

The Thickness of Left-Invariant Metrics on Compact Connected Lie Groups

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Abstract

The subject of the present thesis is to study the thickness

$$\theta(Q) = \frac{\text{dia}(G, Q)^d}{\text{vol}(G, Q)}$$

of left-invariant riemannian metrics Q on compact connected d -dimensional Lie groups G with respect to the existence of globally or locally thinnest metrics. This is a variation of a problem posed by Berger in [Ber03] concerning the identification of particularly good riemannian metrics on compact manifolds.

If the group in question is abelian, the problem is equivalent to the covering problem of the integral lattice of the group. In particular, theorems from the theory of lattices may be applied to answer the question of the existence of a thinnest left-invariant metric in the affirmative. If the dimension of the group is sufficiently small, a classification of locally thinnest left-invariant metrics is obtained additionally. By contrast, the existence of arbitrarily thin left-invariant metrics on all non-abelian groups is proved using the theory of Carnot–Carathéodory metrics.

Special attention is paid to simple Lie groups. It is shown that the bi-invariant metrics of certain simple Lie groups with finitely many antipodes may be continuously deformed into thinner left-invariant metrics by shrinking a maximal torus. The affected class of Lie groups consists of the simply-connected Lie groups of the infinite families A_n , B_n and D_n , the simply-connected exceptional Lie groups E_6 and E_7 as well as $\text{SO}(2n)$ for $n \geq 2$. The existence of locally thinnest left-invariant metrics on these groups remains unsettled.

Finally, it is proved that the bi-invariant metrics of the Lie group $\text{PSU}(n+1)$ for $n \geq 1$ are locally optimal left-invariant metrics, meaning that their thickness does not decrease under any deformations except (possibly) those contained in a certain zero set. To this end, an infinitesimal connection between the covering radii of the integral lattices of the maximal tori and the diameter of the Lie group under volume-preserving deformations is revealed. It remains unknown whether the bi-invariant metrics are locally thinnest left-invariant metrics, except in the affirmative case of $\text{PSU}(2)$. The problematic zero set is described as a subset of deformations that preserve the volume of every maximal torus up to first order. The first derivatives of these deformations may be described as symmetric trace-free endomorphisms of the Lie algebra that are perpendicular to the second Cartan power. Their effect on the covering radii of the integral lattices is studied using an adaptation of Bleicher’s theorem [Ble62].

Zusammenfassung

Gegenstand der vorliegenden Dissertationsschrift ist die Untersuchung der Dichte

$$\theta(Q) = \frac{\text{dia}(G, Q)^d}{\text{vol}(G, Q)}$$

links-invarianter riemannscher Metriken Q auf kompakten zusammenhängenden d -dimensionalen Liegruppen G hinsichtlich der Existenz global oder lokal dünnster Metriken. Es handelt sich dabei um eine Variante des von Berger in [Ber03] formulierten Problems über die Bestimmung besonders gutartiger riemannscher Metriken auf kompakten Mannigfaltigkeiten.

Liegt eine abelsche Gruppe vor, so ist das Problem äquivalent zu dem Überdeckungsproblem des Einheitsgitters der Gruppe. Insbesondere lassen sich Sätze aus der Theorie der Gitter anwenden, um die Frage nach der Existenz einer dünnsten Metrik auf diesen Gruppen positiv zu beantworten. Ist die Dimension der Gruppe ausreichend klein, ergibt sich zusätzlich eine Klassifikation aller lokal dünnsten links-invarianten Metriken. Im Gegensatz dazu wird die Existenz von beliebig dünnen links-invarianten Metriken auf allen nicht-abelschen Liegruppen mit Hilfe der Theorie der Carnot–Carathéodory-Metriken nachgewiesen.

Besondere Aufmerksamkeit wird einfachen Liegruppen gewidmet. Es wird gezeigt, dass die bi-invarianten Metriken bestimmter einfacher Liegruppen mit endlich vielen Antipoden durch das Zusammenziehen eines maximalen Torus in stetiger Weise zu dünneren links-invarianten Metriken verformt werden können. Die betroffene Klasse von Liegruppen besteht aus den einfach-zusammenhängenden Liegruppen der unendlichen Familien A_n, C_n und D_n , den einfach-zusammenhängenden exzeptionellen Liegruppen E_6 und E_7 sowie $SO(2n)$ für $n \geq 2$. Die Existenz lokal dünnster links-invarianter Metriken auf diesen Gruppen bleibt weiter ungeklärt.

Schließlich wird bewiesen, dass die bi-invarianten Metriken der Liegruppe $PSU(n + 1)$ für $n \geq 1$ lokal optimale links-invariante Metriken sind, was bedeutet, dass ihre Dichte unter keiner Deformation kleiner wird, mit der möglichen Ausnahme solcher Deformationen, die in einer bestimmten Nullmenge enthalten sind. Zu diesem Zweck wird ein infinitesimaler Zusammenhang zwischen den Überdeckungsradien der Einheitsgitter maximaler Tori und dem Durchmesser der Liegruppe unter volumenerhaltenden Verformungen hergestellt. Die Frage, ob die bi-invarianten Metriken lokal dünnste Metriken sind, bleibt

unbeantwortet, außer im Falle von $PSU(2)$, in dem sie positiv beantwortet werden kann. Die problematische Nullmenge lässt sich beschreiben als Teilmenge von Deformationen, die in erster Näherung das Volumen eines maximalen Tori ändern. Die ersten Ableitungen solcher Verformungen lassen sich beschreiben als die symmetrischen spurfreien Endomorphismen der Liealgebra, die senkrecht auf der zweiten Cartanpotenz stehen. Ihre Wirkung auf die Überdeckungsradien der Einheitsgitter wird anhand einer Bearbeitung des Satzes von Bleicher [Ble62] untersucht.

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Introduction

In 1958, Thom posed the famous question

Given a compact, differentiable manifold, does it carry a *best*, or a family of best, Riemannian structure(s)? [Ber03, Ch. III, p. 63]

to Berger, the main intricacy of which, as LeBrun observes in [LeB04], does not lie in solving any particular geometric problem, but in interpreting or defining the term *best*. The problem now sits at the top of Yau's list of *Open Problems in Geometry* [Yau93] in the following form: "Find a general method to construct canonical metrics on a compact Riemannian manifold." [Yau93, Sec. I, p. 1]

Berger's own approach to the problem is to take a functional

$$F: \mathcal{RS}(M) \rightarrow \mathbb{R}_+$$

defined on the space $\mathcal{RS}(M) = \mathcal{RM}(M)/\text{Diff}(M)$ of Riemannian structures over a compact manifold M and then study the infimum $\inf_{g \in \mathcal{RS}(M)} F(g)$. He proceeds to ask the following questions:

1. Is $\inf_{\mathcal{RS}(M)} F(g)$ zero or positive?
2. If positive, then is it attained by some Riemannian metric g on M ? Such a g will of course be called a best metric.
3. When these metrics exist, how can we classify best metrics? This is the moduli problem.
4. Compute $\inf_{\mathcal{RS}(M)} F$ for various F and frequently encountered manifolds M .
5. [...] Classify the manifolds M for which $\inf_{\mathcal{RS}(M)} F$ is positive and those for which it is zero. [Ber03, Question 266, p. 502]

The particular choice of the functional F remains deliberately open. Usual choices are products of two counteracting geometric invariants. For instance, Berger's suggestions include $F(g) = \text{dia}(M, g)^2 \cdot \sup |K_g|$. The best metrics described by F are those $g_0 \in \mathcal{RS}(M)$ whose diameter is minimal amongst all $g \in \mathcal{RS}(M)$ with $\sup |K_g| = \sup |K_{g_0}|$, or, equivalently, all those $g_0 \in \mathcal{RS}(M)$ so that $\sup |K_{g_0}|$ is minimal amongst all $g \in \mathcal{RS}(M)$ with $\text{dia}(g) = \text{dia}(g_0)$.

Questions 1 and 5 are answered for this choice of F by the Gromov–Ruh theorem, which asserts that there exists a critical $\varepsilon = \varepsilon(n) > 0$ so that for all n -dimensional compact manifolds M , $\inf_{g \in \mathcal{R}\mathcal{S}(M)} F(g) < \varepsilon$ if and only if M is an infranil manifold, i.e. if there exists a nilpotent Lie group N and a discrete subgroup $\Gamma \subseteq \text{Aut}(N) \rtimes \text{L}(N)$ so that N/Γ is diffeomorphic to M . Moreover, $\inf_{g \in \mathcal{R}\mathcal{S}(M)} F(g) = 0$ in this case, and such a manifold is said to *collapse to a point (with bounded curvature)*.

As Nabutovsky and Weinberger show, for every compact connected manifold M of dimension $n \geq 5$, there are infinitely many local minima of this particular F on the Gromov–Hausdorff closure of the moduli space of riemannian metrics g on M with $|K_g| \leq 1$. In particular, it cannot be expected that there are straightforward answers to questions 2 and 3. For details, see [Wei05] or [Ber03, §11.5].

In addition to the approach to best metrics using least curved metrics, Berger also mentions *purely geometric* functionals like $F(g) = \text{vol}(M, g) / \text{inj}(M, g)^n$ and the Hilbert–Einstein functional $F(g) = \int_M \text{scal}_g / \text{vol}(M, g)^{(n-2)/n}$, whose critical points are the famous Einstein metrics. The reader is encouraged to study [Ber03, Ch. 11] and the references therein to learn more about these topics.

The approach to characterise the best geometric objects as optima of a functional defined on their configuration space is by no means a recent development. In Virgil’s *Aeneid*, Queen Dido “bargained to buy as much land as she could enclose with an oxhide. So she cut the hide into thin strips, and then faced, and presumably solved, the problem of enclosing the largest possible area within a given perimeter—the isoperimetric problem.” [Blå05, Sec. 1, p. 526] The problem that Dido allegedly solved was minimising the ratio of squared curve length to enclosed area $L(c)^2/A(c)$ on the moduli space of piecewise differentiable simply-closed curves. Again, the considered functional is composed of two counteracting geometric invariants, each restraining the other.

In the present thesis, the following modification of Berger’s question is proposed: Replace M , $\mathcal{R}\mathcal{M}(M)$ and $\mathcal{R}\mathcal{S}(M)$ by a d -dimensional compact connected Lie group G with Lie algebra \mathfrak{g} , the space of left-invariant metrics $\tilde{\mathfrak{M}}$ and the moduli space \mathfrak{M} of left-invariant structures on G , i.e. isometry classes of left-invariant metrics on G , respectively, and consider a functional $F: \mathfrak{M} \rightarrow \mathbb{R}_+$. Berger’s questions may then be paraphrased as follows:

- (1) Classify the compact connected Lie groups G for which $\inf_{Q \in \mathfrak{M}} F(Q)$ is positive and those for which it is zero.
- (2) If $\inf_{Q \in \mathfrak{M}} F(Q)$ is positive, does there exist a global minimum of F ? In general, do there exist local minima?
- (3) Classify (local) minima of F .

The question regarding the existence and classification of minima has been relaxed to allow *local* minima—the general scarcity of global minima on the one hand and results like those of Nabutovsky and Weinberger on the other suggest that local minima are a proper subject to study. Note that Berger’s questions 1, 4 and 5 were merged into (2).

In this thesis, a complete answer to (1) and partial answers to (2) and (3) will be given for the particular choice of the *thickness*

$$\theta: \widetilde{\mathfrak{M}} \rightarrow \mathbb{R}_+, \quad \theta(Q) = \frac{\text{dia}(G, Q)^d}{\text{vol}(G, Q)}$$

in place of F .

The definition of θ is inspired by the *normalised thickness* $\theta(\Lambda) = \mu(\Lambda)^n / (\det \Lambda)^{1/2}$ of a lattice Λ in an n -dimensional inner product space V , where

$$\mu(\Lambda) = \max_{x \in V} \min_{v \in \Lambda} \|x - v\|$$

is the *covering radius* and $(\det \Lambda)^{1/2}$ is the volume of a fundamental domain of Λ . But unlike the thickness of their Lie group counterparts, the thickness of lattices has been studied extensively in an attempt to solve the *lattice covering problem*, the search for (local) minima of the normalised thickness, as the reader will notice by a glimpse at [CS99]. The main theme of this thesis is that the thickness of metrics that are almost bi-invariant can be described in terms of the thickness of the integral lattices of *maximal tori*, i.e. maximal connected abelian subgroups of the underlying group. The progress in the lattice covering problem is surveyed in [CS99, Ch. 2, Sec. 1.3], in Sec. 1.1, and further below.

In addition to the adapted questions above, the structure of the moduli space \mathfrak{M} is considered. A space very similar to \mathfrak{M} has already been studied by Kodama, Takahara and Tamaru in [KTT11], where they define \mathfrak{PM} as the quotient $\widetilde{\mathfrak{M}} / \mathbb{R}_+ \times \text{Aut}(\mathfrak{g})$. They hope to discover good left-invariant metrics on (noncompact) Lie groups by studying points $Q \in \widetilde{\mathfrak{M}}$ whose $\mathbb{R}_+ \times \text{Aut}(\mathfrak{g})$ -orbits are well behaved. One might therefore say that their measure of quality of a left-invariant metric is the size of the isometry group. On compact Lie groups, the best metrics in this sense would be the bi-invariant metrics, whose isometry group contains the full automorphism group. Kodama, Takahara and Tamaru prove a characterisation result for Lie groups with $|\mathfrak{PM}| = 1$ previously obtained by Lauret [Lau03] and an existence result for Lie groups with $\dim \mathfrak{PM} = 1$ in all dimensions $d \geq 3$.

Question (1) may be interpreted as a question about collapsing Lie groups to a point through left-invariant metrics of constant volume. In fact, if Q_k is a sequence in \mathfrak{PM} , it may be rescaled to constant volume, so that $\theta(Q_k)$ only measures the diameter of the metric. The question calls for a classification of Lie groups that allow such a collapse.

Returning to the general setting for a moment, F is called differentiable if $\tilde{F}: \tilde{\mathfrak{M}} \rightarrow \mathbb{R}_+$ is differentiable and determines F by descent. A theorem by Bleecker implies that the bi-invariant metric is at least a critical point of \tilde{F} . In fact, Bleecker proved that a riemannian manifold (M, g) is a critical point of all fully-invariant differentiable riemannian functionals $\mathcal{RM}(M) \rightarrow \mathbb{R}_+$ if and only if (M, g) is isometric to a homogeneous space $(G/H, Q)$ whose isotropy group H acts irreducibly on the tangent space of M and which is equipped with the (up to homothety unique) G -invariant metric [Ble79] [Bes08, 7.43–7.44]. Of course, if G is a simple Lie group, this applies to any bi-invariant metric on $G = (G \times G)/G$ [WZ91]. This extremal property, in addition to the maximality property of their isometry groups, makes the bi-invariant metrics canonical candidates for best metrics. But one of the many subtleties of the presented problem is that many functionals on \mathfrak{M} , the thickness θ included, do not fall into the category of differentiable functionals. As it turns out, the partial answer to (2) will reveal that for many compact connected simple Lie groups, the bi-invariant metrics are *not best metrics* for the particular (and apparently rather peculiar) choice of θ as functional.

Methods and Results of the Thesis

The proofs of the main theorems are preceded by a discussion of the general properties of $\tilde{\mathfrak{M}}$ and \mathfrak{M} . Using results of D’Atri and Ziller [DZ79], it is shown that \mathfrak{M} is a quotient of $\tilde{\mathfrak{M}}$ modulo the action of $\text{Aut}(G)$ by pullback (see Sec. 4.3). In particular, $\mathfrak{PM} = \mathfrak{M} / \mathbb{R}_+$ coincides with the space of left-invariant metrics as defined by Kodama, Takahara and Tamaru, except for $G = \text{SO}(8)$ due to the symmetry of the Dynkin diagram D_4 . Moreover, this characterisation of \mathfrak{M} makes it possible to prove slice theorems akin to those obtained by Ebin for $\mathcal{RM}(M)$ in [Ebi68]. That includes a discussion of the dense set of *generic* left-invariant metrics, i.e. those left-invariant metrics whose isometry group is minimal.

The answer to the questions (1)–(3) depends on whether the group G under consideration is abelian or non-abelian. If G is abelian and equipped with a left-invariant metric, there exists a lattice $\Lambda \subseteq \mathfrak{g}$ so that G is isometric to \mathfrak{g}/Λ . The lattice Λ , called the *integral lattice* of G , completely determines the geometry of G .

Consider an arbitrary lattice $\Lambda \subseteq \mathbb{R}^n$. Then $\mu(\Lambda) = \max_{x \in \mathbb{R}^n} \min_{v \in \Lambda} \|x - v\|$, the covering radius, is the smallest number $r > 0$ so that \mathbb{R}^n can be covered by balls of radius r centred at the points of the lattice Λ . The effectiveness of this *lattice covering* is measured by

$$\theta(\Lambda) = \frac{\mu(\Lambda)^n}{(\det \Lambda)^{1/2}},$$

the normalised thickness of Λ . The number $(\det \Lambda)^{1/2}$ is the volume of the Dirichlet fundamental domain \mathcal{V} of Λ , also called the *Voronoi cell* of Λ . The thinner a lattice is, the smaller the overlap between the balls of the covering, i.e. the more effective it is.

The problem of finding the most effective lattice coverings is called the *lattice covering problem*. The reader is cordially invited to learn more about the lattice covering problem from [CS99, Ch. 2, Sec. 1.3], or from the first chapter of this thesis and the references therein.

For the sake of brevity, only the most important results on the lattice covering problem are stated here. *Hlawka's theorem* asserts that in every dimension, there exists a globally thinnest lattice [Hla49]. Such lattices are unknown except in dimensions $n \leq 5$, where the unique thinnest lattice is equal to the dual lattice A_n^* of the root lattice A_n . The lattice A_n^* is also known as *Voronoi's principal lattice* and enjoys many preferable properties. For instance, Bleicher has proved that A_n^* is a locally thinnest lattice for all n [Ble62]. The other locally thinnest lattices are classified in dimensions $n \leq 4$. In higher dimensions, the hope for such a classification is small. For instance, the number of locally thinnest lattices in dimension 5 lies between 216 and 218 [SV06]. Moreover, many canonical root lattices have been revealed to either not be locally thinnest lattices, or, worse, so called *pessima* [DSSV12, Sec. 6, Tab. 4.2] [Sch09, Sec. 5.6]: The measure of the set of directions in which the thickness of these lattices increases is zero. These discoveries have inspired the use of the notion of *optimum* in one of the main theorems of this thesis.

Returning to the abelian Lie group G and its integral lattice Λ , the thickness is determined by Λ , and the questions (1)–(3) are equivalent to aspects of the lattice covering problem of the integral lattice. Hlawka's theorem implies the existence of a globally thinnest left-invariant metric (see Sec. 4.1), and Bleicher's theorem and the classification from [SV06] provide concrete examples. Of course, many more major and minor statements about the thickness of lattices can be translated into statements about the behaviour of θ in this fashion. Once again, the reader must be referred to [CS99, Ch. 2, Sec. 1.3] or the survey given in the present thesis (see Sec. 1.1).

Replacing the abelian Lie group with a non-abelian Lie group changes the behaviour of θ considerably:

Theorem A. *Let G be a compact connected Lie group. Then $\inf_{Q \in \mathfrak{M}} \theta(Q) = 0$ if and only if G is non-abelian.*

Note that Theorem A fully answers question (1). The application of Hlawka's theorem shows that abelian Lie groups satisfy $\inf_{Q \in \mathfrak{M}} \theta(Q) > 0$. Conversely, arbitrarily thin metrics on non-abelian Lie groups are explicitly constructed as follows (see Sec. 4.5): Start with a bi-invariant metric Q , split the Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ into the semisimple part \mathfrak{s} and the centre \mathfrak{z} , take $0 \neq X \in \mathfrak{s}$, expand $\mathbb{R} \cdot X$ by the factor e^s and shrink X^\perp homothetically to keep the volume constant. The theory of Carnot–Carathéodory metrics shows that every point of G is joined by a curve tangent to the distribution X^\perp of horizontal left-invariant vector fields. This implies an upper bound of $O(e^{-s})$ on the thickness.¹

¹ s will always be used as time variable for variations of left-invariant metrics.

Theorem A shows the existence of what might be construed as a badly behaved *boundary* or *end* of \mathfrak{PM} . In Sec. 4.6, the boundary points are interpreted and studied using the notion of *degenerations of the Lie bracket*. More specifically, every sequence $Q_k \in \mathfrak{PM}$ with $\theta(Q_k) \rightarrow 0$ may be transformed into a sequence of Lie algebras (\mathfrak{g}, μ_k) , now equipped with a fixed metric on \mathfrak{g} . The limit behaviour of Q_k is thereby captured in terms of a *degeneration* of the Lie algebra (\mathfrak{g}, μ) of G as defined in [Lau03], i.e. a certain Lie algebra (\mathfrak{g}, λ) so that λ lies in the closure of the orbit \mathcal{O}_μ of μ under the basis change action

$$\varphi.\mu(x, y) = \varphi(\mu(\varphi^{-1}(x), \varphi^{-1}(y)))$$

of $\mathrm{GL}(\mathfrak{g})$ on the variety of Lie brackets on \mathfrak{g} . It is proved that, in the case of the deformation through Carnot–Carathéodory metrics defined above, the limit point is the direct sum of a Heisenberg algebra and an abelian ideal. In particular, the limit is almost flat.

While Theorem A excludes the existence of a thinnest metric on non-abelian Lie groups, the existence of locally thinnest metrics remains unsolved. It seems perfectly reasonable to expect that every bi-invariant metric of a compact connected simple Lie group is a locally thinnest metric. After all, bi-invariant metrics have shown great promise when differentiable functionals were under consideration, and their isometry group is maximal (see above). The next two theorems explore this assertion. Theorem B is quite staggering, as it provides a rich family of counterexamples.

Theorem B. *Let G be one of the following compact connected simple Lie groups: $\mathrm{SU}(n+1)$ for $n \geq 1$, $\mathrm{Spin}(5)$, $\mathrm{Spin}(7)$, $\mathrm{Sp}(n)$ for $n \geq 3$, $\mathrm{Spin}(2n)$ for $n \geq 4$, $\mathrm{SO}(2n)$ for $n \geq 3$, or the simply-connected exceptional Lie groups E_6 or E_7 . Then none of the bi-invariant metrics of G is a locally thinnest left-invariant metric.*

Theorem C. *Every bi-invariant metric Q_0 of $\mathrm{PSU}(n+1)$ is a locally thinnest metric for $n = 1$ and an optimum for all $n \geq 1$.*

An *optimum* is a left-invariant metric Q so that the symmetric endomorphisms X of the Lie algebra \mathfrak{g} of G for whom the thickness of $Q(\exp(sX) \cdot, \cdot)$ decreases form a zero set. In fact, the tangent space of the space of all left-invariant metrics on G may be described by the space $S^2(\mathfrak{g})$ of symmetric endomorphisms of \mathfrak{g} , and, thus, an optimum is, in this sense, an *almost thinnest* left-invariant metric.

Table 1 on p. xx below summarises these results. The paradigm of the method used to prove Theorem B and Theorem C is that maximal tori and their infinitesimal behaviour under a variation of a bi-invariant metric should determine the evolution of the geometry of the group. In fact, every maximal torus T is totally geodesic and geodesically convex with respect to bi-invariant metrics, and $G = \bigcup_{g \in G} gTg^{-1}$. Moreover, the geometry of a maximal torus T is determined by its integral lattice. This lattice is obtained by taking the root system R and adjoining finitely many elements from R^* to the coroot lattice R^\vee . It is conceivable that this relationship between geometry and lattice is preserved infinitesimally under variations of a bi-invariant metric into a merely left-invariant metric.

The groups occurring in Theorem B are compact connected simple Lie groups whose bi-invariant metrics admit only finitely many *antipodes*, i.e. points with maximal distance to the identity element e of G . Under the weaker assumption that G be only semisimple, groups with this property are almost completely classified in Sec. 6.2 by characterising the finiteness of the antipodal set by the condition that all antipodes be located in the centre $Z(G)$, or, equivalently, that for every maximal torus $T \subseteq G$ the *deep holes* of the integral lattice Λ of T , i.e. those points of the Voronoi cell of Λ with maximum distance to Λ , be located in the dual lattice R^* of the root system. The latter condition can easily be verified or refuted, except in the case of proper quotients of $SU(n+1)$, whose integral lattices are the badly understood *Coxeter lattices*. This leads to a classification of simple Lie groups of type $\neq A_n$ whose bi-invariant metrics admit only finitely many antipodes. Beyrer has recently classified antipodal sets of almost all symmetric spaces of compact type [Bey18]. He, too, had to skip most of the quotients of $SU(n+1)$. Using computational methods with rational integers, all Coxeter lattices whose deep holes are contained in A_n^* are determined for $n \leq 14$ in App. D, and the results suggest a certain pattern in which these lattices might occur.

With the characterisation above in mind, the proof of Theorem B (see Sec. 6.3) is almost straightforward: Define a variation by shrinking a maximal torus T while expanding the rest of the group to keep the volume constant. These variations are generalisations of Berger metrics on spheres [Sak97, App. 6, 4°, pp. 307–308].² As T contains the centre, it contains all antipodes. Therefore, it is not surprising that the variation decreases diameter and thickness. Unfortunately, the technical condition that the intersection of the Voronoi cell with the Weyl chamber be a simplex cannot be eliminated here. This makes it impossible to treat most of those quotients of $SU(n+1)$ whose antipodal sets are finite. A notable exception is $SO(6) = SU(4)/\mathbb{Z}_2$.

Note that some of the integral lattices of these groups are locally thinnest lattices, A_2 for instance. Thus, the local behaviour of the integral lattices of the maximal tori has, in general, a very limited effect on the thickness. One of the contributing factors to this divergence effect is that the antipodes of the groups above are conjugate points (of the identity element of G) in the sense of riemannian geometry and are contained in multiple maximal tori. This leads to the consideration of simple Lie groups none of whose antipodes is a first conjugate point or contained in multiple maximal tori. These may be classified by a condition on the integral lattice of the maximal tori (see Sec. 6.4), and once again, this condition is easily verified or refuted, except in the case of the same proper quotients of $SU(n+1)$ that already made an appearance above. The only known family of groups to satisfy this condition turns out to be $PSU(n+1)$ for $n \geq 1$. The other known simple Lie group to admit the property that none of its antipodes are first conjugate points is the centre-free Lie group PE_6 of type E_6 .

²It should be mentioned that the variations used in the proof of Theorem B are obtained by going down a slightly different avenue, but the portrayal of the variation given here is more intuitive, if only due to its reminiscence of Berger metrics.

What makes $\text{PSU}(n+1)$ even more extraordinary is that its integral lattice A_n^* is a locally thinnest lattice, and, moreover, a *lattice of Voronoi's first kind* with a *strictly obtuse superbasis* (see Sec. 1.9). These features play an important role in the proof of Theorem C (see Sec. 6.5), and are not shared by the centre-free group PE_6 . They may be exploited with the use of Lem. 3.5.2, which asserts that for an arbitrary Lie group G equipped with a bi-invariant metric, a maximal torus $T \subseteq G$, a variation $Q(s)$ of the bi-invariant metric $Q(0)$, a family of points $p(s) \in T$, and families of geodesics $\tau_s: \mathbb{R} \rightarrow (T, Q(s))$ and $\gamma_s: \mathbb{R} \rightarrow (G, Q(s))$ with $\dot{\gamma}_s(0) = \dot{\tau}_s(0)$, the infinitesimal behaviour of the families τ_s and γ_s coincides up to first order in $s = 0$. In other words: the geometry of the maximal torus, and therefore the geometry of the integral lattice, dictates the evolution of geodesics in $(G, Q(s))$ up to first order in $s = 0$. Thus, when faced with a variation $Q(s)$ with constant volume (every variation can be rescaled to one of constant volume without changing thickness) so that there exists a maximal torus T whose volume increases *superlinearly*, that is: at least linearly, then one may exploit the capabilities of the locally thinnest integral lattice A_n^* to demonstrate that not only the diameter of $(T, Q(s))$ increases superlinearly, but so does the diameter of $(G, Q(s))$.

This warrants a momentary return to the setting of compact connected simple Lie groups G . Using representation theory, it is discovered in Sec. 5.1 that for every trace-free symmetric endomorphism X of the Lie algebra \mathfrak{g} of G , i.e. an infinitesimal volume-preserving variation, there either exists a maximal torus $T \subseteq G$ with $\partial_X \text{vol}(T) > 0$, or $\partial_X \text{vol}(T) = 0$ holds for all maximal tori $T \subseteq G$. The latter case is problematic, as the argument above may not be applied with any ease. Instead, one must approach this case by characterising it in terms of the representation

$$\Phi: G \rightarrow \text{GL}(S^2(\mathfrak{g})), \quad \Phi(g)(X) = \text{Ad}(g) \circ X \circ \text{Ad}(g)^{-1}$$

on the space $S^2(\mathfrak{g})$ of symmetric endomorphisms of \mathfrak{g} . It turns out that a trace-free symmetric endomorphism X satisfies $\partial_X \text{vol}(T) = 0$ for all maximal tori T if and only if the G -module generated by X is perpendicular to the G -module generated by the trace-free part π_0 of the orthogonal projection $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ mapping to the Lie algebra \mathfrak{h} of an arbitrary maximal torus T . The complexification $S^2(\mathfrak{g}_{\mathbb{C}})$ admits the following *unique* decomposition into irreducible G -modules:

$$S^2(\mathfrak{g}_{\mathbb{C}}) = \mathbb{C} \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma),$$

the *Vogel decomposition* [Vog99].³ Each summand is an eigenspace of the quadratic Casimir operator of the squared adjoint representation on $\text{End}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}})$. In this thesis, a real counterpart of the Vogel decomposition is obtained:

$$S^2(\mathfrak{g}) = \mathbb{R} \oplus \mathfrak{g}^{(2)} \oplus Y_2' \oplus Y_2''.$$

³Strictly speaking, some of the modules may be zero, depending on the rank of \mathfrak{g} . For example, if $\mathfrak{g} = \mathfrak{su}(n+1)$, then $Y_2(\beta) \neq 0$ only if $n \geq 2$ and $Y_2(\gamma) \neq 0$ only if $n \geq 3$.

Moreover, for Lie algebras of ADE type, i.e. A_n, D_n, E_6, E_7 and E_8 , the *Cartan power* $\mathfrak{g}^{(2)}$ is shown to be generated by π_0 . The conclusion is that for all compact connected simple Lie groups G of type ADE, $\partial_X \text{vol}(T) = 0$ holds for all maximal tori T if and only if $X \perp \mathfrak{g}^{(2)}$ (see Sec. 5.2). But this means that unless X is contained in the zero set $Y'_2 \oplus Y''_2$, there exists a torus to which the arguments from above may be applied, yielding nondecreasing thickness for *almost all* directions. Thus, the proof of Theorem C is settled.

These astonishing elements $X \in S_0^2(\mathfrak{g})$ with $X \perp \mathfrak{g}^{(2)}$ cannot be dealt with at the moment, although individual progress has been made using a quantitative adaptation of Bleicher's theorem (which asserts that A_n^* is a locally thinnest lattice, see above). This adaptation classifies the set \mathcal{F} of directions in which A_n^* , $n \geq 3$, may be deformed so that the thickness increases superlinearly, that is: at least linearly (see Sec. 1.10). It turns out that when one is faced with a variation of left-invariant metrics determined by $0 \neq X \in Y'_2$, then it is possible to find some maximal torus $T \subseteq G$ whose diameter grows superlinearly. Therefore the argument sketched earlier may once again be applied. This approach, can, however, not mend the other difficult case, for if $X \in Y''_2$, then the thickness of A_n^* seems to increase only up to second order (quadratically), not up to first.

As simple Lie groups whose antipodes are not contained in multiple maximal tori may be thought of as having the largest possible antipodal set, it may be said that Theorem B and Theorem C deal with diametrically opposite ends of the spectrum of simple Lie groups: The former shows that the bi-invariant metrics of simple Lie groups with finitely many antipodes are not locally thinnest left-invariant metrics, while the latter shows that the bi-invariant metrics on $\text{PSU}(n+1)$, thanks to their humongous antipodal sets, fall only slightly (if at all!) short of this goal. It remains oblique where on this spectrum the tilt from one type of behaviour to the other occurs.

Future Research

As the reader may have gathered from the introduction, those simple Lie groups to which the methods of the thesis may be successfully applied are almost completely classified and, therefore, the possibilities of further applications are almost exhausted. Improving the rather tantalising Theorem C, which asserts that the metrics of projective special unitary groups are *almost* thinnest metrics is certainly the highest priority. It would seem to be a fitting approach to improve the understanding of the divergence between the geodesics of maximal tori and those of the underlying Lie group up to *second or higher* order. This boils down to obtaining an improvement of the aforementioned Lem. 3.5.2. An attempt at this is made in the vista at the end of the last chapter (see Sec. 6.6),

Lie group	constraints	$\pi_1(G)$	$Z(G)$	optimal?
$SU(n+1)$	$n \geq 1$	1	\mathbb{Z}_{n+1}	no
$SU(n+1)/\mathbb{Z}_r$	$r \mid (n+1), r \neq n+1$	\mathbb{Z}_r	$\mathbb{Z}_{(n+1)/r}$?
$PSU(n+1)$	$n \geq 1, n \neq 1$	\mathbb{Z}_{n+1}	1	yes
$PSU(2)$		\mathbb{Z}_4	1	loc. min.
$Spin(5), Spin(7)$		1	\mathbb{Z}_2	no
$Spin(2n+1)$	$n \geq 4$	1	\mathbb{Z}_2	?
$SO(2n+1)$	$n \geq 2$	\mathbb{Z}_2	1	?
$Sp(n)$	$n \geq 3$	1	\mathbb{Z}_2	no
$PSp(n)$	$n \geq 3$	\mathbb{Z}_2	1	?
$Spin(2n)$	$n \geq 4$	1	\mathbb{Z}_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$	no
$SO(2n)$	$n \geq 3$	\mathbb{Z}_2	\mathbb{Z}_2	no
$PSO(2n)$	$n \geq 4$	\mathbb{Z}_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$	1	?
E_6		1	\mathbb{Z}_3	no
E_6/\mathbb{Z}_3		\mathbb{Z}_3	1	?
E_7		1	\mathbb{Z}_2	no
E_7/\mathbb{Z}_2		\mathbb{Z}_2	1	?
E_8		1	1	?
F_4		1	1	?
G_2		1	1	?

Table 1: Results of the thesis regarding the question if bi-invariant metrics are locally thinnest left-invariant metrics.

the results of which strongly suggest that a direct generalisation of Lem. 3.5.2 would be rather far-fetched. Obtaining some kind of second order estimate on this divergence effect might also shed some light on groups whose integral lattices do not behave as extreme as A_n^* , although the usefulness of the real Vogel decomposition seems to be limited to groups of type A_n, F_4 and G_2 (see remarks following Cor. 5.2.12).

The removal of the hypothesis on the integral lattice used in the proof of Theorem B (see Prop. 6.3.1), which prevents the theorem from applying to quotients of $SU(n+1)$ with finitely many antipodes, may also seem feasible to the reader. Together with a proper classification of these quotients, this would yield a few more examples of badly behaved bi-invariant metrics. Such a classification is equivalent to a classification of Coxeter lattices whose deep holes are contained in A_n^* , a problem which has, apparently, not yet been considered, although results due to Yang show the existence of infinitely many of such lattices [Yan07, Yan08]. In App. D, a conjecture generalising Yang's work is made and supported by computer-based calculations. Solving this conjecture would greatly

enrich Beyrer’s classification of the antipodal sets of compact symmetric spaces [Bey18]. A proof might, however, require an almost complete classification of Voronoi cells of Coxeter lattices. Unfortunately, the outlook for such a result is quite bleak, or rather: nonexistent. Increasingly sophisticated algorithms have been used to classify the Delone complexes of Coxeter lattices in dimensions $n \leq 21$ [DSSV09] (cf. App. D), but a methodical approach to a classification of Coxeter lattices in general does simply not exist.

Aside from the above, some other questions are raised by the thesis. They are listed below in no specific order.

Problem 1. *Study the existence of a locally thinnest metric on simple Lie groups. On convex spaces like \mathfrak{M} , or the quotient \mathfrak{M} , this is usually done using some kind of discrete gradient flow method, like the Birkhoff curve shortening algorithm. It might therefore be good to know exactly *how* nondifferentiable θ actually is.*

A good vantage point to figuring this out would be the group $SU(2)$ and, more specifically, the formula for the diameter of axisymmetric left-invariant metrics on $SU(2)$ due to Podobryaev [Pod18]. Denote by \mathfrak{MS} the space of left-invariant axisymmetric riemannian structures of fixed volume. This space is homeomorphic to \mathbb{R} . The diameter function admits two types of singularities when restricted to \mathfrak{MS} , namely, a C^0 -singularity at the bi-invariant metric and a C^1 -singularity somewhere else. Moreover, Podobryaev’s formula reveals the existence of a locally thinnest structure in \mathfrak{MS} and, furthermore, that the bi-invariant metric is a *locally thickest* metric in \mathfrak{MS} . Understanding the behaviour of θ on \mathfrak{M} would be an important step towards the answers to questions (2) and (3). A detailed discussion of \mathfrak{MS} is found in 6.1.2.

Problem 2. *Study the limit points of \mathfrak{PM} . More specifically, take a sequence (Q_k) in \mathfrak{PM} with $\theta(Q_k) \rightarrow 0$ and interpret it as a sequence of Lie algebras (\mathfrak{g}, μ_k) , keeping the metric fixed instead. What is the Lie structure of the limit $\lim_{k \rightarrow \infty} (\mathfrak{g}, \mu_k)$? Is the limit almost flat? For details, see Sec. 4.6.*

Problem 3. *How badly behaved are the bi-invariant metrics on E_6/\mathbb{Z}_3 ? To be more specific, the integral lattice E_6^* of (the maximal tori of) $PE_6 = E_6/\mathbb{Z}_3$ is a pessimum. The Vogel decomposition of $S_0^2(\mathfrak{e}_6)$ implies that for many $X \in S_0^2(\mathfrak{e}_6)$, there exists a maximal torus $T \subseteq PE_6$ so that the integral lattice of T is deformed in a direction belonging to the problematic zero set. Therefore, it would be rather bold to conjecture that the bi-invariant metrics of PE_6 are *locally thickest* metrics⁴, but it might be worthwhile to study the extent of their malignancy.*

⁴Would these metrics be the *worst* left-invariant metrics?

1 Theory of Lattices

The following chapter is a comprehensive account of the basics of the theory of lattices, with a strong focus on the lattice covering problem (see Sec. 1.1) and the geometry of the Voronoi cells of root lattices, i.e. lattices generated by root systems. A discussion of irreducible root systems and their classification by Dynkin diagrams (see Sec. 1.5) is followed by a complete description of the lattices they generate and their duals (see Sec. 1.7), including computations of their Voronoi cells. Particular diligence is taken in the treatment of Voronoi's principal lattice A_n^* using Conway–Sloane's superbasis approach to lattices of Voronoi's first kind in the last three sections. This will yield a classification of variations which increase the covering radius of A_n^* up to first order, essentially a quantitative version of *Bleicher's theorem*, which asserts that A_n^* is a locally thinnest lattice [Ble62]. This result will later be used in the proof of Theorem C.

The chapter is based on Conway–Sloane's *Sphere Packings, Lattices and Groups* [CS99] and Martinet's *Perfect Lattices in Euclidean Space* [Mar03].¹ The section on root systems also relies on Humphrey's *Introduction to Lie Algebras and Representation Theory* [Hum80] and Bröcker–tom Dieck's *Representations of Compact Lie Groups* [BtD85]. Any claim without a citation is assumed to be well-known or so obvious as to not warrant a reference.

1.1 Preliminaries and the Covering Radius Problem

1.1.1 Definitions. Let V be an (\mathbb{R} -)vector space of dimension n . Following [Mar03, 1.1], a *lattice* Λ of V is defined as a subgroup of V for which there exists a basis $B = (b_1, \dots, b_n)$ of V such that Λ is the set of all linear combinations of (b_1, \dots, b_n) over \mathbb{Z} . Equivalently, a lattice may be characterised as a discrete cocompact subgroup of $(V, +)$.²

A basis B as above is called a *basis* of Λ . It is convenient to view a basis B as equivalent to the group isomorphism

$$\phi_B: \mathbb{Z}^n \rightarrow \Lambda, \quad \phi_B(x) = \sum_i x_i b_i.$$

¹Unfortunately, these three authors use the convention that vectors are *row vectors*, not column vectors.

²*Cocompact* means that V/L is a compact topological space.

If $B = (b_1, \dots, b_n)$ and $B' = (b'_1, \dots, b'_n)$ are bases of Λ , there exists an automorphism $\alpha: \Lambda \rightarrow \Lambda$ with $\alpha(b_i) = b'_i$ for all i , or, equivalently, $\phi_{B'} = \alpha \circ \phi_B$.³ If $V = \mathbb{R}^n$, then $\phi_B = B$, provided that B is interpreted as the matrix whose column vectors are b_1, \dots, b_n . In particular, $M = \phi_B^{-1} \circ \alpha \circ \phi_B \in \text{Aut}(\mathbb{Z}^n) = \text{GL}(n, \mathbb{Z})$ satisfies $BM = B'$.

As ϕ_B is an isomorphism, every ‘naked’ lattice is equivalent to \mathbb{Z}^n . By equipping V with an inner product, usually denoted by Q or $\langle \cdot, \cdot \rangle$, lattices are turned into geometric objects with an isometry relation that may be used to distinguish them from one another. In lattice theory, the term *positive quadratic form* is predominantly used in place of *inner product* or *scalar product*. All these terms will be used interchangeably in the present thesis. The notion of isometry is defined as follows: Let V_0, V_1 be inner product spaces. Then $\Lambda_0 \subseteq V_0$ is *isometric* to $\Lambda_1 \subseteq V_1$ if there exists an isometry $f: V_0 \rightarrow V_1$ with $f(\Lambda_0) = \Lambda_1$. They are *similar* if, instead, there exists $a > 0$ so that $af(\Lambda_0) = \Lambda_1$. In other words, two lattices are similar if they are related by a *similarity transformation*, i.e. an isometry preceded or followed by a scalar multiplication.

The geometric structure of a lattice Λ lies hidden in the *Voronoi cell* \mathcal{V} of Λ .⁴ The Voronoi cell may be defined as

$$\mathcal{V} = \{x \in V : \forall v \in \Lambda : \|x\| \leq \|x - v\|\},$$

a compact, convex polytope (see 1.1.2 below), whose translates $v + \mathcal{V}$ by elements $v \in \Lambda$ tessellate V . The vertices of \mathcal{V} are called *holes*. Those holes with maximal distance to the origin are *deep holes*. The translates of the deep holes by lattice elements are exactly those points of \mathbb{R}^n with maximal distance $\mu(\Lambda) = \max_{x \in V} \min_{v \in \Lambda} \|x - v\|$ to the lattice. The number $\mu(\Lambda)$ is called the *covering radius* of Λ and will be discussed in more detail below.

The *determinant* of Λ is defined by picking a basis (b_1, \dots, b_n) and setting

$$\det \Lambda = \det [Q(b_i, b_j)],$$

i.e. $\det \Lambda$ is the squared volume of the parallelotope spanned by the chosen basis. As any two bases are related by an element of $\text{GL}(n, \mathbb{Z})$, this definition does not depend on the particular choice of (b_1, \dots, b_n) . Moreover, it follows that $(\det \Lambda)^{1/2} = \text{vol } \mathcal{V}$. The index $[\Lambda : \Lambda']$ of a sublattice $\Lambda' \subseteq \Lambda$ is equal to $(\det \Lambda' / \det \Lambda)^{1/2}$.

1.1.2 Structure of \mathcal{V} . Let (V, Q) be an inner product space and $\Lambda \subseteq V$ a lattice with Voronoi cell \mathcal{V} . For each $v \in V$, define the *covector* $v^\vee \in V^*$ of v by

$$v^\vee: V \rightarrow \mathbb{R}, \quad v^\vee(w) = 2 \cdot \frac{Q(v, w)}{Q(v, v)}.$$

³Every group automorphism $\Lambda \rightarrow \Lambda$ of Λ extends to a linear isomorphism. In fact, Λ is a \mathbb{Z} -module and $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

⁴Strictly speaking, this should read ‘... of (Λ, Q) ’. The positive quadratic form Q is always part of the structure.

Then $\|w\| \leq \|w - v\|$ if and only if $v^\vee(w) \leq 1$. Furthermore, define

$$\begin{aligned} E(v) &= \{w \in V : v^\vee(w) \leq 1\}, \\ H(v) &= \{w \in V : v^\vee(w) = 1\}. \end{aligned}$$

It follows that \mathcal{V} is the intersection of the closed half-spaces $E(v)$ of all $v \in \Lambda$. If the bisector $H(v)$ intersects \mathcal{V} , then v is called *Voronoi vector*, and \mathcal{V} is the intersection of all those $E(v)$ where v is a Voronoi vector. The fact that \mathcal{V} is bounded shows that there are only finitely many Voronoi vectors. It follows that \mathcal{V} is a polyhedral set in the sense of [Grü03, 2.6]. In particular, \mathcal{V} is a convex polytope, i.e. it admits a unique minimal \mathcal{H} -representation (E_1, \dots, E_k) of closed half-spaces so that $\mathcal{V} = \bigcap E_i$ [Grü03, 3.1.1, 3.1.2, 3.6]. If $E(v)$ belongs to the \mathcal{H} -representation of \mathcal{V} , then v is called a *relevant Voronoi vector*.

1.1.3 Covering radius. The *covering radius* $\mu(\Lambda)$ of Λ is the smallest radius $r > 0$ required to cover V with balls of radius r centred at the points of Λ . A covering of V in this fashion is called a *lattice covering*.⁵ The covering radius may be characterised as the maximum distance of points $x \in V$ to Λ or, equivalently, as the maximum distance of a point of \mathcal{V} to the origin (see above). The efficiency of a lattice covering is measured by the *thickness*

$$\Theta(\Lambda) = \frac{V_n \cdot \mu(\Lambda)^n}{(\det \Lambda)^{1/2}},$$

or the *normalised thickness* $\theta(\Lambda) = \Theta(\Lambda)/V_n$, where V_n denotes the volume of the unit ball of \mathbb{R}^n [CS99, Ch. 2, Sec. 1.1].⁶ The *thinner* a covering, the more effective it is. Note that $\Theta(\Lambda)$ and $\theta(\Lambda)$ only depend on the similarity class of Λ .

1.1.4 Space of lattices. Let $\widetilde{\mathcal{L}}(n)$ be the set of lattices in \mathbb{R}^n and $\mathcal{L}(n)$ the set of lattices up to isometry, i.e. $O(n) \backslash \widetilde{\mathcal{L}}(n)$. Consider the map

$$\mathrm{GL}(n, \mathbb{R}) \rightarrow \widetilde{\mathcal{L}}(n), \quad B \mapsto \phi_B(\mathbb{Z}^n).$$

According to 1.1.1, this map is surjective and factors through the action of $\mathrm{GL}(n, \mathbb{Z})$ by right-multiplication to a bijection $\mathrm{GL}(n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{Z}) \rightarrow \widetilde{\mathcal{L}}(n)$. Factoring by the action of $O(n)$ by left-multiplication yields a bijection

$$O(n) \backslash \mathrm{GL}(n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{Z}) \rightarrow \mathcal{L}(V).$$

Therefore, $\widetilde{\mathcal{L}}(n)$ and its quotient $\mathcal{L}(n)$ inherit topologies from the spaces above.

The space of lattices may also be described using the set $\mathcal{S}_{>0}(n)$ of positive-definite quadratic forms on \mathbb{R}^n . In fact, the map

$$\mathrm{GL}(n, \mathbb{R}) \rightarrow \mathcal{S}_{>0}(n), \quad B \mapsto \{x \mapsto x^\top B^\top B x\}$$

⁵The terms *lattice covering* and *lattice* are often used interchangeably in this context.

⁶Other names for Θ are *(covering) density* and *sparsity*.

factors through $O(n)$ to a bijection $O(n) \backslash GL(n, \mathbb{R}) \rightarrow \mathcal{S}_{>0}(n)$. Here a scalar product is represented by the associated gramian matrix $A = [Q(e_i, e_j)]$. Two positive quadratic forms are said to be *arithmetically equivalent* if they are equivalent under the right-action of $GL(n, \mathbb{Z})$ on $\mathcal{S}_{>0}(n)$ by pullback, i.e.

$$\mathcal{S}_{>0}(n) \times GL(n, \mathbb{Z}) \rightarrow \mathcal{S}_{>0}(n), \quad (Q, M) \mapsto \{M^*Q: (x, y) \mapsto Q(Mx, My)\}.$$

Under the map $O(n) \backslash GL(n, \mathbb{R}) \rightarrow \mathcal{S}_{>0}(n)$, this action corresponds to the action of $GL(n, \mathbb{Z})$ on $O(n) \backslash GL(n, \mathbb{R})$ by right-multiplication. Therefore, $\mathcal{L}(n)$ is bijective to $\mathcal{S}_{>0}(n)/\{\text{arithmetic equivalence}\}$. For details, see [Sch09, 1.1.5].

The underlying geometric idea behind this characterisation of $O(n) \backslash \mathcal{L}(n)$ is brought to light by considering \mathbb{Z}^n as a *reference lattice*. Instead of mapping \mathbb{Z}^n *forward* to other lattices with ϕ_B , a fixed positive quadratic form Q of \mathbb{R}^n is *pulled back* to \mathbb{Z}^n , turning ϕ_B into an isometry $(\mathbb{Z}^n, \phi_B^*Q) \rightarrow (\Lambda, Q)$. In fact, $B^\top B$ is the matrix representing ϕ_B^*Q . In other words, the geometric data are changed in place of the underlying object.⁷ Of course, this means that, strictly speaking, the positive quadratic form Q is part of the reference data.

Suppose that the reference lattice \mathbb{Z}^n is replaced by a lattice Λ in an inner product space (V, Q) . Then the isometry classes $[\Lambda', Q]$ of lattices in V correspond to arithmetic equivalence classes of positive quadratic forms on V , i.e. the elements of $\mathcal{S}_{>0}(V)/\text{Aut}(\Lambda)$, by pulling back Q along an arbitrary isomorphism $\Lambda \rightarrow \Lambda'$, or, equivalently, to elements of the double coset $O(V, Q) \backslash GL(V)/\text{Aut}(\Lambda)$ by associating to $\phi \in GL(V)$ the isometry class $[\phi(\Lambda), Q]$.

As the thickness $\theta(\Lambda)$ of a lattice only depends on its similarity class, it is preferable to work with the space $\mathcal{P}\mathcal{L}(n)$ of similarity classes of n -dimensional lattices. This space is obtained from $\mathcal{L}(n)$ by factoring out the left-action of $\mathbb{R}_+ \times O(n)$, or, equivalently, as the coset space $SO(n) \backslash SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, using the arguments above, m.m.

Note that $\mathcal{S}_{>0}(V)$ is an open set of the vector space $S^2(V)$ of symmetric endomorphisms of the underlying vector space V . In particular, the $\mathcal{S}_{>0}(V)$ is a manifold with $T_{I_n} \mathcal{S}_{>0}(V) = S^2(V)$, where I_n denotes the unit matrix of $\mathbb{R}^{n \times n}$. The space $\mathcal{P}\mathcal{L}(V)$ may therefore be considered as having the ‘tangent space’ $T_\Lambda \mathcal{P}\mathcal{L}(V) = S_0^2(V)$, the space of symmetric trace-free endomorphisms of V , at the reference lattice Λ .

⁷This is similar to the approach of changing, say, the riemannian metric of a submanifold, rather than changing the submanifold itself.

1.1.5 The lattice covering problem.

What are the (locally) thinnest lattice coverings?

A (locally) thinnest lattice is sometimes called a (*local*) *optimiser* or *optimum*. Conway and Sloane have composed an extensive treatise [CS99] on the (lattice) covering problem and its cousin, the sphere packing problem. In [CS99, Ch. 2, Sec. 1.3] they summarise the progress in solving the lattice covering problem up to roughly the end of the previous century. A comprehensive summary involving more recent results may be found in [SV06, Sec. 3]. In the following, the two surveys are summarised. References are only given as far as they pertain to the present thesis. The remaining references are found in [CS99] and [SV06]. Many results regard root lattices, whose definitions may be found in 1.7.1–1.7.10.

Hlawka proved in 1949 that, in every dimension n , a globally thinnest lattice exists [Hla49]. This optimum is known to be A_n^* for $n \leq 5$, as verified by Bambah in 1954 for $n = 3$, by Delone and Ryskov in 1963 for $n = 4$, and by Ryskov and Baranovskii in 1975 for $n = 5$. Dickson and Baranovskii classified locally thinnest lattices in dimensions ≤ 4 . Bambah's conjecture, asserting that the lattice A_n^* is a global optimiser for all n , was soon disproved by Bleicher [Ble62], who proved that there exists an unspecified dimension n_0 , so that $A_{n_0}^*$ is not a global optimum. Ryshkov showed that n_0 may be as small as 114, and later thinner coverings were found in dimensions as low as $n = 24$, amongst them the famous *Leech lattice* A_{24} . Bleicher also proved a theorem asserting that A_n^* is locally thinnest lattice for all n [Ble62, Thm. 3], which will be discussed in detail in Sec. 1.10.⁸

Since then, some of the most important developments have been as follows: Schürmann and Vallentin proved that the Leech lattice A_{24} , conjectured to be the thinnest lattice covering of \mathbb{R}^{24} , is a local optimiser [SV05]. Together with Dutour Sikirić they discovered many Coxeter lattices thinner than the A_n^* of their respective dimension [DSSV09]. Coxeter lattices are discussed in detail in App. D. The root lattices and their duals did not fare so well: In [DSSV12], a sufficient condition for being a local covering *maximum* similar to a well-known description of local packing minima due to Voronoi is obtained and used to show that E_6 and E_7 are local covering maxima. The others, \mathbb{Z}^n , D_n , E_6^* , E_7^* , E_8 , are *local covering pessima*, which means that the measure of the set of directions in which their thickness does increase is zero [DSSV12, Sec. 6, Tab. 4.2] [Sch09, Sec. 5.6].^{9,10}

⁸Schürmann and Vallentin credit Gameckiĭ with this result [SV06, Sec. 3], claiming it was proved in [Gam62] or [Gam63]. Unfortunately, the author was unable to obtain the article in Russian or English language.

⁹Apparently, *pessimum* is not the opposite of *optimum*.

¹⁰In [DSSV12], the authors require that a pessimum also not be a local maximum.

The notion of *pessimum*, as already mentioned in the introduction, is later used to inspire the notion of *optimum* (see 4.2.5), used to describe left-invariant metrics that are *almost locally thinnest* metrics in the same sense as local covering pessima are *almost locally thickest* lattices. Note that such terminology clashes rather harshly with terms such as *optimiser* or (the other) *optimum*, which are (quite unfortunately) also used for proper local minima of the thickness.

The significance of the lattice covering problem for the present thesis lies in the following observations made in a later chapter: The normalised thickness $\theta(\Lambda)$ of a lattice $\Lambda \subseteq V$ is equal to the thickness of the torus V/Λ , as defined in 3.3.3. For details, see Prop. 4.1.2. As a corollary, the lattice covering problem is equivalent to determining the locally and globally thinnest left-invariant metrics in abelian Lie groups (see Cor. 4.1.3). In particular, all the theorems discussed above may be translated into theorems about tori and their thickness.

1.2 Glueing Theory

1.2.1. Let (V, Q) be an inner product space. The *dual lattice* of $\Lambda \subseteq V$ is

$$\Lambda^* = \{\alpha \in V^* : \forall v \in \Lambda : \alpha(v) \in \mathbb{Z}\}.$$

It is straightforward to verify that Λ^* is a lattice of the dual space V^* . Using the canonical isomorphism $\sharp : V^* \rightarrow V$ defined by Q allows one to think of Λ^* as a lattice of V . Furthermore, Λ is *integral* if $Q(v, w) \in \mathbb{Z}$ for all $v, w \in \Lambda$, or, equivalently, if $\Lambda \subseteq \Lambda^*$.

It is straightforward to verify that $\det \Lambda^* = (\det \Lambda)^{-1}$. In particular, $[\Lambda^* : \Lambda] = \det \Lambda$ holds for integral lattices. A lattice is called *unimodular* if it satisfies $\Lambda = \Lambda^*$ or, equivalently, if it is integral and satisfies $\det \Lambda = 1$.

1.2.2. The *glue group* of an integral lattice $L \subseteq V$ is L^*/L . The elements of the glue group and their representatives are called *glue vectors*. The order of the glue group coincides with $\det L$ [CS99, Ch. 2, Sec. 2.4].

The glue group plays a role in describing the intermediate lattices $L \subseteq \Lambda \subseteq L^*$.¹¹ If $\mathcal{R} \subseteq \Lambda$ is a system of representatives of Λ/L , then

$$\Lambda = \bigcup_{v \in \mathcal{R}} (v + L).$$

Therefore, Λ is often denoted as $L\mathcal{R}$ (L with \mathcal{R} *adjoined*) or, if $[v]$ is a generator of Λ/L , as $L[v]$ (L with v *adjoined*).

¹¹The glue group actually accomplishes a lot more than that. See [CS99, Ch. 4, §3].

1.3 Root Systems

1.3.1. Let V be an inner product space of dimension n . A finite set $R \subseteq V^\times = V \setminus \{0\}$ is a *root system* if the following properties are satisfied:¹²

- (1) R spans V .
- (2) For every $\alpha \in R$, the only multiples of α contained in R are $\alpha, -\alpha$.
- (3) For all $\alpha \in R$, the reflection s_α in $\mathcal{H}_\alpha = \alpha^\perp$ maps R to R .
- (4) $\alpha^\vee(\beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

Let V' be another inner product space and $R' \subseteq V'$ be a root system. Then R is *similar* to R' if there exists a similarity transformation $\varphi: V \rightarrow V'$ with $\varphi(R) = R'$ [BtD85, Ch. V, (5.2)] [BtD85, Ch. V, (5.14), Ex. 6,7]. The *product* $R \times R'$ is defined as $R \cup R' \subseteq V \times V'$. A root system is *irreducible* if it is not similar to a product of two or more root systems [BtD85, Ch. V, (5.3)–(5.5)].

Axiom (4) asserts that for every $\alpha \in R$ the reflection s_α , which is defined by

$$s_\alpha(v) = v - \alpha^\vee(v) \cdot \alpha$$

for all $v \in V$ and may be written as $s_\alpha(\beta) = \beta - n_{\alpha\beta} \cdot \beta$ for all $\beta \in R$, satisfies $n_{\alpha\beta} \in \mathbb{Z}$. The numbers $n_{\alpha\beta}$ are called the *Cartan numbers* of R [BtD85, Ch. V, (3.7)]. There are very few possible choices for $n_{\alpha\beta}$. In fact, $n_{\alpha\beta} = 2 \cos \angle(\alpha, \beta) \cdot \|\beta\|/\|\alpha\|$ and $n_{\alpha\beta} \cdot n_{\beta\alpha} = 4 \cos^2 \angle(\alpha, \beta)$ for all $\alpha, \beta \in R$. In particular, the angles between nonproportional roots are restricted and determine (after switching α and β if necessary) the ratio $\|\alpha\|^2/\|\beta\|^2 \in \mathbb{Z}$ unless $\angle(\alpha, \beta) = \pi/2$ [BtD85, Ch. V, (3.8)–(3.10)]. It is also true that at most two different root lengths can occur in a single irreducible root system [Hum80, Sec. 10.4., Lem. C]. One speaks of the *short* and *long roots* of R . An irreducible root system whose roots are all short is called *short*.

1.3.2. Let V be an inner product space and $R \subseteq V$ be a root system. For all $\alpha \in R$, the covector α^\vee is also referred to as *coroot* and is the unique element of V^* with $s_\alpha(v) = v - \alpha^\vee(v) \cdot \alpha$. Applying the canonical isomorphism $\sharp: V^* \rightarrow V$ yields the element

$$(\alpha^\vee)^\sharp = \frac{2\alpha}{\langle \alpha, \alpha \rangle},$$

which is also referred to as coroot and often denoted by α^\vee . When dealing with root systems in the dual space V^* , the coroots may be identified with elements of V , without

¹²Sometimes R is called root system if the axioms (2)–(4) are satisfied, and *reduced root system* if (1) is also satisfied. Nonreduced root systems occur as root systems of non-semisimple Lie algebras.

resorting to the use of a scalar product, by applying the evaluation homomorphism $V^{**} = V$.¹³

1.3.3. A linearly independent set S of a root system R is a *simple basis* if, for all $\alpha \in R$, there exists a number k_β for each $\beta \in S$, all of the same sign, so that $\alpha = \sum_{\beta \in S} k_\beta \cdot \beta$. Every root system admits a simple basis [Hum80, Sec. 10.1, p. 48] [BtD85, Ch. V, (4.5)]. The *Cartan matrix* of R with respect to S is defined as $(n_{\alpha\beta})_{\alpha, \beta \in S}$, cf. 1.3.1. Note that if S is a simple basis of R , then S^\vee is a simple basis of R^\vee .

The sum $\sum_{\beta \in S} k_\beta$ is called the *height* of α . The simple basis S induces an ordering \succ on R : $\alpha = \sum_{\beta \in S} k_\beta \cdot \beta$ is *positive* if $k_\beta \geq 0$ for all $\beta \in S$, and $\alpha \succ \gamma$ if $\alpha - \gamma$ is a multiple of a positive root. The set of positive roots is often denoted by R^+ . If R is irreducible, then there exists a unique *maximal root* ρ with respect to the ordering \succ [Hum80, Sec. 10.4. Lemma A].

1.3.4. The *Weyl group* W of R is the subgroup of $O(V)$ generated by $\{s_\alpha : \alpha \in R\}$. The connected components of $V \setminus \bigcup_{\alpha \in R} \mathcal{H}_\alpha$ are called *Weyl chambers*. The *walls* of a Weyl chamber K are those intersections $\bar{K} \cap \mathcal{H}_\alpha$ that have dimension $n - 1$.^{14,15} The set of Weyl chambers is bijective to the set of simple bases [BtD85, Ch. V, (4.5)]. In fact, for every Weyl chamber K there exists a simple basis S so that

$$K = \{v \in V : \forall \alpha \in S : \langle \alpha, v \rangle > 0\}.$$

This chamber is called the *fundamental Weyl chamber* of R with respect to S . The set of reflections in the walls of the fundamental Weyl chamber, i.e. the reflections associated to the simple roots, generate the Weyl group [BtD85, Ch. V, (4.6)].

The *affine Weyl group* W_a of R is obtained by adjoining the translations by the root vectors $\alpha \in R$ to W [CS99, Ch. 4, p. 98]. The affine Weyl group is an affine reflection group in the sense of [Bro89, Ch. VI, Sec. 1B]. The connected components of $V \setminus \bigcup_{w \in W_a} \mathcal{H}_w$, where \mathcal{H}_w denotes the fixed point set of $w \in W_a$, are bounded convex sets. They are usually called chambers, rooms or *alcoves*. The *fundamental alcove* of R is the closure of the chamber A with $0 \in \bar{A}$ that is contained in the fundamental Weyl chamber. When referring to the topological closures of alcoves of irreducible root systems, Conway–Sloane use the term *simplex*. The walls of an alcove A are the $(n - 1)$ -dimensional intersections of \bar{A} with translates of the hyperplanes \mathcal{H}_α by root vectors. For all alcoves A , the reflections in the walls of A generate W_a .

In the context of root systems of Lie groups, the *extended Weyl group* Ω [BtD85, Ch. V, (7.8)] is used, a notion very similar to the affine Weyl group in the sense of Conway–Sloane, which also gives rise to a notion of alcoves called *Stiefel alcove*, introduced in 2.11.3. The connection between W_a and Ω , and between alcoves and Stiefel alcoves is also discussed there.

¹³Bröcker–tom Dieck refer to the coroot as *inverse root* and denote it by α^* [BtD85, Ch. V, (2.13)]. In particular, their coroot system is denoted by R^* . Generally, there is no connection between the coroot system and the dual root lattice.

¹⁴ \bar{K} denotes the topological closure.

¹⁵These walls are manifolds with boundary and the dimension refers to their dimension as such.

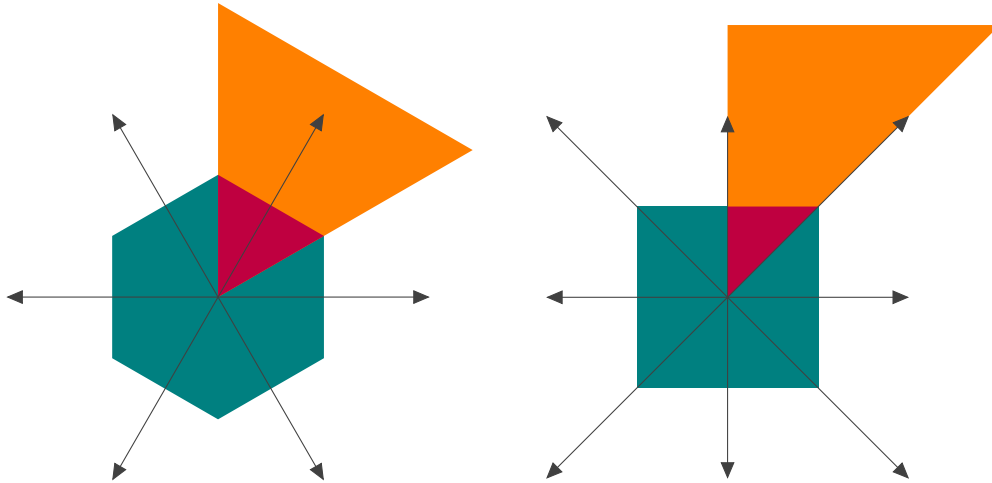


Figure 1.1: The root systems A_2 and B_2 with Voronoi cell \bullet , Weyl chamber \circ and fundamental alcove $\color{red}\bullet$. See [BtD85, Ch. V, Sec. 7, Fig. 24, p. 227].

1.4 Dynkin Diagrams

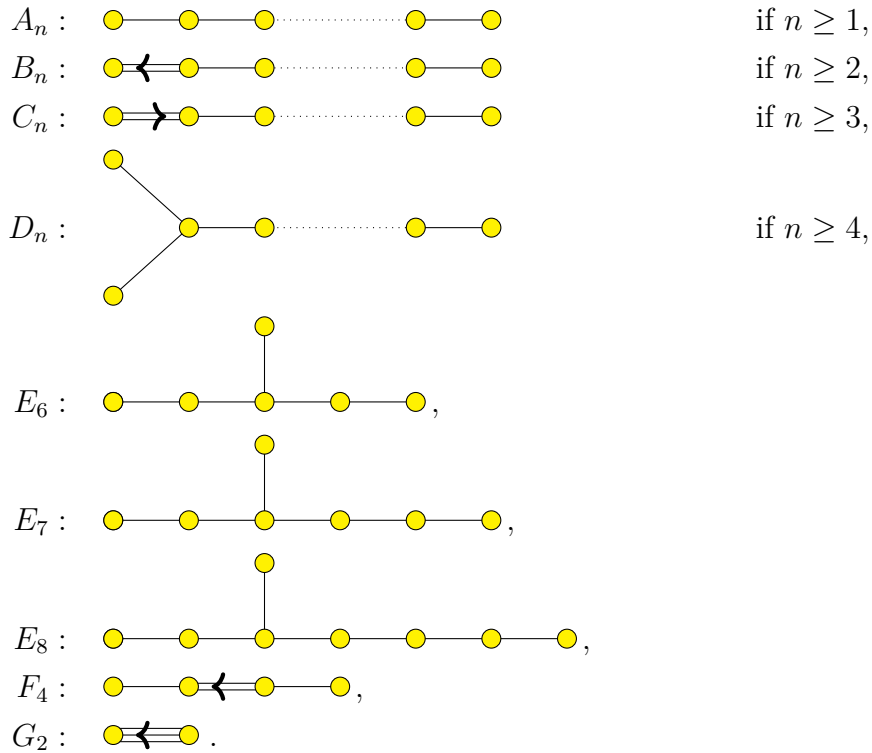
1.4.1. Let R be a root system and $S \subseteq R$ a simple basis. The *Coxeter graph* of R is a *multigraph* (a graph that allows for more than one edge between any two vertices) with vertex set S that admits exactly $e_{\alpha\beta} = n_{\alpha\beta} \cdot n_{\beta\alpha}$ edges between each pair of distinct vertices $\alpha, \beta \in S$ [BtD85, Ch. V, (5.6)].

As the Weyl group W acts isometrically on the underlying inner product space and transitively on the set of simple bases of R , the isomorphism type of the Coxeter graph does not depend on the particular choice of the simple basis S . According to [BtD85, (5.2)], similarity transformations between root systems preserve the Cartan matrix. In particular, the Coxeter graph of R is invariant under similarity transformations [BtD85, Ch. V, (5.6)]

1.4.2. By 1.3.1, (1) $e_{\alpha\beta} \in \{0, 1, 2, 3\}$, (2) $e_{\alpha\beta} = 0$ if and only if $\alpha \perp \beta$ and (3) $\|\alpha\| = \|\beta\|$ if and only if $e_{\alpha\beta} = 1$. The *Dynkin diagram* of R is obtained by decorating every double or triple edge with an arrow pointing toward the shorter of the two roots [BtD85, Ch. V, (5.7)]. As in the case of the Coxeter graph, the isomorphism type of the Dynkin diagram (as decorated multigraph) does not depend on the particular choice of the simple basis. According to (2), the Dynkin diagram is connected if and only if R is irreducible.

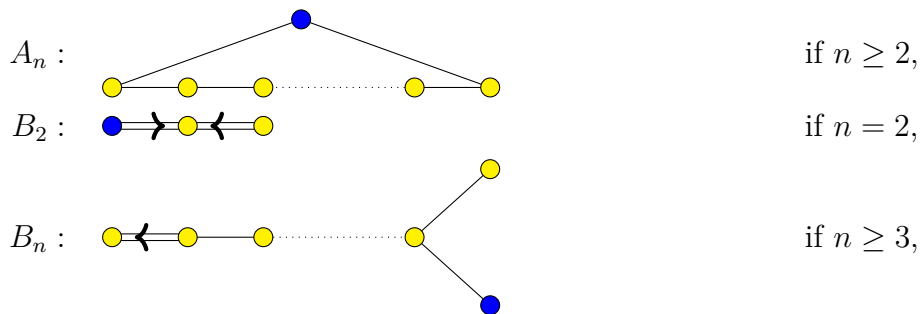
Every root system may be reconstructed from the associated Dynkin diagram [BtD85, (5.9)], yielding a correspondence between similarity classes of (irreducible) root systems and (connected) Dynkin diagrams. The Dynkin diagrams of irreducible root systems

are the following [Hum80, 11.4]:¹⁶



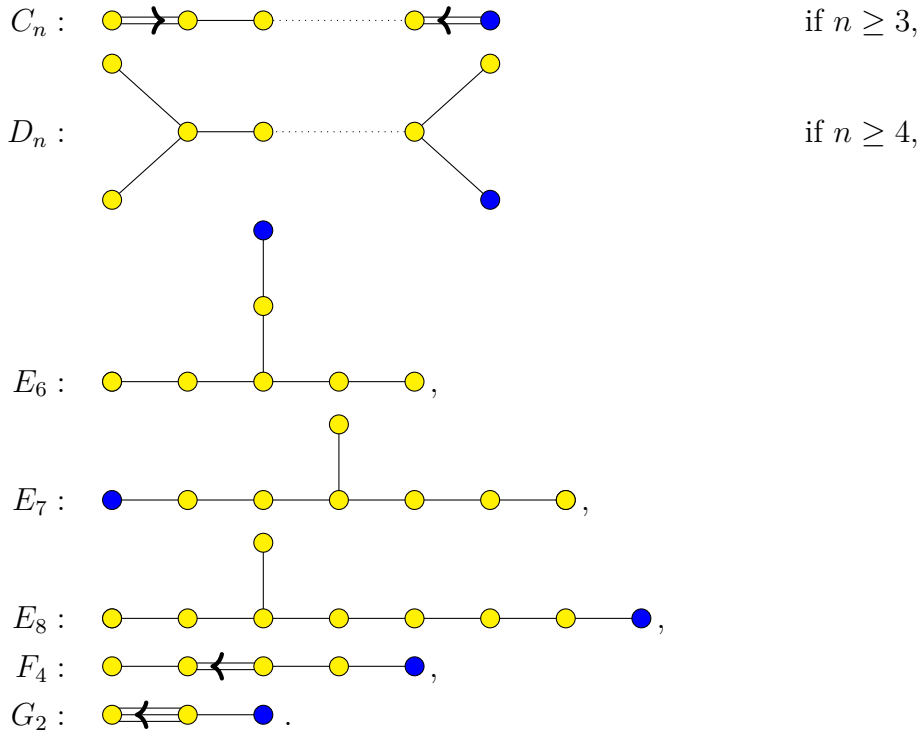
A complete discussion of the irreducible root systems associated to these Dynkin diagrams follows in the next section.

1.4.3. The *affine Dynkin diagram* is the decorated multigraph obtained by adding the maximal root ρ of R to the vertex set of the Dynkin diagram and drawing $\alpha^\vee(\rho) \cdot \rho^\vee(\alpha)$ edges between each simple root α and ρ , with decoration as specified above [CS99, Ch. 4, §2]. In addition, the vertex of the maximal root is marked. In the present thesis, blue (\bullet) is used to indicate the marking:¹⁷



¹⁶Throughout the thesis, a personal adaptation of B. McKay’s latex commands for Dynkin diagrams is used, which he kindly made available on mathoverflow [Mu13]. Recently, McKay’s original set of commands was uploaded to the Comprehensive Tex Archive Network (CTAN).

¹⁷The affine Dynkin diagram of the 1-dimensional root system A_1 makes little sense in the theory and is left out for this reason.



1.5 Table of Irreducible Root Systems

The following is a complete list of all irreducible root systems and their geometric data. It was assembled using [CS99, Fig. 21.1–21.3], [CS99, Ch. 4] and [Hum80, 11.4], but these sources may at times vary from the presentations given here. Simple bases are represented by matrices whose columns are the simple basis vectors. The maximal roots are taken from [BB10, 17.1–17.9].


1.5.1 The root system A_n . Let $n \geq 1$ and define

$$\begin{aligned}
 A_n &= \{(x_0, \dots, x_n) \in \mathbb{Z}^{n+1} : \sum_i x_i = 0, \langle x, x \rangle = 2\} \\
 &= \{e_a - e_b : 0 \leq a \neq b \leq n\}.
 \end{aligned}$$

Then A_n is a root system of $V = \{x \in \mathbb{R}^{n+1} : \sum_i x_i = 0\}$. *Simple basis:*¹⁸

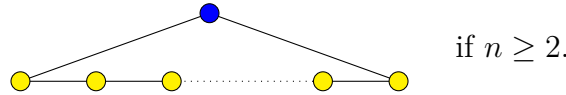
$$\begin{bmatrix} -1 & & & & & & \\ & 1 & -1 & & & & \\ & & & 1 & \ddots & & \\ & & & & \ddots & -1 & \\ & & & & & & 1 \end{bmatrix}. \tag{1.5.1}$$

¹⁸Blank entries in matrices are zero.

Dynkin diagram: . *Weyl group:* $s_{e_a - e_b}$ permutes the a^{th} and b^{th} coordinate. W is therefore equivariantly isomorphic to S_{n+1} acting by permutation of coordinates. *Weyl chamber:*

$$\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 \leq x_1 \leq \dots \leq x_n, \sum_i x_i = 0\}.$$

Maximal root: $\alpha_1 + \dots + \alpha_n = (-1, 0^{n-1}, 1)$.¹⁹ *Affine Dynkin diagram:*



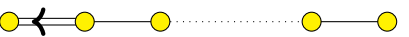
Coroot system: A_n .

1.5.2 The root system B_n . Let $n \geq 2$ and define²⁰

$$B_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : \langle x, x \rangle \in \{1, 2\}\} = \{\pm e_a : a = 1, \dots, n\} \cup \{\pm e_a \pm e_b : a < b\}.$$

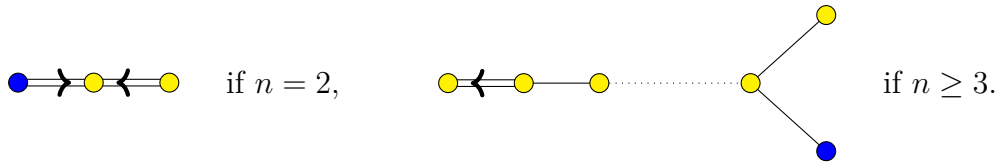
Simple basis:

$$\begin{bmatrix} 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & & 1 \end{bmatrix}.$$

Dynkin diagram: . *Weyl group:* $(\mathbb{Z}_2)^n \rtimes S_n$. *Weyl chamber:*

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \dots \leq x_n\}.$$

Maximal root: $2\alpha_1 + \dots + 2\alpha_{n-1} + \alpha_n = (0^{n-2}, 1, 1)$. *Affine Dynkin diagram:*



Coroot system: C_n .

1.5.3 The root system C_n . Let $n \geq 2$ and define

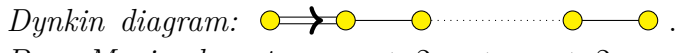
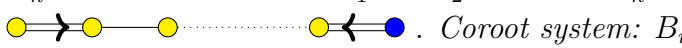
$$C_n = \{\pm 2e_a : a = 1, \dots, n\} \cup \{\pm e_a \pm e_b : a < b\}.$$

Simple basis:

$$\begin{bmatrix} 2 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & & 1 \end{bmatrix}.$$

¹⁹Superscripts denote repeated components. For example, $(0^3, -1, 1, 0^2) = (0, 0, 0, -1, 1, 0, 0)$.

²⁰Here, each \pm stands for an individual choice of sign.

Dynkin diagram: . *Weyl group and chamber:* Same as B_n . *Maximal root:* $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n = (0^{n-1}, 2)$. *Affine Dynkin diagram:* . *Coroot system:* B_n .

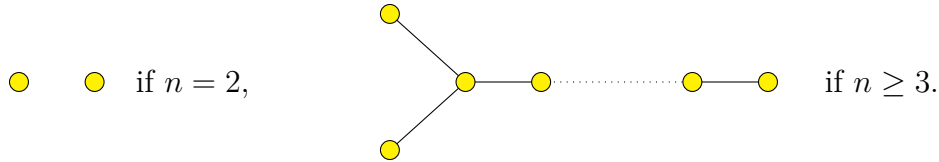
1.5.4 The root system D_n . Let $n \geq 2$ and define

$$D_n = \{x \in \mathbb{Z}^n : \langle x, x \rangle = 2\} = \{\pm e_a \pm e_b : 1 \leq a < b \leq n\}.$$

Simple basis:

$$\begin{bmatrix} 1 & -1 & & & \\ & 1 & 1 & \cdots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix}.$$

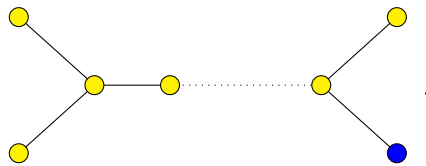
Dynkin diagram:



Note that $D_2 \cong A_1 \times A_1$. In particular, D_n is irreducible only if $n \geq 3$ (cf. Sec. 2.15).
Weyl group: $(\mathbb{Z}_2)^{n-1} \rtimes S_n$. *Weyl chamber:*

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq x_2 \leq \cdots \leq x_n, \quad 0 \leq x_1 + x_2\}.$$

Maximal root: $\alpha_1 + \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n = (0^{n-2}, 1, 1)$. *Affine Dynkin diagram:*



Coroot system: D_n .

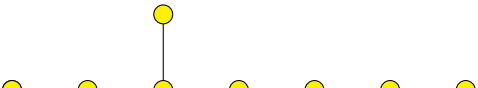
1.5.5 The root system E_8 . The root system E_8 in the even coordinate system is

$$\begin{aligned} E_8 &= \{(x_1, \dots, x_8) \in \mathbb{Z}^8 \cup (1/2 + \mathbb{Z})^8 : \langle x, x \rangle = 2, \sum_i x_i \equiv 0 \pmod{2}\} \\ &= \{\pm e_a \pm e_b : 1 \leq a < b \leq 8\} \cup \{\frac{1}{2} \sum \sigma_i e_i : \sigma \in (\mathbb{Z}_2)^8, \prod_i \sigma_i = 1\}. \end{aligned}$$

Another isomorphic copy of E_8 , namely E_8 in the odd coordinate system, will not be

used here. Both may be found in [CS99, 8.1]. *Simple basis:*

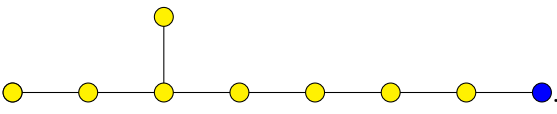
$$\begin{bmatrix} 1 & -1 & & & & & & 1/2 \\ 1 & 1 & -1 & & & & & -1/2 \\ & & 1 & -1 & & & & -1/2 \\ & & & 1 & -1 & & & -1/2 \\ & & & & 1 & -1 & & -1/2 \\ & & & & & 1 & -1 & -1/2 \\ & & & & & & 1 & -1/2 \\ & & & & & & & 1/2 \end{bmatrix}.$$

Dynkin diagram: . *Weyl group:* Order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600$. For a description of the generators see [CS99, Ch. 4, 8.1, p. 123]. See also [Ada96, Ch. 10] and [Wil09, 2.8.4]. *Weyl chamber:*

$$\{(x_1, \dots, x_8) \in \mathbb{R}^8 : x_1 \leq \dots \leq x_7, \quad 0 \leq x_1 + x_2, \quad x_2 + \dots + x_7 \leq x_1 + x_8\}.$$

Maximal root:

$$2\alpha_8 + 2\alpha_7 + 3\alpha_6 + 4\alpha_5 + 5\alpha_4 + 6\alpha_3 + 4\alpha_2 + 3\alpha_1 = (0^6, 1, 1).$$

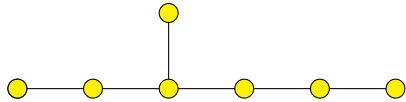
Affine Dynkin diagram: . *Coroot system:* E_8 .

1.5.6 The root system E_7 . The intersection of E_8 with the orthogonal space of $(0^6, 1, 1)$ is

$$\begin{aligned} E_7 &= \{(x_1, \dots, x_8) \in \mathbb{Z}^8 \cup (1/2 + \mathbb{Z})^8 : \langle x, x \rangle = 2, x_7 + x_8 = 0\} \\ &= \{\pm e_a \pm e_b : 1 \leq a < b \leq 6\} \cup \{\pm(e_7 - e_8)\} \\ &\quad \cup \{\frac{1}{2} \sum \sigma_i e_i : \sigma \in (\mathbb{Z}_2)^8, \prod_i \sigma_i = 1, \sigma_7 \neq \sigma_8\}. \end{aligned}$$

Simple basis:

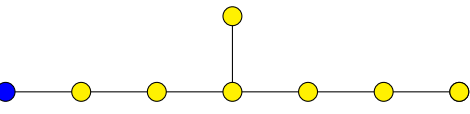
$$\begin{bmatrix} 1 & -1 & & & & & & 1/2 \\ 1 & 1 & -1 & & & & & -1/2 \\ & & 1 & -1 & & & & -1/2 \\ & & & 1 & -1 & & & -1/2 \\ & & & & 1 & -1 & & -1/2 \\ & & & & & 1 & -1 & -1/2 \\ & & & & & & 1 & -1/2 \\ & & & & & & & -1/2 \\ & & & & & & & 1/2 \end{bmatrix}.$$

Dynkin diagram: . *Weyl group:* Order $2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2903040$ [CS99, Ch. 4, 8.1, p. 125] [Wil09, 2.8.4]. *Weyl chamber:*

$$\{(x_1, \dots, x_8) \in \mathbb{R}^8 : x_1 \leq \dots \leq x_6, \quad 0 \leq x_1 + x_2, \\ x_2 + \dots + x_7 \leq x_1 + x_8, \quad x_7 + x_8 = 0\}.$$

Maximal root:

$$2\alpha_7 + \alpha_6 + 2\alpha_5 + 3\alpha_4 + 4\alpha_3 + 3\alpha_2 + 2\alpha_1 = (0^6, -1, 1).$$

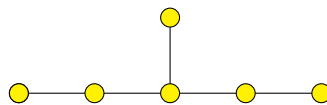
Affine Dynkin diagram: . *Coroot system:* E_7 .

1.5.7 The root system E_6 . The intersection of E_7 with the orthogonal space of $(0^5, 1, 0, 1)$ is

$$E_6 = \{(x_1, \dots, x_8) \in \mathbb{Z}^8 \cup (1/2 + \mathbb{Z})^8 : \langle x, x \rangle = 2, x_6 + x_7 = x_7 + x_8 = 0\} \\ = \{\pm e_a \pm e_b : 1 \leq a < b \leq 5\} \cup \{\frac{1}{2} \sum \sigma_i e_i : \sigma \in (\mathbb{Z}_2)^8, \prod_i \sigma_i = 1, \sigma_6 = \sigma_7 \neq \sigma_8\}.$$

Simple basis:

$$\begin{bmatrix} 1 & -1 & & & & & & 1/2 \\ 1 & 1 & -1 & & & & & -1/2 \\ & & & 1 & -1 & & & -1/2 \\ & & & & 1 & -1 & & -1/2 \\ & & & & & 1 & -1/2 & \\ & & & & & & 1 & -1/2 \\ & & & & & & & -1/2 \\ & & & & & & & -1/2 \\ & & & & & & & 1/2 \end{bmatrix}.$$

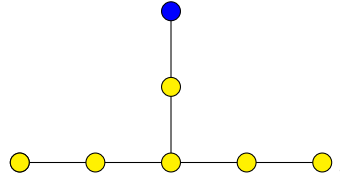
Dynkin diagram: . *Weyl group:* Order $2^7 \cdot 3^4 \cdot 5 = 51840$ [CS99, Ch. 4, 8.1, p. 126] [Wil09, 2.8.4]. *Weyl chamber:*

$$\{(x_1, \dots, x_8) \in \mathbb{R}^8 : x_1 \leq \dots \leq x_7, \quad 0 \leq x_1 + x_2, \\ x_2 + \dots + x_7 \leq x_1 + x_8, \quad x_6 + x_8 = x_7 + x_8 = 0\}.$$

Maximal root:

$$\alpha_6 + \alpha_5 + 2\alpha_4 + 3\alpha_3 + 2\alpha_2 + 2\alpha_1 = ((\frac{1}{2})^5, (-\frac{1}{2})^2, \frac{1}{2}).$$

Affine Dynkin diagram:



Coroot system: E_6 .

1.5.8 The root system F_4 . Define

$$F_4 = \left\{ \frac{1}{2}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) : \sigma \in (\mathbb{Z}_2)^4 \right\} \cup \{ \pm(e_a - e_b) : a < b \} \cup \{ \pm e_a : a = 1, 2, 3, 4 \}.$$

Simple basis:

$$\begin{bmatrix} 1 & -1 & & -1/2 \\ & 1 & -1 & -1/2 \\ & & 1 & -1/2 \\ & & & 1/2 \end{bmatrix}.$$

Dynkin diagram: . *Weyl group:* Group of order $2^7 \cdot 3^2$ [Ada96, Thm. 14.2] [Wil09, 2.8.4]. *Weyl chamber:*

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 0 \leq x_1 \leq x_2 \leq x_3, \quad x_1 + x_2 + x_3 \leq x_4\}.$$

Maximal root:

$$2\alpha_4 + 2\alpha_3 + 3\alpha_2 + 4\alpha_1 = (0, 0, 1, 1).$$

Affine Dynkin diagram: . *Coroot system:* Similar to F_4 .

1.5.9 The root system G_2 . Define

$$G_2 = \{ \pm(e_a - e_b) : a < b \} \cup \{ \pm(-2e_a + e_b + e_c) : \{a, b, c\} = \{1, 2, 3\} \} \subseteq \mathbb{R}^3.$$

Then G_2 is a root system of $V = \{x \in \mathbb{R}^3 : \sum_i x_i = 0\}$. *Simple basis:*

$$\begin{bmatrix} -1 & 1 \\ 1 & -2 \\ & 1 \end{bmatrix}.$$

Dynkin diagram: . *Weyl group:* Dihedral group of order 12 [Wil09, 2.8.4]. *Weyl chamber:* Same as A_2 . *Maximal root:* $2\alpha_1 + \alpha_2 = (-1, 0, 1)$. *Affine Dynkin diagram:* . *Coroot system:* Similar to G_2 .

1.6 Root Lattices

1.6.1. [CS99, Ch. 4, p. 97] Let Λ be a lattice. A vector $r \in \Lambda$ is a *root vector* if r is a shortest vector of $\mathbb{R} \cdot r \cap \Lambda$ and the reflection

$$x \mapsto x - r^\vee(x) \cdot r$$

is an automorphism of Λ . The lattice Λ is a *root lattice* if Λ is generated by its root vectors. The *root system* of a root lattice is defined as the set of its root vectors. It is a root system in the sense of 1.3.1. The list of root lattices generated by irreducible root systems is presented in Sec. 1.7. Any other root lattice is isomorphic to a sum of these lattices.

1.6.2. Let L be a root lattice generated by a root system R and fix a Weyl chamber K of R . Suppose that Λ is a lattice with $\Lambda \supseteq L$ and $w(\Lambda) \subseteq \Lambda$ for all $w \in W$. Then the (deep) holes x of Λ with $x \in K$ are called *typical (deep) holes* with respect to K , and every (deep) hole of Λ is related to a typical (deep) hole by W .

The intersection A of the Voronoi cell of Λ with the topological closure of K is called the *fundamental alcove* of Λ with respect to K .^{21,22} The fundamental alcove of the affine Weyl group of R introduced in 1.3.4 is the fundamental alcove of the root lattice L , which follows by repeating the arguments of [CS99, Ch. 21, Thm. 5]. In particular, the Weyl orbit $\bigcup_{w \in W} w(A)$ of A is equal to the Voronoi cell of Λ . It is therefore sufficient to determine the fundamental alcove in order to determine the Voronoi cell of Λ .

1.6.3 Short vs. long root system. [CS99, p. 99] If Λ is a lattice generated by a root system R , then R is in general only a subset of the root system of Λ . The root systems C_n and D_n , for instance, differ by their first simple root, but they span the same lattice, which, by convention, is designated D_n . It may be similarly observed that F_4 spans D_4 and G_2 spans A_2 . Therefore, the root system of the lattice D_n is C_n and the root system D_n may only be reconstructed from D_n as the set of the short roots of the lattice D_n .

With the exception of B_n , all root lattices generated by irreducible root systems are generated by their short root vectors.²³ Therefore, when dealing with root lattices generated by irreducible root systems $\neq B_n$, it is sufficient to consider those of *ADE type*. This is preferable in some respects. For instance, the short roots of the irreducible root systems listed above have length $\sqrt{2}$. It follows that

$$a - s_r(a) = r^\vee(a) \cdot r = \langle r, a \rangle \cdot r$$

²¹If $\Lambda = L$ is generated by an irreducible root system, then Conway–Sloane call this the *fundamental simplex*.

²²The root system R may not be uniquely determined by L , and different choices of root systems may lead to different fundamental alcoves. For instance, $A_7^* \subseteq E_7 \subseteq A_7$, cf. Sec. D.1.

²³There is no irreducible short root system that generates the lattice B_n , although it is generated by $A_1 \times \cdots \times A_1$.

for all glue vectors $a \in \Lambda^*$ and $r \in R$, where $\langle r, a \rangle \in \mathbb{Z}$ by the definition of the dual lattice. Therefore, the Weyl group of R acts trivially on the elements of the glue group Λ^*/Λ and every Weyl chamber of R contains a representative of each such element.

1.6.4 Voronoi cell. Conway and Sloane observe in the aforementioned [CS99, Ch. 21, Thm. 5] that the Voronoi cell of a root lattice Λ is the union of the images of what they call the fundamental simplex S under the Weyl group W of the generating root system R , i.e. $\mathcal{V} = \bigcup_{w \in W} w(S)$. The proof is based on the simple geometric observation that for every point x of the interior of S the closest point of Λ to x is the origin. It follows from this that the notion of fundamental alcove defined above coincides with that of Conway–Sloane.

Moreover, they gather that the affine Dynkin diagram determines the walls of the fundamental alcove S of the root lattices A_n for $n \geq 1$, D_n for $n \geq 3$, E_6, E_7 and E_8 . In fact, the vertex x of S opposite to a wall $\bar{K} \cap \mathcal{H}_\alpha$ of the selected fundamental chamber K is obtained by intersecting the remaining walls and the *roof* of S , i.e. the bisector $H(\rho)$ of the maximal root ρ of R in K [CS99, Ch. 21, 3.A.]. Using this method of determining the fundamental alcove, they compute the Voronoi cell of all root lattices of ADE type. In the next section, these calculations are repeated with minor changes.

Thanks to the results of these calculations and the remark made in 1.6.3 about short root systems of root lattices, it is possible to determine the Voronoi cell of root lattices generated by irreducible root systems using the Dynkin diagram of the associated short root system, except if $R = B_n$. The fundamental alcove of B_n , whose vertices are $(1/2, 0^{n-1}), (1/2, 1/2, 0^{n-2}), \dots, (1/2)^n$, cannot be determined using only the maximal root and the Weyl chamber. Luckily, as $\Lambda = \mathbb{Z}^n$ if $R = B_n$, no deep theory is required to compute the Voronoi cell in this special case.

1.7 Common Root Lattices and Their Dual Lattices

In the following, the reader will find a table of root lattices generated by irreducible root systems and their dual lattices, are of which all commonly encountered throughout the present thesis. This section uses the conventions of [CS99, Ch. 4, Sec. 6–8] and [CS99, Ch. 21, 3.A–3.F]. Unfortunately, the simple bases presented in [CS99, Ch. 4, Sec. 6–8] are not those used for all calculations later made in [CS99, Ch. 21, Sec. 3]. This problem is mitigated by some sign changes made in this thesis. The choices of basis matrices made for the list of irreducible root system in Sec. 1.5 are read off of the labeled Dynkin diagrams from [CS99, Fig. 21.1–21.3]. They are compatible with the calculations made in [CS99, Ch. 21, Sec. 3], except that the signs of the vertices of the fundamental alcove of E_7 presented here must be reversed to obtain the result from [CS99, Fig. 21.8].

1.7.1 The lattice A_n . [CS99, Ch. 4, Sec. 6] For $n \geq 1$,

$$A_n = \{(x_0, \dots, x_n) \in \mathbb{Z}^{n+1} : x_0 + \dots + x_n = 0\}$$

is the lattice generated by the root system A_n . *Determinant:* $n + 1$. *Glue vectors:* The j^{th} glue vector is

$$[j] = \left[\left(\frac{-j}{n+1} \right)^k, \left(\frac{k}{n+1} \right)^j \right],$$

where $k = n + 1 - j$. *Glue group:* $\mathbb{Z}_{n+1}, [j] + [\ell] = [j + \ell]$. *Typical deep holes:*

$$\left[\frac{n+1}{2} \right] = \left[\left(-\frac{1}{2} \right)^{(n+1)/2}, \left(\frac{1}{2} \right)^{(n+1)/2} \right]$$

if n is odd, and

$$\begin{aligned} \left[\frac{n}{2} \right] &= \left[\left(-\frac{n+2}{2(n+1)} \right)^{n/2}, \left(\frac{n}{2(n+1)} \right)^{(n+2)/2} \right], \\ \left[\frac{n+2}{2} \right] &= \left[\left(-\frac{n}{2(n+1)} \right)^{(n+2)/2}, \left(\frac{n+2}{2(n+1)} \right)^{n/2} \right] \end{aligned}$$

if n is even. *Covering radius:* $\frac{1}{2}\sqrt{n+1}$ if n is odd, $\frac{1}{2}\sqrt{\frac{n(n+2)}{n+1}}$ if n is even. *Coroot lattice:* A_n .

1.7.2 The dual lattice A_n^* . [CS99, Ch. 4, Sec. 6.6]

$$A_n^* = \bigcup_{j=1}^{n+1} ([j] + A_n).$$

Generating matrix:

$$\frac{1}{n+1} \cdot \begin{bmatrix} n & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n \\ -1 & \cdots & \cdots & -1 \end{bmatrix}. \quad (1.7.1)$$

Voronoi cell: Convex closure of the orbit of

$$\sigma = \frac{1}{2(n+1)}(-n, -n+2, \dots, n-2, n)$$

under the Weyl group [CS99, Ch. 21, Thm. 7]. *Typical deep hole:* σ . *Covering radius:* $\sqrt{\frac{n(n+2)}{12(n+1)}}$.

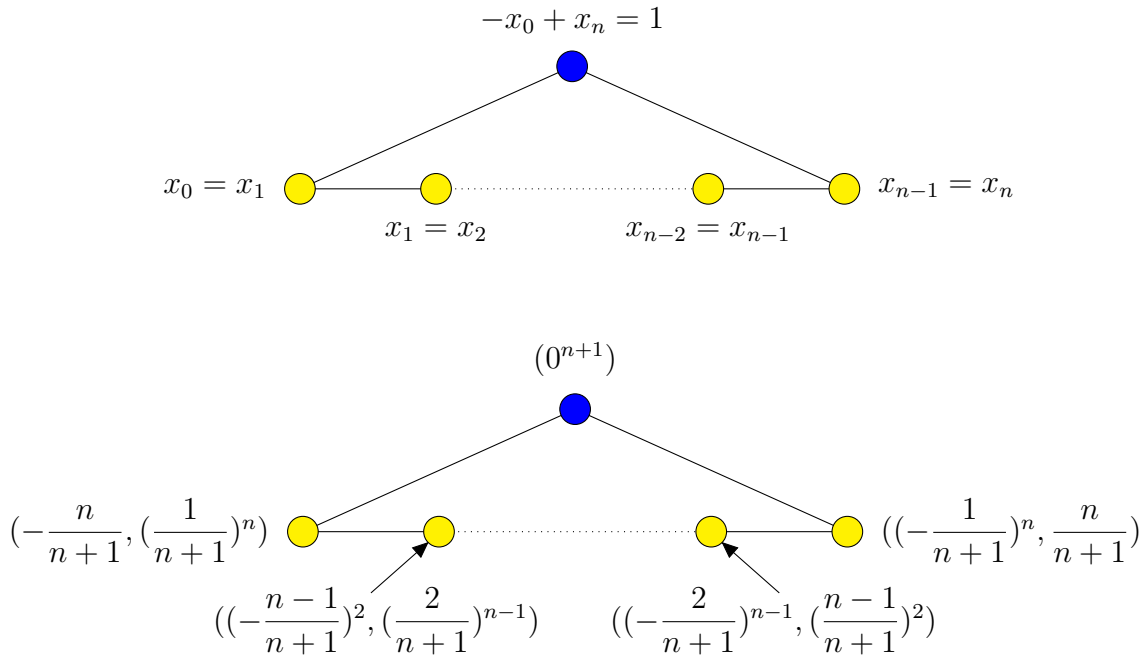


Figure 1.2: Walls and opposite vertices of the fundamental alcove of A_n [CS99, Fig. 21.1] [CS99, Fig. 21.6].

1.7.3 The lattice B_n . For $n \geq 2$, the lattice generated by B_n is \mathbb{Z}^n . *Determinant:* 1 (\mathbb{Z}^n is unimodular). *Dual lattice:* \mathbb{Z}^n . *Glue group:* trivial. *Coroot lattice:* C_n , see 1.7.4. *Typical deep hole:* $(\frac{1}{2})^n$. *Covering radius:* $\sqrt{n}/2$.

1.7.4 The lattice C_n . For $n \geq 2$, the lattice generated by C_n is D_n (see 1.6.3). *Coroot lattice:* B_n .

1.7.5 The lattice D_n . [CS99, Ch. 4, Sec. 7] For $n \geq 2$,

$$D_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : x_1 + \dots + x_n \in 2\mathbb{Z}\}$$

is the lattice generated by the root system D_n . *Determinant:* 4. *Glue group:* cyclic \mathbb{Z}_4 , i.e. $[i] + [j] = [i + j]$ if n is odd, Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $[i] + [i] = 0$, if n is even. *Typical deep holes:* $[2] = (0^{n-1}, 1)$ if $n \leq 4$ and $[1] = (\frac{1}{2})^n$, $[3] = ((\frac{1}{2})^{n-1}, -\frac{1}{2})$ if $n \geq 4$. *Covering radius:* 1 if $n \leq 4$ and $\sqrt{n}/2$ if $n \geq 4$. *Coroot lattice:* D_n .

1.7.6 The dual lattice D_n^* . $D_n \subseteq \mathbb{Z}^n \subseteq D_n^*$ are index-2-inclusions. \mathbb{Z}^n is obtained by adjoining $[2]$ to D_n and D_n^* by adjoining $[1]$ to \mathbb{Z}^n . *Generating matrix:*

$$\begin{bmatrix} 1 & & & 1/2 \\ & \ddots & & \\ & & 1 & 1/2 \\ & & & 1/2 \end{bmatrix}.$$

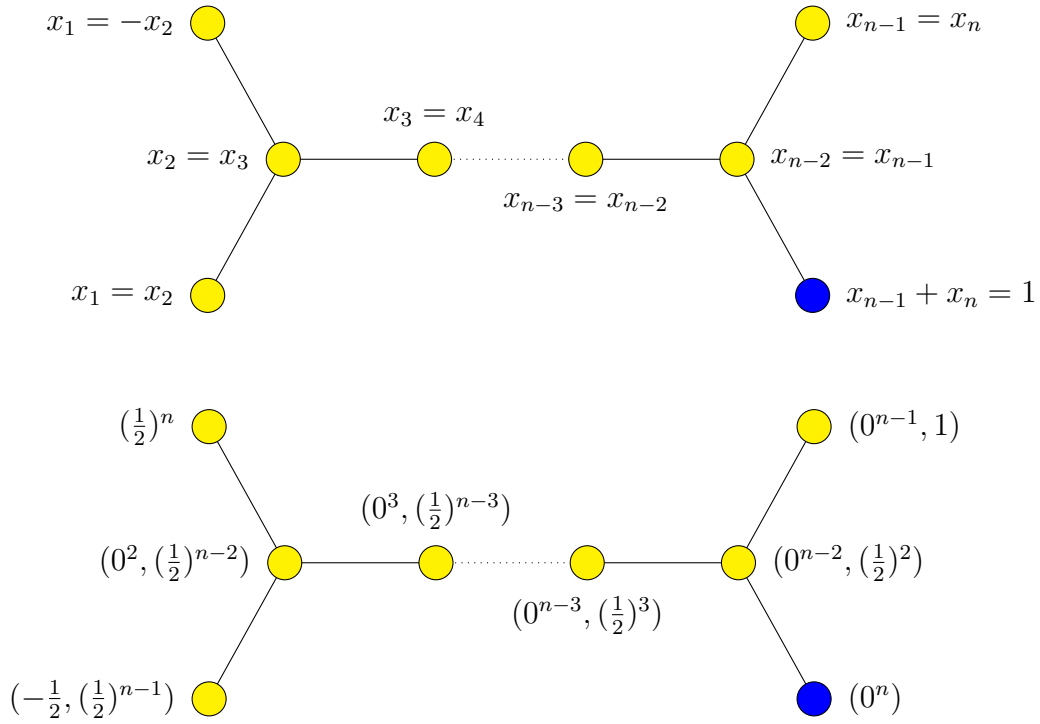


Figure 1.3: Walls and opposite vertices of the fundamental alcove of D_n [CS99, Fig. 21.2, Fig. 21.7].

Typical deep hole: Let $t = n/2$ if n is even and $t = (n - 1)/2$ if n is odd. Then the typical deep hole is $\sigma = ((\frac{1}{2})^t, 0^t)$ if n is even and $\sigma = ((\frac{1}{2})^t, \frac{1}{4}, 0^t)$ if n is odd [CS91, Sec. 7]. *Voronoi cell:* Convex closure of the Weyl orbit of σ [CS91, Sec. 7]. *Covering radius:* $\sqrt{n/8}$ if n is even, and $\sqrt{(2n - 1)/4}$ if n is odd.

1.7.7 The lattice E_8 . [CS99, Ch. 4, Sec. 8] The lattice

$$E_8 = \{(x_1, \dots, x_8) \in \mathbb{Z}^8 \cup (1/2 + \mathbb{Z})^8 : \sum x_i \equiv 0 \pmod{2}\}$$

is called E_8 in the even coordinate system and is the lattice generated by the root system E_8 . *Determinant:* 1 (E_8 is unimodular). *Dual lattice:* $E_8^* = E_8$. *Glue group:* trivial. *Typical deep hole:* $(0^7, 1)$. *Covering radius:* 1. *Coroot lattice:* E_8 .

1.7.8 The lattice E_7 . [CS99, Ch. 4, Sec. 8] The lattice

$$E_7 = \{(x_1, \dots, x_8) \in E_8 : x_7 + x_8 = 0\}$$

is the lattice of vectors in E_8 perpendicular to $(0^6, 1, 1)$. *Determinant:* 2. *Glue group:* $\mathbb{Z}_2, [1] = (0^5, 1, -\frac{1}{2}, \frac{1}{2})$. *Typical deep hole:* $[1]$. *Covering radius:* $\sqrt{3/2}$.

1.7.9 The lattice E_7^* . [CS99, Ch. 4, Sec. 8] $E_7^* = E_7 \cup ([1] + E_7)$. *Generator matrix:*

$$\begin{bmatrix} -1 & & & & & & -3/4 \\ 1 & -1 & & & & & -3/4 \\ & 1 & -1 & & & & 1/4 \\ & & 1 & -1 & & & 1/4 \\ & & & 1 & -1 & & 1/4 \\ & & & & 1 & -1 & 1/4 \\ & & & & & 1 & 1/4 \\ & & & & & & 1/4 \end{bmatrix}.$$

Typical deep hole: $\sigma = (\frac{7}{8}, (-\frac{1}{8})^7)$. Voronoi cell: Convex closure of the Weyl orbit of σ [CS99, Ch. 4, Sec. 8.2]. Covering radius: $\sqrt{7/8}$.

1.7.10 The lattice E_6 . [CS99, Ch. 4, Sec. 8] The lattice

$$E_6 = \{(x_1, \dots, x_8) \in E_8 : x_6 + x_7 = x_7 + x_8 = 0\}$$

is the lattice of vectors in E_8 perpendicular to $(0^6, 1, 1)$ and $(0^5, 1, 0, 1)$. *Glue group:* \mathbb{Z}_3 , $[1] = (0^4, 1, (-\frac{1}{3})^2, \frac{1}{3})$, $[2] = (0^5, (-\frac{2}{3})^2, \frac{2}{3})$. *Typical deep holes:* $[1], [2]$. *Covering radius:* $2/\sqrt{3}$.

1.7.11 The lattice E_6^* . [CS99, Ch. 4, Sec. 8] $E_6^* = E_6 \cup ([1] + E_6) \cup ([2] + E_6)$. *Generating matrix:*

$$\begin{bmatrix} & & & & & & 1/2 \\ -1 & & & & & & 2/3 & 1/2 \\ 1 & -1 & & & & & 2/3 & 1/2 \\ & 1 & -1 & & & & -1/3 & 1/2 \\ & & 1 & -1 & & & -1/3 & -1/2 \\ & & & 1 & -1 & & -1/3 & -1/2 \\ & & & & 1 & -1/3 & -1/2 \\ & & & & & -1/3 & -1/2 \\ & & & & & & -1/2 \end{bmatrix}.$$

Typical deep hole: $\sigma = (0, (\frac{1}{3})^3, (-\frac{1}{3})^3, 0)$. Voronoi cell: Convex closure of the Weyl orbit of σ [CS99, Ch. 4, Sec. 8.3]. Covering radius: $\sqrt{2/3}$.

1.7.12 The lattice F_4 . Equal to $\frac{1}{2}D_4$. *Coroot lattice:* $F_4^\vee = D_4$. *Dual lattice:* $F_4^* = D_4$.

1.7.13 The lattice G_2 . Equal to A_2 . *Coroot lattice:* $G_2^\vee = A_2^*$. *Dual lattice:* $G_2^\vee = G_2^*$.

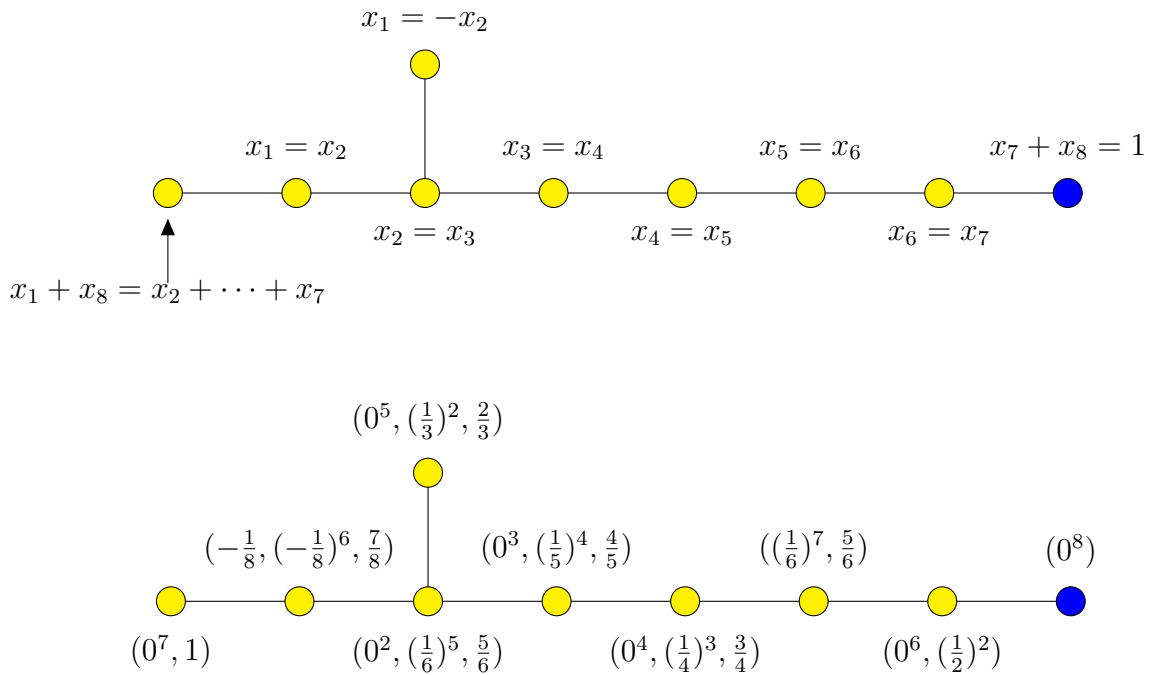


Figure 1.4: Walls and vertices of the fundamental simplices of E_8 [CS99, Fig. 21.3, Fig. 21.8].

1.8 Delone Complex and Circumradius Formula

1.8.1 Delone polytope. Let $v_0, \dots, v_n \in \mathbb{R}^m$. The convex closure P of the points v_0, \dots, v_n is called *L-polyhedron*²⁴ or *Delone polytope*²⁵ if there exists a sphere in \mathbb{R}^m containing v_0, \dots, v_n [GL87, vi, p. 172, (i)]. The radius and centre of the sphere are called the *radius* $\mu(P)$ and *centre* of P , respectively, and the *thickness* of P is $\mu(P)^{\dim P} / \text{vol}(P)$.²⁶

1.8.2 Delone complex. Let Λ be a lattice with Voronoi cell \mathcal{V} . The *Delone cell* of a vertex x of \mathcal{V} is the convex closure of all lattice points with minimal distance to x . Every Delone cell is a Delone polytope. The set of all Delone cells of Λ is the *Delone complex*

²⁴*Polyhedron* is of course the wrong word, the correct one being *polytope*.

²⁵After Russian mathematician and mountaineer B.N. Delone, sometimes transliterated as *Delauney*. Early works of Delone were published under the name *Delauney* [CS99, p. 35].

²⁶Conway and Sloane have already expressed their dissatisfaction with the term *thickness* in the context of lattices [CS99, pp. 31–32], but this particular terminology is even worse. Which of the following simplices is thinner?



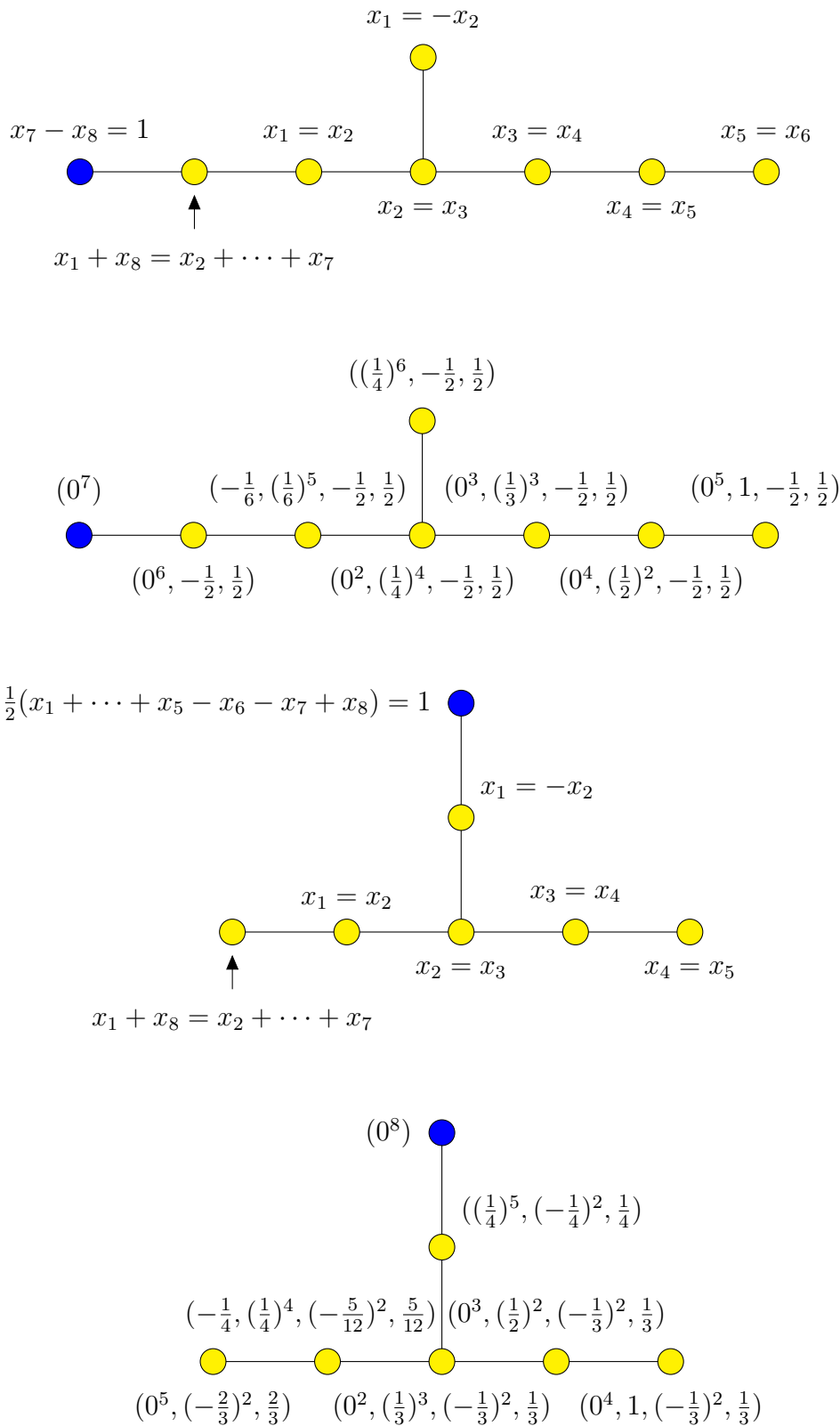


Figure 1.5: Walls and vertices of the fundamental simplices of E_7 and E_6 [CS99, Fig. 21.3, Fig. 21.8].

Del Λ of Λ .²⁷ The structure of the Delone complex can be very convoluted. For instance, the Delone cells of E_8 have 16 (deep holes) or 9 (shallow holes) vertices [CS99, Ch. 4, Sec. 8.1], those of E_6^*, E_7^* have 7, 15, respectively, and those of D_n are either octahedral or cuboid [CS99, Ch. 5, Sec. 4.2]. A complete computation of the Delone cells of lattices generated by irreducible root systems and their dual lattices was achieved by Conway and Sloane in [CS91]. In 1.9.8, the reader may find a computation of the Delone complex of A_n^* using Conway–Sloane’s notion of *superbasis*.

In general, the image $\exp(tC)P$ of a Delone polytope P for $C \in \mathbb{R}^{m \times m}$ is not a Delone polytope. This property is reserved for simplices. In the following paragraphs, the derivative of the circumradius of a simplex P is studied by expressing the circumcentre of P as a differentiable function of the \mathcal{V} -representation of P using a particular notion of *dual basis*:

1.8.3 Dual basis. Let V_0 be a subspace of \mathbb{R}^m . Recall that every basis v_1, \dots, v_n of V_0 admits a *dual basis* v_1^*, \dots, v_n^* of V_0 with $\langle v_i, v_j^* \rangle = \delta_{ij}$. Let $V \in \mathbb{R}^{m \times n}$ denote the matrix with column vectors v_1, \dots, v_n and $V^* \in \mathbb{R}^{n \times m}$ the matrix with row vectors $(v_1^*)^\top, \dots, (v_n^*)^\top$. Then V^* is the unique element $W \in \mathbb{R}^{n \times m}$ so that VW is symmetric and $WV = I_n$. In fact, suppose that W is such an element and note that $WV = (\langle w^i, v_j \rangle)$ implies $\langle w^i, v_j \rangle = \delta_{ij}$, where w^i denotes the i^{th} row of W . Moreover, $P = VW$ is a symmetric idempotent and therefore the orthogonal projection $\pi: \mathbb{R}^m \rightarrow V_0$. It follows that

$$Pw^i = VWW^\top e_i = (VW)^\top W^\top e_i = W^\top V^\top W^\top e_i = W^\top (WV)^\top e_i = W^\top e_i = w^i$$

by using first that VW is symmetric and then $WV = I_n$. Thus, $w^i \in V_0$, and it follows that $w^i = (v_i^*)^\top$, completing the proof. Note that $(v_1, \dots, v_n) \mapsto V^*$ is a differentiable map.

1.8.4 Inner product on $\text{End } V$. If V is an inner product space, then $\text{End } V$ may be equipped with the scalar product

$$\langle A, B \rangle = \text{tr}(A \circ B^\top),$$

where \top denotes transposition with respect to the inner product of V . Thanks to the invariance of the trace under transposition and cyclic permutations, $\langle \cdot, \cdot \rangle$ enjoys these properties as well. Note that the orthogonal space of Id_V is the space of trace-free endomorphisms.

If $V = \mathbb{R}^n$ and $\text{End } V$ is identified with $\mathbb{R}^{n \times n}$, then

$$\langle A, B \rangle = \sum_{i,j=1}^n a_{ij} b_{ij}.$$

²⁷Although this set may be equipped with the structure of a polyhedral complex, this carries no further weight here.

More generally, the typical scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^{k \times \ell}$ is defined as

$$\langle A, B \rangle = \text{tr}(AB^\top) = \sum_{i,j=1}^{k,\ell} a_{ij}b_{ij}.$$

Note that if $\mathbb{R}^{k \times \ell}$ is identified with $\mathbb{R}^{k\ell}$, then $\langle \cdot, \cdot \rangle$ coincides with the standard scalar product.

1.8.5 Lemma. *With notation as in 1.8.3. Let P be a simplex with \mathcal{V} -representation $\{v_0 = 0, v_1, \dots, v_n\}$.*

(1) *The circumcentre of P is*

$$x = \sum_{i=1}^n \frac{\|v_i\|^2}{2} v_i^*.$$

(2) *The circumradius $\mu(V)$ and squared circumradius $f(V) = \mu(P)^2$ of P , viewed as functions of the \mathcal{V} -representation in matrix form V , are differentiable.*

Moreover, if

$$F(V) = 2 \left(\sum_{i=1}^n \langle v_i^*, x \rangle \cdot v_i v_i^\top - x x^\top \right),$$

then $\langle C, F(V) \rangle = (d/ds) f(\exp(sC)V) |_{s=0}$ for all $C \in S^2(\mathbb{R}^m)$ with $[C, \pi] = 0$.

Proof. (1) Recall from 1.8.3 that $v_i^* \in V_0$. Thus, $x \in V_0$. Now observe that $\langle x, v_i \rangle = \|v_i\|^2/2$. It follows that x is the circumcentre.

(2) Note that $[\pi, \exp(tC)] = 0$, using the exponential series formula for $\exp(tC)$. Let $V(t) = \exp(tC)V$. It follows that $V(t)^* = V^* \exp(-tC)$. In fact,

$$V(t)V^* \exp(-tC) = \exp(tC)VV^* \exp(-tC) = \exp(tC)\pi \exp(-tC) = \pi$$

and $V^* \exp(-tC)V(t)$ obviously equals I_n . Note that the i^{th} column of $V(t)$ is $v_i(t) = \exp(tC)v_i$, and the i^{th} row of $V(t)^*$ is $(v_i^*)^\top \exp(-tC)$. It follows, using the symmetry of C , that the vectors $v_i^*(t) = \exp(-tC)v_i^*$ form the dual basis of $v_1(t), \dots, v_n(t)$ in V_0 . Let $y: \mathbb{R} \rightarrow \mathbb{R}^n, y_i(t) = \frac{1}{2}\|v_i(t)\|^2$. Then $x(t) = (V(t)^*)^\top \cdot y(t)$ is the circumcentre of the simplex whose nonzero vertices are $v_1(t), \dots, v_n(t)$, according to (1). It follows that

$$\begin{aligned} \frac{d}{dt} \|x(t)\|^2 \Big|_{t=0} &= 2 \cdot \left\langle \frac{d}{dt} \exp(-tC) \cdot (V^*)^\top \cdot y(t) \Big|_{t=0}, x \right\rangle \\ &= 2 \cdot \langle (V^*)^\top \cdot \dot{y}(0), x \rangle - 2 \cdot \langle C \cdot (V^*)^\top \cdot y(0), x \rangle. \end{aligned}$$

The second summand of the right-hand side is equal to (twice of)

$$-\langle Cx, x \rangle = -\text{tr}(Cxx^\top) = -\langle C, xx^\top \rangle.$$

Note that $\dot{y}_i(0) = \langle Cv_i, v_i \rangle = \text{tr}(Cv_i v_i^\top)$ and conclude that the first summand is equal to (twice of)

$$\langle (V^*)^\top \cdot \dot{y}(0), x \rangle = \sum_{i=1}^n \langle \dot{y}_i(0) \cdot v_i^*, x \rangle = \sum_{i=1}^n \langle v_i^*, x \rangle \text{tr}(Cv_i v_i^\top) = \left\langle C, \sum_{i=1}^n \langle v_i^*, x \rangle \cdot v_i v_i^\top \right\rangle.$$

Thus, it follows that

$$\left. \frac{d}{dt} \|x(t)\|^2 \right|_{t=0} = 2 \cdot \left\langle C, \sum_{i=1}^n \langle v_i^*, x \rangle \cdot v_i v_i^\top - x x^\top \right\rangle. \quad \square$$

1.8.6 The effect of orthogonal matrices. With notation as above. Let $U \in O(V_0)$ and let $\hat{U} \in O(m)$ so that $\hat{U}|_{V_0} = U$ and $\hat{U}|_{V_0^\perp} = \text{Id}_{V_0^\perp}$. Let $\tilde{V} = UV$. Mark all data associated to \tilde{V} with a \sim . Then $\tilde{V}^* = V^* \hat{U}^\top$ follows from the characterisation of V^* in 1.8.3. This implies that $\tilde{v}_i^* = \hat{U} v_i^*$, and, in particular, $\tilde{x} = \hat{U} x$. Thus, $F(\tilde{V}) = \hat{U} F(V) \hat{U}^\top$.

1.9 Lattices of Voronoi's First Kind

Certain well-behaved lattices, later called *lattices of Voronoi's first kind*, were introduced in 1908 by Voronoi, but have already been studied earlier in the 19th century by Selling (see [BK10, B.2.2] and the reference therein). The modern approach allows for a simple description of the Delone complex and the Voronoi vectors of a lattice of Voronoi's first kind in terms of what Conway and Sloane call a strictly obtuse superbasis [CS92], and this approach is adopted in the following exposition.

1.9.1 Superbasis. Let (v_1, \dots, v_n) be a basis of an inner product space V and set $v_0 = -\sum_i v_i$. Then (v_0, \dots, v_n) is called a *superbasis*.

For every $x \in V$, there exists $\xi \in \mathbb{R}^{n+1}$ with $x = \sum_i \xi_i v_i$. The vector $\xi \in \mathbb{R}^{n+1}$ is unique up to addition of multiples of $(1)^{n+1}$. Uniqueness of ξ may be achieved by requiring that $\sum_i \xi_i = 0$.

The *Selling parameters* of a superbasis (v_0, \dots, v_n) are $q_{ij} = \langle v_i, v_j \rangle$ and form the *Selling matrix* $(q_{ij}) \in S^2(\mathbb{R}^{n+1})$. The superbasis (v_0, \dots, v_n) is called *obtuse* (respectively *strictly obtuse*) if $q_{ij} \leq 0$ (respectively $q_{ij} < 0$) for all $i \neq j$. For the sake of simplicity, (v_1, \dots, v_n) will also be called a *(strictly) obtuse superbasis*.

A lattice $\Lambda \subseteq V$ is a *lattice of Voronoi's first kind (with strictly obtuse superbasis)* if Λ admits a basis (v_1, \dots, v_n) that is a (strictly) obtuse superbasis.

1.9.2 Selling formula. Let (v_1, \dots, v_n) be an obtuse superbasis. The *Selling formula* describes the norm of a vector $x \in V$ in terms of the parameter $\xi \in \mathbb{R}^{n+1}$ from 1.9.1 [CS92, Eq. (5)]:²⁸

$$\|x\|^2 = - \sum_{i,j=0,i<j}^n q_{ij} \cdot (\xi_i - \xi_j)^2.$$

1.9.3 Examples. (1) \mathbb{Z}^n is a lattice of Voronoi's first kind, but it does not admit a strictly obtuse superbasis. (2) The basis (1.5.1) of A_n is an obtuse superbasis, but it is not strictly obtuse unless $n = 2$. In fact, A_n does not admit a strictly obtuse superbasis for $n \geq 3$. (3) The basis (1.7.1) of A_n^* is a strictly obtuse superbasis. In particular, A_n^* is a lattice of Voronoi's first kind with strictly obtuse superbasis. (4) D_n is not a lattice of Voronoi's first kind for any $n \geq 4$ [BK10, Prop. B.2.6]. (5) D_n^*, E_6^*, E_7^*, E_8 are not lattices of Voronoi's first kind [BK10, B.2.3].

1.9.4. The Voronoi vectors of lattices of Voronoi's first kind with strictly obtuse superbasis have been determined in terms of their superbasis in [CS92, Thm. 3]:²⁹ If Λ is a lattice of Voronoi's first kind with strictly obtuse superbasis (v_1, \dots, v_n) , define $v_S = \sum_{i \in S} v_i$ for every subset $S \subseteq I = \{0, \dots, n\}$ with $S \neq \emptyset, I$. Then $\{v_S : S \subseteq I, S \neq \emptyset, I\}$ is the set of Voronoi vectors of Λ .

The proposition below gives a complete and simple description of the Delone complex of lattices such as Λ in terms of the strictly obtuse superbasis. This result is implicitly used in many cited sources such as [Ble62], but conspicuously absent from every discussion of lattices of Voronoi's first kind known to the author,³⁰ with the exception of [BK10, Prop. B.3.4]. A proof using induction may be found in App. B of this thesis.

1.9.5 Proposition. *Let Λ be a lattice of Voronoi's first kind with strictly obtuse superbasis v_1, \dots, v_n . Let W be the group of permutations of $\{0, \dots, n\}$. For every $\pi \in W$, define the simplex P_π as the convex closure of*

$$v_{\pi\{1\}}, v_{\pi\{1,2\}}, \dots, v_{\pi\{1,\dots,n\}}.$$

Then the Delone complex of Λ is $\{P_\pi : \pi \in W\}$.

1.9.6 Corollary. *Let Λ be a lattice of Voronoi's first kind with strictly obtuse superbasis (v_1, \dots, v_n) in the inner product space (V, Q_0) . Then there exists an open neighbourhood $\mathcal{U} \subseteq \mathcal{L}(V)$ of Λ so that all $\Gamma \in \mathcal{U}$ are lattices of Voronoi's first kind with strictly obtuse superbasis.*

²⁸Note that the statement of the Selling formula in [CS92, Eq. (5)] has the wrong sign.

²⁹[CS92, Thm. 3] almost completely describes the Voronoi vectors of a nonstrict lattice of Voronoi's first kind, as well.

³⁰Including [CS99] and [CS91]. The computation of the Voronoi cell [CS99, Ch. 21, Thm. 7] can be accomplished without knowledge of the Delone complex.

Using the interpretation of $\mathcal{L}(n)$ as quotient of $\mathrm{GL}(V)$, this means that there exists an open neighbourhood $\mathcal{B} \subseteq \mathrm{GL}(V)$ of Id_V so that $(\Phi(v_1), \dots, \Phi(v_n))$ is a strictly obtuse superbasis for all $\Phi \in \mathcal{B}$. In particular,

$$\mathrm{Del}(\Phi\Lambda) = \{\Phi\beta : \beta \in \mathrm{Del} \Lambda\}.$$

Using the interpretation of $\mathcal{L}(n)$ as quotient of $\mathcal{S}_{>0}(V)$, this means that there exists an open neighbourhood $\mathcal{T} \subseteq \mathcal{S}_{>0}(V)$ of Q_0 so that (v_1, \dots, v_n) is a strictly obtuse superbasis of (Λ, Q) for all $Q \in \mathcal{T}$. In particular, $\mathrm{Del}(\Lambda, Q) = \mathrm{Del} \Lambda$.

Proof of Corollary 1.9.6. It is sufficient to prove the statement in the first interpretation. Since ‘strictly obtuse’ is an open condition, there exists a compact ball $\mathcal{B} \subseteq \mathrm{GL}(n, \mathbb{R})$ with centre I_n so that $w_1 = \Phi v_1, \dots, w_n = \Phi v_n$ is a strictly obtuse superbasis of $\Phi\Lambda$ for all $\Phi \in \mathcal{B}$. It follows that $w_S = \Phi(v_S)$. An application of Prop. 1.9.5 yields the result. \square

1.9.7. The symmetric group S_{n+1} acts combinatorially on the elements of the superbasis (v_0, \dots, v_n) by permutation. According to Prop. 1.9.5, this action ascends to an action on the Delone complex, which may be described by $\sigma(P_\pi) = P_{\sigma\pi}$ for all $\sigma, \pi \in W$. Proposition 1.9.5 shows that this action is simply-transitive. The polytope P_{Id} will be called the *fundamental Delone polytope*.

1.9.8 The Delone complex of A_n^* . Denote by w_1, \dots, w_n the columns of the matrix (1.7.1) and let $w_0 = -\sum_i w_i$. It is straightforward to verify that $\langle w_i, w_j \rangle = -1/(n+1)$ and this implies that this particular basis is a strictly obtuse superbasis. The action of the Weyl group W on the Delone complex of A_n^* is simply-transitive. The fundamental polytope determined by the strictly obtuse superbasis above is spanned by the vectors $v_1 = w_{\{1\}}, \dots, v_n = w_{\{1, \dots, n\}}$. Further computations show that v_i is equal to the glue vector $[i]$.

Let $V \in \mathbb{R}^{(n+1) \times n}$ be the matrix whose columns are v_1, \dots, v_n and $V^* \in \mathbb{R}^{n \times (n+1)}$ be its dual matrix as defined in 1.8.3. This choice of notation is compatible with 1.8.3–1.8.6. The i^{th} row of V^* is

$$v_i^* = (0^{i-1}, -1, 1, 0^{n-i}),$$

implying that V^* is the transpose of the standard generating matrix (1.5.1). The circumcentre x of the Delone polytope may be calculated using Lem. 1.8.5(1). It coincides with the typical deep hole σ from (1.7.1):

$$x = \frac{1}{2(n+1)} (-n, -n+2, \dots, n-2, n).$$

Every other Delone polytope and deep hole is obtained by applying the simply-transitive action of the Weyl group $W = S_{n+1}$.

1.10 Bleicher's Theorem

1.10.1 Bleicher's theorem. In 1.1.5, *Bleicher's theorem* was mentioned, which asserts that the dual lattice A_n^* is a locally thinnest lattice [Ble62, Thm. 3]. Bleicher's original proof exploits the fact that $\Lambda = A_n^*$ and every lattice close to it is a lattice of Voronoi's first kind with strictly obtuse superbasis. Bleicher, however, considers the positive quadratic forms representing these lattices instead of the lattices themselves (cf. 1.1.4). He interprets the radius of each simplex $P_1, \dots, P_{(n+1)!}$ as function of the Selling parameters of the quadratic forms and studies the Taylor series of these differentiable functions $F_1, \dots, F_{(n+1)!}$ (cf. Lem. 1.8.5). His first step is to prove that for every direction v , the sum of the first derivatives $\partial_v F_i$ is zero [Ble62, p. 642] (cf. Lem. 1.10.8). Bleicher concludes that the thickness of the lattice must increase *superlinearly* if $\partial_v F_i \neq 0$ for some i . To be more precise, a continuous function $f: U \rightarrow \mathbb{R}$ on an open set U of a vector space V exhibits *superlinear growth* at $u_0 \in U$ in direction $v \in V$ if there exists a differentiable map $b: \mathbb{R} \rightarrow \mathbb{R}$ with $b(0) = f(u_0)$, $b'(0) > 0$ and $f(u_0 + tv) \geq b(t)$. In the final step, Bleicher studies what happens if the first derivatives of all F_i vanish.

Denote by V the space of zero-sum vectors and recall that $A_n^* \subseteq V$. Bleicher's argument suggests a direct sum decomposition of the 'tangent space' $T_\Lambda \mathcal{P}\mathcal{L}(V) = S_0^2(V)$ (see 1.1.4) into the space of directions \mathcal{F} with superlinear growth of thickness, which is generally useful and straightforward to deal with, and its orthogonal complement \mathcal{F}^\perp , which is difficult to deal with. To put it crudely, there is a decomposition of $T_\Lambda \mathcal{P}\mathcal{L}(V)$ into *good* and *bad* directions. In this section, a complete description of the space \mathcal{F} in terms of the gradient matrices defined in Lem. 1.8.5 is given.

1.10.2 Extension of symmetric endomorphisms. For $C \in S^2(V)$, define the symmetric endomorphism $\tilde{C} \in S^2(\mathbb{R}^{n+1})$ by setting $\tilde{C}(v) = C(v)$ for $v \in V$ and $\tilde{C}(1, \dots, 1) = 0$. Denote by \mathcal{S} the set of symmetric endomorphisms $X \in S^2(\mathbb{R}^{n+1})$ satisfying one of the following equivalent conditions:

- (a) $X(1, \dots, 1) = 0$.
- (b) The row sums of X are zero.
- (c) The column sums of X are zero.
- (d) There exists $C \in S^2(V)$ with $X = \tilde{C}$.

Note that all $X \in \mathcal{S}$ satisfy $XV \subseteq V$, and that $S^2(V) \rightarrow \mathcal{S}, C \mapsto \tilde{C}$ is an isomorphism. Furthermore, observe that $\text{tr } \tilde{C} = \text{tr } C$, and that \tilde{C} satisfies the hypothesis of Lem. 1.8.5(2).

1.10.3 Computation of the gradient matrix. Let v_1, \dots, v_n be the nonzero vertices of the fundamental Delone polytope associated to the basis matrix (1.7.1), (v_1^*, \dots, v_n^*) the dual basis, x the circumcentre of the fundamental Delone polytope and

$$F = 2 \left(\sum_{i=1}^n \langle v_i^*, x \rangle \cdot v_i v_i^\top - x x^\top \right)$$

(the gradient matrix from Lem. 1.8.5(2)). All notation is chosen to be compatible with 1.8.3–1.8.6. The entries of vectors and matrices are indexed from 0 to n . The formula for x (see 1.9.8) immediately implies

$$(x x^\top)_{k\ell} = \frac{1}{4(n+1)^2} (n^2 - 2n(k+\ell) + 4k\ell),$$

for all $k, \ell = 0, \dots, n$, and $\langle v_i^*, x \rangle = 1/(n+1)$ for all $i = 1, \dots, n$. Denote by v^0, \dots, v^n the rows of the matrix whose columns are v_1, \dots, v_n . This means that

$$v^k = \frac{1}{n+1} (-1, \dots, -k, n-k, n-k-1, \dots, 1).$$

It follows that

$$\begin{aligned} \sum_{i=1}^n \langle v_i^*, x \rangle \cdot (v_i v_i^\top)_{k\ell} &= \frac{1}{n+1} (\langle v^k, v^\ell \rangle)_{k\ell} \\ &= \frac{1}{(n+1)^3} \left(\sum_{p=0}^{k-1} (p+1)^2 - \sum_{p=k}^{\ell-1} (n-p)(p+1) + \sum_{p=\ell}^{n-1} (n-p)^2 \right). \end{aligned}$$

Further computations involving the Gaussian summation formulas yield

$$\begin{aligned} & \sum_{p=0}^{k-1} (p+1)^2 - \sum_{p=k}^{\ell-1} (n-p)(p+1) + \sum_{p=\ell}^{n-1} (n-p)^2 \\ &= \sum_{p=1}^k p^2 - n \sum_{p=k+1}^{\ell} p + \sum_{p=k}^{\ell-1} p^2 + \sum_{p=k}^{\ell-1} p + \sum_{p=\ell}^{n-1} n^2 + \sum_{p=\ell}^{n-1} p^2 - 2n \sum_{p=\ell}^{n-1} p \\ &= \sum_{p=1}^k p^2 - n \left(\sum_{p=1}^{\ell} p - \sum_{p=1}^k p \right) + \left(\sum_{p=1}^{\ell-1} p^2 - \sum_{p=1}^{k-1} p^2 \right) + \left(\sum_{p=1}^{\ell-1} p - \sum_{p=1}^{k-1} p \right) \\ & \quad + (n-\ell)n^2 + \left(\sum_{p=1}^{n-1} p^2 - \sum_{p=1}^{\ell-1} p^2 \right) - 2n \left(\sum_{p=1}^{n-1} p - \sum_{p=1}^{\ell-1} p \right) \\ &= k^2 - n \left(\sum_{p=1}^{\ell} p - \sum_{p=1}^k p \right) + \sum_{p=1}^{n-1} p^2 + \left(\sum_{p=1}^{\ell-1} p - \sum_{p=1}^{k-1} p \right) + (n-\ell)n^2 - 2n \left(\sum_{p=1}^{n-1} p - \sum_{p=1}^{\ell-1} p \right) \\ &= k^2 - \frac{n}{2} (\ell(\ell+1) - k(k+1)) + \frac{1}{6} (n-1)n(2n-1) + \frac{(\ell-1)\ell}{2} - \frac{(k-1)k}{2} \\ & \quad + (n-\ell)n^2 - n((n-1)n - (\ell-1)\ell) \end{aligned}$$

for $k \leq \ell$. This gives an algebraic expression for $F_{k\ell}$ (with $k \leq \ell$), which may be simplified to

$$F_{k\ell} = \frac{1}{6(n+1)^2} (6(k-\ell)^2 + 6(k-\ell) + 6n(k-\ell) + n^2 + 2n).$$

1.10.4. Alongside the definition of \mathcal{S} , the following are required: Denote by F^0 the trace-free part of F , and, furthermore, let

$$\begin{aligned} \mathcal{S}^0 &= \{X \in \mathcal{S} : X_{00} = \cdots = X_{nn} = 0\}, \\ \mathcal{T} &= S^2(\mathbb{R}^{n+1}), & \mathcal{T}^0 &= \{X \in \mathcal{T} : X_{00} = \cdots = X_{nn} = 0\}, \\ \mathcal{F} &= \text{span}\{\pi(F^0) : \pi \in W\}, & \mathcal{G} &= \mathcal{F}^\perp \subseteq \mathcal{T}^0, \\ \tilde{F} &= \sum_{\pi \in W} \pi(F^0) \in \mathcal{F} \end{aligned}$$

(where \perp denotes the orthogonal complement in \mathcal{T}^0).

1.10.5 Lemma. (1) F^0 is invariant under cyclic permutations: Let $\gamma = (0\ 1\ 2\ \dots\ n)$. Then $F_{k\ell}^0 = F_{\gamma(k)\gamma(\ell)}^0$ for all k, ℓ . In particular, \tilde{F} is determined by its first row.

(2) $F_{kk}^0 = 0$ for all k .

(3) There exists nonzero $f \in \mathbb{R}$ with

$$\tilde{F} = \begin{bmatrix} 0 & f & \cdots & f \\ f & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f \\ f & \cdots & f & 0 \end{bmatrix}.$$

Proof. (1) and (2) are straightforward. To prove (3), let $\pi \in W$ with $\pi(0) = 0$. Then for every $k \in \{1, \dots, n\}$,

$$\tilde{F}_{0,\pi(k)} = (\pi \cdot \tilde{F})_{0,k} = \tilde{F}_{0,k}.$$

Thus, the first row of \tilde{F} takes the form $(0, f, \dots, f)$. By the symmetries of F^0 , it follows that \tilde{F} is equal to

$$\begin{bmatrix} 0 & f & \cdots & f \\ f & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f \\ f & \cdots & f & 0 \end{bmatrix}. \quad \square$$

1.10.6 Remarks. Recall that F describes the gradient of the squared circumradius of the fundamental simplex D of A_n^* . To obtain the other simplices, one must merely apply the induced action of the Weyl group W on the Delone complex. According to 1.8.6, the corresponding gradient matrix is the conjugate $Q_\pi F Q_\pi^\top$, where Q_π denotes the permutation matrix associated with $\pi \in W$. Therefore, the following orthogonal

representation of W on $S^2(\mathbb{R}^{n+1})$ (and its numerous modules) will play a significant role:

$$\varphi: W \rightarrow \mathrm{O}(S^2(\mathbb{R}^{n+1})), \quad \varphi(\pi)(X) = Q_\pi X Q_\pi^\top,$$

as will the corresponding unitary representation $\varphi_{\mathbb{C}}: W \rightarrow \mathrm{U}(S^2(\mathbb{C}^{n+1}))$.³¹ Note that

$$(\varphi(\pi)(X))_{ab} = X_{\pi(a)\pi(b)}$$

for all $X \in \mathcal{F}^0$.

1.10.7 Proposition. $\mathcal{F}^0 = \mathbb{R} \cdot \tilde{F} \oplus \mathcal{S}^0 \oplus \mathcal{G}$ is a decomposition into irreducible W -modules. In particular, $\mathcal{F} = \mathbb{R} \cdot \tilde{F} \oplus \mathcal{S}^0$.

Proof. First step. \mathcal{F}^0 is the sum of exactly three irreducible W -modules. Let

$$\chi: W \rightarrow \mathbb{C}, \quad \chi(\pi) = \mathrm{tr} \varphi_{\mathbb{C}}(\pi)$$

denote the character of the complexification $\varphi_{\mathbb{C}}$ of φ . By the theory of finite group representations, it is sufficient to prove that $\langle \chi, \chi \rangle = 3$.

Observe that W acts on

$$\mathcal{X} = \left\{ \{i, j\} \in 2^{\{0, \dots, n\}} : i \neq j \right\}$$

by $\pi \cdot \{i, j\} = \{\pi(i), \pi(j)\}$. Denote the fixed point set of $\pi \in W$ by \mathcal{X}^π . Using the orthonormal basis

$$\{S_{ij} = \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}) : 0 \leq i < j \leq n\},$$

where $E_{ij} = (\delta_{ik}\delta_{jl})_{kl}$, it follows that

$$\begin{aligned} \chi(\pi) &= \sum_{i < j} \langle \pi \cdot S_{ij}, S_{ij} \rangle \\ &= \frac{1}{2} \sum_{i < j} (\langle \pi \cdot E_{ij}, E_{ij} \rangle + \langle \pi \cdot E_{ij}, E_{ji} \rangle + \langle \pi \cdot E_{ji}, E_{ij} \rangle + \langle \pi \cdot E_{ji}, E_{ji} \rangle) \\ &= \sum_{i < j} (\langle \pi \cdot E_{ij}, E_{ij} \rangle + \langle \pi \cdot E_{ij}, E_{ji} \rangle) \\ &= \sum_{i < j} (\delta_{\pi(i)i} \delta_{\pi(j)j} + \delta_{\pi(i)j} \delta_{\pi(j)i}) \\ &= |\mathcal{X}^\pi|. \end{aligned}$$

Thus,

$$\langle \chi, \chi \rangle = \frac{1}{|W|} \sum_{\pi \in W} |\mathcal{X}^\pi|^2 = \frac{1}{|W|} \sum_{\pi \in W} |(\mathcal{X} \times \mathcal{X})^\pi|.$$

³¹Subscript \mathbb{C} denotes complexification. See Sec. 2.2 for details.

According to Burnside's lemma, this term is equal to

$$|\mathcal{X} \times \mathcal{X}/W|,$$

the number of W -orbits in $\mathcal{X} \times \mathcal{X}$. A simple counting argument shows that the orbits are

$$O_a = \left\{ (\{i, j\}, \{k, l\}) \in \mathcal{X} \times \mathcal{X} : |\{i, j, k, l\}| = a \right\}$$

for $a = 2, 3, 4$. Thus,

$$\langle \chi, \chi \rangle = 3.$$

Second step. Let's begin by remarking that, by the first step,

$$\mathcal{T}_{\mathbb{C}}^0 = \mathbb{C} \cdot \tilde{F} \oplus \mathcal{S}_{\mathbb{C}}^0 \oplus \mathcal{G}_{\mathbb{C}},$$

where \mathcal{G} is the orthogonal space of $\mathbb{R} \cdot \tilde{F} \oplus \mathcal{S}^0$ in \mathcal{T}^0 , is a decomposition into irreducible W -modules. It remains to show that $\mathbb{C} \cdot \tilde{F} \oplus \mathcal{S}_{\mathbb{C}}^0 = \mathcal{F}_{\mathbb{C}}$. Recall that the row sums of F^0 are all equal. According to Lem. 1.10.5(3), there exists $\lambda \in \mathbb{R}$ with $F^0 - \lambda \tilde{F} \in \mathcal{S}^0$. Thus,

$$\mathcal{F}_{\mathbb{C}} \subseteq \mathbb{C} \cdot \tilde{F} \oplus \mathcal{S}_{\mathbb{C}}^0.$$

Denote by $\hat{\mathcal{F}}$ the orthogonal complement of \hat{F} in \mathcal{F} . Then $\mathcal{F}_{\mathbb{C}} = \mathbb{C} \cdot \tilde{F} \oplus \hat{\mathcal{F}}_{\mathbb{C}}$. It follows that $\hat{\mathcal{F}}_{\mathbb{C}} \subseteq \mathcal{S}_{\mathbb{C}}^0$. Using the irreducibility of $\mathcal{S}_{\mathbb{C}}^0$ this implies $\hat{\mathcal{F}}_{\mathbb{C}} = \mathcal{S}_{\mathbb{C}}^0$. We conclude that $\mathcal{F}_{\mathbb{C}} = \mathbb{C} \cdot \tilde{F} \oplus \mathcal{S}_{\mathbb{C}}^0$, and thus the following decomposition of $\mathcal{T}_{\mathbb{C}}^0$ is obtained:

$$\mathcal{T}_{\mathbb{C}}^0 = \mathbb{C} \cdot \tilde{F} \oplus \mathcal{S}_{\mathbb{C}}^0 \oplus \mathcal{G}_{\mathbb{C}}.$$

By the first step, this is a decomposition into irreducible W -modules, which immediately implies the decomposition

$$\mathcal{T}^0 = \mathbb{R} \cdot \tilde{F} \oplus \mathcal{S}^0 \oplus \mathcal{G},$$

and from this follows $\mathcal{F} = \mathbb{R} \cdot \tilde{F} \oplus \mathcal{S}^0$. □

1.10.8 Lemma. *Let $X \in \mathcal{S}$ with $\text{tr } X = 0$.*

(1) $\tilde{F} \perp X$.

(2) *Suppose that $X \notin \mathcal{G}$. Then there exists $\pi \in W$ with $\langle X, \pi.F^0 \rangle > 0$.*

Proof. (1) As the row sums of X are zero,

$$\langle X, \tilde{F} \rangle = f \cdot \sum_{i \neq j} X_{ij} = -f \cdot \sum_i X_{ii} = -f \cdot \text{tr } X = 0.$$

(2) By (1) and the definition of \tilde{F} ,

$$0 = \langle X, \tilde{F} \rangle = \sum_{\pi \in W} \langle X, \pi.F^0 \rangle.$$

As $X \notin \mathcal{G}$, there exists $\pi \in W$ with $\langle X, \pi.F^0 \rangle \neq 0$. Thus, one of the summands in the sum above must be positive. □

This finally leads to the superlinearity estimate for the thickness of A_n^* under variations in *good* direction, i.e. those not contained in \mathcal{F}^\perp , announced at the start of this section:

1.10.9 Proposition. *Let $\Phi: \mathbb{R} \rightarrow \mathrm{GL}(V)$ be a differentiable map with $\Phi(0) = \mathrm{Id}_V$ and X be the symmetric trace-free part of $\Phi'(0)$. Suppose that $\tilde{X} \notin \mathcal{F}^\perp$. Then there exists a Delone cell P of A_n^* so that*

$$\left. \frac{d}{ds} \mu(\Phi(s)(P)) \right|_{s=0} > 0.$$

In particular, there exist $\varepsilon > 0, M > 0$ so that

$$\theta(\Phi(s)(A_n^*)) \geq \theta(A_n^*) + |s| \cdot M$$

holds for all $s \in (-\varepsilon, \varepsilon)$.

Proof. Let $\Lambda = A_n^*$. Note that there exists a function $g: \mathbb{R} \rightarrow \mathbb{R}_+$ with $g(s)\Phi(s) \in \mathrm{SL}(V)$ for all $s \in \mathbb{R}$. Apply the scaling invariance of θ to obtain that $\theta(g(s)\Phi(s)(\Lambda)) = \theta(\Phi(s)(\Lambda))$. It is therefore sufficient to consider the special case $\Phi(\mathbb{R}) \subseteq \mathrm{SL}(V)$ for the second part of the proposition.

Use $\tilde{X} \notin \mathcal{F}^\perp$ and Lem. 1.10.8 to obtain that there exists $\pi \in W$ so that $\langle \tilde{X}, \pi.F \rangle > 0$. Denote by D the fundamental Delone cell and $P = \pi.D$. As the squared circumradius f satisfies

$$\left. \frac{d}{ds} f(\Phi(s)(P)) \right|_{s=0} = \langle \tilde{X}, \pi.F \rangle > 0,$$

according to Lem. 1.8.5(2), it follows that $(d/ds) \mu(\Phi(s)(P))|_{s=0} > 0$. In particular, there exist ε, M' so that

$$\mu(\Phi(s)(P))^n \geq \mu(P)^n + sM'$$

for $0 \leq s \leq \varepsilon$. By Prop. 1.9.5, $\Phi(s)(\Lambda)$ is a lattice of Voronoi's first kind. It follows, after a correction of ε , that $\Phi(s)(P)$ is a Delone cell of $\Phi(s)(\Lambda)$ for all $s \in (-\varepsilon, \varepsilon)$. Thus,

$$\mu(\Phi(s)(\Lambda))^n \geq \mu(\Phi(s)(P))^n \geq \mu(P)^n + sM' = \mu(\Lambda)^n + sM'.$$

Note that $\det(\Phi(s)(\Lambda))^{1/2}$ is constant by the assumption $\Phi(\mathbb{R}) \subseteq \mathrm{SL}(V)$. Dividing by $\det(\Lambda)^{1/2}$ and letting $M = M'/\det(\Lambda)^{1/2}$ yields

$$\theta(\Phi(s)(\Lambda)) \geq \theta(\Lambda) + sM$$

for $0 \leq s \leq \varepsilon$. By choosing π so that $\langle \tilde{X}, \pi.F \rangle < 0$ instead, the estimates for $-\varepsilon \leq s \leq 0$ are obtained. \square

The methods used here do not immediately lead to a proof of Bleicher's theorem. Up to now, only a description of \mathcal{F} was discovered. Although it carries only a little role in the outcome of Theorem C, the arguments for a complete proof of Bleicher's theorem using the methods above are sketched. One must start by determining the subspace \mathfrak{X} of *bad* directions.

1.10.10 The space of bad directions. (1) The \mathcal{T}^0 -orthogonal space $(\mathcal{S}^0)^\perp$ is generated by

$$\left[\begin{array}{c|ccc} 0 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & & \\ 1 & & & \end{array} \right], \left[\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & & & \\ \vdots & \vdots & & & \\ 0 & 1 & & & \end{array} \right], \dots, \left[\begin{array}{ccc|c} & & & 1 \\ \hline & & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{array} \right].$$

It follows that the dimension of \mathcal{S}^0 is

$$\dim \mathcal{S}^0 = \dim \mathcal{T}^0 - (n+1) = \frac{n(n-1)}{2} - 1.$$

(2) $\mathcal{F} = \mathcal{S}^0 \oplus \mathbb{R} \cdot \tilde{F}$ is the space of $X \in \mathcal{T}^0$ whose row-sums are all equal, i.e. $R^0 X = \cdots = R^n X$.

(3) $\mathcal{G} = \mathcal{F}^\perp$, therefore, has the following basis:

$$U_1 = \left[\begin{array}{cccc} & & -1 & \cdots & -1 \\ & & 1 & \cdots & 1 \\ -1 & 1 & & & \\ \vdots & \vdots & & & \\ -1 & 1 & & & \end{array} \right], \quad U_2 = \left[\begin{array}{cccc} & -1 & 1 & & \\ -1 & & & -1 & \cdots & -1 \\ 1 & & & 1 & \cdots & 1 \\ & -1 & 1 & & & \\ & \vdots & \vdots & & & \\ & -1 & 1 & & & \end{array} \right], \quad \dots,$$

$$U_n = \left[\begin{array}{cccc} & & -1 & 1 \\ & & \vdots & \vdots \\ & & -1 & 1 \\ -1 & \cdots & -1 & \\ 1 & \cdots & 1 & \end{array} \right].$$

In fact, the orthogonal space U_i^\perp is the set of $X \in \mathcal{T}^0$ with $R^{i-1}X = R^iX$. Thus, their intersection $\bigcap U_i^\perp$ must be \mathcal{F} .

(4) The orthogonal space of \mathcal{S}^0 in \mathcal{S} is generated by the matrices

$$\begin{bmatrix} n & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n \end{bmatrix}$$

and

$$\begin{aligned}
 X_1 &= \begin{bmatrix} -(n-1) & & 1 & \cdots & 1 \\ & n-1 & -1 & \cdots & -1 \\ 1 & & -1 & & \\ \vdots & & \vdots & & \\ 1 & & -1 & & \end{bmatrix}, \\
 X_2 &= \begin{bmatrix} & 1 & & -1 & & \\ 1 & -(n-1) & & & 1 & \cdots & 1 \\ -1 & & n-1 & -1 & \cdots & -1 \\ & 1 & & -1 & & \\ & \vdots & & \vdots & & \\ & 1 & & -1 & & \end{bmatrix}, \quad \dots \\
 \dots, \quad X_n &= \begin{bmatrix} & & & 1 & & -1 \\ & & & \vdots & & \vdots \\ & & & 1 & & -1 \\ 1 & \cdots & 1 & -(n-1) & & \\ -1 & \cdots & -1 & & n-1 & \end{bmatrix}.
 \end{aligned}$$

As X_1, \dots, X_n are perpendicular to \tilde{F} , it follows that they generate the orthogonal space \mathfrak{X} of \mathcal{F} in the space of trace-free elements of \mathcal{S} . Observe that the space

$$\mathfrak{X} = \text{span}\{X_1, \dots, X_n\}$$

is the space of directions X in which the thickness of A_n^* does not increase at least linearly.

Thus the space of bad directions \mathfrak{X} is finally known! It can now be shown that the Hessian of the squared circumradius f of the fundamental Delone cell, when represented as matrix with respect to the basis X_1, \dots, X_n , is equal to

$$\begin{bmatrix} a & b & c & \cdots & c \\ b & a & \ddots & \ddots & \vdots \\ c & \ddots & \ddots & \ddots & c \\ \vdots & \ddots & \ddots & \ddots & b \\ c & \cdots & c & b & a \end{bmatrix},$$

where the entries $a, b, c \in \mathbb{R}$ are defined by

$$a = \frac{n^3 + 3n^2 + 2n - 6}{6(n+1)}, \quad b = -\frac{a}{2} + \frac{1}{2} \cdot \frac{n-2}{n+1}, \quad c = -\frac{1}{n+1}.$$

It is straightforward to show that this matrix is *strictly diagonally dominant*, i.e.

$$\sum_{i \neq j} |a_{ij}| < |a_{jj}|$$

holds for all j [HJ90, 6.1.9]. But strictly diagonally dominant matrices with positive diagonal entries are positive-definite [HJ90, 6.1.10]. Thus the Hessian above is positive-definite, and Bleicher's theorem is proved.

The computation that is necessary to obtain the formula for the Hessian is tedious, and the reader will be spared. It should also be mentioned that the mere fact that \mathcal{S} is decomposed into three types of direction (those with growing volume, first order growth of thickness, and second order growth of thickness, respectively) can also be deduced directly from Bleicher's original proof.

2 Representation Theory

This chapter serves purely as a collection of common knowledge concerning the representation theory of Lie algebras and Lie groups, as it pertains to this thesis. Almost everything mentioned in here may be found in the textbooks that the chapter is based on: Bröcker–tom Dieck’s *Representations of Compact Lie Groups* [BtD85], Helgason’s *Differential Geometry, Lie Groups and Symmetric Spaces* [Hel78] and Humphrey’s *Introduction to Lie Algebras and Representation Theory* [Hum80]. Any claim without citation is assumed to be well-known or so obvious as to not warrant a reference.

Collecting results from different sources always means managing conflicts in notation and convention, and this chapter shall serve to represent the subject in a way that causes little friction between the lattice theorist’s, the algebraist’s and the geometer’s treatment of representations, root systems and the classification of simple Lie algebras.

2.1 Lie Group vs. Lie Algebra Representations

2.1.1. Let G be a compact connected Lie group.¹ If $\Phi: G \rightarrow \mathrm{GL}(V)$ is a Lie group representation, then the derivative $\varphi = \mathrm{L}\Phi: \mathrm{LG} \rightarrow \mathfrak{gl}(V)$ at the identity element $e \in G$ is a representation of the Lie algebra LG of G . This process may be reversed if G is simply-connected, which yields a bijective correspondence between representations of G and representations of LG , according to the Ado–Iwasawa theorem [Hal03, Thm. 4.9] (cf. 2.10.1). This correspondence preserves the modules of these representations, even if G is not simply-connected. Due to the lack of a proper source for this result, a detailed proof will be provided here.

2.1.2 Proposition. *Let G be a compact connected Lie group, $\Phi: G \rightarrow \mathrm{GL}(V)$ be a representation of G and $\varphi = \mathrm{L}\Phi$. Then a subspace W of V is a G -module if and only if it is an LG -module.*

¹The term *Lie group* will usually mean ‘compact connected Lie group’.

Proof. Suppose first that W is a G -module. Let $X \in \text{LG}$ and $w \in W$. Then apply that $\Phi(\exp(tX))(w) \in W$ for all t to obtain that

$$\varphi(X)(w) = \left. \frac{d}{dt} \Phi(\exp(tX))(w) \right|_{t=0} \in W.$$

Conversely, suppose that W is an LG -module. Let $g \in G$ and let $X \in \text{LG}$ so that $\exp X = g$.² Equip V with a G -invariant inner product.³ Let $\pi: V \rightarrow W$ be the orthogonal projection and note that $\pi \circ \varphi(X) = \varphi(X) \circ \pi$. Define $c(t) = \Phi(\exp tX)(w)$ and $\tilde{c} = \pi \circ c$. Then c, \tilde{c} solve the same ordinary differential equation. In fact,

$$\begin{aligned} c'(t_0) &= \left. \frac{d}{dt} \Phi(\exp(tX))(w) \right|_{t=t_0} \\ &= \left. \frac{d}{dt} \Phi(\exp(tX) \exp(t_0X))(w) \right|_{t=0} \\ &= \varphi(X) \circ \Phi(\exp(t_0X))(w) \\ &= \varphi(X)(c(t_0)). \end{aligned}$$

It follows that $\tilde{c}'(t_0) = \varphi(X)(\tilde{c}(t_0))$. An application of the Picard–Lindelöf theorem shows that $c = \tilde{c}$. In particular, $\Phi(g)(w) = \tilde{c}(1) \in W$. \square

2.2 Complexification

2.2.1 Complexification of vector spaces. Let V be a real vector space. The *complexification* of V is the complex vector space $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Unless otherwise marked, \otimes denotes tensor products over \mathbb{R} . The natural map $q: V \rightarrow V_{\mathbb{C}}, v \mapsto v \otimes 1$ isomorphically maps V to a real subspace of $V_{\mathbb{C}}$, whose span over \mathbb{C} is $V_{\mathbb{C}}$. It is sometimes useful to skip q in notation and just consider V to be a subspace of $V_{\mathbb{C}}$. Note that, while not every element of $V_{\mathbb{C}}$ is an elementary tensor, there exist $v_0, v_1 \in V$ for every $v \in V_{\mathbb{C}}$ so that $v = v_0 \otimes 1 + v_1 \otimes i$, or, by the convention above, $v = v_0 + iv_1$. In particular, $V_{\mathbb{C}} = V \oplus i \cdot V$. The *complex conjugation* on $V_{\mathbb{C}}$ is the \mathbb{R} -linear involution $V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ induced by the map $(v, z) \mapsto v \otimes \bar{z}$. The decomposition $V_{\mathbb{C}} = V \oplus i \cdot V$ is a decomposition into the eigenspaces of the complex conjugation.

Complexification may be described as functor

$$- \otimes \mathbb{C}: \text{Vect}_{\mathbb{R}} \rightarrow \text{Vect}_{\mathbb{C}}, \quad V \mapsto V_{\mathbb{C}}, \quad f \mapsto f_{\mathbb{C}} = f \otimes \text{Id}_{\mathbb{C}},$$

²The exponential map of a compact connected Lie group is surjective. In particular, this type of argument depends on G being compact.

³A G -invariant inner product on V may be constructed using the method of [War83, 4.12]: Let $\langle \cdot, \cdot \rangle$ be any inner product on V and let $\langle v, w \rangle = \int_G \langle \varphi(g)(v), \varphi(g)(w) \rangle dg$.

which inherits some of the properties of the tensor bifunctor $- \otimes -$, amongst them commutativity with direct sums and tensor products:

$$\left(\bigoplus_i V_i\right)_{\mathbb{C}} \cong \bigoplus_i (V_i)_{\mathbb{C}}, \quad \left(\bigotimes_i V_i\right)_{\mathbb{C}} \cong \bigotimes_i (V_i)_{\mathbb{C}}.$$

These are functorial in the sense that $(\bigotimes_i f_i)_{\mathbb{C}}$ and $(\bigotimes_{\mathbb{C}})_i (f_i)_{\mathbb{C}}$ commute with the natural maps for all \mathbb{R} -linear maps $f_i: V_i \rightarrow W_i$. Moreover, the following diagram of inclusion maps is commutative:

$$\begin{array}{ccc} \bigotimes_i V_i & \longrightarrow & (\bigotimes_i V_i)_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \bigotimes_i (V_i)_{\mathbb{C}} & \longrightarrow & (\bigotimes_{\mathbb{C}})_i (V_i)_{\mathbb{C}}. \end{array}$$

In other words, upon identification of $(\bigotimes_i V_i)_{\mathbb{C}}$ with $(\bigotimes_{\mathbb{C}})_i (V_i)_{\mathbb{C}}$, the natural map $\bigotimes_i V_i \rightarrow (\bigotimes_i V_i)_{\mathbb{C}}$ becomes $\bigotimes_i V_i \rightarrow (\bigotimes_{\mathbb{C}})_i (V_i)_{\mathbb{C}}, x \otimes_{\mathbb{R}} y \mapsto x \otimes_{\mathbb{C}} y$.

2.2.2 Complexification of Lie algebras. If \mathfrak{g} is a Lie algebra, then $\mathfrak{g}_{\mathbb{C}}$ may be turned into a Lie algebra so that $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$ is a real Lie subalgebra: The Lie bracket of $\mathfrak{g}_{\mathbb{C}}$ is obtained by applying the universal property of the tensor product $(\mathfrak{g} \times \mathfrak{g})_{\mathbb{C}}$ to

$$(\mathfrak{g} \times \mathfrak{g}) \times \mathbb{C} \rightarrow \mathfrak{g}_{\mathbb{C}}, \quad (x, y, z) \mapsto [x, y] \otimes z,$$

to obtain a \mathbb{C} -bilinear map $[\cdot, \cdot]: \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} = (\mathfrak{g} \times \mathfrak{g}) \otimes \mathbb{C} \rightarrow \mathfrak{g}_{\mathbb{C}}$, which turns out to be a Lie bracket. The adjoint representation $\mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}})$ is equal to the complexification $\text{ad}_{\mathbb{C}}$ of the adjoint representation $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$.

2.2.3. The natural isomorphism $\mathfrak{gl}(V, \mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}(V_{\mathbb{C}}, \mathbb{C})$ gives rise to the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi(X)} & V \\ \downarrow & & \downarrow \\ V_{\mathbb{C}} & \xrightarrow{\varphi_{\mathbb{C}}(X)} & V_{\mathbb{C}} \end{array}$$

for every real Lie algebra representation $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V, \mathbb{R})$ and $X \in \mathfrak{g}$. Thanks to this isomorphism, each $\varphi(X)$ may be considered as the restriction of $\varphi_{\mathbb{C}}(X)$ to the real subspace V of $V_{\mathbb{C}}$. The map $\varphi_{\mathbb{C}}(X)$ may be recovered from $\varphi(X)$ by the formula

$$\varphi_{\mathbb{C}}(wX)(zv) = wz\varphi(X)(v)$$

for all $X \in \mathfrak{g}, v \in V, z, w \in \mathbb{C}$. This identification will be used in the thesis without further comment.

2.2.4 Real structures. The complexification is used to turn real representations of Lie groups and Lie algebras into complex representations. The inverse process is achieved using the notion of *real structure*. For the sake of simplicity, Lie algebra representations are considered. Fix a representation $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of a real Lie algebra \mathfrak{g} on a complex

vector space V . A *real structure* on V is an involutive *conjugate linear* \mathfrak{g} -map $\mathcal{J} : V \rightarrow V$, i.e. an \mathbb{R} -linear \mathfrak{g} -map with $\mathcal{J}(zv) = \bar{z}\mathcal{J}(v)$ for all $z \in \mathbb{C}, v \in V$ and $\mathcal{J}^2 = \text{Id}$ [BtD85, Ch. 2, Sec. 6, p. 93]. A prime example of a real structure is that of the complex conjugation $\mathcal{J}(z \otimes v) = \bar{z} \otimes v$ of a complexified real vector space $V \otimes \mathbb{C}$, which is a real structure with respect to the complexification of every real representation on V .

Given a real structure \mathcal{J} of a complex representation $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, it is possible to write φ as complexification of a real representation. In fact, the two possible eigenvalues of \mathcal{J} are ± 1 , with eigenspaces V_{\pm} . These eigenspaces are \mathfrak{g} -modules, and therefore a real representation $\varphi_+(X) = \varphi(X)|_{V_+}$ of \mathfrak{g} on V_+ is obtained. It is straightforward to verify that $\mathbb{C} \otimes V_+ \rightarrow V, z \otimes v \mapsto zv$ is an isomorphism between the representations $(\varphi_+)_{\mathbb{C}}$ and φ .

2.3 Semisimple Lie Algebras

2.3.1. A Lie algebra \mathfrak{g} over a field F of characteristic zero is *semisimple* if one of the following equivalent conditions holds:

- (1) Every solvable ideal of \mathfrak{g} is zero [Hum80, Sec. 3.1].⁴
- (2) Every abelian ideal of \mathfrak{g} is zero [Hum80, Sec. 5.1].
- (3) The *Killing form*

$$\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow F, \quad \kappa(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y))$$

is nondegenerate [Hum80, Sec. 5.1].

A Lie algebra is *simple* if it is non-abelian and contains no nontrivial ideals. Weyl's theorem (see 2.3.2 below) implies that every semisimple Lie algebra is the product of simple Lie algebras. In particular, a fourth item may be added to the list:

- (4) \mathfrak{g} is the product of simple Lie algebras [Hum80, Sec. 6.3].

With respect to the Killing form κ , the adjoint representation ad is skew-symmetric. According to the *Schur lemma* [Hum80, Sec. 6.1] applied to $\kappa : \mathfrak{g} \rightarrow \mathfrak{g}^*$, the Killing form is unique up to multiplication with scalars on simple Lie algebras. Therefore, any scalar multiple of the Killing form will be referred to as a *Killing form*.

⁴Humphrey defines a semisimple Lie algebra as one whose *radical*, i.e. the maximal solvable ideal, is zero.

2.3.2. By *Weyl's theorem* [Hum80, Sec. 6.3], every representation $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of a complex semisimple Lie algebra \mathfrak{g} is *completely decomposable*, i.e. V is the direct sum of indecomposable \mathfrak{g} -modules. The proof provided in [Hum80, Sec. 6.3] is completely algebraic. Weyl's original idea, on the other hand, consisted of using the Lie group G belonging to the compact real form of \mathfrak{g} (see 2.7.1) to introduce a \mathfrak{g} -invariant inner product on V (cf. footnote ³). Then, if V is not irreducible, it may be decomposed into the direct sum of a proper submodule V_1 and its orthogonal space $V_2 = V_1^\perp$. Iterating this procedure yields irreducible \mathfrak{g} -submodules $V_i \subseteq V$ so that $V = \bigoplus_i V_i$.

Such a decomposition is unique in the following sense: If $W_j \subseteq V$ are irreducible \mathfrak{g} -submodules of V so that $V = \bigoplus_j W_j$, then the maps $V_i \rightarrow W_j$ defined by composing the inclusion $V_i \rightarrow V$ with the orthogonal projection $V \rightarrow W_j$ are either 0 or isomorphisms, according to another application of the Schur lemma [Hum80, Sec. 6.1]. It follows that there exists an isomorphism $f: V \rightarrow V$ so that $f \circ \varphi(X) = \varphi(X) \circ f$ for all $X \in \mathfrak{g}$ and each $f(V_i)$ is one of the W_j .

2.4 Root Systems of Semisimple Lie Algebras over Algebraically Closed Fields

In this section \mathfrak{g} denotes a semisimple Lie algebra over an algebraically closed field F of characteristic zero.

2.4.1. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *toral* if all elements of $\text{ad}(\mathfrak{h})$ are diagonalisable [Hum80, Sec. 8.1]. A maximal toral Lie subalgebra is called a *Cartan subalgebra* in the context of semisimple Lie algebras [Hum80, Sec. 15.3]. According to the *conjugacy theorem*, any two Cartan subalgebras of \mathfrak{g} are *conjugate* in the sense that there exists an automorphism ϕ of \mathfrak{g} that maps one isomorphically onto the other [Hum80, Ch. 16] (cf. 2.8.1).

Toral subalgebras coincide with the abelian subalgebras in the context of semisimple Lie algebras [Hum80, Sec. 8.1–8.2, 15.3] and therefore the elements of $\text{ad}(\mathfrak{h})$ are *simultaneously diagonalisable* [HJ90, 1.3.19]. If \mathfrak{h} is the Cartan subalgebra, this yields the following *root space decomposition* or *Cartan decomposition* of \mathfrak{g} : For every $\Theta \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\Theta = \{X \in \mathfrak{g} : \forall H \in \mathfrak{h} : \text{ad}(H)(X) = \Theta(H) \cdot X\}.$$

Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\Theta \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\Theta$ [Hum80, Sec. 8.1]. Those nonzero $\Theta \in \mathfrak{h}^*$ with $\mathfrak{g}_\Theta \neq 0$ are called *roots* [Hum80, Sec. 8.1] or *complex roots* in the context of Lie groups [BtD85, Ch. V, (1.3)]. The set of complex roots (corresponding to \mathfrak{h}),

$$R = \{\Theta \in \mathfrak{h}^* : \Theta \neq 0, \mathfrak{g}_\Theta \neq 0\},$$

is called the *root system* of \mathfrak{g} .

Upon first glimpse, it may seem dubious whether R is a root system in the sense of 1.3.1. The arguments in [Hum80, Sec. 8.5], however, show that κ is a positive-definite \mathbb{Q} -form on $E_{\mathbb{Q}} = \text{span}_{\mathbb{Q}}(R)$. In particular, R is a root system of the euclidean vector space $(E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}, \kappa)$.

The *root spaces* \mathfrak{g}_{θ} are one-dimensional [Hum80, Sec. 8.4, (a)].⁵ Note that, according to the conjugacy theorem, the type of R does not depend on the particular choice of the toral subalgebra. To distinguish R from the system of real roots (see 2.6.2), the systems of complex roots is sometimes denoted by $R(\mathbb{C})$ [BtD85, Ch. V, (1.3), p. 185].

The *rank* $\text{rk}(\mathfrak{g})$ of a complex Lie algebra is the dimension of the Cartan subalgebra. Therefore, $\dim \mathfrak{g} = \text{rk}(\mathfrak{g}) + |R|$. The number $|R|$ is even since $\theta \mapsto -\theta$ is a fixpoint-free involution.

2.4.2. Some of the following results will be used later in the thesis:

- (1) The restriction of the Killing form to \mathfrak{h} is nondegenerate [Hum80, Sec. 8.2].
- (2) $\kappa(\mathfrak{g}_{\theta_0}, \mathfrak{g}_{\theta_1}) = 0$ for all $\theta_0, \theta_1 \in \mathfrak{h}^*$ with $\theta_0 + \theta_1 \neq 0$ [Hum80, Sec. 8.1].
- (3) $[X, Y] = \kappa(X, Y) \cdot \theta^{\#}$ for all $\theta \in \mathfrak{h}^*$, $X \in \mathfrak{g}_{\theta}, Y \in \mathfrak{g}_{-\theta}$ [Hum80, Sec. 8.3, (c)].
- (4) For every $\theta \in R$, there exist $X_{\theta} \in \mathfrak{g}_{\theta}, X_{-\theta} \in \mathfrak{g}_{-\theta}$ so that $\theta^{\vee} = [X_{\theta}, X_{-\theta}]$. Moreover, the Lie subalgebra spanned by $\theta^{\vee}, X_{\theta}, X_{-\theta}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ [Hum80, Sec. 8.3, (f)].
- (5) $[\mathfrak{g}_{\theta_0}, \mathfrak{g}_{\theta_1}] = \mathfrak{g}_{\theta_0 + \theta_1}$ for all $\theta_0, \theta_1 \in R$ with $\theta_0 + \theta_1 \neq 0$ [Hum80, Sec. 8.4, (d)].
- (6) \mathfrak{g} is generated (as Lie algebra) by $\cup_{\theta \in R} \mathfrak{g}_{\theta}$ [Hum80, Sec. 8.4, (f)].

2.5 Classification of Semisimple Lie Algebras over Algebraically Closed Fields

Semisimple Lie algebras over an algebraically closed field F of characteristic zero are classified by their root systems, i.e. they are isomorphic if and only if their respective root systems are similar or, equivalently, if and only if their Dynkin diagrams are isomorphic [Hum80, Ch. 14]. Moreover, the root systems of simple Lie algebras are irreducible, or, equivalently, their Dynkin diagrams are connected [Hum80, Sec. 14.1].⁶ The classification of irreducible root systems via connected Dynkin diagrams into the four infinite families A_n, B_n, C_n, D_n and five exceptional root systems E_6, E_7, E_8, F_4 and G_2 therefore

⁵Note that $\mathfrak{g}_0 = \mathfrak{h}$. However, the toral algebra is not a root space.

⁶In fact, the decomposition of the root system of a compact connected semisimple Lie group into simple Lie groups equals the decomposition of its root system into its irreducible components.

implies the incredible classification of simple Lie algebras over F into four infinite families and five exceptional simple Lie algebras over F . In fact, the *existence theorem* that complements the *isomorphism theorem* stated above asserts that for every root system R , there exists a semisimple Lie algebra over F whose root system is R [Hum80, Sec. 18.4]. The classification of complex simple Lie algebras and the resulting classification of compact Lie algebras and compact connected Lie groups is discussed further below. Concrete descriptions of the exceptional simple Lie algebras for $F = \mathbb{C}$ may be found in [Ada96] or [Gar16], and the references therein.

2.6 Root Systems of Compact Semisimple Lie Algebras

2.6.1. A real Lie algebra \mathfrak{g} is called *compact* if it is the Lie algebra of a compact Lie group [BtD85, Ch. V, Sec. 5, p. 209] [Kna02, Ch. IV, Sec. 4]. Real Lie algebras with negative-definite Killing form are compact according to [Kna02, Ch. IV, Prop. 4.27]. Conversely, the Killing form of a compact Lie algebra is negative-semidefinite, but in general not negative-definite [Kna02, Ch. IV, Cor. 4.26].⁷ If \mathfrak{g} is compact, then a maximal abelian subalgebra \mathfrak{h} of \mathfrak{g} is called a *Cartan subalgebra* of \mathfrak{g} .⁸ Although the conjugacy theorem does not hold for real Lie algebras in general, it does hold for compact Lie algebras.⁹

2.6.2. Suppose that \mathfrak{g} is a compact semisimple Lie algebra. Then $\mathfrak{g}_{\mathbb{C}}$ is a complex semisimple Lie algebra [Kna02, §VII, Prop. 7.1]. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . It is straightforward to see that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\Theta \in R(\mathbb{C})} (\mathfrak{g}_{\mathbb{C}})_{\Theta}$ is the Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$. Note that skew-symmetry implies that all eigenvalues of the adjoint representation are imaginary. A nonzero linear function $\Theta: \mathfrak{h} \rightarrow \mathbb{R} \cdot i$ is called *infinitesimal root* if

$$(\mathfrak{g}_{\mathbb{C}})_{\Theta} = \{X \in \mathfrak{g}_{\mathbb{C}}: \forall H \in \mathfrak{h}: \text{ad}(H)(X) = \Theta(H) \cdot X\}$$

is nonzero [BtD85, Ch. V, (1.3)].¹⁰ For every $\alpha \in \mathfrak{h}^*$, let

$$(\mathfrak{g}_{\mathbb{C}})_{\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}}: \forall H \in \mathfrak{h}: \text{ad}(H)(X) = 2\pi i \alpha(H) \cdot X\}.$$

Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} (\mathfrak{g}_{\mathbb{C}})_{\alpha}$. Those nonzero $\alpha \in \mathfrak{h}^*$ with $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \neq 0$ are called *real roots*. Equivalently, $\alpha \in \mathfrak{h}^*$ is a real root if and only if there exists an infinitesimal root Θ so that $\alpha = \Theta/2\pi i$. The *system of real roots* (corresponding to \mathfrak{h}),

$$R(\mathbb{R}) = \{\alpha \in \mathfrak{h}^*: \alpha \neq 0, \mathfrak{g}_{\alpha} \neq 0\},$$

⁷Beware! Some authors define the notion of compact Lie algebra by requiring that the Killing form of \mathfrak{g} be negative-semidefinite.

⁸For a noncompact real Lie algebra \mathfrak{g} , a *Cartan subalgebra* is a nilpotent subalgebra \mathfrak{h} of \mathfrak{g} whose normaliser $N_{\mathfrak{g}}(\mathfrak{h})$ equals \mathfrak{h} [Jac79, Ch. III, Def. 1] [Kna02, Ch. II, Sec. 2, p. 133].

⁹This follows from the conjugation theorem of maximal tori. See 2.8.1 or [BtD85, Ch. IV, (1.6)].

¹⁰Therefore, complex roots are the functions obtained by complexifying infinitesimal roots.

is a root system of \mathfrak{h}^* in the sense of 1.3.1. Note that, according to the conjugacy theorem for compact semisimple real Lie algebras, the type of $R\langle\mathbb{R}\rangle$ does not depend on the particular choice of the Cartan subalgebra. If there is no risk of confusing real and complex roots, then $R\langle\mathbb{R}\rangle$ is denoted by R .

It is straightforward to verify that $\overline{(\mathfrak{g}_{\mathbb{C}})_{\alpha}} = (\mathfrak{g}_{\mathbb{C}})_{-\alpha}$ for all $\alpha \in R\langle\mathbb{R}\rangle$. Define the *real root space* $\mathfrak{g}_{\alpha} = \mathfrak{g} \cap ((\mathfrak{g}_{\mathbb{C}})_{\alpha} \oplus (\mathfrak{g}_{\mathbb{C}})_{-\alpha})$ [BtD85, Ch. V, (2.1)]. Then $\mathfrak{g}_{\alpha} = \mathfrak{g}_{-\alpha}$ and \mathfrak{g}_{α} is two-dimensional for all $\alpha \in R\langle\mathbb{R}\rangle$.¹¹ In fact, if $0 \neq X_{\alpha} \in (\mathfrak{g}_{\mathbb{C}})_{\alpha}$ and $X_{-\alpha} = \overline{X_{\alpha}}$ then $E_{\alpha} = X_{\alpha} + X_{-\alpha}$, $F_{\alpha} = i(X_{\alpha} - X_{-\alpha}) \in \mathfrak{g}_{\alpha}$ (by their conjugation invariance), and (E_{α}, F_{α}) form a basis of \mathfrak{g}_{α} .

The (*real*) *Cartan decomposition* of \mathfrak{g} is $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}$, where R^+ is a set of positive roots of $R\langle\mathbb{R}\rangle$ [BtD85, Ch. V, (2.1)]. This is not a decomposition into simultaneous eigenspaces. In fact, $\text{ad}(H)(E_{\alpha}) = 2\pi\alpha(H) \cdot F_{\alpha}$ and $\text{ad}(H)(F_{\alpha}) = -2\pi\alpha(H) \cdot E_{\alpha}$ for all $H \in \mathfrak{h}$. In other words, $\text{ad}(H)$ is represented by the matrix

$$2\pi\alpha(H) \cdot \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$$

with respect to the basis (E_{α}, F_{α}) . A straightforward calculation using 2.4.2(3) shows that $[E_{\alpha}, F_{\alpha}] = 4\pi\kappa(X_{\alpha}, X_{-\alpha}) \cdot \alpha^{\vee}$. In particular, $\bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}$ generates \mathfrak{g} (as a Lie algebra).

2.7 Classification of Compact Simple Lie Algebras

2.7.1. The complexification of a compact simple real Lie algebra is a complex simple Lie algebra, and, conversely, every complex simple Lie algebra is the complexification of a compact simple real Lie algebra, called its *compact real form* [Kna02, §VII, Prop. 7.1]. Therefore, compact simple real Lie algebras are classified in terms of complex simple Lie algebras, which are in turn classified by connected Dynkin diagrams.

root systems	complex simple Lie algebra	compact real form	dimension
A_n ($n \geq 1$)	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{su}(n+1)$	$n(n+2)$
B_n ($n \geq 2$)	$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n+1)$	$n(2n+1)$
C_n ($n \geq 3$)	$\mathfrak{sp}(2n, \mathbb{C})$	$\mathfrak{sp}(n)$	$n(2n+1)$
D_n ($n \geq 4$)	$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n)$	$n(2n-1)$

Table 2.1: The four infinite families of irreducible root systems corresponding to simple Lie algebras over the complex and real numbers [Hel78, Ch. X, Tab. IV].

¹¹Note that $\mathfrak{g}_0 = \mathfrak{h}$. However, the Cartan subalgebra is not a real root space.

2.8 Maximal Tori of Compact Connected Groups

2.8.1. A compact connected abelian Lie group of dimension d is isomorphic to the standard torus $T^d = \mathbb{R}^d / \mathbb{Z}^d$ and therefore called *torus*, as well [BtD85, Ch. I, (3.7)]. The *integral lattice* of a torus T is $I = \exp^{-1}(e)$, and the exponential map $\exp: LT \rightarrow T$ is a surjective homomorphism whose kernel is I .

2.8.2. Let G be a compact connected Lie group. A *maximal torus* of G is a torus $T \subseteq G$ that is maximal with respect to inclusion amongst other tori $T' \subseteq G$, i.e. $T = T'$ for all tori T' with $T \subseteq T' \subseteq G$. The *conjugation theorem* asserts that any two maximal tori of G are conjugate and (as a corollary) that every element of G is contained in a maximal torus [BtD85, Ch. IV, (1.6)]. In particular, the exponential map $\exp: LG \rightarrow G$ is surjective [BtD85, Ch. IV, (2.2)].

Note that the Lie algebras of maximal tori are the Cartan subalgebras of LG . In particular, the conjugation theorem of Cartan subalgebras may be immediately deduced from the conjugation theorem of maximal tori, and vice versa.

2.8.3. Let G be a compact connected Lie group and T a maximal torus of G . The *Weyl group* of G is $N_G(T)/T$, where $N_G(T)$ denotes the normaliser of T in G [BtD85, Ch. IV, (1.3)]. The Weyl group acts on T by conjugation and on LT by the adjoint representation. With respect to the Killing form, this action is isometric [BtD85, Ch. V, (3.12)]. The conjugation theorem implies that the isomorphism type of W does not depend on the particular choice of the maximal torus T .

2.8.4. Let G be a compact connected Lie group. An element $g \in G$ is *general* if the closed Lie group generated by g is a maximal torus.¹² Otherwise, g is *special*. The element g is called *singular* if it is contained in multiple maximal tori. Otherwise, g is *regular* [BtD85, Ch. IV, (2.10)]. Every general element of G is regular and almost every element (in the sense of Lebesgue) is general [BtD85, Ch. IV, (2.11)]. Singular elements may be characterised in terms of global roots (see 2.9.3).

Some vector (field) $X \in LG$ is *singular* if $\exp(X)$ is a singular element of G . Otherwise, X is *regular*. Singular and regular vector fields may be characterised in terms of Stiefel diagrams (see 2.11).

¹²The image of a dense one-parameter subgroup of T^2 comes to mind.

2.9 Root Systems of Semisimple Compact Connected Groups

2.9.1. A compact connected Lie group G is called *semisimple* if one of the following equivalent conditions holds [BtD85, Ch. V, (7.13)]:¹³

- (1) G contains no connected abelian normal subgroup (i.e. normal torus) other than $\{e\}$.
- (2) The centre $Z(G)$ is finite.
- (3) The fundamental group $\pi_1(G)$ is finite.
- (4) The universal covering space \tilde{G} is compact.
- (5) The Lie algebra LG is semisimple.

A compact connected Lie group is called *simple* if it contains no nontrivial connected normal Lie subgroups,¹⁴ or, equivalently, if the Lie algebra LG is simple.

2.9.2. Let G be a compact connected semisimple Lie group and T a maximal torus of G . The *root system* R of G with respect to T is defined as the root system of LG with respect to the Cartan algebra LT . In the same fashion, the associated data of infinitesimal roots, real roots α , root spaces $LG_{\pm\alpha}$ and Cartan decomposition

$$LG = LT \oplus \bigoplus_{\alpha \in R^+} LG_{\alpha}$$

of LG are associated to G . The similarity type of the root system R does not depend on the particular choice of maximal torus T due to the conjugation theorem, and is commonly referred to as the *type* of G .

The Weyl group of R is equivariantly isomorphic to the Weyl group $N_G(T)/T$ of the maximal torus T , provided that the Cartan subalgebra LT is identified with its dual space using the canonical isomorphisms \sharp and \flat obtained from the Killing form [BtD85, Ch. V, (3.12)].

2.9.3. Let G be a compact connected semisimple Lie group and T a maximal torus of G . A *global root* of T is a homomorphism $\vartheta: T \rightarrow U(1)$ so that

$$(LG_{\mathbb{C}})_{\vartheta} = \{X \in LG_{\mathbb{C}}: \forall g \in T: \text{Ad}(g)(X) = \vartheta(g) \cdot X\}$$

¹³Occasionally, G is allowed to be noncompact or disconnected, in which case *semisimplicity* is usually defined by (5), cf. e.g. [Kna02, Ch. I, Sec. 15, p. 105].

¹⁴Note that a simple Lie group may contain nontrivial subgroups.

is nonzero [BtD85, Ch. II, (8.2)] [BtD85, Ch. V, (1.3)]. The map $\vartheta \mapsto \Theta = L\vartheta$ defines a bijection between global roots and infinitesimal roots, which satisfies $(LG_{\mathbb{C}})_{\vartheta} = (LG_{\mathbb{C}})_{\Theta}$. In particular, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\Theta} & \mathbb{R} \cdot i = \text{LU}(1) \\ \downarrow \text{exp} & & \downarrow \text{it} \mapsto e^{it} \\ T & \xrightarrow{\vartheta} & \text{U}(1). \end{array}$$

For every real root α , define ϑ_{α} as the global root corresponding to the infinitesimal root $\Theta = 2\pi i\alpha$. Let $U_{\alpha} = \ker \vartheta_{\alpha}$ [BtD85, Ch. V, (2.2)]. Note that $U_{\alpha} = U_{-\alpha}$. Then $Z(G) = \bigcap_{\alpha \in R^+} U_{\alpha}$, and the set of singular elements of G is $\bigcup_{\alpha \in R^+} U_{\alpha}$ [BtD85, Ch. V, (2.3)].

2.10 Classification of Compact Connected Lie Groups

2.10.1. The *Ado–Iwasawa theorem* states that every finite-dimensional Lie algebra \mathfrak{g} over a field F of characteristic zero has a faithful finite-dimensional representation [Jac79, Ch. VI, p. 202], i.e. there exists $N \in \mathbb{N}$ and an injective Lie algebra homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(F, N)$. An immediate corollary is that every real Lie algebra \mathfrak{g} is isomorphic to the Lie algebra of a Lie group G , which is defined as the Lie group generated by $\exp G$. In particular, there is a natural bijection between real Lie algebras and simply-connected Lie groups [War83, 3.22–3.27]. Compact Lie algebras do not always correspond to compact simply-connected Lie groups under this bijection, but compact semisimple Lie algebras do.¹⁵

2.10.2 Classification of semisimple Lie groups. Let G be a compact connected semisimple Lie group and $p: \tilde{G} \rightarrow G$ the universal covering. Recall that \tilde{G} carries a natural group structure [War83, 3.24]. Moreover, p is a homomorphism and $N = \ker p \subseteq Z(\tilde{G})$. In fact, a connectedness argument shows that every element of every finite normal subgroup N is central, implying that $N \subseteq Z(G)$. The Lie algebra $L\tilde{G}$ splits into compact simple Lie algebras, and \tilde{G} into compact connected simple Lie groups, accordingly. In particular, every compact connected semisimple Lie group is a quotient of a product of simple Lie groups S_1, \dots, S_t by a subgroup of $Z(S_1) \times \dots \times Z(S_t)$. At the bottom of the resulting covering diagram lies the *projective group* $PG = G/Z(G)$.

2.10.3 Classification of compact connected Lie groups. Every compact group G is finitely covered by a compact connected Lie group \hat{G} which is the product of a compact simply-connected Lie group S and a torus T [BtD85, Ch. V, (8.1)]. In

¹⁵Had compact Lie algebras been defined by requiring that the Killing form be negative-definite, then compact Lie algebras would correspond to simply-connected compact Lie groups.

type	Lie group	centre	dimension
$A_n, n \geq 1$	$SU(n+1)$	\mathbb{Z}_{n+1}	$n(n+2)$
$B_n, n \geq 2$	$\text{Spin}(2n+1)$	\mathbb{Z}_2	$n(2n+1)$
$C_n, n \geq 3$	$\text{Sp}(n)$	\mathbb{Z}_2	$n(2n+1)$
$D_n, n \geq 4$	$\text{Spin}(2n)$	\mathbb{Z}_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$	$n(2n-1)$
E_6	\tilde{E}_6	\mathbb{Z}_3	78
E_7	\tilde{E}_7	\mathbb{Z}_2	133
E_8	E_8	1	248
F_4	F_4	1	52
G_2	G_2	1	14

Table 2.2: Table of irreducible root systems and their associated compact simply-connected Lie groups.

particular, the Lie algebra of G (or \hat{G}) is the direct sum of a semisimple Lie algebra \mathfrak{s} and an abelian Lie algebra \mathfrak{z} . The semisimple part S may further be split into simple Lie groups S_1, \dots, S_t . The quotient G of \hat{G} then takes the form $S_1 \times \dots \times S_t \times T/Z$, where $Z \subseteq Z(S_1) \times \dots \times Z(S_t) \times T$ and $Z \cap T = 1$ [Pro07, Ch. 10, Thm. 4].

2.11 Fundamental Group, Integral Lattice, Centre and Stiefel Diagram

2.11.1. Let G be a compact connected semisimple Lie group, T a maximal torus with inclusion map $j: T \rightarrow G$ and integral lattice I , R the corresponding root system and W the Weyl group. If \mathfrak{h}^{**} is canonically identified with \mathfrak{h} , the coroot system R^\vee is a root system of \mathfrak{h} (see 1.3.2), and the dual lattice R^* is a lattice of \mathfrak{h} . Unless there is cause for confusion between the root system R^\vee and the lattice generated by R^\vee , both will be denoted by the symbol R^\vee . Then $R^\vee \subseteq I$ and $\pi_1(j): \pi_1(T) = I \rightarrow \pi_1(G)$ is a surjective homomorphism with kernel R^\vee [BtD85, Ch. V, (7.1)].¹⁶ In particular, $\pi_1(G) \cong I/R^\vee$.

2.11.2. Let G be a compact connected semisimple Lie group, T a maximal torus and R the corresponding root system. Define $L_{\alpha, \ell} = \alpha^{-1}\{\ell\}$ for all $\alpha \in R, \ell \in \mathbb{Z}$. The *Stiefel diagram* of G with respect to T is $LT_s = \bigcup_{\alpha \in R^+, \ell \in \mathbb{Z}} L_{\alpha, \ell}$ [BtD85, Ch. V, (7.8)]. Thus, the Stiefel diagram is the set of singular left-invariant vector fields (cf. 2.8.4). In Cor. 3.4.4, the *truncated Stiefel diagram* $LT'_s = \bigcup_{\alpha \in R^+, \ell \in \mathbb{Z} \setminus \{0\}} L_{\alpha, \ell}$ will be characterised as the set of tangent conjugate points, i.e. singular points of the exponential map (cf. 3.1.5), and the boundary of $\bigcap_{\alpha \in R^+} \alpha^{-1}([-1, 1])$ as the set of first tangent conjugate points.

¹⁶Bröcker and tom Dieck denote the coroot system by R^* and call them *inverse roots*.

2.11.3. The connected components of $LT \setminus LT_s$ are called the *Stiefel alcoves* of T . The walls of the Stiefel alcoves are defined analogously to the walls of the alcoves of affine Weyl groups [BtD85, Ch. V, (7.8)]. Define the *extended Weyl group* Ω of G as the group generated by the Weyl group W and the translations $\gamma_v: LT \rightarrow LT$ by elements v of the coroot lattice R^\vee . Equivalently, Ω is the group generated by the reflections $s_{\alpha,\ell}$ in the affine hyperplanes $L_{\alpha\ell}$ [BtD85, Ch. V, (7.8), (7.9)].

Note that $\mathcal{H}_\alpha = (\alpha^\vee)^\perp$, and, using $\alpha(\alpha^\vee) = 2$, that the bisector $H(\alpha^\vee)$ is equal to $L_{\alpha,1}$. In particular, it follows that Ω is the affine Weyl group W_a of the coroot system $R^\vee \subseteq \mathfrak{h}$ (cf. 1.3.4), and the alcoves of W_a are the Stiefel alcoves. As a corollary, note that the Voronoi cell of the coroot lattice is the orbit of any given closed Stiefel alcove \bar{A} with $0 \in \bar{A}$ under the Weyl group.

2.11.4. Let G be a compact connected semisimple Lie group, T a maximal torus and R the corresponding root system. The lattice of *central elements* is $\Lambda = \bigcap_{\alpha \in R^+} \bigcup_{\ell \in \mathbb{Z}} L_{\alpha,\ell} \subseteq LT$. It follows from $Z(G) = \bigcap_{\alpha \in R^+} U_\alpha$ that $\Lambda = \exp^{-1}(Z(G))$ [BtD85, Ch. V (7.15)], where the U_α denote the kernels of the global roots.

By definition $\Lambda = R^*$. Denote the universal covering group by \tilde{G} . Then $\tilde{T} = \exp^{-1}(T)_0$ is a maximal torus of \tilde{G} and there exists a subgroup Z of $Z(G)$ so that $G = \tilde{G}/Z$.¹⁷ In particular, the integral lattice I of T is obtained by adjoining $\exp^{-1}(Z)$ to the integral lattice R^\vee of \tilde{T} . Moreover, $I \subseteq \Lambda$. Therefore, the integral lattice is integral in the sense of 1.2.1. In particular, $Z(G)$ is isomorphic to the glue group of R if G is simply-connected and R is of ADE type.

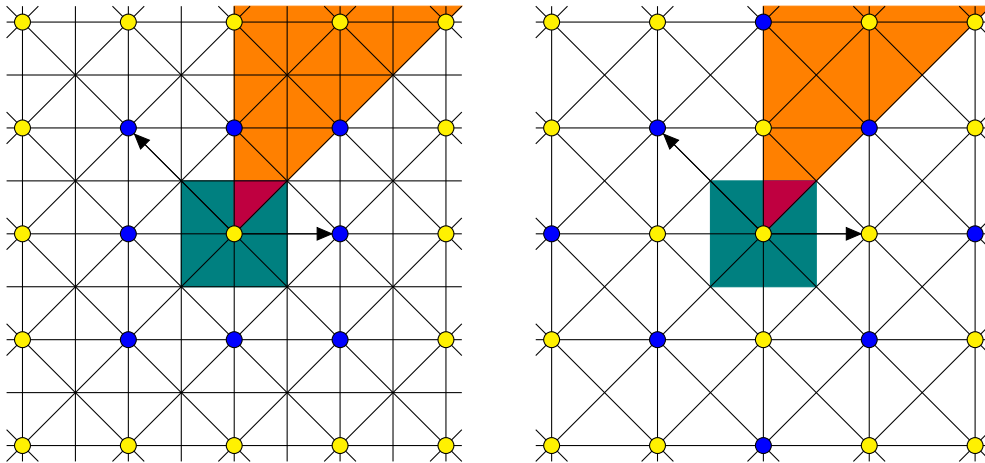


Figure 2.1: Stiefel diagrams of $\mathrm{Sp}(2)$ and $\mathrm{SO}(4)$, with coroots \bullet , integral elements \circ , Weyl chamber $\color{orange}\blacksquare$, Voronoi cell $\color{teal}\blacksquare$ and fundamental alcove $\color{red}\blacksquare$ [Sam90, Fig. 3–5, pp. 113–115] [BtD85, Ch V., Fig. 24, p. 227].

¹⁷ G_0 denotes the identity component of a Lie group G .

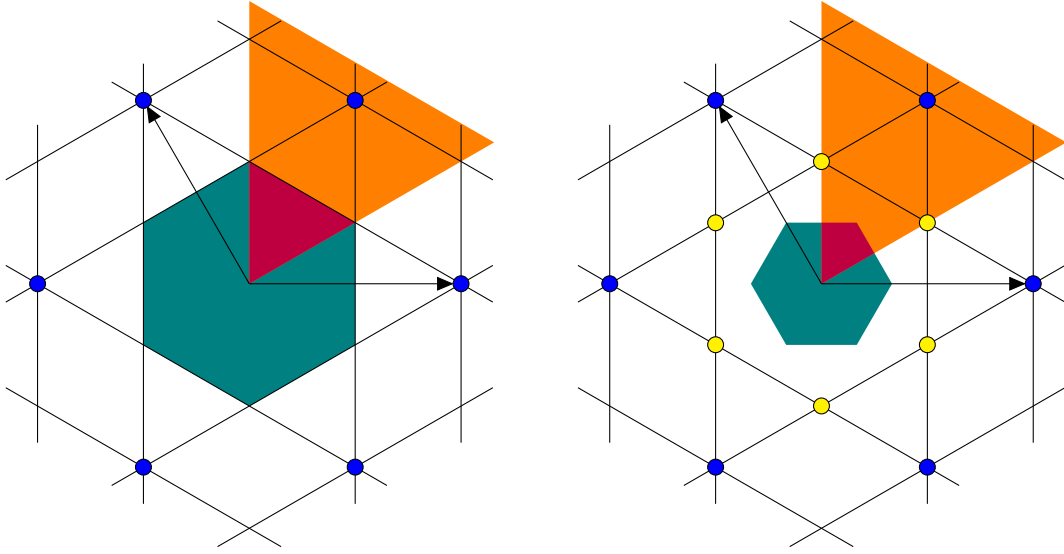


Figure 2.2: Truncated Stiefel diagrams of $SU(3)$ and $PSU(3)$ with coroots \bullet , integral elements \bullet , Weyl chamber $\color{orange}\bullet$, Voronoi cell $\color{teal}\bullet$ and fundamental alcove $\color{red}\bullet$ [Sam90, Fig. 3–5, pp. 133–134] [BtD85, Ch. V, Fig. 24, p. 227].

The following three sections will contain descriptions of the four infinite families of complex simple Lie algebras, compact simple Lie algebras and compact connected simple Lie groups. They are based on [BtD85, Ch. V, Sec. 6] and [Hel78, Ch. III, §8]. All Lie algebras are Lie algebras of endomorphisms and carry the commutator bracket as Lie bracket. Special care will be given to type A_n , which will play a role in the proof of Theorem C. The seven exceptional compact connected simple Lie groups or their Lie algebras are not considered. Concrete descriptions of these may be found in [Ada96], [Gar16], and the references therein.

2.12 Table of Classical Complex Simple Lie Algebras

2.12.1 A_n , the complex special linear Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. For $n \geq 1$, define

$$\mathfrak{sl}(n+1, \mathbb{C}) = \{X \in \mathbb{C}^{(n+1) \times (n+1)} : \operatorname{tr} X = 0\},$$

the *special linear algebra of rank n* . *Dimension:* $n(n+2)$. *Killing form:* $\kappa(X, Y) = 2(n+1) \operatorname{tr}(XY)$. *Cartan subalgebra:*

$$\mathfrak{h} = \left\{ \begin{bmatrix} \theta_0 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix} : \theta_0, \dots, \theta_n \in \mathbb{C}, \sum_j \theta_j = 0 \right\}.$$

Complex roots: For $a \neq b$, each

$$\Theta: \mathfrak{h} \rightarrow \mathbb{C}, \quad \Theta_{ab}(H) = H^{aa} - H^{bb}$$

is a complex root, whose corresponding root space is $\mathbb{C} \cdot E_{ab}$, where E_{ab} is the matrix whose only nonzero entry is the $(a, b)^{\text{th}}$ entry $(E_{ab})^{ab} = 1$. The system of complex roots is

$$R = \{\Theta_{ab} : 1 \leq a \neq b \leq n\}.$$

With respect to the simple basis $(\Theta_{12}, \dots, \Theta_{n, n+1})$, the set of positive roots is

$$R^+ = \{\Theta_{ab} : 1 \leq a < b \leq n\}.$$

Coroots: The coroots of R are $H_{ab} = \Theta_{ab}^\sharp$, i.e. $H_{ab} = E_{aa} - E_{bb}$.

2.12.2 D_n , the even complex special orthogonal Lie algebra $\mathfrak{so}(2n, \mathbb{C})$. For $n \geq 2$, define

$$\mathfrak{so}(2n, \mathbb{C}) = \{X \in \mathbb{C}^{2n \times 2n} : X^\top = -X\},$$

the *even special complex orthogonal algebra of rank n* . *Lie bracket:* $[X, Y] = XY - YX$. *Dimension:* $2n(n+1)$. *Killing form:* $\kappa(X, Y) = (2n-2)\text{tr}(XY)$. *Cartan subalgebra:*

$$\mathfrak{h} = \underbrace{\mathfrak{so}(2, \mathbb{C}) \times \dots \times \mathfrak{so}(2, \mathbb{C})}_n \subseteq \mathfrak{so}(2n, \mathbb{C}).$$

Set $X_{ab} = -E_{ab} + E_{ba}$ and $X_a = X_{2a-1, 2a}$. Then $(X_{ab} : a < b)$ is a basis of $\mathfrak{so}(2n, \mathbb{C})$ and $(X_a : a = 1, \dots, n)$ is a basis of \mathfrak{h} .

Complex roots. Define $\Theta_a : \mathfrak{h} \rightarrow \mathbb{C}$, $\Theta_a(Y) = i(-Y^{a, a+1} + Y^{a+1, a})$ for all a and let

$$\begin{aligned} M_{ab} &= (X_{2a-1, 2b-1} + X_{2a, 2b}) + i(X_{2a-1, 2b} - X_{2a, 2b-1}), \\ N_{ab} &= (X_{2a-1, 2b-1} - X_{2a, 2b}) + i(X_{2a-1, 2b} + X_{2a, 2b-1}). \end{aligned}$$

There are four types of complex roots, displayed in Tab. 2.3.

infinitesimal root	root space
$i(\Theta_a - \Theta_b)$	$\mathbb{C} \cdot M_{ab}$
$i(\Theta_a + \Theta_b)$	$\mathbb{C} \cdot N_{ab}$
$-i(\Theta_a - \Theta_b)$	$\mathbb{C} \cdot \overline{M_{ab}}$
$-i(\Theta_a + \Theta_b)$	$\mathbb{C} \cdot \overline{N_{ab}}$

Table 2.3: Complex roots of $\mathfrak{so}(2n, \mathbb{C})$.

2.12.3 B_n , the odd special complex orthogonal algebra $\mathfrak{so}(2n+1, \mathbb{C})$. For $n \geq 2$, define

$$\mathfrak{so}(2n+1, \mathbb{C}) = \{X \in \mathbb{C}^{(2n+1) \times (2n+1)} : X^\top = -X\},$$

the *odd special complex orthogonal algebra of rank n* . Lie bracket: $[X, Y] = XY - YX$. Dimension: $2n(n+1)$. Killing form: $\kappa(X, Y) = (2n-1) \operatorname{tr}(XY)$. Cartan subalgebra: $\mathfrak{h} \subseteq \mathfrak{so}(2n, \mathbb{C}) \subseteq \mathfrak{so}(2n+1, \mathbb{C})$.

Complex roots: The complex roots are (1) those of $\mathfrak{so}(2n, \mathbb{C})$ found in Tab. 2.3 and (2) the roots $\pm\Theta_a$ with root spaces $\mathbb{C} \cdot Z_a$ and $\mathbb{C} \cdot \bar{Z}_a$ (depending on the sign \pm), where

$$Z_a = X_{2a-1, 2n+1} - iX_{2a, 2n+1}.$$

2.12.4 C_n , the complex symplectic Lie algebra $\mathfrak{sp}(n, \mathbb{C})$. For $n \geq 3$, define

$$\mathfrak{sp}(n, \mathbb{C}) = \left\{ \left[\begin{array}{c|c} U & W \\ \hline V & -U^\top \end{array} \right] : U, V, W \in \mathbb{C}^{n \times n}, \quad V^\top = V, W^\top = W \right\},$$

the *complex symplectic Lie algebra of rank n* .¹⁸ Dimension: $2n(n+1)$. Killing form: $\kappa(X, Y) = (2n+2) \operatorname{tr}(XY)$. Cartan subalgebra:

$$\mathfrak{h} = \left\{ \left[\begin{array}{ccccccc} \theta_1 & & & & & & \\ & \ddots & & & & & \\ & & \theta_n & & & & \\ & & & -\theta_1 & & & \\ & & & & \ddots & & \\ & & & & & & -\theta_n \end{array} \right] : \theta_1, \dots, \theta_n \in \mathbb{C} \right\}.$$

The vectors $X_a = i(E_{aa} - E_{n+a, n+a})$ form a basis of \mathfrak{h} .

Complex roots: Define $\Theta_a = i \cdot X_a^\flat$ and $P_{ab} = E_{n+a, b} + E_{n+b, a}$, $Q_{ab} = E_{ab} - E_{n+b, a}$. The three types of complex roots are displayed in Tab. 2.4.

infinitesimal root	root space
$\Theta_a + \Theta_b$	$\mathbb{C} \cdot P_{ab}$
$-\Theta_a - \Theta_b$	$\mathbb{C} \cdot P_{ab}^\top$
$\Theta_a - \Theta_b$	$\mathbb{C} \cdot Q_{ab}$

Table 2.4: Complex roots of $\mathfrak{sp}(n)$, assuming $1 \leq a \leq b \leq n$. Note that this table includes the roots $2\Theta_a$.

¹⁸This Lie algebra is sometimes denoted by $\mathfrak{sp}(2n, \mathbb{C})$.

2.13 Table of Classical Compact Simple Lie Algebras

2.13.1 The special unitary Lie algebra $\mathfrak{su}(n+1)$. For $n \geq 1$, define

$$\mathfrak{su}(n+1) = \{X \in \mathbb{C}^{(n+1) \times (n+1)} : X^* = -X\},$$

the *special unitary algebra of rank n* . *Dimension:* $n(n+2)$. *Inner product:* $Q(X, Y) = \text{tr}(XY^*)$. *Cartan subalgebra:*

$$\mathfrak{h} = \left\{ i \cdot \begin{bmatrix} \theta_0 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix} : \theta_0, \dots, \theta_n \in \mathbb{R}, \sum_j \theta_j = 0 \right\}.$$

The a^{th} diagonal entry of $H \in \mathfrak{h}$ will be denoted by H^a . *Complexification:* $\mathfrak{sl}(n+1, \mathbb{C})$.

Typical basis: A typical basis of $\mathfrak{su}(n+1)$ consists of (1) some basis H_1, \dots, H_n of \mathfrak{h} , (2) $X_{ab} = E_{ab} - E_{ba}$ and $Y_{ab} = i(E_{ab} + E_{ba})$ for $a < b$. Note that $X_{ab} = -X_{ba}$, $X_{aa} = 0$ and $Y_{ab} = Y_{ba}$, $Y_{aa} = 2iE_{aa}$. Usually, $H_a = \tilde{H}_{a, a+1}$, where $\tilde{H}_{ab} = iH_{ab}$.

Real roots: The real root corresponding to Θ_{ab} is

$$\alpha_{ab} : \mathfrak{h} \rightarrow \mathbb{R}, \quad \alpha_{ab}(H) = \frac{1}{2\pi i} \Theta_{ab}(H) = \frac{H^a - H^b}{2\pi}.$$

The real root space $\mathfrak{g}_{\alpha_{ab}}$ is $\mathfrak{g}_{\pm\alpha_{ab}} = \mathbb{R} \cdot X_{ab} \oplus \mathbb{R} \cdot Y_{ab}$. The root system is

$$R = \{\alpha_{ab} : 1 \leq a \neq b \leq n\}.$$

The set of positive roots with respect to the simple basis $\alpha_1 = -\alpha_{1,2}, \dots, \alpha_n = -\alpha_{n, n+1}$ is

$$R^+ = \{-\alpha_{ab} : a < b\}.$$

If \mathfrak{h} is identified with the vector space $V \subseteq \mathbb{R}^{n+1}$ of zero-sum vectors, then $R^\sharp = \frac{1}{2\pi} A_n$ and $R^* = 2\pi A_n^*$. The coroot of α_{ab} is $2\pi \tilde{H}_{ab}$. In particular, $R^\vee = 2\pi A_n$.

2.13.2 D_n , the special orthogonal Lie algebra $\mathfrak{so}(2n)$. For $n \geq 1$, define

$$\mathfrak{so}(2n) = \{X \in \mathbb{R}^{2n \times 2n} : X^\top = -X\},$$

the *even special orthogonal Lie algebra*. *Dimension:* $2n(n-1)$. *Inner product:* $Q(X, Y) = \text{tr}(XY^\top)/2$. *Cartan subalgebra:*

$$\mathfrak{h} = \underbrace{\mathfrak{so}(2) \times \cdots \times \mathfrak{so}(2)}_n \subseteq \mathfrak{so}(2n).$$

Complexification: $\mathfrak{so}(2n, \mathbb{C})$.

Real roots: Define $\vartheta_a(Y) = (-Y^{a,a+1} + Y^{a+1,a})/2\pi$. The real root corresponding to $\pm\Theta_a \pm \Theta_b$ is $\pm\vartheta_a \pm \vartheta_b$. Define $\alpha_1 = \vartheta_1 + \vartheta_2$, $\alpha_a = -\vartheta_{a-1} + \vartheta_a$. The positive roots corresponding to the simple basis $(\alpha_1, \dots, \alpha_n)$ are $-\vartheta_a + \vartheta_b$ for $1 \leq a < b \leq n$. If \mathfrak{h} is identified with \mathbb{R}^n by mapping X_a to e_a , then R^\sharp equals $\frac{1}{2\pi}D_n$. In particular, $R^\vee = 2\pi D_n$ and $R^* = 2\pi D_n^*$.

2.13.3 B_n , the odd orthogonal Lie algebra $\mathfrak{so}(2n+1)$. For $n \geq 2$, define

$$\mathfrak{so}(2n+1) = \{X \in \mathbb{R}^{(2n+1) \times (2n+1)} : X^\top = -X\},$$

the *odd orthogonal Lie algebra of rank n* . *Dimension:* $2n(n-1)$. *Inner product:* $Q(X, Y) = \text{tr}(XY^\top)/2$. *Cartan subalgebra:* $\mathfrak{h} \subseteq \mathfrak{so}(2n, \mathbb{C}) \subseteq \mathfrak{so}(2n+1, \mathbb{C})$. *Complexification:* $\mathfrak{so}(2n+1, \mathbb{C})$. Complexification commutes with $\mathfrak{so}(2n) \rightarrow \mathfrak{so}(2n+1)$.

Real roots: The real root corresponding to $\pm\Theta_a \pm \Theta_b$ is $\pm\vartheta_a \pm \vartheta_b$. The real root corresponding to $\pm\Theta_a$ is ϑ_a . If \mathfrak{h} is identified with \mathbb{R}^n by mapping X_a to e_a , then R^\vee equals $\frac{1}{2\pi}B_n$. In particular, $R^\vee = 2\pi C_n$ and $R^* = 2\pi B_n^* = \mathbb{Z}^n$.

2.13.4 C_n , the compact symplectic Lie algebra $\mathfrak{sp}(n)$. For $n \geq 3$, define

$$\mathfrak{sp}(n) = \left\{ \left[\begin{array}{c|c} A & -\bar{B} \\ \hline B & \bar{A} \end{array} \right] : A, B \in \mathbb{C}^{n \times n}, A^* = -A, B^\top = B \right\},$$

the *compact symplectic Lie algebra of rank n* . *Dimension:* $2n(n+1)$. *Inner product:* $B(X, Y) = \text{tr}(XY^*)/2$. *Cartan subalgebra:*

$$\mathfrak{h} = \left\{ i \cdot \left[\begin{array}{ccccccc} \theta_1 & & & & & & \\ & \ddots & & & & & \\ & & \theta_n & & & & \\ & & & -\theta_1 & & & \\ & & & & \ddots & & \\ & & & & & & -\theta_n \end{array} \right] : \theta_1, \dots, \theta_n \in \mathbb{R} \right\}.$$

Complexification: $\mathfrak{sp}(n, \mathbb{C})$.

Real roots: Define $\vartheta_a = X_a^b/2\pi$. Then the real root corresponding to $\pm(\Theta_a \pm \Theta_b)$ is $\pm(\vartheta_a \pm \vartheta_b)$. Define $\alpha_1 = 2\vartheta_1$, $\alpha_a = -\vartheta_{a-1} + \vartheta_a$ for $2 \leq a \leq n$. The positive roots corresponding to the simple basis $(\alpha_1, \dots, \alpha_n)$ are $-(\vartheta_a \pm \vartheta_b)$ for $a < b$ and $-2\vartheta_a$ for $1 \leq a \leq n$. If \mathfrak{h} is identified with \mathbb{R}^n by mapping X_a to e_a , then $R^\sharp = \frac{1}{2\pi}C_n$. In particular, $R^\vee = 2\pi B_n$ and $R^* = 2\pi D_n^*$.

2.14 The Classical Compact Lie Groups

2.14.1 The special unitary group $SU(n+1)$. For $n \geq 1$, define

$$SU(n+1) = \{A \in \mathbb{C}^{(n+1) \times (n+1)} : AA^* = I, \det A = 1\},$$

the *special unitary group of rank n* . Lie algebra: $\mathfrak{su}(n+1)$. Maximal torus: $S\Delta(n+1) = \Delta(n+1) \cap SU(n+1)$, where

$$\Delta(n+1) = \left\{ \begin{bmatrix} z_0 & & \\ & \ddots & \\ & & z_n \end{bmatrix} : z_0, \dots, z_n \in S^1 \right\}.$$

Note that $LS\Delta(n+1)$ is the Cartan algebra established in 2.13.1. *Type:* A_n .

Integral lattice: $R^\vee = 2\pi A_n$. *Fundamental group:* 1. *Centre:* The glue group

$$2\pi A_n^* / 2\pi A_n$$

is isomorphic to $A_n^*/A_n = \mathbb{Z}_{n+1}$. The glue vectors are 2π multiples of the elements of A_n^*/A_n :

$$2\pi[j] = \frac{2\pi i}{n+1} \left[\begin{array}{c|c} -kI_j & \\ \hline & jI_k \end{array} \right]$$

(with the convention $j+k = n+1$). Their images are

$$\exp 2\pi[j] = \left[\begin{array}{c|c} \exp\left(\frac{-2\pi ki}{n+1}\right) I_j & \\ \hline & \exp\left(\frac{2\pi ji}{n+1}\right) I_k \end{array} \right].$$

In particular, $Z(SU(n+1))$ is the set of scalar matrices with $(n+1)^{\text{th}}$ roots of unity on the diagonal. *Quotients:* The quotients of $SU(n+1)$ take the form $SU(n+1)/\mathbb{Z}_r$, where $r \mid (n+1)$. Their integral lattices are Coxeter lattices, whose geometry is badly understood (cf. App. D). The exception is the *projective special unitary group* $PSU(n+1) = SU(n+1)/Z(SU(n+1))$, whose integral lattice is A_n^* .

Typical deep holes of the integral lattice: $2\pi \left[\frac{n+1}{2} \right]$ if n is odd, and $2\pi \left[\frac{n}{2} \right], 2\pi \left[\frac{n}{2+1} \right]$ if n is even.

2.14.2 The even special orthogonal group $SO(2n)$. For $n \geq 1$, define

$$SO(2n) = \{A \in \mathbb{R}^{2n \times 2n} : AA^\top = I_{2n}\},$$

the *even special orthogonal group of rank n* . For $n \geq 3$, this Lie group is simple, but $\mathrm{SO}(2)$ is abelian and $\mathrm{SO}(4)$ is semisimple of type $A_1 \times A_1$ (cf. 1.5.4 and Sec. 2.15).
Lie algebra: $\mathfrak{so}(2n)$. *Maximal torus:*

$$T^n = \underbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}_n \subseteq \mathrm{SO}(2n).$$

Integral lattice: $2\pi\mathbb{Z}^n$ under the previously established identification $\mathfrak{h} = \mathbb{R}^n$.
Fundamental group: \mathbb{Z}_2 with generator $t \mapsto [\exp tX_a]$ for any a . *Centre:* $\mathbb{Z}_2 = \{\pm I_{2n}\}$.
Quotients: $\mathrm{PSO}(2n) = \mathrm{SO}(2n)/\mathbb{Z}_2$.

Typical deep holes of the integral lattice: $2\pi[1]$.

2.14.3 The odd special orthogonal group $\mathrm{SO}(2n+1)$. For $n \geq 1$, define

$$\mathrm{SO}(2n+1) = \{A \in \mathbb{R}^{(2n+1) \times (2n+1)} : AA^\top = I_{2n+1}\},$$

the *even special orthogonal group of rank n* . *Lie algebra:* $\mathfrak{so}(2n+1)$. *Maximal torus:* $T^n \subseteq \mathrm{SO}(2n) \subseteq \mathrm{SO}(2n+1)$.

Integral lattice: $2\pi\mathbb{Z}^n$ under the previously established identification $\mathfrak{h} = \mathbb{R}^n$. *Fundamental group:* \mathbb{Z}_2 with generator $t \mapsto [\exp tX_a]$ for any a . *Centre:* trivial. *Quotients:* None.

Typical deep holes of the integral lattice: $2\pi[1]$.

2.14.4 The spin group $\mathrm{Spin}(\ell)$. This paragraph follows [LM89, Ch. I]. Let $\ell \geq 2$, $V = \mathbb{R}^\ell$ and define the *Clifford algebra* $\mathrm{Cl}(V)$ as the quotient algebra of the tensor algebra¹⁹

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

by the two-sided ideal generated by

$$\{v \otimes v + \langle v, v \rangle : v \in \mathbb{R}^\ell\}.$$

Let $\pi: T(V) \rightarrow \mathrm{Cl}(V)$ be the natural projection. The natural map $V \rightarrow T(V) \rightarrow \mathrm{Cl}(V)$ is injective [LM89, p. 8] and will be used to identify V with a subspace of $\mathrm{Cl}(V)$.

The Clifford algebra enjoys the following universal property. For every linear map $f: V \rightarrow A$ into a unital \mathbb{R} -algebra A so that

$$f(v) \cdot f(v) = -\langle v, v \rangle \cdot 1_A$$

¹⁹ $V^{\otimes 0} = \mathbb{R}$, $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_k$.

holds for all $v \in V$, there exists a unique algebra homomorphism $\tilde{f}: \text{Cl}(V) \rightarrow A$ with $\tilde{f}(v) = f(v)$ for all $v \in V$ [LM89, Prop. 1.1]. The canonical vector space isomorphism

$$\Lambda^*V \rightarrow \text{Cl}(V), \quad v_0 \wedge \cdots \wedge v_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} \text{sign}(\sigma) \cdot v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

will be used to identify Λ^*V and $\text{Cl}(V)$ [LM89, Prop. 1.3]. Note that

$$v_0 \wedge v_1 \mapsto \frac{1}{2}(v_0 \otimes v_1 - v_1 \otimes v_0).$$

Define the involution $\alpha: \text{Cl}(V) \rightarrow \text{Cl}(V)$ as the unique extension of $V \rightarrow \text{Cl}(V)$, $v \mapsto -v$. Denote by $\text{Cl}(V) = \text{Cl}^0(V) \oplus \text{Cl}^1(V)$ the corresponding eigenspace decomposition into their *even* and *odd* parts. Note that $\text{Cl}(V)$ thereby becomes a \mathbb{Z}_2 -graded algebra [LM89, p. 9].

Let $\text{Cl}^\times(V)$ the multiplicative group of units of $\text{Cl}(V)$, which is a Lie group of dimension 2^ℓ [LM89, p. 12]. The exponential map is given by the usual exponential series

$$\exp: \mathfrak{cl}^\times(V) \rightarrow \text{Cl}^\times(V), \quad \exp(y) = \sum_{m=0}^{\infty} \frac{1}{m!} y^{\otimes m}$$

[LM89, Ch. I, (2.6)]. Obviously $V^\times \subseteq \text{Cl}^\times(V)$ and $v^{-1} = -v/\langle v, v \rangle$. The *pin group* $\text{Pin}(V)$ of V is the subgroup generated by $\{v \in V: \|v\| = 1\}$. The *spin group* $\text{Spin}(V)$, or $\text{Spin}(\ell)$, of V is defined by

$$\text{Spin}(V) = \text{Pin}(V) \cap \text{Cl}^0(V).$$

The pin and spin groups are represented on V by the *twisted adjoint representation*

$$\widetilde{\text{Ad}}: \text{Pin}(V) \rightarrow \text{O}(V), \quad \widetilde{\text{Ad}}(\varphi)(v) = \alpha(\varphi) \cdot v \cdot \varphi^{-1}$$

[LM89, Ch. I, (2.11)]. For $v \in V^\times$, the twisted adjoint representation $\widetilde{\text{Ad}}(v)$ is the reflection in the hyperplane v^\perp . By the theorem of Cartan–Dieudonné, $\text{O}(V)$ is generated by reflections [LM89, Thm. 2.7]. Thus, $\widetilde{\text{Ad}}$ is surjective.

The Lie algebra $\mathfrak{spin}(\ell)$ of $\text{Spin}(\ell)$ is equal to $\Lambda^2 \mathbb{R}^\ell$ [LM89, Prop. 6.1]. The derivative

$$\Xi = \text{L}\widetilde{\text{Ad}}: \mathfrak{spin}(\ell) \rightarrow \mathfrak{so}(\ell),$$

satisfies $\Xi(e_a \otimes e_b) = 2 \cdot X_{ab}$. It follows that $\text{Spin}(\ell)$ is simple if $\ell \geq 5$ and semisimple if $\ell = 4$. Its type is B_n if ℓ is odd and D_n if ℓ is even. Note that $D_2 \cong A_1 \times A_1$ (cf. 2.14.2 and Sec. 2.15). The inner product of $\mathfrak{spin}(\ell)$ is obtained by pulling back the inner product of $\mathfrak{so}(\ell)$.

Integral lattice: $R^\vee = D_n$ for all ℓ , where $n = \lfloor \ell/2 \rfloor$. *Centre:* Define $\omega = e_1 \otimes \cdots \otimes e_\ell$. If ℓ is odd, then the central elements are $2\pi[a]$ for $a = 0, 2$, using the glue vector notation

of D_n . If ℓ is even, then the central elements are $2\pi[a]$ for $a = 0, 1, 2, 3$. Their images are $\exp(2\pi[1]) = \omega$, $\exp(2\pi[2]) = -1$, $\exp(2\pi[3]) = -\omega$. Thus, $Z(\text{Spin}(\ell)) = \{\pm 1\}$ if ℓ is odd, and $Z(\text{Spin}(\ell)) = \{\pm 1, \pm \omega\}$ if ℓ is even. *Fundamental group:* 1. *Quotients:* $\text{SO}(\ell) = \text{Spin}(\ell)/\{\pm 1\}$ for all ℓ and $\text{PSO}(\ell) = \text{Spin}(\ell)/\{\pm 1, \pm \omega\}$ if ℓ is even.

Typical deep holes: $2\pi[2]$ if $\ell \leq 9$ and $2\pi[1], 2\pi[3]$ if $\ell \geq 8$. Note that this implies that $\frac{1}{2} \text{dia Spin}(\ell) = \text{dia SO}(\ell)$ for $\ell = 2, 3$, $(1/\sqrt{2}) \cdot \text{dia Spin}(\ell) = \text{dia SO}(\ell)$ for $\ell = 4, 5$, $(\sqrt{3}/2) \cdot \text{dia Spin}(\ell) = \text{dia SO}(\ell)$ for $\ell = 6, 7$ and $\text{dia Spin}(\ell) = \text{dia SO}(\ell)$ for $\ell \geq 8$ (cf. 4.4.1).

2.14.5 The compact symplectic group $\text{Sp}(n)$. For $n \geq 1$, define

$$\text{Sp}(n) = \left\{ \left[\begin{array}{c|c} A & -\bar{B} \\ \hline B & \bar{A} \end{array} \right] \in \text{U}(2n) : A, B \in \mathbb{C}^{2n \times 2n} \right\},$$

the compact symplectic group of rank n . *Lie algebra:* $\mathfrak{sp}(n)$. *Maximal torus:*

$$T^n = \left\{ \left[\begin{array}{c|c} A & \\ \hline & \bar{A} \end{array} \right] : A \in \Delta(n) \right\}.$$

Integral lattice: If \mathfrak{h} is identified with \mathbb{R}^n by mapping X_a to e_a , then the integral lattice is $R^\vee = 2\pi \mathbb{Z}^n$. *Fundamental group:* 1. *Centre:* $\mathbb{Z}_2 = \{\pm I_{2n}\}$. In fact, the nontrivial element of the glue group is $2\pi[1]$, using the glue vector notation of B_n . *Quotients:* $\text{PSp}(n) = \text{Sp}(n)/\mathbb{Z}_2$.

Typical deep holes of the integral lattice: $2\pi[1]$.

2.15 Exceptional Isomorphisms

The four infinite families of irreducible root systems and their Lie theoretic counterparts are sometimes only defined under dimensional restrictions with the purpose of excluding isomorphic duplicates. For instance, $A_1: \bullet \cong B_1: \bullet$. Such an isomorphism between elements of two different infinite families is called an *exceptional isomorphism*. These sometimes help to explain odd structural phenomena in low-dimensional Lie theory. Below follows a concise list of all exceptional isomorphisms. For details, see [Hel78, Ch. X, §6].

2.15.1. $A_1: \bullet \cong B_1: \bullet \cong C_1: \bullet$.

$$\begin{aligned} \mathfrak{su}(2) &\cong \mathfrak{so}(3) \cong \mathfrak{sp}(1), & \text{SU}(2) &\cong \text{Spin}(3) \cong \text{Sp}(1), \\ \text{PSU}(2) &\cong \text{SO}(3) \cong \text{PSp}(1). \end{aligned}$$

2.15.2. D_2 : $\bullet \quad \bullet \cong A_1 \times A_1$: $\bullet \quad \bullet$.

$$\begin{aligned} \mathfrak{su}(2) \times \mathfrak{su}(2) &\cong \mathfrak{so}(4), \\ \mathrm{SU}(2) \times \mathrm{SU}(2)/\{\pm(1, 1)\} &\cong \mathrm{SO}(4), \\ \mathrm{SU}(2) \times \mathrm{SU}(2)/(\mathbb{Z}_2 \times \mathbb{Z}_2) &\cong \mathrm{PSO}(4). \end{aligned}$$

$$\begin{aligned} \mathrm{SU}(2) \times \mathrm{SU}(2) &\cong \mathrm{Spin}(4), \\ \mathrm{SU}(2)/\mathbb{Z}_2 \times \mathrm{SU}(2) &\cong \mathrm{Spin}(4)/\{1, \omega\}, \end{aligned}$$

2.15.3. B_2 : $\bullet \leftarrow \bullet \cong C_2$: $\bullet \rightarrow \bullet$.

$$\begin{aligned} \mathfrak{so}(5) &\cong \mathfrak{sp}(2), \\ \mathrm{SO}(5) &\cong \mathrm{PSp}(2). \end{aligned}$$

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2),$$

2.15.4. D_3 : $\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \cong A_3$: $\bullet - \bullet - \bullet$.

$$\begin{aligned} \mathfrak{su}(4) &\cong \mathfrak{so}(6), \\ \mathrm{SU}(4)/\{\mathbb{Z}, 2 + 4\mathbb{Z}\} &\cong \mathrm{SO}(6), \end{aligned}$$

$$\begin{aligned} \mathrm{SU}(4) &\cong \mathrm{Spin}(6), \\ \mathrm{PSU}(4) &\cong \mathrm{PSO}(6). \end{aligned}$$

3 Geometry of Lie Groups and Manifolds

Two strands run through this chapter. The first is the study of the evolution of geodesics under variations of its endpoints or the underlying metric. In Sec. 3.1, a collection of lemmas regarding this topic is presented. The second strand is Lie theoretic in nature: the basics of the differential geometry of Lie groups are retold in Sec. 3.2, in order to define and study the vital notions of *antipodal point* and *thickness* and later characterise antipodal sets and conjugate loci of bi-invariant metrics in terms of the integral lattices of maximal tori in the sections to come. The two strands finally meet in Sec. 3.5, where the evolution of geodesics under variations of left-invariant metrics on compact Lie groups are studied, yielding the most significant Lem. 3.5.2, which will be used in the proof of Theorem C. Some results presented here may have occurred in the works of other mathematicians, and others may have found them too trivial to write down. References are not exhaustive, but provided whenever possible.

A few words on notation: Throughout the following chapters, $c(g)$ denotes the *left conjugation* with the group element g , i.e. $c(g)(h) = ghg^{-1}$. When families of metrics depending on a parameter x are considered, be it riemannian or left-invariant, then their associated geometric properties, like the Levi–Civita connection, are usually denoted by a super- or subscript x , e.g. \exp^x for the riemannian exponential map or ∇^x for the Levi–Civita connection.

3.1 Evolution of Sets of Geodesics on Riemannian Manifolds

This section is concerned with the evolution of geodesics under a differentiable variation of riemannian metrics on a compact manifold M , i.e. a differentiable map into the space of riemannian metrics on M . Specifically, a notion of lower semicontinuity for maps from topological spaces to spaces of sets is introduced and it is proved that the map associating to (x, p, q) the set of $g(x)$ -geodesics joining the points p and q is lower semicontinuous (see Lem. 3.1.9, Prop. 3.1.10). Moreover, it is shown that the invertibility of the exponential

map is preserved under small changes of the parameter x (Prop. 3.1.6). All results are supplied with proof for the sake of completeness.

3.1.1 Notation. For this section, (M, g_0) will be a compact riemannian manifold. Following [Bes08], $\mathcal{M} \subseteq \mathcal{S}^2(M)$ denotes the space of all riemannian metrics on M , an open convex subset of the vector space $\mathcal{S}^2(M)$ of symmetric 2-tensor fields on M . The space $\mathcal{S}^2(M)$ may be equipped with the L^2 -product with respect to g_0 [Bes08, App. B] [Bes08, 1.172] [Ble79, p. 600].¹ The families of riemannian metrics mentioned above will be described as a differentiable map $g: X \rightarrow \mathcal{M}$ defined on a compact manifold X .

3.1.2 Definitions. Let (X, d_X) be a compact metric space. Denote by $H(X)$ the space of nonempty compact subsets of X . This space will be equipped with the *one-sided Hausdorff pseudo-quasi-metric*²

$$d_X^{\subseteq}(A, B) = \max_{a \in A} \min_{b \in B} d_X(a, b).$$

d_X^{\subseteq} satisfies the triangle inequality, but it is in general *neither* symmetric *nor* reflexive, although $d_{H(X)}(A, B) = 0$ always implies $A \subseteq B$. A map $f: Y \rightarrow H(X)$ defined on a metric space Y is *lower semicontinuous*³ if

$$\lim_{n \rightarrow \infty} d_X^{\subseteq}(f(y_n), f(y)) = 0$$

holds for all convergent sequences $(y_n) \in Y^{\mathbb{N}}$ with limit $y \in Y$.

3.1.3 Lemma. *Let (A_n) be a sequence in $H(X, d_X)$ and let $B \in H(X, d_X)$. Then $d_X^{\subseteq}(A_n, B) \rightarrow 0$ if and only if all sequences (a_i) in X , so that there exists a subsequence (A_{n_i}) with $a_i \in A_{n_i}$ for all $i \in \mathbb{N}$, subconverge to an element of B .*

Proof. ‘ \Rightarrow ’: Suppose that $d_X^{\subseteq}(A_n, B) \rightarrow 0$. Let (a_i) be a sequence in X with $a_i \in A_{n_i}$ for all $i \in \mathbb{N}$. Then (a_i) subconverges to some $a \in X$. Let

$$d_X(a_i, B) = \min_{b \in B} d_X(a_i, b).$$

For each $i \in \mathbb{N}$, take $b_i \in B$ so that $d_X(a_i, b_i) = d_X(a_i, B)$. Then

$$d_X(a_i, b_i) \leq d_X^{\subseteq}(A_{n_i}, B) \rightarrow 0.$$

It follows that (b_i) subconverges to a . Therefore, $a \in B$.

‘ \Leftarrow ’: Suppose that $d_X^{\subseteq}(A_n, B)$ does not converge to 0. Then there exists $\varepsilon > 0$ and a strictly increasing sequence (n_i) so that $d_X^{\subseteq}(A_{n_i}, B) \geq \varepsilon$ for all $i \in \mathbb{N}$. Let (a_i) be a sequence with $a_i \in A_{n_i}$ so that

$$d_X^{\subseteq}(A_{n_i}, B) = \min_{b \in B} d_X(a_i, b).$$

¹The choice of the reference metric is irrelevant for the topology of $\mathcal{S}^2(M)$.

²This curious bit of nomenclature is taken from [SS78, Pt. 1, Sec. 5]. Pseudometrics need not satisfy the axiom $(x \neq y \Rightarrow d(x, y) > 0)$ and quasimetrics need not satisfy the symmetry axiom.

³Rouyer calls this ‘upper semicontinuous’ in [Rou03].

By the hypothesis, there exists a subsequence (a_{i_j}) that converges to $a \in B$. Thus,

$$d_X^{\subseteq}(A_{n_{i_j}}, B) = \min_{b \in B} d_X(a_{i_j}, b) \leq d_X(a_{i_j}, a).$$

It follows that $d_X^{\subseteq}(A_{n_{i_j}}, B)$ approaches zero. This is a contradiction to $d_X^{\subseteq}(A_{n_i}, B) \geq \varepsilon$. \square

3.1.4 Lemma. $d_X^{\subseteq}(A_n, B) \rightarrow 0$ if and only if for every neighbourhood N of B in X , there exists $n_0 \in \mathbb{N}$ so that $A_n \subseteq N$ for all $n \geq n_0$.

Proof. ‘ \Rightarrow ’: Let N be a neighbourhood of B in X and let $\varepsilon > 0$ so that the ε -tube

$$U_\varepsilon = \{x \in X : \min_{b \in B} d_X(x, b) < \varepsilon\}$$

is contained in N . Then there exists $n_0 \in \mathbb{N}$ so that $d_X^{\subseteq}(A_n, B) < \varepsilon$ for all $n \geq n_0$. It follows that $A_n \subseteq U_\varepsilon \subseteq N$ for all $n \geq n_0$.

‘ \Leftarrow ’: Define N as U_ε . Then there exists $n_0 \in \mathbb{N}$ with $A_n \subseteq N$ for all $n \geq n_0$. Therefore, $d_X(a, B) < \varepsilon$ for all $n \geq n_0, a \in A_n$. It follows that $d_X^{\subseteq}(A_n, B) < \varepsilon$. \square

3.1.5. Let X be a compact manifold and $g: X \rightarrow \mathcal{M}$ a differentiable map. Let $\exp^x: TM \rightarrow M$ denote the exponential map of $(M, g(x))$. For $p \in M$, the *tangent conjugate locus* $\mathcal{C}_p^x = \mathcal{C}_p(M, g(x)) \subseteq T_p M$ is the set of vectors $v \in T_p M$ so that $D(\exp^x)_v$ is noninvertible. The *first conjugate locus* $\mathcal{C}_p^1(M, g(x))$ consists of all $v \in \mathcal{C}_p^x$ so that $tv \notin \mathcal{C}_p^x$ for all $t \in (0, 1)$, cf. [Sak97, p. 107]. Define

$$\text{Exp}: X \times TM \rightarrow X \times M, \quad \text{Exp}(x, v) = (x, \exp^x(v)).$$

3.1.6 Proposition. (1) $(x, v) \mapsto \exp^x(v)$ is a differentiable map. (2) Let $p \in M$ and suppose that $v \in T_p M \setminus \mathcal{C}_p^{x_0}$. Then there exist neighbourhoods $U \subseteq X$ of x_0 and $V \subseteq T_p M$ of v so that $\exp^x|_V$ is a diffeomorphism for all $x \in U$, or, equivalently, $\text{Exp}|_{U \times V}$ is a diffeomorphism.

Proof. (1) In local coordinates the Christoffel symbols $(x, u) \mapsto \Gamma_{ij}^k(x, u)$ are differentiable maps. It follows that the solutions to the differential equations

$$0 = \ddot{u}^\ell(t) + \sum_{ij} \Gamma_{ij}^\ell(x, u(t)) \dot{u}^i(t) \dot{u}^j(t)$$

with initial condition $\dot{u}(0) = v$ depend differentiably on $(x, u, v) \in X \times \mathbb{R}^n \times \mathbb{R}^n$.

(2) Let $f = \text{Exp}$. The differential of f in (x_0, v) is

$$Df_{(x_0, v)} = \begin{bmatrix} 1 & \\ ? & D(\exp^{x_0})_v \end{bmatrix}.$$

It follows that there exists a neighbourhood $W \subseteq X \times T_p M$ of (x_0, v) so that $f: W \rightarrow \tilde{W} = f(W)$ is a diffeomorphism. Let $g = (g_1, g_2): \tilde{W} \rightarrow W$ be its inverse and assume that $W = U \times V$ using the tubular neighbourhood lemma.

Let us prove that

$$g_2(x, \cdot): \tilde{W} \rightarrow V, \quad g_2(x, \cdot)(q) = g_2(x, q)$$

is the inverse of $\exp^x|_V$ for all $x \in U$. Note first that for all $(x, q) \in \tilde{W}$ the equation

$$(x, q) = (f \circ g)(x, q) = (g_1(x, q), \exp^{g_1(x, q)}(g_2(x, q)))$$

implies $g_1(x, q) = x$. In particular, $q = \exp^x(g_2(x, q))$. Conversely, the equation above also implies

$$(x, v) = (g \circ f)(x, v) = g(x, \exp^x(v)) = (x, g_2(\exp^x(v)))$$

for all $(x, v) \in W$. Thus $v = g_2(\exp^x(v))$ for all $v \in V$. \square

3.1.7. Let X be a compact manifold and $g: X \rightarrow \mathcal{M}$ a differentiable map with $g(x_0) = g_0$. For a curve $c: I \rightarrow M$ defined on either $I = [0, 1]$ or $I = \mathbb{R}$, the *endpoints* of c are $c(0)$ and $c(1)$ and c is said to *join* them.⁴ Denote by $\mathcal{G}_p^q(x)$ the set of minimising geodesics joining p and q in $(M, g(x))$. The *length* $L_x(c)$ of c with respect to $g(x)$ is the length of $c|_{[0,1]}$ with respect to $g(x)$. Use Nash's embedding theorem [Nas56] to embed (M, g_0) isometrically into \mathbb{R}^N . Denote by $C(M)$ the space of piecewise C^∞ -differentiable curves in M , and equip $C(M)$ with the C^k -metric of $C^k(\mathbb{R}, \mathbb{R}^N)$, which induces the C^k -topology on $C(M)$.

3.1.8 Proposition. *Let (x_i, p_i, q_i) be a sequence in $X \times M \times M$ with limit (x_0, p_0, q_0) and $\gamma_i \in \mathcal{G}_{p_i}^{q_i}(x_i)$ for all i . Then (γ_i) subconverges in the C^∞ -topology if and only if (γ_i) subconverges in the C^0 -topology.*

Proof. Suppose that (γ_i) subconverges to some $\gamma_0 \in C(M)$ in the C^0 -topology. Parametrise all geodesics so that their velocity is their length, i.e. $\|\dot{\gamma}_i(0)\|_{x_i} = L_{x_i}(\gamma_i)$. Let $a_i = \|\dot{\gamma}_i(0)\|_{x_i} = L_{x_i}(\gamma_i)$ and note that $a_i = d_{x_i}(p_i, q_i)$. Thus, $a_i \rightarrow a_0$. Let $r = 2a_0$. It follows that almost all $v_i = \dot{\gamma}_i(0)$ are contained in

$$T^r M = \{v \in TM: \|v\|_{x_0} \leq r\}.$$

In particular, v_i subconverges to some $v_0 \in T_{p_0} M$. According to Prop. 3.1.6,

$$F: X \times TM \times \mathbb{R} \rightarrow M, \quad F(x, v, t) = \exp^x(tv)$$

is smooth. Note that $F(x, v, \cdot)$ is the geodesic γ with respect to $g(x)$ with $\dot{\gamma}(0) = v$. It follows that $\gamma_i = F(x_i, v_i, \cdot)$ subconverges to $F(x_0, v_0, \cdot)$ in the C^∞ -topology. As the C^0 -limit of the subsequence of (γ_i) under consideration is γ_0 , it follows that $\gamma_0 = F(x_0, v_0, \cdot)$. In particular, γ_0 is the C^∞ -limit. \square

⁴This allows us to extend geodesics to \mathbb{R} and still speak of its endpoints.

3.1.9 Lemma. *Let $k = 0, 1, 2, \dots, \infty$. Then the map*

$$\mathcal{G}: X \times M \times M \rightarrow H(C(M), d_{C^k}), \quad \mathcal{G}(x, p, q) = \mathcal{G}_p^q(x)$$

is lower semicontinuous.

Proof. Verify the hypothesis of Lem. 3.1.3. Let $(x_n, p_n, q_n) \rightarrow (x_0, p_0, q_0)$ and let (γ_i) be a sequence with $\gamma_i \in \mathcal{G}_{p_{n_i}}^{q_{n_i}}(x_{n_i})$ for all i . According to the Ascoli–Arzelá theorem, (γ_i) subconverges in the C^0 -topology to a minimising geodesic $\gamma \in \mathcal{G}_{p_0}^{q_0}(x_0)$. By Prop. 3.1.8, (γ_i) subconverges in the C^k -topology. Thus, one may apply Lem. 3.1.3. \square

3.1.10 Proposition. *Let $k = 0, 1, 2, \dots, \infty$, $p, q: X \rightarrow M$ be differentiable and define $\mathcal{G}(x) = \mathcal{G}_{p(x)}^{q(x)}(x)$. Fix $x_0 \in X$. Then for all open C^k -neighbourhoods N of $\mathcal{G}(x_0)$ in $C(M)$, there exists an open neighbourhood U of x_0 in X so that $\mathcal{G}(x) \subseteq N$ for all $x \in U$.*

Proof. Follows from Lem. 3.1.9 and Lem. 3.1.4. \square

3.2 Preliminaries on Left-Invariant Metrics

3.2.1 Left-invariant metrics. The left- and right-multiplication of a Lie group with a group element g of a Lie group G are denoted by $L_g(h) = gh$ and $R_g(h) = hg$, respectively. Recall that a riemannian metric Q on G is left- or right-invariant if and only $L_g^*Q = Q$ for all g or $R_g^*Q = Q$, respectively. Obviously, such metrics are determined by their values on the Lie algebra \mathfrak{g} .

3.2.2 Bi-invariant metrics. Recall that a metric on a Lie group G is bi-invariant if and only if the adjoint representation $\text{ad}(X)$ of every $X \in \mathfrak{g}$ is skew-symmetric with respect to the corresponding scalar product on the Lie algebra.

Bi-invariant metrics on compact Lie groups may be constructed using either the averaging process described in footnote ³ of Ch. 2, or, given the background on Lie theory established just there, by using the Killing form and the classification of compact Lie algebras. It follows from the uniqueness of such Killing forms of simple Lie algebras, i.e. $\text{ad}(\mathfrak{g})$ -invariant products, that bi-invariant metrics on simple Lie groups are unique up to scalar multiples. On the other hand, it is obvious that all left-invariant metrics on abelian Lie groups are bi-invariant.

3.2.3 Inertia operator. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The *inertia operator* of another inner product Q is the symmetric positive-definite endomorphism $A: V \rightarrow V$ so that

$$Q(x, y) = \langle Ax, y \rangle$$

for all $x, y \in V$.

3.2.4 Convention. As every left-invariant metric Q on G is determined by its values on the Lie algebra \mathfrak{g} , the notions of *left-invariant metric on G* and *inner product on \mathfrak{g}* will be used interchangeably. Bi-invariant metrics are sometimes denoted by brackets $\langle \cdot, \cdot \rangle$, and, unless otherwise mentioned, *orthogonality*, *symmetry*, *transposition* and similar notions are to be understood with respect to a bi-invariant metric or the metrics derived from it.⁵ In particular, inertia operators are always taken with respect to bi-invariant metrics, unless otherwise mentioned.

3.2.5. Let Q_0 be a left-invariant metric on a compact Lie group G with Lie algebra \mathfrak{g} . Fix an orthonormal basis e_1, \dots, e_d of (\mathfrak{g}, Q_0) , interpreted as global orthonormal left-invariant frame. Take the orientation defined by (e_1, \dots, e_d) . Let $(\omega_1, \dots, \omega_d)$ be the dual frame. It follows that $\omega = \omega_1 \wedge \dots \wedge \omega_d$ is a volume form of (G, Q_0) and

$$\text{vol}(G, Q_0) = \int_G \omega.$$

If there is no danger of confusion regarding the underlying group G , then the volume of (G, Q_0) is denoted by $\text{vol } Q_0$.

Let Q be another left-invariant metric. To compare their volumes, let $A: \mathfrak{g} \rightarrow \mathfrak{g}$ be the inertia operator of Q with respect to Q_0 . This symmetric endomorphism may be diagonalised by an orthonormal basis e_1, \dots, e_d of (\mathfrak{g}, Q_0) with eigenvalues $\lambda_1, \dots, \lambda_d$. Then $\tilde{e}_1 = e_1/\sqrt{\lambda_1}, \dots, \tilde{e}_d = e_d/\sqrt{\lambda_d}$ form an orthonormal basis of (\mathfrak{g}, Q) . Let $(\omega_1, \dots, \omega_d)$ and $(\tilde{\omega}_1, \dots, \tilde{\omega}_d)$ be the corresponding dual frames, and ω and $\tilde{\omega}$ be the volume forms of Q_0 and Q , respectively. Then

$$\tilde{\omega} = \tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_d = \sqrt{\lambda_1 \dots \lambda_d} \cdot \omega_1 \wedge \dots \wedge \omega_d = \sqrt{\det A} \cdot \omega.$$

In particular, $\text{vol } Q_0 = \text{vol } Q$ if and only if $\det A = 1$.

3.3 Geodesics, Conjugate Points and Antipodes of Left-Invariant Metrics

Here, the notions of *antipodal point* and *thickness* are defined. The antipodal sets of Lie groups equipped with bi-invariant metrics are determined in terms of the integral lattice of their maximal tori. Moreover, the first conjugate points of bi-invariant metrics are computed.

3.3.1 Koszul formula. According to [Sak97, App. 2, Prop. 2.1], the Levi–Civita connection ∇ of a left invariant metric Q on a Lie group G with Lie algebra \mathfrak{g} satisfies

$$\nabla_X Y = \frac{1}{2}([X, Y] - \text{ad}_X^\top Y - \text{ad}_Y^\top X)$$

⁵The particular choice of Killing form is irrelevant to these notions.

for all left-invariant vector fields $X, Y \in \mathfrak{g}$, where \top marks the adjoint map with respect to Q , defined by the equation $Q(\text{ad}_Z^\top X, Y) = Q(X, \text{ad}(Z)Y)$ for all $X, Y, Z \in \mathfrak{g}$.

3.3.2. Let G be a compact connected Lie group equipped with a left-invariant metric Q . Let $c: \mathbb{R} \rightarrow G$ and define $X(t) = D(L_{c(t)})_e^{-1}(\dot{c}(t))$. Then c is a geodesic if and only if X solves the *Euler–Arnold equation*

$$\dot{X} = \text{ad}_X^\top X, \quad (\dagger)$$

where ad_X^\top denotes the adjoint map with respect to Q [MPMM11, §2]. If Q is a bi-invariant metric, then $\text{ad}_X^\top = -\text{ad}_X$, and therefore the Euler–Arnold equation is

$$\dot{X} = 0.$$

Thus, there exists $X_0 \in \text{LG}$ so that $X(t) = X_0$ for all t . It follows that $c(t)$ is an integral curve of the left-invariant vector field X_0 . In particular, the exponential map of the Lie group G and the riemannian exponential map of the bi-invariant metric Q coincide.

If G is semisimple, then the Euler–Arnold equation (\dagger) may be rephrased. Let Q_0 be a bi-invariant metric and let A be the inertia operator of Q with respect to Q_0 , i.e. the symmetric endomorphism of LG with $Q(X, Y) = Q_0(X, AY)$. Denote transposition with respect to Q_0 by $*$. It follows that the Q -adjoint map of ad satisfies

$$\text{ad}_X^\top = A^{-1} \circ \text{ad}_X^* \circ A = -A^{-1} \circ \text{ad}_X \circ A. \quad (\dagger\dagger)$$

Then the Euler–Arnold equation (\dagger) reads

$$\dot{X} = A^{-1}(\text{ad}_X^*(AX)) = A^{-1}([AX, X])$$

(cf. e.g. [Gue96, Sec. 2.1]).

3.3.3 Definitions. Let (M, g) be a compact riemannian manifold. The *antipodal set* $A_p M$ of a point $p \in M$ is the set of points of M with maximal distance to p . The elements of $A_p M$ are called *antipodes* or *antipodal points*. The *tangent antipodal set* $\tilde{A}_p M$ is the set of shortest vectors $v \in T_p M$ with $\exp(v) \in A_p M$. Sometimes the notation $\tilde{A}_p^q M$ for the set of shortest vectors $v \in T_p M$ with $\exp(v) = q$ is useful to have for *any two points* $p, q \in M$.

The antipodal set of symmetric spaces of compact type have been studied by Helgason [Hel65, §5] and others (see 6.2.4). Amongst other things, Helgason proves that $A_p M$ is a manifold if M is a symmetric space. This result will be rediscovered in 3.3.7 in the context of Lie groups.

For a compact connected Lie group G equipped with a left-invariant metric Q , define⁶

$$\begin{aligned} A(G, Q) &= A_e(G, Q), & \tilde{A}(G, Q) &= \tilde{A}_e(G, Q), \\ \mathcal{C}(G, Q) &= \mathcal{C}_e(G, Q), & \mathcal{C}^1(G, Q) &= \mathcal{C}_e^1(G, Q). \end{aligned}$$

⁶For definitions of $\mathcal{C}_e(G, Q)$ and $\mathcal{C}_e^1(G, Q)$, see 3.1.5.

Then the antipodal set, tangent antipodal set, tangent conjugate locus and first conjugate locus in $g \in G$ are determined by the sets above according to the formulas

$$\begin{aligned} A_g(G, Q) &= L_g(A(G, Q)), & \tilde{A}_g(G, Q) &= D(L_g)_e(\tilde{A}(G)), \\ \mathcal{C}_g(G, Q) &= D(L_g)_e(\mathcal{C}(G, Q)), & \mathcal{C}_g^1(G, Q) &= D(L_g)_e(\mathcal{C}^1(G, Q)). \end{aligned}$$

If the metric Q is obvious from the context, then $A(G)$ will be used to denote the antipodal set.

Furthermore, define $d_Q: G \rightarrow \mathbb{R}$, $d_Q(g) = d(e, g)$, which determines the riemannian distance function $d: G \times G \rightarrow \mathbb{R}$ associated to Q according to the formula

$$d(g, h) = d_Q(g^{-1}h)$$

for all $g, h \in G$.

The diameter of (G, Q) is equal to $\max_{g \in G} d_Q(g)$, or, using the notation above, $d_Q(g)$ for any $g \in A(G)$. If there is no danger of confusion regarding the underlying group G , then $\text{dia}(G, Q)$ is denoted by $\text{dia } Q$.

3.3.4. The *thickness*^{7,8} of (G, Q) is defined as

$$\theta(G, Q) = \frac{\text{dia}(G, Q)^d}{\text{vol}(G, Q)},$$

where d denotes the dimension of G . If there is no danger of confusion, then $\theta(G, Q)$ is denoted by $\theta(Q)$. Once the space $\tilde{\mathfrak{M}}$ is properly introduced, θ will be viewed as functional $\theta: \tilde{\mathfrak{M}}(G) \rightarrow \mathbb{R}_+$, as established in the introduction. Note that θ is invariant under scaling and pullbacks. In particular, θ may also be interpreted as function on the spaces \mathfrak{M} and \mathfrak{PM} defined below.

3.3.5 Proposition. *Let T be a torus with a left-invariant metric Q and integral lattice Λ . Then the following holds:*

- (1) *The holes of Λ are mapped to the local maxima of the distance function d_T under $\exp: LT \rightarrow T$. In particular, $\text{dia}(T, Q) = \mu(\Lambda, Q)$.⁹*
- (2) *$\text{vol}(T, Q) = (\det(\Lambda, Q))^{1/2}$ and $\theta(T, Q) = \theta(\Lambda, Q)$.*
- (3) *If $p \in T$, $X \in LT$ is a hole of Λ with $\exp X = p$ and P denotes the Delone cell of X , then¹⁰*

$$\tilde{A}_e^p T = \{X - v: v \in P^0\}.$$

⁷This notion of thickness corresponds to the notion of *normalised* thickness of lattices, but there is no natural fashion in which an analogue of Θ may be defined.

⁸Conway–Sloane’s reservations toward the term *thickness* apply here as well, cf. footnote ²⁶ of Ch. 1.

⁹On the left-hand side, Q is a left-invariant metric on T , but on the right-hand side it is a scalar product on LT .

¹⁰Here P^0 denotes the 0-skeleton of P .

Proof. (1) Note that for all $p \in T$ and $X \in LT$ with $\exp X = p$ the distance $d_Q(p)$ equals $d(X, \Lambda)$. Thus, if p is a local maximum of the distance function d_Q , then X is a local maximum of $d(\cdot, \Lambda)$. As X is contained in the Voronoi cell, it follows that X is a hole.

(2) The Voronoi cell \mathcal{V} is the Dirichlet fundamental domain of the action of Λ on LT by translation. It follows that $\text{vol } T = \text{vol } \mathcal{V} = (\det \Lambda)^{1/2}$.

(3) ‘ \supseteq ’: Let $v \in P^0$. It is straightforward to see that $P-v$ is another Delone polytope. In particular, $X-v$ is a hole of Λ . Moreover, $\exp(X-v) = \exp(X)\exp(v)^{-1} = \exp(X) = p$.

‘ \subseteq ’: If $Y \in \tilde{A}_e^p T$, then Y is a hole and there exists $v \in \Lambda$ with $Y = X-v$. It follows that $P-v$ is the Delone cell of Y . Since $0 \in P-v$ it follows that v is a vertex of P . \square

3.3.6 Proposition. *Let G be a compact connected Lie group with maximal torus T and bi-invariant metric Q . Then the following holds.*

(1) T is totally geodesic.

(2) T is geodesically convex, i.e. for every $g \in T$ there exists $\gamma \in \mathcal{G}_e^g(G)$ with $\gamma \subseteq T$. In fact, every $\gamma \in \mathcal{G}_e^g(G)$ is conjugate to some $\gamma' \in \mathcal{G}_e^g(T)$.

(3) $\tilde{A}(G) \cap LT = \tilde{A}(T)$. In particular, $\tilde{A}(G) \cap LT$ is the set of deep holes of the integral lattice of T .

Proof of Proposition 3.3.6. (1) Follows from the fact that T is a subgroup and 3.3.2.

(2) Let $g \in T$ and $X \in LG$ with $\exp X = g$ so that $\gamma(t) = \exp(tX)$ is a minimising geodesic joining e to g . By the conjugation theorem, there exists $a \in G$ with $\text{Ad}(a)(X) \in LT$. Note that $c(a)(g) = c(a)(\exp X) = \exp \text{Ad}(a)(X) \in T$.¹¹ As $c(a)(g)$ and g are conjugate, they are conjugate by an element w of $N_G(T)$: $c(w) \circ c(a)(g) = g$ [BtD85, Ch. IV, (2.5)]. Let $X' = \text{Ad}(wa)(X) \in LT$. Then $\exp X' = c(wa)(\exp X) = c(wa)(g) = g$.

(3) Follows from (1) and Prop. 3.3.5. \square

3.3.7. As a corollary, observe the following: Let G be a compact connected Lie group with bi-invariant metric Q_0 and maximal torus T . Denote the Weyl orbits of the antipodal points of the integral lattice of T by O_1, \dots, O_k and let $M_i = c(G)(O_i)$. Then each M_i is a homogeneous G -manifold and $M_i \cap M_j$ is empty if $i \neq j$ according to [BtD85, Ch. IV, (2.5)], already used in the proof above. It follows that

$$A(G, Q_0) = \bigcup_{i=1}^k M_i.$$

¹¹Recall that $c(a)(g) = aga^{-1}$ (conjugation from the left).

3.4 Jacobi Vector Fields of Bi-Invariant Metrics

3.4.1 Proposition. *Let (G, Q) be a compact connected semisimple Lie group equipped with a bi-invariant metric Q , $T \subseteq G$ a maximal torus with root system R and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ the corresponding real Cartan decomposition. Consider a geodesic $\gamma: \mathbb{R} \rightarrow T$ and let $H = \dot{\gamma}(0) \in \mathfrak{h}$. For every $\alpha \in R$, let $k_\alpha = \pi \cdot \alpha(H)$. Let $\mathfrak{g}_\alpha(t) = D(L_{\gamma(t)})_e(\mathfrak{g}_\alpha)$ for all $\alpha \in R$ and $\mathfrak{h}(t) = D(L_{\gamma(t)})_e(\mathfrak{h})$. Decompose every vector field Y along γ in the form*

$$Y(t) = Y_0(t) + \sum_{\alpha \in R^+} Y_\alpha(t),$$

where $Y_\alpha(t) \in \mathfrak{g}_\alpha(t)$ and $Y_0(t) \in \mathfrak{h}(t)$. Then the Jacobi equation is

$$\nabla^2 Y = - \sum_{\alpha \in R^+} k_\alpha^2 \cdot Y_\alpha.$$

Proof. Note that, according to 2.6.2, $\text{ad}(H)^2|_{\mathfrak{g}_\alpha} = -4k_\alpha^2 \cdot \text{Id}_{\mathfrak{g}_\alpha}$ for all $\alpha \in R^+$. Therefore, the Jacobi operator satisfies $R(\cdot, H, H) = k_\alpha^2 \cdot \text{Id}_{\mathfrak{g}_\alpha}$ on \mathfrak{g}_α (see [Sak97, App. 2]). On the other hand, $R(\cdot, H, H) = 0$ on \mathfrak{h} . Thus, the Jacobi equation is

$$\nabla^2 Y = - \sum_{\alpha \in R^+} k_\alpha^2 \cdot Y_\alpha. \quad \square$$

3.4.2 Proposition. *Assumptions as above. Define*

$$c_\alpha(t) = \cos(k_\alpha t), \quad s_\alpha(t) = \begin{cases} t, & k_\alpha = 0, \\ \frac{1}{k_\alpha} \sin(k_\alpha t), & k_\alpha \neq 0. \end{cases}$$

Let $V, W \in \mathfrak{g}$. Then the Jacobi field $Y: \mathbb{R} \rightarrow TG$ along γ with the initial conditions $Y(0) = V$ and $\nabla Y(0) = W$ is defined by the equation

$$Y(t) = \bar{V}_0(t) + t\bar{W}_0(t) + \sum_{\alpha \in R^+} c_\alpha(t)\bar{V}_\alpha(t) + \sum_{\alpha \in R^+} s_\alpha(t)\bar{W}_\alpha(t),$$

where the overline denotes parallel transport along γ .

3.4.3 Lemma. *Assumptions as above. Let $U \in \mathfrak{g}_\alpha$. Then the parallel transport \bar{U} along γ satisfies $\bar{U}(t) \in \mathfrak{g}_\alpha(t)$.*

Proof of Lemma 3.4.3. It is sufficient to verify this lemma for the elements E_α and F_α (see 2.6.2). But a simple calculation using $\nabla_H E_\alpha = k_\alpha F_\alpha$ and $\nabla_H F_\alpha = -k_\alpha E_\alpha$ shows that

$$\bar{E}_\alpha(t) = \cos(k_\alpha t)E_\alpha(\gamma(t)) - \sin(k_\alpha t)F_\alpha(\gamma(t)).$$

Thus $\bar{E}_\alpha(t) \in \mathfrak{g}_\alpha(t)$. In similar fashion, F_α is treated. □

Proof of Proposition 3.4.2. It is straightforward to verify the following equations:

$$\begin{aligned} c'_\alpha(t) &= -k_\alpha \sin(k_\alpha t), & c''_\alpha(t) &= -k_\alpha^2 c_\alpha(t), \\ s'_\alpha(t) &= \begin{cases} 1, & k_\alpha = 0, \\ \cos(k_\alpha t), & k_\alpha \neq 0, \end{cases} & s''_\alpha(t) &= \begin{cases} 0, & k_\alpha = 0, \\ -k_\alpha^2 s_\alpha(t), & k_\alpha \neq 0. \end{cases} \end{aligned}$$

By the lemma above, $Y_0 = \bar{V}_0 + t\bar{W}_0$ and $Y_\alpha = c_\alpha \bar{V}_\alpha + s_\alpha \bar{W}_\alpha$. The covariant derivatives of Y are

$$\begin{aligned} \nabla Y(t) &= \bar{W}_0(t) + \sum_{\alpha \in R^+} c'_\alpha(t) \bar{V}_\alpha(t) + \sum_{\alpha \in R^+} s'_\alpha(t) \bar{W}_\alpha(t), \\ \nabla^2 Y(t) &= \sum_{\alpha \in R^+} c''_\alpha(t) \bar{V}_\alpha(t) + \sum_{\alpha \in R^+} s''_\alpha(t) \bar{W}_\alpha(t) \\ &= - \sum_{\alpha \in R^+} k_\alpha^2 c_\alpha(t) \bar{V}_\alpha - \sum_{\alpha \in R^+, k_\alpha \neq 0} k_\alpha^2 s_\alpha(t) \bar{W}_\alpha(t) \\ &= - \sum_{\alpha \in R^+} k_\alpha^2 c_\alpha(t) \bar{V}_\alpha - \sum_{\alpha \in R^+} k_\alpha^2 s_\alpha(t) \bar{W}_\alpha(t) \\ &= - \sum_{\alpha \in R^+} k_\alpha^2 Y_\alpha(t). \end{aligned}$$

Thus, Y is a Jacobi vector field along γ . The initial conditions are easily verified. \square

3.4.4 Corollary. *Assumptions as above. The truncated Stiefel diagram*

$$LT'_s = \bigcup_{\alpha \in R^+, \ell \in \mathbb{Z} \setminus \{0\}} \alpha^{-1}\{\ell\}$$

is the intersection of the tangent conjugate locus $\mathcal{C}(G)$ with \mathfrak{h} . The first tangent conjugate locus $\mathcal{C}^1(G)$ is the boundary of the Voronoi cell of the coroot lattice R^\vee .

Proof. Suppose first that $H \in LT'_s$ and $\gamma(t) = \exp(tH)$. Take $\alpha \in R^+$ so that $\alpha(H) \in \mathbb{Z} \setminus \{0\}$, let $W(t) = \bar{E}_\alpha(t)$, and define

$$Y(t) = s_\alpha(t) \cdot W(t).$$

According to Prop. 3.4.1, Y is a Jacobi field along γ . Note, furthermore, that $Y \neq 0$, $Y(0) = 0$, $Y(1) = 0$. Thus $\gamma(1)$ is a conjugate point along γ .

Conversely, suppose that $H \in LT$ and let Y be a nonzero Jacobi vector field γ with $Y(0) = 0$, $Y(1) = 0$. According to Prop. 3.4.2,

$$Y(t) = \bar{V}_0 + t\bar{W}_0 + \sum_{\alpha} c_\alpha(t) \cdot \bar{V}_\alpha(t) + \sum_{\alpha} s_\alpha(t) \cdot \bar{W}_\alpha(t).$$

Note that $Y(0) = 0$ implies $V_0 = 0$ and $V_\alpha = 0$ for all $\alpha \in R^+$, and $Y(1) = 0$ implies $W_0 = 0$. Apply $Y \neq 0$ to see that there exists $\alpha \in R^+$ so that $s_\alpha(t) \cdot \bar{W}_\alpha(t)$ does not

vanish for all t . Then either $k_\alpha = 0$ and $s_\alpha(t) \cdot \bar{W}_\alpha(t) = t\bar{W}_\alpha(t)$ can never vanish again (a contradiction!), or $k_\alpha \neq 0$ and $s_\alpha(t) \cdot \bar{W}_\alpha(t) = (1/k_\alpha) \sin(k_\alpha t) \cdot \bar{W}_\alpha(t)$. Apply $Y(1) = 0$ to see that $k_\alpha \in \pi\mathbb{Z}$. But now recall that $k_\alpha = \pi\alpha(H)$. Thus, $\alpha(H) \in \mathbb{Z} \setminus \{0\}$. Therefore, $H \in LT'_s$.

As for the second claim, note that H lies in the boundary of the Voronoi cell of R^\vee if and only if $H \in LT'_s$ and $tH \notin LT'_s$ for all $t \in (0, 1)$, which is the case if and only if $\gamma(1)$, but none of the $\gamma(t)$ with $t \in (0, 1)$, is a conjugate point of γ . \square

3.5 Evolution of Geodesics on Lie Groups

In the following, the evolution of geodesics under a *variation of left-invariant metrics* $Q(s)$ on a compact connected Lie group G , i.e. a map into the space of left-invariant metrics on G , is studied. The main result of this section will be Lem. 3.5.2, which asserts that for every variation of geodesics $\gamma(s, t)$ whose endpoints $\gamma(s, 1)$ are contained in a maximal torus T , the variational vector field is, at least up to first order and viewed from within T , equal to a variation of geodesics in T .

3.5.1 Lemma. *Let (M, g) be a riemannian manifold with Levi-Civita connection ∇ , $c: \mathbb{R} \rightarrow M$ a smooth curve, T an (r, m) -tensor along c , and let X_1, \dots, X_m be vector fields along c . Suppose that $T(0) = 0$. Then*

$$\nabla T(0)(X_1(s), \dots, X_m(s)) = \left. \frac{\partial}{\partial s} T(s)(X_1(s), \dots, X_m(s)) \right|_{s=0}$$

Proof. Using local coordinates,

$$T(s) = \sum T_{j_1 \dots j_m}^{i_1 \dots i_r}(s) \cdot \partial_{i_1}(c(s)) \otimes \dots \otimes \partial_{i_r}(c(s)) \otimes dx^{j_1}(c(s)) \otimes \dots \otimes dx^{j_m}(c(s)).$$

Define $\nabla T_{j_1 \dots j_m}^{i_1 \dots i_r}$ by the equation

$$\nabla T(s) = \sum \nabla T_{j_1 \dots j_m}^{i_1 \dots i_r}(s) \cdot \partial_{i_1}(c(s)) \otimes \dots \otimes \partial_{i_r}(c(s)) \otimes dx^{j_1}(c(s)) \otimes \dots \otimes dx^{j_m}(c(s)).$$

Then, using the local expression for the covariant derivative of tensors [Sak97, Ch. II, Sec. 1, Ex. 6, p. 30], it follows that

$$\nabla T_{j_1 \dots j_m}^{i_1 \dots i_r}(0) = \frac{\partial T_{j_1 \dots j_m}^{i_1 \dots i_r}}{\partial s}(0).$$

Now let $\omega^1, \dots, \omega^r$ be covector fields. It follows from the equation above that

$$\begin{aligned} & \left. \frac{\partial}{\partial s} T(s)(X_1(s), \dots, X_m(s)) \right|_{s=0} (\omega^1(0), \dots, \omega^r(0)) \\ &= \left. \frac{\partial}{\partial s} \sum T_{j_1 \dots j_m}^{i_1 \dots i_r}(s) \cdot \omega_{i_1}^1(0) \dots \omega_{i_r}^r(0) \cdot X_1^{j_1}(s) \cdot \dots \cdot X_m^{j_m}(s) \right|_{s=0} \\ &= \sum \frac{\partial T_{j_1 \dots j_m}^{i_1 \dots i_r}}{\partial s}(s) \cdot \omega_{i_1}^1(0) \dots \omega_{i_r}^r(0) \cdot X_1^{j_1}(0) \cdot \dots \cdot X_m^{j_m}(0) \\ &= \nabla T(0)(X_1(0), \dots, X_m(0), \omega^1(0), \dots, \omega^r(0)). \quad \square \end{aligned}$$

3.5.2 Lemma. *Let G be a compact connected Lie group with Lie algebra \mathfrak{g} and $T \subseteq G$ a maximal torus with Cartan algebra \mathfrak{h} . Take a variation $Q(s)$ of a bi-invariant metric $Q_0 = \langle \cdot, \cdot \rangle$ with inertia operator $A(s)$ and let $X = A'(0) \in S_0^2(\mathfrak{g})$. Fix a smooth curve $p(s) = \exp H(s)$ in T , given in terms of a smooth map H into the Lie algebra \mathfrak{h} of T . Consider the family of geodesics $\tau(s, t) = \exp(tH(s))$ of $(T, Q(s))$ and suppose that $\gamma(s, t) = \exp^s(tV(s))$ is a variation so that each $\gamma(s, \cdot)$ is a geodesic of $(G, Q(s))$ that connects e with $p(s)$, and, moreover, that*

$$\frac{\partial \tau}{\partial t}(0, 0) = \frac{\partial \gamma}{\partial t}(0, 0).$$

Denote by $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ the orthogonal projection. Then

$$\pi \left(\left. \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial t}(s, 0) \right|_{s=0} \right) = \left. \frac{\partial}{\partial s} \frac{\partial \tau}{\partial t}(s, 0) \right|_{s=0}.$$

Proof. Define $T^s = \nabla^s - \nabla^0$. It is straightforward to verify that this is a symmetric left-invariant $(1, 2)$ -tensor. If each T^s is considered as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$, then the time derivative $T = (\partial/\partial s) T^s|_{s=0}$ satisfies

$$\begin{aligned} T(V, W) &= \left. \frac{\partial}{\partial s} \nabla_V^s W \right|_{s=0} \\ &= \frac{1}{2} (-X \circ \text{ad}(V)(W) + \text{ad}(V) \circ X(W) - X \circ \text{ad}(W)(V) + \text{ad}(W) \circ X(V)) \\ &= \frac{1}{2} ([V, XW] + [W, XV]), \end{aligned}$$

due to 3.3.1 and the formula (††) of 3.3.2. Now note that

$$0 = \frac{\nabla^s}{dt} \frac{\partial \gamma}{\partial t}(s, t) = (\nabla_{\partial \gamma / \partial t}^s \frac{\partial \gamma}{\partial t})(s, t) = T^s \left(\frac{\partial \gamma}{\partial t}(s, t), \frac{\partial \gamma}{\partial t}(s, t) \right) + (\nabla_{\partial \gamma / \partial t}^0 \frac{\partial \gamma}{\partial t})(s, t).$$

Define $\gamma(t) = \gamma(0, t)$ and $U(t) = (\partial \gamma / \partial s)(0, t)$. Let $\tilde{H} = H(0) = (\partial \tau / \partial t)(0, 0)$. Apply the connection $(\nabla^0/ds) \cdot |_{s=0}$ to the equation above, and use 3.5.1 with $c(s) = \gamma(s, t)$ and $T(s) = T^s(\gamma(s, t))$, and furthermore [Sak97, Ch. II, Lem. 2.2], the fact that the

Levi-Civita connection is torsion-free and $V(0) = (\partial\gamma/\partial t)(0,0) = \tilde{H}$ to obtain

$$\begin{aligned} 0 &= \frac{\nabla^0}{ds} T^s \left(\frac{\partial\gamma}{\partial t}(s,t), \frac{\partial\gamma}{\partial t}(s,t) \right) \Big|_{s=0} + \frac{\nabla^0 \nabla^0}{ds du} \frac{\partial\gamma}{\partial t}(s,u) \Big|_{u=t} \Big|_{s=0} \\ &= T \left(\frac{\partial\gamma}{\partial t}(0,t), \frac{\partial\gamma}{\partial t}(0,t) \right) + R^0(U(t), \frac{\partial\gamma}{\partial t}(0,t), \frac{\partial\gamma}{\partial t}(0,t)) + \frac{\nabla^0 \nabla^0}{du ds} \frac{\partial\gamma}{\partial t}(s,u) \Big|_{s=0} \Big|_{u=t} \\ &= [\tilde{H}, X\tilde{H}]_{\gamma(t)} + R^0(U(t), \tilde{H}_{\gamma(t)}, \tilde{H}_{\gamma(t)}) + \nabla\nabla U(t), \end{aligned}$$

where $\tilde{H} \in \mathfrak{h}$ is considered a left-invariant vector field and $\nabla = \nabla^0$ (per the convention of omitting sub- and superscripts if $s = 0$).

Define the following left-invariant vector fields

$$W = \frac{\partial}{\partial s} \frac{\partial\gamma}{\partial t}(s,0) \Big|_{s=0}, \quad \tilde{W} = \frac{\partial}{\partial s} \frac{\partial\tau}{\partial t}(s,0) \Big|_{s=0}.$$

In order to prove $\pi(W) = \tilde{W}$, one must show that $\langle W, A \rangle = \langle \tilde{W}, A \rangle$ holds for all $A \in \mathfrak{h}$. To this end, define the function

$$\rho: \mathbb{R} \rightarrow \mathbb{R}, \quad \rho(t) = \langle U(t), A_{\gamma(t)} \rangle$$

By noting that A is parallel along $\gamma(\cdot)$, and then using the skew-symmetry of the adjoint representation, the symmetries of the curvature tensor and the fact that abelian Lie groups are flat one obtains

$$\begin{aligned} \rho''(t) &= \langle \nabla\nabla U(t), A_{\gamma(t)} \rangle \\ &= -\langle [\tilde{H}, X\tilde{H}]_{\gamma(t)}, A_{\gamma(t)} \rangle - R^0(U(t), \tilde{H}_{\gamma(t)}, \tilde{H}_{\gamma(t)}, A_{\gamma(t)}) \\ &= \langle X\tilde{H}_{\gamma(t)}, [\tilde{H}, A]_{\gamma(t)} \rangle + R^0(\tilde{H}_{\gamma(t)}, A_{\gamma(t)}, \tilde{H}_{\gamma(t)}, U(t)) \\ &= 0. \end{aligned}$$

As $(\partial\gamma/\partial t)(\cdot, 0)$ is a vector field along the constant curve $\gamma(\cdot, 0) = e$, it follows together with an application of [Sak97, Ch. II, Lem. 2.2] that

$$W = \frac{\nabla}{ds} \frac{\partial\gamma}{\partial t}(s,0) \Big|_{s=0} = \frac{\nabla}{dt} \frac{\partial\gamma}{\partial s}(0,t) \Big|_{t=0} = \nabla U(0).$$

Moreover, using that the variation τ maps into the torus T , it follows that

$$U(1) = p'(0) = \frac{\partial\tau}{\partial s}(0,1) = \frac{\partial}{\partial s} \exp H(s) \Big|_{s=0} = \tilde{W}_{p(0)}.$$

Now apply the fact that ρ is affine, the left-invariance of Q , and $U(0) = 0$ to obtain

$$\langle W, A \rangle = \langle \nabla U(0), A_{\gamma(0)} \rangle = \rho'(0) = \rho(1) - \rho(0) = \langle \tilde{W}_{p(0)}, A_{p(0)} \rangle = \langle \tilde{W}, A \rangle. \quad \square$$

4 Moduli Spaces of Left-Invariant Metrics and Structures

In this chapter, spaces of left-invariant metrics are established and studied. The first section will serve as introduction and deal exclusively with tori. It is proved there that if T is a torus, then the problem of finding a (locally) thinnest left-invariant metric on T is equivalent to the (local) lattice covering problem of the integral lattice of T . In particular, basic lattice theory implies that there exists a globally thinnest metric on T , which yields half of Theorem A.

The second half of Theorem A is proved using the notion of Carnot–Carathéodory metric in Sec. 4.5. The proof is followed by a discussion of how the degenerations of these metrics may be captured by degenerations of the underlying Lie algebra in Sec. 4.6.

Before the proof of Theorem A is provided, a comprehensive account of some basic properties of the space of left-invariant metrics $\widetilde{\mathfrak{M}}(G)$ on compact connected simple Lie groups G and the corresponding moduli space

$$\mathfrak{M}(G) = \widetilde{\mathfrak{M}}(G)/\text{isometry}$$

is given in Sec. 4.1–4.3. It is shown that $\mathfrak{M} = \mathfrak{M}(G)$ arises as quotient by the group action of $\text{Aut}(G)$ on $\widetilde{\mathfrak{M}}$ via pullback, just as in the abelian case, and results reminiscent of the slice theorems for the riemannian moduli space are obtained.

4.1 The Moduli Space over Tori

4.1.1. This paragraph uses ideas from [Bes08, Sec. 12.18]. Recall that the moduli space of *riemannian structures* on a manifold M is defined as the quotient $\mathcal{M}/\text{Diff}(M)$ of the space of riemannian metrics \mathcal{M} by the right-action of the diffeomorphism group $\text{Diff}(M)$ by pull-back: $(g.f)(v, w) = f^*g(v, w) = g(Df(v), Df(w))$ for all $f \in \text{Diff}(M)$, $g \in \mathcal{M}$ [Bes08, Sec. 4.2]. The isotropy group of $g \in \mathcal{M}$ is the isometry group I_g . The *moduli space of left-invariant structures on T* should be defined in a similar way.

Denote by T a torus with integral lattice Λ and let $\widetilde{\mathfrak{M}}(T)$ be the space of left-invariant metrics on T , interpreted as the space of positive quadratic forms $\mathcal{S}_{>0}(LT)$. Define the equivalence relation \sim on $\widetilde{\mathfrak{M}}(T)$ by

$$Q_0 \sim Q_1 \Leftrightarrow (T, Q_0) \text{ is isometric to } (T, Q_1).$$

Can $\mathfrak{M}(T) = \widetilde{\mathfrak{M}}(T)/\sim$ be written as quotient of $\widetilde{\mathfrak{M}}(T)$ by an appropriate subgroup of the diffeomorphism group?

Indeed, if $f: (T, Q_0) \rightarrow (T, Q_1)$ is an isometry between left-invariant metrics, it must already be the composition of a left-multiplication with an automorphism. To prove this, let $a = f(e)^{-1}$ and apply the lifting lemma to $f_0 = L_a \circ f$ to obtain an isometry $\hat{f}: (LT, Q_0) \rightarrow (LT, Q_1)$ with $\hat{f}(\Lambda) = \Lambda$. As an isometry between inner product spaces with $\hat{f}(0) = 0$, \hat{f} must be a linear map. Thus $f_0 = L_a \circ f$ is an automorphism. In particular, $Q_0 \sim Q_1$ if and only if there exists an automorphism $f: T \rightarrow T$ with $f^*Q_1 = Q_0$. It follows that

$$\mathfrak{M}(T) = \widetilde{\mathfrak{M}}(T) / \text{Aut}(T).$$

4.1.2 Proposition. *Let T be a torus with integral lattice $\Lambda \subseteq LT$ and denote by $\mathcal{L}(LT)$ the space of lattices in LT with reference lattice Λ (see 1.1.4). Then $\widetilde{\mathfrak{M}}(T) = \mathcal{L}(LT)$. Moreover, $\theta(T, Q) = \Theta(\Lambda, Q)$ for all $Q \in \widetilde{\mathfrak{M}}(T)$. In particular, the problem of finding a (locally) thinnest torus is equivalent to the (local) lattice covering problem.*

4.1.3 Corollary. (1) *On every torus of dimension ≥ 2 , there exists a thinnest left-invariant metric on T .*

(2) *For $n \leq 5$, the thinnest left-invariant metric is unique and isometric to the quotient of the vector space of zero-sum vectors in \mathbb{R}^{n+1} by A_n^* .*

(3) *Let τ_n be the number of locally thinnest left-invariant metrics on a torus of dimension n . Then $\tau_2 = 1, \tau_3 = 1, \tau_4 = 3$ and $216 \leq \tau_5 \leq 218$.*

Of course, the statements made in the corollary above are of an exemplary nature. Every result obtained in the theory of lattices is translatable into a result on left-invariant metrics on tori. Note that, as already mentioned in the introduction, Theorem A shows that the existence of a (globally) thinnest metric is exclusive to abelian Lie groups.

Proof of Proposition 4.1.2. Follows from Prop. 3.3.5 and the remarks preceding Prop. 4.1.2. □

Proof of Corollary 4.1.3. By Prop. 4.1.2, it is sufficient to refer to proofs of the corresponding statements about lattices: (1) Proved by Hlawka in [Hla49, Satz 1].^{1,2} (2) Joint efforts of Bambah, Delone, Ryskov, and Baranovskii (see 1.1.5 or [CS99, Ch. 2, Sec. 1.3]). (3) See [CS99, Ch. 2, Sec. 1.3] ($n = 3, 4$) and [SV06, Thm. 11.1] ($n = 5$). A simple proof of (2) and (3) for $n = 2$ may be found in App. B. \square

4.1.4 Flat structures. It is useful to hint at the distinction between left-invariant structures and flat structures on tori. Let T be a torus. A *flat structure* on T is an isometry class of flat riemannian metrics. More precisely, using the right-action of the diffeomorphism group $\text{Diff}(T)$ on the set of flat riemannian metrics on T by pullback, the flat structures on T are the orbits of this action. If $f \in \text{Diff}(T)$ is sufficiently complicated, the pullback of a left-invariant metric need not be left-invariant. Cause for confusion is given, though: the pullback is a left-invariant metric of a Lie group whose base manifold is T but whose Lie group structure is the pullback f^*T .³ The argument of [Bes08, Sec. 12.18] shows that every flat structure is the pullback of a left-invariant metric (with respect to some fixed Lie group structure on T).

4.2 The Moduli Space over Simple Groups

4.2.1. Let G be a compact connected Lie group of dimension d . Define $\widetilde{\mathfrak{M}} \subseteq S^2(\text{LG})$ as the space of left-invariant metrics on G , which inherits its topology from $S^2(\text{LG})$. Note that $\widetilde{\mathfrak{M}}$ is open and convex. In particular, $\widetilde{\mathfrak{M}}$ is a smooth contractible manifold of dimension $d(d-1)/2$ with $T_Q\widetilde{\mathfrak{M}} = S^2(\text{LG})$.

Define the equivalence relation \sim on $\widetilde{\mathfrak{M}}$ by

$$Q_0 \sim Q_1 \Leftrightarrow (G, Q_0) \text{ is isometric to } (G, Q_1).$$

A *left-invariant riemannian structure* on G is an equivalence class of \sim . The space of left-invariant riemannian structures on G is $\mathfrak{M} = \widetilde{\mathfrak{M}}/\sim$.

4.2.2. Let G be a compact connected simple Lie group with Lie algebra \mathfrak{g} . It was mentioned in 4.1.1 and 4.1.4 that $\text{Diff}(G)$ does not act naturally on the space of left-invariant metrics. The left-invariance of Q merely implies that f^*Q is invariant under the maps $L_a \circ f^{-1}$, but not L_a . Yet, it was possible to show that the isometry classes of left-invariant metrics on a torus T are the elements of $\widetilde{\mathfrak{M}}(T)/\text{Aut}(T)$. The same can be achieved for simple Lie groups like G .

¹In [Hla49], $\varepsilon(f)$ denotes the *eccentricity* of a quadratic form f , the infimum over the covering radii $\varepsilon(\Gamma, f)$ over all lattices Γ , which is now called the thickness of f . Hlawka's *Satz 1* asserts that (for all n and all f) there exists a lattice Γ so that $\varepsilon(\Gamma, f) = \varepsilon(f)$.

²[Hla49] is often erroneously cited as having been published in 1953 in *Monatshefte für mathematische Physik*.

³The confusion is caused by denoting by T both the Lie group and the underlying manifold. This train of thought will be picked up in Sec. 4.6.

As in the case of tori, one must know what the isometries between left-invariant metrics on G are. D'Atri and Ziller proved that $Q_0, Q_1 \in \widetilde{\mathfrak{M}}$ are isometric if and only if there exists $\phi \in \text{Aut}(G)$ with $Q_0 = \phi^*Q_1$ [DZ79, Thm. 5].⁴ It follows immediately that

$$\mathfrak{M}(G) = \widetilde{\mathfrak{M}}(G) / \text{Aut}(G).$$

In terms of quadratic forms this may be expressed as $\mathfrak{M}(G) = \mathcal{S}_{>0}(\mathfrak{g}) / \text{Aut}(G)$ or, equivalently, $\mathfrak{M}(G) = \text{O}(\mathfrak{g}) \backslash \text{GL}(\mathfrak{g}) / \text{Aut}(G)$, cf. 1.1.4.

Kodama, Takahara and Tamaru define the *space of left-invariant metrics up to isometry* $\mathfrak{M}(G)$ as the quotient of $\widetilde{\mathfrak{M}}(G)$ by the action of $\text{Aut}(\mathfrak{g})$ [KTT11]. If G is simple, then their definition of $\mathfrak{M}(G)$ coincides with the one given here, with one rather odd exception. This will be discussed in the next paragraph.

4.2.3 The outer automorphism group. Recall that the *inner automorphism group* of a group G is defined as $\text{Inn}(G) = \{c(g) : g \in G\}$. The quotient $\text{Aut}(G) / \text{Inn}(G)$ is called the *outer automorphism group* $\text{Out}(G)$. Generally, an automorphism α of G is called *outer* if $\alpha \notin \text{Inn}(G)$, i.e. if $[\alpha] \in \text{Out}(G)$ is nontrivial. The inner and outer automorphism group of a Lie algebra are defined m.m.

For a simple Lie algebra \mathfrak{g} , the outer automorphism group is isomorphic to the automorphism group of the Dynkin diagram of the root system of \mathfrak{g} [Hel78, Ch. IX, 5.4] [Hel78, Ch. X, 3.29], i.e. the group of graph automorphisms preserving the numbers of edges and their decoration. Inspection of the Dynkin diagrams of the irreducible root systems reveals the isomorphism type of the respective automorphism groups. Concrete realisations (for the complexification) of these outer automorphisms are given for A_n, D_n in [Jac79, Ch. IX, Sec. 5] and for E_6 in [Jac71, Sec. 7].

Let G be a compact connected simple Lie group. Define $\ell : \text{Aut}(G) \rightarrow \text{Aut}(LG), \phi \mapsto L\phi$. If G is simply-connected, then ℓ is an isomorphism by standard Lie theory [War83, 3.27]. In particular, it follows that \mathfrak{M} is the moduli space of left-invariant metrics in the sense of Kodama, Takahara and Tamaru.

If G is not simply-connected, then ℓ might not be surjective. Let $\alpha \in \text{Aut}(LG)$. Fix the universal covering group \tilde{G} and let $Z \subseteq Z(\tilde{G})$ so that $G = \tilde{G}/Z$. Let $\phi \in \text{Aut}(\tilde{G})$ so that $L\phi = \alpha$. Obviously, ϕ only descends to a map $G \rightarrow G$ if and only if $\phi(Z) \subseteq Z$. In particular, $\alpha \in \text{im}(\ell)$ if and only if $\phi(Z) \subseteq Z$. It is sufficient to consider the case of outermorphisms. If G is not of type A_n or D_n , then $Z(\tilde{G})$ does not have any nontrivial subgroups. In that case, $\phi(Z) \subseteq Z$ is satisfied for trivial reasons.

If G is of type A_n , then $\text{Out}(G) = \mathbb{Z}_2$ with nontrivial element $X \mapsto \bar{X}$. Since complex conjugation preserves the p^{th} roots of unity for all p , it follows that

⁴Moreover, D'Atri and Ziller claim that every isometry between two left-invariant metrics is the composition of a left-translation and an automorphism of G . This is apparently true if neither metric is bi-invariant, but Miatello and Miatello proved that the isometry group of a bi-invariant metric is $L(G) \circ (\text{Aut}(G) \cup \text{Aut}(G)\eta) \circ R(G)$, where $\eta(g) = g^{-1}$ [MM90].

$\phi(Z) = Z$. If G is of type D_n with $n \neq 4$, then $\text{Out}(G) = \mathbb{Z}_2$ with nontrivial element $X \mapsto UXU^\top$, where $U \in \text{Pin}(2n)$ or $U \in \text{O}(2n)$, depending on whether $G = \text{Spin}(2n)$ or $G = \text{SO}(2n), \text{PSO}(2n)$. It follows that $\phi(Z) = Z$. However, if $n = 4$, then $\text{Out}(G) = S_3$, and there exists $\phi \in \text{Out}(G)$ so that ϕ permutes the three nontrivial elements of the Klein four group $Z(\text{Spin}(8)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ [Lou01, Sec. 8]. In particular, $\phi(Z) \neq Z$ if $1 \neq Z \subsetneq Z(\text{Spin}(8))$. Thus, the two definitions of the moduli spaces coincide, except if $G = \text{SO}(8)$.

4.2.4. Let G be a compact connected simple Lie group with Lie algebra \mathfrak{g} . Let \mathbb{R}_+ act on $\widetilde{\mathfrak{M}}$ by multiplication from the right and note that this action commutes with the action of $\text{Aut}(G)$. In analogy to [KTT11], one may therefore define

$$\mathfrak{PM} = \widetilde{\mathfrak{M}} / \mathbb{R}_+ \times \text{Aut}(G),$$

the space of left-invariant structures up to similarity. Equivalently, \mathfrak{PM} may be described as a quotient of a submanifold of left-invariant metrics of equal volume by the automorphism group. In fact, the gradient of the volume functional at $Q \in \widetilde{\mathfrak{M}}$ is a multiple of $Q \in T_Q \widetilde{\mathfrak{M}} = S^2(\mathfrak{g})$ [Ble79, Sec. 3]. Therefore, the subspace

$$\widetilde{\mathfrak{PM}} = \{Q \in \widetilde{\mathfrak{M}} : \text{vol } Q = 1\}$$

is a submanifold of $\widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{M}} / \text{Aut}(G) \rightarrow \mathfrak{PM}$ is bijective when restricted to $\widetilde{\mathfrak{PM}}$. Note that the thickness functional θ is well-defined on \mathfrak{PM} .

Fix a bi-invariant metric Q_0 on G and consider the bijection

$$A: \widetilde{\mathfrak{M}} \rightarrow \mathcal{S}_{>0}(\mathfrak{g}), \quad Q \mapsto A_Q$$

taking a metric Q to its inertia operator A_Q with respect to Q_0 . A simple computation shows that $A_{\phi^*Q} = L\phi^{-1} \circ A_Q \circ L\phi$ for all $Q \in \widetilde{\mathfrak{M}}$ and $\phi \in \text{Aut}(G)$. It follows that A is $\text{Aut}(G)$ -equivariant if $\text{Aut}(G)$ acts from the *right* on $\mathcal{S}_{>0}(\mathfrak{g})$ by conjugation. In particular, \mathfrak{M} is homeomorphic to $\mathcal{S}_{>0}(\mathfrak{g}) / \text{Aut}(G)$. By similar arguments, $\widetilde{\mathfrak{PM}}$ may be identified with $\text{SL}(\mathfrak{g}) \cap \mathcal{S}_{>0}(\mathfrak{g})$ and \mathfrak{PM} is homeomorphic to $\text{SL}(\mathfrak{g}) \cap \mathcal{S}_{>0}(\mathfrak{g}) / \text{Aut}(G)$.

Considering $\widetilde{\mathfrak{PM}}$ in place of $\widetilde{\mathfrak{M}}$ simplifies the treatment of θ . Instead of dealing with arbitrary *variations of left-invariant structures*, i.e. differentiable maps $\mathbb{R} \rightarrow \mathfrak{M}$, only *volume-preserving* variations need to be considered, i.e. those variations with $\text{vol } Q(s) = \text{const}$, and the local behaviour of θ under the variation is dictated entirely by the diameter. This is reflected in the isomorphism between the tangent space $T_{Q_0} \widetilde{\mathfrak{PM}}$ and the space $S_0^2(\mathfrak{g})$ of trace-free endomorphisms of \mathfrak{g} , which is obtained by taking the derivative of A .

In the proof of Theorem C, the following type of polar coordinates will be used: Let $Q_0 \in \mathfrak{PM}$ denote *any* left-invariant metric, and let $\mathfrak{S} \subseteq S_0^2(\mathfrak{g})$ be the unit sphere. Then

$$Q: \mathfrak{S} \times \mathbb{R} \rightarrow \widetilde{\mathfrak{PM}}, \quad Q(X, s)(x, y) = Q_0(x, \exp(sX)(y))$$

maps $\mathfrak{S} \times (-\varepsilon, \varepsilon)$ to a neighbourhood of Q_0 , and $Q(X, s)$ is the left-invariant metric with inertia operator $\exp(sX)$ with respect to Q_0 .

4.2.5. (With notation as in the last paragraph.) An *optimum* is a left-invariant metric Q_0 on G so that the set of $0 \neq X \in S_0^2(\mathfrak{g})$ for whom $\theta(Q(X, s))$ is decreasing is a zero set.

4.3 Slice Theorems for the Moduli Space over Simple Groups

4.3.1 Slices. Ebin proved in [Ebi68] that for every point g of the space of riemannian metrics \mathcal{M} of a manifold M , there exists a *slice* through g , i.e. an analytic⁵ submanifold S with $g \in S$ so that the following holds:

- (1) S is invariant under the isometry group I_g .
- (2) If $\phi \in \text{Diff}(M)$ and $\phi^*S \cap S \neq \emptyset$, then $\phi \in I_g$.
- (3) There exists a neighbourhood $\mathcal{U} \subseteq \text{Diff}(M)/I_g$ of the coset $I_g \in \text{Diff}(M)/I_g$ and a local cross-section $\chi: \mathcal{U} \rightarrow \text{Diff}(M)$ so that $\mathcal{U} \times S \rightarrow \mathcal{M}, (u, s) \mapsto \chi(u)^*s$ is a homeomorphism onto a neighbourhood of g .⁶ In particular, the induced map $S/I_g \rightarrow \mathcal{M}/\text{Diff}(M)$ is a homeomorphism onto an open neighbourhood of the coset of g in $\mathcal{M}/\text{Diff}(M)$.⁷

Ebin concludes that for all $h \in \mathcal{M}$ that are close to g , there exists $\phi \in \text{Diff}(M)$ with $I_h \subseteq \phi I_g \phi^{-1}$. In particular,

$$\mathcal{G} = \{g \in \mathcal{M} : I_g = \{\text{Id}\}\}$$

is an open subset of \mathcal{M} . Ebin proves that if $n \geq 2$, then \mathcal{G} is dense and therefore calls the elements of \mathcal{G} the *generic* riemannian metrics.

The first result is called a *slice theorem*. A *slice* of a G -manifold X through $x \in X$ is a subset $S \subseteq X$ so that the following holds [Bor60, Ch. VIII, 3.2]:⁸

- (1') $x \in S$ and S is invariant under the isotropy group H of x .
- (2') If $g \in G$ and $S.g \cap S = \emptyset$, then $g \in H$.

⁵ \mathcal{M} inherits a differentiable structure as open convex subset of $\mathcal{S}^2(M)$, cf. 3.1.1.

⁶ If G is a Lie group and H is a closed subgroup, then a *local cross-section* of the projection $\pi: G \rightarrow G/H$ is a smooth map $\chi: \mathcal{U} \rightarrow G$ defined on an open neighbourhood $\mathcal{U} \subseteq G/H$ of the coset H so that $\chi(H) = e$ and $\pi \circ \chi = \text{Id}$ [Bor60, Ch. VIII, 3.1]. The existence of local cross sections for (finite-dimensional) Lie groups in this sense follows from [War83, 3.58], where the differentiable structure of G/H is also described.

⁷In [Bes08, Sec. 12.22], the equal-volume version of this theorem is cited.

⁸This definition is equivalent to the one given in [Mic08, 6.14]. This can be seen using the arguments of the proof of [Bor60, Ch. VIII, 3.8].

- (3') If $\chi: \mathcal{U} \rightarrow G/H$ is a local cross-section, then the map $F: \mathcal{U} \times S \rightarrow X, F(u, x) = x \cdot \chi(u)$ is a homeomorphism onto an open set in X .

Note that (1)–(2) are special cases of (1')–(2'), provided $\text{Diff}(M)$ is considered to be a Lie group, while (3) and (3') differ slightly. It seems that Ebin's proof actually yields, alongside the existence of the local cross-section, an analogue of (3'), and that Ebin's formulation only serves the purpose of emphasising the existence of a local cross-section. (In the realms of finite-dimensional spaces, this existence result is well-known, see footnote ⁶.) The existence of slices in the finite-dimensional setting is also well-explored. In fact, according to [Bor60, Ch. VIII, 3.8], there exists a slice through each point of a G -manifold X if G is compact. Moreover, [Bor60, Ch. VIII, 3.6] asserts that $S/H \rightarrow X/G$ is a homeomorphism onto an open set of X/G .⁹ The goal of this section is to prove a similar theorem for the space of left-invariant metrics \mathfrak{M} on simple Lie groups using the action of $\text{Aut}(G)$, and to derive results about *generic* left-invariant metrics.

4.3.2. The structure of the isometry group of left-invariant metrics on compact Lie groups was already considered by Miatello–Miatello in [MM90]. They study the group $U = \{g \in G: c(g) \in I_Q\}$, which is a subgroup of our $\hat{I}_Q = \{\phi \in \text{Aut}(G): \phi^*Q = Q\}$, considered below. One of their theorems is: *Let G be a compact simple Lie group. If $\mathfrak{g} \neq \mathfrak{su}(2)$, then there exists a family of left-invariant metrics with $U = \{e\}$ [MM90, Prop. 3.2(1)], and if $\mathfrak{g} = \mathfrak{su}(2)$, then I_Q is disconnected and U contains a Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [MM90, Prop. 3.2(2)].*

Below in Thm. 4.3.4, a new proof of [MM90, Prop. 3.2(1)] is demonstrated, which relies on the slice theorem. It is also pointed out that the family from [MM90, Prop. 3.2(1)] is dense. Moreover, the new methods yield that, if $\mathfrak{g} = \mathfrak{su}(2)$ (or, equivalently, $\text{rk}(G) \neq 1$), then the generic metrics satisfy $\hat{I}_Q = \mathbb{Z}_2 \times \mathbb{Z}_2$, an improvement of [MM90, Prop. 3.2(2)].

4.3.3 Theorem. *Let G be a compact connected simple Lie group. For every $Q \in \widetilde{\mathfrak{M}}$, there exists a submanifold $S \subseteq \widetilde{\mathfrak{M}}$ with $Q \in S$ so that the following holds:*

- (1'') S is an \hat{I}_Q -manifold, where $\hat{I}_Q = \{\phi \in \text{Aut}(G): \phi^*Q = Q\}$.
- (2'') If $\phi \in \text{Aut}(G)$ and $\phi^*S \cap S \neq \emptyset$, then $\phi \in \hat{I}_Q$.
- (3'') There exists a neighbourhood $\mathcal{U} \subseteq \text{Aut}(G)/\hat{I}_Q$ of the coset $\hat{I}_Q \in \text{Aut}(G)/\hat{I}_Q$ and a local cross-section $\chi: \mathcal{U} \rightarrow \text{Aut}(G)$ so that $F: \mathcal{U} \times S \rightarrow \mathfrak{M}, F(u, s) = \chi(u)^*s$ is a homeomorphism onto an open neighbourhood of Q . In particular, the induced map $S/\hat{I}_Q \rightarrow \mathfrak{M}$ is a homeomorphism onto an open neighbourhood of the coset of Q in \mathfrak{M} .

The same is true if $\widetilde{\mathfrak{M}}$ and \mathfrak{M} are replaced with their normalised counterparts $\widetilde{\mathfrak{PM}}$ and \mathfrak{PM} , respectively.

⁹In terms of Michor's notation, the image of $S/H \rightarrow X/G$ is \mathcal{U}/G .

Proof of Theorem 4.3.3. Obviously the action of $\text{Aut}(G)$ on $\widetilde{\mathfrak{M}}$ is proper. It follows from [Bor60, Ch. VIII, 3.8] or [Mic08, 6.26, 6.27] that there exists a slice through each point of $\widetilde{\mathfrak{M}}$ and from [War83, 3.58] that there exists a local cross-section $\mathcal{U} \rightarrow \text{Aut}(G)$, $\mathcal{U} \subseteq \text{Aut}(G)/\hat{I}_Q$. \square

4.3.4 Theorem. *If G is a compact connected simple Lie group with $\text{rk}(G) \neq 1$, then (1) $\mathfrak{G} = \{Q \in \widetilde{\mathfrak{M}}: \hat{I}_Q = \{\text{Id}_G\}\}$ is open and dense in $\widetilde{\mathfrak{M}}$. If $\text{rk}(G) = 1$ instead, then (2) $\mathfrak{G} = \{Q \in \widetilde{\mathfrak{M}}: \hat{I}_Q = \mathbb{Z}_2 \times \mathbb{Z}_2\}$ is open and dense in $\widetilde{\mathfrak{M}}$.*

4.3.5 Lemma. *Suppose that G is a compact connected simple Lie group with $\text{rk}(G) \neq 1$ and Lie algebra \mathfrak{g} . Let Q_0 be a bi-invariant metric on G . Then there exists an orthonormal basis e_1, \dots, e_d of \mathfrak{g} so that the following holds: For every involution $\theta \in \text{Aut}(\mathfrak{g})$ with fixed point set \mathfrak{k} for which there exists a sequence $i_1 < \dots < i_k$ with $\text{span}\{e_{i_1}, \dots, e_{i_k}\} = \mathfrak{k}$, it follows that $\theta = \text{Id}$, and, in particular, $\mathfrak{k} = \mathfrak{g}$.*

Proof of Lemma 4.3.5. The proof uses E. Cartan's classification of compact symmetric spaces by symmetric pairs, i.e. pairs $(\mathfrak{g}, \mathfrak{k})$ consisting of a compact Lie algebra \mathfrak{g} and a subalgebra \mathfrak{k} that admits a complement \mathfrak{p} so that $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, or, equivalently, pairs (\mathfrak{g}, θ) , where θ is an involutive Lie algebra automorphism [Hel78, Ch. IV, §3] [Hel78, Ch. X, §3]. E. Cartan's list consists of all such pairs (\mathfrak{g}, θ) , where \mathfrak{g} is simple and θ is specified *up to conjugacy*. The resulting symmetric spaces are labeled by the type of the simple Lie algebra \mathfrak{g} , followed by a roman number. For instance, the spaces corresponding to $\mathfrak{g} = \mathfrak{su}(n+1)$ are AI, AII and AIII.

The idea of the proof is to consider orthonormal bases of \mathfrak{g} as points of $O(\mathfrak{g})$. Let \mathfrak{k} be a fixed point space of a nontrivial Lie algebra involution and $\tilde{e}_1, \dots, \tilde{e}_d$ be an orthonormal basis of \mathfrak{g} so that $\text{span}\{\tilde{e}_1, \dots, \tilde{e}_k\} = \mathfrak{k}$. If $(e_1, \dots, e_d) \in O(\mathfrak{g})$ and there exist $i_1 < \dots < i_k$ so that $\text{span}\{e_{i_1}, \dots, e_{i_k}\}$ is equal to some conjugate $\text{Ad}(g)(\mathfrak{k})$ of \mathfrak{k} , then there exists a permutation π , and transformations $U \in O(\mathfrak{k})$, $U' \in O(\mathfrak{p})$ so that

$$\begin{aligned} & (e_1, \dots, e_d) \\ &= (U \text{Ad}(g)\tilde{e}_{\pi(1)}, \dots, U \text{Ad}(g)\tilde{e}_{\pi(k)}, U' \text{Ad}(g)\tilde{e}_{\pi(k+1)}, \dots, \text{Ad}(g)U'\tilde{e}_{\pi(d)}). \end{aligned}$$

Thus, the sought-after basis lies in the complement

$$O(\mathfrak{g}) \setminus \bigcup_{\mathfrak{k}} [\mathfrak{k}],$$

where the union is taken over all \mathfrak{k} from Cartan's list and $[\mathfrak{k}]$ denotes the set of all $(e_1, \dots, e_d) \in O(\mathfrak{g})$ belonging to an orthonormal basis $(\tilde{e}_1, \dots, \tilde{e}_d)$ as above with $\text{span}\{\tilde{e}_1, \dots, \tilde{e}_k\} = \mathfrak{k}$. Let $K \subseteq G$ be the subgroup associated to \mathfrak{k} [Hel78, Ch. VII, §1]. To prove that the above set is nonempty, it is sufficient to prove that

$$\dim(O(\mathfrak{k}) \times O(\mathfrak{p})) + \dim(\text{Ad}(G)/\text{Ad}(K)) < \dim O(\mathfrak{g}).$$

Let $\ell = d - k$ and note that $k > 1$. In fact, if $k \leq 1$, then one may choose a Cartan subalgebra \mathfrak{h} with $\mathfrak{k} \subseteq \mathfrak{h}$ and the corresponding Cartan decomposition $\mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$, and deduce the following contradiction:

$$\mathfrak{k} \subsetneq \mathfrak{h} \subseteq \left[\bigoplus_{\alpha} \mathfrak{g}_{\pm\alpha}, \bigoplus_{\alpha} \mathfrak{g}_{\pm\alpha} \right] \subseteq [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

A straightforward computation shows that

$$\dim(\mathrm{O}(\mathfrak{k}) \times \mathrm{O}(\mathfrak{p})) + \dim(\mathrm{Ad}(G)/\mathrm{Ad}(K)) = \frac{1}{2}(d + d^2 - 2(\ell + 1)k).$$

The assumption on the rank of G implies that $d \geq 6$ (cf. Tab. 2.2). In particular, $(\ell + 1)k > d$, which implies the required estimate. \square

Proof of Theorem 4.3.4. (1) Suppose that $\mathrm{rk}(G) \neq 1$. By the principal orbit theorem, it is sufficient to prove that some left-invariant metric $Q \in \widetilde{\mathfrak{M}}$ with $\hat{I}_Q = \{\mathrm{Id}_G\}$ exists. Fix a bi-invariant metric Q_0 and let e_1, \dots, e_d denote a Q_0 -orthonormal basis as described in Lem. 4.3.5 above. Let $Q \in \widetilde{\mathfrak{M}}$ be the left-invariant metric with inertia operator

$$\begin{bmatrix} a_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & a_d \end{bmatrix},$$

where $a_1 < \dots < a_d$. *Claim.* For every isometry $\phi \in \hat{I}_Q$, the derivative $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ is an involution whose fixed point set \mathfrak{k} is spanned by a subset of $\{e_1, \dots, e_d\}$. According to Lem. 4.3.5, this will imply $\hat{I}_Q = \{\mathrm{Id}_G\}$.

To prove the claim, recall that ϕ is an isometry of the bi-invariant metric Q_0 by the D'Atri–Ziller theorem (see 4.2.2). It follows that

$$\theta(e_1) = \lambda_1 e_1 + \dots + \lambda_d e_d,$$

where $\sum_i \lambda_i^2 = 1$. This implies

$$Q(\theta(e_1), \theta(e_1)) = \sum_i a_i^2 \lambda_i^2 \geq a_1^2 \sum_i \lambda_i^2 = a_1^2,$$

with equality if and only if $|\lambda_1| = 1$. Thus, $\theta(e_1) = \pm e_1$. In particular, θ maps $e_1^\perp = \mathrm{span}\{e_2, \dots, e_n\}$ into itself. Proceed inductively to obtain that $\theta(e_i) = \pm e_i$ for all i . This implies exactly that θ is an involution whose fixed point set \mathfrak{k} is spanned by a subset of $\{e_1, \dots, e_d\}$, as advertised.

(2) Suppose that $\mathrm{rk}(G) = 1$, i.e. $\mathfrak{g} = \mathfrak{su}(2)$. Proceed as above, but with an arbitrary Q_0 -orthonormal basis e_1, e_2, e_3 , to obtain some $Q \in \widetilde{\mathfrak{M}}$ so that the derivative of every $\phi \in \hat{I}_Q$ is an involutive map that—one might say—*flips* some of the basis vectors e_1, e_2, e_3 and fixes the others. According to the list [Hel78, Ch. X, §2, p. 451–455], $\mathfrak{su}(2)$ admits

exactly one nontrivial involution up to conjugation, and it is straightforward to verify that there exist three involutions $\theta_i, i = 1, 2, 3$, whose fixed point sets are generated by e_1, e_2 and e_3 , respectively. In particular, the corresponding (-1) -eigenspaces of θ_i are

$$\mathfrak{p}_i = e_i^\perp = \text{span}\{e_j, e_k\}$$

if $\{i, j, k\} = \{1, 2, 3\}$. It follows that any two of these involutions generate a Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ that contains the remaining third involution. This is exactly the Klein four group that occurs in the theorem. Each of these involutions is an isometry of (\mathfrak{g}, Q) , and corresponds to an isometry of (G, Q) according to the remarks made in 4.2.3. \square

4.4 Ivanov's Diameter Covering Formula

4.4.1. The following estimate was proved by Ivanov, answering a question posed by Petrunin on *mathoverflow* [Iva10]: Let $p: \hat{X} \rightarrow X$ be a k -sheeted locally isometric covering of compact length spaces. Then

$$\text{dia } X \leq \text{dia } \hat{X} \leq k \cdot \text{dia } X.$$

Note that the left-hand side estimate is straightforward to obtain. The proof will not be discussed here. Instead, the following corollary will be drawn from Ivanov's estimate:

4.4.2 Corollary. *Let $p: (\hat{G}, \hat{Q}_0) \rightarrow (G, Q_0)$ be an isometric k -sheeted d -dimensional Lie group covering. Suppose that $\text{dia}(\hat{G}, \hat{Q}_0) = \text{dia}(G, Q_0)$. If Q_0 is a locally thinnest left-invariant metric, then \hat{Q}_0 is a locally thinnest left-invariant metric.*

Proof of Corollary 4.4.2. Consider the map

$$p^*: \widetilde{\mathfrak{M}}(G) \rightarrow \widetilde{\mathfrak{M}}(\hat{G}),$$

taking $Q \in \widetilde{\mathfrak{M}}(G)$ to the lift p^*Q . This map is a homeomorphism.¹⁰ Observe that $\text{vol}(\hat{G}, p^*Q) = k \cdot \text{vol}(G, Q)$ for all $Q \in \widetilde{\mathfrak{M}}(G)$. In particular, $p^*Q_0 = \hat{Q}_0$ and $\theta(\hat{G}, \hat{Q}_0) = \theta(G, Q_0)/k$.

Let \mathcal{U} be a neighbourhood of Q_0 in $\widetilde{\mathfrak{M}}(G)$ so that $\theta(Q) \geq \theta(Q_0)$ for all $Q \in \mathcal{U}$. Define $\hat{\mathcal{U}} = p^*\mathcal{U}$. It follows that

$$\theta(\hat{G}, p^*Q) = \frac{1}{k} \cdot \frac{\text{dia}(\hat{G}, p^*Q)^d}{\text{vol}(G, Q)} \geq \frac{1}{k} \cdot \theta(G, Q) \geq \frac{1}{k} \cdot \theta(G, Q_0) = \theta(\hat{G}, \hat{Q}_0)$$

for all $Q' = p^*Q \in \hat{\mathcal{U}}$ with $Q \in \mathcal{U}$. \square

4.4.3. The corollary above is the only result of this type that can be deduced directly from Ivanov's formula. For simple Lie groups, the hypothesis $\text{dia}(\hat{G}, \hat{Q}_0) = \text{dia}(G, Q_0)$ only applies to the covering $\text{Spin}(\ell) \rightarrow \text{SO}(\ell)$ for $\ell \geq 8$, where the corollary reveals an interesting dynamic between the two groups.

¹⁰If the Lie algebras LG and $L\hat{G}$ are identified using p , then $p^* = \text{Id}$.

4.5 Carnot–Carathéodory Metrics and Proof of Theorem A

This section contains a proof of Theorem A using the notion of *Carnot–Carathéodory metric*, defined below.

Theorem A. *Let G be a compact connected Lie group. Then $\inf_{Q \in \mathfrak{m}} \theta(Q) = 0$ if and only if G is non-abelian.*

4.5.1 Carnot–Carathéodory metrics. Let (M, g) be a connected riemannian manifold and H a subbundle of TM . A differentiable curve c in M whose velocity is at all times contained in H is called *horizontal*. Denote by \mathcal{H} the set of piecewise differentiable horizontal curves. The *Carnot–Carathéodory metric*¹¹ associated with H (or \mathcal{H}) is defined as

$$d_{\mathcal{H}}^g(p, q) = \inf\{L(c) : c \in \mathcal{H}, c(0) = p, c(1) = q\}$$

[Gro96, Sec. 0]. Note that the right-hand set might be empty.

Very little of [Gro96], which serves as main reference here, is required for the proof of Theorem A: *Chow’s theorem*, which asserts that if H satisfies a certain *Chow’s condition*, then $d_{\mathcal{H}}^g(p, q) < \infty$ for all $p, q \in M$, and the *ball-box theorem*, which asserts that (M, d_g) and $(M, d_{\mathcal{H}}^g)$ have the same topology if Chow’s condition is satisfied, where d_g denotes the ordinary distance function of (M, g) .

4.5.2 Chow’s theorem. Let X_1, \dots, X_m be smooth vector fields on the connected riemannian manifold (M, g) , define the sets of successive Lie brackets $\mathfrak{X}^{(0)} = \{X_1, \dots, X_m\}$,

$$\mathfrak{X}^{(k)} = \{[X, X_i] : X \in \mathfrak{X}^{(k-1)}, i = 1, \dots, m\},$$

$\mathfrak{X} = \bigcup_{k=1}^{\infty} \mathfrak{X}^{(k)}$. Define the *degree* $\deg(Z)$ of $Z \in \mathfrak{X}$ as the smallest k with $Z \in \mathfrak{X}^{(k)}$. Suppose that the following condition is satisfied: (C) $\text{span } \mathfrak{X} = TM$.¹² *Chow’s theorem* asserts that under this assumption, any two points of M may be joined by a piecewise smooth horizontal curve with respect to the subbundle generated by X_1, \dots, X_m [Gro96, Sec. 0.4, 1.1] [Bel96, Thm. 2.4].

Condition (C) is called *Chow’s condition* or *Hörmander’s condition* and is the key to approximating the theory of riemannian metrics. There is even a *Hopf–Rinow theorem* for Carnot–Carathéodory metrics belonging to distributions that satisfy Chow’s condition [Bel96, Thm. 2.7].

¹¹Other names for this type of metric are *sub-riemannian metric*, *singular riemannian metric* and *nonholonomic riemannian metric* [Bel96].

¹²Or, to be more precise, $\bigcup_{p \in M} \text{span}\{X(p) : X \in \mathfrak{X}, p \in M\} = TM$.

4.5.3 Ball-box theorem. Suppose that $\text{span}(\cup_{k=1}^d \mathfrak{X}^{(k)}) = TM$ and let $\cup_{k=1}^d \mathfrak{X}^{(k)} = \{Z_1, \dots, Z_m\}$. Define the *box*

$$\text{Box}(r) = \{(t_1, \dots, t_m) \in \mathbb{R}^m : |t_i| \leq r^{\deg(Z_i)}\}.$$

The *exponential map* $\exp_p: \mathbb{R}^m \rightarrow M$ at $p \in M$ is defined by letting $\exp_p(t)$ be the integral curve of $\sum_i t_i Z_i$ through p evaluated at 1. Provided that M is compact, \exp is well-defined. The *ball-box theorem* asserts that there exist strictly positive continuous functions $C, r_0: M \rightarrow \mathbb{R}_+$ so that

$$\exp_p(\text{Box}(p, C(p)^{-1} \cdot r)) \subseteq B_{\mathcal{H}}(p, r) \subseteq \exp_p(\text{Box}(p, C(p) \cdot r))$$

for all $p \in M, r \leq r_0(p)$, where $B_{\mathcal{H}}(p, r)$ denotes the r -ball the Carnot–Carathéodory metric around p . For details, see [Gro96, 0.5.A] and the references therein. It follows immediately that the topologies of (M, d) and $(M, d_{\mathcal{H}})$ coincide [Gro96, 0.5.A(a)].

Proof of Theorem A. By the application of Hlawka’s theorem made in Cor. 4.1.3, $\inf_{Q \in \mathfrak{M}} \theta(Q) \neq 0$ if G is abelian. Therefore, assume that G is non-abelian. Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ be the decomposition of the Lie algebra of G into its semisimple and abelian part, and $G = (S \times T)/Z$ the corresponding group decomposition, where S is a compact simply-connected semisimple Lie group (cf. 2.10.3). Note that, according to the trivial estimate in Ivanov’s formula, discussed in 4.4.1, it is sufficient to construct a variation Q_s on $\hat{G} = S \times T$ so that $\lim_{s \rightarrow \infty} \theta(Q_s) = 0$. In fact, if this is accomplished, then Q_s factors through the action of Z to a left-invariant metric \bar{Q}_s , and $\theta(\bar{Q}_s) \leq \theta(Q_s)/k$. Removing the quotient from the picture is necessary in order to construct Q_s as a product metric.

Since \mathfrak{g} is not abelian, it follows that $\dim \mathfrak{s} \geq 3$. Let $d = \dim G, k = \dim T, t = \dim \mathfrak{s}$. Fix a bi-invariant metric Q_0 on \hat{G} so that the direct sum decomposition above is orthogonal, and a basis X_1, \dots, X_d of \mathfrak{g} so that X_1, \dots, X_t is an orthonormal basis of \mathfrak{s} and X_{t+1}, \dots, X_d is an orthonormal basis of \mathfrak{z} . Let $H = X_1, \ell = d - 1$. Define the variation of left-invariant metrics Q_s by the variation of inertia operators

$$A_s = \begin{bmatrix} e^{2s} & \\ & e^{-2s/\ell} I_{\ell} \end{bmatrix}.$$

Note that Q_s is volume-preserving. Moreover, $\mathfrak{s} \perp \mathfrak{z}$ holds for all s , implying that $Q_s = Q_s^1 + Q_s^2$ is a product metric. The diameter, due to the Pythagorean theorem, is equal to $\sqrt{\text{dia}(S, Q_s^1)^2 + \text{dia}(T, Q_s^2)^2}$. Clearly the diameter of the torus factor approaches zero as $s \rightarrow \infty$. It remains to prove that $\text{dia}(S, Q_s^1) \rightarrow 0$.

For the vector fields X_2, \dots, X_t , let $\mathfrak{X} \subseteq \mathfrak{s}$ denote the span of their successive Lie brackets as in 4.5.2. Let $\mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{s}_{\alpha}$ denote a Cartan decomposition of \mathfrak{s} so that $X_1 \in \mathfrak{h}$. Then $\bigoplus_{\alpha} \mathfrak{s}_{\alpha} \subseteq \text{span } \mathfrak{X}^{(0)}$. According to 2.6.2, $\bigoplus_{\alpha} \mathfrak{s}_{\alpha}$ generates the Lie algebra \mathfrak{s} . It follows

that $\mathfrak{X} = \mathfrak{s}$.¹³ Therefore, the distribution $H(a) = D(L_a)_e(\mathfrak{s})$ satisfies Chow's condition (C). As the sub-riemannian distance $d_{\mathcal{H}}^{Q_0^1}$ is continuous, there exists some $c > 0$, so that $d_{\mathcal{H}}^{Q_0^1}(a) < c$ for all $a \in S$. This immediately implies

$$\text{dia}(S, Q_s^1) \leq \max_{a \in S} d_{\mathcal{H}}^{Q_s^1}(a) < e^{-s/\ell} c. \quad \square$$

4.6 Degenerations of Lie Algebras and Limit Points

If Q_k is a sequence in \mathfrak{PM} with $\theta(Q_k) \rightarrow 0$, then after rescaling, the Gromov–Hausdorff limit of the metric spaces (G, d_k) is a point.¹⁴ It is plain to see that this metric degeneration gobbles up the asymptotic (Lie algebra) structure of (\mathfrak{g}, Q_k) . As an example, consider the variation Q_s from the proof of Theorem A. How could the direction of H possibly be reconstructed from this limit? For studying what the limit points at the ‘bad end’ of \mathfrak{PM} look like, a different notion of convergence is required. This notion is supplied in the form of *Lie bracket degenerations* (or *Lie algebra degenerations*), found, for instance, in [Lau03] or [BS99], on which the following short exposition is based. The theorem below provides a description of how the particular variation (\mathfrak{g}, Q_s) degenerates. It turns out that the direction H may be reconstructed as generator of the centre of a Heisenberg algebra.

4.6.1. Let \mathfrak{g} be a d -dimensional vector space over \mathbb{R} . A Lie bracket μ on \mathfrak{g} is an element of the space $\Lambda^2(\mathfrak{g}^*) \otimes \mathfrak{g}$ of skew-symmetric maps $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Jacobi identity, which, when using the structure constants μ_{ij}^k as coordinates for $\Lambda^2(\mathfrak{g}^*) \otimes \mathfrak{g}$, is a polynomial equation. Therefore, the subspace $\mathcal{L} \subseteq \Lambda^2(\mathfrak{g}^*) \otimes \mathfrak{g}$ of all Lie brackets is a real algebraic variety. The group $\text{GL}(\mathfrak{g})$ acts on \mathcal{L} from the left by *pushforward*:

$$\text{GL}(\mathfrak{g}) \times \mathcal{L} \rightarrow \mathcal{L}, \quad \varphi \cdot \mu(X, Y) = \varphi(\mu(\varphi^{-1}(X), \varphi^{-1}(Y))).$$

The orbit of $\mu \in \mathcal{L}$ is denoted by $\mathcal{O}(\mu)$, and μ is said to *degenerate* to $\lambda \in \mathcal{L}$ if λ lies in the closure $\bar{\mathcal{O}}(\mu)$, or, equivalently, if there exists a sequence $\varphi_k \in \text{GL}(\mathfrak{g})$ so that $\lim_{n \rightarrow \infty} \varphi_k \cdot \mu = \lambda$. The degeneration is *nontrivial* if λ lies in the boundary $\partial \mathcal{O}(\mu)$.

4.6.2. Suppose that G_μ is a compact simply-connected Lie group with Lie algebra (\mathfrak{g}, μ) . Fix a left-invariant metric Q and take $\varphi \in \text{GL}(\mathfrak{g})$. Let $G_{\mu, \varphi}$ denote the compact simply-connected Lie group with Lie algebra $(\mathfrak{g}, \varphi \cdot \mu)$. It is a straightforward computation to show that the homomorphism

$$\Phi: (G_\mu, \varphi^* Q) \rightarrow (G_{\mu, \varphi}, Q)$$

¹³In fact, $\text{span } \mathfrak{X}^{(1)} = \mathfrak{s}$.

¹⁴See [Gro07, Ch. 3A] for a definition of the Gromov–Hausdorff distance.

with $L\Phi = \varphi$ is an isometry between riemannian manifolds, where the metric on the left (respectively right) denotes the left-invariant metric on G_μ (respectively $G_{\varphi.\mu}$) determined by the inner product φ^*Q (respectively Q). Thus, a change in the left-invariant structure may be shifted to a change in the Lie algebra structure.¹⁵

4.6.3 Theorem. *Let G be a non-abelian Lie group with Lie algebra (\mathfrak{g}, μ) . Let Q_s be the variation of left-invariant metric from the proof of Theorem A and let*

$$\varphi_s = \begin{bmatrix} e^s & \\ & e^{-s/\ell} I_\ell \end{bmatrix}$$

be a square root of the inertia operator of Q_s , with respect to the orthonormal basis X_1, \dots, X_d . Assume that (X_1, \dots, X_t) consists of an orthonormal basis X_1, \dots, X_n of a Cartan algebra \mathfrak{h} of the semisimple part \mathfrak{s} of \mathfrak{g} and orthonormal bases (E_α, F_α) of each root space $\mathfrak{s}_{\pm\alpha}$ belonging to the Cartan decomposition $\mathfrak{s} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{s}_{\pm\alpha}$ with respect to \mathfrak{h} .¹⁶ Let $H = X_1$, and define the numbers b_α^i by

$$[E_\alpha, F_\alpha] = \sum_{i=1}^n b_\alpha^i X_i$$

for all $i = 1, \dots, n$ and $\alpha \in R^+$. Then $\lim_{s \rightarrow 0} e^{-s-2(s/\ell)} \varphi_s.\mu = \lambda$, where the Lie bracket λ is defined by the equation

$$\lambda(E_\alpha, F_\alpha) = -\lambda(F_\alpha, E_\alpha) = b_\alpha^1 H,$$

for all $\alpha \in R^+$, and defined as zero on combinations of elements from distinct root spaces and the Cartan algebra.

Proof. Then $\varphi_s.\mu(H_0, H_1) = 0$ for all $H_0, H_1 \in \mathfrak{h}$,

$$\varphi_s.\mu(H', Y) = \begin{cases} e^{-s}[H', Y] & \text{if } H' \in \mathbb{R} \cdot H, \\ e^{s/\ell}[H', Y] & \text{if } H' \perp H, \end{cases}$$

for all $H' \in \mathfrak{h}$, $Y \in \bigoplus_{\alpha} \mathfrak{s}_\alpha$. Let $\alpha, \beta \in R^+$. Then $\varphi_s.\mu(E_\alpha, E_\beta) = \varphi_s.\mu(F_\alpha, F_\beta) = 0$, and, moreover, $\varphi_s.\mu(E_\alpha, F_\beta) = 0$ unless $\alpha = \beta$. Finally,

$$\varphi_s.\mu(E_\alpha, F_\alpha) = e^{s+2(s/\ell)} b_\alpha^1 X_1 + \sum_{i=2}^n e^{s/\ell} b_\alpha^i X_i.$$

Therefore, after correction with the factor $e^{-s-2(s/\ell)}$, all terms approach zero, except the constant term $e^{-s-2(s/\ell)} \varphi_s.\mu(E_\alpha, F_\alpha) = b_\alpha^1 H$, which is nonzero if and only if $b_\alpha^1 \neq 0$. \square

¹⁵Essentially, this is the effect that was avoided in previous sections by replacing $\text{Diff}(G)$ with $\text{Aut}(G)$.

¹⁶According to 2.4.2(3), $\kappa((\mathfrak{s}_{\mathbb{C}})_\alpha, (\mathfrak{s}_{\mathbb{C}})_\beta) \neq 0$ if and only if $\alpha = -\beta$.

4.6.4. A *Heisenberg (Lie) algebra* is a Lie algebra that takes the form $\mathbb{R} \cdot H \oplus \mathfrak{b}$, where H generates the centre and there exists a nondegenerate skew-symmetric bilinear form $l: \mathfrak{b} \times \mathfrak{b} \rightarrow \mathbb{R}$ so that $[U, V] = l(U, V) \cdot H$ [Kna02, p. 231]. According to Thm. 4.6.3, the Lie algebra $(\mathfrak{g}, \lambda, Q_0)$ is the direct sum $\mathfrak{N} \oplus \mathfrak{a}$ of a Heisenberg algebra \mathfrak{N} and an abelian ideal \mathfrak{a} , equipped with a certain inner product Q_0 . The dimension of \mathfrak{N} depends on the position of H relative to R^\vee : If H is not perpendicular to any coroot γ^\vee , i.e. $b_\gamma^1 \neq 0$ for all $\gamma \in R^\vee$, then $\dim \mathfrak{N} = |R^\vee| + 1$. For each coroot γ that H is perpendicular to, E_γ and F_γ slip from \mathfrak{N} into \mathfrak{a} , meaning that

$$\dim \mathfrak{N} = |\{\gamma \in R^\vee : H \perp \gamma\}| + 1.$$

By elementary properties of a root system, the latter case is *singular* in the sense that it only occurs if the element H , on which the construction of Q_s depends alone, is contained in a zero set of \mathfrak{g} .

The resulting algebra $\mathfrak{N} \oplus \mathfrak{a}$ is two-step nilpotent. Although $(\mathfrak{g}, \lambda, Q_0)$ is not flat [Mil76, Thm. 1.5], it is *almost flat* in the following sense: It does admit a variation $(\mathfrak{g}, \lambda_s, Q_0)$ so that the curvature of $(\mathfrak{g}, \lambda_s, Q_0)$, which is defined as the curvature of the simply-connected Lie group H_s with Lie algebra $(\mathfrak{g}, \lambda_s)$ and left-invariant metric Q_0 , is bounded independently of s ,¹⁷ and so that the limit $\lim_{s \rightarrow \infty} (\mathfrak{g}, \lambda_s, Q_0)$ is abelian [Fuk90, Thm.-Ex. 10.8] [Sak97, pp. 293–294]. In particular, compact quotients H_s/Γ_s are *almost flat* in the sense that

$$\text{dia}(H_s/\Gamma_s, Q_0)^2 \cdot \max K(H_s/\Gamma_s, Q_0) \rightarrow 0.$$

Cf. [Sak97, App. 6, Thm. 6.13].

A glance at the construction made in the proof of Theorem A shows that it can be modified by replacing $\mathbb{R} \cdot H$ with an arbitrary subspace of \mathfrak{h} . The resulting Lie algebra limit remains two-step nilpotent.

¹⁷Or, equivalently, the curvature of $(\mathfrak{g}, \lambda_s, Q_0)$ is defined by the same algebraic expressions that determine the curvature of H_s in terms of the adjoint representation [Sak97, App. 2].

5 Volume Variation of Maximal Tori

What is the evolution of the volume of maximal tori in compact connected simple Lie groups G ? More specifically, the following question is addressed in this chapter: Let G be a compact connected simple Lie group with Lie algebra \mathfrak{g} . Fix a bi-invariant metric Q_0 , let $0 \neq X \in S^2(\mathfrak{g})$ be a symmetric and trace-free endomorphism and consider a differentiable variation $Q(s)$ with $Q(0) = Q_0$, whose inertia operator $A(s)$ satisfies $A'(0) = X$. Does there exist a maximal torus so that¹

$$\left. \frac{d}{ds} \text{vol}(T, Q(s)) \right|_{s=0} > 0?$$

An affirmative answer would be quite useful. Whatever the answer, the implications on the problems surrounding Theorem C is significant. The question leads to the problem of understanding the representation

$$\Phi: G \rightarrow \text{GL}(\text{End}(\mathfrak{g})), \quad \Phi(g)(C) = \text{Ad}(g) \circ C \circ \text{Ad}(g)^{-1}.$$

Unfortunately, the answer is affirmative if and only if X is not contained in the orthogonal space of a certain G -module $\mathcal{P} \subseteq S_0^2(\mathfrak{g})$, see Prop. 5.1.5. Here $S_0^2(\mathfrak{g})$ denotes the space of trace-free symmetric endomorphisms of \mathfrak{g} .

The decomposition of $S^2(\mathfrak{g}_{\mathbb{C}})$ into irreducible $\mathfrak{g}_{\mathbb{C}}$ -modules,² called the Vogel decomposition, may therefore be used to acquire a partial answer to the question posed above. Using the natural real structure on $S^2(\mathfrak{g}_{\mathbb{C}})$, it is shown in Sec. 5.2 that $S^2(\mathfrak{g})$ is the sum of as much as sometimes four irreducible G -modules, and that in most cases $\mathcal{P} \neq S_0^2(\mathfrak{g})$. In particular, the answer to the question will often be negative. Luckily, vast computations involving the representation Φ , combined with the quantitative analysis of Bleicher's theorem (see Sec. 1.10), will grant something resembling a handle on the problem (see Sec. 5.3), as they show exactly which trace-free endomorphisms spawn undesirable deformations under which all maximal tori have constant volume up to first order on Lie groups of type A_n .

¹Here $\text{vol}(T, Q(s))$ denotes the volume of T with respect to the restriction $Q(s)|_T$.

²With respect to the complexified derived representation $(L\Phi)_{\mathbb{C}}$. Note that Φ is equivalent to a representation of \mathfrak{g} even if G is not simply-connected since $\Phi(Z(\tilde{G})) = 1$.

5.1 Volume Variation of Maximal Tori

5.1.1 Notation. Denote by G a compact connected Lie group, \mathfrak{g} its Lie algebra and by Φ the representation

$$\Phi: G \rightarrow \mathrm{GL}(\mathrm{End}(\mathfrak{g})), \quad \Phi(g)(C) = \mathrm{Ad}(g) \circ C \circ \mathrm{Ad}(g)^{-1}.$$

Fix a bi-invariant metric Q_0 on G . For every maximal torus T and $X \in S_0^2(\mathfrak{g})$, define the derivative $\partial_X \mathrm{vol}(T)$ by taking a variation $Q(s)$ with $Q(0) = Q_0$ whose inertia operator $A(s)$ satisfies $A'(0) = X$ and define

$$\partial_X \mathrm{vol}(T) = \left. \frac{d}{ds} \mathrm{vol}(T, Q(s)) \right|_{s=0}.$$

5.1.2 Proposition. *Let G be a compact connected Lie group, $T_0, T \subseteq G$ maximal tori, $g \in G$ so that $c(g)(T_0) = T$, $\pi: \mathrm{LG} \rightarrow \mathrm{LT}_0$ the orthogonal projection, $X \in S_0^2(\mathrm{LG})$. Then the following holds:*

- (1) $\Phi(g)(\pi): \mathrm{LG} \rightarrow \mathrm{LT}$ is the orthogonal projection onto LT .
- (2) $\pi \circ \Phi(g^{-1})(A(s))|_{\mathrm{LT}_0}$ is the inertia operator of $(T_0, c(g)^*Q(s))$.
- (3) $\partial_X \mathrm{vol}(T) = \frac{1}{2} \mathrm{vol}(T_0, Q_0) \cdot \mathrm{tr}(\pi \circ \Phi(g^{-1})(X))$.

Proof. (1) and (2) follow from straightforward computations using that $\mathrm{Ad}(g)$ is orthogonal.

To prove (3), recall that $(T_0, c(g)^*Q(s))$ is isometric to $(T, Q(s))$ and that $c(g)^*Q_0 = Q_0$. Apply 3.2.5 and Prop. 3.3.5(2) to get

$$\begin{aligned} \partial_X \mathrm{vol}(T) &= \left. \frac{d}{ds} \mathrm{vol}(T_0, c(g)^*Q(s)) \right|_{s=0} \\ &= \mathrm{vol}(T_0, Q_0) \cdot \left. \frac{d}{ds} \sqrt{\det(\pi \circ \Phi(g)^{-1}(A(s))|_{\mathrm{LT}_0})} \right|_{s=0} \\ &= \frac{1}{2} \mathrm{vol}(T_0, Q_0) \cdot \mathrm{tr}(\pi \circ \Phi(g)^{-1}(X)|_{\mathrm{LT}_0}) \\ &= \frac{1}{2} \mathrm{vol}(T_0, Q_0) \cdot \mathrm{tr}(\pi \circ \Phi(g)^{-1}(X)). \quad \square \end{aligned}$$

5.1.3 Proposition. *Let G be a simple Lie group and let $\mathfrak{g} = \mathrm{LG}$. Let $X \in S_0^2(\mathfrak{g})$, T_0 a maximal torus and $\pi: \mathfrak{g} \rightarrow \mathrm{LT}_0$ the orthogonal projection. Then either there exists a maximal torus T so that $\partial_X \mathrm{vol} T > 0$ or $\partial_X \mathrm{vol} T = 0$ for all maximal tori T . In the latter case, $\mathrm{tr}(\pi \circ \Phi(g)(X)) = 0$ for all $g \in G$.*

5.1.4 Lemma. *Let V be an inner product space, M a compact riemannian manifold and $f: M \rightarrow \mathrm{End}(V)$ a continuous map. Then $\int_M \phi \circ (f(m)) dm = \phi \circ \int_M f$ and $\mathrm{tr} \int_M f = \int_M \mathrm{tr}(f(m)) dm$ for every $\phi \in \mathrm{End}(V)$.*

Proof of Lemma 5.1.4. Let e_1, \dots, e_d be an orthonormal basis of V and let $\phi_{ij}, f_{ij}(m)$ denote the components of ϕ and $f(m)$, respectively. Then $\int_M f_{ij}(m) dm = (\int_M f(m) dm)_{ij}$. Both stated equations follow from this immediately. \square

Proof of Proposition 5.1.3. Let $Q(s)$ be a variation with inertia operator $A(s)$ so that $Q(0)$ is bi-invariant, and suppose that $A'(0) = X$. Note that the derivative of $\Phi(g^{-1})(A(s))$ at $s = 0$ is $\Phi(g^{-1})(X)$. It is sufficient to prove that

$$\int_G \operatorname{tr}(\pi \circ \Phi(g)(X)) dg = 0.$$

According to Prop. 5.1.2, this would immediately imply the desired result.

Define $X_0 = \int_G \Phi(g)(X) dg \in S^2(\mathfrak{g})$. Obviously $[X_0, \operatorname{Ad}(g)] = 0$ for all $g \in G$. Therefore, the adjoint representation of G on \mathfrak{g} preserves the X_0 -eigenspace decomposition of \mathfrak{g} . Since the adjoint representation is irreducible, it follows that X_0 is a multiple of the identity. But according to Lem. 5.1.4,

$$\operatorname{tr} X_0 = \int_G \operatorname{tr} \Phi(g)(X) dg = \int_G \operatorname{tr} X dg = 0.$$

Therefore, $X_0 = 0$. It follows, again using Lem. 5.1.4, that

$$\int_G \operatorname{tr}(\pi \circ \Phi(g)(X)) dg = \operatorname{tr} \int_G \pi \circ \Phi(g)(X) dg = \operatorname{tr}(\pi \circ X_0) = 0. \quad \square$$

Evidently the condition ' $\operatorname{tr}(\pi \circ \Phi(g)(X)) = 0$ for all $g \in G$ ' must be studied. The following lemma shows that this condition holds if and only if X belongs to a certain G -invariant orthogonal subspace \mathcal{P}^\perp of $S_0^2(\mathfrak{g})$ with respect to the scalar product defined in 1.8.4. Hence, the Vogel decomposition comes into play.

5.1.5 Proposition. *Assumptions as in Prop. 5.1.3. Let π_0 be the trace-free part of π and $X \in S_0^2(\mathfrak{g})$. Define $\mathcal{P} = \operatorname{span}\{\Phi(g)(\pi_0) : g \in G\}$.³ Then $\operatorname{tr}(\pi \circ \Phi(g)(X)) = 0$ for all $g \in G$ if and only if X is orthogonal to \mathcal{P} .*

Proof. The invariance of the trace under cyclic permutations yields

$$\begin{aligned} \operatorname{tr}(\pi \circ \Phi(g^{-1})(X)) &= \operatorname{tr}(\pi \circ \operatorname{Ad}(g)^{-1} \circ X \circ \operatorname{Ad}(g)) \\ &= \operatorname{tr}(X \circ \operatorname{Ad}(g) \circ \pi \circ \operatorname{Ad}(g)^{-1}) \\ &= \langle X, \Phi(g)(\pi_0) \rangle \end{aligned}$$

for all $g \in G$. The claim now follows from Prop. 5.1.2. \square

³ \mathcal{P} is the G -module generated by π_0 .

5.2 Vogel Decomposition

5.2.1 Introduction to the Vogel decomposition. Let \mathfrak{g} be a complex simple Lie algebra. In [Vog99], Vogel determined a decomposition of the space of symmetric tensors $S^2(\mathfrak{g})$ into irreducible \mathfrak{g} -modules with respect to the tensor product representation $\varphi = \text{ad} \otimes \text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}^{\otimes 2})$, now called the *Vogel decomposition*. Landsberg and Manivel set out to replace the computer-based proofs of dimension formulas of the irreducible modules due to Vogel in a series of articles and finally obtained the formulas in [LM06]. This short exposition is based on their article.

The gist of the Vogel decomposition is the following (exact formulation given below): The complex simple Lie algebras \mathfrak{g} and their Killing forms κ are parametrised by the *Vogel parameters* α, β, γ . The Vogel decomposition is

$$S^2(\mathfrak{g}) = \mathbb{C} \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma),$$

where \mathbb{C} denotes the trivial representation and each $Y_2(\cdot)$ may be described as an eigenspace of the *Casimir operator* (with respect to κ), whose eigenvalue is determined by the parameters α, β and γ , respectively. The largest module (dimensionwise) is $Y_2(\alpha)$, the *second Cartan power*, also denoted by $\mathfrak{g}^{(2)}$, which is usually defined as the cyclic module generated by the weight space of the highest weight 2ρ , where ρ is the maximal root of \mathfrak{g} .

The following paragraphs serve to introduce the preliminaries required for the exact formulation of the Vogel decomposition.

5.2.2 Tensor products and endomorphisms. If V is a vector space, then

$$V \otimes V^* \rightarrow \text{End}(V), \quad v \otimes \lambda \mapsto \{u \mapsto \lambda(u) \cdot v\}$$

is an isomorphism. If V is additionally equipped with a nondegenerate bilinear form, then V^* may be identified with V using the natural isomorphism $\sharp: V^* \rightarrow V$ and the isomorphism above turns into

$$V \otimes V \rightarrow \text{End}(V), \quad v_0 \otimes v_1 \mapsto \{u \mapsto \langle v_1, u \rangle \cdot v_0\}.$$

The inverse may be expressed in terms of a basis: If v_1, \dots, v_n is a basis of V and v'_1, \dots, v'_n the corresponding dual basis, then the inverse is

$$\text{End}(V) \rightarrow V \otimes V, \quad T \mapsto \sum_i T v_i \otimes v'_i.$$

The decomposition of $\text{End}(V)$ into the space of skew-symmetric and symmetric endomorphisms equals a decomposition of $V^{\otimes 2} = V \otimes V$ into the spaces of *skew-symmetric* and *symmetric tensors*, $\Lambda^2(V)$ and $S^2(V)$, respectively.⁴

⁴The space $\Lambda^2(V)$ is usually defined as a quotient of $V^{\otimes 2}$ by $S^2(V)$ [War83, 2.4].

5.2.3. The representation φ from above occurs, up to isomorphism of representations, as the complexification of the derivative of the representation $\Phi: G \rightarrow \mathrm{GL}(\mathfrak{g}_{\mathbb{R}})$ from the previous section, where G is a Lie group whose Lie algebra is the compact real form $\mathfrak{g}_{\mathbb{R}}$ of \mathfrak{g} . In fact, with the use of 5.2.2, one may show that Φ is isomorphic to

$$\tilde{\Phi}: G \rightarrow \mathrm{GL}(\mathfrak{g}_{\mathbb{R}}^{\otimes 2}), \quad \tilde{\Phi}(g)(x \otimes y) = \mathrm{Ad}(g)(x) \otimes \mathrm{Ad}(g)(y)$$

as follows. Let $f: \mathrm{End}(\mathfrak{g}_{\mathbb{R}}) \rightarrow \mathfrak{g}_{\mathbb{R}}^{\otimes 2}$ be the isomorphism from 5.2.2 and note that f does not depend on the choice of the basis (v_1, \dots, v_d) . Thus, one may use the orthonormal bases (e_1, \dots, e_d) and $(\mathrm{Ad}(g)(e_1), \dots, \mathrm{Ad}(g)(e_d))$ in place of (v_1, \dots, v_n) . This yields

$$\begin{aligned} f(\Phi(g)(T)) &= \sum_i \mathrm{Ad}(g) \circ T \circ \mathrm{Ad}(g)^{-1}(e_i) \otimes e_i \\ &= \sum_i \mathrm{Ad}(g) \circ T \circ \mathrm{Ad}(g)^{-1} \circ \mathrm{Ad}(g)(e_i) \otimes \mathrm{Ad}(g)(e_i) \\ &= \sum_i \mathrm{Ad}(g) \circ T(e_i) \otimes \mathrm{Ad}(g)(e_i) \\ &= (\mathrm{Ad} \otimes \mathrm{Ad})(g)(f(T)) \end{aligned}$$

for all $T \in \mathfrak{g}_{\mathbb{R}}^{\otimes 2}, g \in G$. As $L\tilde{\Phi}_{\mathbb{C}} = \varphi$, it follows that $L\Phi_{\mathbb{C}}$ is equivalent to φ .

5.2.4 Universal enveloping algebra. The *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra \mathfrak{g} over a field F is a unital associative algebra over F equipped with a map $f: \mathfrak{g} \rightarrow U(\mathfrak{g})$ so that

$$f([x, y]) = f(x)f(y) - f(y)f(x) \tag{5.2.1}$$

holds for all $x, y \in \mathfrak{g}$, and so that $(U(\mathfrak{g}), f)$ satisfies the following universal property: For any unital, associative algebra A over F and every vector space homomorphism $g: \mathfrak{g} \rightarrow A$ satisfying (5.2.1) with g in place of f , there exists a unique algebra homomorphism $\hat{g}: U(\mathfrak{g}) \rightarrow A$ so that $\hat{g} \circ f = g$ [Hum80, Sec. 17.2]. The universal enveloping algebra may be constructed explicitly as the quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ by the ideal

$$(x \otimes y - y \otimes x - [x, y]: x, y \in \mathfrak{g}).$$

It follows that every representation $\varphi: \mathfrak{g} \rightarrow A = \mathfrak{gl}(V)$ of \mathfrak{g} on a vector space V may be lifted to a map $\hat{\varphi}: U(\mathfrak{g}) \rightarrow A$ so that $\hat{\varphi}(x \otimes y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in \mathfrak{g}$.

5.2.5 Casimir element. Let v_1, \dots, v_d be a basis of \mathfrak{g} and v'_1, \dots, v'_n be the dual basis of \mathfrak{g} with respect to a Killing form κ of \mathfrak{g} . The *universal (quadratic) Casimir element* Ω of $U(\mathfrak{g})$ may be defined as the image of $\sum_i v_i \otimes v'_i$ in $U(\mathfrak{g})$ [Hum80, Sec. 22.1]. Ω may be described as the image of $\mathrm{Id}_{\mathfrak{g}}$ in $U(\mathfrak{g})$ using the identification $\mathfrak{g} \otimes \mathfrak{g} = \mathrm{End}(\mathfrak{g})$. The natural choice of a Cartan–Weyl basis, on the other hand, yields the following representation:⁵

$$\Omega = \sum_i H_i \otimes H'_i + \sum_{\theta \in R} X_{\theta} \otimes X_{-\theta}.$$

⁵This is the type of basis that Humphrey uses to define Ω [Hum80, Sec. 22.1].

Note that Ω depends on the choice of κ . If κ is replaced by $a\kappa$, then Ω is replaced by $a^{-1}\Omega$. Another important property of Ω is that it lies in the centre of $U(\mathfrak{g})$ [Hum80, Sec. 23.2].

The (*quadratic*) *Casimir operator* of a representation $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is $C(\varphi) = \hat{\varphi}(\Omega)$. Note that $[C(\varphi), \varphi(\mathfrak{g})] = 0$ since $\Omega \in Z(U(\mathfrak{g}))$. In particular, the eigenspaces of $C(\varphi)$ are φ -modules. Note that the particular scalar multiple of Ω does not affect the eigenspace decomposition of the Casimir operator.

5.2.6 Vogel decomposition and Cartan power. Let \mathfrak{g} be a complex simple Lie algebra and consider the representation $\varphi = \text{ad} \otimes \text{ad}$ of \mathfrak{g} on $\mathfrak{g}^{\otimes 2}$. It is straightforward to see that $\mathfrak{g}^{\otimes 2} = \Lambda^2(\mathfrak{g}) \oplus S^2(\mathfrak{g})$ is a decomposition into \mathfrak{g} -modules. According to [LM06, Sec. 1], there exist numbers $\alpha, \beta, \gamma, t = \alpha + \beta + \gamma$ so that

$$S^2(\mathfrak{g}) = \mathbb{C} \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma)$$

is a decomposition into irreducible \mathfrak{g} -modules, where \mathbb{C} is used to denote the trivial representation⁶ and $Y_2(\alpha), Y_2(\beta), Y_2(\gamma)$ are the eigenspaces of the Casimir operator $C = C(\varphi)$ with respect to the virtual⁷ eigenvalues $\lambda_\alpha = 4t - 2\alpha, \lambda_\beta = 4t - 2\beta, \lambda_\gamma = 4t - 2\gamma$ on $S^2(\mathfrak{g})$. The module $Y_2(\alpha)$ is the *Cartan power*, usually denoted by $\mathfrak{g}^{(2)}$, the cyclic module generated by the highest weight vector $X_\rho \otimes X_\rho$ of the highest weight 2ρ , where ρ is the maximal root of \mathfrak{g} [GW09, 5.5.19].⁸

The numbers α, β, γ are called the *Vogel parameters*, and each set of Vogel parameters defines a point of the *Vogel plane* $\mathbb{C}P^2/S_3$, which may be thought to parametrise complex Lie algebras [LM06, Sec. 1]. The Vogel parameters for the complex simple Lie algebras may be found in Tab. 5.1. Note that the families $\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$, etc. lie on affine lines in the Vogel plane. In addition to the Lie algebra structure, the Vogel parameters determine a Killing form κ on \mathfrak{g} . The obvious normalisation $\alpha = -2$ implies that all (complex) roots Θ of \mathfrak{g} satisfy $\kappa(\Theta, \Theta) = 2$, as Mkrtychyan observes [Mkr17, p. 238]. In particular, $\Theta^\sharp = \Theta^\vee$.

5.2.7 Uniqueness of the Vogel decomposition. The Vogel decomposition is the only possible decomposition of $S^2(\mathfrak{g})$ into irreducible \mathfrak{g} -modules, provided that the virtual eigenvalues above are pairwise distinct. In fact, if $S^2(\mathfrak{g}) = \bigoplus_i W_i$ is a decomposition into irreducible \mathfrak{g} -modules, then there exists an isomorphism $f: S^2(\mathfrak{g}) \rightarrow S^2(\mathfrak{g})$ that commutes with each $\varphi(X)$ and that maps each summand of the Vogel decomposition to a summand of $S^2(\mathfrak{g}) = \bigoplus_i W_i$ (see 2.3.2). It follows from the definition of $\hat{\varphi}$ that $C(\varphi)$ commutes with f . In particular, f preserves the eigenspaces of $C(\varphi)$, i.e. $f(Y_2(\alpha)) = Y_2(\alpha)$, and so on.

⁶ \mathbb{C} corresponds to the space generated by the identity with $\text{End}(\mathfrak{g})$.

⁷Landsberg and Manivel use the term *virtual* to emphasise that some of these eigenspaces may be zero, and therefore the corresponding virtual eigenvalue not a true eigenvalue.

⁸Goodman and Wallach do not use the term *Cartan power* or the notation $\mathfrak{g}^{(2)}$.

\mathfrak{g}	α	β	γ	t	λ_α	λ_β	λ_γ
$\mathfrak{sl}(2, \mathbb{C})$	-2	2	2	2	12	4	4
$\mathfrak{sl}(3, \mathbb{C})$	-2	3	2	3	16	6	8
$\mathfrak{sl}(n, \mathbb{C})$	-2	3	n	n	$4(n+1)$	$4(n-1)$	$2n$
$\mathfrak{so}(2n+1, \mathbb{C})$	-2	4	$2n-3$	$2n-1$	$8n$	$4(2n-3)$	$2(2n+1)$
$\mathfrak{sp}(n, \mathbb{C})$	-2	1	$n+2$	$n+1$	$4(n+2)$	$2(n+2)$	$2n$
$\mathfrak{so}(2n, \mathbb{C})$	-2	4	$2n-4$	$2(n-1)$	$4(2n-1)$	$8(n-2)$	$4n$
\mathfrak{e}_6	-2	6	8	12	52	36	32
\mathfrak{e}_7	-2	8	12	18	76	56	48
\mathfrak{e}_8	-2	12	20	30	124	96	80
\mathfrak{f}_4	-2	5	6	9	40	26	24
\mathfrak{g}_2	-2	10/3	8/3	4	20	28/3	32/3

Table 5.1: The Vogel parameters of the complex simple Lie algebras, and the corresponding virtual Casimir eigenvalues.

5.2.8 The real Vogel decomposition. Let $\mathfrak{g}_{\mathbb{R}}$ be the real form of \mathfrak{g} , so that $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{g}$. Note that the conjugation map

$$\mathcal{J} : S^2(\mathfrak{g}) \rightarrow S^2(\mathfrak{g}), \quad \mathcal{J}(z \otimes C) = \bar{z} \otimes C$$

is a *conjugate linear map* with $\mathcal{J}^2 = \text{Id}$. Moreover, \mathcal{J} commutes with $\tilde{\Phi}_{\mathbb{C}}$. Therefore, \mathcal{J} is a real structure of the representation

$$\Phi_{\mathbb{C}} : G \rightarrow \text{GL}(S^2(\mathfrak{g})), \quad \Phi_{\mathbb{C}}(g)(z \otimes C) = z \otimes \Phi(g)(C)$$

in the sense of 2.2.4. It follows immediately that there is a Vogel decomposition

$$S^2(\mathfrak{g}_{\mathbb{R}}) = \mathbb{R} \oplus \mathfrak{g}_{\mathbb{R}}^{(2)} \oplus Y'_2 \oplus Y''_2,$$

which is a unique decomposition of $S^2(\mathfrak{g}_{\mathbb{R}})$ into irreducible Φ -modules, up to permutation of the factors. This allows the usage of the Vogel decomposition in solving the problem of volume variation of maximal tori.

As seen in Prop. 5.1.5, the irreducible module \mathcal{P} generated by the trace-free part π_0 of the orthogonal projection $\pi : \mathfrak{g} \rightarrow \mathfrak{h}$ onto a Cartan subalgebra plays an important role in understanding the volume variation of maximal tori. In the complex setting, this module may be described as follows, using the Vogel decomposition:

\mathfrak{g}	$\dim Y_2(\alpha)$	$\dim Y_2(\beta)$	$\dim Y_2(\gamma)$
$\mathfrak{sl}(2, \mathbb{C})$	5		
$\mathfrak{sl}(3, \mathbb{C})$	27	8	
$\mathfrak{sl}(n, \mathbb{C}), n \geq 4$	$\frac{n^2(n-1)(n+3)}{4}$	$\frac{n^2(n+1)(n-3)}{4}$	$n^2 - 1$
$\mathfrak{so}(2n+1, \mathbb{C}), n \geq 2$	$\frac{(2n+3)(2n+1)(n+1)(n-1)}{3}$	$\frac{(2n+1)(2n-1)n(n-1)}{6}$	$(2n+3)n$
$\mathfrak{sp}(n, \mathbb{C}), n \geq 3$	$\frac{(2n+3)(2n+1)(n+1)n}{6}$	$\frac{(2n+3)(2n-1)n(n-1)}{3}$	$(2n+1)(n-1)$
$\mathfrak{so}(8, \mathbb{C})$	300		
$\mathfrak{so}(2n, \mathbb{C}), n \geq 5$	$\frac{(2n+1)(2n-3)(n+1)n}{3}$	$\frac{(2n-1)(2n-3)n(n-1)}{6}$	$(2n-1)(n+1)$
\mathfrak{e}_6	2430	650	
\mathfrak{e}_7	7371	1539	
\mathfrak{e}_8	27000	3875	
\mathfrak{f}_4	1053	324	
\mathfrak{g}_2	77	27	

Table 5.2: Dimensions of the \mathfrak{g} -modules occurring in the Vogel decomposition. Empty fields denote zero dimension.

5.2.9 Theorem. *Let \mathfrak{g} be a complex simple Lie algebra of rank r , and let \mathfrak{h} be a Cartan subalgebra with complex root system $R = R(\mathbb{C})$. Denote by $\pi_0 \in S_0^2(\mathfrak{g})$ the trace-free part of the orthogonal projection $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ (with respect to a Killing form). Then π_0 is an eigenvector of the Casimir operator $C: S^2(\mathfrak{g}) \rightarrow S^2(\mathfrak{g})$ of $\varphi = \text{ad} \otimes \text{ad}$ if and only if R contains only short roots, i.e. R is of ADE type. Moreover, if this is the case, then $\pi_0 \in Y_2(\alpha)$, i.e. the Cartan power is the \mathfrak{g} -module generated by π_0 .*

5.2.10 Lemma. *There exists a number c so that $\sum_{\theta \in R} \theta(H) \cdot \theta^\vee = c \cdot H$ for all $H \in \mathfrak{h}$.*

Proof of Lemma 5.2.10. The map $f: \mathfrak{h} \rightarrow \mathfrak{h}, f(H) = \sum_{\theta \in R} \theta(H) \cdot \theta^\vee$ is linear and commutes with the action of the Weyl group W : In fact, for each $w \in W$

$$f(w(H)) = \sum_{\theta \in R} \kappa(w^{-1}(\theta)^\sharp, H) \cdot \theta^\vee = \sum_{\theta \in R} \kappa(\theta^\sharp, H) \cdot w(\theta^\vee) = w(f(H)).$$

As the Weyl groups acts irreducibly, it is sufficient to prove that there exists $0 \neq H \in \mathfrak{h}$ with $f(H) = c \cdot H$ for some $c \in \mathbb{R}$. This can be done for each root system individually. The results are found in Tab. 5.3. \square

R	A_n	B_n	C_n	D_n	E_8	E_7	E_6	F_4	G_2
c	$2(n+1)$	$4n$	$4n$	$4(n-1)$	60	36	24	24	12

Table 5.3: Each root system and the dilatation of the map f .

5.2.11 Lemma. *Let H_1, \dots, H_r be a basis of \mathfrak{h} and denote its dual basis by H'_1, \dots, H'_r . Then $\kappa(\Theta^\sharp, \Theta^\sharp) = \sum_k \Theta(H_k)\Theta(H'_k)$ for all $\Theta \in \mathfrak{h}^*$.*

Proof of Lemma 5.2.11. Θ may be written in the following two ways:

$$\Theta = \sum_k \Theta(H_k) \cdot (H'_k)^\flat, \quad \Theta = \sum_k \Theta(H'_k) \cdot (H_k)^\flat.$$

Substituting each formula once yields

$$\kappa(\Theta^\sharp, \Theta^\sharp) = \sum_{k,\ell} \Theta(H_k) \Theta(H'_\ell) \langle H_k, H'_\ell \rangle = \sum_k \Theta(H_k)\Theta(H'_k). \quad \square$$

Proof of Theorem 5.2.9. For each $\Theta \in R$ take $X_\Theta \in \mathfrak{g}_\Theta$ and $X_{-\Theta} \in \mathfrak{g}_{-\Theta}$ with $\kappa(X_\Theta, X_{-\Theta}) = 1$ and let H_1, \dots, H_r be any basis of \mathfrak{h} . Together, these vectors form a basis. Note that $X'_\Theta = X_{-\Theta}$. The Casimir element is $\Omega = \sum_i H_i \otimes H'_i + \sum_{\Theta \in R} X_\Theta \otimes X_{-\Theta}$, and the corresponding Casimir operator is

$$C = \sum_i \varphi(H_i) \circ \varphi(H'_i) + \sum_{\Theta \in R} \varphi(X_\Theta) \circ \varphi(X_{-\Theta}).$$

Note that $\pi = \sum_i H_i \otimes H'_i$ and

$$\pi_0 = \pi - \frac{r}{d} \cdot \Omega = \frac{d-r}{d} \cdot \pi - \frac{r}{d} \cdot \Xi,$$

where $d = \dim \mathfrak{g}$ and $\Xi = \sum_{\Theta \in R} X_\Theta \otimes X_{-\Theta}$. As Ω belongs to the irreducible module previously denoted by \mathbb{C} , it follows that $C(\Omega) = 0$. In particular, $C(\pi_0) = C(\pi)$. Through the usage of 2.4.2(3) and the previous lemmas, it follows that

$$\begin{aligned} C(\pi_0) &= \sum_k \sum_{\Theta \in R} \varphi(X_\Theta) \circ \varphi(X_{-\Theta})(H_k \otimes H'_k) \\ &= \sum_k \sum_{\Theta \in R} [X_\Theta, [X_{-\Theta}, H_k]] \otimes H'_k + H_k \otimes [X_\Theta, [X_{-\Theta}, H'_k]] \\ &\quad + \sum_k \sum_{\Theta \in R} [X_\Theta, H_k] \otimes [X_{-\Theta}, H_k] + [X_{-\Theta}, H_k] \otimes [X_\Theta, H'_k] \\ &= \sum_k \sum_{\Theta \in R} \Theta(H_k) \cdot [X_\Theta, X_{-\Theta}] \otimes H'_k + \Theta(H'_k) \cdot H_k \otimes [X_\Theta, X_{-\Theta}] \\ &\quad - \sum_k \sum_{\Theta \in R} (\Theta(H_k)\Theta(H'_k) \cdot X_\Theta \otimes X_{-\Theta} + \Theta(H_k)\Theta(H'_k) \cdot X_{-\Theta} \otimes X_\Theta) \\ &= \sum_k \sum_{\Theta \in R} (\Theta(H_k) \cdot \Theta^\vee \otimes H'_k + \Theta(H'_k) \cdot H_k \otimes \Theta^\vee) \\ &\quad - 2 \sum_k \sum_{\Theta \in R} \Theta(H_k)\Theta(H'_k) \cdot X_\Theta \otimes X_{-\Theta} \\ &= 2c \cdot \pi - 2 \sum_{\Theta \in R} \kappa(\Theta^\vee, \Theta^\vee) \cdot X_\Theta \otimes X_{-\Theta}. \end{aligned}$$

Furthermore, using $\pi = d/(d-r) \cdot \pi_0 + r/(d-r) \cdot \Xi$, it follows that

$$\begin{aligned} C(\pi_0) &= 2c \cdot \pi - 2 \sum_{\Theta \in R} \kappa(\Theta^\vee, \Theta^\vee) \cdot X_\Theta \otimes X_{-\Theta} \\ &= 2c \cdot \frac{d}{d-r} \cdot \pi_0 + 2 \left(c \cdot \frac{r}{d-r} \cdot \Xi - \sum_{\Theta \in R} \kappa(\Theta^\vee, \Theta^\vee) \cdot X_\Theta \otimes X_{-\Theta} \right). \end{aligned}$$

This implies that if π_0 is an eigenvector of C , then $\sum_{\Theta \in R} \kappa(\Theta^\vee, \Theta^\vee) \cdot X_\Theta \otimes X_{-\Theta}$ is a multiple of Ξ , i.e. $\kappa(\Theta^\vee, \Theta^\vee)$ is the same for all $\Theta \in R$. Note that in this case $\sum_{\Theta \in R} \kappa(\Theta^\vee, \Theta^\vee) \cdot X_\Theta \otimes X_{-\Theta} = 2\Xi$.

Conversely, it is possible to verify for all root systems of ADE type that $c \cdot \frac{r}{d-r} = 2$ and $2c \cdot \frac{d}{d-r} = \lambda_\alpha$. Together, these statements imply that $\pi_0 \in Y_2(\alpha)$. \square

5.2.12 Corollary. *Let \mathfrak{g} be a real simple Lie algebra, and let \mathfrak{h} be a Cartan subalgebra with root system R . Denote by $\pi_0 \in S_0^2(\mathfrak{g})$ the trace-free part of the orthogonal projection $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ (with respect to the Killing form). Then $\pi_0 \in Y_2(\alpha)$ if and only if R contains only short roots, i.e. R is of ADE type.*

It follows from Cor. 5.2.12 that π_0 generates $S_0^2(\mathfrak{g})$ for the exceptional Lie algebras \mathfrak{f}_4 and \mathfrak{g}_2 . In fact, as π_0 is not an eigenvector of the Casimir operator, it follows, using the uniqueness of the Vogel decomposition and $Y_2(\gamma) = 0$, that π_0 must generate $S_0^2(\mathfrak{g})$. For most other Lie algebras, like $\mathfrak{su}(n+1)$ for $n \geq 2$, $\mathfrak{so}(2n)$ for $n \geq 4$, and $\mathfrak{e}_8, \mathfrak{e}_7, \mathfrak{e}_6$, the operator π_0 only generates the proper submodule $Y_2(\alpha)$. In particular, the corresponding Lie groups admit the awkward property that there exists $0 \neq X \in S_0^2(\mathfrak{g})$ so that $\partial_X \text{vol}(T) = 0$ for all maximal tori T , according to Prop. 5.1.5. Other groups, like F_4 or G_2 , do not admit such deformations.

The following proposition summarises the uses of the Vogel decomposition to the representation Φ specifically for Lie groups with $\mathfrak{g} = \mathfrak{su}(n)$. Of course, it is possible to write down similar results for every other Lie group, but since those of type A_{n-1} are the main focus in Theorem C, these will not be stated.

5.2.13 Lemma. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$, $n \geq 4$ and let $S^2(\mathfrak{g}) = \mathbb{R} \oplus \mathfrak{g}^{(2)} \oplus Y' \oplus Y''$ denote the Vogel decomposition of $S^2(\mathfrak{g})$. Let $0 \neq X \in S^2(\mathfrak{g})$ be trace-free. Then (1)*

$$\mathcal{Y} = \text{span}\{\Phi(g^{-1})(X) : g \in G\}$$

is one of the following: $\mathfrak{g}^{(2)}, \mathfrak{g}^{(2)} \oplus Y', \mathfrak{g}^{(2)} \oplus Y'', S_0^2(\mathfrak{g}), Y', Y'', Y' \oplus Y''$, and (2) $\partial_X \text{vol}(T) = 0$ for all maximal tori T if and only if $\mathcal{Y} = Y', Y''$ or $Y' \oplus Y''$.

Proof of Lemma 5.2.13. (1) follows immediately from elementary properties of the Vogel decomposition (see 5.2.6) and the fact that \mathcal{Y} is a G -invariant subspace.

(2) Denote by π_0 the trace-free part of the projection $\mathfrak{g} \rightarrow \mathfrak{h}$. It follows from Prop. 5.1.3 and Prop. 5.1.5 that $\partial_X \text{vol}(T) = 0$ for all maximal tori if and only if X is orthogonal to $\mathcal{P} = \text{span}\{\Phi(g)(\pi_0) : g \in G\}$, which, according to Thm. 5.2.9, is the Cartan power $\mathfrak{g}^{(2)}$. \square

5.3 Elements of $S_0^2(\mathfrak{su}(n))$

As already mentioned, the existence of nonzero $X \in S_0^2$ so that the volume derivative vanishes for all maximal tori poses a problem for the proof of Theorem C. In the case of $\mathfrak{g} = \mathfrak{su}(n)$, those X are the elements of $Y_2(\beta) \oplus Y_2(\gamma)$. The space of directions $S_0^2(\mathfrak{g})$ is therefore classified into two types of directions, referred to as *good* and *bad*, to pick up the terminology of Sec. 1.10. In the following, elements $S' \in Y_2'$ and $S'' \in Y_2''$ are determined. Later, the effect of bad directions on the standard maximal torus of $\text{PSU}(n)$ will be studied (see Lem. 6.5.1): While S' corresponds to a good direction in the sense of Sec. 1.10 (a nonzero element of \mathcal{S}^0), S'' corresponds to a bad direction (a nonzero element of \mathcal{X}).

5.3.1 Proposition. *Let $n \geq 4$. Let π_0 be the trace-free part of the orthogonal projection $\mathfrak{su}(n) \rightarrow \mathfrak{h}$, and*

$$\begin{aligned} S' = & \tilde{H}_1 \otimes \tilde{H}_{n-1} + \tilde{H}_{n-1} \otimes \tilde{H}_1 + \frac{1}{2} \left(-X_{1,n-1}^{\otimes 2} + X_{1,n}^{\otimes 2} + X_{2,n-1}^{\otimes 2} - X_{2,n}^{\otimes 2} \right) \\ & + \frac{1}{2} \left(-Y_{1,n-1}^{\otimes 2} + Y_{1,n}^{\otimes 2} + Y_{2,n-1}^{\otimes 2} - Y_{2,n}^{\otimes 2} \right), \end{aligned} \quad (5.3.1)$$

$$S'' = \sum_{a=1}^{n-2} \left(-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2} - \frac{n}{4} \cdot X_{a,n-1}^{\otimes 2} + \frac{n}{4} \cdot X_{a,n}^{\otimes 2} - \frac{n}{4} \cdot Y_{a,n-1}^{\otimes 2} + \frac{n}{4} \cdot Y_{a,n}^{\otimes 2} \right). \quad (5.3.2)$$

Then $\pi_0 \in \mathfrak{g}^{(2)}$, $S' \in Y_2'$, $S'' \in Y_2''$.

The proof consists of vast calculations and may be found in App. E.

6 Bi-Invariant vs. Locally Thinnest Metrics

6.1 Exposé and Programme

The compact connected simple groups of rank one are $SU(2)$ and $SO(3)$. Moreover, $SU(2)$ is the universal covering group of $SO(3)$. In particular, $SO(3) = PSU(2)$, cf. Sec. 2.15. From the differential geometer's view, these groups may be preferable to all others. And indeed, the discussions of the thickness of these groups will later serve as model cases for the proofs of Theorem B and Theorem C.

6.1.1 The special unitary group $SU(2)$. Let $G = SU(2)$. Fix the bi-invariant metric $Q_0(U, V) = \text{tr}(UV^*)$, cf. 2.14.1. This paragraph will contain a proof that Q_0 is not a locally thinnest metric.

The antipodal set $A(G)$ consists of a single point $p = -I_2$. Observe that $p \in Z(G)$. Moreover,

$$\exp\left(\pi \begin{bmatrix} i & \\ & -i \end{bmatrix}\right) = p.$$

In particular, $\text{dia } Q_0 = \pi\sqrt{2}$. Since $\text{rk}(G) = 1$, every $X \in LG$ is either regular or central.

Let e_1, e_2, e_3 be a $\sqrt{2}$ -orthogonal basis of LG . Consider the *Berger metric* $Q(s)$ defined by the inertia operator

$$A(s) = \begin{bmatrix} s & & \\ & s & \\ & & s^{-2} \end{bmatrix}$$

(cf. [Sak97, App. 6, 4°, pp. 307–308]). Note that Q is a volume-preserving variation of $Q(1) = Q_0$. Define $C(s) = \frac{1}{2}\pi\sqrt{2}(s^{-1} + \sqrt{s})$. Then $d_s(g) \leq C(s)$ for all $g \in G, s \geq 0$. In fact, let T be a maximal torus with $g \in T$ and $H \in LT$ with $\|H\| = \sqrt{2}$. Define the closed broken geodesic

$$\gamma: [0, 2] \rightarrow SU(2), \quad \gamma(t) = \begin{cases} \exp(\pi t H) & \text{if } t \in [0, 1], \\ \exp((2-t)\pi e_3) & \text{if } t \in [1, 2]. \end{cases}$$

Note that $\gamma(0) = \gamma(2) = e, \gamma(1) = p$. Clearly $\|H\|_s$ grows fastest if it is contained in $\text{span}\{e_1, e_2\}$. Thus,

$$L_s(\gamma) = \pi(\|H\|_s + \|e_3\|_s) \leq \pi\sqrt{2}(s^{-1} + \sqrt{s}).$$

It follows that $d_s(g) \leq \frac{1}{2}\pi\sqrt{2}(s^{-1} + \sqrt{s}) = C(s)$.

Now observe that $C(1) = \text{dia } Q_0$ and

$$C'(s) = \frac{1}{2}\pi\sqrt{2} \left(-\frac{1}{s^2} + \frac{1}{2\sqrt{s}} \right).$$

It follows that $C'(s) < 0$ for $s \in [1, \sqrt[3]{4}]$, and C is therefore strictly decreasing on $[1, \sqrt[3]{4}]$. Thus, $\text{dia } Q(s) < \text{dia } Q_0$ for all $s \in (1, \sqrt[3]{4}]$. In particular, $\theta(Q(s)) < \theta(Q_0)$ for all $s \in (1, \sqrt[3]{4}]$.

6.1.2 Axisymmetric metrics. The metric $Q(s)$ from 6.1.1 is axisymmetric, i.e. one of the eigenvalues of the inertia operator occurs with multiplicity two or three. Podobryaev has recently succeeded in computing the diameter of such metrics on $\text{SU}(2)$ [Pod18].¹ Using his formula, the diameter of $(\text{SU}(2), Q(s))$ may be expressed as²

$$d(s) = \begin{cases} \sqrt{2}\pi s^{1/2} & \text{if } s \leq 1, \\ \sqrt{2}\pi \frac{1}{s} & \text{if } 1 \leq s \leq \sqrt[3]{2}, \\ \frac{\pi}{\sqrt{2}} \cdot \frac{s}{\sqrt{s - s^{-2}}} & \text{otherwise.} \end{cases}$$

Moreover, the variation sweeps out the entire moduli space \mathfrak{MS} of axisymmetric left-invariant structures up to scaling on $\text{SU}(2)$. In fact, $Q: \mathbb{R}_+ \rightarrow \mathfrak{MS}$ is a homeomorphism. To prove surjectivity, suppose that Q' is an axisymmetric left-invariant metric, and diagonalise its inertia operator A' with respect to an orthonormal basis e'_1, e'_2, e'_3 of Q_0 with eigenvalues $a_1 = a_2, a_3$. Let $g \in G$ so that $\text{Ad}(g)(e_1) = e'_1$. Set $s = a_1$ and note that $a_3 = s^{-2}$. Moreover, observe that $\text{Ad}(g)$ maps $\text{span}\{e_2, e_3\}$ to $\text{span}\{e'_2, e'_3\}$. It follows that $\text{Ad}(g)^{-1} \circ A' \circ \text{Ad}(g) = A(s)$. In particular, $Q(s)$ and Q' are isometric. Injectivity is straightforward.

Elementary computations show that $Q(\sqrt[3]{4})$ is locally thinnest amongst axisymmetric left-invariant metrics. The diameter function on \mathfrak{MS} is not differentiable at $s = 1$ (the bi-invariant metric), C^1 but not C^2 at $s = \sqrt[3]{2}$ and smooth outside of $\{1, \sqrt[3]{2}\}$. Moreover, the bi-invariant metric is *locally thickest* amongst axisymmetric left-invariant

¹With similar techniques, Podobryaev–Sachkov have previously obtained a formula for the diameter of axisymmetric left-invariant metrics on $\text{SO}(3)$ [PS16]. Sadly, this formula is of little use to us.

²Podobryaev uses eigenvalues with respect to the bi-invariant metric with diameter 2π . Therefore, all his formulas must be multiplied with $1/\sqrt{2}$ to yield the correct result when using Q_0 from above as reference metric.

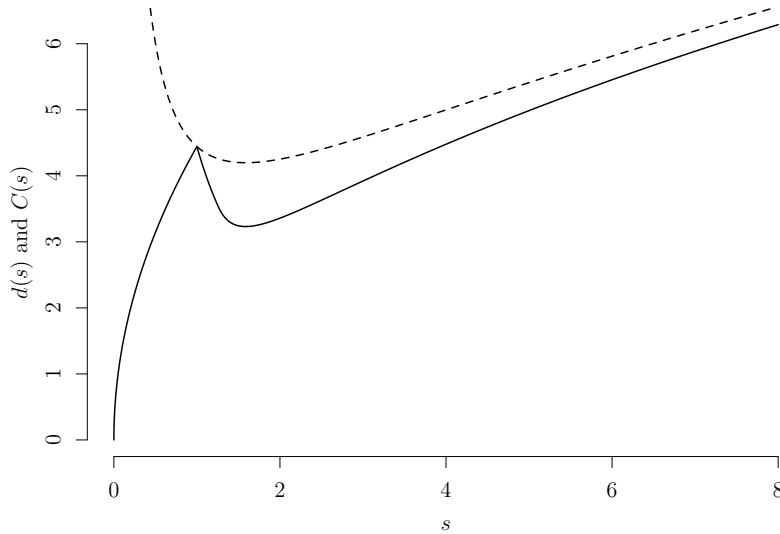


Figure 6.1: The diameter function $d(s)$ of $\mathfrak{MS}(\mathrm{SU}(2))$ and its upper bound $C(s)$. Note that Theorem A is exemplified in this figure, as one can clearly see that $\lim_{s \rightarrow 0} d(s) = 0$.

metrics. Finally, note that the upper bound $C(s)$ is asymptotically sharp. In particular, $\theta(\mathfrak{MS}) = \mathbb{R}_+$.

6.1.3 The odd orthogonal group $\mathrm{SO}(3)$. Let $G = \mathrm{SO}(3)$. Fix the bi-invariant metric $Q_0(U, V) = \mathrm{tr}(UV^\top)/2$. This paragraph will contain a proof that Q_0 is a locally thinnest metric.

Consider the standard action of G on $\mathbb{R}P^3$. For every $L \in \mathbb{R}P^3$, the identity component of the stabiliser is a maximal torus isomorphic to $\mathrm{SO}(2)$, which acts by rotations on L^\perp . The antipodal point of this torus is the reflection in the origin of L^\perp . The conjugation theorem implies that every geodesic emanating from e is a minimising geodesic to an antipodal point.³ This result will be extended in App. C.

Taking $L = \mathbb{R} \cdot e_3$ yields the maximal torus

$$T = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta & \\ \sin \theta & \cos \theta & \\ & & 1 \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

The antipodal point in T is

$$p = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Observe that no other torus contains p . Thus, the only minimising geodesics from e to p are $t \mapsto \exp(\pm tX_{12})$.⁴ In particular, the diameter of G is π .

³The conjugation theorem also implies that every element of $\mathrm{SO}(3)$ is a rotation of \mathbb{R}^3 .

⁴Cf. 2.12.2.

It might help the reader to remember that $\mathrm{SO}(3)$ is isometric to the real projective space $\mathbb{R}P^3$ with diameter π . The closed geodesics through the north pole are the maximal tori. If $\gamma(t)$ is a unit speed geodesic through the north pole $\gamma(0)$, then $\gamma(t)$ corresponds to a rotation by t . The fact that no other torus contains p is reflected by the fact that the equator points are not conjugate points of the north pole.

Denote by \mathfrak{S} the unit sphere of $S_0^2(\mathrm{LG})$. Let $X \in \mathfrak{S}$ and consider the metric $Q(X, s)$ with inertia operator $A(X, s) = \exp(sX)$. Diagonalise X with respect to Q_0 . The eigenvalues satisfy $a_1 + a_2 + a_3 = 0$. Assume without loss of generality that $a_1 > 0$. Let $H \in \mathrm{LG}$ be the corresponding eigenvector with $\|H\| = \pi$. Consider the geodesics $\tau_1(t) = \exp(tH)$, $\tau_2(t) = \exp(-tH)$ that pass through the antipodal point p . Then⁵

$$\left. \frac{d}{ds} L_{X,s}(\tau_i)^2 \right|_{s=0} = \left. \frac{d}{ds} \|H\|_{X,s}^2 \right|_{s=0} = Q_0(XH, H) = a_1 \cdot \pi > 0.$$

Take a neighbourhood $\mathcal{U} = \mathcal{U}(X)$ of X in \mathfrak{S} so that there exists $c > 0$ with

$$\left. \frac{d}{ds} L_{Y,s}(\tau_i) \right|_{s=s_0} > c$$

for all $Y \in \mathcal{U}$, $s_0 \in (-\varepsilon, \varepsilon)$, where $\varepsilon = \varepsilon(X)$. Since p is not a conjugate point, \mathcal{U} and $\varepsilon(X)$ may be chosen in such a way that there exist open neighbourhoods $W_i \subseteq \mathrm{LG}$ of $\pm H$ so that

$$\mathrm{Exp}: (\mathfrak{S} \times \mathbb{R}) \times \mathrm{LG} \rightarrow (\mathfrak{S} \times \mathbb{R}) \times G, \quad (Y, s, V) \mapsto (Y, s, \exp^{Y,s} V)$$

is invertible on $\mathcal{U} \times (-\varepsilon, \varepsilon) \times W_i$, according to Prop. 3.1.6. In particular, there exist differentiable maps $V_i: \mathcal{U} \times (-\varepsilon, \varepsilon) \rightarrow W_i$ with $V_1(Y, 0) = H, V_2(Y, 0) = -H$ for all $Y \in \mathcal{U}$ so that $\gamma_i(Y, s, t) = \exp^{Y,s}(tV_i(Y, s))$ are the only candidates for minimising $Q(Y, s)$ -geodesics joining e and p .

Now observe that applying Lem. 3.5.2 to the variation $(s, t) \mapsto \gamma_i(Y, s, t)$ with fixed endpoints implies that

$$\begin{aligned} \left. \frac{d}{ds} \|V_i(Y, s)\|_{Y,s}^2 \right|_{s=0} &= Q_0(YH, H) + 2 \cdot Q_0(V_i'(Y, 0), \pm H) \\ &= Q_0(YH, H) \\ &= \left. \frac{d}{ds} Q(Y, s)(H, H) \right|_{s=0} \\ &> c. \end{aligned}$$

Adjust $\varepsilon = \varepsilon(X)$ so that

$$\left. \frac{d}{ds} \|V_i(Y, s)\|_{Y,s}^2 \right|_{s=s_0} > 0$$

for all $Y \in \mathcal{U}$, $s_0 \in (-\varepsilon, \varepsilon)$. Thus, $d_{Y,s}(p) = L_{Y,s}(\gamma_i(Y, s)) > d(p)$ for all $0 < s < \varepsilon$.

⁵Here, notation from Sec. 3.1 is used. For instance, $L_{X,s}(\tau_i)$ denotes the length of τ_i with respect to $Q(X, s)$ and $\exp^{Y,s}$ the exponential map of $(G, Q(Y, s))$.

Now cover the compact space \mathfrak{S} with finitely many $\mathcal{U}(X_1), \dots, \mathcal{U}(X_k)$ and let $\varepsilon_0 = \min\{\varepsilon(X_1), \dots, \varepsilon(X_k)\}$. Define

$$\mathcal{U}_0 = \exp\{sX : s \in (-\varepsilon_0, \varepsilon_0), X \in \mathfrak{S}\},$$

the open ε_0 -ball in $\mathcal{S}_{>0}(\mathrm{LG})$. The image of \mathcal{U}_0 in $\mathfrak{PM} = \widetilde{\mathfrak{M}}/\mathbb{R} \times \mathrm{Aut}(G)$ is an open neighbourhood of Q_0 , and $\mathrm{dia} Q' \geq \mathrm{dia} Q_0$ for all $Q' \in \mathcal{U}_0$. In particular, $\theta(Q') \geq \theta(Q_0)$ for all $Q' \in \mathcal{U}_0$. Equality holds if and only if $Q' = Q_0$.

6.1.4. The reader may perceive it as odd that the minimisation property of $\mathrm{SO}(3)$ would not lift to $\mathrm{SU}(2)$. As a remedy, recall that Ivanov's formula (see 4.4.1) asserts that $\mathrm{dia} X \leq \mathrm{dia} \hat{X} \leq k \cdot \mathrm{dia} X$ for every locally isometric k -sheeted covering $\hat{X} \rightarrow X$ of compact length spaces. Both estimates are sharp, as one can see from the diameter calculations in 2.14.4. Moreover, Cor. 4.4.2 is the only type of inheritance property that can be derived from Ivanov's estimate. But since $\frac{1}{2} \mathrm{dia} \mathrm{SU}(2) = \mathrm{dia} \mathrm{SO}(3)$, at least after one of the metrics is rescaled so that the covering homomorphism is a local isometry, none of those results apply.

6.1.5 Two blueprints. Another thing that has certainly not escaped the reader's attention is that the two examples above are opposite in their antipodal behaviour. In the case of $\mathrm{SU}(2)$ the antipodal points lie in the centre, and, thus, only one maximal torus must be shrunk to decrease the diameter, leaving enough space to compensate the volume loss. The abundance of geodesics leading to the antipode p is used to prove the diameter estimate. Later, this procedure will be generalised to prove Theorem B.

Theorem B. *Let G be one of the following compact connected simple Lie groups: $\mathrm{SU}(n+1)$ for $n \geq 1$, $\mathrm{Spin}(5)$, $\mathrm{Spin}(7)$, $\mathrm{Sp}(n)$ for $n \geq 3$, $\mathrm{Spin}(2n)$ for $n \geq 4$, $\mathrm{SO}(2n)$ for $n \geq 3$, or the simply-connected exceptional Lie groups E_6 or E_7 . Then none of the bi-invariant metrics of G is a locally thinnest left-invariant metric.*

In fact, all groups occurring in Theorem B are Lie groups with finite antipodal set (with respect to the bi-invariant structure, to be understood). These will be characterised by the condition $A(G) \subseteq \bigcap_T T = Z(G)$.⁶ Thus, shrinking $\mathrm{rk}(G)$ many directions is sufficient, leaving $\dim G - \mathrm{rk}(G) \approx \mathrm{rk}(G)^2$ directions to compensate the volume loss. Note that the question about the existence of a locally thinnest metric remains unsolved for these groups.

On the other hand, antipodal points are abundantly available in the case of $\mathrm{SO}(3)$, i.e. $A(G) = S^1$, but each $p \in A(G)$ and all geodesics leading to p are contained in *only one* maximal torus. Lemma 3.5.2 above asserts that the evolution of geodesic length in the Lie group and the maximal torus coincide up to first order. Thus, given a variation of the bi-invariant metric, it is sufficient to find a maximal torus of superlinear growth (see 1.10.1), which is conveniently supplied in the form of the eigenvector H above. This procedure serves as a blueprint for the proof of Theorem C.

⁶This is reminiscent of the fact that finite normal subgroups of G are contained in the centre.

Theorem C. *Every bi-invariant metric Q_0 of $\text{PSU}(n+1)$ is a locally thinnest metric for $n = 1$ and an optimum for all $n \geq 1$.*

The properties of $\text{SO}(3)$ that are exploited above will be condensed into the notion of compact connected Lie groups G all of whose antipodes are regular elements. The simple groups with this property are $\text{PSU}(n+1)$ for $n \geq 1$ and possibly some proper quotients of $\text{SU}(n+1)$. To be able to properly control the antipodes of the maximal tori and prevent possible bifurcation effects⁷, attention must be restricted to the case of $\text{PSU}(n+1)$, whose integral lattice A_n^* is a lattice of Voronoi's first kind with strictly obtuse superbasis. Unfortunately, the search for a *good* torus, i.e. one whose diameter increases at least linearly, will fail if the direction of the deformation is contained in a certain zero set of $S_0^2(\mathfrak{su}(n+1))$. Thus, it can only be said that the bi-invariant metrics of $\text{PSU}(n+1)$ are *optima* (see 4.2.5). The proof involves the Vogel decomposition and the approach to Bleicher's theorem given in Sec. 1.10. The Vogel composition also yields a description of the problematic zero set.

6.2 Compact Connected Simple Lie Groups with Finite Antipodal Set

As mentioned above, all groups that occur in Theorem B are simple Lie groups G whose antipodal set $A(G)$ for the bi-invariant structure is finite, i.e. $|A(G)| < \infty$. This section is concerned with the verification of this claim, which is achieved by employing the following Lie theoretic characterisation of the condition $|A(G)| < \infty$.

6.2.1 Proposition. *Let G be a simple Lie group. Then the following are equivalent.*

- (1) $A(G)$ is finite.
- (2) There exists a maximal torus T with $A(G) \subseteq T$.
- (3) All maximal tori T satisfy $A(G) \subseteq T$.
- (4) $A(G) \subseteq Z(G)$.
- (5) $\tilde{A}(T) \subseteq R^*$ for all maximal tori T with root system R .
- (6) There exists a maximal torus T with $\tilde{A}(T) \subseteq R^*$.

Proof. (1) \Leftrightarrow (4): If $A(G) \subseteq Z(G)$, then $|A(G)| \leq |Z(G)| < \infty$. Suppose that $A(G)$ is finite. Let $p \in A(G)$. Then $c(G)(p)$ is connected. It follows that $c(G)(p) \subseteq \{p\}$. Thus, $p \in Z(G)$.

⁷An example of such a bifurcation of B_2 can be seen in Fig. B.1.

(2) \Rightarrow (1): If there exists a maximal torus T with $A(G) \subseteq T$, then according to Prop. 3.3.6, $A(G) = A(T)$. Moreover, $A(T)$ is finite by an application of Prop. 3.3.5(1).

(2) \Leftrightarrow (3): Follows from the conjugation theorem.

(3) \Leftrightarrow (4): Follows from $Z(G) = \bigcap_T T$, see 2.11.4.

(4) \Leftrightarrow (5): Follows from $\exp^{-1} Z(G) = R^*$, see 2.11.4.

(5) \Leftrightarrow (6): Follows from the conjugation theorem. \square

6.2.2 Corollary. *If $Z(G) = 1$, then $A(G)$ is infinite.*

6.2.3 Corollary. (1) *The antipodal sets of the following compact connected simple Lie groups are finite:*

$$\begin{array}{ll} \text{SU}(n+1) & \text{if } n \geq 1, \\ \text{Sp}(n) & \text{if } n \geq 3, \\ E_6, E_7. & \end{array} \quad \begin{array}{l} \text{Spin}(5), \text{Spin}(7), \\ \text{Spin}(2n), \text{SO}(2n) \quad \text{if } n \geq 4, \end{array}$$

(2) *The antipodal sets of the following compact connected simple Lie groups are infinite:*

$$\begin{array}{llll} \text{PSU}(n+1) & \text{if } n \geq 1, & \text{Spin}(2n+1) & \text{if } n \geq 4, & \text{SO}(2n+1) & \text{if } n \geq 2, \\ \text{PSp}(n) & \text{if } n \geq 3, & & & \text{PSO}(2n) & \text{if } n \geq 4, \\ PE_6, PE_7, E_8, & & & & F_4, & \\ G_2. & & & & & \end{array}$$

Proof. All groups occurring in (2) are centre-free (cf. Tab. 2.2), and may therefore be treated by an application of Cor. 6.2.2, except for $\text{Spin}(2n+1)$. It remains to check $\text{Spin}(2n+1)$ for $n \geq 2$ and the groups in list (1). Let G be one of these. Fix a maximal torus T and the corresponding root system R . Denote by I the integral lattice. It is sufficient to test if the typical deep holes of I are elements of R^* .

Case 1: $R = A_n, D_n, E_6, E_7, E_8$. It follows that $R^\vee = R$. Assume that $G \neq \text{SO}(2n)$. Then G is simply-connected. Therefore, $I = R^\vee = R$. According to the list of root lattices from Sec. 1.7, the typical deep holes of I are representatives of the glue group. In particular, they are central elements.

Suppose now that $G = \text{SO}(2n)$. Then $I = \mathbb{Z}^n$ and $R^* = D_n^*$. Thus, the typical deep hole of I is $[1]$. According to 1.7.5, $R^* = \mathbb{Z}^n[1, 3]$. Thus, $[1] \in R^*$.

Case 2: $R = C_n$. Then $G = \text{Sp}(n)$ is simply-connected. Thus, $I = R^\vee = B_n = \mathbb{Z}^n$. Of course $C_n \subsetneq \mathbb{Z}^n \subsetneq C_n^*$. Moreover, $\det C_n = 4$ implies $[C_n^*: C_n] = 4$. It follows that $[C_n^*: \mathbb{Z}^n] = 2$. The nontrivial element of C_n^*/\mathbb{Z}^n is easily identified as the typical deep hole $[1] = \left(\frac{1}{2}\right)^n$.

Case 3: $R = B_n$. Then $G = \text{Spin}(2n+1)$ is simply-connected. Thus, $R^* = \mathbb{Z}^n$, $R^\vee = C_n$ and $I = D_n$. A quick glance at the simple basis of C_n shows that $[\mathbb{Z}^n : C_n] = 2$. The nontrivial element of \mathbb{Z}^n / D_n is $[2] = (0^{n-1}, 1)$. If $n \leq 3$, then $[2]$ is the unique typical deep hole of D_n . Thus, the antipodal sets of $\text{Spin}(3)$, $\text{Spin}(5)$ and $\text{Spin}(7)$ are finite, but since $\text{Spin}(3) \cong \text{SU}(2)$, this only adds $\text{Spin}(5)$ and $\text{Spin}(7)$ to the list.

Conversely, if $n \geq 4$, then $(\frac{1}{2})^n$ and $(-\frac{1}{2}, (\frac{1}{2})^{n-1})$ are deep holes, and neither of these is contained in \mathbb{Z}^n . In particular, the antipodal set of $\text{Spin}(2n+1)$ is infinite. \square

6.2.4 A note on the computation of antipodal sets. The only simple Lie groups that do not appear in the lists (1) or (2) above are the quotients $\text{SU}(n+1)/\mathbb{Z}_r$ for $r \mid (n+1)$, except $\text{SU}(n+1)$, $\text{SO}(6) = \text{SU}(4)/\mathbb{Z}_2$ and $\text{PSU}(n+1)$. Their integral lattices are obtained by adjoining some $[s] \in A_n^*/A_n$ defined by $rs = n+1$ to A_n . The lattices $A_n^r = A_n[s]$ thus obtained are called *Coxeter lattices*. Their Voronoi cells are poorly understood, with minor exceptions. An algorithm for computing their Delone cells and the results up to $n = 29$ are presented in [DSSV09]. Yang has managed to compute the deep holes of A_n^r for $2 \mid (n+1)$ and $r = 2$ [Yan07, Yan08]. It follows from his work that the antipodal set of $\text{SU}(n+1)/\mathbb{Z}_2$ is finite if and only if $4 \mid (n+1)$ [Bey18, Tab. 2].

Using Yang's results, Beyrer recently calculated the antipodal sets of most symmetric spaces of compact type, leaving only the antipodal sets of $\text{SU}(n+1)/\mathbb{Z}_r$ for $r \neq 2, n+1$ undetermined [Bey18]. The method presented here is a straightforward way of obtaining their results in the case of simple Lie groups. Note also that Prop. 6.2.1 is a wide extension of [LD14, Lem. 4.2] in this special case.

Appendix D includes a classification of all quotients $\text{SU}(n+1)/\mathbb{Z}_r$ with finite antipodal set for $n \leq 14$. In addition to those obtained by Yang, these are $\text{SU}(6)/\mathbb{Z}_3$ and $\text{SU}(12)/\mathbb{Z}_3$. The proof is computer-based and leads to Conj. D.3.2, a special case of which asserts that the antipodal set of $\text{SU}(n+1)/\mathbb{Z}_3$ intersects $Z(G)$ nontrivially if and only if $6 \mid (n+1)$.

6.3 The Proof of Theorem B

Recall the statement of Theorem B.

Theorem B. *Let G be one of the following compact connected simple Lie groups: $\text{SU}(n+1)$ for $n \geq 1$, $\text{Spin}(5)$, $\text{Spin}(7)$, $\text{Sp}(n)$ for $n \geq 3$, $\text{Spin}(2n)$ for $n \geq 4$, $\text{SO}(2n)$ for $n \geq 3$, or the simply-connected exceptional Lie groups E_6 or E_7 . Then none of the bi-invariant metrics of G is a locally thinnest left-invariant metric.*

The proof is split into two parts. The first step is to uncover a sufficient condition on the fundamental alcove of the integral lattice of a maximal torus (see 1.3.4 and 1.6.2) in order to successfully reuse the technique that was already applied to

SU(2) in 6.1.1. This is done in Prop. 6.3.1, the proof of which entails Lemmas 6.3.3 and 6.3.4.

The second step is to verify that each Lie group occurring in Theorem B satisfies the hypothesis of this sufficient condition. This means making a number of calculations, which may be found in the paragraphs 6.3.5–6.3.9.

6.3.1 Proposition. *Let G be a compact connected simple Lie group. Suppose that G satisfies the following properties:*

- (1) $|A(G)| < \infty$.
- (2) *For every maximal torus T with root system R and integral lattice I , the fundamental alcoves K of the integral lattice (see 1.3.4 and 1.6.2) are simplices.*
- (3) *For every $H \in K$,*

$$\frac{|A(G)|}{\dim G - |A(G)|} + \max_{X \in K \setminus \{H\}} \cos \angle(X - H, H) < 0.$$

Then no bi-invariant metric Q_0 on G is a locally thinnest left-invariant metric.

6.3.2 Remark. Note that (1)–(3) only depend on the interplay between the root system and the integral lattice I . In fact, $|A(G)|$ is the number of typical deep holes of I and $\dim G = \dim \mathfrak{h} + |R|$.⁸

Proof. Let $k = |A(G)|$, $A(G) = \{p_1, \dots, p_k\}$, T_0 a maximal torus, $\mathfrak{h}_0 = LT_0$, $Z_1, \dots, Z_k \in \tilde{A}(T_0)$ so that $t \mapsto \exp tZ_j$ is a minimising geodesic from e to p_j , $\mathcal{Z} = \text{span}\{Z_1, \dots, Z_k\}$, $m = \dim G - k$. Define the variation $Q(s)$ by $Q(s)|_{\mathcal{Z}} = s^{-2}Q_0$, $Q(s)|_{\mathcal{Z}^\perp} = s^{2k/m}Q_0$ and $Q(s)(\mathcal{Z}, \mathcal{Z}^\perp) = 0$. Denote $Q(s)$ by $\langle \cdot, \cdot \rangle_s$ and discard the subscript if $s = 1$ to emphasise that $Q(1) = Q_0$. The inertia operator with respect to an orthonormal basis compatible with the decomposition $LG = \mathcal{Z}^\perp \oplus \mathcal{Z}$ is

$$A(s) = \left[\begin{array}{c|c} s^{2k/m} I_m & \\ \hline & s^{-2} I_k \end{array} \right].$$

Clearly $\det Q(s) = 1$ for all s . Set $d = \text{dia}(G, Q_0)$. The goal is to prove that there exists $s_0 > 1$ so that $d_s(p) < d$ for all $p \in G$, $s \in (1, s_0)$. The proof will encompass the Lemmas 6.3.3 and 6.3.4.

Let M be the set of all pairs $(X, H) \in LG \times LG$ so that there exists a maximal torus T and a fundamental alcove K of the integral lattice $I \subseteq LT$ so that $H \in K$ is a deep

⁸Cf. 2.4.1.

hole of K and $X \in K \setminus \{H\}$. Define

$$r(s, X, H) = -\frac{\langle H, X - H \rangle_s}{\|X - H\|_s},$$

$$r(s) = \inf_{(X, H) \in M} \left(-\frac{\langle H, X - H \rangle_s}{\|X - H\|_s} \right).$$

The infimum in the definition of $r(s)$ is a minimum. In fact, note that $r(s, \cdot, H)$ is constant along open rays emanating from H into K for every fundamental alcove K so that $X, H \in K$. The infimum may therefore be taken over the compact intersection of K and a small sphere with centre H :

$$r(s) = \min_{\substack{(X, H) \in M, \\ \|X - H\| = \rho_0}} \left(-\frac{\langle H, X - H \rangle_s}{\|X - H\|_s} \right).$$

In particular, $r(s)$ is continuous (see Cor. A.0.3).

6.3.3 Lemma. (1) $r(1) > 0$. (2) Let $s > 0$. If $r(s) > 0$ and $(X, H) \in M$ with $X \in B_{r(s)}^s(H) \setminus \{H\}$, then the function $\gamma(t) = \|H + t(X - H)\|_s$ is strictly decreasing on $[0, 1]$.⁹

Proof. (1) Let T be a maximal torus and $\mathfrak{h} = LT$. Note that

$$r(1) = \min_{(X, H) \in M \cap (\mathfrak{h} \times \mathfrak{h})} r(1, X, H).$$

It is sufficient to prove

$$\min_{X \in K \setminus \{H\}} r(1, X, H) > 0$$

for all deep holes $H \in \mathfrak{h}$ and all fundamental simplices $K \subseteq \mathfrak{h}$ containing H , since there are only finitely many deep holes in \mathfrak{h} .

Let $H \in \mathfrak{h}$ be a deep hole, K the alcove with $H \in K$, $X \in K \setminus \{0\}$. Apply the Cauchy–Schwarz inequality and then $\|X\| \leq \|H\|$ to obtain

$$\langle H, X - H \rangle = \langle H, X \rangle - \|H\|^2 \leq \|H\| \cdot \|X\| - \|H\|^2 \leq 0.$$

Equality can only hold if X, H are linearly dependent (first inequality) and X is a deep hole (second inequality). This would imply $X = H$, a contradiction. Therefore, $\max_{\xi} \langle H, \xi - H \rangle < 0$. It follows that

$$r(1) = \min_{(\xi, \eta) \in M} \left(-\frac{\langle \eta, \xi - \eta \rangle}{\|\xi - \eta\|} \right) = -\max_{(\xi, \eta) \in M} \frac{\langle \eta, \xi - \eta \rangle}{\|\xi - \eta\|} > 0.$$

⁹The condition $r(s) > 0$ is obviously unnecessary and only included to make the statement nonvacuous.

(2) Consider the derivative of the squared function

$$\begin{aligned} \frac{d}{dt} \gamma(t)^2 &= 2\langle X - H, H + t(X - H) \rangle_s \\ &= 2\left(\langle X - H, H \rangle_s + t\|X - H\|_s^2\right) \\ &\leq 2\left(\langle X - H, H \rangle_s + \|X - H\|_s^2\right). \end{aligned}$$

This number is negative if and only if

$$\|X - H\|_s < -\frac{\langle H, X - H \rangle_s}{\|X - H\|_s}.$$

Using $X \in B_{r(s)}^s(H)$, or, equivalently, $X - H \in B_{r(s)}^s(0)$, this condition is easily verified:

$$\|X - H\|_s < r(s) = \min_{(\xi, \eta) \in M} r(s, \xi, \eta) \leq r(s, X, H) = -\frac{\langle H, X - H \rangle_s}{\|X - H\|_s}.$$

Therefore, $\gamma(t)^2$ and $\gamma(t)$ are strictly decreasing on $[0, 1]$. \square

Let $0 < 4r_0 < r(1)$, $s_1 > 1$ be so that $3r_0 < r(s)$ for all $s \in [1, s_1]$. As $B_{3r_0}(H) \subseteq B_{r(s)}(H)$ and $\tilde{A}(G)$ is compact, there exists $s_2 > 1$ so that $B_{2r_0}(H) \subseteq B_{r(s)}^s(H)$ for all $H \in \tilde{A}(G)$, $s \in [1, s_2]$. Define $M_{r_0} = \{(X, H) \in M : X \in \overline{B_{r_0}(H)}\}$,

$$\lambda: \mathbb{R}_+ \times M_{r_0} \rightarrow \mathbb{R}, \quad \lambda(s, X, H) = \frac{d(1 - s^{-1})}{\|X - H\|_s},$$

and $F: \mathbb{R}_+ \times M_{r_0} \rightarrow LG$ by

$$F(s, X, H) = H + \lambda(s, X, H) \cdot (X - H).$$

6.3.4 Lemma. *Let $(X, H) \in M_{r_0}$, $s \in [1, \min\{s_1, s_2\}]$. If $\|X - H\|_s \geq (1 - s^{-1})d$, then $\|X\|_s \leq \|F(s, X, H)\|_s$.*

Proof. Note that $X \in \overline{B_{r_0}(H)} \subseteq B_{2r_0}(H) \subseteq B_{r(s)}^s(H)$. Lemma 6.3.3 implies that $t \mapsto \|H + t(X - H)\|_s$ is decreasing on $[0, 1]$. But the hypothesis implies that $\lambda(s) \leq 1$. Therefore,

$$\|F(s, X, H)\|_s = \|H + \lambda(s)(X - H)\|_s \geq \|H + (X - H)\|_s = \|X\|_s. \quad \square$$

The next step is to obtain an estimate on $(d/ds) \|F(s, X, H)\|_s^2|_{s=1}$. Let T be a maximal torus, $\mathfrak{h} = LT$, $K \subseteq \mathfrak{h}$ a fundamental alcove, H_1, \dots, H_n the nonzero vertices of K and μ so that H_μ is a deep hole. If $(X, H_\mu) \in M_{r_0}$, then $X - H_\mu$ is contained in the translated alcove $K - H_\mu$, whose nonzero vertices are $\tilde{H}_i = H_i - H_\mu$ ($i \neq \mu$) and $\tilde{H}_\mu = -H_\mu$.

Therefore, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$ with $X - H_\mu = \sum_i \alpha_i \tilde{H}_i$. Define the functions $\tilde{\alpha}_i(s)$ by $F(s, X, H_\mu) = \sum_i \tilde{\alpha}_i(s) H_i$ and $\lambda(s) = \lambda(s, X, H_\mu)$. Then

$$\begin{aligned} F(s, X, H_\mu) &= H_\mu + \lambda(s) \cdot (X - H_\mu) = H_\mu + \lambda(s) \sum_i \alpha_i \tilde{H}_i \\ &= H_\mu - \lambda(s) \alpha_\mu H_\mu + \lambda(s) \sum_{i \neq \mu} \alpha_i (H_i - H_\mu) \\ &= (1 - \lambda(s) \sum_j \alpha_j) H_\mu + \sum_{i \neq \mu} \lambda(s) \alpha_i H_i. \end{aligned}$$

Let $\Lambda = \lambda'(1) = \frac{d}{\|X - H_\mu\|}$. Use $\lambda(1) = 0$ to obtain

$$\begin{aligned} \tilde{\alpha}_i(s) &= \begin{cases} 1 - \lambda(s) \sum_j \alpha_j & \text{if } i = \mu, \\ \lambda(s) \alpha_i & \text{if } i \neq \mu, \end{cases} & \tilde{\alpha}_i(1) &= \begin{cases} 1 & \text{if } i = \mu, \\ 0 & \text{if } i \neq \mu, \end{cases} \\ \tilde{\alpha}'_i(1) &= \begin{cases} -\Lambda \sum_j \alpha_j & \text{if } i = \mu, \\ \Lambda \alpha_i & \text{if } i \neq \mu. \end{cases} \end{aligned}$$

Note that the decomposition $LG = \mathcal{Z}^\perp \oplus \mathcal{Z}$ is orthogonal for all s . Let $H_i = H_i^\perp + H_i^\top \in \mathcal{Z}^\perp \oplus \mathcal{Z}$. Then

$$\begin{aligned} \|F(s, X, H_\mu)\|_s^2 &= \sum_{ij} \tilde{\alpha}_i(s) \tilde{\alpha}_j(s) \cdot \langle H_i, H_j \rangle_s \\ &= \sum_{ij} \tilde{\alpha}_i(s) \tilde{\alpha}_j(s) \cdot (s^{2k/m} \langle H_i^\perp, H_j^\perp \rangle + s^{-2} \langle H_i^\top, H_j^\top \rangle). \end{aligned}$$

Therefore, the derivative of $s \mapsto \|F(s, X, H_\mu)\|_s$ satisfies the following estimate:

$$\begin{aligned} \frac{d}{ds} \|F(s, X, H_\mu)\|_s^2 \Big|_{s=1} &= \sum_{ij} \left(\tilde{\alpha}'_i(1) \tilde{\alpha}_j(1) + \tilde{\alpha}_i(1) \tilde{\alpha}'_j(1) \right) \cdot \langle H_i, H_j \rangle \\ &\quad + \sum_{ij} \tilde{\alpha}_i(1) \tilde{\alpha}_j(1) \left(\frac{2k}{m} \cdot \langle H_i^\perp, H_j^\perp \rangle - 2 \langle H_i^\top, H_j^\top \rangle \right) \\ &= 2 \sum_i \tilde{\alpha}'_i(1) \langle H_i, H_\mu \rangle + \frac{2k}{m} \cdot \langle H_\mu^\perp, H_\mu^\perp \rangle - 2 \langle H_\mu^\top, H_\mu^\top \rangle \\ &= 2\Lambda \sum_{i \neq \mu} \alpha_i \langle H_i, H_\mu \rangle - 2\Lambda \sum_j \alpha_j \langle H_\mu, H_\mu \rangle \\ &\quad + \frac{2k}{m} \|H_\mu^\perp\|^2 - 2 \|H_\mu^\perp\|^2 \\ &= 2\Lambda \cdot \langle X - H_\mu, H_\mu \rangle + \frac{2k}{m} \|H_\mu^\perp\|^2 - 2 \|H_\mu^\top\|^2 \\ &\leq 2\Lambda \cdot \langle X - H_\mu, H_\mu \rangle + \frac{2k}{m} \|H_\mu\|^2 \\ &= 2d^2 \frac{\langle X - H_\mu, H_\mu \rangle}{\|X - H_\mu\| \|H_\mu\|} + \frac{2k}{m} \cdot d^2 \end{aligned}$$

$$\begin{aligned}
&= 2d^2 \left(\frac{\langle X - H_\mu, H_\mu \rangle}{\|X - H_\mu\| \|H_\mu\|} + \frac{k}{m} \right) \\
&= 2d^2 \left(\frac{|A(G)|}{\dim G - |A(G)|} + \cos \angle(X - H_\mu, H_\mu) \right).
\end{aligned}$$

By hypothesis (3), this upper bound is negative.

Define $\phi: \mathbb{R}_+ \times M_{r_0} \rightarrow \mathbb{R}$, $\phi(s, X, H) = (d/d\sigma) \|F(\sigma, X, H)\|_\sigma^2|_{\sigma=s}$ and note that, since $F(s, \cdot, H_\mu)$ is constant along half-open rays emanating from H_μ into K , so is ϕ . Therefore,

$$\phi(s) = \sup_{(X, H) \in M_{r_0}} \phi(s, X, H) = \max_{\substack{(X, H) \in M_{r_0} \\ \|X - H\| = r_0}} \phi(s, X, H).$$

In particular, $\phi(s)$ is continuous (see Cor. A.0.3). Using that $\phi(1) < 0$, one may take $s_3 > 1$ so that $\phi(s) < 0$ for all $s \in [1, s_3]$. This means that $s \mapsto \|F(s, X, H)\|_s$ is strictly decreasing on $[1, \min\{s_1, s_2, s_3\}]$ for all $(X, H) \in M_{r_0}$.

There exists $s_4 > 1$ so that for all $s \in (1, s_4)$ and all points $p \in G$ with $d_s(p) \geq d$ there exists $q \in A(G)$ with $p \in B_{r_0}(q)$. In fact, for every $r > 0$ there exists $\varepsilon > 0$ so that for all $s \in (0, \varepsilon)$ and all $p \in A(G, Q(s))$ there exists $q \in A(G, Q_0)$ so that $p \in B_r(q)$, for if this were not the case, there would exist $r > 0$ and sequences $s_i \rightarrow 0$ and $p_i \in A(G, Q(s_i))$ so that $p_i \notin B_r(q)$ for all $q \in A(G, Q_0)$. Apply the compactness of G to obtain a sublimit p_0 of (p_i) . But since $\text{dia } Q(s_i) \rightarrow \text{dia } Q_0$ and $d_s(p_i) \rightarrow d_0(p_0)$, it follows that $p_0 \in A(G, Q_0)$, a contradiction. Let $s_0 = \min\{s_1, s_2, s_3, s_4\}$.

Let $s \in (1, s_0)$ and suppose that $p \in G$ satisfies $d_s(p) \geq d$. Fix a maximal torus T with Cartan subalgebra \mathfrak{h} satisfying $p \in T$. Let $X \in \mathfrak{h}$ so that $t \mapsto \exp tX$ is a minimising geodesic to p with respect to Q_0 . Let $q \in A(G, Q_0)$ with $p \in B_{r_0}(q)$. Then there exists $H \in \tilde{A}(T)$ with $X \in B_{r_0}(H)$. Thus, $(X, H) \in M_{r_0}$.

Now establish a relatively short path γ to p : Take a geodesic from e to q along one of the curves $t \mapsto \exp tZ_i$, then from q to p along $q \cdot \exp t(X - H)$. The total length of this path is $s^{-1}d + \|X - H\|_s$. Therefore, $\|X - H\|_s \geq (1 - s^{-1})d$. According to Lem. 6.3.4, $\|X\|_s \leq \|F(s, X, H)\|_s$. But $\sigma \mapsto \|F(\sigma, X, H)\|_\sigma$ is strictly decreasing! It follows that

$$d \leq d_s(p) \leq \|X\|_s \leq \|F(s, X, H)\|_s < \|F(1, X, H)\| = \|H\| = d.$$

This is a contradiction. □

With the proof of Prop. 6.2.1 completed, it remains to verify that each of the Lie groups occurring in Theorem B satisfies the sufficient condition (3). This will be done in the paragraphs 6.3.5–6.3.9, one integral lattice at a time. Note that the maximal angle is always assumed in a vertex of the fundamental alcove. It is therefore sufficient to consider $\sigma_{i\mu} = \cos \angle(H_i - H_\mu, H_\mu)$. For the impatient reader, the results are summarised in Tab. 6.1. Once the hypothesis is successfully verified, the proof concludes.

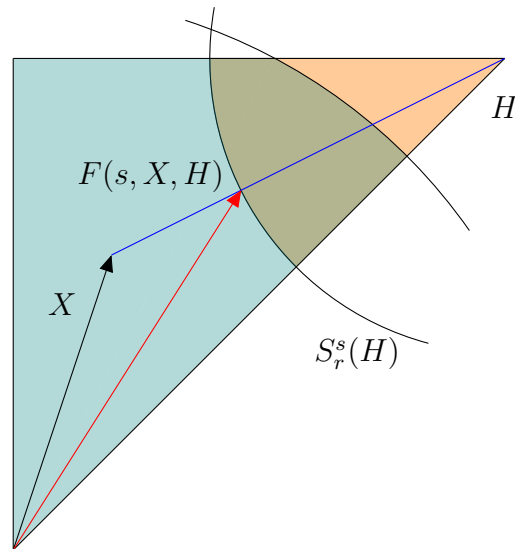


Figure 6.2: A fundamental alcove of a maximal torus. The light orange area $K \cap B_r^s(H)$ is the set of points at distance at most d along shortcuts like γ , where $r = (1 - s^{-1})d$. The map $F(s, \cdot, H)$ projects some point $X \in K$ outside of $B_r^s(H)$ onto a point of the distance sphere $S_r^s(H)$. Although $F(s, X, H)$ is longer than X , it is still contained in $K \cap B_d^s(0)$, displayed in light teal. This means that any point that cannot be reached sufficiently quickly by means of a shortcut may still be connected by a path of length $< d$ inside T .

I	angle	G	$ A(G) /(\dim(G) - A(G))$
A_n, n odd	$-1/\sqrt{n+2}$	$SU(n+1)$	$2/(n^2 + 2n - 2)$
A_n, n even	$-2/\sqrt{2n+1}$		$1/(n^2 + 2n - 1)$
B_n	$-1/\sqrt{n}$	$\left\{ \begin{array}{l} \text{Sp}(n) \\ \text{SO}(2n) \end{array} \right.$	$\left\{ \begin{array}{l} 1/(2n^2 + n - 1) \\ 2/(2n^2 - n - 2) \end{array} \right.$
		D_n	$\left\{ \begin{array}{l} \text{Spin}(5) \\ \text{Spin}(7) \\ \text{Spin}(8) \\ \text{Spin}(2n) \end{array} \right.$
E_6	E_6		$1/38$
E_7	E_7		$1/132$

Table 6.1: All groups occurring in Theorem B satisfy the hypothesis of Prop. 6.3.1.

6.3.5 The lattice A_n . Let $H_1 = [1], \dots, H_n = [n]$. Suppose that n is odd. Then $\mu = (n+1)/2$. Let $i+j = n+1$ and note that $\sigma_{i\mu} = \sigma_{j\mu}$. Assume that $i < \mu$. It follows that

$$\sigma_{i\mu} = \sigma_{j\mu} = \frac{\frac{i}{2} - \frac{n+1}{4}}{\sqrt{\frac{ij}{n+1} + \frac{n+1}{4} - i} \cdot \sqrt{\frac{n+1}{4}}} = \frac{2i - (n+1)}{\sqrt{(n+1)^2 - 4i^2}}.$$

Moreover, $i \mapsto \sigma_{i\mu}$ is increasing if $i < \mu$. Thus,

$$\sigma_{i\mu} \leq \sigma_{\mu-1,\mu} = -\frac{2}{\sqrt{2n+1}}.$$

Conversely, suppose that n is even. Set $\nu = n/2$ and note that $\mu = \nu$ or $\mu = \nu + 1$. If $i < \nu$, then

$$\sigma_{i\nu} = \sigma_{j,\nu+1} = \sqrt{\frac{n+2}{n}} \cdot \frac{2i - n}{\sqrt{n^2 + 2n - 4i^2 - 4i}}.$$

If $i < \nu + 1$, then

$$\sigma_{i,\nu+1} = \sigma_{j,\nu} = \sqrt{\frac{n}{n+2}} \cdot \frac{2i - (n+2)}{\sqrt{n^2 + 2n - 4i^2 + 4i}}.$$

The functions $i \mapsto \sigma_{i,\nu}$, $i \mapsto \sigma_{i,\nu+1}$ are increasing if $i < \nu$, $i < \nu + 1$, respectively. Thus,

$$\sigma_{i\nu} \leq \sigma_{\nu-1,\nu} = -\frac{\sqrt{n+2}}{n}, \quad \sigma_{i,\nu+1} \leq \sigma_{\nu,\nu+1} = -\frac{1}{\sqrt{n+2}}.$$

6.3.6 The lattice B_n . Let $H_1 = (0^{n-1}, \frac{1}{2}), \dots, H_n = (\frac{1}{2})^n$. Then $\mu = n$ and

$$\sigma_{i\mu} = -\frac{\sqrt{n-i}}{\sqrt{n}} \leq \sigma_{n-1,n} = -\frac{1}{\sqrt{n}}.$$

6.3.7 The lattice D_n . Let $H_i = (0^i, (\frac{1}{2})^{n-i})$ for $i = 2, \dots, n-2$,

$$\begin{aligned} H_1 &= (\frac{1}{2})^n, & H_{n-1} &= (-\frac{1}{2}, (\frac{1}{2})^{n-1}), \\ H_n &= (0^{n-1}, 1). \end{aligned}$$

Then $\mu = 1, n-1, n$ if $n = 4$ and $\mu = 1, n-1$ if $n > 4$. The results of calculating the numbers $\sigma_{\mu,i}$ are found in Tab. 6.2.

i	1	$2, \dots, n-2$	$n-1$	n
$\sigma_{i,11}$		$-\sqrt{i/n}$	$-\sqrt{2/n}$	$-(n-2)/n$
$\sigma_{i,n-1}$	$-1/\sqrt{n}$	$-\sqrt{i/n}$		$-(n-2)/n$
$\sigma_{i,n}$	$-1/\sqrt{n}$	$-1/\sqrt{n-i}$	$-(n-2)/n$	

Table 6.2: The numbers $\sigma_{\mu i}$ for the lattice D_n . Their maximum is $-1/\sqrt{n}$.

6.3.8 The lattice E_7 . Define

$$\begin{aligned} H_1 &= (0^5, 1, -\frac{1}{2}, \frac{1}{2}), & H_2 &= (0^4, (\frac{1}{2})^2, -\frac{1}{2}, \frac{1}{2}), \\ H_3 &= ((\frac{1}{4})^6, -\frac{1}{2}, \frac{1}{2}), & H_4 &= (0^3, (\frac{1}{3})^3, -\frac{1}{2}, \frac{1}{2}), \\ H_5 &= (0^2, (\frac{1}{4})^4, -\frac{1}{2}, \frac{1}{2}), & H_6 &= (-\frac{1}{6}, (\frac{1}{6})^5, -\frac{1}{2}, \frac{1}{2}), \\ H_7 &= (0^6, -\frac{1}{2}, \frac{1}{2}). \end{aligned}$$

Then $\mu = 1$. The results of calculating the numbers $\sigma_{\mu,i}$ are found in Tab. 6.3.

i	2	3	4	5	6	7
$\sigma_{i,1}$	$-1/\sqrt{3}$	$-\sqrt{3/7}$	$-2/3$	$-1/\sqrt{2}$	$-\sqrt{5}/3$	$-\sqrt{2/3}$

Table 6.3: The numbers $\sigma_{\mu i}$ for the lattice E_7 . Their maximum is $-1/\sqrt{3}$.

6.3.9 The lattice E_6 . Define

$$\begin{aligned} H_1 &= \left(0^4, 1, \left(-\frac{1}{3}\right)^2, \frac{1}{3}\right), & H_2 &= \left(0^5, \left(-\frac{2}{3}\right)^2, \frac{2}{3}\right), \\ H_3 &= \left(-\frac{1}{4}, \left(\frac{1}{4}\right)^4, \left(-\frac{5}{12}\right)^2, \frac{5}{12}\right), & H_4 &= \left(0^2, \left(\frac{1}{3}\right)^3, \left(-\frac{1}{3}\right)^2, \frac{1}{3}\right), \\ H_5 &= \left(0^3, \left(\frac{1}{2}\right)^2, \left(-\frac{1}{3}\right)^2, \frac{1}{3}\right), & H_6 &= \left(\left(\frac{1}{4}\right)^5, \left(-\frac{1}{4}\right)^2, \frac{1}{4}\right). \end{aligned}$$

Then $\mu = 1, 2$. The results of calculating the numbers $\sigma_{\mu,i}$ are found in Tab. 6.4.

i	1	2	3	4	5	6
$\sigma_{i,1}$		$-1/2$	$-\sqrt{2/5}$	$-1/\sqrt{2}$	$-\sqrt{3/8}$	$-\sqrt{5/8}$
$\sigma_{i,2}$	$-1/2$		$-\sqrt{3/8}$	$-1/\sqrt{2}$	$-\sqrt{2/5}$	$-\sqrt{5/8}$

Table 6.4: The numbers $\sigma_{\mu i}$ for the lattice E_6 . Their maximum is $-\sqrt{3/8}$.

6.3.10 Corollary. *Let G be a compact simply-connected Lie group which is not isomorphic to a product of the simple Lie groups $\text{Spin}(2n+1)$ for $n \geq 4$, E_8, F_4 or G_2 . Then no bi-invariant metric on G is a locally thinnest left-invariant metric.*

Proof. The classification of compact Lie groups asserts that $G = S_1 \times \cdots \times S_t$, a product of simply-connected compact Lie groups at least one of which, say S_1 , is isomorphic to $\text{SU}(n+1), \text{Spin}(2n), E_6$ or E_7 , and a torus T .

Let \tilde{Q} be a bi-invariant metric. Then $\tilde{Q} = Q_1 \times \cdots \times Q_t \times Q'$, where Q_i is a bi-invariant metric on S_i . By Prop. 6.3.1, there exists a volume-preserving variation $Q_1(s)$ of Q_1 with $\text{dia } Q_1(s) < \text{dia } Q_1$ for all $s > 0$. Define

$$\tilde{Q}(s) = Q_1(s) \times Q_2 \times \cdots \times Q_t \times Q'.$$

Then the Pythagorean theorem implies

$$d(\tilde{Q}(s)) = \sqrt{d(Q_1(s))^2 + \sum_{i \geq 2} d(Q_i)^2 + d(Q')^2} < \sqrt{d(Q_1)^2 + \sum_{i \geq 2} d(Q_i)^2 + d(Q')^2} = d(\tilde{Q}).$$

□

6.4 Compact Connected Simple Lie Groups with Regular Antipodal Points

6.4.1. Recall that a regular element g of a compact connected Lie group G is one that is contained in only one maximal torus. It was established in 6.1.3 and 6.1.5 that the

key property of $\mathrm{SO}(3)$ that was used to prove that every bi-invariant metric on $\mathrm{SO}(3)$ is a locally thinnest left-invariant metric is that all its antipodal points are regular. This notion lies opposite to the notion of a Lie group with finite antipodal set, whose antipodal points are contained in *all* maximal tori. Below follows a characterisation of this property in terms of the Stiefel diagram and a proof that $\mathrm{PSU}(n+1)$ for $n \geq 1$ is the only family of Lie groups to which it applies, other quotients of $\mathrm{SU}(n+1)$ excepted.¹⁰

In fact, the characterisation asserts that the antipodal points of G are all regular if and only if for every maximal torus T the deep holes X of the integral lattice I are not contained in the Stiefel diagram LT_s (see 2.11). If the requirement is slightly relaxed to merely X not being contained in the truncated Stiefel diagram LT'_s (see 2.11.2), or, equivalently, to none of the antipodal points being a first conjugate point, then the notion is also satisfied by PE_6 .

6.4.2 Proposition. *Let G be a simple Lie group. Then the following are equivalent:*

- (1) *None of the antipodal points of G are first conjugate points.*
- (2) *For every maximal torus T , the set of deep holes $\tilde{A}(T)$ does not meet the truncated Stiefel diagram LT'_s .*
- (3) *For every maximal torus T with root system R , no Delone polytope P of a deep hole of the integral lattice I of T meets the coroot system R^\vee outside of $\{0\}$.*

Moreover, the equivalences remain intact if each ‘For every maximal torus $T\dots$ ’ is replaced with ‘There exists a maximal torus T so that...’.

6.4.3 Proposition. *Let G be a simple Lie group. Then the following statements are equivalent:*

- (1) *Every antipodal point of G is regular.*
- (2) *For every maximal torus T , the set of deep holes $\tilde{A}(T)$ does not meet the Stiefel diagram LT_s .*
- (3) *For every maximal torus T and every $p \in A(T)$, the centraliser $Z(p)$ is contained in the normaliser $N_G(T)$.*

Moreover, the equivalences remain intact if each ‘For every maximal torus $T\dots$ ’ is replaced with ‘There exists a maximal torus T so that...’.

Proof of Proposition 6.4.2. The ‘moreover’-part of the proposition follows from the conjugation theorem.

(1) \Leftrightarrow (2): Follows from Cor. 3.4.4.

¹⁰Recall that $\mathrm{PSU}(2) = \mathrm{SO}(3)$.

(2) \Leftrightarrow (3): Let T be a maximal torus with root system R , and X a deep hole of the integral lattice I of T . Denote by P the Delone polytope of X and recall that $R^\vee \subseteq I$. Thus, X is contained in the Voronoi cell of R^\vee . According to 2.11.3, LT'_s is a union of bisectors of elements of the lattice R^\vee . It follows that $X \in LT'_s$ if and only if there exists some $v \in R^\vee$ so that $X \in H(v)$. As $P^0 \setminus \{0\}$ (0-skeleton without the origin) is the set of all elements of I on whose bisectors X lies, $X \in LT'_s$ if and only if $P \cap R^\vee \neq \{0\}$. \square

Proof of Proposition 6.4.3. The ‘moreover’-part of the proposition follows from the conjugation theorem.

(1) \Leftrightarrow (2): An element $p \in G$ is regular if and only if every $X \in \exp^{-1}(p)$ is regular, i.e. $X \notin LT_s$ for all maximal tori T .

(1) \Leftrightarrow (3): Follows from the conjugation theorem. \square

6.4.4 Corollary. *Suppose that G is a compact simply-connected simple Lie group. Then all antipodal points of G are first conjugate points.*

Proof. Let $T \subseteq G$ be a maximal torus with corresponding integral lattice I and root system R . Then $\pi_1(G) = 1$ implies $I = R^\vee$. Recall that, according to 1.6.4, the Voronoi cell of I is the Weyl orbit of a simplex whose roof is the bisector of a coroot. Thus, if $X \in LG$ lies on the boundary of the Voronoi cell of I , there exists a coroot $\gamma = \alpha^\vee \in R^\vee$ so that X lies on the bisector $H(\gamma) = \alpha^{-1}(1) \subseteq LT'_s$. Thus, Prop. 6.4.2(3) is violated. \square

6.4.5 Corollary. *The antipodal points of $\text{PSU}(n+1)$ are regular for all $n \geq 1$, and none of the antipodal points of PE_6 is a first conjugate point. None of the other simple Lie groups of type $\neq A_n$ has either property.*

Proof. Once $\text{PSU}(n+1)$ is treated, the remaining Lie groups that must be considered are $\text{SO}(2n+1)$ for $n \geq 2$, $\text{PSp}(n)$ for $n \geq 3$, $\text{PSO}(2n)$ for $n \geq 4$, E_6^* and E_7^* . Thanks to the isometric operation of the Weyl group, it is sufficient to treat typical deep holes.

Consider $\text{PSU}(n+1)$. In 1.9.8, a Delone polytope of the typical deep hole

$$X = \frac{1}{2(n+1)}(-2n, -2n+2, \dots, 2n-2, 2n)$$

is specified. Recall that the Delone polytope contains all points of A_n^* on whose bisectors X lies. One can immediately see that $P \cap A_n = \{0\}$. Moreover, it is straightforward to see that X is not perpendicular to any (co)root, as no two entries of X coincide. Therefore, $X \notin LT_s$.

The proof will utilise Prop. 6.4.2(3) to show that none of the other Lie groups satisfies said properties. The integral lattice of $\text{SO}(2n+1)$ is \mathbb{Z}^n . The Delone cell of the typical deep hole $\left(\frac{1}{2}\right)^n$ contains, for instance, the point $(1, 1, 0^{n-2})$, which belongs to $R^\vee = D_n$.

The integral lattice of $\mathrm{PSp}(n)$, $n \geq 3$ is D_n^* , the coroot system is $R^\vee = B_n$. The typical deep hole is $((\frac{1}{2})^t, 0^t)$ if $n = 2t$, and $((\frac{1}{2})^t, \frac{1}{4}, 0^t)$ if $n = 2t + 1$ [CS91, Sec. 7]. The Delone cell of both of these contains $(1, 0^{n-1}) \in B_n$.

The integral lattice of $\mathrm{PSO}(2n)$, $n \geq 4$, is D_n^* (with deep holes as above), the coroot system is $R^\vee = D_n$. Note that $t \geq 2$ (t defined as above). Therefore, the Delone cell of the typical deep hole contains $(1^2, 0^{n-2})$.

It is easy to verify that the typical deep hole $X = (\frac{7}{8}, (-\frac{1}{8})^7)$ of E_7^* lies on the bisector of $(1, 0^4, -1, 0^2) \in E_7$, which solves the case of E_7^* . The typical deep hole $X = (0, (\frac{1}{3})^3, (-\frac{1}{3})^3, 0)$ of E_6^* is perpendicular to the coroot $(0, -1, 1, 0^5)$, which solves the case of E_6^* . Finally, the proof that $X \notin LT'_s$ is accomplished by checking that X is not contained on the bisector of an element v of the coroot system $R^\vee = R$. This is done using the Fincke–Pohst algorithm [FP85] in App. D. \square

6.4.6 Remark. Let G be a simple Lie group. Fix a maximal torus $T \subseteq G$. According to 3.3.7, the antipodal set $A(G) = \bigcup_i M_i$ is a union of compact connected homogeneous G -manifolds M_i , the G -orbits of antipodal points of T . In particular, $M_i = Z(p_i) \backslash G$ for every $p_i \in T \cap M_i$. Note that $T \subseteq Z(p_i)$. It follows that $\dim M_i \leq \dim G - \mathrm{rk} G$. Thus, the contrast between those simple Lie groups with finite antipodal set and those with regular antipodal set may be described as the contrast between those with minimal and maximal dimension of the antipodal manifold.

6.5 Proof of Theorem C

Theorem C. *Every bi-invariant metric Q_0 of $\mathrm{PSU}(n+1)$ is a locally thinnest metric for $n = 1$ and an optimum for all $n \geq 1$.*

As advertised in 6.1.5, the proof of Theorem C works similarly to the treatment of $\mathrm{SO}(3)$ in 6.1.3. Let us sketch the proof: To prove that each of the bi-invariant metrics Q_0 is an optimum, one must find a zero set $\mathcal{N} \subseteq S_0^2(\mathfrak{g})$ and show that for every $X \in S_0^2 \setminus \mathcal{N}$ the diameter is nondecreasing in the direction of X , where \mathfrak{g} denotes the Lie algebra of $G = \mathrm{PSU}(n+1)$. It was already verified in Cor. 6.4.5 that the antipodal set of G is regular, and thus, the situation is comparable with that of $\mathrm{SO}(3)$. Let $0 \neq X \in S_0^2(\mathfrak{g})$. The problem is, as mentioned earlier, that for some X , it seems as if there exists no *good* maximal torus T if $n \geq 2$, that is: one whose diameter $\mathrm{dia}(T, Q(X, s))$ grows superlinearly (see 1.10.1), where $Q(X, s)$, as always, denotes the left-invariant metric with inertia operator $A(s) = \exp(sX)$. In fact, the real Vogel decomposition is $S^2(\mathfrak{g}) = \mathbb{R} \oplus \mathfrak{g}^{(2)} \oplus Y_2'$ for $n = 2$ and $S^2(\mathfrak{g}) = \mathbb{R} \oplus \mathfrak{g}^{(2)} \oplus Y_2' \oplus Y_2''$ for $n \geq 3$, and if $X \in Y_2'$ (if $n = 2$) or $X \in Y_2''$ (if $n \geq 3$), then such a torus may (at the moment) not be found. As the reader may already have guessed, the zero set \mathcal{N} will be Y_2' (for $n = 2$) or Y_2'' (for $n \geq 3$).

The remaining directions can be dealt with as follows: If $X \notin \mathfrak{g}^{(2)}$, then there exists a maximal torus T with positive volume derivative (see Prop. 5.1.3). This, together with Bleicher's theorem (see Sec. 1.10), which, as the reader may recall, asserts that the integral lattice A_n^* is a locally thinnest lattice, implies that the diameter of the torus increases superlinearly. If, however, $X \in Y_2'$ (and $n \geq 3$), then one must face the curious phenomenon that *all* maximal tori of G have unchanging volume up to first order. Fortunately, manual calculations in App. E reveal that there exists a maximal torus T so that $\pi \circ X|_{LT}$,¹¹ the derivative of the inertia operator of $(T, Q(X, s))$, points into a good direction in the sense of Sec. 1.10, implying that $\text{dia}(T, Q(X, s))$ grows superlinearly. This is part 1 of the proof of Theorem C.

When this deed is finally done, it remains to repeat some the arguments of 6.1.3, m.m., to obtain the result. This will be done in part 2. For good measure, let us recall that this meant applying Lem. 3.5.2 (which claims that the evolution of geodesics in T and the evolution of geodesics in G coincide up to first order), and using the regularity of the antipodes to show that, like the diameter of $(T, Q(X, s))$, the diameter of $(G, Q(X, s))$ grows superlinearly. The covering argument from 6.1.3 is no longer required. As a corollary to the proof, note that superlinear growth immediately implies that the thickness function $\theta: \widetilde{\mathfrak{M}}(\text{PSU}(n+1)) \rightarrow \mathbb{R}$ is not differentiable in Q_0 .

Proof of Theorem C, part 1. For the sake of convenience, assume $n \geq 3$. Fix a bi-invariant metric Q_0 , let \mathfrak{S} be the unit sphere of $S_0^2(\mathfrak{g})$, $X \in \mathfrak{S} \setminus Y_2''$, T_0 the standard maximal torus of G whose Lie algebra $\mathfrak{h} = LT_0$ consists of trace-free diagonal matrices with imaginary entries (cf. 2.13.1 and 2.14.1). Fix the left-invariant metric $Q(X, s)$ with inertia operator $A(X, s) = \exp(sX)$. It is sufficient to show that there exists $g \in G$ so that the diameter of T_0 with respect to the pullback metric $Q_g(X, s) = c(g)^*Q(X, s)$ grows superlinearly. Let Φ be as in 5.1.1 and recall that the inertia operator of $Q_g(X, s)$ is $\Phi(g^{-1})(A(X, s))$. Moreover,

$$\left. \frac{d}{ds} \Phi(g^{-1})(A(X, s)) \right|_{s=0} = \Phi(g^{-1})(X).$$

In particular,

$$\mathcal{Y} = \text{span}\{\Phi(g)(X) : g \in G\} \subseteq S_0^2(\mathfrak{g})$$

is the subspace generated by the derivatives of the inertia operators of the metrics $Q_g(X, s)$, $g \in G$. Let

$$S^2(\mathfrak{g}) = \mathbb{R} \oplus \mathfrak{g}^{(2)} \oplus Y_2' \oplus Y_2''$$

denote the Vogel decomposition from Prop. 5.1.3. According to Lem. 5.2.13, there are two cases to consider: (1) $\mathfrak{g}^{(2)} \subseteq \mathcal{Y}$,¹² (2) $Y_2' \subseteq \mathcal{Y}$.

¹¹As always, π denotes the orthogonal projection $\mathfrak{g} \rightarrow LT$.

¹²This is the aforementioned preferable case.

(1) According to Lem. 5.2.13, there exists a maximal torus T with $\partial_X \text{vol}(T) > 0$. Define

$$\tilde{Q}(s) = \left(\frac{\text{vol}(T, Q_0)}{\text{vol}(T, Q(X, s))} \right)^{2/n} \cdot Q(X, s)|_{LT} \in \tilde{\mathfrak{M}}(T).$$

Then $\text{vol}(T, \tilde{Q}(s)) = \text{vol}(T, Q_0)$. Use Bleicher's theorem (see 1.10.1) followed by the scaling property of the diameter to obtain

$$\text{dia}(T, Q_0) \leq \text{dia}(T, \tilde{Q}(X, s)) = \left(\frac{\text{vol}(T, Q_0)}{\text{vol}(T, Q(X, s))} \right)^{1/n} \cdot \text{dia}(T, Q(X, s))$$

for $|s| < \varepsilon$. Reordering the terms yields

$$\left(\frac{\text{vol}(T, Q(X, s))}{\text{vol}(T, Q_0)} \right)^{1/n} \cdot \text{dia}(T, Q_0) \leq \text{dia}(T, Q(X, s)).$$

Thus, $s \mapsto \text{dia}(T, Q(X, s))$ is superlinear.

Solving the second case requires some preparation. Denote by $V \subseteq \mathbb{R}^{n+1}$ the space of zero-sum vectors and by $\Lambda \subseteq \mathfrak{h}$ the integral lattice of T_0 . Let

$$f: V \rightarrow (\mathfrak{h}, Q_0), \quad f(x_0, \dots, x_n) = 2\pi \cdot \begin{bmatrix} ix_0 & & \\ & \ddots & \\ & & ix_n \end{bmatrix}.$$

Then f is a similarity transformation with $f(A_n^*) = \Lambda$. Consider the pullback map

$$f^*: S^2(\mathfrak{h}) \rightarrow S^2(V), \quad Y \mapsto f^{-1} \circ Y \circ f.$$

According to Prop. 5.1.2(2), the inertia operator of $Q_g(X, s)|_{\mathfrak{h}}$ is $A_g(X, s) = \pi \circ \Phi(g^{-1})(A(X, s))$. It is straightforward to see that $f^*A_g(X, s)$ is the inertia operator of the pullback metric $f^*Q_g(X, s)$ with respect to $\langle \cdot, \cdot \rangle = f^*Q_0$. Observe that $(A_n^*, f^*Q_g(X, s)) \cong (\Lambda, Q_g(X, s))$. Thus, it is sufficient to show that there exists a Delone cell β of A_n^* so that¹³

$$\left. \frac{d}{ds} \mu(\beta, f^*Q_g(X, s)) \right|_{s=0} > 0,$$

in order to prove that the covering radius of $(\Lambda, Q_g(X, s))$, and therefore the diameter of $(T_0, Q_g(X, s))$, grows superlinearly.

To accomplish this, note that $(d/ds) f^*A_g(X, s)|_{s=0} = f^*(\pi \circ \Phi(g^{-1}))(X)$ and define $X_g = f^*(\pi \circ \Phi(g^{-1}))(X)$. Therefore, it is sufficient to prove that (for some $g \in G$) there exists a Delone cell β of A_n^* so that

$$\left. \frac{d}{ds} \mu \left(\beta, \langle \exp(sX_g) \cdot, \cdot \rangle \right) \right|_{s=0} > 0.$$

¹³Here $\mu(P, Q)$ denotes the circumradius of the simplex P with respect to Q (cf. 1.8.2).

Therefore, TN is the matrix representing f^*X with respect to (v_1, \dots, v_n) . Thus, the matrix representing f^*X^\sim with respect to $(v_1, \dots, v_n, (1)^{n+1})$ is

$$\left[\begin{array}{c|c} TN & 0 \\ \hline 0 & 0 \end{array} \right].$$

The result now follows from the obvious fact that

$$\left[\begin{array}{ccc|cc} 1 & & & & 1 \\ -1 & \ddots & & & \vdots \\ & & \ddots & 1 & 1 \\ & & & -1 & 1 \end{array} \right]$$

is the matrix of the basis change $(e_0, \dots, e_n) \mapsto (v_1, \dots, v_n, (1)^{n+1})$. \square

Let $S' \in Y'_2$ be defined as in equation (5.3.1) found in Prop. 5.3.1.¹⁴ According to the lemma above,

$$f^*(\pi \circ S')^\sim = \left[\begin{array}{cc|cc} & & 1 & -1 \\ & & -1 & 1 \\ 1 & -1 & & \\ -1 & 1 & & \end{array} \right],$$

which is a nonzero element of \mathcal{F} , according to 1.10.10(2). \square

6.5.2. Let $S'' \in Y''_2$ be defined as in equation (5.3.2) found in Prop. 5.3.1. According to the lemma above,

$$f^*(\pi \circ S'')^\sim = \left[\begin{array}{cccc|cc} & & & & 1 & -1 \\ & & & & \vdots & \vdots \\ & & & & 1 & -1 \\ 1 & \cdots & 1 & -(n-1) & & \\ -1 & \cdots & -1 & & & n-1 \end{array} \right].$$

But this is an old acquaintance, the matrix X_n from 1.10.10(4), one of the *bad* directions from the discussion of Bleicher's theorem. Thus, without any knowledge of an improvement of 3.5.2 from first order to second order, these directions cannot be treated with the methods at hand.

Proof of Theorem C, part 2. Let \mathfrak{S} , $X \in \mathfrak{S} \setminus Y''_2$ and T be as above. Denote the integral lattice of T by Λ . Let $p \in T$ be an antipodal point. Then Λ is a lattice of Voronoi's first kind with strictly obtuse superbasis. In particular, the Delone cells of Λ are simplices,

¹⁴Beware! In Sec. 5.2, $\mathfrak{su}(n)$ is considered in place of $\mathfrak{su}(n+1)$.

according to 1.9.5. Thus, there exist exactly $n+1$ deep holes H_0, \dots, H_n with $\exp H_i = p$, according to Prop. 3.3.5. Denote their Delone cells by β_0, \dots, β_n . An application of Cor. 1.9.6 shows that there exists $\varepsilon_1 > 0$ so that β_0, \dots, β_n are Delone cells of $(\Lambda, Q(Y, s))$ for all $Y \in \mathfrak{S}$ and $s \in (-\varepsilon_1, \varepsilon_1)$, where $Q(Y, s)$ is defined as in 4.2.4.

Denote by $H_i(Y, s)$ the centre of β_i with respect to $Q(Y, s)$. Then $(Y, s) \mapsto H_i(Y, s)$ is differentiable and $H_i(Y, 0) = H_i$ for all $Y \in \mathfrak{S}$ and i . An application of Prop. 3.3.5(3) implies that $\exp H_i(Y, s) = \exp H_j(Y, s)$ for all i, j . In particular, $p(Y, s) = \exp H_i(Y, s)$ is the same local maximum of the distance function of $(T, Q(Y, s))$ for all i , $p(Y, 0) = p$ for all $Y \in \mathfrak{S}$, and $\tilde{A}_e^{p(Y, s)}(T, Q(Y, s)) = \{H_0(Y, s), \dots, H_n(Y, s)\}$ is exactly the set of holes that map to $p(Y, s)$. Define $\tau_i(Y, s, t) = \exp(tH_i(Y, s))$. Then $t \mapsto \tau_i(Y, s, t)$ is a geodesic of $(T, Q(Y, s))$ joining e and $p(Y, s)$ with velocity vector $H_i(Y, s)$.

By Cor. 6.4.4, none of the H_i is contained in the Stiefel diagram LT'_s , which, according to Cor. 3.4.4, is equal to $\mathcal{C}(G)$, the conjugate locus of Q_0 . Apply Prop. 3.1.6 to obtain $0 < \varepsilon_2 < \varepsilon_1$ and neighbourhoods $W_i \subseteq \mathfrak{g}$ of H_i so that

$$\text{Exp}: (\mathfrak{S} \times \mathbb{R}) \times \mathfrak{g} \rightarrow (\mathfrak{S} \times \mathbb{R}) \times G, \quad (Y, s, V) \mapsto (Y, s, \exp^{Y, s} V)$$

is invertible on $\mathfrak{S} \times (-\varepsilon_2, \varepsilon_2) \times W_i$. Define

$$V_i(Y, s) = (\exp^{Y, s} |_{W_i})^{-1}(p(Y, s))$$

and $\gamma_i(Y, s, t) = \exp^{Y, s}(tV_i(Y, s))$. Then $\gamma_i(Y, s, \cdot)$ is a geodesic of $(G, Q(Y, s))$ joining e and $p(Y, s)$. Note that $\dot{\gamma}_i(Y, 0, 0) = V_i(Y, 0) = H_i$.¹⁵

Let us prove that¹⁶

$$\mathcal{G}_e^{p(Y, s)}(G, Q(Y, s)) \subseteq \{\gamma_i(Y, s, \cdot) : i = 0, \dots, n\}.$$

Take a neighbourhood N of $\{\tau_1(0, 0, \cdot), \dots, \tau_n(0, 0, \cdot)\}$ in the space of C^1 -curves $C^1(G)$ so that $\dot{\gamma}(0) \in \bigcup_i W_i$ for all $\gamma \in N$. An application of Prop. 3.1.10 shows that there exists $0 < \varepsilon_3 < \varepsilon_2$ so that $\gamma \in N$ for all $\gamma \in \mathcal{G}_e^{p(Y, s)}(G, Q(Y, s))$ and all $|s| < \varepsilon_3$. Since there is exactly one $V \in W_i$ with $\exp^{Y, s}(V) = p(Y, s)$, namely $V_i(Y, s)$, it follows that each $\gamma \in \mathcal{G}_e^{p(Y, s)}(G, Q(Y, s))$ is one of the $\gamma_i(Y, s, \cdot)$, provided that $|s| < \varepsilon_3$.

Apply Lem. 3.5.2 to the variation $(s, t) \mapsto \gamma(X, s, t)$ to obtain

$$\begin{aligned} \frac{d}{ds} \|V_i(X, s)\|_{X, s}^2 \Big|_{s=0} &= 2 \cdot Q_0(V_i'(X, 0), H_i) + Q_0(H_i, XH_i) \\ &= 2 \cdot Q_0(H_i'(X, 0), H_i) + Q_0(H_i, XH_i) \\ &= \frac{d}{ds} Q(X, s)(H_i(X, s), H_i(X, s)) \Big|_{s=0} \\ &= \frac{d}{ds} \|H_i(X, s)\|_{X, s}^2 \Big|_{s=0}. \end{aligned}$$

¹⁵ $\dot{\gamma}_i(Y, s, t_0) = (d/dt) \gamma_i(Y, s, t) |_{t=t_0}$.

¹⁶Where $\mathcal{G}_e^{p(Y, s)}(G, Q(Y, s))$ is defined as the set of minimising geodesics from e to $p(Y, s)$, cf. 3.1.7.

Recall that up to now p was an arbitrary antipodal point. As the thickness of T is superlinear in the direction X , it follows that

$$\left. \frac{d}{ds} \|H_i(X, s)\|_{X,s} \right|_{s=0} > 0$$

for some choice of the antipodal point $p \in T$ and all i . Thus, after making the appropriate change,

$$\left. \frac{d}{ds} \|V_i(X, s)\|_{X,s} \right|_{s=0} > 0$$

holds for all i . It follows that

$$d_{Q(X,s)}(p(X, s)) \geq \min_i L_{X,s}(\gamma_i(X, s, \cdot)) = \min_i \|V_i(X, s)\|_{X,s}.$$

The lower bound $f(s) = \min_i \|V_i(X, s)\|_{X,s}$ is the minimum of a set of functions all of which are strictly increasing at $s = 0$, and therefore strictly increasing at $s = 0$, as well. As $f(0) = d_{Q_0}(p)$, it follows that $d_{Q(X,s)}(p(X, s))$ cannot be decreasing at $s = 0$. \square

6.6 Vista: The Riddle of Y_2''

(With notation as in the proof of Theorem C.) It has become apparent in 6.5.2 that the map

$$\psi: S^2(\mathfrak{g}) \rightarrow S^2(\mathbb{R}^{n+1}), \quad \psi(X) = f^*(\pi \circ X)^\sim,$$

which, as the reader may recall from the proof of Theorem C, measures the effect of X on the standard maximal torus, maps at least one nonzero element of Y_2'' into the space \mathfrak{X} of bad directions as defined in Sec. 1.10. Of course, a single element is not such a big deal (remember that X may, at a whim, be replaced with any element from the $\Phi(G)$ -orbit of X , cf. 5.1.1), but computer-based calculations using rational numbers suggest that ψ determines a curious correspondence between the Vogel decomposition

$$S_0^2(\mathfrak{g}) = \mathfrak{g}^{(2)} \oplus Y_2' \oplus Y_2''$$

and the decomposition of the space \mathcal{S} (defined in 1.10.2, may be identified with the space of symmetric endomorphisms of \mathfrak{h}) into good and bad directions (see Prop. 1.10.7, Prop. 1.10.9, and 1.10.10):

$$\mathcal{S} = \mathbb{R} \cdot \tilde{F} \oplus \mathcal{S}^0 \oplus \mathfrak{X}.$$

In fact, it would appear as if ψ mapped $\mathfrak{g}^{(2)}$ onto the sum of the first two modules, Y_2' onto \mathcal{S}^0 and Y_2'' onto \mathfrak{X} . So the elements of the last Vogel module not only fix the volume of all maximal tori (up to first order), but their diameter (expressed as curve lengths of geodesics) as well (up to first order). One cannot help but wonder what the second derivative might look like. In this section, the proof of a formula for the second derivative of a different type of length function is sketched.

Fix $X \in Y_2''$, $X \neq 0$, $A(s) = \exp(sX)$, a bi-invariant metric Q_0 , let $Q(s) = Q_0(A(s) \cdot, \cdot)$, and a maximal torus T . As observed in the proof of Theorem C, there exist $n + 1$ deep holes H_0, \dots, H_n above every antipode p_0 of G . They satisfy $H_0 + \dots + H_n = 0$. As H_0, \dots, H_n are located outside the conjugate locus, there exist neighbourhoods $V_i \subseteq \mathfrak{g}$ of each H_i and $W \subseteq G$ of p_0 so that $\exp^s|_{V_i}: V_i \rightarrow W$ is a diffeomorphism for small s . It is therefore safe to define the *generalised distance functions*

$$d_i^s: W \rightarrow \mathbb{R}, \quad d_i^s(q) = \|(\exp^s|_{V_i})^{-1}(q)\|_s$$

and let $q(s)$ denote the implicitly defined differentiable path in T with

$$d_0^s(q(s)) = \dots = d_n^s(q(s))$$

(the dependence of d_i^s on X has deliberately been left out to simplify the notation). This is a significant variation to the approach taken above in the proof of Theorem C. Instead of considering a differentiable path $p(s)$ in T so that each $p(s) = p(X, s)$ is an antipode of $(T, Q(s))$, one instead lets $q(s)$ satisfy a certain *midpoint* property, which will turn out to be useful further below.

By appealing once more to the fact that the conjugate locus is far away, one obtains $Q(s)$ -geodesics γ_i^s in G that join e and $q(s)$ and whose velocity vectors satisfy $T_i(s, t) = \dot{\gamma}_i^s(s) \in V_i$ for sufficiently small s . Using the definition of $q(s)$, they are minimising geodesics, and, thus, their length $L_i(s)$ is $d_i^s(q(s))$. Recall that $T_i(0, 0) = H_i$, and, furthermore, note that $L_0(s) = \dots = L_n(s)$. There is a variational vector field

$$U_i(s, t) = \frac{\partial \gamma_i}{\partial s}(s, t)$$

with respect to the variation $\gamma_i(s, t) = \gamma_i^s(t)$. Let $\tilde{U}_i(s, t) = D(\mathbf{L}_{\gamma_i(s, t)}^{-1})_e(U_i(s, t))$. If $s = 0$, the first parameter will usually be omitted in notation. Our point of departure is the formula

$$\frac{\nabla^2 U_i}{\partial t}(t) = \frac{1}{4} \text{ad}(H_i)^2(\tilde{U}_i(t)) - \text{ad}(H_i)(XH_i) \Big|_{\gamma_i(t)}$$

that was established in the proof of Lem. 3.5.2. If $\nabla \nabla U_i(t)$ is not calculated in the roundabout way of Lem. 3.5.2, but straightforwardly, then one obtains

$$\frac{\nabla^2 U_i}{\partial t}(t) = \tilde{U}_i''(t) + \text{ad}(H_i)(\tilde{U}_i'(t)) + \frac{1}{4} \text{ad}(H_i)^2(\tilde{U}_i(t)) \Big|_{\gamma_i(t)}.$$

instead. By combining both equations, one finds that \tilde{U}_i must solve a homogenous second order differential equation, which might be dubbed a *transmetric Jacobi equation*:

$$\tilde{U}_i'' + \text{ad}(H_i)(\tilde{U}_i') + \text{ad}(H_i)(XH_i) = 0, \quad \tilde{U}_i(0) = 0, \quad \tilde{U}_i(1) = \bar{q} = D(\mathbf{L}_{p_0}^{-1})_e(q'(0)).$$

The solution may be expressed by involving power series. Given a formal expression and its corresponding power series expansion as a function of the formal parameter z , for example

$$\frac{1 - e^{-tz}}{z} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{(k+1)!} z^k,$$

the term

$$\frac{I - \exp(-t \operatorname{ad}(H_i))}{\operatorname{ad}(H_i)} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{(k+1)!} \operatorname{ad}(H_i)^k$$

is defined by evaluating the power series on the right hand side after substituting the formal parameter $z = \operatorname{ad}(H_i)$, proper convergence provided. A popular example for this is [Hel78, Thm. 1.7]. (For aesthetic reasons, 1 is replaced by I , the identity, and e by \exp .) Thus, the solution to the equation above reads

$$\tilde{U}_i(t) = t \cdot (\bar{q} - XH_i) + \left(\frac{I - \exp(-t \operatorname{ad}(H_i))}{\operatorname{ad}(H_i)} \circ \frac{\operatorname{ad}(H_i)}{I - \exp(-\operatorname{ad}(H_i))} \right) (XH_i).$$

The power series of the right fraction is

$$\frac{z}{1 - e^{-z}} = 1 + \frac{z}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\beta_k}{(2k)!} z^{2k},$$

where β_k denote the *Bernoulli numbers*.¹⁷ The series converges for $|z| < 2\pi$ [O⁺10, Sec. 24.2]. Given the special circumstances of H_i , this property is satisfied. In fact, using the operator norm, it follows that

$$\|\operatorname{ad}(H_i)\|_0^2 = \|\operatorname{ad}(H_i)^2\|_0 = 4 \cdot \|R(\cdot, H_i, H_i)\|_0.$$

As there are no conjugate points on (the relevant part of) γ_i , it follows that the absolute value of every eigenvalue of $R(\cdot, H_i, H_i)$ must be $< \pi^2$ (see Prop. 3.4.1). Thus, $\|\operatorname{ad}(H_i)\|_0 < 2\pi$.

A couple of observations can be made using this explicit formula for \tilde{U}_i .

6.6.1 Lemma. (1)

$$\int_0^1 [H_i, \tilde{U}_i(t)] dt = -\frac{1}{2} \operatorname{ad}(H_i)(XH_i) + \left(\frac{\operatorname{ad}(H_i)}{I - \exp(-\operatorname{ad}(H_i))} - I \right) (XH_i).$$

¹⁷The Bernoulli numbers are often defined by the equation

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$$

[O⁺10, Sec. 24.1–24.2, p. 588]. The relation of the two definitions is given by the formula $B_{2k} = (-1)^{k+1} \beta_k$.

(2)

$$\frac{d}{dt} Q_0(U_i(t), U_i(t)) \Big|_{t=1} = 2 \cdot \|\bar{q}\|_0^2.$$

The computation of the first derivative of the length function $L(s)$, defined by the equation $L(s) = L_0(s) = \dots = L_n(s)$, eventually leads to the following formula:

$$\frac{d}{ds} L(s)^2 = \int_0^1 Q_0(A(s)XT_i(s, t), T_i(s, t)) dt + 2 \cdot Q_0(A(s)T_i(s, 1), U_i(s, 1)). \quad (6.6.1)$$

At $s = 0$, this is the equation

$$\frac{d}{ds} L(s)^2 \Big|_{s=0} = Q_0(XH_i, H_i) + 2 \cdot Q_0(H_i, \bar{q}).$$

Assuming that $\psi(X)$ is a bad direction in the sense of Bleicher (as conjectured above), it follows, using the path $p(s)$ defined in the proof of Theorem C, that

$$Q_0(XH_i, H_i) + 2 \cdot Q_0(H_i, p'(0)) = 0.$$

Summing over all i yields

$$\sum_{i=0}^n Q_0(XH_i, H_i) = 0.$$

On the other hand, summing the formula for $\frac{d}{ds} L(s)^2 \Big|_{s=0}$ and combining with this last equation shows that

$$(n+1) \cdot \frac{d}{ds} L(s)^2 \Big|_{s=0} = \sum_i Q_0(XH_i, H_i) = 0.$$

Thus, $q(s)$ suffers the same fate as the path $p(s)$. Higher derivatives are required!

The derivative of the first summand of (6.6.1) at $s = 0$ is

$$2 \cdot Q_0(XH_i, \bar{q}) + \|XH_i\|_0^2 - 2 \int_0^1 Q_0(U_i, [T_i, XT_i]) dt.$$

The last summand of the resulting expression may be determined using Lem. 6.6.1(1) and some shifting of $\text{ad}(H_i)$ terms:

$$\begin{aligned} -2 \int_0^1 Q_0(U_i, [T_i, XT_i]) dt &= -2 \cdot Q_0 \left(\int_0^1 \tilde{U}_i(t) dt, [H_i, XH_i] \right) \\ &= 2 \cdot Q_0 \left(\left(\frac{\text{ad}(H_i)}{I - \exp(-\text{ad}(H_i))} - I \right) (XH_i), XH_i \right) \\ &= 2 \cdot Q_0 \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\beta_k}{(2k)!} \text{ad}(H_i)^{2k} (XH_i), XH_i \right) \\ &= -2 \sum_{k=1}^{\infty} \frac{\beta_k}{(2k)!} \|\text{ad}(H_i)^k (XH_i)\|_0^2. \end{aligned}$$

The sum of the derivate of the second summand of (6.6.1) at $s = 0$ over all $i = 0, \dots, n$ is, due to $\sum_i H_i = 0$ and Lem. 6.6.1(2),

$$\frac{1}{n+1} \cdot \sum_{i=0}^n \frac{d}{dt} Q_0(U_i(0, t), U_i(0, t)) \Big|_{t=1} = 2 \cdot \|\bar{q}\|_0^2.$$

Altogether, using $\sum_i H_i = 0$ once more, this yields a formula for the second derivative:

$$\frac{d^2}{ds^2} L^2(s) \Big|_{s=0} = 2 \cdot \|\bar{q}\|_0^2 + \frac{1}{n+1} \cdot \sum_{i=0}^n \left(\|XH_i\|_0^2 - 2 \sum_{k=1}^{\infty} \frac{\beta_k}{(2k)!} \|\text{ad}(H_i)^k(XH_i)\|_0^2 \right).$$

Now, this expression seems difficult to gauge in theory. But numerical computations made in low dimensions suggest to us that the quadratic form

$$\begin{aligned} X &\mapsto \left(\frac{d^2}{ds^2} L^2(s) \Big|_{s=0} \right) (X) \\ &= 2 \cdot \|\bar{q}\|_0^2 + \frac{1}{n+1} \cdot \sum_{i=0}^n \left(\|XH_i\|_0^2 - 2 \sum_{k=1}^{\infty} \frac{\beta_k}{(2k)!} \|\text{ad}(H_i)^k(XH_i)\|_0^2 \right) \end{aligned}$$

is *not positive-definite* on Y_2'' , and, therefore, no conclusions about the diameter of $(G, Q(s))$ can be drawn from this. Although this sounds troublesome, the reader must remember that this state of affairs might change after switching to a different maximal torus $\tilde{T} = \text{Ad}(g)(T)$, or, equivalently, to another direction $\tilde{X} \in \Phi(g)(X)$. So, the question really becomes: *Does there exists, for every $X \in Y_2'', X \neq 0$, some $g \in G$ with $((d/ds)L^2(0)|_{s=0})(\Phi(g)(X)) > 0$?*

Appendices

A Continuous Maps

A.0.1 Proposition. *Let X, Y, Z be metric spaces, X, Y compact,*

$$\Phi: C(X \times Y, Z) \times Y \rightarrow C(X, Z), \quad \Phi(f, y)(x) = f(x, y).$$

Then Φ is a continuous map.

A.0.2 Proposition. *Let X be a compact metric space. Then*

$$\mu: C(X) \rightarrow \mathbb{R}, \quad \mu(f) = \min_{x \in X} f(x)$$

is continuous.

A.0.3 Corollary. *Let X, Y be compact metric spaces, $f: X \times Y \rightarrow \mathbb{R}$ continuous. Then*

$$F: Y \rightarrow \mathbb{R}, \quad y \mapsto \min_{x \in X} f(x, y)$$

is continuous.

Proof of Proposition A.0.1. Let $(f_0, y_0) \in C(X \times Y, Z) \times Y$ and $\varepsilon > 0$. Fix $\delta > 0$ so that $d(f_0(x, y), f_0(x', y')) < \varepsilon/2$ for all $(x, y), (x', y')$ with $d_{X \times Y}((x, y), (x', y')) < \delta$. Let $(f, y) \in B_{\varepsilon/2}(f_0) \times B_{\delta}(y_0)$. Then $d_{X \times Y}((x, y_0), (x, y)) < \delta$ for all $x \in X$ and it follows that

$$\begin{aligned} d_{C(X, Z)}(f_0(\cdot, y_0), f(\cdot, y)) &\leq d_{C(X, Z)}(f_0(\cdot, y_0), f_0(\cdot, y)) + d_{C(X, Z)}(f_0(\cdot, y), f(\cdot, y)) \\ &\leq \max_{x \in X} d_Z(f_0(x, y_0), f_0(x, y)) + d_{C(X \times Y, Z)}(f_0, f) \\ &< \varepsilon. \end{aligned} \quad \square$$

Proof of Proposition A.0.2.

$$|\mu(f_0 - f_1)| = \left| \max_{x \in X} (f_0(x) - f_1(x)) \right| \leq \max_{x \in X} |f_0(x) - f_1(x)| = \|f_0 - f_1\|_{C(X, Y)}. \quad \square$$

Proof of Corollary A.0.3. Let

$$i: Y \rightarrow C(X \times Y) \times Y, \quad i(y) = (f, y).$$

Then $F = \mu \circ \Phi \circ i$. □

B Lattices of Voronoi's First Kind

B.1 Proof Sketch of Proposition 1.9.5

Recall that Proposition 1.9.5 asserts the following:

1.9.5 Proposition. *Let Λ be a lattice of Voronoi's first kind with strictly obtuse superbasis v_1, \dots, v_n . Let W be the group of permutations of $\{0, \dots, n\}$. For every $\pi \in W$, define the simplex P_π as the convex closure of*

$$v_{\pi\{1\}}, v_{\pi\{1,2\}}, \dots, v_{\pi\{1,\dots,n\}}.$$

Then the Delone complex of Λ is $\{P_\pi : \pi \in W\}$.

As already mentioned, the only reference to or proof of this result that the author knows of may be found in [BK10, Prop. B.3.4], where Bost remarks that some of the relevant equations occur in Voronoi's publications. The proof below shows similarities to the proof given in the reference above, but uses induction:

Proof sketch of Proposition 1.9.5. Let $X = \{P_\pi : \pi \in W\}$. It is sufficient to prove that for every Voronoi vector u of Λ and $P \in X$ the circumcentre x of P satisfies $u^\vee(x) \leq 1$, with equality if and only if $u \in P$, where $u^\vee(x) = 2\langle u, x \rangle / \langle u, u \rangle$ is defined as in 1.1.2. Equality would imply that x is a hole of Λ whose Delone cell is P . The case of $n = 1$ is straightforward.

Suppose that n is arbitrary. Let w_1, \dots, w_n be the nonzero vertices of P . It is easy to see that one may exchange the given strictly obtuse superbasis for one that satisfies $w_i = \sum_{j=1}^i v_j$ (using the action of the symmetric group W). By 1.9.4, there exists I so that $u = v_I$. Let $W_0 = \text{span}\{w_1, \dots, w_{n-1}\}$ and $\Lambda_0 = \Lambda \cap W_0$. It is straightforward to verify that Λ_0 is a lattice of Voronoi's first kind with strictly obtuse superbasis w_1, \dots, w_{n-1} . Apply the induction hypothesis to obtain that the simplex P_0 with nonzero vertices $\{w_1, \dots, w_{n-1}\}$ is a Delone polytope of Λ_0 . Let x_0 be the circumcentre of P_0 .

Let $b \in W_0^\perp$ and note that $x \in \bigcap_{i=1}^{n-1} H(w_i) = x_0 + \mathbb{R} \cdot b$. Let $t \in \mathbb{R}$ with $x = x_0 + t \cdot b$. Apply w_n^b to obtain that

$$t = \frac{1}{\langle b, w_n \rangle} \cdot \left(\frac{\|w_n\|^2}{2} - \langle x_0, w_n \rangle \right).$$

Note that if $\{0, n\} \subseteq I$ or $\{0, n\} \cap I = \emptyset$, then $u \in W_0$. In that case, the result (including the discussion of equality) follows by induction. Therefore, one may assume one of the following cases: $I \cap \{0, n\} = \{n\}$ and $I \cap \{0, n\} = \{0\}$. As these cases may be treated very similarly, only the first one will be considered here.

Set $u_0 = u_{I \setminus \{n\}} = u - v_n \in W_0$. A straightforward computation shows that

$$\begin{aligned} \langle x, u \rangle &= \langle x_0, u_0 \rangle - \frac{\|w_{n-1}\|^2}{2} + \frac{\|w_n\|^2}{2} \\ &\leq \frac{1}{2}(\|u_0\|^2 + \|w_n\|^2 - \|w_{n-1}\|^2). \end{aligned}$$

Apply the Selling formula (see 1.9.2) to obtain

$$\begin{aligned} \|u_0\|^2 &= - \sum_{(i,j) \in \bar{I} \times I \setminus \{n\}} q_{ij} - \sum_{j \in I \setminus \{n\}} q_{nj}, \\ \|w_n\|^2 &= \sum_{j \neq 0, n} q_{0j} - q_{0n}, \\ -\|w_{n-1}\|^2 &= \sum_{j \neq 0, n} q_{0j} + \sum_{j \in I \setminus \{n\}} q_{jn} + \sum_{j \in \bar{I} \setminus \{0\}} q_{nj}, \\ \|u\|^2 &= - \sum_{(i,j) \in \bar{I} \times I \setminus \{n\}} q_{ij} - q_{0n} - \sum_{j \in \bar{I} \setminus \{0\}} q_{nj}. \end{aligned}$$

As the off-diagonal entries of the Selling matrix are negative, it follows that

$$\begin{aligned} \|u_0\|^2 + \|w_n\|^2 - \|w_{n-1}\|^2 &= - \sum_{(i,j) \in \bar{I} \times I \setminus \{n\}} q_{ij} - q_{0n} + \sum_{j \in \bar{I} \setminus \{0\}} q_{nj} \\ &\leq - \sum_{(i,j) \in \bar{I} \times I \setminus \{n\}} q_{ij} - q_{0n} - \sum_{j \in \bar{I} \setminus \{0\}} q_{nj} \\ &= \|u\|^2. \end{aligned}$$

In particular, it follows that $u^\vee(x) \leq 1$.

Now suppose $u^\vee(x) = 1$. Then $\sum_{j \in \bar{I} \setminus \{0\}} q_{nj} = 0$. As all off-diagonal entries of the Selling matrix are negative, it follows that $\bar{I} \setminus \{0\} = \emptyset$, which implies that $I = \{1, \dots, n\}$ and therefore $u = w_n \in P$. \square

B.2 Application to Two-Dimensional Lattices

B.2.1 Short basis vs. superbasis. The usual approach to two-dimensional lattices utilises *short bases*: Let Λ be a two-dimensional lattice. Let x be a shortest vector of $\Lambda \setminus \{0\}$ and y be a shortest vector of $\Lambda \setminus \mathbb{R} \cdot x$. Then (x, y) is a basis of Λ , and any basis that can be constructed in this particular fashion is called a *short basis*. It is easy to see that $|\langle x, y \rangle| \leq \|x\|^2/2$ [GHL04, Sec. 2.24].

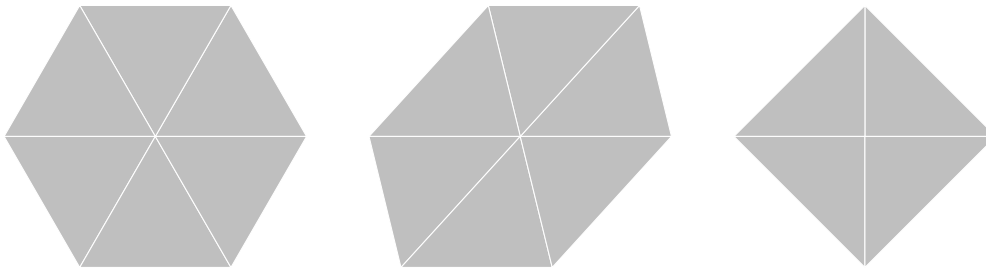


Figure B.1: Delone complexes of three two-dimensional lattices. The hexagonal lattice A_2 on the left, the rectangular lattice \mathbb{Z}^2 on the right. Note how the Delone complex degenerates from six simplices to four simplices upon the lattice becoming rectangular. In particular, the Voronoi cell degenerates from a hexagon into a rectangle.

Obtuse superbases are closely related to short bases, or rather a generalisation thereof. In fact, these types of bases correspond to one-another as follows:

$$\begin{aligned} \{\text{superbases}\} &\leftrightarrow \{\text{short bases}\} \\ (v_1, v_2) &\rightarrow \left(v_1, v_1 - \text{sign} \left(\frac{\|v_2\|^2}{2} + \langle v_1, v_2 \rangle \right) v_2 \right) \\ \left. \begin{array}{l} (x, y) \quad \text{if } \langle x, y \rangle \leq 0 \\ (x, y - x) \quad \text{otherwise} \end{array} \right\} &\leftarrow (x, y). \end{aligned}$$

As every two-dimensional lattice Λ clearly admits a short basis, Λ admits an obtuse superbasis (v_1, v_2) . It may be observed that (v_1, v_2) is strictly obtuse if and only if every obtuse superbasis of Λ is strictly obtuse. If (v_1, v_2) is not strictly obtuse, then Λ is called *rectangular*. Note that just as the space of lattices $\mathcal{L}(2)$ may be parametrised in terms of short bases [GHL04, Sec. 2.24], it may be parametrised in terms of superbases.

B.2.2 Delone complex of two-dimensional lattices. If Λ is rectangular the relevant Voronoi vectors are $\pm v_1, \pm v_2$, the deep holes $(\pm v_1 \pm v_2)/2$ and the vertices of the corresponding Delone cell $0, \pm v_1, \pm v_2, \pm v_1 \pm v_2$. If Λ is not rectangular, the relevant Voronoi vectors are $\pm v_1, \pm v_2, \pm(v_1 + v_2)$, according to Prop. 1.9.5. The Delone cells are six mutually congruent triangles $\Delta_1, \dots, \Delta_6$ and the deep holes their circumcentres. In particular, the covering radius of Λ is the circumradius of any of these triangles: $\mu(\Lambda) = \mu(\Delta_i)^2$. Moreover, the covering radius is differentiable if expressed as function of the superbasis.

Thanks to the superbasis approach, it is now possible to give a simple proof of the following theorem, apparently well-known even before Bambah proved it (see 1.1.5 and the references specified there).

B.2.3 Theorem. A_2 is the unique globally optimal lattice covering of the plane (up to similarity).

Proof of Theorem B.2.3. According to Prop. 1.9.5, all lattices close of A_2 are lattices of Voronoi's first kind with strictly obtuse superbases. Thus, the thickness is determined entirely by any one of the triangles $\Delta_1, \dots, \Delta_6$ in a neighbourhood of A_2 . The theorem will therefore follow from the following more general result.

B.2.4 Proposition. Let Δ be a simplex in \mathbb{R}^n and define the thickness

$$\theta(\Delta) = \mu(\Delta)^n / (\det \Delta)^{1/2}.$$

Then Δ is a locally thinnest simplex if and only if Δ is equilateral.

Proof of Proposition B.2.4. Assume that $V = \mathbb{R}^n$ and imagine Δ as inscribed in the sphere S^{n-1} , thereby making its circumradius the origin. A thinnest simplex is one of maximal determinant amongst all such simplices.¹ Let

$$h: M = S^{n-1} \times \dots \times S^{n-1} \rightarrow \mathbb{R}, \quad h(x_0, \dots, x_n) = \det(x_1 - x_0, \dots, x_n - x_0).$$

The determinant of the simplex with \mathcal{V} -representation $\{x_0, \dots, x_n\}$ is $|h(x)|$. Note that the local maxima of $|h|$ are local extrema of h . For the *only if* part, it is therefore sufficient to prove that every critical point $v \in M$ of h is an equilateral simplex.

The symmetric group W acts on M via permutation of coordinates, and $h(\pi.v) = \text{sign } \pi \cdot h(v)$. It immediately follows that (1) $h(M) = [-c, c]$ for some $c > 0$, (2) $v \in M$ is a critical point if and only if πv is a critical point for all $\pi \in W$, (3)

$$\begin{aligned} h(v_0, \dots, v_n) &= \pm h(v_\ell, v_0, \dots, \widehat{v}_\ell, \dots, v_n) \\ &= \pm \det(v_0 - v_\ell, \dots, \widehat{v_\ell - v_\ell}, \dots, v_n - v_\ell) \end{aligned}$$

for all $\ell \in \{0, \dots, n\}$.²

Suppose that $v \in M$ is a critical point. Fix k , take $w \in T_{v_k} S^{n-1} = v_k^\perp$ and a curve $v_k: \mathbb{R} \rightarrow S^{n-1}$ with $\dot{v}_k(0) = w$. Fix $\ell \neq k$. Assuming that $\ell > k$ for convenience, use the fact that \det is alternating to obtain

$$\begin{aligned} 0 &= \frac{d}{dt} h(v_0, \dots, v_k(t), \dots, v_n) \\ &= \pm \frac{d}{dt} h(v_\ell, \dots, v_k(t), \dots, \widehat{v}_\ell, \dots, v_n) \\ &= \pm \frac{d}{dt} \det(v_0 - v_\ell, \dots, v_k(t) - v_\ell, \dots, v_n - v_\ell) \\ &= \pm \det(v_0 - v_\ell, \dots, w, \dots, v_n - v_\ell). \end{aligned}$$

¹Cf. footnote ²⁶ of Ch. 1.

²Where the superscript $\widehat{}$ marks a deleted component.

It follows that $w \in E_k = \text{span}\{v_0 - v_\ell, \dots, \widehat{v_k - v_\ell}, \dots, v_n - v_\ell\}$. Because w was an arbitrary element of $T_{v_k}S^{n-1}$, it follows that

$$v_k^\perp = T_{v_k}S^{n-1} = E_k.$$

The subspace E_k is parallel to the minimal affine subspace containing the k^{th} facet v^k of v . Therefore, $v_k \in E_k^\perp = H(v^k)$, where $H(v^k)$ denotes the bisector of the facet v^k . Since k was arbitrary, this holds for all k . It easily follows that v is equilateral. This completes the first part of the proof.

Conversely, it remains to prove that v is a global maximum of $|h|$ if v is equilateral. This follows from (1) above. In fact, $h(v) = \pm c$ and therefore $|h(v)| = c$. \square

C The Antipodal Set of $\mathrm{SO}(2n + 1)$

It is the purpose of this appendix to give a description of the antipodal set of $\mathrm{SO}(2n + 1)$, with respect to bi-invariant structure, naturally extending the description of $A(\mathrm{SO}(3))$ from 6.1.3. Note that Beyrer has already obtained a nearly complete classification of antipodal sets of compact symmetric spaces [Bey18]. He proves that the antipodal sets of $\mathrm{Spin}(2n + 1)$ for $n \geq 5$, E_8 and G_2 consist of one orbit of dimension $2n$, 128 and 6, respectively. The antipodal sets of these and other simple Lie groups can likely be calculated in the same fashion as below.

Let $G = \mathrm{SO}(2n + 1)$ and $T = T(n) \subseteq G$ the maximal torus. Then $T(n)$ is contained in the subgroup

$$U = \left\{ \left[\begin{array}{c|c} A & \\ \hline & 1 \end{array} \right] : A \in \mathrm{SO}(2n) \right\},$$

which is isomorphic to $\mathrm{SO}(2n)$. Therefore,

$$A_0 = \left[\begin{array}{c|c} -I_{2n} & \\ \hline & 1 \end{array} \right]$$

is an antipodal point of G , where I_m denotes the unit matrix of dimension m .

C.0.1 Proposition. *Let $m \geq 3$ and $p + q = m$. Define*

$$A_{pq} = \left[\begin{array}{c|c} -I_p & \\ \hline & I_q \end{array} \right].$$

Then the stabiliser of A_{pq} satisfies¹

$$Z(A_{pq}) = \left\{ \left[\begin{array}{c|c} A & \\ \hline & D \end{array} \right] : A \in \mathrm{O}(p), D \in \mathrm{O}(q), \det A \cdot \det D = 1 \right\}.$$

C.0.2 Corollary. $Z(A_0) = \left\{ \left[\begin{array}{c|c} A & \\ \hline & \det A \end{array} \right] : A \in \mathrm{O}(2n) \right\}.$

¹This stabiliser is isomorphic to the special part of $\mathrm{O}(p) \times \mathrm{O}(q)$, and usually denoted by $S(\mathrm{O}(p) \times \mathrm{O}(q))$.

Proof of Proposition C.0.1. ‘ \supseteq ’: Obvious. ‘ \subseteq ’: Let $P \in Z(A_{pq})$. There exist $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{q \times p}$, $D \in \mathbb{R}^{q \times q}$ so that

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

As $P \in \mathrm{SO}(m)$, it follows that

$$I_{2n+1} = P^\top \cdot P = \left[\begin{array}{c|c} A^\top A + C^\top C & A^\top B + C^\top D \\ \hline B^\top A + D^\top C & B^\top B + D^\top D \end{array} \right].$$

Extract from this the following equations:

$$A^\top A + C^\top C = I_p, \tag{C.0.1}$$

$$B^\top B + D^\top D = I_q. \tag{C.0.2}$$

As $P \in Z(A_{pq})$, it follows that

$$\begin{aligned} \left[\begin{array}{c|c} -I_p & \\ \hline & I_q \end{array} \right] &= \left[\begin{array}{c|c} A & C^\top \\ \hline B & D \end{array} \right] \left[\begin{array}{c|c} -I_p & \\ \hline & I_q \end{array} \right] \left[\begin{array}{c|c} A^\top & B \\ \hline C & D^\top \end{array} \right] \\ &= \left[\begin{array}{c|c} -A^\top A + C^\top C & -A^\top B + C^\top D \\ \hline -B^\top A + D^\top C & -B^\top B + D^\top D \end{array} \right]. \end{aligned}$$

Extract from this the following equations:

$$-A^\top A + C^\top C = -I_p, \tag{C.0.3}$$

$$-B^\top B + D^\top D = I_q. \tag{C.0.4}$$

Add (C.0.1) and (C.0.3) to obtain that $C^\top C = 0$ and deduce $C = 0$. Apply that to (C.0.1) to obtain $A^\top A = I_p$. In other words: $A \in \mathrm{O}(p)$. A similar argument using (C.0.2) and (C.0.4) shows that $B = 0$ and $D \in \mathrm{O}(q)$. As $P \in \mathrm{SO}(m)$, it follows that $\det A \cdot \det D = 1$. \square

Define $\tilde{U} = Z(A_0)$. Then $\tilde{U} \cong \mathrm{O}(2n)$ by virtue of the map

$$\mathrm{O}(2n) \rightarrow \tilde{U}, \quad A \mapsto \left[\begin{array}{c|c} A & \\ \hline & \det A \end{array} \right].$$

In particular, U is one of the two connected components of \tilde{U} .

C.0.3 Proposition. (1) $\mathrm{SO}(2n + 1)$ acts transitively on $A(\mathrm{SO}(2n + 1))$.

(2) The set $\tilde{A}_{I_{2n+1}}^{A_0}(\mathrm{SO}(2n + 1))$ of all shortest $X \in \mathfrak{so}(2n + 1)$ with $\exp X = A_0$ is contained in LU .

- (3) For every $A \in A(\mathrm{SO}(2n + 1))$, there exists a conjugate U' of U so that $\tilde{A}_e^A(\mathrm{SO}(2n + 1)) \subseteq LU'$. In particular, every minimising geodesic from e to A is contained in U' .

C.0.4 Corollary. *The antipodal set of $\mathrm{SO}(2n + 1)$ is isometric to $\tilde{U} \setminus \mathrm{SO}(2n + 1)$ with the metric induced from $\mathrm{SO}(2n + 1)$. In particular, the antipodal set is isometric to the real projective plane $\mathbb{R}P^{2n}$ of diameter $\pi/2$.*

Proof of Proposition C.0.3. (1) Let $G = \mathrm{SO}(2n + 1)$ and $A \in A(G)$. By the conjugation theorem, there exists $B \in G$ so that $c(B)(A) \in U$. But then $c(B)(A) \in A(U)$. It follows that $c(B)(A) = A_0$. Thus, $A \in c(G)(A_0)$.

(2) Let $H_0 \in \tilde{A}(U)$. Note that Prop. 3.3.6(2), combined with the fact that the Weyl group of B_n acts transitively on the set of deep holes of \mathbb{Z}^n , implies that $\tilde{A}_e^{A_0}(G) = \mathrm{Ad}(Z(A_0))(H_0)$. According to (1) and the remark preceding Cor. C.0.2,

$$\mathrm{Ad}(Z(A_0))(H_0) = \mathrm{Ad}(\tilde{U})(H_0) \subseteq L\tilde{U} = LU.$$

Thus, $\tilde{A}_e^{A_0}(G) \subseteq LU$.

- (3) This follows immediately from (1) and (2). □

Proof of Corollary C.0.4. Let $G = \mathrm{SO}(2n + 1)$. It follows from Prop. C.0.3 that the map

$$G \rightarrow A(G), \quad A \mapsto c(A)(A_0) = AA_0A^{-1}$$

induces the homeomorphism

$$G/\tilde{U} \rightarrow A(G), \quad A\tilde{U} \mapsto c(A)(A_0) = AA_0A^{-1}.$$

According to [War83, 3.65(c)], the coset space G/\tilde{U} is diffeomorphic to $\mathbb{R}P^{2n}$. In particular, $A(G)$ is a submanifold of G and inherits a homogeneous metric as described in [Sak97, App. 2].

Note that G/U is diffeomorphic to S^{2n} [War83, 3.65(a)]. The natural map $G/U \rightarrow G/\tilde{U}$ is equivalent to the covering $S^{2n} \rightarrow \mathbb{R}P^{2n}$. This yields the following commutative diagram with constant rows: The map f is $G \rightarrow S^{2n}$, $A \mapsto Ae_{2n+1}$. To compute the

$$\begin{array}{ccccc} U & \longrightarrow & G & \xrightarrow{f} & S^{2n} \\ \downarrow & & \parallel & & \downarrow \\ \tilde{U} & \longrightarrow & G & \longrightarrow & \mathbb{R}P^{2n}. \end{array}$$

diameter of $\mathrm{SO}(2n + 1)/\tilde{U}$, it must be proved that the induced map $\tilde{f}: M = G/U \rightarrow S^{2n}$

is an isometry to the round sphere of diameter π . Observe that $T_U M = \mathfrak{so}(2n+1)/\mathfrak{so}(2n)$ is generated by $\{X_{i,2n+1} : i = 1, \dots, 2n\}$.² It follows that

$$f(\exp(tX_{i,2n+1})) = \sin t \cdot e_i + \cos t \cdot e_{2n+1}.$$

Therefore, $D\tilde{f}_U(X_{i,2n+1}) = e_i$, i.e. $D\tilde{f}_U$ is a linear isometry. Thus, $\mathrm{dia}(G/U) = \pi$. It follows that $\mathrm{dia}(G/\tilde{U}) = \pi/2$. \square

C.0.5 Geometric interpretation. Let $G = \mathrm{SO}(2n + 1)$. Note that U consists of those $A \in \mathrm{SO}(2n + 1)$ that leave e_{2n+1} fixed and \tilde{U} of those that leave $L_0 = \mathbb{R} \cdot e_{2n+1}$ invariant. According to the conjugation theorem, for every $A \in G$, there exists $B \in G$ so that $c(B)(A) \in U$. This can immediately be interpreted geometrically: For every $A \in G$, there exists a one-dimensional subspace $L \subseteq \mathbb{R}^{2n+1}$ with $A|_L = \mathrm{Id}_L$. Let $B \in G$ so that B maps L_0 to L [dC92, Ch. 9, 3.8]. Then BAB^{-1} leaves L_0 fixed. It follows that $BAB^{-1} \in U$.

The antipodal point of U is the point reflection of the hyperspace L_0^\perp in the origin. In particular, the antipodal point of $c(B^{-1})(U)$ is the point reflection of the hyperspace L^\perp in the origin.

²Cf. 2.13.1.

D Fincke–Pohst Algorithm and Applications

D.1 Coxeter Lattices

D.1.1 Coxeter lattices. Each of the irreducible root lattices L from the first chapter is integral, including A_n . The intermediate lattices $A_n \subseteq \Lambda \subseteq A_n^*$ are called *Coxeter lattices*. Therefore, a Coxeter lattice is obtained by self-glueing the subgroup Λ/A_n of the glue group $A_n^*/A_n = \mathbb{Z}_{n+1}$ to A_n . Thus, the Coxeter lattices are the lattices A_n^r constructed by taking a divisor r of $n + 1$, setting $s = (n + 1)/r$ and

$$A_n^r = \bigcup_{i=0}^{r-1} ([is] + A_n),$$

where $[t]$ denotes the t^{th} glue vector. Note that $A_n^r = A_n[s]$. Obviously $[A_n^r : A_n] = r$, $[A_n^* : A_n^r] = s$. Using both formulas yields $\sqrt{\det A_n^r} = s/\sqrt{n+1} = \sqrt{n+1}/r$ and therefore $\det A_n^r = s/r$. It follows that the dual lattice of A_n^r is A_n^s . Some familiar lattices are similar to proper Coxeter lattices: $A_7^2 \cong E_7$, $A_7^4 \cong E_7^*$, $A_8^3 \cong E_8$ [Cox51, 12.3]. A typically useful basis of A_n^r in matrix form is

$$B_r = \left[\begin{array}{cccc|c} & & & & -\frac{s}{n+1} \\ & & & & \vdots \\ & \frac{1}{n+1} & & & -\frac{s}{n+1} \\ & & \ddots & & \frac{t}{n+1} \\ & & & \ddots & \vdots \\ & & & & -\frac{1}{n+1} \\ & & & & \vdots \\ \frac{1}{n+1} & \dots & \dots & \frac{1}{n+1} & \frac{t}{n+1} \end{array} \right],$$

where $s + t = n + 1$ and $-s/(n + 1)$ occurs t times and $t/(n + 1)$ occurs s times, i.e. the right-most column vector is the s^{th} glue vector $[s]$. The left part of the basis is a

basis of a A_{n-1} . Note that if $r \mid u$ then $A_n^r \subseteq A_n^u$. This inclusion can be read off of the basis B_u as follows: If $(x_1, \dots, x_n) \in \mathbb{Z}^n$, then $B_u x \in A_n^r$ if and only if $(s/v) \mid x_n$, where $v = (n+1)/u$.

A complete description of the Voronoi cell V_n^r of A_n^r has not yet been obtained. The latest results are algorithmic computations determining V_n^r for all $n \leq 31$ and most divisors r of $n+1$ in terms of their Delone cells [DSSV09]. As mentioned earlier in 6.2.4, Yang has obtained a description of V_n^2 in the case $4 \mid (n+1)$. The Voronoi cells of certain Coxeter lattices have also been studied in [Anz02, Anz06, Anz15].¹

D.2 The Fincke–Pohst Algorithm

D.2.1. In [FP85], Fincke and Pohst present an algorithm FP that takes as input a positive quadratic form $A: \mathbb{R}^n \rightarrow \mathbb{R}$ and $C > 0$ and returns all elements $v \in \mathbb{Z}^n$ with $A(v) \leq C$, the *Fincke–Pohst algorithm*. For a lattice Λ of an n -dimensional subspace V of \mathbb{R}^m , specified either by a basis $B \in \mathbb{R}^{m \times n}$ or a positive quadratic form $A = B^\top B \in \mathbb{R}^{n \times n}$, this algorithm may be used to determine the set $\text{FP}(\Lambda, C)$ of all $v \in \Lambda$ with $\|v\| \leq C$. The author has written a C++ implementation of FP that uses only rational integers, so that calculations made by the algorithm are exact and may be used in mathematical proof. As a byproduct, a naive algorithm for computing the Voronoi cell of lattices $\Lambda \subseteq V \cap \mathbb{Q}^m$ is specified.

Rather than working with the matrix A itself, the FP algorithm uses the *quadratic completion* Q of A . The matrix $Q \in \mathbb{R}^{n \times n}$ satisfies

$$A(x) = \sum_{i=1}^n q_{ii}(x_i + \sum_{j=i+1}^n q_{ij}x_j)^2$$

and may be obtained from the Cholesky decomposition $A = R^\top R$ of A by elementary computation. To improve the tempo of the algorithm, R may be reconditioned to a *row reduced* version, i.e. transformed into some $S \in \mathbb{R}^{n \times n}$ with $A = S^\top S$ that has particularly short column vectors in the sense of [HPS08, Thm. 6.66]. The row reduced version S^{-1} is obtained by transforming R^{-1} with a unimodular matrix U^{-1} ,² using the LLL algorithm due to Lenstra, Lenstra and Lovász [LLL82] to R . Once Q is determined, either from R or S , the algorithm enters the while loop Algorithm D.1 below, which is paraphrased from [FP85, (2.8)] in pseudocode using zero-based numbering. It is noteworthy that the set $\text{FP}(\Lambda, C)$ is returned element by element. Algorithms that do not require access to the entire set $\text{FP}(\Lambda, C)$ at once may exploit this to reduce their memory usage.

¹The author was not able to obtain the translations of these articles cited in [DSSV09].

²A matrix is *unimodular* if $U \in \text{GL}(n, \mathbb{Z})$.

Algorithm D.1 The Fincke–Pohst algorithm [FP85, (2.8)].

```

1: (step 1)  $i = n - 1$ ,  $t_i = C$ ,  $u_i = 0$ 
2: while true do
3:   (step 2)  $z = (t_i/q_{ii})^{1/2}$ ,  $ub_i = \lfloor z - u_i \rfloor$ ,  $x_i = \lceil -z - u_i \rceil - 1$ 
4:   (step 3)  $x_i = x_i + 1$ 
5:   if  $x_i \leq ub_i$  then
6:     goto step 5
7:   else
8:     goto step 4
9:   (step 4)  $i = i + 1$ , goto step 3
10:  (step 5)
11:  if  $i == 0$  then
12:    goto step 6
13:  else
14:     $i = i - 1$ ,  $u_i = \sum_{j=i+1}^{n-1} q_{ij}x_j$ ,  $t_i = t_{i+1} - q_{i+1,i+1}(x_{i+1} + u_{i+1})^2$ 
15:    goto step 2
16:  (step 6)
17:  if  $x == 0$  then
18:    return
19:  else
20:    print  $x, -x$ 

```

D.2.2. As advertised, the Fincke–Pohst algorithm may be used to compute the Voronoi cell of L . In fact, let $C \geq 4 \cdot \mu(\Lambda)^2$.³ Then the output $\text{FP}(\Lambda, C)$ is a superset of the relevant Voronoi vectors. If Λ is a Coxeter lattice, then $\mu(A_n) = a(n + 1 - a)/(n + 1)$, where $a = \lfloor n + 1 \rfloor$ [CS99, Ch. 4, (54)], may be used as upper bound on $\mu(\Lambda)$. Once the superset $\text{FP}(\Lambda, C)$ is determined, the Voronoi cell is obtained by computing the polytope whose \mathcal{H} -representation consists of the halfplanes $E(v)$ for $v \in \text{FP}(\Lambda, C)$.

Note that if $\Lambda = A_n^r$, then it is sufficient to compute $K_n^r = V_n^r \cap K$ in place of the entire Voronoi cell \mathcal{V}_n^r , where K is a Weyl chamber of A_n , by intersecting K_n^1 with the halfplanes $E(v)$ for $v \in \text{FP}(\Lambda, C)$. Of course, the sharper the choice of C , the faster FP will terminate.

Examples of algorithms more sophisticated than the one presented here are the *diamond cutting algorithm*, which is also based on the Fincke–Pohst algorithm, and the algorithm presented in [DSSV09], which computes the Delone cells of a lattices, rather than the Voronoi cell.

³An upper bound on the covering radius $\mu(\Lambda)$ may always be obtained using the *successive minima* $\lambda_1(\Lambda) \leq \dots \leq \lambda_n$ of L . Namely, $2 \cdot \mu(\Lambda) \leq \sqrt{n} \cdot \lambda_n(\Lambda)$ [MG02, Thm. 7.9].

D.2.3 The C++ implementation, I. The C++ library RatFP [Kli17] contains an implementation of the Fincke–Pohst algorithm using the `boost::rational<int>` class from the *Boost Library* [boo] for computations with rational numbers. Classes for rational vectors and matrices are provided. As a result, all computations are exact. This requires no modifications of the original algorithm, except that care has to be taken when computing $\lfloor z - U_i \rfloor$ and $\lceil -z - U_i \rceil$ from step 2 of the Fincke–Pohst algorithm, as z is, in general, a nonrational algebraic number. This is done with the help of Lem. D.2.5 below.

To execute the algorithm, an object of the class FP must be constructed from some $A \in \mathcal{S}_{>0}(\mathbb{Q}^n)$. The quadratic completion is computed upon construction of the object. Every time the method `FP::next()` is called, Algorithm D.1 is looped through until a solution x is found. If $x \neq 0$, `FP::next()` returns `true` and x is obtained by the method `FP::x()`. Otherwise, `FP::next()` returns `false`. As an example, the code below shows that none of the antipodal points of PE_6 is a first conjugate point, which finally settles the proof of Cor. 6.4.5:

```
namespace PPL = Parma_Polyhedra_Library;
using namespace RatFP;
using rat=rat_type;

//E_6-program
RMatrix b{vector<vector<rat>>{
vector<rat >{-1,1,0,0,0,0,0,0},
vector<rat >{0,-1,1,0,0,0,0,0},
vector<rat >{0,0,-1,1,0,0,0,0},
vector<rat >{0,0,0,-1,1,0,0,0},
vector<rat >{0,0,0,0,-1,1,0,0},
vector<rat >{rat(1,2),-rat(1,2),-rat(1,2),-rat(1,2),
-rat(1,2),-rat(1,2),-rat(1,2),rat(1,2)},
}}};
b = b.transpose(); // column vectors, not row vectors, were specified!
RMatrix a = b.transpose()*b; // Cartan matrix
RVector deep_hole{vector<rat >{0,rat(1,3),rat(1,3),rat(1,3),
-rat(1,3),-rat(1,3),-rat(1,3),0}}};
rat mu = rat(2, 3); // squared covering radius
FP fp{a, 4*mu}; // all Voronoi relevant vectors have squared norm <= 4*mu

while (fp.next())
{
    RVector x = b*RVector(fp.x());
    RVector y = deep_hole - x;
    if (mu == y.dot()) // check if deep_hole lies on the bisector of x
    {
        cout << x << endl;
    }
}
return 0;
```


D.2.4 The C++ implementation, II. In addition to the implementation of the Fincke–Pohst algorithm, RatFP also contains a makeshift interface with the *Parma Polyhedra Library* [BHZ08], which may be used to compute intersections of halfplanes in euclidean space. In particular, one may use it to execute the algorithm for computing the Voronoi cell of Coxeter lattices, as described above. The Voronoi cell of A_n^r is obtained by calling `voronoi(n,r)`, which returns the polytope K_n^r using PPL’s `ascii_dump` format.

Unfortunately, most of the classes of PPL may not be initialised using rational numbers, only integers. This pertains, amongst others, to the instances of the classes `Linear_Expression` and `Constraint_System`. The problem is circumvented by considering $(n+1) \cdot A_n^r$ in place of A_n^r . One of the lucky exceptions is the `Generator` class, which is used to represent the points of the \mathcal{V} -representation of a polytope. This means that, even though the affine halfplanes intersecting a polytope can only be specified using integer arithmetic, the \mathcal{V} - or \mathcal{H} -representation of the resulting intersection may be returned as rational numbers. An assessment of the effectiveness of the algorithm is not available.

D.2.5 Lemma. *Let $q \in \mathbb{Q}_+$, $z = \sqrt{q}$, $u \in \mathbb{Q}_+$, $u = v + w$, where $v \in \mathbb{Z}$ and $w \in [0, 1)$. Moreover, let y be the maximal integer satisfying $y^2 \leq q$. Then*

$$\lfloor z - u \rfloor = \begin{cases} y - v & \text{if } (y + w)^2 \leq q, \\ y - v - 1 & \text{otherwise,} \end{cases}$$

and

$$\lceil -z - u \rceil = \begin{cases} -(y + v) & \text{if } (y + 1 - w)^2 > q, \\ -(y + v) - 1 & \text{otherwise.} \end{cases}$$

In particular, $\lfloor z - u \rfloor$ and $\lceil -z - u \rceil$ may be calculated without resorting to the use of floating point numbers.

Proof. Note that $z = y + x$ where $x \in [0, 1)$, i.e. $y = \lfloor z \rfloor$. Moreover, $(y + w)^2 \leq q$ if and only if $w \leq z - y = x$. Apply that $x - w \in (-1, 1)$ to obtain

$$\lfloor z - u \rfloor = \lfloor y - v + x - w \rfloor = \begin{cases} y - v & \text{if } x - w \geq 0, \\ y - v - 1 & \text{otherwise.} \end{cases}$$

Using $x = z - \lfloor z \rfloor$ yields that $x - w \geq 0$ is equivalent to $z \geq y + w$. Squaring this inequality implies the first equation.

Similarly, $(y + 1 - w)^2 > q$ if and only if $y + w - 1 < 0$. Apply that $x + w \in [0, 2)$ to obtain

$$\lceil -z - u \rceil = \lceil -(y + v) - (x + w) \rceil = \begin{cases} -(y + v) & \text{if } x + w - 1 < 0, \\ -(y + v) - 1 & \text{otherwise.} \end{cases}$$

Rewriting the inequality $x + w - 1 < 0$ as above implies the second equation. \square

D.3 Results and Conjectures

The results from computations with RatFP may be found in Tab. D.2. The corollary below follows immediately:

D.3.1 Corollary. *The following simple Lie groups with rank $n \leq 14$ have only finitely many antipodal points: The groups listed in Theorem B, $SU(4)/\mathbb{Z}_2$, $SU(6)/\mathbb{Z}_3$, $SU(8)/\mathbb{Z}_2$, $SU(12)/\mathbb{Z}_2$, $SU(12)/\mathbb{Z}_3$.*

Define $\Theta_n^r[t] = \|[t]\|^n \cdot V_n \cdot r^2 / (n + 1)$ and recall that $\det A_n^r = (n + 1)/r^2$. It follows that if a glue vector $[t]$ is a deep hole of A_n^r , then $\Theta(A_n^r) = \Theta_n^r[t]$. It is straightforward to write a program that returns the values $\Theta_n^r[t]$ as floating point numbers. Comparing the results with those found in [DSSV09] gives the table of Coxeter lattices A_n^r with $11 \leq n \leq 29$ whose thickness⁴ is equal to some $\Theta_n^r[t]$ for $t \leq 29$, up to machine precision, with the exception of $A_{23}^2, A_{25}^2, A_{27}^2, A_{27}^{14}, A_{29}^2$, which do not occur in [DSSV09, Tab. 1].

A	$[t]$	Θ_n^r
A_{11}^2	[3]	94.0909960562798
A_{11}^3	[2]	27.0896627506872
A_{15}^2	[4]	722.452642424053
A_{15}^4	[2]	25.3638592364654
A_{17}^3	[3]	240.511580360328
A_{23}^3	[4]	2405.03274678518
A_{23}^4	[3]	205.561225892535
A_{29}^5	[3]	25664.6441142646

Table D.1: Coxeter lattices of dimension $n \leq 29$ whose thickness is equal to some $\Theta_n^r[t]$.

Given this small window into a classification of Coxeter lattices by their Voronoi cell, one cannot help but make the following conjecture:

Conjecture D.3.2. The deep holes of A_n^r contain glue vectors if and only if $2r \mid (n + 1)$ and $r^2 \leq n + 1$. Moreover, if this is the case, then those glue vectors are $[(n + 1)/2r]$ and $[n + 1 - (n + 1)/2r]$.

Note that Yang’s result cited in 6.2.4 affirms the conjecture for $r = 2$. The odd occurrence of both glue and nonglue vectors as the deep holes of A_7^4 may be brushed off as singular occurrence ascribed to the ‘exceptional similarity’ $A_7^4 \cong E_7^*$,⁵ or seen as harbinger of

⁴In [DSSV09] the term *covering density* is used.

⁵Beware! The deep holes in Tab. D.2 are typical with respect to the root system A_n . This is why there occur *four* deep holes in Tab. D.2, not only the two typical deep holes contained in the narrower Weyl chamber of E_7 .

a general phenomenon. The lattices in Tab. D.1 may strike the reader as defining rather uneconomical coverings, at least when compared with the other Coxeter lattices in their dimensions. One should expect this pattern to continue.

Λ	typical deep holes
A_3^2	$[1] = (-3, (1)^3), [3] = ((-1)^3, 3)$
A_5^2	$((\frac{-3}{2})^4, \frac{3}{2}, \frac{9}{2}), (\frac{-9}{2}, \frac{-3}{2}, (\frac{3}{2})^4)$
A_5^3	$[1] = (-5, (1)^5), [5] = ((-1)^5, 5)$
A_7^2	$[2] = ((-2)^6, (6)^2), [6] = ((-6)^2, (2)^6)$
A_7^4	$[1] = (-7, (1)^7), [7] = ((-1)^7, 7), (-5, (-1)^4, (3)^3), ((-3)^3, (1)^4, 5)$
A_8^3	$((\frac{-3}{2})^7, 3, \frac{15}{2}), ((-3)^4, (\frac{3}{2})^4, 6), ((\frac{-9}{2})^2, (0)^5, (\frac{9}{2})^2), (-6, (\frac{-3}{2})^4, (3)^4),$ $(\frac{-15}{2}, -3, (\frac{3}{2})^7)$
A_9^2	$((\frac{-5}{2})^7, \frac{5}{2}, (\frac{15}{2})^2), ((\frac{-15}{2})^2, \frac{-5}{2}, (\frac{5}{2})^7)$
A_9^5	$((-4)^3, (0)^3, (2)^3, 6), ((-4)^2, (-2)^3, (2)^4, 6), (-6, (-2)^4, (2)^3, (4)^2),$ $(-6, (-2)^3, (0)^3, (4)^3), ((-4)^3, (0)^4, (4)^3)$
A_{11}^2	$[3] = ((-3)^9, (9)^3), [9] = ((-9)^3, (3)^9)$
A_{11}^3	$[2] = ((-2)^{10}, (10)^2), [10] = ((-10)^2, (2)^{10})$
A_{11}^4	$((\frac{-9}{2})^4, \frac{-3}{2}, (\frac{3}{2})^5, \frac{9}{2}, \frac{15}{2}), (\frac{-15}{2}, \frac{-9}{2}, (\frac{-3}{2})^5, \frac{3}{2}, (\frac{9}{2})^4)$
A_{11}^6	$((-4)^5, (2)^6, 8), (-8, (-2)^6, (4)^5)$
A_{13}^2	$((\frac{-7}{2})^{10}, \frac{7}{2}, (\frac{21}{2})^3), ((\frac{-21}{2})^3, \frac{-7}{2}, (\frac{7}{2})^{10})$
A_{13}^7	$((-6)^3, (-2)^4, (2)^3, (4)^3, 8), ((-6)^3, (-2)^3, (0)^3, (4)^4, 8),$ $((-6)^2, (-4)^3, (0)^4, (4)^4, 8), (-8, (-4)^3, (-2)^3, (2)^4, (6)^3),$ $(-8, (-4)^4, (0)^3, (2)^3, (6)^3), (-8, (-4)^4, (0)^4, (4)^3, (6)^2),$ $((-6)^3, (-2)^4, (2)^4, (6)^3)$

Table D.2: Typical deep holes of $(n+1) \cdot A_n^r$ for $n \leq 13$, $r \mid (n+1)$, $r \neq 1, n+1$, calculated using RatFP.

E Elements of $S_0^2(\mathfrak{g})$

This section contains the proof of Prop. 5.3.1, which asserts the following:

5.3.1 Proposition. *Let $n \geq 4$. Let π_0 be the trace-free part of the orthogonal projection $\mathfrak{su}(n) \rightarrow \mathfrak{h}$, and*

$$S' = \tilde{H}_1 \otimes \tilde{H}_{n-1} + \tilde{H}_{n-1} \otimes \tilde{H}_1 + \frac{1}{2} \left(-X_{1,n-1}^{\otimes 2} + X_{1,n}^{\otimes 2} + X_{2,n-1}^{\otimes 2} - X_{2,n}^{\otimes 2} \right) + \frac{1}{2} \left(-Y_{1,n-1}^{\otimes 2} + Y_{1,n}^{\otimes 2} + Y_{2,n-1}^{\otimes 2} - Y_{2,n}^{\otimes 2} \right), \quad (5.3.1)$$

$$S'' = \sum_{a=1}^{n-2} \left(-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2} - \frac{n}{4} \cdot X_{a,n-1}^{\otimes 2} + \frac{n}{4} \cdot X_{a,n}^{\otimes 2} - \frac{n}{4} \cdot Y_{a,n-1}^{\otimes 2} + \frac{n}{4} \cdot Y_{a,n}^{\otimes 2} \right). \quad (5.3.2)$$

Then $\pi_0 \in \mathfrak{g}^{(2)}$, $S' \in Y_2'$, $S'' \in Y_2''$.

Above and below, the notation from 2.12.1 and 2.13.1 is used, where the Lie algebras $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{su}(n)$ are described. The root spaces \mathfrak{g}_{ab} of $\mathfrak{g} = \mathfrak{su}(n)$ are generated by the skew-symmetric $X_{ab} = E_{aa} - E_{bb}$ and the symmetric $Y_{ab} = i(E_{aa} + E_{bb})$. The action of the Cartan algebra \mathfrak{h} on \mathfrak{g}_{ab} is described by the functionals $\hat{\alpha}_{ab}: \mathfrak{h} \rightarrow \mathfrak{h}$, $\hat{\alpha}(H) = H^{aa} - H^{bb}$ and the elements $\tilde{H}_{ab} = iH_{ab} \in \mathfrak{h}$, where $H_{ab} = E_{aa} - E_{bb}$ and superscripts denote matrix entries. The Lie brackets of the generators are found in Tab. E.1.

	X_{ab}	X_{ad}	X_{cd}	Y_{ab}	Y_{ad}	Y_{cd}	H
X_{ab}		$-X_{bd}$		$2\tilde{H}_{ab}$	$-Y_{bd}$		$-\hat{\alpha}_{ab}(H) \cdot Y_{ab}$
Y_{ab}	$-2\tilde{H}_{ab}$	Y_{bd}			$-X_{bd}$		$\hat{\alpha}_{ab}(H) \cdot X_{ab}$

Table E.1: The Lie bracket relations between the generators of the root spaces of $\mathfrak{su}(n)$. Here a, b, c, d denote pairwise distinct indices and H an arbitrary element of \mathfrak{h} .

Moreover, the tensor product $A \otimes A$ for $A \in \mathfrak{g}$ is abbreviated by $A^{\otimes 2}$. For $A, B \in \mathfrak{g}$ define $A \odot B = A \otimes B + B \otimes A$.¹ The proof of Prop. 5.3.1 is based on the characterisation

¹Beware! Some authors define $A \odot B = \frac{1}{2}(A \otimes B + B \otimes A)$, which has the advantage that $A \otimes B \mapsto A \odot B$ is a projection $\mathfrak{g}^{\otimes 2} \rightarrow S^2(\mathfrak{g})$.

of the \mathfrak{g} -modules as eigenspaces of the Casimir operator C and will make frequent use of Lem. E.0.1, given below. For the reader's convenience the Casimir operator is later split into its H -, X - and Y -parts, denoted by C_H , C_X and C_Y , respectively. This only serves to solve a spacial problem. Without this splitting, some of the terms would otherwise become too big to be displayed on a book page.

E.0.1 Lemma. *Let $1 \leq a < b \leq n$,*

$$I_{ab} = \{(c, d) : 1 \leq c < d \leq n, \quad |\{a, b\} \cap \{c, d\}| = 1\}$$

and $\hat{\alpha}_{cd} \in \mathfrak{h}^*$, $\hat{\alpha}_{cd}(H) = \Theta_{cd}(H)/i$. Then the following holds:

$$\begin{aligned} C(H_0 \otimes H_1) &= 4n \cdot H_0 \otimes H_1 - \sum_{c < d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot X_{cd} \otimes X_{cd} \\ &\quad - \sum_{c < d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot Y_{cd} \otimes Y_{cd} \quad \forall H_0, H_1 \in \mathfrak{h}, \end{aligned} \quad (\text{E.0.1})$$

$$\begin{aligned} C(X_{ab} \otimes X_{ab}) &= -4 \cdot \tilde{H}_{ab} \otimes \tilde{H}_{ab} + 4n \cdot X_{ab} \otimes X_{ab} - 4 \cdot Y_{ab} \otimes Y_{ab} \\ &\quad - \sum_{(c,d) \in I_{a,b}} X_{cd} \otimes X_{cd} - \sum_{(c,d) \in I_{a,b}} Y_{cd} \otimes Y_{cd}, \end{aligned} \quad (\text{E.0.2})$$

$$\begin{aligned} C(Y_{ab} \otimes Y_{ab}) &= -4 \cdot \tilde{H}_{ab} \otimes \tilde{H}_{ab} - 4 \cdot X_{ab} \otimes X_{ab} + 4n \cdot Y_{ab} \otimes Y_{ab} \\ &\quad - \sum_{(c,d) \in I_{ab}} X_{cd} \otimes X_{cd} - \sum_{(c,d) \in I_{ab}} Y_{cd} \otimes Y_{cd}. \end{aligned} \quad (\text{E.0.3})$$

Proof of Lemma E.0.1. Note that

$$C = \sum_a \hat{\varphi}(\tilde{H}_c \otimes \tilde{H}'_c) - \frac{1}{2} \sum_{c < d} \hat{\varphi}(X_{cd} \otimes X_{cd}) - \frac{1}{2} \sum_{c < d} \hat{\varphi}(Y_{cd} \otimes Y_{cd}).$$

The proof will make extensive use of the formulas from 2.12.1. Further cumbersome reference to them will be avoided from now on.

Sub-proof of (E.0.1). Obviously $\varphi(\tilde{H}_c \otimes \tilde{H}'_c)(H_0 \otimes H_1) = 0$. The summand of the second sum yields

$$\begin{aligned} \hat{\varphi}(X_{cd} \otimes X_{cd})(H_0 \otimes H_1) &= \varphi(X_{cd}) ([X_{cd}, H_0] \otimes H_1 + H_0 \otimes [X_{cd}, H_1]) \\ &= [X_{cd}, [X_{cd}, H_0]] \otimes H_1 \\ &\quad + H_0 \otimes [X_{cd}, [X_{cd}, H_1]] \\ &\quad + 2 \cdot [X_{cd}, H_0] \otimes [X_{cd}, H_1] \\ &= -\hat{\alpha}_{cd}(H_0) \cdot [X_{cd}, Y_{cd}] \otimes H_1 \\ &\quad - \hat{\alpha}_{cd}(H_1) \cdot H_0 \otimes [X_{cd}, Y_{cd}] \\ &\quad + 2\hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot Y_{cd} \otimes Y_{cd} \\ &= -2 \cdot \hat{\alpha}_{cd}(H_0) \cdot \tilde{H}_{cd} \otimes H_1 \\ &\quad - 2\hat{\alpha}_{cd}(H_1) \cdot H_0 \otimes \tilde{H}_{cd} \\ &\quad + 2\hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot Y_{cd} \otimes Y_{cd}. \end{aligned}$$

Summing these terms over all $c < d$, and applying the opening remark and Lem. 5.2.10 to the resulting first two terms gives

$$\begin{aligned} \sum_{c<d} \hat{\varphi}(X_{ab} \otimes X'_{ab})(H_0 \otimes H_1) &= \sum_{c<d} \hat{\alpha}_{cd}(H_0) \tilde{H}_{cd} \otimes H_1 \\ &\quad + \sum_{c<d} \hat{\alpha}_{cd}(H_1) \cdot H_0 \otimes \tilde{H}_{cd} \\ &\quad - \sum_{c<d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot Y_{cd} \otimes Y_{cd} \\ &= 2n \cdot H_0 \otimes H_1 - \sum_{c<d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot Y_{cd} \otimes Y_{cd}. \end{aligned}$$

Virtually the same computation, with the roles of X and Y reversed, yields

$$\sum_{c<d} \hat{\varphi}(Y_{ab} \otimes Y'_{ab})(H_0 \otimes H_1) = 2n \cdot H_0 \otimes H_1 - \sum_{c<d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot X_{cd} \otimes X_{cd}.$$

The sum of the previous two equations is

$$\begin{aligned} C(H_0 \otimes H_1) &= 4n \cdot H_0 \otimes H_1 - \sum_{c<d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot X_{cd} \otimes X_{cd} \\ &\quad - \sum_{c<d} \hat{\alpha}_{cd}(H_0) \hat{\alpha}_{cd}(H_1) \cdot Y_{cd} \otimes Y_{cd}. \end{aligned}$$

This completes the proof of the first formula.

Sub-proof of (E.0.2). It is best to treat the three expressions

$$\left\{ \begin{array}{l} \sum_c \hat{\varphi}(H_c \otimes H'_c)(X_{ab} \otimes X_{ab}), \\ \sum_{c<d} \hat{\varphi}(X_{cd} \otimes X_{cd})(X_{ab} \otimes X_{ab}), \\ \sum_{c<d} \hat{\varphi}(Y_{cd} \otimes Y_{cd})(X_{ab} \otimes Y_{ab}), \end{array} \right.$$

separately.

The summand of the *first expression* is

$$\begin{aligned} \hat{\varphi}(H_c \otimes H'_c)(X_{ab} \otimes X_{ab}) &= [H_c, [H'_c, X_{ab}]] \otimes X_{ab} + X_{ab} \otimes [H_c, [H'_c, X_{ab}]] \\ &\quad + [H_c, X_{ab}] \otimes [H'_c, X_{ab}] + [H'_c, X_{ab}] \otimes [H_c, X_{ab}] \\ &= \hat{\alpha}_{ab}(H'_c) \cdot [H_c, Y_{ab}] \otimes X_{ab} + \hat{\alpha}_{ab}(H'_c) \cdot X_{ab} \otimes [H_c, Y_{ab}] \\ &\quad + 2\hat{\alpha}_{ab}(H_c) \hat{\alpha}_{ab}(H'_c) \cdot Y_{ab} \otimes Y_{ab} \\ &= -2\hat{\alpha}_{ab}(H_c) \hat{\alpha}_{ab}(H'_c) \cdot X_{ab} \otimes X_{ab} + 2\hat{\alpha}_{ab}(H_c) \hat{\alpha}_{ab}(H'_c) \cdot Y_{ab} \otimes Y_{ab}. \end{aligned}$$

The entire sum is now dealt with by an application of Lem. 5.2.11 and the use of $\kappa(\hat{\alpha}_{ab}^\#, \hat{\alpha}_{ab}^\#) = -2$ (cf. 5.2.6):

$$\begin{aligned} \sum_c \hat{\varphi}(H_c \otimes H'_c)(X_{ab} \otimes X_{ab}) &= -2 \sum_c \hat{\alpha}_{ab}(H_c) \hat{\alpha}_{ab}(H'_c) \cdot X_{ab} \otimes X_{ab} \\ &\quad + 2 \sum_c \hat{\alpha}_{ab}(H_c) \hat{\alpha}_{ab}(H'_c) \cdot Y_{ab} \otimes Y_{ab} \\ &= 4X_{ab} \otimes X_{ab} - 4Y_{ab} \otimes Y_{ab}. \end{aligned} \quad (\text{E.0.4})$$

The summand of the *second expression*

$$\begin{aligned} \hat{\varphi}(X_{cd} \otimes X_{cd})(X_{ab} \otimes X_{ab}) &= [X_{cd}, [X_{cd}, X_{ad}]] \otimes X_{ad} + X_{ab} \otimes [X_{cd}, [X_{cd}, X_{ab}]] \\ &\quad + 2 \cdot [X_{cd}, X_{ab}] \otimes [X_{cd}, X_{ab}] \end{aligned}$$

must be treated in two cases. If $\{a, b\} = \{c, d\}$ or $\{a, b\} \cap \{c, d\} = \emptyset$, then

$$\hat{\varphi}(X_{cd} \otimes X_{cd})(X_{ab} \otimes X_{ab}) = 0.$$

The remaining case is $(c, d) \in I_{ab}$. Thanks to the symmetries of X_{ab} one may assume $a = c, b \neq d$ and easily obtain

$$[X_{ad}, [X_{ad}, X_{ab}]] = -X_{ab}.$$

It follows that

$$\hat{\varphi}(X_{ad} \otimes X_{ad})(X_{ab} \otimes X_{ab}) = -2 \cdot X_{ab} \otimes X_{ab} + 2 \cdot X_{bd} \otimes X_{bd}. \quad (\text{E.0.5})$$

Together with multiple applications of the skew-symmetry of X_{ab} , this yields

$$\begin{aligned} \sum_{c < d} \hat{\varphi}(X_{cd} \otimes X_{cd})(X_{ab} \otimes X_{ab}) &= \sum_{d > a, d \neq b} \hat{\varphi}(X_{ad} \otimes X_{ad})(X_{ab} \otimes X_{ab}) \\ &\quad + \sum_{c < b, c \neq a} \hat{\varphi}(X_{bc} \otimes X_{bc})(X_{ab} \otimes X_{ab}) \\ &= -2 \sum_{d > a, d \neq b} X_{ab} \otimes X_{ab} + 2 \sum_{d > a, d \neq b} X_{bd} \otimes X_{bd} \\ &\quad - 2 \sum_{c < b, c \neq a} X_{ab} \otimes X_{ab} + 2 \sum_{c < b, c \neq a} X_{ac} \otimes X_{ac} \\ &= -4(n-2) \cdot X_{ab} \otimes X_{ab} + 2 \sum_{(c,d) \in I_{ab}} X_{cd} \otimes X_{cd}. \end{aligned} \quad (\text{E.0.6})$$

Thus,

$$\sum_{c < d} \hat{\varphi}(X_{cd} \otimes X'_{cd})(X_{ab} \otimes X_{ab}) = 2(n-2) \cdot X_{ab} \otimes X_{ab} - \sum_{(c,d) \in I_{a,b}} X_{cd} \otimes X_{cd}. \quad (\text{E.0.7})$$

The summand of the *third expression*

$$\begin{aligned} \hat{\varphi}(X_{cd} \otimes X_{cd})(X_{ab} \otimes X_{ab}) &= [Y_{cd}, [Y_{cd}, X_{ad}]] \otimes X_{ad} + X_{ab} \otimes [Y_{cd}, [Y_{cd}, X_{ab}]] \\ &\quad + 2[Y_{cd}, X_{ab}] \otimes [Y_{cd}, X_{ab}] \end{aligned} \quad (\text{E.0.8})$$

must be treated in three cases. If $\{a, b\} \cap \{c, d\} = \emptyset$, then

$$\hat{\varphi}(Y_{cd} \otimes Y_{cd})(X_{ab} \otimes X_{ab}) = 0.$$

If $\{a, b\} = \{c, d\}$, then thanks to the (skew-)symmetries of X_{ab} and Y_{ab} it may be assumed that $(a, b) = (c, d)$, and it follows that

$$\begin{aligned} C(Y_{ab} \otimes Y_{ab})(X_{ab} \otimes X_{ab}) &= -2[Y_{ab}, \tilde{H}_{ab}] \otimes X_{ab} - 2 \cdot X_{ab} \otimes [Y_{ab}, \tilde{H}_{ab}] \\ &\quad + 8 \cdot \tilde{H}_{ab} \otimes \tilde{H}_{ab} \\ &= -4\hat{\alpha}_{ab}(\tilde{H}_{ab}) \cdot X_{ab} \otimes X_{ab} + 8 \cdot \tilde{H}_{ab} \otimes \tilde{H}_{ab} \\ &= -8 \cdot X_{ab} \otimes X_{ab} + 8 \cdot H_{ab} \otimes H_{ab}. \end{aligned} \quad (\text{E.0.9})$$

The remaining case is $|\{a, b\} \cap \{c, d\}| = 1$, and one may assume that $a = c, b \neq d$ and obtain

$$[Y_{ad}, [Y_{ad}, X_{ab}]] = -X_{ab}.$$

It follows that

$$\hat{\varphi}(Y_{ad} \otimes Y_{ad})(X_{ab} \otimes X_{ab}) = -2 \cdot X_{ab} \otimes X_{ab} + 2 \cdot Y_{bd} \otimes Y_{bd}. \quad (\text{E.0.10})$$

Together with multiple applications of the (skew-)symmetries of X_{ab} and Y_{ab} , this yields

$$\begin{aligned} \sum_{c < d} \hat{\varphi}(Y_{cd} \otimes Y_{cd})(X_{ab} \otimes X_{ab}) &= \sum_{d > a, d \neq b} \hat{\varphi}(Y_{ad} \otimes Y_{ad})(X_{ab} \otimes X_{ab}) \\ &\quad + \sum_{c < b, c \neq a} \hat{\varphi}(Y_{bc} \otimes Y_{bc})(X_{ab} \otimes X_{ab}) \\ &= -8 \cdot X_{ab} \otimes X_{ab} + 8 \cdot H_{ab} \otimes H_{ab} \\ &\quad - 2 \sum_{d > a, d \neq b} Y_{ab} \otimes Y_{ab} + 2 \sum_{d > a, d \neq b} Y_{bd} \otimes Y_{bd} \\ &\quad - 2 \sum_{c < b, c \neq a} Y_{ab} \otimes Y_{ab} + 2 \sum_{c < b, c \neq a} Y_{ac} \otimes Y_{ac} \\ &= -4n \cdot X_{ab} \otimes X_{ab} + 2 \sum_{(c,d) \in I_{a,b}} Y_{cd} \otimes X_{cd}. \end{aligned} \quad (\text{E.0.11})$$

Thus,

$$\sum_{c < d} \hat{\varphi}(Y_{cd} \otimes Y'_{cd})(X_{ab} \otimes X_{ab}) = 2n \cdot X_{ab} \otimes X_{ab} - \sum_{(c,d) \in a,b} Y_{cd} \otimes X_{cd}. \quad (\text{E.0.12})$$

Now add the equations (E.0.4), (E.0.7) and (E.0.12) to obtain the second formula of Lem. E.0.1.

This completes the proof of the second formula. The proof of the third formula is completely analogous to the proof of the second formula. The fact that Y_{ab} is symmetric and X_{ab} is skew-symmetric in (a, b) accounts for the differences in sign. \square

Proof of Proposition 5.3.1. Obviously $\pi_0, S', S'' \in S^2(\mathfrak{g})$. It remains to prove that they are eigenvectors of the Casimir operator.

‘ $\pi_0 \in \mathfrak{g}^{(2)}$ ’: Most of the work for this case is already accomplished. Observe that $\pi = \sum_a (iH_a) \otimes (-iH'_a) \in S^2(\mathfrak{g})$. It follows immediately that π is, as an element of $\mathfrak{g}_{\mathbb{C}}^{\otimes 2}$, equal to

$$\pi = \sum_a H_a \otimes H'_a,$$

the orthogonal projection $\pi: \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{h}_{\mathbb{C}}$. By Thm. 5.2.9, $\pi_0 \in Y_2(\alpha)$.

‘ $S' \in Y_2'$ ’: It is sufficient to prove that $C(S') = 4(n-1)S'$ in order to verify that $S' \in Y_2(\beta)$. The following calculations make excessive use of Lem. E.0.1. The image of $H_1 \odot H_{n-1}$ is straightforward to calculate in its entirety:

$$\begin{aligned} C(H_1 \odot H_{n-1}) &= 4n \cdot H_1 \odot H_{n-1} \\ &\quad + 2X_{1,n-1}^{\otimes 2} - 2X_{1,n}^{\otimes 2} - 2X_{2,n-1}^{\otimes 2} + 2X_{2,n}^{\otimes 2} \\ &\quad + 2Y_{1,n-1}^{\otimes 2} - 2Y_{1,n}^{\otimes 2} - 2Y_{2,n-1}^{\otimes 2} + 2Y_{2,n}^{\otimes 2}. \end{aligned} \quad (\text{E.0.13})$$

The images of the X - and Y -part will be split into their H -, X - and Y -part and calculated separately.

H-part. The H -parts of the X -summands are

$$\begin{aligned} C_H(X_{1,n-1}^{\otimes 2}) &= -4\tilde{H}_{1,n-1}^{\otimes 2}, & C_H(X_{1n}^{\otimes 2}) &= -4\tilde{H}_{1n}^{\otimes 2}, \\ C_H(X_{2,n-1}^{\otimes 2}) &= -4\tilde{H}_{2,n-1}^{\otimes 2}, & C_H(X_{2n}^{\otimes 2}) &= -4\tilde{H}_{2n}^{\otimes 2}. \end{aligned}$$

Let $E_a = iE_{aa}$. Then the formulas above yield

$$\begin{aligned} &C_H \left(-\frac{1}{2}X_{1,n-1}^{\otimes 2} + \frac{1}{2}X_{1,n}^{\otimes 2} + \frac{1}{2}X_{2,n-1}^{\otimes 2} - \frac{1}{2}X_{2,n}^{\otimes 2} \right) \\ &= 2 \left(\tilde{H}_{1,n-1}^{\otimes 2} - \tilde{H}_{1n}^{\otimes 2} - \tilde{H}_{2,n-1}^{\otimes 2} + \tilde{H}_{2n}^{\otimes 2} \right) \\ &= 2(E_1 \otimes E_1 - E_1 \otimes E_{n-1} - E_{n-1} \otimes E_1 + E_{n-1} \otimes E_{n-1} \\ &\quad - E_1 \otimes E_1 + E_1 \otimes E_n + E_n \otimes E_1 - E_n \otimes E_n \\ &\quad - E_2 \otimes E_2 + E_2 \otimes E_{n-1} + E_{n-1} \otimes E_2 - E_{n-1} \otimes E_{n-1} \\ &\quad + E_2 \otimes E_2 - E_2 \otimes E_n - E_n \otimes E_2 + E_n \otimes E_n). \\ &= -2 \cdot H_1 \otimes H_{n-1} - 2 \cdot H_{n-1} \otimes H_1 \end{aligned} \quad (\text{E.0.14})$$

and, by virtually the same computation,

$$C_H \left(-\frac{1}{2}Y_{1,n-1}^{\otimes 2} + \frac{1}{2}Y_{1,n}^{\otimes 2} + \frac{1}{2}Y_{2,n-1}^{\otimes 2} - \frac{1}{2}Y_{2,n}^{\otimes 2} \right) = -2 \cdot H_1 \otimes H_{n-1} - 2 \cdot H_{n-1} \otimes H_1. \quad (\text{E.0.15})$$

Adding (E.0.13), (E.0.14) and (E.0.15) yields

$$C_H(S') = 4(n-1) \cdot \tilde{H}_1 \odot \tilde{H}_{n-1} = 4(n-1) \cdot (H\text{-part of } S').$$

X-part. Proceed with the $C_X(S')$ by computing all four terms $C_X(X_{ab})$ separately:

$$\begin{aligned} C_X\left(-\frac{1}{2}X_{1,n-1}^{\otimes 2}\right) &= -2n \cdot X_{1,n-1}^{\otimes 2} + \frac{1}{2} \sum_{(c,d) \in I_{1,n-1}} X_{cd}^{\otimes 2} \\ &= -2n \cdot X_{1,n-1}^{\otimes 2} + \frac{1}{2}X_{1,n}^{\otimes 2} + \frac{1}{2}X_{2,n-1}^{\otimes 2} \\ &\quad + \frac{1}{2} \sum_{d>1, d \neq n-1, n} X_{1,d}^{\otimes 2} + \frac{1}{2} \sum_{c<n-1, c \neq 1, 2} X_{c,n-1}^{\otimes 2} + \frac{1}{2}X_{n-1,n}^{\otimes 2}. \end{aligned} \quad (\text{E.0.16})$$

The others are obtained by the same method:

$$\begin{aligned} C_X\left(\frac{1}{2}X_{1,n}^{\otimes 2}\right) &= 2n \cdot X_{1,n}^{\otimes 2} - \frac{1}{2}X_{1,n-1}^{\otimes 2} - \frac{1}{2}X_{2,n}^{\otimes 2} \\ &\quad - \frac{1}{2} \sum_{d>1, d \neq n-1, n} X_{1,d}^{\otimes 2} - \frac{1}{2} \sum_{c<n, d \neq 1, 2} X_{c,n}^{\otimes 2}, \end{aligned} \quad (\text{E.0.17})$$

$$\begin{aligned} C_X\left(\frac{1}{2}X_{2,n-1}^{\otimes 2}\right) &= 2n \cdot X_{2,n-1}^{\otimes 2} - \frac{1}{2}X_{1,n-1}^{\otimes 2} - \frac{1}{2}X_{2,n}^{\otimes 2} \\ &\quad - \frac{1}{2}X_{12}^{\otimes 2} - \frac{1}{2} \sum_{d>2, d \neq n-1, n} X_{1,d}^{\otimes 2} - \frac{1}{2} \sum_{c<n-1, d \neq 1, 2} X_{c,n-1}^{\otimes 2} - \frac{1}{2}X_{n-1,n}^{\otimes 2}, \end{aligned} \quad (\text{E.0.18})$$

$$\begin{aligned} C_X\left(-\frac{1}{2}X_{2,n}^{\otimes 2}\right) &= -2n \cdot X_{2,n}^{\otimes 2} + \frac{1}{2}X_{1,n}^{\otimes 2} + \frac{1}{2}X_{2,n-1}^{\otimes 2} \\ &\quad + \frac{1}{2}X_{12}^{\otimes 2} + \frac{1}{2} \sum_{d>2, d \neq n-1, n} X_{1,d}^{\otimes 2} + \frac{1}{2} \sum_{c<n, c \neq 1, 2} X_{c,n}^{\otimes 2}. \end{aligned} \quad (\text{E.0.19})$$

When these four terms are summed up, the second lines cancel each other out. Therefore,

$$\begin{aligned} C_X\left(-\frac{1}{2}X_{1,n-1}^{\otimes 2} + \frac{1}{2}X_{1,n}^{\otimes 2} + \frac{1}{2}X_{2,n-1}^{\otimes 2} - \frac{1}{2}X_{2,n}^{\otimes 2}\right) \\ = (4n+2) \left(-\frac{1}{2}X_{1,n-1}^{\otimes 2} + \frac{1}{2}X_{1,n}^{\otimes 2} + \frac{1}{2}X_{2,n-1}^{\otimes 2} - \frac{1}{2}X_{2,n}^{\otimes 2}\right). \end{aligned} \quad (\text{E.0.20})$$

The formulas for the C_X -image of the Y -terms from the definition of S' are found to be very similar to (E.0.16)–(E.0.19). In fact, if ‘ X ’ is replaced by ‘ Y ’ and ‘ n ’ by ‘ -1 ’, the correct equations are obtained:

$$\begin{aligned} C_X\left(-\frac{1}{2}Y_{1,n-1}^{\otimes 2}\right) &= 2 \cdot X_{1,n-1}^{\otimes 2} + \frac{1}{2}X_{1,n}^{\otimes 2} + \frac{1}{2}X_{2,n-1}^{\otimes 2} \\ &\quad + \frac{1}{2} \sum_{d>1, d \neq n-1, n} X_{1,d}^{\otimes 2} + \frac{1}{2} \sum_{c<n-1, c \neq 1, 2} X_{c,n-1}^{\otimes 2} + \frac{1}{2}X_{n-1,n}^{\otimes 2}, \\ C_X\left(\frac{1}{2}Y_{1,n}^{\otimes 2}\right) &= -2 \cdot X_{1,n}^{\otimes 2} - \frac{1}{2}X_{1,n-1}^{\otimes 2} - \frac{1}{2}X_{2,n}^{\otimes 2} \\ &\quad - \frac{1}{2} \sum_{d>1, d \neq n-1, n} X_{1,d}^{\otimes 2} - \frac{1}{2} \sum_{c<n, d \neq 1, 2} X_{c,n}^{\otimes 2}, \end{aligned}$$

$$\begin{aligned}
C_X \left(\frac{1}{2} Y_{2,n-1}^{\otimes 2} \right) &= -2 \cdot X_{2,n-1}^{\otimes 2} - \frac{1}{2} X_{1,n-1}^{\otimes 2} - \frac{1}{2} X_{2,n}^{\otimes 2} \\
&\quad - \frac{1}{2} X_{12}^{\otimes 2} - \frac{1}{2} \sum_{d>2, d \neq n-1, n} X_{1,d}^{\otimes 2} - \frac{1}{2} \sum_{c<n-1, d \neq 1, 2} X_{c,n-1}^{\otimes 2} - \frac{1}{2} X_{n-1,n}^{\otimes 2} \\
C_X \left(-\frac{1}{2} Y_{2,n}^{\otimes 2} \right) &= 2 \cdot X_{2,n}^{\otimes 2} + \frac{1}{2} X_{1,n}^{\otimes 2} + \frac{1}{2} X_{2,n-1}^{\otimes 2} \\
&\quad + \frac{1}{2} X_{12}^{\otimes 2} + \frac{1}{2} \sum_{d>2, d \neq n-1, n} X_{1,d}^{\otimes 2} + \frac{1}{2} \sum_{c<n, c \neq 1, 2} X_{c,n}^{\otimes 2}.
\end{aligned}$$

An immediate consequence is that

$$\begin{aligned}
C_X \left(-\frac{1}{2} Y_{1,n-1}^{\otimes 2} + \frac{1}{2} Y_{1,n}^{\otimes 2} + \frac{1}{2} Y_{2,n-1}^{\otimes 2} - \frac{1}{2} Y_{2,n}^{\otimes 2} \right) \\
= -2 \left(-\frac{1}{2} X_{1,n-1}^{\otimes 2} + \frac{1}{2} X_{1,n}^{\otimes 2} + \frac{1}{2} X_{2,n-1}^{\otimes 2} - \frac{1}{2} X_{2,n}^{\otimes 2} \right). \tag{E.0.21}
\end{aligned}$$

Now add (E.0.13), (E.0.20) and (E.0.21) to obtain

$$C_X(S') = 4(n-1) \cdot (X\text{-part of } S').$$

Y-part. Thanks to the symmetries in the formulas in Lem. E.0.1, the calculation of the *Y*-part of the Casimir operator is completely analogous to the computation of the *X*-part above and yields

$$C_Y(S') = 4(n-1) \cdot (Y\text{-part of } S').$$

Combining all equations obtained above shows that $S' \in Y_2(\beta)$.

' $S'' \in Y_2''$ ': It is sufficient to prove that $C(S'') = 2nS''$ in order to verify that $S'' \in Y_2(\gamma)$. The procedure of showing this is the same as above, but requires the following lemma:

E.0.2 Lemma. *Let $\{b, f\} = \{n-1, n\}$. Then*

$$\sum_{a=1}^{n-2} \sum_{(c,d) \in I_{a,b}} X_{cd}^{\otimes 2} - \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{a,f}} X_{cd}^{\otimes 2} = -(n-4) \sum_{a=1}^{n-2} (-X_{a,b}^{\otimes 2} + X_{a,f}^{\otimes 2}).$$

Proof of Lemma E.0.2. Decompose the sum $\sum_{(c,d) \in I_{ab}} X_{cd}^{\otimes 2}$ into its upper and lower summands, and those with mixed indices:

$$\sum_{(c,d) \in I_{a,b}} X_{cd}^{\otimes 2} = X_{n-1,n}^{\otimes 2} + \sum_{1=c<d}^{n-2} X_{cd}^{\otimes 2} + X_{a,f}^{\otimes 2} + \sum_{c \leq n-2, c \neq a} X_{cd}^{\otimes 2}.$$

This implies that

$$\begin{aligned}
\sum_{a=1}^{n-2} \sum_{(c,d) \in I_{ab}} X_{cd}^{\otimes 2} &= \sum_{a=1}^{n-2} X_{n-1,n}^{\otimes 2} + \sum_{1=c<d}^{n-2} X_{cd}^{\otimes 2} + \sum_{a=1}^{n-2} X_{a,f}^{\otimes 2} + \sum_{a=1}^{n-2} \sum_{c \leq n-2, c \neq a} X_{cd}^{\otimes 2} \\
&= (n-2) \cdot X_{n-1,n}^{\otimes 2} + \sum_{1=c<d}^{n-2} X_{cd}^{\otimes 2} + \sum_{a=1}^{n-2} X_{a,f}^{\otimes 2} + (n-3) \sum_{a=1}^{n-2} X_{ab}^{\otimes 2}.
\end{aligned}$$

Thus,

$$\begin{aligned} \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{ab}} X_{cd}^{\otimes 2} - \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{af}} X_{cd}^{\otimes 2} &= \sum_{a=1}^{n-2} ((n-3) \cdot X_{a,b} + X_{a,f} - (n-3) \cdot X_{a,f} - X_{a,b}) \\ &= -(n-4) \sum_{a=1}^{n-2} (-X_{a,b}^{\otimes 2} + X_{a,f}^{\otimes 2}). \quad \square \end{aligned}$$

Apply Lem. E.0.2 to the HX -part \tilde{C} of C :

$$\begin{aligned} \sum_{a=1}^{n-2} \tilde{C}(-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) &= 4n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) \\ &\quad + \sum_{a=1}^{n-2} \sum_{c < d} \hat{\alpha}_{cd} (\tilde{H}_{a,n-1})^2 X_{cd} - \sum_{a=1}^{n-2} \sum_{c < d} \hat{\alpha}_{cd} (\tilde{H}_{a,n})^2 X_{cd} \\ &= 4n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) - 4 \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) \\ &\quad + \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{1,n-1}} X_{cd}^{\otimes 2} - \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{1,n}} X_{cd}^{\otimes 2} \\ &= 4n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) - n \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}). \quad (\text{E.0.22}) \end{aligned}$$

Proceed with the remaining parts:

$$\begin{aligned} \frac{n}{4} \sum_{a=1}^{n-2} \tilde{C}(-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) &= -n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) + 4n \frac{n}{4} \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) \\ &\quad + \frac{n}{4} \sum_{(c,d) \in I_{a,n-1}} X_{cd}^{\otimes 2} - \frac{n}{4} \sum_{(c,d) \in I_{a,n}} X_{cd}^{\otimes 2}, \\ \frac{n}{4} \sum_{a=1}^{n-2} \tilde{C}(-Y_{a,n-1}^{\otimes 2} + Y_{a,n}^{\otimes 2}) &= -n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) - 4 \frac{n}{4} \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) \\ &\quad + \frac{n}{4} \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{a,n-1}} X_{cd}^{\otimes 2} - \frac{n}{4} \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{a,n}} X_{cd}^{\otimes 2}, \end{aligned}$$

and apply Lem. E.0.2 to their sum:

$$\begin{aligned} \frac{n}{4} \tilde{C}(-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2} - Y_{a,n-1}^{\otimes 2} + Y_{a,n}^{\otimes 2}) &= -2n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) \\ &\quad + 4(n-1) \frac{n}{4} \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) \\ &\quad + \frac{n}{4} \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{a,n-1}} X_{cd}^{\otimes 2} - \frac{n}{4} \sum_{a=1}^{n-2} \sum_{(c,d) \in I_{a,n}} X_{cd}^{\otimes 2} \end{aligned}$$

$$\begin{aligned}
&= -2n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) \\
&\quad + 4(n-1) \frac{n}{4} \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) \\
&\quad - 2(n-4) \frac{n}{4} \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}) \\
&= -2n \sum_{a=1}^{n-2} (-H_{a,n-1}^{\otimes 2} + H_{a,n}^{\otimes 2}) \\
&\quad + (2n+1) \frac{n}{4} \sum_{a=1}^{n-2} (-X_{a,n-1}^{\otimes 2} + X_{a,n}^{\otimes 2}). \quad (\text{E.0.23})
\end{aligned}$$

Summing (E.0.22) and (E.0.23) yields

$$\tilde{C}(S'') = 2n \cdot (HX\text{-part of } S'').$$

Equations similar to (E.0.22) and (E.0.23), obtained by replacing ‘ X ’ on the right-hand side by ‘ Y ’, are also correct. Summing these equations up yields

$$C_{HY}(S'') = 2n \cdot (HY\text{-part of } S'').$$

Both equations combined finally show that $S'' \in Y_2(\gamma)$. □

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Erklärung

Hiermit erkläre ich an Eides statt, dass die vorliegende Abhandlung - abgesehen von der Beratung durch Prof. Dr. Jens Heber - nach Inhalt und Form eigenständig und unter der Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft erstellt wurde, sie weder ganz noch zum Teil einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen hat, veröffentlicht wurde oder zur Veröffentlichung eingereicht wurde und mir noch nie ein akademischer Grad entzogen wurde.