

## $\sigma$ -LOCALIZATION AND $\sigma$ -MARTINGALES\*

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**Abstract.** This paper introduces the concept of  $\sigma$ -localization, which is a generalization of localization in the general theory of stochastic processes. The  $\sigma$ -localized class derived from the set of martingales is the class of  $\sigma$ -martingales, which plays an important role in mathematical finance. These processes and the corresponding  $\sigma$ -martingale measures are considered in detail. By extending the stochastic integral with respect to compensated random measures, a canonical representation of  $\sigma$ -martingales as for local martingales is derived.

**Key words.**  $\sigma$ -localization,  $\sigma$ -martingale, stochastic integral, canonical representation,  $\sigma$ -martingale measure

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**1. Introduction.**  $\sigma$ -martingales have been introduced by Chou [3] and were investigated further by Emery [6]. They play a key role in the general statement of the fundamental theorems of asset pricing in [5], [12], and [2].  $\sigma$ -martingales can be interpreted quite naturally as semimartingales with vanishing drift. Similar to local martingales, the set of  $\sigma$ -martingales may be obtained from the class of martingales by a localization procedure, but here localization has to be understood in a broader sense than usual (cf. [11, section I.1d]). This concept of  $\sigma$ -localization is introduced in section 2. The subsequent section treats the set of  $\sigma$ -martingales and their properties. By extending the stochastic integral relative to compensated random measures, the canonical local martingale representation  $X = X_0 + X^c + x * (\mu - \nu)$  is generalized to  $\sigma$ -martingales in section 4. Finally,  $\sigma$ -martingale measures are characterized in terms of semimartingale characteristics.

Throughout the paper, we use the notation of [11], [9], [10]. In particular, we work with a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}_+}, P)$ . The transposition of a vector  $x$  or matrix is denoted by  $x^\top$  and its components by superscripts. Increasing processes are identified with their corresponding Lebesgue–Stieltjes measures.

**2.  $\sigma$ -localization.** For any semimartingale  $X$  and any predictable set  $D \subset \Omega \times \mathbf{R}_+$ , we write  $X^D := X_0 1_D(0) + 1_D \cdot X$ , where  $1_D(0)(\omega) := 1_D((\omega, 0))$  for  $\omega \in \Omega$ . In particular, we have  $X^{[0, T]} = X^T$  for any stopping time  $T$  (cf. [11, Relations I.4.37]).

**DEFINITION 2.1.** For any class  $\mathcal{C}$  of semimartingales we define the  $\sigma$ -localized class  $\mathcal{C}_\sigma$  as follows: A process  $X$  belongs to  $\mathcal{C}_\sigma$  if and only if there exists an increasing sequence  $(D_n)_{n \in \mathbf{N}}$  of predictable sets such that  $D_n \uparrow \Omega \times \mathbf{R}_+$  up to an evanescent set and  $X^{D_n} \in \mathcal{C}$  for any  $n \in \mathbf{N}$ .

**DEFINITION 2.2.** A class  $\mathcal{C}$  of semimartingales is called stable under  $\sigma$ -stopping if  $X^D \in \mathcal{C}$  for any  $X \in \mathcal{C}$  and any predictable set  $D$ .

**LEMMA 2.1.** If  $\mathcal{C}$  is stable under stopping, then  $(\mathcal{C}_{\text{loc}})_\sigma = \mathcal{C}_\sigma$ .

*Proof.* We mimic the proof of a similar statement in [11, Lemma I.1.35]. Let  $X \in (\mathcal{C}_{\text{loc}})_\sigma$  and  $(D_n)_{n \in \mathbf{N}}$  be a localizing sequence of predictable sets such that  $X^{D_n} \in \mathcal{C}_{\text{loc}}$  for any  $n \in \mathbf{N}$ . Since  $\mathcal{C}_{\text{loc}}$  is stable under stopping, we may assume that  $D_n = D_n \cap \llbracket 0, n \rrbracket$ . For any  $n \in \mathbf{N}$  there exists a localizing sequence of stopping times  $(T(n, p))_{p \in \mathbf{N}}$  and  $p_n \in \mathbf{N}$  such that  $(X^{D_n})^{T(n, p)} \in \mathcal{C}$  for any  $p \in \mathbf{N}$  and  $P(T(n, p_n) < n) \leq 2^{-n}$ . Let  $\tilde{D}_n := D_n \cap$

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$(\bigcap_{m \geq n} [0, T(m, p_m)])$ . Observe that  $(\tilde{D}_n)_{n \in \mathbf{N}}$  is an increasing sequence of predictable sets. Let  $k \in \mathbf{N}$  and  $(\omega, t) \in \Omega \times [0, k]$  with  $(\omega, t) \in (\bigcup_{n \in \mathbf{N}} D_n) \setminus (\bigcup_{n \in \mathbf{N}} \tilde{D}_n) = \limsup_{n \rightarrow \infty} (D_n \setminus \tilde{D}_n)$ . Obviously, this holds (up to evanescence) also for  $(\omega, k)$  instead of  $(\omega, t)$ . Since

$$\begin{aligned} P\left(\{\omega \in \Omega : (\omega, k) \in D_n \setminus \tilde{D}_n\}\right) &\leq \sum_{m \geq n} P(T(m, p_m) < n) \leq \sum_{m \geq n} P(T(m, p_m) < m) \\ &\leq \sum_{m \geq n} 2^{-m} = 2^{-(n-1)}, \end{aligned}$$

the Borel–Cantelli lemma yields  $P(\{\omega \in \Omega : (\omega, k) \in \limsup_{n \rightarrow \infty} (D_n \setminus \tilde{D}_n)\}) = 0$  and hence  $\Omega \times [0, k] \subset \bigcup_{n \in \mathbf{N}} \tilde{D}_n$  up to an evanescent set. Since  $\mathcal{C}$  is stable under stopping and  $X^{\tilde{D}_n} = ((X^{D_n})^{T(n, p_n)})^{\wedge_{m \geq n+1} T(m, p_m)}$ , it follows that  $X^{\tilde{D}_n} \in \mathcal{C}$  for any  $n \in \mathbf{N}$ . Hence  $X \in \mathcal{C}_\sigma$ . Lemma 2.1 is proved.

The following result serves primarily as a preparation to Corollary 2.1 because most classes of interest (in particular  $\mathcal{M}$ ) are not stable under  $\sigma$ -stopping.

PROPOSITION 2.1. *If  $\mathcal{C}$  is stable under  $\sigma$ -stopping, then  $(\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$ .*

*Proof.* The proof is similar to the proof of Lemma 2.1. Let  $X \in (\mathcal{C}_\sigma)_\sigma$  and  $(D_n)_{n \in \mathbf{N}}$  be a localizing sequence of predictable sets such that  $X^{D_n} \in \mathcal{C}_\sigma$  for any  $n \in \mathbf{N}$ . Let the characteristics of  $X$  be given in the form (3.1) below. Without loss of generality we may modify  $A$  so that  $A_{0-} = 0$ ,  $\Delta A_0 = A_0 - A_{0-} > 0$ , and  $A_\infty = 1$ , which implies that  $P \otimes A$  is a probability measure on  $(\Omega \times \mathbf{R}_+, \mathcal{P})$ . For any  $n \in \mathbf{N}$  there exists a localizing sequence of predictable sets  $(D(n, p))_{p \in \mathbf{N}}$  and  $p_n \in \mathbf{N}$  such that  $(X^{D_n})^{D(n, p)} \in \mathcal{C}$  for any  $p \in \mathbf{N}$  and  $(P \otimes A)(D(n, p_n)^C) \leq 2^{-n}$ . Let  $\tilde{D}_n := D_n \cap (\bigcap_{m \geq n} D(m, p_m))$ . Observe that  $(\tilde{D}_n)_{n \in \mathbf{N}}$  is an increasing sequence of predictable sets and hence  $(\bigcup_{n \in \mathbf{N}} D_n) \setminus (\bigcup_{n \in \mathbf{N}} \tilde{D}_n) = \limsup_{n \rightarrow \infty} (D_n \setminus \tilde{D}_n)$ . Since  $(P \otimes A)(D_n \setminus \tilde{D}_n) \leq \sum_{m \geq n} (P \otimes A)(D(m, p_m)^C) \leq \sum_{m \geq n} 2^{-m} = 2^{-(n-1)}$ , the Borel–Cantelli lemma yields  $(P \otimes A)(\limsup_{n \rightarrow \infty} (D_n \setminus \tilde{D}_n)) = 0$ . Therefore,  $D := (\Omega \times \mathbf{R}_+) \setminus (\bigcup_{n \in \mathbf{N}} \tilde{D}_n)$  is a  $(P \otimes A)$ -null set. By [14, Lemma 2.5], this implies that  $X_0 1_D(0) + 1_D \cdot X = 0$  and hence  $X^{\tilde{D}_n \cup D} = X^{\tilde{D}_n}$  up to indistinguishability. Since  $\mathcal{C}$  is stable under  $\sigma$ -stopping and  $X^{\tilde{D}_n \cup D} = X^{\tilde{D}_n} = ((X^{D_n})^{D(n, p_n)})^{\cap_{m \geq n+1} D(m, p_m)}$ , it follows that  $X^{\tilde{D}_n \cup D} \in \mathcal{C}$  for any  $n \in \mathbf{N}$ . Hence  $X \in \mathcal{C}_\sigma$ . Proposition 2.1 is proved.

COROLLARY 2.1. *If  $\mathcal{C}$  is stable under stopping and  $\mathcal{C}_{\text{loc}}$  is stable under  $\sigma$ -stopping, then  $\mathcal{C}_\sigma$  is stable under  $\sigma$ -stopping and  $(\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$ .*

*Proof.* It is easy to see that  $(\mathcal{C}_{\text{loc}})_\sigma$  is stable under  $\sigma$ -stopping, which implies that  $\mathcal{C}_\sigma$  is stable under  $\sigma$ -stopping (cf. Lemma 2.1). By Lemma 2.1 and Proposition 2.1, we have  $(\mathcal{C}_\sigma)_\sigma = ((\mathcal{C}_{\text{loc}})_\sigma)_\sigma = (\mathcal{C}_{\text{loc}})_\sigma = \mathcal{C}_\sigma$ .

The following result shows that the general stochastic integral in [10] can easily be interpreted in terms of  $\sigma$ -localization.

LEMMA 2.2. *Let  $X$  be an  $\mathbf{R}^d$ -valued semimartingale and let  $H$  be an  $\mathbf{R}^d$ -valued predictable process. Then  $H \in L(X)$  if and only if there exist a semimartingale  $Z$  and an increasing sequence  $(D_n)_{n \in \mathbf{N}}$  of predictable sets such that  $D_n \uparrow \Omega \times \mathbf{R}_+$  up to an evanescent set,  $H 1_{D_n}$  is bounded, and  $Z^{D_n} = (H 1_{D_n})^\top \cdot X$  for any  $n \in \mathbf{N}$ . In this case  $Z = H^\top \cdot X$ .*

*Proof.*  $\Rightarrow$ : Let  $D_n := \{|H| \leq n\}$  for any  $n \in \mathbf{N}$ . The statement follows from [7, Proposition A.1].

$\Leftarrow$ : On any  $D_n$  we have  $\Delta Z = \Delta Z^{D_n} = (H 1_{D_n})^\top \Delta X = H^\top \Delta X$ , which implies that  $D := \{|\Delta X| > 1\} \cup \{|H^\top \Delta X| > 1\}$  is a discrete set in the sense of [10]. Let  $M$  be the local martingale part of the  $\mathbf{R}^d$ -valued special semimartingale  $\bar{X} := X - X_0 - \sum_{t \leq \cdot} \Delta X_t 1_D(t)$  and  $A := X - M$ , which has components in  $\mathcal{V}$ . Choose an  $\mathbf{R}^{d \times d}$ -valued process  $c$  and  $C \in \mathcal{V}$  such that  $[M^i, M^j] = c^{ij} \cdot C$  for  $i, j = 1, \dots, d$ . Moreover, choose an  $\mathbf{R}^d$ -valued optional process  $a$  and  $\bar{A} \in \mathcal{V}$  with  $A^i = a^i \cdot \bar{A}$  for  $i = 1, \dots, d$ . Finally, let  $N$  be the local martingale

part of the special semimartingale  $\bar{Z} := Z - \sum_{t \leq \cdot} \Delta Z_t 1_D(t)$  and  $B := Z - N \in \mathcal{V}$ . Since  $\bar{Z}$  has bounded jumps, we have  $N \in \mathcal{H}_{\text{loc}}^2$  (cf. [11, Lemma I.4.24]). Fix  $n \in \mathbf{N}$ . Since  $H1_{D_n}$  is locally bounded and  $(H1_{D_n})^\top \cdot \bar{X} = (H1_{D_n})^\top \cdot X - \sum_{t \leq \cdot} (H_t 1_{D_n}(t))^\top \Delta X_t 1_D(t) = Z^{D_n} - \sum_{t \leq \cdot} \Delta Z_t^{D_n} 1_D(t) = \bar{Z}^{D_n}$ , Proposition 2 in [10] yields that  $(H1_{D_n})^\top \cdot M = N^{D_n}$  and hence  $(H1_{D_n})^\top \cdot A = B^{D_n}$ . Therefore  $1_{D_n} \cdot [N, N] = [N^{D_n}, N^{D_n}] = [(H1_{D_n})^\top \cdot M, (H1_{D_n})^\top \cdot M] = (H^\top c H 1_{D_n}) \cdot C$ , which implies that  $(H^\top c H) \cdot C = [N, N] \in \mathcal{A}_{\text{loc}}^+$  by monotone convergence. In particular,  $H \in L_{\text{loc}}^2(M) \subset L_{\text{loc}}^1(M)$ . Moreover,  $1_{D_n} \cdot \text{Var}(B) = \text{Var}(B^{D_n}) = |\sum_{i=1}^d (H^i 1_{D_n} a^i)| \cdot \bar{A}$ , where  $\text{Var}(B)$  denotes the variation process of  $B \in \mathcal{V}$ . Again by monotone convergence, this implies  $|\sum_{i=1}^d (H^i a^i)| \cdot \bar{A} = \text{Var}(B) \in \mathcal{V}$  and hence  $H \in L_s(A)$ . Altogether, we have  $H \in L(X)$ . The equality  $Z = H^\top \cdot X$  follows from the first part of the proof. Lemma 2.2 is proved.

**3.  $\sigma$ -martingales.** Of course, we have  $\mathcal{C} \subset \mathcal{C}_{\text{loc}} \subset \mathcal{C}_\sigma$  for any class  $\mathcal{C}$  of semimartingales. As a particular case, we obtain the set  $\mathcal{M}_\sigma$  of  $\sigma$ -martingales. Denote by  $\mathcal{D}$  the set of semimartingales such that the stopped process  $X^t$  is of class (D) for any  $t \in \mathbf{R}_+$  (cf. [11, Definition I.1.46]). Below we will consider the corresponding localized classes  $\mathcal{D}_{\text{loc}}$  and  $\mathcal{D}_\sigma$ .

The name  $\sigma$ -martingale was used in [5] to refer to the set of *semimartingales of the class*  $(\Sigma_m)$  introduced by Chou [3] and Emery [6], who also consider  $\mathcal{D}_\sigma$  as *semimartingales of the class*  $(\Sigma)$ . To be more precise,  $X \in \mathcal{M}_\sigma$  (respectively,  $X \in \mathcal{D}_\sigma$ ) holds if and only if  $X - X_0$  is a semimartingale of the class  $(\Sigma_m)$  (respectively, semimartingale of the class  $(\Sigma)$ ). Some researchers prefer the older name *martingale transform*, which was originally applied in discrete-time settings. Equivalent definitions can be found in [6, Proposition 2].

The  $\sigma$ -martingale property can easily be read from the characteristics of a semimartingale as it was observed by Kabanov [12, Lemma 3]. To this end, fix a real-valued semimartingale  $X$  with  $X_0 = 0$ . Let its characteristics  $(B, C, \nu)$  relative to some truncation function  $h: \mathbf{R} \rightarrow \mathbf{R}$  be given in the form

$$(3.1) \quad B = b \cdot A, \quad C = c \cdot A, \quad \nu = A \otimes F,$$

where  $A \in \mathcal{A}_{\text{loc}}^+$  and  $b$  are predictable processes,  $c$  is a predictable  $\mathbf{R}_+$ -valued process, and  $F$  is a transition kernel from  $(\Omega \times \mathbf{R}_+, \mathcal{P})$  into  $(\mathbf{R}, \mathcal{B})$  (cf. [11, Proposition II.2.9]).

LEMMA 3.1.

1.  $X$  is a  $\sigma$ -martingale if and only if  $\int_{\{|x|>1\}} |x| F(dx) < \infty$  and

$$(3.2) \quad b + \int (x - h(x)) F(dx) = 0$$

hold  $(P \otimes A)$ -almost everywhere on  $\Omega \times \mathbf{R}_+$ .

2.  $X$  is a local martingale if and only if  $\int_{\{|x|>1\}} |x| F(dx) \in L(A)$  and equation (3.2) holds  $(P \otimes A)$ -almost everywhere on  $\Omega \times \mathbf{R}_+$ .

3.  $X$  is a martingale if and only if  $X \in \mathcal{D}$  and equation (3.2) holds  $(P \otimes A)$ -almost everywhere on  $\Omega \times \mathbf{R}_+$ .

*Proof.*

2. This follows immediately from [11, Proposition II.2.29, Relations II.2.13, I.3.10].

1. This follows from statement 2 because  $\mathcal{M}_\sigma = (\mathcal{M}_{\text{loc}})_\sigma$  by Lemma 2.1 (cf. also [14, Lemma 2.5]).

3.  $\Rightarrow$ : Since martingales are local martingales that belong to  $\mathcal{D}$  (cf. [11, Proposition I.1.47]), this inclusion follows from statement 2.

$\Leftarrow$ : By [9, Exercise 2.1], we have that  $\mathcal{D}_{\text{loc}}$  coincides with the set of special semimartingales, which in turn satisfy

$$\int_{\{|x|>1\}} |x| F(dx) \in L(A)$$

(cf. [11, Proposition II.2.29]). Using statement 2 we see that  $X$  is a local martingale of class (D) on  $[0, t]$  for any  $t \in \mathbf{R}_+$  and hence a martingale (cf. [11, Proposition I.1.47]). Lemma 3.1 is proved.

Consequently, martingales, local martingales, and  $\sigma$ -martingales can all be interpreted as processes with vanishing drift (3.2) that differ only in the extent of uniform integrability. If we drop equation (3.2) and keep the integrability conditions in Lemma 3.1, then we end up with the sets  $\mathcal{D}, \mathcal{D}_{loc}, \mathcal{D}_\sigma$  as the following lemma shows.

LEMMA 3.2.

- 1.  $X \in \mathcal{D}_\sigma$  if and only if  $\int_{\{|x|>1\}} |x| F(dx) < \infty$  ( $P \otimes A$ )-almost everywhere on  $\Omega \times \mathbf{R}_+$ .
- 2.  $X \in \mathcal{D}_{loc}$  if and only if  $X$  is a special semimartingale if and only if

$$\int_{\{|x|>1\}} |x| F(dx) \in L(A).$$

*Proof.*

2. The first equivalence is [9, Exercise 2.1]. The second equivalence follows from [11, Proposition II.2.29, Relations II.2.13, I.3.10].

- 1. Since  $\mathcal{D}_\sigma = (\mathcal{D}_{loc})_\sigma$  by Lemma 2.1, this follows from statement 2.

We summarize some relationships between  $\sigma$ -martingales and other classes of processes.

COROLLARY 3.1.

1.  $X$  is a local martingale if and only if it is both a  $\sigma$ -martingale and a special semimartingale.

2.  $X$  is a martingale if and only if it is both a  $\sigma$ -martingale and in  $\mathcal{D}$ .

3.  $X$  is a uniformly integrable martingale if and only if it is a  $\sigma$ -martingale of class (D).

*Proof.* This follows immediately from Lemmas 3.1, 3.2 and [11, Proposition I.1.47].

COROLLARY 3.2.

1. Any locally bounded  $\sigma$ -martingale is a local martingale.

2. Any bounded  $\sigma$ -martingale is a uniformly integrable martingale.

*Proof.* Corollary 3.1, [11, section I.4.23].

Note that any discrete-time  $\sigma$ -martingale starting in 0, i.e., of the form

$$X = \sum_{t \leq \cdot} \Delta X_t 1_{\mathbf{N} \setminus \{0\}}(t),$$

is a local martingale because the integrability conditions in statements 1 and 2 of Lemma 3.2 are equivalent for  $A = \sum_{t \leq \cdot} 1_{\mathbf{N} \setminus \{0\}}(t)$ .

The following result generalizes a parallel statement for local supermartingales (cf. [9, (5.17)]). It is needed, e.g., in [8].

PROPOSITION 3.1. *Let  $X$  be a nonnegative  $\sigma$ -supermartingale with  $E(X_0) < \infty$ . Then  $X$  is a supermartingale.*

*Proof.* Let the characteristics of  $X$  be of the form (3.1). Since any supermartingale is locally of class (D) (cf. [9, Exercise 2] or the proof of [9, (2.18)]), there exists a sequence  $(D_n)_{n \in \mathbf{N}}$  of predictable sets with  $D_n \uparrow \Omega \times \mathbf{R}_+$  such that  $1_{D_n} \cdot X$  is a supermartingale of class (D) for any  $n$ .

Fix  $n \in \mathbf{N}$ . By [11, Proposition IX.5.3] and [14, Lemma 2.5], the characteristics of  $1_{D_n} \cdot X$  are of the form (3.1) as well, but with  $b' := 1_{D_n} b$ ,  $c' := 1_{D_n} c$ ,  $F' := 1_{D_n} F$  instead of  $b, c, F$ . Since  $1_{D_n} \cdot X$  is a supermartingale, it is a special semimartingale and its unique predictable part of bounded variation is given by  $(b' + \int (x - h(x)) F'(dx)) \cdot A = ((b + \int (x - h(x)) F(dx)) 1_{D_n}) \cdot A$  (cf. [11, Proposition II.2.29]). This process is actually decreasing because  $1_{D_n} \cdot X$  is a supermartingale of class (D) (cf. [11, Theorem I.3.38]). This implies that  $b + \int (x - h(x)) F(dx) \leq 0$  ( $P \otimes A$ )-almost everywhere on  $D_n$  and hence on  $\Omega \times \mathbf{R}_+$  because  $n$  was arbitrarily chosen.

The nonnegativity of  $X$  implies that  $\int_{\{|x|>1\}}(-x \vee 0) F(dx) \leq X_- \int_{\{|x|>1\}} F(dx)$ . By [11, Proposition II.2.13] this implies

$$\int_{\{|x|>1\}} (-x \vee 0) F(dx) \in L(A).$$

Since  $b + \int(x - h(x)) F(dx) \leq 0$ , it follows that  $\int_{\{|x|>1\}} |x| F(dx) \in L(A)$ . By Lemma 3.2 and [11, Relation II.2.29], this means that  $X$  is a special semimartingale whose predictable part of bounded variation  $(b + \int(x - h(x)) F(dx)) \cdot A$  is decreasing. Hence,  $X$  is in fact a nonnegative local supermartingale. In view of [9, (5.17)], we are done.

As opposed to local martingales and special semimartingales, the corresponding  $\sigma$ -localized classes are stable relative to stochastic integration.

LEMMA 3.3. *Let  $X$  be an  $\mathbf{R}^d$ -valued semimartingale and  $H \in L(X)$ . Then the following statements hold:*

1. *If  $X^i \in \mathcal{M}_\sigma$  for  $i = 1, \dots, d$ , then  $H^\top \cdot X \in \mathcal{M}_\sigma$ ;*
2. *If  $X^i \in \mathcal{D}_\sigma$  for  $i = 1, \dots, d$ , then  $H^\top \cdot X \in \mathcal{D}_\sigma$ .*

*Proof.* This follows from Lemmas 3.1, 3.2, and [14, Lemma 2.5].

So far, we have concentrated on real-valued processes. As for local martingales, it makes sense to define  $\mathbf{R}^d$ -valued  $\sigma$ -martingales as semimartingales whose components are  $\sigma$ -martingales. Following [5] one may also call an  $\mathbf{R}^d$ -valued process  $X$   $\sigma$ -martingale if there exist an  $\mathbf{R}^d$ -valued martingale  $M$  and a predictable, nonnegative process  $\varphi$  such that  $\varphi \in L(M^i)$  and  $X^i = X_0^i + \varphi \cdot M^i$  for  $i = 1, \dots, d$ . By [2, Theorem 5.6] and [6, Proposition 2], the two definitions are consistent.

The existence of *Doob–Meyer decompositions* is a delicate issue. Whereas any  $X \in \mathcal{D}$  (respectively,  $X \in \mathcal{D}_{\text{loc}}$ ) can be written as a sum  $X = M + A$  of a martingale (respectively, local martingale)  $M$  and a predictable process  $A \in \mathcal{V}$ , a process  $X \in \mathcal{D}_\sigma$  does not necessarily allow a decomposition  $X = M + A$  with  $M \in \mathcal{M}_\sigma$  and a predictable process  $A \in \mathcal{V}$ . Such processes are called *almost special semimartingales* by Emery [6].

**4. Canonical representation of  $\sigma$ -martingales.** Local martingales can be uniquely written as a sum of continuous and purely discontinuous local martingales. More precisely, we have the canonical representation  $X = X_0 + X^c + x * (\mu - \nu)$ . The goal of this section is to derive an analogous statement for  $\sigma$ -martingales. To this end, we generalize the definition of stochastic integrals relative to compensated random measures (cf. Definition 4.3 below).

Let  $\mu$  be an integer-valued random measure on  $(\mathbf{R}_+ \times E, \mathcal{B}_+ \otimes \mathcal{E})$  with compensator  $\nu$ . Moreover, let  $A \in \mathcal{A}_{\text{loc}}^+$  be a predictable process and  $F$  a transition kernel from  $(\Omega \times \mathbf{R}_+, \mathcal{P})$  into  $(E, \mathcal{E})$  with  $\nu = A \otimes F$  (cf. [11, Theorem II.1.8]). Recall that we use the notation of [11], in particular Chapter II.

DEFINITION 4.1. *We denote by  $L_\sigma(\nu)$  the set of all  $\tilde{\mathcal{P}}$ -measurable functions  $W$  on  $\tilde{\Omega}$  such that  $\int |W(t, x)| F_t(dx) < \infty$  ( $P \otimes A$ )-a.e. and  $(\int W(t, x) F_t(dx))_{t \in \mathbf{R}_+} \in L(A)$ . For  $W \in L_\sigma(\nu)$  we define  $W * \nu := \int_0^\cdot \int W(t, x) F_t(dx) dA_t$ .*

Alternatively, one could define this stochastic integral by  $\sigma$ -localization.

LEMMA 4.1. *Let  $W$  be a  $\tilde{\mathcal{P}}$ -measurable function on  $\tilde{\Omega}$ . Then  $W \in L_\sigma(\nu)$  if and only if there exist a semimartingale  $X$  and an increasing sequence  $(D_n)_{n \in \mathbf{N}}$  of predictable sets such that  $D_n \uparrow \Omega \times \mathbf{R}_+$  up to an evanescent set,  $|W 1_{D_n}| * \nu \in \mathcal{V}$ , and  $X^{D_n} = W 1_{D_n} * \nu$  for any  $n \in \mathbf{N}$ . In this case  $X = W * \nu$ .*

*Proof.*  $\Rightarrow$ : Let  $D_n := \{(\omega, t) \in \Omega \times \mathbf{R}_+ : (\int |W(t, x)| F_t(dx))(\omega) \leq n\}$  for any  $n \in \mathbf{N}$  and  $X := W * \nu$ .

$\Leftarrow$ : By [9, (2.74)],  $X$  is indistinguishable from a predictable process and hence it is a special semimartingale (e.g., by [11, Proposition I.4.23 and Lemma I.3.10]). If we denote its local martingale part by  $M$ , we have  $M^{D_n} = 0$  for any  $n \in \mathbf{N}$  and hence  $M = 0$  up to indistinguishability (again by [9, (2.74)]). Therefore,  $X \in \mathcal{V}$ . For its variation process, we

have

$$\begin{aligned} 1_{D_n} \cdot \text{Var}(X) &= \text{Var}(X^{D_n}) = \text{Var}(W1_{D_n} * \nu) = \text{Var}\left(\int_0^\cdot 1_{D_n}(t) \int W(t, x) F_t(dx) dA_t\right) \\ &= \int_0^\cdot 1_{D_n}(t) \left| \int W(t, x) F_t(dx) \right| dA_t, \end{aligned}$$

for any  $n \in \mathbf{N}$ , which implies that  $\int_0^\cdot | \int W(t, x) F_t(dx) | dA_t = \text{Var}(X) \in \mathcal{V}$  by monotone convergence. In particular,  $\int W(t, x) F_t(dx) \in L(A)$ . The equality  $X = W * \nu$  follows from the first part of the proof.

Lemma 4.1 is proved.

DEFINITION 4.2. A  $\sigma$ -martingale  $M$  is called purely discontinuous if  $M_0 = 0$  and  $MN \in \mathcal{M}_\sigma$  for any continuous  $\sigma$ -martingale  $N$ .

PROPOSITION 4.1.

1. A local martingale is purely discontinuous in the sense of Definition 4.2 if and only if it is purely discontinuous in the sense of [11, Definition I.4.11].

2.  $M \in \mathcal{M}_\sigma$  is purely discontinuous if and only if  $M_0 = 0$  and  $[M, N] \in \mathcal{M}_\sigma$  for any continuous  $N \in \mathcal{M}_\sigma$ .

3.  $M$  is a purely discontinuous  $\sigma$ -martingale if and only if it belongs  $\sigma$ -locally to the class of purely discontinuous local martingales.

4. Suppose that  $M \in \mathcal{V}$  can be written as  $M = \sum_{t \leq \cdot} \Delta M_t 1_D(t)$  for some predictable thin set  $D$ . If  $E(|\Delta M_T| | \mathcal{F}_{T-}) < \infty$  and  $E(\Delta M_T | \mathcal{F}_{T-}) = 0$   $P$ -almost everywhere on  $\{T < \infty\}$  for any predictable stopping time  $T$ , then  $M$  is a purely discontinuous  $\sigma$ -martingale.

Proof.

1.  $\Rightarrow$ : We have to show that  $MN \in \mathcal{M}_{\text{loc}}$  if  $N$  is continuous,  $M \in \mathcal{M}_{\text{loc}}$ ,  $M_0 = 0$ ,  $MN \in \mathcal{M}_\sigma$ . In view of Corollary 3.1, it suffices to prove that  $MN$  is a special semimartingale. Define stopping times  $T_n := \inf\{t \in \mathbf{R}_+ : |N_t| > n\}$  for any  $n \in \mathbf{N}$ . Since  $\sup_{t \leq \cdot} |(MN)_t^{T_n}| \leq n \sup_{t \leq \cdot} |M_t^{T_n}| \in \mathcal{A}_{\text{loc}}^+$ , it follows that  $(MN)^{T_n}$  is special (cf. [11, Proposition I.4.23]). Since locally special semimartingales are special (cf. [11, Proposition I.4.25]), we are done.

$\Leftarrow$ : Note that  $N - N_0 \in \mathcal{M}_{\text{loc}}$  for any continuous  $\sigma$ -martingale  $N$  (cf. Corollary 3.2). Moreover,  $MN_0 \in \mathcal{M}_\sigma$  for any  $M \in \mathcal{M}_\sigma$  since  $N_0$  is  $\sigma$ -locally bounded.

2. This follows immediately from  $MN = M_0N_0 + M_- \cdot N + N_- \cdot M + [M, N]$  and Lemma 3.3.

3. Since  $[M, N]^D = 1_D \cdot [M, N] = [1_D \cdot M, N] = [M^D, N]$  for any predictable set  $D$ , this follows easily from statement 2.

4. Let  $(T_n)_{n \in \mathbf{N}}$  be a sequence of predictable stopping times with  $D = \bigcup_{n \in \mathbf{N}} \llbracket T_n \rrbracket$  up to an evanescent set (cf. [11, Lemma I.2.23]). For any  $n \in \mathbf{N}$  define a predictable set  $D_n := D^C \cup (\bigcup_{m \leq n} \llbracket T(m, n) \rrbracket)$ , where

$$T(m, n)(\omega) := \begin{cases} T_m(\omega) & \text{if } E(|\Delta M_{T_m}| | \mathcal{F}_{T_m-})(\omega) \leq n, \\ \infty & \text{otherwise.} \end{cases}$$

( $D_n$  is predictable by [11, Proposition I.2.10].) Since

$$M^{D_n} = \sum_{m \leq n} \Delta M_{T_m} 1_{\llbracket T(m, n), \infty \rrbracket},$$

we have that  $M^{D_n}$  is a purely discontinuous local martingale by [9, (1.45)] and [11, Lemma I.4.14]. In view of statement 3, we are done.

DEFINITION 4.3. Let  $W$  be a  $\tilde{\mathcal{P}}$ -measurable function on  $\tilde{\Omega}$ . We say that  $W \in G_\sigma(\mu)$  if there exists a purely discontinuous  $\sigma$ -martingale  $M$  such that  $\Delta M$  and  $\tilde{W}$  are indistinguishable. In this case, we set  $W * (\mu - \nu) := M$ .

LEMMA 4.2. *Let  $W \in G_\sigma(\mu)$ . Then*

- (1) *the stochastic integral in the previous definition is well defined;*
- (2)  *$W \in G_{loc}(\mu)$  if and only if  $W * (\mu - \nu) \in \mathcal{M}_{loc}$ .*

*Proof.* (1) Let  $\bar{M}$  be a purely discontinuous  $\sigma$ -martingale with the same jumps as  $M$ . Then  $M - \bar{M}$  is a continuous  $\sigma$ -martingale and hence a local martingale which is both continuous and purely discontinuous (cf. Corollary 3.2). By [11, Lemma I.4.13], it follows that  $M = \bar{M}$  up to indistinguishability.

(2)  $\Rightarrow$ : This is obvious.

$\Leftarrow$ : By [11, Theorem I.4.52 and Corollary I.4.55] we have that

$$\sqrt{\sum_{t \leq \cdot} (\widetilde{W}_t)^2} = \sqrt{\sum_{t \leq \cdot} (\Delta(W * (\mu - \nu))_t)^2} \leq \sqrt{[W * (\mu - \nu), W * (\mu - \nu)]} \in \mathcal{A}_{loc}^+,$$

which yields the claim.

Lemma 4.2 is proved.

The following result corresponds to Lemmas 2.2 and 4.1.

LEMMA 4.3. *Let  $W$  be a  $\widetilde{\mathcal{P}}$ -measurable function on  $\widetilde{\Omega}$ . Then  $W \in G_\sigma(\mu)$  if and only if there exist a semimartingale  $X$  and an increasing sequence  $(D_n)_{n \in \mathbf{N}}$  of predictable sets such that  $D_n \uparrow \Omega \times \mathbf{R}_+$  up to an evanescent set,  $W1_{D_n} \in G_{loc}(\mu)$ , and  $X^{D_n} = W1_{D_n} * (\mu - \nu)$  for any  $n \in \mathbf{N}$ . In this case  $X = W * (\mu - \nu)$ .*

*Proof.*  $\Rightarrow$ : Let  $(D_n)_{n \in \mathbf{N}}$  be a  $\sigma$ -localizing sequence of  $W * (\mu - \nu)$  in the sense of Proposition 4.1, statement 3. Then  $(W * (\mu - \nu))^{D_n}$  is a purely discontinuous local martingale with jumps  $\widetilde{W}1_{D_n} = (W1_{D_n})^\sim$ , which implies that  $W1_{D_n} \in G_{loc}(\mu)$  and  $(W * (\mu - \nu))^{D_n} = W1_{D_n} * (\mu - \nu)$  (cf. Lemma 4.2).

$\Leftarrow$ : Obviously,  $X$  has jumps  $\Delta X = (W1_{D_n})^\sim = \widetilde{W}1_{D_n} = \widetilde{W}$  on any  $D_n$ , which implies that  $\Delta X = \widetilde{W}$  up to indistinguishability. By statement 3 of Proposition 4.1,  $X$  is a purely discontinuous  $\sigma$ -martingale.

Lemma 4.3 is proved.

Definition 4.3 and Lemma 4.3 do not provide an explicit condition for the integrability of a predictable function  $W$ . Such a condition should ideally be expressed only in terms of the compensator  $\nu$  of  $\mu$ . It is provided by the following lemma.

LEMMA 4.4. *Let  $W$  be a  $\widetilde{\mathcal{P}}$ -measurable function on  $\widetilde{\Omega}$ . Let*

$$\begin{aligned} W' &:= (W - \widehat{W}) 1_{\{|W - \widehat{W}| \leq 1\}} + \widehat{W} 1_{\{|\widehat{W}| \leq 1\}}, \\ W'' &:= (W - \widehat{W}) 1_{\{|W - \widehat{W}| > 1\}}, \\ W''' &:= W'' - (1 - a) W 1_{\{|\widehat{W}| > 1\}}. \end{aligned}$$

*Then we have equivalence between*

- 1.  $W \in G_\sigma(\mu)$ ;
- 2. (a)  $\widehat{W}$  is finite outside some evanescent set;
- (b)  $C(W') \in \mathcal{V}$ ; where  $C(W') := (W' - \widehat{W}')^2 * \nu + \sum_{t \leq \cdot} (1 - a_t)(\widehat{W}'_t)^2$ ;
- (c)  $1_{\{W''(x) \neq 0\}} * \nu \in \mathcal{V}$ ;
- (d)  $\sum_{t \leq \cdot} (1 - a_t) 1_{\{|\widehat{W}_t| > 1\}} \in \mathcal{V}$ ;
- (e)  $W''' \in L_\sigma(\nu)$ .

*In this case*

$$W * (\mu - \nu) = W' * (\mu - \nu) + W'' * \mu - \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) \widehat{W}_t 1_{\{|\widehat{W}_t| > 1\}} - W''' * \nu.$$

*Proof.*  $1 \Rightarrow 2$ : Let  $X := W * (\mu - \nu)$ . By [11, Theorem II.2.34], the canonical semimartingale representation of  $X$  is

$$(4.1) \quad X = X^c + h * (\mu^X - \nu^X) + \bar{h} * \mu^X + B,$$

where  $\mu^X, \nu^X$  denote the measure of jumps of  $X$  and its compensator,  $B \in \mathcal{V}$  is some predictable process, and  $h(x) := x1_{\{|x| \leq 1\}}$ ,  $\bar{h}(x) := x - h(x)$  for  $x \in \mathbf{R}$ .

Let  $(D_n)_{n \in \mathbf{N}}$  be a  $\sigma$ -localizing sequence for  $X$  as in Lemma 4.3. Fix  $n \in \mathbf{N}$ . From  $W1_{D_n} \in G_{\text{loc}}(\mu)$  it follows that  $\widehat{W}$  is finite on  $D_n$ .

Since  $\Delta X = \widehat{W}$  and by definition of  $\widehat{W}$  in [11, Definitions II.1.27], we have

$$\begin{aligned}
 \bar{h}1_{D_n} * \mu^X &= \sum_{t \leq \cdot} \bar{h}(\Delta X_t) 1_{D_n}(t) = \sum_{t \leq \cdot} \bar{h}(\widehat{W}_t) 1_{D_n}(t) \\
 &= \bar{h}(W - \widehat{W}) 1_{D_n} * \mu + \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) \bar{h}(-\widehat{W}_t) 1_{D_n}(t) \\
 (4.2) \quad &= W'' 1_{D_n} * \mu - \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) \widehat{W}_t 1_{\{|\widehat{W}_t| > 1\}} 1_{D_n}(t).
 \end{aligned}$$

From  $X^{D_n} = (X^c)^{D_n} + h1_{D_n} * (\mu^X - \nu^X) + \bar{h}1_{D_n} * \mu^X + B^{D_n} \in \mathcal{M}_{\text{loc}}$  it follows that  $\bar{h}1_{D_n} * \mu^X + B^{D_n} \in \mathcal{M}_{\text{loc}} \cap \mathcal{A}_{\text{loc}}$ , and hence  $\bar{h}1_{D_n} * \mu^X \in \mathcal{A}_{\text{loc}}$  by [11, Proposition I.4.23 and Lemma I.3.10]. By [11, Theorem II.1.8], this implies  $-B^{D_n} = \bar{h}1_{D_n} * \nu^X$ . This process in turn equals  $W'' 1_{D_n} * \nu - \sum_{t \leq \cdot} (1 - a_t) \widehat{W}_t 1_{\{|\widehat{W}_t| > 1\}} 1_{D_n}(t) = (W'' - (1 - a)W 1_{\{|\widehat{W}| > 1\}}) 1_{D_n} * \nu = W''' 1_{D_n} * \nu$  by equation (4.2) and [9, (3.67)]. It follows that  $W''' \in L_\sigma(\nu)$  and  $B = -W''' * \nu$  by Lemma 4.1.

Similarly to equation (4.2), it follows that  $1_{\{\bar{h}(x) \neq 0\}} * \mu^X = 1_{\{W''(x) \neq 0\}} * \mu + \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) 1_{\{|\widehat{W}_t| > 1\}}$ . This process counts the “big” jumps of  $X$ . Since  $X$  has only finitely many big jumps on any finite interval, it follows that the process  $1_{\{\bar{h}(x) \neq 0\}} * \mu^X$  and hence both terms on the right-hand side are in  $\mathcal{A}_{\text{loc}}$ . By [9, (3.67)], this implies  $1_{\{W''(x) \neq 0\}} * \nu \in \mathcal{V}$  and  $\sum_{t \leq \cdot} (1 - a_t) 1_{\{|\widehat{W}_t| > 1\}} \in \mathcal{V}$ .

Similarly to equation (4.2), it follows that

$$(4.3) \quad h 1_D * \mu^X = h(W - \widehat{W}) 1_D * \mu + \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) h(-\widehat{W}_t) 1_D(t)$$

for any predictable set  $D$  with  $|h| 1_D * \mu^X \in \mathcal{V}$ . Fix  $t \in \mathbf{R}_+$ . Since  $\int |h| \nu^X(\{t\} \times dx) \leq 1$ , we have that  $|h 1_D| * \nu^X \in \mathcal{A}_{\text{loc}}^+$  and hence  $|h 1_D| * \mu^X \in \mathcal{A}_{\text{loc}}^+$  for  $D := \Omega \times \{t\}$ . In view of [9, (3.67)], calculating the compensator at time  $t$  of both sides of equation (4.3) yields

$$\begin{aligned}
 \widehat{h}_t &:= \int h(x) \nu^X(\{t\} \times dx) = \int h(W(t, x) - \widehat{W}_t) \nu(\{t\} \times dx) + (1 - a_t) h(-\widehat{W}_t) \\
 &= \int (h(W(t, x) - \widehat{W}_t) - h(-\widehat{W}_t)) \nu(\{t\} \times dx) + h(-\widehat{W}_t) = \widehat{W}'_t + h(-\widehat{W}_t).
 \end{aligned}$$

Similarly to equation (4.2), it follows that

$$\begin{aligned}
 (h - \widehat{h})^2 * \mu^X + \sum_{t \leq \cdot} (1 - \mu^X(\{t\} \times \mathbf{R})) \widehat{h}_t^2 &= \sum_{t \leq \cdot} (h(\Delta X_t) - \widehat{h}_t)^2 \\
 &= (h(W - \widehat{W}) - \widehat{h})^2 * \mu + \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) (h(-\widehat{W}_t) - \widehat{h}_t)^2 \\
 (4.4) \quad &= (W' - \widehat{W}')^2 * \mu + \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) (-W'_t)^2.
 \end{aligned}$$

Since  $h * (\mu^X - \nu^X)$  has bounded jumps, it is in  $\mathcal{H}_{\text{loc}}^2$ . By [11, Theorem II.1.33], this implies that  $(h - \widehat{h})^2 * \nu^X + \sum_{t \leq \cdot} (1 - \nu^X(\{t\} \times \mathbf{R})) \widehat{h}_t^2 \in \mathcal{A}_{\text{loc}}^+$ . By [9, (3.67)], this process is the compensator of the left-hand side of equation (4.4). Also by [9, (3.67)], we obtain



$(W' - \widehat{W}')^2 * \nu + \sum_{t \leq \cdot} (1 - a_t) (\widehat{W}'_t)^2 \in \mathcal{A}_{\text{loc}}^+$  for the compensator of the right-hand side of equation (4.4).

2 $\Rightarrow$ 1: By [11, Theorem II.1.33] and [9, (3.67)],

$$X := W' * (\mu - \nu) + W'' * \mu - W''' * \nu - \sum_{t \leq \cdot} (1 - \mu(\{t\} \times E)) \widehat{W}_t 1_{\{|\widehat{W}_t| > 1\}}$$

is a well-defined semimartingale. Let  $(D_n)_{n \in \mathbf{N}}$  be a  $\sigma$ -localizing sequence for  $W''' \in L_\sigma(\nu)$  in the sense of Lemma 4.1. Fix  $n \in \mathbf{N}$ . By [11, Proposition II.1.28], we have that  $W''' 1_{D_n} \in G_{\text{loc}}(\mu)$  and  $W''' 1_{D_n} * \mu - W''' 1_{D_n} * \nu = W''' 1_{D_n} * (\mu - \nu)$ . Observe that

$$\begin{aligned} X^{D_n} - (W' + W''') 1_{D_n} * (\mu - \nu) &= \sum_{t \leq \cdot} \left( (1 - a_t) \int W(t, x) \mu(\{t\} \times dx) - (1 - \mu(\{t\} \times E)) \widehat{W}_t \right) 1_{\{|\widehat{W}_t| > 1\}} 1_{D_n}(t). \end{aligned}$$

If we denote this process by  $M$ , then we have  $E(\Delta M_T | \mathcal{F}_{T-}) = ((1 - a_T) \widehat{W}_T - (1 - a_T) \widehat{W}_T) 1_{\{|\widehat{W}_T| > 1\}} 1_{D_n}(T) = 0$  on  $\{T < \infty\}$  for any predictable stopping time  $T$  (cf. [11, Relation II.1.26]). From Proposition 4.1 it follows that  $M$  and hence  $X^{D_n}$  is a purely discontinuous  $\sigma$ -martingale. By Propositions 4.1 and 2.1 applied to the class of purely discontinuous local martingales, this is also true for  $X$ . For the jumps we have

$$\begin{aligned} \Delta X_t &= (W' + W''')_t \widetilde{\phantom{W}} + (1 - a_t) \int W(t, x) \mu(\{t\} \times dx) 1_{\{|\widehat{W}_t| > 1\}} \\ &\quad - (1 - \mu(\{t\} \times E)) \widehat{W}_t 1_{\{|\widehat{W}_t| > 1\}} \\ &= (W' + W''')_t \widetilde{\phantom{W}} + (1 - a_t) \widetilde{W}_t 1_{\{|\widehat{W}_t| > 1\}} + \widetilde{1}_t \widehat{W}_t 1_{\{|\widehat{W}_t| > 1\}} \\ &= \left( W' + W''' + (1 - a) W 1_{\{|\widehat{W}| > 1\}} + \widehat{W} 1_{\{|\widehat{W}| > 1\}} \right)_t \widetilde{\phantom{W}} = \widetilde{W}_t \end{aligned}$$

on  $D_n$  and hence outside some evanescent set. It follows that  $W \in G_\sigma(\mu)$  and  $X = W * (\mu - \nu)$ .

Lemma 4.4 is proved.

The following lemma extends [11, Proposition II.1.30] and [7, Proposition A.1.5] to general stochastic integration.

LEMMA 4.5. *Let  $W = (W^1, \dots, W^d)$ , where  $W^i \in G_\sigma(\mu)$  for  $i = 1, \dots, d$ . Moreover let  $H$  be an  $\mathbf{R}^d$ -valued predictable process. Then  $H^\top W \in G_\sigma(\mu)$  if and only if  $H \in L(W * (\mu - \nu))$ . In this case  $(H^\top W) * (\mu - \nu) = H^\top \cdot (W * (\mu - \nu))$ , where the stochastic integral of  $W$  is to be understood componentwise.*

*Proof.*  $\Rightarrow$ : Define  $X := (H^\top W) * (\mu - \nu)$  and let  $(D_n)_{n \in \mathbf{N}}, (D'_n)_{n \in \mathbf{N}}$  be  $\sigma$ -localizing sequences as in Lemma 4.3 for  $H^\top W \in G_\sigma(\mu)$  and  $W \in G_\sigma(\mu)$ , respectively. Without loss of generality,  $H$  is bounded on  $D_n$  for any  $n \in \mathbf{N}$ . Then we have

$$\begin{aligned} X^{D_n \cap D'_n} &= 1_{D'_n} \cdot ((H^\top W 1_{D_n}) * (\mu - \nu)) = (H^\top W 1_{D_n \cap D'_n}) * (\mu - \nu) \\ &= (H 1_{D_n})^\top \cdot ((W 1_{D'_n}) * (\mu - \nu)) = (H 1_{D_n})^\top \cdot (1_{D'_n} \cdot (W * (\mu - \nu))) \\ &= (H 1_{D_n \cap D'_n})^\top \cdot (W * (\mu - \nu)) \end{aligned}$$

by [7, Proposition A.1]. In view of Lemma 2.2, the claim follows.

$\Leftarrow$ : Define  $X := H^\top \cdot (W * (\mu - \nu))$ . Let  $(D_n)_{n \in \mathbf{N}}, (D'_n)_{n \in \mathbf{N}}$  be  $\sigma$ -localizing sequences as in Lemma 2.2 for  $H \in L(W * (\mu - \nu))$  and as in Lemma 4.3 for  $W \in G_\sigma(\mu)$ , respectively. Then we have

$$\begin{aligned} X^{D_n \cap D'_n} &= 1_{D'_n} \cdot ((H 1_{D_n})^\top \cdot (W * (\mu - \nu))) = (H 1_{D_n \cap D'_n})^\top \cdot (W * (\mu - \nu)) \\ &= (H 1_{D_n})^\top \cdot (1_{D'_n} \cdot (W * (\mu - \nu))) = (H 1_{D_n})^\top \cdot ((W 1_{D'_n}) * (\mu - \nu)) \\ &= (H^\top W 1_{D_n \cap D'_n}) * (\mu - \nu) \end{aligned}$$

by [7, Proposition A.1]. In view of Lemma 4.3, the claim follows.

Lemma 4.5 is proved.

Finally, we state and prove the main result of this section.

**THEOREM 4.1.** *Any  $\mathbf{R}^d$ -valued  $\sigma$ -martingale  $X$  can be written as*

$$X = X_0 + X^c + x * (\mu - \nu),$$

where  $\mu$  denotes the measure of jumps of  $X$ , and the stochastic integral of  $x = (x^1, \dots, x^d)$  is to be understood componentwise.

*Proof.* Without loss of generality  $d = 1$ . The canonical semimartingale representation of  $X$  is  $X = X_0 + X^c + h * (\mu - \nu) + (x - h(x)) * \mu + B$  for some predictable process  $B \in \mathcal{V}$  and some truncation function  $h: \mathbf{R}^d \rightarrow \mathbf{R}^d$  (cf. [11, Theorem II.2.34]). Let  $(D_n)_{n \in \mathbf{N}}$  denote a  $\sigma$ -localizing sequence of  $X$ . Fix  $n \in \mathbf{N}$ . Since  $X^{D_n} = X_0^{D_n} + (X^c)^{D_n} + h1_{D_n} * (\mu - \nu) + (x - h(x))1_{D_n} * \mu + B^{D_n}$  is a local martingale, we have that  $(x - h(x))1_{D_n} * \mu \in \mathcal{A}_{\text{loc}}$ , and its compensator  $(x - h(x))1_{D_n} * \nu \in \mathcal{A}_{\text{loc}}$  coincides with  $-B^{D_n}$  (cf. [11, section I.4.23]). By [11, Proposition II.1.28],  $(x - h(x))1_{D_n} \in G_{\text{loc}}(\mu)$  and  $(x - h(x))1_{D_n} * \mu + B^{D_n} = (x - h(x))1_{D_n} * (\mu - \nu)$ , which implies that  $(X - X_0 - X^c)^{D_n} = x1_{D_n} * (\mu - \nu)$ . In view of Lemma 4.3, it follows that  $x \in G_\sigma(\mu)$  and  $x * (\mu - \nu) = X - X_0 - X^c$ . Theorem 4.1 is proved.

**5. Equivalent  $\sigma$ -martingale measures.** Delbaen and Schachermayer [5] relate the absence of free lunches to the existence of equivalent  $\sigma$ -martingale measures. Below we show how these measures can be characterized in terms of semimartingale characteristics. To this end, let  $S = (S^1, \dots, S^d)$  be an  $\mathbf{R}^d$ -valued semimartingale which stands for the discounted prices of securities  $1, \dots, d$ . Moreover, let  $P^* \stackrel{\text{loc}}{\sim} P$  be a probability measure with density process  $Z$ . By  $N := \mathcal{L}(Z) := (1/Z_-) \cdot Z$  we denote the stochastic logarithm of  $Z$ , i.e., the unique semimartingale  $N$  with  $\mathcal{E}(N) = Z$  (cf. [13] for details on the stochastic logarithm; note that  $Z$  and  $Z_-$  are positive by  $P^* \stackrel{\text{loc}}{\sim} P$  and [11, Definition I.2.27]). Suppose that the characteristics  $(B, C, \nu)$  of the  $\mathbf{R}^{d+1}$ -valued semimartingale  $(S, N)$  relative to  $h: \mathbf{R}^{d+1} \rightarrow \mathbf{R}^{d+1}$  are given in the form

$$(5.1) \quad B = b \cdot A, \quad C = c \cdot A, \quad \nu = A \otimes F,$$

where  $A \in \mathcal{A}_{\text{loc}}^+$  is a predictable process,  $b$  is a predictable  $\mathbf{R}^{d+1}$ -valued process,  $c$  is a predictable  $\mathbf{R}^{(d+1) \times (d+1)}$ -valued process whose values are nonnegative, symmetric matrices, and  $F$  is a transition kernel from  $(\Omega \times \mathbf{R}_+, \mathcal{P})$  into  $(\mathbf{R}^{d+1}, \mathcal{B}^{d+1})$ .

The Girsanov–Jacod–Mémén theorem as stated in [11, Theorem III.3.24] indicates how the characteristics change if  $P$  is replaced with  $P^*$ . Here, we formulate this result in terms of the joint characteristics of  $(S, N)$ , which is convenient for applications.

**LEMMA 5.1.** *The  $P^*$ -characteristics of  $(S, N)$  are of the form (5.1), but with*

$$b^* = b + c \cdot^{d+1} + \int h(x) x^{d+1} F(dx), \quad c^* = c, \quad \frac{dF^*}{dF}(x) = 1 + x^{d+1}$$

instead of  $b, c, F$ .

*Proof.* First, according to [11, Lemma III.3.31], there exists a predictable  $\mathbf{R}^{d+1}$ -valued process  $\beta$  such that  $\beta^\top c \beta \in L(A)$  and  $c^{i,d+1} \cdot A = \langle S^{i,c}, N^c \rangle = (c^i \beta) \cdot A$  for  $i = 1, \dots, d$ . It follows that  $\langle Z^c, S^{i,c} \rangle = Z_- \cdot \langle N^c, S^{i,c} \rangle = (Z_- c^i \beta) \cdot A$ .

Second, we have  $Z_t = Z_{t-}(1 + \Delta N_t) = Z_{t-}(1 + x^{d+1})$  for  $\mu^{(S,N)}$ -almost all  $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^{d+1}$ , which implies that  $E(ZU * \mu_\infty^{(S,N)}) = E((1 + x^{d+1})Z_-U * \mu_\infty^{(S,N)})$  for any nonnegative  $(\mathcal{P} \otimes \mathcal{B}^{d+1})$ -measurable function  $U$ . Put differently, we have  $YZ_- = M_{\mu^{(S,N)}}^P(Z|\tilde{\mathcal{P}})$  in the sense of [11, section III.3c] for  $Y(\omega, t, x) := 1 + x^{d+1}$ . By [11, Theorem III.3.24], we are done.

**DEFINITION 5.1.** *We call  $P^*$  an equivalent martingale measure (EMM) (respectively, equivalent local martingale measure (ELMM), equivalent  $\sigma$ -martingale measure (E $\sigma$ MM)) if  $S^i$  is a martingale (respectively, local martingale,  $\sigma$ -martingale) relative to  $P^*$ , for  $i = 1, 2, \dots, d$ .*

*Remark.* In the paper by Delbaen and Schachermayer [4] the term *equivalent local martingale measure* is used in the above sense. Kramkov and Schachermayer [15] and Becherer [1], however, apply the same name to denote measures  $P^* \sim P$  such that  $1 + \varphi^\top \cdot S$  is a  $P^*$ -local martingale for any  $\varphi \in L(S)$  with  $1 + \varphi^\top \cdot S \geq 0$ . In order to avoid confusion, we prefer to call measures of such kind *equivalent weak local martingale measures*. One easily shows that any  $\sigma$ -martingale measure is a weak local martingale measure while the converse is not true. Nevertheless, the existence of an equivalent weak local martingale measure suffices to ensure condition NFLVR (cf. [1, Proposition 2.3]).

For the proof of Lemma 5.2 we need the following proposition, which extends [11, Proposition III.3.8] to the  $\sigma$ -martingale case.

**PROPOSITION 5.1.** *A real-valued semimartingale  $X$  is a  $P^*$ -martingale (respectively,  $P^*$ -local martingale,  $P^*$ - $\sigma$ -martingale) if and only if  $XZ$  is a  $P$ -martingale (respectively,  $P$ -local martingale,  $P$ - $\sigma$ -martingale).*

*Proof.* By [11, Proposition III.3.8], it suffices to prove the assertion for  $\sigma$ -martingales. Let  $(D_n)_{n \in \mathbf{N}}$  be an increasing sequence of predictable sets. By partial integration we have

$$\begin{aligned} 1_{D_n} \cdot (XZ) &= 1_{D_n} \cdot (X_- \cdot Z) + 1_{D_n} \cdot (Z_- \cdot X) + 1_{D_n} \cdot [X, Z] \\ &= (1_{D_n} X_-) \cdot Z + (1_{D_n} \cdot X) Z - (1_{D_n} \cdot X)_- \cdot Z. \end{aligned}$$

$\Rightarrow$ : Let  $(D_n)_{n \in \mathbf{N}}$  be a  $\sigma$ -localizing sequence for the  $P^*$ - $\sigma$ -martingale  $X$ . Then  $1_{D_n} \cdot (XZ)$  is a  $P$ -local martingale. Therefore,  $XZ \in (\mathcal{M}_{\text{loc}}^P)_\sigma = \mathcal{M}_\sigma^P$ , i.e., it is a  $P$ - $\sigma$ -martingale.

$\Leftarrow$ : This follows similarly if  $(D_n)_{n \in \mathbf{N}}$  now denotes a  $\sigma$ -localizing sequence for the  $P$ - $\sigma$ -martingale  $XZ$ .

We are now ready to characterize EMM's, ELMM's, and E $\sigma$ MM's in terms of semimartingale characteristics.

**LEMMA 5.2.** 1.  *$S$  is a  $P^*$ -martingale if and only if  $SZ \in \mathcal{D}$  relative to  $P$  and*

$$(5.2) \quad b^i + c^{i,d+1} + \int (x^i(1+x^{d+1}) - h^i(x)) F(dx) = 0 \quad \text{for } i = 1, \dots, d$$

*holds  $(P \otimes A)$ -almost everywhere on  $\Omega \times \mathbf{R}_+$ .*

2.  *$S$  is a  $P^*$ -local martingale if and only if  $\int_{\{|x^i|>1\}} |x^i(1+x^{d+1})| F(dx) \in L(A)$  for  $i = 1, \dots, d$  and condition (5.2) holds  $(P \otimes A)$ -almost everywhere on  $\Omega \times \mathbf{R}_+$ .*

3.  *$S$  is a  $P^*$ - $\sigma$ -martingale if and only if  $\int_{\{|x^i|>1\}} |x^i(1+x^{d+1})| F(dx) < \infty$  for  $i = 1, \dots, d$  and condition (5.2) hold  $(P \otimes A)$ -almost everywhere on  $\Omega \times \mathbf{R}_+$ .*

*Proof.* Statements 2 and 3 follow from Lemmas 5.1 and 3.1.

1.  $\Rightarrow$ : Condition (5.2) follows from statement 2. By Proposition 5.1,  $SZ$  is a  $P$ -martingale and hence in  $\mathcal{D}$  (cf. Lemma 3.1).

$\Leftarrow$ : Fix  $i \in \{1, \dots, d\}$ . Since  $S^i Z$  and  $Z$  are special semimartingales, this is also true for  $Y := (1/Z_-) \cdot (S^i Z - S^i_- \cdot Z) = (1/Z_-) \cdot (Z_- \cdot S^i + [S^i, Z]) = S^i + [S^i, N]$ . From  $\Delta Y = \Delta S^i(1 + \Delta N)$  it follows that  $|x^i(1+x^{d+1})| 1_{\{|x^i(1+x^{d+1})|>1\}} * \nu = |x| 1_{\{|x|>1\}} * \nu^Y \in \mathcal{A}_{\text{loc}}$ , where  $\nu^Y$  denotes the compensator of the measure of jumps of  $Y$ . By [11, Relation II.2.13], we have that  $|x^i(1+x^{d+1})| 1_{\{|x^i|>1\}} * \nu \leq |x^i(1+x^{d+1})| 1_{\{|x^i(1+x^{d+1})|>1\}} * \nu + 1_{\{|x|>1\}} * \nu \in \mathcal{A}_{\text{loc}}$ . Statement 2 yields that  $S$  is a  $P^*$ -local martingale, which in turn implies that  $SZ$  is a  $P$ -local martingale. By Corollary 3.1, we have that  $SZ$  is a  $P$ -martingale. Applying Proposition 5.1 once more yields the claim.

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