .

So suppose now that $h^{-1}\nu_t(a_j)h \in M$. Then there exists an element $x \in \omega(A_e) \leq A_u$ of infinite order, such that $\tau(x) = \iota_{i+1} \circ \eta_i(x) \in M$. Hence $h^{-1}\nu_t(a_j)h$ and $\iota_{i+1} \circ \eta_i(x)$ lie in the same maximal virtually abelian subgroup of W_t . It follows that $\nu_t(a_j)$ and $c_h \circ \iota_{i+1} \circ \eta_i(x)$ lie in the same maximal virtually abelian subgroup of W_t . We can consider x also as an element of A_w and since by the induction hypothesis $\nu_t|_{A_w} = c_h \circ \iota_{i+1} \circ \eta_i$ and ν_t is injective on $\pi_1(\mathbb{A}_t)$ we have that a_j and x lie in the same maximal virtually abelian subgroup of $\pi_1(\mathbb{A}_t)$. Hence $a_j \in A_e$, a contradiction to the reducedness of y .

Now assume that $w = \alpha(e_{t+1}) \in \mathbb{A}_t$ is the virtually abelian vertex. In this case $\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e} A_u$ for some rigid vertex group A_u or $\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e}$. In both cases $W_{t+1} = W_t$ has a decomposition as $W_{t+1} = H *_{M} A$, where A is a virtually abelian group containing A_w , M is the maximal virtually abelian subgroup containing $\iota_{i+1} \circ \eta_i(A_e)$ and $\iota_{i+1} \circ \eta_i(A_u) \leq \text{Comp}(L_{i+1}) \leq H$.

Suppose we are in the first case. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = \iota_{i+1} \circ \eta_i|_{A_u}.$$

Now suppose we are in the second case, i.e. $\pi_1(\mathbb{A}_{t+1}) = \pi_1(\mathbb{A}_t) *_{A_e}$. Let t be the stable letter of this HNN-extension. By induction hypothesis there exists an element $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t(A_u) = c_h \circ \iota_{i+1} \circ \eta_i.$$

We then define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(t) = \iota_{i+1} \circ \eta_i(t)h^{-1}.$$

In both cases one can check as before that ν_{t+1} is an injective homomorphism.

(2) Assume that A_e is finite. Then by Lemma 8.1 the vertex groups of the adjacent vertices of e_{t+1} are either rigid, QH subgroups, isolated QH vertex groups or isolated virtually abelian vertex groups. We again distinguish several cases.

(a) First assume that A_w is rigid and $u = \omega(e_{t+1}) \notin \mathbb{A}_t$, i.e. $\mathbb{A}_{t+1} = \mathbb{A}_t *_{A_e} A_u$. By construction there exists a distinguished edge f in \mathbb{B}_{t+1} , whose edge group is isomorphic to A_e (we ignore the isomorphism and consider them equal). Denote by $\alpha_f : A_e \rightarrow B_{v'} := B_{\alpha(f)}$ and $\omega_f : A_e \rightarrow B_{u'} := B_{\omega(f)}$ the corresponding

boundary monomorphisms. By the induction hypothesis there exists an element $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t(A_w) = c_h \circ \iota_{i+1} \circ \eta_i.$$

- Suppose that A_u is rigid. Note that then $\iota_{i+1} \circ \eta_i(A_u)$ is contained in $B_{\omega(f)}$. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_u}.$$

- Suppose that A_u is a QH vertex group, an isolated QH vertex group or an isolated virtually abelian vertex group. Then $B_{u'}$ is an isomorphic copy of A_u witnessed by the isomorphism $\tau : A_u \rightarrow B_{u'}$. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = c_h \circ \tau.$$

In both cases it follows straightforward from the construction that ν_{t+1} defines a monomorphism.

- (b) Suppose that A_w is either a QH vertex group in one of the non-trivial JSJ decompositions \mathbb{A}_j^i , an isolated QH vertex group or an isolated virtually abelian vertex group in \mathbb{A} . Assume moreover that $u = \omega(e_{t+1}) \notin \mathbb{A}_t$, i.e. $\mathbb{A}_{t+1} = \mathbb{A}_t *_{A_e} A_u$. By construction there exists a distinguished edge f in \mathbb{B}_{t+1} , whose edge group is isomorphic to A_e (we ignore the isomorphism and consider them equal). Denote by $\alpha_f : A_e \rightarrow B_{v'} := B_{\alpha(f)}$ and $\omega_f : A_e \rightarrow B_{u'} := B_{\omega(f)}$ the corresponding boundary monomorphisms

- Suppose that A_u is rigid. Note that then $\iota_{i+1} \circ \eta_i(A_u)$ is contained in $B_{\omega(f)}$. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = \iota_{i+1} \circ \eta_i|_{A_u}.$$

- Suppose that A_u is a QH vertex group, an isolated QH vertex group or an isolated virtually abelian vertex group. Then $B_{u'}$ is an isomorphic

copy of A_u witnessed by the isomorphism $\tau : A_u \rightarrow B_{u'}$. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}|_{A_u} = \tau.$$

Again in both cases it follows straightforward from the construction that ν_{t+1} defines a monomorphism which satisfies the claim.

- (c) Assume that A_w is rigid and $u = \omega(e_{t+1}) \in \mathbb{A}_t$, i.e. $\mathbb{A}_{t+1} = \mathbb{A}_t *_{A_e}$. Denote by t the stable letter of this HNN-extension. By construction there exists a distinguished edge f in \mathbb{B}_{t+1} , whose edge group is isomorphic to A_e (we ignore the isomorphism and consider them equal). Denote by $\alpha_f : A_e \rightarrow B_{v'} := B_{\alpha(f)}$ and $\omega_f : A_e \rightarrow B_{u'} := B_{\omega(f)}$ the corresponding boundary monomorphisms. Furthermore denote by s the generator of $\pi_1(\mathbb{B}_{t+1})$ corresponding to f . By the induction hypothesis there exists an element $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_w} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_w}.$$

- Suppose that A_u is rigid. Note that then $\iota_{i+1} \circ \eta_i(A_u)$ is contained in $B_{\omega(f)}$. By the induction hypothesis there exists an element $\tilde{h} \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_u} = c_{\tilde{h}} \circ \iota_{i+1} \circ \eta_i|_{A_u}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(t) = \tilde{h} s h^{-1}.$$

We show that this is a well-defined homomorphism. Let $x \in A_e$. Then

$$\begin{aligned} \nu_{t+1}(t \alpha_{e_{t+1}}(x) t^{-1}) &= \nu_{t+1}(t) h (\iota_{i+1} \circ \eta_i(\alpha_{e_{t+1}}(x))) h^{-1} \nu_{t+1}(t)^{-1} \\ &= \tilde{h} s h^{-1} h (\iota_{i+1} \circ \eta_i(\alpha_{e_{t+1}}(x))) h^{-1} h s^{-1} \tilde{h}^{-1} \\ &= \tilde{h} s (\iota_{i+1} \circ \eta_i(\alpha_{e_{t+1}}(x))) s^{-1} \tilde{h}^{-1} \\ &= \tilde{h} (\iota_{i+1} \circ \eta_i(\omega_{e_{t+1}}(x))) \tilde{h}^{-1} \\ &= \nu_{t+1}(\omega_{e_{t+1}}(x)) \end{aligned}$$

and hence we have shown that ν_{t+1} is a homomorphism. The injectivity is now an easy computation.

- Suppose that A_u is a QH vertex group, an isolated QH vertex group or an isolated virtually abelian vertex group. Then $B_{u'}$ is an isomorphic copy of A_u witnessed by the isomorphism $\tau : A_u \rightarrow B_{u'}$. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(t) = sh^{-1}.$$

As in the case before one can show that ν_{t+1} is an injective homomorphism.

- (d) Suppose that A_w is either a QH vertex group in one of the non-trivial JSJ decompositions \mathbb{A}_j^i , an isolated QH vertex group or an isolated virtually abelian vertex group in \mathbb{A} . Assume moreover that $u = \omega(e_{t+1}) \in \mathbb{A}_t$, i.e. $\mathbb{A}_{t+1} = \mathbb{A}_t *_{A_e}$. Denote by t the stable letter of this HNN-extension. By construction there exists a distinguished edge f in \mathbb{B}_{t+1} , whose edge group is isomorphic to A_e (we ignore the isomorphism and consider them equal). Furthermore denote by s the generator of $\pi_1(\mathbb{B}_{t+1})$ corresponding to f . Denote by $\alpha_f : A_e \rightarrow B_{v'} := B_{\alpha(f)}$ and $\omega_f : A_e \rightarrow B_{u'} := B_{\omega(f)}$ the corresponding boundary monomorphisms

- Suppose that A_u is rigid. Note that then $\iota_{i+1} \circ \eta_i(A_u)$ is contained in $B_{\omega(f)}$. By the induction hypothesis there exists an element $h \in (W_t \setminus \text{Comp}(L_{i+1})) \cup \{1\}$ such that

$$\nu_t|_{A_u} = c_h \circ \iota_{i+1} \circ \eta_i|_{A_u}.$$

We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(t) = hs.$$

- Suppose that A_u is a QH vertex group, an isolated QH vertex group or an isolated virtually abelian vertex group. Then $B_{u'}$ is an isomorphic copy of A_u witnessed by the isomorphism $\tau : A_u \rightarrow B_{u'}$. We define $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ by

$$\nu_{t+1}|_{\pi_1(\mathbb{A}_t)} = \nu_t$$

and

$$\nu_{t+1}(t) = s.$$

Again in both cases it follows straightforward from the construction that ν_{t+1} defines a monomorphism.

Therefore in all cases we have constructed an embedding $\nu_{t+1} : \pi_1(\mathbb{A}_{t+1}) \rightarrow W_{t+1}$ which coincides on rigid vertex groups with $\nu_{i+1} \circ \eta_i$ up to conjugation by an element which is either trivial or does not belong to $\text{Comp}(L_{i+1})$.

After finitely many steps of this iterative procedure we have constructed a graph of groups \mathbb{B}' with $W := \pi_1(\mathbb{B}')$ such that $L_i = \pi_1(\mathbb{A})$ embeds into W . For the purpose of a possible trial and error procedure for quantifier elimination similar to the torsion-free case in [Se7], we have to remove some vertices and edges from \mathbb{B}' .

Let Q be a non-isolated QH vertex group in \mathbb{A} and assume that $\eta_i(Q)$ has infinite intersection with $\pi_1(\mathbb{U})$ for some connected subgraph of groups \mathbb{U} of \mathbb{H}_{i+1} . Then $\eta_i(Q)$ splits as a subgraph of groups of the Dunwoody decomposition $\mathbb{D}^{(i+1)}$ of L_{i+1} and we denote the vertex groups of this graph of groups which belong to \mathbb{H}_{i+1} by P_1, \dots, P_k . Since \mathbb{H}_{i+1} is isomorphic to a subgraph of groups of \mathbb{B}_{i+1} , by the construction of \mathbb{B}' , there exists for all $s \in \{1, \dots, k\}$ a distinguished vertex group \tilde{P}_s in \mathbb{B}' isomorphic to P_s . We then remove all the vertices with vertex groups $\tilde{P}_1, \dots, \tilde{P}_k$ together with adjacent edges from \mathbb{B}' and call this new graph of groups \mathbb{B}_i . Note that \mathbb{B}_i is still connected and there exists an embedding $\nu_i : L_i \rightarrow \pi_1(\mathbb{B}_i)$ (namely the same embedding as before). This completes the proof of Theorem 8.2. \square

Definition 8.3. *Keeping the notation from the above construction, we say that*

$$\text{Comp}(L) := \text{Comp}(L_0)$$

is the completed limit group or the completion of the well-structured resolution $\text{Res}(L)$ and the sequence of completed limit groups $\text{Comp}(L_i)$, $1 \leq i \leq l$, together with the quotient maps $\text{Comp}(\eta_i)$ is the completed resolution of $\text{Res}(L)$, which we denote by $\text{Comp}(\text{Res}(L))$.

The following Lemma is now a straightforward computation, making use of the construction before.

Lemma 8.4. *$\text{Comp}(L)$ is a Γ -limit tower and for all $i \in \{0, \dots, l-1\}$ the quotient map $\eta_i : L_i \rightarrow L_{i+1}$ can be naturally extended to a quotient map $\text{Comp}(\eta_i) : \text{Comp}(L_i) \rightarrow \text{Comp}(L_{i+1})$.*

8.2 Closures of completions

The next lemma follows immediately from the construction of the completion.

Lemma 8.5. *Let $\text{Res}(L)$ be a well-structured resolution of a restricted Γ -limit group L and let $\text{Comp}(\text{Res}(L))$ be its completed resolution. Denote by $\iota : L \hookrightarrow \text{Comp}(L)$ the embedding of L into its completion. Then:*

- (1) *The completed resolution $\text{Comp}(\text{Res}(L))$ is well-structured.*

- (2) The rank of the completed resolution $\text{Comp}(\text{Res}(L))$ is at most the rank of the well-structured resolution $\text{Res}(L)$.
- (3) Any homomorphism $\varphi : L \rightarrow \Gamma$ that factors through the resolution $\text{Res}(L)$ can be extended to a homomorphism $\bar{\varphi} : \text{Comp}(L) \rightarrow \Gamma$ that factors through the completed resolution $\text{Comp}(\text{Res}(L))$, such that $\varphi = \bar{\varphi} \circ \iota$.

Given a Γ -limit tower T , we can associate a slightly larger tower to it, a so-called closure of T .

Definition 8.6. (1) Let $T = T_n \geq \dots \geq T_1 \geq T_0 = \Gamma$ be a Γ -limit tower. A tower closure of T is a Γ -limit tower $\text{Cl}(T)$ together with an embedding $\iota : T \rightarrow \text{Cl}(T)$ such that the following diagram commutes

$$\begin{array}{ccccccc}
 T = T_n & \geq & T_{n-1} & \geq & \dots & \geq & T_1 & \geq & T_0 = \Gamma \\
 \downarrow \iota & & \downarrow \iota_{n-1} & & & & \downarrow \iota_1 & & \downarrow \iota_0 \\
 \text{Cl}(T) = \bar{T}_n & \geq & \bar{T}_{n-1} & \geq & \dots & \geq & \bar{T}_1 & \geq & \bar{T}_0 = \Gamma
 \end{array}$$

where for all $i \in \{0, \dots, n-1\}$ the map $\iota_i = \iota|_{T_i}$ is an embedding and the horizontal lines are the tower decompositions. Moreover the following hold:

- If $T_i = T_{i-1} *_M A$ is a virtually abelian flat over T_{i-1} then $\bar{T}_i = \bar{T}_{i-1} *_{\iota(M)} \bar{A}$ is a virtually abelian flat over \bar{T}_{i-1} , where $\iota|_A : A \rightarrow \bar{A}$ is an embedding which is an isomorphism when restricted to M , such that $\iota(A)$ is a subgroup of finite index in \bar{A} and $Z_{\iota(M)}$ is a direct factor of $Z_{\bar{A}}$. Here we denote by $Z_{\bar{A}}$ and $Z_{\iota(M)}$ the maximal free abelian subgroups of \bar{A} and $\iota(M)$ respectively.
- If $T_i = T_{i-1} *_E$ is an HNN-extension along a finite subgroup, then $\bar{T}_i = \bar{T}_{i-1} *_E'$, where E' is a finite subgroup containing $\iota(E)$.
- If $T_i *_E H$ is an amalgamated product with a subgroup H of Γ along a finite group E , then $\bar{T}_i = \bar{T}_{i-1} *_E' V$, where E' is a finite group containing $\iota(E)$, V is a subgroup of Γ and $\iota(H) \leq V$.
- If T_i is an orbifold flat over T_{i-1} witnessed by the graph of groups \mathbb{A} with QH vertex group Q fitting into the short exact sequence

$$1 \rightarrow E \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1,$$

then \bar{T}_i is an orbifold flat over \bar{T}_{i-1} witnessed by the graph of groups \mathbb{B} with QH vertex group Q' fitting into the short exact sequence

$$1 \rightarrow E' \rightarrow Q' \rightarrow \pi_1(\mathcal{O}) \rightarrow 1,$$

where E' is a finite group containing $\iota(E)$ and $\iota(Q) \leq Q'$. Moreover the underlying graphs of \mathbb{A} and \mathbb{B} are isomorphic and edge groups of \mathbb{B} are finite index supergroups of the corresponding edge groups in \mathbb{A} .

- (2) Let $\text{Res}(L)$ be a well-structured resolution of a Γ -limit group L and $\text{Comp}(\text{Res}(L))$ its completed resolution. A closure of the resolution $\text{Res}(L)$, denoted by $\text{Cl}(\text{Res}(L))$, is defined as follows: We replace the completed limit group $\text{Comp}(L)$ by a tower closure of itself and all other limit groups which appear along the completed resolution by the tower closure which is induced by the tower closure of $\text{Comp}(L)$.

Definition 8.7. *Let*

$$\text{Cl}(\text{Res}(L)) : \text{Cl}(\text{Comp}(L)) = L_0 \xrightarrow{\pi_1} L_1 \xrightarrow{\pi_2} L_2 \xrightarrow{\pi_3} \dots \xrightarrow{\pi_k} L_k$$

be a closure of a well-structured resolution of a Γ -limit group L . Let G be a Γ -limit group and

$$\text{GCl}(\text{Res}(L)) : G = G_0 \xrightarrow{\eta_1} G_1 \xrightarrow{\eta_2} G_2 \xrightarrow{\eta_3} \dots \xrightarrow{\eta_k} G_k$$

a resolution of G with the following properties:

- (a) G_0 admits a decomposition as a graph of groups \mathbb{D}_0 with finite edge groups, such that one vertex group is isomorphic to L_0 and all other vertex groups H_1, \dots, H_l are isomorphic to subgroups of Γ .
- (b) For all $i \in \{1, \dots, k\}$, G_i admits a decomposition as a graph of groups \mathbb{D}_i which arises from the graph of groups \mathbb{D}_0 by replacing the vertex group L_0 by L_i and adjusting the adjacent boundary morphisms in the obvious way.
- (c) $\eta_i|_{L_i} = \pi_i$ for all $i \in \{1, \dots, k\}$.
- (d) $\eta_i|_{H_j} = \text{id}$ for all $j \in \{1, \dots, l\}$, for all $i \in \{1, \dots, k\}$.

Then we call $\text{GCl}(\text{Res}(L))$ a generalized closure of the resolution $\text{Res}(L)$ and

$$\text{GCl}(\text{Comp}(L)) := G$$

a generalized closure of the completion of L .

Closures of the completion of a well-structured resolution of a Γ -limit group L will appear in a natural way in the proof of the Generalized Merzlyakov's Theorem.

Definition 8.8. *Let $\text{Res}(L)$ be a well-structured resolution of a Γ -limit group L . We call a finite set $\{\text{Cl}(\text{Res}(L))_1, \dots, \text{Cl}(\text{Res}(L))_q\}$ of closures of $\text{Res}(L)$ a covering closure if for any homomorphism $\varphi : L \rightarrow \Gamma$ that factors through $\text{Res}(L)$ there exists an extension $\bar{\varphi} : \text{Cl}(\text{Comp}(L))_i \rightarrow \Gamma$ of φ that factors through $\text{Cl}(\text{Res}(L))_i$ for (at least) one index $i \in \{1, \dots, q\}$.*

In order to understand what it means for a finite set $\{\text{Cl}(\text{Res})_1, \dots, \text{Cl}(\text{Res})_q\}$ to be a covering closure we look at the following toy example.

Example 8.9. Let $F = F(a_1, a_2)$ and $L = \langle a_1, a_2, c \mid [a_2, z] \rangle$. Then

$$L = F *_Z \mathbb{Z}^2 = \langle a_1, a_2 \rangle *_{\{a_2=c_1\}} \langle c_1, c_2 \mid [c_1, c_2] \rangle$$

is a restricted F -limit group that admits a well-structured resolution $\text{Res}(L) : L \xrightarrow{\eta} F$. Clearly $\text{Comp}(L) = L$ and $\text{Comp}(\text{Res}(L)) = \text{Res}(L)$.

Let $C := \langle c_1, c_2 \rangle \leq L$, $Z = \langle c_1, z_2 \mid [c_1, z_2] \rangle \cong \mathbb{Z}^2$ and consider the homomorphism $f : C \rightarrow Z$ given by $f(c_1) = c_1$ and $f(c_2) = c_1^2 z_2^3$. Then since $\langle c_1, c_1^2 z_2^3 \rangle$ is of index 3 in Z , the map f defines a closure

$$\text{Cl}(L) = F *_{\{a_2=c_1\}} Z$$

of L . To the closure embedding f we can assign the following system of linear equation(s) with two variables and one equation:

$$\Sigma(x, y) : \quad x = 2 + 3y.$$

For any $p \in \mathbb{Z}$ the system of equations $\Sigma(p, y)$ (in one variable y) has a solution if and only if $p \in 2 + 3\mathbb{Z}$.

Let now $\varphi \in \text{Hom}(L, F)$ be a restricted homomorphism and assume that φ can be extended to a homomorphism $\tilde{\varphi} : \text{Cl}(L) \rightarrow F$. Clearly $\varphi(c_2) = a_2^p$ for some $p \in \mathbb{Z}$. Since φ can be extended to $\text{Cl}(L)$, this implies that

$$a_2^p = \varphi(c_2) = \tilde{\varphi} \circ f(c_2) = \tilde{\varphi}(c_1^2 z_2^3) = a_2^{2+3k}$$

for some $k \in \mathbb{Z}$. In particular the system of equations $\Sigma(p, y)$ has a solution.

On the contrary suppose we are given a finite index subgroup $U = m\mathbb{Z}$ of \mathbb{Z} . Then for every integer k the coset $k + U$ defines a closure embedding $h : C \rightarrow Z$ by $h(c_1) = c_1$ and $h(c_2) = c_1^k z_2^m$. A homomorphism $\varphi : L \rightarrow F$ can be extended to a homomorphism of $\text{Cl}_h(L) \rightarrow F$ if and only if $\varphi(c_2) = a_2^p$ for some $p \in k + U$ (here $\text{Cl}_h(L)$ denotes the closure given by h).

Therefore a set $\{\text{Cl}(\text{Res}(L))_1, \dots, \text{Cl}(\text{Res}(L))_q\}$ is a covering closure of L if and only if the union of the cosets

$$\bigcup_{i=1}^q k_i + U_i = \bigcup_{i=1}^q k_i + m_i \mathbb{Z}$$

associated to the (systems of equations of the) closure embeddings cover \mathbb{Z} .

The following observation is now immediate.

Remark 8.10. (1) Let $A = \langle a_1, \dots, a_m \rangle$ and $\bar{A} = \langle z_1, \dots, z_m \rangle$ be free abelian groups of rank m and $f : A \rightarrow \bar{A}$ be an embedding which maps A to a finite index subgroup

of \bar{A} . By the Elementarteilersatz we can assume that for all $i \in \{1, \dots, m\}$ there exists $k_i \in \mathbb{Z}$ such that $f(a_i) = k_i z_i$. Hence

$$f(A) \cong k_1 \mathbb{Z} \oplus k_2 \mathbb{Z} \oplus \dots \oplus k_m \mathbb{Z} =: U_f \leq \mathbb{Z}^m.$$

Let Γ be a torsion-free hyperbolic group and $\varphi : A \rightarrow \Gamma$ an arbitrary homomorphism. Clearly $\varphi(A) \leq \langle u \rangle \cong \mathbb{Z}$ for some element $u \in \Gamma$ with no root. For all $i \in \{1, \dots, m\}$ there exists $p_i \in \mathbb{Z}$ such that $\varphi(a_i) = u^{p_i}$. Then the following hold:

$$\begin{aligned} & \varphi \text{ can be extended to } \bar{A} \\ \iff & \text{ there exists a homomorphism } \bar{\varphi} : \bar{A} \rightarrow \Gamma \text{ such that } \varphi = \bar{\varphi} \circ f \\ \iff & u^{p_i} = \varphi(a_i) = \bar{\varphi} \circ f(a_i) = \bar{\varphi}(k_i z_i) = \bar{\varphi}(z_i)^{k_i} \text{ for all } i \in \{1, \dots, m\} \\ \iff & p_i \in k_i \mathbb{Z} \text{ for all } i \in \{1, \dots, m\} \\ \iff & (p_1, \dots, p_m) \in U_f. \end{aligned}$$

Hence given finitely many embeddings f_1, \dots, f_k of A into finite index abelian supergroups $\bar{A}_1, \dots, \bar{A}_k$ the following holds:

Any homomorphism $\varphi : A \rightarrow \Gamma$ can be extended to one of the groups $\bar{A}_1, \dots, \bar{A}_k$ if and only if the union $\bigcup_{i=1}^k U_{f_i}$ of the associated subgroups of \mathbb{Z}^m covers \mathbb{Z}^m .

(2) Let now $A = \langle c, a_1, \dots, a_m \rangle$ and $\bar{A} = \langle c, z_1, \dots, z_m \rangle$ be free abelian groups of rank $m+1$ and $f : A \rightarrow \bar{A}$ be an embedding which maps A to a finite index subgroup of \bar{A} and stays the identity on c . Let $\varphi : A \rightarrow \Gamma$ be an arbitrary homomorphism that maps c to an element $u \in \Gamma$ with no root. Again $\varphi(A) = \langle u \rangle \cong \mathbb{Z}$ and for all $i \in \{1, \dots, m\}$ there exists $p_i \in \mathbb{Z}$ such that $\varphi(a_i) = u^{p_i}$. Similar to case (1) there exist $K_f \in \mathbb{Z}^m$ and a subgroup $U_f \leq \mathbb{Z}^m$ such that $\varphi : A \rightarrow \Gamma$ can be extended to \bar{A} if and only if $(p_1, \dots, p_m) \in K_f + U_f$.

Hence given finitely many embeddings f_1, \dots, f_k of A into finite index abelian supergroups $\bar{A}_1, \dots, \bar{A}_k$ that stay the identity on c the following holds:

Any homomorphism $\varphi : A \rightarrow \Gamma$ can be extended to one of the groups $\bar{A}_1, \dots, \bar{A}_k$ if and only if the union $\bigcup_{i=1}^k K_{f_i} + U_{f_i}$ of the associated cosets of subgroups of \mathbb{Z}^m covers \mathbb{Z}^m .

We will use the ideas from this Remark later in the proof of the Generalized Merzlyakov's Theorem. But before we can start proving the theorem we have to define so-called test sequences for Γ -limit groups.

9 Test sequences

We are going to define test sequences for the completion of a good well-structured resolution of a Γ -limit group L . We are doing this iteratively by defining these test sequences from bottom to top of the Γ -limit tower associated to the completion. We

have to look at all the possible cases how the tower can be build, namely if we add virtually abelian or orbifold flats or we perform an HNN-extension or an amalgamated product along some finite subgroup. The idea of a test sequence (λ_n) for a Γ -limit tower

$$T = T_0 \geq T_1 \geq \dots \geq T_l = \Gamma$$

is the following:

Let G be a group containing T and $(\varphi_n) \subset \text{Hom}(G, \Gamma)$ a stable sequence of extensions of (λ_n) , which converges into the action of a Γ -limit group $L = G/\varinjlim \varphi_n$ on some real tree Y such that:

- $T \leq L$,
- L is one-ended relative T and
- T acts without a global fixed point on Y .

Then the splitting of T coming from the top level tower decomposition is "visible" in a decomposition of L associated to its action on the real tree Y . To achieve this we need to control which elements of $T = T_0$ act hyperbolically on Y , while we have to guarantee that all elements of T_1 act elliptically on T .

Definition 9.1. Let G, Γ be groups with fixed finite generating sets and $(\lambda_n) \subset \text{Hom}(G, \Gamma)$ be a sequence of homomorphisms. For $g, g' \in G$ we say that $\lambda_n(g)$ dominates the growth of $\lambda_n(g')$ if

$$\lim_{n \rightarrow \infty} \frac{|\lambda_n(g')|_\Gamma}{|\lambda_n(g)|_\Gamma} = 0.$$

For two subgroups $A, B \leq G$ we say that $\lambda_n|_A$ dominates the growth of $\lambda_n|_B$ if

$$\lim_{n \rightarrow \infty} \frac{|\lambda_n(b)|_\Gamma}{|\lambda_n(a)|_\Gamma} = 0$$

for all $a \in A \setminus \{1\}, b \in B \setminus \{1\}$.

We are now ready to define test sequences.

Definition 9.2. Let Γ be a non-elementary hyperbolic group,

$$L = T_m \geq T_{m-1} \geq \dots \geq T_0 = \Gamma$$

a Γ -limit tower and $(\varphi_n) \subset \text{Hom}(L, \Gamma)$ a stably injective sequence. For $i \in \{1, \dots, m\}$ let \mathbb{B}_i be the graph of groups which is given by the i -th level of the tower decomposition of L . We call (φ_n) a test sequence for the tower L if the following holds for any restricted Γ -limit group M_i which contains T_i and is one-ended relative T_i :

For any stably injective convergent sequence $(\Psi_n) \subset \text{Hom}(M_i, \Gamma)$ such that

- for all $n \in \mathbb{N}$, Ψ_n is an extension of $\varphi_n|_{T_i}$ which is short relative to T_i with respect to the word metric on Γ and
- T_i does not act with a global fixpoint on the real tree T into which the sequence (Ψ_n) converges,

the following hold:

- (1) If T_i has the structure of an orbifold flat over T_{i-1} with QH vertex group Q fitting into the short exact sequence

$$1 \rightarrow E \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1,$$

then M_i admits an orbifold flat decomposition \mathbb{A} over a group V , which has one vertex v with QH vertex group Q' fitting into the short exact sequence

$$1 \rightarrow E' \rightarrow Q' \rightarrow \pi_1(\mathcal{O}) \rightarrow 1$$

and several vertices with vertex groups V_1, \dots, V_r corresponding to the various orbits of point stabilizers in the action of M_i on T . Moreover the underlying graphs of \mathbb{A} and \mathbb{B}_i are isomorphic and the edge groups of \mathbb{A} are finite index supergroups of the corresponding edge groups in \mathbb{B}_i , and $T_{i-1} \leq V$.

- (2) If T_i has the structure of a virtually abelian flat $T_i = T_{i-1} *_C K$ over T_{i-1} then there exist finitely many Γ -limit quotients N_1, \dots, N_p of the Γ -limit group M_i and each N_i admits a decomposition \mathbb{D} relative T_i along finite groups with one distinguished vertex group U and all other vertex groups of \mathbb{D} are isomorphic to subgroups of Γ . Moreover U admits a decomposition as an amalgamated product \mathbb{A} of the form $U = V *_A A$, where the subgroup A is virtually abelian and contains the virtually abelian subgroup K as a subgroup of finite index, A_1 contains C as a subgroup of finite index and $T_{i-1} \leq V$.
- (3) If $T_i = T_{i-1} *_E K$ is an extension of T_{i-1} along a finite subgroup and K is isomorphic to a one-ended subgroup of Γ , then there exist finitely many Γ -limit quotients N_1, \dots, N_p of the Γ -limit group M_i and each N_i admits a decomposition \mathbb{D} relative T_i along finite groups with one distinguished vertex group U and all other vertex groups of \mathbb{D} are isomorphic to subgroups of Γ . Moreover U admits a splitting as a graph of groups \mathbb{A} of the form $U = V *_E P$, where E is a finite subgroup containing E' , $T_{i-1} \leq V$ and P is isomorphic to a subgroup of Γ containing K .
- (4) If $T_i = T_{i-1} *_E K$ is an extension of T_{i-1} along a finite subgroup then M_i admits a splitting \mathbb{A} as an HNN-extension $M_i = V *_E$, where E is a finite subgroup containing E' and $T_{i-1} \leq V$.

In all cases the induced decomposition of T_i inherited from \mathbb{A} yields a graph of groups which is isomorphic to \mathbb{B}_i .

9.1 Orbifold flats

Theorem 9.3. *Assume that G has the structure of an orbifold flat over some subgroup H with QH vertex group Q , and there exists a test sequence λ_n^H for H . Assume moreover that for all $n \in \mathbb{N}$ there exists a homomorphism $\Theta_n : Q \rightarrow \Gamma$ with non-virtually abelian image, such that (up to conjugacy) $\Theta_n(C) = \lambda_n^H(C)$ for all edge groups C adjacent to Q in the orbifold flat decomposition of G . Then there exists a test sequence λ_n for G .*

Before we start, we observe that extensions of some orbifold groups cannot appear as QH subgroups of Γ -limit groups. Recall that a QH subgroup is a finite extension of a compact hyperbolic cone-type 2-orbifold group.

Lemma 9.4. *Let L be a one-ended Γ -limit group and \mathbb{A} be a virtually abelian JSJ decomposition of L . Let Q be a finite extension of the fundamental group of a cone-type orbifold \mathcal{O} (possibly with boundary) such that the complement of the cone points \mathcal{O}^c of \mathcal{O} is a pair of pants, i.e. \mathcal{O} is either a sphere with 3 cone points, a disc with 2 cone points, an annulus with 1 cone point or a pair of pants. Then Q is not isomorphic to a QH subgroup of L . In particular if Q is a vertex group of \mathbb{A} , then Q is rigid.*

Proof. This follows from Proposition 5.12 and Proposition 5.20 in [GuLe]. \square

The following result for hyperbolic groups is well-known so we omit its proof.

Lemma 9.5. *Let Γ be a hyperbolic group. Then there exists $N > 0$ such that for all $g, h \in \Gamma$ the following holds:
If g has finite order and $\langle h, g \rangle$ is virtually cyclic, then $gh^N = h^N g$.*

Before we start constructing test sequences for orbifold flats, we will prove some preliminary results. Throughout the whole chapter we will not distinguish between a simple closed curve on an orbifold and the corresponding element of the fundamental group.

Lemma 9.6. *Let Q be the fundamental group of a compact hyperbolic cone-type orbifold \mathcal{O} , such that the complement \mathcal{O}^c of the cone points has Euler characteristic at most -2 or is a once-punctured torus. Let $\mu : Q \rightarrow \Gamma$ be a homomorphism such that the following hold:*

- $\mu(Q)$ is an infinite non-virtually abelian (i.e. non-virtually cyclic) subgroup of Γ ,
- μ is injective on boundary components and finite subgroups of Q and
- there exists a Γ -limit group H , a stably injective sequence $(\varphi_n) \subset \text{Hom}(H, \Gamma)$ and an epimorphism $\pi : Q \rightarrow H$ with non-virtually abelian image, such that $\mu = \varphi_n \circ \pi$ for some $n \in \mathbb{N}$.

- *There exists a collection of non-homotopic, non-boundary parallel, essential simple closed curves $C = \{c_1, \dots, c_m\}$ on \mathcal{O}^c , which get mapped to elements of infinite order by μ , such that every connected component Σ of $\mathcal{O} \setminus C$ is either a once-punctured sphere with 2 cone points of order 2, or the complement Σ^c of its cone points is a pair of pants and $\pi_1(\Sigma)$ is non-virtually abelian.*

Then there exists a collection $B = \{b_1, \dots, b_m\}$ of essential, non-homotopic, non-boundary parallel simple closed curves on \mathcal{O}^c such that the following hold:

- (1) *Every connected component Σ obtained by cutting \mathcal{O} along B is either a once-punctured sphere with 2 cone points of order 2, or its complement Σ^c of the cone points has Euler characteristic -1 and $\mu(\Sigma)$ is non-virtually abelian in Γ .*
- (2) *If x corresponds to a cone point on a connected component Σ of $\mathcal{O} \setminus B$, whose fundamental group is not isomorphic to D_∞ , and the curve b_i is a boundary component of Σ , then for all $k \geq 1$ either $x^k = 1$ or $\langle \mu(x^k), \mu(b_i) \rangle$ is non-virtually abelian.*

Remark 9.7. (a) *The assumptions of the lemma hold for example if Q is a QH subgroup of a one-ended Γ -limit group L and $\mu = \varphi_n \circ \pi|_Q$ where $\pi : L \rightarrow M$ is a quotient map onto a strict shortening quotient M and $(\varphi_n) \subset \text{Hom}(M, \Gamma)$ is a test sequence (i.e. in particular stably injective). In particular these assumptions are fulfilled in the setting of Theorem 9.3.*

- (b) *To guarantee that a collection of curves C as in the assumption of the lemma exists, we might have to precompose μ by a properly chosen automorphism $\beta \in \text{Mod}(Q)$. We then denote $\mu \circ \beta$ in abuse of notation again by μ .*

Proof. To improve readability we restrict ourselves to the case that the underlying surface of \mathcal{O} is orientable. The proof can easily be adapted to the non-orientable case though. Let $C = \{c_1, \dots, c_m\}$ be the collection of curves which exists by assumption.

Since Γ is hyperbolic there exist up to isomorphism only finitely many virtually cyclic subgroups of Γ . For every virtually cyclic subgroup of Γ , say A , fix a maximal cyclic subgroup Z_A of finite index. Then $|A : Z_A| =: k_A < \infty$ and there exist only finitely many of these indices. So let $K \in \mathbb{N}$ be a positive integer such that for every element g in a virtually cyclic subgroup A of Γ , g^K is contained in Z_A .

For every connected component Σ of $\mathcal{O} \setminus C$, whose fundamental group is not isomorphic to D_∞ , choose elements $x, y \in \pi_1(\Sigma)$ such that $w_\Sigma := [x^K, y^K] \neq 1$. Such elements exist since $\pi_1(\Sigma) \not\cong D_\infty$ and hence contains a non-abelian free group.

Since $\pi(Q)$ is non-virtually abelian there exists an automorphism $\alpha \in \text{Aut}(Q)$ such that the following hold:

- $\mu \circ \alpha(w_\Sigma) = \varphi_n \circ \pi \circ \alpha(w_\Sigma) \neq 1$ for all connected components Σ of $\mathcal{O} \setminus C$, whose fundamental groups are not isomorphic to D_∞ .

- If $l \geq 1$ and x is an element corresponding to a cone point on a connected component Σ of $\mathcal{O} \setminus C$, whose fundamental group is not isomorphic to D_∞ , such that $x^l \neq 1$ then

$$[\mu \circ \alpha(x^l), \mu \circ \alpha(c_i^p)] \neq 1$$

for all c_i corresponding to boundary components of Σ and all $p \leq N$, where N is chosen according to Lemma 9.5.

Let Σ be a connected component of $\mathcal{O} \setminus C$, such that $\pi_1(\Sigma) \neq D_\infty$ and suppose that $\mu \circ \alpha(\pi_1(\Sigma))$ is virtually abelian. It follows that

$$\mu \circ \alpha(w_\Sigma) = [\mu \circ \alpha(x)^K, \mu \circ \alpha(y)^K] \in [\mathbb{Z}, \mathbb{Z}] = 1,$$

a contradiction.

Now suppose that x corresponds to a cone point on Σ and let $l \geq 1$ such that $x^l \neq 1$. Since $[\mu \circ \alpha(x^l), \mu \circ \alpha(c_i^N)] \neq 1$ it follows from Lemma 9.5 that $\langle \mu \circ \alpha(x^l), \mu \circ \alpha(c_i) \rangle$ is non-virtually abelian in Γ . Now $B := \{\alpha(c_1), \dots, \alpha(c_m)\}$ is the desired collection of curves. \square

Proposition 9.8. *Let Q be the fundamental group of a compact hyperbolic cone-type orbifold \mathcal{O} , such that the complement \mathcal{O}^c of the cone points has Euler characteristic at most -2 or is a once-punctured torus. Let $\mu : Q \rightarrow \Gamma$ be a homomorphism such that the following hold:*

- $\mu(Q)$ is an infinite non-virtually abelian subgroup of Γ ,
- μ is injective on boundary components and finite subgroups of Q and
- there exists a Γ -limit group H , a stably injective sequence $(\varphi_n) \subset \text{Hom}(H, \Gamma)$ and an epimorphism $\pi : Q \rightarrow H$ with non-virtually abelian image, such that $\mu = \varphi_n \circ \pi$ for some $n \in \mathbb{N}$.

Then there exist two collections of essential, non-homotopic, non-boundary parallel, disjoint simple closed curves on \mathcal{O}^c : $A := \{a_1, \dots, a_n\}$ and $B := \{b_1, \dots, b_m\}$, and an automorphism $\alpha \in \text{Aut}(Q)$ with the following properties:

- (1) Each complement of the cone points of a connected component obtained by cutting \mathcal{O} along A has Euler characteristic -1 , and the homomorphism $\mu \circ \alpha : Q \rightarrow \Gamma$ maps the fundamental group of each of these connected components isomorphically onto a quasi-convex subgroup of Γ .
- (2) Each of the curves b_i intersects at least one of the curves a_j .
- (3) $A \cup B$ fills \mathcal{O} , i.e. $\mathcal{O}^c \setminus (A \cup B)$ is a disjoint collection of discs and annuli.

Proof. We again restrict ourselves to the case that the underlying surface of \mathcal{O} is orientable. Let B be the collection of curves given by Lemma 9.6. Hence each of the complements of the cone points of a connected component Σ obtained by cutting \mathcal{O} along B is a pair of pants and has non-virtually abelian image in Γ , or $\pi_1(\Sigma) \cong D_\infty$. Note that by construction every component of $\mathcal{O} \setminus B$ has at most 2 cone points.

Denote by H the fundamental group of such a component. Then either $H \cong F_2 = \langle a, b \rangle$, where a, b correspond to two boundary components, or $H \cong \mathbb{Z} * \mathbb{Z}_p = \langle a, c \mid c^p \rangle$, where a corresponds to a boundary component, while c corresponds to a cone point of order p , or $H \cong \mathbb{Z}_p * \mathbb{Z}_q = \langle x, y \mid x^p, y^q \rangle$, where x, y correspond to cone points of order p and q respectively and moreover either $p \neq 2$ or $q \neq 2$, or $H \cong D_\infty$.

Before we proceed with the proof we need the following criterion for elements in hyperbolic groups to be non-trivial. The equivalent result in free groups was proved by G. Baumslag in [G.Ba].

Lemma 9.9. *Let Γ be a hyperbolic group and $z \in \Gamma$ be an element of infinite order. Let moreover $g \in \Gamma$ be an element of the form*

$$g = a_0 z^{i_1} a_1 z^{i_2} a_2 \dots a_{n-1} z^{i_n} a_n,$$

where $n \geq 1$ and whenever $0 < k < n$ ($a_0, a_n = 1$ is allowed), $\langle a_k, z \rangle$ is not virtually cyclic. Then there exists $K \in \mathbb{N}$ (depending on z and $\max_i |a_i|_\Gamma$) such that $g \neq 1$ if $|i_k| \geq K$ for all $k \in \{1, \dots, n\}$.

Proof. This result is well-known. See for example Lemma 2.4 in [Ol]. □

Now we are ready to continue the proof of Proposition 9.8. Let \mathbb{B} be the graph of groups given by the decomposition of \mathcal{O} along the collection of simple closed curves $B = \{b_1, \dots, b_m\}$. Denote by $\varphi_1, \dots, \varphi_m$ the corresponding Dehn-twists along the edges of this decomposition corresponding to b_1, \dots, b_m respectively.

Lemma 9.10. *Let D be an arbitrary finite collection of essential simple closed curves on \mathcal{O} , such that each $d \in D$ has non-trivial intersection with at least one of the curves b_1, \dots, b_m . Then for sufficiently large powers n_1, \dots, n_m of the Dehn-twists $\varphi_1, \dots, \varphi_m$, the image of every $d \in D$ under $\mu \circ \varphi_m^{n_m} \circ \dots \circ \varphi_1^{n_1}$ is non-trivial in Γ .*

Proof. By assumption the Euler characteristic of \mathcal{O}^c is at most -2 or \mathcal{O}^c is a once-punctured torus.

We first assume that no fundamental group of a component of $\mathcal{O} \setminus B$ is isomorphic to D_∞ and consider the general case afterwards.

Let $d \in D$. We first consider the case that d intersects precisely one curve from B , say b_1 . Then d and b_1 are contained in a suborbifold Σ of \mathcal{O} such that either Σ^c is a once-punctured torus or a 4th-punctured sphere. Suppose that Σ^c is a 4th-punctured sphere. If d either corresponds to a cone point or a boundary component of \mathcal{O} or a curve

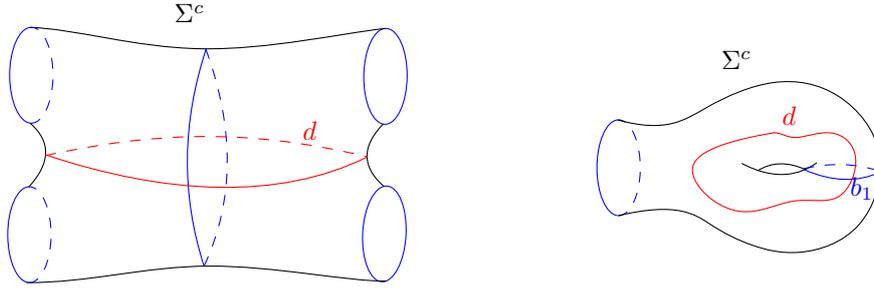


Figure 16: d and b_1 are contained in a suborbifold Σ of \mathcal{O} such that either Σ^c is a once-punctured torus or a 4th-punctured sphere

in B then $\mu(d) \neq 1$ by assumption. So we can assume that the connected components P_1^c and P_2^c of $\Sigma^c \setminus \{b_1\}$ are pairs of pants (see Figure 16) and there exist $p \geq 1$ and $s_0, \dots, s_p \in \pi_1(P_1) \cup \pi_1(P_2) \subset \pi_1(\mathcal{O})$ such that

$$d = [s_0, e^{\epsilon_1}, s_1, e^{\epsilon_2}, s_2, \dots, e^{\epsilon_p}, s_p]$$

is a \mathbb{B} -path in normal form for d , where $\epsilon_j \in \{\pm 1\}$ for all $j \in \{1, \dots, p\}$. Then $(P_1 \setminus \{s_i\})^c$ (respectively $(P_2 \setminus \{s_i\})^c$) is an annulus for all $i \in \{0, \dots, p\}$ and therefore $\{b_1, s_i\}$ is a generating set for $\pi_1(P_1)$ (respectively $\pi_1(P_2)$). Since $\mu(\pi_1(P_1))$ and $\mu(\pi_1(P_2))$ are non-virtually abelian by construction of the set B , it follows that $\langle \mu(b_1), \mu(s_i) \rangle$ is non-virtually abelian for all $i \in \{0, \dots, p\}$. Hence applying Lemma 9.9 to

$$\mu \circ \alpha(d) = \mu \circ \varphi_1^{n_1}(d) = \mu(s_0)\mu(b_1)^{\epsilon_1 n_1} \mu(s_1)\mu(b_1)^{\epsilon_2 n_1} \mu(s_2) \dots \mu(b_1)^{\epsilon_p n_1} \mu(s_p)$$

yields that $\mu \circ \alpha(d)$ is non-trivial for n_1 chosen large enough.

So now we assume that d is contained in a suborbifold Σ of \mathcal{O} such that Σ^c is a once-punctured torus and cutting Σ along b_1 yields an orbifold P , such that P^c is a pair of pants. By assumption d has non-trivial intersection with b_1 . If $d = b_1$ the claim is trivial, so let $p \geq 1$ and $s_0, \dots, s_p \in \pi_1(P) \subset \pi_1(\mathcal{O})$ such that

$$d = [s_0, e^{\epsilon_1}, s_1, e^{\epsilon_2}, s_2, \dots, e^{\epsilon_p}, s_p]$$

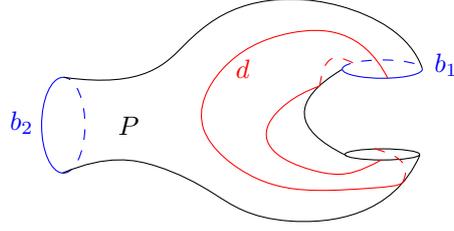
is a \mathbb{B} -path in normal form for d , where $\epsilon_j \in \{\pm 1\}$ for all $j \in \{1, \dots, p\}$. Then $(P \setminus \{s_i\})^c$ is an annulus for all $i \in \{0, \dots, p\}$ and therefore $\{b_1, s_i\}$ is a generating set for $\pi_1(P)$. Since $\mu(\pi_1(P))$ is non-virtually abelian by construction of the set B , it follows that $\langle \mu(b_1), \mu(s_i) \rangle$ is non-virtually abelian for all $i \in \{0, \dots, p\}$. Hence applying Lemma 9.9 to

$$\mu \circ \alpha(d) = \mu \circ \varphi_1^{n_1}(d) = \mu(s_0)\mu(b_1)^{\epsilon_1 n_1} \mu(s_1)\mu(b_1)^{\epsilon_2 n_1} \mu(s_2) \dots \mu(b_1)^{\epsilon_p n_1} \mu(s_p)$$

yields that $\mu \circ \alpha(d)$ is non-trivial for n_1 chosen large enough.

So from now on we can assume that d intersects at least two curves, say b_1, b_2 , from B . Let

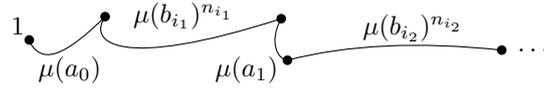
$$d = [a_0, e_1, a_1, e_2, a_2, \dots, e_p, a_p]$$


 Figure 17: d has non trivial intersection with b_1

be a \mathbb{B} -path in normal form for d . Then

$$\mu \circ \alpha(d) = \mu(a_0)\mu(b_{i_1})^{n_{i_1}}\mu(a_1)\mu(b_{i_2})^{n_{i_2}}\mu(a_2) \dots \mu(b_{i_p})^{n_{i_p}}\mu(a_p)$$

where $i_1, \dots, i_p \in \{1, \dots, m\}$. Let $j \in \{1, \dots, p-1\}$. Then $\{b_{i_{j-1}}, a_j\}$ and $\{a_j, b_{i_{j+1}}\}$ generate the respective fundamental group of a component of $\mathcal{O} \setminus B$ and therefore $\langle \mu(b_{i_{j-1}}), \mu(a_j) \rangle$ and also $\langle \mu(a_j), \mu(b_{i_{j+1}}) \rangle$ is non-virtually abelian. Therefore it follows


 Figure 18: The path corresponding to $\mu \circ \alpha(d)$ in $\text{Cay}(\Gamma)$

that for large enough powers of the Dehn-twists along the curves in B , the path in $\text{Cay}(\Gamma)$ corresponding to $\mu \circ \alpha(d)$ contains no essential backtracking and hence $\mu \circ \alpha(d)$ is non-trivial.

The only thing that remains to be considered is the case that the fundamental group, say H , of a connected component Σ of $\mathcal{O} \setminus B$ is isomorphic to D_∞ . In particular Σ is a once-punctured sphere with two cone points x, y of order 2 and

$$H = \langle x, y \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

Let as before

$$d = [a_0, e_1, a_1, e_2, a_2, \dots, e_p, a_p]$$

be a \mathbb{B} -path in normal form for d . Then

$$\mu \circ \alpha(d) = \mu(a_0)\mu(b_{i_1})^{n_{i_1}}\mu(a_1)\mu(b_{i_2})^{n_{i_2}}\mu(a_2) \dots \mu(b_{i_p})^{n_{i_p}}\mu(a_p)$$

where $i_1, \dots, i_p \in \{1, \dots, m\}$. Let $j \in \{1, \dots, p-1\}$ and suppose that

$$b_{i_{j-1}} = b_{i_j} \leq H \cong \mathbb{D}_\infty.$$

Then $b_{i_j} = xy$ and either $a_{i_{j-1}} = x$ or $a_{i_{j-1}} = y$ (see Figure 19). It follows that either

$$\mu(b_{i_{j-1}})^{n_{i_{j-1}}}\mu(a_{i_{j-1}})\mu(b_{i_j})^{n_{i_j}} = \mu((xy)^{n_{i_{j-1}}}x(yx)^{n_{i_j}}) = \mu(xy)^{n_{i_{j-1}}+n_{i_j}}\mu(x)$$

or

$$\mu(b_{i_{j-1}})^{n_{i_{j-1}}}\mu(a_{i_{j-1}})\mu(b_{i_j})^{n_{i_j}} = \mu((xy)^{n_{i_{j-1}}}y(yx)^{n_{i_j}}) = \mu(xy)^{n_{i_{j-1}}+n_{i_j}-1}\mu(x).$$

Hence there exists no essential backtracking in the path corresponding to

$$\mu(b_{i_{j-1}})^{n_{i_{j-1}}}\mu(a_{i_{j-1}})\mu(b_{i_j})^{n_{i_j}}$$

in $\text{Cay}(\Gamma)$ and therefore the proof from before generalizes to the case when there exists

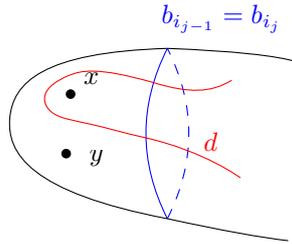


Figure 19: The case $H = \langle x, y \rangle \cong D_\infty$

a connected component Σ of $\mathcal{O} \setminus B$ with $\pi_1(\Sigma) \cong D_\infty$. □

The aim now is to choose a new collection A of simple closed curves on \mathcal{O} such that the fundamental groups of connected components after cutting along these curves get mapped isomorphically onto quasi-convex subgroups after precomposing μ by high powers of Dehn-twists along the collection of curves B .

Lemma 9.11. *There exists a collection A of simple closed curves on \mathcal{O} such that each of the components of $\mathcal{O}^c \setminus A$ has Euler characteristic -1 and the induced decomposition of the fundamental group of a component of $\mathcal{O} \setminus A$ with respect to \mathbb{B} is a finite graph of groups with trivial edge groups and finite or infinite cyclic vertex groups (the non-trivial finite ones corresponding to cone points, while the infinite cyclic ones correspond to boundary components of \mathcal{O}). In particular $A \cup B$ fills the surface \mathcal{O}^c .*

Proof. Denote by g the genus of the surface $S := \mathcal{O}^c$ and by r the number of boundary components of S . First assume that S is neither the sphere with 4 punctures nor the once-punctured torus, i.e. neither $g = 0, r = 4$, nor $g = 1, r = 1$. Let α and β be two non-homotopic distinct essential simple closed curves on S . We denote the geometric intersection number of α and β by $i(\alpha, \beta)$. Note that curves homotopic to the boundary are inessential.

Then the complex of curves $C(S)$ of the surface S is defined as follows: The k simplices of $C(S)$ are $(k + 1)$ -tuples of homotopy classes of distinct, essential simple closed curves, which have pairwise geometric intersection number zero. The 1-skeleton of $C(S)$ is usually referred to as the curve graph $C^1(S)$ of the surface S . There is one vertex in $C^1(S)$ for each homotopy class of essential simple closed curves in S and an edge between

two vertices α and β if and only if $i(\alpha, \beta) = 0$.

In the case that S is the 4-punctured sphere or the once-punctured torus, the above defined complex is just an infinite collection of vertices without any edges. Therefore in these cases we alter the definition of $C^1(S)$ in the following way: The vertices are still the homotopy classes of essential simple closed curves on S , while there is now an edge between two vertices α and β if and only if the geometric intersection number $i(\alpha, \beta)$ is minimal. In this case $C^1(S)$ is the well-known Farey-graph and in all cases Theorem 1.1 in [MaMi] shows that $C^1(S)$ has infinite diameter.

It is obvious that two (homotopy classes of) essential simple closed curves α and β which have distance ≥ 3 in $C^1(S)$ fill the surface, since an essential simple closed curve on S which has trivial intersection with both curves would yield a vertex in $C^1(S)$ connected to both α and β , contradicting the assumption on the distance. Moreover the collection of curves B is a subset of diameter ≤ 1 in $C^1(S)$. Hence the infinite diameter of $C^1(S)$ yields the claim. \square

We continue with the proof of Proposition 9.8. Recall that throughout the proof we do not distinguish between a simple closed curve on \mathcal{O} and the corresponding element of the fundamental group. Let A be the collection of curves chosen according to Lemma 9.11 and denote by \mathbb{A} the corresponding graph of groups decomposition of $\pi_1(\mathcal{O})$ induced by cutting \mathcal{O} along A .

So let $v \in VA$ with vertex group A_v and denote by \mathcal{O}_v the underlying orbifold. Then \mathcal{O}_v^c is a pair of pants and \mathcal{O}_v has at least 1 boundary component and at most 2 cone points. Hence $A_v = \langle x_1, x_2 \rangle$, where x_1 and x_2 either correspond to two boundary components or to two cone points. Let $n_1, \dots, n_m \in \mathbb{N}$ be large enough such that the conclusion of Lemma 9.10 holds for the set of curves A . We define $\alpha := \varphi_m^{n_m} \circ \dots \circ \varphi_1^{n_1}$. We assume that every fundamental group of a connected component of $\mathcal{O} \setminus B$ (i.e. the corresponding vertex group in \mathbb{B}) is non-virtually abelian, that is not isomorphic to D_∞ . If some vertex group of \mathbb{B} is isomorphic to D_∞ , we can fairly straightforwardly adjust the arguments of the following proof precisely as we did in the proof of Lemma 9.10 to also hold in this case.

First suppose that \mathcal{O}_v^c has two boundary components which are also boundary components of \mathcal{O}^c . Let x and y be the elements of A_v corresponding to the boundary components of \mathcal{O}_v^c which are also boundary components of \mathcal{O}^c . That is x (and also y) either corresponds to a cone point or to a proper boundary component of \mathcal{O} . In particular $A_v = \langle x \rangle * \langle y \rangle$. Since \mathcal{O}_v is a pair of pants and $A \cup B$ fill the orbifold \mathcal{O} there exists a curve, say b_1 , in B such that x and y lie in different components of $\mathcal{O}_v \setminus \{b_1\}$, as shown in Figure 20. For presentation purposes we assume that b_1 is the only curve in B which intersects \mathcal{O}_v non-trivially. The general case (i.e. if there are either multiple curves in B , or multiple times the same curve, which separate x and y in \mathcal{O}_v) follows from the arguments given in the other cases below.

Then y and b_1 correspond to boundary components of a component of $(\mathcal{O} \setminus B)^c$, say Σ^c . Since by the construction of the set of curves B , $\mu(\pi_1(\Sigma))$ is non-virtually abelian,

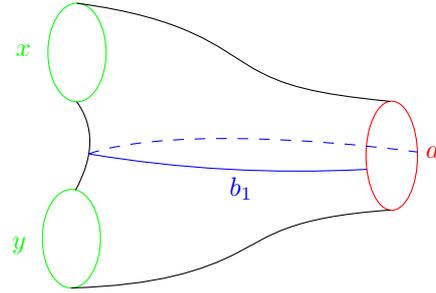


Figure 20: Two boundary components x, y of \mathcal{O}_v^c are boundary components of \mathcal{O}^c

it clearly follows that $\langle \mu(y), \mu(b_1) \rangle$ is non-virtually abelian. By the same argument it follows that $\langle \mu(x), \mu(b_1) \rangle$ is non-virtually abelian. Moreover μ is injective on $\langle x \rangle$ and $\langle y \rangle$. Lemma 9.6 yields that $\langle \mu(x^l), \mu(b_1) \rangle$ and $\langle \mu(y^k), \mu(b_1) \rangle$ is non-virtually abelian whenever $x^l \neq 1$ (respectively $y^k \neq 1$) and x, y correspond to cone points. If x corresponds to a boundary component of \mathcal{O} then $\langle \mu(x^l), \mu(b_1) \rangle$ is non-virtually abelian since $\langle \mu(x), \mu(b_1) \rangle$ is non-virtually abelian and elements of infinite order are contained in a unique maximal virtually abelian subgroup of Γ .

Let now $w \in A_v$ be a non-trivial element with normal form

$$w = x^{i_1} y^{j_1}, \dots, x^{i_k} y^{j_k}$$

with respect to the free decomposition $A_v = \langle x \rangle * \langle y \rangle$. We get that

$$\mu \circ \alpha(w) = \mu(x^{i_1}) \mu(b_1)^{n_1} \mu(y^{j_1}) \mu(b_1)^{-n_1} \mu(x^{i_2}) \dots \mu(b_1)^{n_1} \mu(y^{j_k}) \mu(b_1)^{-n_1}.$$

Hence after possibly enlarging n_1 there is no essential backtracking in the path corresponding to $\mu \circ \alpha(w)$ in $\text{Cay}(\Gamma)$ and therefore $\mu \circ \alpha$ is injective on A_v .

Now assume that \mathcal{O}_v^c has precisely one boundary component which is also a boundary component of \mathcal{O}^c and denote the corresponding curve on \mathcal{O}_v by x . Let a be an element of

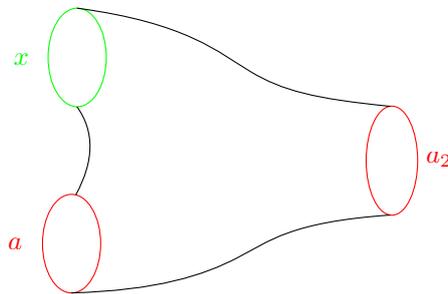


Figure 21: Precisely one boundary component of \mathcal{O}_v^c is a boundary component of \mathcal{O}^c

A_v corresponding to a boundary component of \mathcal{O}_v^c which is not a boundary component

of \mathcal{O}^c , i.e. a corresponds to a curve in A . Hence

$$A_v = \pi_1(\mathcal{O}_v) = \langle x, a \rangle = \langle x \rangle * \langle a \rangle.$$

Let $w = x^{k_1} a^{k_2} x^{k_3} \dots a^{k_l}$ be a non-trivial element of A_v . If either $w = x^{k_1}$ or $w = a^{k_1}$ then $\mu \circ \alpha(w) \neq 1$ since $\mu \circ \alpha$ is injective on $\langle x \rangle$ by assumption and on $\langle a \rangle$ by Lemma 9.10.

So we can assume that $l \geq 2$. First consider the case that $|B| = |\{b\}| = 1$, i.e. \mathcal{O}^c is a once punctured torus. Let Σ_B be the orbifold obtained from cutting \mathcal{O} along b . Let

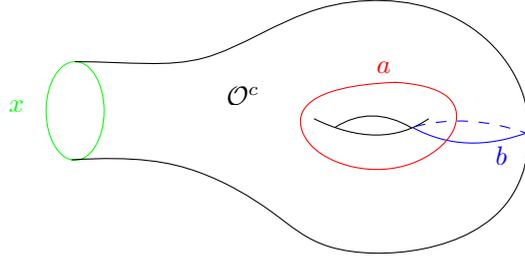


Figure 22: The case that \mathcal{O}^c is a once punctured torus

$a = [s_1, e^{\epsilon_1}, s_2, e^{\epsilon_2}, s_3, \dots, e^{\epsilon_{p-1}}, s_p]$ be a \mathbb{B} -path in normal form for a , where $\epsilon_j \in \{\pm 1\}$ for all $j \in \{1, \dots, p-1\}$. Then $(\Sigma_B \setminus \{s_i\})^c$ is an annulus for all $i \in \{1, \dots, p\}$ since a is a simple closed curve, and therefore $\{b, s_i\}$ is a generating set for $\pi_1(\Sigma_B)$. Moreover $s_i = b^{n_i} x$ for some $n_i \in \mathbb{Z}$.

Since $\mu(\pi_1(\Sigma_B))$ is non-virtually abelian it follows that $\langle \mu(b), \mu(s_i) \rangle$ is non-virtually abelian, for all $i \in \{1, \dots, p\}$. Moreover Lemma 9.6 yields for all $l \geq 1$, such that $x^l \neq 1$, that $\langle \mu(x)^l, \mu(b) \rangle$ is non-virtually abelian if $\mu(x)$ has finite order (i.e. x corresponds to a cone point on \mathcal{O}). If $\mu(x)$ has infinite order (i.e. x corresponds to a boundary component of \mathcal{O}), then clearly $\langle \mu(x), \mu(b) \rangle$ and therefore also $\langle \mu(x)^l, \mu(b) \rangle$ is non-virtually abelian for all $l \geq 1$, since every element of infinite order is contained in a unique maximal virtually abelian subgroup of Γ . Hence for n_1 chosen large enough, the path in $\text{Cay}(\Gamma)$ corresponding to

$$\begin{aligned} \mu \circ \alpha(w) &= \mu \circ \varphi_1^{n_1}(w) \\ &= \mu(x)^{k_1} \left(\mu(s_1) \mu(b)^{\epsilon_1 n_1} \mu(s_2) \mu(b)^{\epsilon_2 n_1} \mu(s_3) \dots \mu(b)^{\epsilon_{p-1} n_1} \mu(s_p) \right)^{k_2} \mu(x)^{k_3} \dots \\ &\quad \left(\mu(s_1) \mu(b)^{\epsilon_1 n_1} \mu(s_2) \mu(b)^{\epsilon_2 n_1} \mu(s_3) \dots \mu(b)^{\epsilon_{p-1} n_1} \mu(s_p) \right)^{k_l} \\ &= \mu(x)^{k_1} \left(\mu(b^{n_1}) \mu(x) \mu(b)^{\epsilon_1 n_1} \mu(s_2) \mu(b)^{\epsilon_2 n_1} \mu(s_3) \dots \mu(b)^{\epsilon_{p-1} n_1} \mu(b^{n_p}) \mu(x) \right)^{k_2} \mu(x)^{k_3} \\ &\quad \dots \left(\mu(b^{n_1}) \mu(x) \mu(b)^{\epsilon_1 n_1} \mu(s_2) \mu(b)^{\epsilon_2 n_1} \mu(s_3) \dots \mu(b)^{\epsilon_{p-1} n_1} \mu(b^{n_p}) \mu(x) \right)^{k_l} \end{aligned}$$

contains no essential backtracking and therefore $\mu \circ \alpha(w)$ is non-trivial. So now we can assume that $B = \{b_1, \dots, b_m\}$ for some $m \geq 2$. Let

$$a = [s_0, e_1^{\epsilon_1}, s_1, e_2^{\epsilon_2}, s_2, \dots, e_p^{\epsilon_p}, s_p]$$

be a \mathbb{B} -path in normal form for a , where $\epsilon_j \in \{\pm 1\}$ for all $j \in \{1, \dots, p\}$. To simplify notation we assume without loss of generality that e_i corresponds to the curve b_i in B . Note that a crosses at least two different simple closed curves $b_i, b_j \in B$. Hence

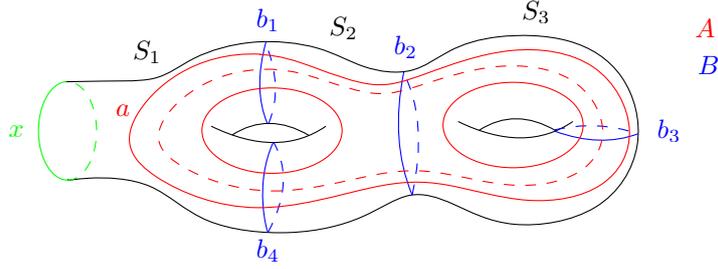


Figure 23: The curve a crosses multiple curves from B

$$\begin{aligned} \mu \circ \alpha(w) &= \mu(x)^{k_1} \left(\mu(s_0) \mu(b_1)^{\epsilon_1 n_1} \mu(s_1) \mu(b_2)^{\epsilon_2 n_2} \mu(s_2) \dots \mu(b_p)^{\epsilon_p n_p} \mu(s_p) \right)^{k_2} \mu(x)^{k_3} \dots \\ &\quad \left(\mu(s_0) \mu(b_1)^{\epsilon_1 n_1} \mu(s_1) \mu(b_2)^{\epsilon_2 n_2} \mu(s_2) \dots \mu(b_p)^{\epsilon_p n_p} \mu(s_p) \right)^{k_l}. \end{aligned}$$

Let $i \in \{1, \dots, p-1\}$. Since a is a simple closed curve and components of $(\mathcal{O} \setminus B)^c$ are pairs of pants, $\{b_i, s_i\}$ and $\{s_i, b_{i+1}\}$ generate the respective fundamental group of a component of $\mathcal{O} \setminus B$ and therefore $\langle \mu(b_i), \mu(s_i) \rangle$ and also $\langle \mu(s_i), \mu(b_{i+1}) \rangle$ are non-virtually abelian. Moreover by the same arguments as before we can assume that $s_0 = x b_1^k$, $s_p = b_p^{k'} x$ and $\langle \mu(x^l), \mu(b_1) \rangle$ and $\langle \mu(b_p), \mu(x^l) \rangle$ are non-virtually abelian for all $l \geq 1$ such that $x^l \neq 1$. Therefore it follows that there is no essential backtracking in the path

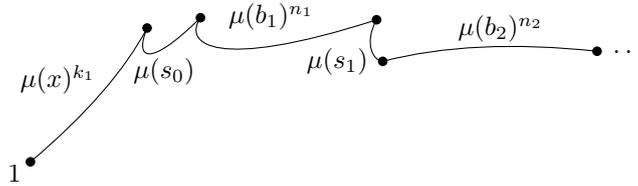


Figure 24: The path corresponding to $\mu \circ \alpha(w)$ in $\text{Cay}(\Gamma)$

corresponding to $\mu \circ \alpha(w)$ in $\text{Cay}(\Gamma)$ and hence $\mu \circ \alpha(w)$ is non-trivial for large enough powers of the Dehn-twists along the set of simple closed curves B .

So for the last case assume that no boundary component of \mathcal{O}_v^c is a boundary component of \mathcal{O} . In particular every boundary component of \mathcal{O}_v^c corresponds to one of the

curves in A and $A_v = \pi_1(\mathcal{O}_v) \cong F_2$. Without loss of generality let a_1 and a_2 be the curves corresponding to two of the boundary components of \mathcal{O}_v . Then clearly $A_v = \langle a_1, a_2 \rangle$. Let

$$a_1 = [s_0, e_1, s_1, e_2, s_2, \dots, e_p, s_p]$$

be a \mathbb{B} -path in normal form for a_1 and

$$a_2 = [r_0, \bar{e}_1, r_1, \bar{e}_2, r_2, \dots, \bar{e}_q, r_q]$$

be a \mathbb{B} -path in normal form for a_2 . Then

$$\mu \circ \alpha(a_1) = \mu(s_0)\mu(b_{i_1})^{n_{i_1}}\mu(s_1)\mu(b_{i_2})^{n_{i_2}}\mu(s_2) \dots \mu(b_{i_p})^{n_{i_p}}\mu(s_p)$$

and

$$\mu \circ \alpha(a_2) = \mu(r_0)\mu(b_{j_1})^{n_{j_1}}\mu(r_1)\mu(b_{j_2})^{n_{j_2}}\mu(r_2) \dots \mu(b_{j_q})^{n_{j_q}}\mu(r_q),$$

where $i_1, \dots, i_p, j_1, \dots, j_q \in \{1, \dots, m\}$. Let

$$w = a_1^{k_1} a_2^{l_1} \dots a_1^{k_p} a_2^{l_p}$$

be a reduced element in $F(a_1, a_2) \cong A_v$. Then

$$\begin{aligned} \mu \circ \alpha(w) &= \left(\mu(s_0)\mu(b_{i_1})^{n_{i_1}}\mu(s_1)\mu(b_{i_2})^{n_{i_2}}\mu(s_2) \dots \mu(b_{i_p})^{n_{i_p}}\mu(s_p) \right)^{k_1} \cdot \\ &\quad \left(\mu(r_0)\mu(b_{j_1})^{n_{j_1}}\mu(r_1)\mu(b_{j_2})^{n_{j_2}}\mu(r_2) \dots \mu(b_{j_q})^{n_{j_q}}\mu(r_q) \right)^{l_1} \cdot \dots \cdot \\ &\quad \left(\mu(r_0)\mu(b_{j_1})^{n_{j_1}}\mu(r_1)\mu(b_{j_2})^{n_{j_2}}\mu(r_2) \dots \mu(b_{j_q})^{n_{j_q}}\mu(r_q) \right)^{l_p}. \end{aligned}$$

Since a_1 is a simple closed curve, $\{b_{i_j}, s_j\}$ is a generating set of the fundamental group of

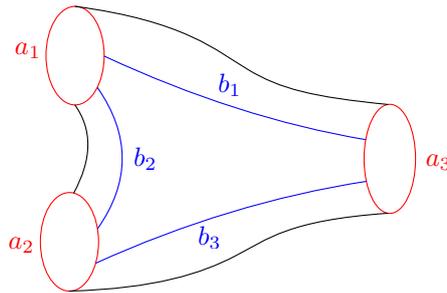


Figure 25: All boundary components of \mathcal{O}_v^c correspond to curves in A

a component of $\mathcal{O} \setminus B$ and hence it follows in particular that $\langle \mu(b_{i_j}), \mu(s_j) \rangle$ is non-virtually abelian for all $i \in \{1, \dots, p\}$. By the same arguments $\langle \mu(s_j), \mu(b_{i_{j+1}}) \rangle$, $\langle \mu(b_{j_k}), \mu(r_k) \rangle$ and $\langle \mu(r_k), \mu(b_{j_{k+1}}) \rangle$ are also non-virtually abelian.

In addition since $a_1 \neq a_2$ there exists $k \geq 1$ such that $e_k \neq \bar{e}_k$. The corresponding

elements b_{i_k} and b_{j_k} are two different boundary curves of a pair of pants which is a component of $\mathcal{O} \setminus B$. It follows in particular that $\langle \mu(b_{i_k}), \mu(b_{j_k}) \rangle$ is non-virtually abelian. Hence for appropriately chosen powers of the Dehn-twists along the set of simple closed curves B the path in $\text{Cay}(\Gamma)$ corresponding to $\mu \circ \alpha(w)$ contains no essential backtracking and therefore $\mu \circ \alpha$ is injective when restricted to A_v .

Hence in all cases the fundamental group of every component of $\mathcal{O} \setminus A$ gets mapped isomorphically onto a subgroup of Γ by $\mu \circ \alpha$, where α is the product of sufficiently high powers of Dehn-twists along the curves $b_1, \dots, b_m \in B$. By possibly further increasing the powers of the Dehn-twists we can furthermore guarantee that the images of the fundamental groups of components of $\mathcal{O} \setminus A$ are quasi-convex subgroups of Γ . We have therefore completed the proof of Proposition 9.8. \square

Now we are ready to proof Theorem 9.3.

Proof (Theorem 9.3). We first assume that Q is just the fundamental group of the cone-type orbifold and deal with the general case later.

Applying Proposition 9.8 to the homomorphisms $\Theta_n : Q \rightarrow \Gamma$ yields automorphisms $\alpha_n \in \text{Aut}(Q)$ and collections of simple closed curves $A = \{a_1, \dots, a_q\}$ and $B = \{b_1, \dots, b_t\}$ such that $\mu_n := \Theta_n \circ \alpha_n$ satisfies the conclusion of Proposition 9.8. We start by introducing some notation.

Let $\varphi_1, \dots, \varphi_q$ be the automorphisms of Q that correspond to Dehn-twists along the simple closed curves a_1, \dots, a_q , and let Ψ_1, \dots, Ψ_t be the automorphisms of Q corresponding to Dehn-twists along b_1, \dots, b_t . We define the following sequences of automorphisms of Q iteratively:

For all $n \in \mathbb{N}$ let $l^n \in \mathbb{N}^t, m^n \in \mathbb{N}^q$ be integer tuples. We set

- $\tau_1 := \text{id}$ and
- $\nu_1 := (\Psi_1)^{l_1^1} \circ (\Psi_2)^{l_2^1} \circ \dots \circ (\Psi_t)^{l_t^1}$.

For $n > 1$ we define:

- $\tau_n = (\varphi_1)^{m_1^n} \circ (\varphi_2)^{m_2^n} \circ \dots \circ (\varphi_q)^{m_q^n} \circ \nu_{n-1}$ and
- $\nu_n := (\Psi_1)^{l_1^n} \circ (\Psi_2)^{l_2^n} \circ \dots \circ (\Psi_t)^{l_t^n} \circ \tau_n$.

Let $\{y_1, \dots, y_s\}$ be a fixed finite generating set of Q . Let X be the Cayley graph of Γ with respect to some fixed finite generating set and Y the Cayley graph of Q with respect to $\{y_1, \dots, y_s\}$. Let (T_A, t_A) be the Bass-Serre tree corresponding to the decomposition \mathbb{A} of Q along the collection of simple closed curves A with base vertex t_A , and let (T_B, t_B) be the Bass-Serre tree corresponding to the decomposition \mathbb{B} of Q along the collection of simple closed curves B . We denote by d_X, d_Y, d_{T_A} and d_{T_B} the natural simplicial metric

on X, Y, T_A and T_B respectively.

For every element $g \in Q$ we set

$$l_A(g) = d_{T_A}(gt_A, t_A), \quad l_B(g) = d_{T_B}(gt_B, t_B).$$

If g acts hyperbolically on T_A we denote by $\text{tr}_A(g)$ the translation length of the action of g on T_A , and similarly if g acts hyperbolically on T_B we denote its translation length by $\text{tr}_B(g)$. For an element $f \in \Gamma$, let $\text{tr}(f)$ be the translation length of f under its action on X . Let

$$(a_0^{(y_i)}, e_1, a_1^{(y_i)}, e_2, \dots, e_{l(y_i)}, a_{l(y_i)}^{(y_i)})$$

be a \mathbb{B} -path in normal form for y_i , i.e. $[(a_0^{(y_i)}, e_1, a_1^{(y_i)}, e_2, \dots, e_{l(y_i)}, a_{l(y_i)}^{(y_i)})] = y_i$, that is the corresponding element in $\pi_1(\mathbb{B})$ is equal to y_i . Let

$$P^{\nu_1} = \left\{ [(a_0^{(y_i)}, e_1, a_1^{(y_i)}, e_2, \dots, e_j, a_j^{(y_i)})] \mid i \in \{1, \dots, s\}, j \in \{1, \dots, l(y_i)\} \right\}$$

be the set of all prefixes of the normal forms of the generators y_i . We set

$$R^{\tau_1} := 1$$

and

$$R^{\nu_1} := 2 \cdot \max_{u \in P^{\nu_1}} |u|_Y.$$

Moreover we define the constants R^{τ_n} and R^{ν_n} for $n \geq 2$ iteratively. For each $n \geq 2$ and $g \in Q$ for which $|g|_Y \leq R^{\nu_{n-1}}$ let

$$(a_0^{(\nu_{n-1}(g))}, e_1, a_1^{(\nu_{n-1}(g))}, e_2, \dots, e_{l(\nu_{n-1}(g))}, a_{l(\nu_{n-1}(g))}^{(\nu_{n-1}(g))})$$

be a \mathbb{A} -path in normal form for $\nu_{n-1}(g)$. Let

$$P^{\tau_n} = \left\{ [(a_0^{(\nu_{n-1}(g))}, e_1, a_1^{(\nu_{n-1}(g))}, e_2, \dots, e_j, a_j^{(\nu_{n-1}(g))})] \mid |g| \leq R^{\nu_{n-1}}, j \in \{1, \dots, l(\nu_{n-1}(g))\} \right\}$$

be the set of all prefixes of normal forms of elements $g \in Q$ for which $|g| \leq R^{\nu_{n-1}}$. We then define

$$R^{\tau_n} := 2 \cdot \max_{u \in P^{\tau_n}} |\nu_{n-1}^{-1}(u)|_Y.$$

Similarly for each $n \geq 2$ and $g \in Q$ for which $|g|_Y \leq R^{\tau_n}$ let

$$(a_0^{(\tau_n(g))}, e_1, a_1^{(\tau_n(g))}, e_2, \dots, e_{l(\tau_n(g))}, a_{l(\tau_n(g))}^{(\tau_n(g))})$$

be a \mathbb{B} -path in normal form for $\tau_n(g)$. Let

$$P^{\nu_n} = \left\{ [(a_0^{(\tau_n(g))}, e_1, a_1^{(\tau_n(g))}, e_2, \dots, e_j, a_j^{(\tau_n(g))})] \mid |g| \leq R^{\tau_n}, j \in \{1, \dots, l(\tau_n(g))\} \right\}$$

be the set of all prefixes of normal forms of elements $g \in Q$ for which $|g| \leq R^{\tau_n}$. We then define

$$R^{\nu_n} := 2 \cdot \max_{u \in P^{\nu_n}} |\tau_n^{-1}(u)|_Y.$$

We set for all $n \geq 2$:

- $HY^{\tau_n} = \{g \in Q \mid |g|_Y \leq R^{\tau_n} \wedge 0 < \text{tr}_A(\nu_{n-1}(g))\}$
- $NF^{\tau_n} = \{g \in Q \mid |g|_Y \leq R^{\tau_n} \wedge 0 < l_A(\nu_{n-1}(g))\}$

and for all $n \geq 1$:

- $HY^{\nu_n} = \{g \in Q \mid |g|_Y \leq R^{\nu_n} \wedge 0 < \text{tr}_B(\tau_n(g))\}$
- $NF^{\nu_n} = \{g \in Q \mid |g|_Y \leq R^{\nu_n} \wedge 0 < l_B(\tau_n(g))\}$.

Lemma 9.12. *We keep the notation from above. Let $\lambda_Q^{(n)} := \mu_n \circ \varphi_1^{e_n} \circ \dots \circ \varphi_q^{e_n} \circ \nu_n$. Then for all $n \in \mathbb{N}$ the integer tuples $l^n \in \mathbb{N}^t$, $m^n \in \mathbb{N}^q$ and $e_n \in \mathbb{N}$ can be chosen such that $(\lambda_Q^{(n)}, \tau_n, \nu_n)_{n \in \mathbb{N}}$ satisfy the following conditions:*

To simplify notation, for every index n and every $g \in NF^{\tau_n}$ let

$$\nu_{n-1}(g) = a_0^{(\nu_{n-1}(g))} a_1^{(\nu_{n-1}(g))} \dots a_{l(\nu_{n-1}(g))}^{(\nu_{n-1}(g))}$$

be the normal form of $\nu_{n-1}(g)$ with respect to \mathbb{A} . For every $h \in NF^{\nu_n}$ let

$$\tau_n(h) = a_0^{(\tau_n(h))} \dots a_{l(\tau_n(h))}^{(\tau_n(h))}$$

be the normal form of $\tau_n(h)$ with respect to \mathbb{B} .

(1) For $n > 1$ and every a_i , $1 \leq i \leq q$:

$$\text{tr}_B((a_i)^{m_i^n}) > 100 \cdot 2^n \cdot \max_{1 \leq i \leq q} l_B(a_i) \cdot \sum_{|g|_Y \leq R^{\tau_n}, j \leq l(\nu_{n-1}(g))} l_B(a_j^{(\nu_{n-1}(g))})$$

(2) For $n > 1$ and every b_i , $1 \leq i \leq t$:

$$\text{tr}_A((b_i)^{l_i^n}) > 100 \cdot 2^n \cdot \max_{1 \leq i \leq t} l_A(b_i) \cdot \sum_{|h|_Y \leq R^{\nu_n}, j \leq l(\tau_n(h))} l_A(a_j^{(\tau_n(h))})$$

(3) For every $n > 1$ and every $g, g' \in NF^{\tau_n}$:

$$\left| \frac{l_B(\tau_n(g)) l_A(\nu_{n-1}(g'))}{l_B(\tau_n(g')) l_A(\nu_{n-1}(g))} - 1 \right| < \frac{1}{100 \cdot q \cdot 2^n}$$

(4) For every $n > 1$ and every $g \in HY^{\tau_n}$:

$$\left| \frac{\text{tr}_B(\tau_n(g)) l_A(\nu_{n-1}(g))}{l_B(\tau_n(g)) \text{tr}_A(\nu_{n-1}(g))} - 1 \right| < \frac{1}{100 \cdot q \cdot 2^n}$$

(5) For every $n \geq 1$ and every $h, h' \in NF^{\nu_n}$:

$$\left| \frac{l_A(\nu_n(h))l_B(\tau_n(h'))}{l_A(\nu_n(h'))l_B(\tau_n(h))} - 1 \right| < \frac{1}{100 \cdot q \cdot 2^n}$$

(6) For every $n \geq 1$ and every $h \in HY^{\nu_n}$:

$$\left| \frac{\text{tr}_A(\nu_n(h))l_B(\tau_n(h))}{l_A(\nu_n(h))\text{tr}_B(\tau_n(h))} - 1 \right| < \frac{1}{100 \cdot q \cdot 2^n}$$

(7) There exist constants $c_1, c_2 > 0$, so that for every $n \geq 1$ and every $h, h' \in NF^{\nu_n}$:

$$c_1 < \frac{l_A(\nu_n(h))d_X(\lambda_Q^{(n)}(h'), 1)}{d_X(\lambda_Q^{(n)}(h), 1)l_A(\nu_n(h'))} < c_2$$

(8) There exist constants $c_3, c_4 > 0$, so that for every $n \geq 1$ and every $h \in HY^{\nu_n}$:

$$c_3 < \frac{\text{tr}_A(\nu_n(h))d_X(\lambda_Q^{(n)}(h), 1)}{\text{tr}_X(\lambda_Q^{(n)}(h))l_A(\nu_n(h))} < c_4$$

(9) For every index n , the homomorphism $\lambda_Q^{(n)} : Q \rightarrow \Gamma$ cannot be extended to an homomorphism $\gamma_n : Q_1 \rightarrow \Gamma$, such that $\lambda_Q^{(n)} = \gamma_n \circ \iota$ where $\iota : Q \rightarrow Q_1$ is an embedding of Q into the fundamental group of a cone-type orbifold \mathcal{O}_1 , finitely covered by the orbifold \mathcal{O} .

Moreover $\lambda_Q^{(n)}$ dominates the growth of λ_n^H .

Proof. Recall that \mathbb{A} is the graph of groups obtained by decomposing Q along the cyclic groups corresponding to the simple closed curves a_1, \dots, a_q .

(1) – (6) For the proof of the inequalities (1) – (6) we refer the reader to the proof of Claim 4.7 of [RiSe2].

(7) We claim that given any finite set of elements $M = \{m_1, \dots, m_k\} \subset Q$, there exist constants $c_1, c_2 > 0$ and some constant $e \in \mathbb{N}$ so that if $e_n > e$, for every $m_1, m_2 \in M$ for which both $\nu_n(m_1)$ and $\nu_n(m_2)$ do not fix the point $t_A \in T_A$, i.e. $l_A(\nu_n(m_i)) > 0$ for $i = 1, 2$, the following holds

$$c_1 < \frac{l_A(\nu_n(m_1))d_X(\lambda_Q^{(n)}(m_2), 1)}{d_X(\lambda_Q^{(n)}(m_1), 1)l_A(\nu_n(m_2))} < c_2.$$

To see this choose $e > 0$ large enough, such that for $e_n > e$ for all $j \in \{1, \dots, k\}$:

$$d_X(\lambda_Q^{(n)}(m_j), 1) = 2e_n \left(\sum_{i=1}^{l_A(\nu_n(m_j))} \mu_n(a_{p_i}) \right) + \epsilon_j$$

and $|\epsilon_j| < \frac{1}{1000}e_n$. Then for $e_n > e$ the following hold:

$$\begin{aligned} d_X(\lambda_Q^{(n)}(m_j), 1) &< 2 \cdot e_n \cdot l_A(\nu_n(m_j)) \cdot \max_{i \in \{1, \dots, q\}} \mu_n(a_i) + \epsilon_j \\ d_X(\lambda_Q^{(n)}(m_j), 1) &> 2 \cdot e_n \cdot l_A(\nu_n(m_j)) \cdot \min_{i \in \{1, \dots, q\}} \mu_n(a_i) + \epsilon_j \end{aligned}$$

Since $\epsilon_j < \frac{1}{1000}e_n$ for all $j \in \{1, \dots, k\}$

$$\begin{aligned} d_X(\lambda_Q^{(n)}(m_j), 1) &< 3 \cdot e_n \cdot l_A(\nu_n(m_j)) \cdot \max_{i \in \{1, \dots, q\}} \mu_n(a_i) \\ d_X(\lambda_Q^{(n)}(m_j), 1) &> e_n \cdot l_A(\nu_n(m_j)) \cdot \min_{i \in \{1, \dots, q\}} \mu_n(a_i) \end{aligned}$$

Applying both inequalities to

$$\frac{l_A(\nu_n(m_1))d_X(\lambda_Q^{(n)}(m_2), 1)}{d_X(\lambda_Q^{(n)}(m_1), 1)l_A(\nu_n(m_2))}$$

yields the desired upper and lower bounds c_1 and c_2 and inequality (7) follows, since c_1 and c_2 are independent of n and the finite set M . Note that for all $n \geq 1$ the sets NF^{ν_n} and HY^{ν_n} are finite sets.

(8) This inequality follows similar to (7).

(9) For the rest of the proof we fix some $n > 1$ and the sequence of automorphisms $\{\tau_n, \nu_n\}$. Suppose that there exists an increasing sequence of exponents $(e_j)_{j \in \mathbb{N}}$ for which the homomorphisms

$$\lambda^j := \mu_n \circ \varphi_1^{e_j} \circ \dots \circ \varphi_q^{e_j} \circ \nu_n$$

can be extended to the fundamental group Q_1 of a cone-type orbifold \mathcal{O}_1 finitely covered by \mathcal{O} . Since up to isomorphism there are only finitely many possibilities for a couple (Q, Q_1) , where $Q \leq Q_1$ and Q_1 is the fundamental group of a cone-type orbifold finitely covered by \mathcal{O} , after passing to a subsequence we may assume that all the homomorphisms $\{\lambda^j\}$ can be extended to the fundamental group Q_1 of a fixed orbifold \mathcal{O}_1 finitely covered by \mathcal{O} . So each homomorphism λ^j can be extended to a homomorphism $\gamma_j : Q_1 \rightarrow \Gamma$, such that $\lambda^j = \gamma_j \circ \iota$ where $\iota : Q \rightarrow Q_1$ is the canonical embedding.

By Theorem 3.8 the sequence (γ^j) converges into an action of Q_1 on a limit real tree (T, t_0) . Since each homomorphism γ^j is an extension of the homomorphism $\lambda^j : Q \rightarrow \Gamma$, and since by the structure of the sequence of homomorphisms (λ^j) the finite index subgroup $Q < Q_1$ acts discretely on the limit tree T , the group Q_1 also acts discretely on T , i.e. T is a simplicial tree.

Let $c_i := \nu_n^{-1}(a_i)$ for all $i \in \{1, \dots, q\}$. Since ν_n is an automorphism of Q , c_1, \dots, c_m are simple closed curves which induce a splitting of Q along cyclic subgroups, denoted by \mathbb{C} . Clearly every edge stabilizer in the action of Q on T is conjugate to a cyclic

subgroup generated by one of the c_i 's, hence each edge stabilizer in the action of Q_1 on T is conjugate to a cyclic subgroup containing the element corresponding to c_i in the orbifold fundamental group Q_1 .

Let \mathbb{A}_{Q_1} be the graph of groups corresponding to the action of Q_1 on T . By the structure of the homomorphisms λ^j , every vertex group of \mathbb{C} fixes a point in T , hence is contained in a vertex group of \mathbb{A}_{Q_1} .

We denote the covering map corresponding to the embedding $\iota : Q \rightarrow Q_1$ by $\pi : \mathcal{O} \rightarrow \mathcal{O}_1$. Since Q_1 is the fundamental group of the orbifold \mathcal{O}_1 and since \mathbb{A}_{Q_1} is a graph of groups with cyclic edge stabilizers, each vertex group in \mathbb{A}_{Q_1} is isomorphic to the fundamental group of a suborbifold of \mathcal{O}_1 . Let $\hat{\mathcal{O}}$ be a connected component of $\mathcal{O} \setminus \bigcup\{c_1, \dots, c_m\}$ and denote by $\hat{\mathcal{O}}^c$ the complement of the cone points of $\hat{\mathcal{O}}$. By construction $\chi(\hat{\mathcal{O}}^c) = -1$, and $\hat{Q} := \pi_1(\hat{\mathcal{O}})$ is conjugate to a vertex stabilizer in \mathbb{C} . Moreover \hat{Q} fixes a point in T so it can be conjugated into a vertex group of \mathbb{A}_{Q_1} . Let \hat{Q}_1 be this vertex group and let $\hat{\mathcal{O}}_1$ be the suborbifold of \mathcal{O}_1 for which $\hat{Q}_1 = \pi_1(\hat{\mathcal{O}}_1)$. The covering map $\pi : \mathcal{O} \rightarrow \mathcal{O}_1$ maps the suborbifold $\hat{\mathcal{O}}$ into the suborbifold $\hat{\mathcal{O}}_1$, and the boundary of $\hat{\mathcal{O}}^c$ is mapped to the boundary of $\hat{\mathcal{O}}_1^c$. Since in addition $\chi(\hat{\mathcal{O}}^c) = -1$, then, necessarily, π maps $\hat{\mathcal{O}}$ homeomorphically onto $\hat{\mathcal{O}}_1$.

Since the restriction of π to each suborbifold $\hat{\mathcal{O}}$ is a homeomorphism, and since every edge stabilizer in \mathbb{C} is also contained in some edge stabilizer of \mathbb{A}_{Q_1} , the orbifold \mathcal{O} must be a suborbifold of the orbifold \mathcal{O}_1 , such that \mathcal{O}^c has the same boundary components as \mathcal{O}_1^c , which clearly implies that $\mathcal{O} = \mathcal{O}_1$. Hence $Q = Q_1$, a contradiction.

Therefore we can choose an exponent e_n large enough such that the homomorphisms

$$\lambda_Q^{(n)} = \mu_n \circ \varphi_1^{e_n} \circ \dots \circ \varphi_q^{e_n} \circ \nu_n$$

satisfy properties (1) – (9) and moreover $\lambda_Q^{(n)}$ dominates the growth of λ_n^H . \square

Up to now we have only considered the case that Q is the fundamental group of a cone-type orbifold. So from now on we assume that Q is an extension of a finite group E by the fundamental group $\pi_1(\mathcal{O})$ of a cone-type orbifold \mathcal{O} , i.e. there exists a short exact sequence

$$1 \rightarrow E \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1.$$

If $\pi : Q \rightarrow \pi_1(\mathcal{O})$ is the canonical projection, then we say that an automorphism $\alpha \in \text{Aut}(\pi_1(\mathcal{O}))$ lifts to Q if there exists $\tilde{\alpha} \in \text{Aut}(Q)$ such that $\pi \circ \tilde{\alpha} = \alpha \circ \pi$.

Let \mathcal{H} be the collection of peripheral subgroups of $\pi_1(\mathcal{O})$ and $\tilde{\mathcal{H}}$ the corresponding collection of peripheral subgroups of Q , i.e. groups of the form $\pi^{-1}(Z)$, where Z corresponds to a boundary component of \mathcal{O} . Let $S \leq \text{Aut}_{\mathcal{H}}(\pi_1(\mathcal{O}))$ be the subgroup of those automorphisms that lift to $\text{Aut}_{\tilde{\mathcal{H}}}(Q)$. Then, by Lemma 4.17 in [ReWe], S has finite index in $\text{Aut}_{\mathcal{H}}(\pi_1(\mathcal{O}))$.

Hence there exists subsequences of the sequences of automorphisms $(\tau_n), (\nu_n)$ (still denoted $(\tau_n), (\nu_n)$), such that all τ_n and ν_n are in the same left coset $C := \beta \circ S$ of S . Therefore the automorphisms of the sequences $(\beta^{-1} \circ \tau_n)$ and $(\beta^{-1} \circ \nu_n)$ lie in S and

hence lift from $\pi_1(\mathcal{O})$ to Q and therefore the properties of Lemma 9.12 also hold for Q in the general case. We define the sequence $(\lambda_n) \subset \text{Hom}(G, \Gamma)$ by $\lambda_n|_H = \lambda_n^H$ and $\lambda_n|_Q = \lambda_Q^{(n)}$ for all $n \in \mathbb{N}$. It remains to show that the properties from Lemma 9.12 imply that (λ_n) is a test sequence for G .

We briefly recall our setup. G is a Γ -limit group which has the structure of an orbifold flat over H and $(\lambda_n) \subset \text{Hom}(G, \Gamma)$ is a sequence, such that $(\lambda_n|_Q)$ fulfills the properties of Lemma 9.12, where Q is the extension of a finite group by the fundamental group of a cone-type orbifold \mathcal{O} corresponding to the orbifold flat. Let \tilde{G} be a group which contains G as a subgroup and in abuse of notation let moreover $\lambda_n : \tilde{G} \rightarrow \Gamma$ be an extension of λ_n for all $n \in \mathbb{N}$. Then (a subsequence of) (λ_n) converges into the action of a Γ -limit group L on some real tree T , $G \leq L$ and we assume that G does not act with a global fixpoint on T . Assume moreover that L is one-ended relative G .

Let \mathbb{B} be the graph of groups corresponding to the orbifold flat, i.e. the following hold:

- There exists a vertex $v \in VB$ with a QH subgroup $Q = B_v$ as vertex group, which fits into the short exact sequence

$$1 \rightarrow E \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1.$$

- Every edge $e \in EB$ is of the form $e = (v, u)$ for some $u \in VB \setminus \{v\}$ and for every edge e with $\alpha(e) = v$, either there exists a peripheral subgroup O_e of $\pi_1(\mathcal{O})$ such that $\alpha_e(B_e)$ is in B_v conjugate to a finite index subgroup of $\pi^{-1}(O_e)$, where $\pi : B_v \rightarrow \pi_1(\mathcal{O})$ denotes the canonical projection, or B_e is finite. Moreover every boundary component of \mathcal{O} corresponds to an edge in such a way.
- H admits a decomposition as a graph of groups along finite subgroups, such that the vertex groups of this decomposition are precisely the vertex groups of the vertices in $VB \setminus \{v\}$.

Since $\lambda_n|_Q$ dominates the growth of $\lambda_n|_H$, every vertex group of \mathbb{B} except for Q acts elliptically on T . Since G and therefore also L is not necessarily one-ended relative to its elliptic subgroups we can not simply apply the Rips machine to analyze the action of L on T . In order to still be able to get information about the action of L on T we first prove the following:

Lemma 9.13. *Keeping the notation from before, let Q be a finite extension of the fundamental group of some cone-type orbifold \mathcal{O} , i.e. there exists a finite group E and a short exact sequence*

$$1 \rightarrow E \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1.$$

Assume that sequences of automorphisms of Q , $(\nu_n), (\tau_n)$ and homomorphisms $\lambda_n : Q \rightarrow \Gamma$ satisfy the properties of Lemma 9.12. Let $g, g' \in Q$ be elements of infinite order, which do not correspond to boundary components of \mathcal{O} , and suppose that

$$\max(d_Y(g, 1), d_Y(g', 1)) \leq R^{\nu_s}$$

for some $s \in \mathbb{N}$. Then there exist constants $C_{g,g'}^1, C_{g,g'}^2, C_g^3, C_g^4 > 0$ (depending on g, g') so that for every index $n > s + 1$ the following holds:

$$C_{g,g'}^1 < \frac{d_X(\lambda_n(g), 1)}{d_X(\lambda_n(g'), 1)} < C_{g,g'}^2 \quad (1)$$

$$C_g^3 < \frac{\text{tr}_X(\lambda_n(g))}{d_X(\lambda_n(g), 1)} < C_g^4 \quad (2)$$

$$C_{g,g'}^1 < \frac{\text{tr}_X(\lambda_n(g))}{\text{tr}_X(\lambda_n(g'))} < C_{g,g'}^2 \quad (3)$$

Proof. If $\max(d_Y(g, 1), d_Y(g', 1)) \leq R^{\nu_s}$ then

$$\max(d_Y(g, 1), d_Y(g', 1)) \leq \min(R^{\nu_n}, R^{\tau_n})$$

for every $n > s$. Let \mathcal{O}^c be the complement of the cone points of the orbifold \mathcal{O} . Since the curves $a_1, \dots, a_q, b_1, \dots, b_t$ were chosen to fill the surface \mathcal{O}^c , every element $u \in \pi_1(\mathcal{O})$ of infinite order, which does not correspond to a boundary component, acts hyperbolically on either the Bass-Serre tree T_A or the Bass-Serre tree T_B . Since in addition $\max(d_Y(g, 1), d_Y(g', 1)) \leq \min(R^{\tau_n}, R^{\nu_n})$ for every $n > s$, each of the elements g, g' belongs to either $HY^{\tau_{s+1}}$ or $HY^{\nu_{s+1}}$. Therefore the inequalities (3) – (8) in Lemma 9.12 hold for the couple (g, g') for every $n > s + 1$. Hence inequality (7) yields constants $c_1, c_2 > 0$ such that

$$c_1 < \frac{l_A(\nu_n(g'))d_X(\lambda_n(g), 1)}{d_X(\lambda_n(g'), 1)l_A(\nu_n(g))} < c_2.$$

It follows in particular that

$$\frac{d_X(\lambda_n(g), 1)}{d_X(\lambda_n(g'), 1)} < c_2 \cdot \frac{l_A(\nu_n(g))}{l_A(\nu_n(g'))}. \quad (4)$$

From (5) and (3) in Lemma 9.12 follows:

$$\frac{l_A(\nu_n(g))}{l_A(\nu_n(g'))} < \left(1 + \frac{1}{100 \cdot q \cdot 2^n}\right) \cdot \frac{l_B(\tau_n(g))}{l_B(\tau_n(g'))}$$

and

$$\frac{l_B(\tau_n(g))}{l_B(\tau_n(g'))} < \left(1 + \frac{1}{100 \cdot q \cdot 2^n}\right) \cdot \frac{l_A(\nu_{n-1}(g))}{l_A(\nu_{n-1}(g'))}.$$

If we plug in the above two inequalities in (4) we get:

$$\frac{d_X(\lambda_n(g), 1)}{d_X(\lambda_n(g'), 1)} < c_2 \cdot \left(1 + \frac{1}{100 \cdot q \cdot 2^n}\right)^2 \frac{l_A(\nu_{n-1}(g))}{l_A(\nu_{n-1}(g'))}.$$

So by induction:

$$\frac{d_X(\lambda_n(g), 1)}{d_X(\lambda_n(g'), 1)} < c_2 \cdot \left(1 + \frac{1}{100 \cdot q \cdot 2^n}\right)^n \frac{l_A(g)}{l_A(g')} < 2 \cdot c_2 \frac{l_A(g)}{l_A(g')}$$

which proves the existence of the desired upper bound. In a complete analogous way we get a lower bound and hence the first part of the claim follows. Plugging inductively the inequalities (4), (6) of Lemma 9.12 in inequality (8), we get as before the second part of the claim. Combining the first two inequalities yields the third one and the proof is complete. \square

By using Lemma 9.13 we will prove the following result, which describes the action of L on T . As mentioned before, if L is one-ended relative its elliptic subgroups, we could get the following a lot easier by simply applying the relative version of the Rips machine to the action of L on T .

Lemma 9.14. *L has a decomposition as a graph of groups \mathbb{A} such that the underlying graph of \mathbb{A} is isomorphic to the underlying graph of \mathbb{B} . Moreover every edge group of \mathbb{A} is a finite index supergroup of the corresponding edge group of \mathbb{B} . And for the QH vertex group Q of \mathbb{B} the corresponding vertex group in \mathbb{A} is a finite extension of the underlying orbifold of Q and contains Q as a subgroup of finite index.*

Proof. We are first going to show that every $g \in Q$ of infinite order, which does not correspond to a boundary component of the underlying orbifold acts hyperbolically on T . Let $\{y_1, \dots, y_m\}$ be a generating set for Q and

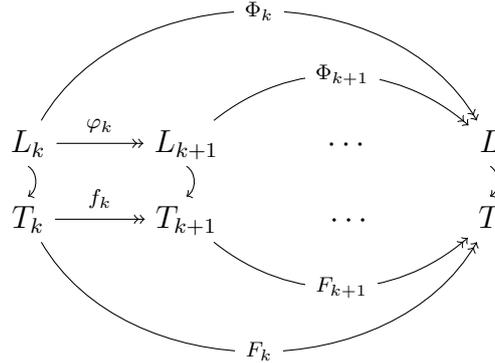
$$\mu_n = \max_{i=1, \dots, m} d_X(1, \lambda_n(y_i)) = \max_{i=1, \dots, n} |\lambda_n(y_i)|_X$$

for all $n \in \mathbb{N}$, where d_X denotes the metric on the Cayleygraph of Γ with respect to some fixed generating set X . Since G does not act with a global fixed point on T , $\mu_n \geq 1$. Assume without loss of generality that $\mu_n = |\lambda_n(y_1)|_X$. Suppose for the sake of contradiction that there exists some $g \in Q \setminus E$, not corresponding to a boundary component or a cone point, that acts elliptically on T . Then there exists some point t in the real tree T which is fixed by g . Let $(t_n)_{n \in \mathbb{N}} \in \text{Cay}(\Gamma, X)^{\mathbb{N}}$ be an approximating sequence for t . Recall that we denote the scaled Cayleygraph by $X_n = (\text{Cay}(\Gamma, X), d_n)$ where $d_n = \frac{d_X}{\mu_n}$. Then

$$\begin{aligned} 0 = d_T(t, gt) &= \lim_{n \rightarrow \infty} d_n(t_n, \lambda_n(g)t_n) \\ &= \lim_{n \rightarrow \infty} d_n(1, t_n^{-1} \lambda_n(g)t_n) \\ &= \lim_{n \rightarrow \infty} \frac{d_X(1, t_n^{-1} \lambda_n(g)t_n)}{\mu_n} \\ &\geq \lim_{n \rightarrow \infty} \frac{\text{tr}_X(\lambda_n(g))}{d_X(1, \lambda_n(y_1))} \\ &\stackrel{9.13(2)}{>} \lim_{n \rightarrow \infty} \frac{1}{C_{y_1}^3} \cdot \frac{\text{tr}_X(\lambda_n(g))}{\text{tr}_X(\lambda_n(y_1))} \\ &\stackrel{9.13(3)}{>} \frac{C_{g, y_1}^1}{C_{y_1}^3} > 0, \end{aligned}$$

a contradiction.

Let H_1, \dots, H_k be the vertex groups of \mathbb{B} which do not correspond to the QH vertex. Since $\lambda_n|_Q$ dominates the growth of $\lambda_n|_{H_i}$ for all $i \in \{1, \dots, k\}$, H_i fixes a point in the real tree T . Let $\mathcal{H} = \{H_1, \dots, H_k\}$. Then by Theorem 2.15 the pair action $(L, \mathcal{H}) \curvearrowright T$ is the strong limit of a direct system of geometric pair actions



such that

- φ_k and Φ_k are one-to-one in restriction to arc stabilizers of T_k .
- φ_k (resp. Φ_k) restricts to an isomorphism between \mathcal{H}_k and \mathcal{H}_{k+1} (resp. \mathcal{H}).
- The action of L_k on T_k splits as a graph of actions where each non-degenerate vertex action is either axial, thin or of orbifold type and therefore indecomposable, or is an arc containing no branch point except at its endpoints.

Since G is finitely presented relative \mathcal{H} there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $G \leq L_k$. Denote by T_{\min} the minimal G -invariant subtree of T and by T_{\min}^k the minimal G -invariant subtree of T_k for all $k \geq k_0$. Clearly the action of G on T_{\min} and on T_{\min}^k is minimal. Hence the direct system of pair actions $(G, \mathcal{H}) \curvearrowright T_{\min}^k$ converges strongly to $(G, \mathcal{H}) \curvearrowright T_{\min}$.

Denote by $T_{\min}^Q \subset T_{\min}$ the minimal Q -invariant subtree. Since every element of Q of infinite order, not corresponding to a boundary component, acts hyperbolically on T_{\min}^Q , it follows that the action of Q on T_{\min}^Q is of orbifold type, i.e. there exists a finite normal subgroup E of Q such that the action of Q/E is dual to an arational measured foliation on the underlying orbifold. In particular this action is geometric.

By Theorem 2.5 in [LePa] a geometric action of a finitely generated group cannot be a non-trivial strong limit of actions. This implies that there exists $k_1 \geq k_0$ such that F_k is an isometry when restricted to the minimal Q -invariant subtree of T_{\min}^k for all $k \geq k_1$. In abuse of notation for $k \geq k_1$ we denote this minimal Q -invariant subtree of T_k also by T_{\min}^Q .

Since the action of L_k on T_k splits as a graph of actions \mathbb{A}_k where each non-degenerate vertex action is either axial, thin or of orbifold type and therefore indecomposable, or is an arc containing no branch point except at its endpoints, T_{\min}^Q is contained in an orbifold component $T_{\mathcal{O}'}$ of this graph of actions. Denote by $Q' := \text{Stab}_{L_k}(T_{\mathcal{O}'})$ the vertex group of \mathbb{A}_k corresponding to this orbifold component. In particular $Q \leq Q'$.

Since Q' is the stabilizer of an orbifold component, there exists a finite subgroup E' of Q' and a short exact sequence

$$1 \rightarrow E' \rightarrow Q' \rightarrow \pi_1(\mathcal{O}') \rightarrow 1,$$

where E' is the kernel of the action of Q' on $T_{\mathcal{O}'}$ and the faithful action of Q'/E' on $T_{\mathcal{O}'}$ is dual to an arational measured foliation on \mathcal{O}' . From property (9) of Lemma 9.12 follows that the underlying orbifolds \mathcal{O} and \mathcal{O}' are identical and therefore Q has finite index in Q' .

It follows that the setwise stabilizer of $T_{\min}^Q \subset T$ is a group (also denoted by) Q' fitting into a short exact sequence

$$1 \rightarrow E' \rightarrow Q' \rightarrow \pi_1(\mathcal{O}) \rightarrow 1,$$

for some finite group E' containing E . In particular Q has finite index in $Q' = \text{Stab}_L(T_{\min}^Q)$.

We define a family \mathcal{Z} of subtrees of T consisting of:

- L -translates of T_{\min}^Q and
- closures of connected components of $T \setminus L \cdot T_{\min}^Q$.

Since in any of the approximation trees T_k for $k \geq k_1$, either $gT_{\min}^Q \cap T_{\min}^Q$ is at most one point or $g \in Q' = \text{Stab}_{L_k}(T_{\min}^Q)$, the same holds for $L \curvearrowright T$. Hence the family \mathcal{Z} gives rise to an equivariant covering of T such that any two subtrees intersect in at most one point. If we can show that every arc of T is covered by finitely many subtrees of \mathcal{Z} then \mathcal{Z} is precisely what V. Guirardel in [Gu] calls a transverse covering of the tree T . So suppose for the sake of contradiction that there exists some arc $I \subset T$ which is not covered by finitely many subtrees of \mathcal{Z} . Then there exists $k_2 \geq k_1$ and an arc $J \subset F_k^{-1}(I)$ such that F_k maps J isometrically onto I for all $k \geq k_2$. Let now $k \geq k_2$. Then the action of L_k on T_k splits as a graph of actions \mathbb{A}_k and therefore every finite subtree of T_k is covered by finitely many translates of the vertex trees of \mathbb{A}_k . In particular there exists a decomposition of J into non-trivial subsegments

$$J = J_1 \cup \dots \cup J_l$$

such that each J_i is contained in a unique translate of a vertex tree of \mathbb{A}_k . But this implies that J intersects only finitely many translates of T_{\min}^Q in a segment (i.e. not just in a point). But since F_k is an isometry when restricted to T_{\min}^Q , it follows that I also

intersects only finitely many translates of $T_{\min}^Q \subset T$ in more than a point. So assume that $I_0 \subset I$ is a non-trivial subsegment of I that intersects translates of T_{\min}^Q in at most one point. Then I_0 is contained in the closure of a connected component of $T \setminus L \cdot T_{\min}^Q$. Hence we conclude that I is covered by finitely many subtrees of \mathcal{Z} , a contradiction.

Therefore \mathcal{Z} is a transverse covering of T and it now follows from Lemma 1.5 in [Gu] that the action of L on T splits as a graph of actions \mathbb{A} such that the vertex trees are precisely the trees contained in \mathcal{Z} . Associated to this graph of actions there is a simplicial tree S on which L acts (in [Gu] this tree is called the skeleton of the graph of actions). For convenience we recall the definition of S . The vertex set $V(S)$ is $V_0(S) \cup V_1(S)$, where $V_1(S)$ contains a vertex x_Y for every subtree Y of \mathcal{Z} and $V_0(S)$ is the set of points $x \in T$ lying in the intersection of two distinct subtrees in \mathcal{Z} . There is an edge $e = (x, x_Y)$ between a vertex $x \in V_0(S)$ and a vertex $x_Y \in V_1(S)$ if and only if $x \in Y$.

The stabilizer of a vertex $x_Y \in V_1(S)$ is the setwise stabilizer of Y , while the stabilizer of a vertex $x \in V_0(S)$ is the stabilizer of the corresponding point $x \in T$. It follows from the discussion above that the vertex $v_{T_{\min}^Q} \in V_1(S)$ is stabilized by Q' .

Let $e = (x, v_{T_{\min}^Q})$ be an edge in S adjacent to $v_{T_{\min}^Q}$. Then

$$\text{Stab}_L(e) = \text{Stab}_L(x) \cap \text{Stab}_L(v_{T_{\min}^Q}) = \text{Stab}_L(x) \cap Q'.$$

Since L is one-ended relative G , either $\text{Stab}_L(e)$ corresponds to a boundary component of \mathcal{O} or is a finite group N containing a finite edge group adjacent to Q in \mathbb{B} . Moreover every edge group of an edge in \mathbb{B} adjacent to Q appears as a finite index subgroup of an edge group of \mathbb{A} adjacent to Q' . \square

Lemma 9.14 completes the proof of Theorem 9.3. \square

9.2 Extensions along finite groups

We start with the following definition.

Definition 9.15. Let $F_m = \langle a_1, \dots, a_m \rangle$ be a non-abelian free group.

- (a) Let $x = (x_1, \dots, x_k)$, $x_1, \dots, x_k \in F_m$. We call a word $w \in F_m$ a piece of x if w is either a subword of $x_i^{\pm 1}$ and $x_j^{\pm 1}$ for $i \neq j \in \{1, \dots, k\}$, or w appears twice in $x_j^{\pm 1}$ for some $j \in \{1, \dots, k\}$.
- (b) Let $0 < p < 1$. We say that $x = (x_1, \dots, x_k)$ satisfies the small cancellation property $C'(p)$ in F_m , if for any piece w of x , if w is a subword of say x_i , then we have $|w|_{F_m} < p \cdot |x_i|_{F_m}$.
- (c) Let $(x^{(n)})_{n \in \mathbb{N}}$ be a sequence of tuples $x^{(n)} = (x_1^{(n)}, \dots, x_k^{(n)}) \in F_m^k$. We say that $(x^{(n)})$ satisfies a very small cancellation property if for all $n \geq 1$, $x^{(n)}$ satisfies $C'(1/n)$ in F_m .

Theorem 9.16. *Suppose G has the structure of an amalgamated product $H *_E K$ where E is some finite group and K is isomorphic to a one-ended subgroup of Γ . Assume that there exists a test sequence (λ_n^H) for H . Assume moreover that for all $n \in \mathbb{N}$ there exists an embedding $\tau_n : K \rightarrow \Gamma$ such that $\lambda_n^H(E) = \tau_n(E)$ and $C_\Gamma(\tau_n(E))$ contains a non-abelian free subgroup which is quasi-convex in Γ . Then there exists a test sequence (λ_n) for G .*

Before we can start the proof, we need the following result about hyperbolic groups, which is well-known to the experts, so we only sketch a proof here.

Lemma 9.17. *Let Γ be a hyperbolic group and $E \leq E' \leq \Gamma$ two finite subgroups of Γ . Suppose that the centralizer $C_\Gamma(E)$ contains a quasi-convex non-abelian free group. Then one of the following holds:*

(a) $|C_\Gamma(E) : C_\Gamma(E')| < \infty$.

(b) $C_\Gamma(E) \setminus (C_\Gamma(E') \setminus \{1\})$ contains a quasi-convex non-abelian free group.

Proof. Let $C := C_\Gamma(E)$ and $C' := C_\Gamma(E')$ and suppose that $|C : C'| = \infty$. Then there exists a hyperbolic element $w \in C$ such that the diameter of the intersection of the axis $\text{Ax}(w)$ with the convex hull $\text{CH}(\Lambda(C'))$ of the limit set of C' is finite. Moreover there exists by assumption a quasi-convex non-abelian free group, say F , of infinite index in C . Hence again there exists a hyperbolic element $v \in C$ such that the diameter of the intersection of the axis $\text{Ax}(v)$ with the convex hull $\text{CH}(\Lambda(F))$ of the limit set of F is finite.

After possibly conjugating w or v , we can assume that the limit points of $\text{Ax}(w)$ and $\text{Ax}(v)$ are disjoint. Hence we can find an element $u \in C$ such that the path corresponding to the element $w^k u v^{-n}$ is a quasi-geodesic for large $k, n \in \mathbb{N}$. Therefore for k, n large enough $(w^k u v^{-n}) F (w^k u v^{-n})^{-1}$ is the desired quasi-convex non-abelian free group. \square

Proof (Theorem 9.16). Denote by $|\cdot|_\Gamma$ and $|\cdot|_G$ the respective word metric on Γ and G with respect to some fixed finite generating sets. Denote by X the Cayley graph of Γ with respect to this generating set. For all $n \in \mathbb{N}$ let $B_H(n) = \{h \in H \mid |h|_G \leq n\}$ and

$$L(n) = \max\{|\lambda_n^H(h)|_\Gamma \mid h \in B_H(n)\}.$$

Fix $n \in \mathbb{N}$ and denote by $C_n := C_\Gamma(\lambda_n^H(E))$ the centralizer of the image of E under the test sequence for H . We distinguish two cases:

First assume that for every finite group $E' \leq H$, containing E ,

$$C_n \setminus (C_\Gamma(\lambda_n^H(E')) \setminus \{1\})$$

contains a quasi-convex non-abelian free group. Let $F(a, b)$ be this quasi-convex non-abelian free subgroup of C_n which exists by assumption. Let $F = F(x, y) \leq F(a, b)$ be a non-abelian free subgroup which is quasi-convex in Γ such that any segment lying in

the 10δ -neighborhood of both axis $\text{Ax}(a)$ and $\text{Ax}(cbc^{-1})$ in X of a and any conjugate of b by an element $c \in \Gamma$ has length at most

$$\frac{1}{1000} \min\{|x|_\Gamma, |y|_\Gamma\}.$$

This can be achieved by taking $x = a^k$ and $y = b^k a^k$ for a large integer k , since it follows from Proposition 4.1 in [We] that there exists a bound on the length of how long the axis of a and b can lie in the 10δ -neighborhood of each other. For all $n \geq 1$ we define

$$w_n := xyxy^2xy^3x \cdots xy^{n \cdot L(n)}x$$

and a homomorphism $\lambda_n : G \rightarrow \Gamma$ by

- $\lambda_n(H) = \lambda_n^H(H)$ and
- $\lambda_n(K) = w_n \tau_n(K) w_n^{-1}$.

Since $w_n \in C_n$ this is a well-defined homomorphism. Moreover by the construction of w_n we have that

$$n \cdot |\lambda_n(h)|_\Gamma \leq n \cdot L(n) < |\lambda_n(k)|_\Gamma$$

for all $n \in \mathbb{N}$, $h \in B_H(n)$, $k \in K \setminus E$. Hence $(\lambda_n|_K)$ dominates the growth of $(\lambda_n|_H) = (\lambda_n^H)$. Note that (w_n) satisfies the small cancellation property $C'(1/n)$ for all $n \in \mathbb{N}$. For the second case assume that there exists some maximal finite subgroup $E' \leq H$, containing E such that

$$C_n \setminus (C_\Gamma(\lambda_n^H(E')) \setminus \{1\})$$

does not contain a quasi-convex non-abelian free group. By Lemma 9.17 this implies that $|C_n : C_\Gamma(\lambda_n^H(E'))| < \infty$. Let now $(\varphi_n) \subset \text{Hom}(E' *_E K, \Gamma)$ be a stably injective sequence. Then for n large enough

$$\varphi_n|_{E'} = c_{g_n} \circ \iota_n \circ \alpha_n$$

for one of finitely many embeddings $\iota_n : E' \rightarrow \Gamma$, an automorphism $\alpha_n \in \text{Aut}(E')$ and some element $g_n \in C_n$ (see [ReWe]). Since $\text{Aut}(E')$ is finite and $|C_n : C_\Gamma(\lambda_n^H(E'))| < \infty$ we can assume that (after passing to a subsequence) $\varphi_n|_{E'} = \lambda_n^H$. Hence we can extend the homomorphisms $\lambda_n^H : H \rightarrow \Gamma$ and $\varphi_n : E' *_E K \rightarrow \Gamma$ to a homomorphism

$$\mu_n : H *_E (E' *_E K) \rightarrow \Gamma.$$

Let $F(a, b)$ be the quasi-convex non-abelian free subgroup of $C'_n := C_\Gamma(\lambda_n^H(E'))$ which exists by assumption. Let again $F = F(x, y) \leq F(a, b)$ be a non-abelian free subgroup which is quasi-convex in Γ such that any segment lying in the 10δ -neighborhood of both axis $\text{Ax}(a)$ and $\text{Ax}(cbc^{-1})$ in X of a and any conjugate of b by an element $c \in \Gamma$ has length at most

$$\frac{1}{1000} \min\{|x|_\Gamma, |y|_\Gamma\}.$$

Let

$$L'(n) = \max\{|\mu_n(g)|_\Gamma \mid g \in H \cup (E' *_E K) \text{ with } |g|_G \leq n\}.$$

For all $n \geq 1$ we define

$$w_n := xyxy^2xy^3x \cdots xy^{n \cdot L'(n)}x$$

and a homomorphism $\lambda_n : G \rightarrow \Gamma$ by

- $\lambda_n|_H = \mu_n|_H$ and
- $\lambda_n|_{E' *_E K} = c_{w_n} \circ \mu_n|_{E' *_E K}$.

Since $w_n \in C'_n$ this is again a well-defined homomorphism. Moreover by the construction of w_n we have that

$$n \cdot |\lambda_n(h)|_\Gamma \leq n \cdot L'(n) < |\lambda_n(k)|_\Gamma$$

for all $n \in \mathbb{N}$, $h \in B_H(n)$, $k \in (E' *_E K) \setminus E'$. Hence $(\lambda_n|_{E' *_E K})$ dominates the growth of $(\lambda_n|_H) = (\lambda_n^H)$. Note that (w_n) satisfies the small cancellation property $C'(1/n)$ for all $n \in \mathbb{N}$. For simplicity we assume from now on that we are in the first case from above. The second one is similar.

Let \tilde{G} be a group containing G and in abuse of notation let $\lambda_n : \tilde{G} \rightarrow \Gamma$ be an extension of λ_n which is short relative to G for all $n \in \mathbb{N}$. Let T be the real tree into which (a convergent subsequence of) (λ_n) converges and let $L := \tilde{G}/\overline{\ker \lambda_n}$. Assume that L is one-ended relative G and moreover that G does not act elliptically on T . Since G does not fix a point when acting on T , there exists a non-trivial minimal G -invariant subtree T_{\min} of T on which G acts.

Lemma 9.18. *The following hold:*

- (1) T_{\min} is equivariantly isomorphic as a G -tree to the Bass-Serre tree of the splitting $G = H *_E K$.
- (2) For every $g \in L$ either $T_{\min} \cap gT_{\min}$ is at most one point or g is contained in the setwise stabilizer of T_{\min} .
- (3) T_{\min} intersects any indecomposable subtree of T in at most one point.
- (4) The setwise stabilizer of T_{\min} in L is $\langle H, E_L \rangle *_E \langle K, E_L \rangle$ for some finite group E_L , containing E .
- (5) Edges in T_{\min} are stabilized by conjugates of E_L and vertices by conjugates of either $\langle H, E_L \rangle$ or $\langle K, E_L \rangle$.

Proof. By Lemma 1.10(3) in [ReWe] for all $k \in K$, $(1)_{n \in \mathbb{N}}$ and $(\lambda_n(k)) = (w_n \tau_n(k) w_n^{-1})$ are approximating sequences for t_0 and kt_0 respectively. Hence there exists a point w on the arc $[t_0, kt_0]$ such that (w_n) is an approximating sequence for w . It follows that for all $k \in K$:

$$d_T(kw, w) = \lim_{n \rightarrow \infty} d_n(\lambda_n(k)w_n, w_n) = \lim_{n \rightarrow \infty} d_n(w_n \tau_n(k), w_n) = \lim_{n \rightarrow \infty} d_n(\tau_n(k), 1).$$

Here d_n denotes the scaled metric on the Cayleygraph of Γ . Since w_n dominates the growth of τ_n , K fixes the point w in the limit tree T and therefore acts elliptically on T . Since $\Gamma t_0 = t_0$ and $Kw = w$, it follows from Lemma 1.14 in [Gu] that T_{\min} is covered by translates of the convex hull of the points $\{kt_0 \mid k \in K\}$, i.e. translates of the arcs $[t_0, kt_0], k \in K$. The first claim is now obvious. For (2) – (5) we need the following lemma.

Lemma 9.19. *Let $I = [a, b] \subset [t_0, w]$ be an arc of $T_{\min} \subset T$. There exists a finite group $E_L \leq L$ such that for any $g \in L \setminus E_L$, gI intersects $[t_0, w]$ in at most one point. Moreover E_L stabilizes the segment $[1, w] \subset T_{\min}$, where w is the point in T approximated by (w_n) .*

Proof. We suppose that there exists some $g \in L$ such that $gI \cap [t_0, w]$ is non-trivial and not a point. Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be approximating sequences for a and b respectively, and $f \in \tilde{G}$, such that $\eta(f) = g$, where $\eta : \tilde{G} \rightarrow L$ is the limit quotient map. Let $u_n \in F(x, y)$ be the unique reduced word such that $b_n = a_n u_n$, i.e. the segment $[a_n, b_n] \subset (X, d_n)$ is labeled by the word u_n . Since $I \subset [t_0, w]$, u_n is a subword of w_n (since (w_n) is an approximating sequence of w). Clearly the segments $J_n := [\lambda_n(f)a_n, \lambda_n(f)b_n]$ approximate the segment gI and are labeled by u_n .

Let t_{a_n} and t_{b_n} be points on the segment $[1, w_n] \leq X$, such that (t_{a_n}) and (t_{b_n}) are approximating sequences for ga and gb respectively. The subsegment

$$I_n := [t_{a_n}, t_{b_n}] \subset [1, w_n]$$

is labeled by some word $z_n \in F(x, y)$ and we want to show that $u_n = z_n$ as words in $F(x, y)$.

Since $(\lambda_n(f)a_n)$ and (t_{a_n}) are both approximating sequences for ga it follows from

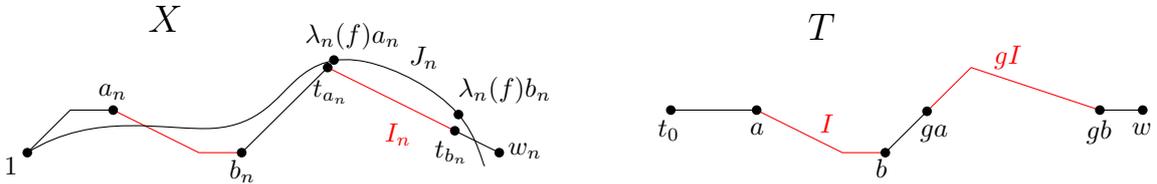


Figure 26: Approximating the arc $[1, w]$

Lemma 1.10 in [ReWe] that

$$\lim_{n \rightarrow \infty} d_n(\lambda_n(f)a_n, t_{a_n}) = 0.$$

From the same argument also follows that

$$\lim_{n \rightarrow \infty} d_n(\lambda_n(f)b_n, t_{b_n}) = 0.$$

Therefore we can assume that for large n the segment J_n is in the 2δ -neighborhood of I_n , where δ is the hyperbolicity constant of X . Suppose now that for large n the first letter of u_n is x and the first letter of z_n is y . Since $x = a^k$ and $y = b^k a^k$ there exists $c_n \in \Gamma$ such that

$$\text{Ax}(t_{a_n} a t_{a_n}^{-1}) \subset N_{10\delta}(\text{Ax}(t_{a_n} c_n b (t_{a_n} c_n)^{-1}))$$

for length at least $\frac{1}{2} \min\{|x|_\Gamma, |y|_\Gamma\}$. But this implies that

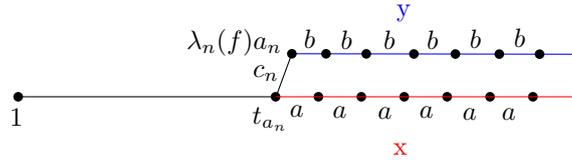


Figure 27: $\text{Ax}(t_{a_n} a t_{a_n}^{-1}) \subset N_{10\delta}(\text{Ax}(t_{a_n} c_n b (t_{a_n} c_n)^{-1}))$

$$\text{Ax}(a) \cap N_{10\delta}(\text{Ax}(c_n b c_n^{-1}))$$

has diameter at least $\frac{1}{2} \min\{|x|_\Gamma, |y|_\Gamma\}$, a contradiction to the assumption that the axis $\text{Ax}(a)$ and $\text{Ax}(c_n b c_n^{-1})$ fellow travel for length at most

$$\frac{1}{1000} \min\{|x|_\Gamma, |y|_\Gamma\}$$

for any $c \in \Gamma$. Essentially the same argument guarantees that the first letter of z_n can also not be y^{-1} or x^{-1} . Therefore both u_n and z_n start with the letter x and by induction we conclude that $z_n = u_n$ as words in $F(x, y)$.

Note that in the case that Γ is a free group and therefore X is a tree, $I_n = J_n$ and the above argument becomes trivial.

So either $gI = I$ or u_n appears twice in the label of the segment $[1, w_n]$. In the second case u_n is a piece of (w_n) . Since (w_n) satisfies the small cancellation property $C'(1/n)$ it follows that

$$\lim_{n \rightarrow \infty} d_n(1, u_n) < \lim_{n \rightarrow \infty} \frac{1}{n} d_n(1, w_n) = \lim_{n \rightarrow \infty} \frac{1}{n} d_T(t_0, w) = 0,$$

a contradiction to the assumption that $gI \cap [t_0, w]$ is not a point.

So suppose we are in the case that $gI = I$. Hence $g \in \text{Stab}(I)$. If $g \notin \text{Stab}([1, w])$ then $[1, w]$ is an unstable arc and therefore $\text{Stab}([1, w])$ is finite by Theorem 3.15. Otherwise $g \in \text{Stab}([1, w])$ and $[1, \lambda_n(f)w_n] \subset N_{2\delta}([1, w_n])$ for large n . This implies that $\lambda_n(f)$ lies in a ball $B_r(1)$ of radius r around the identity for large n and the diameter of the ball does not depend on n . Moreover $d_X(w_n, \lambda_n(f)w_n)$ is also globally bounded

for all n , implying that $w_n^{-1}\lambda_n(f)w_n \in B_r(1)$ (after possibly enlarging r). Hence after passing to a subsequence we can assume that $w_n^{-1}\lambda_n(f)w_n = \lambda_n(f)$, which implies that $w_n \in C_\Gamma(\lambda_n(f))$ for n sufficiently large. But by the same arguments every subword of w_n lies in the centralizer of $\lambda_n(f)$. Therefore $\langle \lambda_n(f) \rangle$ is finite. Hence $\langle g \rangle$ is a finite subgroup of L . The claim follows. \square

First of all it follows from Lemma 9.19 that all arcs in T_{\min} have finite stabilizer. Let now $l \in L \setminus \{1\}$ and $g_1, g_2 \in G$ such that

$$l[g_1t_0, g_1w] \cap [g_2t_0, g_2w]$$

is non-trivial and not a point. Lemma 9.19 yields that $g_2^{-1}lg_1$ stabilizes the segment $[t_0, w]$ and in particular $g_2^{-1}lg_1 \in E_L$. Hence

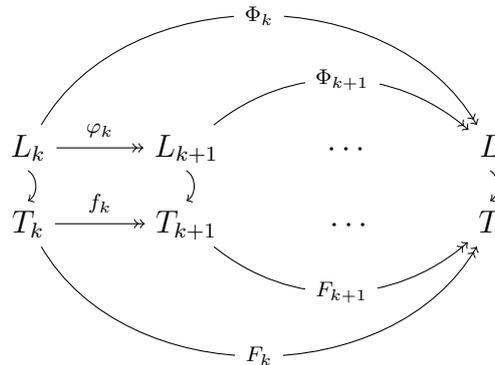
$$l \in \langle E_L, G \rangle = \langle H, E_L \rangle *_{E_L} \langle K, E_L \rangle.$$

This finally proves (2) – (5). \square

We define a family \mathcal{Z} of subtrees of T consisting of:

- L -translates of T_{\min} and
- closures of connected components of $T \setminus L \cdot T_{\min}$.

By (2) of Lemma 9.18 the family \mathcal{Z} gives rise to an equivariant covering of T such that any two subtrees intersect in at most one point. If we can show that every arc of T is covered by finitely many subtrees of \mathcal{Z} then \mathcal{Z} is precisely what V. Guirardel in [Gu] calls a transverse covering of the tree T . So suppose for the sake of contradiction that there exists some arc $I \subset T$ which is not covered by finitely many subtrees of \mathcal{Z} . Let $\mathcal{H} = \{H, K\}$. Then by Theorem 2.15 the pair action $(L, \mathcal{H}) \curvearrowright T$ is the strong limit of a direct system of geometric pair actions



such that

- φ_k and Φ_k are one-to-one in restriction to arc stabilizers of T_k .
- φ_k (resp. Φ_k) restricts to an isomorphism between \mathcal{H}_k and \mathcal{H}_{k+1} (resp. \mathcal{H}).
- The action of L_k on T_k splits as a graph of actions where each non-degenerate vertex action is either axial, thin or of orbifold type and therefore indecomposable, or is an arc containing no branch point except at its endpoints.

Since G is finitely presented relative \mathcal{H} there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $G \leq L_k$. Denote by T_{\min} the minimal G -invariant subtree of T and by T_{\min}^k the minimal G -invariant subtree of T_k for all $k \geq k_0$. Clearly the action of G on T_{\min} and on T_{\min}^k is minimal. Hence the direct system of pair actions $(G, \mathcal{H}) \curvearrowright T_{\min}^k$ converges strongly to $(G, \mathcal{H}) \curvearrowright T_{\min}$.

Clearly the action of G on T_{\min} is geometric and since by Theorem 2.5 in [LePa] a geometric action of a finitely generated group cannot be a non-trivial strong limit of actions, this implies that there exists $k_1 \geq k_0$ such that F_k is an isometry when restricted to T_{\min}^k for all $k \geq k_1$.

By Lemma 9.18(3) T_{\min} intersects any indecomposable subtree of T in at most one point. It follows that T_{\min}^k intersects each indecomposable subtree of T_k in at most one point for $k \geq k_1$. Since the action of L_k on T_k splits as a graph of actions \mathbb{A}_k where each non-degenerate vertex action is either axial, thin or of orbifold type and therefore indecomposable, or is an arc containing no branch point except at its endpoints, it follows that T_{\min}^k is contained in the simplicial part of T_k .

Now there exists $k_2 \geq k_1$ and an arc $J \subset F_k^{-1}(I)$ such that F_k maps J isometrically onto I for all $k \geq k_2$. Let now $k \geq k_2$. Since the action of L_k on T_k splits as a graph of actions, every finite subtree of T_k is covered by finitely many translates of the vertex trees of \mathbb{A}_k . In particular there exists a decomposition of J into non-trivial subsegments

$$J = J_1 \cup \dots \cup J_l$$

such that each J_i is contained in a unique translate of a vertex tree of \mathbb{A}_k . But this implies that J intersects only finitely many translates of T_{\min}^k in a segment (i.e. not just in a point). But since F_k is an isometry when restricted to T_{\min}^k , it follows that I also intersects only finitely many translates of $T_{\min} \subset T$ in more than a point. So assume that $I_0 \subset I$ is a non-trivial subsegment of I that intersects translates of T_{\min} in at most one point. Then I_0 is contained in the closure of a connected component of $T \setminus L \cdot T_{\min}$. Hence we conclude that I is covered by finitely many subtrees of \mathcal{Z} , a contradiction.

Therefore \mathcal{Z} is a transverse covering of T and it now follows from Lemma 1.5 in [Gu] that the action of L on T splits as a graph of actions such that the vertex trees are precisely the trees contained in \mathcal{Z} . Associated to this graph of actions there is a simplicial tree S on which L acts (in [Gu] this tree is called the skeleton of the graph of actions). For convenience we recall the definition of S .

The vertex set $V(S)$ is $V_0(S) \cup V_1(S)$, where $V_1(S)$ contains a vertex x_Y for every subtree

Y of \mathcal{Z} and $V_0(S)$ is the set of points $x \in T$ lying in the intersection of two distinct subtrees in \mathcal{Z} . There is an edge $e = (x, x_Y)$ between a vertex $x \in V_0(S)$ and a vertex $v_Y \in V_1(S)$ if and only if $x \in Y$.

The stabilizer of a vertex $v_Y \in V_1(S)$ is the setwise stabilizer of Y , while the stabilizer of a vertex $x \in V_0(S)$ is the stabilizer of the corresponding point $x \in T$. By Lemma 9.18 (4) the vertex $v_{T_{\min}} \in V_1(S)$ is stabilized by

$$\langle H, E_L \rangle *_{E_L} \langle K, E_L \rangle.$$

Let $e = (x, v_{T_{\min}})$ be an edge in S adjacent to $v_{T_{\min}}$. Then

$$\text{Stab}_L(e) = \text{Stab}_L(x) \cap \text{Stab}_L(v_{T_{\min}}) = \text{Stab}_L(x) \cap (\langle H, E_L \rangle *_{E_L} \langle K, E_L \rangle).$$

By Lemma 9.18(5) either $\text{Stab}_L(x) = \langle H, E_L \rangle =: H'$ or $\text{Stab}(x) = \langle K, E_L \rangle =: K'$. Therefore we can refine S by replacing each vertex $v_{gT_{\min}}$ corresponding to a translate of T_{\min} by gT_{\min} . By Lemma 9.18(1) this yields again a simplicial tree, say S' . Hence from the action of L on S' , L admits a splitting as a graph of groups \mathbb{A}' , such that there exists an edge $e \in EA'$ with edge group $A'_e = E_L$ and $H' \in A'_{\alpha(e)}$, $K' \in A'_{\omega(e)}$. Let \mathbb{A} be the graph of groups which we get by collapsing all edges in \mathbb{A}' except for e .

Suppose that \mathbb{A} has only one vertex group, i.e. L splits as the HNN-extension $L = M *_{E_L}$. Since L is one-ended relative G and $H', K' \in M$, we conclude that M is one-ended relative $\{H', K'\}$.

Let $\{m_1, \dots, m_l\}$ be a generating set of M . We can extend this to a generating set $\{m_1, \dots, m_l, t\}$ of L , where t is the stable letter of the HNN-extension $L = M *_{E_L}$. For every homomorphism $\varphi : \tilde{G} \rightarrow \Gamma$ that factors through the limit map $\eta : \tilde{G} \rightarrow L$, we denote by $\bar{\varphi} : L \rightarrow \Gamma$ the unique homomorphism such that $\varphi = \bar{\varphi} \circ \eta$. Let A_n be the following set:

$$A_n = \{ \varphi_n \in \text{Hom}(L, \Gamma) \mid \varphi_n|_{\langle t \rangle} = \bar{\lambda}_n|_{\langle t \rangle}, \varphi_n|_G = \bar{\lambda}_n|_G, \\ \varphi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\} \}.$$

Since $\bar{\lambda}_n \in A_n$ the set A_n is non-empty. Let U be the set of all converging sequences (φ_n) , such that for all $n \in \mathbb{N}$, $\varphi_n \in A_n$ and

$$\sum_{i=1}^l |\varphi_n(m_i)|_{\Gamma}$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal Γ -limit groups. By abuse of notation let $(\varphi_n) \in U$ be a sequence that converges into such a maximal limit group, say N .

In particular $(\varphi_n|_M)$ converges into the action of a Γ -limit group M_1 on some real tree T without a global fixpoint. Suppose that M_1 is not one-ended relative H, K . In this

case we can pass to finitely many Γ -limit quotients L_1, \dots, L_k of N , each admitting a decomposition along finite subgroups with one vertex group Y which has a decomposition either as $Y = N_1 *_{E_L}$ or as $Y = N_1 *_{E_L} N_2$, and all other vertex groups are isomorphic to subgroups of Γ . Moreover either $H, K \leq N_1$ and N_1 is one-ended relative $\{H, K\}$ (in the first case) or $H \leq N_1, K \leq N_2$ and N_1 is one-ended relative H and N_2 is one-ended relative K (in the second case).

This procedure of passing to finitely many Γ -limit quotients with the desired properties is straightforward but rather lengthy and we explain it in great detail in the proof of the main theorem (Theorem 10.1). So we omit the proof here and just note that, to simplify notation, we can assume without loss of generality that N already is of this type and moreover that in the decomposition of N along finite subgroups there are no vertices with vertex group isomorphic to a subgroup of Γ . That is we can assume that N admits a decomposition either as $N = M_1 *_{E_L}$, where $H, K \leq M_1$ and M_1 is one-ended relative $\{H, K\}$, or as $N = M_1 *_{E_L} M_2$, where $H \leq M_1, K \leq M_2$ and M_1 is one-ended relative H and M_2 is one-ended relative K .

Suppose we are in the first case., i.e. M_1 is one-ended relative $\{H, K\}$. Since φ_n is an extension of λ_n , the subgroups H and K act elliptically on T . Hence we can apply the relative version of the Rips machine (Theorem 2.11) to analyze the action of M_1 on T . By Theorem 2.11 the action of M_1 on T splits as a graph of actions with corresponding graph of groups \mathbb{B} . Suppose that there exist a vertex in \mathbb{B} corresponding to an axial or orbifold component. By the shortening argument (Corollary 5.6) we can shorten the action of M_1 on T while keeping $\varphi_n|_G$ fixed, a contradiction to the assumption that φ_n was chosen short relative to G . Hence there are no axial or orbifold components in \mathbb{B} and therefore the action of M_1 on T is simplicial.

Suppose that an edge e in T has infinite stabilizer. Again in this case we can shorten the length of φ_n while keeping $\varphi_n|_G$ fixed, a contradiction. Hence all edges in T have finite stabilizer. Therefore an edge $e \in \mathbb{B}$ induces a splitting of M_1 along a finite subgroup relative H, K a contradiction to the assumption that M_1 is one-ended relative H, K .

We conclude that there exists a decomposition of N as an amalgamated product

$$N = V *_{E_L} P$$

along some finite subgroup E_L containing E and $H \leq V, K \leq P$. Moreover V and P are one-ended relative H, K respectively.

To simplify notation we assume that L is already of this type, i.e. $L = V *_{E_L} P$ with the properties above. Let $\{u_1, \dots, u_s\}$ be a generating set of V . We can extend this to a generating set $\{u_1, \dots, u_s, p_1, \dots, p_l\}$ of L . Recall that for every homomorphism $\varphi : \tilde{G} \rightarrow \Gamma$ that factors through the limit map $\eta : \tilde{G} \rightarrow L$, we denote by $\bar{\varphi} : L \rightarrow \Gamma$ the unique homomorphism such that $\varphi = \bar{\varphi} \circ \eta$. Let B_n be the following set:

$$B_n = \{\varphi_n \in \text{Hom}(L, \Gamma) \mid \varphi_n|_V = \bar{\lambda}_n|_V, \varphi_n|_G = \bar{\lambda}_n|_G, \\ \varphi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\}\}.$$

Since $\bar{\lambda}_n \in B_n$ the set B_n is non-empty. Let U be the set of all converging sequences (φ_n) , such that for all $n \in \mathbb{N}$, $\varphi_n \in B_n$ and

$$\sum_{i=1}^l |\varphi_n(p_i)|_\Gamma$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal Γ -limit quotients. By abuse of notation let $(\varphi_n) \in U$ be a sequence that converges into such a maximal limit group, say N .

In particular $(\varphi_n|_P)$ converges into the action of a Γ -limit group P_1 on some real tree T without a global fixpoint. As noted before we can assume without loss of generality that P_1 is one-ended relative K .

Since φ_n is an extension of λ_n , the subgroup K acts elliptically on T . Hence we can once again apply the relative version of the Rips machine (Theorem 2.11) to analyze the action of P_1 on T . By Theorem 2.11 the action of P_1 on T splits as a graph of actions with corresponding graph of groups \mathbb{P} . Suppose that there exist an vertex in \mathbb{P} corresponding to an axial or orbifold component. By the shortening argument (Corollary 5.6) we can shorten the action of P_1 on T while keeping $\varphi|_K$ fixed, a contradiction to the assumption that $\varphi_n|_P$ was chosen to be short. Hence there are no axial or orbifold components in \mathbb{P} and therefore the action of P_1 on T is simplicial.

Suppose that an edge e in T has infinite stabilizer. Again in this case we can shorten the length of $\varphi_n|_P$ while keeping $\varphi_n|_K$ fixed, a contradiction. Hence all edges in T have finite stabilizer. Therefore an edge $e \in \mathbb{P}$ induces a splitting of P_1 along a finite subgroup relative K a contradiction to the assumption that P_1 is one-ended relative K . But this contradicts our assumption that P_1 acts without a global fixed point on T . Hence there exists no sequence of homomorphisms $(\varphi_n) \in U$ such that the $\varphi_n|_P$ are pairwise distinct. Therefore the sequence $(\varphi_n|_P)$ has a constant subsequence and hence after passing to this subsequence, say (φ) , we conclude that $P_1 = P/\ker(\varphi_n) = P/\ker(\varphi)$ is isomorphic to a subgroup of Γ . Hence $N = V *_{E_L} P_1$, where E_L is some finite group containing E and P_1 is isomorphic to a subgroup of Γ and contains K . \square

Theorem 9.20. *Suppose G has the structure of an HNN-extension*

$$\langle H, s \mid s\alpha(e)s^{-1} = \beta(e) \forall e \in E \rangle$$

over H along some finite group E . Assume that there exists a test sequence (λ_n^H) for H and for all $n \in \mathbb{N}$ a homomorphism $\Psi_n : G \rightarrow \Gamma$ such that $\Psi_n(\alpha(E)) = \lambda_n^H(\alpha(E))$ and $\Psi_n(\omega(E)) = \lambda_n^H(\omega(E))$. Assume moreover that for all $n \in \mathbb{N}$, $C_\Gamma(\lambda_n^H(\alpha(E)))$ contains a non-abelian free subgroup which is quasi-convex in Γ . Then there exists a test sequence (λ_n) for G .

Proof. Denote by $|\cdot|_\Gamma$ and $|\cdot|_G$ the word metric on Γ and G with respect to some fixed generating set respectively. Let X be the Cayley graph of Γ with respect to this

generating set. For all $n \in \mathbb{N}$ let $B_H(n) = \{h \in H \mid |h|_G \leq n\}$ and

$$L(n) = \max\{|\lambda_n^H(h)|_\Gamma \mid h \in B_H(n)\}.$$

Let $n \in \mathbb{N}$. Denote by $C_n := C_\Gamma(\lambda_n^H(\alpha(E)))$ the centralizer of $\lambda_n^H(E)$ in Γ . Let $F(a, b)$ be the quasi-convex non-abelian free subgroup of C_n which exists by assumption. As in the proof of Theorem 9.16 let $F(x, y) \leq F(a, b)$ be a non-abelian free subgroup of $F(a, b)$ such that any segment lying in the 10δ -neighborhood of both axis $\text{Ax}(a)$ and $\text{Ax}(cbc^{-1})$ in X of a and any conjugate of b by an element $c \in \Gamma$ has length at most

$$\frac{1}{1000} \min\{|x|_\Gamma, |y|_\Gamma\}.$$

We define

$$w_n := xyxy^2xy^3x \cdots xy^{n \cdot L(n)}x$$

and a homomorphism $\lambda_n : G \rightarrow \Gamma$ by

- $\lambda_n(H) = \lambda_n^H(H)$ and
- $\lambda_n(s) = \Psi_n(s)w_n$.

Then since $w_n \in C_\Gamma(\lambda_n^H(\alpha(E)))$:

$$\begin{aligned} \lambda_n(s\alpha(E)s^{-1}\omega(E)^{-1}) &= \Psi_n(s)w_n\lambda_n^H(\alpha(E))w_n^{-1}\Psi_n(s)^{-1}\lambda_n^H(\omega(E))^{-1} \\ &= \Psi_n(s)\lambda_n^H(\alpha(E))\Psi_n(s)^{-1}\lambda_n^H(\omega(E))^{-1} \\ &= \Psi_n(s\alpha(E)s^{-1}\omega(E)^{-1}) = 1 \end{aligned}$$

and hence λ_n is a well-defined homomorphism. Moreover by the construction of w_n we have that

$$n \cdot |\lambda_n(h)|_\Gamma \leq n \cdot L(n) < |\lambda_n(s)|_\Gamma$$

for all $n \in \mathbb{N}$, $h \in H$. Hence $\lambda_n(s)$ dominates the growth of $\lambda_n|_H = \lambda_n^H$.

Let \tilde{G} be a group containing G and in abuse of notation $\lambda_n : \tilde{G} \rightarrow \Gamma$ an extension of λ_n which is short relative G for all $n \in \mathbb{N}$. Let (T, t_0) be the real tree into which (a convergent subsequence of) (λ_n) converges and let $L := \tilde{G}/\ker \lambda_n$. Assume that L is one-ended relative G and moreover that G does not act elliptically on T . Since G does not fix a point when acting on T , there exists a non-trivial minimal G -invariant subtree T_{\min} of T on which G acts.

Lemma 9.21. *The following holds:*

- (1) T_{\min} is equivariantly isomorphic as a G -tree to the Bass-Serre tree of the splitting $G = H *_E$.

- (2) For every $g \in L$ either $T_{\min} \cap gT_{\min}$ is at most one point or g is contained in the setwise stabilizer of T_{\min} .
- (3) T_{\min} intersects any indecomposable subtree of T in at most one point.
- (4) The setwise stabilizer of T_{\min} in L is $\langle H, E_L \rangle *_{E_L}$ for some finite group E_L , containing E .
- (5) Edges in T_{\min} are stabilized by conjugates of E_L and vertices by conjugates of $\langle H, E_L \rangle$.

Proof. Identical to the proof of Lemma 9.18. □

We again define a family \mathcal{Z} of subtrees of T consisting of:

- L -translates of T_{\min} and
- closures of connected components of $T \setminus L \cdot T_{\min}$.

By (2) of Lemma 9.21 the family \mathcal{Z} gives rise to an equivariant covering of T such that any two subtrees intersect in at most one point. It follows from Lemma 1.5 in [Gu] that the action of L on T splits as a graph of actions such that the vertex trees are precisely the trees contained in \mathcal{Z} . Associated to this graph of actions there is a simplicial tree S on which L acts (in [Gu] this tree is called the skeleton of the graph of actions). Refining this tree precisely as we did in the proof of Theorem 9.16, finally yields a decomposition of L as $L = V *_{E_L}$, where $H \leq V$ and E_L is a finite group containing E . □

9.3 Virtually abelian flats

We distinguish several cases how a virtually abelian flat can appear in the construction of the completion. First consider the case that the virtually abelian flat is coming from an edge connecting two rigid vertices, i.e. is of the form $G = H *_C C \oplus \mathbb{Z}^n$, with C infinite finite-by-abelian. This case is also covered by the more general Proposition 9.26 below, but we still prefer to present a proof here, because the underlying ideas needed to prove the existence of test sequences for virtually abelian flats are easier to grasp in this simpler case.

Proposition 9.22. *Suppose $G = H *_C C \oplus A$ has the structure of a virtually abelian flat over some subgroup H , such that $A \cong \mathbb{Z}^k$, C is an infinite finite-by-abelian subgroup of H , and there exists a test sequence (λ_n^H) for H . Then there exists a test sequence (λ_n) for G .*

Proof. Since λ_n^H is a test sequence, $\lambda_n^H(C)$ is infinite finite-by-abelian for n sufficiently large. By Proposition 3.16 the center $Z(\lambda_n^H(C))$ is infinite and hence contains an element, say w , of infinite order which generates a quasi-convex subgroup of Γ . We define the test sequence $(\lambda_n) \subset \text{Hom}(G, \Gamma)$ as follows. For all $n \in \mathbb{N}$

- we set $\lambda_n|_H = \lambda_n^H$.
- Let $\{a_1, \dots, a_k\}$ be a basis of A and $B(n)$ the intersection of H with the ball of radius n around the identity in the Cayleygraph of G (i.e. all elements in H of word length at most n in the word metric of G with respect to some fixed finite generating set). Further let

$$\text{dist}(n) = \max\{|\lambda_n^H(h)|_\Gamma \mid h \in B(n)\}.$$

We then choose $k_n^1 \in \mathbb{N}$ such that

$$|w^{k_n^1}|_\Gamma > n \cdot \text{dist}(n)$$

and define

$$\lambda_n(a_1) = w^{k_n^1}$$

Moreover for all $i \in \{2, \dots, k\}$ we choose inductively $k_n^i \in \mathbb{N}$ such that

$$|w^{k_n^i}|_\Gamma > n \cdot |w^{k_n^{i-1}}|_\Gamma$$

and define

$$\lambda_n(a_i) = w^{k_n^i}.$$

Clearly λ_n is a homomorphism and $\lambda_n|_A$ dominates the growth of λ_n^H . It remains to show that (λ_n) is indeed a test sequence.

So let \tilde{G} be a group containing G and in abuse of notation let $\lambda_n : \tilde{G} \rightarrow \Gamma$ be an extension of λ_n which is short relative G for all $n \in \mathbb{N}$. Let (T, t_0) be the real tree into which (a convergent subsequence of) (λ_n) converges and let $L := \tilde{G}/\ker \lambda_n$. Assume that L is one-ended relative G and moreover that G does not act elliptically on T . Let $\{a_1, \dots, a_k\}$ be the basis of A chosen above and c a generating set of C . Then $C \oplus A = \langle c, a_1, \dots, a_k \rangle$.

We claim that L admits a splitting as an amalgamated product $L = V *_P M$, where $H \leq V$, the subgroups P and M are virtually abelian and $\langle c, a_1, \dots, a_{k-1} \rangle \leq P$, while $a_k \in M \setminus P$.

To see this we now take a closer look at the action of $C \oplus A$ on T . Since $\lambda_n|_A$ dominates the growth of λ_n^H , H fixes the base point of T .

Lemma 9.23. (1) *The finite-by-abelian subgroup $B := \langle a_1, \dots, a_{k-1}, c \rangle$ fixes the base point t_0 of T .*

(2) *a_k acts hyperbolically on T .*

(3) *B fixes the segment $[t_0, a_k t_0] \subset T$.*

Proof. Let (X, d_X) be the Cayley graph of Γ with respect to some fixed finite generating set. Let Y be a generating set of L containing $\{c, a_1, \dots, a_k\}$ and set

$$\mu_n := \max_{y \in Y} d_X(1, \lambda_n(y)) = |\lambda_n(y)|_\Gamma.$$

In particular $\mu_n \geq d_X(1, \lambda_n(a_k))$. Denote by d_T the metric on the real tree T and by d_n the scaled metric on X , i.e. $(X_n, d_n) = (X, \frac{d_X}{\mu_n})$. We assume without loss of generality that (1) is an approximating sequence for t_0 . By definition of the test sequence (λ_n) it holds for all $i \in \{1, \dots, k-1\}$ that

$$|\lambda_n(a_k)|_\Gamma > n^{k-i} \cdot |\lambda_n(a_i)|_\Gamma$$

and therefore:

$$\begin{aligned} d_T(t_0, a_i t_0) &= \lim_{n \rightarrow \infty} d_n(1, \lambda_n(a_i) \cdot 1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\mu_n} d_X(1, \lambda_n(a_i)) \\ &\leq \lim_{n \rightarrow \infty} \frac{|\lambda_n(a_i)|_\Gamma}{|\lambda_n(a_k)|_\Gamma} \\ &< \lim_{n \rightarrow \infty} \frac{1}{n^{k-i}} = 0. \end{aligned}$$

We have shown (1) since C is contained in H and therefore fixes t_0 .

Since G does not act with a global fixed point on T , we have that a_k does not fix t_0 and it is easy to see that a_k acts by translation on the infinite axis $\bigcup_{i \in \mathbb{Z}} [a_k^i t_0, a_k^{i+1} t_0]$ and hence (2) holds.

By (1) the group $B = \langle c, a_1, \dots, a_{k-1} \rangle$ fixes t_0 , hence it remains to show that B also fixes $a_k t_0$. Since (1) is an approximating sequence for t_0 , $(\lambda_n(a_k))$ is an approximating sequence for $a_k t_0$ by Lemma 1.10 in [ReWe]. For all $i \in \{1, \dots, k-1\}$ we have that $[\lambda_n(a_i), \lambda_n(a_k)] = 1$ and therefore

$$\begin{aligned} d_T(a_i \cdot a_k t_0, a_k t_0) &= \lim_{n \rightarrow \infty} \frac{d_X(\lambda_n(a_i) \lambda_n(a_k), \lambda_n(a_k))}{\mu_n} \\ &= \lim_{n \rightarrow \infty} \frac{|\lambda_n(a_k)^{-1} \lambda_n(a_i) \lambda_n(a_k)|_\Gamma}{\mu_n} \\ &\leq \lim_{n \rightarrow \infty} \frac{|\lambda_n(a_i)|_\Gamma}{|\lambda_n(a_k)|_\Gamma} \\ &< \lim_{n \rightarrow \infty} \frac{1}{n^{k-i}} = 0. \end{aligned}$$

The same argument yields that C fixes $a_k t_0$ and hence the claim follows. \square

By assumption L is one-ended relative G . In particular L is one-ended relative $\langle B, H \rangle$. It follows from Lemma 9.23 that $\langle H, B \rangle$ acts elliptically on the real tree T and hence we can apply the relative version of the Rips machine (Theorem 2.11). This yields a graph of actions with corresponding graph of groups, say \mathbb{A} . By Lemma 9.23

$$B \leq \text{Stab}_L([t_0, a_k t_0])$$

and hence the stabilizer of the segment $[t_0, a_k t_0] \subset T$ contains a finite-by-abelian group. Therefore this segment has trivial intersection with orbifold components, since segments in orbifold components have finite stabilizer. So suppose there exists a proper subsegment J of $[t_0, a_k t_0]$ which is contained in an axial component, while $[t_0, a_k t_0] \setminus J$ is contained in the discrete part of T . Denote by Z the stabilizer of the whole axial component. Since the subgroup B stabilizes in particular J , it also stabilizes the whole axial component and therefore $B \leq Z$. Since L is a Γ -limit group and therefore $N(\Gamma)$ -CSA and Z is a maximal virtually abelian subgroup of L (since it is the stabilizer of an axial component) this implies that $a_k \in Z$. But the axis of a_k contains $[t_0, a_k t_0]$, hence is not contained in the axial component stabilized by Z , a contradiction.

Therefore $[t_0, a_k t_0]$ is either contained in an axial component or contained in the discrete part of T .

Lemma 9.24. *If $[t_0, a_k t_0]$ is contained in an axial component of T , then*

$$L = V *_P M,$$

where M, P are finitely generated virtually abelian groups, $a_k \in M \setminus P$ and

$$B = \langle c, a_1, \dots, a_{k-1} \rangle \leq P.$$

Proof. Let v be the vertex in the graph of groups \mathbb{A} , which corresponds to the axial component of T containing $[t_0, a_k t_0]$. Let $M := A_v$ be the associated vertex group. In particular M is the stabilizer of the axial component and therefore virtually abelian. Let P be the kernel of the action of M on the axial component. Let e_1, \dots, e_k be the edges in \mathbb{A} connected to v , with corresponding edge groups A_{e_1}, \dots, A_{e_k} . Hence A_{e_1}, \dots, A_{e_k} are subgroups of P . Therefore we can refine \mathbb{A} by adding a new vertex w with vertex group $A_w := P$ and a new edge f with edge group $A_f := P$ connecting v and w . Moreover the edges e_1, \dots, e_k are now connected to w instead of v . Since $A_{e_1}, \dots, A_{e_k} \leq P = A_w$ this yields a graph of groups \mathbb{A}' , with $\pi_1(\mathbb{A}') = \pi_1(\mathbb{A})$. In particular

$$L = \pi_1(\mathbb{A}') = V *_P M.$$

By Lemma 9.23 the remaining properties of the Lemma follow. □

Lemma 9.25. *If $[t_0, a_k t_0]$ is contained in the discrete part of T , then*

$$L = V *_P M,$$

where M, P are finitely generated virtually abelian groups, $a_k \in M \setminus P$ and

$$B = \langle c, a_1, \dots, a_{k-1} \rangle \leq P.$$

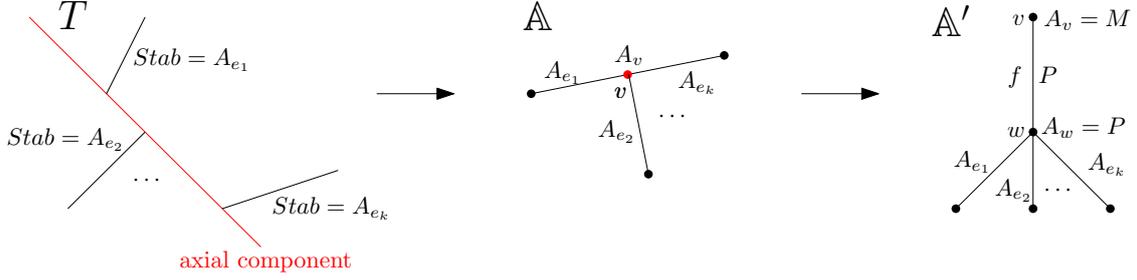


Figure 28: The axial case

Proof. Let e_1, \dots, e_m be the edges contained in $e := [t_0, a_k t_0]$ and let $i \in \{1, \dots, m\}$. Clearly $B \leq \text{Stab}_L(e_i) =: A_{e_i}$ by Lemma 9.23. Let $\text{Ax}(a_k)$ be the axis of a_k in T . We claim that A_{e_i} stabilizes pointwise the entire axis $\text{Ax}(a_k)$. First we note that since

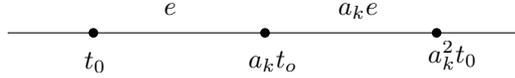


Figure 29: Axis of a_k in T

$$B = \langle c, a_1, \dots, a_{k-1} \rangle \leq \text{Stab}(e)$$

and $[B, a_k] = 1$ it follows that

$$B = a_k B a_k^{-1} \leq a_k \text{Stab}(e) a_k^{-1} = \text{Stab}(a_k e)$$

and therefore $B \leq \text{Stab}(e) \cap \text{Stab}(a_k e)$. Hence B stabilizes pointwise the entire axis of a_k in T . Now suppose that there exists a segment $I \subset \text{Ax}(a_k)$ such that $A_{e_i} \not\leq \text{Stab}(I)$. But this implies that $\text{Ax}(a_k)$ is an unstable subtree of T . It follows from Theorem 3.15(4) that $\text{Stab}(\text{Ax}(a_k))$ is finite, a contradiction to the fact that $B \leq \text{Stab}(\text{Ax}(a_k))$. Therefore A_{e_i} stabilizes the entire axis of a_k for all $i \in \{1, \dots, m\}$.

Denote by M the setwise stabilizer of $\text{Ax}(a_k)$. By Theorem 3.15 (4) M is finitely generated virtually abelian and contains a finite-by-abelian subgroup of index at most 2 which leaves $\text{Ax}(a_k)$ invariant and fixes its ends. As shown above there exists a finite-by-abelian group P' such that $\text{Stab}(f) = P'$ for all edges f contained in $\text{Ax}(a_k)$. Hence L admits a decomposition as $L *_P M$, where P is finitely generated virtually abelian and contains P' as a subgroup of index at most 2. Moreover it follows from Lemma 9.23 that $B \leq P$ and $a_k \in M \setminus P$. \square

Therefore in both cases (axial and discrete), $L = V *_P M$, where $B \leq P$, $a_k \in M \setminus P$ and P, M are finitely generated virtually abelian. The remaining part of the proof is identical to the equivalent part of the proof of Theorem 9.26, and we prefer to do this proof there in the more general case. \square

Now suppose that the virtually abelian flat comes from a virtually abelian vertex group A' , i.e. is of the form $G = H *_C A$, where C is the maximal virtually abelian subgroup of H containing a peripheral subgroup $P(A')$ of A' and A is a virtually abelian group containing A' .

Theorem 9.26. *Suppose $G = H *_C A$ has the structure of a virtually abelian flat over some subgroup H coming from a virtually abelian vertex group A' , i.e. A is virtually abelian, C is the maximal virtually abelian subgroup of H containing a peripheral subgroup $P(A')$ of A' , and there exists a test sequence (λ_n^H) for H . Then there exists a test sequence (λ_n) for G .*

Proof. For every infinite virtually abelian subgroup M of a Γ -limit group there exists by Proposition 3.16 a unique finite-by-abelian subgroup of index at most 2 which we denote by M^+ .

Let E be the torsion subgroup of A^+ . It follows from Lemma 3.13 that E is a finite normal subgroup of A . Denote by $\pi : A \rightarrow A/E$ the canonical projection. Recall that $P(A')^+$ consists of all elements of A'^+ which vanish under every homomorphism from A'^+ to \mathbb{Z} . It follows from the construction of the completion that $E \leq P(A')^+ \leq C^+$. Let $W := \pi(A) = A/E$ and set $W^+ := \pi(A^+)$. Since A^+ is finitely generated finite-by-abelian it follows immediately that W^+ is finitely generated free abelian.

Moreover since $E \leq C^+$ it follows from the construction of the completion that

$$A^+/C^+ \cong \mathbb{Z}^n$$

for some $n \in \mathbb{N}$. From the second isomorphism theorem then follows

$$W^+/(C^+/E) = (A^+/E)/(C^+/E) \cong A^+/C^+ \cong \mathbb{Z}^n.$$

Hence there exists a short exact sequence

$$1 \rightarrow C^+/E \rightarrow A^+/E \xrightarrow{\tau} \mathbb{Z}^n \rightarrow 1$$

and we claim that this sequence splits. To see this choose a basis $\{z_1, \dots, z_n\}$ of \mathbb{Z}^n and define a map $s_0 : \{z_1, \dots, z_n\} \rightarrow A^+/E$ such that $\tau \circ s_0 = \text{id}_{\{z_1, \dots, z_n\}}$. Since A^+/E is abelian there exists a unique extension of s_0 to a homomorphism $s : \mathbb{Z}^n \rightarrow A^+/E$ by the universal property of the free abelian group \mathbb{Z}^n . By construction clearly $\tau \circ s = \text{id}_{\mathbb{Z}^n}$ and therefore the sequence splits. It follows that

$$W^+ = A^+/E \cong \mathbb{Z}^n \oplus C^+/E,$$

i.e. $K := \pi(C^+)$ is a direct summand of the free abelian group $W^+ = \pi(A^+)$. Hence there exists a subgroup $B \cong \mathbb{Z}^n$ of W^+ such that $W^+ = K \oplus B$.

By construction of the completion, G comes equipped with a retract $\eta : G \rightarrow H$. Since by assumption C is the maximal virtually abelian subgroup of W containing $P(A')$ and

$\pi(A)$ is virtually abelian, we conclude that $\eta(A) = C$.

Let $\{k_1, \dots, k_m\}$ be a basis of K and $\{b_1, \dots, b_k\}$ a basis of B . We define the following sequence of automorphisms of W^+ :

For $n \in \mathbb{N}$ let $\alpha_n : W^+ \rightarrow W^+$ be defined by:

$$\begin{aligned} \alpha_n(k_j) &= k_j \quad \text{for all } j \in \{1, \dots, m\}, \\ \alpha_n(b_j) &= \sum_{i=1}^j n^{j-i} b_i + n^j \sum_{l=1}^m k_l \quad \text{for all } j \in \{1, \dots, k\}. \end{aligned}$$

Set $\tilde{K} = \pi^{-1}(K)$, $\tilde{B} = \pi^{-1}(B)$. Then $A = \langle \tilde{K}, \tilde{B}, s \rangle$ for some element s , where either $\pi(s)$ has order 2 (see [ReWe] Lemma 4.9) or $s = 1$. Denote by $\text{Aut}_s(A)$ the subgroup of $\text{Aut}(A)$ consisting of those automorphisms which restrict to the identity on $\langle \tilde{K}, s \rangle$ and preserve \tilde{B} .

It follows from Corollary 4.13 in [ReWe] that after passing to a subsequence we can extend α_n to an automorphism (still denoted) $\alpha_n \in \text{Aut}_s(A)$ for all $n \in \mathbb{N}$.

We then define our test sequence $\lambda_n : G \rightarrow \Gamma$ on A as $\lambda_n|_A = \lambda_n^H \circ \eta \circ \alpha_n$ and on H as $\lambda_n|_H = \lambda_n^H$ for all $n \in \mathbb{N}$.

Clearly $\lambda_n|_A$ dominates the growth of $\lambda_n|_H = \lambda_n^H$. Moreover for n sufficiently large it holds that

$$|\lambda_n(\tilde{b}_i)|_\Gamma > n \cdot |\lambda_n(\tilde{b}_{i-1})|_\Gamma$$

and

$$|\lambda_n(\tilde{b}_1)|_\Gamma > n \cdot |\lambda_n(\tilde{k})|_\Gamma$$

for all $i \in \{2, \dots, k\}$, $\tilde{b}_i \in \pi^{-1}(b_i)$ and $\tilde{k} \in \pi^{-1}(K)$. It remains to show that λ_n is indeed a test sequence.

So let \tilde{G} be a group containing G and in abuse of notation let $\lambda_n : \tilde{G} \rightarrow \Gamma$ be an extension of λ_n which is short relative G for all $n \in \mathbb{N}$. Let (T, t_0) be the real tree into which (a convergent subsequence of) (λ_n) converges and let $L := \tilde{G}/\overline{\ker \lambda_n}$. Assume that L is one-ended relative G and moreover that G does not act elliptically on T .

We claim that L admits a splitting as an amalgamated product $L = V *_P M$, where $H \leq V$, the subgroups P, M are virtually abelian and

$$N := \langle k_1, \dots, k_m, b_1, \dots, b_{k-1} \rangle \leq P^+ / E_P,$$

while $b_k \in (M^+ / E_M) \setminus (P^+ / E_M)$, here E_P, E_M denote the torsion subgroups of P and M respectively.

To see this we now take a closer look at the action of A on T . Since $\lambda_n|_A$ dominates the growth of λ_n^H , H fixes the base point of T . Recall that $\pi : A \rightarrow A/E$ denotes the canonical projection and that $A = \langle \tilde{K}, \tilde{B}, \tilde{s} \rangle$, where $\tilde{K} = \pi^{-1}(K)$, $\tilde{B} = \pi^{-1}(B)$ and either $\pi(\tilde{s}) = s$ has order 2 or $\tilde{s} = 1$.

Lemma 9.27. (1) *The subgroup $\tilde{N} := \pi^{-1}(N)$ fixes the base point t_0 of T .*

- (2) Elements from $\pi^{-1}(b_k)$ act hyperbolically on T .
- (3) If $\tilde{b}_k \in \pi^{-1}(b_k)$ then N fixes the segment $[t_0, \tilde{b}_k t_0] \subset T$.
- (4) If $\tilde{s} \neq 1$ then \tilde{s} exchanges the ends of the axis of \tilde{b}_k in T .

Proof. Let (X, d_X) be the Cayley graph of Γ with respect to some fixed generating set. Let Y be a generating set of L containing the generators of $\pi^{-1}(K \oplus B)$ and \tilde{s} and set

$$\mu_n := \max_{y \in Y} d_X(1, \lambda_n(y)) = |\lambda_n(y)|_\Gamma.$$

Let $\tilde{b}_k \in \pi^{-1}(b_k)$. Then in particular $\mu_n \geq d_X(1, \lambda_n(\tilde{b}_k))$. Denote by d_T the metric on the real tree T and by d_n the scaled metric on X , i.e. $(X_n, d_n) = (X, \frac{d_X}{\mu_n})$. We assume without loss of generality that (1) is an approximating sequence for t_0 . Since $\tilde{K} \leq C \leq H$, \tilde{K} fixes t_0 . In particular E fixes t_0 . By definition of the test sequence (λ_n) it holds for all $i \in \{1, \dots, k-1\}$ and $\tilde{b}_i \in \pi^{-1}(b_i)$ that

$$|\lambda_n(\tilde{b}_k)|_\Gamma > n^{k-i} \cdot |\lambda_n(\tilde{b}_i)|_\Gamma$$

and therefore:

$$\begin{aligned} d_T(t_0, \tilde{b}_i t_0) &= \lim_{n \rightarrow \infty} d_n(1, \lambda_n(\tilde{b}_i) \cdot 1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\mu_n} d_X(1, \lambda_n(\tilde{b}_i)) \\ &\leq \lim_{n \rightarrow \infty} \frac{|\lambda_n(\tilde{b}_i)|_\Gamma}{|\lambda_n(\tilde{b}_k)|_\Gamma} \\ &< \lim_{n \rightarrow \infty} \frac{1}{n^{k-i}} = 0. \end{aligned}$$

We have shown (1).

Since A does not act with a global fixed point on T , \tilde{b}_k does not fix t_0 and it is easy to see that \tilde{b}_k acts by translation on the infinite axis

$$Y := \bigcup_{i \in \mathbb{Z}} [\tilde{b}_k^i t_0, \tilde{b}_k^{i+1} t_0].$$

By (1) the group $\tilde{N} = \pi^{-1}(N)$ fixes t_0 , hence it remains to show that \tilde{N} also fixes $\tilde{b}_k t_0$. Since (1) is an approximating sequence for t_0 , $(\lambda_n(\tilde{b}_k))$ is an approximating sequence for $\tilde{b}_k t_0$ by Lemma 1.10 in [ReWe]. For all $i \in \{1, \dots, k-1\}$, $\tilde{k} \in \tilde{K}$, there exists $e_i, e_i^{(\tilde{k})} \in E$

such that $[\lambda_n(\tilde{b}_i), \lambda_n(\tilde{b}_k)] = \lambda_n(e_i)$ and $[\lambda_n(\tilde{b}_i), \lambda_n(\tilde{k})] = \lambda_n(e_i^{(\tilde{k})})$ and therefore

$$\begin{aligned} d_T(\tilde{b}_i \cdot \tilde{b}_k t_0, \tilde{b}_k t_0) &= \lim_{n \rightarrow \infty} \frac{d(\lambda_n(\tilde{b}_i) \lambda_n(\tilde{b}_k), \lambda_n(\tilde{b}_k))}{\mu_n} \\ &= \lim_{n \rightarrow \infty} \frac{|\lambda_n(\tilde{b}_k)^{-1} \lambda_n(\tilde{b}_i) \lambda_n(\tilde{b}_k)|_\Gamma}{\mu_n} \\ &\leq \lim_{n \rightarrow \infty} \frac{|\lambda_n(\tilde{b}_i) \lambda_n(e_i)|_\Gamma}{|\lambda_n(\tilde{b}_k)|_\Gamma} \\ &< \lim_{n \rightarrow \infty} \frac{1}{n^{k-i}} = 0. \end{aligned}$$

The same argument yields that \tilde{K} fixes $\tilde{b}_k t_0$ and hence (3) follows.

Assume that $\tilde{s} \neq 1$. It remains to show that \tilde{s} exchanges the ends of Y . This follows from Lemma 4.9 in [ReWe]. There it is shown that $W = \pi(A) = W^+ \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2 = \langle s \rangle$ and the action of \mathbb{Z}_2 on W^+ is given by $s w s^{-1} = w^{-1}$ for all $w \in W^+$. In particular $\tilde{b}_k \tilde{s} \tilde{b}_k \tilde{s}^{-1} \in E$. Moreover by construction of the test sequence \tilde{s} fixes t_0 . The claim now follows from

$$\tilde{s} \tilde{b}_k t_0 = \tilde{b}_k^{-1} \tilde{s} t_0 = \tilde{b}_k^{-1} t_0.$$

□

Lemma 9.27 shows in particular that A acts invariantly on Y and E lies in the kernel of this action. Hence the action factors through π and induces an action of

$$W = \pi(A) = (K \oplus B) \rtimes \mathbb{Z}_2$$

on Y .

By assumption L is one-ended relative G . In particular L is one-ended relative $\langle H, \tilde{N} \rangle$. It follows from Lemma 9.27 that $\langle \tilde{N}, H \rangle$ acts elliptically on the real tree T and hence we can apply the relative version of the Rips machine (Theorem 2.11). This yields a graph of actions with corresponding graph of groups, say \mathbb{A} . As in the proof of Proposition 9.22 we conclude that Y is either contained in an axial component or in the discrete part of T .

Lemma 9.28. *If Y is contained in an axial component of T , then $L = V *_P M$, where M, P are finitely generated virtually abelian groups, where $H \leq V$, $\pi^{-1}(N) \subset P$ and*

$$b_k \in (M/E_M) \setminus (P/E_M)$$

(here E_P, E_M denote the torsion subgroups of P and M respectively).

Proof. The proof is identical to the one of Lemma 9.24. □

Lemma 9.29. *If Y is contained in the discrete part of T , then $L = V *_P M$, where M, P are finitely generated virtually abelian groups, where $H \leq V$, $\pi^{-1}(N) \subset P$ and $b_k \in (M/E_M) \setminus (P/E_M)$.*

Proof. Since E lies in the kernel of the action of A on Y we get an induced action of $W = (K \oplus B) \rtimes \mathbb{Z}_2$ on Y . Let e_1, \dots, e_p be the edges contained in $e := [t_0, \tilde{b}_k t_0]$. Let $i \in \{1, \dots, m\}$. Clearly $\pi^{-1}(N) \leq \text{Stab}_L(e_i) =: A_{e_i}$ by Lemma 9.27. We claim that A_{e_i} stabilizes pointwise the entire axis $Y = \text{Ax}(\tilde{b}_k)$ of \tilde{b}_k in T . First note that since $[N, b_k] = 1_W$ it follows that

$$N = b_k N b_k^{-1} \leq b_k \text{Stab}_W(e) b_k^{-1} = \text{Stab}_W(b_k e)$$

and therefore $\pi^{-1}(N) \leq \text{Stab}_L(e) \cap \text{Stab}_L(\tilde{b}_k e)$. Hence \tilde{N} stabilizes pointwise the entire axis of \tilde{b}_k in T . Now suppose that there exists a segment $I \subset Y$ such that $A_{e_i} \not\leq \text{Stab}(I)$. But this implies that Y is an unstable subtree of T . It follows from Theorem 3.15(4) that $\text{Stab}(Y)$ is finite, a contradiction to the fact that $\pi^{-1}(N) \leq \text{Stab}(Y)$. Therefore A_{e_i} stabilizes the entire axis of \tilde{b}_k for all $i \in \{1, \dots, p\}$.

Denote by M the setwise stabilizer of Y . By Theorem 3.15(4) M is finitely generated virtually abelian and contains a finite-by-abelian subgroup of index at most 2 which leaves Y invariant and fixes its ends. As shown above there exists a finite-by-abelian group P' , containing \tilde{N} , such that $\text{Stab}(f) = P'$ for all edges f contained in Y . Hence L admits a decomposition as $L *_P M$, where P is finitely generated virtually abelian and contains P' as a subgroup of index at most 2. In particular $P = \langle P', \tilde{s} \rangle$ by Lemma 9.27(4). The remaining part of the claim follows from Lemma 9.27. \square

In the following we denote the maximal free abelian subgroup of a virtually abelian group P by Z_P . Let $\{m_1, \dots, m_l\}$ be a generating set of M and $\{v_1, \dots, v_q\}$ a generating set for V . Then clearly $\{m_1, \dots, m_l, v_1, \dots, v_q\}$ is a generating set of L . For every homomorphism $\varphi : \tilde{G} \rightarrow \Gamma$ that factors through the Γ -limit map $\eta : \tilde{G} \rightarrow L$, we denote by $\bar{\varphi} : L \rightarrow \Gamma$ the unique homomorphism such that $\varphi = \bar{\varphi} \circ \eta$. Let A_n be the following set:

$$A_n = \{ \varphi_n \in \text{Hom}(L, \Gamma) \mid \varphi_n|_M = \bar{\lambda}_n|_M, \varphi_n|_G = \bar{\lambda}_n|_G, \\ \varphi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\} \}.$$

Since $\bar{\lambda}_n \in A_n$ the set C_n is non-empty. Let U be the set of all converging sequences (φ_n) , such that for all $n \in \mathbb{N}$, $\varphi_n \in C_n$ and

$$\sum_{i=1}^q |\varphi_n(v_i)|_\Gamma$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal limit groups. Let $(\varphi_n) \in U$ be a sequence that converges into such a maximal limit group, say N .

In particular $(\varphi_n|_V)$ converges into the action of a Γ -limit group V_1 on some real tree T without a global fixpoint. Moreover $(\varphi_n|_V)$ is the extension of $\lambda_n|_Y$ where

$Y = H *_C A_1$ and $A_1 \leq A$ is the subgroup of the original virtually abelian group A such that $A_1 = \langle \tilde{N}, \tilde{s} \rangle$, i.e. $A_1^+ = \tilde{N}$. In particular $\text{rk}(Z_{A_1}) = \text{rk}(Z_A) - 1$.

As in the proof of Theorem 9.16 (where we referred to the proof of Theorem 10.1) we can assume without loss of generality that N is one-ended relative G and moreover that Y does not act elliptically on the limit tree T . Hence we can apply the same procedure to analyze the action of Y on T as we did before for the action of A on the corresponding real tree.

Therefore (in general) we end up with finitely many Γ -limit quotients of L which admit a decomposition \mathbb{D} relative G along finite subgroups with one vertex with vertex group U_1 , which contains G , while all other vertices are isomorphic to subgroups of Γ . Moreover $U_1 = V_1 *_P M_1$, where $H \leq V_1$, P_1 is virtually abelian and contains C and M_1 is virtually abelian and contains A . In addition $b_{k-1}, b_k \in (M_1/E_{M_1}) \setminus (P_1/E_{M_1})$ and $\tilde{K}, \tilde{b}_1, \dots, \tilde{b}_{k-2} \subset P_1$.

Hence after repeating these steps finitely many times we end up with finitely many Γ -limit quotients of \tilde{G} which admit a decomposition \mathbb{D} relative G along finite subgroups with one vertex with vertex group \hat{U} , which contains G , while all other vertices are isomorphic to subgroups of Γ . Moreover $\hat{U} = \hat{V} *_P \hat{M}$, where $H \leq \hat{V}$, \hat{P} is virtually abelian and contains C and \hat{M} is virtually abelian and contains A . In addition

$$B = \langle b_1, \dots, b_k \rangle \subset (\hat{M}/E_{\hat{M}}) \setminus (\hat{P}/E_{\hat{M}})$$

and $\tilde{K} \subset \hat{P}$.

To simplify notation we assume that L is already of this form. We moreover assume without loss of generality that $L = \hat{U} = \hat{V} *_P \hat{M}$, i.e. that the decomposition \mathbb{D} of L along finite subgroups is trivial. Let $(\varphi_n) \subset \text{Hom}(L, \Gamma)$ be a stably injective sequence. After passing to a subsequence we can assume that $\varphi_n|_{\hat{M}} \subset \text{Hom}(\hat{M}, Z)$ for some virtually cyclic group $Z \leq \Gamma$ and that moreover φ_n is injective on a virtually cyclic subgroup of \hat{M} , containing \hat{E} , and $\varphi_n(v_j) \neq 1$ for all $j \in \{1, \dots, r\}$ and $n \in \mathbb{N}$. In particular \hat{M} is a Z -limit group.

Let us briefly recall the structure of A and \hat{M} . $E \leq A$ is the torsion subgroup of A and $\pi : A \rightarrow A/E$ denotes the canonical projection. Then $\pi(A) = (K \oplus B) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle s \rangle$, $K = \pi(C^+)$ and K, B are finitely generated free abelian. Let $\tilde{X} = \pi^{-1}(X)$ for all $X \leq \pi(A)$. Then $A = \langle \tilde{K}, \tilde{B}, \tilde{s} \rangle$. As shown before \hat{M} is finitely generated virtually abelian,

$$A \leq \hat{M}, B \leq (\hat{M}/E_{\hat{M}}) \setminus (\hat{P}/E_{\hat{M}}) \text{ and } C \leq \hat{P},$$

where $E_{\hat{M}}$ is the torsion subgroup of \hat{M} . Denote by $\hat{\pi} : \hat{M} \rightarrow \hat{M}/\hat{E}$ the canonical projection.

Then there exists a decomposition of $\hat{\pi}(\hat{M}^+)$ as $\hat{\pi}(\hat{M}^+) = R \oplus S$ such that $\hat{\pi}(A)$ is a subgroup of finite index of R . In particular $\hat{M} = \langle \hat{\pi}^{-1}(R), \hat{\pi}^{-1}(S), \hat{s} \rangle$ for some element \hat{s} such that either $\hat{\pi}(\hat{s})$ has order 2 or $\hat{s} = 1$.

Let $\text{Aut}_{\hat{s}}(\hat{M}) \leq \text{Aut}(\hat{M})$ be the subgroup consisting of those automorphisms that restrict

to the identity on $\langle \hat{\pi}^{-1}(S), \hat{s} \rangle$ and preserve $\hat{\pi}^{-1}(R)$. Any $\alpha \in \text{Aut}_{\hat{s}}(\hat{M})$ restricts to an automorphism of $\hat{\pi}^{-1}(R)$ and therefore induces an automorphism of R . Denote the subgroup of $\text{Aut}(R)$ induced in this fashion by $K_{\hat{s}}$. By Lemma 4.12 in [ReWe] $K_{\hat{s}}$ has finite index in $\text{Aut}(R)$. Therefore for every sequence of automorphisms $(\alpha_n) \subset \text{Aut}(R)$ we can assume, after passing to a subsequence, that $(\alpha_n) \subset K_{\hat{s}}$.

Let $(\alpha_n) \subset \text{Aut}(R)$. Then (after passing to a subsequence) there exists a lift $\hat{\alpha}_n \in \text{Aut}_{\hat{s}}(\hat{M})$ of α_n for all $n \in \mathbb{N}$. Define $\Psi_n := \varphi_1 \circ \hat{\alpha}_n$ for all $n \in \mathbb{N}$ and let $M' := \hat{M} / \ker \Psi_n$ be the corresponding Γ -limit group. Since M' is also a Z -limit group, it is in particular virtually abelian. We now choose the sequence $(\alpha_n) \subset \text{Aut}(R)$ such that $(\Psi_n|_{\langle \pi^{-1}(R), \hat{s} \rangle})$ is stably injective. In particular it follows that $A \leq M'$ is a subgroup of finite index. Let $L' := \hat{V} *_{\hat{p}} M'$ and denote by $\eta : L \rightarrow L'$ the corresponding Γ -limit map. The claim follows. \square

Now suppose that the virtually abelian flat comes from an isolated virtually abelian vertex group A , i.e. is of the form $G = H *_E A$ for some finite group E .

Theorem 9.30. *Suppose $G = H *_E A$ where A is virtually abelian, E is finite and there exists a test sequence (λ_n^H) for H . Let F be the torsion subgroup of A^+ . Denote by $\pi : A \rightarrow A/F$ the canonical projection. Let $s \in A$ be an element such that $A = \langle A^+, s \rangle$ and let $\{z_1, \dots, z_k\}$ be a basis of $\pi(A^+) \cong \mathbb{Z}^k$. Assume that there exists a sequence of homomorphisms $(\lambda_n^A) \subset \text{Hom}(A, \Gamma)$ such that $\lambda_n^H(E) = \lambda_n^A(E)$ for all $n \in \mathbb{N}$. Assume moreover that λ_n^A is injective when restricted to the virtually cyclic subgroup $Z_1 := \langle s, \pi^{-1}(\langle z_1 \rangle) \rangle$, $\lambda_n^A(A) = \lambda_n^A(Z_1)$ and that $C_{\Gamma}(\lambda_n^A(E))$ contains a non-abelian free subgroup which is quasi-convex in Γ . Then there exists a test sequence (λ_n) for G .*

Proof. Denote by $|\cdot|_{\Gamma}$ and $|\cdot|_G$ the word metric on Γ and G respectively. For all $n \in \mathbb{N}$ let $B_H(n) = \{h \in H \mid |h|_G \leq n\}$ and

$$L(n) = \max\{|\lambda_n^H(h)|_{\Gamma} \mid h \in B_H(n)\}.$$

Denote by $C_n := C_{\Gamma}(\lambda_n^A(E))$ the centralizer of $\lambda_n^A(E)$ in Γ . Let $F(a, b)$ be the quasi-convex non-abelian free subgroup of C_n which exists by assumption. As in the proof of Theorem 9.16 let $F(x, y) \leq F(a, b)$ be a non-abelian free subgroup of $F(a, b)$ such that any segment lying in the 10δ -neighborhood of both axis $Ax(a)$ and $Ax(abc^{-1})$ in X of a and any conjugate of b by an element $c \in \Gamma$ has length at most

$$\frac{1}{1000} \min\{|x|_{\Gamma}, |y|_{\Gamma}\}.$$

For all $n \geq 1$ we define

$$w_n := xyxy^2xy^3x \cdots xy^{n \cdot L(n)}x.$$

Let $W := \pi(A) = A/E$ and set $W^+ := \pi(A^+)$. We define the following sequence of automorphisms of W^+ :

For $n \in \mathbb{N}$ let $\alpha_n : W^+ \rightarrow W^+$ be defined by:

$$\alpha_n(z_j) = \sum_{i=1}^j n^{j-i} z_i \quad \text{for all } j \in \{1, \dots, k\}.$$

It follows from Corollary 4.13 in [ReWe] that after passing to a subsequence we can lift α_n to an automorphism (still denoted) $\alpha_n \in \text{Aut}(A)$ for all $n \in \mathbb{N}$.

We then define our test sequence $\lambda_n : G \rightarrow \Gamma$ on A as $\lambda_n|_A = c_{w_n} \circ \lambda_n^A \circ \alpha_n$ and on H as $\lambda_n|_H = \lambda_n^H$ for all $n \in \mathbb{N}$.

By further modifying the automorphisms (α_n) (if necessary) we can assume that $\lambda_n|_A$ dominates the growth of λ_n^H . It remains to show that λ_n is indeed a test sequence.

So let \tilde{G} be a group containing G and in abuse of notation let $\lambda_n : \tilde{G} \rightarrow \Gamma$ be an extension of λ_n which is short relative G for all $n \in \mathbb{N}$. Let (T, t_0) be the real tree into which (a convergent subsequence of) (λ_n) converges and let $L := \tilde{G}/\ker \lambda_n$. Assume that L is one-ended relative G and moreover that G does not act elliptically on T .

After applying the same same procedure as we did in the proof of Theorem 9.26 we end up with finitely many Γ -limit quotients of \tilde{G} which admit a decomposition \mathbb{D} relative G along finite subgroups with one vertex with vertex group Q , which contains G , while all other vertices are isomorphic to subgroups of Γ . Moreover $Q = V *_P M$, where $H \leq V$, P is virtually abelian and contains E and M is virtually abelian and contains A . In addition $z_1 \leq P^+/E_P$ and $\{z_2, \dots, z_k\} \leq (M^+/E_M) \setminus (P^+/E_P)$, where E_M, E_P denote the respective torsion subgroups of M and P .

To simplify notation we assume without loss of generality that $L = V *_P M$ is already of this type and moreover that L is one-ended relative G , i.e. \mathbb{D} contains only a single vertex. Let $\{u_1, \dots, u_l\}$ be a generating set for V . For every homomorphism $\varphi : \tilde{G} \rightarrow \Gamma$ that factors through the Γ -limit map $\eta : \tilde{G} \rightarrow L$, we denote by $\bar{\varphi} : L \rightarrow \Gamma$ the unique homomorphism such that $\varphi = \bar{\varphi} \circ \eta$. Let A_n be the following set:

$$A_n = \{\varphi_n \in \text{Hom}(L, \Gamma) \mid \varphi_n|_M = \bar{\lambda}_n|_M, \varphi_n|_G = \bar{\lambda}_n|_G, \\ \varphi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\}\}.$$

Since $\bar{\lambda}_n \in A_n$ the set A_n is non-empty. Let U be the set of all converging sequences (φ_n) , such that for all $n \in \mathbb{N}$, $\varphi_n \in A_n$ and

$$\sum_{i=1}^l |\varphi_n(u_i)|_\Gamma$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal limit groups. Let $(\varphi_n) \in U$ be a sequence that converges into such a maximal Γ -limit group, say N .

In particular $(\varphi_n|_V)$ converges into the action of a Γ -limit group V_1 on some real tree T

without a global fixpoint. Recall that $Z_1 = \langle s, \pi^{-1}(\langle z_1 \rangle) \rangle$. As in the proof of Theorem 9.16 (where we referred to the proof of Theorem 10.1) we can assume without loss of generality that N is one-ended relative G and moreover that $X := H *_E Z_1$ does not act elliptically on the limit tree T .

But now we are in the case of Theorem 9.16. $\lambda_n|_{Z_1}$ is by assumption an embedding postcomposed by conjugation with an element satisfying a $C'(1/n)$ small-cancellation property, while H acts elliptically on T . Hence by the same arguments as in the proof of Theorem 9.16 it follows that (finitely many quotients of) V_1 (still denoted V_1) have the structure $V_1 = R *_E C_1$ where E' is a finite group containing E , and C_1 is virtually cyclic and contains Z_1 as a subgroup of finite index.

Applying the same procedure as in the end of the proof of Theorem 9.26 yields (finitely many) Γ -limit quotients L_1, \dots, L_m of \tilde{G} such that every L_i has a decomposition \mathbb{D} along finite subgroups with one distinguished vertex group M and all other vertex group are isomorphic to subgroups of Γ . Moreover $M = V_1 *_E A_1$, where $H \leq V_1$, E' is a finite group containing E , and A_1 is virtually abelian and contains A as a subgroup of finite index. This completes the proof of the theorem. \square

9.4 Test sequences for Completions of Resolutions

Theorem 9.31. *Let $\text{Comp}(L)$ be the completion of a good well-structured resolution with completed resolution*

$$\text{Comp}(L) = \text{Comp}(L)_0 \xrightarrow{\eta_1} \text{Comp}(L)_1 \xrightarrow{\eta_2} \dots \xrightarrow{\eta_l} \text{Comp}(L)_l.$$

Then there exists a test sequence for $\text{Comp}(L)$.

Proof. We construct a test sequence for $\text{Comp}(L)$ iteratively from bottom to top along the completed resolution by essentially combining Theorem 9.3, Theorem 9.16, Theorem 9.20, Theorem 9.26 and Theorem 9.30.

Denote by \mathbb{B}_i the completed decomposition of $\text{Comp}(L)_i$ for all $i \in \{1, \dots, l\}$. Then \mathbb{B}_l is a Dunwoody decomposition of $\text{Comp}(L)_l$ with one vertex group, say H , isomorphic to Γ and all other vertex groups isomorphic to subgroups of Γ . By Theorem 5.18 there exists a restricted locally injective homomorphism $\varphi : \text{Comp}(L)_l \rightarrow \Gamma$ such that the centralizer in Γ of the image of each edge group of \mathbb{B}_l contains a quasi-convex non-abelian free group. We then construct the test sequence λ_n of $\text{Comp}(L)_l$ by setting $\lambda_n|_H = \text{id}$ and applying Theorem 9.16 and Theorem 9.20 finitely many times.

So now let $i < l$ and assume that we already have constructed a test sequence

$$(\lambda_n) \subset \text{Hom}(\text{Comp}(L)_{i+1}, \Gamma).$$

Let us first recall how we have constructed $\text{Comp}(L)_i$ and its completed decomposition \mathbb{B}_i out of $\text{Comp}(L)_{i+1}$ and the corresponding completed decomposition \mathbb{B}_{i+1} .

First we have replaced subgraphs of groups whose vertex groups were isomorphic to

subgroups of Γ by either isolated QH groups or by isolated virtually abelian groups. Afterwards we have added several virtually abelian flats to this graph of groups. Then we have added several (non-isolated) QH vertex groups, all amalgamated along their extended boundary subgroups and finally we have removed some subgraphs which were contained in the image of these QH subgroups under η_{i+1} . The resulting graph of groups \mathbb{B}_i is the completed decomposition of $\text{Comp}(L)_i$.

So let \mathbb{A}_1 be the (possibly disconnected) subgraph of \mathbb{B}_{i+1} which "survives" in \mathbb{B}_i , that is all vertices and edges which neither get replaced by isolated QH or virtually abelian groups nor which get removed after the addition of QH vertex groups. We then define our test sequence φ_n on \mathbb{A}_1 to be equal to λ_n . Let \mathbb{A}_2 be the (still possibly disconnected) subgraph of groups of \mathbb{B}_i which we get out of \mathbb{A}_1 after having added all the isolated QH and virtually abelian vertex groups. We define the test sequence φ_n on $\pi_1(\mathbb{A}_2)$ by applying finitely many times Theorem 9.30 and Theorem 9.3.

Now let \mathbb{A}_3 be the graph of groups which we get after having added all the virtually abelian flats to the graph of groups \mathbb{A}_3 . Note that \mathbb{A}_3 can still be disconnected. Applying Theorem 9.26 finitely many times yields a test sequence (still denoted) φ_n for $\pi_1(\mathbb{A}_3)$. The only thing left is to extend the test sequence onto the (non-isolated) QH vertex groups added in the construction of the test sequence. This is done by applying Theorem 9.3. Note that $\eta_{i+1} \circ \lambda_n$ plays the role of the homomorphism Θ_n in the assumption of Theorem 9.3. Hence we end up with a test sequence $\varphi_n : \text{Comp}(L)_i \rightarrow \Gamma$. \square

The following Lemma follows from the construction of the test sequences.

Lemma 9.32. *Let $\text{Comp}(\text{Res}(L))$ be a completed resolution of a completed Γ -limit group $\text{Comp}(L)$ and $s, t \in \mathbb{Z}$. Let moreover A be a virtually abelian vertex group in one of the completed decompositions appearing along $\text{Comp}(\text{Res}(L))$. Denote by Z_A the maximal free abelian subgroup of A . Let a be a basis element of Z_A . Then there exists a test sequence (λ_n) for $\text{Comp}(L)$, integers $m_n \in \mathbb{Z}$ and an element $g \in \Gamma$ which has infinite order and no non-trivial roots, such that $\lambda_n(a) = g^{m_n s + t}$.*

10 The Generalized Merzlyakov's Theorem over hyperbolic groups

Having constructed test sequences for completions of well-structured resolutions we are finally ready to prove a generalization of Theorem 1.18 in [Se2], respectively Theorem 2.3 in [Se7], that is a Generalized Merzlyakov's Theorem over hyperbolic groups with torsion.

Theorem 10.1 (Generalized Merzlyakov's Theorem). *Let $\Gamma = \langle a \rangle$ be a hyperbolic group, and let $R := \langle y, a \mid V(y, a) \rangle$ be a restricted Γ -limit group. Denote by V_y the basic definable set corresponding to R . Let $\text{Res}(R)$ be a good well-structured resolution of R , and let*

$\text{Comp}(\text{Res}(R))$ be the completion of the resolution $\text{Res}(R)$ with corresponding completed Γ -limit group $\text{Comp}(R)$. Let

$$\Sigma(x, y, a) : w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1$$

be a system of equations over Γ and let $v_1(x, y, a), \dots, v_r(x, y, a)$ be a collection of words in the alphabet $\{x, y, a\}$. Let

$$G_\Sigma := \langle x, y, a \mid V(y, a), w_1(x, y, a), \dots, w_s(x, y, a) \rangle.$$

Suppose that

$$\Gamma \models \forall y \in V_y \exists x : w_1(x, y, a) = 1, \dots, w_s(x, y, a) \wedge v_1(x, y, a) \neq 1, \dots, v_r(x, y, a) \neq 1.$$

Then there exist closures

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

of $\text{Res}(R)$ and for each index $1 \leq i \leq q$, there exists a generalized closure $\text{GCl}(\text{Res}(R))_i$ and a retraction

$$\pi_i : G_\Sigma *_R \text{GCl}(\text{Comp}(R))_i \rightarrow \text{GCl}(\text{Comp}(R))_i.$$

In addition for each $1 \leq i \leq q$ there exists a homomorphism

$$\Psi_i : \text{GCl}(\text{Comp}(R))_i \rightarrow \Gamma$$

that factors through the resolution $\text{GCl}(\text{Res}(R))_i$ such that $\Psi_i \circ \pi_i(v_j) \neq 1$ for all $j \in \{1, \dots, r\}$. $\pi_i(x)$ is called a formal solution.

If Γ is torsion-free then in addition $\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$ form a covering closure.

Remark 10.2. (1) We are still not sure if one can possibly always replace the generalized closure in the above theorem by an ordinary closure. In the torsion-free case this is always true. On the other hand if Γ is torsion-free hyperbolic and the words

$$w_1(x, y), \dots, w_s(x, y), v_1(x, y), \dots, v_r(x, y)$$

are coefficient free, then there exist coefficient free formal solutions but only in a generalized closure and not in an ordinary closure of the corresponding well-structured resolution.

(2) We expect the last claim of the theorem to also be true in the case with torsion, but the proof from the torsion-free case does no longer work in this setting.

(3) We expect that the assumption that $\text{Res}(R)$ is a good resolution can be dropped by either refining the construction of the MR diagram in [ReWe] to guarantee that every strict resolution in this diagram is already good, or by defining a new type of test sequences in the case that $\text{Res}(R)$ is not good.

Remark 10.3. An alternative way to formulate Theorem 10.1 is as follows:

Let $M_i := \text{GCl}(\text{Comp}(R))_i$, $\iota_i : R \rightarrow M_i$ the canonical embeddings into the closures of the completion of R and $\nu : R \rightarrow G_\Sigma$ the canonical (injective) map.

For all restricted homomorphisms $\varphi : R \rightarrow \Gamma$ there exists an extension $\tilde{\varphi} : M_i \rightarrow \Gamma$ of φ for some i and under the assumption of the theorem there exists an extension $\bar{\varphi} : G_\Sigma \rightarrow \Gamma$ making the following diagram commute.

$$\begin{array}{ccc} & G_\Sigma & \\ & \nearrow \nu & \searrow \exists \bar{\varphi} \\ R & \xrightarrow{\iota_i} M_i & \xrightarrow{\tilde{\varphi}} \Gamma \end{array}$$

Now the theorem states that there exists a map (not necessarily a retraction) $\pi : G_\Sigma \rightarrow M_i$ such that both sides of the following diagram commute:

$$\begin{array}{ccc} & G_\Sigma & \\ & \nearrow \nu & \searrow \exists \bar{\varphi} \\ R & \xrightarrow{\iota_i} M_i & \xrightarrow{\tilde{\varphi}} \Gamma \\ & \downarrow \pi & \end{array}$$

To prove Theorem 10.1 we have to plug together all results from the previous chapters.

Proof of Theorem 10.1. By Theorem 8.2 $\text{Comp}(R)$ is a Γ -limit tower. We proceed by induction on the height of the tower $\text{Comp}(R)$ over Γ . If the height is zero then necessarily $\text{Comp}(R)$ is of the form $\text{Comp}(R) = \Gamma$ and the claim is trivial. Hence we may assume that the height of $\text{Comp}(R)$ is at least one.

By Theorem 9.31 there exists a test sequence $(\lambda_n) \subset \text{Hom}(\text{Comp}(R), \Gamma)$ for $\text{Comp}(R)$. For all $n \geq 1$ by the assumption of the theorem we can extend $\lambda_n|_R : R \rightarrow \Gamma$ to a homomorphism $\bar{\lambda}_n : G_\Sigma \rightarrow \Gamma$, where

$$G_\Sigma = \langle x, y, a \mid V(y, a), w_1(x, y, a), \dots, w_s(x, y, a) \rangle,$$

such that $\bar{\lambda}_n(v_j) \neq 1$ for all $j \in \{1, \dots, r\}$. Note that we can view R as a subgroup of G_Σ . In particular $\bar{\lambda}_n|_R = \lambda_n$. Moreover for all $n \geq 1$ we can choose $\bar{\lambda}_n$ in such a way that $\bar{\lambda}_n(x)$ is shortest possible with respect to the word metric on Γ for the generating set $\{a\}$. Clearly $\bar{\lambda}_n$ can be extended to a homomorphism

$$\varphi_n : G := G_\Sigma *_R \text{Comp}(R) \rightarrow \Gamma,$$

such that $\varphi_n|_{\text{Comp}(R)} = \lambda_n$ for all $n \in \mathbb{N}$.

Definition 10.4. *If such a sequence of homomorphisms $(\varphi_n) \subset \text{Hom}(G, \Gamma)$ corresponding to a test sequence (and some shortest possible specializations $\{\varphi_n(x)\}$) converges, we call the obtained Γ -limit group T a test limit group with corresponding test limit map $\eta : G \rightarrow T$.*

On the set of test limit groups, we define a partial order \geq , by setting $T_1 \geq T_2$ if there exists an epimorphism $\pi : T_1 \rightarrow T_2$, such that $\eta^2 = \pi \circ \eta^1$, where $\eta^i : G \rightarrow T_i$ is the corresponding test limit map for $i \in \{1, 2\}$. We further say that $T_1 > T_2$ if $T_1 \geq T_2$ and $T_2 \not\geq T_1$.

Lemma 10.5. *There exist maximal elements in the set of all test limit groups, with the partial order defined above.*

Proof. Assume that there exists an infinite ascending sequence of test limit groups

$$T_1 < T_2 < T_3 < T_4 < \dots$$

For all $i \geq 1$, we denote the test limit map corresponding to T_i by η^i and the sequence of homomorphisms from G to Γ which converges into T_i by $\varphi_n^{(i)}$. Let B_i be the ball of radius i in $\text{Cay}(G)$ around the identity with respect to some fixed finite generating set. Since for all $i \geq 1$, $(\varphi_n^{(i)})$ is a stable sequence, there exists $n^{(i)} \in \mathbb{N}$, such that

$$\ker(\varphi_{n^{(i)}}^{(i)}) \cap B_i = \ker(\eta^i) \cap B_i.$$

After possibly increasing some of the $n^{(i)}$'s, the sequence $(\varphi_{n^{(i)}}^{(i)})_{i \in \mathbb{N}}$ is an extension of the test sequence (λ_n) . We denote the corresponding test limit group into which (a subsequence of) $(\varphi_{n^{(i)}}^{(i)})$ converges by T and the test limit map by $\eta : G \rightarrow T$. It remains to show that $T_i < T$ for all $i \in \mathbb{N}$. Assume that there exists $i_0 \in \mathbb{N}$ such that $T_{i_0} \not< T$, i.e. there exists no proper epimorphism $\pi : T \rightarrow T_{i_0}$ such that $\eta^{i_0} = \pi \circ \eta$. Hence there exists some element $g \in \ker \eta$, such that $g \notin \ker \eta^{i_0}$. Let $m = \max\{i_0, |g|\}$, where $|\cdot|$ denotes the metric on the Cayley graph of G with respect to the fixed generating set chosen before. Since $T_1 < T_2 < T_3 < T_4 < \dots$ is an infinite ascending sequence of test limit groups, for all $i \geq m$, $g \notin \ker \varphi_{n^{(i)}}^{(i)}$ and hence $g \notin \ker \eta$, a contradiction. \square

Lemma 10.6. *There exist only finitely many equivalence classes (with respect to the equivalence relation \leq) of maximal test limit groups.*

Proof. Assume that there exist infinitely many equivalence classes of maximal test limit maps T_1, T_2, T_3, \dots . For all $i \geq 1$, we again denote the test limit map corresponding to T_i by η^i and the sequence of homomorphisms from G to Γ which converges into T_i by $\varphi_n^{(i)}$. Let B_i be the ball of radius i in $\text{Cay}(G)$ around the identity with respect to some fixed finite generating set.

Since for all $i \geq 1$, B_i contains only finitely many elements, there exists a sequence

of pairwise distinct maximal test limit maps (which we denote for simplicity again by) $(\eta^i)_{i \in \mathbb{N}}$ such that for each $m, n \geq i$ the following hold:

$$\ker \eta^m \cap B_i = \ker \eta^n \cap B_i.$$

As in the proof of Lemma 10.5 for all $i \geq 1$ there exists $n^{(i)} \in \mathbb{N}$, such that

$$\ker(\varphi_{n^{(i)}}^{(i)}) \cap B_i = \ker(\eta^i) \cap B_i$$

and the sequence $(\varphi_{n^{(i)}}^{(i)})_{i \in \mathbb{N}}$ is an extension of the test sequence (λ_n) after possibly enlarging some of the $n^{(i)}$'s. Let T be the associated test limit group and $\eta : G \rightarrow T$ the corresponding test limit map. Since the sequence $(\eta^i)_{i \in \mathbb{N}}$ consists of pairwise distinct maximal test limit maps, after possibly removing η from the sequence, we have that $T \not\cong T_i$ for all $i \geq 1$.

Hence for all $i \geq 1$ there exists some element $g_i \in G$, such that $\eta^i(g_i) \neq 1$ and $\eta(g_i) = 1$. Since for all $i \geq 1$, $(\varphi_n^{(i)})$ is a stable sequence, by possibly enlarging the $n^{(i)}$'s we can assume that

$$\ker(\varphi_{n^{(i)}}^{(i)}) \cap B_i = \ker(\eta^i) \cap B_i$$

and $\varphi_{n^{(i)}}^{(i)}(g_i) \neq 1$. In particular none of the $\varphi_{n^{(i)}}^{(i)}$'s factors through η , since $\eta(g_i) = 1$ for all $i \geq 1$. By Theorem 6.5 in [ReWe] a subsequence of $(\varphi_{n^{(i)}}^{(i)})$ factors through η (recall that η is the limit map corresponding to the sequence $(\varphi_{n^{(i)}}^{(i)})$), a contradiction. \square

By Lemma 10.6 there exist only finitely many maximal test limit groups, which we denote by M_1, \dots, M_m . For all $i \in \{1, \dots, m\}$ we denote the test limit map corresponding to M_i by η^i . Moreover for all $i \in \{1, \dots, m\}$ let $(\varphi_n^{(i)})_{n \in \mathbb{N}} \subset \text{Hom}(G, \Gamma)$ be the extension of a test sequence, such that

$$M_i = G / \underline{\ker}(\varphi_n^{(i)}).$$

Clearly the images of the words v_j under the test limit maps η^i are non-trivial in the maximal test limit groups M_1, \dots, M_m .

Let $M \in \{M_1, \dots, M_m\}$ be one of these maximal test limit groups. For simplicity we denote the extension of a test sequence which converges into the action of M on the associated real tree T by (φ_n) and the corresponding test limit map by $\eta : G \rightarrow M$. We identify $\text{Comp}(R)$ and the words v_j with their images in M under η . Then there exists a sequence of homomorphisms $\bar{\varphi}_n : M \rightarrow \Gamma$ such that $\varphi_n = \bar{\varphi}_n \circ \eta$ for all $n \geq 1$. Let \mathbb{D} be a Dunwoody decomposition of M relative $\text{Comp}(R)$, i.e. $\pi_1(\mathbb{D}) = M$ and $\text{Comp}(R)$ is contained in a vertex group.

Suppose that $\text{Comp}(R) \leq M$ fixes a point in T . Denote by H the vertex group of \mathbb{D} containing $\text{Comp}(R)$ and by D_1, \dots, D_t the vertex group of \mathbb{D} which do not contain $\text{Comp}(R)$ and are not isomorphic to subgroups of Γ . Suppose that $t \geq 1$, i.e. there exists a vertex group in \mathbb{D} not isomorphic to a subgroup of Γ , which does not contain $\text{Comp}(R)$. Let D_{t+1}, \dots, D_s be the vertex groups of \mathbb{D} isomorphic to subgroups of Γ .

Let $\Gamma_1, \dots, \Gamma_s$ be isomorphic copies of Γ . Let moreover \mathbb{D}' be the graph of groups which we get by replacing the vertex groups D_1, \dots, D_s in \mathbb{D} by $\Gamma_1, \dots, \Gamma_s$ respectively and the vertex group H by Γ . For all $i \in \{t+1, \dots, s\}$ denote by $\iota_i : D_i \rightarrow \Gamma_i$ the canonical embedding. By the Bestvina-Feighn combination theorem $\pi_1(\mathbb{D}')$ is hyperbolic. Clearly $\bar{\varphi}_n \in \text{Hom}(M, \Gamma)$ induces a homomorphism (still denoted by)

$$\bar{\varphi}_n \in \text{Hom}(M, \pi_1(\mathbb{D}')).$$

For all $n \in \mathbb{N}$ we define the following set:

$$\begin{aligned} A_n = \{ \Psi_n \in \text{Hom}(M, \pi_1(\mathbb{D}')) \mid & \Psi_n|_H = \bar{\varphi}_n|_H, \Psi_n|_{\pi_1(D)} = \text{id}, \\ & \Psi_n|_{D_i} = \iota_i \text{ for } i \in \{t+1, \dots, s\} \text{ i.e. if } D_i \text{ is isomorphic to a subgroup of } \Gamma, \\ & \Psi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\} \}. \end{aligned}$$

Since $\bar{\varphi}_n \in A_n$, the set A_n is non-empty. We fix a finite generating set B of D_t . Let U be the set of all converging sequences (Ψ_n) , such that for all $n \in \mathbb{N}$, $\Psi_n \in A_n$ and

$$\sum_{b \in B} |\Psi_n(b)|_{\Gamma_t}$$

is minimal (with respect to the word metric on Γ_t). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal test limit groups. Let $(\Psi_n) \in U$ be a sequence that converges into such a maximal test limit group, say N .

In particular $(\Psi_n|_{D_t})$ converges either into a finite group or into the action of an infinite Γ -limit group \tilde{D}_t on some real tree T without a global fixpoint. Assume that \tilde{D}_t is one-ended. By the shortening argument T contains no axial or orbifold components and all stabilizers of edges in the discrete part of the output of the Rips machine are finite. Since \tilde{D}_t does not act with a global fixpoint on T , \tilde{D}_t admits a decomposition along a finite subgroup, a contradiction to the one-endedness assumption. Hence \tilde{D}_t is either finite or admits a decomposition along a finite subgroup and therefore \tilde{D}_t is a proper quotient of D_t . It follows that N is a proper quotient of the original maximal test limit group M .

It follows from the descending chain condition for Γ -limit groups (Proposition 5.14) that after repeating this procedure finitely many times, we end up with finitely many maximal test limit groups N_1, \dots, N_k such that for all $N \in \{N_1, \dots, N_k\}$ the Dunwoody decomposition \mathbb{D} relative $\text{Comp}(R)$ of N consists of one vertex with vertex group H which contains $\text{Comp}(R)$ and all other vertex groups are isomorphic to subgroups of Γ . To simplify notation we assume that our original test limit groups M_1, \dots, M_k are already of this type.

We continue with all the maximal test limit groups M_1, \dots, M_k in parallel. So let $M \in \{M_1, \dots, M_k\}$ be such a maximal test limit group with Dunwoody decomposition \mathbb{D} relative $\text{Comp}(R)$. Denote by v_0 the unique vertex of \mathbb{D} whose vertex group H

contains $\text{Comp}(R)$. Recall that $(\varphi_n) \subset \text{Hom}(G, \Gamma)$ is the extension of a test sequence which converges into the action of M on the associated real tree T and $\eta : G \rightarrow M$ is the corresponding test limit map. Then there exists a sequence of homomorphisms $\bar{\varphi}_n : M \rightarrow \Gamma$ such that $\varphi_n = \bar{\varphi}_n \circ \eta$ for all $n \geq 1$. (In general, i.e. without the simplifying assumption made above, we would have to precompose by a modular automorphism $\alpha_n \in \text{Mod}(M)$, i.e. $\varphi_n = \bar{\varphi}_n \circ \pi \circ \alpha_n \circ \eta$, where $\pi : M \rightarrow N$ is the limit map from M to N).

Suppose that $\text{Comp}(R)$ is still elliptic when acting on the limit tree T . Let \mathbb{D}' be the graph of groups which we get by replacing the vertex group H in \mathbb{D} by Γ and all other vertex groups by isomorphic copies of Γ . By the Bestvina-Feighn combination theorem $\pi_1(\mathbb{D}')$ is hyperbolic. Again $\bar{\varphi}_n \in \text{Hom}(M, \Gamma)$ induces a homomorphism (still denoted by)

$$\bar{\varphi}_n \in \text{Hom}(M, \pi_1(\mathbb{D}')).$$

For all $n \in \mathbb{N}$ we define the following set:

$$B_n = \{\Psi_n \in \text{Hom}(M, \pi_1(\mathbb{D}')) \mid \Psi_n|_{D_v} = \text{id for all } v \in VD \setminus \{v_0\}, \\ \Psi_n|_{\pi_1(D)} = \text{id}, \Psi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\}, \Psi_n|_{\text{Comp}(R)} = \bar{\varphi}_n|_{\text{Comp}(R)}\}.$$

Since $\bar{\varphi}_n \in B_n$, the set B_n is non-empty. We fix a generating set $\{h_1, \dots, h_m\}$ of H . Let U be the set of all converging sequences (Ψ_n) , such that for all $n \in \mathbb{N}$, $\Psi_n \in B_n$ and

$$\sum_{i=1}^m |\Psi_n(h_i)|_{\Gamma}$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal test limit groups. Let $(\Psi_n) \in U$ be a sequence that converges into such a maximal test limit group, say N .

In particular $(\Psi_n|_H)$ converges into the action of a Γ -limit group \tilde{H} on some real tree T without a global fixpoint. Assume that \tilde{H} is one-ended relative $\text{Comp}(R)$. Suppose that $\text{Comp}(R)$ fixes a point in T . Then it follows from the shortening argument (relative $\text{Comp}(R)$) that T contains no axial or orbifold components and all stabilizers of edges in the discrete part in the output of the Rips machine are finite. Since \tilde{H} does not act with a global fixpoint on T , \tilde{H} admits a decomposition along a finite subgroup relative $\text{Comp}(R)$, a contradiction to the one-endedness assumption. Hence \tilde{H} is a proper quotient of H . It follows that N is a proper quotient of the original maximal test limit group M .

It follows from the descending chain condition for Γ -limit groups (Proposition 5.14) that after repeating this procedure finitely many times we end up with finitely many maximal test limit groups N_1, \dots, N_k such that for all $N \in \{N_1, \dots, N_k\}$ the Dunwoody decomposition \mathbb{D} relative $\text{Comp}(R)$ of N consists of one vertex with vertex group H which contains $\text{Comp}(R)$, all other vertex groups are isomorphic to subgroups of Γ and

$\text{Comp}(R)$ does not fix a point when acting on the corresponding limit tree. To simplify notation we assume that our original test limit groups M_1, \dots, M_k are already of this type.

We continue with all the maximal test limit groups M_1, \dots, M_k in parallel. So let $M \in \{M_1, \dots, M_k\}$ be such a maximal test limit group with Dunwoody decomposition \mathbb{D} relative $\text{Comp}(R)$. Recall that $(\varphi_n) \subset \text{Hom}(G, \Gamma)$ is the extension of a test sequence which converges into the action of M on the associated real tree T and $\eta : G \rightarrow M$ is the corresponding test limit map. Then there exists a sequence of homomorphisms $\bar{\varphi}_n : M \rightarrow \Gamma$ such that $\varphi_n = \bar{\varphi}_n \circ \eta$ for all $n \geq 1$.

The graph of groups \mathbb{D} consists of a unique vertex v_0 whose vertex group H contains $\text{Comp}(R)$ and all other vertex groups are isomorphic to subgroups of Γ . Note that $\bar{\varphi}_n$ is still short with respect to the fixed generating set of H . Moreover $\text{Comp}(R)$ does not fix a point when acting on T .

We can describe the action of M on T more precisely. To do this we exploit the fact that $\bar{\varphi}_n|_{\text{Comp}(R)}$ is a test sequence. Recall that $\text{Comp}(R)$ has the structure of a Γ -limit tower

$$\Gamma = T_0 \leq T_1 \leq \dots \leq T_n = \text{Comp}(R).$$

Denote by \mathbb{B} the graph of groups decomposition of T_n corresponding to the top level tower structure, i.e. the graph of groups corresponding to the orbifold/virtually abelian flat or extension along a finite group which witnesses the tower structure of T_n over T_{n-1} .

Lemma 10.7. (1) *Assume that T_n has the structure of an orbifold flat over T_{n-1} and let*

$$1 \rightarrow E \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1$$

be the short exact sequence corresponding to the QH vertex group Q of \mathbb{B} . Then H inherits from its action on T an orbifold flat decomposition \mathbb{A} over a group V , which has one vertex v with vertex group Q' which contains the subgroup $Q \leq \text{Comp}(R)$ as a subgroup of finite index, and several vertices with vertex groups V_1, \dots, V_r corresponding to the various orbits of point stabilizers in the action of H on T . In addition Q' fits into the short exact sequence

$$1 \rightarrow E' \rightarrow Q' \rightarrow \pi_1(\mathcal{O}) \rightarrow 1.$$

Moreover the underlying graphs of \mathbb{A} and \mathbb{B} are isomorphic, edge groups in \mathbb{A} are isomorphic to finite index supergroups of the corresponding edge groups in \mathbb{B} and $T_{n-1} \leq V$.

(2) *Assume that T_n has the structure of a virtually abelian flat $T_n = T_{n-1} *_C K$ over T_{n-1} . Then there exist finitely many Γ -limit quotients N_1, \dots, N_p of the test limit group M and each N_i admits a decomposition \mathbb{D} along finite groups with one distinguished vertex group U and all other vertex groups of \mathbb{D} are isomorphic to subgroups of Γ . Moreover U admits a decomposition as an amalgamated product of the form*

$U = V *_{A_1} A$, where the subgroup A is virtually abelian and contains the virtually abelian subgroup K as a subgroup of finite index, A_1 contains C as a subgroup of finite index and $T_{n-1} \leq V$.

(3) If $T_n = T_{n-1} *_{E'} K$ is an extension of T_{n-1} along a finite subgroup and K is isomorphic to a one-ended subgroup of Γ , then there exist finitely many Γ -limit quotients N_1, \dots, N_p of the test limit group M and each N_i admits a decomposition \mathbb{D} along finite groups with one distinguished vertex group U and all other vertex groups of \mathbb{D} are isomorphic to subgroups of Γ . Moreover U admits a splitting of the form $U = V *_E P$, where E is a finite subgroup containing E' , $T_{n-1} \leq V$ and P is isomorphic to a subgroup of Γ containing K .

(4) If $T_n = T_{n-1} *_{E'}$ is an extension of T_{n-1} along a finite subgroup E' then H admits a splitting \mathbb{A} as an HNN-extension $M = V *_E$, where E is a finite subgroup containing E' and $T_{n-1} \leq V$.

Proof. This follows immediately from the definition of test sequences (Definition 9.2). \square

To improve readability we split the remaining part of the proof of Theorem 10.1 into several lemmas.

Lemma 10.8. *If the action of the maximal test limit group M satisfies the properties of case (3) or (4) in Lemma 10.7, i.e. $\text{Comp}(R) = T_n$ is an extension of T_{n-1} along a finite subgroup, then there exists finitely many closures*

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

of $\text{Res}(R)$ and for each index $1 \leq i \leq q$, there exists a generalized closure $\text{GCl}(\text{Res}(R))_i$ and a retraction

$$\pi_i : G_\Sigma *_{R} \text{GCl}(\text{Comp}(R))_i \rightarrow \text{GCl}(\text{Comp}(R))_i.$$

Proof. By assumption either $T_n = T_{n-1} *_{E'} K$ or $T_n = T_{n-1} *_{E'}$, where E' is some finite group and K is a one-ended subgroup of Γ . By Lemma 10.7 there exist finitely many Γ -limit quotients N_1, \dots, N_k of the test limit group M and each N_i admits a decomposition \mathbb{D} along finite groups with one distinguished vertex group H and all other vertex groups of \mathbb{D} are isomorphic to subgroups of Γ . Moreover H admits a splitting as $H = V *_E P$ in the first case and as $H = V *_E$ in the second case, where $T_{n-1} \leq V$, $K \leq P$, P is isomorphic to a subgroup of Γ and $E' \leq E$, where E is some finite group. To simplify notation we assume that $\{M_1, \dots, M_k\}$ is already of this type. So let $M \in \{M_1, \dots, M_k\}$. We only consider the amalgamated product case, the HNN case is similar. Let C_n be the following set:

$$C_n = \left\{ \Psi_n \in \text{Hom}(M, \pi_1(\mathbb{D}')) \mid \begin{aligned} &\Psi_n|_{D_v} = \text{id} \text{ for all } D_v \text{ isomorphic to a subgroup of } \Gamma, \\ &\Psi_n|_{\pi_1(D)} = \text{id}, \Psi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\}, \Psi_n|_P = \bar{\varphi}_n|_P, \\ &\Psi_n|_{\text{Comp}(R)} = \bar{\varphi}_n|_{\text{Comp}(R)} \end{aligned} \right\}.$$

Since $\bar{\varphi}_n \in C_n$ the set C_n is non-empty. We fix a generating set $\{u_1, \dots, u_n\}$ of V . Let U be the set of all converging sequences (Ψ_n) , such that for all $n \in \mathbb{N}$, $\Psi_n \in C_n$ and

$$\sum_{i=1}^n |\Psi_n(u_i)|_\Gamma$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal test limit groups. Let $(\Psi_n) \in U$ be a sequence that converges into such a maximal test limit group, say N .

In particular $(\Psi_n|_V)$ converges into the action of a Γ -limit group \tilde{V} on some real tree T without a global fixpoint. Moreover $(\Psi_n|_V)$ is the extension of a test sequence of T_{n-1} . Since T_{n-1} is a tower of height $n-1$, we can apply the induction hypothesis. Denote by \tilde{H} the limit group into which $\Psi_n|_H$ converges. It follows that \tilde{H} admits a decomposition along finite edge groups with one distinguished vertex h_0 whose vertex group H_0 is a closure of the tower $T_n = \text{Comp}(R)$ and all other vertices have vertex groups isomorphic to subgroups of Γ . Therefore N admits a decomposition along finite edge groups with one distinguished vertex whose vertex group is a closure of the tower $T_n = \text{Comp}(R)$ and all other vertices have vertex groups isomorphic to subgroups of Γ . Hence we can assume that every extension of a test sequence to a convergent sequence of homomorphisms from G to Γ factors through one of finitely many maximal test limit groups (still denoted) M_1, \dots, M_m with corresponding quotient maps η_1, \dots, η_m . Each M_i admits a decomposition \mathbb{A} along finite edge groups with one vertex v_0 with vertex group $\text{Cl}(\text{Comp}(R))_i$ which is a closure of $\text{Comp}(R)$ and all other vertex groups are isomorphic to subgroups of Γ . By construction the images of the words v_j under the maps η_1, \dots, η_m are non-trivial in these test limit groups.

The map $\eta_i : G \rightarrow \text{GCl}(\text{Comp}(R))_i$ can be extended in the obvious way to a retraction

$$G_\Sigma *_R \text{GCl}(\text{Comp}(R))_i \rightarrow \text{GCl}(\text{Comp}(R))_i$$

which maps the words v_j to non-trivial elements. □

Lemma 10.9. *If the action of the maximal test limit group M satisfies the properties of case (1) in Lemma 10.7, i.e. $\text{Comp}(R) = T_n$ is an orbifold flat over T_{n-1} , then there exist finitely many closures*

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

of $\text{Res}(R)$ and for each index $1 \leq i \leq q$, there exists a generalized closure $\text{GCl}(\text{Res}(R))_i$ and a retraction

$$\pi_i : G_\Sigma *_R \text{GCl}(\text{Comp}(R))_i \rightarrow \text{GCl}(\text{Comp}(R))_i.$$

Proof. By assumption T_n is an orbifold flat over T_{n-1} , i.e. there exists some finite group N , a cone-type orbifold \mathcal{O} and a short exact sequence

$$1 \rightarrow N \hookrightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1,$$

such that T_n admits a graph of groups decomposition \mathbb{B} with one vertex q with vertex group Q and several vertices v_1, \dots, v_r with vertex groups H_1, \dots, H_r . Moreover T_{n-1} admits a decomposition as a graph of groups with finite edge groups, whose vertex groups are precisely H_1, \dots, H_r . Edges adjacent to q in \mathbb{B} are either finite or have virtually cyclic edge groups which correspond to boundary components of \mathcal{O} and every boundary component of \mathcal{O} corresponds to precisely one adjacent edge.

Since we are in case (1) of Lemma 10.7 H inherits from its action on T an orbifold flat decomposition \mathbb{A} over V with the same underlying graph as \mathbb{B} . One vertex q' in \mathbb{A} has vertex group Q' , which fits into the short exact sequence

$$1 \rightarrow N' \rightarrow Q' \rightarrow \pi_1(\mathcal{O}) \rightarrow 1,$$

where N' is a finite group, $N \leq N'$ and the other vertices of \mathbb{A} have vertex groups V_1, \dots, V_r and $H_i \leq V_i$ for all $i \in \{1, \dots, r\}$. Edges adjacent to q' are isomorphic to finite index supergroups of the corresponding ones in \mathbb{B} . In particular $T_{n-1} \leq V$.

Let C_n be the following set:

$$C_n = \{ \Psi_n \in \text{Hom}(M, \pi_1(\mathbb{D}')) \mid \Psi_n|_{D_v} = \text{id for all } v \in VD \setminus \{v_0\}, \\ \Psi_n|_{\pi_1(D)} = \text{id}, \Psi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\}, \Psi_n|_{Q'} = \bar{\varphi}_n|_{Q'}, \\ \Psi_n|_{\text{Comp}(R)} = \bar{\varphi}_n|_{\text{Comp}(R)} \}.$$

Since $\bar{\varphi}_n \in C_n$ the set C_n is non-empty. We fix a generating set $\{u_1, \dots, u_n\}$ of V . Let U be the set of all converging sequences (Ψ_n) , such that for all $n \in \mathbb{N}$, $\Psi_n \in C_n$ and

$$\sum_{i=1}^n |\Psi_n(u_i)|_\Gamma$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal test limit groups. Let $(\Psi_n) \in U$ be a sequence that converges into such a maximal test limit group, say N .

In particular $(\Psi_n|_V)$ converges into the action of a Γ -limit group \tilde{V} on some real tree T without a global fixpoint. Moreover $(\Psi_n|_V)$ is the extension of a test sequence of T_{n-1} . Since T_{n-1} is a tower of height $n - 1$, we can apply the induction hypothesis and the claim follows in the same way as in the proof of Lemma 10.8. \square

Lemma 10.10. *If the action of the maximal test limit group M satisfies the properties of case (2) in Proposition 10.7, i.e. $\text{Comp}(R) = T_n$ is a virtually abelian flat over T_{n-1} , then there exist finitely many closures*

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

of $\text{Res}(R)$ and for each index $1 \leq i \leq q$, there exists a generalized closure $\text{GCl}(\text{Res}(R))_i$ and a retraction

$$\pi_i : G_\Sigma *_R \text{GCl}(\text{Comp}(R))_i \rightarrow \text{GCl}(\text{Comp}(R))_i.$$

Proof. By assumption T_n has the structure of a virtually abelian flat $T_n = T_{n-1} *_C K$ over T_{n-1} . It follows from Lemma 10.7 that there exist finitely many Γ -limit quotients N_1, \dots, N_k of the test limit group M and each N_i admits a decomposition \mathbb{D} along finite groups with one distinguished vertex group H_0 and all other vertex groups of \mathbb{D} are isomorphic to subgroups of Γ . Moreover H_0 admits a decomposition as an amalgamated product of the form $H_0 = V *_A A$, where the subgroup A is virtually abelian and contains the virtually abelian subgroup K as a subgroup of finite index, A_1 contains C as a subgroup of finite index and $T_{n-1} \leq V$. To simplify notation we assume that $\{M_1, \dots, M_k\}$ is already of this type. So let $M \in \{M_1, \dots, M_k\}$.

Let C_n be the following set:

$$\begin{aligned} C_n = \{ & \Psi_n \in \text{Hom}(M, \pi_1(\mathbb{D}')) \mid \Psi_n|_{D_v} = \text{id} \text{ for all } D_v \text{ isomorphic to a subgroup of } \Gamma, \\ & \Psi_n|_{\pi_1(D)} = \text{id}, \Psi_n(v_j) \neq 1 \text{ for all } j \in \{1, \dots, r\}, \Psi_n|_A = \bar{\varphi}_n|_A, \\ & \Psi_n|_{\text{Comp}(R)} = \bar{\varphi}_n|_{\text{Comp}(R)} \} \end{aligned}$$

Since $\bar{\varphi}_n \in C_n$ the set C_n is non-empty. We fix a generating set $\{u_1, \dots, u_n\}$ of V . Let U be the set of all converging sequences (Ψ_n) , such that for all $n \in \mathbb{N}$, $\Psi_n \in C_n$ and

$$\sum_{i=1}^n |\Psi_n(u_i)|_\Gamma$$

is minimal (with respect to the word metric on Γ). By the same arguments as in the proof of Lemma 10.5 and Lemma 10.6 the collection of all sequences from U factor through a finite collection of maximal test limit groups. Let $(\Psi_n) \in U$ be a sequence that converges into such a maximal test limit group, say N .

In particular $(\Psi_n|_V)$ converges into the action of a Γ -limit group \tilde{V} on some real tree T without a global fixpoint. Moreover $(\Psi_n|_V)$ is the extension of a test sequence of T_{n-1} . Since T_{n-1} is a tower of height $n - 1$, we can apply the induction hypothesis. Denote by \tilde{H}_0 the limit group into which $\Psi_n|_{H_0}$ converges. It follows that \tilde{H}_0 admits a decomposition along finite edge groups with one distinguished vertex h_1 whose vertex group H_1 is a closure of the tower $T_n = \text{Comp}(R)$ and all other vertices have vertex groups isomorphic to subgroups of Γ . Therefore N admits a decomposition along finite edge groups with one distinguished vertex whose vertex group is a closure of the tower $T_n = \text{Comp}(R)$ and all other vertices have vertex groups isomorphic to subgroups of Γ . Hence we can assume that every extension of a test sequence to a convergent sequence of homomorphisms from G to Γ factors through one of finitely many maximal test limit groups (still denoted) M_1, \dots, M_q , with corresponding quotient maps η_1, \dots, η_q . Each M_i admits a decomposition \mathbb{A}_i along finite edge groups with one vertex v_0 with vertex

group $\text{Cl}(\text{Comp}(R))_i$ which is a closure of $\text{Comp}(R)$ and all other vertex groups are isomorphic to subgroups of Γ . By construction the images of the words v_j under the maps η_1, \dots, η_q are non-trivial in these test limit groups. The claim follows. \square

Hence in all cases there exists for every $i \in \{1, \dots, m\}$ some homomorphism

$$\Psi_i : \text{GCl}(\text{Comp}(R))_i \rightarrow \Gamma,$$

such that $\Psi_i \circ \eta_i(v_j) \neq 1$ in Γ . It remains to show that if Γ is torsion-free the finitely many closures $\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$ form a covering closure.

So let from now on Γ be a torsion-free hyperbolic group and suppose that

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

do not form a covering closure. Note that in particular R and $\text{Comp}(R)$ are torsion-free. Then there exists a homomorphism $\varphi : R \rightarrow \Gamma$ that factors through $\text{Res}(R)$ which cannot be extended to a homomorphism of any of the closures $\text{Cl}(\text{Res}(R))_i$. By Lemma 8.5 φ can be extended to a homomorphism (still denoted φ) from $\text{Comp}(R)$ to Γ which factors through the completed resolution $\text{Comp}(\text{Res}(R))$. $\text{Comp}(R)$ has the structure of a Γ -limit tower

$$\text{Comp}(R) = T_l \geq \dots \geq T_1 \geq T_0 = \Gamma$$

and hence there are two possibilities:

- (a) There exists $i \in \{1, \dots, l\}$ such that $T_i = T_{i-1} *_C A$ is an abelian flat, i.e. either C is trivial or infinite free abelian and $A = \langle a_1, \dots, a_m \rangle$ is a free abelian subgroup of rank $m \geq 2$, and $\varphi|_A$ cannot be extended to the closure of A corresponding to any of the closures $\text{Cl}(\text{Comp}(R))_1, \dots, \text{Cl}(\text{Comp}(R))_q$.
- (b) There exists $i \in \{1, \dots, l\}$ such that $T_i = T_{i-1} * H$, where H is isomorphic to a subgroup of Γ and $\varphi|_H$ cannot be extended to the closure of H corresponding to any of the closures $\text{Cl}(\text{Comp}(R))_1, \dots, \text{Cl}(\text{Comp}(R))_q$.

Assume we are in the first case. Denote by f_1, \dots, f_q the respective embeddings of the free abelian group A into $\text{Cl}(\text{Comp}(R))_1, \dots, \text{Cl}(\text{Comp}(R))_q$. For all $i \in \{1, \dots, q\}$ there exists a maximal free abelian subgroup $\tilde{A}_i \leq \text{Cl}(\text{Comp}(R))_i$ that contains $f_i(A)$ as a subgroup of finite index. Following Remark 8.10 we can either associate a subgroup $U_{f_i} \leq \mathbb{Z}^m$ to every embedding f_i (if $C = 1$) or we can associate a coset $K_{f_i} + U_{f_i} \leq \mathbb{Z}^m$ to every embedding f_i (if $C \neq 1$). We assume without loss of generality that we can associate a subgroup $U_{f_i} \leq \mathbb{Z}^m$ to every embedding f_i (the other case is identical). Since $\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$ do not form a covering closure, the union of the subgroups $\bigcup_{i=1}^q U_{f_i}$ does not cover \mathbb{Z}^m by Remark 8.10. Hence there exists an element

$$p = (p_1, \dots, p_m) \in \mathbb{Z}^m \setminus \bigcup_{i=1}^q U_{f_i}.$$

Let $d_i := |\mathbb{Z}^m : U_{f_i}|$ for all $i \in \{1, \dots, q\}$. Then clearly for every element $v \in \mathbb{Z}^m$

$$d_1 \cdots d_q v + p \notin U_{f_1, \dots, f_q}.$$

By Lemma 9.32 for all $n \in \mathbb{N}$ there exist elements $v^{(n)} \in \mathbb{Z}^m$ and a test sequence $(\lambda_n) \subset \text{Hom}(\text{Comp}(R), \Gamma)$, such that

$$\lambda_n(a_i) = u^{d_1 \cdots d_q v_i^{(n)} + p_i}$$

for all $i \in \{1, \dots, m\}$ and some element $u \in \Gamma$ with no root. By the assumption of the theorem λ_n can be extended to a homomorphism (still denoted) $\lambda_n : G_\Sigma \rightarrow \Gamma$ such that $\varphi_n(v_j) \neq 1$ for all $j \in \{1, \dots, r\}$. By the construction of $\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$ in the proof, there exist $i \in \{1, \dots, q\}$, quotient maps η_1, \dots, η_p , modular automorphism $\alpha_1, \dots, \alpha_p$ and a homomorphism $\bar{\lambda}_n : \text{Cl}(\text{Comp}(R))_i \rightarrow \Gamma$ such that

$$\lambda_n = \bar{\lambda}_n \circ \eta_p \circ \alpha_p \circ \dots \circ \eta_1 \circ \alpha_1.$$

Moreover $\eta_p \circ \alpha_p \circ \dots \circ \eta_1 \circ \alpha_1|_A = f_i$. Hence $\lambda_n = \bar{\lambda}_n \circ f_i$ and therefore by Remark 8.10 $d_1 \cdots d_q v^{(n)} + p \in U_{f_i}$ a contradiction to the choice of p . Hence

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

is a covering closure.

So now assume that we are in the second case. Again denote by f_1, \dots, f_q the respective embeddings of the subgroup H into $\text{Cl}(\text{Comp}(R))_1, \dots, \text{Cl}(\text{Comp}(R))_q$. For all $i \in \{1, \dots, q\}$ denote by V_i the maximal subgroup of $\text{Cl}(\text{Comp}(R))_i$ which is isomorphic to a subgroup of Γ and contains $f_i(H)$. By assumption $\varphi|_H$ cannot be extended to V_1, \dots, V_q . Let $(\lambda_n) \subset \text{Hom}(\text{Comp}(R), \Gamma)$ be a test sequence. Then $\lambda_n|_H$ is an embedding of H into Γ . Changing $\lambda_n|_H$ to $\varphi|_H$ yields a new test sequence, say (Ψ_n) .

Again by the construction of $\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$ in the proof, there exist $i \in \{1, \dots, q\}$, quotient maps η_1, \dots, η_p , modular automorphism $\alpha_1, \dots, \alpha_p$ and a homomorphism $\bar{\Psi}_n : \text{Cl}(\text{Comp}(R))_i \rightarrow \Gamma$ such that

$$\Psi_n = \bar{\Psi}_n \circ \eta_p \circ \alpha_p \circ \dots \circ \eta_1 \circ \alpha_1.$$

Moreover $\eta_p \circ \alpha_p \circ \dots \circ \eta_1 \circ \alpha_1|_H = f_i$. Hence $\Psi_n|_H = \bar{\Psi}_n \circ f_i$ and therefore

$$\varphi|_H = \Psi_n|_H = \bar{\Psi}_n|_{V_i} \circ f_i,$$

a contradiction to the assumption that $\varphi|_H$ cannot be extended to V_i . We have shown that

$$\text{Cl}(\text{Res}(R))_1, \dots, \text{Cl}(\text{Res}(R))_q$$

is a covering closure. □

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Einige der Ergebnisse der Dissertation sind in folgender Publikation enthalten:

- Heil, Simon, *JSJ decompositions of doubles of free groups*. Preprint, arXiv:1611.01424. Zur Publikation eingereicht.

Kiel, den 26. September 2018

(Simon Heil)