

Dimension-free maximal and
 r -variational L^p -estimates of averaging
operators for products of Euclidean
balls and spheres

Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität
zu Kiel

vorgelegt von
Frederic Sommer

Kiel,
Juli 2019

Referent: Prof. Dr. Detlef Müller
Korreferent: Prof. Dr. Markus Haase

Tag der mündlichen Prüfung: 14.10.2019
Zum Druck genehmigt: 14.10.2019

Dekan: Prof. Dr. Frank Kempken

Dedicated to Tobias Löffler, beloved flatmate and formidable mathematician.
(1989 - 2018)

Abstract

This thesis provides a further contribution to the topic of dimension-free L^p -bounds for the centered Hardy-Littlewood maximal operator M_B , where B is a convex symmetric body in \mathbb{R}^n . It is well-known that M_B is of weak type $(1, 1)$ and of type (p, p) for all $p \in (1, \infty]$. The standard approach to this result uses the Vitali covering lemma, however, the constants occurring there grow exponentially with the dimension n .

In 1982, Stein gave a significant improvement by showing that if $p > 1$ and B is the Euclidean ball, the operator norm of M_B on $L^p(\mathbb{R}^n)$ can be bounded independently of n . Stein obtains these bounds by showing a similar result for spherical maximal operators. Stein's proof is followed by several further works on dimension-free bounds of M_B . In 1986, Bourgain showed that for $p = 2$, the operator norm of M_B can be bounded independently of n and B . In the same year, Bourgain and Carbery independently improved that result by proving that for every $p > 3/2$, M_B is bounded independently of n and B . For arbitrary B and $p \leq 3/2$, it is widely unknown whether such a general bound exists. Despite that, the range of p could be improved for some specific convex symmetric bodies. In 1990, Müller could solve the case where B is the ℓ^q -ball for $1 \leq q < \infty$. He showed that in this setting, the norm of M_B can be bounded for $p > 1$, by a constant that only depends on p and q . The case $q = \infty$, where B is a cube, remained open for a long time, until Bourgain solved it in 2014. Here, one can again find a dimension-free bound for every $p > 1$.

The first topic of this thesis builds on the latter results. We study the case where B is a cartesian product of Euclidean balls of arbitrary dimensions. By generalizing Bourgain's methods for the cube and Stein's methods for the ball, and combining them, we show that for $p > 1$, M_B can be bounded by a constant that only depends on p , but not on any occurring dimension. Similarly as in Stein's proof for one ball, this result is connected with a related result for products of spheres, which we also prove in this work.

As a second topic of this work, we consider the r -variational seminorm associated to the averaging operators which underlie M_B . Estimates of variational type are, in general, stronger than their maximal counterparts, and a work by Bourgain from 1989 reveals that such estimates have a wide field of applications, for instance in ergodic theory. Bourgain's fundamental ideas for r -variations led to several further works on inequalities for r -variations; among the most notable of these is an article by Jones, Seeger, and Wright. For $r > 2$, the latter work provides variational estimates for a large class of averaging operators.

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Inspired by the aforementioned works, Bourgain, Mirek, Stein, and Wróbel showed in 2017 that for $r > 2$, the r -variation of the averaging operators that underlie M_B can also be bounded from L^p to L^p , independently of the dimension. For variations, the dimension-free bounds transfer from the range $1 < p \leq \infty$ to the range $1 < p < \infty$, and from the range $3/2 < p \leq \infty$ to the range $3/2 < p < 4$. However, one can reobtain the aforementioned maximal bounds for their respective ranges of p from their variational counterparts. Based on their proof, we generalize our results for maximal operators to the corresponding results in the variational setting.

This thesis is structured as follows. In Chapter 1, we will introduce the centered Hardy-Littlewood maximal operator and provide a summary of the known dimension-free results. After that, we will present our results and an outline of our proofs. We conclude the first chapter by presenting some applications and some open problems.

Chapter 2 gives a comprehensive overview of the essential mathematical tools and methods from these topics, on which we will rely later on.

Chapter 3 will be dedicated to proving our results on maximal operators, and in Chapter 4, we will show our variational estimates.

Zusammenfassung

In dieser Dissertation setzen wir die Forschung auf dem Gebiet der dimensionsunabhängigen L^p -Schranken für den zentrierten Hardy-Littlewoodschen Maximaloperator M_B fort, wobei B ein konvexer, symmetrischer Körper im \mathbb{R}^n ist. Es ist bekannt, dass M_B vom schwachen Typ $(1, 1)$ und vom Typ (p, p) für alle $p \in (1, \infty]$ ist. Der gängigste Ansatz für dieses Resultat nutzt den Überdeckungssatz von Vitali, jedoch wachsen die Konstanten in diesem Beweis exponentiell in der Dimension n an.

Eine signifikante Verbesserung gelang Stein im Jahr 1982, indem er bewiesen hat, dass sich die Operatornorm von M_B unabhängig von n auf $L^p(\mathbb{R}^n)$ beschränken lässt, falls $p > 1$ und B die euklidische Kugel ist. Diesen Beweis führt Stein auf ein analoges Resultat für sphärische Maximaloperatoren zurück. Steins Beweis folgen mehrere weitere Ergebnisse mit dimensionsunabhängigen Schranken für M_B . 1986 zeigte Bourgain, dass die Operatornorm von M_B für $p = 2$ unabhängig von n und B beschränkt werden kann. Dieses Ergebnis wurde noch im selben Jahr von Bourgain und Carbery separat voneinander verbessert, indem sie bewiesen haben, dass M_B sogar für $p > 3/2$ unabhängig von n und B beschränkt ist. Für beliebiges B und $p \leq 3/2$ ist weitgehend unbekannt, ob es eine solch allgemeine Schranke gibt. Im Jahr 1990 konnte Müller den Fall, dass B die ℓ^q -Kugel für $1 \leq q < \infty$ ist, lösen und zeigen, dass sich die Norm von M_B dann für $p > 1$ durch eine Konstante beschränken lässt, welche nur von p und q abhängt. Der Fall $q = \infty$, also dass B ein Würfel ist, ist lange offen geblieben und wurde erst im Jahr 2014 von Bourgain gelöst. Auch hier lässt sich für jedes $p > 1$ eine dimensionsunabhängige Schranke finden.

Das erste Thema dieser Arbeit knüpft an das letztgenannte Resultat an. Wir untersuchen den Fall, dass B ein kartesisches Produkt verschiedendimensionaler euklidischer Kugeln ist. Mit einer Verallgemeinerung der Methoden von Bourgain für den Würfel und denen von Stein für die Kugel zeigen wir, dass sich M_B für $p > 1$ auch in diesem Fall durch eine Konstante beschränken lässt, welche nur von p abhängt, und von keiner hierbei auftretenden Dimension. Ähnlich wie in Steins Beweis für eine Kugel hängt dieses Resultat mit einem analogen Resultat für Produkte von Sphären zusammen.

Als zweites Thema dieser Arbeit betrachten wir die r -variationelle Halbnorm bezüglich der M_B zugrundeliegenden Mittelungsoperatoren. Abschätzungen variationellen Typs sind im Allgemeinen stärker als entsprechende Maximalabschätzungen. Eine Arbeit von Bourgain aus dem Jahr 1989 zeigt auf, dass derartige Ungleichungen einen breit gefächerten Anwendungsbereich haben, zum Beispiel in der Ergodentheorie. Bourgain's grundlegende Ansätze zu den r -Variationen haben zu diversen weiteren Arbeiten über

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r -variationelle Abschätzungen geführt. Unter den erwähnenswertesten Arbeiten ist ein Artikel von Jones, Seeger und Wright. Für $r > 2$ liefert die letztgenannte Arbeit r -variationelle Ungleichungen für eine umfassende Klasse von Mittelungsoperatoren.

Durch die oben erwähnten Arbeiten angeregt, haben Bourgain, Mirek, Stein und Wróbel 2017 gezeigt, dass die r -Variation bezüglich der M_B zugrundeliegenden Mittelungsoperatoren für $r > 2$ ebenfalls unabhängig von der Dimension von L^p nach L^p beschränkt werden kann. Die dimensionsunabhängigen Schranken übertragen sich für die Variationen dabei vom Bereich $1 < p \leq \infty$ auf den Bereich $1 < p < \infty$ und vom Bereich $3/2 < p \leq \infty$ auf den Bereich $3/2 < p < 4$. Dennoch kann man die zugehörigen Maximalabschätzungen daraus für ihre jeweiligen Bereiche von p wiedererhalten. Basierend auf deren Beweis konnten wir unsere Ergebnisse für Maximaloperatoren entsprechend für die Variationen verallgemeinern.

Diese Dissertation ist wie folgt strukturiert: In Kapitel 1 werden wir den zentrierten Hardy-Littlewoodschen Maximaloperator einführen und eine Zusammenfassung der bisherigen dimensionsunabhängigen Ergebnisse liefern. Anschließend werden wir unsere Ergebnisse sowie einen Überblick der Ideen für unsere Beweise präsentieren. Wir schließen das erste Kapitel ab, indem wir mehrere Anwendungen und offene Probleme aufzeigen. Kapitel 2 liefert eine umfassende Übersicht der wesentlichen mathematischen Werkzeuge und Ideen für die bisherigen Beweise auf diesem Gebiet, da wir von diesen später stark Gebrauch machen werden.

Kapitel 3 ist unseren Resultaten über Maximaloperatoren gewidmet, während wir in Kapitel 4 unsere variationellen Ergebnisse zeigen werden.

Danksagung

Zuerst möchte Herrn Prof. Dr. Detlef Müller für die Ermöglichung und Betreuung dieser Arbeit danken, dafür, dass er mich bereits im Masterstudium für das Thema dieser Arbeit begeistert hat, für seine ständige Unterstützung sowie für die zahlreichen interessanten Gespräche, für die er sich immer Zeit genommen hat.

Zudem danke ich Herrn Prof. Dr. Markus Haase dafür, dass er sich bereit erklärt hat, das Zweitgutachten zu erstellen.

Weiterhin danke ich meinen Kolleg*innen und ehemaligen Kolleg*innen Alexander, Florian, Ljudevit, Marco, Mareike, Nico, Olaf, Patrick und Stefan für die zahlreichen tollen Erlebnisse während meiner Promotionszeit sowie für diverse Gespräche mathematischer und nichtmathematischer Natur. Hervorheben möchte ich hierbei meinen Dank an Florian und Marco.

Meiner Familie gilt mein Dank dafür, dass sie während dieser Zeit stets hinter mir standen und mich in guten wie in weniger guten Zeiten ermutigt haben, am Ball zu bleiben.

Abschließend danke ich noch meinen Freund*innen Caroline und Moritz für ihre große moralische Unterstützung sowie Peer dafür, dass er mich dazu gebracht hat, mich für ein Studium der Mathematik zu entscheiden und somit die Schönheit dieser für mich zu entdecken.

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1 Introduction

In this chapter, we introduce the centered Hardy-Littlewood averaging operators M_t^B and summarize the history of dimension-free L^p -bounds for the corresponding centered maximal operator M_B and the associated r -variations $V_r(M_t^B f : t > 0)$. After that, we will present the results of this thesis for products of Euclidean balls and spheres, and provide an outline of the basic ideas of our proofs. This chapter concludes with some applications.

1.1 Averages and the centered Hardy-Littlewood maximal function

Maximal functions play a fundamental rôle in a wide variety of problems in harmonic analysis, the most prominent application perhaps being results about (almost everywhere) convergence and differentiation theory, leading to a rather simple proof of the Lebesgue differentiation theorem, as well as bounds for some singular integral operators.

Let $B \subset \mathbb{R}^n$ be a convex symmetric body, i.e. a closed, bounded, convex, and centrally symmetric set with nonempty interior. For such a set B , one can always find a norm on \mathbb{R}^n such that B is the unit ball of this norm. For that reason, it makes sense to write $B_t(x)$ for the set $x + tB$, whenever $x \in \mathbb{R}^n$ and $t > 0$. For any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $t > 0$, we define the averaging operator M_t^B pointwise by setting

$$M_t^B f(x) := \frac{1}{|B_t(x)|} \int_{B_t(x)} f(y) \, dy = \frac{1}{|B|} \int_B f(x + ty) \, dy, \quad (1.1.1)$$

where $|A|$ denotes the n -dimensional volume of any Lebesgue-measurable set $A \subset \mathbb{R}^n$. The *centered Hardy-Littlewood maximal function* $M_B f$ of f associated to B is then defined as

$$M_B f(x) := \sup_{t>0} M_t^B |f|(x) = \sup_{t>0} \frac{1}{|B_t(x)|} \int_{B_t(x)} |f(y)| \, dy. \quad (1.1.2)$$

M_B is called the centered Hardy-Littlewood maximal operator associated to B . We are interested in finding a bound for the operator norm $\|M_B\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)}$, $1 \leq p \leq \infty$. The first ones to study this problem were Hardy and Littlewood [16], who found bounds for one dimension. In their work, they gave the following explanation: “The problem is most easily grasped when stated in the language of cricket, or any other game in which a player compiles a series of scores of which an average is recorded.” In higher dimensions,

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the first result is due to Wiener [32], who considered the case where B is a Euclidean ball, and even adapted his methods to solve some problems in ergodic theory. One of Wiener's core results is that M_B is of weak type $(1, 1)$, i.e. there is a constant C_n such that for every $f \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$, we have

$$|\{x \in \mathbb{R}^n : M_B f(x) > \lambda\}| \leq \frac{C_n}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}. \quad (1.1.3)$$

This bound can be interpolated with the trivial bound from $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, which arises from the fact that $\|M_B f\|_\infty \leq \|f\|_\infty$ for every $f \in L^\infty(\mathbb{R}^n)$. By the Marcinkiewicz interpolation theorem, we thus get that for $1 < p \leq \infty$, there is a constant $C_{p,n}$ such that for every $f \in L^p(\mathbb{R}^n)$, we have

$$\|M_B f\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.1.4)$$

For $p = 1$, one can show that if for some positive $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we have $M_B f \in L^1(\mathbb{R}^n)$, then $f = 0$ almost everywhere.

As of today, the most common approach to show (1.1.3) uses the Vitali covering lemma. This lemma holds in much wider generality, for instance for spaces of homogeneous type. This approach also allows us to extend the supremum in definition (1.1.2) to range over any balls that contain the point x , resulting in a weak type $(1, 1)$ -bound for the uncentered maximal function

$$f_B^*(x) := \sup_{B \in \mathcal{B}, x \in B} \frac{1}{|B|} \int_B |f(y)| dy,$$

where $\mathcal{B} = \{B_t(y) : y \in \mathbb{R}^n, t > 0\}$, and B is any convex symmetric body.

It is not possible to find a dimension-free bound for f_B^* , but for the centered maximal operator, it turned out that in general, the constants $C_{p,n}$ in (1.1.4) are far from optimal. The original proof by Wiener and the covering lemma approach both lead to a constant C_n in (1.1.3) which grows exponentially in n , and thus, the Marcinkiewicz interpolation also delivers an exponential growth in n for the constant $C_{p,n}$ in (1.1.4). The problem of finding better L^p -bounds for M_B , especially bounds that are independent of n , has brought up a lot of interest and is one of the core topics of this thesis.

1.2 History of maximal and variational bounds

1.2.1 Dimension-free maximal bounds

In the following, denote by $C_p(n, B)$ the best possible constant for which (1.1.4) holds if $p > 1$, and the best constant for (1.1.3) if $p = 1$.

More than 40 years after the works of Hardy-Littlewood and Wiener, Stein [27] achieved a remarkable improvement of the known constants in a very captivating way. He showed that if B is a Euclidean ball and $p > 1$, one can bound $C_p(n, B)$ by a constant that only depends on p , but not on the dimension n . The main ideas for this proof are that for fixed

$p > 1$, one can replace the averages over balls by averages over spheres in sufficiently high dimensions, and bound the corresponding maximal function. By averaging over rotations of lower dimensional spheres, Stein then showed that one can increase the dimension in this case without increasing the bound on the operator norm of the corresponding maximal operator. The idea of averaging over rotations was originated by Calderón in the context of singular integrals. Stein's method of rotations is presented in Section 2.2, while for a detailed proof, we refer to [31]. In the latter work, Stein and Strömberg also showed that for a general convex body B , the constants $C_1(n, B)$ can be bounded by a constant of growth order $\mathcal{O}(n \log n)$, independently of B and n . Furthermore, they showed that for $p > 1$, the growth of the constants $C_p(n, B)$ is at most of order $\mathcal{O}(n^{\frac{p}{p-1}})$, also with an implicit constant that is independent of B and n .

These works have been followed up by the discovery of several more dimension-free bounds. The first general result is due to Bourgain [4], who showed that for any convex symmetric body B , the constants $C_2(n, B)$ can be bounded by some universal constant. This has been improved to the range $p > 3/2$ independently by Bourgain [5] and Carbery [12] shortly afterwards, meaning that in this case, each $C_p(n, B)$ can be bounded by a constant which depends only on p . They also showed that if the supremum in (1.1.2) only ranges over the dyadic numbers 2^k , $k \in \mathbb{Z}$, the resulting operator can be bounded independently of B and n for every $p > 1$. As of today, this is the best known range for p where $C_p(n, B)$ can be bounded independently of both n and B . However, a further improvement in the field of dimension-free results was due to Müller [21] in 1989, who introduced two geometric invariants $\sigma(B)$ and $Q(B)$ and showed that for each $p > 1$, $C_p(n, B) \leq C'(p, \sigma(B), Q(B))$, where the constant $C'(p, \sigma(B), Q(B))$ grows with $\sigma(B)$ and $Q(B)$, but bears no further dependence on n . Thus we get a dimension-free bound if $\sigma(B)$ and $Q(B)$ can both be bounded independently of n . Müller also showed that if $1 \leq q < \infty$ and B is an ℓ^q -ball, then one can indeed find bounds on $\sigma(B)$ and $Q(B)$ that only depend on q , so that for $p > 1$, we find a constant $C_{p,q}$ such that $C_p(n, B) \leq C_{p,q}$. We will present many of the main ideas of these results throughout Sections 2.1 - 2.7.

The case $q = \infty$, when B is a cube, remained open for a long time. Müller had shown in [21] that for the cube, $Q(B) = \sqrt{n}$. However, in 2014, Bourgain [7] showed that nevertheless, $C_p(n, B)$ can be bounded independently of n for $p > 1$. A surprising counterpart to this work is a preceding work by Aldaz [1] about the weak type $(1, 1)$ constant of M_B for the cube, in which he showed that $C_1(n, B) \rightarrow \infty$ as $n \rightarrow \infty$.

The nice survey article [13] gives a very detailed overview about the aforementioned dimension-free results. In my master's thesis [23], I have elaborated on Bourgain's work on the cube. Some of the major arguments for the cube will play an important rôle in Section 3.1.

We summarize the maximal dimension-free bounds for M_B that have been established so far in the following Theorem.

Theorem (Maximal results from [4, 5, 7, 12, 21, 27]). *Let B be a convex symmetric body*

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in \mathbb{R}^n .

(i) If $3/2 < p \leq \infty$, then there is a constant C_p independent of n and B such that

$$\|M_B f\|_{L^p} \leq C_p \|f\|_{L^p} \quad (1.2.1)$$

for every $f \in L^p(\mathbb{R}^n)$. Consequently, $C_p(n, B) \leq C_p$.

(ii) If $1 < p \leq \infty$, then there is a constant C'_p independent of n and B such that

$$\left\| \sup_{k \in \mathbb{Z}} M_{2^k}^B f \right\|_{L^p} \leq C'_p \|f\|_{L^p} \quad (1.2.2)$$

for every $f \in L^p(\mathbb{R}^n)$.

(iii) If $1 < p \leq \infty$, $1 \leq q \leq \infty$, and $B = \{x \in \mathbb{R}^n : \|x\|_q = (\sum_{j=1}^n |x_j|^q)^{1/q} \leq 1\}$, then there is a constant $C_{p,q}$ independent of n such that

$$\|M_B f\|_{L^p} \leq C_{p,q} \|f\|_{L^p} \quad (1.2.3)$$

for every $f \in L^p(\mathbb{R}^n)$. Consequently, $C_p(n, B) \leq C_{p,q}$.

1.2.2 Spherical maximal functions

Spherical maximal functions are a key object in Stein's approach for the case of the Euclidean ball, and we need to make use of a generalization of them as well. Let $n \geq 2$ and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the Euclidean sphere of radius 1, and let $\sigma = \sigma^{n-1}$ be the surface measure on S^{n-1} , so that $\sigma(S^{n-1})$ is the $(n-1)$ -dimensional volume of S^{n-1} . As an abbreviation, we write $|S^{n-1}| = \sigma(S^{n-1})$. For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, we consider the averaging operator

$$M_t^{S^{n-1}} f(x) := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x + t\omega) d\sigma^{n-1}(\omega) \quad (1.2.4)$$

and the associated maximal function

$$M_{S^{n-1}} f(x) := \sup_{t>0} M_t^{S^{n-1}} |f|(x). \quad (1.2.5)$$

Since the operators in (1.2.4) contain integrals over a set of n -dimensional Lebesgue measure 0, we need to restrict our function space to a dense subspace of $L^p(\mathbb{R}^n)$ in which each function is continuous. It is known that for $p > \frac{n}{n-1}$, there is a constant $C_{p,n}$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|M_{S^{n-1}} f\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.2.6)$$

The lower bound for p has been shown to be sharp. For $n \geq 3$, the inequality (1.2.6) has been derived by Stein [28] by means of Littlewood-Paley theory. The case $n = 2$ turned

out to be much harder, as one has to restrict to the range $p > 2$, so that no L^2 -theory can be applied. A proof for $n = 2$ has been given by Bourgain [3]. His methods based on many geometrical arguments. Later on, Mockenhaupt, Seeger, and Sogge [20] developed a different proof for the case $n = 2$, based on some new local smoothing estimates and techniques from microlocal analysis.

In his work [29], Stein showed that if for some $p > 1$, there is a constant C such that

$$\|M_{S^{n-1}}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, then also

$$\|M_{S^n}f\|_{L^p(\mathbb{R}^{n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{n+1})}$$

with the same constant C . This implies that the constant $C_{p,n}$ in (1.2.6) can be chosen independently of n . Together with the pointwise estimate

$$M_B f(x) \leq M_{S^{n-1}} f(x) \tag{1.2.7}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and B being the Euclidean ball, this leads to the dimension-free bound established in [27].

1.2.3 Variational seminorms and dimension-free variational bounds

More recently, dimension-free r -variational bounds have been established for the averaging operators M_t^B . These results are stronger than their corresponding maximal counterparts and allow some better convergence results and applications in ergodic theory.

The study of r -variational seminorms and their applications is based on ideas by Bourgain [6] and Lépingle [18]. Many techniques and ideas for r -variations have been developed in the work of Jones, Seeger, and Wright [17], which is a good reference on this topic.

Let $1 \leq r < \infty$, $Z \subseteq \mathbb{R}$, and $a: Z \rightarrow \mathbb{C}$, $t \mapsto a_t$, be a complex-valued function. Then we define the r -variational seminorm of a , often referred to as r -variation, as

$$V_r(a_t : t \in Z) := \sup_{\substack{t_0 < \dots < t_M \\ t_j \in Z, M \in \mathbb{N}}} \left(\sum_{l=0}^{M-1} |a_{t_{j+1}} - a_{t_j}|^r \right)^{1/r}. \tag{1.2.8}$$

For $r = \infty$, we also define

$$V_\infty(a_t : t \in Z) := \sup_{s, t \in Z} |a_t - a_s| \leq 2 \sup_{t \in Z} |a_t|.$$

The only purpose of the latter definition is for a few interpolation arguments. The r -variation allows us to deduce convergence easily. If for some $r \in [1, \infty)$,

$$V_r(a_t : t \in (0, \infty)) < \infty, \tag{1.2.9}$$

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then it follows that the limits $\lim_{t \rightarrow 0} a_t$ and $\lim_{t \rightarrow \infty} a_t$ both exist, since one can easily see that for any strictly positive sequence $(t_n)_{n \in \mathbb{N}}$ that tends to 0 or ∞ , $(a_{t_n})_{n \in \mathbb{N}}$ must be a Cauchy sequence. Such convergence statements also follow for general sets $Z \subset \mathbb{R}$ and their accumulation points in $\mathbb{R} \cup \{-\infty, \infty\}$.

Clearly, one can also deduce that for every $Z \subset \mathbb{R}$, $r \in [1, \infty)$, and $t_0 \in Z$, we have

$$\sup_{t \in Z} |a_t| \leq |a_{t_0}| + V_r(a_t : t \in Z). \quad (1.2.10)$$

Furthermore, $V_r(a_t : t \in Z)$ is decreasing in r , and if Z is countable, we have

$$V_r(a_t : t \in Z) \leq 2 \left(\sum_{t \in Z} |a_t|^r \right)^{1/r}. \quad (1.2.11)$$

More generally than in (1.2.8), we can assume that each a_t is a function from \mathbb{R}^n to \mathbb{C} , and obtain a new function through

$$x \mapsto V_r(a_t(x) : t \in Z). \quad (1.2.12)$$

The setting we are interested in is the following. Given a convex symmetric body B in \mathbb{R}^n , and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we want to find L^p -bounds for the function defined through

$$(V_r(M_t^B f : t \in (0, \infty)))(x) := V_r(M_t^B f(x) : t \in (0, \infty)). \quad (1.2.13)$$

It has been shown in [17] that it is not possible to find such bounds for $r \leq 2$.

For the range $r > 2$ though, Bourgain, Mirek, Stein, and Wróbel [9] showed that if $r > 2$, one can find such bounds in many cases where a dimension-free maximal bound has already been found. The only differences are that there is no known bound for $p = \infty$, and that the range $3/2 < p \leq \infty$ needs to be restricted to $3/2 < p < 4$. Among other things, the following results have been established so far.

Theorem (Variational results from [9]). *Let B be a convex symmetric body in \mathbb{R}^n and $2 < r < \infty$.*

(i) *If $3/2 < p < 4$, then there is a constant $C_{p,r}$ independent of n and B such that*

$$\|V_r(M_t^B f : t \in (0, \infty))\|_{L^p} \leq C_{p,r} \|f\|_{L^p} \quad (1.2.14)$$

for every $f \in L^p(\mathbb{R}^n)$.

(ii) *If $1 < p < \infty$, then there is a constant $C'_{p,r}$ independent of n and B such that*

$$\|V_r(M_{2^k}^B f : k \in \mathbb{Z})\|_{L^p} \leq C'_{p,r} \|f\|_{L^p} \quad (1.2.15)$$

for every $f \in L^p(\mathbb{R}^n)$.

(iii) *If $1 < p < \infty$, $1 \leq q \leq \infty$, and $B = \{x \in \mathbb{R}^n : \|x\|_q \leq 1\}$, then there is a constant $C_{p,q,r}$ independent of n such that*

$$\|V_r(M_t^B f : t \in (0, \infty))\|_{L^p} \leq C_{p,q,r} \|f\|_{L^p} \quad (1.2.16)$$

for every $f \in L^p(\mathbb{R}^n)$.

Despite the restriction $3/2 < p < 4$, these results are stronger than their maximal counterparts, which easily follows from (1.2.10). While nothing can be said about the existence of general dimension-free variational bounds for $p \geq 4$, the methods of the proof for (1.2.16) easily extend to Müller's geometrical bounds, which means that for $p > 1$, $r > 2$, and general B , we have

$$\|V_r(M_t^B f : t \in (0, \infty))\|_{L^p} \leq C(p, r, \sigma(B), Q(B))\|f\|_{L^p}, \quad (1.2.17)$$

where $C(p, r, \sigma(B), Q(B))$ is a constant that grows with $\sigma(B)$ and $Q(B)$.

There also have been variational bounds for the spherical averaging operator defined in (1.2.4). Jones, Seeger, and Wright [17] showed that if $n \geq 2$, $r > 2$ and $\frac{n}{n-1} < p \leq 2n$, then

$$\|V_r(M_t^{S^{n-1}} f(x) : t \in (0, \infty))\|_{L^p} \leq C_{p,r}\|f\|_{L^p} \quad (1.2.18)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. Again, we have an upper bound for p because we cannot interpolate with the endpoint $p = \infty$, however, an interpolation provides that (1.2.18) also holds for $2n < p < \infty$ if $r > p/n$. They also showed that this bound is sharp in the following sense: If (1.2.18) holds for some $p > 2n$, then it follows that $r \geq p/n$.

A similar interpolation could be done with the dimension-free bound (1.2.14) to achieve a dimension-free bound for $p \geq 4$ and $r > p/2$, but nothing can be said about the sharpness of this range for dimension-free bounds.

In the appendix of [9], a generalization of Stein's rotation method was established to show that the bound in (1.2.18) can be chosen to be independent of the dimension n , too. This method provides a much more direct proof of (1.2.16) in case of the Euclidean ball, but this does not lead to (1.2.18).

1.3 A short remark on measurability

We would like to address a technical issue that we silently pushed under the rug in the previous sections. Since the suprema in (1.1.2) and (1.2.13) range over sets that have the same cardinality as \mathbb{R} , we have no a priori guarantee that these expressions deliver measurable functions. A simple way to work around this issue is to take f to be in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$, the function $t \mapsto M_t^B f(x)$ is continuous, and by density, it suffices to let t range over the countable dense set $\mathbb{Q}_{>0}$, the set of strictly positive rationals. For r -variations, we will show this in Section 2.8.

A posteriori, an inequality of the form

$$\|M_B f\|_{L^p} \leq C\|f\|_{L^p}$$

will imply, by density, that there is a unique continuation $M_B: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$. Since the constant C need not be independent of any parameters to guarantee this continuation, the Vitali covering approach suffices to let us work in $L^p(\mathbb{R}^n)$ instead of $\mathcal{S}(\mathbb{R}^n)$. However, we need to be more careful in the setting of r -variations, as there have

not been many known bounds in the case we considered before. This is why we will often restrict our function space to $\mathcal{S}(\mathbb{R}^n)$ in the variational setting.

1.4 Results of this thesis

1.4.1 Maximal results

The first theorem of this thesis follows Bourgain's work for the cube [7]. We generalize his approach to show the following result.

Theorem 1. *Let $B := B_1 \times \cdots \times B_\ell$ be a direct product of $\ell \geq 1$ Euclidean balls B_k in \mathbb{R}^{n_k} , $n_k \geq 1$, and put $n := n_1 + \cdots + n_\ell$. Then for $1 < p \leq \infty$, we have*

$$\|M_B f\|_p \leq C_p \|f\|_p \tag{1.4.1}$$

for every $f \in L^p(\mathbb{R}^n)$, where C_p depends only on p , but not on ℓ , n , or the dimensions n_k of the factors B_k .

This result cannot be obtained from the bounds from [21] since, similarly to the case of the cube, we have $Q(B) \approx \sqrt{\ell}$.

A key tool for the proof of Theorem 1 is the maximal operator obtained by replacing each B_k with its boundary. Let $S := S_1 \times \cdots \times S_\ell$ be a product of Euclidean spheres S_k , with $S_k \subset \mathbb{R}^{n_k}$ for each k . Let σ_k be the spherical Lebesgue measure of S_k , so that $\sigma_k(S_k) = |S_k|$, and let σ_S be the product measure of the σ_k , so that $|S| = \sigma_S(S)$ is the $(n - \ell)$ -dimensional volume of S . For $f \in \mathcal{S}(\mathbb{R}^n)$, we define the averaging operator M_t^S by

$$M_t^S f(x) := \frac{1}{|S|} \int_S f(x + t\omega) d\sigma_S(\omega).$$

The next theorem provides a bound for the corresponding maximal function $M_S f$, given by

$$M_S f(x) := \sup_{t>0} M_t^S |f|(x).$$

Theorem 2. *Let S be as above and $n_k \geq 3$ for each k . Put $N := \min_{1 \leq k \leq \ell} n_k$ and assume that $p > \frac{N}{N-1}$. Then we have*

$$\|M_S f\|_p \leq C_p \|f\|_p \tag{1.4.2}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, where C_p only depends on p .

As in the case $\ell = 1$, the lower bound for p in Theorem 2 is optimal. While Theorem 2 generalizes the results from [28], the independence of ℓ remains open when $n_k = 2$ for some k , e.g. when $S = (S^1)^\ell$, $\ell \geq 2$. It is not clear how the methods from [3] or from [20] can be applied in this case.

The maximal results are proven in Chapter 3, which is mostly based on the preprint [24].

1.4.2 Variational results

Chapter 4 will focus on the variational counterparts of Theorems 1 and 2.

In view of the dimension-free variational results from [9] and Theorem 1, it is not surprising, but also not obvious, that the variational counterpart of Theorem 1 also holds.

Theorem 3. *Let $B := B_1 \times \dots \times B_\ell$ be a direct product of $\ell \geq 1$ Euclidean balls B_k in \mathbb{R}^{n_k} , $n_k \geq 1$, and put $n := n_1 + \dots + n_\ell$. Furthermore, let $2 < r < \infty$ and $1 < p < \infty$. Then*

$$\|V_r(M_t^B f : t > 0)\|_p \leq C_{p,r} \|f\|_p \quad (1.4.3)$$

for every $f \in L^p(\mathbb{R}^n)$, where $C_{p,r}$ depends only on p and r , but not on ℓ , n , or the dimensions n_k of the factors B_k .

The proof will be heavily based on the arguments from [9], but we also need to use several methods and results from Chapter 3. One of them leads to the variational counterpart of Theorem 2.

Theorem 4. *Let S be as in Theorem 2, $n_k \geq 3$ for each k , and $2 < r < \infty$. Put $N := \min_{1 \leq k \leq \ell} n_k$ and assume that $\frac{N}{N-1} < p < N$. Then we have*

$$\|V_r(M_t^S f : t > 0)\|_p \leq C_{p,r} \|f\|_p \quad (1.4.4)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, where $C_{p,r}$ only depends on p and r .

While the lower bound on p in Theorem 4 is optimal, the upper bound is most likely not. For $\ell = 1$, optimal bounds have been found in [17] even for $n = 2$. The estimate (1.2.18) suggests the conjecture that Theorem 4 even holds if $N \geq 2$ and $\frac{N}{N-1} < p \leq 2N$. For $\ell \geq 2$ however, the independence of ℓ for $N \leq p \leq 2N$ or $S = (S^1)^\ell$ is mostly unknown. The only improvement that we could achieve is that if $N = 3$, the methods to show (1.2.14) allow us to prove (1.4.4) for $3/2 < p < 4$ instead of $3/2 < p < 3$, leaving the range $4 \leq p \leq 6$ open.

1.5 Outline of the proofs

The proofs of Theorems 1 and 2 from Section 1.4 rely strongly on the methods from [4, 7, 12, 13, 21, 27, 31], and those of Theorems 3 and 4 rely on the methods from [9] and Chapter 3. These methods and the ideas behind them will be explained to some extent in Chapter 2, with the exception of [7]. The approach from [7] will be generalized in Section 3.1, and even though this will require a significant amount of extra work, the essential ideas from [7] will not be changed. For this reason, the methods for the cube will be explained in Section 3.1 and not in Chapter 2.

1.5.1 On Theorems 1 and 2

As mentioned before, we will prove Theorems 1 and 2 in Chapter 3. Let $B = B_1 \times \cdots \times B_\ell$, ℓ , and n be as in Theorem 1. Our main idea for the proof of Theorem 1 is the following. A naïve approach suggests that for $\ell = 2$, we can estimate

$$\begin{aligned}
M_{B_1 \times B_2} f(x) &= \frac{1}{|B_1|} \frac{1}{|B_2|} \sup_{t>0} \int_{B_1} \int_{B_2} |f(x' + ty', x'' + ty'')| dy'' dy' \\
&\leq \frac{1}{|B_1|} \frac{1}{|B_2|} \sup_{t_1, t_2 > 0} \int_{B_1} \int_{B_2} |f(x' + t_1 y', x'' + t_2 y'')| dy'' dy' \\
&\leq \frac{1}{|B_1|} \frac{1}{|B_2|} \sup_{t_1 > 0} \int_{B_1} \left(\sup_{t_2 > 0} \int_{B_2} |f(x' + t_1 y', x'' + t_2 y'')| dy'' \right) dy' \\
&= M_{B_1}(M_{B_2} f)(x', x''),
\end{aligned}$$

where we write $x = (x', x'')$ with $x' \in \mathbb{R}^{n_1}$, $x'' \in \mathbb{R}^{n_2}$, M_{B_1} acts on the x' variable, and M_{B_2} acts on the x'' variable.

For a general $\ell \in \mathbb{N}_{>0}$, we can use induction on ℓ and Stein's dimension-free bound for the Euclidean ball to get

$$\|M_B f\|_p \leq C_p^\ell \|f\|_p \quad (1.5.1)$$

for $1 < p < \infty$.

On the other hand, let B^N be the Euclidean ball in \mathbb{R}^N whose Lebesgue measure is 1. We will make use of the helpful decay properties of $\widehat{\chi_{B^N}}$. More precisely, we know that the Fourier transform satisfies $\widehat{\chi_{B^N}}(\xi) = \mathcal{O}(|\xi|^{-\frac{N+1}{2}})$ as $|\xi| \rightarrow \infty$. Thus if we choose $N = \min_{1 \leq k \leq \ell} n_k$, we see that also $\widehat{\chi_B}(\xi) = \mathcal{O}(|\xi|^{-\frac{N+1}{2}})$ as $|\xi| \rightarrow \infty$. This bound is only sharp on regions where there is $k \in \{1, \dots, \ell\}$ such that $n_k = N$, and for each $k' \neq k$, $|\xi^{(k')}|$ is small. If, however, none of the $|\xi^{(k)}|$ is small, the decay rate is significantly better. This was first exploited by Bourgain [7] for the cube. Hence, in order to show Theorem 1, we shall make use of some of the key arguments in Bourgain's approach for the cube to obtain a bound that does not depend on ℓ . From here on, we aim to combine this with (1.5.1) to achieve a bound as in Theorem 1.

The proof consists of three steps, the first one being indeed the adaption of Bourgain's arguments for the cube [7] to our setting, and this leads to a bound as described above. This step is done in Section 3.1. However, it turns out that the bound obtained by this approach will depend on $\max_{1 \leq k \leq \ell} n_k$. The remaining two steps provide a way to work around this problem.

The second step is a generalization of Stein's rotation method from [31] and will be done in Section 3.2. Let $S_k = \partial B_k$ and $S = S_1 \times \cdots \times S_\ell$. We cannot obtain a pointwise bound as in (1.2.7), but we can show that if for some $p > 1$ and every $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|M_S f\|_p \leq C \|f\|_p, \quad (1.5.2)$$

then we also have the estimate

$$\|M_B f\|_p \leq C \|f\|_p$$

for each $f \in \mathcal{S}(\mathbb{R}^n)$, with the same constant C . Furthermore, the rotation argument allows to increase the dimension of each of the S_k while keeping the same constant as in estimate (1.5.2).

At this point, our generalization of Stein's arguments shows that we only need to prove Theorem 2 for the case $S = (S^{N-1})^\ell$, where $N = N(p)$ is chosen to be minimal such that $p > \frac{N}{N-1}$ (with $p > 1$ fixed). This is the final step to show Theorem 1. This problem will be dealt with in Section 3.3, where we proceed with applying an interpolation similar as in Carbery's proof for $p > 3/2$. Carbery makes use of the fact that for a general convex symmetric body B , the multiplier operator associated to $\langle \xi, \nabla \widehat{\chi}_B(\xi) \rangle$ has a bounded operator norm from L^2 to L^2 . In the concluding remark of his work [12], he states that if this derivative has bounded L^q -multiplier norm for a bigger range of q , we can obtain a better lower bound on p than $3/2$. This is the case if we can bound the higher fractional derivatives

$$\left(\frac{d}{dr} \right)^z \widehat{\sigma}_S(r\xi),$$

where $\operatorname{Re}(z) = \frac{N+1}{2}$. To bound these, we will make use of the ideas from Müller's article [21], where he proceeded similarly for arbitrarily high derivatives, and bounded them in terms of the aforementioned geometric invariants. Also, a lot of the calculations will rely on the explicit forms and the decay of $\widehat{\sigma}_{S^{N-1}}$ and $\widehat{\sigma}_S$.

Hence, if we fix $p > 1$, we can achieve a dimension-free bound, for N large enough, by generalizing Stein's and Carbery's ideas. By rearranging the variables, we may split $B = B' \times B''$ such that each factor of B' has a dimension smaller than N , while each factor of B'' has a dimension larger than or equal to N . Then $M_{B'}$ is bounded by the arguments in Section 3.1, while $M_{B''}$ is bounded by Theorem 2, implying a bound for M_B that only depends on p .

1.5.2 On Theorems 3 and 4

We will deal with the variational results in Chapter 4. One of the key ideas is to show that in high dimensions, we can reduce Theorem 3 to Theorem 4, just as in Chapter 3 with Theorems 1 and 2. For this, we generalize Stein's ideas from [31] again in Section 4.1.

In Section 4.3, we adapt [9, Lemmas 4.1 and 4.2] to the settings of Theorems 3 and 4, which is another important part of the proofs, with the most essential estimate being

$$\|M_{t+h}^S f - M_t^S f\|_p \leq C_{p,N} \left(\frac{h}{t} \right)^\alpha \|f\|_p \quad (1.5.3)$$

for every $t, h > 0$, $f \in \mathcal{S}(\mathbb{R}^{N^\ell})$, $\frac{N}{N-1} < p < N$, some suitable $\alpha \in (0, 1)$, and $S = (S^{N-1})^\ell$. This result relies on the multiplier bounds for fractional derivatives achieved in Section

3.3.

A major difficulty that occurs in the variational setting is that after splitting $B = B' \times B''$ as above, we cannot iterate as for the maximal estimates. Instead, we have to go through the arguments of Section 3.3 and [9, Lemmas 4.1 and 4.2] for a structure of the form $B' \times S$, with S as above. While the rotation method will be essentially the same as in Section 3.2, our variants of [9, Lemmas 4.1 and 4.2] require some more refined estimates as in [9], which we will provide in Section 4.3. After that, the remaining arguments for the proofs in Chapter 4 are essentially as in [9], and they are just included for the sake of completeness. In Section 4.2, we deal with the long variation estimate for Theorem 4, while in Section 4.4, we show the short variation estimate for Theorem 4 after having verified [9, Lemmas 4.1 and 4.2] in our setting, and prove Theorem 3 by using Theorem 4, similar as in Chapter 3.

1.6 Applications

In this section, we present some standard applications of maximal and variational estimates without proofs. Many other operators in harmonic analysis can be controlled by bounding certain maximal functions. However, we shall mention that some of the following applications do not require dimension-free bounds, while other do, and that dimension-free constants themselves often serve the purpose of optimizing and leading to further dimension-free bounds in other topics. For instance, the dimension-free maximal bounds are essential in proving their variational counterparts, which lead to some dimension-free bounds in ergodic theory.

For proofs of the following applications, we refer to [30, Chapter I] or [15, Sections 2.1 and 4.3], [9, Section 5], and [8, 10, 11, 19].

An important application of maximal operators is the following almost everywhere convergence result.

Proposition 1.6.1. *Let (X, μ) be a σ -finite measure space, $1 \leq p \leq \infty$, and $(T_t)_{t>0}$ be a family of linear operators mapping functions in $L^p(X)$ into the space of μ -measurable functions on X . Let T_* be the corresponding maximal operator, i.e.*

$$T_*f(x) := \sup_{t>0} |T_t f(x)|.$$

Assume that T_ is of weak type (p, q) , with $1 \leq q \leq \infty$, and that there is a dense subspace V of $L^p(X)$ such that for each $\varphi \in V$, the limit*

$$T\varphi(x) := \lim_{t \rightarrow 0} T_t \varphi(x)$$

exists and is finite for almost every $x \in X$. Then for every $f \in L^p(X)$, $\lim_{t \rightarrow 0} f(x)$ exists for almost every $x \in X$, and the operator T is also of weak type (p, q) .

As a corollary, we obtain Lebesgue's differentiation theorem, by taking (X, μ) to be \mathbb{R}^n endowed with the Lebesgue measure, $T_t = M_t^B$ for any convex symmetric body B in \mathbb{R}^n , and $V = \mathcal{D}(\mathbb{R}^n)$.

Corollary 1.6.2. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then for almost every $x \in \mathbb{R}^n$, we have*

$$\lim_{t \rightarrow 0} M_t^B f(x) = \lim_{t \rightarrow 0} \frac{1}{|B_t(x)|} \int_{B_t(x)} f(y) \, dy = f(x).$$

The Hardy-Littlewood maximal function for the Euclidean ball arises in several estimates for singular integrals. Let K be a kernel defined on $\mathbb{R}^n \setminus \{0\}$ that satisfies the following size condition

$$|K(x)| \leq C_1 |x|^{-n} \tag{1.6.1}$$

for $x \in \mathbb{R}^n \setminus \{0\}$ and a constant C_1 . For any $\varepsilon > 0$, let $K^{(\varepsilon)}(x) = K(x)\chi_{|x| \geq \varepsilon}$. The size condition ensures that $K^{(\varepsilon)}$ is in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$, and thus, the operator

$$T_\varepsilon f := f * K^{(\varepsilon)}$$

is well-defined on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. Proposition 1.6.1 suggests that we shall study the maximal function T_* associated to $(T_\varepsilon)_{\varepsilon > 0}$. Indeed, under some additional conditions, we can bound T_* in terms of M_B , where B is the Euclidean ball.

Lemma 1.6.3 (Cotlar's inequality). *Let K be a kernel defined on $\mathbb{R}^n \setminus \{0\}$ satisfying the size condition (1.6.1), the smoothness condition*

$$|K(x-y) - K(x)| \leq C_2 |y|^\delta |x|^{-n-\delta}$$

for $|x| \geq 2|y| > 0$ and some $\delta > 0$, and the cancellation condition

$$\sup_{0 < r < R < \infty} \left| \int_{r < |x| < R} K(x) \, dx \right| \leq C_3.$$

Let W be a tempered distribution such that $W|_{\mathbb{R}^n \setminus \{0\}} = K$ and let $Tf = f * W$ on $\mathcal{S}(\mathbb{R}^n)$. Then for $1 < p < \infty$, T and T_* extend continuously to $L^p(\mathbb{R}^n)$, and there is a constant $C_{n,\delta}$ such that for every $f \in L^p(\mathbb{R}^n)$, the inequality

$$T_* f(x) \leq M_B(Tf)(x) + C_{n,\delta}(C_1 + C_2 + C_3)M_B f(x)$$

holds for almost every x .

In several ergodic theoretic settings, almost everywhere convergence on a dense subspace in the settings of Proposition 1.6.1 is a priori not given. However, if one even has a variational bound, one can deduce almost-everywhere convergence without showing this property on a dense subspace first. Indeed, take a σ -finite measure space (X, μ) , $1 \leq p \leq \infty$, and a family of linear operators $(T_t)_{t > 0}$ mapping functions in $L^p(X)$ into the space of μ -measurable functions on X , and assume that for some $r \in [1, \infty)$ and

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some $f \in L^p(X)$, $V_r(T_t f : t > 0)$ is in $L^q(X)$ for some $q \in [1, \infty]$ (and thus measurable). In that case, $V_r(T_t f(x) : t > 0) < \infty$ for almost every $x \in X$. But for each such x , the limits $\lim_{t \rightarrow 0} T_t f(x)$ and $\lim_{t \rightarrow \infty} T_t f(x)$ both exist.

The following ergodic setting, where convergence on a dense subspace is significantly harder to achieve than for $(M_t^B)_{t>0}$ is presented in [9]. Again, let (X, μ) be a σ -finite measure space and take families $(T_j^t)_{t \in \mathbb{R}}$, $j \in \{1, \dots, n\}$, of mutually commuting, measure-preserving transformations from X to itself. For a convex symmetric body B in \mathbb{R}^n and $f \in L^1(X)$, we define the averaging operator $A_t^B f$ pointwise by

$$A_t^B f(x) := \frac{1}{|B_t|} \int_{B_t} f(T_1^{y_1} \circ T_2^{y_2} \circ \dots \circ T_n^{y_n} x) \, d(y_1, y_2, \dots, y_n), \quad (1.6.2)$$

where $B_t = B_t(0)$. In [9], the following transference principle is proven to allow to reduce this setting to the operators M_t^B .

Proposition 1.6.4. *Let $Z \subset \mathbb{R}_{>0}$, $1 < p < \infty$, and $2 < r \leq \infty$. Suppose that there is a constant $C_{p,r}$ such that for every $n \in \mathbb{N}_{>0}$ and for every convex symmetric body B , the estimate*

$$\|V_r(M_t^B g : t \in Z)\|_{L^p(\mathbb{R}^n)} \leq C_{p,r} \|g\|_{L^p(\mathbb{R}^n)} \quad (1.6.3)$$

holds for every $g \in L^p(\mathbb{R}^n)$. Let X , $(T_j^t)_{t \in \mathbb{R}}$ for $j \in \{1, \dots, n\}$, and A_t^B be as above. Then for every $f \in L^p(X)$, we have

$$\|V_r(A_t^B f : t \in Z)\|_{L^p(X)} \leq C_{p,r} \|f\|_{L^p(X)} \quad (1.6.4)$$

with the same constant $C_{p,r}$ as in (1.6.3).

This proposition allows us to deduce dimension-free variational bounds for the operators A_t^B from those for M_t^B . More precisely, we can deduce the ergodic counterparts of the dimension-free variational bounds (1.2.14) and (1.2.15) directly from Proposition 1.6.4, and a slight modification, which will not affect the proof, allows us to find ergodic counterparts to (1.2.16) and Theorem 3. These bounds are thus independent of the number of transformations in the definition of A_t^B , and for any exponent p for which (1.6.4) holds, the limits $\lim_{t \rightarrow 0} A_t^B f(x)$ and $\lim_{t \rightarrow \infty} A_t^B f(x)$ both exist almost everywhere for every $f \in L^p(X)$.

The methods from [9] also provide a new, easier way to show the dimension-free maximal results for $p > 3/2$. This was mentioned in the recent work [11] and proven in [19]. For the range $1 \leq p_0 < p \leq 3/2$, the methods from [9] allow to reduce the problem of finding a dimension-free (p, p) -bound for M_B to finding a bound of the form

$$\|M_{t+h}^B f - M_t^B f\|_{p_0} \leq C_{p_0} \left(\frac{h}{t}\right)^\alpha \|f\|_{p_0},$$

for every $t, h > 0$ and $f \in L^{p_0}(\mathbb{R}^n)$, which is similar to (1.5.3). By using a certain bootstrapping argument, it was shown in [19] that if C_{p_0} is independent of n , then one

can bound $C_p(n, B)$ independently of n for every $p > p_0$.

These methods for (1.2.14) and (1.2.15) also allow to find some dimension-free bounds for discrete Hardy-Littlewood maximal functions, where the averages are sums of the form

$$\mathcal{M}_t^B f(x) = \frac{1}{|B_t \cap \mathbb{Z}^n|} \sum_{y \in B_t \cap \mathbb{Z}^n} f(x - y).$$

The analysis of these averages is much more difficult than in the continuous setting, but since they are discrete, they are much closer to the setting described by Hardy and Littlewood in their original work to explain the maximal problem. So far, results on maximal functions in the discrete setting have been established for the cases where B is the Euclidean ball or the cube, and they are also due to Bourgain, Mirek, Stein, and Wróbel. For this, we refer to the very recent works [8, 10, 11].

1.7 Open Problems

We conclude this chapter by presenting some open problems related to the topics of this thesis. The major problem this work contributes to is to find a general dimension-free bound for the centered Hardy-Littlewood maximal function.

Problem 1. *For $1 < p \leq 3/2$, is there a constant C_p such that for any $n \in \mathbb{N}_{>0}$ and any convex symmetric body $B \subset \mathbb{R}^n$, we have*

$$\|M_B f\|_p \leq C_p \|f\|_p$$

for every $f \in L^p(\mathbb{R}^n)$, with C_p independent of B and n ?

One can weaken this problem in several ways, for instance by allowing a dependence on q if B is an ℓ^q -ball, with $1 \leq q \leq \infty$. The most natural special cases that this work suggest to consider next are more general products of convex bodies.

Problem 2. *Let $B \subset \mathbb{R}^n$ be a convex symmetric body such that for all $1 < p \leq \infty$, there is a constant $C_p(B)$ independent of n such that*

$$\|M_B f\|_p \leq C_p(B) \|f\|_p$$

for every $f \in L^p(\mathbb{R}^n)$. Put $\tilde{B} := \underbrace{B \times \cdots \times B}_{\ell \text{ times}}$ for some $\ell \in \mathbb{N}_{>0}$. Can we bound

$$\|M_{\tilde{B}}\|_p \leq C'_p(B) \|f\|_p$$

for every $f \in L^p(\mathbb{R}^{n\ell})$, where $C'_p(B)$ only depends on p , B , and $C_p(B)$, but not on n or ℓ ?

We will see in Section 3.1 that even in the case that B is a Euclidean ball, an a priori estimate will provide a bound independent of ℓ , but depending on n . Under a few additional assumptions, one can apply the arguments from Section 3.1 to products

1 Introduction

of more general bodies B , but the bound will depend on n . For products of Euclidean balls, we can work around this problem by generalizing Stein's rotation method in high dimensions, but even if B is an ℓ^q -ball for some $q \notin \{2, \infty\}$, we cannot apply this method.

In view of Theorem 2 and the works [3] and [20], one can ask what happens for products of circles.

Problem 3. *Let $S := (S^1)^\ell$. Can one find a bound*

$$\|M_S f\|_p \leq C_p \|f\|_p$$

for $2 < p \leq \infty$ and every $f \in \mathcal{S}(\mathbb{R}^{2\ell})$, where C_p only depends on p ?

The major difficulty with this problem is that our methods in Section 3.3 rely on a strong L^2 -bound, while in the setting of Problem 3, such a bound is not given. Even the essential problem of finding an L^4 -bound, which does not depend on ℓ , comes with major difficulties that require new ideas and dimension-free estimates not only for products of Bessel functions, but also for certain Fourier integral operators.

In the case of variational estimates, the range $1 < p < \infty$ is linked to the range $1 < p \leq 3/2$ in the maximal setting, but one can ask the following question.

Problem 4. *Let B a convex symmetric body in \mathbb{R}^n . Can one estimate*

$$\|V_r(M_t^B f : t \in (0, \infty))\|_{L^p} \leq C_{p,r} \|f\|_{L^p}$$

for $r > 2$, $4 \leq p < \infty$, and every $f \in \mathcal{S}(\mathbb{R}^n)$, especially under the assumption that the answer to Problem 1 is negative? If not, what is the optimal upper bound for p ?

In view of [17], the case of spherical averages suggests that if a general dimension-free estimate does not hold for the full range $1 < p \leq \infty$, there may be a finite upper bound for p in the variational setting. Moreover the work [17] suggests that the range for p in Theorem 4 is not optimal for $\ell \geq 2$.

Problem 5. *Let $n \geq 3$, $\ell \geq 2$, and $S := (S^{n-1})^\ell$. For $r > 2$ and $n \leq p \leq 2n$, can one estimate*

$$\|V_r(M_t^S f : t \in (0, \infty))\|_{L^p} \leq C_{p,r} \|f\|_{L^p}$$

for every $f \in \mathcal{S}(\mathbb{R}^{n\ell})$, where $C_{p,r}$ does not depend on ℓ ? If not, what is the optimal upper bound for p ?

Here, a minor improvement on the upper bound of p can be made in the case $n = 3$. In this case, one can apply the methods for the range $3/2 < p < 4$ from [9] to see that one has

$$\|V_r(M_t^S f : t \in (0, \infty))\|_{L^p} \leq C_{p,r} \|f\|_{L^p}$$

if $3/2 < p < 4$. This will not be done in this work.

Notation

The set of natural numbers is denoted by $\mathbb{N} = \{0, 1, \dots\}$. The number $n \in \mathbb{N}_{>0} = \{1, 2, \dots\}$ usually refers to the dimension of the space we work in.

For any $x \in \mathbb{R}$, let $\lfloor x \rfloor := \sup\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil := \inf\{k \in \mathbb{Z} : k \geq x\}$.

To defray the notation, we will use “variable constants”, i.e. a constant C occurring in a chain of inequalities may differ from line to line, unless stated otherwise.

For any Lebesgue-measurable subset $A \subset \mathbb{R}^n$, $|A|$ denotes the n -dimensional volume of A . In abuse of notation, but without any ambiguities, the cardinality of a finite set X shall be denoted by $|X|$.

A set $B \subset \mathbb{R}^n$, which is closed, bounded, convex, has nonempty interior, and satisfies for every $x \in \mathbb{R}^n$ that $-x \in B$ whenever $x \in B$, is referred to as convex symmetric body (in \mathbb{R}^n). Since such sets are precisely the unit balls of norms on \mathbb{R}^n , we will put $B_t(x) := x + tB$ ($x \in \mathbb{R}^n$, $t > 0$) and $B_t := B_t(0)$.

For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x . In particular, we write

$$B_r^n = \{x \in \mathbb{R}^n : |x| \leq r\}, \quad S_r^{n-1} = \{x \in \mathbb{R}^n : |x| = r\}$$

for the Euclidean ball and sphere in \mathbb{R}^n of radius $r > 0$, and with $B^n = B_1^n$, $S^{n-1} = S_1^{n-1}$. For the spheres, $|S_r^{n-1}|$ shall be the $(n-1)$ -dimensional volume (not the n -dimensional volume, which would be 0).

For any convex symmetric body B , $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $t > 0$, the operators M_t^B and M_B shall be defined as in (1.1.1) and (1.1.2), and for Euclidean spheres, $M_t^{S^{n-1}}$ and $M_{S^{n-1}}$ shall be defined as in (1.2.4) and (1.2.5).

In our setting, we often consider products of Euclidean balls $B = B_1 \times \dots \times B_\ell$ or products of spheres $S = S_1 \times \dots \times S_\ell$, with $B_k, S_k \subset \mathbb{R}^{n_k}$. Here, ℓ always denotes the number of factors, and $n = n_1 + \dots + n_\ell$. In particular, if $n_k = N$ for all $k \in \{1, \dots, \ell\}$ and some $N \in \mathbb{N}_{>0}$, $n = N\ell$.

For S as above, let σ_k be the surface measure of S_k . The measure σ_S shall always be the product measure $\sigma_1 \otimes \dots \otimes \sigma_\ell$, so that $\sigma_S(S)$ is the $(n-\ell)$ -dimensional volume of S . We will write $|S| = \sigma_S(S)$.

Notation

As we work with these particular choices of B and S , we will divide our variables $x \in \mathbb{R}^n$ and $\omega \in S$ into blocks of the dimensions of the corresponding factors. We will do that in the following way:

$$\begin{aligned} x &= (x^{(1)}, x^{(2)}, \dots, x^{(\ell)}), \\ \omega &= (\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(\ell)}) \end{aligned}$$

with $x^{(k)} \in \mathbb{R}^{n_k}$ and $\omega^{(k)} \in S_k$. The same notation shall apply to every variable in \mathbb{R}^n .

For $r \in (\mathbb{R}_{>0})^\ell$, $x = (x^{(1)}, \dots, x^{(\ell)}) \in \mathbb{R}^n$, and $f: \mathbb{R}^n \rightarrow \mathbb{C}$, we write $r.x := (r_1 x^{(1)}, \dots, r_\ell x^{(\ell)})$ and $r.f(x) = f(r.x)$. Similarly to the averaging operator M_t^S , defined in Section 1.4 by

$$M_t^S f(x) := \frac{1}{|S|} \int_S |f(x + t\omega)| d\sigma_S(\omega),$$

we will also define an averaging operator that dilates the blocks ω_k separately. For $r \in (\mathbb{R}_{>0})^\ell$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we put

$$M_r^S f(x) := \int_S f(x + r.\omega) d\sigma(\omega) = \int_S f(x + (r_1 \omega^{(1)}, \dots, r_\ell \omega^{(\ell)})) d\sigma_S(\omega).$$

For any f in $L^1(\mathbb{R}^n)$ and $t > 0$, we will denote the dilated function f_t by

$$f_t(x) := \frac{1}{t^n} \cdot f\left(\frac{x}{t}\right).$$

The Fourier transform of f is defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx,$$

also denoted by $\mathcal{F}f(\xi)$ instead of $\hat{f}(\xi)$. The inverse operation is given by

$$\check{f}(x) := \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi,$$

also denoted by $\mathcal{F}^{-1}f(x)$.

For any multiplier $m \in L^\infty(\mathbb{R}^n)$, we write T_m for the corresponding Fourier multiplier operator, i.e.

$$T_m f = \mathcal{F}^{-1}(m \hat{f}).$$

If $T_m: L^p \rightarrow L^p$ is bounded, then the L^p -multiplier norm $\|m\|_{p \rightarrow p}$ of m is defined to be the operator norm of T_m , i.e.

$$\|m\|_{p \rightarrow p} := \|T_m\|_{p \rightarrow p}.$$

2 Mathematical methods and results used in this thesis

The sections of this chapter give an overview of all the known tools and ideas that we shall employ in our proofs later on. Most of them are key ideas for proofs of known dimension-free maximal or variational bounds. This chapter mainly serves as a reference chapter to look into when some known methods for dimension-free results are used in our proofs, and the reader familiar with the known results may skip this chapter or parts of it for now. However, we would like to draw some attention to Section 2.6, especially Proposition 2.6.3, which is an original result of this work.

2.1 Interpolation of analytic families

We begin with Stein's celebrated interpolation theorem for analytic families of linear operators. This has been used in [12], [21], and [9], and it will also be used several times in this thesis. In all of these cases, fractional derivative operators will be considered, see Section 2.6 for more details.

Let (X, μ) and (Y, ν) be measure spaces. For every $z \in \mathbb{S} := \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$, let T_z be an operator that maps simple functions on (X, μ) to measurable functions on (Y, ν) , and suppose that for all simple functions f on X and g on Y , we have

$$\int_Y |T_z(f)g| \, d\nu < \infty.$$

The family $(T_z)_{z \in \mathbb{S}}$ is called *analytic* if the function

$$z \mapsto \langle T_z(f), g \rangle = \int_Y T_z(f)g \, d\nu$$

is analytic on $\mathbb{S}^\circ = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on \mathbb{S} , whenever f and g are simple functions on X resp. Y . We call $(T_z)_{z \in \mathbb{S}}$ to be of *admissible growth* if there are constants $a < \pi$ and $C_{f,g}$ such that

$$e^{-a|\operatorname{Im} z|} \log |\langle T_z(f), g \rangle| \leq C_{f,g}$$

uniformly in $z \in \mathbb{S}$ (with f, g being simple functions again). With these definitions, Stein's interpolation theorem can be stated as follows.

Theorem 2.1.1. *Let X, Y , and $(T_z)_{z \in \mathbb{S}}$ be as above and assume that $(T_z)_{z \in \mathbb{S}}$ is analytic and of admissible growth. Furthermore, let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that there are functions $M_0, M_1: \mathbb{R} \rightarrow \mathbb{R}_{>0}$ such that*

$$\|T_{it}f\|_{q_0} \leq M_0(t)\|f\|_{p_0} \quad \text{and} \quad \|T_{1+it}f\|_{q_1} \leq M_1(t)\|f\|_{p_1}$$

for every simple function f on X , and that we have $b < \pi$ so that

$$\sup_{t \in \mathbb{R}} e^{-b|t|} \log M_j(t) < \infty, \quad j = 0, 1. \quad (2.1.1)$$

Choose $0 < \theta < 1$ and define p, q by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we get

$$\|T_\theta f\|_q \leq M(\theta)\|f\|_p,$$

where for $0 < t < 1$,

$$M(t) = \exp \left[\frac{\sin(\pi t)}{2} \int_{\mathbb{R}} \left(\frac{\log M_0(y)}{\cosh(\pi y) - \cos(\pi t)} + \frac{\log M_1(y)}{\cosh(\pi y) + \cos(\pi t)} \right) dy \right]. \quad (2.1.2)$$

By interpolating each of the families $(T_{z+it})_{z \in \mathbb{R}}, t \in \mathbb{R}$, we can deduce the following Corollary.

Corollary 2.1.2. *Under the same hypotheses and notation as in Theorem 2.1.1, we get*

$$\|T_z f\|_q \leq B(z)\|f\|_p$$

for $0 < \operatorname{Re} z < 1$, where

$$B(t + is) = \exp \left[\frac{\sin(\pi t)}{2} \int_{\mathbb{R}} \left(\frac{\log M_0(y + s)}{\cosh(\pi y) - \cos(\pi t)} + \frac{\log M_1(y + s)}{\cosh(\pi y) + \cos(\pi t)} \right) dy \right]. \quad (2.1.3)$$

Theorem 2.1.1 and its proof can be found in [15, pp. 37-41]. For Corollary 2.1.2, see [15, Exercise 1.3.6]. For If X and Y are \mathbb{R}^n and \mathbb{R}^m equipped with their respective Lebesgue measures, one can replace simple functions by smooth functions with compact support. This is what we will use in most cases. Also, we will always have $p_0 = q_0$ and $p_1 = q_1$. By mapping $z \mapsto \frac{z-a}{b-a}$, one can generalize this interpolation to the range $a \leq \operatorname{Re} z \leq b$ for real numbers $a < b$.

2.2 Stein's rotation method

A beautiful mathematical argument, which arose in the context of dimension-free bounds for maximal functions, is the rotation method that Stein used in [29] to show that for $M_{S^{n-1}}$, one can increase the dimension n while maintaining the known L^p -bounds.

Proposition 2.2.1. *Let $N \leq n$ and assume that for some $p > 1$, there is a constant C such that for every $f \in \mathcal{S}(\mathbb{R}^N)$,*

$$\|M_{S^{N-1}} f\|_{L^p(\mathbb{R}^N)} \leq C \|f\|_{L^p(\mathbb{R}^N)}. \quad (2.2.1)$$

Then we also get

$$\|M_{S^{n-1}} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad (2.2.2)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, with the same constant C .

Combining that with the facts that $M_{S^{n-1}}$ is bounded on $L^p(\mathbb{R}^n)$ iff $\frac{n}{n-1} < p \leq \infty$, and that we have the pointwise estimate

$$M_{B^n} f(x) \leq M_{S^{n-1}} f(x) \quad (2.2.3)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, the rotation method shows that the bounds for M_{B^n} and $M_{S^{n-1}}$ can be chosen independently of n . As we need to adapt the method of rotations to our case, we will show Stein's argument in this section.

Proof of Proposition 2.2.1. Denote by $O(n)$ the orthogonal group, i.e. the group of rotations, on \mathbb{R}^n , and write $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^N \times \mathbb{R}^{n-N}$. For $f \in \mathcal{S}(\mathbb{R}^n)$, let

$$M'_t f(x) := M_t^{S^{N-1}}(f_{x^{(2)}})(x^{(1)}) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} f(x^{(1)} + t\omega^{(1)}, x^{(2)}) d\sigma^{N-1}(\omega^{(1)}),$$

where $f_{x^{(2)}}(x^{(1)}) = f(x^{(1)}, x^{(2)})$. Of course, we have

$$\| \sup_{t>0} M'_t f \|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad (2.2.4)$$

with the same C as in our assumption. Next, let $\rho \in O(n)$ and put

$$M_t^\rho f := [M'_t(f \circ \rho)] \circ \rho^{-1}.$$

Using (2.2.4), this definition directly yields

$$\| \sup_{t>0} M_t^\rho f \|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (2.2.5)$$

Our goal is to show

$$M_t^{S^{n-1}} f(x) = \int_{O(n)} M_t^\rho f(x) d\lambda(\rho), \quad (2.2.6)$$

where λ is the normalized Haar measure on $O(n)$. For this, we shall first show the pointwise identity

$$M_t^\rho f(x) = \int_{S^{N-1}} f(x + t\rho(\omega^{(1)}, 0)) d\sigma^{N-1}(\omega^{(1)}). \quad (2.2.7)$$

This holds because we have

$$\begin{aligned}
 M_t^p f(x) &= \frac{1}{|S^{N-1}|} \int_{S^{N-1}} f(\rho((\rho^{-1}x)^{(1)} + t\omega^{(1)}, (\rho^{-1}x)^{(2)})) d\sigma^{N-1}(\omega^{(1)}) \\
 &= \frac{1}{|S^{N-1}|} \int_{S^{N-1}} f(\rho\rho^{-1}x + \rho(t(\omega^{(1)}, 0))) d\sigma^{N-1}(\omega^{(1)}) \\
 &= \frac{1}{|S^{N-1}|} \int_{S^{N-1}} f(x + t\rho(\omega^{(1)}, 0)) d\sigma^{N-1}(\omega^{(1)}).
 \end{aligned}$$

To prove (2.2.6), we define a new measure $\tilde{\sigma}$ on S^{n-1} by setting

$$\int_{S^{n-1}} f(\omega) d\tilde{\sigma}(\omega) = \int_{O(n)} \int_{S^{N-1}} f(\rho(\omega^{(1)}, 0)) d\sigma^{N-1}(\omega^{(1)}) d\lambda(\rho).$$

Now it suffices to show that $\frac{1}{|S^{N-1}|}\tilde{\sigma} = \frac{1}{|S^{n-1}|}\sigma^{n-1}$. But for any fixed rotation τ , we have

$$\begin{aligned}
 \int_{S^{n-1}} f(\tau\omega) d\tilde{\sigma}(\omega) &= \int_{O(n)} \int_{S^{N-1}} f(\tau\rho(\omega^{(1)}, 0)) d\sigma^{N-1}(\omega^{(1)}) d\lambda(\rho) \\
 &= \int_{S^{N-1}} \int_{O(n)} f(\tau\rho(\omega^{(1)}, 0)) d\lambda(\rho) d\sigma^{N-1}(\omega^{(1)}) \\
 &= \int_{S^{N-1}} \int_{O(n)} f(\rho(\omega^{(1)}, 0)) d\lambda(\rho) d\sigma^{N-1}(\omega^{(1)}) \\
 &= \int_{O(n)} \int_{S^{N-1}} f(\rho(\omega^{(1)}, 0)) d\sigma^{N-1}(\omega^{(1)}) d\lambda(\rho)
 \end{aligned}$$

due to the invariance property of λ , and

$$\tilde{\sigma}(S^{n-1}) = \sigma^{N-1}(S^{N-1}) \cdot \lambda(O(n)) = |S^{N-1}|.$$

Thus $\frac{1}{|S^{N-1}|}\tilde{\sigma}$ is a rotation invariant normalized measure on S^{n-1} , and the uniqueness of the Haar measure implies $\frac{1}{|S^{N-1}|}\tilde{\sigma} = \frac{1}{|S^{n-1}|}\sigma^{n-1}$. So we get indeed

$$\begin{aligned}
 M_t^{S^{n-1}} f(x) &= \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x + t\omega) d\sigma^{n-1}(\omega) \\
 &= \frac{1}{|S^{N-1}|} \int_{O(n)} \int_{S^{N-1}} f(x + t(\rho(\omega^{(1)}, 0))) d\sigma^{N-1}(\omega^{(1)}) d\lambda(\rho) \\
 &= \int_{O(n)} M_t^p f(x) d\lambda(\rho).
 \end{aligned}$$

Hence, by (2.2.5), (2.2.6), and Minkowski's inequality, we finally get

$$\|M_{S^{n-1}}f\|_{L^p(\mathbb{R}^n)} \leq \int_{O(n)} \left\| \sup_{t>0} M_t^\rho f \right\|_{L^p(\mathbb{R}^n)} d\lambda(\rho) = \left\| \sup_{t>0} M_t^\rho f \right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

again with the C from our assumption. **q.e.d.**

2.3 Bourgain's L^2 -estimates

This section provides some of the ideas for Bourgain's L^2 -result [4]. Let B be a convex symmetric body in \mathbb{R}^n with $|B| = 1$. Bourgain showed that there is a linear mapping $U \in SL(\mathbb{R}^n)$ such that $U(B)$ is in isotropic position, i.e. there is a constant L such that for every $\theta \in S^{n-1}$, we have

$$\int_{U(B)} \langle x, \theta \rangle^2 dx = L^2. \quad (2.3.1)$$

By a simple use of the transformation formula, one can see that $\|M_B\|_{p \rightarrow p} = \|M_{U(B)}\|_{p \rightarrow p}$, so we can assume that B itself is in isotropic position, and we will write $L = L(B)$ for the constant in (2.3.1).

In that case, Bourgain showed that the Fourier transform $\widehat{\chi}_B$ has some useful decay properties; namely there is a constant C such that for every $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$|\widehat{\chi}_B(\xi)| \leq C|L\xi|^{-1}, \quad |1 - \widehat{\chi}_B(\xi)| \leq C|L\xi|, \quad |\langle \xi, \nabla \widehat{\chi}_B(\xi) \rangle| \leq C. \quad (2.3.2)$$

We will include the proof, but we will replace the middle inequality by a slightly stronger one that has been shown in [13].

Lemma 2.3.1. *Let $B \subset \mathbb{R}^n$ be a convex symmetric body in isotropic position. Then there is a constant C such that for every $\xi \in \mathbb{R}^n \setminus \{0\}$, $\theta \in S^{n-1}$, and $t \in \mathbb{R}$, we have*

$$|\widehat{\chi}_B(\xi)| \leq C|L\xi|^{-1}, \quad |\langle \theta, \nabla \widehat{\chi}_B(t\theta) \rangle| \leq CL, \quad |\langle \xi, \nabla \widehat{\chi}_B(\xi) \rangle| \leq C. \quad (2.3.3)$$

The middle inequality from (2.3.3) implies the middle one from (2.3.2), because if $\xi = |\xi|\theta$, then

$$|1 - \widehat{\chi}_B(\xi)| = |\widehat{\chi}_B(0) - \widehat{\chi}_B(|\xi|\theta)| \leq |\xi| \cdot |\langle \theta, \nabla \widehat{\chi}_B(|\xi|\theta) \rangle| \leq C|L\xi|.$$

For the proof, we introduce the volumes of $(n-1)$ -dimensional sections of B . For $\theta \in S^{n-1}$ and $u \in \mathbb{R}$, let

$$\varphi_\theta(u) := \text{Vol}_{n-1}(\{x \in B : \langle x, \theta \rangle = u\}). \quad (2.3.4)$$

If there is no ambiguity, we will write φ instead of φ_θ . If $g: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $g \cdot \varphi$ is integrable on \mathbb{R} , Fubini's theorem implies that

$$\int_{-\infty}^{\infty} g(u)\varphi_\theta(u) du = \int_B g(\langle x, \theta \rangle) dx. \quad (2.3.5)$$

In particular, we have

$$\int_{-\infty}^{\infty} \varphi(u) \, du = |B| = 1. \quad (2.3.6)$$

Clearly, φ is even. It follows from Brunn's theorem that $\varphi^{\frac{1}{n-1}}$ is concave (see [2]). Hence φ is differentiable almost everywhere, increasing on $\mathbb{R}_{\leq 0}$, and decreasing on $\mathbb{R}_{\geq 0}$. Bourgain shows that there are universal constants $a, A > 0$ such that for every $\theta \in S^{n-1}$ and every $u \in \mathbb{R}$, we have

$$\varphi(u) \leq A\varphi_\theta(0)e^{-a\varphi_\theta(0)|u|}. \quad (2.3.7)$$

Furthermore, there exists another universal constant $a_1 \geq 1$ so that

$$a_1^{-1} \leq \varphi_\theta(0) \cdot L(B) \leq a_1. \quad (2.3.8)$$

With (2.3.8), we can show Lemma 2.3.1.

Proof of Lemma 2.3.1. Let $\xi \in \mathbb{R}^n \setminus \{0\}$, $\theta = \xi/|\xi|$, and consider $\varphi = \varphi_\theta$. As for the first inequality, (2.3.5) implies that

$$m(\xi) := \widehat{\chi_B}(\xi) = \int_B e^{-2\pi i \langle x, \xi \rangle} \, dx = \int_{\mathbb{R}} \varphi(u) e^{-2\pi i |\xi| u} \, du. \quad (2.3.9)$$

Since φ is increasing on $\mathbb{R}_{\leq 0}$ and decreasing on $\mathbb{R}_{\geq 0}$, integration by parts thus implies

$$|m(\xi)| \leq \frac{1}{2\pi|\xi|} \int_{\mathbb{R}} |\varphi'(u)| \, du = \frac{1}{2\pi|\xi|} \cdot 2\varphi(0) \leq C(L|\xi|)^{-1}.$$

For the middle inequality let $t \in \mathbb{R}$, $\theta \in S^{n-1}$, and again $\varphi = \varphi_\theta$. Then by (2.3.9), we have

$$\langle \theta, \nabla m(t\theta) \rangle = -2\pi i \int_B \langle x, \theta \rangle e^{-2\pi i \langle x, t\theta \rangle} \, dx = -2\pi i \int_{-\infty}^{\infty} u\varphi(u) e^{-2\pi i ut} \, du. \quad (2.3.10)$$

Due to (2.3.6), we can apply Jensen's inequality and get

$$|\langle \theta, \nabla m(t\theta) \rangle| \leq 2\pi \int_{-\infty}^{\infty} |u|\varphi(u) \, du \leq 2\pi \left(\int_{-\infty}^{\infty} u^2 \varphi(u) \, du \right)^{1/2}.$$

But by (2.3.1) and (2.3.5), we have

$$\int_{-\infty}^{\infty} u^2 \varphi(u) \, du = \int_B \langle x, \theta \rangle^2 \, dx = L^2,$$

concluding the middle inequality. As for the last inequality, write $\xi = t\theta$, i.e. $t = |\xi|$. By (2.3.10) and integration by parts, we get

$$\langle \nabla m(\xi), \xi \rangle = t \langle \theta, \nabla m(t\theta) \rangle = -2\pi i t \int_{-\infty}^{\infty} u \varphi(u) e^{-2\pi i t u} du = \int_{-\infty}^{\infty} \frac{d}{du} (u \varphi(u)) e^{-2\pi i t u} du.$$

This leads to

$$|\langle \nabla m(\xi), \xi \rangle| \leq \int_{-\infty}^{\infty} \left| \frac{d}{du} (u \varphi(u)) \right| du \leq \int_{-\infty}^{\infty} |\varphi(u)| du + \int_{-\infty}^{\infty} |u \varphi'(u)| du.$$

Since by (2.3.6) and another integration by parts,

$$\int_{-\infty}^{\infty} |u \varphi'(u)| du = - \int_{-\infty}^{\infty} u \varphi'(u) du = \int_{-\infty}^{\infty} \varphi(u) du = 1,$$

it follows that

$$|\langle \nabla m(\xi), \xi \rangle| \leq 2.$$

q.e.d.

In his paper, Bourgain provided a general tool on how to make use of the Fourier decay of an L^1 -kernel K with differentiable Fourier transform, to get an L^2 -estimate on the corresponding maximal function $f \mapsto \sup_{t>0} |f * K_t|$. The proof of this result only uses

properties of \hat{K} (essentially), and hence it remains exactly the same if K is replaced by a finite Borel measure ν with differentiable Fourier transform.

Lemma 2.3.2 (Lemma 3 from [4]). *Let ν be a finite Borel measure on \mathbb{R}^n so that $\hat{\nu}$ is differentiable, and put*

$$\alpha_j := \sup_{2^j \leq |\xi| \leq 2^{j+2}} |\hat{\nu}(\xi)|, \quad \beta_j := \sup_{2^j \leq |\xi| \leq 2^{j+2}} |\langle \nabla \hat{\nu}(\xi), \xi \rangle|$$

for every $j \in \mathbb{Z}$. Then for every $f \in L^2(\mathbb{R}^n)$, we have

$$\left\| \sup_{t>0} |f * \nu_t| \right\|_2 \leq C \Gamma(\nu) \|f\|_2 \quad (2.3.11)$$

where

$$\Gamma(\nu) := \sum_{j \in \mathbb{Z}} \alpha_j^{1/2} (\alpha_j^{1/2} + \beta_j^{1/2}) = \sum_{j \in \mathbb{Z}} (\alpha_j + \alpha_j^{1/2} \beta_j^{1/2}), \quad (2.3.12)$$

and C does not depend on ν or n .

Remark. We have $f * (\chi_B)_t = M_t^B f$, so this lemma gives an estimate for the pointwise supremum of $|M_t^B f|$ instead of $M_t^B |f|$. However, by considering positive functions, we can see that this difference is not relevant for our purposes.

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As some similar estimates will occur in this work, we will explain how this provides a dimension-free L^2 -estimate for M_B . We cannot estimate $\Gamma(\chi_B)$ directly. To work around this problem, we consider the Poisson kernel $\widehat{P}_t(\xi) = e^{-2\pi t|\xi|}$. The kernel $P = P_1$ satisfies similar estimates as in Lemma 2.3.1. More precisely, if $\xi \in \mathbb{R}^n \setminus \{0\}$, $\theta \in S^{n-1}$, and $t \in \mathbb{R}$, we have

$$|\widehat{P}(\xi)| \leq C|\xi|^{-1}, \quad |\langle \theta, \nabla \widehat{P}(t\theta) \rangle| \leq C, \quad |\langle \xi, \nabla \widehat{P}(\xi) \rangle| \leq C. \quad (2.3.13)$$

If K is the difference

$$K = \chi_B - P_L,$$

where $L = L(B)$ is the isotropic constant as in (2.3.1), we can estimate $\Gamma(K)$ independently of the dimension. By Stein's maximal theorem for semigroups, which we will prove in Section 2.9 along with its variational counterpart, we have

$$\|M_B f\|_2 \leq \left\| \sup_{t>0} |f * K_t| \right\|_2 + \left\| \sup_{t>0} |f * P_{Lt}| \right\|_2 \leq (\Gamma(K) + C)\|f\|_2,$$

so that it will suffice to find a bound for $\Gamma(K)$, which is independent of n . Let $j \in \mathbb{Z}$. We will estimate the corresponding α_j based on two cases, using Lemma 2.3.1.

If $2^j \leq L^{-1}$, we have

$$\widehat{K} = (1 - \widehat{P}_L) - (1 - \widehat{\chi}_B),$$

and thus

$$\alpha_j \leq \sup_{2^j \leq |\xi| \leq 2^{j+2}} (|1 - \widehat{\chi}_B(\xi)| + |1 - \widehat{P}_L(\xi)|) \leq C2^j L + (1 - e^{-2\pi \cdot 2^{j+2}L}) \leq C2^j L. \quad (2.3.14)$$

If $2^j > L^{-1}$, we have

$$\alpha_j \leq \sup_{2^j \leq |\xi| \leq 2^{j+2}} (|\widehat{\chi}_B(\xi)| + |\widehat{P}_L(\xi)|) \leq (C2^j L)^{-1} + e^{-2\pi \cdot 2^j L} \leq C(2^j L)^{-1}. \quad (2.3.15)$$

Also, Lemma 2.3.1 directly provides that

$$\beta_j \leq C$$

for every $j \in \mathbb{Z}$. Thus we have

$$\sum_{j \in \mathbb{Z}} \alpha_j \leq \sum_{2^j \leq L^{-1}} C2^j L + \sum_{2^j > L^{-1}} C(2^j L)^{-1} \leq 2C \sum_{k=0}^{\infty} 2^{-k} = 4C,$$

and

$$\sum_{j \in \mathbb{Z}} \alpha_j^{1/2} \beta_j^{1/2} \leq C \sum_{2^j \leq L^{-1}} (2^j L)^{1/2} + C \sum_{2^j > L^{-1}} (2^j L)^{-1/2} \leq 2C \sum_{k=0}^{\infty} 2^{-k/2} = 2(2 + \sqrt{2})C.$$

This means that we can estimate $\Gamma(K)$ by a constant independent of B and n .

2.4 Carbery's almost orthogonality principle

Carbery's almost orthogonality principle in [12] reduces the analysis of the Hardy-Littlewood maximal function to the case where the corresponding supremum ranges over the interval $[1, 2]$, by involving a suitable square function. It also gives us a dimension-free bound for the lacunary maximal function, where the supremum ranges over the dyadic numbers 2^k , $k \in \mathbb{Z}$, for the full range $1 < p \leq \infty$.

Let V be a subspace of the space of measurable functions from \mathbb{R}^n to \mathbb{C} . We call a linear operator T on V *essentially positive* if there are positive operators R and S on V , i.e. $Rf \geq 0$ and $Sf \geq 0$ for every $f \in V$ with $f \geq 0$, such that $T = S - R$.

Theorem 2.4.1 (from [12]). *Let I be an arbitrary index set, and let $(T_{j,\nu})_{\substack{j \in \mathbb{Z} \\ \nu \in I}}$ be a family of essentially positive linear operators defined on $\mathcal{S}(\mathbb{R}^n)$. Put*

$$T_j f := \sup_{\nu \in I} |T_{j,\nu} f|.$$

Assume that there is $p_0 \in [1, 2)$ so that the following conditions hold.

- For each $f \in \mathcal{S}(\mathbb{R}^n)$ and each $x \in \mathbb{R}^n$, $t \mapsto T_t f(x)$ is continuous.
- There is a family of linear operators $(S_j)_{j \in \mathbb{Z}}$ on $\bigcup_{1 \leq p \leq \infty} L^p(\mathbb{R}^n)$ such that for every $f \in L^2(\mathbb{R}^n)$, we have $\sum_{j \in \mathbb{Z}} S_j f = f$ on $L^2(\mathbb{R}^n)$, and for every $p \in (1, 2]$, there is a constant $C_{1,p}$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C_{1,p} \|f\|_{L^p}. \quad (2.4.1)$$

- For each $p \in (p_0, 2]$, there is a constant $C_{2,p}$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\sup_{j \in \mathbb{Z}} \|T_j f\|_{L^p} \leq C_{2,p} \|f\|_{L^p}. \quad (2.4.2)$$

- $(T_{j,\nu})_{j,\nu}$ is strongly bounded on L^2 with respect to $(S_n)_{n \in \mathbb{Z}}$, i.e. there is a sequence $(a_k)_{k \in \mathbb{Z}}$, $a_k > 0$, such that $\sum_{k \in \mathbb{Z}} a_k^\rho = A_\rho < \infty$ for every $\rho > 0$, and there is a constant C_3 so that for every $k \in \mathbb{Z}$ and every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\| \sup_{j,\nu} |T_{j,\nu} S_{j+k} f| \right\|_{L^2} \leq C_3 a_k \|f\|_{L^2}. \quad (2.4.3)$$

Then for every $p \in (p_0, 2]$, there is a constant C_p such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\| \sup_{j,\nu} |T_{j,\nu} f| \right\|_{L^p} \leq C_p \|f\|_{L^p}. \quad (2.4.4)$$

If we get back to the setting of maximal functions for a convex symmetric body B , we can simply set

$$\widehat{S}_j f := \widehat{P_{2^{j+1}} f} - \widehat{P_{2^j} f}$$

to satisfy (2.4.1). If we choose $I = [1, 2]$ and $T_{j,\nu} = M_{2^j\nu}^B$, it has been shown in [12] that $(T_{j,\nu})_{j,\nu}$ indeed satisfies the strong boundedness condition (2.4.3). For a very detailed proof of this, see [13, Chapter 6]. Because of that, we can reduce the problem to bounding

$$\left\| \sup_{1 \leq t \leq 2} |M_t^B f| \right\|_{L^p} \leq C_p \|f\|_{L^p} \quad (2.4.5)$$

in the case $p < 2$. This is justified by the following scaling principle. For any L^1 -kernel (or finite Borel measure) K on \mathbb{R}^n and $s > 0$, we have

$$K_t * f(x) = s^n (K_{ts} * f_s)(sx). \quad (2.4.6)$$

If we assume that (2.4.5) holds, we thus have for $j \in \mathbb{Z}$:

$$\begin{aligned} \left\| \sup_{2^j \leq t \leq 2^{j+1}} |M_t^B f| \right\|_{L^p} &\leq 2^{-jn} \left\| \sup_{1 \leq t \leq 2} |M_t^B f_{2^{-j}}(2^{-j}x)| \right\|_{L^p} \\ &\leq C_p 2^{-jn} \|f_{2^{-j}}(2^{-j}x)\|_{L^p} \\ &= C_p \|f\|_{L^p}. \end{aligned}$$

Thus (2.4.2) holds if (2.4.5) holds.

The lacunary case (1.2.2) follows directly, since we can omit the estimate (2.4.5). A general way to establish (2.4.3) and (2.4.5) is provided in the next two sections.

2.5 Fractional derivatives

Fractional derivatives play a key role in many of the proofs that this work is based on, and we will need similar methods in most of our proofs as well. There are several types of fractional derivatives that we need to consider, based on which one gives a more convenient analysis.

2.5.1 Fractional derivatives defined by the Fourier transform

For the first type of fractional derivatives which we introduce, we make use of the following property. If $h \in \mathcal{S}(\mathbb{R})$, then we have

$$h^{(k)}(t) = (-1)^k \mathcal{F}_s((2\pi i s)^k \check{h}(s))(t) \quad (2.5.1)$$

for every $t \in \mathbb{R}$. Thus for any complex number $z \in \mathbb{C}$ with $\operatorname{Re} z > -1$, we shall follow [13, Section 6.2] and [9, Section 4], and put

$$\mathcal{D}^z h(t) = \mathcal{D}_t^z h(t) := \mathcal{F}_s((2\pi i s)^z \check{h}(s))(t). \quad (2.5.2)$$

Here, we use the main branch of the complex logarithm, together with the conventions $0^z = 0$ for $z \neq 0$ and $0^0 = 1$, so that for $z = a + ib$ and $s \neq 0$, we have

$$(2\pi i s)^z = e^{(a+ib)(\log |2\pi s| + \frac{i\pi}{2} \operatorname{sgn}(s))} = e^{a \log |2\pi s| - b \frac{\pi}{2} \operatorname{sgn}(s)} e^{i(b \log |2\pi s| + a \frac{\pi}{2} \operatorname{sgn}(s))}.$$

The condition $\operatorname{Re} z > -1$ is necessary for this pointwise definition to make sense: For $h \in \mathcal{S}(\mathbb{R})$, we have of course $\check{h} \in \mathcal{S}(\mathbb{R})$. But we have

$$|(2\pi i s)^z| = |2\pi s|^a e^{-b \frac{\pi}{2} \operatorname{sgn}(s)} \leq |2\pi s|^a e^{|b| \frac{\pi}{2}}.$$

This means that we can only ensure that $(2\pi i s)^z \check{h}(s)$ is an L^1 -function if $\operatorname{Re} z > -1$. Definition (2.5.2) provides an L^∞ -function. Moreover, (2.5.2) extends naturally to each function h such that there is a function $k \in L^1$ with $h = \hat{k}$ and $(1 + |s|^{\operatorname{Re} z})k(s) \in L^1$. Since for any $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, we have

$$\langle \mathcal{D}^z \varphi, \psi \rangle = \langle \varphi, (2\pi i s)^z \hat{\psi}(s) \rangle, \quad (2.5.3)$$

definition (2.5.2) can also be extended to compactly supported, bounded Radon measures ν on \mathbb{R} in place of h . In this case, $\mathcal{D}^z \nu$ is a tempered distribution, and for any $k \in \mathbb{N}$, (2.5.1) gives us the identity

$$\mathcal{D}^k \nu = (-1)^k \nu^{(k)}. \quad (2.5.4)$$

Note that even if ν is a function, but not in $\mathcal{S}(\mathbb{R})$, this identity need not hold pointwise. Due to the dilating sign in (2.5.4), it may seem more natural to consider the fractional derivative operator defined by

$$\tilde{\mathcal{D}}^z h(t) := \mathcal{F}_s^{-1}((2\pi i s)^z \hat{h}(s))(t), \quad (2.5.5)$$

as in [12] and [21], but the definition (2.5.2) has some advantages, one of them being the formula in the following lemma.

Lemma 2.5.1 (Lemma 6.6 from [9]). *Let $0 < \alpha < 1$, $t_0 \in \mathbb{R}$, and let h be the Fourier transform of an L^1 -function k , such that $s \mapsto (1 + |s|^\alpha)k(s)$ is integrable on \mathbb{R} . Furthermore, assume that h is Lipschitz on $[t_0, \infty)$ with $h'(t) \leq \kappa(1 + |t|)^{-1}$ for almost every $t \geq t_0$. Then we have the following formula for $\operatorname{Re} z = \alpha$ and $t \geq t_0$:*

$$\mathcal{D}^z h(t) = \frac{-1}{\Gamma(1-z)} \int_t^\infty (u-t)^{-z} h'(u) du = \frac{-1}{\Gamma(1-z)} \int_0^\infty u^{-z} h'(t+u) du. \quad (2.5.6)$$

The proof of Lemma 2.5.1 remains the same if we replace the function k by a compactly supported, bounded Borel measure ν .

If we defined the fractional derivative as in (2.5.5), the identity (2.5.6) would change slightly, so that under the hypotheses of Lemma 2.5.1, we would get

$$\tilde{\mathcal{D}}^z h(t) = \frac{1}{\Gamma(1-z)} \int_t^\infty (u-t)^{-z} \frac{d}{du} (h^-)(u) du,$$

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where h^- is defined by $h^-(t) := h(-t)$. However, we will need to apply Lemma 2.5.1 to some functions which are not symmetric in t , so that definition (2.5.2) bears some easier estimates.

An important tool for us is to use this definition of fractional derivatives to generalize the derivative $\langle \xi, \nabla m(\xi) \rangle$, where either $m = \widehat{\chi}_B$ or $m = \widehat{\sigma}_S$. In this case, we have

$$\frac{d}{dt} m(t\xi)|_{t=1} = \langle \xi, \nabla m(\xi) \rangle.$$

We first take a multiplier $m = \widehat{K}$, where $K \in \mathcal{S}(\mathbb{R}^n)$. Then we can set

$$\langle \xi, \nabla \rangle^z m(\xi) := \mathcal{D}_t^z m(t\xi)|_{t=1}, \quad \xi \neq 0, \operatorname{Re} z > -1, \quad (2.5.7)$$

and

$$\langle \xi, \nabla \rangle^z m(\xi) = 0, \quad \xi = 0.$$

We will show that in this case,

$$\langle \xi, \nabla \rangle^z m(\xi) = \int_{\mathbb{R}^n} (2\pi i \langle x, \xi \rangle)^z K(x) e^{-2\pi i \langle x, \xi \rangle} dx. \quad (2.5.8)$$

For $\xi \neq 0$, write again $\theta = \xi/|\xi|$. We extend the definition of φ_θ from (2.3.4) by setting

$$\varphi_\theta(u) := \int_{\langle x, \theta \rangle=0} K(x + u\theta) dx, \quad (2.5.9)$$

with dx being the $(n-1)$ -dimensional Lebesgue measure over the hyperplane orthogonal to θ . If $K = \chi_B$, this coincides with φ as in (2.3.4). Furthermore, we have

$$\int_{-\infty}^{\infty} g(u) \varphi_\theta(u) du = \int_{\mathbb{R}^n} g(\langle x, \theta \rangle) K(x) dx,$$

which is a generalization of (2.3.5), and thus, (2.3.9) still holds with definition (2.5.9). It follows that for $t \in \mathbb{R}$,

$$m(t\xi) = \widehat{\varphi_\theta}(t|\xi|),$$

and hence,

$$\begin{aligned} \mathcal{D}_t^z m(t\xi) &= \int_{\mathbb{R}} (2\pi i u)^z \mathcal{F}_t^{-1}(\widehat{\varphi_\theta}(t|\xi|)) e^{-2\pi i u t} du \\ &= \int_{\mathbb{R}} (2\pi i u)^z \frac{1}{|\xi|} \varphi_\theta\left(\frac{u}{|\xi|}\right) e^{-2\pi i u t} du \\ &= \int_{\mathbb{R}} (2\pi i |\xi| s)^z \varphi_\theta(s) e^{-2\pi i |\xi| s t} ds \\ &= \int_{\mathbb{R}^n} (2\pi i \langle x, \xi \rangle)^z K(x) e^{-2\pi i t \langle x, \xi \rangle} dx. \end{aligned}$$

Once again, we can achieve the formula (2.5.8) for compactly supported, bounded Radon measures ν . For this, consider a positive, symmetric function $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } \psi \subset B_1^n$ and $\|\psi\|_1 = 1$, and let $E = B_1^n + \text{supp } \nu$. For $0 < \varepsilon < 1$, put $\nu_\varepsilon := \nu * \psi_\varepsilon$ and $m_\varepsilon := \widehat{\nu}_\varepsilon$. Then we have

$$\text{supp } \nu_\varepsilon \subset B_\varepsilon^n + \text{supp } \nu \subset E,$$

and

$$\nu = \lim_{\varepsilon \rightarrow 0} \nu_\varepsilon$$

in the sense of distributions. We will show that there is a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ so that

$$\lim_{k \rightarrow \infty} \mathcal{D}_t^z m_{\varepsilon_k}(t\xi) = \int_{\mathbb{R}^n} (2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle} d\nu(x) \quad (2.5.10)$$

for every $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Indeed, since each ν_ε is a smooth, compactly supported function, (2.5.8) and Fubini's theorem imply that

$$\begin{aligned} \mathcal{D}_t^z m_\varepsilon(t\xi) &= \int_{\mathbb{R}^n} (2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle} \nu_\varepsilon(x) dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle} \psi_\varepsilon(x - y) d\nu(y) dx \\ &= \int_E \int_E (2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle} \psi_\varepsilon(y - x) dx d\nu(y). \end{aligned} \quad (2.5.11)$$

Write again $z = a + ib$. Since E is compact and $a > -1$, we get

$$\int_E |(2\pi i \langle x, \xi \rangle)^z| dx \leq (2\pi)^a e^{b|\frac{\pi}{2}|} \int_E |\langle x, \xi \rangle|^a dx \leq (2\pi)^a e^{b|\frac{\pi}{2}|} |\xi|^a \int_E |x|^a dx \leq C < \infty,$$

and thus

$$\left| \int_E (2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle} \psi_\varepsilon(y - x) dx \right| \leq \|\psi_\varepsilon\|_1 \int_E |(2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle}| dx < C.$$

If we set $g_{\xi,t}(y) := \chi_E(y) \cdot (2\pi i \langle y, \xi \rangle)^z e^{-2\pi i t \langle y, \xi \rangle}$, we find a subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ so that $g_{\xi,t} * \psi_{\varepsilon_k} \rightarrow g_{\xi,t}$ almost everywhere as $k \rightarrow \infty$. Since E is compact, it follows by the dominated convergence theorem and (2.5.11) that

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathcal{D}_t^z m_{\varepsilon_k}(t\xi) &= \lim_{k \rightarrow \infty} \int_E g_{\xi,t} * \psi_{\varepsilon_k}(y) d\nu(y) \\ &= \int_E \lim_{k \rightarrow \infty} g_{\xi,t} * \psi_{\varepsilon_k}(y) d\nu(y) \\ &= \int_E g_{\xi,t}(y) d\nu(y) \\ &= \int_{\mathbb{R}^n} (2\pi i \langle y, \xi \rangle)^z e^{-2\pi i t \langle y, \xi \rangle} d\nu(y), \end{aligned}$$

so that (2.5.10) holds. This justifies the definitions

$$\mathcal{D}_t^z m(t\xi) := \int_{\mathbb{R}^n} (2\pi i \langle x, \xi \rangle)^z e^{-2\pi i t \langle x, \xi \rangle} d\nu(x)$$

and

$$\langle \xi, \nabla \rangle^z m(\xi) := \mathcal{D}_t^z m(t\xi) \Big|_{t=1}, \quad (2.5.12)$$

including the cases $\nu = \chi_B$ and $\nu = \sigma_S$.

The operators defined in (2.5.2) and (2.5.12) provide another key result from [12], which connects them to maximal operators.

Proposition 2.5.2. *Let K be a Schwartz function or a compactly supported, bounded Borel measure, such that $m := \hat{K}$ is differentiable and there is a constant C so that for every $\theta \in S^{n-1}$ and every $u \in \mathbb{R} \setminus \{0\}$,*

$$|m(u\theta)| \leq C \cdot \min\{|u|, |u|^{-1}\}, \quad (2.5.13)$$

$$|\langle \theta, \nabla m(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\}. \quad (2.5.14)$$

(i) *If for some $\alpha > 1/2$, we have*

$$C_\alpha(m) := \sup_{\omega \in S^{n-1}} \left(\int_0^\infty \left| u^{\alpha+1} \mathcal{D}_u^\alpha \left(\frac{m(u\omega)}{u} \right) \right|^2 \frac{du}{u} \right)^{1/2} < \infty, \quad (2.5.15)$$

then

$$\left\| \sup_{t>0} |K_t * f| \right\|_{L^2} \leq C'_\alpha \|f\|_{L^2}, \quad (2.5.16)$$

with C'_α only depending on α and $C_\alpha(m)$.

(ii) *If for some $\alpha > 1/p$ (or $\alpha = 1$ if $p = 1$), both m and $\langle \xi, \nabla \rangle^\alpha m$ are L^p -multipliers, then we have*

$$\left\| \sup_{1 \leq t \leq 2} |K_t * f| \right\|_{L^p} \leq C_{\alpha,p} (2 \|T_m f\|_p + \|T_{\langle \xi, \nabla \rangle^\alpha m} f\|_p), \quad (2.5.17)$$

where $C_{\alpha,p}$ only depends on α and p .

In [12] and [13, Chapter 6], it is shown that we can apply part (i) of Proposition 2.5.2 to achieve the strong boundedness property (2.4.3) in Proposition 2.4.1, if m satisfies (2.5.13) and (2.5.14). Furthermore, the properties (2.3.3) ensure that $m = \widehat{\chi_B} - \widehat{P_{L(B)}}$ always satisfies (2.5.13) and (2.5.14). Besides the proof in [12], a very detailed proof is provided in [13, Chapter 6], including several more properties about fractional derivative operators and a way to fix a technical difficulty, which occurs because part (i) of Proposition 2.5.2 does not exactly imply strong boundedness, even though this can be achieved without any further assumptions, see [12] and [13, Sections 6.4 and 6.5]. Because of that, we will not explain the proof of Proposition 2.5.2, but we shall still point

out a few details from it for later use.

If h is a function that satisfies the conditions from Lemma 2.5.1, one can easily see that for any $\lambda > 0$, we have

$$\mathcal{D}_s^z h(s\lambda) \Big|_{s=t} = \lambda^z \mathcal{D}_s^z h(s) \Big|_{s=\lambda t}. \quad (2.5.18)$$

For K and m as in Proposition 2.5.2, it follows from [13, Remark 6.10] that we can write

$$\frac{m(t\xi)}{t} = \mathcal{D}_t^{-\alpha} \mathcal{D}_t^\alpha \left(\frac{m(t\xi)}{t} \right) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (u-t)^{\alpha-1} \mathcal{D}_u^\alpha \left(\frac{m(u\xi)}{u} \right) du,$$

with $0 < \alpha < 1$. This leads to

$$K_t * f(x) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{t}{u} \left(1 - \frac{t}{u} \right)^{\alpha-1} P_u^\alpha f(x) \frac{du}{u}, \quad (2.5.19)$$

where

$$\widehat{P_u^\alpha f}(\xi) = p_u^\alpha(\xi) \hat{f}(\xi) := u^{\alpha+1} \mathcal{D}_u^\alpha \left(\frac{m(u\xi)}{u} \right) \hat{f}(\xi). \quad (2.5.20)$$

Hence, to bound $\| \sup_{1 \leq t \leq 2} |K_t * f| \|_{L^p}$, with K as in Proposition 2.5.2, Hölder's inequality implies that we only need to bound $\|P_u^\alpha f\|_p$, but from the assumption of (ii), one can derive the uniform estimate

$$\|P_u^\alpha f\|_p \leq 2\|T_m f\|_p + \|T_{\langle \xi, \nabla \rangle^{\alpha m}} f\|_p. \quad (2.5.21)$$

Thus our analysis of maximal functions boils down to bounding the multiplier operator $T_{\langle \xi, \nabla \rangle^{\alpha m}}$ in L^p .

2.5.2 Riesz fractional derivatives

In Section 2.6, we will use fractional derivatives to construct an analytic family of multiplier operators. However, this requires us to consider a different family of fractional derivative operators, the Riesz fractional derivatives. In [21], it has been shown that in our setting, we can bound the fractional derivative operators defined in (2.5.2) if we can bound the Riesz fractional derivative operators.

For a function $f \in C((0, 2])$ and a complex number z with $\operatorname{Re} z < 0$, the Riesz fractional derivative with base point 2 is defined as

$$I^{-z} f(t) = I_t^{-z} f(t) = \frac{-1}{\Gamma(-z)} \int_t^2 (u-t)^{-z-1} f(u) du,$$

for $0 < t \leq 2$. Through analytic continuation, this family of operators can be extended to the complex plane: For any $k \in \mathbb{N}_{>0}$, assume that f as above is k times differentiable and let $\operatorname{Re} z < k$. Then

$$I^{-z} f(t) = E_{k,f}(z, t) + (-1)^k \frac{1}{\Gamma(k-z)} \int_t^2 (u-t)^{-z+k-1} f^{(k)}(u) du, \quad (2.5.22)$$

where

$$E_{k,f}(z, t) = \sum_{j=0}^{k-1} (-1)^j \frac{(2-t)^{-z+j} f^{(j)}(2)}{\Gamma(j+1-z)}. \quad (2.5.23)$$

One can see this by using integration by parts (see [13, Section 7.1]). It also follows that for $f \in \mathcal{S}(\mathbb{R})$, we have

$$I^{-k} f(t) = (-1)^k f^{(k)}(t)$$

for each $k \in \mathbb{N}$. We conclude this section by the connection between \mathcal{D}^z and I^{-z} that allows us to study the latter one. The proof is from [21, pp. 307-308].

Lemma 2.5.3. *Let $1 \leq p \leq \infty$, $0 < \alpha < 1$, and let m be an L^p -multiplier on \mathbb{R}^n . Let*

$$\tilde{m}(\xi) = [I_t^{-\alpha} m(t\xi) - \mathcal{D}_t^\alpha m(t\xi)]|_{t=1}.$$

Then \tilde{m} is an L^p -multiplier and

$$\|\tilde{m}\|_{p \rightarrow p} \leq \|m\|_{p \rightarrow p}$$

Proof. By (2.5.6) and integration by parts, we have

$$\begin{aligned} \tilde{m}(\xi) &= \frac{m(2\xi)}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_2^\infty (u-1)^{-\alpha} \frac{d}{du} m(u\xi) du \\ &= \frac{-1}{\Gamma(-\alpha)} \int_2^\infty (u-1)^{-\alpha-1} m(u\xi) du. \end{aligned}$$

Since each $m(u\xi)$ has the same multiplier norm as m , Minkowski's inequality implies

$$\|\tilde{m}\|_{p \rightarrow p} \leq \frac{1}{\Gamma(-\alpha)} \int_2^\infty (u-1)^{-\alpha-1} \|m\|_{p \rightarrow p} du \leq \frac{1}{\Gamma(1-\alpha)} \|m\|_{p \rightarrow p} < \|m\|_{p \rightarrow p},$$

since $\Gamma(x) > 1$ for $0 < x < 1$.

q.e.d.

2.6 Fourier decay and interpolation between derivatives

As we already mentioned in Sections 2.1 and 2.5, we will use interpolation of analytic families to achieve the bound necessary for part (ii) of Proposition 2.5.2. We start with a very simplified approach: In the case $m = \widehat{\chi}_B$, with B in isotropic position, m is an L^1 -multiplier, while the decay properties (2.3.2) of m yield that $\langle \xi, \nabla m(\xi) \rangle$ is bounded uniformly, thus $\langle \xi, \nabla \rangle^1 m$ is an L^2 -multiplier. We will consider the family of functions $m_z := \langle \xi, \nabla \rangle^z m$, $z \in \mathbb{S}$, where $\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$. Then the multiplier operators $T_{m_0} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ and $T_{m_1} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ are bounded.

If we suppose that we may use Stein's interpolation theorem for the family of operators $(T_{m_z})_{z \in \mathbb{S}}$, we would get that for each $\alpha \in (0, 1)$, the operator

$$T_{m_\theta}: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$$

would be bounded, by choosing θ , as in Theorem 2.1.1, such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \theta/2, \quad \text{i.e.} \quad \theta = 2 - \frac{2}{p}.$$

By Proposition 2.5.2 (ii), we have to find $\alpha > 1/p$ such that $\langle \xi, \nabla \rangle^\alpha m$ is an L^p -multiplier, so by setting $\alpha = \theta$, we need to have

$$\frac{1}{p} < 2 - \frac{2}{p}.$$

But this is equivalent to $3/p < 2$, or $p > 3/2$.

However, we cannot show that we may interpolate the family $(T_{m_z})_{z \in \mathbb{S}}$. This is where the operators I^{-z} come into play, using Lemma 2.5.3. Fix $k \in \mathbb{N}_{>0}$. We redefine the functions m_z by setting

$$m_z(\xi) := I_t^{-z} m(t\xi)|_{t=1}$$

for $\operatorname{Re} z < k$, and (2.5.22) and (2.5.23) allow us to break down the boundedness of m_z to the boundedness of the classic derivatives $\frac{d^j}{dt^j} m(t\xi)$ for $1 \leq t \leq 2$, $1 \leq j \leq k$. A problem of rather technical nature is that a direct approach to bound m_{1+it} for $t \in \mathbb{R}$ would require the boundedness of $\frac{d^2}{dt^2} m(t\xi)$, but in general, we will not be able to obtain such a bound independently of n . However, we can work around that by taking the range $-\varepsilon \leq \operatorname{Re} z \leq 1 - \varepsilon$ for some sufficiently small $\varepsilon > 0$ instead. We will show that the requirements for Theorem 2.1.1 are fulfilled in an even more generalized setting, where we also require the boundedness of some higher derivatives.

This follows the concluding remark from [12], in which Carbery implies that if for some $p \neq 2$, $\langle \xi, \nabla m \rangle$ has an L^p -multiplier norm that does not depend on n , we can improve the lower bound $3/2$ for p . This can be achieved by bounding some higher derivatives. The boundedness of these derivatives seems to be linked with the decay of m itself: In our setting, where we will have $m = \widehat{\sigma_S}$ with $S = (S^{N-1})^\ell$ for some $N \geq 3$, it turns out that $m(\xi) = \mathcal{O}(|\xi|^{-\frac{N-1}{2}})$ as $|\xi| \rightarrow \infty$, and we can use this to show that $\langle \xi, \nabla \rangle^{\frac{N-1}{2}} m$ is bounded.

While the interpolation argument has been mentioned in [12], carried out for a different family of multiplier operators in [21], and repeated in [13] with a lot of details, it has not been stated in form of a lemma or proposition in any of the works we rely on.

Proposition 2.6.1. *Let ν be a bounded Radon measure on \mathbb{R}^n and let $m = \widehat{\nu}$. Suppose that for some $k \in \mathbb{N}_{>0}$, there is a constant C such that*

$$\left| \left(\frac{d}{dt} \right)^j m(t\xi) \right| < C$$

uniformly for $j \in \{0, \dots, k\}$ and $1 \leq t \leq 2$. Then for each $\frac{2k+1}{2k} < p < \frac{2k}{2k-1}$, there is an $\alpha > 1/p$ such that m_α is an L^p -multiplier, and $\|m_\alpha\|_{p \rightarrow p}$ is bounded by a constant depending only on C , p , and k .

We shall prove this using Stein's interpolation theorem. To ensure that we can indeed apply Stein's interpolation theorem, we will use some estimates on the gamma function, given by the following lemma taken from [13, Section 3.1].

Lemma 2.6.2. *Let $a \in \mathbb{R}$. Then for each $z \in \mathbb{C}$ with $\operatorname{Re} z \geq a$, we have the bound*

$$\left| \frac{1}{\Gamma(z)} \right| \leq \beta_a (1 + |\operatorname{Im} z|^2)^{1/4 - a/2} e^{\frac{\pi}{2} |\operatorname{Im} z|}, \quad (2.6.1)$$

where $\beta_a = 2\Gamma(|a| + 1)$ if $a < -1$, and $\beta_a = 2$ if $a \geq -1$.

A proof of Lemma 2.6.2 is given in [13, Section 3.1] as well.

Proof of Proposition 2.6.1. We fix $1 < p < 2$ and $0 < \varepsilon < 1$ first and designate some more precise values later. For $-\varepsilon \leq \operatorname{Re} z \leq k - \varepsilon$, put

$$T_z = T_{m_z}.$$

We shall show first that the family $(T_z)_z$ satisfies the conditions of Theorem 2.1.1. We start with verifying the admissible growth property. For $j \in \{0, \dots, k\}$, write

$$m_t^j(\xi) := \frac{d^j}{dt^j} m(t\xi),$$

and note that we have

$$\frac{d^j}{dt^j} T_{m_t^0} f = T_{m_t^j} f. \quad (2.6.2)$$

Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then by (2.5.22), (2.5.23), and (2.6.2), one can see that

$$\begin{aligned} \langle T_z f, g \rangle &= \sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^n} \frac{T_{m_2^j} f(x)}{\Gamma(j+1-z)} g(x) dx \\ &\quad + \frac{(-1)^k}{\Gamma(k-z)} \int_{\mathbb{R}^n} \int_1^2 (u-1)^{-z+k-1} T_{m_u^k} f(x) g(x) du dx. \end{aligned} \quad (2.6.3)$$

For the first term in (2.6.3), take $j \in \{0, \dots, k-1\}$. Then by the Cauchy-Schwarz inequality, we have

$$\left| \int_{\mathbb{R}^n} T_{m_2^j} f(x) g(x) dx \right| \leq \|T_{m_2^j} f\|_2 \|g\|_2 \leq \|m_2^j\|_\infty \|f\|_2 \|g\|_2. \quad (2.6.4)$$

For the second term in (2.6.3), we can apply Fubini's theorem and Cauchy-Schwarz again

to get

$$\begin{aligned}
 \left| \int_{\mathbb{R}^n} \int_1^2 (u-1)^{-z+k-1} T_{m_u^k} f(x) g(x) \, du \, dx \right| &\leq \int_1^2 |(u-1)^{-z+k-1}| \int_{\mathbb{R}^n} |T_{m_u^k} f(x) g(x)| \, dx \, du \\
 &\leq \int_1^2 (u-1)^{-\operatorname{Re} z+k-1} \, du \max_{1 \leq t \leq 2} \|m_t^k\|_\infty \|f\|_2 \|g\|_2 \\
 &= \frac{1}{k - \operatorname{Re} z} \max_{1 \leq t \leq 2} \|m_t^k\|_\infty \|f\|_2 \|g\|_2 \\
 &\leq \frac{1}{\varepsilon} \max_{1 \leq t \leq 2} \|m_t^k\|_\infty \|f\|_2 \|g\|_2. \tag{2.6.5}
 \end{aligned}$$

Since the factor $\Gamma(k-z)^{-1}$ occurs twice in (2.6.3), combining (2.6.4) and (2.6.5) yields

$$|\langle T_z f, g \rangle| \leq \frac{2}{\varepsilon} \max_{\substack{0 \leq j \leq k \\ 1 \leq t \leq 2}} \|m_t^j\|_\infty \cdot \|f\|_2 \|g\|_2 \sum_{j=0}^{k-1} \frac{1}{|\Gamma(j-z+1)|}. \tag{2.6.6}$$

Since for each j , $\operatorname{Re}(j-z+1) > -k+1$, Lemma 2.6.2 and the assumption that $\|m_t^j\|_\infty \leq C$ uniformly in j and t imply that

$$|\langle T_z f, g \rangle| \leq C \frac{2}{\varepsilon} k! \cdot (1 + |\operatorname{Im} z|^2)^{k/2-1/4} e^{\frac{\pi}{2} |\operatorname{Im} z|} \|f\|_2 \|g\|_2.$$

Hence there is $a < \pi$ such that

$$e^{-a |\operatorname{Im} z|} \log |\langle T_z f, g \rangle| < C_k C_{f,g},$$

so that $(T_z)_z$ is indeed of admissible growth.

Next, we show that we can estimate

$$\|T_{-\varepsilon+i\tau} f\|_1 \leq M_1(\tau) \|f\|_1 \quad \text{and} \quad \|T_{k-\varepsilon+i\tau} f\|_2 \leq M_2(\tau) \|f\|_2,$$

with

$$e^{-b\tau} \log M_j(\tau) \leq C_k \tag{2.6.7}$$

for some $b < \pi$, $j = 1, 2$, and uniformly in $\tau \in \mathbb{R}$, which will allow us to apply Stein's interpolation theorem.

We have

$$T_{-\varepsilon+i\tau} f(x) = \frac{1}{\Gamma(\varepsilon - i\tau)} \int_1^2 (u-1)^{\varepsilon-i\tau-1} T_{m_u^0} f(x) \, du. \tag{2.6.8}$$

Since m is an L^1 -multiplier, m_u^0 is also an L^1 -multiplier, and since

$$\left| \int_1^2 (u-1)^{\varepsilon-i\tau-1} \, du \right| \leq \int_1^2 (u-1)^{\varepsilon-1} \, du = \frac{1}{\varepsilon},$$

Minkowski's inequality and Lemma 2.6.2 yield that

$$\|T_{-\varepsilon+i\tau}f\|_1 \leq \frac{1}{|\Gamma(\varepsilon-i\tau)|} \frac{1}{\varepsilon} \|T_m f\|_1 \leq \frac{1}{\varepsilon} \cdot 2(1+\tau^2)^{1/4-\varepsilon/2} e^{\frac{\pi}{2}|\tau|} \|\nu\|_1 \|f\|_1. \quad (2.6.9)$$

For the L^2 -estimate, we have to bound

$$\sum_{j=0}^{k-1} (-1)^k \frac{m_2^j(\xi)}{\Gamma(j+1-k+\varepsilon-i\tau)} + (-1)^k \frac{1}{\Gamma(\varepsilon-i\tau)} \int_1^2 (u-1)^{\varepsilon-i\tau-1} m_u^k(\xi) du$$

uniformly in ξ . But by arguing as before, this expression can be bounded by

$$\frac{1}{\varepsilon} \max_{\substack{0 \leq j \leq k \\ 1 \leq t \leq 2}} \|m_t^j\|_\infty \cdot 2 \sum_{j=0}^{k-1} \frac{1}{|\Gamma(\varepsilon-i\tau-j)|} \leq \frac{1}{\varepsilon} C_k (1+\tau^2)^{k/2-1/4} e^{\frac{\pi}{2}|\tau|}.$$

Hence we can put

$$M_1(\tau) = \frac{1}{\varepsilon} \cdot 2(1+\tau^2)^{1/4-\varepsilon/2} e^{\frac{\pi}{2}|\tau|} \|\nu\|_1, \quad M_2(\tau) = \frac{1}{\varepsilon} C_k (1+\tau^2)^{k/2-1/4} e^{\frac{\pi}{2}|\tau|}$$

for some suitable C_k , and (2.6.7) is indeed fulfilled.

Choose $\theta = 2 - 2/p$. Then $1/p = 1 - \theta + \theta/2$, and by Stein's interpolation theorem,

$$\|T_{\alpha_\theta}\|_{p \rightarrow p} \leq M(\theta) \leq C_{p,k,\varepsilon},$$

where $M(\theta)$ is defined as in (2.1.2), and $\alpha_\theta = (1-\theta) \cdot (-\varepsilon) + \theta(k-\varepsilon) = k\theta - \varepsilon$.

Now fix p such that $\frac{2k+1}{2k} < p < \frac{2k}{2k-1}$, and $\alpha = \alpha_\theta$. To conclude the lemma, we simply need to find some $0 < \varepsilon < 1$ such that

$$\frac{1}{p} < k\theta - \varepsilon < 1. \quad (2.6.10)$$

Before we choose ε , let us mention that $k\theta - \varepsilon = k(2 - 2/p) - \varepsilon > 1/p$ holds iff $2k - \varepsilon > \frac{2k+1}{p}$, which is equivalent to

$$p > \frac{2k+1}{2k-\varepsilon}, \quad (2.6.11)$$

while $k\theta - \varepsilon < 1$ holds iff $2k - \varepsilon - 1 < 2k/p$, which can be restated as

$$p < \frac{2k}{2k-1-\varepsilon}. \quad (2.6.12)$$

Since $\frac{2k+1}{2k-\varepsilon} > \frac{2k+1}{2k}$, we will thus pick ε such that (2.6.11) holds. (2.6.11) is equivalent to

$$\varepsilon < 2k - \frac{2k+1}{p},$$

and since $p > \frac{2k+1}{2k}$, we have $\frac{2k+1}{p} < 2k$, and so, we can pick $\varepsilon > 0$ satisfying (2.6.11). Due to the condition $p < \frac{2k}{2k-1}$, we also get (2.6.12) and furthermore,

$$\varepsilon < 2k - \frac{4k^2 - 1}{2k} = \frac{1}{2k} \leq \frac{1}{2}.$$

As the choice of ε only depends on k and p , this concludes the proof. **q.e.d.**

Remark. *The proof shows that the condition $\frac{2k+1}{2k} < p$ is indeed necessary. The lower bound comes from the stronger condition (2.6.11). The only purpose of the upper bound $p < \frac{2k}{2k-1}$ is to make it more easy to pick suitable values for ε and α , and by duality, one can see that with $\alpha = \alpha_\theta$ as above, $\|m_\alpha\|_{p \rightarrow p}$ is bounded for $\frac{2k+1}{2k} < p < 2k + 1$.*

If $m = \widehat{\sigma_S}$ and $S = (S^{N-1})^\ell$ again, then $(N-1)/2$ is not an integer if N is even. In this case, we cannot bound the multiplier $\langle \xi, \nabla \rangle^{N/2}$, but the bound on $\langle \xi, \nabla \rangle^{N/2-1}$ will not be sufficient for our proofs of Theorems 2 and 4. In general, our Fourier decay need not be of integer order. Thus we will also provide a way that allows us to replace k in Lemma 2.6.1 by a positive real number. This has not been done in [12], [13], or [21], and the following proposition is an original result of this thesis.

Proposition 2.6.3. *Let ν be a finite Borel measure, $m = \hat{\nu}$, $\eta \in (1, \infty) \setminus \mathbb{N}$, and put $k = \lceil \eta \rceil$. Assume that there is a constant C such that*

$$(1 + |\xi|)^\eta |m(t\xi)| < C, \quad \text{and} \quad (1 + |\xi|)^{\eta-k} \left| \frac{d^j}{dt^j} m(t\xi) \right| < C$$

uniformly for $\xi \in \mathbb{R}^n$, $1 \leq t \leq 2$, and $j \in \{0, \dots, k\}$. Then for each $\frac{2\eta+1}{2\eta} < p < \frac{2\eta}{2\eta-1}$, there is $\alpha > 1/p$ such that m_α is an L^p -multiplier, and $\|m_\alpha\|_{p \rightarrow p}$ is bounded by a constant depending only on C , p , and η .

Proof. Let $0 < \varepsilon < \min\{2\eta - \frac{2\eta+1}{p}, \eta - \lceil \eta \rceil\}$. We will just show that for $-\varepsilon \leq \operatorname{Re} z \leq \eta - \varepsilon$, the family $(T_z)_z$, with $T_z = T_{m_z}$, satisfies the conditions to be interpolated between L^1 for $\operatorname{Re} z = -\varepsilon$ and L^2 for $\operatorname{Re} z = \eta - \varepsilon$. Then, the interpolation part will be the same as in Proposition 2.6.1, with η instead of k . For the endpoint $\operatorname{Re} z = -\varepsilon$, nothing changes as well, so we only have show that

$$\|T_{\eta-\varepsilon+i\tau} f\|_2 \leq M_2(\tau) \|f\|_2 \tag{2.6.13}$$

for each $\tau \in \mathbb{R}$, such that there is some $b < \pi$ with

$$e^{-b\tau} \log M_2(\tau) \leq C_\eta.$$

We can not show directly that $m_{\eta-\varepsilon+i\tau} \in L^\infty$, as this would require uniform bounds on $\frac{d^k}{dt^k} m(t\xi)$, $1 \leq t \leq 2$, but in general, such bounds are not given. Instead, our proof requires an additional interpolation and depends on keeping track of

the constants which bound the respective operators, so that we need to do some rather technical calculations. For this, write

$$m_z^\varepsilon(\xi) = (1 + |\xi|)^{\eta - \varepsilon - z} m_z, \quad T_z^\varepsilon = T_{m_z^\varepsilon}.$$

Note that $m_{\eta - \varepsilon + i\tau}^\varepsilon(\xi) = (1 + |\xi|)^{-i\tau} m_{\eta - \varepsilon + i\tau}$, for each $\tau \in \mathbb{R}$, so that $\|m_{\eta - \varepsilon + i\tau}^\varepsilon\|_\infty = \|m_{\eta - \varepsilon + i\tau}\|_\infty$.

We will apply Corollary 2.1.2 to the family $(T_z^\varepsilon)_{-\varepsilon \leq \operatorname{Re} z \leq k - \varepsilon}$ on L^2 to achieve (2.6.13). First, we show that our assumptions provide that we have admissible growth. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and define the multipliers

$$m_{t,z}^j(\xi) = (1 + |\xi|)^{\eta - \varepsilon - z} \frac{d^j}{dt^j} m(t\xi).$$

Then the arguments for the proof of (2.6.6) show that

$$|\langle T_z^\varepsilon f, g \rangle| \leq \frac{2}{\varepsilon} \max_{\substack{0 \leq j \leq k \\ 1 \leq t \leq 2}} \|T_{m_{t,z}^j} f\|_2 \cdot \|g\|_2 \sum_{j=0}^{k-1} \frac{1}{|\Gamma(j - z + 1)|}.$$

But by our assumption, we can bound

$$\begin{aligned} \|T_{m_{t,z}^j} f\|_2^2 &= \int_{\mathbb{R}^n} \left| (1 + |\xi|)^{\eta - \varepsilon - z} \frac{d^j}{dt^j} m(t\xi) \hat{f}(\xi) \right|^2 d\xi \\ &= \int_{\mathbb{R}^n} \left| (1 + |\xi|)^{\eta - k} \frac{d^j}{dt^j} m(t\xi) \right|^2 \cdot |(1 + |\xi|)^{k - \varepsilon - z} \hat{f}(\xi)|^2 d\xi \\ &< C^2 \int_{\mathbb{R}^n} |(1 + |\xi|)^{k - \varepsilon - z} \hat{f}(\xi)|^2 d\xi. \end{aligned}$$

It follows that

$$|\langle T_z^\varepsilon f, g \rangle| \leq C \frac{2}{\varepsilon} k! (1 + |\operatorname{Im} z|^2)^{k/2 - 1/4} e^{\frac{\pi}{2} |\operatorname{Im} z|} \|g\|_2 \left(\int_{\mathbb{R}^n} |(1 + |\xi|)^{k - \varepsilon - z} \hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

and since for $-\varepsilon \leq \operatorname{Re} z \leq k - \varepsilon$, we have

$$|(1 + |\xi|)^{k - \varepsilon - z}| = (1 + |\xi|)^{k - \varepsilon - \operatorname{Re} z} \leq (1 + |\xi|)^k,$$

we can deduce that for each $a > 0$,

$$e^{-a |\operatorname{Im} z|} \log |\langle T_z^\varepsilon f, g \rangle| \leq C_{k,f,g} e^{-a |\operatorname{Im} z|} (\log(1 + |\operatorname{Im} z|^2) + \frac{\pi}{2} |\tau + \operatorname{Im} z|),$$

so that the admissible growth condition is fulfilled. The next step is to find some real functions \tilde{M}_1, \tilde{M}_2 such that for each $\tau \in \mathbb{R}$,

$$\|T_{-\varepsilon + i\tau}^\varepsilon f\|_2 \leq \tilde{M}_1(\tau) \|f\|_2, \quad \|T_{k - \varepsilon + i\tau}^\varepsilon f\|_2 \leq \tilde{M}_2(\tau) \|f\|_2,$$

and some $b < \pi$ such that

$$e^{-b\tau} \log \tilde{M}_j(\tau) \leq C_\eta,$$

for $j = 1, 2$. For $\operatorname{Re} z = -\varepsilon$, we first bound

$$\|T_{m_{t,z}^0} f\|_2 \leq \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^\eta |m(t\xi)| \|f\|_2 < C \|f\|_2$$

for $1 \leq t \leq 2$, which is possible due to our assumptions. Now we can use the identity (2.6.8) and Minkowski's inequality, just as for (2.6.9), to see that

$$\|T_{-\varepsilon+i\tau}^\varepsilon f\|_2 \leq C \frac{1}{\varepsilon} \cdot 2(1 + \tau^2)^{1/4 - \varepsilon/2} e^{\frac{\pi}{2}|\tau|} \|f\|_2 \leq C_\varepsilon e^{\frac{3\pi}{4}|\tau|} \|f\|_2.$$

Similarly, the case $\operatorname{Re} z = k - \varepsilon$ breaks down to bounding

$$\sum_{j=0}^{k-1} (-1)^k \frac{m_{2,k-\varepsilon+i\tau}^j(\xi)}{\Gamma(j+1-k+\varepsilon-i\tau)} + (-1)^k \frac{1}{\Gamma(\varepsilon-i\tau)} \int_1^2 (u-1)^{\varepsilon-i\tau-1} m_{u,k-\varepsilon+i\tau}^k(\xi) du$$

uniformly in ξ . But since the occurring multipliers are uniformly bounded by our assumptions, we can argue again as in the proof of Proposition 2.6.1 to see that

$$\|T_{k-\varepsilon+i\tau}^\varepsilon f\|_2 \leq C_k \frac{1}{\varepsilon} \cdot 2(1 + \tau^2)^{k/2 - 1/4} e^{\frac{\pi}{2}|\tau|} \|f\|_2 \leq C_{k,\varepsilon} e^{\frac{3\pi}{4}|\tau|} \|f\|_2.$$

Altogether, we can put

$$\tilde{M}_1(\tau) = \tilde{M}_2(\tau) = C_{k,\varepsilon} e^{\frac{3\pi}{4}|\tau|}. \quad (2.6.14)$$

Until now, the proof was very similar to the one before, but from here on, we need to keep a sharp eye on the growth of our constants. The L^2 -interpolation of the family $(T_z^\varepsilon)_{-\varepsilon \leq \operatorname{Re} z \leq k-\varepsilon}$ is indeed possible, as

$$\log \tilde{M}_1(\tau) = C'_{k,\varepsilon} \frac{3\pi}{4} |\tau|,$$

so that for any $0 < b < \pi$, we get

$$e^{-b|\tau|} \log \tilde{M}_1(\tau) \leq C_{k,\varepsilon} < \infty.$$

By Corollary 2.1.2, the interpolation gives us indeed a bound as in (2.6.13): If we choose $0 < \theta < 1$ such that

$$\underbrace{(1-\theta) \cdot (-\varepsilon) + \theta(k-\varepsilon)}_{=k\theta-\varepsilon} = \eta - \varepsilon,$$

then $\theta = \eta/k > 1/2$, and

$$M_2(\tau) = \exp \left[\frac{\sin(\pi\theta)}{2} \int_{\mathbb{R}} \left(\frac{\log \tilde{M}_1(y+\tau)}{\cosh(\pi y) - \cos(\pi\theta)} + \frac{\log \tilde{M}_2(y+\tau)}{\cosh(\pi y) + \cos(\pi\theta)} \right) dy \right]. \quad (2.6.15)$$

It remains to find $b < \pi$ so that we have

$$e^{-b\tau} \log M_2(\tau) < C_\eta < \infty. \quad (2.6.16)$$

But since our choices of ε , k and θ depend only on η , and $|\cos(\pi\theta)| < 1$ (as $1/2 < \theta < 1$), we do have

$$\begin{aligned} \log M_2(\tau) &\leq C_\eta \int_{\mathbb{R}} \left(\frac{|y + \tau|}{\cosh(\pi y) - \cos(\pi\theta)} + \frac{|y + \tau|}{\cosh(\pi y) + \cos(\pi\theta)} \right) dy \\ &= C_\eta \int_{\mathbb{R}} \frac{2 \cosh(\pi y) |y + \tau|}{\cosh(\pi y)^2 - \cos(\pi\theta)^2} dy \\ &\leq 2C_\eta \int_{\mathbb{R}} \frac{|y + \tau|}{\cosh(\pi y) - |\cos(\pi\theta)|} dy \\ &= 2C_\eta \int_{\mathbb{R}} \frac{|y + \tau|}{\cosh(\pi y)} \cdot \underbrace{\frac{\cosh(\pi y)}{\cosh(\pi y) - |\cos(\pi\theta)|}}_{\leq 1/(1-|\cos(\pi\theta)|)} dy \\ &\leq C'_\eta \int_{\mathbb{R}} \frac{|y| + |\tau|}{e^{\pi y} + e^{-\pi y}} dy \\ &\leq C''_\eta (1 + |\tau|). \end{aligned}$$

Thus $\log M_2(\tau)$ grows linearly in $|\tau|$, allowing us to proceed with the interpolation as in Proposition 2.6.1. **q.e.d.**

2.7 Müller's geometric invariants and interpolation ideas

In this section, we shall point out how Müller [21] exploited the geometric properties of a convex symmetric body B even further. If B is in isotropic position, we define the two geometric invariants $\sigma = \sigma(B)$ and $Q = Q(B)$ as

$$\sigma(B)^{-1} := \sup\{\text{Vol}_{n-1}(\{x \in B : \langle x, \xi \rangle = 0\}) : \xi \in S^{n-1}\} = \sup_{\xi \in S^{n-1}} \varphi_\xi(0), \quad (2.7.1)$$

and

$$Q(B) := \max\{\text{Vol}_{n-1}(\pi_\xi(B)) : \xi \in S^{n-1}\}, \quad (2.7.2)$$

where π_ξ is the orthogonal projection onto the hyperplane orthogonal to ξ . Due to (2.3.8), we have $\sigma(B) \approx L(B)$.

Write again $m(\xi) = \widehat{\chi_B}(\xi)$. From Carbery's arguments, especially Proposition 2.5.2, it suffices to show that for $1 < p < \infty$ and $1/2 < \alpha < 1$, we have

$$\|T_{\langle \xi, \nabla \rangle}^{\alpha m} f\|_p \leq C_\alpha(p, \sigma, Q) \|f\|_p. \quad (2.7.3)$$

For this, choose $N = N(p) \in \mathbb{N}_{>0}$ large and define, for $0 \leq \operatorname{Re} z \leq N$, the multipliers

$$m_z(\xi) = (1 + |\xi|)^{1-z} \langle \nabla, \xi \rangle^z m(\xi).$$

Morally, the key ideas lie in estimating the multiplier norms $\|m_N\|_{2 \rightarrow 2}$ and $\|m_0\|_{p \rightarrow p}$, $1 < p < 2$, in terms of σ and Q . We can not interpolate between these endpoints, but arguments very similar to those for Proposition 2.6.1 lead to an interpolation that provides the desired estimate (2.7.3). Fix $\xi \in \mathbb{R}^n \setminus \{0\}$, and let $\theta = \xi/|\xi|$ and $\varphi = \varphi_\theta$. For the endpoint $z = N$, we have by (2.3.5) and partial integration

$$\begin{aligned} (1 + |\xi|)^{N-1} m_N(\xi) &= (-i2\pi|\xi|)^N \int_{-\infty}^{\infty} e^{-2\pi i|\xi|u} u^N \varphi(u) \, du \\ &= (-i2\pi|\xi|)^{N-1} \int_{-\infty}^{\infty} e^{-2\pi i|\xi|u} \frac{d}{du} (u^N \varphi(u)) \, du. \end{aligned}$$

The properties of φ and similar arguments as in Section 2.3 thus give us

$$|(1 + |\xi|)^{N-1} m_N(\xi)| \leq C_N |\xi|^N \int_B |\langle x, \theta \rangle|^N \, dx \quad (2.7.4)$$

and

$$|(1 + |\xi|)^{N-1} m_N(\xi)| \leq C_N |\xi|^{N-1} \int_B |\langle x, \theta \rangle|^{N-1} \, dx. \quad (2.7.5)$$

Suppose that for each $\tilde{\theta} \in S^{n-1}$ and $k \in \mathbb{N}_{>0}$, we have uniform bounds of the form

$$\int_B |\langle x, \tilde{\theta} \rangle|^k \, dx \leq C_k. \quad (2.7.6)$$

Then (2.7.4) and (2.7.5) imply

$$|(1 + |\xi|)^{N-1} m_N(\xi)| \leq C_N \frac{|\xi|^N}{1 + |\xi|},$$

resulting in m_N being bounded. Indeed, we can find bounds for (2.7.6) in terms of σ . Inequality (2.3.7) and the substitution $t = u\varphi_\theta(0)$ imply

$$\begin{aligned} \int_B |\langle x, \theta \rangle|^k \, dx &= \int_{\mathbb{R}} |u|^k \varphi_\theta(u) \, du \\ &\leq 2A \int_0^\infty u^k \varphi_\theta(0) e^{-a\varphi_\theta(0)u} \, du \\ &= 2A\varphi_\theta(0)^{-k} \int_0^\infty t^k e^{-at} \, dt \\ &\leq C_k \sigma(B)^k. \end{aligned}$$

2 Mathematical methods and results used in this thesis

For the other endpoint, $\operatorname{Re} z = 0$, we have $m_0(\xi) = m(\xi) + |\xi|m(\xi)$, and since we know that $\|m\|_{p \rightarrow p} = 1$, the crucial part lies in estimating the multiplier norm of $|\xi|m(\xi)$. We decompose

$$-2\pi|\xi|m(\xi) = \sum_{j=1}^n \left(-i \frac{\xi_j}{|\xi|} \right) (-2\pi i \xi_j m(\xi)),$$

which leads to

$$-2\pi T_{|\xi|m(\xi)} f = \sum_{j=1}^n R_j(f * \mu_j),$$

where R_j is the j -th Riesz transform and $\mu_j = \frac{\partial}{\partial x_j} \chi_B$ in the distributional sense. A result by Stein (see [29]) states that we have

$$\left\| \left(\sum_{j=1}^n |R_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p$$

for $1 < p < \infty$, with C_p independent of n . By a duality argument, we thus get

$$\|T_{|\xi|m(\xi)} f\|_{p'} \leq C_p \|g(f)\|_{p'}$$

for $1 < p' < \infty$, where

$$g(f) = \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2}.$$

To conclude the estimate, we can interpolate between $\|g(f)\|_2$ and $\|g(f)\|_\infty$. We have

$$\|g(f)\|_2 \leq C \|m_0\|_\infty \|f\|_2 \leq C(1 + L(B)^{-1}) \|f\|_2 = C(\sigma) \|f\|_2. \quad (2.7.7)$$

For $p' = \infty$, let $\mu_\xi = \langle \xi, \nabla \rangle \chi_B$ for each $\xi \in S^{n-1}$, where the derivatives are taken in the distributional sense. Then [21, Lemma 3] implies that

$$\|g(f)\|_\infty \leq \left\| \sup_{\xi \in S^{n-1}} |\mu_\xi * f| \right\|_\infty \leq 2 \sup_{\xi \in S^{n-1}} \operatorname{Vol}_{n-1}(\pi_\xi(B)) \|f\|_\infty = 2Q(B) \|f\|_\infty, \quad (2.7.8)$$

and interpolation leads to the bound (2.7.3). As we mentioned before, we need to modify these arguments in a similar way as in the proof of Proposition 2.6.1, but even then, the most crucial part of the proof is to control the L^p -multiplier norms of $|\xi|m(\xi)$.

2.8 Some properties of r -variations

Up to this point, we presented the most important tools that we need for our maximal estimates. In the remaining sections of this chapter, we provide the main tools that we need in addition to establish the variational estimates, starting with some more facts about r -variations, most of them being taken from [9]. Let $1 \leq r < \infty$, $Z \subseteq \mathbb{R}_{>0}$, and

$a: Z \rightarrow \mathbb{C}$, $t \mapsto a_t$, a complex-valued function. Similarly to r -variations, we can define r -variations of finite length. For $L \in \mathbb{N}_{>0}$, let

$$V_r^L(a_t : t \in Z) = \sup_{\substack{t_0 < \dots < t_L \\ t_j \in Z}} \left(\sum_{j=0}^{L-1} |a_{t_{j+1}} - a_{t_j}|^r \right)^{1/r}.$$

It follows immediately that we have

$$V_r(a_t : t \in Z) = \sup_{L \in \mathbb{N}} V_r^L(a_t : t \in Z),$$

and for $L_1 < L_2$, we have

$$V_r^{L_1}(a_t : t \in Z) \leq V_r^{L_2}(a_t : t \in Z).$$

Similarly to estimate (1.2.10), we have

$$\sup_{t \in Z} |a_t| = |a_{t_0}| + V_r^2(a_t : t \in Z)$$

for any $t_0 \in Z$. If a is continuous, we can replace Z by a dense subset, as stated in the following lemma.

Lemma 2.8.1. *Let $D \subset Z$ be dense in Z . Then for every continuous $a: Z \rightarrow \mathbb{C}$, $t \mapsto a_t$, and every $L \in \mathbb{N}_{>0}$, we have*

$$V_r^L(a_t : t \in Z) = V_r^L(a_t : t \in D).$$

Consequently, we have

$$V_r(a_t : t \in Z) = V_r(a_t : t \in D).$$

Proof. Since a is continuous, the map $F: Z^{L+1} \rightarrow \mathbb{R}$,

$$F(s) = \sum_{j=0}^{L-1} |a_{s_{j+1}} - a_{s_j}|^r,$$

where $s = (s_0, \dots, s_L)$, is continuous. Furthermore, D^{L+1} is dense in Z^{L+1} . Fix $\varepsilon > 0$ and take $s \in Z^{L+1}$ such that $s_0 < \dots < s_L$ and

$$F(s) > V_r^L(a_t : t \in Z)^r - \varepsilon. \quad (2.8.1)$$

By density, we can find $\tilde{s} = (\tilde{s}_0, \dots, \tilde{s}_L) \in D^{L+1}$ such that $\tilde{s}_0 < \dots < \tilde{s}_L$ and

$$|F(s) - F(\tilde{s})| < \varepsilon. \quad (2.8.2)$$

Combining (2.8.1) and (2.8.2), we get

$$V_r^L(a_t : t \in Z)^r < F(s) + \varepsilon \leq F(\tilde{s}) + |F(s) - F(\tilde{s})| + \varepsilon < F(\tilde{s}) + 2\varepsilon.$$

This concludes the lemma. **q.e.d.**

If $D = \{d_1, d_2, \dots\} \subset Z$ is countable and dense, (1.2.11) thus implies that

$$V_r(a_t : t \in Z) \leq 2 \left(\sum_{t \in D} |a_t|^r \right)^{1/r}.$$

Furthermore, if we let $D_k = \{d_1, \dots, d_k\}$, it follows that

$$V_r^L(a_t : t \in Z) = V_r^L(a_t : t \in D) = \sup_{k > 0} V_r^L(a_t : t \in D_k), \quad (2.8.3)$$

which also holds for $V_r(a_t : t \in Z)$.

Next, we will split the variation into two terms which will make it easier to establish any theorems on dimension-free variational bounds.

Lemma 2.8.2. *For every $r \in [1, \infty)$, there is a constant C_r such that for every (not necessarily continuous) function $a: \mathbb{R}_{>0} \rightarrow \mathbb{C}$, we have*

$$V_r(a_t : t > 0) \leq C_r V_r(a_{2^j} : j \in \mathbb{Z}) + C_r \left(\sum_{j \in \mathbb{Z}} V_r(a_t : t \in [2^j, 2^{j+1}))^r \right)^{1/r}. \quad (2.8.4)$$

Remark. *The term $V_r(a_{2^j} : j \in \mathbb{Z})$ is called long variation seminorm, and*

$$\left(\sum_{j \in \mathbb{Z}} V_r(a_t : t \in [2^j, 2^{j+1}))^r \right)^{1/r}$$

is called short variation seminorm.

Proof. To observe this, we take $t > s > 0$ and simply estimate each difference $|a_t - a_s|$ as follows.

- If $s = 2^j$, $t = 2^k$ for some $j, k \in \mathbb{Z}$, the difference will belong to the long variation on the right hand side of (2.8.4).
- If $2^j \leq s < t < 2^{j+1}$ for some j , the difference will belong to the short variation.
- If however $s \in [2^j, 2^{j+1})$ and $t \in [2^k, 2^{k+1})$ with $j < k$, we estimate

$$|a_t - a_s| \leq |a_t - a_{2^k}| + |a_{2^k} - a_{2^j}| + |a_s - a_{2^j}|,$$

so that by Hölder's inequality, it follows that

$$|a_t - a_s|^r \leq 3^{r-1} (|a_t - a_{2^k}|^r + |a_{2^k} - a_{2^j}|^r + |a_s - a_{2^j}|^r).$$

This allows us to easily conclude the lemma by applying these estimates to the differences in any finite sum in the supremum of the left hand side. **q.e.d.**

The inequality in Lemma 2.8.2 allows us to reduce the proof of Theorem 3 to the analysis of short variations, since the long variation estimate for M_t^B has already been established in [9]. To control the short variations with respect to M_t^B and M_t^S , we will present a simple geometric inequality from [9], which will allow us to bound each dyadic block by a suitable square function.

Lemma 2.8.3. *Let $1 \leq r < \infty$, $j \in \mathbb{Z}$, and $a : [2^j, 2^{j+1}) \rightarrow \mathbb{C}$ be continuous. Then we have*

$$\begin{aligned} V_r(a_t : t \in [2^j, 2^{j+1})) &\leq 2^{1-1/r} \sum_{m \geq -j} \left(\sum_{k=0}^{2^{m+j}-1} |a_{2^j+2^{-m}(k+1)} - a_{2^j+2^{-m}k}|^r \right)^{1/r} \\ &= 2^{1-1/r} \sum_{l \geq 0} \left(\sum_{k=0}^{2^l-1} |a_{2^j+2^{j-l}(k+1)} - a_{2^j+2^{j-l}k}|^r \right)^{1/r}. \end{aligned} \quad (2.8.5)$$

Proof. For $m \in \mathbb{Z}$, let

$$\mathcal{I}_m = \{ [2^j + 2^{-m}k, 2^j + 2^{-m}(k+1)) : 0 \leq k \leq 2^{m+j} - 1 \}$$

be the collection of dyadic intervals in $[2^j, 2^{j+1})$ of length 2^{-m} . Then for all dyadic numbers $s, t \in \{2^l k : k, l \in \mathbb{Z}\}$ with $2^j \leq s < t < 2^{j+1}$, we can write the interval $[s, t)$ as a disjoint union of dyadic intervals

$$[s, t) = \bigcup_{\ell \in \mathbb{Z}} [w_\ell, w_{\ell+1}),$$

such that each $[w_\ell, w_{\ell+1}) \in \mathcal{I}_m$ for some $m \geq -j$, at most two intervals in this union have the same length, and only finitely many intervals are nonempty. This construction has been described for general s, t in [9] (with a possibly infinite number of nonempty intervals), but it is achieved more easily for dyadic endpoints.

Now fix any dyadic numbers t_0, \dots, t_L such that $2^j \leq t_0 < \dots < t_L < 2^{j+1}$. For each $0 \leq l \leq L$, divide

$$[t_l, t_{l+1}) = \bigcup_{\ell \in \mathbb{Z}} [w_\ell^l, w_{\ell+1}^l)$$

as described above. We thus get

$$|a_{t_{l+1}} - a_{t_l}| \leq \sum_{\ell \in \mathbb{Z}} |a_{w_{\ell+1}^l} - a_{w_\ell^l}| = \sum_{m \geq -j} \sum_{\substack{\ell \in \mathbb{Z} \\ [w_\ell^l, w_{\ell+1}^l) \in \mathcal{I}_m}} |a_{w_{\ell+1}^l} - a_{w_\ell^l}|.$$

Note that $|\{\ell \in \mathbb{Z} : [w_\ell^l, w_{\ell+1}^l) \in \mathcal{I}_m\}| \leq 2$ for every $m \geq -j$. This gives us

$$\left(\sum_{l=0}^L |a_{t_{l+1}} - a_{t_l}|^r \right)^{1/r} \leq \left(\sum_{l=0}^L \left(\sum_{m \geq -j} \sum_{\substack{\ell \in \mathbb{Z} \\ [w_\ell^l, w_{\ell+1}^l) \in \mathcal{I}_m}} |a_{w_{\ell+1}^l} - a_{w_\ell^l}| \right)^r \right)^{1/r}$$

$$\begin{aligned} &\leq \sum_{m \geq -j} \left(\sum_{l=0}^L \left(\sum_{\substack{\ell \in \mathbb{Z} \\ [w_\ell^l, w_{\ell+1}^l] \in \mathcal{I}_m}} |a_{w_{\ell+1}^l} - a_{w_\ell^l}| \right)^r \right)^{1/r} \\ &\leq 2^{1-1/r} \sum_{m \geq -j} \left(\sum_{l=0}^L \sum_{\substack{\ell \in \mathbb{Z} \\ [w_\ell^l, w_{\ell+1}^l] \in \mathcal{I}_m}} |a_{w_{\ell+1}^l} - a_{w_\ell^l}|^r \right)^{1/r}, \end{aligned}$$

where the first estimate is an application of Minkowski's inequality, and the last estimate follows by applying Hölder's inequality to the innermost sum, which consists of at most two summands. One can see that the last sum is dominated by the right hand side of (2.8.5), and thus by density and continuity, the lemma holds. **q.e.d.**

The final result of this section presents a way to estimate r -variations against fractional derivatives, which is some sort of generalization of Propostion 2.5.2.

Proposition 2.8.4 (Proposition 2.1 (i) from [9]). *Let $F: \mathbb{R} \rightarrow \mathbb{C}$ have compact support, and assume that for some r, α with $1 < r < \infty$ and $\alpha > 1/r$, both F and $\mathcal{D}^\alpha F$ are in $L^r(\mathbb{R})$. Then there is a constant C_r such that*

$$V_r(F(t) : t \in \mathbb{R}) \leq C_r (\|F\|_{L^r(\mathbb{R})} + \|\mathcal{D}^\alpha F\|_{L^r(\mathbb{R})}). \quad (2.8.6)$$

Proof. We will make use of the Bessel potential G_α on \mathbb{R} , defined by

$$G_\alpha(u) = \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\pi u^2}{v}} e^{-\frac{v}{4\pi}} v^{\frac{\alpha-1}{2}} \frac{dv}{v}.$$

From [26, Chapter 5, Section 3.1], we have the following properties: $G_\alpha \in L^1$ and

$$\widehat{G_\alpha}(s) = (1 + (2\pi s)^2)^{-\alpha/2}, \quad s \in \mathbb{R}.$$

Furthermore, if $0 < \alpha < 1$, we have $G_\alpha(u) = \mathcal{O}(|u|^{-1+\alpha})$ for $u \rightarrow 0$, $G'_\alpha(u) = \mathcal{O}(|u|^{-2+\alpha})$ for $u \rightarrow 0$, and both G_α and G'_α are rapidly decreasing at ∞ . Since $\widehat{G_\alpha}$ has no zeros, we find a distribution u such that

$$F = G_\alpha * u. \quad (2.8.7)$$

We will show that there is $f \in L^r(\mathbb{R})$ such that $f = u$ in the distributional sense. By [26, Chapter 5, Section 3.2], there exist finite measures ν_α and λ_α on \mathbb{R} such that

$$(1 + (2\pi s)^2)^{\alpha/2} = \hat{\nu}_\alpha(s) + (2\pi|s|)^\alpha \hat{\lambda}_\alpha(s).$$

This implies that

$$\hat{u}(s) = \hat{\nu}_\alpha(s) \hat{F}(s) + (2\pi|s|)^\alpha \hat{\lambda}_\alpha(s) \hat{F}(s),$$

and thus

$$\check{u}(s) = \check{\nu}_\alpha(s) \check{F}(s) + (2\pi|s|)^\alpha \check{\lambda}_\alpha(s) \check{F}(s) = \check{\nu}_\alpha(s) \check{F}(s) + \check{\lambda}_\alpha(s) e^{-i\frac{\pi\alpha}{2} \operatorname{sgn}(s)} (2\pi i s)^\alpha \check{F}(s).$$

Since $\mathcal{D}^\alpha F \in L^r$ by our assumption, the Marcinkiewicz multiplier theorem in one dimension ensures that

$$\mathcal{F}_s(e^{-i\frac{\pi\alpha}{2}\operatorname{sgn}(s)}(2\pi is)^\alpha \check{F}(s))$$

also lies in L^r , and it follows immediately that we can identify $u = f$ for some $f \in L^r$. The identity (2.8.7) and the properties of G_α thus yield that F can be chosen to be continuous, which ensures that $V_r(F(t) : t \in \mathbb{R})$ is well-defined. Also, we have

$$\|f\|_r \leq C_r(\|F\|_r + \|\mathcal{D}^\alpha F\|_r). \quad (2.8.8)$$

To estimate $\|f\|_r$, we define a family of operators $(A_z)_{z \in \mathbb{S}}$ on $\mathcal{S}(\mathbb{R})$ by setting

$$A_z(f)(t) := e^{(z-1+1/r)^2} \int_{\mathbb{R}} G_\alpha(u) |u|^{1-z-1/r} f(t-u) du.$$

Our goal is to find a bound

$$V_r(A_z(f)(t) : t \in \mathbb{R}) \leq C_r \|f\|_r \quad (2.8.9)$$

in the case $z = 1 - 1/r$. If $f \in L^r$ with $F = G_\alpha * f$, we have $A_z(f) = F$ for $z = 1 - 1/r$, and by (2.8.8), it suffices to show (2.8.9) to conclude the proposition.

Since $\alpha > 1/r$, the bounds on G_α mentioned above imply that for each $z \in \mathbb{S}$, A_z maps \mathcal{S} continuously into L^1 and into L^∞ , and by interpolation, we can extend A_z to each $L^p(\mathbb{R})$.

In order to prove (2.8.9), we show the following two estimates.

$$V_1(A_z(f)(t) : t \in \mathbb{R}) \leq C_0 \|f\|_{L^1}, \quad \operatorname{Re} z = 0, \quad (2.8.10)$$

$$V_\infty(A_z(f)(t) : t \in \mathbb{R}) \leq C_1 \|f\|_{L^\infty}, \quad \operatorname{Re} z = 1 \quad (2.8.11)$$

for $f \in \mathcal{S}(\mathbb{R})$. The second bound follows directly from the fact that A_z maps $\mathcal{S}(\mathbb{R})$ continuously into L^∞ . For the first bound, we show that for $\operatorname{Re} z = 0$, $\frac{d}{dt} A_z(f)$ is also in L^1 and use that for any differentiable $a : \mathbb{R} \rightarrow \mathbb{C}$, we have

$$V_1(a_t : t \in \mathbb{R}) \leq \int_{\mathbb{R}} \left| \frac{d}{dt} a_t \right| dt.$$

Indeed, if $z = i\tau$, $\tau \in \mathbb{R}$, we have

$$\begin{aligned} V_1(A_z(f)(t) : t \in \mathbb{R}) &\leq \int_{\mathbb{R}} \left| \frac{d}{dt} A_z(f)(t) \right| dt \\ &\leq e^{(-1+1/r)^2} e^{-\tau^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{d}{du} (G_\alpha(u) |u|^{1-i\tau-1/r}) \right| \cdot |f(t-u)| du dt \\ &\leq c \|f\|_{L^1} e^{-\tau^2} \int_{\mathbb{R}} \left| \frac{d}{du} (G_\alpha(u) |u|^{1-i\tau-1/r}) \right| du \end{aligned}$$

$$\begin{aligned} &\leq c \|f\|_{L^1} e^{-\tau^2} \int_{\mathbb{R}} |(1 - i\tau - 1/r)G_\alpha(u)| |u|^{-1/r} + |G'_\alpha(u)| |u|^{1-1/r} \, du \\ &\leq c \|f\|_{L^1} e^{-\tau^2} (1 + |\tau|) \left(\int_{|u| \leq 1} |u|^{-1+\alpha-1/r} \, du + \int_{|u| > 1} |u|^{-2} \, du \right). \end{aligned}$$

This implies (2.8.10). To show the estimate (2.8.9), we fix a sequence $t_0 < t_1 < \dots < t_L$ of real numbers, and for $f \in \mathcal{S}$, $z \in \mathbb{S}$, we set

$$T_z(f) = \left(A_z(f)(t_k) - A_z(f)(t_{k-1}) \right)_{k=1}^L.$$

Each T_z maps into \mathbb{R}^L , the family $(T_z)_{z \in \mathbb{S}}$ is analytic, and the factor $e^{(z-1+1/r)^2}$ ensures that the growth conditions for Stein's interpolation theorem are satisfied. Additionally, the bounds (2.8.10) and (2.8.11) imply that

$$\begin{aligned} \|T_z(f)\|_{\ell^1} &\leq C_0 \|f\|_{L^1}, & \operatorname{Re} z = 0, \\ \|T_z(f)\|_{\ell^\infty} &\leq C_1 \|f\|_{L^\infty}, & \operatorname{Re} z = 1, \end{aligned}$$

with C_0 and C_1 being independent of the choice of t_0, \dots, t_L . For $\theta = 1 - 1/r$, we then have $1/r = 1 - \theta$, and by interpolation, we get

$$\|T_\theta(f)\|_{\ell^r} \leq C_r \|f\|_{L^r}.$$

In other words, if $F = G_\alpha * f$, we have

$$\left(\sum_{k=1}^L |F(t_k) - F(t_{k-1})|^r \right)^{1/r} \leq C_r \|f\|_{L^r},$$

where C_r is also independent of t_0, \dots, t_L . This implies (2.8.9) and concludes the proposition. **q.e.d.**

2.9 Maximal and variational bounds for semigroups

In this section, we show how the dimension-free maximal and variational estimates for symmetric diffusion semigroups are obtained from Rota's theorem and the corresponding martingale estimates. The maximal result is from [25], while the variational result is from [9]. Let (X, \mathcal{B}, μ) be a σ -finite measure space and $(T_t)_{t>0}$ be a family of linear operators such that each T_t maps μ -measurable functions to μ -measurable functions (on X), we have the semigroup property $T_s T_t = T_{s+t}$ for $s, t > 0$, whenever $f \in L^2(X)$, we have $T_t f \in L^2(X)$, and in $L^2(X)$, we have $\lim_{t>0} T_t f = f$. Furthermore, $(T_t)_{t>0}$ shall satisfy the following axioms.

- (i) For each $t > 0$ and $p \in [1, \infty]$, $\|T_t f\|_{L^p} \leq \|f\|_{L^p}$,
- (ii) Each T_t is self-adjoint on $L^2(X)$,

- (iii) Each T_t is a positive operator, i.e. $T_t f \geq 0$ if $f \geq 0$,
- (iv) For each $t > 0$, $T_t 1 = 1$.

Such a semigroup is called a symmetric diffusion semigroup, and the Poisson semigroup satisfies these axioms. Under these conditions, the following estimates hold. If $1 < p \leq \infty$, we have

$$\left\| \sup_{t>0} |T_t f| \right\|_{L^p(X)} \leq C_p \|f\|_{L^p(X)} \quad (2.9.1)$$

for each $f \in L^p(X)$, where C_p only depends on p . Furthermore, if $1 < p < \infty$ and $2 < r < \infty$, we have

$$\left\| V_r(T_t f : t > 0) \right\|_{L^p(X)} \leq C_{p,r} \|f\|_{L^p(X)}, \quad (2.9.2)$$

for every $f \in L^p(X)$, with $C_{p,r}$ only depending on p and r . We can apply Rota's theorem to connect the semigroup $(T_t)_{t>0}$ to martingales, and apply some martingale theoretic results like Doob's maximal inequality to achieve (2.9.1). A similar variational martingale estimate by Lépingle [18] leads to the variational semigroup result (2.9.2). We shall mention that the maximal theorem for semigroups still holds if the conditions (iii) and (iv) are dropped, which Stein also proved in [25].

Theorem 2.9.1 (Rota's theorem). *Let (X, \mathcal{B}, μ) be a σ -finite measure space and Q be a linear operator defined on $\bigcup_{1 \leq p \leq \infty} L^p(X, \mathcal{B}, \mu)$, that satisfies the following conditions.*

- (a) $\|Qf\|_{L^p(X)} \leq \|f\|_{L^p(X)}$ for $p \in [1, \infty]$,
- (b) Q is self-adjoint on $L^2(X)$,
- (c) Q is a positive operator,
- (d) $Q1 = 1$.

Then there exist a large measure space $(\Omega, \mathcal{F}, \nu)$, a collection of σ -algebras $\dots \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \dots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 \subseteq \mathcal{F}$, and another σ -algebra $\hat{\mathcal{F}} \subseteq \mathcal{F}$ with the following properties.

- (1) *There is a measure space isomorphism between $(\Omega, \hat{\mathcal{F}}, \nu)$ and (X, \mathcal{B}, μ) , given through a natural mapping $\iota: \Omega \rightarrow X$, which induces an isomorphism between $L^p(\Omega, \hat{\mathcal{F}}, \nu)$ and $L^p(X, \mathcal{B}, \mu)$, also denoted by ι and given through the formula $\iota(f)(x) = f(\iota^{-1}(x))$.*
- (2) *For $f \in L^p(\Omega, \hat{\mathcal{F}}, \nu)$, denote by \hat{E} and E_n the conditional expectation operators*

$$\hat{E}f = E(f|\hat{\mathcal{F}}) \quad \text{and} \quad E_n f = E(f|\mathcal{F}_n).$$

Then for every $F \in L^p(X, \mathcal{B}, \mu)$, we have

$$Q^{2n}(\iota f) = \hat{E}(E_n f). \quad (2.9.3)$$

Thus the operator Q is associated with a reverse martingale $(E_n f)_{n \in \mathbb{N}}$.

For a proof, see [25, pp. 107-110]. We will directly show the variational result, since by (1.2.10), one can obtain (2.9.1) from (2.9.2). Let $1 < p < \infty$, fix $k \in \mathbb{N}$, and set $Q = T_{1/2^{k+1}}$. Then Q satisfies the conditions of Rota's theorem. With the notation from Theorem 2.9.1, let $F \in L^p(\mathbb{R}^n)$ such that $F = \iota(f)$. Then for $r > 2$, we apply (2.9.3) to get

$$\begin{aligned} \|V_r(T_{n/2^k}F : n \in \mathbb{N})\|_{L^p(\mathbb{R}^n)} &= \|V_r(Q^{2n}F : n \in \mathbb{N})\|_{L^p(\mathbb{R}^n)} \\ &= \|V_r(\hat{E}(E_n f) : n \in \mathbb{N})\|_{L^p(\Omega, \hat{\mathcal{F}}, \nu)} \\ &\leq \|\hat{E}(V_r(E_n f : n \in \mathbb{N}))\|_{L^p(\Omega, \hat{\mathcal{F}}, \nu)} \\ &\leq \|V_r(E_n f : n \in \mathbb{N})\|_{L^p(\Omega, \hat{\mathcal{F}}, \nu)} \\ &\leq C_{p,r} \|f\|_{L^p(\Omega, \hat{\mathcal{F}}, \nu)} = C_{p,r} \|F\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where the last inequality follows from Lépingle's inequality for martingales, see [18], and $C_{p,r}$ is independent of k . From [25, III, §1, Section 2], we know that for $f \in L^p(X)$, $t \mapsto T_t f$ is continuous (indeed, it even extends to some analytic function on some complex sector containing $\mathbb{R}_{>0}$). By letting $k \rightarrow \infty$, the monotone convergence theorem and Lemma 2.8.1 imply (2.9.2). Lépingle's inequality can fail for $r = 2$, as has been shown in [17], resulting in the restriction $r > 2$ for the dimension-free variational theorems. Instead of concluding (2.9.1) from (2.9.2), we could also show (2.9.1) directly by arguing as above and using Doob's martingale inequality, resulting in the proof from [25, pp. 106-107].

2.10 A variational version of Carbery's almost orthogonality principle

For our variational results, we have to adapt the almost orthogonality result from Section 2.4 to the variational setting. Essentially, we just add a slight modification, and the idea of the proof is basically the same as in [12], but since it is slightly more complicated, we will present this tool and its proof from [9] here.

Proposition 2.10.1 (Proposition 2.2 from [9]). *Let $Z \subseteq \mathbb{R}_{>0}$, and $(T_t)_{t \in Z}$ be a family of positive linear operators defined on $\mathcal{S}(\mathbb{R}^n)$. Write $Z_j = Z \cap [2^j, 2^{j+1})$. Assume that there is $p_0 \in [1, 2)$ such that the following conditions hold true:*

- *There is a family of linear operators $(S_j)_{j \in \mathbb{Z}}$ on $\bigcup_{1 \leq p \leq \infty} L^p(\mathbb{R}^n)$ such that for every $f \in L^2(\mathbb{R}^n)$, we have $\sum_{j \in \mathbb{Z}} S_j f = f$ on $L^2(\mathbb{R}^n)$, and for every $p \in (1, 2]$, there is a constant $C_{1,p}$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C_{1,p} \|f\|_{L^p}. \quad (2.10.1)$$

- *For each $p \in (p_0, 2]$, there is a constant $C_{2,p}$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\left\| \sup_{t \in Z} |T_t f| \right\|_{L^p} \leq C_{2,p} \|f\|_{L^p}. \quad (2.10.2)$$

2.10 A variational version of Carbery's almost orthogonality principle

- For each $p \in (p_0, 2]$, there is a constant $C_{3,p}$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\sup_{j \in \mathbb{Z}} \|V_p(T_t f : t \in Z_j)\|_{L^p} \leq C_{3,p} \|f\|_{L^p}. \quad (2.10.3)$$

- There is a sequence $(a_k)_{k \in \mathbb{Z}}$, $a_k > 0$, such that $\sum_{k \in \mathbb{Z}} a_k^\rho = A_\rho < \infty$ for every $\rho > 0$, and there is a constant C_4 so that for every $k \in \mathbb{Z}$ and every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j \in \mathbb{Z}} V_2(T_t S_{j+k} f : t \in Z_j)^2 \right)^{1/2} \right\|_{L^2} \leq C_4 a_k \|f\|_{L^2}. \quad (2.10.4)$$

Then for every $p \in (p_0, 2]$, there is a constant C_p such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j \in \mathbb{Z}} V_2(T_t f : t \in Z_j)^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}. \quad (2.10.5)$$

Proof. Let $p_0 < p \leq 2$. To achieve (2.10.5), it suffices to show that for every $L, N \in \mathbb{N}_{>0}$, we have

$$\left\| \left(\sum_{|j| \leq N} V_2^L(T_t f : t \in Z_j)^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}, \quad (2.10.6)$$

with C_p being independent of L and N . To establish (2.10.6), we will apply a linearization and establish a slightly stronger estimate. For every $L \in \mathbb{N}_{>0}$ and $j \in \mathbb{Z}$, let

$$\mathfrak{T}_L^j = \{t_L^j = (t_l^j)_{l=0}^L : t_l^j : \mathbb{R}^n \rightarrow Z_j \text{ is measurable and } \forall x \in \mathbb{R}^n : t_0^j(x) < \dots < t_L^j(x)\}.$$

We will show that there is a constant C_p such that for every $L, N \in \mathbb{N}_{>0}$, we have

$$\sup_{t_L^N \in \mathfrak{T}_L^N} \dots \sup_{t_L^N \in \mathfrak{T}_L^N} \left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{l+1}^j} - T_{t_l^j})f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}. \quad (2.10.7)$$

To see how (2.10.7) implies (2.10.6), we will choose a $t_{L,f}^j \in \mathfrak{T}_L^j$ for each $j \in \mathbb{Z}$, such that for every $x \in \mathbb{R}^n$, we have

$$V_2^L(T_t f(x) : t \in Z_j)^2 \leq 4 \sum_{l=0}^{L-1} |(T_{t_{l+1}^j, f(x)} - T_{t_l^j, f(x)})f(x)|^2. \quad (2.10.8)$$

For this, we fix a countable dense subset $D_j \subset Z_j$, and let

$$D_{j,L} = \{(t_0, \dots, t_L) \in D_j^{L+1} : t_0 < \dots < t_L\}.$$

Then $D_{j,L}$ is countable as well. Write $D_{j,L} = \{d^1, d^2, \dots\}$, and for each $x \in \mathbb{R}^n$, choose $t_{L,f}^j(x)$ to be the first d^k for which (2.10.8) holds. The countability of $D_{j,L}$ ensures the

measurability of $\mathfrak{t}_{L,f}^j$. Now if we assume that (2.10.7) holds, we have

$$\begin{aligned} & \left\| \left(\sum_{|j| \leq N} V_2^L(T_t f : t \in Z_j)^2 \right)^{1/2} \right\|_{L^p} \\ & \leq 2 \left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{i+1,f}^j} - T_{t_{i,f}^j})f|^2 \right)^{1/2} \right\|_{L^p} \\ & \leq 2 \sup_{\mathfrak{t}_L^{-N} \in \mathfrak{X}_L^{-N}} \cdots \sup_{\mathfrak{t}_L^N \in \mathfrak{X}_L^N} \left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{i+1}^j} - T_{t_i^j})f|^2 \right)^{1/2} \right\|_{L^p} \\ & \leq C_p \|f\|_{L^p}. \end{aligned}$$

Since for each $j \in \mathbb{Z}$ and $\mathfrak{t}_L^j \in \mathfrak{X}_L^j$, we have

$$\left(\sum_{l=0}^{L-1} |(T_{t_{i+1}^j(x)} - T_{t_i^j(x)})f(x)|^2 \right)^{1/2} \leq V_2^L(T_t f(x) : t \in Z_j),$$

(2.10.3) implies that (2.10.7) holds with the constant $(2N+1)C_{3,p}$, but we still need to find a constant independent of L and N . To do that, we fix $\mathfrak{t}_L^{-N} \in \mathfrak{X}_L^{-N}, \dots, \mathfrak{t}_L^N \in \mathfrak{X}_L^N$, and for $p \in (p_0, 2)$, $r \in [p, \infty]$, and $s \in [1, \infty]$, denote by $A_N(p, r, s)$ the best possible constant such that we have

$$\left\| \left(\sum_{|j| \leq N} \left(\sum_{l=0}^{L-1} |(T_{t_{i+1}^j} - T_{t_i^j})g_j|^r \right)^{s/r} \right)^{1/s} \right\|_{L^p} \leq A_N(p, r, s) \left\| \left(\sum_{|j| \leq N} |g_n|^s \right)^{1/s} \right\|_{L^p}. \quad (2.10.9)$$

Our first purpose is to bound $A_N(u, 2, 2)$ for some $u \in (p_0, p)$. For this, we will use interpolation between $L^p(\ell^s(\ell^r))$ spaces. Let $p_0 < q < p$ and let $0 < \theta < 1$ be so that

$$\frac{1}{2} = \frac{1-\theta}{q} + \frac{\theta}{\infty},$$

i.e. $\theta = 1 - q/2$. Then we find u with $q < u < p$ such that

$$\frac{1}{u} = \frac{1-\theta}{q} + \frac{\theta}{p}.$$

Thus if we can bound $A_N(q, q, q)$ and $A_N(p, \infty, \infty)$, we can interpolate to bound $A_N(u, 2, 2)$. We claim that

$$A_N(q, q, q) \leq C_{3,q}, \quad \text{and} \quad A_N(p, \infty, \infty) \leq 2C_{2,p}.$$

For the first bound, we use (2.10.3) and get

$$\left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{i+1}^j} - T_{t_i^j})g_j|^q \right)^{1/q} \right\|_{L^q} \leq \left\| \left(\sum_{|j| \leq N} V_q(T_t g_j : t \in Z_j)^q \right)^{1/q} \right\|_{L^q}$$

$$\begin{aligned}
 &= \left(\int_{\mathbb{R}^n} \sum_{|j| \leq N} V_q(T_t g_j(x) : t \in Z_j)^q dx \right)^{1/q} \\
 &\leq C_{3,q} \left(\int_{\mathbb{R}^n} \sum_{|j| \leq N} |g_j(x)|^q dx \right)^{1/q} \\
 &= C_{3,q} \left\| \left(\sum_{|j| \leq N} |g_j|^q \right)^{1/q} \right\|_{L^q}.
 \end{aligned}$$

The bound on $A_N(p, \infty, \infty)$ is achieved more easily. Write $g = \sup_{|j| \leq N} |g_j|$. Then by (2.10.2), we get

$$\left\| \sup_{|j| \leq N} \sup_{0 \leq l \leq L-1} |(T_{t_{l+1}^j} - T_{t_l^j})g_j| \right\|_{L^p} \leq 2 \left\| \sup_{t \in Z} T_t g \right\|_{L^p} \leq 2C_{2,p} \|g\|_{L^p}.$$

Now by interpolation, we get

$$A_N(u, 2, 2) \leq A_N(q, q, q)^{1-\theta} A_N(p, \infty, \infty)^\theta \leq C_{3,q}^{1-\theta} 2^\theta C_{2,p}^\theta. \quad (2.10.10)$$

For every $k \in \mathbb{Z}$, we apply (2.10.1), with $g_j = S_{k+j}f$, and thus get

$$\begin{aligned}
 \left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{l+1}^j} - T_{t_l^j})S_{k+j}f|^2 \right)^{1/2} \right\|_{L^u} &\leq A_N(u, 2, 2) \left\| \left(\sum_{|j| \leq N} |S_{k+j}f|^2 \right)^{1/2} \right\|_{L^u} \\
 &\leq C_{1,u} C_{3,q}^{1-\theta} 2^\theta C_{2,p}^\theta \|f\|_{L^u}. \quad (2.10.11)
 \end{aligned}$$

Also, (2.10.4) implies that

$$\begin{aligned}
 \left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{l+1}^j} - T_{t_l^j})S_{k+j}f|^2 \right)^{1/2} \right\|_{L^2} &\leq \left\| \left(\sum_{|j| \leq N} V_2(T_t S_{k+j}f : t \in Z_j)^2 \right)^{1/2} \right\|_{L^2} \\
 &\leq C_4 a_k \|f\|_{L^2}. \quad (2.10.12)
 \end{aligned}$$

We will interpolate again, between (2.10.11) and (2.10.12). Let $0 < \rho < 1$ be so that

$$\frac{1}{p} = \frac{1-\rho}{u} + \frac{\rho}{2}.$$

Then we get

$$\left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{l+1}^j} - T_{t_l^j})S_{k+j}f|^2 \right)^{1/2} \right\|_{L^p} \leq (C_{1,u} C_{3,q}^{1-\theta} (2C_{2,p}^\theta))^{1-\rho} C_4^\rho a_k^\rho \|f\|_{L^p},$$

and summing over $k \in \mathbb{Z}$ gives us

$$\left\| \left(\sum_{|j| \leq N} \sum_{l=0}^{L-1} |(T_{t_{l+1}^j} - T_{t_l^j})f|^2 \right)^{1/2} \right\|_{L^p} \leq (C_{1,u} C_{3,q}^{1-\theta} (2C_{2,p}^\theta))^{1-\rho} C_4^\rho A_\rho \|f\|_{L^p},$$

which gives us a constant for (2.10.7) which is independent of L and N .

q.e.d.

Compared to Theorem 2.4.1, the main differences are that r variations are taken instead of suprema, the addition of the estimate (2.10.3), and that the supremum in (2.10.2) ranges over the set Z , rather than over each Z_j . This means that whenever we want to use Proposition 2.10.1 to establish a variational estimate, it is necessary that we have already shown the corresponding maximal estimate. Also, Proposition 2.10.1 is trivial in the lacunary case $Z = \{2^j : j \in \mathbb{Z}\}$. However, using the maximal lacunary estimate, it is possible to show the bound (1.2.15) directly.

Similarly as in Section 2.4, we have to establish the slightly stronger L^2 -estimate (2.10.4), but then we can reduce the problem to bounding

$$\|V_p(T_t f : 1 \leq t < 2)\|_{L^p} \leq C_p \|f\|_{L^p} \quad (2.10.13)$$

for $p < 2$. Again, this follows from the dilation principle (2.4.6): If we assume that (2.10.13) holds, then for every $j \in \mathbb{Z}$, we have

$$\begin{aligned} \|V_p(M_t^B f : 2^j \leq t < 2^{j+1})\|_{L^p} &= \|V_p(2^{-jn} M_t^B f_{2^{-j}}(2^{-j}x) : 1 \leq t < 2)\|_{L^p} \\ &\leq C_p 2^{-jn} \|f_{2^{-j}}(2^{-j}x)\|_{L^p} \\ &= C_p \|f\|_{L^p}. \end{aligned}$$

3 Maximal bounds for products of Euclidean balls and spheres

As mentioned in Section 1.5, we will show Theorems 1 and 2 in this Chapter.

3.1 Independence of the number of factors

The generalization of the arguments in [7] is most important for the proof of Theorem 1, as we will obtain a constant that will not depend on the number of factors ℓ . We will omit any proof from [7] that does not need any further modification. A very detailed description of Bourgain's proof for the cube is given in [23]. For more details that may miss here, we also refer to [13, Chapter 8].

Throughout this section, put $r_N > 0$ for every $N \in \mathbb{N}_{>0}$ so that $|B_{r_N}^N| = 1$, and hence $r_N^N = \Gamma(N/2 + 1)/\pi^{N/2}$.

Let B , ℓ , and n be as in Theorem 1, i.e. $B_1 \times \dots \times B_\ell$, where for each $k \in \{1, \dots, \ell\}$, $B_k \subset \mathbb{R}^{n_k}$ is a Euclidean ball, and let $n = n_1 + \dots + n_\ell$. Furthermore, let $m(\xi) = \widehat{\chi_B}(\xi)$. We group the variables by setting

$$V_k := \left\{ \sum_{j=1}^{k-1} n_j + 1, \dots, \sum_{j=1}^k n_j \right\}, \quad k \in \{1, \dots, \ell\}. \quad (3.1.1)$$

The goal of this section is to show the following weaker result.

Proposition 3.1.1. *Let $N := \max_{1 \leq k \leq \ell} n_k$. Then*

$$\|M_B f\|_p \leq C_{p,N} \|f\|_p \quad (3.1.2)$$

for every $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$.

It is enough to consider the case $B = (B_{r_N}^N)^\ell$: Since $\|M_B\|_{p \rightarrow p}$ is invariant under linear transformations of B (as already mentioned in Section 2.3), we can assume that $|B_k| = 1$ for every $k \in \{1, \dots, \ell\}$ (hence $B_k = B_{r_{n_k}}^{n_k}$). By change of coordinates, we can assume that $n_1 \leq \dots \leq n_\ell$, so that $B = \prod_{j=1}^N (B_{r_j}^j)^{\ell_j}$ for certain $\ell_j \in \mathbb{N}$, where, without loss of generality, we allow that $\ell_j = 0$. Suppose that we already found constants $C_{p,j}$ independent of ℓ_j such that

$$\|M_{(B_{r_j}^j)^{\ell_j}} f\|_p \leq C_{p,j} \|f\|_p.$$

3 Maximal bounds for products of Euclidean balls and spheres

Then we can iterate similarly as in (1.5.1) to get

$$\|M_B f\|_p \leq \prod_{j=1}^N C_{p,j} \|f\|_p.$$

Let $B = (B_{r_N}^N)^\ell$. Then B is in isotropic position, since for each $\theta \in S^{n-1}$,

$$\begin{aligned} L(B)^2 &= \int_B |\langle x, \theta \rangle|^2 dx = \sum_{k=1}^{\ell} \int_B |\langle x^{(k)}, \theta^{(k)} \rangle|^2 dx \\ &= \sum_{k=1}^{\ell} \int_{B_{r_N}^N} |\langle y, \theta^{(k)} \rangle|^2 dy \\ &= \sum_{k=1}^{\ell} |\theta^{(k)}|^2 L(B_{r_N}^N)^2 \\ &= L(B_{r_N}^N)^2, \end{aligned}$$

where we used that $|B_{r_N}^N| = 1$ and that each Euclidean ball is rotation invariant. By [21], it thus follows that there is a constant $a > 1$, which is independent of N and ℓ , such that

$$a^{-1} \leq L(B) \leq a. \quad (3.1.3)$$

We will first show that Müller's bounds do not yield any estimates independent of ℓ .

Lemma 3.1.2. *We have*

$$Q(B) = \sqrt{\ell} \cdot Q(B_{r_N}^N) \geq C\sqrt{\ell}.$$

Proof. Let $\theta \in S^{n-1}$. We will estimate $\text{Vol}_{n-1}(\pi_\theta(B))$. The geometric arguments from [2] and a limiting argument show that

$$\text{Vol}_{n-1}(\pi_\theta(B)) = \int_{\partial B} (\langle n(x), \theta \rangle)_+ d\sigma(x),$$

where σ is the Lebesgue surface measure of ∂B and $n(x)$ is the corresponding normal vector (well-defined σ -almost everywhere on ∂B). We decompose ∂B into σ -almost disjoint components by writing

$$\partial B = \bigcup_{k=1}^{\ell} H_k, \quad H_k := (B_{r_N}^N)^{k-1} \times S_{r_N}^{N-1} \times (B_{r_N}^N)^{\ell-k-1}.$$

Then the H_k are the counterparts to the faces of a cube. If $x \in H_k$ and $x \notin H_j$ for any $j \neq k$, we have $n(x) = x^{(k)}/|x^{(k)}|$, and by the arguments from [2, pp. 41-45] and Fubini's

theorem, we get

$$\begin{aligned}
 \text{Vol}_{n-1}(\pi_\theta(B)) &= \sum_{k=1}^{\ell} \int_{H_k} (\langle n(x), \theta \rangle)_+ d\sigma(x) \\
 &= \sum_{k=1}^{\ell} \int_{H_k} (\langle x^{(k)} / |x^{(k)}|, \theta \rangle)_+ d\sigma(x) \\
 &= \sum_{k=1}^{\ell} \int_{S_{r_N}^{N-1}} r_N^{-1} (\langle \omega, \theta^{(k)} \rangle)_+ d\sigma_{S_{r_N}^{N-1}}(\omega) \\
 &= \sum_{k=1}^{\ell} |\theta^{(k)}| Q(B_{r_N}^N).
 \end{aligned}$$

It follows that we can maximize $\text{Vol}_{n-1}(\pi_\theta(B))$ by choosing $\theta^{(k)} = \ell^{-1/2} e_1 \in \mathbb{R}^N$. By the arguments from [21], we thus get

$$Q(B) = \ell \cdot \ell^{-1/2} Q(B_{r_N}^N) \geq C\sqrt{\ell},$$

concluding the lemma. **q.e.d.**

Since the constant $C(p, \sigma(B), Q(B))$ from [21] grows with $Q(B)$, we shall therefore rather follow Bourgain's ideas from [7].

Consider the same decomposition as in [7]: Let $H: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\hat{H}(\xi) := e^{-|\xi|^2}$, and put $\Omega^{(s)} = \chi_B * (H_{2^s} - H_{2^{s-1}})$ for $s \geq 1$, where again

$$H_R(x) = \frac{1}{R^n} H(x/R), \quad R > 0.$$

Then

$$\chi_B = (\chi_B * H) + \sum_{s=1}^{\infty} \Omega^{(s)}.$$

Furthermore, let

$$m^{(s)}(\xi) := e^{-4^{-s}|\xi|^2} m(\xi), \quad s \in \mathbb{N}.$$

Then

$$\widehat{\Omega^{(s)}} = m^{(s)} - m^{(s-1)}$$

for each $s \in \mathbb{N}_{>0}$. We will slightly modify Bourgain's approach: For $1 < p < \infty$ and $1/2 < \alpha < 1$, we will establish estimates of the form

$$\|T_{\langle \xi, \nabla \rangle^\alpha m^{(s)}}\|_{p \rightarrow p} \leq C_{p, \alpha, N, s}, \quad (3.1.4)$$

and show how these imply a bound

$$\|T_{\langle \xi, \nabla \rangle^\alpha m}\|_{p \rightarrow p} \leq C_{p, \alpha, N}. \quad (3.1.5)$$

The reason for this is that later on in Chapter 4, we need to use the estimate (3.1.5) directly, but while it has been mentioned in [9] that this inequality is justified in [7], it has not been shown explicitly in any of the works [7, 9, 13].

By the arguments from [12], the estimate (3.1.5) directly implies Proposition 3.1.1. For the cube, bounds of the form (3.1.4) are implicitly given by [7, Lemma 3]. For this, Bourgain takes Müller's ideas from Section 2.7 to conclude that it suffices to find an L^p -bound for the operator T , defined by

$$\widehat{Tf}(\xi) = |\xi| m(\xi) e^{-4^{-s}|\xi|^2} \widehat{f}(\xi), \quad (3.1.6)$$

and to estimate

$$\sup_{|\xi|=1} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^k (\chi_B * H_{2^{-s}})(x) dx < C_k, \quad k \geq 1. \quad (3.1.7)$$

For (3.1.7), we can make use of the fact that B is symmetric in each coordinate and of the convolution factor $H_{2^{-s}}$, applying Khinchin's inequality as in [7, p. 279].

To estimate the operator T in (3.1.6), Bourgain uses a duality argument and Stein's dimension-free bound on the Riesz transforms as in [21] (see also Section 2.7), which leaves him with proving [7, Lemma 3]. In our situation, the estimates (3.1.4) follow from a similar Lemma.

Lemma 3.1.3. *Fix $N \geq 1$ and let $B = (B_{r_N}^N)^\ell$, so that $n = N\ell$. For $R \geq 2$ and $j \in \{1, \dots, n\}$, let $\mu_j = \partial_j(\chi_B * H_{1/R})$. Then for every $2 \leq p < \infty$, $0 < \varepsilon < 1$, and $f \in L^p$ we have*

$$\left\| \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon,N} R^{24N \cdot \varepsilon} \|f\|_p, \quad (3.1.8)$$

with $C_{p,\varepsilon,N}$ independent of R and ℓ .

In particular, if $1 < p < \infty$, $1/2 < \alpha < 1$, $s \in \mathbb{N}_{>0}$, and $0 < \varepsilon < 1$, then we have the estimate

$$\|T_{\langle \xi, \nabla \rangle^{\alpha m(s)}}\|_{p \rightarrow p} \leq C_{p,\varepsilon,\alpha,N} 2^{24N \cdot s \cdot \varepsilon}, \quad (3.1.9)$$

where $C_{p,\varepsilon,\alpha,N}$ is independent of s and ℓ .

Before we get to the proof of Lemma 3.1.3, we will show how this implies (3.1.5).

Lemma 3.1.4. *Let $N \geq 1$, $B = (B_{r_N}^N)^\ell$, and $m = \widehat{\chi_B}$. Then for $1 < p < \infty$ and $1/2 < \alpha < 1$, we have*

$$\|T_{\langle \xi, \nabla \rangle^{\alpha m}}\|_{p \rightarrow p} \leq C_{p,\alpha,N}, \quad (3.1.10)$$

where $C_{p,\alpha,N}$ is independent of ℓ .

Proof. Assume that the estimate (3.1.9) is already proven. For $s \in \mathbb{N}_{>0}$, our aim is to establish a suitable L^2 -bound for the multiplier operator associated to $\langle \xi, \nabla \rangle^{\alpha \widehat{\Omega}^{(s)}}$. More precisely, we will show that for every $\varepsilon \in (0, 1)$, we have

$$\|\langle \xi, \nabla \rangle^{\alpha \widehat{\Omega}^{(s)}}\|_\infty \leq C_{\varepsilon,\alpha} 2^{-(1-\alpha)s/2} \cdot 2^{\varepsilon s/2}, \quad (3.1.11)$$

where $C_{\varepsilon, \alpha}$ does not depend on s or ℓ . Then by interpolation with (3.1.9), one can see that for every $p \in (1, 2)$, there is some $\theta = \theta_p \in (0, 1)$ such that for every $\varepsilon \in (0, 1)$, we have

$$\|T_{\langle \xi, \nabla \rangle^\alpha \widehat{\Omega}^{(s)}}\|_{p \rightarrow p} \leq C_{p, \varepsilon, \alpha, N} 2^{24N s \varepsilon (1-\theta)} \cdot 2^{-(1-\alpha)s\theta/2} \cdot 2^{\varepsilon s \theta/2}. \quad (3.1.12)$$

From [7, (1.28)] and the arguments from [21], we also know that

$$\|T_{\langle \xi, \nabla \rangle^\alpha m^{(0)}}\|_{p \rightarrow p} \leq C_{p, \alpha}. \quad (3.1.13)$$

Thus if we choose $\varepsilon = \varepsilon(p, \alpha, N) > 0$ small enough such that

$$24N(1-\theta)\varepsilon + \frac{\theta}{2}\varepsilon - \frac{1}{2}(1-\alpha)\theta < 0,$$

(3.1.12) and (3.1.13) yield

$$\begin{aligned} \|T_{\langle \xi, \nabla \rangle^\alpha m}\|_{p \rightarrow p} &\leq \|T_{\langle \xi, \nabla \rangle^\alpha m^{(0)}}\|_{p \rightarrow p} + \sum_{s=1}^{\infty} \|T_{\langle \xi, \nabla \rangle^\alpha \widehat{\Omega}^{(s)}}\|_{p \rightarrow p} \\ &\leq C_{p, \alpha} + C_{p, \varepsilon, \alpha, N} \sum_{s=1}^{\infty} 2^{(24N(1-\theta)\varepsilon + \frac{\theta}{2}\varepsilon - \frac{1}{2}(1-\alpha)\theta)s} \\ &< C_{p, \alpha, N}. \end{aligned}$$

In order to show (3.1.11), note that for each $t > 0$, we have

$$e^{-t^2} - e^{-4t^2} \leq C \min\{t, t^{-1}\}.$$

For $|\xi| \leq 2^{s/2}$, we thus get

$$e^{-4^{-s}|\xi|^2} - e^{-4^{-s+1}|\xi|^2} \leq C 2^{-s/2},$$

while for $|\xi| > 2^{s/2}$, we have

$$|m(\xi)| \leq (L(B)|\xi|)^{-1} \leq C 2^{-s/2}.$$

This yields

$$\|\widehat{\Omega}^{(s)}\|_{\infty} \leq C 2^{-s/2}. \quad (3.1.14)$$

Moreover, since the function $t \mapsto t(e^{-t^2} - e^{-4t^2})$ is bounded on \mathbb{R} , one can easily see that

$$\sup_{\substack{\xi \in \mathbb{R}^n \\ 1 \leq t \leq 2}} \left| \frac{d}{dt} \widehat{\Omega}^{(s)}(t\xi) \right| < C, \quad (3.1.15)$$

where C does not depend on s or ℓ . Let $0 < \varepsilon < 1$. For $-\varepsilon \leq \operatorname{Re} z \leq 1 - \varepsilon$, define the multipliers

$$m_{z, s}^{\varepsilon}(\xi) := 2^{(1-z-\varepsilon)s/2} I_t^{-z} \left(\widehat{\Omega}^{(s)}(t\xi) \right) \Big|_{t=1},$$

and write

$$T_z^{\varepsilon} := T_{m_{z, s}^{\varepsilon}}.$$

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By arguing as in Section 2.6, one can see that the analytic family of operators $(T_z)_{-\varepsilon \leq \operatorname{Re} z \leq 1-\varepsilon}$ is of admissible growth, and that for each $\tau \in \mathbb{R}$, we have

$$\|T_{-\varepsilon+i\tau}^\varepsilon\|_{2 \rightarrow 2} \leq C_\varepsilon e^{\frac{3\pi}{4}|\tau|}, \quad \text{and} \quad \|T_{1-\varepsilon+i\tau}^\varepsilon\|_{2 \rightarrow 2} \leq C_\varepsilon e^{\frac{3\pi}{4}|\tau|},$$

which is achieved in a similar way as (2.6.14). Thus we may apply Theorem 2.1.1 to get

$$\|m_{\alpha,s}^\varepsilon\|_\infty = \|T_\alpha^\varepsilon\|_{2 \rightarrow 2} \leq C_{\varepsilon,\alpha}$$

for $0 < \alpha < 1$, which is equivalent to

$$\|I_t^{-\alpha}(\widehat{\Omega^{(s)}}(t\xi))\big|_{t=1}\|_\infty \leq C_{\varepsilon,\alpha} 2^{-(1-\alpha)s/2} \cdot 2^{\varepsilon s/2}.$$

Since $\|m^{(s)}\|_\infty \leq C 2^{-s/2}$, Lemma 2.5.3 yields

$$\|\langle \xi, \nabla \rangle^\alpha \widehat{\Omega^{(s)}}\|_\infty \leq \|I_t^{-\alpha}(\widehat{\Omega^{(s)}}(t\xi))\big|_{t=1}\|_\infty + \|m^{(s)}\|_\infty \leq C_{\varepsilon,\alpha} 2^{-(1-\alpha)s/2} \cdot 2^{\varepsilon s/2},$$

which is (3.1.11). The case $p > 2$ follows by duality. **q.e.d.**

The rest of this section will be devoted to the proof of Lemma 3.1.3.

Fix $2 \leq p < \infty$, $R \geq 2$, and $0 < \varepsilon < 1$. The direct interpolation from [7, p. 280] with bounds similar to (2.7.7) and (2.7.8) shows the weaker estimate

$$\left\| \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_p R^{1-2/p} \|f\|_p. \quad (3.1.16)$$

We proceed with Bourgain's Fourier localization. By means of Pisier's result on contractive semigroups [22, p. 390] and an averaging argument, we get the following.

Lemma 3.1.5 (Lemma 5 from [7]). *Let $\eta(x) = (1 - |x|)_+$ on \mathbb{R} , $t > 0$, and let $T_j: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ be the convolution by η_t in the j -th variable, where $\eta_t(x) = t^{-1}\eta(t^{-1}x)$. For every $k \in \{1, \dots, n\}$ let*

$$A_k := \sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k}} \prod_{j \notin A} T_j \prod_{j \in A} (\operatorname{Id} - T_j). \quad (3.1.17)$$

Then for $1 < q < \infty$, we have $\|A_k\|_{q \rightarrow q} \leq C_q^k$, with C_q independent of k .

Fix $t := R^{-\varepsilon}$ and let A_k be as in (3.1.17). Then each A_k is a positive operator and $\sum_{k=0}^\ell A_k = \operatorname{Id}$. For some $K = K(p, \varepsilon) \geq 1$ to be chosen later, we decompose $f \in L^p$ as

$$f = \sum_{k=0}^K A_k f + g. \quad (3.1.18)$$

To achieve a good L^2 -estimate on $(\sum_{j=1}^n |g * \mu_j|^2)^{1/2}$, we prove a variant of [7, Lemma 6]. For this, we write

$$\tilde{m}(\zeta) := \widehat{\chi_{B_{r_N}^N}}(\zeta), \quad \zeta \in \mathbb{R}^N.$$

Furthermore, put $\hat{\xi}_j := \xi^{(k)}$ whenever $j \in V_k$, where V_k is defined as in (3.1.1), so that

$$V_k = \{(k-1)N+1, \dots, kN\}.$$

Lemma 3.1.6. *Let $A \geq 1$ be the best constant independent of ℓ and N such that*

$$|\tilde{m}(\xi)| \leq A|\xi|^{-1}$$

holds for every $\xi \neq 0$ (such a constant exists by (3.1.3)). Then for every $\delta > 0$ and $k \geq 1$, we have

$$|m(\xi)| \leq C_{k,N} \left(1 + \sum_{j:|\xi_j| \leq AR^\delta} |\xi_j|^2\right)^{-k/2} R^{\delta k}. \quad (3.1.19)$$

Proof. To adapt the original proof from [7] to our setting, we need to show that

$$|\tilde{m}(\zeta)| \leq e^{-c|\zeta|^2} \quad (3.1.20)$$

for $|\zeta| \leq A$ and

$$|\tilde{m}(\zeta)| \leq C = e^{-c'} \quad (3.1.21)$$

for $|\zeta| > A$, with $c' > 0$, $C < 1$, $\zeta \in \mathbb{R}^N$. By [15, Appendix B.5], we have

$$\tilde{m}(\zeta) = r_N^{N/2} |\zeta|^{-N/2} J_{N/2}(2\pi r_N |\zeta|)$$

with J_ν being the Bessel function of order ν . Since for each $k \geq 0$, we have

$$\Gamma(k+N/2+1) = (N/2+1) \cdots (N/2+k) \Gamma(N/2+1), \quad \Gamma(k+1/2) = \frac{1}{2} \cdot \frac{3}{2} \cdots (k-1/2) \Gamma(1/2),$$

we use the well-known series expansion (see also [15, Appendix B.2])

$$J_\nu(x) = \pi^{-1/2} \frac{x^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+\nu+1)} \frac{x^{2k}}{(2k)!} \quad (3.1.22)$$

to calculate

$$\begin{aligned} \tilde{m}(\zeta) &= \pi^{\frac{N-1}{2}} r_N^N \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2+1)} \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!} \\ &= \pi^{-1/2} \Gamma(N/2+1) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2+1)} \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2+1}. \end{aligned} \quad (3.1.23)$$

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Then we have

$$\tilde{m}(\zeta) = 1 - \frac{(2\pi r_N)^2}{N+2} |\zeta|^2 + \mathcal{O}(|\zeta|^4) \quad \text{as } |\zeta| \rightarrow 0,$$

while for every $c > 0$

$$e^{-c|\zeta|^2} = 1 - c|\zeta|^2 + \mathcal{O}(|\zeta|^4) \quad \text{as } |\zeta| \rightarrow 0.$$

By fixing $0 < c < (2\pi r_N)^2/(N+2)$, we can see that (3.1.20) holds near 0, say, for $|\zeta| < \delta'$, $\delta' > 0$. Next, we show that for $\delta' \leq |\zeta| \leq 2A$, we have $\tilde{m}(\zeta) < 1$, so that by compactness,

$$\sup_{\delta' \leq |\zeta| \leq 2A} |\tilde{m}(\zeta)| \leq C < 1. \quad (3.1.24)$$

For that, we use the integral representation for Bessel functions from [15, Appendices B.1 and B.2], to see that

$$\begin{aligned} \tilde{m}(\zeta) &= \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2})\pi^{1/2}} \int_{-1}^1 e^{it \cdot 2\pi r_N |\zeta|} (1-t^2)^{\frac{N-1}{2}} dt \\ &= \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2})\pi^{1/2}} \cdot 2 \int_0^1 \cos(t \cdot 2\pi r_N |\zeta|) (1-t^2)^{\frac{N-1}{2}} dt. \end{aligned} \quad (3.1.25)$$

Since the integrand in (3.1.25) is continuous and $|\cos(t \cdot 2\pi r_N |\zeta|)| < 1$ for all but finitely many $t \in [0, 1]$ (if $|\zeta| \neq 0$), we get

$$|\tilde{m}(\zeta)| < \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2})\pi^{1/2}} \cdot 2 \int_0^1 (1-t^2)^{\frac{N-1}{2}} dt = 1.$$

If $|\zeta| > 2A$, it follows immediately that

$$|\tilde{m}(\zeta)| \leq \frac{1}{2}.$$

By choosing $c < (2\pi r_N)^2/(N+2)$ so small that

$$e^{-cA^2} \geq C,$$

with C as in (3.1.24), one can see that (3.1.20) and (3.1.21) hold, allowing us to conclude the Lemma as in [7]. We will recall the argument.

Note that

$$|m(\xi)| = \prod_{k=1}^{\ell} |\tilde{m}(\xi^{(k)})|.$$

Define the sets

$$I_0 := \{j \in \{1, \dots, n\} : |\hat{\xi}_j| > A\} \quad \text{and} \quad J_0 := \{k \in \{1, \dots, \ell\} : |\xi^{(k)}| > A\}.$$

Then we clearly have $V_k \subset I_0$ if $k \in J_0$, and $V_k \cap I_0 = \emptyset$ if $k \notin J_0$, so that

$$I_0 = \bigcup_{k \in J_0} V_k. \quad (3.1.26)$$

Since for $k \notin I_0$, we have $|\xi^{(k)}| \leq A$ and thus $|m(\xi^{(k)})| < e^{-c|\xi^{(k)}|^2}$ by (3.1.20), we get

$$\prod_{k \notin J_0} |\tilde{m}(\xi^{(k)})| \leq \exp\left(-c \sum_{k \notin J_0} |\xi^{(k)}|^2\right) = \exp\left(-c \sum_{j \notin I_0} |\xi_j|^2\right),$$

and also, by (3.1.21),

$$\prod_{k \in J_0} |\tilde{m}(\xi^{(k)})| \leq e^{-c'|J_0|} = e^{-\frac{c'}{N}|I_0|},$$

resulting in

$$|m(\xi)| \leq \exp\left(-c_N \left(|I_0| + \sum_{j \notin I_0} |\xi_j|^2\right)\right). \quad (3.1.27)$$

Together with the obvious estimate

$$\sum_{|\hat{\xi}_j| \leq AR^\delta} |\xi_j|^2 \leq A^2 R^{2\delta} |I_0| + \sum_{j \notin I_0} |\xi_j|^2 \leq A^2 R^{2\delta} \left(|I_0| + \sum_{j \notin I_0} |\xi_j|^2\right),$$

this leads to

$$|m(\xi)| \leq \exp\left(-c_N A^{-2} R^{-2\delta} \sum_{|\xi_j| \leq AR^\delta} |\xi_j|^2\right). \quad (3.1.28)$$

But $e^{-C|x|} = \mathcal{O}((1+|x|)^{-k/2})$ for every $C > 0$ and $k \in \mathbb{N}_{>0}$, and since A is independent of ℓ and N , (3.1.28) implies

$$|m(\xi)| \leq C_{k,N} \left(1 + R^{-2\delta} \sum_{|\xi_j| \leq R^\delta} |\xi_j|^2\right)^{-k/2} \leq C_{k,N} \left(R^{-2\delta} + R^{-2\delta} \sum_{|\xi_j| \leq R^\delta} |\xi_j|^2\right)^{-k/2},$$

immediately concluding the Lemma. **q.e.d.**

In view of the decomposition (3.1.18), we can use Lemma 3.1.6 to establish the following L^2 -bound (see [13] or [23], rather than [7]).

Lemma 3.1.7. *We have*

$$\left\| \left(\sum_{j=1}^n |g * \mu_j|^2 \right)^{1/2} \right\|_2 \leq C_K R^{1 - \frac{\varepsilon K}{10}} \|f\|_2, \quad (3.1.29)$$

with C_K only depending on K .

To achieve (3.1.29), one basically has to replace the set I_1 from [7] by $I_1 := \{j \in \{1, \dots, n\} : |\hat{\xi}_j| > AR^{\varepsilon/5}\}$, with A as in Lemma 3.1.6. However, there is a slight error in [7] concerning the bound in (3.1.29), which is why we will provide a detailed proof.

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Proof. Since each A_k is a convolution operator, we will define the function $\widehat{A}_k: \mathbb{R}^n \rightarrow \mathbb{R}^n$ through

$$\widehat{A}_k f(\xi) =: \widehat{A}_k(\xi) \widehat{f}(\xi).$$

By the Plancherel theorem, we have

$$\begin{aligned} \left\| \left(\sum_{j=1}^n |g * \mu_j|^2 \right)^{1/2} \right\|_2^2 &= \int \sum_{j=1}^n |g * \mu_j(x)|^2 dx \\ &= \int \sum_{j=1}^n |\widehat{g}(\xi) \widehat{\mu}_j(\xi)|^2 d\xi \\ &= \int \sum_{j=1}^n \left(\xi_j m(\xi) e^{-R^{-2}|\xi|^2} \left(1 - \sum_{k=0}^K \widehat{A}_k(\xi) \right) |\widehat{f}(\xi)| \right)^2 d\xi \\ &= \int |\xi|^2 m(\xi)^2 e^{-2R^{-2}|\xi|^2} \left(1 - \sum_{k=0}^K \widehat{A}_k(\xi) \right)^2 |\widehat{f}(\xi)|^2 d\xi \\ &\leq \sup_{x \in \mathbb{R}^n} |x|^2 m(x)^2 e^{-2R^{-2}|x|^2} \left(1 - \sum_{k=0}^K \widehat{A}_k(x) \right)^2 \cdot \int |\widehat{f}(\xi)|^2 d\xi \\ &= \|f\|_2^2 \cdot \sup_{x \in \mathbb{R}^n} |x|^2 m(x)^2 e^{-2R^{-2}|x|^2} \left(1 - \sum_{k=0}^K \widehat{A}_k(x) \right)^2. \quad (3.1.30) \end{aligned}$$

Thus we have to bound

$$|\xi| |m(\xi)| e^{-R^{-2}|\xi|^2} \left| 1 - \sum_{k=0}^K \widehat{A}_k(\xi) \right|$$

uniformly in ξ . We can estimate $|\xi| e^{-R^{-2}|\xi|^2}$ by simply replacing ξ with $R\xi$, getting

$$R|\xi| e^{-|\xi|^2} < R,$$

and due to (3.1.30), it suffices to show that

$$|m(\xi)| \cdot \left(1 - \sum_{k=0}^K \widehat{A}_k(\xi) \right) < C_K R^{-\frac{\varepsilon K}{10}} \quad (3.1.31)$$

for every $\xi \in \mathbb{R}^n$. For that, let

$$I_1 := \{ \{j \in \{1, \dots, n\} : |\widehat{\xi}_j| > AR^{\varepsilon/5} \} \} \quad \text{and} \quad J_1 := \{ \{k \in \{1, \dots, \ell\} : |\xi^{(k)}| > AR^{\varepsilon/5} \} \}.$$

If we assume that $|I_1| > K/2$, the choice of A and the fact that $|\tilde{m}(\zeta)| \leq 1$ for $\zeta \in \mathbb{R}^N$ yield

$$|m(\xi)| \cdot \left(1 - \sum_{k=0}^K \widehat{A}_k(\xi) \right) \leq |m(\xi)| \leq \prod_{k \in J_1} |\tilde{m}(\xi^{(k)})| \leq (R^{-\varepsilon/5})^{|I_1|} < R^{-\frac{\varepsilon K}{10}}. \quad (3.1.32)$$

Otherwise, $|I_1| \leq K/2$. In this case, we want to apply Lemma 3.1.6. We make use of the identity

$$1 - \sum_{k=0}^K \widehat{A}_k(\xi) = \sum_{\substack{A \subset \{1, \dots, n\} \\ |A| > K}} \prod_{j \notin A} \hat{\eta}(t\xi_j) \prod_{j \in A} (1 - \hat{\eta}(t\xi_j)) \quad (3.1.33)$$

First, we use an estimate that allows us to deal only with the variables with indices not in I_1 . Every set $A \subset \{1, \dots, n\}$ with $|A| > K$ has the unique form $A = S \cup T$ with $S \subset \{1, \dots, n\} \setminus I_1$, $|S| \geq \lceil K/2 \rceil$, and $T \subset I_1$. We can simply increase the number of summands in (3.1.33) by taking every possible $S \cup T$ with S, T as above, and decrease the number of factors in each summand of (3.1.33) using that $0 \leq \hat{\eta}(\tau) \leq 1$ for each $\tau \in \mathbb{R}$. This gives us

$$\begin{aligned} 1 - \sum_{k=0}^K \widehat{A}_k(\xi) &= \sum_{\substack{A \subset \{1, \dots, n\} \\ |A| > K}} \prod_{j \notin A} \hat{\eta}(t\xi_j) \prod_{j \in A} (1 - \hat{\eta}(t\xi_j)) \\ &\leq \sum_{\substack{S \subset \{1, \dots, n\} \setminus I_1 \\ |S| \geq \lceil \frac{K}{2} \rceil, T \subset I_1}} \prod_{j \notin S \cup T} \hat{\eta}(t\xi_j) \prod_{j \in S \cup T} (1 - \hat{\eta}(t\xi_j)) \\ &\leq \sum_{\substack{S \cap I_1 = \emptyset \\ |S| \geq \lceil \frac{K}{2} \rceil, T \subset I_1}} \prod_{j \notin S \cup I_1} \hat{\eta}(t\xi_j) \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) \\ &\leq 2^{\lceil \frac{K}{2} \rceil} \sum_{\substack{S \cap I_1 = \emptyset \\ |S| \geq \lceil \frac{K}{2} \rceil}} \prod_{j \notin S \cup I_1} \hat{\eta}(t\xi_j) \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)). \end{aligned} \quad (3.1.34)$$

Since we have

$$\sum_{S \cap I_1 = \emptyset} r^{|S|} \prod_{j \notin S \cup I_1} \hat{\eta}(t\xi_j) \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) = \prod_{\substack{1 \leq j \leq n \\ j \notin I_1}} (\hat{\eta}(t\xi_j) + r(1 - \hat{\eta}(t\xi_j)))$$

for each $r > 0$, we can use the standard rules for derivatives to see that

$$\begin{aligned} &\sum_{\substack{S \cap I_1 = \emptyset \\ |S| \geq \lceil \frac{K}{2} \rceil}} r^{|S| - \lceil \frac{K}{2} \rceil} \prod_{l=0}^{\lceil \frac{K}{2} \rceil - 1} (|S| - l) \prod_{j \notin S \cup I_1} \hat{\eta}(t\xi_j) \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) \\ &= \left(\frac{d}{dr} \right)^{\lceil \frac{K}{2} \rceil} \left(\sum_{S \cap I_1 = \emptyset} r^{|S|} \prod_{j \notin S \cup I_1} \hat{\eta}(t\xi_j) \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) \right) \\ &= \left(\frac{d}{dr} \right)^{\lceil \frac{K}{2} \rceil} \left(\prod_{\substack{1 \leq j \leq n \\ j \notin I_1}} (\hat{\eta}(t\xi_j) + r(1 - \hat{\eta}(t\xi_j))) \right) \\ &= \sum_{\substack{S \cap I_1 = \emptyset \\ |S| = \lceil \frac{K}{2} \rceil}} \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) \prod_{j \notin S \cup I_1} (\hat{\eta}(t\xi_j) + r(1 - \hat{\eta}(t\xi_j))). \end{aligned} \quad (3.1.35)$$

By applying (3.1.35) to the right hand side term in (3.1.34), one can see that

$$\begin{aligned}
1 - \sum_{k=0}^K \widehat{A}_k(\xi) &\leq C_K \sum_{\substack{S \cap I_1 = \emptyset \\ |S| \geq \lceil \frac{K}{2} \rceil}} \prod_{l=0}^{\lceil \frac{K}{2} \rceil - 1} (|S| - l) \prod_{j \notin S \cup I_1} \hat{\eta}(t\xi_j) \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) \\
&= C_K \left(\frac{d}{dr} \right)^{\lceil \frac{K}{2} \rceil} \left(\prod_{\substack{1 \leq j \leq n \\ j \notin I_1}} (\hat{\eta}(t\xi_j) + r(1 - \hat{\eta}(t\xi_j))) \right) \Big|_{r=1} \\
&= C_K \sum_{\substack{S \cap I_1 = \emptyset \\ |S| = \lceil \frac{K}{2} \rceil}} \prod_{j \in S} (1 - \hat{\eta}(t\xi_j)) \\
&\leq C_K \sum_{j_1, \dots, j_{\lceil \frac{K}{2} \rceil} \notin I_1} \prod_{l=1}^{\lceil \frac{K}{2} \rceil} (1 - \hat{\eta}(t\xi_{j_l})) \\
&= C_K \left(\sum_{j \notin I_1} (1 - \hat{\eta}(t\xi_j)) \right)^{\lceil \frac{K}{2} \rceil}. \tag{3.1.36}
\end{aligned}$$

It is well-known that $1 - \hat{\eta}(\lambda) < C\lambda^2$ for $|\lambda| < 1$, and if $j \notin I_1$, then $|t\xi_j| \leq R^{-\varepsilon} \cdot R^{\varepsilon/5} < 1$ (since $t = R^{-\varepsilon}$). Thus

$$\left(\sum_{j \notin I_1} (1 - \hat{\eta}(t\xi_j)) \right)^{\lceil \frac{K}{2} \rceil} \leq C t^{2\lceil \frac{K}{2} \rceil} \left(\sum_{j \notin I_1} \xi_j^2 \right)^{\lceil \frac{K}{2} \rceil}. \tag{3.1.37}$$

On the other hand, an application of Lemma 3.1.6 with $\varepsilon/5$ and $2\lceil K/2 \rceil$ yields

$$|m(\xi)| < C_K \left(1 + \sum_{j \notin I_1} \xi_j^2 \right)^{-\lceil \frac{K}{2} \rceil} R^{\frac{2\varepsilon}{5}\lceil \frac{K}{2} \rceil}.$$

Combining that with (3.1.36) and (3.1.37) leads to

$$|m(\xi)| \cdot \left(1 - \sum_{k=0}^K \widehat{A}_k(\xi) \right) \leq C_K t^{2\lceil \frac{K}{2} \rceil} R^{\frac{2\varepsilon}{5}\lceil \frac{K}{2} \rceil} = C_K R^{(2/5-2)\varepsilon\lceil \frac{K}{2} \rceil} \leq C_K R^{-\frac{\varepsilon K}{2}}. \tag{3.1.38}$$

Since $R^{-\frac{\varepsilon K}{2}} < R^{-\frac{\varepsilon K}{10}}$, combining (3.1.32) and (3.1.38) concludes Lemma 3.1.7. **q.e.d.**

By interpolation with (3.1.16), the choice $K = \lceil \frac{10(p-1)}{\varepsilon} \rceil$ gives us

$$\left\| \left(\sum_{j=1}^n |g * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon} \|f\|_p.$$

In order to conclude the proof of Lemma 3.1.3, it thus only remains to estimate

$$\left\| \left(\sum_{j=1}^n |A_k f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon} R^{24N \cdot \varepsilon} \|f\|_p \tag{3.1.39}$$

for $0 \leq k \leq K$. For $S \subset \{1, \dots, n\}$, put

$$\Gamma_S := \prod_{j \notin S} T_j \prod_{j \in S} (\text{Id} - T_j).$$

Then, by the triangle inequality, we have

$$\left(\sum_{j=1}^n |A_k f * \mu_j|^2 \right)^{1/2} \leq \left(\sum_{j=1}^n \left| \sum_{\substack{|S|=k \\ j \notin S}} \Gamma_S f * \mu_j \right|^2 \right)^{1/2} \quad (3.1.40)$$

$$+ \left(\sum_{j=1}^n \left| \sum_{\substack{|S|=k \\ j \in S}} \Gamma_S f * \mu_j \right|^2 \right)^{1/2}. \quad (3.1.41)$$

Bourgain applies a stochastic method to decouple the variables, which reduces (3.1.40) to the case $k = 0$ and (3.1.41) to the case $k = 1$. With that, we only have to find suitable constants $b_0 = b_0(R)$, $b_1 = b_1(R)$ such that

$$\left\| \left(\sum_{j=1}^n |A_0 f * \mu_j|^2 \right)^{1/2} \right\|_p \leq b_0 \|f\|_p \quad (3.1.42)$$

and

$$\left\| \left(\sum_{j=1}^n |\Gamma_j f * \mu_j|^2 \right)^{1/2} \right\|_p \leq b_1 \|f\|_p, \quad (3.1.43)$$

where $\Gamma_k := \Gamma_{\{k\}} = (\text{Id} - T_k) \prod_{j \neq k} T_j$.

Let $B_p = B_{p,R,N}$ be minimal such that

$$\left\| \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq B_p \|f\|_p. \quad (3.1.44)$$

We will be able to estimate b_0 and b_1 exactly as in [7].

However, to estimate b_1 , we need to rely on [7, Lemmas 7-9]. The proof of [7, Lemma 8] will be essentially the same, only with some additional estimates that will depend on N . For this, we will only provide the necessary estimates that change compared to the original proofs of [7, Lemmas 8 and 9]. Instead of using the properties of the convolution operators T_j , we need to convolve with a function that is roughly stable under small translations. Bourgain considers the function

$$\varphi(x) := \frac{c}{1 + x^4}$$

with c so that $\int_{\mathbb{R}} \varphi(x) dx = 1$. Then $C^{-1}\varphi(x - y) \leq \varphi(x) \leq C\varphi(x - y)$ for every $x \in \mathbb{R}$ and $|y| \leq 1$, $\hat{\varphi}(x) = \mathcal{O}(e^{-C|x|})$ as $|x| \rightarrow \infty$, and $|1 - \hat{\varphi}(x)| < Cx^2$ for every $x \in \mathbb{R}$.

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Fix $t_0 := R^{-3\varepsilon}$, and let $L_j: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ be the convolution by φ_{t_0} in the j -th variable, $j \in \{1, \dots, n\}$. For $k \in \{1, \dots, n\}$, let also

$$L^{(k)} = \prod_{j \neq k} L_j.$$

We shall recall [7, Lemma 7].

Lemma 3.1.8. *Let $M \in \mathbb{N}$, $q = 2^M$, and let $f_1, \dots, f_n \in L^q(\mathbb{R}^n)$ be positive functions. Then*

$$\begin{aligned} \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q &\leq C_q \sum_{k=0}^{M-1} \left\| \left(\prod_{j=1}^n L_j \right) \left(\sum_{j=1}^n f_j^{2^k} \right) \right\|_{2^{M-k}}^{2^{-k}} + C_q \left(\sum_{j=1}^n \|f_j\|_q^q \right)^{1/q} \\ &\leq C_q \left\| \sum_{j=1}^n f_j \right\|_q. \end{aligned} \quad (3.1.45)$$

With that, we directly show [7, Lemma 9], which is a corollary of [7, Lemma 8], for our setting.

Lemma 3.1.9. *Let $M \geq 1$, $q = 2^M$, and $f_1, \dots, f_\ell \in L^q(\mathbb{R}^n)$. Then*

$$\left\| \left(\sum_{j=1}^n |L^{(j)} f_j * \mu_j|^2 \right)^{1/2} \right\|_q \leq C_{q,N} R^{24N-\varepsilon} \left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\|_q. \quad (3.1.46)$$

Proof. Note that from the properties of φ , we can deduce

$$\varphi_{t_0}(x + \rho) = \frac{1}{t_0} \frac{c}{1 + \frac{(x+\rho)^4}{t_0^4}} \leq \frac{1}{t_0} \varphi(x + \rho) \leq \frac{C^{\lceil r_N \rceil}}{t_0} \varphi(x) = \frac{C_N}{t_0^5} \frac{1}{\frac{1+x^4}{t_0^4}} \leq \frac{C_N}{t_0^4} \varphi_{t_0}(x)$$

for $x \in \mathbb{R}$ and $\rho \in [-r_N, r_N]$. Define the function

$$\Phi_{t_0}: \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad \Phi_{t_0}(y) = \varphi_{t_0}(y_1) \cdots \varphi_{t_0}(y_N).$$

Then

$$\Phi_{t_0}(x - y) \leq C_N t_0^{-4N} \Phi_{t_0}(x) \quad (3.1.47)$$

for every $x, y \in \mathbb{R}^N$ with $|y| \leq r_N$. For the proof, we have to consider the distributional derivatives $\partial_j \chi_B$ for every $j \in \{1, \dots, n\}$. First, assume $j = \ell = 1$, so that $B = B_{r_N}^N$. By [21], we then have

$$\|T_{\partial_1 \chi_B}\|_{1 \rightarrow 1} \leq Q(B_{r_N}^N) \leq C, \quad (3.1.48)$$

and the distribution $\partial_1 \chi_B$ is an average of shift operators over $S_{r_N}^{N-1}$. By (3.1.47) and (3.1.48), one can thus see that

$$(\Phi_{t_0} * \partial_1 \chi_B)(x) = \langle \Phi_{t_0}(x - \cdot), \partial_1 \chi_B \rangle \leq C_N t_0^{-4N} \Phi_{t_0}(x) \leq C'_N t_0^{-8N} \Phi_{t_0} * \chi_B(x).$$

leading to

$$\left(\prod_{j=1}^N L_j \right) (f * \partial_1 \chi_B) \leq C_N t_0^{-8N} \left(\prod_{j=1}^N L_j \right) (f * \chi_B) \quad (3.1.49)$$

for every positive function $f \in L^q(\mathbb{R}^N)$.

In the general case, we also achieve the estimate (3.1.49) with the same constant C_N , independently of ℓ . Thus we may proceed as in the proof of [7, Lemma 8]. Regarding that $t_0^{-8N} = R^{24\varepsilon}$, the lemma follows. **q.e.d.**

Now, we can estimate (3.1.42) and (3.1.43) by arguing exactly as in [7, Section 4]. We already mentioned that nothing changes in the estimate for b_0 , so that

$$b_0 \leq C_p (R^{3\varepsilon} + B_p R^{-\frac{2\varepsilon}{p}})$$

with B_p as in (3.1.44). For (3.1.43), we can use Lemmas 3.1.8 and 3.1.9 with $f_k = \Gamma_k f$ to obtain

$$b_1 \leq C_{p,N} (R^{24N\varepsilon} + B_p R^{-\frac{2\varepsilon}{p}}).$$

This leads to

$$B_p \leq C_{p,\varepsilon} (1 + b_0 + b_1) < C_{p,\varepsilon,N} (R^{24N\varepsilon} + B_p R^{-\frac{2\varepsilon}{p}}),$$

giving us

$$B_p \leq C_{p,\varepsilon,N} R^{24N\varepsilon}, \quad (3.1.50)$$

and thus proving Lemma 3.1.3 and Proposition 3.1.1.

3.2 Stein's rotation method for products of spheres and decoupling of high and low dimensions

Let $B = B_1 \times \cdots \times B_\ell$, $n = n_1 + \cdots + n_\ell$ be as in Theorem 1. Let $N := \min_{1 \leq k \leq \ell} n_k$. Fix $1 < p \leq \infty$. We show that if $N > \frac{p}{p-1}$, i.e. $p > \frac{N}{N-1}$, we can deduce (1.4.1) in Theorem 1 from Theorem 2, and that (1.4.2) from Theorem 2 holds if we show the following proposition. This will be done in Section 3.3.

Proposition 3.2.1. *Let $N \geq 3$ and $S := (S_R^{N-1})^\ell$, with R so that $|S| = 1$. Let $m = \widehat{\sigma}_S$. Then for every $p \in (\frac{N}{N-1}, 2)$ there is $\alpha \in (1/p, 1)$ such that*

$$\|T_{\langle \xi, \nabla \rangle^{\alpha m}}\|_{p \rightarrow p} \leq C_{p,N}. \quad (3.2.1)$$

Consequently, for every $p > \frac{N}{N-1}$ and every $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|M_S f\|_p \leq C_{p,N} \|f\|_p. \quad (3.2.2)$$

We can freely change the radii of each sphere because for $r, s > 0$ and $n > 1$, we have

$$\int_{S_r^{n-1}} f(s\omega) d\sigma_{S_r^{n-1}}(\omega) = \frac{1}{s^{n-1}} \int_{S_{sr}^{n-1}} f(\omega) d\sigma_{S_{sr}^{n-1}}(\omega). \quad (3.2.3)$$

More precisely, we have the following lemma.

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Lemma 3.2.2. *Let $S = S_1 \times \cdots \times S_\ell$, and $S' = S'_1 \times \cdots \times S'_\ell$, where $S_k = S^{n_k-1}$, $S'_k = S^{n_k-1}_{r_k}$, and each $r_k > 0$. Assume that for some $p > 1$, we have*

$$\|M_S f\|_p \leq C \|f\|_p.$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. Then we also have

$$\|M_{S'} f\|_p \leq C \|f\|_p$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, with the same constant C .

Proof. Let $r = (r_1, \dots, r_\ell)$. We use that $|S_k| = r_k^{-n_k+1} |S'_k|$ for each k and apply (3.2.3) to each spherical integral to get

$$\begin{aligned} M_{S'} f(x) &= \frac{1}{|S'|} \sup_{t>0} \int_{S'_1} \cdots \int_{S'_\ell} |f(x + t(\omega^{(1)}, \dots, \omega^{(\ell)}))| d\sigma_{S'_\ell}(\omega^{(\ell)}) \cdots d\sigma_{S'_1}(\omega^{(1)}) \\ &= \frac{1}{|S'|} \sup_{t>0} \prod_{k=1}^{\ell} r_k^{n_k-1} \int_{S_1} \cdots \int_{S_\ell} |f(x + t(r_1\omega^{(1)}, \dots, r_\ell\omega^{(\ell)}))| d\sigma_\ell(\omega^{(\ell)}) \cdots d\sigma_1(\omega^{(1)}) \\ &= \frac{1}{|S|} \sup_{t>0} \int_S |f(x + t(r.\omega))| d\sigma_S(\omega) \\ &= \sup_{t>0} M_{tr}^S f(x). \end{aligned}$$

By the transformation formula, we have

$$\int_{\mathbb{R}^n} g(x) dx = \prod_{k=1}^{\ell} r_k^{n_k} \int_{\mathbb{R}^n} g(r.x) dx$$

for every $g \in L^1(\mathbb{R}^n)$. This leads to

$$\begin{aligned} \|M_{S'} f(x)\|_p^p &= \int_{\mathbb{R}^n} \left(\frac{1}{|S'|} \sup_{t>0} \int_S |f(x + t(r.\omega))| d\sigma_S(\omega) \right)^p dx \\ &= \prod_{k=1}^{\ell} r_k^{n_k} \int_{\mathbb{R}^n} \left(\frac{1}{|S|} \sup_{t>0} \int_S |f(r.x + t(r.\omega))| d\sigma_S(\omega) \right)^p dx \\ &= \prod_{k=1}^{\ell} r_k^{n_k} \|M_S(r.f)\|_p^p \\ &\leq C^p \prod_{k=1}^{\ell} r_k^{n_k} \|r.f\|_p^p = C^p \|f\|_p^p. \end{aligned}$$

q.e.d.

Remark. From this proof, we can also see that

$$M_t^{S'} f(x) = M_{tr}^S f(x). \quad (3.2.4)$$

We can not get a pointwise estimate $M_B f(x) \leq M_S f(x)$ as in the case $\ell = 1$, but we can indeed get an L^p -estimate by a similar argument. Assume that each B_k and each S_k has radius 1 (thus $S_k = S^{n_k-1}$), and let σ_k be the respective surface measure for each S_k .

Lemma 3.2.3. Let $S = S_1 \times \cdots \times S_\ell$. Then we have

$$M_t^B f(x) = \int_{[0,1]^\ell} \left[\prod_{k=1}^{\ell} n_k r_k^{n_k-1} \right] M_{tr}^S f(x) dr. \quad (3.2.5)$$

Furthermore, if we assume that for some $p > 1$ and every $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|M_S f\|_p \leq C \|f\|_p,$$

then we also have the estimate

$$\|M_B f\|_p \leq C \|f\|_p \quad (3.2.6)$$

for each $f \in \mathcal{S}(\mathbb{R}^n)$, with the same constant C .

Proof. Using polar coordinates, we write

$$\begin{aligned} M_t^B f(x) &= \frac{1}{|B|} \int_{B_1} \cdots \int_{B_\ell} f(x + t(y_1, \dots, y_\ell)) dy_\ell \cdots dy_1 \\ &= \frac{1}{|B|} \int_{[0,1]^\ell} \prod_{k=1}^{\ell} r_k^{n_k-1} \int_S |f(x + t(r_1 \omega^{(1)}, \dots, r_\ell \omega^{(\ell)}))| d\sigma_S((\omega^{(1)}, \dots, \omega^{(\ell)})) dr \\ &= \frac{|S|}{|B|} \int_{[0,1]^\ell} \prod_{k=1}^{\ell} r_k^{n_k-1} M_{tr}^S f(x). \end{aligned}$$

By the fact that $|S_k| = n_k |B_k|$, we have

$$\frac{|S|}{|B|} = \prod_{k=1}^{\ell} n_k,$$

and thus (3.2.5) follows. A simple application of Minkowski's integral inequality together with Lemma 3.2.2 now yields

$$\|M_B f\|_p = \left\| \sup_{t>0} \int_{[0,1]^\ell} \prod_{k=1}^{\ell} n_k r_k^{n_k-1} M_{tr}^S f dr \right\|_p \leq \int_{[0,1]^\ell} \prod_{k=1}^{\ell} n_k r_k^{n_k-1} C \|f\|_p dr = C \|f\|_p,$$

which is (3.2.6). **q.e.d.**

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We now generalize Stein's method of rotations from [29] (see Section 2.2) for our situation with the following Lemma.

Lemma 3.2.4. *Let $S' = S^{n_1-1} \times \dots \times S^{n_\ell-1}$ and $S = (S^{N-1})^\ell$. Furthermore, assume that there are $1 < p \leq \infty$ and a constant $C > 0$ such that*

$$\|M_S f\|_{L^p(\mathbb{R}^{N\ell})} \leq C \|f\|_{L^p(\mathbb{R}^{N\ell})}$$

for every $f \in \mathcal{S}(\mathbb{R}^{N\ell})$. Then also

$$\|M_{S'} f\|_p \leq C \|f\|_p \tag{3.2.7}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, again with the same constant C .

Proof. For $x \in \mathbb{R}^n$, write $x = (x^{(1,1)}, x^{(1,2)}, \dots, x^{(\ell,1)}, x^{(\ell,2)})$, with $x^{(k,1)} \in \mathbb{R}^N$ and $x^{(k,2)} \in \mathbb{R}^{n_k-N}$ for each $k \in \{1, \dots, \ell\}$. For $f \in \mathcal{S}(\mathbb{R}^n)$, let

$$M'_t f(x) := \frac{1}{|S|} \int_S f(x^{(1,1)} - t\omega^{(1,1)}, x^{(1,2)}, \dots, x^{(\ell,1)} - t\omega^{(\ell,1)}, x^{(\ell,2)}) d\sigma_S(\omega^{(1,1)}, \dots, \omega^{(\ell,1)}). \tag{3.2.8}$$

Just as in Section 2.2, we have

$$\|\sup_{t>0} M'_t f(x)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.2.9}$$

with the same C as in our assumption. Let $G := O(n_1) \times \dots \times O(n_\ell)$. For $\rho \in G$, write $\tilde{\rho}(x) := (\rho_1 x^1, \dots, \rho_\ell x^\ell)$, and let

$$M_t^\rho f := [M'_t(f \circ \tilde{\rho})] \circ \tilde{\rho}^{-1}. \tag{3.2.10}$$

Again, we get

$$\|\sup_{t>0} M_t^\rho f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.2.11}$$

and Fubini's theorem and the very same arguments as in Section 2.2 imply that

$$M_t^{S'} f(x) = \int_G M_t^\rho f(x) d\lambda(\rho), \tag{3.2.12}$$

where λ is the product measure of the normalized Haar measures of the $O(n_k)$. Minkowski's integral inequality, as well as (3.2.11) and (3.2.12), then yield

$$\|M_{S'} f\|_{L^p(\mathbb{R}^n)} \leq \int_G \|\sup_{t>0} M_t^\rho f\|_{L^p(\mathbb{R}^n)} d\lambda(\rho) \leq \int_G C \|f\|_{L^p(\mathbb{R}^n)} d\lambda(\rho) = C \|f\|_{L^p(\mathbb{R}^n)},$$

again with the C from our assumption. This concludes the lemma. **q.e.d.**

3.3 Higher Fourier derivatives of spherical measures

Now assume that we have already shown Proposition 3.2.1 and take B as in Theorem 1. Fix $1 < p < 2$ and let $N = N(p) = \lfloor \frac{p}{p-1} \rfloor + 1 \geq 3$. By change of coordinates, we can assume $B = B' \times B''$, where

$$B' = \prod_{n_k < N} B_k \quad \text{and} \quad B'' = \prod_{n_k \geq N} B_k.$$

Then

$$\|M_B\|_{p \rightarrow p} \leq \|M_{B'}\|_{p \rightarrow p} \cdot \|M_{B''}\|_{p \rightarrow p}.$$

Furthermore, assume that for $n_k \geq N$, each B_k has radius 1, while for $n_k < N$, each B_k has volume 1. Since N only depends on p , we get

$$\|M_{B'}\|_{p \rightarrow p} \leq C_p$$

from Proposition 3.1.1. Let $S = \prod_{n_k \geq N} S^{n_k-1}$. Then by Lemmas 3.2.3 and 3.2.4, we obtain

$$\|M_{B''}\|_{p \rightarrow p} \leq \|M_S\|_{p \rightarrow p} \leq \|M_{(S^{N-1})^{\ell''}}\|_{p \rightarrow p},$$

where $\ell'' = |\{k \in \{1, \dots, \ell\} : n_k \geq N\}|$. But since we can freely vary the radius of each sphere, Proposition 3.2.1 implies

$$\|M_{(S^{N-1})^{\ell''}}\|_{p \rightarrow p} \leq C_N = C_p.$$

This concludes Theorem 1.

3.3 Higher Fourier derivatives of spherical measures

In the rest of this chapter, we deal with Proposition 3.2.1. Let $N \geq 3$, $p > \frac{N}{N-1}$, $S = (S_R^{N-1})^\ell$ with R so that $|S| = 1$, i.e. $R^{N-1} = \frac{\Gamma(N/2)}{2\pi^{N/2}}$, and

$$m(\xi) := \widehat{\sigma_S}(\xi).$$

Let P be the Poisson kernel, i.e. $\hat{P}(\xi) = e^{-|\xi|}$. If we can show that the Borel measure $\sigma_S - P dx$ satisfies (2.5.13) and (2.5.14), then by Sections 2.4 and 2.5, we only have to bound

$$\left\| \sup_{1 \leq t \leq 2} \int_S |f(x + t\omega)| d\sigma_S(\omega) \right\|_p \leq C_{p,N} \|f\|_p \quad (3.3.1)$$

for $p < 2$. According to Proposition 2.5.2, we need to show that there is α with $1/p < \alpha < 1$ such that the fractional derivative $\langle \xi, \nabla \rangle^\alpha m$ has bounded L^p -multiplier norm independent of ℓ . For this, we will show that $\langle \xi, \nabla \rangle^{\frac{N-1}{2}} m$ is an L^2 -multiplier and interpolate as in Section 2.6. It turns out that we can also establish the necessary estimates (2.5.13) and (2.5.14) while bounding these fractional derivatives. It follows that we need to bound the classical derivatives $\frac{d^k}{dr^k} m(r\xi)$ for $1 \leq r \leq 2$.

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Let us give a quick idea of our approach to bound these derivatives. We have

$$m(\xi) = \prod_{j=1}^{\ell} \tilde{m}(\xi^{(j)}),$$

writing again $\xi = (\xi^{(1)}, \dots, \xi^{(\ell)})$ with each $\xi^{(j)} \in \mathbb{R}^N$, and

$$\tilde{m}(\zeta) := \widehat{\sigma_{S_R^{N-1}}(\zeta)}.$$

By the general Leibniz formula, we have

$$\left(\frac{d}{dr}\right)^k m(r\xi) = \sum_{\substack{\alpha \in \mathbb{N}^\ell \\ |\alpha|=k}} \binom{k}{\alpha} \prod_{j=1}^{\ell} \left(\frac{d}{dr}\right)^{\alpha_j} \tilde{m}(r\xi^{(j)}). \quad (3.3.2)$$

It turns out that the well known bound $\tilde{m}(\zeta) = \mathcal{O}(|\zeta|^{-\frac{N-1}{2}})$, as $|\zeta| \rightarrow \infty$, extends to $\frac{d^\alpha}{dr^\alpha} \tilde{m}(r\zeta) = \mathcal{O}(|\zeta|^{\alpha - \frac{N-1}{2}})$, $|\zeta| \rightarrow \infty$. Hence we can get a good estimate on the factors for which $|\xi^{(j)}|$ is sufficiently large. One can also use the integral representation of Bessel functions to see that for $k > 0$, $\frac{d^k}{dr^k} \tilde{m}(r\zeta)$ is close to 0 if ζ is close to 0.

However, we also need to get some estimate for the derivatives of \tilde{m} if $|\zeta|$ is of intermediate size. All the necessary estimates are established in the following lemma.

Lemma 3.3.1. *For each $\alpha \in \mathbb{N}$ and $\zeta \in \mathbb{R}^N$, the following estimates hold.*

(i) *There are constants $C_{\alpha,N}, \tilde{C}_{\alpha,N}$ such that for every $|\zeta| \geq C_{\alpha,N}$ and every $r \in [1, 2]$, we have*

$$\left| \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) \right| \leq \tilde{C}_{\alpha,N} (2\pi R |\zeta|)^{-\frac{N-1}{2} + \alpha}.$$

(ii) *If $0 \leq 2\pi R |\zeta| \leq C$ for some constant $C > 1$, then there is $c > 0$, depending only on C , such that for every $r \in [1, 2]$, we have*

$$|\tilde{m}(r\zeta)| \leq e^{-c(2\pi R |\zeta|)^2}.$$

(iii) *If $\alpha > 0$ and $1 \leq 2\pi R |\zeta| \leq C$ for some constant $C > 1$, then for every $r \in \mathbb{R}_{>0}$, we have*

$$\left| \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) \right| \leq C' (2\pi R |\zeta|)^{2\alpha} \leq (2\pi R |\zeta|)^{2\alpha} \cdot e^{-c'(2\pi R |\zeta|)^2},$$

with $0 < C' < 1$, C' only depending on C , and c' chosen so that $C' \leq e^{-c'C^2}$.

(iv) *If $\alpha > 0$ and $2\pi R |\zeta| < 1$, then there is $C' < 1$ such that for $0 < r \leq 2$,*

$$\left| \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) \right| \leq C' (2\pi R |\zeta|)^2.$$

Proof. (i) We will show that

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = 2\pi^{\alpha+1} r^{-\frac{N-2}{2}} R^{\frac{N}{2}+\alpha} |\zeta|^{-\frac{N-2}{2}+\alpha} \sum_{k=0}^{\alpha} \binom{\alpha}{k} |\pi r R \zeta|^{-k} B_{\alpha,k}(2\pi r R |\zeta|), \quad (3.3.3)$$

with

$$B_{\alpha,k}(t) = \frac{\Gamma(\frac{N-2}{2} + k)}{\Gamma(\frac{N-2}{2})} \sum_{j=0}^{\alpha-k} (-1)^{j+k} \binom{\alpha-k}{j} J_{\frac{N-2}{2}-\alpha+k+2j}(t). \quad (3.3.4)$$

Since

$$\tilde{m}(\zeta) = 2\pi R^{\frac{N}{2}} |\zeta|^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(2\pi R |\zeta|),$$

we obtain

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = (2\pi)^{\frac{N}{2}} R^{N-1+\alpha} |\zeta|^\alpha \left(\frac{d}{dt}\right)^\alpha \left[(2\pi t)^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(2\pi t) \right] \Big|_{t=rR|\zeta|}. \quad (3.3.5)$$

It is well-known that for every $\nu \in \mathbb{R}$, we have

$$\frac{d}{dt} J_\nu(t) = \frac{1}{2} (J_{\nu-1}(t) - J_{\nu+1}(t)),$$

which extends to

$$\left(\frac{d}{dt}\right)^\alpha J_\nu(t) = \frac{1}{2^\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} J_{\nu-\alpha+2j}(t).$$

By the Leibniz formula, we thus get

$$\begin{aligned} \left(\frac{d}{dt}\right)^\alpha [t^{-\nu} J_\nu(t)] &= \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu)} t^{-\nu-k} J_\nu^{(\alpha-k)}(t) \\ &= \sum_{k=0}^{\alpha} \sum_{j=0}^{\alpha-k} 2^{-\alpha+k} (-1)^{j+k} \binom{\alpha}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu)} t^{-\nu-k} \binom{\alpha-k}{j} J_{\nu-\alpha+k+2j}(t), \end{aligned}$$

which we can insert into (3.3.5). For $\nu = \frac{N-2}{2}$, combining this equation with (3.3.4) yields

$$\left(\frac{d}{dt}\right)^\alpha \left[(2\pi t)^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(2\pi t) \right] = (2\pi)^\alpha \sum_{k=0}^{\alpha} 2^{-\alpha+k} \binom{\alpha}{k} (2\pi t)^{-\frac{N-2}{2}-k} B_{\alpha,k}(2\pi t).$$

Thus (3.3.5) gives us

$$\begin{aligned} \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) &= (2\pi)^{\frac{N}{2}+\alpha} R^{N-1+\alpha} |\zeta|^\alpha \sum_{k=0}^{\alpha} 2^{-\alpha+k} \binom{\alpha}{k} (2\pi r R |\zeta|)^{-\frac{N-2}{2}-k} B_{\alpha,k}(2\pi r R |\zeta|) \\ &= 2\pi^{\alpha+1} r^{-\frac{N-2}{2}} R^{\frac{N}{2}+\alpha} |\zeta|^{-\frac{N-2}{2}+\alpha} \sum_{k=0}^{\alpha} \binom{\alpha}{k} |\pi r R \zeta|^{-k} B_{\alpha,k}(2\pi r R |\zeta|), \end{aligned}$$

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which is (3.3.3). Since R depends only on N , all the parameters in (3.3.3) depend only on N and α . Since for every half-integer ν , $J_\nu(t) = \mathcal{O}(t^{-1/2})$ as $t \rightarrow \infty$, (i) follows for each fixed $r \in [1, 2]$. Since $\frac{d^\alpha}{dr^\alpha} \tilde{m}(r\zeta)$ is continuous in r and ζ , and $[1, 2]$ is a compact interval, (i) holds true with a bound that is uniform in r .

(ii) From the series expansion for Bessel functions (3.1.22), we argue similarly as for (3.1.23) to get

$$\begin{aligned}
 \tilde{m}(\zeta) &= 2\pi R^{\frac{N}{2}} \frac{|\zeta|^{-\frac{N-2}{2}}}{\Gamma(1/2)} (\pi R|\zeta|)^{\frac{N-2}{2}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2)} \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \\
 &= 2\pi^{\frac{N-1}{2}} R^{N-1} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2)} \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \\
 &= \frac{\Gamma(N/2)}{\Gamma(1/2)} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2)} \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2}.
 \end{aligned} \tag{3.3.6}$$

Hence we have

$$\tilde{m}(\zeta) = 1 + \pi R|\zeta|^2 + \mathcal{O}(|\zeta|^4)$$

as $|\zeta| \rightarrow 0$. Again, it follows that

$$|\tilde{m}(\zeta)| \leq e^{-c'(2\pi R|\zeta|)^2}$$

holds for some $c' > 0$ if $|\zeta|$ is close to 0, say $2\pi R|\zeta| < \varepsilon$ for some $\varepsilon > 0$. Otherwise, the integral representation of Bessel functions gives us

$$\tilde{m}(\zeta) = \frac{\Gamma(N/2)}{\Gamma(\frac{N-1}{2})\sqrt{\pi}} \int_{-1}^1 e^{2\pi i R|\zeta|t} (1-t^2)^{\frac{N-3}{2}} dt. \tag{3.3.7}$$

But we have

$$\begin{aligned}
 \left| \int_{-1}^1 e^{2\pi i R|\zeta|t} (1-t^2)^{\frac{N-3}{2}} dt \right| &\leq 2 \int_0^1 |e^{2\pi i R|\zeta|t} (1-t^2)^{\frac{N-3}{2}}| dt \\
 &\leq 2 \int_0^1 (1-t^2)^{\frac{N-3}{2}} dt \\
 &= \frac{\Gamma(1/2)\Gamma(\frac{N-1}{2})}{\Gamma(N/2)},
 \end{aligned}$$

with equality holding if and only if $\zeta = 0$. Combining this with (3.3.7) leads to

$$|\tilde{m}(\zeta)| < 1$$

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for $\zeta \neq 0$, implying $|\tilde{m}(\zeta)| \leq C' < 1$ for $\varepsilon \leq 2\pi R|\zeta| \leq 2C$. Choosing a positive $c < c'$ such that $C' \leq e^{-c \cdot 4C^2}$ implies

$$|\tilde{m}(r\zeta)| \leq e^{-c(2\pi R|\zeta|)^2}$$

for $2\pi R|\zeta| \leq C$ and $1 \leq r \leq 2$.

(iii) Similarly as in (ii), we get

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = \frac{\Gamma(N/2)}{\Gamma(\frac{N-1}{2})\sqrt{\pi}} (2\pi i R|\zeta|)^\alpha \int_{-1}^1 e^{2\pi i Rr|\zeta|t} t^\alpha (1-t^2)^{\frac{N-3}{2}} dt. \quad (3.3.8)$$

But we have

$$\begin{aligned} \left| \int_{-1}^1 e^{2\pi i Rr|\zeta|t} t^\alpha (1-t^2)^{\frac{N-3}{2}} dt \right| &\leq 2 \int_0^1 t^\alpha (1-t^2)^{\frac{N-3}{2}} dt \\ &= \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N+\alpha}{2})}. \end{aligned}$$

Together with (3.3.8), this gives us

$$\left| \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) \right| \leq (2\pi R|\zeta|)^\alpha \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(N/2)}{\Gamma(1/2)\Gamma(\frac{N+\alpha}{2})}.$$

Hence (iii) follows if we can estimate

$$\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(N/2)}{\Gamma(1/2)\Gamma(\frac{N+\alpha}{2})} < 1,$$

But this can be established using Stirling's formula: For each $x > 0$, we have

$$\left(\frac{2\pi}{x}\right)^{1/2} \left(\frac{x}{e}\right)^x \leq \Gamma(x) \leq \left(\frac{2\pi}{x}\right)^{1/2} \left(\frac{x}{e}\right)^x e^{\frac{1}{12x}},$$

see for instance [14, Proposition IV.1.14]. Since $N \geq 3$ and $\alpha \geq 1$, we thus get

$$\begin{aligned} \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(N/2)}{\Gamma(1/2)\Gamma(\frac{N+\alpha}{2})} &\leq \frac{1}{\sqrt{\pi}} \left(\frac{4\pi}{\alpha+1}\right)^{1/2} \left(\frac{\alpha+1}{2e}\right)^{\frac{\alpha+1}{2}} \left(\frac{4\pi}{N}\right)^{1/2} \\ &\quad \cdot \left(\frac{N}{2e}\right)^{\frac{N}{2}} \left(\frac{N+\alpha}{4\pi}\right)^{1/2} \left(\frac{2e}{N+\alpha}\right)^{\frac{N+\alpha}{2}} e^{\frac{1}{6(\alpha+1)} + \frac{1}{6N}} \\ &= \sqrt{2} \cdot e^{\frac{1}{6(\alpha+1)} + \frac{1}{6N} - \frac{1}{2}} \cdot \underbrace{(\alpha+1)^{\frac{\alpha}{2}} N^{\frac{N-1}{2}} (N+\alpha)^{-\frac{N+\alpha-1}{2}}}_{<1} \\ &< \sqrt{2} \cdot e^{-\frac{13}{36}} < 1. \end{aligned}$$

(iv) By deriving the series expansion (3.3.6) in the proof of (ii), we get

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = (2\pi R|\zeta|)^{2\lfloor \frac{\alpha+1}{2} \rfloor} \sum_{k=\lfloor \frac{\alpha+1}{2} \rfloor}^{\infty} (-1)^k r^{2k-\alpha} \frac{(2\pi R|\zeta|)^{2(k-\lfloor \frac{\alpha+1}{2} \rfloor)}}{(2k-\alpha)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2}.$$

Since $\alpha > 0$, we use that

$$(2\pi R|\zeta|)^{2\lfloor \frac{\alpha+1}{2} \rfloor} \leq (2\pi R|\zeta|)^2,$$

since $2\pi R|\zeta| < 1$. To conclude the proof, we have to estimate

$$\sum_{k=\lfloor \frac{\alpha+1}{2} \rfloor}^{\infty} (-1)^k r^{2k-\alpha} \frac{t^{2(k-\lfloor \frac{\alpha+1}{2} \rfloor)}}{(2k-\alpha)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2} \leq C' < 1 \quad (3.3.9)$$

for $t = 2\pi R|\zeta| < 1$. If α is odd, then $\lfloor \frac{\alpha+1}{2} \rfloor = \frac{\alpha+1}{2}$, and we get

$$\sum_{k=\frac{\alpha+1}{2}}^{\infty} (-1)^k r^{2k-\alpha} \frac{t^{2(k-\frac{\alpha+1}{2})}}{(2k-\alpha)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2} = r \sum_{k=0}^{\infty} (-1)^{k+\frac{\alpha+1}{2}} \frac{(rt)^{2k}}{(2k+1)!} \prod_{j=0}^{k+\frac{\alpha-1}{2}} \frac{j+1/2}{j+N/2}. \quad (3.3.10)$$

But since $rt < 2$, one can use $(rt)^{2k+2} \leq 4(rt)^{2k}$ and $(2k+3)! \geq 6 \cdot (2k+1)!$ for each $k \in \mathbb{N}$ to see that the summands of the alternating series on the right hand side of (3.3.10) are strictly decreasing in k . Hence

$$\left| r \sum_{k=0}^{\infty} (-1)^{k+\frac{\alpha+1}{2}} \frac{(rt)^{2k}}{(2k+1)!} \prod_{j=0}^{k+\frac{\alpha-1}{2}} \frac{j+1/2}{j+N/2} \right| \leq r \prod_{j=0}^{\frac{\alpha-1}{2}} \frac{j+1/2}{j+N/2} \leq \frac{2}{N} < 1.$$

If α is even, then $\lfloor \frac{\alpha+1}{2} \rfloor = \frac{\alpha}{2}$, and similarly, we get

$$\sum_{k=\alpha/2}^{\infty} (-1)^k r^{2k-\alpha} \frac{t^{2k-\alpha}}{(2k-\alpha)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2} = \sum_{k=0}^{\infty} (-1)^{k+\alpha/2} \frac{(rt)^{2k}}{(2k)!} \prod_{j=0}^{k+\alpha/2-1} \frac{j+1/2}{j+N/2}. \quad (3.3.11)$$

Here, a similar argument provides that since $rt < 2$, the summands of the alternating series on the right hand side of (3.3.11) are strictly decreasing from $k \geq 1$.

For a general alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$, where $a_k > 0$ for each $k \in \mathbb{N}$, and $a_k > a_{k+1}$ for each $k \geq 1$, we have

$$a_0 - a_1 < \sum_{k=0}^{\infty} (-1)^k a_k < a_0,$$

so that

$$\left| \sum_{k=0}^{\infty} (-1)^k a_k \right| \leq \max\{a_0, |a_0 - a_1|\} \leq \max\{a_0, a_1\}.$$

Thus

$$\left| \sum_{k=0}^{\infty} (-1)^{k+\frac{\alpha}{2}} \frac{(rt)^{2k}}{(2k)!} \prod_{j=0}^{k+\alpha/2-1} \frac{j+1/2}{j+N/2} \right| \leq \max \left\{ \frac{1}{N}, \frac{(rt)^2}{2!} \frac{1}{N} \frac{3}{N+2} \right\}.$$

But since $N \geq 3$ and $rt < 2$, we have

$$\frac{1}{N} \leq \frac{1}{3} < \frac{2}{3} \quad \text{and} \quad \frac{(rt)^2}{2!} \frac{1}{N} \frac{3}{N+2} < \frac{2}{N} \leq \frac{2}{3},$$

allowing us to estimate

$$\left| \sum_{k=0}^{\infty} (-1)^{k+\frac{\alpha}{2}} \frac{(rt)^{2k}}{(2k)!} \prod_{j=0}^{k+\alpha/2-1} \frac{j+1/2}{j+N/2} \right| \leq \frac{2}{3}.$$

q.e.d.

Besides the Leibniz formula (3.3.2), we need another form of $\frac{d^j}{dr^j} m(r\xi)$ to apply the estimates from Lemma 3.3.1.

Lemma 3.3.2. *For each $j \in \mathbb{N}$, and $a = (a_1, \dots, a_j) \in \{1, \dots, \ell\}^j$, define the finite sequence $\alpha(a) \in \mathbb{N}^\ell$ by*

$$\alpha(a)_k = |\{j' : a_{j'} = k\}|.$$

Then

$$\left(\frac{d}{dr} \right)^j m(r\xi) = \sum_{a \in \{1, \dots, \ell\}^j} \prod_{k=1}^{\ell} \left(\frac{d}{dr} \right)^{\alpha(a)_k} \tilde{m}(r\xi^{(k)}). \quad (3.3.12)$$

Proof. We use induction on j . For the case $j = 0$, we have $\{1, \dots, \ell\}^j = \{\emptyset\}$, and $\alpha(\emptyset) = (0, \dots, 0)$. Hence (3.3.12) is just the product definition of m .

Now assume that (3.3.12) is true for some $j \in \mathbb{N}$. Then

$$\begin{aligned} \left(\frac{d}{dr} \right)^{j+1} m(r\xi) &= \sum_{a \in \{1, \dots, \ell\}^j} \frac{d}{dr} \left[\prod_{k=1}^{\ell} \left(\frac{d}{dr} \right)^{\alpha(a)_k} \tilde{m}(r\xi^{(k)}) \right] \\ &= \sum_{a \in \{1, \dots, \ell\}^j} \sum_{k=1}^{\ell} \left(\frac{d}{dr} \right)^{\alpha(a)_k+1} \tilde{m}(r\xi^{(k)}) \prod_{k' \neq k} \left(\frac{d}{dr} \right)^{\alpha(a)_{k'}} \tilde{m}(r\xi^{(k')}) \\ &= \sum_{a \in \{1, \dots, \ell\}^j} \sum_{k=1}^{\ell} \prod_{k'=1}^{\ell} \left(\frac{d}{dr} \right)^{\alpha(a)_{k'} + \delta_{k,k'}} \tilde{m}(r\xi^{(k')}). \end{aligned}$$

Since for each $a \in \{1, \dots, \ell\}^j$, $k \in \{1, \dots, \ell\}$, the sequence $(a, k) \in \{1, \dots, \ell\}^{j+1}$ has the property

$$\alpha((a, k))_{k'} = \begin{cases} \alpha(a)_{k'} + 1, & k = k', \\ \alpha(a)_{k'}, & k \neq k', \end{cases}$$

it follows that

$$\left(\frac{d}{dr}\right)^{j+1} m(r\xi) = \sum_{a \in \{1, \dots, \ell\}^{j+1}} \prod_{k=1}^{\ell} \left(\frac{d}{dr}\right)^{\alpha(a)_k} \tilde{m}(r\xi^{(k)}).$$

q.e.d.

With this, we can estimate the derivatives of m straight forward to show that we can apply Proposition 2.6.1 and Proposition 2.6.3.

Lemma 3.3.3. *Let $K = \lceil \frac{N-1}{2} \rceil$. Then the following holds.*

(a) *For every $\xi \in \mathbb{R}^n$ and every $r \in [1, 2]$, we have*

$$(1 + |\xi|)^{\frac{N-1}{2}} |m(r\xi)| \leq C_N. \quad (3.3.13)$$

(b) *If N is even, i.e. $K = \frac{N}{2}$, then for every $\xi \in \mathbb{R}^n$, $r \in [1, 2]$, and $L \in \{0, \dots, K\}$, we have*

$$(1 + |\xi|)^{\frac{N-1}{2} - K} \left| \left(\frac{d}{dr}\right)^L m(r\xi) \right| \leq C_N. \quad (3.3.14)$$

(c) *If N is odd, i.e. $K = \frac{N-1}{2}$, then for every $\xi \in \mathbb{R}^n$, $r \in [1, 2]$, and $L \in \{0, \dots, K\}$, we have*

$$\left| \left(\frac{d}{dr}\right)^L m(r\xi) \right| \leq C_N. \quad (3.3.15)$$

Proof. (a) Fix $0 \neq \xi = (\xi^{(1)}, \dots, \xi^{(\ell)})$ and put $t = (t_1, \dots, t_\ell)$, $t_k := 2\pi R |\xi^{(k)}|$, $1 \leq k \leq \ell$. By Lemma 3.3.1 (i), we can find constants $C_{0,N}$ and $\tilde{C}_{0,N}$ such that for $|2\pi R \zeta| \geq C_{0,N}$, we have

$$\tilde{m}(r\zeta) \leq \tilde{C}_{0,N} (2\pi R |\zeta|)^{-\frac{N-1}{2}} \leq C'_{0,N} |\zeta|^{-\frac{N-1}{2}}. \quad (3.3.16)$$

We will assume $C_{0,N}$ to be large enough that if $|2\pi R \zeta| \geq C_{0,N}$, then

$$C'_{0,N} {}^{-\frac{4}{N-1}} |\zeta|^2 \geq 2. \quad (3.3.17)$$

By rearranging the order of variables, we can assume that $\{k : t_k \leq C_{N,0}\} = \{1, \dots, \ell'\}$ for some $\ell' \leq \ell$. We may also assume $0 < \ell' < \ell$, since otherwise, the following estimates hold in a more simple form. Split $\xi = (\xi', \xi'')$, where $\xi' = (\xi^{(1)}, \dots, \xi^{(\ell')})$ and $\xi'' = (\xi^{(\ell'+1)}, \dots, \xi^{(\ell)})$. Since $|\tilde{m}(\zeta)| \leq 1$ for every $\zeta \in \mathbb{R}^N$, we can use Lemma 3.3.1 (ii) to estimate

$$\begin{aligned} (1 + |\xi|)^{\frac{N-1}{2}} |m(r\xi)| &\leq C_N (1 + |\xi'|^{\frac{N-1}{2}} + |\xi''|^{\frac{N-1}{2}}) |m(r\xi)| \\ &\leq C_N \left(1 + |\xi'|^{\frac{N-1}{2}} \prod_{k=1}^{\ell'} |\tilde{m}(r\xi^{(k)})| + |\xi''|^{\frac{N-1}{2}} \prod_{k=\ell'+1}^{\ell} |\tilde{m}(r\xi^{(k)})| \right) \\ &\leq C_N \left(1 + \underbrace{|\xi'|^{\frac{N-1}{2}} e^{-c_N |2\pi R \xi'|^2}}_{\leq C_N} + |\xi''|^{\frac{N-1}{2}} \prod_{k=\ell'+1}^{\ell} (C'_{0,N} |\xi^{(k)}|^{-\frac{N-1}{2}}) \right) \\ &\leq C_N + |\xi''|^{\frac{N-1}{2}} \prod_{k=\ell'+1}^{\ell} (C'_{0,N} |\xi^{(k)}|^{-\frac{N-1}{2}}), \end{aligned}$$

where c_N only depends on $C_{N,0}$. Thus we have to show that

$$|\xi''|^{\frac{N-1}{2}} \leq C_N \prod_{k=\ell'+1}^{\ell} \left(\frac{|\xi^{(k)}|^{\frac{N-1}{2}}}{C'_{0,N}} \right).$$

By taking the $\frac{4}{N-1}$ -th powers of both sides, this is equivalent to

$$C'_{0,N}{}^{\frac{4}{N-1}} \frac{|\xi''|^2}{C'_{0,N}{}^{\frac{4}{N-1}}} \leq C_N{}^{\frac{4}{N-1}} \prod_{k=\ell'+1}^{\ell} \frac{|\xi^{(k)}|^2}{C'_{0,N}{}^{\frac{4}{N-1}}}. \quad (3.3.18)$$

But from (3.3.17), this estimate follows with $C_N = C'_{0,N}$, since for any sequence $(a_k)_{k \in \mathbb{N}}$ with $a_k \geq 2$ and any $M \in \mathbb{N}$, we have

$$\sum_{k=0}^M a_k \leq \prod_{k=0}^M a_k.$$

Thus (3.3.18) holds with $C_N \geq C'_{0,N}$, which shows (3.3.13).

(b) The proof of (3.3.14) is more technical. By Lemma 3.3.1 (i), we can find a constant A_N such that for $|2\pi R\zeta| > A_N$ and $0 \leq j < K$,

$$\left| \left(\frac{d}{dr} \right)^j \tilde{m}(r\zeta) \right| \leq \frac{1}{2}, \quad (3.3.19)$$

since $j - \frac{N-1}{2} < 0$ for each such j . By the Leibniz formula, we get

$$\left(\frac{d}{dr} \right)^L m(r\xi) = \sum_{j=0}^L \binom{L}{j} \left(\frac{d}{dr} \right)^j \left[\prod_{t_k \leq A_N} \tilde{m}(r\xi^{(k)}) \right] \cdot \left(\frac{d}{dr} \right)^{L-j} \left[\prod_{t_k > A_N} \tilde{m}(r\xi^{(k)}) \right]. \quad (3.3.20)$$

Once again, we can assume that $\{k : t_k \leq A_N\} = \{1, \dots, \ell'\}$ for some $\ell' \leq \ell$. Then we have

$$\begin{aligned} & (1 + |\xi|)^{\frac{N-1}{2} - K} \left(\frac{d}{dr} \right)^L m(r\xi) \\ &= \sum_{j=0}^L \binom{L}{j} \left(\frac{d}{dr} \right)^j \left[\prod_{k=1}^{\ell'} \tilde{m}(r\xi^{(k)}) \right] \cdot (1 + |\xi|)^{-1/2} \left(\frac{d}{dr} \right)^{L-j} \left[\prod_{k=\ell'+1}^{\ell} \tilde{m}(r\xi^{(k)}) \right]. \end{aligned}$$

We thus have to find bounds

$$(1 + |\xi|)^{-1/2} \left| \left(\frac{d}{dr} \right)^j \left[\prod_{k=\ell'+1}^{\ell} \tilde{m}(r\xi^{(k)}) \right] \right| \leq C_N \quad (3.3.21)$$

and

$$\left| \left(\frac{d}{dr} \right)^j \left[\prod_{k=1}^{\ell'} \tilde{m}(r\xi^{(k)}) \right] \right| \leq C_N \quad (3.3.22)$$

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for each $j \in \{0, \dots, L\}$, with C_N being independent of ℓ and ℓ' . We can assume $L = K$ and $j > 0$ for the rest of the proof. As of (3.3.22), we shall also assume that for some $\ell'' \leq \ell'$, we have $\{k : t_k < 1\} = \{1, \dots, \ell''\}$. Then by using the Leibniz formula as in (3.3.20), we can establish (3.3.22) by estimating

$$\left| \left(\frac{d}{dr} \right)^j \left[\prod_{k=\ell'+1}^{\ell'} \tilde{m}(r\xi^{(k)}) \right] \right| \leq C_N \quad (3.3.22')$$

and

$$\left| \left(\frac{d}{dr} \right)^j \left[\prod_{k=1}^{\ell''} \tilde{m}(r\xi^{(k)}) \right] \right| \leq C_N, \quad (3.3.23)$$

$j \in \{1, \dots, K\}$, C_N independent of ℓ , ℓ' , and ℓ'' . One further assumption will be $\ell'' < \ell' < \ell$, since otherwise, at least one of our required estimates would be trivial. We will now show (3.3.21). First, we have

$$\begin{aligned} \left(\frac{d}{dr} \right)^j \left[\prod_{k=\ell'+1}^{\ell} \tilde{m}(r\xi^{(k)}) \right] &= \sum_{\substack{\alpha \in \mathbb{N}^{\ell-\ell'} \\ |\alpha|=j}} \binom{j}{\alpha} \prod_{k=1}^{\ell-\ell'} \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\xi^{(k+\ell')}) \\ &= \sum_{\substack{|\alpha|=j \\ \forall k \alpha_k < K}} \binom{j}{\alpha} \prod_{k=1}^{\ell-\ell'} \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\xi^{(k+\ell')}) \\ &\quad + \sum_{\substack{|\alpha|=j \\ \exists k \alpha_k = K}} \binom{j}{\alpha} \prod_{k=1}^{\ell-\ell'} \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\xi^{(k+\ell')}). \end{aligned} \quad (3.3.24)$$

Due to the choice of A_N , (3.3.19) and the multinomial theorem yield

$$\underbrace{(1 + |\xi|)^{-1/2}}_{\leq 1} \sum_{\substack{|\alpha|=j \\ \forall k \alpha_k < K}} \binom{j}{\alpha} \prod_{k=1}^{\ell-\ell'} \underbrace{\left| \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\xi^{(k+\ell')}) \right|}_{\leq 1/2} \leq \sum_{\substack{|\alpha|=j \\ \forall k \alpha_k < K}} \binom{j}{\alpha} \frac{1}{2^{\ell-\ell'}} < \frac{(\ell - \ell')^j}{2^{\ell-\ell'}} < C. \quad (3.3.25)$$

The second sum in (3.3.24) is equal to 0 if $j < K$. If $j = K$, we use part (i) of Lemma 3.3.1 to get

$$\begin{aligned} &(1 + |\xi|)^{-1/2} \sum_{\substack{|\alpha|=j \\ \exists k \alpha_k = K}} \binom{j}{\alpha} \prod_{k=1}^{\ell-\ell'} \left| \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\xi^{(k+\ell')}) \right| \\ &= (1 + |\xi|)^{-1/2} \sum_{k=\ell'+1}^{\ell} \left| \left(\frac{d}{dr} \right)^j \tilde{m}(r\xi^{(k)}) \right| \cdot \prod_{\substack{k' > \ell' \\ k' \neq k}} \underbrace{|\tilde{m}(r\xi^{(k')})|}_{\leq 1/2} \\ &\leq \sum_{k=\ell'+1}^{\ell} (1 + |\xi|)^{-1/2} \frac{\tilde{C}_N |\xi^{(k)}|^{1/2}}{2^{\ell-\ell'-1}} \end{aligned}$$

$$\begin{aligned}
 &< \frac{\tilde{C}_N(\ell - \ell')}{2^{\ell - \ell' - 1}} \\
 &< C_N.
 \end{aligned} \tag{3.3.26}$$

Combining (3.3.24), (3.3.25), and (3.3.26) gives us (3.3.21).

Coming to (3.3.22'), let $t' = (t_{\ell''+1}, \dots, t_{\ell'})$. By the multinomial theorem and Lemma 3.3.1 (ii) and (iii), we get

$$\begin{aligned}
 \left| \left(\frac{d}{dr} \right)^j \left[\prod_{k=\ell''+1}^{\ell'} \tilde{m}(r\xi^{(k)}) \right] \right| &= \sum_{\substack{\alpha \in \mathbb{N}^{\ell' - \ell''} \\ |\alpha| = j}} \binom{j}{\alpha} \prod_{k=1}^{\ell' - \ell''} \left| \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\xi^{(k+\ell'')}) \right| \\
 &\leq \sum_{|\alpha|=j} \binom{j}{\alpha} \prod_{k=1}^{\ell' - \ell''} t_{k+\ell''}^{2\alpha_k} e^{-c_N t_{k+\ell''}^2} \\
 &= e^{-c_N |t'|^2} |t'|^{2j} \\
 &< C_N.
 \end{aligned}$$

To show (3.3.23), we apply Lemma 3.3.2 to get

$$\left(\frac{d}{dr} \right)^j \left[\prod_{k=1}^{\ell''} \tilde{m}(r\xi^{(k)}) \right] = \sum_{a \in \{1, \dots, \ell''\}^j} \prod_{k=1}^{\ell''} \left(\frac{d}{dr} \right)^{\alpha(a)_k} \tilde{m}(r\xi^{(k)}).$$

Let $t'' = (t_1, \dots, t_{\ell''})$. Lemma 3.3.1 (ii) and (iv) now lead to

$$\left| \left(\frac{d}{dr} \right)^j m(r\xi) \right| \leq \sum_{a \in \{1, \dots, \ell''\}^j} e^{-c_N |t''|^2} \prod_{\alpha(a)_k \neq 0} t_k^2. \tag{3.3.27}$$

Note that for all $a \in \{1, \dots, \ell''\}^j$, we have $k \in \text{ran}(a)$ iff $\alpha(a)_k \neq 0$, where

$$\text{ran}(a) = \{a_1, \dots, a_j\}$$

is the range of a . To conclude the estimate (3.3.14), we will partition the set $\{1, \dots, \ell''\}^j$ as follows. Since we assumed $j \geq 1$, we have $1 \leq |\text{ran}(a)| \leq j$ for each $a \in \{1, \dots, \ell''\}^j$. For $i \in \{1, \dots, j\}$, put

$$A_i := \{a \in \{1, \dots, \ell''\}^j : |\text{ran}(a)| = i\}.$$

Then for any given set $S \subset \{1, \dots, \ell''\}$ with $|S| = i$, the number of sequences $a \in A_i$ with $\text{ran}(a) = S$ is equal to the number of surjective maps from $\{1, \dots, j\}$ to $\{1, \dots, i\}$. This number only depends on i and j , and is thus smaller than some constant C_N . These

properties yield

$$\begin{aligned}
 \sum_{a \in A_i} \prod_{\alpha(a)_k \neq 0} t_k^2 &= \sum_{\substack{S \subset \{1, \dots, \ell''\} \\ |S|=i}} \sum_{\substack{a \in A_i \\ \text{ran}(a)=S}} \prod_{k \in \text{ran}(a)} t_k^2 \\
 &\leq C_N \sum_{\substack{S \subset \{1, \dots, \ell''\} \\ |S|=i}} \prod_{k \in S} t_k^2 \\
 &\leq C_N \sum_{k_1, \dots, k_i=1}^{\ell''} t_{k_1}^2 \cdots t_{k_i}^2.
 \end{aligned}$$

Hence by (3.3.27), summing over i gives us

$$\begin{aligned}
 \left(\frac{d}{dr} \right)^j \left[\prod_{k=1}^{\ell''} \tilde{m}(r\xi^{(k)}) \right] &\leq e^{-c_N |t''|^2} \sum_{i=1}^j \sum_{a \in A_i} \prod_{\alpha(a)_k \neq 0} t_k^2 \\
 &\leq C_N e^{-c_N |t''|^2} \left(\sum_{k_1=1}^{\ell''} t_{k_1}^2 + \sum_{k_1, k_2=1}^{\ell''} t_{k_1}^2 t_{k_2}^2 + \dots + \sum_{k_1, \dots, k_j=1}^{\ell''} t_{k_1}^2 \cdots t_{k_j}^2 \right) \\
 &= C_N e^{-c_N |t''|^2} \sum_{k=1}^j |t''|^{2k} \\
 &< C_N.
 \end{aligned}$$

This concludes the proof for $K = N/2$.

(c) The arguments for part (c) are basically the same as for part (b). We also have to estimate (3.3.22) as before, but we have to drop the factor $(1 + |\xi|)^{-1/2}$ in (3.3.21). Then the only difference is that the sum

$$\sum_{k=\ell'+1}^{\ell} (1 + |\xi|)^{-1/2} \frac{\tilde{C}_N |\xi^{(k)}|^{1/2}}{2^{\ell-\ell'-1}}$$

in (3.3.26) becomes

$$\sum_{k=\ell'+1}^{\ell} \frac{\tilde{C}_N}{2^{\ell-\ell'-1}},$$

which makes the estimate even slightly easier. **q.e.d.**

Now, we are able to show that $\sigma_S - P dx$ fulfills (2.5.13) and (2.5.14), which allows us to apply Carbery's interpolation argument.

Lemma 3.3.4. *There is a constant C_N such that for every $\xi \in \mathbb{R}^n$, $\theta \in S^{n-1}$, and $u \in \mathbb{R} \setminus \{0\}$, we have*

$$|\xi| |m(\xi)| \leq C_N, \quad |\langle \theta, \nabla m(u\theta) \rangle| \leq C_N, \quad \text{and} \quad |\langle \xi, \nabla m(\xi) \rangle| \leq C. \quad (3.3.28)$$

3.3 Higher Fourier derivatives of spherical measures

Consequently, there is a constant C_N such that for every $\theta \in S^{n-1}$ and every $u \in \mathbb{R} \setminus \{0\}$, we have

$$|m(u\theta) - \hat{P}(u\theta)| \leq C_N \cdot \min\{|u|, |u|^{-1}\}, \quad (3.3.29)$$

$$|\langle \theta, \nabla(m - \hat{P})(u\theta) \rangle| \leq C_N \cdot \min\{1, |u|^{-1}\}. \quad (3.3.30)$$

Proof. Since $N \geq 3$, Lemma 3.3.3 (a) yields

$$|\xi| |m(\xi)| \leq (1 + |\xi|)^{\frac{N-1}{2}} |m(\xi)| \leq C_N.$$

For the second inequality, we have

$$\langle \theta, \nabla m(u\theta) \rangle = \sum_{k=1}^{\ell} \langle \theta^{(k)}, \tilde{m}(u\theta^{(k)}) \rangle \prod_{j \neq k} \tilde{m}(u\theta^{(j)}).$$

Thus by Lemma 3.3.1 (iii), we get

$$|\langle \theta, \nabla m(u\theta) \rangle| \leq \sum_{k=1}^{\ell} |\langle \theta^{(k)}, \tilde{m}(u\theta^{(k)}) \rangle| \leq C(2\pi R |\theta^{(k)}|^2) = C2\pi R^2 = C_N.$$

The last estimate in (3.3.28) follows from Lemma 3.3.3 (b) and (c), since we have

$$\langle \xi, \nabla m(\xi) \rangle = \frac{d}{dt} m(t\xi)|_{t=1}.$$

For N even, the estimate Lemma 3.3.3 (b) does not seem to provide the estimate

$$\left| \frac{d}{dt} m(t\xi)|_{t=1} \right| \leq C_N,$$

but an inspection of the proof yields that, using the notation of Lemma 3.3.3, the factor $(1 + |\xi|)^{\frac{N-1}{2}-K}$ in (3.3.14) can be omitted if $L < K = N/2 \geq 2$.

Regarding (2.3.2) and (2.3.3), it follows that

$$|1 - m(\xi)| \leq C_N |\xi|.$$

From (2.3.13), we can deduce that

$$|u \cdot \hat{P}(u\theta)| \leq C, \quad |\langle \theta, \nabla \hat{P}(t\theta) \rangle| \leq C, \quad \text{and} \quad |\langle \theta, \nabla \hat{P}(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\}.$$

To show (3.3.29), we first consider the case $|u| > 1$. Then

$$|m(u\theta) - \hat{P}(u\theta)| \leq |m(u\theta)| + |\hat{P}(u\theta)| \leq C_N |u|^{-1} + C |u|^{-1}.$$

If $|u| \leq 1$, we have

$$|m(u\theta) - \hat{P}(u\theta)| \leq |1 - m(u\theta)| + |1 - \hat{P}(u\theta)| \leq C_N |u| + C |u|,$$

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so that (3.3.29) follows. For (3.3.30), we also consider the case $|u| > 1$ first. Then

$$|u| |\langle \theta, \nabla(m - \hat{P})(u\theta) \rangle| \leq |\langle u\theta, \nabla m(u\theta) \rangle| + |\langle u\theta, \nabla \hat{P}(u\theta) \rangle| \leq C_N,$$

while for $|u| \leq 1$,

$$|\langle \theta, \nabla(m - \hat{P})(u\theta) \rangle| \leq |\langle \theta, \nabla m(u\theta) \rangle| + |\langle \theta, \nabla \hat{P}(u\theta) \rangle| \leq C_N.$$

This concludes the estimates.

q.e.d.

This leaves us with finding some α with $1/p < \alpha < 1$ such that m_α is an L^p -multiplier. If N is odd, $\frac{N-1}{2}$ is an integer, and we can apply Lemma 2.6.1 to interpolate between fractional derivatives. Otherwise, N is even and we apply Proposition 2.6.3 instead. This proves Proposition 3.2.1 and thus Theorems 1 and 2.

4 Variational bounds for products Euclidean balls and spheres

In the final chapter of this work, we will use the results from [9] along with Theorems 1 and 2 to prove Theorems 3 and 4.

4.1 Stein's arguments and decoupling of high and low dimensions for r -variations

The basic idea of this section is similar to what we did in Section 3.2, by showing that we can replace each ball B_k by the sphere ∂B_k , and that we can increase the dimensions of our spheres again, using a generalization of Stein's rotation method. This means that Theorem 4 only needs to be proven for $(S^{N-1})^\ell$, with a constant that may even depend on N (but not on ℓ).

However, the attempt to consider high and low dimensions separately and replace the high-dimensional balls by spheres bears a major difficulty: If, for some fixed N , we put again $B = B' \times B''$ with $B' = \prod_{n_k < N} B_k$ and $B'' = \prod_{n_k \geq N} B_k$, we need to estimate

$$\|V_r(M_t^{B'} M_t^{B''} f : t > 0)\|_{p \rightarrow p}, \quad (4.1.1)$$

where $M_t^{B'}$ and $M_t^{B''}$ are applied to their respective range of variables. In the case of maximal functions, we could simply consider the iterated maximal operator $M_{B'} M_{B''}$, but we cannot compare (4.1.1) to the iterated r -variation $V_r(V_r(T_t U_s f : s > 0) : t > 0)$, and we have no direct way to estimate (4.1.1) against the r -variations for $(M_t^{B'})_{t>0}$ and $(M_t^{B''})_{t>0}$. We can work around this issue by showing first that we may replace $B' \times B''$ by $B' \times S$, with $S = (S^{N-1})^{\ell''}$ and $\ell'' = |\{k \in \{1, \dots, \ell\} : n_k \geq N\}|$, and by applying some more refined estimates than the essential Lemmas 4.1 and 4.2 in [9].

We start by showing an r -variational version of Minkowski's integral inequality.

Lemma 4.1.1. *Let (X, μ) be a σ -finite measure space, $1 \leq r < \infty$, $Z \subset \mathbb{R}$, and $(a_t)_{t \in Z}$ be a family of μ -integrable functions from X to \mathbb{C} . Moreover, assume that the function $x \mapsto V_r(a_t(x) : t \in Z)$ is also μ -integrable. Then we have*

$$V_r\left(\int_X a_t(x) \, d\mu(x) : t \in Z\right) \leq \int_X V_r(a_t(x) : t \in Z) \, d\mu(x). \quad (4.1.2)$$

Proof. Fix any finite sequence $t_0 < \dots < t_L$, $t_j \in Z$, and for $j \in \{1, \dots, L\}$, define $f_j: X \rightarrow \mathbb{C}$, $f_j(x) = a_{t_j}(x) - a_{t_{j-1}}(x)$. Then we can use Minkowski's integral inequality for $\ell^r(\{1, \dots, L\})$ to estimate

$$\begin{aligned} & \left(\sum_{j=0}^{L-1} \left| \int_X a_{t_{j+1}}(x) d\mu(x) - \int_X a_{t_j}(x) d\mu(x) \right|^r \right)^{1/r} \\ &= \left(\sum_{j=1}^L \left| \int_X f_j(x) d\mu(x) \right|^r \right)^{1/r} \\ &\leq \int_X \left(\sum_{j=1}^L |f_j(x)|^r \right)^{1/r} d\mu(x) \\ &= \int_X \left(\sum_{j=0}^{L-1} |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r} d\mu(x). \end{aligned}$$

Thus we get

$$\begin{aligned} V_r \left(\int_X a_t(x) d\mu(x) : t \in Z \right) &= \sup_{\substack{t_0 < \dots < t_L \\ t_j \in Z, L \in \mathbb{N}}} \left(\sum_{j=0}^{L-1} \left| \int_X a_{t_{j+1}}(x) d\mu(x) - \int_X a_{t_j}(x) d\mu(x) \right|^r \right)^{1/r} \\ &\leq \sup_{\substack{t_0 < \dots < t_L \\ t_j \in Z, L \in \mathbb{N}}} \int_X \left(\sum_{j=0}^{L-1} |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r} d\mu(x) \\ &\leq \int_X \sup_{\substack{t_0 < \dots < t_L \\ t_j \in Z, L \in \mathbb{N}}} \left(\sum_{j=0}^{L-1} |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r} d\mu(x) \\ &= \int_X V_r(a_t(x) : t \in Z) d\mu(x), \end{aligned}$$

concluding the proof. **q.e.d.**

From this point on, fix $B = B_1 \times \dots \times B_\ell$ as in Theorem 3, $N \geq 3$ to be chosen later (depending on p , $1 < p < \infty$), write $S_k = \partial B_k$ for each k , and assume again without loss of generality that $B = B' \times B''$, where

$$B' = \prod_{n_k < N} B_k, \quad B'' = \prod_{n_k \geq N} B_k, \quad S'' = \prod_{n_k \geq N} S_k.$$

Furthermore, let $n' = \dim B'$, $n'' = \dim B''$, $\ell' = |\{k : n_k < N\}|$, $\ell'' = \ell - \ell'$, and $S = (S^{N-1})^{\ell''}$. Write

$$H := B' \times S, \quad H'' := B' \times S''.$$

Furthermore, let $d := n' + N\ell'' = \dim H + \ell''$. The aim of this section is to show that it suffices to bound

$$\|V_r(M_t^H f : t > 0)\|_{L^p(\mathbb{R}^d)} \leq C_{p,N} \|f\|_{L^p(\mathbb{R}^d)}, \quad (4.1.3)$$

4.1 Stein's arguments and decoupling of high and low dimensions for r -variations

where

$$M_t^H f(x) = M_t^{B'} M_t^S f(x', x'') = \frac{1}{|B'|} \int_{B'} \frac{1}{|S|} \int_S f(x' + ty, x'' + t\omega) d\sigma_S(\omega) dy.$$

Lemma 4.1.2. *Let $1 < p < \infty$, $1 \leq r < \infty$, and assume that there is a constant C such that for every $f \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|V_r(M_t^{H''} f : t > 0)\|_p \leq C \|f\|_p.$$

Then for any $s \in [0, 1]^{\ell''}$, we have

$$\|V_r(M_t^{B'} M_{ts}^{S''} f : t > 0)\|_p \leq C \|f\|_p, \quad (4.1.4)$$

and furthermore,

$$\|V_r(M_t^B f : t > 0)\|_p \leq C \|f\|_p, \quad (4.1.5)$$

both with the same constant C .

Proof. Similar arguments as in Section 3.2 provide that

$$\begin{aligned} & \|V_r(M_t^{B'} M_{ts}^{S''} f : t > 0)\|_p^p \\ &= \int_{\mathbb{R}^n} V_r \left(\frac{1}{|B'|} \int_{B'} \frac{1}{|S''|} \int_{S''} f(x' + ty, x'' + ts.\omega) d\sigma_{S''}(\omega) dy : t > 0 \right)^p dx \\ &= \prod_{k=\ell'+1}^{\ell} s_k^{n_k} \int_{\mathbb{R}^n} V_r \left(\frac{1}{|B'|} \int_{B'} \frac{1}{|S''|} \int_{S''} f(x' + ty, s.x'' + t(s.\omega)) d\sigma_S(\omega) dy : t > 0 \right)^p dx \\ &\leq C^p \prod_{k=\ell'+1}^{\ell} s_k^{n_k} \int_{\mathbb{R}^n} |f(x', s.x'')|^p dx \\ &= C^p \|f\|_p^p, \end{aligned}$$

which proves (4.1.4). As for (4.1.5), we use the fact that

$$\int_{[0,1]^{\ell''}} \prod_{k=\ell'+1}^{\ell} n_k s_k^{n_k-1} ds = 1.$$

By the identity (3.2.5) and Lemma 4.1.1, we have

$$V_r(M_t^B f(x) : t > 0) \leq \int_{[0,1]^{\ell''}} \prod_{k=\ell'+1}^{\ell''} n_k s_k^{n_k-1} V_r(M_t^{B'} M_{ts}^{S''} f(x) : t > 0) ds.$$

Thus we can apply Minkowski's integral inequality to get

$$\|V_r(M_t^B f : t > 0)\|_p \leq \int_{[0,1]^{\ell''}} \prod_{k=\ell'+1}^{\ell''} n_k s_k^{n_k-1} \|V_r(M_t^{B'} M_{ts}^{S''} f : t > 0)\|_p ds \leq C \|f\|_p,$$

just as in Section 3.2.

q.e.d.

4 Variational bounds for products Euclidean balls and spheres

As we stated at the beginning of this section, the increase of the dimension of any spherical component also works very similar to before.

Lemma 4.1.3. *Assume that there are $1 < p \leq \infty$, $1 \leq r < \infty$, and a constant $C > 0$ such that*

$$\|V_r(M_t^H f : t > 0)\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. Then also

$$\|V_r(M_t^{H''} f : t > 0)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (4.1.6)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, with the same constant C .

Proof. The proof is very similar to Lemma 3.2.4, with S'' in place of S' and ℓ'' in place of ℓ . Define the operators M_t' on $\mathcal{S}(\mathbb{R}^{n''})$ as in (3.2.8), put $G := O(n_{\ell'+1}) \times \cdots \times O(n_\ell)$, and define M_t^ρ on $\mathcal{S}(\mathbb{R}^{n''})$ as in (3.2.10) for each $\rho \in G$. By our assumption, we have

$$\|V_r(M_t^{B'} M_t^\rho : t > 0)\|_{p \rightarrow p} = \|V_r(M_t^{B'} M_t' : t > 0)\|_{p \rightarrow p} \leq C$$

for each $\rho \in G$, and thus by (3.2.12), Lemma 4.1.1, and Minkowski's inequality, we get

$$\begin{aligned} \|V_r(M_t^{H''} f : t > 0)\|_{L^p(\mathbb{R}^n)} &\leq \int_G \|V_r(M_t^{B'} M_t^\rho f : t > 0)\|_{L^p(\mathbb{R}^n)} d\lambda(\rho) \\ &\leq \int_G C \|f\|_{L^p(\mathbb{R}^n)} d\lambda(\rho) \\ &= C \|f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

with the same C .

q.e.d.

It thus suffices to show the following proposition.

Proposition 4.1.4. *Let $N \geq 3$, $\ell', \ell'' \in \mathbb{N}$, $B' = B_1 \times \cdots \times B_{\ell'}$, where for each $k \in \{1, \dots, \ell'\}$, $B_k \subset \mathbb{R}^{n_k}$ is a Euclidean ball of dimension $n_k < N$, let $S = (S^{N-1})^{\ell''}$, and $H = B' \times S$. Furthermore, let $n' = n_1 + \cdots + n_{\ell'} = \dim B'$ and $d = n' + N\ell''$. If $\frac{N}{N-1} < p < N$, we have*

$$\|V_r(M_t^H f : t > 0)\|_{L^p(\mathbb{R}^d)} \leq C_{p,r,N} \|f\|_{L^p(\mathbb{R}^d)} \quad (4.1.7)$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$, where $C_{p,r,N}$ only depends on p , r , and N .

4.2 The long variation estimate for H

The long variation seminorm for $M_t^B f$ has already been bounded for every $p > 1$ in [9, Theorem 1.2], but for the proof of Theorems 3 and 4, we also need to bound

$$V_r(M_{2^j}^H f : j \in \mathbb{Z}).$$

For the long variation, the essential argument in [9] uses that the multiplier $\widehat{\chi}_B - \widehat{P}_{L(B)}$, where P is the Poisson kernel, satisfies the conditions

$$|\widehat{\chi}_B(\xi) - \widehat{P}_L(\xi)| \leq C \cdot \min\{L|\xi|, (L|\xi|)^{-1}\}, \quad |\langle \xi, \nabla(\widehat{\chi}_B - \widehat{P}_L)(\xi) \rangle| \leq C, \quad (4.2.1)$$

which follow from (2.3.2) and (2.3.13). Write $\xi = (\xi', \xi'')$ and let

$$m_1(\xi') := \frac{1}{|B'|} \widehat{\chi}_{B'}(\xi'), \quad m_2(\xi'') := \frac{1}{|S|} \widehat{\sigma}_S(\xi''), \quad m(\xi) := m_1(\xi') m_2(\xi''),$$

so that m is the Fourier transform of the normalized measure that underlies H . We can show that m satisfies conditions similar to (4.2.1).

Lemma 4.2.1. *There is a constant C_N such that for every $\theta \in \mathbb{S}^{d-1}$ and every $u \in \mathbb{R} \setminus \{0\}$, we have*

$$|m(u\theta) - \widehat{P}(u\theta)| \leq C_N \cdot \min\{|u|, |u|^{-1}\}, \quad (4.2.2)$$

$$|\langle \theta, \nabla(m - \widehat{P})(u\theta) \rangle| \leq C_N \cdot \min\{1, |u|^{-1}\}. \quad (4.2.3)$$

Proof. By Lemma 3.3.4, it follows that we only need to verify that m satisfies the estimates (3.3.28), i.e.

$$|\xi| |m(\xi)| \leq C_N, \quad |\langle \theta, \nabla m(u\theta) \rangle| \leq C_N, \quad |\langle \xi, \nabla m(\xi) \rangle| \leq C \quad (4.2.4)$$

for every $\xi \in \mathbb{R}^d$, $\theta \in S^{d-1}$, and $u \in \mathbb{R} \setminus \{0\}$. From Section 3.1, we know that $L(B')$ does not depend on ℓ' or N , and by (4.2.1), m_1 thus fulfills (4.2.4). From Lemma 3.3.4, we know that m_2 also fulfills (4.2.4). Write $\xi = u\theta$. We can achieve the first estimate in (4.2.4) by

$$|\xi| |m(\xi)| \leq |\xi'| |m_1(\xi')| |m_2(\xi'')| + |\xi''| |m_1(\xi')| |m_2(\xi'')| \leq |\xi'| |m_1(\xi')| + |\xi''| |m_2(\xi'')| \leq C_N,$$

the third estimate by

$$\begin{aligned} |\langle \xi, \nabla m(\xi) \rangle| &= |m_2(\xi'') \langle \xi', \nabla m_1(\xi') \rangle + m_1(\xi') \langle \xi'', \nabla m_2(\xi'') \rangle| \\ &\leq |\langle \xi', \nabla m_1(\xi') \rangle| + |\langle \xi'', \nabla m_2(\xi'') \rangle| \\ &\leq 2C, \end{aligned}$$

and the second estimate, writing $\theta = (\theta', \theta'')$ through

$$\begin{aligned} |\langle \theta, \nabla m(u\theta) \rangle| &\leq |\langle \theta', \nabla m_1(u\theta') \rangle| + |\langle \theta'', \nabla m_2(u\theta'') \rangle| \\ &= \left| |\theta'| \left\langle \frac{\theta'}{|\theta'|}, \nabla m_1\left(\left(u|\theta'\right) \frac{\theta'}{|\theta'|}\right) \right\rangle \right| + \left| |\theta''| \left\langle \frac{\theta''}{|\theta''|}, \nabla m_2\left(\left(u|\theta''\right) \frac{\theta''}{|\theta''|}\right) \right\rangle \right| \\ &\leq (|\theta'| + |\theta''|) C_N \\ &\leq 2C_N. \end{aligned}$$

q.e.d.

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With the estimates from Lemma 4.2.1, the proof for the long variation will basically be the same as in [9], just with $M_{2^j}^H$ instead of $M_{2^j}^B$ and with P instead of P_L . As described in Section 2.4, we set

$$\widehat{S_j f}(\xi) = \widehat{P_{2^{j+1}} f}(\xi) - \widehat{P_{2^j} f}(\xi) = (e^{-2\pi 2^{-j-1}|\xi|} - e^{-2\pi 2^{-j}|\xi|}) \widehat{f}(\xi).$$

We will include the proof of the long variation estimate for the sake of completeness, but a reader familiar with the original proof can skip the rest of this section.

Lemma 4.2.2. *Let $1 < p < \infty$, $2 < r < \infty$. Then there is a constant $C_{p,r}$ such that for every $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|V_r(M_{2^j}^H f : j \in \mathbb{Z})\|_{L^p} \leq C_{p,r} \|f\|_{L^p}. \quad (4.2.5)$$

Proof. The proof relies on the maximal and variational estimates on semigroups, see Section 2.9, and on the maximal lacunary result

$$\left\| \sup_{j \in \mathbb{Z}} M_{2^j}^H f \right\|_p \leq C_{p,\infty} \|f\|_p, \quad (4.2.6)$$

which follows from Lemma 4.2.1 and Proposition 2.4.1. We have

$$\|V_r(M_{2^j}^H f : j \in \mathbb{Z})\|_p \leq \|V_r(P_{2^j} f : j \in \mathbb{Z})\|_p + \|V_r(M_{2^j}^H f - P_{2^j} f : j \in \mathbb{Z})\|_p,$$

and since $r > 2$, (1.2.11) and the triangle inequality show that

$$\begin{aligned} \|V_r(M_{2^j}^H f - P_{2^j} f : j \in \mathbb{Z})\|_p &\leq \|V_2(M_{2^j}^H f - P_{2^j} f : j \in \mathbb{Z})\|_p \\ &\leq 2 \left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H f - P_{2^j} f|^2 \right)^{1/2} \right\|_p \\ &\leq 2 \sum_{k \in \mathbb{Z}} \left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H S_{j+k} f - P_{2^j} S_{j+k} f|^2 \right)^{1/2} \right\|_p. \end{aligned} \quad (4.2.7)$$

Due to the variational semigroup estimate, we only have to bound the last sum in (4.2.7). For that purpose, we will show that there is some $\delta_p > 0$ such that for every $k \in \mathbb{Z}$,

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H S_{j+k} f - P_{2^j} S_{j+k} f|^2 \right)^{1/2} \right\|_p \leq C_p 2^{-\delta_p |k|} \|f\|_p. \quad (4.2.8)$$

This can be achieved by interpolating an L^p -bound of the form

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H S_{j+k} f|^2 \right)^{1/2} \right\|_p + \left\| \left(\sum_{j \in \mathbb{Z}} |P_{2^j} S_{j+k} f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p \quad (4.2.9)$$

with a favourable L^2 -bound

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H S_{j+k} f - P_{2^j} S_{j+k} f|^2 \right)^{1/2} \right\|_2 \leq C 2^{-\frac{|j|}{2}} \|f\|_2. \quad (4.2.10)$$

The inequality (4.2.8) follows directly from that. For (4.2.9), we establish a vector-valued bound

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H g_j|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^2 \right)^{1/2} \right\|_p \quad (4.2.11)$$

for every family of functions $(g_j)_{j \in \mathbb{Z}} \in L^2(\mathbb{R}^{N\ell}, \ell^2(\mathbb{Z}))$. For $1 < p < \infty$ and $1 \leq s \leq \infty$, let $A(p, s)$ the best constant such that

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H g_j|^s \right)^{1/s} \right\|_p \leq A(p, s) \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^s \right)^{1/s} \right\|_p$$

holds. Since $M_t^H = (M_t^H)^*$ for every $t > 0$, duality then yields $A(p, s) = A(p', s')$. Due to (4.2.6), we thus have

$$A(p, 2) \leq (A(p, 1)A(p, \infty))^{1/2} = (A(p', \infty)A(p, \infty))^{1/2} = C_{p', \infty}^{1/2} C_{p, \infty}^{1/2},$$

and (4.2.11) follows. By similar means, the maximal semigroup estimate implies that

$$\left\| \left(\sum_{j \in \mathbb{Z}} |P_{2^j} g_j|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^2 \right)^{1/2} \right\|_p.$$

By choosing $g_j = S_{j+k}f$, this gives us

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H S_{j+k}f|^2 \right)^{1/2} \right\|_p + \left\| \left(\sum_{j \in \mathbb{Z}} |P_{2^j} S_{j+k}f|^2 \right)^{1/2} \right\|_p \\ & \leq C_p \left\| \left(\sum_{j \in \mathbb{Z}} |S_{j+k}f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p. \end{aligned}$$

For (4.2.10), we use Plancherel's identity to get

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j}^H S_{j+k}f - P_{2^j} S_{j+k}f|^2 \right)^{1/2} \right\|_2 = \left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{F}(M_{2^j}^H S_{j+k}f - P_{2^j} S_{j+k}f)|^2 \right)^{1/2} \right\|_2.$$

The fact that for each $t > 0$, both of the terms $t(e^{-t} - e^{-2t})$ and $\frac{1}{t}(e^{-t} - e^{-2t})$ are smaller than 1 implies that

$$\min\{2^j|\xi|, (2^j|\xi|)^{-1}\} \cdot (e^{-2\pi 2^{j+k-1}|\xi|} - e^{-2\pi 2^{j+k}|\xi|}) \leq C 2^{-|k|}. \quad (4.2.12)$$

Thus by using Lemma 4.2.1, we have

$$\begin{aligned} & |\mathcal{F}(M_{2^j}^H S_{j+k}f - P_{2^j} S_{j+k}f)|^2 \\ & \leq C_N \min\{2^j|\xi|, (2^j|\xi|)^{-1}\}^2 \cdot (e^{-2\pi 2^{j+k-1}|\xi|} - e^{-2\pi 2^{j+k}|\xi|})^2 \cdot |\hat{f}(\xi)|^2 \\ & \leq C'_N 2^{-|k|} \min\{2^j|\xi|, (2^j|\xi|)^{-1}\} \cdot (e^{-2\pi 2^{j+k-1}|\xi|} - e^{-2\pi 2^{j+k}|\xi|}) \cdot |\hat{f}(\xi)|^2 \\ & \leq C''_N 2^{-|k|} \min\{2^j|\xi|, (2^j|\xi|)^{-1}\} |\hat{f}(\xi)|^2 \end{aligned} \quad (4.2.13)$$

for every $k \in \mathbb{Z}$. Since

$$\sum_{j \in \mathbb{Z}} \min\{2^j |\xi|, (2^j |\xi|)^{-1}\} \leq 2 \sum_{j=0}^{\infty} 2^{-j} = 4,$$

we thus get

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{F}(M_{2^j}^H S_{j+k} f - P_{2^j} S_{j+k} f)|^2 \right)^{1/2} \right\|_2 \leq 4C'_N 2^{-\frac{|k|}{2}} \|f\|_2,$$

which concludes the proof of the lemma. q.e.d.

4.3 Variational estimates involving spherical averages

In this section, we will provide counterparts of [9, Lemmas 4.1 and 4.2], which are very crucial for proving dimension-free r -variational estimates. In the case of a general body B , [9, Lemma 4.1] allows to verify the condition (2.10.13), i.e.

$$\|V_p(M_t^B f : 1 \leq t < 2)\|_{L^p} \leq C_p \|f\|_{L^p}$$

for $p < 2$, so that one can apply Proposition 2.10.1. For the case $p > 2$, the Hölder property [9, Lemma 4.2] is shown. For both of these lemmas, the multiplier properties of $\langle \xi, \nabla \rangle^\alpha \widehat{\chi_B}$ are of the same fundamental importance as for the maximal estimates.

Again, fix $N \geq 3$. If m is the Fourier transform of the measure underlying H , as in Section 4.2, we technically did not show that $\langle \xi, \nabla \rangle^\alpha m$ is an L^p -multiplier for some $p < 2$, $1/p < \alpha < 1$. Instead, Lemma 3.1.4, Proposition 3.2.1, and a short duality argument directly imply the following properties.

Lemma 4.3.1. (i) Let $j \in \mathbb{N}_{>0}$, $\tilde{B} = (B_{r_j}^j)^{\ell_j}$ for some $\ell_j \in \mathbb{N}_{>0}$ (so that $|\tilde{B}| = 1$), and $\tilde{m} = \widehat{\chi_{\tilde{B}}}$. Then for every $p \in (1, \infty)$ and every $\alpha \in (1/2, 1)$ there is $C_{p,\alpha,j}$ such that for every $f \in L^p(\mathbb{R}^{j \cdot \ell_j})$, we have

$$\|T_{\langle \xi, \nabla \rangle^\alpha \tilde{m}} f\|_{L^p} \leq C_{p,\alpha,j} \|f\|_{L^p}. \quad (4.3.1)$$

(ii) For every $p \in (\frac{N}{N-1}, N)$, there are $\alpha \in (\max\{1/p, 1/p'\}, 1)$ and $C_{p,N}$ such that for every $f \in L^p(\mathbb{R}^{N\ell''})$, we have

$$\|T_{\langle \xi'', \nabla \rangle^\alpha m_2} f\|_{L^p} \leq C_{p,N} \|f\|_{L^p}, \quad (4.3.2)$$

We need to show the Hölder property [9, Lemma 4.2] first, because we will need it to prove our counterpart of [9, Lemma 4.1]. The Hölder property has only been shown for ℓ^q -balls, but we can use the multiplier estimates in Lemma 4.3.1 to show it in our settings. This is where the restriction $p < N$ for Theorem 4 comes from.

Lemma 4.3.2. *Let $N \geq 3$.*

(a) *For every $p \in (1, \infty)$ and every $\alpha \in (1/2, 1)$, there is a constant $C_{p,\alpha,N}$ such that for all $t, h > 0$, $f \in L^p(\mathbb{R}^{n'})$, we have*

$$\|M_{t+h}^{B'} f - M_t^{B'} f\|_p \leq C_{p,\alpha,N} \left(\frac{h}{t}\right)^\alpha \|f\|_p. \quad (4.3.3)$$

(b) *If $\frac{N}{N-1} < p < N$, then there exist α with $\max\{1/p, 1/p'\} < \alpha < 1$ and a constant $C_{p,N}$ such that for all $t, h > 0$, $f \in \mathcal{S}(\mathbb{R}^{N\ell''})$, we have*

$$\|M_{t+h}^S f - M_t^S f\|_p \leq C_{p,N} \left(\frac{h}{t}\right)^\alpha \|f\|_p. \quad (4.3.4)$$

(c) *If $\frac{N}{N-1} < p < N$, then there exist α with $\max\{1/p, 1/p'\} < \alpha < 1$ and a constant $C_{p,N}$ such that for all $t, h > 0$, $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\|M_{t+h}^H f - M_t^H f\|_p \leq C_{p,N} \left(\frac{h}{t}\right)^\alpha \|f\|_p. \quad (4.3.5)$$

In all cases, $C_{p,N}$ and $C_{p,\alpha,N}$ are independent of ℓ .

Proof. We will show (b) first. Obviously, we can assume $h < t$. We will first provide a scaling argument that shows that we can assume $t = 1$. In the case of S , assume that (4.3.4) already holds in the case of $t = 1$. Then for general t and $0 < h < t$, a simple substitution in x yields

$$\begin{aligned} \|M_{t+h}^S f - M_t^S f\|_p^p &= \int_{\mathbb{R}^{N\ell''}} \left| \int_S (f(x + (t+h)\omega) - f(x + t\omega)) \, d\sigma_S(\omega) \right|^p dx \\ &= t^{N\ell''} \int_{\mathbb{R}^{N\ell''}} \left| \int_B (f(tx + t(1+h/t)\omega) - f(tx + t\omega)) \, d\sigma_S(\omega) \right|^p dx \\ &\leq t^{N\ell''} C_{p,N}^p \left(\frac{h}{t}\right)^{\alpha p} \int_{\mathbb{R}^{N\ell''}} |f(tx)|^p dx \\ &= C_{p,N}^p \left(\frac{h}{t}\right)^{\alpha p} \|f\|_p^p. \end{aligned}$$

To show (4.3.4) for $t = 1$, choose α such that (4.3.2) holds. We use formula (2.5.19) to write

$$M_t^S f(x) = \int_0^\infty A(t, u) P_u^\alpha f(x) \, du, \quad (4.3.6)$$

where

$$A(t, u) := \frac{1}{\Gamma(\alpha)} \frac{t}{u} \left(1 - \frac{t}{u}\right)_+^{\alpha-1} \frac{1}{u},$$

and P_u^α is defined through

$$\widehat{P_u^\alpha f}(\xi'') = p_u^\alpha(\xi'') \hat{f}(\xi'') := u^{\alpha+1} \mathcal{D}_u^\alpha \left(\frac{m_2(u\xi'')}{u} \right) \hat{f}(\xi''),$$

similarly as in (2.5.20). Put

$$X(u, h) := A(1+h, u) - A(1, u), \quad u \geq 1, h > 0.$$

The conditions $h > 0$ and $\alpha < 1$ ensure that $X(u, h)$ is always positive. We obtain

$$|M_{1+h}^S f(x) - M_1^S f(x)| \leq \int_1^\infty X(u, h) \cdot |P_u^\alpha f(x)| \, du,$$

and by Minkowski's inequality, we thus get

$$\|M_{1+h}^S f - M_1^S f\|_p \leq \sup_{u \geq 1} \|P_u^\alpha f\|_p \int_1^\infty X(u, h) \, du. \quad (4.3.7)$$

for $t = 1$ and $0 < h < 1$. Using (2.5.21), the fact that m_2 is the Fourier transform of a normalized measure, and (4.3.2), we have

$$\|P_u^\alpha f\|_p \leq 2\|T_{m_2} f\|_p + \|T_{\langle \xi, \nabla \rangle^\alpha m_2} f\|_p \leq 2\|f\|_p + C_{p,N}\|f\|_p = C_{p,N}\|f\|_p. \quad (4.3.8)$$

This means that we are left with finding a bound of the form

$$\int_1^\infty X(u, h) \, du \leq Ch^\alpha. \quad (4.3.9)$$

Such a bound is given in the proof of [9, Lemma 4.2], so that part (b) of the lemma is proven.

For part (a), we argue as in the beginning of Section 3.1 and assume without loss of generality that $n_1 \leq \dots \leq n_{\ell'}$, i.e. $B' = \prod_{j=1}^K (B_{r_j}^j)^{\ell_j}$ for some $K \leq N$ and certain $\ell_j \in \mathbb{N}$.

We use induction on $K \geq 1$.

If $K = 1$, we have $B' = [-1/2, 1/2]^{\ell_1}$, and the estimate follows either from [9, Lemma 4.2], or by applying (4.3.1) in place of (4.3.2), and replacing m_2 by m_1 and M_t^S by $M_t^{B'}$.

Now assume that $B' = \tilde{B} \times \tilde{\tilde{B}}$, where

$$\tilde{B} = \prod_{j=1}^K (B_{r_j}^j)^{\ell_j} \quad \text{and} \quad \tilde{\tilde{B}} = (B_{r_{K+1}}^{K+1})^{\ell_{K+1}}.$$

Then our induction hypothesis provides that (4.3.3) holds for \tilde{B} , while (4.3.1) for $j = K + 1$ implies that (4.3.3) holds for $\tilde{\tilde{B}}$. Thus we get

$$\begin{aligned}
 \|M_{t+h}^{B'} f - M_t^{B'} f\|_p &= \|M_{t+h}^{\tilde{B}} M_{t+h}^{\tilde{\tilde{B}}} f - M_{t+h}^{\tilde{B}} M_t^{\tilde{\tilde{B}}} f + M_{t+h}^{\tilde{B}} M_t^{\tilde{\tilde{B}}} f - M_t^{\tilde{B}} M_t^{\tilde{\tilde{B}}} f\|_p \\
 &\leq \|M_{t+h}^{\tilde{B}} M_{t+h}^{\tilde{\tilde{B}}} f - M_{t+h}^{\tilde{B}} M_t^{\tilde{\tilde{B}}} f\|_p + \|M_{t+h}^{\tilde{B}} M_t^{\tilde{\tilde{B}}} f - M_t^{\tilde{B}} M_t^{\tilde{\tilde{B}}} f\|_p \\
 &\leq \|M_{t+h}^{\tilde{\tilde{B}}} f - M_t^{\tilde{\tilde{B}}} f\|_p + \|M_{t+h}^{\tilde{B}} f - M_t^{\tilde{B}} f\|_p \\
 &\leq C_{p,\alpha,N} \left(\frac{h}{t}\right)^\alpha \|f\|_p,
 \end{aligned} \tag{4.3.10}$$

where we used that $M_{t+h}^{\tilde{B}}$ and $M_t^{\tilde{\tilde{B}}}$ are contractive and commuting operators on L^p .

Part (c) follows from combining parts (a) and (b). Indeed, in order to show (4.3.5), fix $\frac{N}{N-1} < p < N$, $h, t > 0$, and choose α such that (4.3.2) holds. Then $\alpha > 1/2$, so that (4.3.1) also holds. By using that $M_t^H = M_t^{B'} M_t^S$, a similar argument as for (4.3.10) provides

$$\|M_{t+h}^H f - M_t^H f\|_p \leq C_{p,\alpha,N} \left(\frac{h}{t}\right)^\alpha \|f\|_p,$$

concluding the lemma. **q.e.d.**

As we said before, Lemma 4.3.2 is necessary to show our variant of [9, Lemma 4.1] for H . In [9], this lemma is an r -variational counterpart to Proposition 2.5.2, and its proof also relies on Proposition 2.8.4. By the use of Lemma 4.3.2 however, the second estimate in the following lemma can be expressed in terms of $\langle t\xi'', \nabla \rangle^\alpha m_2(t\xi'')$, rather than $\langle t\xi, \nabla \rangle^\alpha m(t\xi)$.

Lemma 4.3.3. *Let $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta(t) \leq 1$, $|\eta'(t)| \leq 3$, $\eta(t) = 1$ for $t \in [1, 2]$, and $\eta(t) = 0$ for $t \notin (1/2, 3)$. If $N \geq 3$, $\frac{N}{N-1} < p < 2$, and $1/p < \alpha < 1$, then there is a constant $C_{\alpha,p}$ such that for every $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\begin{aligned}
 &\|V_p(\eta(t)M_t^H f : t \in \mathbb{R})\|_{L^p(\mathbb{R}^d)} \\
 &\leq C_{p,\alpha,N} (\|f\|_{L^p(\mathbb{R}^d)} + \|\eta(t)M_t^{B'}(\mathcal{F}_\xi^{-1}(\mathcal{D}_t^\alpha m_2(t\xi)\hat{f}(\xi)))\|_{L^p(\mathbb{R} \times \mathbb{R}^d)}).
 \end{aligned} \tag{4.3.11}$$

Furthermore, we have

$$\|V_p(\eta(t)M_t^H f : t \in \mathbb{R})\|_{L^p(\mathbb{R}^d)} \leq C_{p,\alpha,N} (\|f\|_{L^p(\mathbb{R}^d)} + \sup_{t>0} \|T_{\langle t\xi'', \nabla \rangle^\alpha m_2(t\xi'')} f\|_{L^p(\mathbb{R}^d)}). \tag{4.3.12}$$

Proof. Write $F(t, x) = \eta(t)M_t^H f(x)$. Then by Fubini's theorem and Proposition 2.8.4,

we have

$$\begin{aligned} \|V_p(F(t, \cdot) : t \in \mathbb{R})\|_{L^p(\mathbb{R}^{N\ell})} &= \left(\int_{\mathbb{R}^{N\ell}} V_p(F(t, x) : t \in \mathbb{R})^p dx \right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}^{N\ell}} C_p(\|F(\cdot, x)\|_{L^p(\mathbb{R})} + \|\mathcal{D}_t^\alpha F(\cdot, x)\|_{L^p(\mathbb{R})})^p dx \right)^{1/p} \\ &\leq C_p(\|F\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} + \|\mathcal{D}_t^\alpha F\|_{L^p(\mathbb{R} \times \mathbb{R}^d)}). \end{aligned}$$

Since $\text{supp } \eta \subset [1/2, 3]$ implies that

$$\begin{aligned} \|F\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{N\ell}} |\eta(t) M_t^H f(x)|^p dx dt \right)^{1/p} \\ &\leq 2.5^{1/p} \sup_{1/2 < t < 3} \left(\int_{\mathbb{R}^{N\ell}} |\eta(t) M_t^H f(x)|^p dx \right)^{1/p} \\ &\leq 2.5^{1/p} \|f\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

it suffices to estimate $\|\mathcal{D}_t^\alpha F\|_{L^p(\mathbb{R} \times \mathbb{R}^d)}$. For that, we will first show that

$$\|\mathcal{D}_t^\alpha F\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \leq C_p(\|f\|_{L^p(\mathbb{R}^d)} + \|\eta(t) M_t^{B'}(\mathcal{D}_t^\alpha M_t^S f)\|_{L^p(\mathbb{R} \times \mathbb{R}^d)}). \quad (4.3.13)$$

Put

$$E(t, x) = \mathcal{D}_t^\alpha F(t, x) - \eta(t) M_t^{B'}(\mathcal{D}_t^\alpha M_t^S f(x)).$$

We can find some estimates on $\|E\|_{L^p(\mathbb{R} \times \mathbb{R}^d)}$ by using formula (2.5.6) for $\mathcal{D}_t^\alpha F(t, x)$ and $\mathcal{D}_t^\alpha M_t^S f(x)$. Fix $x \in \mathbb{R}^d$. If we put $h_1(t) = F(t, x)$, then h_1 is C^∞ and compactly supported, so one can easily see that h_1 satisfies the conditions of Lemma 2.5.1.

In order to show that we may apply formula (2.5.6) for $\mathcal{D}_t^\alpha M_t^S f(x)$, we need to put in some more work. Let $h_2(t) = M_t^S f(x)$, let $k(s) = \mathcal{F}^{-1} h_2(s)$. We will show the properties

$$\sup_{s \in \mathbb{R}} |(1 + s^2)k(s)| \leq C_{h_2}, \quad (4.3.14)$$

and

$$\left| \frac{d}{dt} h_2(t) \right| \leq C_{h_2} (1 + |t|)^{-1}, \quad (4.3.15)$$

which are stronger than the conditions of Lemma 2.5.1. The integrability of $(1 + |s|^\alpha)k(s)$ then follows since $\alpha < 1$. To show (4.3.14), we define for $x = (x', x'') \in \mathbb{R}^d$, $\omega \in S$, and $t \in \mathbb{R}$ the functions

$$g_{x, \omega}(u) := f(x', x'' + t\omega).$$

4.3 Variational estimates involving spherical averages

Since $f \in \mathcal{S}(\mathbb{R}^d)$ and $\omega \neq 0$ for $\omega \in S$, we have $g_{x,\omega} \in \mathcal{S}(\mathbb{R})$ for every x and ω . By Fubini's theorem, we get

$$\begin{aligned} k(s) &= \int_{\mathbb{R}} e^{2\pi i s t} M_t^S f(x) dt \\ &= \int_{\mathbb{R}} e^{2\pi i s t} \frac{1}{|S|} \int_S f(x', x'' + t\omega) d\sigma_S(\omega) dt \\ &= \frac{1}{|S|} \int_S \int_{\mathbb{R}} e^{2\pi i s t} f(x', x'' + t\omega) dt d\sigma_S(\omega) \\ &= \frac{1}{|S|} \int_S \mathcal{F}^{-1} g_{x,\omega}(s) d\sigma_S(\omega). \end{aligned}$$

But since each $g_{x,\omega}$ is a Schwartz function, so is each $(1+s^2)\mathcal{F}^{-1}g_{x,\omega}(s)$. Furthermore, for fixed x , the map $\omega \mapsto \mathcal{F}^{-1}g_{x,\omega}(s)$ is continuous. Since S is compact, we find a uniform bound

$$\sup_{s \in \mathbb{R}, \omega \in S} |(1+s^2)\mathcal{F}^{-1}g_{x,\omega}(s)| < C(x, f),$$

and thus

$$|(1+s^2)k(s)| \leq C(x, f).$$

We shall mention that the constants C_{h_2} in (4.3.14) and (4.3.15) may depend on n or ℓ'' , but since they are only used to show that we may apply the equation in Lemma 2.5.1 for $h_1(t) = F(t, x)$ and $h_2(t) = M_t^S f(x)$, this will not have any influence on our dimension-free estimates.

If h is any of the functions h_1 or h_2 as above, we apply Lemma 2.5.1, and through integration by parts, we get

$$\begin{aligned} \mathcal{D}^\alpha h(t) &= \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha} h'(t+u) du \\ &= \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha} \frac{d}{du} (h(t+u) - h(t)) du \\ &= \frac{-1}{\Gamma(1-\alpha)} \lim_{\varepsilon \rightarrow 0} \left(-\varepsilon^{-\alpha} (h(t+\varepsilon) - h(t)) + \alpha \int_\varepsilon^\infty u^{-\alpha-1} (h(t+u) - h(t)) du \right). \end{aligned}$$

By (4.3.15), we have

$$\varepsilon^{-\alpha} (h(t+\varepsilon) - h(t)) = \varepsilon^{1-\alpha} \frac{h(t+\varepsilon) - h(t)}{\varepsilon} \rightarrow 0$$

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as $\varepsilon \rightarrow 0$. Furthermore, for $0 < \varepsilon < 1$, we have

$$\begin{aligned} & \left| \chi_{(\varepsilon, \infty)}(u) \cdot u^{-\alpha-1} (h(t+u) - h(t)) \right| \\ &= \left| \chi_{(\varepsilon, 1)}(u) \cdot u^{-\alpha} \frac{h(t+u) - h(t)}{u} + \chi_{[1, \infty)}(u) \cdot u^{-\alpha-1} (h(t+u) - h(t)) \right| \\ &\leq \chi_{(0, 1)}(u) \cdot u^{-\alpha} \|h'\|_{L^\infty(\mathbb{R})} + 2\chi_{[1, \infty)}(u) \cdot u^{-\alpha-1} \|h\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Since

$$\int_0^1 u^{-\alpha} \|h'\|_{L^\infty} du + \int_1^\infty u^{-\alpha-1} \cdot 2\|h\|_{L^\infty} du = \frac{1}{1-\alpha} \|h'\|_{L^\infty} + \frac{2}{\alpha} \|h\|_{L^\infty} < \infty,$$

it follows by the dominated convergence theorem that

$$u \mapsto u^{-\alpha-1} (h(t+u) - h(t))$$

is integrable on $\mathbb{R}_{>0}$, with

$$\mathcal{D}^\alpha h(t) = \frac{-\alpha}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha-1} (h(t+u) - h(t)) du. \quad (4.3.16)$$

By using this formula together with Fubini's theorem, we get

$$\begin{aligned} E(t, x) &= \mathcal{D}_t^\alpha F(t, x) - \eta(t) M_t^{B'} (\mathcal{D}_t^\alpha M_t^S f(x)) \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \left(\eta(t) M_t^{B'} \left(\int_0^\infty u^{-\alpha-1} (M_{t+u}^S f(x) - M_t^S f(x)) du \right) \right. \\ &\quad \left. - \int_0^\infty u^{-\alpha-1} (\eta(t+u) M_{t+u}^{B'} M_{t+u}^S f(x) - \eta(t) M_t^{B'} M_t^S f(x)) du \right) \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_0^\infty u^{-\alpha-1} (\eta(t) M_t^{B'} M_{t+u}^S f(x) - \eta(t) M_t^{B'} M_t^S f(x)) du \right. \\ &\quad \left. - \int_0^\infty u^{-\alpha-1} (\eta(t+u) M_{t+u}^{B'} M_{t+u}^S f(x) - \eta(t) M_t^{B'} M_t^S f(x)) du \right) \\ &= \int_0^\infty K(t, u) (M_{t+u}^S f(x)) du, \end{aligned}$$

where the operator $K(t, u)$ is defined on $L_{\text{loc}}^1(\mathbb{R}^{n'})$ through

$$K(t, u)(g) := \frac{-\alpha}{\Gamma(1-\alpha)} \frac{\eta(t+u) M_{t+u}^{B'} g - \eta(t) M_t^{B'} g}{u^{\alpha+1}}, \quad t \in \mathbb{R}, u \in \mathbb{R}_{>0}.$$

4.3 Variational estimates involving spherical averages

To estimate $\|E\|_{L^p(\mathbb{R} \times \mathbb{R}^{N\ell})}$, we will estimate $\|K(t, u)\|_{p \rightarrow p}$ first. One can easily see that

$$\|K(t, u)\|_{p \rightarrow p} \leq \frac{2\alpha}{\Gamma(1-\alpha)} u^{-\alpha-1}. \quad (4.3.17)$$

Furthermore, if $t \leq 1/4$, we have $K(t, u) = \frac{-\alpha}{\Gamma(1-\alpha)} u^{-\alpha-1} \eta(t+u) M_{t+u}^{B'}$. Assume that $\eta(t+u) \neq 0$. Then $u \geq 1/4$, and by (4.3.17) we get

$$\|K(t, u)\|_{p \rightarrow p} \leq C_\alpha u^{-\alpha-1} \leq C_\alpha 4^{\alpha+1}. \quad (4.3.18)$$

If on the other hand, $t > 1/4$, let $\tilde{\alpha} = (1+\alpha)/2$. Then $\alpha < \tilde{\alpha} < 1$, and we can apply Minkowski's inequality and Lemma 4.3.2 (a) to get

$$\begin{aligned} \|\eta(t+u)M_{t+u}^{B'}g - \eta(t)M_t^{B'}g\|_p &\leq \|\eta(t+u)M_{t+u}^{B'}g - \eta(t+u)M_t^{B'}g\|_p \\ &\quad + \|\eta(t+u)M_t^{B'}g - \eta(t)M_t^{B'}g\|_p \\ &= \|\eta(t+u)(M_{t+u}^{B'}g - M_t^{B'}g)\|_p \\ &\quad + \|(\eta(t+u) - \eta(t))M_t^{B'}g\|_p \\ &\leq |\eta(t+u)|C_{p,\tilde{\alpha},N}\left(\frac{u}{t}\right)^{\tilde{\alpha}} \|g\|_p + |\eta(t+u) - \eta(t)| \|g\|_p. \end{aligned}$$

Since $|\eta'(t)| \leq 3$, $t > 1/4$ and thus $t^{-\tilde{\alpha}} < 4^{\tilde{\alpha}} < 4$, and $\tilde{\alpha} - \alpha - 1 = \frac{\alpha+1}{2} - (\alpha+1) = -\tilde{\alpha}$, it follows that

$$\begin{aligned} \|K(t, u)\|_{p \rightarrow p} &\leq \frac{\alpha}{\Gamma(1-\alpha)} \left(\frac{1}{u^{\alpha+1}} |\eta(t+u)| C_{p,\tilde{\alpha},N} \left(\frac{u}{t}\right)^{\tilde{\alpha}} + u^{-\alpha} \left| \frac{\eta(t+u) - \eta(t)}{u} \right| \right) \\ &\leq C_\alpha (C_{p,\tilde{\alpha},N} u^{\tilde{\alpha}-\alpha-1} t^{-\tilde{\alpha}} + 3u^{-\alpha}) \\ &\leq C_{p,\alpha,N} (u^{-\tilde{\alpha}} + u^{-\alpha}). \end{aligned} \quad (4.3.19)$$

Since for $t > 3$, $K(t, u) = 0$ due to the choice of η , we can estimate

$$\|E\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \leq \|E\|_{L^p((-\infty, -3) \times \mathbb{R}^d)} + \|E\|_{L^p([-3, 1/4] \times \mathbb{R}^d)} + \|E\|_{L^p((1/4, 3] \times \mathbb{R}^d)}.$$

For $t < -3$, we always have $\eta(t) = 0$, while $\eta(t+u) \neq 0$ implies that $1/2 - t < u < 3 - t$. Hence by Minkowski's integral inequality and (4.3.18), we get

$$\begin{aligned} \|E\|_{L^p((-\infty, -3) \times \mathbb{R}^d)} &\leq \left(\int_{-\infty}^{-3} \left(\int_0^\infty \|K(t, u)\|_{p \rightarrow p} \cdot \sup_{t>0} \|M_t^S f\|_{L^p} du \right)^p dt \right)^{1/p} \\ &\leq \|f\|_{L^p} \left(\int_{-\infty}^{-3} \left(\int_{1/2-t}^{3-t} u^{-\alpha-1} du \right)^p dt \right)^{1/p} \\ &= \|f\|_{L^p} \left(\int_{-\infty}^{-3} \frac{1}{\alpha^p} ((1/2-t)^{-\alpha} - (3-t)^{-\alpha})^p dt \right)^{1/p} \end{aligned}$$

4 Variational bounds for products Euclidean balls and spheres

$$\begin{aligned}
&\leq \|f\|_{L^p} \left(\int_{-\infty}^{-3} \frac{1}{\alpha^p} (1/2 - t)^{-\alpha p} dt \right)^{1/p} \\
&\leq C_p \|f\|_{L^p} \left(\int_{-\infty}^{-3} \frac{1}{\alpha^p} |t|^{-\alpha p} dt \right)^{1/p} \\
&\leq C_{p,\alpha} \|f\|_{L^p},
\end{aligned}$$

since $\alpha > 1/p$. For $-3 \leq t \leq 1/4$, we can use (4.3.18) again, and a similar argument provides

$$\|E\|_{L^p([-3, 1/4] \times \mathbb{R}^d)} \leq \|f\|_{L^p} \left(\int_{-3}^{1/4} \left(\int_{1/2-t}^{3-t} C_\alpha 4^{\alpha+1} du \right)^p dt \right)^{1/p} \leq C_{p,\alpha} \|f\|_{L^p}.$$

As for $1/4 < t \leq 3$, we use (4.3.17) and (4.3.19) and by another application of Minkowski's inequality, we get

$$\begin{aligned}
\|E\|_{L^p([1/4, 3] \times \mathbb{R}^d)} &\leq \|f\|_{L^p} \left(\int_{1/4}^3 \left(\int_0^\infty \|K(t, u)\|_{p \rightarrow p} du \right)^p dt \right)^{1/p} \\
&\leq C_\alpha \|f\|_{L^p} \left(\int_{1/4}^3 \left(\int_1^\infty u^{-\alpha-1} du \right)^p dt \right)^{1/p} \\
&\quad + C_{p,\alpha,N} \|f\|_{L^p} \left(\int_{1/4}^3 \left(\int_0^1 (u^{-\tilde{\alpha}} + u^{-\alpha}) du \right)^p dt \right)^{1/p} \leq C_{p,\alpha,N} \|f\|_{L^p},
\end{aligned}$$

which finally implies

$$\|E\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \leq C_{p,\alpha,N} \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.3.20}$$

The estimate on E directly implies (4.3.13). A further application of (2.5.6) together with Fubini's theorem implies that

$$\begin{aligned}
\mathcal{D}_t^\alpha M_t^S f(x) &= \frac{-1}{\Gamma(1-\alpha)} \int_t^\infty (u-t)^{-\alpha} \frac{d}{du} M_u^S f(x) du \\
&= \frac{-1}{\Gamma(1-\alpha)} \int_t^\infty (u-t)^{-\alpha} \mathcal{F}_\xi^{-1} \left(\frac{d}{du} m_2(u\xi'') \hat{f}(\xi) \right) (x) du \\
&= \mathcal{F}_\xi^{-1} \left(\frac{-1}{\Gamma(1-\alpha)} \int_t^\infty (u-t)^{-\alpha} \frac{d}{du} m_2(u\xi'') du \cdot \hat{f}(\xi) \right) (x) \\
&= \mathcal{F}_\xi^{-1} (\mathcal{D}_t^\alpha m_2(t\xi'') \hat{f}(\xi)) (x).
\end{aligned}$$

Combining this with (4.3.13) leads to (4.3.11). By (2.5.18), we obtain

$$\mathcal{D}_s^\alpha m_2(s\xi'') \Big|_{s=t} = t^{-\alpha} \mathcal{D}_s^\alpha m_2(s(t\xi'')) \Big|_{s=1} = t^{-\alpha} \langle t\xi'', \nabla \rangle^\alpha m_2(t\xi'').$$

Thus, since η has compact support,

$$\begin{aligned} & \left\| \eta(t) M_t^{B'} \left(\mathcal{F}_\xi^{-1} \left(\mathcal{D}_t^\alpha m_2(t\xi'') \hat{f}(\xi) \right) \right) \right\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \\ &= \left\| \eta(t) M_t^{B'} \left(\mathcal{F}_\xi^{-1} \left(t^{-\alpha} \langle t\xi'', \nabla \rangle^\alpha m_2(t\xi'') \hat{f}(\xi) \right) \right) \right\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \\ &\leq C_\alpha \sup_{t>0} \left\| \eta(t) M_t^{B'} \left(\mathcal{F}_\xi^{-1} \left(t^{-\alpha} \langle t\xi'', \nabla \rangle^\alpha m_2(t\xi'') \hat{f}(\xi) \right) \right) \right\|_{L^p(\mathbb{R}^d)} \\ &\leq C_\alpha \sup_{t>0} \| T_{\langle t\xi'', \nabla \rangle^\alpha m_2(t\xi'')} f \|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

which allows us to deduce (4.3.12) from (4.3.11). **q.e.d.**

4.4 Proof of Theorems 3 and 4

Here, we will first use the remaining arguments from [9] to show Proposition 4.1.4, and then show how to derive Theorems 3 and 4 from it.

Note first that by Lemma 2.8.3, we can estimate the short variation seminorm for $r > 2$ by

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbb{Z}} V_r(M_t^H f : t \in [2^j, 2^{j+1})) \right)^r \right\|_{L^p}^{1/r} \\ & \leq C \left\| \left(\sum_{j \in \mathbb{Z}} V_2(M_t^H f : t \in [2^j, 2^{j+1})) \right)^2 \right\|_{L^p}^{1/2} \\ & \leq \sum_{k \in \mathbb{Z}} \sum_{L \in \mathbb{N}} \left\| \sum_{j \in \mathbb{Z}} \sum_{l=0}^{2^L-1} |M_{2^{j+(l+1)2^j-L}}^H S_{j+k} f - M_{2^{j+l2^j-L}}^H S_{j+k} f|^2 \right\|_{L^p}^{1/2}, \quad (4.4.1) \end{aligned}$$

as in [9]. For the case $p = 2$, we will show an analogue to the estimate [9, (4.23)] for M_t^H , which will not only imply Proposition 4.1.4 for $p = 2$, but also provide a bound that allows us to use an interpolation argument for larger values of p .

Lemma 4.4.1. *For every $\varepsilon \in (0, 1)$, there is a constant $C_{\varepsilon, N}$ such that for every $k \in \mathbb{Z}$, $L \in \mathbb{N}$, and $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \sum_{l=0}^{2^L-1} |M_{2^{j+(l+1)2^j-L}}^H S_{j+k} f - M_{2^{j+l2^j-L}}^H S_{j+k} f|^2 \right)^{1/2} \right\|_{L^2} \leq C_{\varepsilon, N} 2^{-\frac{(1-\varepsilon)L}{2}} 2^{-\varepsilon \frac{|k|}{4}} \|f\|_{L^2}, \quad (4.4.2)$$

where $C_{\varepsilon, N}$ does not depend on k , L , or ℓ .

Proof. Fix $0 < \varepsilon < 1$. We will use Plancherel's theorem with the help of two auxiliary estimates. By the fundamental theorem of calculus and the estimates (4.2.4), we have

$$\begin{aligned}
& \sum_{l=0}^{2^L-1} \left| m((2^j + (l+1)2^{j-L})\xi) - m((2^j + l2^{j-L})\xi) \right|^{2-\varepsilon} \\
& \leq \sum_{l=0}^{2^L-1} \left(\int_{2^j+l2^{j-L}}^{2^j+(l+1)2^{j-L}} |\langle \xi, \nabla m(s\xi) \rangle| ds \right)^{2-\varepsilon} \\
& \leq \sum_{l=0}^{2^L-1} C_{\varepsilon,N} \left(\int_{2^j+l2^{j-L}}^{2^j+(l+1)2^{j-L}} \frac{1}{s} ds \right)^{2-\varepsilon} \\
& = C_{\varepsilon,N} \sum_{l=0}^{2^L-1} \log \left(\frac{2^j + (l+1)2^{j-L}}{2^j + l2^{j-L}} \right)^{2-\varepsilon} \\
& = C_{\varepsilon,N} \sum_{l=0}^{2^L-1} \log \left(1 + \frac{1}{2^L + l} \right)^{2-\varepsilon} \\
& \leq C_{\varepsilon,N} 2^L \log \left(1 + \frac{1}{2^L} \right)^{2-\varepsilon} \\
& \leq C_{\varepsilon,N} 2^L \left(\frac{1}{2^L} \right)^{2-\varepsilon} \\
& = C_{\varepsilon,N} 2^{-L(1-\varepsilon)}. \tag{4.4.3}
\end{aligned}$$

Write

$$\widehat{S}_{j+k}(\xi) := e^{-2\pi 2^{j+k-1}|\xi|} - e^{-2\pi 2^{j+k}|\xi|}, \quad \xi \in \mathbb{R}^d,$$

and note that for $0 \leq l \leq 2^L$, we have

$$2^j \leq 2^j + l2^{j-L} \leq 2^j + (l+1)2^{j-L} \leq 2^{j+1}.$$

Using the estimates (4.2.4), (4.2.12), and the fact that $0 \leq \widehat{S}_{j+k}(\xi) \leq 1$, this yields

$$\begin{aligned}
& \left| m((2^j + (l+1)2^{j-L})\xi) - m((2^j + l2^{j-L})\xi) \right|^\varepsilon \cdot \left| \widehat{S}_{j+k}(\xi) \right|^2 \\
& \leq C_{\varepsilon,N} \min\{|2^j \xi|, |2^j \xi|^{-1}\}^\varepsilon \cdot \left| e^{-2\pi 2^{j+k-1}|\xi|} - e^{-2\pi 2^{j+k}|\xi|} \right|^2 \\
& \leq C_{\varepsilon,N} 2^{-\frac{|k|\varepsilon}{2}} \min\{|2^j \xi|, |2^j \xi|^{-1}\}^{\varepsilon/2} \cdot \left| e^{-2\pi 2^{j+k-1}|\xi|} - e^{-2\pi 2^{j+k}|\xi|} \right|^{(2-\varepsilon)/2} \\
& \leq C_{\varepsilon,N} 2^{-\frac{|k|\varepsilon}{2}} \min\{|2^j \xi|, |2^j \xi|^{-1}\}^{\varepsilon/2}. \tag{4.4.4}
\end{aligned}$$

If we combine (4.4.3) and (4.4.4), we obtain

$$\begin{aligned}
& \sum_{l=0}^{2^L-1} \left| m((2^j + (l+1)2^{j-L})\xi) - m((2^j + l2^{j-L})\xi) \right|^2 \cdot |\widehat{S_{j+k}}(\xi)|^2 \\
& \leq C_{\varepsilon, N} 2^{-\frac{|k|\varepsilon}{2}} \min\{|2^j \xi|, |2^j \xi|^{-1}\}^{\varepsilon/2} \\
& \quad \cdot \sum_{l=0}^{2^L-1} \left(|m((2^j + (l+1)2^{j-L})\xi) - m((2^j + l2^{j-L})\xi)|^{2-\varepsilon} \right) \\
& \leq C_{\varepsilon, N} 2^{-\frac{|k|\varepsilon}{2}} 2^{-L(1-\varepsilon)} \min\{|2^j \xi|, |2^j \xi|^{-1}\}^{\varepsilon/2}. \tag{4.4.5}
\end{aligned}$$

By Plancherel's theorem and (4.4.5), we thus get

$$\begin{aligned}
& \left\| \left(\sum_{j \in \mathbb{Z}} \sum_{l=0}^{2^L-1} |M_{2^j+(l+1)2^{j-L}}^H S_{j+k} f - M_{2^j+l2^{j-L}}^H S_{j+k} f|^2 \right)^{1/2} \right\|_{L^2}^2 \\
& \leq C_{\varepsilon, N} 2^{-\frac{|k|\varepsilon}{2}} 2^{-L(1-\varepsilon)} \int_{\mathbb{R}^{N\ell}} \sum_{j \in \mathbb{Z}} \min\{|2^j \xi|, |2^j \xi|^{-1}\}^{\varepsilon/2} |\hat{f}(\xi)|^2 d\xi \\
& \leq C_{\varepsilon, N} 2^{-\frac{|k|\varepsilon}{2}} 2^{-L(1-\varepsilon)} \|f\|_{L^2}^2,
\end{aligned}$$

which implies the claimed L^2 -estimate. **q.e.d.**

For $p \neq 2$, the proof of Proposition 4.1.4 uses the same as the proof of [9, Theorem 1.3], only with the smaller range for p .

Proof of Proposition 4.1.4 for $2 < p < N$. Pick $\alpha \in (1/p', 1)$ such that the estimate in Lemma 4.3.2 (c) holds. We will show that

$$\left\| \left(\sum_{j \in \mathbb{Z}} \sum_{l=0}^{2^L-1} |M_{2^j+(l+1)2^{j-L}}^H S_{j+k} f - M_{2^j+l2^{j-L}}^H S_{j+k} f|^2 \right)^{1/2} \right\|_{L^p} \leq C_{p, N} 2^{\frac{L}{2} - \frac{p'}{2} \alpha L} \|f\|_{L^p}, \tag{4.4.6}$$

which is similar to [9, (4.36)]. If we then interpolate (4.4.6) with (4.4.2) and use (4.4.1), this concludes the proof. Since $p > 2$, we can see that

$$\begin{aligned}
& \left\| \left(\sum_{j \in \mathbb{Z}} \sum_{l=0}^{2^L-1} |M_{2^j+(l+1)2^{j-L}}^H S_{j+k} f - M_{2^j+l2^{j-L}}^H S_{j+k} f|^2 \right)^{1/2} \right\|_{L^p} \\
& = \left\| \sum_{j \in \mathbb{Z}} \sum_{l=0}^{2^L-1} |M_{2^j+(l+1)2^{j-L}}^H S_{j+k} f - M_{2^j+l2^{j-L}}^H S_{j+k} f|^2 \right\|_{L^{p/2}}^{1/2} \\
& \leq \left(\sum_{l=0}^{2^L-1} \left\| \sum_{j \in \mathbb{Z}} |M_{2^j+(l+1)2^{j-L}}^H S_{j+k} f - M_{2^j+l2^{j-L}}^H S_{j+k} f|^2 \right\|_{L^{p/2}} \right)^{1/2} \\
& \leq 2^{L/2} \max_{0 \leq l \leq 2^L-1} \left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j+(l+1)2^{j-L}}^H S_{j+k} f - M_{2^j+l2^{j-L}}^H S_{j+k} f|^2 \right)^{1/2} \right\|_{L^p}.
\end{aligned}$$

4 Variational bounds for products Euclidean balls and spheres

We will argue by duality. Let $(g_j)_{j \in \mathbb{Z}} \in L^{p'}(\ell^2)$ (note that $\frac{N}{N-1} < p' < 2$), so that

$$\left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^2 \right)^{1/2} \right\|_{L^{p'}} \leq 1.$$

Since each M_t^H is self-adjoint, (4.4.6) follows if we can show

$$\begin{aligned} \max_{0 \leq l \leq 2^L - 1} \left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j + (l+1)2^{j-L}}^H g_j - M_{2^j + l2^{j-L}}^H g_j|^2 \right)^{1/2} \right\|_{L^{p'}} \\ \leq C_{q,N} 2^{-\frac{p'}{2}\alpha L} \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^2 \right)^{1/2} \right\|_{L^{p'}}. \end{aligned} \quad (4.4.7)$$

More generally, we will show that for every $q \in (\frac{N}{N-1}, 2)$ and every $l \in \{0, \dots, 2^L - 1\}$, we have

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j + (l+1)2^{j-L}}^H g_j - M_{2^j + l2^{j-L}}^H g_j|^2 \right)^{1/2} \right\|_{L^q} \leq C_{q,N} 2^{-\frac{q}{2}\alpha L} \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^2 \right)^{1/2} \right\|_{L^q}, \quad (4.4.8)$$

so that (4.4.7) follows by taking $q = p'$ in (4.4.8). We can achieve (4.4.8) through interpolation between $L^q(\ell^q)$ and $L^q(\ell^\infty)$, by taking $\theta \in (0, 1)$ such that

$$\frac{1}{2} = \frac{\theta}{q} + \frac{1-\theta}{\infty}.$$

Hence $\theta = q/2$. In the $L^q(\ell^q)$ case, we use Lemma 4.3.2 (c) to get

$$\|M_{2^j + (l+1)2^{j-L}}^H f - M_{2^j + l2^{j-L}}^H f\|_q \leq C_{q,N} \left(\frac{2^{j-L}}{2^j + l2^{j-L}} \right)^\alpha \|f\|_q \leq C_{q,N} 2^{-\alpha L} \|f\|_q,$$

which implies

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbb{Z}} |M_{2^j + (l+1)2^{j-L}}^H g_j - M_{2^j + l2^{j-L}}^H g_j|^q \right)^{1/q} \right\|_{L^q}^q \\ &= \int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} |M_{2^j + (l+1)2^{j-L}}^H g_j(x) - M_{2^j + l2^{j-L}}^H g_j(x)|^q dx \\ &\leq C_{q,N}^q 2^{-q\alpha L} \sum_{j \in \mathbb{Z}} \|g_j\|_{L^q}^q \\ &= C_{q,N}^q 2^{-q\alpha L} \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^q \right)^{1/q} \right\|_{L^q}^q. \end{aligned}$$

As for $L^q(\ell^\infty)$, we write $g(x) = \sup_{j \in \mathbb{Z}} |g_j(x)|$. Since $q > \frac{N}{N-1}$, we can use the maximal estimate for H (by combining Theorems 1 and 2) to obtain

$$\begin{aligned} \left\| \sup_{j \in \mathbb{Z}} |M_{2^j + (l+1)2^{j-L}}^H g_j - M_{2^j + l2^{j-L}}^H g_j| \right\|_{L^q} &\leq \left\| \sup_{j \in \mathbb{Z}} M_{2^j + (l+1)2^{j-L}}^H g \right\|_{L^q} + \left\| \sup_{j \in \mathbb{Z}} M_{2^j + l2^{j-L}}^H g \right\|_{L^q} \\ &\leq C_q \|g\|_{L^q}. \end{aligned}$$

The interpolation between $L^q(\ell^q)$ and $L^q(\ell^\infty)$ concludes the estimate (4.4.6), and by (4.4.1), Proposition 4.1.4 follows for $2 < p < N$. **q.e.d.**

For the case $\frac{N}{N-1} < p < 2$, we can also argue as in [9], by using Lemma 4.3.3.

Proof of Proposition 4.1.4 for $\frac{N}{N-1} < p < 2$. We will apply Proposition 2.10.1. Observe that by summing over $L \in \mathbb{N}$ in Lemma 4.4.1, the condition (2.10.4) in Proposition 2.10.1 is satisfied, with $a_k = 2^{-\varepsilon \frac{|k|}{4}}$ for some suitable $\varepsilon \in (0, 1)$. Since the maximal condition (2.10.2) follows from Theorems 1 and 2, we are left with verifying (2.10.3). As we showed in Section 2.10, we can apply a scaling argument, so that it suffices to show (2.10.13) in our setting, i.e.

$$\|V_p(M_t^H f : 1 \leq t < 2)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

But if we choose η as in Lemma 4.3.3, we see that for every $\alpha \in (1/p, 1)$, we have

$$\begin{aligned} \|V_p(M_t^H f : 1 \leq t < 2)\|_{L^p(\mathbb{R}^d)} &\leq \|V_p(\eta(t)M_t^H f : t \in \mathbb{R})\|_{L^p(\mathbb{R}^d)} \\ &\leq C_{\alpha,p} (\|f\|_{L^p(\mathbb{R}^{N\ell})} + \sup_{t>0} \|T_{\langle t\xi'', \nabla \rangle^{\alpha} m_2(t\xi'')} f\|_{L^p(\mathbb{R}^d)}). \end{aligned}$$

But since we already know that there is $\alpha \in (1/p, 1)$ such that $\langle \xi'', \nabla \rangle^{\alpha} m_2$ is an L^p -multiplier, we have

$$\sup_{t>0} \|T_{\langle t\xi'', \nabla \rangle^{\alpha} m_2(t\xi'')} f\|_{L^p(\mathbb{R}^d)} \leq \sup_{t>0} \|T_{\langle \xi'', \nabla \rangle^{\alpha} m_2(\xi'')} \|_{p \rightarrow p} \|f\|_{L^p(\mathbb{R}^d)} = C_{p,N} \|f\|_{L^p(\mathbb{R}^d)}, \quad (4.4.9)$$

and thus, Proposition 4.1.4 and Theorem 4 are proven. **q.e.d.**

For Theorem 3, we also use the arguments in Section 4.2 and choose $N = N(p)$ accordingly.

Proof of Theorem 3. Fix $1 < p < \infty$. For B as in Theorem 3, we use the notation as in Section 4.1, i.e. fix $N = N(p) \geq 3$ to be chosen later, write $S_k = \partial B_k$ for each k , and assume again that $B = B' \times B''$, where

$$B' = \prod_{n_k < N} B_k \quad \text{and} \quad B'' = \prod_{n_k \geq N} B_k.$$

Furthermore, let $\ell'' = \{k : n_k \geq N\}$, $S = (S^{N-1})^{\ell''}$, and $H = B \times S$. If $p > 2$, put $N(p) = \lfloor p + 1 \rfloor$, and if $p \leq 2$, put $N(p) = \lfloor \frac{p}{p-1} \rfloor + 1$. In both cases, we have $N \geq 3$ and $\frac{N}{N-1} < p < N$, so that by Proposition 4.1.4, we get

$$\|V_r(M_t^H f : t > 0)\|_{L^p} \leq C_{p,r,N(p)} \|f\|_{L^p} = C_{p,r} \|f\|_{L^p}, \quad (4.4.10)$$

By Lemmas 4.1.2 and 4.1.3, Theorem 3 is thus proven. **q.e.d.**

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit - abgesehen von der Beratung durch den Betreuer meiner Promotion - unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe, diese Arbeit nicht im Rahmen eines anderen Prüfungsverfahrens ganz oder zum Teil vorgelegen hat, veröffentlicht worden ist oder zur Veröffentlichung eingereicht wurde, mir bisher kein akademischer Grad aberkannt worden ist und ich keine anderen als die angegebenen Hilfsmittel verwendet habe.

Kiel, den 18.07.2019