Semidirect Products of Finite Group Schemes: Gabriel Quivers and Auslander-Reiten Components

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Jan-Niclas Thiel

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Betreuer und erster Gutachter: Prof. Dr. Rolf Farnsteiner
Zweiter Gutachter: Prof. Dr. Jörg Feldvoss

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Zusammenfassung

Sei G eine zusammenhängende algebraische Gruppe über einem algebraisch abgeschlossenen Körper k. Gilt $\mathrm{char}(k) = 0$, so spiegelt sich die Darstellungstheorie von G sehr gut in derer ihrer Lie-Algebra $\mathfrak{g}$ wider. Im Falle $\mathrm{char}(k) = p > 0$ hingegen ist dies nicht mehr der Fall. Es hat sich hier als nützlich erwiesen, G durch die aufsteigende Folge $(G_r)_{r\geq 1}$ von ihren sogenannten Frobeniuskernen zu studieren. G_r ist ein sogenanntes infinitesimales Gruppenschema, welches nicht mehr eindeutig durch seine k-wertigen Punkte bestimmt ist, es gilt sogar $G_r(k) = \{1\}$ für alle r. Seine Darstellungstheorie ist äquivalent zu der Darstellungstheorie der dualen Hopf-Algebra $kG_r := k[G_r]^*$ des endlich-dimensionalen Koordinatenrings $k[G_r]$. Ist $r = 1$, so ist kG_r isomorph zur universellen restringierten Einhüllenden $\mathrm{U}_0(\mathfrak{g})$ von $\mathfrak{g}$. Somit ist die Darstellungstheorie von G_1 zur Darstellungstheorie der restringierten Lie-Algebra $\mathfrak{g}$ äquivalent. Die Darstellungstheorie von $\mathfrak{g}$ selbst kann grob gesprochen durch dessen Familie $\{\mathrm{U}_\chi(\mathfrak{g}) : \chi \in \mathfrak{g}^*\}$ reduzierter einhüllender Algebren approximiert werden.

Viele Resultate sind bekannt im Fall von reduktiven Gruppen (siehe beispielsweise [41, II]). Im Allgemeinen ist jede Gruppe G eine Erweiterung einer reduktiven Gruppe H durch eine unipotente Gruppe U. Ist U nicht trivial, so ist der 'nächstbessere' Fall jener einer spaltenden Erweiterung. Dies führt zu einem semidirekten Produkt $G = U \rtimes H$. Da der Funktor $G \mapsto G_r$ linksexakt ist, ist der r-te Frobeniuskern von G dann auch das semidirekte Produkt $U_r \rtimes H_r$ der r-ten Frobeniuskerne von U und H. Dies ist genau der Blickwinkel, auf dem diese Arbeit aufbaut. Einfache G_r-Moduln korrespondieren dann zu einfachen H_r-Moduln mittels dem durch die Projektion $G_r \to H_r$ definierten Inflations-Funktor $\mathbf{mod}(H_r) \to \mathbf{mod}(G_r)$ und die projektiv unzerlegbaren G_r-Moduln sind genau von den projektiv unzerlegbaren H_r-Moduln induziert. Wir werden eine konkrete Formel für den Gabriel-Köcher der Hopf-Algebra kG_r herleiten sowie den Inflationsfunktor im Hinblick auf Auslander-Reiten-Folgen untersuchen. Des Weiteren werden wir zeigen, dass der stabile Auslander-Reiten-Köcher von kG_r keine Komponenten von euklidischem Typ aufweisen kann. Diese Resultate werden wir allgemeiner für endliche Gruppenschemata beziehungsweise für gewisse reduzierte Einhüllende von restringierten Lie-Algebren formulieren.

Als Beispiel werden wir die Schrödinger-Gruppe $\mathcal{S} := H \rtimes \mathrm{SL}(2)$, das semidirekte Produkt von der reduktiven Gruppe $\mathrm{SL}(2)$ mit der Heisenberg-Gruppe $H \subseteq \mathrm{SL}(3)$, und ihren Quotient $\overline{\mathcal{S}} \cong \mathbb{G}_a^2 \rtimes \mathrm{SL}(2)$ modulo des Zentrums studieren. Die Gruppe $\mathcal{S}$ wurde bereits näher über dem Körper der komplexen Zahlen untersucht und entspringt aus der Physik. Wir werden zeigen, dass die Gabriel-Köcher der Hopf-Algebren der Frobeniuskerne von $\mathcal{S}$ und $\overline{\mathcal{S}}$ stets zusammenhängend sind und uns reduzierte einhüllende Algebren $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ der Lie-Algebra $\bar{\mathfrak{s}}$ von $\overline{\mathcal{S}}$ näher ansehen.

Abstract

Let G be a connected algebraic group over an algebraically closed field k. If $\mathrm{char}(k) = 0$, then there is a strong correspondence between representations of G and those of its Lie algebra $\mathfrak{g}$. This changes dramatically in the situation $\mathrm{char}(k) = p > 0$ of positive characteristic. In this case, it turned out to be useful to approximate G by the ascending sequence $(G_r)_{r\geq 1}$ consisting of its so-called Frobenius kernels. Each G_r is an infinitesimal group scheme, it is not uniquely determined by its group of k-rational points anymore; in fact, the latter are trivial. Its representation theory is equivalent to that of the dual Hopf algebra $kG_r := k[G_r]^*$ of its finite-dimensional coordinate ring $k[G_r]$. The Hopf algebra kG_1 is isomorphic to the restricted universal enveloping algebra $\mathrm{U}_0(\mathfrak{g})$ of $\mathfrak{g}$ which shows that the representation theory of G_1 is equivalent to that of $\mathfrak{g}$ as a restricted Lie algebra. The representation theory of $\mathfrak{g}$ itself can be approximated by studying the family $\{\mathrm{U}_\chi(\mathfrak{g}) : \chi \in \mathfrak{g}^*\}$ of its reduced enveloping algebras.

Many results are known in case of reductive groups (see for instance [41, II]). In general, every group G is an extension of a reductive group H by a unipotent group U. If U is non-trivial, then the 'next best' case is that this extension splits, which in turn leads to a semidirect product $G = U \rtimes H$. Since the functor $G \mapsto G_r$ is left exact, the rth Frobenius kernel of G is then the semidirect product $U_r \rtimes H_r$ of the Frobenius kernels of U and H. It is exactly this point of view on which this thesis is build upon. Simple G_r-modules correspond to simple H_r-modules via the inflation functor $\mathbf{mod}(H_r) \to \mathbf{mod}(G_r)$ defined by pullback along the projection $G_r \to H_r$ and the principal indecomposable G_r-modules are induced by principal indecomposable H_r-modules. We will also establish a formula for the Gabriel quiver of the Hopf algebra kG_r and analyze the behaviour of the inflation functor in terms of Auslander-Reiten sequences. Furthermore, we will show that the stable Auslander-Reiten quiver of kG_r does not admit components of Euclidean type. We will formulate these results more general for finite group schemes and certain reduced enveloping algebras of restricted Lie algebras.

As a major example, we will consider the Schrödinger group $\mathcal{S} := H \rtimes \mathrm{SL}(2)$, the semidirect product of the reductive group $\mathrm{SL}(2)$ with the Heisenberg group $H \subseteq \mathrm{SL}(3)$, along with its quotient $\overline{\mathcal{S}} \cong \mathbb{G}_a^2 \rtimes \mathrm{SL}(2)$ by the center. The group $\mathcal{S}$ has already been considered over the field of complex numbers and is related to physics. We will show that the Gabriel quivers of the Hopf algebras of the Frobenius kernels of $\mathcal{S}$ and $\overline{\mathcal{S}}$ are all connected and take a closer look at reduced enveloping algebras $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ of the restricted Lie algebra $\bar{\mathfrak{s}} = \mathrm{Lie}(\overline{\mathcal{S}})$.

Contents

1 Outline and Notation

We fix the following notation:

- The set of natural numbers $\mathbb{N}$ does not include 0. We put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- Given $n, m \in \mathbb{N}_0$, we denote by $\delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$ the Kronecker delta.
- Throughout, k denotes a field of positive characteristic $p > 0$ with prime field $\mathbb{F}_p$. Sometimes k is assumed to be perfect or algebraically closed; we will make explicit in the relevant sections, when we require these additional properties.
- Every vector space V is understood to be defined over the field k. Moreover, we denote its dual space $\operatorname{Hom}_k(V, k)$ by V^*.
- The word algebra means associative k-algebra with a unit element.
- If A is an algebra ($\mathcal{G}$ an affine group scheme), we denote by $\mathbf{mod}(A)$ ($\mathbf{mod}(\mathcal{G})$) the category of finite-dimensional left A-modules ($\mathcal{G}$-modules). The unit of A will always act as the identity operator.
- Given a commutative ring R, we denote by $\mathbf{Comm}_R$ the category of commutative R-algebras.

In chapter 2 we introduce everything which is needed to understand this thesis. This also includes a proof of the fact that modules of a smash product $A\#H$ correspond to coherent A-H-modules. Moreover, we sketch a proof of the fact that that the Hopf algebra $k\mathcal{G}$ of a semidirect product $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$ of finite group schemes is isomorphic to the smash product $k\mathcal{N}\#k\mathcal{H}$ of the Hopf algebras of $\mathcal{N}$ and $\mathcal{H}$, respectively.

The main result of chapter 3 is stated in 3.1.8: The presence of a (non-trivial) unipotent normal subgroup $\mathcal{N} \trianglelefteq \mathcal{G}$ of some finite group scheme $\mathcal{G}$ over a perfect field k of characteristic $p \geq 3$ entails the non-existence of Euclidean components of the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$ of the finite-dimensional Hopf algebra $k\mathcal{G} = k[\mathcal{G}]^*$. The proof uses a certain invariant of AR-components of $\Gamma_s(\mathcal{G})$ introduced by Farnsteiner in [23] which relies on the Noetherian topological space $\Pi(\mathcal{G})$ defined by Friedlander and Pevtsova in [29]. Afterwords, we will show that this invariant and others, analogous to those defined in [23], are also valid in the context of reduced enveloping algebras $\mathrm{U}_\chi(\mathfrak{g})$ of restricted Lie algebras $\mathfrak{g}$ over algebraically closed fields. Here, the space $\Pi(\mathcal{G})$ is replaced by the nullcone $V(\mathfrak{g}) := \{x \in \mathfrak{g} : x^{[p]} = 0\}$ of $\mathfrak{g}$. We will then formulate 3.3.11, a version of 3.1.8 in the abovementioned context. Finally, we finish the chapter by giving various applications of our theorem we just developed. In particular, we will see that trigonalizable finite group schemes do not admit components of Euclidean type, we will show (with additional general theory) that the presence of the latter inside the AR-quiver of a Frobenius kernel of a connected algebraic group G readily forces G being reductive of a very special form. In addition, we will give an alternative proof of Okuyama's theorem [49] in case k is algebraically closed.

In chapter 4, we consider an extension $\mathcal{G}$ of $\mathcal{H}$ by $\mathcal{N}$ in the category of finite group schemes. The results of section 4.1 are stated more generally, but mainly we are interested in the case where $\mathcal{N}$ is unipotent and $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$ is a split extension. Here simple $\mathcal{G}$-modules are precisely the images of the simple $\mathcal{H}$-module under the inflation functor, which is the pullback along the projection $\mathcal{G} \to \mathcal{H}$. We will show in 4.1.4 that principal indecomposables of $\mathcal{G}$ are all induced by those of $\mathcal{H}$. Moreover, we will show in 4.1.8 that the Gabriel quiver of $k\mathcal{G}$ is shown to coincide with the generalized McKay quiver $\Gamma_V(\mathcal{H})$, where $V := H^1(\mathcal{N}, k)^*$ is the dual of the first cohomology group of $\mathcal{N}$ with coefficients in k. We will give a more concrete form of the $\mathcal{H}$-module structure of V in case $\mathcal{N}$ is just a vector group on which $\mathcal{H}$ acts linearly. Finally, we will show that the inflation functor usually not sends almost split sequences to almost split sequences. In section 4.2 we will consider the special case of infinitesimal group schemes of height ≤ 1, the restricted Lie algebras. All of the results of section 4.1 will be stated more generally for reduced enveloping algebras $\mathrm{U}_\chi(\mathfrak{g})$, where the defining linear form $\chi \in \mathfrak{g}^*$ vanishes on $\mathfrak{n}$. In section 4.3 we will develop a tool for finding simple modules of reduced enveloping algebras $\mathrm{U}_\chi(\mathfrak{g})$, where $\mathfrak{g}$ is again a certain extension of $\mathfrak{h}$ by $\mathfrak{n}$ ($\mathfrak{g} = \mathrm{Lie}(\mathcal{G}), \mathfrak{n} = \mathrm{Lie}(\mathcal{N})$. Finally, we will prove in section 4.4 a certain 'BGG reciprocity formula' (similar to [41, 11.2/4]) for the category of $(V_a \rtimes G)_r T$-modules, where G is a reductive group with maximal torus T and V a G-module with certain additional properties. The latter properties do not seem too contrived, we will also give an example where these are satisfied.

In our last chapter, we wish to give an application of 4.1.8. As a testing ground, we will consider in section 5.1 semidirect products of closed subgroups $G \subseteq \mathrm{GL}(2)$ with the three-dimensional Heisenberg group $H \subseteq \mathrm{SL}(3)$ and show that the Gabriel quivers of their Frobenius kernels coincide with those of the (Frobenius kernels of the) group $\mathbb{G}_a^2 \rtimes G$ with G acting naturally on $\mathbb{G}_a^2$. We will then explicitly show in section 5.2 that the mentioned quivers are all connected in case $G = \mathrm{SL}(2)$. In section 5.3, we will determine reduced enveloping algebras of the Lie algebras $\overline{\mathfrak{g}}, \overline{\mathfrak{s}}$ of the groups $\overline{\mathcal{G}} := \mathbb{G}_a^2 \rtimes \mathrm{GL}(2), \overline{\mathcal{S}} := \mathbb{G}_a^2 \rtimes \mathrm{SL}(2)$ up to Morita equivalence, where the defining linear form does not vanish on $\mathrm{Lie}(\mathbb{G}_a^2)$. This will also tell us that all Gabriel quivers of reduced enveloping algebras of $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{s}}$ will be connected, no matter how the defining linear form looks like. In the final section 5.4 we then compute the Gabriel quivers explicitly in the remaining cases for $G = \mathrm{SL}(2)$ and for $G = \mathrm{GL}(2)$, where, by way of example, the defining linear form is entirely zero in the latter case.

2 Preliminaries

2.1 Finite dimensional algebras

Throughout this section, vector spaces are understood to be finite-dimensional over k if not otherwise mentioned. We briefly recall standard facts from the representation theory of finite-dimensional k-algebras. These will be used frequently (and mostly) without further reference in this thesis. The reader may consult [2] for more information. Throughout, Λ denotes such an algebra.

Definition 2.1.1. A module $(0) \neq M \in \mathbf{mod}(\Lambda)$ is called

- *simple*, provided $(0), M$ are the only submodules of M.
- *indecomposable*, provided $(0), M$ are the only direct summands of M.

Theorem 2.1.2. *(Krull-Remak-Schmidt)*
A module $M = \bigoplus_{i=1}^{n} a_i M_i$ decomposes into multiples of pairwise non-isomorphic indecomposable modules M_i. If $\bigoplus_{i=1}^{m} b_i N_i$ is another such decomposition, then $m = n$ and there is a permutation $\sigma \in S_n$ such that $N_i \cong M_{\sigma(i)}$ and $b_i = a_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$.

In order to give a classification of $\mathbf{mod}(\Lambda)$, one therefore needs to determine all indecomposable modules and and the morphisms between them. By work of Drozd [8] and Crawley-Boevey [6], finite-dimensional algebras over algebraically closed fields may be divided into three disjoint classes:

Definition 2.1.3. Let k be algebraically closed. The algebra Λ is called

- *representation-finite*, provided there exist only finitely many iso-classes of indecomposable Λ-modules.
- *tame*, provided Λ is not representation-finite and given $d > 0$, there exist $(\Lambda, k[X])$-bimodules $Q_1, \ldots, Q_{m(d)}$ that are finitely-generated free right $k[X]$-modules such that all but finitely many d-dimensional indecomposable Λ-modules are of the form $Q_i \otimes_{k[X]} S$ for some simple $k[X]$-module S and $i \in \{1, \ldots, m(d)\}$.
- *wild*, provided there exists a full embedding $\mathbf{mod}(k\langle x, y\rangle) \hookrightarrow \mathbf{mod}(\Lambda)$, where $k\langle x, y\rangle$ is the free algebra in two generators x, y.

In the wild case, the classification of $\mathbf{mod}(\Lambda)$ is a hopeless endeavour.

Definition 2.1.4. Two finite-dimensional algebras Λ and Γ are *Morita-equivalent*, provided there exists an equivalence $\mathbf{mod}(\Lambda) \cong \mathbf{mod}(\Gamma)$ of categories (with the defining functor being additive).

Example 2.1.5. Let $n \in \mathbb{N}$ be a natural number, then the algebras $\mathrm{Mat}_n(\Lambda)$ and Λ are Morita-equivalent. To see this, one may check that the functors $F : \mathbf{mod}(\mathrm{Mat}_n(\Lambda)) \to \mathbf{mod}(\Lambda)$, $X \mapsto E_{11}.X$ and $G : \mathbf{mod}(\Lambda) \to \mathbf{mod}(\mathrm{Mat}_n(\Lambda))$, $X \mapsto X^n$ are mutually inverse to each other (in the sense that their composites are naturally equivalent to the relevant identities). Here E_{11} denotes the matrix with entry 1 at index $(1,1)$ and all other entries zero. Note that Λ acts on $E_{11}.X$ via $(\Lambda E_{11}).X$ for all $X \in \mathbf{mod}(\mathrm{Mat}_n(\Lambda))$.

Lemma 2.1.6. *(Schur's Lemma)*
Let $S, T \in \mathbf{mod}(\Lambda)$ be simple. Then $\mathrm{Hom}_\Lambda(S,T) = (0)$ provided $S \not\cong T$ and $\mathrm{End}_\Lambda(S)$ is a division algebra over k. If k is algebraically closed, then $\mathrm{End}_\Lambda(S) = k \cdot \mathrm{id}_S$.

Definition 2.1.7. Let M be a Λ-module.

(i) A *filtration of M* is an ascending sequence $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of submodules.

(ii) A *composition series of M* is a filtration $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i-1} is simple for $1 \leq i \leq n$.

Theorem 2.1.8. *(Jordan-Hölder)*
Let $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a composition series of some Λ-module M. Let $S \in \mathbf{mod}(\Lambda)$ be simple. The numbers n and $|\{i \in \{1,\ldots,n\} : M_i/M_{i-1} \simeq S\}|$ do not depend on the choice of the composition series.

In view of the above theorem, the following definition makes sense.

Definition 2.1.9. Let $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a composition series of some Λ-module M. Then n is called the *length $\ell(M)$ of M* and $[M : S] := |\{i \in \{1,\ldots,n\} : M_i/M_{i-1} \cong S\}|$ the *multiplicity of S in M*.

An induction on the length shows:

Lemma 2.1.10. *Let $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a filtration of some Λ-module M. Then $[M : S] = \sum_{i=1}^n [M_i/M_{i-1} : S]$ for all simple Λ-modules S.*

Definition 2.1.11. Let M be a Λ-module.

(i) The *radical, top and socle* of M are defined as follows:

$$\mathrm{Rad}(M) = \bigcap_{N \subseteq M \text{ maximal}} N, \quad \mathrm{Top}(M) := M/\mathrm{Rad}(M), \quad \mathrm{Soc}(M) = \sum_{S \subseteq M \text{ simple}} S$$

(ii) The free Λ-module Λ is also called the *regular Λ-module* and we call $J(\Lambda) := \mathrm{Rad}(\Lambda)$ the *Jacobson radical of Λ*.

(iii) M is called *semi-simple*, provided $M = \mathrm{Soc}(M)$.

(iv) M is called *local*, provided $\mathrm{Top}(M)$ is simple.

(v) The algebra Λ is called *semi-simple* or *local*, provided the regular module has this property.

(vi) M is called *projective*, provided the functor $\mathrm{Hom}_\Lambda(M, -) : \mathbf{mod}(\Lambda) \to$ Abelian Groups is exact.

(vii) M is called *injective*, provided the functor $\mathrm{Hom}_\Lambda(-, M)$ is exact.

The following characterizes the abovementioned properties.

Theorem 2.1.12. *Let M be a Λ-module and denote by $J := \mathrm{Rad}(\Lambda)$ the Jacobson radical of Λ.*

(1) $J \trianglelefteq \Lambda$ is a two-sided ideal.

(2) We have $\mathrm{Rad}(M) = J.M$.

(3) M is semi-simple if and only if $\mathrm{Rad}(M) = (0)$ if and only if there exist simple submodules $S_1, \dots, S_n$ of M such that $M = S_1 \oplus \cdots \oplus S_n$.

(4) Λ is semi-simple if and only if every Λ-module is semi-simple if and only if $J = (0)$.

(5) M is local if and only if M possesses a unique maximal submodule. In particular, local modules are indecomposable.

(6) M is projective if and only if M is a direct summand of a free Λ-module.

(7) Let $S \in \mathbf{mod}(\Lambda)$ be simple. Then $[\mathrm{Top}(M) : S] = \frac{\dim_k \mathrm{Hom}_\Lambda(M,S)}{\dim_k \mathrm{End}_\Lambda(S)}$.

Definition 2.1.13. Let M be a Λ-module. Put $\mathrm{Rad}^0(M) = M$ and $\mathrm{Rad}^i(M) := \mathrm{Rad}(\mathrm{Rad}^{i-1}(M))$ for all $i \in \mathbb{N}$. The descending sequence $(\mathrm{Rad}^i(M))_{i \in \mathbb{N}_0}$ is called the *Loewy series of M*. The minimal n such that $\mathrm{Rad}^n(M) = 0$ is called the *Loewy length* $\ell\ell(M)$ of M. We call M *uniserial*, provided the Loewy series is in fact a composition series of M.

Definition 2.1.14. Let M be a Λ-module.

(i) A pair $(P(M), \varepsilon_M)$ of a projective module $P(M)$ and a surjective Λ-linear map $\varepsilon : P \twoheadrightarrow M$ such that $\ker(\varepsilon_M)$ is *superfluous*, i.e., we have $\ker(\varepsilon_M) \subseteq \mathrm{Rad}(P(M))$, is called a *projective cover of M*.

(ii) We put $\Omega_\Lambda(M) := \ker(\varepsilon_M)$ and call $\Omega_\Lambda : \mathbf{mod}(\Lambda) \to \mathbf{mod}(\Lambda)$ the *Heller operator* of Λ (see also [36]).

(iii) A pair (I_M, ι_M) consisting of an injective Λ-module I_M and an injection $\iota_M : M \hookrightarrow I_M$ of Λ-modules is called an *injective envelope of* M provided $\mathrm{im}(\iota_M) \subseteq I_M$ is *essential*, i.e., it meets every non-zero submodule of I_M.

We take for granted that projective covers and injective envelopes always exist in $\mathbf{mod}(\Lambda)$ and that they are unique up to isomorphism.

Lemma 2.1.15. *Let $(P(M), \varepsilon_M)$ be a projective cover of $M \in \mathbf{mod}(\Lambda)$ and P a projective Λ-module. Then we have $P \cong P(M)$ if and only if* $\mathrm{Top}(P) \cong \mathrm{Top}(M)$.

Definition 2.1.16. Let $M \in \mathbf{mod}(\Lambda)$.

(i) A *projective resolution* $(P_n, \delta_n)_{n\geq 0}$ *of* M is an exact sequence

$$\ldots \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

such that P_n is projective for all $n \geq 0$. If (P_n, δ_n) is a projective cover of $\mathrm{im}(\delta_n)$ for all $n \geq 0$, then $(P_n, \delta_n)_{n\geq 0}$ is called *minimal*.

(ii) An *injective resolution* $(I^n, \delta^n)_{n\geq 0}$ *of* M is an exact sequence

$$0 \longrightarrow M \xrightarrow{\delta^0} I^0 \xrightarrow{\delta^1} I^1 \xrightarrow{\delta^2} I^2 \longrightarrow \ldots$$

such that I^n is injective for all $n \geq 0$. If (I^0, δ^0) is an injective envelope of M and $(I^n, \overline{\delta}^n)$ is an injective envelope of $\mathrm{coker}(\delta^{n-1})$ for all $n \geq 1$ (here $\overline{\delta}^n : \mathrm{coker}(\delta^{n-1}) = I^{n-1}/\mathrm{im}(\delta^{n-1}) \to I^n,\ x + \mathrm{im}(\delta^{n-1}) \mapsto \delta^n(x)$), then $(I^n, \delta^n)_{n\geq 0}$ is called *minimal*.

(iii) Let $(P_n, \delta_n)_{n\geq 0}$ be a minimal projective resolution of M, then the *complexity* $\mathrm{cx}_\Lambda(M)$ *of* M is defined as

$$\mathrm{cx}_\Lambda(M) := \min\{c \in \mathbb{N}_0 \cup \{\infty\} \mid \exists \lambda > 0 : \forall n \geq 1 : \dim_k P_n \leq \lambda n^{c-1}\}$$

Remark 2.1.17. (a) The complexity $\mathrm{cx}_\Lambda(M)$ of M can also be defined via a minimal injective resolution.

(b) If $(P_n, \delta_n)_{n\geq 0}$ is a minimal projective resolution of M, then $\Omega_\Lambda^{n+1}(M) \cong \ker(\delta_n)$ for all $n \geq 0$.

The regular module $\Lambda = \bigoplus_{i=1}^n m_i P_i$ decomposes into projective indecomposable modules ($P_i \not\cong P_j$ for $i \neq j$). The modules P_i give rise to a complete list of representatives for the isomorphism classes of projective indecomposable Λ-modules and therefore also called *principal indecomposable* Λ*-modules* (short: PIM's). Simple modules correspond to the latter:

Lemma 2.1.18. *Let $P_1, \dots, P_n$ and $m_1, \dots, m_n$ be as above.*

(1) P_i is a local module for $1 \leq i \leq n$.

(2) $S_i := P_i/\mathrm{Rad}(P_i)$ for $1 \leq i \leq n$ is a full set of representatives for the iso-classes of simple Λ-modules.

(3) We have $m_i = \frac{\dim_k S_i}{\dim_k \mathrm{End}_\Lambda(S_i)}$.

(4) We have $[M : S_i] = \frac{\dim_k \mathrm{Hom}_\Lambda(P_i,M)}{\dim_k \mathrm{End}_\Lambda(S_i)}$ for all $M \in \mathbf{mod}(\Lambda)$ and all $1 \leq i \leq n$.

Definition 2.1.19. Let $P_1, \dots, P_n$ and $m_1, \dots, m_n$ be as above. Then Λ is called *basic*, provided $m_i = \frac{\dim_k S_i}{\dim_k \mathrm{End}_\Lambda(S_i)} = 1$ for all $1 \leq i \leq n$.

Remark 2.1.20. Two basic algebras Λ and Γ are Morita-equivalent if and only if $\Lambda \cong \Gamma$.

There exists a unique decomposition $\Lambda = \bigoplus_{i=1}^m \mathcal{B}_i$ into two-sided indecomposable ideals $\mathcal{B}_i \trianglelefteq \Lambda$, the *block decomposition of* Λ. Given $i \in \{1, \dots, m\}$, there exists a *central primitive idempotent* $e_i \in C(\Lambda)$ (the center of Λ) such that $\mathcal{B}_i = \Lambda e_i$. Each $\mathcal{B}_i$ is itself a finite-dimensional k-algebra with identity e_i. Given an indecomposable Λ-module M, there exists exactly one $i \in \{1, \dots, m\}$ such that $e_i.M \neq (0)$ and we say that M *belongs to the block* $\mathcal{B}_i$. The category $\mathbf{mod}(\Lambda)$ decomposes into the direct sum of the categories $\mathbf{mod}(\mathcal{B}_i)$.

Given $M, N \in \mathbf{mod}(\Lambda)$ and $n \in \mathbb{N}_0$, we let $\mathrm{Ext}^n_\Lambda(M,N)$ denote the nth extension group of M by N. By definition, it coincides with the nth cohomology group of the complex $\mathrm{Hom}_\Lambda(M, \mathbb{E}_N)$ where $\mathbb{E}_N$ is a 'deleted' injective resolution of N (one simply cuts of the morphism originating in N in an injective resolution $\mathbb{E}$ of N, see also [59, Def. 2.52]). It can also be defined as the nth cohomology group of the complex $\mathrm{Hom}_\Lambda(\mathbb{P}_M, N)$, where $\mathbb{P}_M$ is a 'deleted' projective resolution of M (see [59, Theorem 2.7.6]). We have $\mathrm{Ext}^0_\Lambda(M,N) \cong \mathrm{Hom}_\Lambda(M,N)$ and $\mathrm{Ext}^1_\Lambda(M,N)$ can be identified with equivalence-classes of *extensions of M by N*, where the latter are by definition exact sequences $0 \to N \to E \to M \to 0$ (see [59, Theorem 3.4.3]).

Definition 2.1.21. Let $S(\Lambda)$ be a full set of representatives for the iso-classes of simple Λ-modules. The *Gabriel quiver Q_Λ of* Λ has $S(\Lambda)$ as vertex set and there are $\dim_k \mathrm{Ext}^1_\Lambda(S,T)$ arrows $[S] \to [T]$. We say that Λ *is connected*, provided the underlying graph of Q_Λ enjoys this property.

The connected components of Q_Λ correspond to blocks of Λ:

Lemma 2.1.22. Λ *is connected if and only if* Λ *has exactly one block.*

The following type of algebras is very important.

Definition 2.1.23. Let $Q = (Q_0, Q_1)$ be a *quiver*, i.e., a directed graph with set of vertices Q_0 and set of arrows Q_1.

(i) We denote by kQ the vector space with basis the set of all paths in Q; here we also interpret vertices $e \in Q_0$ as paths of length zero. Given two paths p, q with starting points $s(p), s(q)$ and endpoints $t(s), t(p)$, we define their product pq to be the concatenation if $t(q) = s(p)$ and zero otherwise. Extending this multiplication linearly to all of kQ, one obtains an associative algebra with identity element $1 := \sum_{e \in Q_0} e$, the *path algebra* kQ *relative to* Q.

(ii) If kQ is as above, we denote by $kQ_{\geq n}$ the subspace generated by all paths of length $\geq n$ for all $n \in \mathbb{N}_0$.

Remark 2.1.24. kQ is finite-dimensional if and only if Q_0, Q_1 are finite and Q does not contain an oriented circle. If kQ is finite-dimensional, then it is a basic algebra. Moreover, its Jacobson radical is given by $kQ_{\geq 1}$ and $\{kQe : e \in Q_0\}$ is a complete set of principal indecomposable modules.

It turns out that, over an algebraically closed field, every algebra can be described by some (quotient of a) path algebra. More precisely:

Theorem 2.1.25. *Let k be algebraically closed and $P_1, \ldots, P_n$ be a complete set of principal indecomposable Λ-modules. Put $P := \bigoplus_{i=1}^n P_i \in \mathbf{mod}(\Lambda)$ and consider the algebra $B := \mathrm{End}_\Lambda(P)^{op}$.*

(1) B is a basic algebra.

(2) The algebras Λ and B are Morita-equivalent.

(3) Denote by Q the Gabriel quiver of Λ and let $n := \ell\ell(\Lambda)$. Then there exists an ideal $I \trianglelefteq k[Q]$ of the path algebra $k[Q]$ relative to Q such that $k[Q]_{\geq n} \subseteq I \subseteq k[Q]_{\geq 2}$ and $B \cong k[Q_\Lambda]/I$.

Remark 2.1.26. Part (2) is due to *K. Morita*, while (3) is due to *P. Gabriel*. In view of 2.1.20, any other basic algebra B', which is Morita equivant to Λ, is isomorphic to B. We call B *the basic algebra of* Λ.

Homomorphisms between algebras induce certain exact functors between their module categories.

Definition 2.1.27. Let $\varphi : \Gamma \to \Lambda$ be a homomorphism of algebras and N a Λ-module.

(i) N obtains the structure of a Γ-module via $a.n := \varphi(a).n \;\; \forall\, a \in \Gamma, n \in N$. The corresponding Γ-module is denoted $\varphi^*(N)$. In this way, we obtain an exact functor $\varphi^* : \mathbf{mod}(\Lambda) \to \mathbf{mod}(\Gamma)$, the *pullback along* φ.

(ii) If $\Gamma \subseteq \Lambda$ is a subalgebra and φ is the inclusion $\Gamma \hookrightarrow \Lambda$, we put $\mathrm{Res}_\Gamma^\Lambda := \mathrm{Res}_\Gamma := \varphi^*$ and call this functor the *restriction to* Γ.

(iii) If $\Gamma = \Lambda$ and φ is bijective, we put $N^\varphi := (\varphi^{-1})^*(N)$ and call this module the *twist of N by φ*.

In view of (iii) from above, the automorphism group $\mathrm{Aut}(\Lambda)$ of Λ acts on the category $\mathbf{mod}(\Lambda)$ by auto-equivalences. In particular, twisting will commute with the Heller operator, takes projectives to projectives, simples to simples, and indecomposables to indecomposables.

Lemma 2.1.28. *Let $\varphi : \Gamma \to \Lambda$ be a homomorphism of algebras with kernel $I := \ker(\varphi)$.*

(1) If $\psi : \Lambda \to \Upsilon$ is another homomorphism, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

(2) If φ is surjective, then φ^* induces an equivalence $\mathbf{mod}(\Lambda) \longrightarrow \mathbf{mod}_I(\Gamma)$, where $\mathbf{mod}_I(\Gamma)$ denotes the full subcategory of $\mathbf{mod}(\Gamma)$ consisting of all $M \in \mathbf{mod}(\Gamma)$ such that $I.M = (0)$.

There are two important constructions which enable us to construct modules in a natural way out of modules of subalgebras.

Lemma 2.1.29. *Let $\Gamma \subseteq \Lambda$ be a subalgebra and N a Γ-module.*

(1) The vector space $\Lambda \otimes_\Gamma N$ obtains the structure of a Λ-module via

$$a.(b \otimes n) := ab \otimes n, \quad \forall a, b \in \Lambda, n \in N$$

*The module is denoted $\mathrm{Ind}_\Gamma^\Lambda(N)$ and called an **induced module**. In this way, we obtain a functor $\mathrm{Ind}_\Gamma^\Lambda(-) : \mathbf{mod}(\Gamma) \to \mathbf{mod}(\Lambda)$, the **induction functor**.*

*(2) There is a natural equivalence $\mathrm{Hom}_\Lambda(\mathrm{Ind}_\Gamma^\Lambda(N), -) \cong \mathrm{Hom}_\Gamma(N, -) \circ \mathrm{Res}_\Gamma$ of functors. For any $M \in \mathbf{mod}(\Lambda)$, this assignment is defined via $\varphi \mapsto (n \mapsto \varphi(1 \otimes n))$ for all $\varphi \in \mathrm{Hom}_\Lambda(\mathrm{Ind}_\Gamma^\Lambda(N), M)$ **(Frobenius reciprocity or Adjoint Isomorphism Theorem)**. In particular, induction takes projectives to projectives.*

(3) The vector space $\mathrm{Hom}_\Gamma(\Lambda, N)$ obtains the structure of a Λ-module via

$$(a.\varphi)(b) = \varphi(ba) \; \forall a, b \in \Lambda, \varphi \in \mathrm{Hom}_\Gamma(\Lambda, N).$$

*The module is denoted $\mathrm{Coind}_\Gamma^\Lambda(N)$ and called a **coinduced module**. In this way, we obtain a functor $\mathrm{Coind}_\Gamma^\Lambda(-) : \mathbf{mod}(\Gamma) \to \mathbf{mod}(\Lambda)$, the **coinduction functor**.*

*(4) There is a natural equivalence $\mathrm{Hom}_\Lambda(-, \mathrm{Coind}_\Gamma^\Lambda(N)) \cong \mathrm{Hom}_\Gamma(-, N) \circ \mathrm{Res}_\Gamma$ of functors. For any $M \in \mathbf{mod}(\Lambda)$, this assignment is defined via $\varphi \mapsto (m \mapsto \varphi(m)(1))$ for all $\varphi \in \mathrm{Hom}_\Lambda(M, \mathrm{Coind}_\Gamma^\Lambda(N))$ **(Frobenius reciprocity II)**. In particular, coinduction takes injectives to injectives.*

Lemma 2.1.30. *Let Λ and Γ be algebras and denote by $\Delta := \Lambda \otimes_k \Gamma$ their tensor product.*

(1) Let M be a Λ-module, N a Γ-module. Then the vector space $M \otimes_k N$ obtains the structure of a Δ-module via

$$(\lambda \otimes \gamma).(m \otimes n) := \lambda.m \otimes \gamma.n, \quad \forall m \in M, n \in N, \lambda \in \Lambda, \gamma \in \Gamma$$

(2) Let $S_1, \ldots, S_n$ and $T_1, \ldots, T_m$ be complete sets of representatives for the iso-classes of simple Λ and Γ-modules, respectively. Then $\{S_i \otimes_k T_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a complete set of representatives for the iso-classes of simple Δ-modules.

Proof. (1) Follows from the universal property of Δ (the actions of Λ and Γ on $M \otimes_k N$ clearly commute).

(2) See [7, Section 10.E].

□

An important class of algebras is the following.

Definition 2.1.31. An algebra Λ is called *self-injective*, provided the regular module is injective.

Theorem 2.1.32. *Let Λ be self-injective and M a Λ-module.*

(1) M is injective if and only if M is projective.

(2) The ***Nakayama functor*** *$\mathcal{N}_\Lambda : \mathbf{mod}(\Lambda) \to \mathbf{mod}(\Lambda)$, $M \mapsto \mathrm{Hom}_\Lambda(M, \Lambda)^*$ permutes the simple Λ-modules. We have $S \cong \mathcal{N}_\Lambda(\mathrm{Soc}(P(S)))$ for every simple Λ-modules S.*

(3) The Heller operator Ω_Λ induces a bijection $\mathrm{ind}(\Lambda)_{\mathrm{pf}} \to \mathrm{ind}(\Lambda)_{\mathrm{pf}}$ on the set $\mathrm{ind}(\Lambda)_{\mathrm{pf}}$ of iso-classes of non-projective indecomposable Λ-modules. The inverse of this assignment is given by $\mathrm{ind}(\Lambda)_{\mathrm{pf}} \to \mathrm{ind}(\Lambda)_{\mathrm{pf}}, [M] \mapsto \mathrm{coker}(\iota_M)$, where (I_M, ι_M) is an injective hull of M.

Lemma 2.1.33. *Let Λ be self-injective and $M \in \mathbf{mod}(\Lambda)$ projective free (no non-zero projective direct summands). Then $\mathrm{Ext}^n_\Lambda(M, S) \cong \mathrm{Hom}_\Lambda(\Omega^n(M), S)$ for all $n \in \mathbb{N}_0$ amd every simple Λ-module S. In particular, we have*

$$\mathrm{Ext}^1_\Lambda(S, T) \cong \mathrm{Hom}_\Lambda(\mathrm{Rad}(P(S))/\mathrm{Rad}^2(P(S)), T)$$

for all simple modules S, T.

We now introduce the following class of self-injective algebras. All algebras, that play a role in this thesis, are of this kind.

Definition 2.1.34. An algebra Λ is called *Frobenius (symmetric)*, provided there exists a non-degenerate, bilinear (symmetric) form $(\cdot,\cdot) : \Lambda \times \Lambda \to k$, which is *associative*, i.e., we have $(ab,c) = (a,bc) \ \forall\, a,b,c \in \Lambda$.

Let Λ be Frobenius. The departure from symmetry is measured by the *Nakayama automorphism* $\mu : \Lambda \to \Lambda$, which is given by $(b,a) = (\mu(a),b) \ \forall\, a,b \in \Lambda$. By [50, Satz 1], passage to another non-degenerate, associative form changes the Nakayama automorphism by an inner automorphism (conjugation by a unit $c \in \Lambda^\times$). As twisting by inner automorphisms reproduces modules (in the sense that a twist results in an isomorphic module), all Nakayama automorphisms act on $\mathbf{mod}(\Lambda)$ in the same fashion. For this reason, one speaks of 'the' Nakayama automorphism of Λ.

Remark 2.1.35. Let Λ be Frobenius.

(1) Some authors define a Nakayama automorphism ν via $(b,a) = (a,\nu(b))$. Then we have $\nu = \mu^{-1}$.

(2) The map $a \mapsto (\cdot,a)$ is an isomorphism $\Lambda \cong \Lambda^*$ of Λ-modules, where Λ^* is the dual relative to the right regular representation of Λ. In particular, Λ is self-injective.

Example 2.1.36. The following are examples of Frobenius algebras.

(1) Let G be a finite group. Then the form (below we only describe its values on basis vectors)
$$kG \times kG \to k,\ (g,h) \mapsto \delta_{gh,e}$$
renders kG a symmetric algebra.

(2) Let $n \in \mathbb{N}$, consider the truncated polynomial ring $\Lambda_n := k[X]/(X^n)$ along with its canonical basis $\{1, X + (X^n), \dots, X^{n-1} + (X^n)\}$. Then the form
$$\Lambda_n \times \Lambda_n \to k,\ (X^i + (X^n), X^j + (X^n)) \mapsto \delta_{n-1,i+j}$$
renders Λ_n a symmetric algebra.

(3) Let $n \in \mathbb{N}$, then $\Lambda := \mathrm{Mat}_n(k)$ is a symmetric algebra with corresponding form $(x,y) \mapsto \mathrm{tr}(xy)$ for all $x,y \in \Lambda$. Here $\mathrm{tr}(x) = x_{11} + \cdots + x_{nn}$ is the trace of $x = (x_{ij}) \in \mathrm{Mat}_n(k)$.

(4) Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra (soon more details on that) and $\chi \in \mathfrak{g}^*$ a linear form. Then [25, V.4.3] shows that the reduced enveloping algebra $\mathrm{U}_\chi(\mathfrak{g})$ is a Frobenius algebra. Since J. Schue's computation in [54, Lemma 3] remains

true for arbitrary linear forms, the Nakayama automorphism $\mu : \mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_\chi(\mathfrak{g})$ is uniquely determined by $\mu(x) = x + \mathrm{tr}(\mathrm{ad}(x))$ for all $x \in \mathfrak{g}$.

(5) We will later see in 2.3.9, that every finite-dimensional Hopf algebra is Frobenius.

One should note that associative forms of an arbitrary algebra Λ correspond to linear forms $\lambda \in \Lambda^*$. Each such λ is mapped to the form $(\cdot, \cdot)_\lambda : \Lambda \times \Lambda \to k,\ (x, y) \mapsto \lambda(xy)$. For instance, the corresponding linear forms in 2.1.36 (1),(2),(3) are the elements $e^* : kG \to k,\ g \mapsto \delta_{g,e}$, $(X^{n-1} + (X^n))^* : \Lambda_n \to k,\ X^i + (X^n) \mapsto \delta_{i,n-1}$ and $\mathrm{tr} : \mathrm{Mat}_n(k) \to k$, respectively.

In the following, all material we need is covered in [1].

Definition 2.1.37. A finite-dimensional algebra Λ is called a *Nakayama algebra*, provided all projective and injective indecomposable modules are uniserial (that is, their Loewy series is a composition series).

Lemma 2.1.38. *Let Λ be a Nakayama algebra, M an indecomposable Λ-module. Then there is a principal indecomposable module P and $1 \leq t \leq \ell\ell(P)$, such that $M \cong P/\mathrm{Rad}^t(P)$. In particular, M is uniserial and Λ is representation-finite.*

Lemma 2.1.39. *Let Λ be a connected Nakayama algebra of Loewy length $\ell\ell(\Lambda) = m$ and with exactly n simple modules up to isomorphism. Then Λ is self-injective if and only if Λ is Morita-equivalent to the bound quiver algebra $k\tilde{A}_{n,0}/(k\tilde{A}_{n,0})_{\geq m}$, here $\tilde{A}_{n,0}$ denotes the circle with n vertices (see also p.14 for a more general definition).*

Easiest examples of Nakayama algebras are truncated polynomial rings $\Lambda_n := k[X]/(X^n)$ for $n \in \mathbb{N}$. The algebra Λ_n is a local symmetric algebra with unique maximal ideal $(X + (X^n)) \trianglelefteq \Lambda_n$. There is a bijection between Λ_n-modules and pairs (V, f) consisting of a finite-dimensional vector space V and a linear map $f : V \to V$ such that $f^n = 0$. The theory of Jordan canonical forms of nilpotent endomorphisms then shows that the modules

$$[i] := \Lambda_n/(X^i + (X^n)) \ \ \forall\, 1 \leq i \leq n$$

give rise to a complete list of indecomposable Λ_n-modules up to isomorphism. Here $k = [1]$ is the trivial module and $[n] = \Lambda_n$ is the regular module. Given a Λ_n-module V, one just has to determine the Jordan decomposition of the nilpotent operator

$$l_{X+(X^n)} : V \to V, v \mapsto (X + (X^n)).v,$$

each Jordan block of size k corresponds to one summand of type $[k]$ in the Krull-Remak-Schmidt decomposition of V.

[1] https://www.math.uni-bielefeld.de/ sek/select/Nakayama-alg1.pdf

Lemma 2.1.40. *Let $n \in \mathbb{N}$ and put $\Lambda_n := k[X]/(X^n)$.*

(1) We have $\Omega([i]) \cong [n-i]$ for all $1 \leq i \leq n-1$.

(2) We have $\operatorname{Ext}^m_{\Lambda_n}(k,k) \cong k$ for all $m \geq 1$.

Proof. (1) Let $i \in \{1, \dots, n\}$. Clearly, $\Lambda_n \to [i]$ is a projective cover of $[i]$. It follows easily from the definition (see the above discussion), that the kernel $(X^i + (X^n)) \trianglelefteq \Lambda_n$ is isomorphic to $[n-i]$.

(2) We have $\operatorname{Ext}^m_{\Lambda_n}(k,k) \cong \operatorname{Hom}_{\Lambda_n}(\Omega^m(k), k)$ (see Lemma 2.1.33). By (1), $\Omega^m(k)$ is isomorphic to k if m is even and isomorphic to the local Λ_n-module $[n-1]$ otherwise. The assertion follows.

□

2.2 Auslander-Reiten theory

Let Λ be a finite-dimensional self-injective algebra. We denote by $\Gamma_s(\Lambda)$ the *(valued) stable Auslander-Reiten quiver of* Λ. The vertices are given by the iso-classes of non-projective indecomposable modules and there is an arrow $[M] \to [N]$ if and only if there is an *irreducible map* $f : M \to N$, that is,

(i) f is neither a split epimorphism nor a split monomorphism.

(ii) If f can be written as $g \circ h$, then h is a split monomorphism or g a split epimorphism.

The quiver $\Gamma_s(\Lambda)$ is equipped with a certain automorphism, the *Auslander-Reiten translation* $\tau = \tau_\Lambda$ which is given by $\Omega^2_\Lambda \circ \mathcal{N}_\Lambda$, where $\Omega = \Omega_\Lambda$ and $\mathcal{N} = \mathcal{N}_\Lambda$ denote the Heller operator and the Nakayama functor of Λ, respectively. If Λ is Frobenius with Nakayama automorphism μ, then the functor $\mathcal{N}$ is equivalent to $(-)^{\mu^{-1}}$, the twist by μ^{-1} (see [2] for instance). If M is a non-projective indecomposable Λ-module, we will denote by

$$\mathfrak{E}_M : 0 \longrightarrow \tau_\Lambda(M) \longrightarrow E_M \xrightarrow{\pi_M} M \longrightarrow 0$$

the unique (up to isomorphism) *Auslander-Reiten sequence terminating in M*; by definition, $\mathfrak{E}_M$ does not split and every morphism $Y \to M$, which is not a split epimorphism, factors through π_M (see also [2, V.1.14]). If $X \to M$ is an arrow and k is algebraically closed, then it carries the valuation (m, m) if and only if X occurs in E_M with multiplicity m (see [2, V.1.3]).

Given a quiver $Q = (Q_0, Q_1)$ with no multiple arrows and loops, where Q_0 is the set of vertices and $Q_1 \subseteq Q_0 \times Q_0$ the set of arrows, respectively, we let $\mathbb{Z}[Q]$ be the stable

[2]https://www.math.uni-bielefeld.de/ sek/select/rf2.pdf

translation quiver with vertex set $\mathbb{Z} \times Q_0$ and arrows

$$(n, s) \to (n, t) \;\; (n, t) \to (n + 1, s)$$

for every arrow $s \to t$ in Q and all $n \in \mathbb{Z}$. The translation τ is given by $\tau(n, x) = (n - 1, x)$ for all $n \in \mathbb{Z}, x \in Q_0$. According to Riedtmann's structure theorem (cf. [52, Struktursatz]), every connected component $\Theta \subseteq \Gamma_s(\Lambda)$ is isomorphic to $\mathbb{Z}[T_\Theta]/\Pi$, where $\Pi \subseteq \mathrm{Aut}(\mathbb{Z}[T_\Theta])$ is an admissible subgroup and T_Θ is a directed tree. The undirected tree $\overline{T}_\Theta$ is uniquely determined by Θ and called the *tree class of* Θ.

Given $(p, q) \in \mathbb{N}_0^2 \setminus \{(1, 1), (0, 0)\}$, we let $\tilde{A}_{p,q}$ denote the quiver, whose underlying graph is the circle with $p + q$ vertices and with exactly p consecutive arrows being clockwise oriented and the remaining q arrows being counterclockwise oriented.

Lemma 2.2.1. *Let $(p, q) \in \mathbb{N}_0^2 \setminus \{(1, 1), (0, 0)\}$, then the stable translation quiver $\mathbb{Z}[\tilde{A}_{p,q}]$ is isomorphic to $\mathbb{Z}[Q]/\langle\alpha_{p,q}\rangle$, where Q is the quiver with underlying graph A_∞^∞ and repeated orientation*

$$\cdots - (p + q) \longrightarrow \ldots \longrightarrow -q \longleftarrow \ldots \longleftarrow 0 \longrightarrow 1 \longrightarrow \ldots \longrightarrow p \longleftarrow \ldots \longleftarrow p + 1 \ldots,$$

and $\alpha_{p,q} \subseteq \mathrm{Aut}(\mathbb{Z}[Q])$ is induced by the automorphism $Q \to Q, x \mapsto x + (p + q)$ given by translation by $p + q$. In particular, the tree class of $\mathbb{Z}[\tilde{A}_{p,q}]$ is A_∞^∞.

Definition 2.2.2. Let Λ be self-injective. A component $\Theta \subseteq \Gamma_s(\Lambda)$ is called

- of *Euclidean type* provided its tree class is a Euclidean diagram (see [58, p.98]) or $\Theta \cong \mathbb{Z}[\tilde{A}_{p,q}]$ for $(p, q) \in \mathbb{N}_0^2 \setminus \{(1, 1), (0, 0)\}$.
- *regular*, provided the middle-terms of the almost split sequences terminating in the vertices of Θ have no non-zero projective summands.

If $\Theta \subseteq \Gamma_s(\Lambda)$ is not regular, there is a non-simple principal indecomposable module P such that Θ contains the *standard almost split sequence*

$$\mathfrak{E}_{P/\mathrm{Soc}(P)} : 0 \longrightarrow \mathrm{Rad}(P) \longrightarrow P \oplus (\mathrm{Rad}(P)/\mathrm{Soc}(P)) \longrightarrow P/\mathrm{Soc}(P) \longrightarrow 0$$

involving P. If $M \in \Gamma_s(\Lambda)$ such that $P|E_M$ (P is a direct summand of E_M), then $\mathfrak{E}_M \cong \mathfrak{E}_{P/\mathrm{Soc}(P)}$ are isomorphic Auslander-Reiten sequences (see [2, V.5.5]). We record the following well-known result.

Lemma 2.2.3. *Let Λ be self-injective. If $\Theta \subseteq \Gamma_s(\Lambda)$ is of Euclidean type, then Θ is not regular.*

Proof. Assume that $\Theta \cong \mathbb{Z}[T_\Theta]/\Pi$ is regular. Then the length-function $\Theta \to \mathbb{N}$, $M \mapsto \ell(M)$ is positive, additive, of bounded growth and unbounded (cf. [4, p.153-154]). Composition with the projection $\mathbb{Z}[T_\Theta] \to \Theta$ then yields a positive, additive and unbounded function on $\mathbb{Z}[T_\Theta]$. By assumption, $\overline{T}_\Theta$ is either a Euclidean tree or $\overline{T}_\Theta = A_\infty^\infty$ and $\Pi \neq \{1\}$. In the former case, [58, Corollary 2.4] yields a contradiction and in the latter one, [4, Proposition p. 155] does the same. Thus, Θ is not regular. $\square$

For the following fact, we refer to [2, X.1.3] for more details (Ω defines a stable auto-equivalence of the category $\mathbf{mod}(\Lambda)$).

Lemma 2.2.4. *Let Λ be self-injective, $\Theta \subseteq \Gamma_s(\Lambda)$ be a component. Then $\Omega(\Theta)$ is another component, which is isomorphic to Θ.*

Definition 2.2.5. Let Λ be a finite-dimensional self-injective algebra. A Λ-module M is called *Ω-periodic (τ-periodic)*, provided there is $n \in \mathbb{N}$ such that $\Omega_\Lambda^n(M) \cong M$ ($\tau_\Lambda^n(M) \cong M$).

If Λ is Frobenius and the Nakayama automorphism μ has finite order (which will be the case for the algebras we are interested in), the notions of periodicity relative to τ and Ω obviously coincide and one just speaks of 'periodic modules'.

In [44], a generalization of Webb's theorem [58, Theorem A] has been proven for self-injective algebras over algebraically closed fields:

Theorem 2.2.6. *Let Λ be a finite-dimensional self-injective algebra over an algebraically closed field. If $\Theta \subseteq \Gamma_s(\Lambda)$ is a component of the stable Auslander Reiten quiver, such that*

(a) Every $[M] \in \Theta$ has finite complexity.

(b) Every $[M] \in \Theta$ is not τ-periodic.

Then $\Theta \cong \mathbb{Z}[\mathcal{T}]$, where $\mathcal{T} \in \{A_\infty, D_\infty, A_\infty^\infty, \tilde{A}_{12}, \tilde{A}_{p,q}\ ((p,q) \neq (1,1),(0,0)), \tilde{D}_n\ (n \geq 4), \tilde{E}_r\ (6 \leq r \leq 8)\}$ (see again [58] for a definition of the latter trees).

Definition 2.2.7. Let Θ be a component of the stable Auslander-Reiten quiver of a finite-dimensional self-injective algebra Λ. Let $M \in \Theta$ be a vertex.

(1) M is *located at the end of* Θ, provided M has exactly one predecessor in Θ.

(2) If $T_\Theta = A_\infty$, then there exists a unique path from the end of Θ towards M. The number of vertices on that path is referred to as the *quasi-length* $\mathrm{ql}(M)$ *of* M. We call M *quasi-simple*, provided $\mathrm{ql}(M) = 1$ (equivalently, M is located at an end of Θ).

2.3 Hopf algebras and smash products

Given an algebra A, an A-module M and linear maps $\varphi : V \to A, \psi : W \to M$ for some vector spaces V, W, we denote by

$$\varphi \hat{\otimes} \psi : V \otimes_k W \to M, \quad v \otimes w \mapsto \varphi(v).\psi(w)$$

the k-linear map given by composition of the usual tensor product $\varphi \otimes \psi$ and the structure map $A \otimes_k M \to M$ of the A-module M. Recall that an algebra A is given by a linear map $m_A : A \otimes_k A \to A$ (associative multiplication) and a homomorphism $\eta_A : k \to A,\ \alpha \mapsto \alpha.1_A$ of k-algebras (unit). There is a dual concept to that and also one which combines both:

Definition 2.3.1. Let H be a vector space over the field k.

(i) H is called a *k-coalgebra* provided there are linear maps $\Delta : H \to H \otimes_k H$ (comultiplication) and $\varepsilon : H \to k$ (counit) such that the following identities hold

$$(\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta \tag{1}$$

$$\mathrm{id}_H \hat{\otimes} \varepsilon = \mathrm{id}_H = \varepsilon \hat{\otimes} \mathrm{id}_H \tag{2}$$

(ii) If H is both, a k-algebra and a k-coalgebra such that one of the following two (equivalent) conditions

- Δ_H and ε_H are homomorphisms of k-algebras.
- m_H and η_H are homomorphisms of k-coalgebras.

holds, then H is called a *bialgebra*.

(iii) A bialgebra H is called a *Hopf algebra* provided there is a k-linear map $S : H \to H$ (usually called *antipode*) such that

$$(S \hat{\otimes} \mathrm{id}_H) \circ \Delta = \varepsilon.1 = (\mathrm{id}_H \hat{\otimes} S) \circ \Delta \tag{3}$$

Remark 2.3.2. Expressions like $\mathrm{id}_H \hat{\otimes} \varepsilon$ fit into the situation explained above: We consider H as a k-module.

We will use the *Sweedler notation* for coproducts and their iterates:

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}, \quad ((\Delta \otimes \mathrm{id}_H) \circ \Delta)(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = ((\mathrm{id}_H \otimes \Delta) \circ \Delta)(h).$$

Identities (2) and (3) then read as follows:

$$\sum_{(h)} h_{(1)}\varepsilon(h_{(2)}) = h = \sum_{(h)} \varepsilon(h_{(1)})h_{(2)}, \quad \sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h).1 = \sum_{(h)} h_{(1)}S(h_{(2)})$$

The Hopf algebras we are interested in, shall all have the following property:

Definition 2.3.3. A coalgebra H is called *cocommutative* provided $\tau_H \circ \Delta_H = \Delta_H$. Here we denote, for any vector space V, by $\tau_V : V \otimes_k V \to V \otimes_k V,\ v \otimes w \mapsto w \otimes v$ the flip.

Let H be a Hopf algebra over k and M, N be H-modules. One advantage of a Hopf algebra is, that there are two additional module constructions available. The vector spaces $M \otimes_k N$ and $\mathrm{Hom}_k(M, N)$ obtain the structure of H-modules via

$$h.m \otimes n = \sum_{(h)} h_{(1)}.m \otimes h_{(2)}.n \qquad\qquad (h.\varphi)(m) = \sum_{(h)} h_{(1)}.\varphi(S(h_{(2)}).m)$$

Lemma 2.3.4. *Let M, N be finite-dimensional H-modules. Then the standard k-linear isomorphism $M^* \otimes_k N \cong \mathrm{Hom}_k(M, N)$, $\varphi \otimes n \mapsto (m \mapsto \varphi(m).n)$ is an isomorphism of H-modules.*

As $\varepsilon : H \to k$ is an algebra homomorphism, the ground field k is an H-module, the action given by $h.1 := \varepsilon(h).1$. The resulting module is often called *trivial module*, we will also denote it by k. If M is an H-module, then (2) implies that the standard k-linear isomorphism $M \otimes_k k \cong M, m \otimes \alpha \mapsto \alpha.m$ is H-linear. Given $n \in \mathbb{N}_0$, we can also consider the cohomology groups

$$H^n(H, M) := \mathrm{Ext}^n_H(k, M)$$

of the augmented algebra (H, ε). Let P be another H-module. We know from standard (bi)linear algebra that there is a natural equivalence $\mathrm{Hom}_k(P \otimes_k M, -) \cong \mathrm{Hom}_k(P, -) \circ \mathrm{Hom}_k(M, -)$ of functors (of type $\mathbf{mod}(H) \to \mathbf{mod}(k)$) which amounts to thinking about a bilinear map $P \times M \to N$ as a family of linear maps $M \to N$ indexed by elements of P. A direct computation shows that this induces an equivalence $\mathrm{Hom}_H(P \otimes_k M, -) \cong \mathrm{Hom}_H(P, -) \circ \mathrm{Hom}_k(M, -)$. Since $\mathrm{Hom}_k(M, -)$ is an exact functor, this implies that $\mathrm{Hom}_H(P \otimes_k M, -)$ enjoys this property if $\mathrm{Hom}_H(P, -)$ does. We have proven the first part of the following lemma:

Lemma 2.3.5. *Let H be a finite-dimensional Hopf algebra and $M, N \in \mathbf{mod}(H)$.*

(1) If $P \in \mathbf{mod}(H)$ is projective, so is $P \otimes_k M$.

(2) There are natural isomorphisms $H^n(H, \mathrm{Hom}_k(M, N)) \cong \mathrm{Ext}^n_H(M, N)$ for all $n \in \mathbb{N}_0$.

Proof.

(2) Let $\mathbb{P}$ be a projective resolution of the trivial H-module k. By (1) and the exactness of $\otimes_k$, we get that $\mathbb{P} \otimes_k M$ is a projective resolution of $k \otimes_k M \cong M$. In view of our remarks above, the two complexes $\mathrm{Hom}_H(\mathbb{P}_k \otimes_k M, N) \cong \mathrm{Hom}_H(\mathbb{P}_k, \mathrm{Hom}_k(M, N))$ are isomorphic. Hence there result isomorphisms in cohomology. □

Remark 2.3.6. The Lemma is still valid if the modules or even the Hopf algebra is infinite-dimensional.

We define $H^\dagger := \ker(\varepsilon)$, the *augmentation ideal of* H. When H is finite-dimensional, the block $\mathcal{B}_0(H)$ to which the trivial module k belongs is called the *principal block of* H.

Let $\varphi, \psi : H \to \Lambda$ be two linear maps whose codomain is some k-algebra Λ. Then we define their *convolution* $\varphi * \psi : H \to \Lambda$ via

$$(\varphi * \psi)(h) = \sum_{(h)} \varphi(h_{(1)})\psi(h_{(2)}) \ \ \forall h \in H$$

Recall that the Hopf algebra H is determined by structure maps

$$\begin{array}{lll} \eta : k \to H \text{ (unit)}, & m : H \otimes_k H \to H \text{ (multiplication)}, & \varepsilon : H \to k \text{ (counit)} \\ \Delta : H \to H \otimes_k H \text{ (comultiplication)} & S : H \to H \text{ (antipode)}, & \end{array}$$

which have to fulfill laws that can be expressed in commutative diagrams. Let H be finite-dimensional. Dualizing all these structure maps, we obtain after the usual identifications $k^* \to k, \varphi \mapsto \varphi(1)$, $H^* \otimes_k H^* \to (H \otimes_k H)^*$, $\varphi \otimes \psi \mapsto (h \otimes k \mapsto \varphi(h).\psi(k))$ linear maps

$$\begin{array}{lll} \eta^* : H^* \to k, & m^* : H^* \to H^* \otimes_k H^*, & \varepsilon^* : k \to H^* \\ \Delta^* : H^* \otimes_k H^* \to H^*, & S^* : H^* \to H^* & \end{array}$$

Dualization of the abovementioned diagrams then shows that $H^* = \mathrm{Hom}_k(H, k)$ also obtains the structure of a finite-dimensional Hopf algebra with multiplication Δ^*, unit ε^*, comultiplication $m^* =: \Delta'$, counit η^* and antipode S^*. The multiplication coincides with the convolution product $*$; the identity element of the algebra H^* is given by the counit $\varepsilon : H \to k$ of H. Explicitly, we have

$$\begin{aligned} (\varphi * \psi)(h) &= \sum_{(h)} \varphi(h_{(1)})\psi(h_{(2)}) && \forall\, h \in H, \\ \Delta'(\varphi) &= \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)} \Longleftrightarrow \varphi(ab) = \sum_{(\varphi)} \varphi_{(1)}(a)\varphi_{(2)}(b) && \forall a, b \in H. \end{aligned}$$

Remark 2.3.7. Let H be a Hopf algebra.

(1) Even if H is infinite-dimensional, the dual space H^* still becomes a k-algebra with multiplication given by convolution.

(2) As the dual of the flip is given by $\tau_H^* = \tau_{H^*}$, we see that H^* is commutative (cocommutative) if and only if H is cocommutative (commutative).

Definition 2.3.8. Let H be a Hopf algebra, then the space $\int_H := \{x \in H \mid \forall h \in H : hx = \varepsilon(h)x\}$ is called the *space of left integrals of* H.

Part (1) of the following has been proven by Sweedler [56], part (2) can be found in [27, Lemma 1.5].

Theorem 2.3.9. *Let H be a finite-dimensional Hopf algebra.*

*(1) We have $\dim_k \int_H = 1$. In particular, there exists an algebra homomorphism $\zeta : H \to k$ such that $xh = \zeta(h)x$ for all $h \in H, x \in \int_H$. We call ζ the **left modular function of** H.*

(2) H is a Frobenius algebra with Nakayama automorphism (of finite order) given by $\mu := S^{-2} \circ (\mathrm{id}_H * \zeta)$.

Until the end of this section, we fix a Hopf algebra H.

Definition 2.3.10. (i) A k-algebra (coalgebra, bialgebra) A, which is an H-module such that the multiplication $m_A : A \otimes_k A \to A$ and the unit $\eta_A : k \to A$ (comultiplication and counit, (co)multiplication and (co)unit) are homomorphisms of H-modules is referred to as an *H-module algebra (coalgebra, bialgebra)*.

(ii) If B is another H-module algebra (coalgebra, bialgebra), then $\mathrm{Alg}_H(A, B)$ denotes the set of all *morphisms of H-module algebras (coalgebras, bialgebras)*. By definition, it consists of H-linear algebra (coalgebra, bialgebra)-homomorphisms.

Let A be a k-algebra. By [47, Proposition 2.5 (a)], every algebra homomorphism $f : H \to A$ turns A into an H-module algebra with operation given by

$$h.a = \sum_{(h)} f(h_{(1)}) \cdot a \cdot f(S(h_{(2)})).$$

The corresponding H-module algebra is denoted by A_f and we will call the corresponding action *adjoint action of H on A with respect to f*. In particular, taking $f = \mathrm{id}_H$, we see that H itself is an H-module algebra, the *adjoint representation of H*.

Some details of the following fact can be found in [45, p.50 below], [47, 2.9].

Theorem 2.3.11. *Let A be an H-module algebra. Then the vector space $A \otimes_k H$ obtains the structure of a k-algebra with multiplication determined by*

$$(a \otimes h) * (b \otimes h') = \sum_{(h)} a(h_{(1)}.b) \otimes h_{(2)}.h'$$

*and identity element $1 \otimes 1$. The corresponding algebra is called the **smash product of A and H.***

The smash product will be denoted by $A\#H$ and one writes $a\#h$ for $a \otimes h$. There are injective algebra homomorphisms

$$\iota_A : A \hookrightarrow A\#H,\ a \mapsto a\#1, \qquad \iota_H : H \hookrightarrow A\#H,\ h \mapsto 1\#h$$

and a direct computation reveals that $\iota_A \in \mathrm{Alg}_H(A, (A\#H)_{\iota_H})$.

We proceed by proving a universal property of the smash product.

Lemma 2.3.12. *Let A be an H-module algebra. If B is any k-algebra with maps $g \in \mathrm{Alg}(H, B)$ and $f \in \mathrm{Alg}_H(A, B_g)$, then there exists a unique map $f\#g \in \mathrm{Alg}(A\#H, B)$ such that $f = (f\#g) \circ \iota_A$ and $g = (f\#g) \circ \iota_H$.*

Proof. Using the universal property of the tensor product, we obtain a linear map defined through

$$f\#g : A\#H \to B,\ a\#h \mapsto f(a)g(h)$$

which clearly has the required properties. To check that this map is an algebra homomorphism, we only need to consider products of simple tensors. We have

$$\begin{aligned}
(f\#g)((a\#h) * (b\#k)) &= \sum_{(h)} f(a)f(h_{(1)}.b)g(h_{(2)})g(k) \\
&= f(a)\left(\sum_{(h)} g(h_{(1)})f(b)g(S(h_{(2)}))g(h_{(3)})\right)g(k) && \text{as } f \in \mathrm{Alg}_H(A, B_g) \\
&= f(a)\left(\sum_{(h)} g(h_{(1)})\varepsilon(h_{(2)})f(b)\right)g(k) && \text{as } g \in \mathrm{Alg}(H, B), (3) \\
&= f(a)g(h)f(b)g(k) = (f\#g)(a\#h)(f\#g)(b\#k) && \text{by } (2)
\end{aligned}$$

Uniqueness: Let φ be another algebra homomorphisms satisfying the two mentioned conditions. Then

$$\varphi(a\#h) = \varphi((a\#1) * (1\#h)) = \varphi(a\#1)\varphi(1\#h) = f(a)g(h) = (f\#g)(a\#h)$$

Hence $\varphi = f\#g$. $\square$

Definition 2.3.13. Let A be an H-module algebra and let M be an H-module and an A-module. If the structure map $A \otimes M \to M$ is a homomorphism of H-modules, then M is called a *coherent A-H-module.*

We define the category $\mathbf{cohmod}(A, H)$ as follows:

- The objects are the coherent A-H-modules.
- We put $\mathrm{Hom}_{\mathbf{cohmod}(A,H)}(M, N) := \mathrm{Hom}_A(M, N) \cap \mathrm{Hom}_H(M, N)$ for given $M, N \in \mathbf{cohmod}(A, H)$.

The universal property tells us exactly, which conditions a vector space M needs to be a module for the smash product $A\#H$. We shall see below, that this is the one we just defined.

Theorem 2.3.14. *Let A be an H-module algebra.*

(1) If M is a coherent A-H-module, then M obtains the structure of an $A\#H$-module, denoted $F(M)$, via $(a\#h).m := a.(h.m)$.

(2) If M is an $(A\#H)$-module, then M obtains the structure of a coherent A-H module, denoted $G(M)$, via $a.m := (a\#1).m$ and $h.m := (1\#h).m$.

(3) The functors $F : \mathbf{cohmod}(A, H) \to \mathbf{mod}(A\#H)$ and $G : \mathbf{mod}(A\#H) \to \mathbf{cohmod}(A, H)$, defined as above, are inverse to each other. In particular, the categories $\mathbf{mod}(A\#H)$ and $\mathbf{cohmod}(A, H)$ are equivalent.

Proof. An $(A\#H)$-module M corresponds to an algebra homomorphism $\rho : A\#H \to \mathrm{End}_k(M)$. Thus, taking B as the algebra $\mathrm{End}_k(M)$ consisting of endomorphisms of the k-vector space M, we use Lemma 2.3.12 to see that this is equivalent to

(i) M is an H-module via $\rho_H := \rho \circ \iota_H$.

(ii) M is an A-module via $\rho_A := \rho \circ \iota_A$.

(iii) $\rho_A \in \mathrm{Alg}_H(A, B_{\rho_H})$.

We are left to show that (iii) is equivalent to $A \otimes M \to M,\ a \otimes m \mapsto \rho_A(a)(m)$ being H-linear, which means

$$(iv)\ \ \rho_H(h) \circ \rho_A(a) = \sum_{(h)} \rho_A(h_{(1)}.a) \circ \rho_H(h_{(2)}) \ \ \forall h \in H, a \in A$$

Writing out (iii) means

$$\rho_A(h.a) = h.\rho_A(a) = \sum_{(h)} \rho_H(h_{(1)}) \circ \rho_A(a) \circ \rho_H(S(h_{(2)})) \ \ \forall h \in H, a \in A$$

$(iii) \implies (iv)$: We have

$$\begin{aligned}
\sum_{(h)} \rho_A(h_{(1)}.a) \circ \rho_H(h_{(2)}) &= \sum_{(h)} h_{(1)}.\rho_A(a) \circ \rho_H(h_{(2)}) && (iii)\\
&= \sum_{(h)} \rho_H(h_{(1)}) \circ \rho_A(a) \circ \underbrace{\rho_H(S(h_{(2)})) \circ \rho_H(h_{(3)})}_{=\rho_H(S(h_{(2)})h_{(3)})} && (iii)\\
&= \sum_{(h)} \rho_H(h_{(1)}) \circ \rho_A(a) \circ \varepsilon(h_{(2)})\mathrm{id}_B && \text{by (3)}\\
&= \rho_H(h) \circ \rho_A(a) && \text{by (2)}
\end{aligned}$$

$(iv) \implies (iii)$: In that case, we have

$$\begin{aligned}
h.\rho_A(a) &= \sum_{(h)} \rho_H(h_{(1)}) \circ \rho_A(a) \circ \rho_H(S(h_{(2)}))\\
&= \sum_{(h)} \rho_A(h_{(1)}.a) \circ \rho_H(h_{(2)}) \circ \rho_H(S(h_{(3)})) && (iv)\\
&= \rho_A(h.a) && \text{by (3) and (2)}
\end{aligned}$$

The assertions follow directly from this discussion. □

So far, we didn't require A to be a Hopf algebra. If we add this (and some other) condition, then [47, Theorem 2.13] shows:

Theorem 2.3.15. *Let H be a cocommutative Hopf algebra and A a Hopf algebra, which is an H-module bialgebra. Then the smash product $A\#H$ obtains the structure of a Hopf algebra with structure maps*

$$\begin{aligned}
&\varepsilon : A\#H \longrightarrow k,\ a\#h \mapsto \varepsilon_A(a).\varepsilon_H(h)\\
&\Delta : A\#H \longrightarrow A\#H \otimes_k A\#H,\ a\#h \mapsto \sum_{(a),(h)} a_{(1)}\#h_{(1)} \otimes a_{(2)}\#h_{(2)}\\
&S : A\#H \longrightarrow A\#H,\ \ a\#h \mapsto \sum_{(h)} S_H(h_{(2)}).S_A(a) \# S_H(h_{(1)})
\end{aligned}$$

The natural maps ι_H, ι_A are maps of Hopf algebras.

We finish this section with the following useful observation. One may recall that if A is commutative, then every left A-module X obtains the structure of a right A-module via setting $x.a := a.x$ for all $a \in A, x \in X$ and vice versa.

Lemma 2.3.16. *The following statements holds:*

(a) A is an $A\#H$-module via the left regular representation of A and the given H-module structure.

(b) If M is an $A\#H$-module and N an H-module, then the H-module $M \otimes_k N$ obtains the structure of an $A\#H$-module via $a.(m \otimes n) := a.m \otimes n$.

(c) We have a natural transformation

$$\tau : A \otimes_k - \longrightarrow (A\#H) \otimes_H -, \;\; \tau_N(a \otimes n) = \iota_A(a) \otimes n$$

If A is finite-dimensional and the extension $A\#H : \iota_H(H)$ is free of rank $\dim_k A$, then τ is a natural equivalence (The relevant functors are considered as functors $\mathbf{mod}(H) \longrightarrow \mathbf{mod}(A\#H)$).

(d) Let M, N be $(A\#H)$-modules. If A is commutative, then $M \otimes_A N$ has the structure of an $(A\#H)$-module, where A acts via $a.(m \otimes n) := a.m \otimes n (= m \otimes a.n)$ and the H-module structure is induced by the tensor product $M \otimes_k N$.

Proof. (a) If we let A act via the left regular representation, then the structure map is the multiplication m_A, which is H-linear as A is an H-module algebra.

(b) Both, the given structure map $\mu : A \otimes_k M \to M$ and $\mathrm{id}_N : N \to N$, are H-linear. Thus, their tensor product

$$\mu \otimes \mathrm{id}_N : A \otimes_k M \otimes_k N \to M \otimes_k N$$

is H-linear as well.

(c) By (a) and (b), the transformation τ is well-defined. Given $N \in \mathbf{mod}(H)$, the map

$$\varphi : A \otimes_k N \to (A\#H) \otimes_H N, \; a \otimes n \mapsto \iota_A(a) \otimes n$$

is k-linear and surjective. Moreover, it is clearly A-linear so that we need to show the H-linearity. We have

$$\begin{aligned}
\varphi(h.a \otimes n) &= \sum_{(h)} \iota_A(h_{(1)}.a) \otimes h_{(2)}.n \\
&= \sum_{(h)} \iota_H(h_{(1)}) * \iota_A(a) * \iota_H(S(h_{(2)})) \otimes h_{(3)}.n \quad \iota_A \in Alg_H(A, (A\#H)_{\iota_H}) \\
&= \sum_{(h)} \iota_H(h_{(1)}) * \iota_A(a) \otimes \iota_H(S(h_{(2)})h_{(3)}).n \\
&= \iota_H(h) * \iota_A(a) \otimes n = h.\varphi(a \otimes n) \qquad (3), (2)
\end{aligned}$$

The additional condition ensures that both spaces have the same dimension so that φ must be an isomorphism in that case.
If now $f : N \to N'$ is H-linear, then $\mathrm{id}_A \otimes f : A \otimes N \to A \otimes N$ is also H-linear

and obviously A-linear. Consequently, $A \otimes_k - : \mathbf{mod}(H) \longrightarrow \mathbf{mod}(A\#H)$ defines a functor and it is easily seen now that τ is a natural transformation.

(d) Let $\nu : A \otimes_k N \to N$ and $\mu : M \otimes_k A \to M$ be the corresponding structure maps of the left A-module N and the right A-module M (recall that A is commutative). Then $M \otimes_A N$, the cokernel of the H-linear map

$$(\mu \otimes \mathrm{id}_N - \mathrm{id}_M \otimes \nu) : M \otimes_k A \otimes_k N \to M \otimes_k N,$$

is itself an H-module in a natural way. As in (a), we see that we obtain a coherent A-H-module in this way.

□

Remark 2.3.17. The proofs of (b) and (d) are taken from [45, Lemma 2.2, 2.6].

2.4 Affine group schemes

We will mainly use the books [41, 57] written by J.Jantzen and W.Waterhouse, respectively, as a standard reference for group schemes and their representations.

A *k-functor* is a functor from the category $\mathbf{Comm}_k$ of commutative k-algebras to the category **Sets** of sets. Given $A \in \mathbf{Comm}_k$, we will denote the functor $B \mapsto \mathrm{Hom}(A, B)$ by $\mathrm{Spec}_k(A)$. A functor $F : \mathbf{Comm}_k \to \mathbf{Sets}$ is called *representable*, provided there is $A \in \mathbf{Comm}_k$ and a natural equivalence $F \cong \mathrm{Spec}_k(A)$. In this case, F is also called an *affine scheme over k* and, if (additionally) A is finitely generated as an algebra, then F is called *algebraic*. The following is also known as *Yoneda's Lemma* (cf. [57, Theorem 1.3]):

Theorem 2.4.1. *Let $A, B \in \mathbf{Comm}_k$ and put $F := \mathrm{Spec}_k(A), G := \mathrm{Spec}_k(B)$. The assignments*

$$\Gamma : \mathrm{Mor}(F, G) \to \mathrm{Hom}(B, A),\ \tau \mapsto \tau_A(\mathrm{id}_A),$$
$$\Phi : \mathrm{Hom}(B, A) \to \mathrm{Mor}(F, G),\ \Phi(\varphi)_C : F(C) \to G(C),\ f \mapsto f \circ \varphi\ \forall\ C \in \mathbf{Comm}_k,$$

are inverse to each other. In this way, natural equivalences correspond to isomorphisms and composites go over to composites in the reverse direction.

Let F, A be as above. Yoneda's Lemma implies that the algebra A is uniquely determined by F up to isomorphism. We write $A = k[F]$ and call A the *coordinate ring of F*. If G is another affine scheme represented by B and $\tau : F \to G$ a natural transformation, then we denote the corresponding homomorphism $\Gamma(\tau) : B \to A$ by τ^* and call it the *comorphism of τ*.

Definition 2.4.2. A representable functor $\mathcal{G} : \mathbf{Comm}_k \longrightarrow \mathbf{Sets}$ is called an *affine group scheme over* k, provided $\mathcal{G}$ takes values in the category **Groups** of groups.

If $\mathcal{G}$ is an affine group scheme, then the coordinate ring $k[\mathcal{G}]$ is a commutative Hopf algebra (cf. [57, 1.4]).

Given $n \in \mathbb{N}$, we denote by $\mathbb{A}^n : \mathbf{Comm}_k \to \mathbf{Sets},\ A \mapsto A^n$ the *full affine space of dimension* n with coordinate ring given by the polynomial ring $k[X_1, \dots, X_n]$ in n variables. Every vector space V gives rise to a k-functor

$$V_a = V \otimes_k - : \mathbf{Comm}_k \longrightarrow \mathbf{Sets},\ V_a(A) = V \otimes_k A,\ V_a(\lambda) = \mathrm{id}_V \otimes \lambda.$$

If $V = k^n$, then of course $V_a \cong \mathbb{A}^n$.

Lemma 2.4.3. *Let V be a finite-dimensional vector space.*

(1) The functor V_a is an affine algebraic scheme with coordinate ring $S(V^*)$, the symmetric algebra of the dual space V^*.

(2) Every linear map $f : V \to W$ of finite-dimensional vector spaces induces a morphism $\tau_f : V_a \to W_a$ of affine schemes. Moreover, f is an isomorphism if and only if τ_f enjoys this property.

(3) If $B = \{b_1, \dots, b_n\}$ is a basis of V, then the coordinate isomorphism $C_B : V \to k^n$, $\sum_{i=1}^n \lambda_i b_i \mapsto (\lambda_1, \dots, \lambda_n)$ induces an isomorphism $V_a \cong \mathbb{A}^n$ of schemes.

Proof. (1) Since V is finite-dimensional, there are natural equivalences (the last one being the universal property of the symmetric algebra over V^*)

$$V \otimes_k - \cong (V^*)^* \otimes_k - \cong \mathrm{Hom}_k(V^*, -) \cong \mathrm{Spec}_k(S(V^*))$$

As any basis of V^* generates $S(V^*)$ as an algebra, the claim follows.

(2) The collection of maps $f \otimes \mathrm{id}_A : V \otimes_k A \to W \otimes_k A$ for all $A \in \mathbf{Comm}_k$ gives rise to a natural transformation $V_a \to W_a$ which coincides with f at k. The additional statement is clear.

(3) We clearly have $(k^n)_a \cong \mathbb{A}^n$, now apply (2).

□

Remark 2.4.4. (1) The lemma shows that $(-)_a : \mathbf{mod}(k) \to \{\text{affine schemes over } k\}$ is a functor.

(2) Equipped with addition, each V_a is clearly an affine algebraic group scheme. In particular, $(-)_a$ takes values in the category of (commutative) affine algebraic group schemes.

Let $\mathcal{G}$ be an affine group scheme. If $A \in \mathbf{Comm}_k$, we will denote the extended scheme $\mathbf{Comm}_A \to \mathbf{Groups}$, $B \mapsto \mathcal{G}(B)$, $\lambda \mapsto \mathcal{G}(\lambda)$ by $\mathcal{G}_A$ (cf. [57, 1.6]). Its coordinate ring is given by $k[\mathcal{G}_A] = k[\mathcal{G}] \otimes_k A$. If $\mathcal{H}$ is a group scheme over A such that $\mathcal{H} = \mathcal{G}_A$, we will say that $\mathcal{H}$ *is defined over* k.

Definition 2.4.5. Let $\mathcal{G}$ be an affine group scheme.

(i) Let X be an affine scheme. A *(left) action of* $\mathcal{G}$ *on* X is a morphism $\mathcal{G} \times X \longrightarrow X$ of affine schemes such that each $\mathcal{G}(A) \times X(A) \to X(A)$ is an action of the group $\mathcal{G}(A)$ on the set $X(A)$ for all $A \in \mathbf{Comm}_k$.

(ii) A vector space V is called a $\mathcal{G}$*-module*, provided there exists an action $\mathcal{G} \times V_a \to V_a$ such that the group $\mathcal{G}(A)$ acts on the A-module $V_a(A) = V \otimes_k A$ via A-linear maps for all $A \in \mathbf{Comm}_k$.

Every $\mathcal{G}$-module V corresponds to a homomorphism $\mathcal{G} \to \mathrm{GL}(V)$. If V is finite-dimensional, then of course $\mathrm{GL}(V) \cong \mathrm{GL}(\dim_k V)$ after choice of a basis. Thus, one-dimensional $\mathcal{G}$-modules correspond to homomorphisms $\mathcal{G} \to \mathrm{GL}(1) = \mathbb{G}_m$, the *characters of* $\mathcal{G}$. Here $\mathbb{G}_m$ denotes the *multiplicative group*:

$$\mathbb{G}_m : \mathbf{Comm}_k \to \mathbf{Groups},\ A \mapsto A^\times = \{a \in A \mid \text{ There is } b \in A \text{ such that } ab = 1\}$$

The collection of all characters $X(\mathcal{G})$ forms an abelian group with pointwise multiplication. One writes this group additively and it is easy to see that (identifying characters with their corresponding one-dimensional $\mathcal{G}$-module) $\lambda \otimes_k \mu = \lambda + \mu$ for $\lambda, \mu \in X(\mathcal{G})$. Consequently, the group $X(\mathcal{G})$ acts on $\mathbf{mod}(\mathcal{G})$ with auto-equivalences, each λ corresponds to the equivalence $F_\lambda : \mathbf{mod}(\mathcal{G}) \to \mathbf{mod}(\mathcal{G})$, $V \mapsto V \otimes_k \lambda$.

Definition 2.4.6. A homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ of affine group schemes is referred to as a *closed embedding*, provided the comorphism $\varphi^* : k[\mathcal{H}] \to k[\mathcal{G}]$ is surjective.

In fact, for every algebraic group scheme $\mathcal{G}$ exists $n \in \mathbb{N}$ and a closed embedding $\mathcal{G} \to \mathrm{GL}(n)$ (cf. [57, Theorem 3.4]).

Mainly, we are interested in the following type of group schemes. A survey, which elaborates more on representation theoretical background, can be found in [21].

Definition 2.4.7. An affine group scheme $\mathcal{G}$ is called *finite (infinitesimal)*, provided $k[\mathcal{G}]$ is finite-dimensional (finite-dimensional and local). We call $k\mathcal{G} := k[\mathcal{G}]^*$ the *Hopf algebra of* $\mathcal{G}$.

The assignment $\mathcal{G} \mapsto k\mathcal{G}$ is in fact an equivalence

$$\{\text{finite group schemes over } k\} \longrightarrow \{\text{finite-dimensional cocommutative Hopf algebras over } k\}$$

of categories. Given a finite group scheme $\mathcal{G}$, the categories $\mathbf{mod}(\mathcal{G})$ and $\mathbf{mod}(k\mathcal{G})$ are also equivalent. Basic module constructions for $\mathcal{G}$ as in [41, I.2.7] will yield the corresponding Hopf-theoretic constructions for $k\mathcal{G}$ as explained in the section before (see [41, I.8]).

Example 2.4.8. The following are two important examples of finite group schemes, the second one will be discussed in more detail later on.

(a) Let G be a finite group. Then the group algebra kG is a finite-dimensional Hopf algebra. Given a basis-element $g \in G \subseteq kG$, the co-multiplication, co-unit and antipode are determined by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$, respectively. Thus, $\mathcal{G}_G := \mathrm{Spec}_k(kG^*)$ is a finite group scheme which is uniquely determined by G, the *constant group scheme of* G (cf. [57, 2.3]).

(b) Let $\mathfrak{g}$ be a d-dimensional restricted Lie algebra defined over the field k of characteristic $p > 0$. Then the restricted universal enveloping algebra $\mathrm{U}_0(\mathfrak{g})$ is a p^d-dimensional Hopf algebra. Given an element $x \in \mathfrak{g} \subseteq \mathrm{U}_0(\mathfrak{g})$, the co-multiplication, co-unit and antipode are determined by $\Delta(x) = 1 \otimes x + x \otimes 1, \varepsilon(x) = 0$ and $S(x) = -x$. Thus, $\mathcal{G}_{\mathfrak{g}} := \mathrm{Spec}_k(\mathrm{U}_0(\mathfrak{g})^*)$ is a finite group scheme which is uniquely determined by $\mathfrak{g}$. Using the PBW-theorem, one can show that $\mathrm{U}_0(\mathfrak{g})^* \cong k[X_1, \dots, X_d]/(X_1^p, \dots, X_d^p)$ is a local algebra. Hence $\mathcal{G}_{\mathfrak{g}}$ is infinitesimal.

Definition 2.4.9. A finite group scheme $\mathcal{G}$ is called *linearly reductive*, provided the algebra $k\mathcal{G}$ is semi-simple.

Infinitesimal group schemes arise from the following construction:

Definition 2.4.10. Let $\mathcal{G} = \mathrm{Spec}_k(A)$ be an algebraic group scheme over our field k of positive characteristic $p > 0$. Given $r \geq 0$, the quotient

$$A_r := A/(\{x^{p^r} : x \in A^\dagger\})$$

is a commutative local finite-dimensional Hopf algebra. The infinitesimal group scheme $\mathcal{G}_r := \mathrm{Spec}_k(A_r)$ is called the *rth Frobenius kernel of* $\mathcal{G}$.

In view of the equivalent definition [41, I.9.4] as the kernel of the rth Frobenius morphism $F^r : \mathcal{G} \to \mathcal{G}^{(r)}$, we see that $\mathcal{G} \mapsto \mathcal{G}_r$ is clearly a left exact functor. If $\mathcal{G}$ is infinitesimal, there exists $r \in \mathbb{N}_0$ such that $\mathcal{G}_r = \mathcal{G}$ and the minimal such r is called the *height of* $\mathcal{G}$.

Given an affine group scheme $\mathcal{G}$, the space

$$\mathrm{Der}_k(k[\mathcal{G}], k) = \{d \in \mathrm{Hom}_k(k[\mathcal{G}], k) \mid d(fg) = \varepsilon(f)d(g) + \varepsilon(g)d(f) \ \forall\, f, g \in k[\mathcal{G}]\}$$

of ε-derivations is a restricted Lie subalgebra of the commutator algebra $(k[\mathcal{G}]^*)^-$ and called the *Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ *of* $\mathcal{G}$ (cf [57, § 12]). Given a morphism $\varphi : \mathcal{G} \to \mathcal{H}$ of affine group schemes, the map $\mathrm{Lie}(\varphi) : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{H}),\ d \mapsto d \circ \varphi^*$ is a homomorphism of restricted Lie algebras. We thus have a functor

$$\mathrm{Lie} : \ \{\text{affine group schemes over } k\} \to \{\text{restricted Lie algebras over } k\}.$$

Remark 2.4.11. Let $\mathcal{G}$ be an affine group scheme.

(1) If $\mathcal{N} \trianglelefteq \mathcal{G}$ is an normal subgroup, then $\mathrm{Lie}(\mathcal{N}) \trianglelefteq \mathrm{Lie}(\mathcal{G})$ is an ideal.

(2) If $\mathcal{G}$ is algebraic, then $\mathrm{Lie}(\mathcal{G}_r) \cong \mathrm{Lie}(\mathcal{G})$ for all $r \geq 1$ (see [41, I.9.6(3)]).

(3) If $\mathcal{G}$ is finite, then $\mathfrak{g}$ coincides with the space $\{f \in k\mathcal{G} \mid \Delta(f) = 1 \otimes f + f \otimes 1\}$ of primitive elements of the Hopf algebra $k\mathcal{G}$.

We let $\dim \mathcal{G} := \dim k[\mathcal{G}]$ be the Krull-dimension of its coordinate ring.

Lemma 2.4.12. *Let k be perfect and $\mathcal{G}$ be an affine group scheme over k. Then* $\mathrm{Lie}(\mathcal{G}) = (0)$ *if and only if $\mathcal{G}$ is finite and reduced.*

Proof. This follows on one hand from the fact that $\dim \mathcal{G} \leq \dim_k \mathfrak{g}$ (cf. [57, p.99 exercise 7]) and on the other hand, we have $\dim(\mathcal{G}) = 0$ if and only if $\mathcal{G}$ is finite ([1, Exercise 8.3]), so that we can apply [57, 11.3, 11.2 (e), Theorem 6.2 (6)] in this case. □

If $\mathcal{G}$ is algebraic, then $\mathcal{G}_1$ corresponds to $\mathfrak{g}$: there is an isomorphism $k\mathcal{G}_1 \cong \mathrm{U}_0(\mathfrak{g})$ of Hopf algebras (cf. [41, I.9.6(4)]). In particular, restricted Lie algebras correspond to infinitesimal group schemes of height ≤ 1.

Remark 2.4.13. This also implies that the functor Lie is left exact provided its 'domain of definition' is the category of affine algebraic group schemes.

Definition 2.4.14. An affine group scheme $\mathcal{G}$ is called *trigonalizable*, provided all simple $\mathcal{G}$-modules are one-dimensional.

Two important classes of trigonalizable group schemes are the following:

Definition 2.4.15. An affine group scheme $\mathcal{G}$ is called

- *unipotent*, provided the trivial module k is (up to isomorphism) the only simple $\mathcal{G}$-module.
- *diagonalizable*, provided $k[\mathcal{G}] \cong kX(\mathcal{G})$ (the group algebra of the character group).

Remark 2.4.16. Let $\mathcal{G}$ be an affine group scheme.

(1) If $\mathcal{G}$ is algebraic and unipotent (diagonalizable), every closed subgroup enjoys the same property (cf. [57, Corollary 8.3, 2.2]).

(2) If $\mathcal{G}$ is finite, then $\mathcal{G}$ is unipotent if and only if $k\mathcal{G}$ is a local algebra. In particular, the scheme $\mathcal{G}_G$ ($\mathcal{G}_{\mathfrak{g}}$) corresponding to a finite group G (finite-dimensional restricted Lie algebra $\mathfrak{g}$) is unipotent if and only if G is a p-group ($\mathfrak{g}$ is unipotent, proven later in 2.5.10).

We denote by e_k the *trivial group scheme*, whose coordinate ring is the field k.

Definition 2.4.17. Let $\mathcal{G}$ be an affine algebraic group scheme. Then $\mathcal{G}$ admits a unique maximal, closed, connected, unipotent, normal subgroup $R_u(\mathcal{G})$, the *unipotent radical of* $\mathcal{G}$. We call $\mathcal{G}$ *reductive*, provided $R_u(\mathcal{G}) = e_k$.

Remark 2.4.18. If $\mathcal{G}$ is infinitesimal of height ≤ 1, then $R_u(\mathcal{G})$ corresponds to the p-radical $\mathrm{Rad}_p(\mathfrak{g})$, the largest unipotent p-ideal of the Lie algebra $\mathfrak{g}$ (see [25, p.68]).

One may recall thatIf $\varphi : \mathcal{G} \to \mathcal{H}$ is a homomorphism of group schemes, then the kernel $\ker(\varphi)$ (defined in the obvious fashion) is a closed, normal subgroup of $\mathcal{G}$. In contrast, the image of φ is defined for $A \in \mathbf{Comm}_k$ by means of (see [41, p. 85], [57, Theorem 15.5] and [41, p.67] for the term 'fppf-A-algebra')

$$\mathrm{im}(\varphi)(A) := \{h \in \mathcal{H}(A) \mid \text{There is an fppf-}A\text{-algebra } B \text{ and } g \in \mathcal{G}(B) \text{ such that } h_B = \varphi_B(g)\}$$

It coincides with the so-called associated k-group faisceaux to the k-group functor $A \mapsto \mathrm{im}(\varphi_A)$.

Given a closed, normal subgroup $\mathcal{N} \trianglelefteq \mathcal{G}$ of an affine algebraic group scheme $\mathcal{G}$, there exists a quotient $\mathcal{G}/\mathcal{N}$ which itself is affine algebraic (cf. [41, I.6.5(1)]). Every homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ such that $\mathcal{N} \subseteq \ker(\varphi)$ will factor through $\mathcal{G}/\mathcal{N}$ and if $\mathcal{N} = \ker(\varphi)$, the map φ will induce an isomorphism $\mathcal{G}/\mathcal{N} \cong \mathrm{im}(\varphi)$. In particular, $\mathrm{im}(\varphi)$ is affine algebraic whenever $\mathcal{G}$ enjoys this property (for more details, see [41, I.5/6]).

If $\mathcal{G}$ is a group scheme and V a $\mathcal{G}$-module, then

$$V^{\mathcal{G}} := \{v \in V \mid g.v \otimes 1 = v \otimes 1 \;\; \forall g \in \mathcal{G}(A), A \in \mathbf{Comm}_k\}$$

is the space of *fixed points of* $\mathcal{G}$ *on* V.

Lemma 2.4.19. *If $\mathcal{N} \trianglelefteq \mathcal{G}$ is a closed, normal subgroup and V a $\mathcal{G}$-module, then $V^{\mathcal{N}} \subseteq V$ is a $\mathcal{G}$-submodule. In particular, if $\mathcal{N}$ is unipotent, then $V^{\mathcal{N}} = V$ provided V is simple.*

Proof. The first assertion follows from [41, Lemma I.3.2]. For the second, we combine the first with [41, I.2.14(8)]. □

We denote by $\mathbf{mod}_{\mathcal{N}}(\mathcal{G})$ the full subcategory of all finite-dimensional $\mathcal{G}$-modules V with $V^{\mathcal{N}} = V$. Thus, we have a functor, the *fixed point functor*:

$$(-)^{\mathcal{N}} : \mathbf{mod}(\mathcal{G}) \longrightarrow \mathbf{mod}_{\mathcal{N}}(\mathcal{G}).$$

We now give a proof of the fact that algebraic trigonalizable groups are in fact extensions of diagonalizable groups by unipotent ones.

Lemma 2.4.20. *The following statements are equivalent for an algebraic group scheme $\mathcal{G}$:*

(a) $\mathcal{G}$ is trigonalizable.

(b) The image of every representation $\rho : \mathcal{G} \to \mathrm{GL}(n)$ can be conjugate into T_n, the group of upper triangular matrices.

(c) There exists a closed embedding $\mathcal{G} \hookrightarrow T_n$.

(d) There exists a unipotent normal subgroup $\mathcal{U} \trianglelefteq \mathcal{G}$ such that $\mathcal{G}/\mathcal{U}$ is diagonalizable.

Proof. $(a) \Longrightarrow (b)$: Consider a composition series $(0) = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ of the corresponding $\mathcal{G}$-module $V = k^n$ (cf. [41, p.34]). By assumption, V_i/V_{i-1} is one-dimensional for all $1 \leq i \leq n$. Hence, there exists a basis $\{v_1, \ldots, v_n\}$ of V such that $\{v_1, \ldots, v_i\}$ is one of V_i for all $1 \leq i \leq n$. Denoting by $S \in \mathrm{GL}(n,k)$ the matrix sending e_i to v_i, the representation $\rho' := \rho \circ \kappa_{S^{-1}} : \mathcal{G} \to \mathrm{GL}(n)$ has image in T_n (base change). Here $\kappa_{S^{-1}} : \mathrm{GL}(n) \to \mathrm{GL}(n)$ is the automorphism of the group scheme $\mathrm{GL}(n)$ given by conjugation with S^{-1}.

$(b) \Longrightarrow (c)$: Apply (b) to a faithful representation of $\mathcal{G}$, which exists by [57, Theorem 3.4] because $\mathcal{G}$ is algebraic.

$(c) \Longrightarrow (d)$: We may assume $\mathcal{G} \subseteq T_n$. Denote by $U_n \trianglelefteq T_n$ the unipotent, normal subgroup consisting of unitriangular matrices. Put $\mathcal{U} := U_n \cap \mathcal{G}$, this is a unipotent normal subgroup of $\mathcal{G}$. As the canonical homomorphism $\mathcal{G} \to T_n/U_n \cong D_n$ (diagonal matrices) has kernel $\mathcal{U}$, we get (d).

$(d) \Longrightarrow (a)$. By Lemma 2.4.19, $\mathcal{U}$ acts trivially on any simple $\mathcal{G}$-module V. Hence V is a simple $\mathcal{G}/\mathcal{U}$-module, which is one-dimensional by [41, I.2.11]. $\square$

Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a morphism of group schemes. Then φ induces an exact functor

$$\varphi^* : \mathbf{mod}(\mathcal{H}) \to \mathbf{mod}(\mathcal{G}),\ V \mapsto \varphi^*(V),$$

the *pullback along* φ (composition of the structure homomorphism with φ). One defines $H^\bullet(\mathcal{G}, -) = (R^n(-)^{\mathcal{G}})_{n \in \mathbb{N}_0}$ to be collection of the right derived functors of the fixed point functor. As $H^\bullet(\mathcal{G}, -)$ is a cohomological δ-functor (see [59, §2]), so is $H^\bullet(\mathcal{G}, -) \circ \varphi^* := (R^n(-)^{\mathcal{G}} \circ \varphi^*)_{n \in \mathbb{N}_0}$ by exactness of φ^*.

Lemma 2.4.21. *Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a morphism of affine group schemes.*

(a) φ induces a morphism $\varphi^\bullet : H^\bullet(\mathcal{H}, -) \to H^\bullet(\mathcal{G}, -) \circ \varphi^*$ of cohomological δ-functors.

(b) Let $\psi : \mathcal{N} \to \mathcal{G}$ be another morphism. Then $(\varphi \circ \psi)^\bullet = \psi^\bullet \circ \varphi^\bullet$.

(c) $\mathrm{id}_{\mathcal{H}}^\bullet = \mathrm{id}_{H^\bullet(\mathcal{H},-)}$.

(d) If there exists a homomorphism $\gamma : \mathcal{H} \to \mathcal{G}$ such that $\varphi \circ \gamma = \mathrm{id}_{\mathcal{H}}$, then the maps $\varphi^n : H^n(\mathcal{H}, X) \to H^n(\mathcal{G}, X)$ are injective for all $n \in \mathbb{N}_0, X \in \mathbf{mod}(\mathcal{H})$.

Proof. (a) We have a natural transformation given by inclusion

$$(-)^{\mathcal{H}} \to (-)^{\mathcal{G}} \circ \varphi^*,\ M^{\mathcal{H}} \mapsto M^{\mathcal{H}} \subseteq \varphi^*(M)^{\mathcal{G}}$$

As $(H^n(\mathcal{H}, -))_{n\in\mathbb{N}_0}$ is universal, this lifts to a morphism of cohomological δ-functors (a natural transformation at each stage $n \in \mathbb{N}_0$, commuting with connecting homomorphisms)

$$\varphi^\bullet : H^\bullet(\mathcal{H}, -) \to H^\bullet(\mathcal{G}, -) \circ \varphi^*.$$

(b) As $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$, the morphisms $(\varphi \circ \psi)^\bullet$ and $\psi^\bullet \circ \varphi^\bullet$ are equal at stage 0. Hence they are equal at any stage by the uniqueness of the lifting.

(c) Use the same argument as in (b).

(d) This follows from (b), (c).

□

If we apply the above to an inclusion $\mathcal{N} \subseteq \mathcal{G}$ of group schemes, then we get *restriction morphisms* $\mathrm{Res}^n(M) : H^n(\mathcal{G}, M) \to H^n(\mathcal{N}, M|_{\mathcal{H}})$ for every $\mathcal{G}$-module M and $n \in \mathbb{N}_0$. If $\mathcal{N} \trianglelefteq \mathcal{G}$ is a normal subgroup and $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{N}$ is the canonical projection, then we get *inflation homomorphisms* $\mathrm{Inf}^n(M) : H^n(\mathcal{G}/\mathcal{N}, M) \to H^n(\mathcal{G}, \pi^*(M))$ for each $\mathcal{G}/\mathcal{N}$-module M and $n \in \mathbb{N}_0$.

The cohomology $H^\bullet(\mathcal{G}, V)$ of an affine group scheme $\mathcal{G}$ with V being a $\mathcal{G}$-module can be computed using the *Hochschild complex* $C^\bullet(\mathcal{G}, V)$ (cf. [41, I.4.14]). Here, we just give the corresponding form of the first cohomology group $H^1(\mathcal{G}, V)$. It is the quotient of the space of *crossed homomorphisms*

$$\{f \in \mathrm{Mor}(\mathcal{G}, V_a) \mid f(gh) = g.f(h) + f(g),\ \forall g, h \in \mathcal{G}(A), A \in \mathbf{Comm}_k\}$$

by the space of *principal crossed homomorphisms*

$$\{f \in \mathrm{Mor}(\mathcal{G}, V_a) \mid \exists v \in V : f(g) = g.v \otimes 1 - v \otimes 1,\ \forall g \in \mathcal{G}(A), A \in \mathbf{Comm}_k\}.$$

We denote by $\mathcal{DG}$ the *derived subgroup of* $\mathcal{G}$. It is a closed, normal subgroup of $\mathcal{G}$ (cf. [57, 10.1]) and coincides with the k-group faisceaux associated to the k-group-functor that maps each $A \in \mathbf{Comm}_k$ to the ordinary derived subgroup $[\mathcal{G}(A), \mathcal{G}(A)]$ of the abstract group $\mathcal{G}(A)$. In the following, given a finite-dimensional vector space V and $r \geq 1$, $V_{a(r)}$ denotes the rth Frobenius kernel of the algebraic group scheme V_a (recall Lemma 2.4.3).

Lemma 2.4.22. *Let V be a finite-dimensional $\mathcal{G}$-module on which $\mathcal{G}$ acts trivially.*

(1) There are isomorphisms $H^1(\mathcal{G}, V) \cong \mathrm{Hom}(\mathcal{G}, V_a) \cong \mathrm{Hom}(\mathcal{G}/\mathcal{DG}, V_a)$.

(2) If $\mathcal{G}$ is infinitesimal of height r, then $H^1(\mathcal{G}, V) \cong \mathrm{Hom}(\mathcal{G}, V_{a(r)})$. In particular, if $r = 1$, we have $H^1(\mathcal{G}, k) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] + \langle \mathfrak{g}^{[p]} \rangle)^*$, where $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$.

Proof. (1) The isomorphism $H^1(\mathcal{G}, V) \cong \mathrm{Hom}(\mathcal{G}, V_a)$ is clear from the above description of $H^1(\mathcal{G}, V)$. As V_a being abelian implies that every homomorphism $f : \mathcal{G} \to V_a$ factors through $\mathcal{G}/\mathcal{DG}$, we get the second isomorphism.

(2) As each homomorphism $\varphi : \mathcal{G} \to V_a$ maps $\mathcal{G}_r$ to $V_{a(r)}$, we get $\mathrm{Hom}(\mathcal{G}, V_a) \cong \mathrm{Hom}(\mathcal{G}, V_{a(r)})$ by assumption. For $r = 1$ and $V = k$, we get $\mathrm{Hom}(\mathcal{G}, \mathbb{G}_{a(1)}) \cong \mathrm{Hom}_{\mathrm{Lie}_p}(\mathfrak{g}, \mathfrak{e}_1) \cong \big(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] + \mathfrak{g}^{[p]}\big)^*$. Here $\mathfrak{e}_1 = \mathrm{Lie}(\mathbb{G}_a)$ denotes the elementary abelian Lie algebra of dimension 1.

□

Remark 2.4.23. If $\mathcal{G}/\mathcal{DG}$ is affine (for instance when $\mathcal{G}$ is algebraic), then the above lemma also shows $H^1(\mathcal{G}, V) \cong H^1(\mathcal{G}/\mathcal{DG}, V)$ for every trivial $\mathcal{G}$-module V.

Lemma 2.4.24. *Let V be a $\mathcal{G}$-module and $\mathcal{N} \trianglelefteq \mathcal{G}$ a closed, normal subgroup and $n \in \mathbb{N}_0$. Then the $\mathcal{G}/\mathcal{N}$-module structure on $V^{\mathcal{N}} = H^0(\mathcal{N}, V)$ can be lifted (in some sense explained below) to $H^n(\mathcal{N}, V)$ for all $n \in \mathbb{N}$.*

Proof. Given $A \in \mathbf{Comm}_k$, each $g \in \mathcal{G}(A)$ yields an endomorphism of the functor

$$(-)^{\mathcal{N}}(A) = (- \otimes_k A) \circ (-)^{\mathcal{N}} \circ \mathrm{Res}^{\mathcal{G}}_{\mathcal{N}} : \mathbf{mod}(\mathcal{G}) \to \mathbf{mod}(A)$$

The theory of δ-functors thus yields a unique endomorphism of each $R^n((- \otimes_k A) \circ (-)^{\mathcal{N}} \circ \mathrm{Res}^{\mathcal{G}}_{\mathcal{N}})$ for all $n \in \mathbb{N}_0$ that specializes for $n = 0$ to the above endomorphism. A consecutive application of [41, I.4.1(2)] and [41, I.4.1(3)] shows

$$\begin{aligned} R^n((- \otimes_k A) \circ (-)^{\mathcal{N}} \circ \mathrm{Res}^{\mathcal{G}}_{\mathcal{N}}) &\cong (- \otimes_k A) \circ R^n((-)^{\mathcal{N}} \circ \mathrm{Res}^{\mathcal{G}}_{\mathcal{N}}) \\ &\cong (- \otimes_k A) \circ H^n(\mathcal{N}, -) \circ \mathrm{Res}^{\mathcal{G}}_{\mathcal{N}}. \end{aligned}$$

Hence $\mathcal{G}$ acts on each $H^n(\mathcal{N}, V)$ if V is a $\mathcal{G}$-module. Moreover, $\mathcal{N}$ must act trivially because the identity on stage 0 lifts to the identity at each stage $n \in \mathbb{N}_0$. □

Remark 2.4.25. Given the situation of the above lemma, the group $\mathcal{G}$ acts on the Hochschild complex $C^\bullet(\mathcal{N}, V)$: If $A \in \mathbf{Comm}_k$ and $f : \mathcal{N}_A^m \to (V \otimes_k A)_a \in \mathrm{Mor}(\mathcal{N}_A^m, (V \otimes_k A)_a) \cong \mathrm{Mor}(\mathcal{N}^m, V) \otimes_k A$, then each $g \in \mathcal{G}(A)$ acts via

$$(g.f)_B(n_1, \ldots, n_m) := g_B.f_B(g_B^{-1}.n_1, \ldots, g_B^{-1}.n_m) \ \ \forall B \in \mathbf{Comm}_A$$

Thus, we get an action on each $H^m(\mathcal{N}, V)$. By [41, I.6.7], this action coincides with the action of $\mathcal{G}$ in the above lemma.

Fix an action $\mathcal{G} \times X \to X$ of an affine group scheme $\mathcal{G}$ on another affine scheme X. Let $A \in \mathbf{Comm}_k$ and $g \in \mathcal{G}(A)$. Let $B \in \mathbf{Comm}_A$, then the structure map $i : A \to B$ is a homomorphism of k-algebras. We put $g_B := \mathcal{G}(i)(g)$. In this way, each $g \in \mathcal{G}(A)$ gives rise to an automorphism

$$g : X_A \longrightarrow X_A, \ x \mapsto g_B.x \ \ \forall x \in X(B), B \in \mathbf{Comm}_A$$

of the affine scheme X_A which gives rise to a comorphism $\lambda(g) : k[X] \otimes_k A \longrightarrow k[X] \otimes_k A$. Letting each g act via $\lambda(g^{-1})$, $k[X]$ obtains the structure of a $\mathcal{G}$-module. Recalling that $k[X] \otimes_k A = \mathrm{Mor}(X_A, \mathbb{A}_A^1)$ is the algebra of natural transformations from X_A into the affine line $\mathbb{A}_A^1$, the action is given via (cf. [41, p. 26])

$$(g.f)(x) = f(g_B^{-1}.x) \ \ \forall f \in k[X] \otimes_k A, x \in X(B), B \in \mathbf{Comm}_A.$$

Remark 2.4.26. If $A \to B$ is injective, then $\mathcal{G}(A) \to \mathcal{G}(B)$ enjoys the same property by left exactness of $\mathrm{Spec}_k(k[\mathcal{G}])$. This is of course true when $A = k$, but also more generally when B is an fppf-A-algebra (take $M = A$ in [57, 13.1 (3)]).

Let $\mathcal{H}$ and $\mathcal{N}$ be group schemes. Assume that there is an action $\tau : \mathcal{H} \times \mathcal{N} \longrightarrow \mathcal{N}$ of $\mathcal{H}$ on the affine scheme $\mathcal{N}$ such that each $\tau_A(h, -) : \mathcal{N}(A) \longrightarrow \mathcal{N}(A)$ is a group automorphism for all $h \in \mathcal{H}(A)$ and all $A \in \mathbf{Comm}_k$, then we say $\mathcal{H}$ *acts on* $\mathcal{N}$ *via automorphisms*. If this is case, then we can form the *semi-direct product*

$$\mathcal{N} \rtimes_\tau \mathcal{H} : \mathbf{Comm}_k \longrightarrow \mathbf{Groups}, \ A \mapsto \mathcal{N}(A) \rtimes \mathcal{H}(A), \ \lambda \mapsto \mathcal{N}(\lambda) \times \mathcal{H}(\lambda)$$

of $\mathcal{N}$ and $\mathcal{H}$. The underlying k-functor is the direct product of the two k-functors $\mathcal{N}$ and $\mathcal{H}$. One usually drops the action τ, when it is clear from the context. If $\mathcal{G}$ is an affine group scheme and $\mathcal{H}, \mathcal{N} \subseteq \mathcal{G}$ are closed subgroups such that $\mathcal{N}$ is normal, then we define the *product subgroup* $\mathcal{N}\mathcal{H}$ to be the image of the morphism $\mu : \mathcal{N} \rtimes \mathcal{H} \to \mathcal{G}$ given by multiplication. Its kernel is isomorphic to $\mathcal{H} \cap \mathcal{N}$ under $h \mapsto (h, h^{-1})$ for all $h \in \mathcal{H}(A), A \in \mathbf{Comm}_k$. If μ is an isomorphism, we call $\mathcal{G}$ the (inner) semi-direct product of $\mathcal{N}$ and $\mathcal{H}$ and write $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$.

Recall that $e_k = \mathrm{Spec}_k(k)$ denotes the trivial group scheme. If there exists an exact sequence

$$e_k \longrightarrow \mathcal{N} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow e_k$$

of affine group schemes, then we call $\mathcal{G}$ *an extension of* $\mathcal{H}$ *by* $\mathcal{N}$. Note that ι is necessarily a closed embedding (see [57, 15.3]) and π induces an isomorphism $\mathcal{G}/\mathrm{im}(\iota) \cong \mathcal{H}$. We say that the above extension *splits*, provided there exists a morphism $\varphi : \mathcal{H} \to \mathcal{G}$ such that $\pi \circ \varphi = \mathrm{id}_{\mathcal{H}}$.

Lemma 2.4.27. *Split extensions correspond to semi-direct products.*

Proof. Let

$$e_k \longrightarrow \mathcal{N} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow e_k$$

be a split extension. By definition, there is $\varphi : \mathcal{H} \to \mathcal{G}$ such that $\pi \circ \varphi = \mathrm{id}_{\mathcal{H}}$. In view of [57, Theorem 15.3], φ is a closed embedding. Let $g \in \mathcal{G}(A)$, then

$$g = (g \cdot \varphi_A(\pi_A(g))^{-1}) \cdot \varphi_A(\pi_A(g)) \in \ker(\pi)(A) \cdot \mathrm{im}(\varphi)(A)$$

Thus, $\mathcal{G} = \ker(\pi) \cdot \mathrm{im}(\varphi)$. Let $g \in (\ker(\pi) \cap \mathrm{im}(\varphi))(A)$. Then $g_B = \varphi_B(h)$ for some fppf-A-algebra B and $h \in \mathcal{H}(B)$. Thus

$$h = \pi_B \circ \varphi_B(h) = \pi_B(g_B) = \pi_B \circ \mathcal{G}(\iota_A)(g) = \mathcal{H}(\iota_A) \circ \pi_A(g) = \mathcal{H}(\iota_A)(e) = e$$

Hence $g_B = \varphi_B(e) = e$, so that $g = e$ as $g \mapsto g_B$ is injective. Thus, $\ker(\pi) \cap \mathrm{im}(\varphi) = e_k$. Hence $\mathcal{G} = \ker(\pi) \rtimes \mathrm{im}(\varphi) \cong \mathcal{N} \rtimes \mathcal{H}$.

If conversely $\mathcal{G} = \mathcal{N} \rtimes_\tau \mathcal{H}$, we can put $\iota(n) = (n, e), \pi(n, h) = h, \varphi(h) = (e, h)$ to obtain a split-exact sequence. ☐

Let $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$. The action of $\mathcal{H}$ on $\mathcal{N}$ induces an action of $\mathcal{H}$ on $k[\mathcal{N}]$. Since $\mathcal{H}$ acts on $\mathcal{N}$ via automorphisms, it follows that each $\lambda(h) : k[\mathcal{N}] \otimes_k A \to k[\mathcal{N}] \otimes_k A$ is an automorphism of the A-Hopf algebra $k[\mathcal{N}] \otimes_k A$ for all $h \in \mathcal{H}(A)$. We also say, that $\mathcal{H}$ *acts on* $k[\mathcal{N}]$ *via automorphisms of Hopf algebras.* A consequence is the following: All the structure maps

$$\varepsilon : k[\mathcal{N}] \to k, \ \ \eta : k \to k[\mathcal{N}], \ \ m : k[\mathcal{N}] \otimes_k k[\mathcal{N}] \to k[\mathcal{N}]$$

$$\Delta : k[\mathcal{N}] \to k[\mathcal{N}] \otimes_k k[\mathcal{N}], \ \ S : k[\mathcal{N}] \to k[\mathcal{N}]$$

are maps of $\mathcal{H}$-modules, where we consider k as the trivial $\mathcal{H}$-module. If $\mathcal{N}$ and $\mathcal{H}$ are finite, this implies that $k[\mathcal{N}]$ is a $k\mathcal{H}$-module bialgebra. Dualizing, we see that $k\mathcal{N}$ is also a $k\mathcal{H}$-module bialgebra. Hence we can form the smash product $k\mathcal{N}\#k\mathcal{H}$ which has the structure of a Hopf algebra by Theorem 2.3.15. Next, we give an (almost) complete proof of the following well-known fact.

Theorem 2.4.28. *Let $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$ be a semidirect product of finite group schemes, then the Hopf algebras $k\mathcal{G}$ and $k\mathcal{N}\#k\mathcal{H}$ are isomorphic.*

Proof. We consider the natural emdeddings $i := \mathrm{d}(\iota_{\mathcal{N}}) : k\mathcal{N} \to k\mathcal{G}$, $j := \mathrm{d}(\iota_{\mathcal{H}}) : k\mathcal{H} \to k\mathcal{G}$. The group $\mathcal{H}$ acts on $\mathcal{G}$ and $\mathcal{N}$ via conjugation. As discussed before, this yields actions on $k[\mathcal{G}], k[\mathcal{N}]$. Moreover, by [41, I.7.18] the induced action of $\mathcal{H}$ on $k\mathcal{G}$ is given by restriction of the adjoint representation of the Hopf algebra $k\mathcal{G}$ to $k\mathcal{H}$.
The comorphism $k[\mathcal{G}] \longrightarrow k[\mathcal{N}]$ of the inclusion $\mathcal{N} \to \mathcal{G}$ is given by the restriction of functions. This clearly defines an $\mathcal{H}$-linear map. Consequently, its dual $i : k\mathcal{N} \to k\mathcal{G}$ enjoys the same property. As an upshot of the above, we obtain $i \in \mathrm{Alg}_{k\mathcal{H}}(k\mathcal{N}, (k\mathcal{G})_j)$. The universal property of the smash product thus provides a homomorphism

$$\varphi : k\mathcal{N}\#k\mathcal{H} \to k\mathcal{G}, \;\; u\#v \mapsto i(u) \cdot j(v).$$

By definition of the semidirect product, the multiplication $\mathcal{N} \times \mathcal{H} \to \mathcal{G}$ is an isomorphism of schemes and thus induces an isomorphism $k\mathcal{N} \otimes_k k\mathcal{H} \to k\mathcal{G}$ of vector spaces given by multiplication. Consequently, φ is surjective, hence an isomorphism of k-algebras for dimension reasons.
We are left to show that φ is a Hopf algebra map. By way of example, we show that φ respects the comultiplication, that is, $(\varphi \otimes \varphi) \circ \Delta_{k\mathcal{N}\#k\mathcal{H}} = \Delta_{k\mathcal{G}} \circ \varphi$. In the following computation, we will suppress the embeddings i, j for notational reasons. We have

$$\begin{aligned}(\varphi \otimes \varphi) \circ \Delta_{k\mathcal{N}\#k\mathcal{H}}(u\#v) &= \sum_{(u),(v)} u_{(1)}v_{(1)} \otimes u_{(2)}v_{(2)} \\ &= \sum_{(u)} u_{(1)} \otimes u_{(2)} \cdot \sum_{(v)} v_{(1)} \otimes v_{(2)} \quad \text{(multiplication inside } k\mathcal{G} \otimes_k k\mathcal{G}) \\ &= \Delta_{k\mathcal{G}}(u) \cdot \Delta_{k\mathcal{G}}(v) \\ &= \Delta_{k\mathcal{G}}(uv) = \Delta_{k\mathcal{G}}(\varphi(u\#v)).\end{aligned}$$

□

As extensions of finite-dimensional Hopf algebras are known to be free (see [48]), an application of 2.3.16 yields

Corollary 2.4.29. *Let $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$ be a semidirect product of finite group schemes. Then $\mathcal{G}$-modules correspond to coherent $k\mathcal{N}$-$k\mathcal{H}$-modules. Moreover,*

(a) $k\mathcal{N}$ is a $\mathcal{G}$-module via the left regular representation of $k\mathcal{N}$ and the given $\mathcal{H}$-module structure.

(b) If $M \in \mathbf{mod}(\mathcal{G})$ and $N \in \mathbf{mod}(\mathcal{H})$, then the $\mathcal{H}$-module $M \otimes_k N$ is a $\mathcal{G}$-module with $k\mathcal{N}$ acting via $u.(m \otimes n) := u.m \otimes n$.

(c) *The transformation* $\tau : k\mathcal{N} \otimes_k - \longrightarrow k\mathcal{G} \otimes_{k\mathcal{H}} -$ *given by* $\tau_N(a \otimes n) = \mathrm{d}(\iota_{\mathcal{N}})(a) \otimes n$ *for all* $N \in \mathbf{mod}(\mathcal{H})$ *is a natural equivalence (the relevant functors are considered as functors* $\mathbf{mod}(\mathcal{H}) \to \mathbf{mod}(\mathcal{G})$*).*

(d) *Let* $M, N \in \mathbf{mod}(\mathcal{G})$*. If* $\mathcal{N}$ *is commutative, then* $M \otimes_{k\mathcal{N}} N$ *has a natural structure of a* $\mathcal{G}$*-module. The* $\mathcal{H}$*-module structure is induced by the tensor product* $M \otimes_k N$ *and* $k\mathcal{N}$ *acts via* $a.(m \otimes n) := m \otimes a.n$*.*

Let k be algebraically closed. An *algebraic group* G is an affine variety over k which has a group structure such that the multiplication $m_G : G \times G \to G$ and inversion $\iota_G : G \to G$ are morphisms of affine varieties. The assignments $V \mapsto \mathrm{Spec}_k(k[V])$ and $\mathcal{V} \mapsto \mathcal{V}(k)$ are 'inverse' to each other and therefore yield an equivalence

$$\{\text{Affine varieties over } k\} \to \{\text{Reduced affine algebraic schemes over } k\},$$

of categories, which in turn induces an equivalence

$$\{\text{Algebraic groups over } k\} \to \{\text{Reduced affine algebraic group schemes over } k\},$$

cf. [57, §4]. A k-vector space V together with a homomorphism $G \to GL(V)$ of algebraic groups is referred to as a *rational representation of* G *or a* G*-module.* Then G-modules correspond to modules of the affine algebraic group scheme $\mathcal{G} = \mathrm{Spec}_k(k[G])$ (see [41, p.28]).

Let V be an affine variety with coordinate ring $k[V]$, $x \in V$ a point and denote by k_x the one-dimensional $k[V]$-module corresponding to the evaluation homomorphism $k[V] \to k$ at x. Then the vector space

$$T_x(V) := \mathrm{Der}_k(k[V], k_x) = \{d \in \mathrm{Hom}_k(k[V], k) \mid d(fg) = g(x)d(f) + f(x)d(g)\ \forall f, g \in k[V]\}$$

is called the *tangent space of* V *at* x. Given a morphism $\varphi : V \to W$, its *differential* $\mathrm{d}_x(\varphi)$ *at* x is defined as the linear map

$$\mathrm{d}_x(\varphi) : T_x(V) \to T_{\varphi(x)}(W),\ d \mapsto d \circ \varphi^*.$$

Taking differentials is functorial in the following sense:

Lemma 2.4.30. *Let* $\varphi : V \to W$ *be a morphism of affine varieties. If* $\psi : W \to X$ *is another morphism, then* $\mathrm{d}_x(\psi \circ \varphi) = \mathrm{d}_{\varphi(x)}(\psi) \circ \mathrm{d}_x(\varphi)$*. Moreover, we have* $\mathrm{d}_x(\mathrm{id}_V) = \mathrm{id}_{T_x(V)}$*.*

Lemma 2.4.31. *Let G be an algebraic group with inversion $\iota_G : G \to G$. Then $\mathrm{d}_g(\iota_G) = -\mathrm{d}_e(r_{g^{-1}}) \circ \mathrm{d}_g(l_{g^{-1}})$, where we denote by l_g and r_g the left and right multiplication effected by $g \in G$, respectively.*

Proof. Using the functorial property 2.4.30 of taking differentials, this follows from the identity $\iota_G = r_{g^{-1}} \circ \iota_G \circ l_{g^{-1}}$ and the fact that $d_e(\iota_G) = -\mathrm{id}_{T_e(G)}$ (see [55]). □

Let G be an algebraic group with corresponding reduced group scheme $\mathcal{G}$, then the vector spaces $T_e(G)$ and $\mathrm{Lie}(\mathcal{G})$ coincide. We put $\mathrm{Lie}(G) := \mathrm{Lie}(\mathcal{G})$ and call this the *Lie algebra of* G. If $\varphi : G \to H$ is a homomorphism of algebraic groups, then $\mathrm{d}(\varphi) := \mathrm{d}_e(\varphi) : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ coincides with the homomorphism $\mathrm{Lie}(\psi) : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{H})$ associated to the morphism $\psi : \mathrm{Spec}_k(k[G]) \to \mathrm{Spec}_k(k[H])$ of group schemes with the property that $\psi_k = \varphi$. In particular, if $G \to \mathrm{GL}(V)$ is a rational representation of G, then its differential $\mathfrak{g} \to \mathfrak{gl}(V)$ is a restricted representation of its Lie algebra $\mathfrak{g}$.

We take the following facts for granted.

Lemma 2.4.32. *Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Given $g \in G$, we denote by $\kappa_g : G \to G$ the automorphism given by conjugation effected by g.*

(1) *We have* $\dim G = \dim_k \mathfrak{g}$.

(2) *The map* $\mathrm{Ad}_G : G \to \mathrm{Aut}_p(\mathfrak{g}), g \mapsto \mathrm{d}_e(\kappa_g)$ *is a rational representation, the* ***adjoint representation of*** G ***on*** $\mathfrak{g}$. *Its differential coincides with the adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *of the Lie algebra* $\mathfrak{g}$. *Given an element* $g \in G$ *and* $x \in \mathfrak{g}$, *we write* $g.x := \mathrm{Ad}_G(g)(x)$.

(3) *Let* $\rho : G \to \mathrm{GL}(V)$ *be a rational representation. Then we have* $(g.x).(g.v) = g.(x.v)$ *for all* $g \in G, x \in \mathfrak{g}, v \in V$.

(4) *If* $G \subseteq \mathrm{GL}_n$ *is a closed subgroup, then the differential of the natural representation* $G \to \mathrm{GL}_n$ *is the natural representation* $\mathfrak{g} \to \mathfrak{gl}_n$ *of its Lie algebra* $\mathfrak{g} \subseteq \mathfrak{gl}_n$.

(5) *If* $G \subseteq \mathrm{GL}_n$ *is a closed subgroup, then the differential of the determinant* $\det : G \to \mathrm{GL}(1) =: \mathbb{G}_m$ *is the trace* $\mathrm{tr} : \mathfrak{g} \to k, x \mapsto \mathrm{tr}(x)$.

Let $N \trianglelefteq G$ be a closed normal subgroup of some algebraic group G. If $H \subseteq G$ is a closed subgroup such that $NH = HN = G$ and $H \cap N = \{1\}$, then G is isomorphic to the semidirect product $N \rtimes H$ if and only if $\mathrm{Lie}(N) \cap \mathrm{Lie}(H) = (0)$. This follows from the fact that the multiplication $N \rtimes H \to G$ is an isomorphism if and only if its differential

$$\mathrm{Lie}(N) \oplus \mathrm{Lie}(L) \to \mathrm{Lie}(G), \quad (x, y) \mapsto x + y$$

is bijective (see [55][Corollary 5.3.3(ii)]).

Definition 2.4.33. Let G be an algebraic group.

- There exists a unique maximal closed connected unipotent normal subgroup $R_u(G) \subseteq G$, the *unipotent radical of* G.
- We call G *reductive*, provided $R_u(G) = \{e\}$.

Lemma 2.4.34. *Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$ and put $\mathcal{G} := \mathrm{Spec}_k(k[G])$.*

(a) If G is reductive, then $G_r := \mathcal{G}_r$ is reductive for all $r \geq 1$.

(b) The following statements are equivalent:

(1) G is reductive.

(2) $\mathcal{G}$ is reductive.

Proof. (a) In view of [38, Proposition 11.8], we have $\mathrm{Rad}_p(\mathfrak{g}) = (0)$. If $\mathcal{U} \trianglelefteq G_r$ was a non-trivial closed connected unipotent normal subgroup then $(0) \neq \mathrm{Lie}(\mathcal{U}) \trianglelefteq \mathrm{Lie}(G_r)$ would be a unipotent p-ideal of $\mathrm{Lie}(G_r) \cong \mathfrak{g}$ (see remark 2.4.11, Lemma 2.4.12), which is a contradiction.

(b) (2) $\Rightarrow$ (1): If $\{e\} \neq U \trianglelefteq G$ was a closed, connected, unipotent, normal subgroup of G, then its reduced scheme $e_k \neq \mathcal{U}$ would be a closed connected unipotent normal subgroup of $\mathcal{G}$.

(1) $\Rightarrow$ (2): Assume that $e_k \neq \mathcal{U} \trianglelefteq \mathcal{G}$ is a closed, connected, unipotent, normal subgroup of $\mathcal{G}$. Then $(0) \neq \mathrm{Lie}(\mathcal{U}) \trianglelefteq \mathfrak{g}$ would be a a unipotent p-ideal of $\mathfrak{g}$ (see again 2.4.11, 2.4.12). In view of (a), this is a contradiction.

□

Definition 2.4.35. Let G be an algebraic group with Lie algebra $\mathfrak{g}$.

- A closed subgroup $T \subseteq G$ is called a *torus*, provided its associated reduced group scheme $\mathcal{T}$ is diagonalizable and connected.
- A maximal closed connected solvable subgroup B of G is called a *Borel subgroup of* G. Moreover, $\mathrm{Lie}(B) \subseteq \mathfrak{g}$ is called a *Borel subalgebra of* $\mathfrak{g}$.

Lemma 2.4.36. *Let G be a connected algebraic group and $U := R_u(G)$. Suppose that there is a connected, closed subgroup $K \subseteq G$ such that $G = U \rtimes K$.*

(1) Any Borel subgroup of G is of the form $U \rtimes B_K$, where B_K is a Borel subgroup of K.

(2) Maximal tori of K are maximal tori of G.

Proof. Throughout, we denote by

$$\pi : G \twoheadrightarrow K, \quad u \cdot k \mapsto k$$

the corresponding surjective morphism of algebraic groups with kernel U.

(1) In view of [31, 2.4.6], the product UB_K is clearly a closed, connected solvable subgroup of G. Now suppose UB_K is contained in some Borel subgroup $B \subseteq G$. Then

$$B_K = \pi(UB_K) \subseteq \pi(B)$$

Hence $B_K = \pi(B)$ by maximality, so that

$$UB_K = \pi^{-1}(B_K) = \pi^{-1}(\pi(B)) = UB = B, \quad (*)$$

as every Borel of G contains the unipotent radical U. Now [39, 21.3 C] shows that the map

$$\pi^* : \{\text{Borels of G}\} \to \{\text{Borels of K}\}, \quad B \mapsto \pi(B)$$

is surjective. If $\pi(B) = \pi(B')$, then the same argument as in $(*)$ implies $B = B'$. Consequently, π^* is bijective and hence (1).

(2) Let T_K be a maximal torus of K and let T be a torus of G such that $T_K \subseteq T$. Hence $T_K = \pi(T_K) \subseteq \pi(T)$ and therefore, by maximality of T_K, we arrive at $\pi(T_K) = \pi(T)$. This means for all $t' \in T$ there is $t \in T_K$ such that $t't^{-1} \in \ker(\pi) = U$. It follows that $t't^{-1} \in T \cap U = \{e\}$ (see [57, p.65 Corollary (b)]). Hence $t = t'$ and therefore $T \subseteq T_K$.

$\square$

2.5 Restricted Lie algebras and reduced enveloping algebras

The reader may consult [25] for standard facts concerning restricted Lie algebras. We briefly recall some of these, beginning with the definitions.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k and T be an indeterminate over k. Then the tensor product $\mathfrak{g}[T] := \mathfrak{g} \otimes_k k[T]$ has the structure of a Lie algebra with bracket determined by

$$[x \otimes f, y \otimes g] := [x, y] \otimes fg \qquad \forall x, y \in \mathfrak{g}, f, g \in k[T].$$

Given $x, y \in \mathfrak{g}$, we let the elements $s_i(x, y) \in \mathfrak{g}$ be defined via the equation

$$\mathrm{ad}(x \otimes T + y \otimes 1)^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i \cdot s_i(x, y) \otimes T^{i-1}$$

inside the Lie algebra $\mathfrak{g}[T]$.

Definition 2.5.1. A map $[p] : \mathfrak{g} \to \mathfrak{g}$ is called a *p-map*, provided

$$\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p \qquad \forall x \in \mathfrak{g} \tag{1}$$

$$(\alpha x)^{[p]} = \alpha^p x^{[p]} \qquad \forall \alpha \in k, x \in \mathfrak{g} \tag{2}$$

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y) \qquad \forall x, y \in \mathfrak{g} \tag{3}$$

We then call the pair $(\mathfrak{g}, [p])$ a *restricted Lie algebra*.

Example 2.5.2. Given an associative algebra A, we can consider its (restricted) commutator Lie algebra A^- with bracket and p-map determined by $[a, b] := ab - ba$, $a^{[p]} := a^p$ for all $a, b \in A$.

From now on, we assume that $\mathfrak{g}$ is restricted.

Definition 2.5.3. Let $\chi \in \mathfrak{g}^*$ be a linear form. A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is said to have *character* χ provided the identity $\rho(x)^p - \rho(x^{[p]}) = \chi(x)^p \cdot \mathrm{id}_V$ holds for all $x \in \mathfrak{g}$.

We denote by $\mathbf{mod}_\chi(\mathfrak{g})$ the full subcategory of finite-dimensional $\mathfrak{g}$-modules with character χ. The algebra

$$\mathrm{U}_\chi(\mathfrak{g}) := \mathrm{U}(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p \cdot 1 \mid x \in \mathfrak{g})$$

is called a *reduced enveloping algebra of* $\mathfrak{g}$. Every such algebra $\mathrm{U}_\chi(\mathfrak{g})$ satisfies a universal property, which can roughly be described by means of

$$\mathrm{Hom}_{Alg}(\mathrm{U}_\chi(\mathfrak{g}), A) \cong \{f \in \mathrm{Hom}_{Lie}(\mathfrak{g}, A^-) \mid f(x)^p - f(x^{[p]}) = \chi(x)^p.1_A \;\; \forall x \in \mathfrak{g}\}$$

for every (associative) algebra A with corresponding commutator (Lie) algebra A^-. This property induces an equivalence $\mathbf{mod}_\chi(\mathfrak{g}) \longrightarrow \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$ of categories. The algebra $\mathrm{U}_0(\mathfrak{g})$ is also called the *restricted enveloping algebra of* $\mathfrak{g}$.

The following can be seen as a motivation to study reduced enveloping algebras. If k is algebraically closed and S a simple $\mathfrak{g}$-module, then [25, Theorem 2.5] implies that there exists $\chi \in \mathfrak{g}^*$ such that S is a simple module for the (in view of 2.5.4(1) below) finite-dimensional algebra $\mathrm{U}_\chi(\mathfrak{g})$. This also implies that S is finite-dimensional.

Here are some standard properties of reduced enveloping algebras, the reader may consult [25, p.210ff] for more information.

Lemma 2.5.4. *Let $\chi \in \mathfrak{g}^*$ be a linear form and $\mathfrak{l} \subseteq \mathfrak{g}$ be a restricted subalgebra.*

(1) ('PBW' Poincaré-Birkhoff-Witt) The algebra $\mathrm{U}_\chi(\mathfrak{g})$ is a finite-dimensional algebra of dimension $p^{\dim_k(\mathfrak{g})}$.

(2) The inclusion $\mathfrak{l} \hookrightarrow \mathfrak{g}$ lifts via universal properties to an embedding $(\mathrm{U}_{\chi|_\mathfrak{l}}(\mathfrak{l}) =)\mathrm{U}_\chi(\mathfrak{l}) \hookrightarrow \mathrm{U}_\chi(\mathfrak{g})$. We will identify $\mathrm{U}_\chi(\mathfrak{l})$ with its image in $\mathrm{U}_\chi(\mathfrak{g})$.

(3) $\mathrm{U}_\chi(\mathfrak{g})$ is a free left (resp. right) module over $\mathrm{U}_\chi(\mathfrak{l})$ of rank $p^{\dim_k \mathfrak{g}/\mathfrak{l}}$.

(4) The restriction functor

$$\mathrm{Res}_{\mathrm{U}_\chi(\mathfrak{l})} : \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g})) \longrightarrow \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{l}))$$

takes projectives to projectives.

(5) If $N \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{l}))$, then the induced module (see also 2.1.29)

$$\mathrm{Ind}_\mathfrak{l}^\mathfrak{g}(N, \chi) := \mathrm{U}_\chi(\mathfrak{g}) \otimes_{\mathrm{U}_\chi(\mathfrak{l})} N \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$$

is $(p^{\dim_k \mathfrak{g}/\mathfrak{l}} \cdot \dim_k\ N)$-dimensional. The induction functor

$$\mathrm{Ind}_\mathfrak{l}^\mathfrak{g}(-, \chi) : \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{l})) \longrightarrow \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$$

is exact and sends projectives to projectives.

(6) If $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$ and $N \in \mathbf{mod}(\mathrm{U}_{\chi'}(\mathfrak{g}))$, then $M \otimes_k N \in \mathbf{mod}(\mathrm{U}_{\chi+\chi'}(\mathfrak{g}))$, $M^* \in \mathbf{mod}(\mathrm{U}_{-\chi}(\mathfrak{g}))$ and $\mathrm{Hom}_k(M, N) \in \mathbf{mod}(\mathrm{U}_{\chi'-\chi}(\mathfrak{g}))$. Moreover, if M is $\mathrm{U}_\chi(\mathfrak{g})$-projective, then $M \otimes_k N$ is $\mathrm{U}_{\chi+\chi'}(\mathfrak{g})$-projective.

(7) There are natural isomorphisms $\mathrm{Ext}^n_{\mathrm{U}_\chi(\mathfrak{g})}(M, N) \cong H^n(\mathrm{U}_0(\mathfrak{g}), M^* \otimes_k N)$ for all $n \geq 0$ (cf. [11, Corollary 2.5]).

In general, different linear forms can give rise to the same algebra (up to isomorphism). Denote by $\mathrm{Aut}_p(\mathfrak{g})$ the automorphism group of the restricted Lie algebra $\mathfrak{g}$. It is in fact an algebraic group as a closed subgroup of $\mathrm{GL}(\mathfrak{g})$.

Lemma 2.5.5. *Let $G \subseteq \mathrm{Aut}_p(\mathfrak{g})$ be a subgroup. Then G acts on $\mathfrak{g}^*$ via*

$$f.\chi := \chi \circ f^{-1}, \ \ \forall f \in G, \chi \in \mathfrak{g}^*.$$

Moreover, every $f \in \mathrm{Aut}_p(\mathfrak{g})$ induces an isomorphism $\mathrm{U}_\chi(\mathfrak{g}) \cong \mathrm{U}_{f.\chi}(\mathfrak{g})$ of algebras.

In particular, if $\mathfrak{g} = \mathrm{Lie}(G)$ for an algebraic group G, then one can consider the so-called *coadjoint orbits*, which are the orbits under the subgroup $\mathrm{Ad}(G) \subseteq \mathrm{Aut}_p(\mathfrak{g})$ corresponding to the above action. The name of these orbits comes from the fact that the G-module $\mathfrak{g}^*$ corresponds to the dual of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_p(\mathfrak{g})$.

Lemma 2.5.6. *Let $\chi \in \mathfrak{g}^*$ be a linear form.*

(1) One-dimensional $\mathrm{U}_\chi(\mathfrak{g})$-modules correspond to linear forms $\lambda \in \mathfrak{g}^*$ with $\lambda([\mathfrak{g},\mathfrak{g}]) = 0$ and $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{g}$.

(2) If $\mathrm{U}_\chi(\mathfrak{g})$ admits a one-dimensional module, then $\mathrm{U}_\chi(\mathfrak{g}) \cong \mathrm{U}_0(\mathfrak{g})$ as algebras.

Proof. (1) Since a linear map $\lambda : \mathfrak{g} \to k^-$ is a homomorphism of Lie algebras if and only if $\lambda([\mathfrak{g},\mathfrak{g}]) = 0$ and one-dimensional $\mathrm{U}_\chi(\mathfrak{g})$-modules correspond to homomorphisms $\mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{End}(k) \cong k$ of algebras, this follows from the universal property of $\mathrm{U}_\chi(\mathfrak{g})$.

(2) We consider the homomorphism $\varphi : \mathfrak{g} \to \mathrm{U}_0(\mathfrak{g})^-, \ x \mapsto x + \lambda(x).1$ of Lie algebras. Given $x \in \mathfrak{g}$, we have

$$\begin{aligned}
\varphi(x)^p - \varphi(x^{[p]}) &= (x + \lambda(x).1)^p - (x^{[p]} + \lambda(x^{[p]}).1) \\
&= x^p + \lambda(x)^p.1 - x^{[p]} - \lambda(x^{[p]}).1 && \mathrm{char}(k) = p \\
&= \lambda(x)^p.1 - (\lambda(x)^p.1 - \chi(x)^p.1) && x^p = x^{[p]} \text{ inside } \mathrm{U}_0(\mathfrak{g}) \\
&= \chi(x)^p.1
\end{aligned}$$

Hence φ lifts to a homomorphism $\mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_0(\mathfrak{g})$. For similar reasons, the map $\psi : \mathfrak{g} \to \mathrm{U}_\chi(\mathfrak{g}), \ x \mapsto x - \lambda(x).1$ lifts to a homomorphism $\mathrm{U}_0(\mathfrak{g}) \to \mathrm{U}_\chi(\mathfrak{g})$ of algebras. Since the image of $\mathfrak{g}$ generates any reduced enveloping algebra, it readily follows that these homomorphisms of algebras are inverse to each other.

□

For details of the following, we refer to [28]. Let again $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra and k be algebraically closed. The set

$$V(\mathfrak{g}) := \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$$

is a closed, conical subset of the full affine space $\mathfrak{g}$, the so-called *nullcone* of $\mathfrak{g}$. Let $\chi \in \mathfrak{g}^*$ be a linear form and $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$. Then we can consider the *rank variety*

$$V(\mathfrak{g})_M := \{x \in V(\mathfrak{g}) \mid M|_{\mathrm{U}_\chi(kx)} \text{ is not projective}\} \cup \{0\}$$

which is always a closed, conical subvariety of $V(\mathfrak{g})$. Note that

$$k[X]/(X^p) \to \mathrm{U}_\chi(kx) = k[x], \quad X + (X^p) \mapsto x - \chi(x) \cdot 1$$

is an isomorphism of k-algebras for every $x \in V(\mathfrak{g}) \setminus \{0\}$, so that the module $M|_{\mathrm{U}_\chi(kx)}$ is uniquely determined by the Jordan canonical form of the linear operator

$$x_M : M \to M, m \mapsto x.m - \chi(x).m.$$

We denote by $\Omega := \Omega_{\mathrm{U}_\chi(\mathfrak{g})}$ the Heller operator of $\mathrm{U}_\chi(\mathfrak{g})$ and by $\mathrm{cx}_\mathfrak{g}(M)$ the complexity of the $\mathrm{U}_\chi(\mathfrak{g})$-module M. Recall that M is called periodic, provided there exists $n \in \mathbb{N}$ such that $\Omega^n(M) \cong M$. We now list some important properties concerning rank varieties (cf. [28, p.1081 ff]).

Lemma 2.5.7. *The following statements hold:*

(1) We have $V(\mathfrak{g})_M = \{0\}$ *if and only if* M *is a projective* $\mathrm{U}_\chi(\mathfrak{g})$*-module.*

(2) $V(\mathfrak{g})_{M \oplus N} = V(\mathfrak{g})_M \cup V(\mathfrak{g})_N$.

(3) $V(\mathfrak{g})_{M \otimes_k N} = V(\mathfrak{g})_M \cap V(\mathfrak{g})_N$.

(4) (J.Carlson) If M *is indecomposable, then the projectivization* $\mathbb{P}(V(\mathfrak{g})_M)$ *is connected. In particular, if* $\dim V(\mathfrak{g})_M = 1$*, then* $V(\mathfrak{g})_M$ *is a line.*

(5) We have $\dim V(\mathfrak{g})_M = \mathrm{cx}_\mathfrak{g}(M)$ *(recall Definition 2.1.16). Moreover,* M *is periodic if and only if* $\dim V(\mathfrak{g})_M = 1$ *and, if* M *is indecomposable, in the latter case we already have* $\Omega^2(M) \cong M$ *(see [12, Theorem 2.5]).*

Remark 2.5.8. Periodic modules of finite-dimensional algebras always have complexity 1, but the converse is not true in general: Example 2.3 in [53] provides a counterexample.

Next, we introduce the following two important classes of restricted Lie algebras.

Definition 2.5.9. A restricted Lie algebra $\mathfrak{g}$ is called

(i) a *torus*, provided $\mathfrak{g}$ is abelian and every $x \in \mathfrak{g}$ is *semisimple*, that is, there exist $\alpha_1, \ldots, \alpha_n \in k$ such that $x = \sum_{i=1}^n \alpha_i x^{[p]^i}$.

(ii) *unipotent*, provided every $x \in \mathfrak{g}$ is $[p]$-*nilpotent*, that is, there exists $r \in \mathbb{N}$ such that $x^{[p]^r} = 0$.

(iii) *elementary abelian*, provided $\mathfrak{g}$ is abelian and *p-trivial*, that is, $[p] : \mathfrak{g} \to \mathfrak{g}$ is identically zero.

The lemmas below will be used later on.

Lemma 2.5.10. *Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra.*

(1) $\mathfrak{g}$ is unipotent if and only if the corresponding infinitesimal group scheme $\mathcal{G}_{\mathfrak{g}} = \mathrm{Spec}_k(\mathrm{U}_0(\mathfrak{g})^*))$ is unipotent (that is, $\mathrm{U}_0(\mathfrak{g})$ is local).

(2) Let $\mathcal{G}$ be a unipotent, affine algebraic group scheme. Then $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ is unipotent.

(3) Let $\mathfrak{g}$ be unipotent. If $\chi \in \mathfrak{g}^*$ is a linear form, then $\mathrm{U}_\chi(\mathfrak{g})$ possesses exactly one simple module up to isomorphism.

(4) The infinitesimal group scheme $\mathcal{G}_{\mathfrak{g}}$ is linearly reductive if and only if $\mathfrak{g}$ is a torus.

Proof. (1) Recall that the augmentation ideal $\mathrm{U}_0(\mathfrak{g})^\dagger = \ker(\varepsilon : \mathrm{U}_0(\mathfrak{g}) \to k) = \mathrm{U}_0(\mathfrak{g})\mathfrak{g}$ of the Hopf algebra $\mathrm{U}_0(\mathfrak{g})$ is necessarily maximal. Hence $\mathrm{U}_0(\mathfrak{g})$ is local if and only if $\mathrm{U}_0(\mathfrak{g})^\dagger$ is the unique maximal ideal of $\mathrm{U}_0(\mathfrak{g})$.

If $\mathfrak{g}$ is unipotent, then [25, Corollary I.3.7] implies that $\mathrm{U}_0(\mathfrak{g})^\dagger$ is nilpotent, hence $\mathrm{U}_0(\mathfrak{g})$ is local. If conversely $\mathrm{U}_0(\mathfrak{g})$ is local, then $\mathrm{U}_0(\mathfrak{g})^\dagger$ must be the unique maximal ideal and is therefore nilpotent since it coincides with the Jacobson radical of $\mathrm{U}_0(\mathfrak{g})$. Since $\mathfrak{g} \subseteq \mathrm{U}_0(\mathfrak{g})^\dagger$, it follows easily that $\mathfrak{g}$ is unipotent.

(2) The first Frobenius kernel $\mathcal{G}_1 \trianglelefteq \mathcal{G}$ is unipotent, now apply (1).

(3) Let S, T be simple $\mathrm{U}_\chi(\mathfrak{g})$-modules. Consider the $\mathrm{U}_0(\mathfrak{g})$-module $M := \mathrm{Hom}_k(S, T)$. Its socle $\mathrm{Soc}_{\mathrm{U}_0(\mathfrak{g})}(M) = M^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(S, T)$ is non-zero. Hence $S \cong T$ by Schur's Lemma.

(4) See [25, Proposition II.3.3, Theorem V.5.8].

□

Remark 2.5.11. In case of (3), each $\mathrm{U}_\chi(\mathfrak{g})$ is isomorphic to a matrix algebra over a local ring.

Lemma 2.5.12. *Let $\mathfrak{u} \trianglelefteq \mathfrak{g}$ be a p-ideal of a restricted Lie algebra $\mathfrak{g}$, $\chi \in \mathfrak{g}^*$ a linear form and S a $\mathrm{U}_\chi(\mathfrak{g})$-module.*

(a) The space $S^{\mathfrak{u}} := \{s \in S \mid x.s = 0 \ \ \forall x \in \mathfrak{u}\}$ of $\mathfrak{u}$-invariants *is a $\mathrm{U}_\chi(\mathfrak{g})$-submodule of S.*

(b) If $\chi(\mathfrak{u}) = 0$, $\mathfrak{u}$ is unipotent and S is simple, then $S^{\mathfrak{u}} = S$.

Proof. (a) Let $s \in S^{\mathfrak{u}}, u \in \mathfrak{u}$ and $x \in \mathfrak{g}$, then we have (using that $\mathfrak{u}$ is an ideal)

$$u.(x.s) = x.(u.s) + [x, u].s = 0 + 0 = 0$$

so that $x.s \in S^{\mathfrak{u}}$.

(b) As $\mathfrak{u}$ is unipotent, $\mathrm{U}_\chi(\mathfrak{u}) = \mathrm{U}_0(\mathfrak{u})$ is a local algebra with unique simple module being the trivial module k. Hence $S^{\mathfrak{u}} = \mathrm{Soc}_{\mathrm{U}_0(\mathfrak{u})}(S)$ is non-zero. The assertion follows from (a) and the simplicity of S.

□

Lemma 2.5.13. *Let* $d \in \{1, \ldots, \dim_k \mathfrak{g}\}$*, then*

$$\mathbb{E}(d, \mathfrak{g}) := \{\mathfrak{e} \subseteq \mathfrak{g} \mid \mathfrak{e} \textit{ is a d-dimensional elementary abelian subalgebra}\}$$

is a closed subset of the (projective) Grassmannian variety

$$\mathrm{Gr_d}(\mathfrak{g}) := \{V \subseteq \mathfrak{g} \mid \ V \textit{ is a d-dimensional subspace of } \mathfrak{g}\}.$$

The group $\mathrm{Aut}_p(\mathfrak{g})$ of p-automorphisms operates naturally on the varieties $\mathbb{E}(d, \mathfrak{g})$ and $V(\mathfrak{g})$. In particular, if $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of an algebraic group G, then G acts via the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_p(\mathfrak{g})$ (see 2.4.32).

We proceed with a discussion involving semidirect products. Recall that if A is a (not necessarily Lie or associative) k-algebra, then

$$\mathrm{Der}_k(A) = \{D \in \mathrm{Hom}_k(A, A) \mid D(ab) = D(a)b + aD(b) \ \ \forall a, b \in A\}$$

denotes the space of *derivations of* A. In particular, if $\mathfrak{g}$ is a Lie algebra, then we put

$$\mathrm{Der}(\mathfrak{g}) = \{d \in \mathrm{End}_k(\mathfrak{g}) \mid d([x, y]) = [d(x), y] + [x, d(y)] \ \ \forall x, y \in \mathfrak{g}\},$$

which in fact is a p-subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

Definition 2.5.14. Let $\mathfrak{g}$ be a restricted Lie algebra. A derivation $d : \mathfrak{g} \to \mathfrak{g}$ is called *restricted* provided

$$d(x^{[p]}) = \mathrm{ad}(x)^{p-1}(d(x)) \ \ \forall x \in \mathfrak{g}$$

We denote the space of such derivations by $\mathrm{Der}_p(\mathfrak{g})$. Note that (1) of Definition 2.5.1 implies that every inner derivation is restricted.

Recall that for every associative algebra A, we can consider its (restricted) commutator Lie algebra A^- with bracket and p-map determined by $[a, b] := ab - ba, a^{[p]} := a^p$ for all $a, b \in A$.

Lemma 2.5.15. *Let $D \in \mathrm{Der}_k(A)$ be a derivation of an associative k-algebra A. Then the following statements hold:*

(1) We have $D(x^m) = \sum_{j=0}^{m-1} x^j D(x) x^{m-1-j}$ for all $x \in A, m \in \mathbb{N}$.

(2) $D : A^- \to A^-$ is a restricted derivation of the commutator algebra A^-.

Proof. (1) This can be shown inductively.

(2) It is easy to see, that D is a derivation of A^-. The assertion now follows from (1) in conjunction with the Cartan-Weyl formula [25, Proposition I.1.3(a)] and the fact that $\binom{p-1}{j} \equiv (-1)^{p-1-j} \operatorname{mod}(p)$ for all $0 \leq j \leq p-1$.

□

It is well known that derivations of Lie algebras can be uniquely lifted to derivations of their enveloping algebras (see [25, Theorem I.8.1(5)]). We now show that restricted ones can be lifted to restricted enveloping algebras. It turns out to be enough to modify the proof of [25, Theorem I.8.1(5)] a little.

Lemma 2.5.16. *Let $\mathfrak{g}$ be a restricted Lie algebra and let $d : \mathfrak{g} \to \mathfrak{g} \in \mathrm{Der}_k(\mathfrak{g})$ be a derivation. The following statements are equivalent:*

(1) d is a restricted derivation.

(2) There exists a unique derivation $D \in \mathrm{Der}_k(\mathrm{U}_0(\mathfrak{g}))$ of the restricted enveloping algebra $\mathrm{U}_0(\mathfrak{g})$ such that $D \circ \iota = \iota \circ d$. Here we denote by $\iota : \mathfrak{g} \hookrightarrow \mathrm{U}_0(\mathfrak{g})$ the canonical embedding.

Proof. (1) $\Rightarrow$ (2): Denote the algebra of upper triangular $(2 \times 2)-$matrices with entries in $\mathrm{U}_0(\mathfrak{g})$ by A. Now consider

$$f : \mathfrak{g} \to A^-, \qquad x \mapsto \begin{pmatrix} \iota(x) & \iota(d(x)) \\ 0 & \iota(x) \end{pmatrix}$$

Direct computation shows that f is a homomorphism of Lie algebras and

$$f(x^{[p]}) = \begin{pmatrix} \iota(x^{[p]}) & \iota(d(x^{[p]})) \\ 0 & \iota(x^{[p]}) \end{pmatrix} = \begin{pmatrix} \iota(x)^p & \iota(\mathrm{ad}(x)^{p-1}(d(x))) \\ 0 & \iota(x)^p \end{pmatrix} = \begin{pmatrix} \iota(x)^p & \mathrm{ad}(\iota(x))^{p-1}(\iota(d(x))) \\ 0 & \iota(x)^p \end{pmatrix}$$

$$f(x)^p = \begin{pmatrix} \iota(x)^p & \sum_{j=0}^{p-1} \iota(x)^j \cdot \iota(d(x)) \cdot \iota(x)^{p-1-j} \\ 0 & \iota(x)^p \end{pmatrix}.$$

Using that $\binom{p-1}{j} \equiv (-1)^{p-1-j} \bmod(p)$, we conclude

$$\begin{aligned}\mathrm{ad}(\iota(x))^{p-1}(\iota(d(x))) &= \sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}\iota(x)^j \cdot \iota(d(x)) \cdot \iota(x)^{p-1-j} \qquad [25, \mathrm{I}.1.3]\\ &= \sum_{j=0}^{p-1} \iota(x)^j \cdot \iota(d(x)) \cdot \iota(x)^{p-1-j}.\end{aligned}$$

Hence f is a homomorphism of restricted Lie algebras and we can proceed as in [25, I.8.1(5)].

(2) $\Rightarrow$ (1): Let $D \in \mathrm{Der}(\mathrm{U}_0(\mathfrak{g}))$ be a derivation such that $D \circ \iota = \iota \circ d$. Recall that $\iota : \mathfrak{g} \to \mathrm{U}_0(\mathfrak{g})^-$ is a homomorphism of restricted Lie algebras. Let $x \in \mathfrak{g}$, then

$$\begin{aligned}\iota(d(x^{[p]})) = D(\iota(x^{[p]})) = D(\iota(x)^p) &= \mathrm{ad}_{\mathrm{U}_0(\mathfrak{g})}(\iota(x))^{p-1}(D(\iota(x)) \qquad \text{Lemma 2.5.15(2)}\\ &= \mathrm{ad}_{\mathrm{U}_0(\mathfrak{g})}(\iota(x))^{p-1}(\iota(d(x)))\\ &= \iota(\mathrm{ad}_{\mathfrak{g}}(x)^{p-1}(d(x))).\end{aligned}$$

It now follows from the injectivity of ι, that d is a restricted derivation. $\square$

Remark 2.5.17. Using that $[D, D'], D^p$ are derivations of $\mathrm{U}_0(\mathfrak{g})$ if D, D' are, one can conclude from the above that $\mathrm{Der}_p(\mathfrak{g})$ is in fact a restricted subalgebra of $\mathrm{Der}(\mathfrak{g})$.

Let $\mathfrak{n}$ be a Lie algebra and consider a homomorphism $\tau : \mathfrak{h} \to \mathrm{Der}_k(\mathfrak{n})$ of Lie algebras. Then the vector space $\mathfrak{n} \oplus \mathfrak{h}$ obtains the structure of a Lie algebra with bracket

$$[(x, y), (x', y')] := ([x, x']_{\mathfrak{n}} + \tau_y(x') - \tau_{y'}(x), [y, y']_{\mathfrak{h}}) \;\; \forall x, x' \in \mathfrak{n}, y, y' \in \mathfrak{h}.$$

It is called the *semidirect product of* $\mathfrak{n}$ *and* $\mathfrak{h}$ *(with respect to* τ*)* and denoted by $\mathfrak{n} \rtimes_\tau \mathfrak{h}$ or simply $\mathfrak{n} \rtimes \mathfrak{h}$ when the homomorphism τ is clear from the context.

Lemma 2.5.18. *The following statements hold:*

(1) Let $\mathfrak{n}$ *be a restricted Lie algebra and* $\tau : \mathfrak{h} \to \mathrm{Der}_p(\mathfrak{n})$ *be a homomorphism of restricted Lie algebras. Then* $\mathfrak{n} \rtimes_\tau \mathfrak{h}$ *admits a unique p-map with the property that the canonical embeddings* $\iota_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n} \rtimes_\tau \mathfrak{h}$, $x \mapsto (x, 0)$ *and* $\iota_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{n} \rtimes_\tau \mathfrak{h}$, $y \mapsto (0, y)$ *are homomorphisms of restricted Lie algebras. This p-map is given by*

$$(x, y)^{[p]} = \left(x^{[p]_{\mathfrak{n}}} + \sum_{i=1}^{p-1} s_i((x, 0), (0, y)),\; y^{[p]_{\mathfrak{h}}} \right) \;\; \forall x \in \mathfrak{n}, y \in \mathfrak{h}.$$

(2) Let $\mathfrak{g}$ *be a restricted Lie algebra. If* $\mathfrak{n} \trianglelefteq \mathfrak{g}$ *is a p-ideal and* $\mathfrak{h} \subseteq \mathfrak{g}$ *a p-subalgebra, then*

$\mathfrak{h}$ *acts on* $\mathfrak{n}$ *via restricted derivations. Moreover, if* $\mathfrak{n} \cap \mathfrak{h} = (0)$ *and* $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$*, then* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{n} \rtimes \mathfrak{h}$.

Proof. (1) This follows from the proof of [25, II, Theorem 2.5], which is based on Jacobson's theorem [25, II, Theorem 2.3].

(2) Clearly $\mathfrak{h}$ acts on $\mathfrak{n}$ via derivations, we have to show that these derivations are restricted. Let $y \in \mathfrak{h}$ and $x \in \mathfrak{n}$, then

$$[y, x^{[p]}] = -[x^{[p]}, y] = -\mathrm{ad}_{\mathfrak{g}}(x)^{p-1}(\underbrace{[x, y]}_{\in \mathfrak{n}}) = \mathrm{ad}_{\mathfrak{n}}(x)^{p-1}([y, x]).$$

The requirements $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and $\mathfrak{n} \cap \mathfrak{h} = (0)$ imply that $\mathfrak{g}$ is the direct sum $\mathfrak{n} \oplus \mathfrak{h}$ of vector spaces. A desired isomorphism is now given by the map $\mathfrak{g} \to \mathfrak{n} \rtimes_\tau \mathfrak{h}$, $x + y \mapsto (x, y)$.

□

Remark 2.5.19. Let $\mathfrak{g} = \mathfrak{n} \rtimes_\tau \mathfrak{h}$ be a semidirect product of restricted Lie algebras. Recall that $V(\mathfrak{g}) := \{x \in \mathfrak{g} : x^{[p]} = 0\}$ is the nullcone of $\mathfrak{g}$.

(1) If $\mathfrak{n}$ is elementary abelian, then $\mathfrak{n}$ is nothing but a restricted $\mathfrak{h}$-module.

(2) We clearly have $V(\mathfrak{h}) \cup V(\mathfrak{n}) \subseteq V(\mathfrak{g}) \subseteq \mathfrak{n} \times V(\mathfrak{h})$.

(3) The restricted homomorphism $\tau : \mathfrak{h} \to \mathrm{Der}_p(\mathfrak{n})$ induces a restricted homomorphism $\tau' : \mathfrak{h} \to \mathrm{Der}(\mathrm{U}_0(\mathfrak{n}))$, where $\tau'(x) \in \mathrm{Der}_k(\mathrm{U}_0(\mathfrak{n}))$ is the unique derivation of $\mathrm{U}_0(\mathfrak{n})$ which lifts $\tau(x)$ (see Lemma 2.5.16). Hence, $\mathrm{U}_0(\mathfrak{n})$ is a $\mathrm{U}_0(\mathfrak{h})$-module.

By way of example, and for later reference, we discuss two cases where the p-map on a semidirect product can be written down more explicitly.

Lemma 2.5.20. *Let* $\mathfrak{g} = \mathfrak{n} \rtimes_\tau \mathfrak{h}$ *be a semidirect product of restricted Lie algebras.*

(1) If $\tau \equiv 0$ *(leading to a direct product* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$*), then* $(x, y)^{[p]} = (x^{[p]}, y^{[p]})$ *for all* $x \in \mathfrak{n}, y \in \mathfrak{h}$*. In particular,* $V(\mathfrak{g}) = V(\mathfrak{n}) \times V(\mathfrak{h})$.

(2) If $\mathfrak{n}$ *is an abelian Lie algebra, then we have* $(x, y)^{[p]} = (x^{[p]} + \tau(y)^{p-1}(x), y^{[p]})$ *for all* $x \in \mathfrak{n}, y \in \mathfrak{h}$.

Proof. (1) The assumption $\tau \equiv 0$ implies easily that $s_i((x, 0), (0, y)) = 0$ for all $x \in \mathfrak{n}, y \in \mathfrak{h}$. The assertion follows from the definition of the p-map on $\mathfrak{g}$.

(2) Using that $\mathfrak{n}$ is abelian, a direct computation shows that $s_1((x, 0), (0, y)) = (\tau(y)^{p-1}(x), 0)$ as well as $0 = s_i((x, 0), (0, y))$ for $2 \leq i \leq p - 1$. Hence, the assertion again follows from the definition of the p-map on $\mathfrak{g}$.

□

Lemma 2.5.21. *Let $\mathfrak{g}$ be a restricted Lie algebra and $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an automorphism of the ordinary Lie algebra $\mathfrak{g}$.*

(1) If the center $C(\mathfrak{g})$ is zero, then σ is restricted.

(2) If $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{h}$ is a semidirect product of restricted Lie algebras and the restriction of σ to $\mathfrak{n}$ and $\mathfrak{h}$ induces restricted automorphisms of $\mathfrak{n}$ and $\mathfrak{h}$, respectively, then σ is restricted.

Proof. (1) The elements $\sigma(x^{[p]}) - \sigma(x)^{[p]}$ for $x \in \mathfrak{g}$ are contained in the kernel of ad : $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, which is exactly the center $C(\mathfrak{g})$ of $\mathfrak{g}$.

(2) This follows from 2.5.18 and [25, Exercise 4 on p.69].

□

Given an exact sequence

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$$

of restricted Lie algebras, we call $\mathfrak{g}$ *an extension of* $\mathfrak{h}$ *by* $\mathfrak{n}$. This extension is then referred to as a *split extension*, provided there is a homomorphism of restricted Lie algebras $\varphi : \mathfrak{h} \to \mathfrak{g}$ such that $\pi \circ \varphi = \mathrm{id}_{\mathfrak{h}}$. Note that φ is automatically injective so that $\mathrm{im}(\varphi) \cong \mathfrak{h}$. In analogy to Lemma 2.4.27, one can show

Lemma 2.5.22. *Split extensions correspond to semidirect products of restricted Lie algebras.*

As a consequence, we can easily obtain the following fact:

Lemma 2.5.23. *If $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$ is a semidirect product of affine algebraic group schemes, then (denoting their Lie algebras with gothic letters) $\mathfrak{g}$ is isomorphic to the semidirect product $\mathfrak{n} \rtimes \mathfrak{h}$.*

Proof. Consider the corresponding exact sequence

$$e_k \longrightarrow \mathcal{N} \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow e_k.$$

By definition, there is $\varphi : \mathcal{H} \to \mathcal{G}$ such that $\pi \circ \varphi = \mathrm{id}_{\mathcal{H}}$. Consequently, $\mathrm{Lie}(\pi) \circ \mathrm{Lie}(\varphi) = \mathrm{id}_{\mathfrak{h}}$. In particular, $\mathrm{Lie}(\pi)$ is surjective. Since $\mathcal{G}$ is algebraic (hence $\mathcal{N}$ and $\mathcal{H}$ are), we obtain a split-exact sequence

$$0 \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{g} \xrightarrow{\mathrm{Lie}(\pi)} \mathfrak{h} \longrightarrow 0$$

of restricted Lie algebras. □

Recall that a direct sum $Z := X \oplus Y$ of finite-dimensional vector spaces induces a direct sum decomposition $Z^* = X^* \oplus Y^*$ of its dual, where X^* is identified with $\{\lambda \in Z^* : \lambda(Y) = (0)\}$ and Y^* with $\{\lambda \in Z^* : \lambda(X) = (0)\}$. Recall that every algebraic group acts on its Lie algebra via the adjoint representation (cf. 2.4.32(2)).

Lemma 2.5.24. *Let $G = N \rtimes H$ be a semidirect product of algebraic groups. Then the following statements hold:*

(a) The H-module $\mathfrak{g}$ ($\mathfrak{g}^*$) is the direct sum $\mathfrak{n} \oplus \mathfrak{h}$ ($\mathfrak{n}^* \oplus \mathfrak{h}^*$) of H-modules. Moreover, N acts on $\mathfrak{n} \subseteq \mathfrak{g}$ ($\mathfrak{n}^* \subseteq \mathfrak{g}^*$) via Ad_N (its dual).

(b) Let $n \in N$. Consider the left translation $l_n : N \to N$ effected by n and the morphism $\eta_n : H \to N,\ h \mapsto h.n$. Put

$$\varphi : N \times \mathfrak{h} \longrightarrow \mathfrak{n},\quad (n, x) \mapsto \mathrm{Ad}_N(n) \circ \mathrm{d}_n(l_{n^{-1}}) \circ \mathrm{d}_e(\eta_n)(x)$$
$$\psi : N \times \mathfrak{n}^* \longrightarrow \mathfrak{h}^*,\quad (n, \lambda) \mapsto \lambda \circ \varphi(n, -).$$

Then

$$n.x = (-\varphi(n, x),\ x)\ \ \forall n \in N, x \in \mathfrak{h}.$$

Moreover, if $\chi = (\chi_{\mathfrak{n}}, \chi_{\mathfrak{h}}) \in \mathfrak{g}^*$ is a linear form, then

$$n.\chi = (n.\chi_{\mathfrak{n}},\ \chi_{\mathfrak{h}} - \psi(n^{-1}, \chi_{\mathfrak{n}}))\ \ \forall n \in N.$$

Proof. We only determine the action of N on elements of $\mathfrak{h} \subseteq \mathfrak{g}$ (using this, all others can be obtained via direct computation): Denote the conjugation effected by $n \in N$ as

$$\kappa_n : H \to N \rtimes H = G;\ h \mapsto nhn^{-1} = (n(h.n)^{-1}, h)$$

The second coordinate function of this morphism is the identity id_H on H and the first is the composite $l_n \circ \iota_N \circ \eta_n$, where $\iota_N : N \to N$ denotes the inversion of the group N and the maps l_n, η_n are from above. Then, using the functorial property of taking differentials (Lemma 2.4.30), we obtain

$$\begin{aligned}
\mathrm{d}_e(l_n \circ \iota_N \circ \eta_n) &= \mathrm{d}_{(\iota_N \circ \eta_n)(e)}(l_n) \circ \mathrm{d}_e(\iota_N \circ \eta_n) \\
&= \mathrm{d}_{n^{-1}}(l_n) \circ \mathrm{d}_n(\iota_N) \circ \mathrm{d}_e(\eta_n) \\
&= \mathrm{d}_{n^{-1}}(l_n) \circ (-\mathrm{d}_e(r_{n^{-1}}) \circ \mathrm{d}_n(l_{n^{-1}})) \circ \mathrm{d}_e(\eta_n) && \text{Lemma 2.4.31} \\
&= -\mathrm{d}_e(l_n \circ r_{n^{-1}}) \circ \mathrm{d}_n(l_{n^{-1}}) \circ \mathrm{d}_e(\eta_n) \\
&= -\mathrm{Ad}_N(n) \circ \mathrm{d}_n(l_{n^{-1}}) \circ \mathrm{d}_e(\eta_n).
\end{aligned}$$

□

Remark 2.5.25. If $H' \trianglelefteq H$ is a closed, normal subgroup, then $G' := N \rtimes H'$ is a closed, normal subgroup of $G = N \rtimes H$. The algebraic group G therefore acts on $\mathfrak{g}' = \mathfrak{n} \rtimes \mathfrak{h}'$ with p-automorphisms. Thus, also every element of H gives rise to an automorphism of the restricted Lie algebra $\mathfrak{g}'$. This will be used later on in case $H = \mathrm{GL}(2), H' = \mathrm{SL}(2)$ and N is the three-dimensional Heisenberg group or $\mathbb{G}_a^2$, the direct product of the additive group $\mathbb{G}_a = (\mathbb{A}^1, +)$ with itself.

We want to record a version of the above lemma in a special case. Recall that, equipped with addition, we can view any finite-dimensional vector space V as an algebraic group and that linear maps $V \to W$ are morphisms (see 2.4.3). We record the following well-known fact, which essentially states that linear maps can be identified with their differentials.

Lemma 2.5.26. *Let V be a finite-dimensional vector space and $x \in V$. Then the map $\Gamma_{V,x} : V \to \mathrm{Der}_k(S(V^*), k_x) = T_x(V)$ described below is an isomorphism of vector spaces. If W is another finite-dimensional vector space and $f : V \to W$ a linear map, then $\mathrm{d}_x(f) \circ \Gamma_{V,x} = \Gamma_{W,f(x)} \circ f$.*

Proof. In view of 2.4.3(3), we may after a choice of a basis assume that $V = k^n$. Then $k[V] = k[X_1, \dots, X_n]$ is the polynomial ring in n variables. Given $x = (x_1, \dots, x_n) \in V$, the asserted map $\Gamma_{V,x}$ is now given as follows: starting with a vector $v = (v_1, \dots, v_n) \in V$, we obtain a derivation $d_v := \sum_{j=1}^n v_j \frac{\partial}{\partial X_j} \in \mathrm{Der}_k(k[V], k[V])$ and then composition with the evaluation homomorphism $k[V] \to k$ at x yields the desired derivation $\Gamma_{V,x}(v) \in \mathrm{Der}_k(k[V], k_x)$. For the additional claim, one may also assume $W = k^m$ by picking a basis of W and observe that the linear map $f : V \to W$ is then given by left multiplication with the representing matrix relative to these bases. Then it is enough to show $(\mathrm{d}_x(f)(\Gamma_{V,x}(v))(Y_k) = (\Gamma_{W,f(x)}(f(v))(Y_k)$ for all $1 \leq k \leq m$, which is direct computation. $\square$

Let now V be a restricted representation of some restricted Lie algebra $\mathfrak{g}$, then we can consider the semidirect product $V \rtimes \mathfrak{g}$ (see 2.5.19(1)). If $\mathfrak{g}$ is the Lie algebra of an algebraic group G, we say that V *is integrable to* G, provided there is some rational G-module V' such that $V' \cong V$ as $\mathfrak{g}$-modules.

Lemma 2.5.27. *Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of an algebraic group G and V a finite-dimensional restricted $\mathfrak{g}$-module.*

(1) If V is integrable to G, then $V \rtimes \mathfrak{g} = \mathrm{Lie}(V \rtimes G)$.

(2) If (1) holds, then

(a) The G-module $V \rtimes \mathfrak{g}$ $((V \rtimes \mathfrak{g})^*)$ is the direct sum $V \oplus \mathfrak{g}$ $(V^* \oplus \mathfrak{g}^*)$ of G-modules.

(b) *The group V acts on $V \subseteq V \rtimes \mathfrak{g}$ trivially and we have*

$$v.x = (-x.v, x) \;\; \forall v \in V, x \in \mathfrak{g}.$$

(c) *If $\chi = (\chi_V, \chi_\mathfrak{g}) \in (V \rtimes \mathfrak{g})^*$ is a linear form, then we have*

$$v.\chi = (\chi_V \;,\; \chi_\mathfrak{g} - \psi(v, \chi_V))$$

for all $v \in V$, where $\psi : V \times V^* \to \mathfrak{g}^*$, $(v, \lambda) \mapsto (x \mapsto (x.\lambda)(v) = -\lambda(x.v))$.

Proof. This is a special case of 2.5.23 and 2.5.24. It is not hard to see, that the differential of left translation $l_v : V \to V$, $w \mapsto v + w$ by some vector v can be identified with the identity id_V (see also 2.5.26). Moreover, the map $\eta_v : G \to V, g \mapsto g.v$ is nothing but the composite of the structure homomorphism $G \to \mathrm{GL}(V)$ and the restriction of the evaluation map $\gamma_v : \mathrm{End}_k(V) \to V$, $f \mapsto f(v)$ to $\mathrm{GL}(V)$. Since γ_v is linear, it follows from 2.5.26 and 2.4.30 that the differential of η_v at the identity element e of G can be identified with $\mathfrak{g} \to V$, $x \mapsto x.v$. □

Remark 2.5.28. The mapping ψ is bilinear. Moreover, 2.4.32(3) implies that ψ is G-invariant.

Given a finite group scheme $\mathcal{G}$, we let $\delta_\mathcal{G} \in X(\mathcal{G})$ be the character corresponding to the left modular function of its Hopf algebra $k\mathcal{G}$.

Lemma 2.5.29. *Let G be an algebraic group. Assume that G acts on some unipotent algebraic group H and denote by $\rho : G \to \mathrm{GL}(\mathfrak{h})$ the corresponding action of G on $\mathfrak{h} = \mathrm{Lie}(H)$. Let $r \geq 1$.*

(a) *The modular function $\delta_{H_r \rtimes G_r}$ of the infinitesimal group $H_r \rtimes G_r$ is the restriction of the character*

$$X(H \rtimes G) \ni \delta : H \rtimes G \to \mathbb{G}_m,\; (h, g) \mapsto (\det(\mathrm{Ad}_G(g)) \cdot \det(\rho(g)))^{p^r - 1}$$

to $H_r \rtimes G_r$, where Ad_G denotes the adjoint representation of G on $\mathfrak{g}$.

(b) *If G is reductive, then $\delta(h, g) = \det(\rho(g))^{p^r - 1}$ for all $(h, g) \in H \rtimes G$.*

(c) *Let $\rho : G \to GL(V)$ be a rational representation. The modular function $\delta_{G_{V,r}}$ of the infinitesimal group $G_{V,r} = V_r \rtimes G_r$ is the restriction of the character*

$$X(G_V) \ni \delta : V \rtimes G \to \mathbb{G}_m,\; (v, g) \mapsto (\det(\mathrm{Ad}_G(g)) \cdot \det(\rho(g)))^{p^r - 1}$$

to $G_{V,r}$. If G is reductive, then $\delta(v, g) = \det(\rho(g))^{p^r - 1}$ for all $(v, g) \in G_V$.

Proof. Combining [41, I.8.19, I.9.7], we see that the modular function of $G_{V,r}$ is given by restricting the character

$$\delta : H \rtimes G \to \mathbb{G}_m,\ (h,g) \mapsto \det(\mathrm{Ad}_{H\rtimes G}(h,g))^{p^r-1}$$

of $H \rtimes G$ to $H_r \rtimes G_r$, where $\mathrm{Ad}_{H\rtimes G}$ denotes the adjoint representation of $H \rtimes G$ on its Lie algebra $\mathfrak{h} \rtimes \mathfrak{g}$ (see also 2.5.23, 2.5.24). Since H is unipotent, we have $\delta(h,1) = 1$ for all $h \in H$. Claim (a) now follows directly from the fact that the G-module $\mathfrak{h} \rtimes \mathfrak{g}$ is the direct sum $\mathfrak{h} \oplus \mathfrak{g}$. Part (b) follows from the fact that $\det \circ \mathrm{Ad}_G = 1$ for reductive G (see the proof of [41, Proposition II.3.4 a)]) and (c) is a special instance of (a) and (b). $\square$

3 Existence of Euclidean components

3.1 Relationship to the existence of unipotent, normal subgroups

Given a finite group scheme $\mathcal{G}$ over k, we denote by $\Gamma_s(\mathcal{G})$ the stable Auslander-Reiten quiver of its Hopf algebra $k\mathcal{G}$. Recall from 2.3.9, that $k\mathcal{G}$ is a Frobenius algebra with Nakayama automorphism given by $\mu_{\mathcal{G}} := \mathrm{id}_{k\mathcal{G}} * \zeta_{\mathcal{G}}$ ($k\mathcal{G}$ being cocommutative implies that its antipode has order 2), the convolution product of the identity with the left modular function $\zeta_{\mathcal{G}}$ of $\mathcal{G}$. The following will be used later on.

Lemma 3.1.1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of some finite group scheme $\mathcal{G}$. Then $\mu_G|_{k\mathcal{N}} = \mu_{\mathcal{N}}$.*

Proof. See the proof of [20, Lemma 3.1.1]. $\square$

In [29], Friedlander and Pevtsova introduced the noetherian topological space $\Pi(\mathcal{G})$ of equivalence classes of π-points and attached to every $\mathcal{G}$-module M its Π-support

$$\Pi(\mathcal{G})_M := \{[\alpha_K] \in \Pi(\mathcal{G}) : \alpha_K^*(M_K) \text{ is not projective}\}.$$

These subsets of $\Pi(\mathcal{G})$ are precisely the closed subsets. Here $k \subseteq K$ is a field-extension, $M_K := M \otimes_k K$ and, if $\mathfrak{A}_{p,K} := K[t]/(t^p)$ and $K\mathcal{G} := K\mathcal{G}_K$, a *$\pi$-point* $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ is a left flat map of k-algebras which factors through some abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$. If β_L is another π-point, then α_K and β_L are equivalent, provided the following equivalence holds for any $\mathcal{G}$-module M:

$$\alpha_K^*(M_K) \text{ is projective} \iff \beta_L^*(M_L) \text{ is projective}.$$

Theorem 3.1.2. *Let M be a finite-dimensional module of some finite group scheme $\mathcal{G}$.*

(1) We have $\dim(\Pi(\mathcal{G})_M) = \mathrm{cx}_{\mathcal{G}}(M) - 1$, where $\mathrm{cx}_{\mathcal{G}}(M)$ denotes the complexity of the $k\mathcal{G}$-module M (recall Definition 2.1.16).

(2) M is projective if and only if $\Pi(\mathcal{G})_M = \emptyset$.

(3) Let M be non-projective indecomposable, then $\dim(\Pi(\mathcal{G})_M) = 0$ if and only if M is periodic.

Proof. For (1) and (2), we refer to [29, Corollary 5.5, Proposition 5.8]. Part (3) can be shown analogously to [18, Theorem 3.3(2)]. $\square$

Given $M \in \mathbf{mod}(\mathcal{G})$ and $[\alpha_K] \in \Pi(\mathcal{G})$, we write

$$\alpha_K^*(M_K) \cong \bigoplus_{i=1}^{p} \alpha_{K,i}(M)[i],$$

where $[i] \in \mathbf{mod}(\mathfrak{A}_{p,K})$ denotes the unique indecomposable module of dimension i (see p.13).

Definition 3.1.3. Let $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point and $M \in \mathbf{mod}(\mathcal{G})$.

(i) The iso-class $\bigoplus_{i=1}^{p} \alpha_{K,i}(M)[i] \in \mathbf{mod}(\mathfrak{A}_{p,K})$ is called the *Jordan type* $\mathrm{Jt}(M, \alpha_K)$ *of M with respect to α_K.*

(ii) The iso-class $\bigoplus_{i=1}^{p-1} \alpha_{K,i}(M)[i] \in \mathbf{mod}(\mathfrak{A}_{p,K})$ is called the *stable Jordan type* $\mathrm{StJt}(M, \alpha_K)$ *of M with respect to α_K.*

(iii) The set $\mathrm{supp}_{\alpha_K}(M) := \{i \in \{1, \dots, p-1\} \mid \alpha_{K,i}(M) \neq 0\}$ is called the *α_K-support of M.*

Recall that a finite group scheme $\mathcal{G}$ is called *linearly reductive*, provided its Hopf algebra $k\mathcal{G}$ is semi-simple. We record the following trivial observation.

Lemma 3.1.4. *Let $\mathcal{G}$ be a finite group scheme. Then the following statements are equivalent:*

(a) $\mathcal{G}$ is linearly reductive.

(b) The trivial module k is a projective $\mathcal{G}$-module.

(c) $\Pi(\mathcal{G}) = \emptyset$.

In particular, if $\mathcal{G}$ is unipotent and non-trivial, then there exists a π-point $\alpha_K : K[t]/t^p \to K\mathcal{U}$.

Proof. By 3.1.2, we have $\Pi(\mathcal{G}) = \Pi(\mathcal{G})_k = \emptyset$ if and only if k is a projective $k\mathcal{G}$-module. This shows $(b) \iff (c)$. The implication $(a) \implies (b)$ is clear. If (b) holds, then $k\mathcal{G}$ being a Hopf-algebra implies that $M \otimes_k k \cong M$ is projective for any $M \in \mathbf{mod}(\mathcal{G})$. Thus, $k\mathcal{G}$ is semi-simple. As $\mathcal{G}$ is unipotent if and only if the Hopf algebra $k\mathcal{G}$ is local, which means that the regular $k\mathcal{G}$-module is the projective cover of k, the additional statement is also clear. □

Definition 3.1.5. A component $\Theta \subseteq \Gamma_s(\mathcal{G})$ is called *locally split*, provided the exact sequence $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ splits for every $[\alpha_K] \in \Pi(\mathcal{G})$ and every module $M \in \Theta$.

Analogously to [18, Lemma 3.1] one can show that the Π-support is constant inside Auslander-Reiten components $\Theta \subseteq \Gamma_s(\mathcal{G})$. Thus, we shall speak of the Π-support $\Pi(\mathcal{G})_\Theta$ of Θ.

Lemma 3.1.6. *If $\Theta \subseteq \Gamma_s(\mathcal{G})$ is a Euclidean component, then* $\dim \Pi(\mathcal{G})_\Theta \geq 1$. *In particular, Θ is locally split.*

Proof. If $\dim \Pi(\mathcal{G})_\Theta = 0$, then Theorem 3.1.2(3) implies that every module in Θ is periodic. In view of [35, 4.16.2], this is a contradiction. It now follows from [23, Proposition 2.3(2)] that Θ is locally split. □

In [23, §3], Farnsteiner attached some invariants to locally split components $\Theta \subseteq \Gamma_s(\mathcal{G})$, which help to decide whether two given modules belong to the same component or not (having the same Π-support is in general not enough for a positive answer).

Lemma 3.1.7. *Let $M \in \mathbf{mod}(\mathcal{G})$ be a $\mathcal{G}$-module and $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ a π-point. Then*

$$\alpha_{K,i}(\Omega^n(M)) = \begin{cases} \alpha_{K,i}(M), & n \text{ is even} \\ \alpha_{K,p-i}(M), & \text{otherwise} \end{cases}$$

for all $1 \leq i \leq p-1$.

Proof. By [42, Corollary 3.5 (c)], the Heller-operator commutes with base extensions. As α_K^* is exact and sends projectives to projectives, general properties of the Heller operator yield

$$\alpha_K^*(\Omega^n_{\mathcal{G}}(M)_K) \cong \alpha_K^*(\Omega^n_{\mathcal{G}_K}(M_K)) \cong \Omega^n_{\mathfrak{A}_{p,K}}(\alpha_K^*(M_K)) \oplus \text{ (proj)}$$

As $\Omega_{\mathfrak{A}_{p,K}}([i]) = [p-i]$ for $1 \leq i \leq p-1$ (see Lemma 2.1.40), the lemma follows. □

As the author states in the introduction of [18], the main results of the first three sections are still valid, when the ground field k is perfect. With this in hand, we can now verify the following theorem.

Theorem 3.1.8. *Let $\mathcal{G}$ be a finite group scheme over a perfect field k of characteristic $p \geq 3$. If the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$ admits a component of Euclidean type, then $\mathcal{G}$ has no non-trivial normal, unipotent subgroups.*

Proof. Let $\mathcal{U} \trianglelefteq \mathcal{G}$ be a non-trivial unipotent, normal subgroup. By Lemma 3.1.4, there is $[\alpha_K] \in \Pi(\mathcal{U})$ which we treat as an element of $\Pi(\mathcal{G})$ under the natural map $\Pi(\mathcal{U}) \to \Pi(\mathcal{G})$ (cf. [29, Corollary 2.7]). As $\alpha_K(t) \in K\mathcal{U}$ is nilpotent, the image of α_K is contained in the augmentation ideal of $K\mathcal{U}$. In view of Lemma 2.4.19, $\mathcal{U}$ acts trivially on any simple $\mathcal{G}$-module S. Hence, for all simple $\mathcal{G}$-modules S:

$$\mathrm{Jt}(S, \alpha_K) = \dim_k S \cdot [1] \ (*).$$

Now assume $\Theta \subseteq \Gamma_s(\mathcal{G})$ is a component of Euclidean type. In view of Lemma 3.1.6, Θ is locally split. Moreover, Lemma 2.2.3 shows that $P/\mathrm{Soc}(P)$ belongs to Θ for some principal indecomposable $\mathcal{G}$-module P. Hence the simple module $S = \mathrm{Soc}(P) = \Omega(P/\mathrm{Soc}(P))$ belongs to the component $\Omega(\Theta)$. In particular $\Omega(S^\mu)$ belongs to $\tau_{\mathcal{G}}(\Theta) = \Theta$. As $\Omega(\Theta)$

is isomorphic to Θ (see 2.2.4), it is therefore also attached to some projective indecomposable module Q. Hence, the simple module $T := \Omega(Q/\mathrm{Soc}(Q))^{\mu}$ also belongs to $\tau_{\mathcal{G}}(\Theta) = \Theta$. A consecutive application of [23, Corollary 3.2.3] and Lemma 3.1.7 then yields

$$\{1\} \overset{(*)}{=} \mathrm{supp}_{\alpha_K}(T) = \mathrm{supp}_{\alpha_K}(\Omega(S^{\mu})) = \{p-1\},$$

which contradicts the assumption $p \geq 3$. □

Remark 3.1.9. (1) If $\mathcal{G}$ is infinitesimal, then the theorem shows that if $\mathcal{G}$ admits a Euclidean component, then $\mathcal{G}$ is reductive.

(2) If the characteristic of k equals $p = 2$, our theorem fails. Here the restricted enveloping algebra of the two-dimensional elementary abelian Lie algebra $\mathfrak{e}_2$ is known to have components of type $\mathbb{Z}[\tilde{A}_{12}]$ ($\tilde{A}_{12} = \bullet \xrightarrow{(2,2)} \bullet$) in that case (cf. [10, Theorem 2]).

(3) The converse of our theorem is also not true. In view of Lemma 2.4.34(1) and Corollary 3.2.2 below, one could take a Frobenius kernel of the simple reductive group $\mathrm{SL}(n)$ as a counterexample, where $n \geq 3$ does not divide p.

3.2 Some consequences

As a first consequence of our theorem, the characterization of trigonalizable group schemes 2.4.20 now yields

Corollary 3.2.1. *Let $\mathcal{G}$ be a finite trigonalizable group scheme over a perfect field k of characteristic $p \geq 3$. Then $\Gamma_s(k\mathcal{G})$ does not afford components of Euclidean type.*

In the following, we will again consider the Euclidean tree $\tilde{A}_{12} = \bullet \xrightarrow{(2,2)} \bullet$. Recall from 2.4.34, that an algebraic group G is reductive of and only if its reduced group scheme $\mathcal{G} = \mathrm{Spec}_k(k[G])$ enjoys this property.

Corollary 3.2.2. *Let $\mathcal{G}$ be an affine algebraic group scheme over a field k of positive characteristic $p \geq 3$ and let $r \geq 1$ be an arbitrary natural number. Then the following statements hold:*

(1) If k is perfect and there exists a unipotent, normal subgroup $\mathcal{U} \trianglelefteq \mathcal{G}$, which is not finite and reduced, then the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G}_r)$ of the rth Frobenius kernel $\mathcal{G}_r$ does not afford components of Euclidean type.

(2) If k is algebraically closed, $\mathcal{G}$ is reduced and $\Gamma_s(\mathcal{G}_r)$ admits a component of Euclidean type, then $\mathcal{G}$ is reductive.

(3) If k is algebraically closed of characteristic $p > 3$ and $\mathcal{G}$ is connected and reduced, then the following statements are equivalent for a component $\Theta \subseteq \Gamma_s(\mathcal{G}_r)$ of the stable Auslander-Reiten quiver of the rth Frobenius kernel of $\mathcal{G}$:

(a) Θ is a component of Euclidean type.

(b) Θ is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$, the group $\mathcal{G}$ is reductive and $\mathrm{SL}(2)$ or $\mathrm{PSL}(2) = \mathrm{SL}(2)/\pm E_2$ is a direct factor of $\mathcal{G}$.

Proof. (1) As $\mathcal{U} \trianglelefteq \mathcal{G}$ is a normal subgroup, $\mathcal{G}$ stabilizes all its Frobenius kernels. By our current assumption, 2.4.12 implies that the first Frobenius kernel $\mathcal{U}_1$ is a non-trivial, unipotent normal subgroup of $\mathcal{G}_r$. Now apply Theorem 3.1.8.

(2) By (1), every unipotent, normal subgroup of $\mathcal{G}$ is finite and reduced. Thus, this holds for the unipotent radical $R_u(\mathcal{G})$. Since $R_u(\mathcal{G})$ is connected, we get $R_u(\mathcal{G}) = e_k$. Hence $\mathcal{G}$ is reductive.

(3) The implication $(b) \Rightarrow (a)$ is clear.

$(a) \Rightarrow (b)$: In view of (2), $\mathcal{G}$ is reductive. The statement now follows from the proof of [15, Theorem 4.1] (here we need $p > 3$).

□

We denote by $\mathcal{K}$ the quotient of the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left=60, "a"] \arrow[r, bend left=20, "b"] & 2 \arrow[l, bend left=20, "c"] \arrow[l, bend left=60, "d"] \end{tikzcd}$$

by the relations

$$ad = bc, \qquad da = cb, \qquad ac = bd = ca = db = 0.$$

The algebra $\mathcal{K}$ is isomorphic to the *trival extension* $(kQ)^* \rtimes kQ$ of the path algebra of the *Kronecker quiver* $Q = \bullet \rightrightarrows \bullet$. It is known to be a domestic algebra that affords exactly two components of type $\mathbb{Z}[\tilde{A}_{12}]$ and infinitely many homogeneous tubes $\mathbb{Z}[A_\infty]/(\tau)$.

Given a finite group scheme $\mathcal{G}$, we denote by $C(\mathcal{G})$ its center (see [57, p.27] for the definition). Since $C(\mathcal{G})$ is abelian, we have a decomposition $C(\mathcal{G}) = M(\mathcal{G}) \times U(\mathcal{G})$ into a multiplicative group $M(\mathcal{G})$, the *multiplicative center*, and a unipotent group $U(\mathcal{G})$ (see [57, Theorem 9.5]). Since $\mathcal{G}$ acts trivially on its center, both $M(\mathcal{G})$ and $U(\mathcal{G})$ are necessary normal subgroups of $\mathcal{G}$.

Corollary 3.2.3. *Let $\mathcal{G}$ be an infinitesimal group scheme of height r over an algebraically closed field k of characteristic $p \geq 3$ such that the principal block $\mathcal{B}_0(\mathcal{G})$ is tame.*

(1) If $\Gamma_s(\mathcal{G})$ admits a component of Euclidean type, then $\mathcal{B}_0(\mathcal{G})$ is domestic and $\mathcal{G}/M(\mathcal{G}) \cong \mathrm{SL}(2)_1 T_r$, where $T \subseteq \mathrm{SL}(2)$ denotes the torus of diagonal matrices.

(2) If $\Gamma_s(\mathcal{G})$ admits a component Θ of Euclidean type and $r = 1$, then every block $\mathcal{B} \subseteq k\mathcal{G}$ is either simple, or Morita-equivalent to Λ, where $\Lambda \in \{k[X]/(X^2), \mathcal{K}\}$. In particular, $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$. Moreover, $\mathrm{U}_\chi(\mathrm{Lie}(\mathcal{G}))$ is tame or representation-finite for all $\chi \in \mathfrak{g}^*$.

Proof. Part (1) directly follows from [20, Proposition 3.2.1] in conjunction with Theorem 3.1.8. The second part then follows from (1) and [24, Theorem 8.10, Corollary 8.12]. □

Recall that every finite group G corresponds to a reduced finite group scheme $\mathcal{G}_G$ (cf. 2.4.8(a)). In [49], Okuyama has shown that G does not admit components of Euclidean tree class unless the characteristic p of the base field k equals $p = 2$. We want to give an alternative proof in case of an algebraically closed field, which combines 3.1.8 and Kawata's theorem [43, 4.6]. Therefore, we need to recall the definition of a vertex and some properties of these.

Definition 3.2.4. Let $H \subseteq G$ be a subgroup of some finite group G and M a G-module.

(i) If M is a direct summand of $kG \otimes_{kH} M|_H$, then M is called *relatively H-projective.*

(ii) If M is indecomposable and H is a minimal subgroup having the property that M is relatively H-projective, then H is called a *vertex of M*. We denote by $vx(M)$ the set of all vertices of M.

Lemma 3.2.5. *Let M be an indecomposable module of some finite group G over k.*

(1) Each vertex $H \in vx(M)$ is a p-group and if $H' \in vx(M)$ is another vertex of M, then there exists $g \in G$ such that $gHg^{-1} = H'$.

(2) M is projective if and only if $vx(M) = \{e\}$.

Proof. Part (1) has been proven by J.A. Green, see [34, p.435 above].

(2): It is obvious that $kG \otimes_{ke} M|_e \cong kG \otimes_k M$, where the tensor product is build with M considered as a trivial G-module. If $vx(M) = \{e\}$, then M is a direct summand of $kG \otimes_k M \cong \dim_k M \cdot kG$. Hence M is projective as a direct summand of some free G-module. If conversely M is projective, then the surjective G-linear map $kG \otimes_k M \to M$, $a \otimes m \mapsto a.m$ is split surjective. Hence $vx(M) = \{e\}$. □

If Q is a stable translation quiver, then the *orbit graph* $Q/\langle\tau\rangle$ is given by $(Q/\langle\tau\rangle)_0 := Q_0/\langle\tau\rangle$ and there is a bond between $[x]$ and $[y]$ if and only if there exists $x_0 \in [x]$ such that $y \in x_0^- \cup x_0^+$ if and only if there exists $y_0 \in [y]$ such that $x \in y_0^- \cup y_0^+$.

Lemma 3.2.6. *Let Q be a quiver with no multiple arrows and no loops.*

(1) The graphs $\overline{Q}$ and $\mathbb{Z}[Q]/\langle\tau\rangle$ are isomorphic.

(2) Every (injective) morphism $f : Z[Q] \to \mathbb{Z}[Q']$ induces a (an injective) morphism $\overline{f} : \overline{Q} \to \overline{Q'}$.

Proof. (1) The map $\overline{Q} \to \mathbb{Z}[Q]/\langle\tau\rangle,\ x \mapsto [(0,x)]$ is readily checked to be an isomorphism.

(2) We define $\overline{f} : \mathbb{Z}[Q]/\langle\tau\rangle \to \mathbb{Z}[Q']/\langle\tau'\rangle,\ [(n,x)] \mapsto [f(n,x)]$. As $f \circ \tau = \tau' \circ f$, the map $\overline{f}$ is well defined. Moreover, it is clearly a homomorphism. Now assume that f is injective and let $[(n,x)], [(m,y)] \in (\mathbb{Z}[Q]/\langle\tau\rangle)_0$ such that $[f(n,x)] = [f(m,y)]$. Then there is $z \in \mathbb{Z}$ such that $(\tau')^z(f(n,x)) = f(m,y)$. As $(\tau')^z(f(n,x)) = f(\tau^z(n,x))$, we obtain $\tau^z(n,x) = (m,y)$ by injectivity. Hence we obtain $[(n,x)] = [(m,y)]$. This shows that $\overline{f}$ is injective as well. Now apply (1).

□

Given a finite group G, we denote by $\Gamma_s(G) := \Gamma_s(\mathcal{G}_G) = \Gamma_s(kG)$ the stable Auslander-Reiten quiver of the reduced finite group scheme $\mathcal{G}_G = \mathrm{Spec}_k(kG)$ which is associated to G. Recall that the Auslander-Reiten translation is given by $\tau = \Omega^2$ (cf. 2.1.36(1)).

Theorem 3.2.7. *Let G be a finite group and assume that k is an algebraically closed field of positive characteristic $p \geq 3$. Then $\Gamma_s(kG)$ does not admit components of Euclidean type.*

Proof. Let $\Theta \subseteq \Gamma_s(kG)$ be a component. Consider the set $vx(\Theta) := \bigcup_{M\in\Theta} vx(M)$, choose a minimal element $H \in vx(\Theta)$ and denote by $N := N_G(H)$ its normalizer. Then Kawata's theorem [43, Theorem 4.6] yields a component $\Delta \subseteq \Gamma_s(kN)$, a valued subquiver $\Lambda \subseteq \Delta$ and an isomorphism

$$\psi : \Lambda \to \Theta$$

of valued quivers such that ψ^{-1} is given by the Green correspondence on vertices. As the latter commutes with the Heller operator (cf. [9, V.1.4]), we thus obtain an injective homomorphism $f : \Theta \to \Delta$ of valued stable translation quivers.

Now assume that Θ is of Euclidean type. Then Lemma 3.1.6 in conjunction with 3.1.2(1) implies that $\mathrm{cx}_G(M) \geq 2$ for all $M \in \Theta$. Upon application of the generalization of Webb's theorem 2.2.6, we obtain $\Theta \cong \mathbb{Z}[S]$ where $\overline{S} \in \{\tilde{A}_{12}, \tilde{A}_{p,q}, \tilde{D}_n\ (n \geq 4), \tilde{E}_r\ (6 \leq r \leq 8)\}$. We proceed in several steps.

(i) We have $\mathrm{cx}_N(X) \geq 2$ for all $X \in \Delta$: Indeed, if $L \in \Lambda \subseteq \Delta$, then $\psi(L) \in \Theta$ is (by definition) a direct summand of the induced module $kG \otimes_{kN} (L|_N)$. Thus, we have

$$2 \leq \mathrm{cx}_G(\psi(L)) \leq \mathrm{cx}_G(kG \otimes_{kN} (L|_N)) \leq \mathrm{cx}_N(L).$$

(ii) We have $\Delta \cong \mathbb{Z}[T]$, where T is a directed tree such that $\overline{T} \in \{A_\infty, D_\infty, A_\infty^\infty\}$: In view of 3.2.5(1), $H \trianglelefteq N$ is a non-trivial, normal p-subgroup of N. Thus, an application of Theorem 3.1.8 to the constant group scheme $\mathcal{G}_N$ shows that Δ is not Euclidean (see also 2.4.16(2)). Now apply Theorem 2.2.6 again in conjunction with (i) and 3.1.2(3).

(iii) We now arrive at a contradiction: By the Lemma above, f induces an injective homomorphism $\overline{S} \to \overline{T}$ of (valued) graphs. But the trees $A_\infty, D_\infty, A_\infty^\infty$ do (obviously) neither contain any Euclidean tree of type $\tilde{A}_{12}, \tilde{D}_n$ $(n \geq 4), \tilde{E}_r$ $(6 \leq r \leq 8)$ nor a circle (which is the underlying graph of any quiver of type $\tilde{A}_{p,q}$).

□

3.3 An analogous result for reduced enveloping algebras

We want to prove the analogue of Theorem 3.1.8 for reduced enveloping algebras, where the role of Π-supports is here taken by rank varieties (as introduced before). Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over an algebraically closed field k and $\chi \in \mathfrak{g}^*$ a linear form. Recall from 2.1.36(4), that the algebra $\mathrm{U}_\chi(\mathfrak{g})$ is a Frobenius algebra with Nakayama automorphism given by

$$\mu : \mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_\chi(\mathfrak{g}), \quad \mathfrak{g} \ni x \mapsto x + \mathrm{tr}(\mathrm{ad}(x)).$$

Lemma 3.3.1. *Let* $\mathfrak{n} \trianglelefteq \mathfrak{g}$ *be a* p*-ideal of some restricted Lie algebra* $\mathfrak{g}$*. Then* $\mathrm{tr}(\mathrm{ad}_\mathfrak{g}(x)) = \mathrm{tr}(\mathrm{ad}_\mathfrak{n}(x))$ *for all* $x \in \mathfrak{n}$*. In particular, the restriction of the Nakayama automorphism* $\mu : \mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_\chi(\mathfrak{g})$ *to* $\mathrm{U}_\chi(\mathfrak{n})$ *coincides with the Nakayama automorphism of* $\mathrm{U}_\chi(\mathfrak{n})$ *for all linear forms* $\chi \in \mathfrak{g}^*$*.*

We denote by $\Gamma_s(\mathfrak{g}, \chi)$ the *stable Auslander-Reiten quiver* of $\mathrm{U}_\chi(\mathfrak{g})$. Recall that $\mathrm{U}_\chi(\mathfrak{g})$ being Frobenius implies that the Auslander-Reiten translation τ is given by $\Omega^2 \circ (-)^{\mu^{-1}}$, where Ω denotes the Heller operator of $\mathrm{U}_\chi(\mathfrak{g})$.

The structure of components $\Theta \subseteq \Gamma_s(\mathfrak{g}, \chi)$ has been studied in [12, §5]. We recall some facts which have been proven in the latter reference. For instance, we have

$$V(\mathfrak{g})_M = V(\mathfrak{g})_N, \quad \forall [M], [N] \in \Theta,$$

so that we can speak of the rank variety $V(\mathfrak{g})_\Theta$. The periodic components are precisely those having a one-dimensional rank variety.

Theorem 3.3.2. *Let $\Theta \subseteq \Gamma_s(\mathfrak{g}, \chi)$ be a component.*

(1) If Θ is finite, then Θ is periodic and there exists a block $\mathcal{B} \subseteq \mathrm{U}_\chi(\mathfrak{g})$ such that $\Theta = \Gamma_s(\mathcal{B})$. This block $\mathcal{B}$ is a Nakayama algebra and we have $\Theta \cong \mathbb{Z}[A_{\ell\ell(\mathcal{B})}]/(\tau^k)$, where $k = \operatorname{ord}(\mu) \in \{1, p\}$.

(2) If Θ is an infinite component containing a periodic module, then $\Theta \cong \mathbb{Z}[\mathbb{A}_\infty]/(\tau^k)$ is a k-tube, where $k = \operatorname{ord}(\mu) \in \{1, p\}$. Moreover, Θ is regular.

We let $x \in V(\mathfrak{g}) \setminus \{0\}$ and denote by

$$[i] := k[x]/(x^i) \;\; 1 \leq i \leq p-1$$

the unique indecomposable $k[x]$-module of dimension i. As $k[x]$ is local, the regular module $[p] := k[x]$ is the only projective indecomposable indecomposable module. If $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$, then we can write (corresponding to the Jordan canonical form of the nilpotent linear operator x_M)

$$M|_{k[x]} \cong \bigoplus_{i=1}^{p} a_{M,i}(x)[i]$$

Definition 3.3.3. Let $x \in V(\mathfrak{g}) \setminus \{0\}$.

(i) The isoclass $\bigoplus_{i=1}^{p} a_{M,i}(x)[i] \in \mathbf{mod}(k[x])$ is called the *Jordan type* $\mathrm{Jt}(M, x)$ *of M at x.*

(ii) The *stable Jordan type* $\mathrm{StJt}(M, x)$ of M at x is defined to be the isoclass of $\bigoplus_{i=1}^{p-1} a_{M,i}(x)[i] \in \mathbf{mod}(k[x])$.

(iii) The set $\operatorname{supp}_x(M) := \{i \in \{1, \ldots, p-1\} : a_{M,i}(x) \neq 0\}$ is referred to as the *support of M at x.*

Our goal is to make sure, that $\operatorname{supp}_x(M) = \operatorname{supp}_x(N)$ for all $M, N \in \Theta$.

Definition 3.3.4. A component $\Theta \subseteq \Gamma_s(\mathfrak{g}, \chi)$ is called *locally split*, provided $\mathfrak{E}_M|_{k[x]}$ splits for every $M \in \Theta$ and $x \in V(\mathfrak{g})$.

Lemma 3.3.5. *Let $\chi \in \mathfrak{g}^*$ be a linear form, $\mathfrak{h} \subseteq \mathfrak{g}$ be a p-subalgebra.*

(1) We have $V(\mathfrak{g})_{\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(N,\chi)} \subseteq V(\mathfrak{h})$ for all $N \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))$.

(2) If M is a non-projective indecomposable $\mathrm{U}_\chi(\mathfrak{g})$-module such that $V(\mathfrak{g})_M \nsubseteq V(\mathfrak{h})$, then the exact sequence $\mathfrak{E}_M|_{\mathrm{U}_\chi(\mathfrak{h})}$ splits.

Proof. (1) This follows from [13, Proposition 3.4].

(2) We consider the surjective $\mathrm{U}_\chi(\mathfrak{g})$-linear map

$$\mu : \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M,\chi) \to M, u \otimes m \mapsto u.m$$

which corresponds to $\mathrm{id}_M \in \mathrm{End}_{\mathrm{U}_\chi(\mathfrak{h})}(M)$ under adjoint isomorphism (see 2.1.29). If μ would be split surjective, then standard properties of rank varieties (see 2.5.7) and (1) yield

$$V(\mathfrak{g})_M \subseteq V(\mathfrak{g})_{\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M,\chi)} \subseteq V(\mathfrak{h}),$$

contradicting the assumption. Consequently, μ is not split surjective and the almost split property of $\mathfrak{E}_M$ provides $\varphi \in \mathrm{Hom}_{\mathrm{U}_\chi(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M,\chi), E_M)$ such that $\mu = \pi_M \circ \varphi$, where $\pi_M : E_M \to M$ denotes the corresponding projection of the exact sequence $\mathfrak{E}_M$. Frobenius reciprocity then yields $\mathrm{id}_M = \eta \circ \pi_M$ for $\eta \in \mathrm{Hom}_{\mathrm{U}_\chi(\mathfrak{h})}(M, E_M)$ corresponding to φ, so that $\mathfrak{E}_M|_{\mathrm{U}_\chi(\mathfrak{h})}$ splits.

□

Corollary 3.3.6. *If $\Theta \subseteq \Gamma_s(\mathfrak{g},\chi)$ is a component with $\dim V(\mathfrak{g})_\Theta \geq 2$, then Θ is locally split.*

Proof. Given $x \in V(\mathfrak{g}) \setminus \{0\}$, we consider the one-dimensional elementary abelian subalgebra $kx \subseteq \mathfrak{g}$. As $V(kx) = kx$ is one-dimensional, the assertion is now a direct consequence of 3.3.5. □

The following can be proven analogous to 3.1.7.

Lemma 3.3.7. *Let $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$, $i \in \{1, \dots, p-1\}$ and $x \in V(\mathfrak{g}) \setminus \{0\}$. Then, if $N := \Omega^n_{\mathrm{U}_\chi(\mathfrak{g})}(M)$, we have*

$$a_{N,i}(x) = \begin{cases} a_{M,i}(x), & n \equiv 0 \ mod\ 2 \\ a_{M,p-i}(x), & n \equiv 1 \ mod\ 2 \end{cases}$$

We now trace through the main arguments of [23, §2,3] and we will see, that they still work in our context. In particular, we will see, that supports at points of $V(\mathfrak{g})$ give rise to invariants of AR-components of reduced enveloping algebras.

Theorem 3.3.8. *Let $\Theta \subseteq \Gamma_s(\mathfrak{g},\chi)$ be a component with $\dim V(\mathfrak{g})_\Theta \geq 2$ and $x \in V(\mathfrak{g}) \setminus \{0\}$.*

(a) For every $i \in \{1, \dots, p-1\}$, the map

$$a_{-,i}(x) : \Theta \to \mathbb{N}_0, \quad M \mapsto a_{M,i}(x)$$

is an additive function on Θ such that $a_{-,i}(x) \circ \tau_{\mathrm{U}_\chi(\mathfrak{g})} = a_{-,i}(x)$.

(b) The map

$$a_{-,<p}(x) : \Theta \to \mathbb{N}_0, \quad M \mapsto \dim_k M - p \cdot a_{M,p}(x)$$

is an additive function on Θ *such that* $a_{-,<p}(x) \circ \tau_{\mathrm{U}_\chi(\mathfrak{g})} = a_{-,<p}(x)$.

(c) There are functions $d^\Theta : V(\mathfrak{g}) \to \mathbb{N}_0^p$, $f_\Theta : \Theta \to \mathbb{N}_0$ *such that for all* $[M] \in \Theta$ *and* $x \in V(\mathfrak{g}) \setminus \{0\}$

$$a_{M,i}(x) = d_i^\Theta(x) f_\Theta(M) \;\; 1 \leq i \leq p-1, \;\; a_{M,<p}(x) = d_p^\Theta(x) f_\Theta(M).$$

Proof. By Corollary 3.3.6, Θ is locally split. Moreover, Lemma 3.3.7 in conjunction with the fact that $\mu(x) = x + \mathrm{tr}(\mathrm{ad}(x)) = x$ implies that $a_{-,i}(x) \circ \tau_{\mathrm{U}_\chi(\mathfrak{g})} = a_{-,i}(x)$. The first two statements follow completely analogous to the proof of [23, Thm 2.4]. For the third one note that Θ is infinite by 2.2.6, so that we can use the same proof as in [23, Theorem 3.1.1]. □

Corollary 3.3.9. *Let* $\Theta \subseteq \Gamma_s(\mathfrak{g}, \chi)$ *be a component with* $\dim V(\mathfrak{g})_\Theta \geq 2$ *and* $M, N \in \Theta$.

(a) $|\mathrm{Jt}(M)| = |\mathrm{Jt}(N)|$

(b) If $\Theta \not\cong \mathbb{Z}[A_\infty]$, *then* $|\{\mathrm{StJt}(M, x); M \in \Theta\}| = |\mathrm{im} f_\Theta| \leq 6$ *for all* $x \in V(\mathfrak{g})$.

(c) We have $\mathrm{supp}_x(M) = \mathrm{supp}_x(N)$ *for all* $x \in V(\mathfrak{g}) \setminus \{0\}$.

Proof. (a) Owing to part (c) of the foregoing theorem we have $|\mathrm{Jt}(M)| = |\mathrm{im}(d^\Theta)|$.

(b) Owing to part (c) of the foregoing theorem we have $|\{\mathrm{StJt}(M, x) \mid M \in \Theta\}| = |\mathrm{im} f_\Theta|$.

(c) Owing to part (c) of the foregoing theorem we have $|\mathrm{supp}_x(M)| = \{i \in \{1, \ldots, p-1\} \mid d_i^\Theta(x) \neq 0\}$.

□

Remark 3.3.10. The functions f_Θ are well-known and only depend on the tree class T_Θ. For instance, if $T_\Theta = A_\infty$, then $f_\Theta(M) = \mathrm{ql}(M)$ is the *quasi-length of* M (see 2.2.7) or if $T_\Theta = A_\infty^\infty$, then $f_\Theta \equiv 1$. We refer to [35, p.326-328] for the other cases.

We now have collected all ingredients making it possible to adopt the arguments of 3.1.8 and arrive at the following result.

Theorem 3.3.11. *Let* $\mathfrak{g}$ *be a finite-dimensional restricted Lie algebra over an algebraically closed field of characteristic* $p \geq 3$ *and* $\chi \in \mathfrak{g}^*$ *a linear form vanishing on some non-zero unipotent p-ideal* $\mathfrak{u} \trianglelefteq \mathfrak{g}$. *Then* $\Gamma_s(\mathfrak{g}, \chi)$ *does not afford components of Euclidean type.*

4 Further Results on Extensions and Semidirect Products

Throughout this chapter, vector spaces are understood to have finite-dimension over our field k of positive characteristic $p > 0$.

4.1 Finite group schemes

We consider an extension

$$e_k \longrightarrow \mathcal{N} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow e_k$$

in the category of finite group schemes and for ease of notation:

- We treat ι as an inclusion $\mathcal{N} \subseteq \mathcal{G}$ and $\pi : \mathcal{G} \to \mathcal{H} = \mathcal{G}/\mathcal{N}$ as the canonical projection. Moreover, $I^\dagger \trianglelefteq k\mathcal{N}$ denotes the augmentation ideal of the Hopf algebra $k\mathcal{N}$ and $I \trianglelefteq k\mathcal{G}$ denotes the ideal generated by $I^\dagger$ inside $k\mathcal{G}$.
- If $\mathcal{G}$ is a split extension, then we pick a (necessarily mono)morphism $i : \mathcal{H} \to \mathcal{G}$ with the property that $\pi \circ i = \mathrm{id}_{\mathcal{H}}$ and write $\mathrm{Res}_{\mathcal{H}} := i^*$ (see 2.4.27).

Recall that a flat morphism $\psi : \mathcal{L} \to \mathcal{M}$ of finite group schemes induces a map $\psi_* : \Pi(\mathcal{L}) \to \Pi(\mathcal{M})$, $[\alpha_K] \mapsto [\psi_K \circ \alpha_K]$ on the spaces of equivalence-classes of Π-points (cf. [29, Corollary 2.7, Theorem 3.6]). In particular, we can consider the subset $\iota_*(\Pi(\mathcal{N}))$ and, if $\mathcal{G}$ is a split extension, the subset $i_*(\Pi(\mathcal{H}))$ of $\Pi(\mathcal{G})$.

Lemma 4.1.1. *Pullback along the projection $\pi : \mathcal{G} \to \mathcal{H}$ induces an equivalence*

$$\mathrm{Inf} := \mathrm{Inf}_{\mathcal{H}}^{\mathcal{G}} := \pi^* : \mathbf{mod}(\mathcal{H}) \to \mathbf{mod}_{\mathcal{N}}(\mathcal{G}) \text{ (Inflation)}$$

If $\mathcal{G}$ is a split extension, then we have $\mathrm{Res}_{\mathcal{H}} \circ \mathrm{Inf} = \mathrm{id}_{\mathbf{mod}(\mathcal{H})}$. If $\mathcal{N}$ is not linearly reductive, then $\mathrm{Inf}(M)$ is not $\mathcal{G}$-projective for all $M \in \mathbf{mod}(\mathcal{H})$.

Proof. For the first assertion, we refer to [41, I.6.3]. The identity $\mathrm{Res}_{\mathcal{H}} \circ \mathrm{Inf} = \mathrm{id}_{\mathbf{mod}(\mathcal{H})}$ follows from the analogue of Lemma 2.1.28(1). Finally, since $\mathcal{N}$ acts trivially on $\mathrm{Inf}(M)$ for all $M \in \mathbf{mod}(\mathcal{H})$, we have that $\iota_*(\Pi(\mathcal{N})) \subseteq \Pi(\mathcal{G})_{\mathrm{Inf}(M)}$. Now apply Lemma 3.1.4 and Theorem 3.1.2. □

Remark 4.1.2. The morphism π induces a surjective homomorphism $k\mathcal{G} \to k\mathcal{H}$ of Hopf algebras with kernel I. Hence, on the level of Hopf algebras, the above equivalence is given as an equivalence $\mathbf{mod}(k\mathcal{H}) \to \mathbf{mod}_I(k\mathcal{G})$, where $\mathbf{mod}_I(k\mathcal{G})$ is the full subcategory of all finite-dimensional $k\mathcal{G}$-modules V such that $I.V = (0)$.

Lemma 4.1.3. *Let $i : \mathcal{H} \hookrightarrow \mathcal{G}$ be a closed subgroup of some arbitrary finite group scheme $\mathcal{G}$. Then the following statements hold:*

(1) We have $\Pi(\mathcal{G})_M = \Pi(\mathcal{G})_{M \otimes_k k_\lambda}$ for every character $\lambda \in X(\mathcal{G})$ and every $M \in$ $\mathbf{mod}(\mathcal{G})$.

(2) We have $\Pi(\mathcal{G})_{k\mathcal{G} \otimes_{k\mathcal{H}} M} \subseteq i_*(\Pi(\mathcal{H})_M)$ for all $M \in \mathbf{mod}(\mathcal{H})$.

Proof. (1) This follows from the fact that π-points factor through unipotent subgroups which all have trivial character group.

(2) This follows from (1) in conjunction with [30, Lemma 4.12] and [41, Proposition I.8.17].

□

We let $S(1), \ldots, S(n)$ be a complete set of isoclasses of simple $\mathcal{H}$-modules and denote by $P(1), \ldots, P(n)$ the corresponding projective covers over $\mathcal{H}$. Recall that, if $\mathcal{G}$ is a split extension, then the induction functor

$$\mathbf{mod}(\mathcal{H}) \to \mathbf{mod}(\mathcal{G}),\ N \mapsto k\mathcal{G} \otimes_{k\mathcal{H}} N$$

is equivalent to the functor $N \mapsto k\mathcal{N} \otimes_k N$ (cf. 2.4.29).

Theorem 4.1.4. *Let $\mathcal{G}$ be an extension of $\mathcal{H}$ by some unipotent group $\mathcal{N}$. Then the following statements hold:*

(1) The groups $\mathcal{G}$ and $\mathcal{H}$ have the same simple modules, i.e., the modules $T(i) :=$ $\mathrm{Inf}(S(i))$ for $1 \leq i \leq n$ form a complete set of iso-classes of simple $\mathcal{G}$-modules.

(2) If $\mathcal{G}$ is a split extension, then the following additional statements hold:

(a) If $M = k\mathcal{N} \otimes_k N$ for some $\mathcal{H}$-module N, then $\mathrm{Top}(M) \cong \mathrm{Inf}(\mathrm{Top}_{\mathcal{H}}(N))$. Moreover, $\Pi(\mathcal{G})_M = i_*(\Pi(\mathcal{H})_N)$.

(b) The induced module $Q(i) := k\mathcal{N} \otimes_k P(i)$ is the projective cover of the simple module $T(i)$ over $\mathcal{G}$ for all $1 \leq i \leq n$.

(c) If $M \in \mathbf{mod}(\mathcal{G})$, then we have $[M : T(i)] = [M|_{\mathcal{H}} : S(i)]$ for all $1 \leq i \leq n$

(d) Let $i \in \{1, \ldots, n\}$, then we have $\mathrm{Rad}(Q(i)) = (I^\dagger \otimes_k P(i)) + (k.1 \otimes_k \mathrm{Rad}_{k\mathcal{H}}(P(i)))$. Moreover, $\Omega_{\mathcal{G}}(\mathrm{Inf}(P(i))) \cong I^\dagger \otimes_k P(i)$.

(e) We have $\mathrm{Rad}(k\mathcal{G}) = I^\dagger \otimes_k k\mathcal{H} + k\mathcal{N} \otimes \mathrm{Rad}(k\mathcal{H})$.

Proof. (1) This follows from Lemma 2.4.19.

(2) Now let $\mathcal{G}$ be a split extension.

(a) Let $M = k\mathcal{N} \otimes_k N$. We have $\mathrm{Hom}_{\mathcal{G}}(M, T(i)) \cong \mathrm{Hom}_{\mathcal{H}}(N, S(i))$ by Frobenius reciprocity 2.1.29. It now follows from Lemma 2.1.12(7), that $[\mathrm{Top}(M) : T(i)] = [\mathrm{Top}(N) : S(i)]$, thereby implying the first assertion. Applying Lemma 4.1.3, we get $\Pi(\mathcal{G})_M \subseteq \iota_*(\Pi(\mathcal{H})_N)$. For the reverse inclusion, we note that $\mathcal{H}$ acting on $k\mathcal{N}$ by automorphisms of Hopf algebras forces a decomposition $k\mathcal{N} = k.1 \oplus I^\dagger$ of $\mathcal{H}$-modules. Thus, $M \cong N \oplus (I^\dagger \otimes_k N)$ as $\mathcal{H}$-modules. Now [29, Proposition 3.3(4)] implies that

$$\Pi(\mathcal{H})_N \cup \Pi(\mathcal{H})_{I^\dagger \otimes_k N} = \Pi(\mathcal{H})_M,$$

so that $i_*(\Pi(\mathcal{H})_N) \subseteq \Pi(\mathcal{G})_M$.

(b) Let $i \in \{1, \ldots, n\}$. Since induction takes projectives to projectives, the module $Q(i) = k\mathcal{N} \otimes_k P(i)$ is $\mathcal{G}$-projective. Moreover, $Q(i)$ has simple top $T(i)$ by (a). Hence it is isomorphic to the projective cover of $T(i)$ over $\mathcal{G}$ (see Lemma 2.1.15).

(c) Let again $i \in \{1, \ldots, n\}$ and put $d := \dim_k \mathrm{End}_{\mathcal{G}}(T(i)) = \dim_k \mathrm{End}_{\mathcal{H}}(S(i)) \in \mathbb{N}$. According to (b), Lemma 2.1.18(4) and Frobenius reciprocity imply that

$$\begin{aligned} d \cdot [M : T(i)] &= \dim_k \mathrm{Hom}_{\mathcal{G}}(Q(i), M) = \dim_k \mathrm{Hom}_{\mathcal{H}}(P(i), M|_{\mathcal{H}}) \\ &= d \cdot [M|_{\mathcal{H}} : S(i)]. \end{aligned}$$

The assertion follows.

(d) For ease of notation, we put

$$P := P(i), \quad Q := Q(i), \quad R := I^\dagger \otimes_k P + k.1 \otimes_k \mathrm{Rad}_{k\mathcal{H}}(P).$$

As R is a $k\mathcal{G}$-submodule of Q, the first claim follows if we can show that $Q/R \cong T(i)$. Let $\varepsilon : k\mathcal{N} \to k$ and $\pi : P \longrightarrow P/\mathrm{Rad}(P)$ be the co-unit of the Hopf-algebra $k\mathcal{N}$ and the canonical projection of $k\mathcal{H}$-modules, respectively. As both maps are $\mathcal{H}$-linear and surjective, so is their tensor product

$$f : Q \longrightarrow k \otimes_k P/\mathrm{Rad}(P) \cong S(i), \quad u \otimes p \mapsto \varepsilon(u) \otimes \pi(p).$$

Clearly, $R \subseteq \ker(f)$, hence this induces a surjective $k\mathcal{H}$-linear map

$$\overline{f} : Q/R \longrightarrow k \otimes_k P/\mathrm{Rad}(P).$$

As $R = (I^\dagger \otimes_k P) \oplus (k.1 \otimes_k \mathrm{Rad}_{k\mathcal{H}}(P))$ as vector spaces, we get

$$\begin{aligned}\dim_k Q/R = \dim_k Q - \dim_k R &= \dim_k Q - (\dim_k I^\dagger \cdot \dim_k P + \dim_k \mathrm{Rad}_{k\mathcal{H}}(P))\\ &= (\dim_k I^\dagger + 1) \cdot \dim_k P - (\dim_k I^\dagger \cdot \dim_k P + \dim_k \mathrm{Rad}_{k\mathcal{H}}(P))\\ &= \dim_k P - \dim_k \mathrm{Rad}(P).\end{aligned}$$

It follows that $\overline{f}$ is an isomorphism of $\mathcal{H}$-modules. Upon application of the inflation functor, we obtain the desired isomorphism.

For the second assertion, we let $f : Q(i) \to \mathrm{Inf}(P(i))$ be the $\mathcal{G}$-linear map corresponding to the identity map $\mathrm{id}_{P(i)} : P(i) \to P(i) = \mathrm{Res}_{\mathcal{H}}(\mathrm{Inf}(P(i)))$ under adjoint isomorphism. Then f is surjective and $\ker(f) = I^\dagger \otimes_k P(i)$ (as can be seen by an easy dimension argument) is contained in $\mathrm{Rad}(Q(i))$ by the above.

(e) Put $J := I^\dagger \otimes_k k\mathcal{H} + k\mathcal{N} \otimes \mathrm{Rad}(k\mathcal{H}) \subseteq k\mathcal{G}$. By (1), J annihilates any simple $k\mathcal{G}$-module, therefore $J \subseteq \mathrm{Rad}(k\mathcal{G})$. Moreover, for every projective indecomposable $k\mathcal{G}$-module $Q = k\mathcal{N} \otimes_k P$, we have (observing Theorem 2.1.12(2) and $k.1 \otimes \mathrm{Rad}(P) = k \otimes_k \mathrm{Rad}(P) \cong \mathrm{Rad}(P)$ as $k\mathcal{H}$-modules)

$$k.1 \otimes \mathrm{Rad}(P) = \mathrm{Rad}(k\mathcal{H}).(k.1 \otimes_k P) \subseteq J.Q, \quad I^\dagger \otimes_k P = I^\dagger.(k\mathcal{N} \otimes_k P) \subseteq J.Q.$$

Hence $\mathrm{Rad}(Q) \subseteq J.Q$ by (d). Consequently, $Q/J.Q$ is a semisimple $k\mathcal{G}/J$-module. Hence

$$k\mathcal{G}/J \cong \bigoplus_{i=1}^{n} n_i \cdot Q(i)/J.Q(i)$$

is a semi-simple algebra, so that $\mathrm{Rad}(k\mathcal{G}) \subseteq J$.

□

Let $\mathcal{G}$ be a split extension and consider the composite of $\mathrm{Res}_{\mathcal{H}} : \mathbf{mod}_{\mathcal{N}}(\mathcal{G}) \to \mathbf{mod}(\mathcal{H})$ ('the' inverse of the inflation functor) and the fixed point functor relative to $\mathcal{N}$:

$$\mathfrak{F} := \mathrm{Res}_{\mathcal{H}} \circ (-)^{\mathcal{N}} : \mathbf{mod}(\mathcal{G}) \to \mathbf{mod}(\mathcal{H})$$

Consider the $\mathcal{G}$-module $k\mathcal{N}$. Then

$$(k\mathcal{N})^{\mathcal{N}} = \int_{k\mathcal{N}} = \{x \in k\mathcal{N} \mid ux = \varepsilon_{k\mathcal{N}}(u).x \ \ \forall u \in k\mathcal{N}\}.$$

is the space of *(left) integrals* of the Hopf algebra $k\mathcal{N}$. By Theorem 2.3.9, this space is one-dimensional. We denote by $\lambda_{\mathcal{N}} \in X(\mathcal{H})$ the character corresponding to the one-dimensional $\mathcal{H}$-module $\mathfrak{F}(k\mathcal{N})$.

Corollary 4.1.5. *Let $\mathcal{G}$ be a split extension of $\mathcal{H}$ by $\mathcal{N}$. Then the composite*

$$\mathfrak{F} \circ (k\mathcal{N} \otimes_k -) = \operatorname{Res}_{\mathcal{H}} \circ (-)^{\mathcal{N}} \circ (k\mathcal{N} \otimes_k -) : \mathbf{mod}(\mathcal{H}) \to \mathbf{mod}(\mathcal{H})$$

is naturally equivalent to $\lambda_{\mathcal{N}} \otimes_k -$. In particular, the functor $\mathfrak{F}$ takes projective indecomposables to projective indecomposables.

Proof. Let N be an $\mathcal{H}$-module. It follows directly from the definition of the $\mathcal{N}$-action, that the $\mathcal{N}$-invariants of $k\mathcal{N} \otimes_k N$ are given by $\int_{k\mathcal{N}} \otimes_k N$. As all projective indecomposable $k\mathcal{G}$-modules are - (by 4.1.4) - induced by projective indecomposable $k\mathcal{H}$-modules, the additional statement follows. □

Suppose, we know the Gabriel quiver $Q_{\mathcal{H}}$ of the Hopf algebra $k\mathcal{H}$. We are interested in a formula involving $Q_{\mathcal{H}}$ for computing the full subquiver of $Q_{\mathcal{G}}$ whose vertices correspond to simple $\mathcal{H}$-modules. It turns out that the following definition is the right one for this intention. It is inspired by [46], [17, p.38].

Definition 4.1.6. Let $\mathcal{H}$ be a finite group scheme and V be a finite-dimensional $\mathcal{H}$-module. We then define the *generalized McKay quiver $\Gamma_V(\mathcal{H})$ of $\mathcal{H}$ relative to V* as follows:

- The edges are labelled by the simple $\mathcal{H}$-modules $S(1), \dots, S(n)$.
- There are $\dim_k \operatorname{Hom}_{\mathcal{H}}(V \otimes_k S(i), S(j)) + \dim_k \operatorname{Ext}^1_{\mathcal{H}}(S(i), S(j))$ arrows $S(i) \longrightarrow S(j)$.

Remark 4.1.7. If $\mathcal{H}$ is linearly reductive and V is a faithful $\mathcal{H}$-module, then [17, Lemma 3.1] shows that the opposite quiver of $Q := \Gamma_V(\mathcal{H})$ is connected, hence so is Q.

Recall that $H^1(\mathcal{N}, k)$ has the structure of an $\mathcal{H}$-module (see Lemma 2.4.24).

Theorem 4.1.8. *Let $\mathcal{G}$ be an extension of $\mathcal{H}$ by $\mathcal{N}$ and consider the $\mathcal{H}$-module $V := H^1(\mathcal{N}, k)^*$. Then the following statements hold:*

(1) We have

$$\dim_k \operatorname{Ext}^1_{\mathcal{H}}(M,N) \leq \dim_k \operatorname{Ext}^1_{\mathcal{G}}(\operatorname{Inf}(M), \operatorname{Inf}(N)) \leq \dim_k \operatorname{Hom}_{\mathcal{H}}(V \otimes_k M, N) + \dim_k \operatorname{Ext}^1_{\mathcal{H}}(M,N)$$

for all $M, N \in \mathbf{mod}(\mathcal{H})$. If the second inflation map $\operatorname{Ext}^2_{\mathcal{H}}(M,N) \longrightarrow \operatorname{Ext}^2_{\mathcal{G}}(\operatorname{Inf}(M), \operatorname{Inf}(N))$ is injective, then the right-hand inequality is in fact an equality.

(2) If $\mathcal{G}$ is a split extension or there exists a $\mathcal{G}$-module Q such that $Q|_{\mathcal{N}}$ is projective and $Q^{\mathcal{N}} = Q^{\mathcal{G}} = k$, then

$$\operatorname{Ext}^1_{\mathcal{G}}(\operatorname{Inf}(M), \operatorname{Inf}(N)) \cong \operatorname{Hom}_{\mathcal{H}}(V \otimes_k M, N) \oplus \operatorname{Ext}^1_{\mathcal{H}}(M,N)$$

for all $M, N \in \mathbf{mod}(\mathcal{H})$. In particular, if $\mathcal{N}$ is unipotent, then the Gabriel quiver of $\mathcal{G}$ is the generalized McKay quiver $\Gamma_V(\mathcal{H})$ of $\mathcal{H}$ relative to V.

Proof. (1) Let $M, N \in \mathbf{mod}(\mathcal{H})$ and consider the $\mathcal{G}$-module $W := \mathrm{Inf}(M)^* \otimes_k \mathrm{Inf}(N) = \mathrm{Inf}(M^* \otimes_k N)$. In view of Lemma 2.3.5, we have a natural isomorphism

$$\mathrm{Ext}^n_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \cong H^n(\mathcal{G}, W).$$

Hence the *five-term exact sequence* associated to the *Lyndon-Hochschild-Serre spectral sequence* involving $\mathcal{G}$ and $\mathcal{N}$ (cf. [41, I.4.1(4)] and [41, I.6.6(3)]) yields an exact sequence

$$\begin{aligned}\delta_{M,N} : (0) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(M, N) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \longrightarrow H^1(\mathcal{N}, W)^{\mathcal{H}} \longrightarrow \mathrm{Ext}^2_{\mathcal{H}}(M, N)\\ \longrightarrow \mathrm{Ext}^2_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N)).\end{aligned}$$

The maps of this sequence are known (cf. [41, I.6.6/6.10]):

- The maps $\mathrm{Ext}^n_{\mathcal{H}}(M, N) \longrightarrow \mathrm{Ext}^n_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N))$ for $n \in \{1, 2\}$ are the inflation maps. We note that, considering Ext^1 as equivalence classes of extensions, the first inflation map is given by just applying Inf to an extension in $\mathbf{mod}(\mathcal{H})$.
- $\mathrm{Ext}^1_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \longrightarrow H^1(\mathcal{N}, W)^{\mathcal{H}} = \mathrm{Ext}^1_{\mathcal{N}}(M, N)^{\mathcal{H}}$ is the restriction map. Considering again Ext^1 as equivalence classes of extensions, this map is given by applying $\mathrm{Res}_{\mathcal{N}}$ to an extension in $\mathbf{mod}(\mathcal{G})$.

As $\mathcal{N}$ acts trivially on W, we have isomorphisms of $\mathcal{H}$-modules

$$H^1(\mathcal{N}, W) \cong H^1(\mathcal{N}, k) \otimes_k W \cong \mathrm{Hom}_k(H^1(\mathcal{N}, k)^*, W) = \mathrm{Hom}_k(V, W).$$

Consequently, the $\mathcal{H}$-invariants are given by

$$\mathrm{Hom}_{\mathcal{H}}(V, W) = \mathrm{Hom}_{\mathcal{H}}(V, M^* \otimes_k N) \cong \mathrm{Hom}_{\mathcal{H}}(V \otimes_k M, N).$$

The assertion now follows from the exactness of $\delta_{M,N}$.

(2) Return to the above proof. By [41, Remark (1) on p.91], the existence of a $\mathcal{G}$-module Q with the abovementioned properties ensures that the second inflation map is injective. By Lemma 2.4.21(d), this also happens, when $\mathcal{G}$ is a split extension. Thus, in both cases, $\delta_{M,N}$ induces a short exact sequence

$$(0) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(M, N) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(V \otimes_k M, N) \longrightarrow 0$$

of k-vector spaces, which necessarily splits (If $\mathcal{G}$ is a split extension, then one could take $\mathrm{Res}_{\mathcal{H}} : \mathrm{Ext}^1_{\mathcal{G}}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \to \mathrm{Ext}^1_{\mathcal{H}}(M, N)$ to get such a splitting).

□

Remark 4.1.9. Let $\mathcal{G}$ be an extension of $\mathcal{H}$ by $\mathcal{N}$.

(1) If $\mathcal{N}$ is unipotent, then $k\mathcal{N}$ is the projective cover of k. Then (see Lemma 2.1.33)

$$H^1(\mathcal{N},k) = \mathrm{Ext}^1_{\mathcal{N}}(k,k) \cong \mathrm{Hom}_{k\mathcal{N}/I^\dagger}(I^\dagger/(I^\dagger)^2,k) = \mathrm{Hom}_k(I^\dagger/(I^\dagger)^2,k) = (I^\dagger/(I^\dagger)^2)^*,$$

so that $H^1(\mathcal{N},k)^* \cong I^\dagger/(I^\dagger)^2$ as $\mathcal{H}$-modules.

(2) If $\mathcal{G}$ is a split extension and $\mathcal{H}$ acts trivially on the left integral $\int_{k\mathcal{N}}$ of the Hopf algebra $k\mathcal{N}$, then the $\mathcal{G}$-module $Q := k\mathcal{N}$ satisfies the mentioned conditions in part (2) of the above theorem.

(3) If k is perfect, then every finite group scheme $\mathcal{G}$ can be written as a semidirect product $\mathcal{G}^0 \rtimes \mathcal{G}_{red}$ of an infinitesimal, normal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{red}$ (see [57, Theorem 6.8]), hence our theorem applies.

(4) If $\mathcal{G} = \mathcal{N} \times \mathcal{H}$ is a *direct product* (that is, $\mathcal{H}$ acts trivially on $\mathcal{N}$), we have the *Künneth-formula* available: Let $M, N \in \mathbf{mod}(\mathcal{H})$ and $V, W \in \mathbf{mod}(\mathcal{N})$, then (here the expressions in the braces on the LHS are tensor products of inflated modules)

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{G}}(M \otimes_k V, N \otimes_k W) \cong{} & (\mathrm{Ext}^1_{\mathcal{H}}(M,N) \otimes_k \mathrm{Hom}_{\mathcal{N}}(V,W)) \\ & \oplus\ (\mathrm{Hom}_{\mathcal{H}}(M,N) \otimes_k \mathrm{Ext}^1_{\mathcal{N}}(V,W)).\end{aligned}$$

Taking $V = W = k$, we get

$$\mathrm{Ext}^1_{\mathcal{G}}(M,N) \cong \mathrm{Ext}^1_{\mathcal{H}}(M,N)\ \oplus\ (\mathrm{Hom}_{\mathcal{H}}(M,N) \otimes H^1(\mathcal{N},k)).$$

As $\mathcal{H}$ acts trivially on $H^1(\mathcal{N},k)$, we observe

$$\begin{aligned}\mathrm{Hom}_{\mathcal{H}}(M,N) \otimes H^1(\mathcal{N},k) &= \mathrm{Hom}_k(M,N)^{\mathcal{H}} \otimes_k H^1(\mathcal{N},k) = (\mathrm{Hom}_k(M,N) \otimes_k H^1(\mathcal{N},k))^{\mathcal{H}} \\ &\cong \mathrm{Hom}_k(H^1(\mathcal{N},k)^* \otimes_k M, N))^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}}(H^1(\mathcal{N},k)^* \otimes_k M, N).\end{aligned}$$

Hence, we recover our above result.

We have shown that the determination of the Gabriel quiver of a split extension $\mathcal{G}$ of $\mathcal{H}$ by some unipotent group $\mathcal{N}$ rests on the following data:

- The Gabriel quiver $Q_{k\mathcal{H}}$ of $k\mathcal{H}$.
- The top of every tensor product $H^1(\mathcal{N},k)^* \otimes_k S$ of $\mathcal{H}$-modules for every simple $\mathcal{H}$-module S.

Next, we give an example, where the $\mathcal{H}$-module-structure of the cohomology group $H^1(\mathcal{N},k)^*$ can be written down more explicitly.

Let $\mathcal{H}$ be an arbitrary affine group scheme and V a finite-dimensional $\mathcal{H}$-module. By definition, $\mathcal{H}$ acts on the algebraic group scheme V_a via automorphisms, so that we can form the semidirect product $\mathcal{H}_V := V_a \rtimes \mathcal{H}$. This is not a finite group scheme, but its rth Frobenius kernel $\mathcal{H}_{V,r} = V_{a(r)} \rtimes \mathcal{H}_r$ enjoys this property. In what follows, we denote by $V^{(i)}$ the $\mathcal{H}$-module with underlying scalar multiplication (see [41, I.2.16])

$$a.v := a^{p^{-i}}.v \;\; \forall a \in k, v \in V.$$

Since the $\mathcal{H}$-module structure of $H^1(V_{a(r)}, k)$ is obtained by its $\mathrm{GL}(V)$-module structure via pullback along the structure homomorphism $\mathcal{H} \to \mathrm{GL}(V)$ of V, and pullback commutes with the functor $(-)^{(i)}$, an application of [41, Proposition I.4.27(a)] yields:

Lemma 4.1.10. *Let k be a perfect field, then there is an isomorphism $H^1(V_{a(r)}, k) \cong \bigoplus_{i=0}^{r-1} (V^*)^{(i)}$ of $\mathcal{H}$-modules.*

Remark 4.1.11. In the above setting:

(1) If $\mathcal{H}$ as well as the $\mathcal{H}$-module V are defined over $\mathbb{F}_p$ and $F : \mathcal{H} \to \mathcal{H}$ denotes a Frobenius endomorphism of $\mathcal{H}$, then $V^{(i)}$ is isomorphic to $V^{[i]}$, the twist of V by the i-th power of F (see [41, I.9.10]). Since pullback commutes with taking duals, we get $H^1(V_{a(r)}, k)^* \cong \bigoplus_{i=0}^{r-1} V^{[i]}$ in that case.

(2) If $\mathcal{H}$ is reductive and $V = L(\lambda)$ is a simple $\mathcal{H}$-module (soon more details on that), then we get $H^1(V_{a(r)}, k)^* \cong \bigoplus_{i=0}^{r-1} L(p^i\lambda)$, a completely reducible module. This follows from the first part of this remark (which we can apply because of [41, Corollary II.2.9]) in conjunction with [41, Proposition II.3.16].

We return to the case of an extension $\mathcal{G}$ of $\mathcal{H}$ by $\mathcal{N}$ and finish this section by investigating the behaviour of the inflation functor $\mathrm{Inf} : \mathbf{mod}(\mathcal{H}) \to \mathbf{mod}(\mathcal{G})$ when applied to almost split sequences. It seems to be a natural question to ask, whether Inf transforms almost split sequences into almost split sequences or not. Intuitively, one would expect a negative answer and we will show that this will mostly be the case, when $\mathcal{G}$ is a split extension. First, we prove a lemma.

Lemma 4.1.12. *Let $i : \mathcal{H} \hookrightarrow \mathcal{G}$ be a closed subgroup of some finite group scheme $\mathcal{G}$.*

(1) If $M \in \mathbf{mod}(\mathcal{G})$ is a non-projective indecomposable such that $\Pi(\mathcal{G})_M \not\subseteq i_*(\Pi(\mathcal{H}))$, then the exact sequence $\mathfrak{E}_M|_{\mathcal{H}}$ splits.

(2) If $M, N \in \mathbf{mod}(\mathcal{G})$ are non-projective indecomposables, which lie in the same AR-component, then $\Pi(\mathcal{H})_M = i_*^{-1}(\Pi(\mathcal{G})_M) = \Pi(\mathcal{H})_N$.

(3) If $\mathcal{G}$ is a split extension of $\mathcal{H}$ by $\mathcal{N}$, then

(a) *The map* $i_* : \Pi(\mathcal{H}) \to \Pi(\mathcal{G})$ *is injective.*

(b) $\iota_*(\Pi(\mathcal{N})) \cap i_*(\Pi(\mathcal{H})) = \emptyset$.

Proof. (1) We consider the surjective $\mathcal{G}$-linear map

$$\mu : k\mathcal{G} \otimes_{k\mathcal{H}} M|_{\mathcal{H}} \to M, u \otimes m \mapsto u.m$$

which corresponds to $\mathrm{id}_M \in \mathrm{End}_{\mathcal{H}}(M)$ under adjoint isomorphism. If μ was split surjective, then [29, Proposition 3.3] and Lemma 4.1.3 would imply

$$\Pi(\mathcal{G})_M \subseteq \Pi(\mathcal{G})_{k\mathcal{G}\otimes_{k\mathcal{H}}M} \subseteq i_*(\Pi(\mathcal{H})_{M|_{\mathcal{H}}}) \subseteq i_*(\Pi(\mathcal{H})),$$

contradicting the assumption. Consequently, μ is not split surjective and we can proceed as in the proof of Lemma 3.3.5.

(2) Using that $\Pi(\mathcal{G})_M = \Pi(\mathcal{G})_N$, this follows from [29, Proposition 5.6].

(3) (a) We let $\alpha_K : K[t]/(t^p) \to K\mathcal{H}$ and $\beta_L : L[t]/(t^p) \to L\mathcal{H}$ be π-points of $\mathcal{H}$ such that $[i_K \circ \alpha_K] = [i_L \circ \beta_L]$ as elements of $\Pi(\mathcal{G})$. Let M be an $\mathcal{H}$-module and consider the pullback $\pi^*(M) \in \mathbf{mod}(\mathcal{G})$ along the projection $\pi : \mathcal{G} \to \mathcal{H}$. Taking into account that the equation $\pi \circ i = \mathrm{id}_{\mathcal{H}}$ remains true after base field extension, we conclude

$$\begin{aligned}(i_K \circ \alpha_K)^*(\pi^*(M)_K) &= (i_K \circ \alpha_K)^*(\pi_K^*(M_K)) = (\pi_K \circ i_K \circ \alpha_K)^*(M_K)\\ &= \alpha_K^*(M_K).\end{aligned}$$

It now easily follows $[\alpha_K] = [\beta_L]$ from the definition of the underlying equivalence relation.

(b) Consider a projective $\mathcal{H}$-module P as well as its inflation $\pi^*(P) = \mathrm{Inf}(P) \in \mathbf{mod}(\mathcal{G})$. Let $\alpha_K : K[t]/t^p \to K\mathcal{N}$ and $\beta_L : L[t]/t^p \to L\mathcal{H}$ be π-points of $\mathcal{N}$ and $\mathcal{H}$, respectively. Since $\mathcal{N}$ acts trivially on $\pi^*(P)$, the action of $K[t]/t^p$ on

$$(\iota_K \circ \alpha_K)^*(\pi^*(P)_K) = (\iota_K \circ \alpha_K)^*(\pi_K^*(P_K)) = \alpha_K^*(\iota_K^*(\pi_K^*(P_K)))$$

is trivial as well, while $(i_L \circ \beta_L)^*(\pi^*(P)_L) = \beta_L^*(P_L)$ is a projective $L[t]/t^p$-module. Thus, it follows - again from the definition - that $[\iota_K \circ \alpha_K] \neq [i_L \circ \beta_L]$.

□

We denote by $\mathfrak{S}$ and $\mathfrak{E}$ the almost split sequences terminating in a non-projective $\mathcal{H}$ or $\mathcal{G}$-module, respectively. If M is such a $\mathcal{G}$-module and $\mathcal{G}$ is a split extension, then $\mathcal{N}$ acting trivially on $\tau_{\mathcal{G}}(M), E_M$ and M will force the identity $\mathrm{Res}_{\mathcal{H}}(\mathfrak{E}_M) = \mathfrak{S}_{\mathrm{Res}_{\mathcal{H}}(M)}$ by definition of an almost split sequence. For the following, one may recall Definition 2.2.7 of a quasi-simple module and Definition 2.2.2 of a regular component of the AR-quiver.

Theorem 4.1.13. *Let $\mathcal{G}$ be an extension of $\mathcal{H}$ by some non-linearly reductive group scheme $\mathcal{N}$ over a perfect field k of characteristic $p \geq 3$. Let $M \in \mathbf{mod}(\mathcal{H})$ be indecomposable and denote by Θ_M the AR-component of the (in view of 4.1.1) non-projective indecomposable $\mathcal{G}$-module* $\mathrm{Inf}(M)$.

(1) If there exists a unipotent, normal subgroup scheme $\mathcal{U} \trianglelefteq \mathcal{G}$ such that $\mathcal{N} \cap \mathcal{U} \neq e_k$, then the component Θ_M is regular.

(2) If $\mathcal{G}$ is a split extension, then the following statements hold:

(i) The inflation functor $\mathrm{Inf} = \pi^* : \mathbf{mod}(\mathcal{H}) \to \mathbf{mod}(\mathcal{G})$, $M \mapsto \pi^*(M)$ will never transform AR-sequences into AR-sequences.

(ii) We have $\Pi(\mathcal{H})_M = \Pi(\mathcal{H})_{X|_{\mathcal{H}}}$ for all $X \in \Theta_M$. In particular, X is $\mathcal{H}$-projective if and only if M enjoys this property.

(iii) If there exists a unipotent, normal subgroup $\mathcal{U} \trianglelefteq \mathcal{G}$ such that $\mathcal{U} \cap \mathcal{N} \neq e_k$ and if $\Theta_M \cong \mathbb{Z}[A_\infty]$, then the following statements hold:

- $\mathrm{Inf}(M) \in \Theta_M$ *is quasi-simple.*
- *The set $\Theta_{M,\mathcal{N}} := \{X \in \Theta_M \mid X^{\mathcal{N}} = X\}$ is finite. Moreover, if $X \in \Theta_{M,\mathcal{N}}$, then $\dim_k X = \dim_k M$ and there exists $n \in \mathbb{Z}$ such that $\tau_{\mathcal{G}}^n(\mathrm{Inf}(M)) \cong X$. If $|\Theta_{M,\mathcal{N}}| \geq 2$, then $\dim \Pi(\mathcal{N}) = 0$.*

(iv) If M is not $\mathcal{H}$-projective, then $\mathcal{N}$ does not act trivially on the Auslander-Reiten shift $\tau_{\mathcal{G}}(\mathrm{Inf}(M))$.

Proof. (1) Let $j : \mathcal{U} \hookrightarrow \mathcal{G}$ be the corresponding inclusion. Then Lemma 3.1.4 yields a π-point β_K of the group $\mathcal{N} \cap \mathcal{U}$. We denote by $[\alpha_K]$ the image of $[\beta_K]$ under the natural map $\Pi(\mathcal{U} \cap \mathcal{N}) \to \Pi(\mathcal{G})$. Now assume that Θ_M is not regular. Then, by definition, $\mathrm{Rad}(Q) \in \Theta_M$ for some principal indecomposable $\mathcal{G}$-module Q with top S. As $K(\mathcal{U} \cap \mathcal{N}) = K\mathcal{U} \cap K\mathcal{N}$ inside $K\mathcal{G}$, we have $\alpha_K(t) \in K\mathcal{N} \cap K\mathcal{U}$. Hence, as $\alpha_K(t)$ is nilpotent, it is contained in the augmentation ideals $K\mathcal{N}^\dagger$ and $K\mathcal{U}^\dagger$, respectively. As $M^{\mathcal{N}} = M$ and $S^{\mathcal{U}} = S$, [23, Corollary 3.2.3] and Lemma 3.1.7 yield

$$\{1\} = \mathrm{supp}_{\alpha_K}(\mathrm{Inf}(M)) = \mathrm{supp}_{\alpha_K}(\mathrm{Rad}(Q)) = \mathrm{supp}_{\alpha_K}(\Omega_{\mathcal{G}}(S)) = \{p-1\},$$

which contradicts the assumption $p \geq 3$. Hence Θ_M is regular.

(2) (i) Let $M \in \mathbf{mod}(\mathcal{H})$ be non-projective and assume that $\mathfrak{E}_{\mathrm{Inf}(M)} \cong \mathrm{Inf}(\mathfrak{S}_M)$. In view of Lemma 3.1.4, our assumption yields an element $[\alpha_K] \in \Pi(\mathcal{N})$. Then $\mathrm{Jt}(\mathrm{Inf}(M), \iota_K \circ \alpha_K) = \dim_k M \cdot [1]$. It follows, that $\iota_*(\Pi(\mathcal{N})) \subseteq \Pi(\mathcal{G})_{\mathrm{Inf}(M)}$. In view of Lemma 4.1.12(3b), this yields $\Pi(\mathcal{G})_{\mathrm{Inf}(M)} \nsubseteq i_*(\Pi(\mathcal{H}))$. Hence Lemma 4.1.12(1) implies that $\mathrm{Res}_{\mathcal{H}}(\mathfrak{E}_{\mathrm{Inf}(M)}) = (\mathrm{Res}_{\mathcal{H}} \circ \pi^*)(\mathfrak{S}_M) = \mathfrak{S}_M$ splits, which contradicts the definition of an AR-sequence.

(ii) This follows from Lemma 4.1.12(2) and Theorem 3.1.2(2).

(iii) Assume that $\mathrm{ql}(\mathrm{Inf}(M)) > 1$. As Θ_M is regular by (1), arrows pointing towards infinity are injective. Hence we find $X \in \Theta$ with $X^{\mathcal{N}} = X$ and $\mathrm{ql}(X) = 2$. There results an almost split sequence

$$\mathfrak{E}_Y : 0 \to \tau_{\mathcal{G}}(Y) \to X \to Y \to 0.$$

Consequently, $\tau_{\mathcal{G}}(Y)^{\mathcal{N}} = \tau_{\mathcal{G}}(Y)$ and as Y is a factor module of X, we also have $Y^{\mathcal{N}} = Y$. Hence $\mathfrak{E}_Y \cong \mathrm{Inf}(\mathfrak{S}_{\mathrm{Res}_{\mathcal{H}}(Y)})$, which is impossible by (i). Hence $\mathrm{ql}(\mathrm{Inf}(M)) = 1$, so that $\mathrm{Inf}(M) \in \Theta_M$ is quasi-simple.

Let now $X \in \Theta_M$ be a $\mathcal{G}$-module such that $X^{\mathcal{N}} = X$. Then the above shows that X is quasi-simple. Hence there is $n \in \mathbb{Z}$ such that $X \cong \tau_{\mathcal{G}}^n(\mathrm{Inf}(M))$. Now let $[\beta_K] \in \Pi(\mathcal{N})$ and consider the π-point $\alpha_K := \iota_K \circ \beta_K$ of $\mathcal{G}$. As AR-shifts do not change the local stable Jordan type (cf. [23, Proposition 2.3(1)]), we get

$$\begin{aligned}\dim_k X \cdot [1] &= \mathrm{StJt}(X, \alpha_K) = \mathrm{StJt}(\tau_{\mathcal{G}}^n(\mathrm{Inf}(M)), \alpha_K)\\ &= \mathrm{StJt}(\mathrm{Inf}(M), \alpha_K) = \dim_k M \cdot [1]\end{aligned}$$

so that $\dim_k X = \dim_k M$. Now [15, Theorem 3.2] implies that there are at most finitely many such X in Θ_M. Put $d := \dim_k M$. Observing Lemma 3.1.1, basic properties of the Heller operator yield isomorphisms of $\mathcal{N}$-modules

$$\begin{aligned}d \cdot k &\cong X|_{\mathcal{N}} \cong \tau_{\mathcal{G}}^n(\mathrm{Inf}(M))|_{\mathcal{N}} \cong \Omega_{\mathcal{G}}^{2n}(\mathrm{Inf}(M)^{\mu_{\mathcal{G}}^{-n}})|_{\mathcal{N}}\\ &\cong \Omega_{\mathcal{N}}^{2n}(\mathrm{Inf}(M)^{\mu_{\mathcal{G}}^{-n}}|_{\mathcal{N}}) \oplus (\mathrm{proj}) \cong \Omega_{\mathcal{N}}^{2n}(d \cdot k^{\mu_{\mathcal{N}}^{-n}}) \oplus (\mathrm{proj})\\ &\cong d \cdot \tau_{\mathcal{N}}^n(k) \oplus (\mathrm{proj}).\end{aligned}$$

Thus, if $n \neq 0$ ($X \ncong \mathrm{Inf}(M)$ as $\mathcal{G}$-modules), then the trivial $\mathcal{N}$-module k is periodic, so that Theorem 3.1.2(1) implies $\dim(\Pi(\mathcal{N})) = 0$.

(iv) We consider the almost split sequence $\mathfrak{E}_{\mathrm{Inf}(M)}$ terminating in $\mathrm{Inf}(M)$. We clearly have $\emptyset \neq i_*(\Pi(\mathcal{H})_M) \subseteq \Pi(\mathcal{G})_{\mathrm{Inf}(M)}$. In view of Lemma 4.1.12(3b), this yields $\Pi(\mathcal{G})_{\mathrm{Inf}(M)} \nsubseteq \iota_*(\Pi(\mathcal{N}))$. Hence Lemma 4.1.12(1) implies that $\mathfrak{E}_{\mathrm{Inf}(M)}|_{\mathcal{N}}$ splits. Then $\mathcal{N}$ acting trivially on $\tau_{\mathcal{G}}(\mathrm{Inf}(M))$ would imply $\mathfrak{E}_{\mathrm{Inf}(M)} \cong \mathrm{Inf}(\mathfrak{S}_M)$, which is again impossible by (i).

□

4.2 Restricted Lie algebras

Throughout, we let k be an algebraically closed field of positive characteristic $p > 0$. Making use of the fact that tensoring $\mathrm{U}_0(\mathfrak{g})$-modules with $\mathrm{U}_\chi(\mathfrak{g})$-modules yields $\mathrm{U}_\chi(\mathfrak{g})$-

modules (see 2.5.4(6)), as well as the description 2.5.4(7) of cohomology groups of reduced enveloping algebras and the results of Section 5.2, we will be able to formulate the results of the last section in the context of reduced enveloping algebras, where the defining linear form is not necessary entirely zero. Let

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$$

be an extension of restricted Lie algebras $\mathfrak{n}, \mathfrak{h}$. As before, we have the same conventions:

- We treat ι as an inclusion $\mathfrak{n} \subseteq \mathfrak{g}$ and take π to be the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n} = \mathfrak{h}$.
- If $\mathfrak{g}$ is a split extension, then we pick a (necessarily mono)morphism $i : \mathfrak{h} \to \mathfrak{g}$ such that $\pi \circ i = \mathrm{id}_{\mathfrak{h}}$ and identify $\mathfrak{h}$ with its image $i(\mathfrak{h})$ in $\mathfrak{g}$. Note that this renders $\mathfrak{g}$ a semidirect product $\mathfrak{n} \rtimes \mathfrak{h}$.

If $\mathfrak{g}$ is a split extension, then a vector space M is a $\mathfrak{g}$-module if and only if M is a module for both constituents and additionally

$$x.(y.m) = (x.y).m + y.(x.m) \ \ \forall x \in \mathfrak{h}, y \in \mathfrak{n}.$$

Since $\mathfrak{g}^* = \mathfrak{n}^* \oplus \mathfrak{h}^*$, where we identify $\mathfrak{h}^* = \{\chi \in \mathfrak{g}^* : \chi(\mathfrak{n}) = (0)\}$ and $\mathfrak{n}^* = \{\chi \in \mathfrak{g}^* : \chi(\mathfrak{h}) = (0)\}$, we can write any linear form $\chi \in \mathfrak{g}^*$ as a tuple $(\chi_{\mathfrak{n}}, \chi_{\mathfrak{h}})$ with each component being a linear form for the corresponding Lie algebra.

Lemma 4.2.1. *Let $\mathfrak{g}$ be a split extension of $\mathfrak{h}$ by $\mathfrak{n}$, M be a $\mathfrak{g}$-module and $\chi = (\chi_{\mathfrak{n}}, \chi_{\mathfrak{h}}) \in \mathfrak{n}^* \oplus \mathfrak{h}^* = \mathfrak{g}^*$ a linear form. The following two statements are equivalent:*

(a) M has character χ.

(b) $M|_{\mathfrak{l}}$ has character $\chi_{\mathfrak{l}}$ for $\mathfrak{l} \in \{\mathfrak{n}, \mathfrak{h}\}$.

Proof. $(a) \Rightarrow (b)$: This is clear.

$(b) \Rightarrow (a)$: We have to show that

$$(x+y)^p.m = (x+y)^{[p]}.m + \chi(x+y)^p.m \ \ \forall x \in \mathfrak{n}, y \in \mathfrak{h}, m \in M$$

According to Jacobsons formula, we have (inside $\mathrm{U}(\mathfrak{g})$) $(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x,y)$, hence (note that $s_i(x,y) \in \mathfrak{g} \subseteq \mathrm{U}(\mathfrak{g})$)

$$\begin{aligned}(x+y)^p.m &= x^p.m + y^p.m + \sum_{i=1}^{p-1} s_i(x,y).m \\ &= (x^{[p]} + \chi_{\mathfrak{n}}(x)^p).m + (y^{[p]} + \chi_{\mathfrak{h}}(y)^p).m + \sum_{i=1}^{p-1} s_i(x,y).m \\ &= (x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)).m + (\chi_{\mathfrak{n}}(x)^p + \chi_{\mathfrak{h}}(y)^p).m \\ &= (x+y)^{[p]}.m + \chi(x+y)^p.m\end{aligned}$$

□

Given a linear form $\chi \in \mathfrak{g}^*$ such that $\chi(\mathfrak{n}) = 0$, we denote by $I^\dagger$ the augmentation ideal of the Hopf algebra $\mathrm{U}_0(\mathfrak{n})$ and by $I \trianglelefteq \mathrm{U}_\chi(\mathfrak{g})$ the ideal generated by $I^\dagger$ inside $\mathrm{U}_\chi(\mathfrak{g})$. Then χ induces a linear form $\hat{\chi} \in \mathfrak{h}^* = (\mathfrak{g}/\mathfrak{n})^*$ (if $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{h}$, then $\hat{\chi} = \chi_{\mathfrak{h}}$). Using the universal property of a reduced enveloping algebra, the projection π induces a surjective homomorphism $\mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_{\hat{\chi}}(\mathfrak{h})$ of algebras (by abuse of notation, we will denote this homomorphism also by π) with kernel I. Hence (see 2.1.28 and recall that tori correspond to linearly reductive infinitesimal group schemes of height ≤ 1, see 2.5.10):

Lemma 4.2.2. *Let $\mathfrak{g}$ be an extension of $\mathfrak{h}$ by $\mathfrak{n}$ and $\chi \in \mathfrak{g}^*$ a linear form such that $\chi(\mathfrak{n}) = (0)$.*

(1) The pullback along $\pi : \mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_{\hat{\chi}}(\mathfrak{h})$ induces an equivalence

$$\mathrm{Inf} := \mathrm{Inf}_{\mathfrak{h}}^{\mathfrak{g}}(-,\chi) := \pi^* : \mathbf{mod}(\mathrm{U}_{\hat{\chi}}(\mathfrak{h})) \to \mathbf{mod}_I(\mathrm{U}_\chi(\mathfrak{g})),$$

the ***inflation from*** $\mathfrak{h}$ ***to*** $\mathfrak{g}$.

(2) If $\mathfrak{n}$ is not a torus, then $\mathrm{Inf}(M)$ is not $\mathrm{U}_\chi(\mathfrak{g})$-projective for all $M \in \mathbf{mod}(\mathrm{U}_{\hat{\chi}}(\mathfrak{h}))$.

(3) If $\mathfrak{g}$ is a split extension, then the composite $\mathrm{Res}_{\mathrm{U}_\chi(\mathfrak{h})} \circ \mathrm{Inf}$ is the identity $\mathrm{id}_{\mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))}$.

Let $\mathfrak{g}$ be a split extension. Since restricted Lie algebras correspond to infinitesimal group schemes of height ≤ 1, it follows that the finite group scheme $\mathrm{Spec}(\mathrm{U}_0(\mathfrak{g})^*)$ is a split extension of $\mathrm{Spec}(\mathrm{U}_0(\mathfrak{h})^*)$ by $\mathrm{Spec}(\mathrm{U}_0(\mathfrak{n})^*)$. Thus, Lemma 2.4.27 implies

$$\mathrm{Spec}(\mathrm{U}_0(\mathfrak{g})^*) \cong \mathrm{Spec}(\mathrm{U}_0(\mathfrak{n})^*) \rtimes \mathrm{Spec}(\mathrm{U}_0(\mathfrak{h})^*),$$

so that $\mathrm{U}_0(\mathfrak{g}) \cong \mathrm{U}_0(\mathfrak{n})\#\mathrm{U}_0(\mathfrak{h})$. The corresponding representation

$$\mathrm{U}_0(\mathfrak{h}) \to \mathrm{End}_{\mathrm{k}}(\mathrm{U}_0(\mathfrak{n}))$$

which turns $\mathrm{U}_0(\mathfrak{n})$ into an $\mathrm{U}_0(\mathfrak{h})$-module algebra is described in 2.5.19(3). Viewing $\mathrm{U}_0(\mathfrak{n}) \subseteq \mathrm{U}_0(\mathfrak{g})$, the action of $x \in \mathfrak{h}$ is given by the commutator $[x, -]$: We have

$$x.u = xu - ux \ \ \forall u \in \mathrm{U}_0(\mathfrak{n}).$$

Lemma 4.2.3. *Let $\mathfrak{g}$ be a split extension of $\mathfrak{h}$ by $\mathfrak{n}$ and $\chi \in \mathfrak{g}^*$ be a linear form such that $\chi(\mathfrak{n}) = 0$. Given $N \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))$, the space $\mathrm{U}_0(\mathfrak{n}) \otimes_k N$ has the structure of a $\mathrm{U}_\chi(\mathfrak{g})$-module via the tensor product of $\mathfrak{h}$-modules and $\mathrm{U}_0(\mathfrak{n})$ acts via $u.(v \otimes m) := uv \otimes m$. Moreover, the induction functor*

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(-, \chi) : \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h})) \to \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$$

is naturally equivalent to $\mathrm{U}_0(\mathfrak{n}) \otimes_k -$.

Proof. By Lemma 2.3.16(a), $\mathrm{U}_0(\mathfrak{n})$ is a $\mathrm{U}_0(\mathfrak{g})$-module. Thus, $\mathrm{U}_0(\mathfrak{n}) \otimes_k \mathrm{Inf}_{\mathfrak{h}}^{\mathfrak{g}}(N, \chi) = \mathrm{U}_0(\mathfrak{n}) \otimes_k N$ is a $\mathrm{U}_\chi(\mathfrak{g})$-module and the $\mathfrak{g}$-action is exactly the one described as above. For the remaining statements, we can argue as in 2.3.16(c). □

Let $S_\chi(1), \dots, S_\chi(n)$ be a complete list of simple $\mathrm{U}_{\hat{\chi}}(\mathfrak{h})$-modules (up to isomorphism) and denote by $P_\chi(i)$ the corresponding projective covers over $\mathrm{U}_{\hat{\chi}}(\mathfrak{h})$. If $\mathfrak{n}$ is unipotent, then 2.5.12 shows that all simple $\mathrm{U}_\chi(\mathfrak{g})$-modules lie in the essential image of the inflation functor. We will now collect the corresponding results for Lie algebras, all proofs can be easily adopted.

Theorem 4.2.4. *Let $\mathfrak{g}$ be an extension of $\mathfrak{h}$ by some unipotent restricted Lie algebra $\mathfrak{n}$ and let $\chi \in \mathfrak{g}^*$ be a linear form such that $\chi(\mathfrak{n}) = (0)$. Then the following statements hold:*

(1) The algebras $\mathrm{U}_\chi(\mathfrak{g})$ and $\mathrm{U}_{\hat{\chi}}(\mathfrak{h})$ have the same simple modules, i.e. the modules $T_\chi(i) := \mathrm{Inf}(S_\chi(i))$ for $1 \leq i \leq n$ form a complete set of iso-classes of simple $\mathrm{U}_\chi(\mathfrak{g})$-modules.

(2) If $\mathfrak{g}$ is a split extension, then the following additional statements hold:

(a) If $M = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(N, \chi)$ for some $\mathrm{U}_\chi(\mathfrak{h})$-module N, then $\mathrm{Top}(M) \cong \mathrm{Inf}(\mathrm{Top}(N))$. Moreover, we have $V(\mathfrak{g})_M = V(\mathfrak{h})_N$.

(b) Let $i \in \{1, \dots, n\}$, then the induced module $Q_\chi(i) := \mathrm{U}_0(\mathfrak{n}) \otimes_k P_\chi(i)$ is the projective cover of the simple module $T_\chi(i)$ over $\mathrm{U}_\chi(\mathfrak{g})$.

(c) If $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$, then we have $[M : T_\chi(i)] = [M|_{\mathrm{U}_\chi(\mathfrak{h})} : S_\chi(i)]$ for all $i \in \{1, \dots, n\}$.

(d) Denote by $I^\dagger$ the augmentation ideal of $\mathrm{U}_0(\mathfrak{n})$. We have for all $i \in \{1, \dots, n\}$

$$\mathrm{Rad}(Q_\chi(i)) = (I^\dagger \otimes_k P_\chi(i)) + (k.1 \otimes_k \mathrm{Rad}_{\mathrm{U}_\chi(\mathfrak{h})}(P_\chi(i))).$$

Moreover, $\Omega_{\mathrm{U}_\chi(\mathfrak{g})}(\mathrm{Inf}(P_\chi(i))) = I^\dagger \otimes_k P_\chi(i)$.

(e) If $J_{\mathfrak{h}} \trianglelefteq \mathrm{U}_\chi(\mathfrak{g})$ *is the ideal generated by* $\mathrm{Jac}(\mathrm{U}_\chi(\mathfrak{h}))$*, then* $\mathrm{Jac}(\mathrm{U}_\chi(\mathfrak{g})) = I + J_{\mathfrak{h}}$.

If $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$, then the space of $\mathfrak{n}$-invariants $M^{\mathfrak{n}} = \{m \in M \mid x.m = 0 \;\; \forall x \in \mathfrak{n}\}$ is by Lemma 2.5.12(a) a $\mathrm{U}_\chi(\mathfrak{g})$-submodule. Composition with $\mathrm{Res}_{\mathrm{U}_\chi(\mathfrak{h})}$ (the inverse of the inflation) thus provides a functor

$$\mathfrak{F} := \mathrm{Res}_{\mathrm{U}_\chi(\mathfrak{h})} \circ (-)^{\mathfrak{n}} : \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g})) \to \mathbf{mod}(\mathrm{U}_{\hat{\chi}}(\mathfrak{h})).$$

If $\mathfrak{g}$ is a split extension, then $\mathrm{U}_0(\mathfrak{n})$ is a $\mathrm{U}_0(\mathfrak{g})$-module and $\mathrm{U}_0(\mathfrak{n})^{\mathfrak{n}} = \int_{\mathfrak{n}}$ is the *(left) integral* of the Hopf algebra $\mathrm{U}_0(\mathfrak{n})$. We denote by $\lambda_{\mathfrak{n}} : \mathrm{U}_0(\mathfrak{h}) \to k$ the algebra homomorphism corresponding to this operation.

Corollary 4.2.5. *Let* $\mathfrak{g}$ *be a split extension of* $\mathfrak{h}$ *by* $\mathfrak{n}$ *and* $\chi(\mathfrak{n}) = 0$. *The composite*

$$\mathfrak{F} \circ \mathrm{U}_0(\mathfrak{n}) \otimes_k - : \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h})) \to \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))$$

is naturally equivalent to $\lambda_{\mathfrak{n}} \otimes_k -$. *In particular,* $\mathfrak{F}$ *takes projective indecomposables to projective indecomposables.*

Definition 4.2.6. Let $\mathfrak{h}$ be a finite-dimensional restricted Lie algebra, $\chi \in \mathfrak{h}^*$ be a linear form and V a restricted representation. We then define the *generalized McKay quiver* $\Gamma_V(\mathfrak{h}, \chi)$ *of* $\mathrm{U}_\chi(\mathfrak{h})$ *relative to* V as follows:

- The edges are labelled by the simple $\mathrm{U}_\chi(\mathfrak{h})$ modules $S_1, \ldots, S_n$.
- There are $\dim_k \mathrm{Hom}_{\mathrm{U}_\chi(\mathfrak{h})}(V \otimes_k S_i, S_j) \; + \; \dim_k \mathrm{Ext}^1_{\mathrm{U}_\chi(\mathfrak{h})}(S_i, S_j)$ arrows from $S_i \to S_j$

Theorem 4.2.7. *Let* $\mathfrak{g}$ *be an extension of* $\mathfrak{h}$ *by* $\mathfrak{n}$ *and* $\chi \in \mathfrak{g}^*$ *a linear form such that* $\chi(\mathfrak{n}) = 0$. *Consider the* $\mathrm{U}_0(\mathfrak{h})$*-module* $V := \mathfrak{n}/([\mathfrak{n}, \mathfrak{n}] + \langle \mathfrak{n}^{[p]} \rangle)$.

(1) Let M, N *be* $\mathrm{U}_{\hat{\chi}}(\mathfrak{h})$*-modules, then*

$$\begin{aligned} \dim_k \mathrm{Ext}^1_{\mathrm{U}_{\hat{\chi}}(\mathfrak{h})}(M, N) &\leq \dim_k \mathrm{Ext}^1_{\mathrm{U}_\chi(\mathfrak{g})}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \leq \\ &\leq \dim_k \mathrm{Hom}_{\mathrm{U}_{\hat{\chi}}(\mathfrak{h})}(V \otimes_k M, N) \; + \; \dim_k \mathrm{Ext}^1_{\mathrm{U}_{\hat{\chi}}(\mathfrak{h})}(M, N). \end{aligned}$$

If the second inflation map

$$(\mathrm{Ext}^2_{\mathrm{U}_{\hat{\chi}}(\mathfrak{h})}(M, N) \cong) \; H^2(\mathrm{U}_0(\mathfrak{h}), M^* \otimes_k N) \longrightarrow H^2(\mathrm{U}_0(\mathfrak{g}), \mathrm{Inf}(M^* \otimes_k N))$$

is injective, then the right-hand inequality is in fact an equality.

(2) If there exists a $\mathrm{U}_0(\mathfrak{g})$*-module* Q *such that* $Q|_{\mathrm{U}_0(\mathfrak{n})}$ *is projective and* $Q^{\mathfrak{n}} = Q^{\mathfrak{g}} = k$ *or if* $\mathfrak{g}$ *is a split extension, then we have isomorphisms*

$$\mathrm{Ext}^1_{\mathrm{U}_\chi(\mathfrak{g})}(\mathrm{Inf}(M), \mathrm{Inf}(N)) \cong \mathrm{Hom}_{\mathrm{U}_{\hat{\chi}}(\mathfrak{h})}(V \otimes_k M, N) \oplus \mathrm{Ext}^1_{\mathrm{U}_{\hat{\chi}}(\mathfrak{h})}(M, N)$$

for all $\mathrm{U}_{\hat{\chi}}(\mathfrak{h})$*-modules* M, N*. In particular, if* $\mathfrak{n}$ *is unipotent, then the Gabriel quiver of* $\mathrm{U}_\chi(\mathfrak{g})$ *is the generalized McKay quiver* $\Gamma_V(\mathfrak{h}, \hat{\chi})$*.*

Proof. By Lemma 2.4.22(2), we get an isomorphism $H^1(\mathrm{U}_0(\mathfrak{n}), k) \cong V^*$ of $\mathrm{U}_0(\mathfrak{h})$-modules. Using the description 2.5.4(7) of cohomology groups of reduced enveloping algebras, we can now adopt the arguments of the proof of Theorem 4.1.8. □

For the following, one may recall the definition of the induced p-map of a semidirect product of restricted Lie algebras (see 2.5.18).

Lemma 4.2.8. *Let* $\mathfrak{g}$ *be a split extension of* $\mathfrak{h}$ *by* $\mathfrak{n}$ *and* $\chi \in \mathfrak{g}^*$ *be a linear form such that* $\chi(\mathfrak{n}) = (0)$*.*

(1) If M *is a* $\mathrm{U}_\chi(\mathfrak{h})$*-module, then*

$$V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{g}) \cap (\mathfrak{n} \times V(\mathfrak{h})_M).$$

In particular, M *is projective if and only if* $V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{n}) \times \{0\}$*.*

(2) If $\mathfrak{n}$ *is an abelian Lie algebra and* $\mathrm{Jt}(y, \mathfrak{n}) = \mathrm{StJt}(y, \mathfrak{n})$ *for all* $y \in V(\mathfrak{h})$ *(recall Definition 3.3.3) holds, then we have* $V(\mathfrak{g}) = V(\mathfrak{n}) \times V(\mathfrak{h})$*. In particular, we have* $V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{n}) \times V(\mathfrak{h})_M$ *for all* $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))$ *in that case. Moreover, if* $\dim V(\mathfrak{n}) \geq 2$*, then every block* $\mathcal{B} \subseteq \mathrm{U}_\chi(\mathfrak{g})$*, which contains a simple* $\mathrm{U}_\chi(\mathfrak{g})$*-module on which* $\mathfrak{n}$ *acts trivially and whose restriction to* $\mathrm{U}_\chi(\mathfrak{h})$ *is non-projective, is of wild representation type.*

Proof. (1) Let $0 \neq (x, y) \in V(\mathfrak{g}) \subseteq \mathfrak{n} \times V(\mathfrak{h})$. As $\mathfrak{n}$ acts trivially on $\mathrm{Inf}(M)$, we conclude that

$$\mathrm{Jt}(\mathrm{Inf}(M), (x, y)) = \begin{cases} \dim_k M \cdot [1] & y = 0 \\ \mathrm{Jt}(M, y) & \text{otherwise.} \end{cases}$$

This implies $V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{g}) \cap (\mathfrak{n} \times V(\mathfrak{h})_M)$. Recall that M is $\mathrm{U}_\chi(\mathfrak{h})$-projective if and only if $V(\mathfrak{h})_M = \{0\}$ (see Lemma 2.5.7(1)). Since $V(\mathfrak{g}) \cap (\mathfrak{n} \times \{0\}) = V(\mathfrak{n}) \times \{0\}$, the additional statement follows.

(2) Denote by $\tau : \mathfrak{h} \to \mathrm{Der}_p(\mathfrak{n})$ the corresponding representation of $\mathfrak{h}$ on $\mathfrak{n}$. Recall that, if $\mathfrak{n}$ is abelian, then we have

$$(x, y)^{[p]} = (x^{[p]} + \tau(y)^{p-1}(x), y^{[p]}) \quad (*)$$

for all $x \in \mathfrak{n}, y \in \mathfrak{h}$ (see 2.5.20). Given $y \in V(\mathfrak{h})$, we have $0 = \tau(y^{[p]}) = \tau(y)^p$. But then the condition $\mathrm{Jt}(y, \mathfrak{n}) = \mathrm{StJt}(y, \mathfrak{n})$ for all $y \in V(\mathfrak{h})$ implies that $\tau(y)^{p-1} = 0$ for all $y \in V(\mathfrak{h})$, so that $(*)$ implies $V(\mathfrak{g}) = V(\mathfrak{n}) \times V(\mathfrak{h})$. The equation $V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{n}) \times V(\mathfrak{h})_M$ (for all $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))$) now follows from (1).

Now assume $\dim V(\mathfrak{n}) \geq 2$ and let $T = \mathrm{Inf}(S) \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$ be the inflation of a non-projective simple $\mathrm{U}_\chi(\mathfrak{h})$-module S. Using Lemma 2.5.7(1) and the above, we conclude $\dim V(\mathfrak{g})_T = \dim V(\mathfrak{n}) \times V(\mathfrak{h})_S \geq 2 + 1 = 3$. Now apply [19, Theorem 4.1]. □

Remark 4.2.9. If $\dim_k \mathfrak{n} \leq p - 1$, then clearly $\mathrm{Jt}(y, \mathfrak{n}) = \mathrm{StJt}(y, \mathfrak{n})$ for all $y \in V(\mathfrak{h})$.

Theorem 4.2.10. *Let $\mathfrak{g}$ be an extension of $\mathfrak{h}$ by a non-toral restricted Lie algebra $\mathfrak{n}$ over an algebraically closed field k of characteristic $p \geq 3$ and $\chi \in \mathfrak{g}^*$ a linear form such that $\chi(\mathfrak{n}) = (0)$. Given an indecomposable $\mathrm{U}_{\hat{\chi}}(\mathfrak{h})$-module M, we denote by Θ_M the AR-component of $\Gamma_s(\mathfrak{g}, \chi)$, which contains the (in view of 4.2.2(2)) non-projective indecomposable $\mathrm{U}_\chi(\mathfrak{g})$-module $\mathrm{Inf}(M)$.*

(1) If $\mathfrak{n} \cap \mathrm{Rad}_p(\mathfrak{g}) \neq 0$, then the component Θ_M is regular.

(2) If $\mathfrak{g}$ is a split extension, then the following additional statements hold:

(a) The inflation functor $\mathrm{Inf} : \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h})) \to \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$ will never transform AR-sequences into AR-sequences.

(b) If $\mathfrak{n} \cap \mathrm{Rad}_p(\mathfrak{g}) \neq (0)$ and if $\Theta_M \cong \mathbb{Z}[A_\infty]$, then

- *$\mathrm{Inf}(M)$ is quasi-simple.*
- *The set $\Theta_{M,\mathfrak{n}} := \{X \in \Theta_M \mid X^{\mathfrak{n}} = X\}$ is finite. If $X \in \Theta_{M,\mathfrak{n}}$, then $\dim_k X = \dim_k M$ and there exists $n \in \mathbb{Z}$ such that $\tau^n_{\mathrm{U}_\chi(\mathfrak{g})}(\mathrm{Inf}(M)) \cong X$. If $|\Theta_{M,\mathfrak{n}}| \geq 2$, then $\dim V(\mathfrak{n}) = 1$.*

(c) If M is not $\mathrm{U}_\chi(\mathfrak{h})$-projective, then $\mathfrak{n}.\tau_{\mathrm{U}_\chi(\mathfrak{g})}(\mathrm{Inf}(M)) \neq 0$.

(d) If M is $\mathrm{U}_\chi(\mathfrak{h})$-projective, then $|\mathrm{Jt}(\mathrm{Inf}(M))| = 2$. Moreover, N is $\mathrm{U}_\chi(\mathfrak{h})$-projective and $N|_{\mathrm{U}_0(\mathfrak{n})}$ has constant Jordan type for all $N \in \Theta_M$.

Proof. Using 3.3.1 and 3.3.5, the assertions (1), (2)(a) - (c) can be shown analogous to Theorem 4.1.13(2).

(2)(d): Let $N \in \Theta_M$. As obviously $|\mathrm{Jt}(\mathrm{Inf}(M))| = 2$, it follows that $|\mathrm{Jt}(N)| = 2$ for all $N \in \Theta_M$ (see Corollary 3.3.9(a)). Moreover, we have $V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{n}) \times \{0\}$ (see Lemma 4.2.8), so that N is $\mathrm{U}_\chi(\mathfrak{h})$-projective (we have $V(\mathfrak{h})_N = V(\mathfrak{h})_{\mathrm{Inf}(M)} = \{0\}$). Hence we must have $\mathrm{Jt}(N, x) = \mathrm{Jt}(N, y)$ for all $x, y \in V(\mathfrak{n})$ by the above. □

Remark 4.2.11. If $\dim V(\mathfrak{n}) = 1$, then every $\mathrm{U}_0(\mathfrak{n})$-module is periodic (cf. 2.5.7(5)). In view of [12, Theorem 3.2(2)], $\mathrm{U}_0(\mathfrak{n})$ is representation-finite and then [12, Theorem 4.3(2)] tells us exactly, how $\mathfrak{n}$ looks like. In particular, if $\mathfrak{n}$ is unipotent, then there exists a $[p]$-nilpotent element $x \in \mathfrak{n}$ such that $\mathfrak{n} = (kx)_p$ is a nil-cyclic Lie algebra.

Since AR-components of reduced enveloping algebras with at least three-dimensional rank variety are known to be isomorphic to $\mathbb{Z}[A_\infty]$ (see [14, Theorem 2.1]), we get the following corollary:

Corollary 4.2.12. *Let k be an algebraically closed field of characteristic $\mathrm{char}(k) = p \geq 3$ and $\mathfrak{g}$ be a split extension of $\mathfrak{h}$ by $\mathfrak{n}$. Moreover, let $\chi \in \mathfrak{g}^*$ a linear form such that $\chi(\mathfrak{n}) = (0)$ and assume that the following conditions hold:*

- *$\mathfrak{n}$ is abelian and unipotent,*
- *$\dim V(\mathfrak{n}) \geq 2$,*
- *$\mathrm{Jt}(y, \mathfrak{n}) = \mathrm{StJt}(y, \mathfrak{n})$ for all $y \in V(\mathfrak{h})$.*

If M is a non-projective indecomposable $\mathrm{U}_\chi(\mathfrak{h})$-module, then the component $\Theta \subseteq \Gamma_s(\mathfrak{g}, \chi)$ containing $\mathrm{Inf}(M) \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{g}))$ is isomorphic to $\mathbb{Z}[A_\infty]$. Moreover, $\mathrm{Inf}(M)$ is quasi-simple and if N is a non-projective indecomposable $\mathrm{U}_\chi(\mathfrak{h})$-module such that $\mathrm{Inf}(N) \in \Theta_M$, then $N \cong M$.

Proof. As M is non-projective, we have $\dim V(\mathfrak{h})_M \geq 1$ (see 2.5.7(1)). Then Lemma 4.2.8(2) implies that $V(\mathfrak{g})_{\mathrm{Inf}(M)} = V(\mathfrak{n}) \times V(\mathfrak{g})_M$ is at least three-dimensional. Hence $\Theta \cong \mathbb{Z}[A_\infty]$ by the above remark, so that the assertion follows from Theorem 4.2.10(2b). □

Remark 4.2.13. The conditions of the above corollary are given whenever $\mathfrak{n}$ is just a restricted representation of $\mathfrak{h}$ such that $2 \leq \dim_k \mathfrak{n} \leq p - 1$. For instance (since $p \geq 3$), one may consider semidirect products $V \rtimes \mathfrak{g}$, where $\mathfrak{g} \subseteq \mathfrak{gl}(2)$ and V is the natural representation.

Lemma 4.2.14. *Let k be an algebraically closed field of characteristic $\mathrm{char}(k) = p \geq 3$, V be a two-dimensional restricted module of some restricted Lie algebra $\mathfrak{g}$ and $\chi \in (V \rtimes \mathfrak{g})^*$ a linear form such that $\chi(V) = (0)$. If S is a projective simple $\mathrm{U}_\chi(\mathfrak{g})$-module, then the component $\Theta \subseteq \Gamma_s(V \rtimes \mathfrak{g}, \chi)$ containing $\mathrm{Inf}(S)$ is either isomorphic to $\mathbb{Z}[D_\infty]$ or to $\mathbb{Z}[A_\infty]$. Moreover, if*

- *$V \otimes_k S$ is (projective) indecomposable or*
- *if $\Theta \cong \mathbb{Z}[D_\infty]$ and 2 does not divide $\dim_k S$ or*
- *if $\Theta \cong \mathbb{Z}[A_\infty]$,*

then $\mathrm{Inf}(S)$ is located at an end of Θ.

Proof. Let $\{r, s\}$ be a basis of V and identify $\mathrm{U}_0(V) = k[r, s]/(r^p, s^p)$ with a truncated polynomial ring in the variables r, s. We put

$$I^\dagger := \mathrm{U}_0(V)^\dagger, \quad \int_V := \int_{\mathrm{U}_0(V)} = k(r^{p-1}s^{p-1}),$$

the augmentation ideal and the left integral of the Hopf algebra $\mathrm{U}_0(V)$, respectively. Moreover, we denote by Q the projective cover of the simple $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$-module $\mathrm{Inf}(S)$. In view of 4.2.4 we have $Q \cong \mathrm{U}_0(V) \otimes_k S$ and $\mathrm{Rad}(Q) = I^\dagger \otimes_k S$. Since $\int_V \subseteq \mathrm{U}_0(V)$ is a one-dimensional $\mathrm{U}_0(\mathfrak{h})$-submodule (cf. 2.3.9(1)), we get $\mathrm{Soc}(Q) = \int_V \otimes_k S$. We denote by

$$M := \mathrm{ht}(Q) = \mathrm{Rad}(Q)/\mathrm{Soc}(Q) = I^\dagger \otimes S \ / \ (\int_V \otimes S)$$

the heart of Q.

- In view of 4.2.8(2), $V(V \rtimes \mathfrak{g})_\Theta = V$ is two-dimensional. Combining Theorem 3.1.8 and 2.2.6, we get that $\Theta \cong \mathbb{Z}[\mathcal{T}]$, where $\mathcal{T} \in \{A_\infty, A_\infty^\infty, D_\infty\}$. We proceed by showing $\Theta \not\cong \mathbb{Z}[A_\infty^\infty]$: Consider the component $\tilde{\Theta} := \Omega(\Theta) \cong \Theta$. By the above, we have $M \in \tilde{\Theta}$. We compute the Jordan type of M at $r \in V$. First, we clearly have an isomorphism
$$M|_{\mathrm{U}_0(kr)} \cong \dim_k S \cdot I^\dagger \ / \ \int_V .$$
We clearly have $\mathrm{Jt}(r, I^\dagger) = [p-1] \oplus (p-1)[p] = \mathrm{Jt}(r, \mathrm{U}_0(V)/\int_V)$. Since the standard almost split sequence
$$0 \longrightarrow I^\dagger \longrightarrow \mathrm{U}_0(V) \oplus (I^\dagger/\int_V) \longrightarrow \mathrm{U}_0(V)/\int_V \longrightarrow 0$$
of the local algebra $\mathrm{U}_0(V)$ splits upon restriction to $\mathrm{U}_0(kr)$ (see Corollary 3.3.6), we get
$$(I^\dagger \ / \ \int_V)|_{\mathrm{U}_0(kr)} \cong 2[p-1] \oplus (p-2)[p-1].$$
Hence
$$\mathrm{Jt}(M, r) = 2 \dim_k S \ [p-2] \oplus \dim_k S \ (p-2)[p-1].$$
Analogous, one can determine the Jordan type of $\mathrm{Rad}(Q) = I^\dagger \otimes_k S \in \tilde{\Theta}$ at r, we have
$$\mathrm{Jt}(\mathrm{Rad}(Q), r) = \dim_k S \ [p-1] \oplus \dim_k S \ (p-1)[p].$$
Hence $|\{\mathrm{StJt}(N, r) \mid N \in \tilde{\Theta}\}| \geq 2$, so that 3.3.9(b) implies $\Theta \not\cong \mathbb{Z}[A_\infty^\infty]$.

- We now show that $\mathrm{Inf}(S)$ is located at an end provided $V \otimes_k S$ is indecomposable: Let I be the ideal generated by $I^\dagger$ inside $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$. It is enough to show that the heart
$$M := \mathrm{ht}(Q) = \mathrm{Rad}(Q)/\mathrm{Soc}(Q) = I^\dagger \otimes S/\int_V \otimes S.$$
is a local $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$-module (hence indecomposable). To that end, we need to show that its top is simple. Since I is contained in the Jacobson radical of $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$, it suffices to show that $M/I.M$ has a simple top. We clearly have $I.M = (I^\dagger)^2 \otimes$

$S/\int_V \otimes S$, so that there results a sequence of canonical isomorphisms of $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$-modules

$$\begin{aligned} M/I.M &\cong (I^\dagger \otimes S \;/\; \int_V \otimes S) \;/\; ((I^\dagger)^2 \otimes S \;/\; \int_V \otimes S) \\ &\cong I^\dagger \otimes S \;/\; ((I^\dagger)^2 \otimes S) \\ &\cong I^\dagger \;/\; (I^\dagger)^2 \otimes S \cong V \otimes S. \end{aligned}$$

Thus, if $V \otimes_k S$ is indecomposable, we get that $\mathrm{ht}(Q)$ is local.

- If $\Theta \cong \mathbb{Z}[D_\infty]$, then $\mathrm{im}(f_\Theta) = \{1, 2\}$. Since $\mathrm{Jt}(\mathrm{Inf}(S), r) = \dim_k S[1]$ and 2 does not divide $\dim_k S$, we get $f_\Theta(\mathrm{Inf}(S)) = 1$ (see 3.3.8(c)). Hence the picture in [35, p. 328] implies that $\mathrm{Inf}(S)$ is located at an end of Θ.
- If $\Theta \cong \mathbb{Z}[A_\infty]$, then it follows from [14, Proposition 4.1] that $\mathrm{Inf}(S)$ is located at an end of Θ (i.e. quasi-simple).

□

Remark 4.2.15. The result can be applied in the case where $\mathfrak{g} = \mathfrak{sl}(2)$ and $V = k^2$ is the natural representation. For instance, if $\chi = 0$, then the *Steinberg module* $L(p-1)$ is the unique projective simple $\mathrm{U}_0(\mathfrak{sl}(2))$-module and the modular Clebsch-Gordan rule (see [26, Kap. 5]) implies that $V \otimes L(p-1) \cong P(p-2)$ is the projective cover of $L(p-2)$ (see also 5.3.2 for the case of a non-zero linear form).

For any restricted Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{n}_\mathfrak{g}$ the p-ideal generated by the nullcone $V(\mathfrak{g})$ and, given $j \in \{1, \ldots, p-1\}$, we denote by $\mathrm{EIP}^j(\mathfrak{g}) \subseteq \mathbf{mod}(\mathrm{U}_0(\mathfrak{g}))$ the full subcategory consisting of all $M \in \mathbf{mod}(\mathrm{U}_0(\mathfrak{g}))$ having the *equal j-images property*, i.e. $\mathrm{im}(x_M^j) = \mathrm{im}(y_M^j)$ for all $x, y \in V(\mathfrak{g}) \setminus \{0\}$. The following can be seen as a corollary of [5, Proposition 5.2.3], but will not play an important role later on.

Lemma 4.2.16. *Let $p \geq 3$ and let $\mathfrak{s} = \mathfrak{u} \rtimes \mathfrak{g}$ be a semidirect product of restricted Lie algebras. If there is an elementary abelian subalgebra $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{u})$ on which $\mathfrak{g}$ acts such that $\mathfrak{n}_\mathfrak{g}.\mathfrak{e} \neq (0)$, then the following statements hold:*

(1) The restricted Lie algebra $\mathfrak{s}$ contains a three-dimensional p-trivial Heisenberg algebra.

(2) Given $j \in \{1, \ldots, p-1\}$, the category $\mathrm{EIP}^j(\mathfrak{s})$ is equivalent to $\mathbf{mod}(\mathrm{U}_0(\mathfrak{s}/\mathfrak{n}_\mathfrak{s}))$.

(3) If $\mathfrak{u}$ is elementary abelian, then $\mathfrak{s}/\mathfrak{n}_\mathfrak{s} \cong \mathfrak{g}/\mathfrak{n}_\mathfrak{g}$.

Proof. (1) It suffices to show that the subalgebra $\mathfrak{g}_\mathfrak{e} = \mathfrak{e} \rtimes \mathfrak{g} \subseteq \mathfrak{s}$ contains a Heisenberg subalgebra. Since $\mathfrak{n}_\mathfrak{g}$ acts non-trivially on $\mathfrak{e}$, there necessarily exists an element

$x \in V(\mathfrak{g}) \setminus \{0\}$ such that $x.\mathfrak{e} \neq (0)$. Thus, there exists $v_0 \in \mathfrak{e} \setminus \{0\}$ such that $x.v_0 \neq 0$. Since x acts nilpotently on the two-dimensional space $\mathfrak{e}$, we observe that $x^2.v = 0$ for all $v \in \mathfrak{e}$. Hence $x.v_0 = \lambda v_0$ for some $\lambda \in k \setminus \{0\}$ is not possible. Thus,

$$\mathfrak{h} := kx \oplus kv_0 \oplus k(x.v_0) \subseteq \mathfrak{g}_{\mathfrak{e}}$$

is clearly isomorphic to the three-dimensional Heisenberg Lie algebra and p-trivial by 4.2.8(2) (here we use $p \geq 3$).

(2) Using (1), [5, Proposition 5.2.3] implies that $\mathfrak{n}_{\mathfrak{s}}$ acts trivially on every $M \in \mathrm{EIP}^j(\mathfrak{g}_V)$. Conversely, if $\mathfrak{n}_{\mathfrak{s}}$ acts trivially on $M \in \mathbf{mod}(\mathrm{U}_0(\mathfrak{g}))$, then M has the equal j-images property for all $1 \leq j \leq p-1$, as all relevant images are zero.

(3) The assumption means, that $\mathfrak{u}$ is just a restricted representation of $\mathfrak{g}$. Hence, 4.2.8(2) and $p \geq 3$ imply $\mathfrak{n}_{\mathfrak{s}} = \mathfrak{u} \rtimes \mathfrak{n}_{\mathfrak{g}}$. Now the composition

$$\mathfrak{s} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}_{\mathfrak{g}}, \; u + x \mapsto x + \mathfrak{n}_{\mathfrak{g}}$$

of projections is a surjective homomorphism of restricted Lie algebras with kernel $\mathfrak{u} \rtimes \mathfrak{n}_{\mathfrak{g}}$. This implies the result.

□

Remark 4.2.17. The requirements of the above lemma are given, whenever $\mathfrak{u}$ is (isomorphic to) the two-dimensional elementary abelian Lie algebra and $\mathfrak{g}$ acts faithfully on $\mathfrak{u}$.

4.3 Simple modules of reduced enveloping algebras of certain extensions

An application of the results in [25, § 5.7] will provide a tool to find simple $\mathrm{U}_\chi(\mathfrak{g})$-modules in some special case for an extension $\mathfrak{g}$ of $\mathfrak{h}$ by $\mathfrak{n}$. Given a linear form $\lambda \in \mathfrak{n}^*$ with $\lambda([\mathfrak{n},\mathfrak{n}]) = 0$, we consider the stabilizer $\mathfrak{g}^\lambda := \{x \in \mathfrak{g} \mid \lambda([x,y]) = 0 \; \forall y \in \mathfrak{n}\}$ of λ inside the $\mathfrak{g}$-module $\mathfrak{n}^*$. Moreover, if M is a $\mathrm{U}_\chi(\mathfrak{g})$-module, then we put

$$M^\lambda := \{m \in M \mid x.m = \lambda(x).m \; \forall x \in \mathfrak{n}\}.$$

Theorem 4.3.1. *Let $\mathfrak{g}$ be an extension of $\mathfrak{h}$ by a unipotent restricted Lie algebra $\mathfrak{n}$ and $\chi \in \mathfrak{g}^*$ a linear form such that $\chi([\mathfrak{n},\mathfrak{n}]_p) = 0$ (here $[\mathfrak{n},\mathfrak{n}]_p$ is the smallest p-ideal of $\mathfrak{n}$ that contains $[\mathfrak{n},\mathfrak{n}]$). Then the following statements hold:*

(a) There exists a linear form $\lambda \in \mathfrak{n}^*$ with $\lambda([\mathfrak{n},\mathfrak{n}]_p) = 0$ such that the one-dimensional module k_λ is the only simple $\mathrm{U}_\chi(\mathfrak{n})$-module up to isomorphism. If $\mathfrak{n}$ is p-trivial, then $\lambda = \chi|_{\mathfrak{n}}$.

(b) Let λ be as in (a). Denote by A_S^χ, B_S^χ the sets of isomorphism classes of simple $\mathrm{U}_\chi(\mathfrak{g})$ and simple $\mathrm{U}_\chi(\mathfrak{g}^\lambda)$-modules, respectively. Then the map

$$\Gamma : B_S^\chi \longrightarrow A_S^\chi,\ [S] \mapsto [\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(S, \chi)]$$

is a bijection. In particular, if $\mathfrak{g}^\lambda$ is unipotent, then $\mathrm{U}_\chi(\mathfrak{g})$ admits exactly one simple module.

(c) Let $\lambda \in \mathfrak{n}^*$ with $\lambda([\mathfrak{n}, \mathfrak{n}]) = (0)$. If $\mathfrak{g}$ is a split extension, then $\mathfrak{g}^\lambda = \mathfrak{n} \rtimes \mathfrak{h}^\lambda$, where $\mathfrak{h}^\lambda$ is the stabilizer of λ inside the $\mathfrak{h}$-module $\mathfrak{n}^*$. In particular, $\mathfrak{g}^\lambda$ is unipotent if and only if $\mathfrak{h}^\lambda$ has this property.

Proof. (a) By Lemma 2.5.10(3), the algebra $\mathrm{U}_\chi(\mathfrak{n})$ possesses a unique simple module S. As $\chi([\mathfrak{n}, \mathfrak{n}]_p) = 0$, we can apply 4.2.2 to see that S is a simple module for the commutative algebra $\mathrm{U}_{\hat{\chi}}(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]_p)$. Hence $\dim_k S = 1$ and the first assertion follows. The second assertion follows from the (in view of Lemma 2.5.6(1) necessary) equation $\lambda(x)^p - \lambda(x^{[p]}) - \chi(x)^p = 0$ for all $x \in \mathfrak{n}$.

(b) Let $\lambda \in \mathfrak{n}^*$ be the linear form of (a). Then $S^\lambda = \mathrm{Soc}_{\mathrm{U}_\chi(\mathfrak{n})}(S) \neq 0$ for every simple $\mathrm{U}_\chi(\mathfrak{g})$-module S, so that S has eigenvalue function λ in the sense of [25, p.234]. As $\lambda([\mathfrak{n}, \mathfrak{n}]) = 0$, we have $\mathfrak{n} \trianglelefteq \mathfrak{g}^\lambda$, so that - by the same token - λ is also an eigenvalue function for every simple $\mathrm{U}_\chi(\mathfrak{g}^\lambda)$-module. Now apply [25, §5, Theorem 7.7]. The additional statement now follows from Lemma 2.5.10(3).

(c) The equation $\mathfrak{g}^\lambda = \mathfrak{n} \rtimes \mathfrak{h}^\lambda$ is clear. The additional statement follows from the fact that an extension of affine algebraic group schemes is unipotent if and only if the corresponding extreme terms both enjoy this property.

□

The conditions above are satisfied when $\mathfrak{n}$ is abelian and unipotent; for instance when $\mathfrak{n}$ is elementary abelian. We record a corollary pertaining to the case of a split extension. Recall from Lemma 2.4.3(1), that a finite-dimensional Lie algebra (as well as its dual space) can be viewed as a full affine space.

Corollary 4.3.2. *Let $\mathfrak{g}$ be the Lie algebra of an algebraic group G, $0 \neq V$ a G-module and consider the restricted Lie algebra $V \rtimes \mathfrak{g} = \mathrm{Lie}(V \rtimes G)$. Consider the open subset $\mathcal{O}_V := \{\chi \in (V \rtimes \mathfrak{g})^* : \chi(V) \neq 0\}$ of the dual space $(V \rtimes \mathfrak{g})^*$.*

(1) If there is $\mu \in \mathcal{O}_V$ such that the stabilizer $\mathfrak{g}^{\mu_V} = \{x \in \mathfrak{g} : \mu(x.v) = 0\ \forall v \in V\}$ of the element $\mu|_V \in V^*$ inside the $\mathfrak{g}$-module V^* is unipotent, then $\mathrm{U}_\mu(V \rtimes \mathfrak{g})$ admits exactly one simple module up to isomorphism.

(2) If (1) holds and G acts on $V^* \setminus \{0\}$ transitively, then

(a) $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$ *admits exactly one simple module up to isomorphism for any* $\chi \in \mathcal{O}_V$*.*

(b) $\mathrm{U}_\chi(V \rtimes \mathfrak{g})$ *is connected for any linear form* $\chi \in (V \rtimes \mathfrak{g})^*$*.*

Proof. The first assertion is a consequence of 4.3.1 (c).

(a) Let $\chi = (\chi_V, \chi_\mathfrak{g}) \in \mathcal{O}_V \subseteq V^* \oplus \mathfrak{g}^* = (V \rtimes \mathfrak{g})^*$ be arbitrary. As G acts transitively on $V^* \setminus \{0\}$, we can assume $\chi_V = \mu_V$ after some G-conjugation (cf. 2.5.27(2a) and 2.5.5). Hence $\mathfrak{g}^{\chi_V} = \mathfrak{g}^{\mu_V}$ is unipotent, so that we can apply Theorem 4.3.1(b).

(b) As $\mathcal{O}_V \subseteq (V \rtimes \mathfrak{g})^*$ is non-empty and open, this follows from Lemma 4.3.3 below and (a).

□

We have used a geometric argument:

Lemma 4.3.3. *Let* $\mathcal{O} \subseteq \mathfrak{g}^*$ *be a non-empty open subset. If* $\mathrm{U}_\chi(\mathfrak{g})$ *is connected for all* $\chi \in \mathcal{O}$*, then* $\mathrm{U}_\chi(\mathfrak{g})$ *is connected for all* $\chi \in \mathfrak{g}^*$

Proof. Put $C := \{\chi \in \mathfrak{g}^* : \mathrm{U}_\chi(\mathfrak{g}) \text{ is connected}\}$. Owing to [12, Theorem 4.5], $C \subseteq \mathfrak{g}^*$ is a closed subset. By assumption, we have $\mathcal{O} \subseteq C$. As $\mathcal{O} \subseteq \mathfrak{g}^*$ is non-empty and open, it is dense. Hence $\mathfrak{g}^* = \overline{\mathcal{O}} \subseteq \overline{C} = C$. □

4.4 Split extensions of a reductive group by a vector group

We first explain some terminology. Let G be an algebraic group and $T \subseteq G$ a (not necessarily maximal) torus whose (free abelian) character group is denoted by $X(T)$. Let $r \geq 1$, then T acts on kG_r via the adjoint representation which turns kG_r into a $X(T)$-graded algebra. In particular, one can study the category $\mathbf{mod}_{X(T)}(kG_r)$ of finite-dimensional $X(T)$-graded kG_r-modules and degree zero homomorphisms. We have $M \in \mathbf{mod}_{X(T)}(kG_r)$ if and only if

(i) M is a G_r-module.

(ii) M is a T-module.

(iii) The compatibility condition $t.(u.m) = (t.u).(t.m)$ holds for all $t \in T, u \in kG_r, m \in M$.

In fact, the latter category is equivalent to the module category of the algebraic group $G_r \rtimes T$. We are interested in modules of the product subgroup G_rT. Since the latter is isomorphic to $(G_r \rtimes T)/T_r$, where the image of T_r consists of 'elements' of the form (t, t^{-1}), we may identify the category $\mathbf{mod}(G_rT)$ with the full subcategory of $\mathbf{mod}(G_r \rtimes T)$ consisting of all M such that (in addition to (i)-(iii)):

(iv) $\operatorname{Res}^{T}_{T_r}(M) = \operatorname{Res}^{G_r}_{T_r}(M)$.

The group $G_r \rtimes T$ affords a block decomposition (see [41, II.7.1]) and then the arguments of [16, Lemma 2.1] show that the full subcategory $\mathbf{mod}(G_rT)$ is a sum of blocks of $G_r \rtimes T$. We denote by $\mathfrak{F} : \mathbf{mod}(G_rT) \to \mathbf{mod}(G_r)$ the restriction functor, it simply forgets the $X(T)$-grading. Then

(a) $M \in \mathbf{mod}(G_rT)$ is indecomposable (resp. projective, simple) $\Longleftrightarrow \mathfrak{F}(M) \in \mathbf{mod}(G_r)$ is indecomposable (resp. projective, simple). Since kG_r is self-injective, this also shows that $\mathbf{mod}(G_rT)$ is a Frobenius category, i.e., the notions of projectivity and injectivity coincide (see also [41, Lemma II.9.4]).

(b) If $M, N \in \mathbf{mod}(G_rT)$ are indecomposable, then $\mathfrak{F}(M) \cong \mathfrak{F}(N) \Longleftrightarrow M \cong N \otimes_k k_\lambda$ for some $\lambda \in p^rX(T)$ (the shifts sending $\mathbf{mod}(G_rT)$ onto itself are precisely those which belong to the subgroup $p^rX(T)$). Here k_λ is viewed as a (G_rT)-module via the projection $G_rT \to T$.

(c) Every projective or simple G_r-module has a G_rT-structure extending the given G_r-structure.

(d) Projective covers exist in $\mathbf{mod}(G_rT)$ and if $\hat{Q}$ is the projective cover of a simple module $\hat{S}$, then $Q := \mathfrak{F}(\hat{Q})$ is the projective cover of the simple object $S := \mathfrak{F}(\hat{S})$ in $\mathbf{mod}(G_r)$.

These properties have first been discovered in [32, 33] in the context of $\mathbb{Z}$-graded artin algebras and later in [40] it has been observed that these results are valid, when $\mathbb{Z}$ is replaced by $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Many results are known in case G is reductive and T a maximal torus (see [41, II]) and we want to show that the 'BGG reciprocity formulas' [41, Proposition II.11.2/4] still hold in a certain way when considering a split extension of G by some vector group on which G acts linearly. Below, we will use the following notation:

- If G is an algebraic group and V a G-module, then we put $G_V := V_a \rtimes G$ as well as $G_{V,r} := (G_V)_r = V_{a(r)} \rtimes G_r$ for all $r \geq 1$.

Definition 4.4.1. Let X be a torsion-free abelian group. A submonoid $P \subseteq X$ is called *pointed*, provided $x, -x \in P$ implies $x = 0$ for every $x \in X$.

As the union of an ascending collection of pointed submonoids is pointed, Zorn's Lemma shows that maximal (w.r.t. inclusion) pointed submonoids exist. By the same token, given a pointed submonoid P, there will exist a maximal one containing P.

Lemma 4.4.2. *Given a maximal, pointed submonoid $P \subseteq X$ of a torsion-free abelian group X, the rule*

$$x \leq y :\Longleftrightarrow y - x \in P$$

turns X into a totally ordered group.

Proof. See the answer of David E. Speyer in [3] $\square$

Let now G be a reductive algebraic group with Lie algebra $\mathfrak{g}$ and $T \subseteq G$ a maximal torus. Its character group $X(T)$ is a free abelian group and the weights $R \subseteq X(T)$ of the adjoint action of T on $\mathfrak{g}$ are called *roots*. After choice of a *positive system* $R^+ \subseteq R$ with corresponding set S of simple roots, there is a partial order $\preccurlyeq$ on $X(T)$ (see [41, II.1.5])

$$\lambda \preccurlyeq \mu :\Longleftrightarrow \mu - \lambda \in P := \sum_{\alpha \in S} \mathbb{N}_0 \alpha \ \ \forall\, \lambda, \mu \in X(T).$$

Let $x \in P$ and assume that $-x \in P$. Then $0 \preccurlyeq -x \preccurlyeq 0$, so that $-x = 0$ and hence $x = 0$ by the antisymmetry of $\preccurlyeq$. Hence, $P \subseteq X(T)$ is a pointed submonoid, so that we can turn $X(T)$ into a totally ordered group $(X(T), \leq)$ such that $\leq$ extends $\preccurlyeq$.

Let V be a G-module and let $\Lambda := \Lambda_V \subseteq X(T)$ be the set of weights of T on V. Put

$$\Lambda^+ := \{\lambda \in \Lambda : \lambda > 0\}, \qquad V^+ := \bigoplus_{\lambda \in \Lambda^+} V_\lambda$$

$$\Lambda^- := \{\lambda \in \Lambda : \lambda < 0\}, \qquad V^- := \bigoplus_{\lambda \in \Lambda^-} V_\lambda.$$

We denote by $B^\pm = U^\pm \rtimes T$ the corresponding Borel subgroups of G with their unipotent radicals $U^\pm$ (see [41, II.1.8]) and make the following assumptions

- $0 \notin \Lambda$.
- The Borel subgroup $B^\pm$ stabilizes the subspace $V^\pm$.

We shall consider V as well as $V^\pm$ as algebraic groups, isomorphic to $d^\pm$ copies of the additive group $\mathbb{G}_a$ where $d^\pm := \dim_k V^\pm$. Then we have a decomposition $V_a = V_a^+ \times V_a^-$ ($V_{a(r)} = V_{a(r)}^+ \times V_{a(r)}^-$ for all $r \geq 1$) of algebraic groups. Put

$$U_V^+ := U_{V^+}^+ = V_a^+ \rtimes U^+ \ (U_V^- := U_{V^-}^-) \quad B_V^+ := B_{V^+}^+ = U_V^+ \rtimes T \ (B_V^- := B_{V^-}^- = U_V^- \rtimes T).$$

Then $B_V^+ \subseteq G_V = V_a \rtimes G$ (B_V^-) is a closed, connected solvable algebraic subgroup with unipotent radical U_V^+ (U_V^-).

[3] https://math.stackexchange.com/questions/108165/why-do-torsion-free-abelian-groups-admit-linear-orders

Remark 4.4.3. The group $B_V^{\pm}$ is not a Borel subgroup of G_V. In fact, every Borel of G_V is of the form $B_V = V \rtimes B$ for $B \subseteq G$ being a Borel of G (see Lemma 2.4.36).

Let $r \geq 1$ and denote by $G_{V,r} = (G_V)_r = V_{a(r)} \rtimes G_r$ the rth Frobenius kernel of the group $G_V = V_a \rtimes G$. Note that our torus $T \subseteq G_V$ is still maximal and gives rise to the category $\mathbf{mod}(G_{V,r}T)$ as explained above. General theory (see [41, Lemma II.3.2]) implies that the morphism

$$\psi = \psi_{G,r} : U_r^+ \times T_r \times U_r^- \longrightarrow G_r$$

given by multiplication is an isomorphism of schemes. We now show that we will get a similar decomposition of our group $G_{V,r}$.

Lemma 4.4.4. *Given $r \geq 1$, the morphism $\mu : U_{V,r}^+ \times T_r \times U_{V,r}^- \longrightarrow G_{V,r}$ given by multiplication is an isomorphism of schemes. In particular, we have an isomorphism $kG_{V,r} \cong kU_{V,r}^+ \otimes_k kT_r \otimes_k kU_{V,r}^-$ given by multiplication such that*

$$kU_{V,r}^- \subseteq k.1 + (kG_{V,r})_{<0}, \qquad kT_r \subseteq (kG_{V,r})_0, \qquad kU_{V,r}^+ \subseteq k.1 + (kG_{V,r})_{>0}.$$

Proof. Let $A \in \mathbf{Comm}_k$ and $(v^{\pm}, x^{\pm}) \in U_{V,r}^{\pm}(A) = V^{\pm}(A)_r \rtimes U_r^{\pm}(A), t \in T_r(A)$ be arbitrary. Then we have

$$\mu_A((v^+, x^+), t, (v^-, x^-)) = (v^+ + (x^+ t.v^-), \psi_A(x^+, t, x^-))$$

by definition. Let $(w, g) \in G_{V,r}(A) = V_r(A) \rtimes G_r(A)$ be arbitrary. As ψ_A is surjective, there exist $x^{\pm} \in U_r^{\pm}(A)$ and $t \in T_r(A)$ such that $g = \psi_A(x^+, t, x^-)$. We write $w = w^+ + w^-$ with $w^{\pm} \in V_r^{\pm}(A)$ and put $b := x^+ t \in B_r^+(A)$. Moreover, we write $b^{-1}.w^- = z^+ + z^-$ with $z^{\pm} \in V_r^{\pm}(A)$. It follows that

$$w = w^+ + w^- = w^+ + b.(b^{-1}.w^-) = (w^+ + b.z^+) + b.z^-.$$

Consequently, setting $v^+ := w^+ + b.z^+ \in V_r^+(A), v^- := z^- \in V_r^-(A)$, we get $(w, g) = \mu_A((v^+, x^+), t, (v^-, x^-))$. Hence, μ_A is surjective. Let now $(w^{\pm}, y^{\pm}), (v^{\pm}, x^{\pm}) \in U_{V,r}^{\pm}(A)$ and $s, t \in T_r(A)$ be such that

$$\mu_A((v^+, x^+), t, (v^-, x^-)) = \mu_A((w^+, y^+), s, (w^-, y^-)).$$

As ψ_A is injective, we get $x^{\pm} = y^{\pm}, s = t$. Hence, again for $b := x^+ t = y^+ s \in B_r^+(A)$:

$$v^+ + b.v^- = w^+ + b.w^- \Longrightarrow b^{-1}.v^+ + v^- = b^{-1}.w^+ + w^-,$$

so that $b^{-1}.v^+ = b^{-1}.w^+$ and $v^- = w^-$. As $b^{-1} : V_r(A) \to V_r(A)$ is bijective, we get $v^+ = w^+$. Consequently, μ_A is injective and hence bijective. It follows that μ is an isomorphism of schemes.

As T is commutative, the action of T is trivial on T_r so that $kT_r \subseteq (kG_{V,r})_0$. As the dual space $(V^+)^*$ $((V^-)^*)$ generates the coordinate ring $S((V^+)^*) = k[V^+]$ $(S((V^-)^*) = k[V^-])$ and as T acts via algebra automorphisms, we conclude that all non-zero weights of T on $k[V^+]$ $(k[V^-])$ are negative (positive) and that the zero weight space is $k.1 \subseteq k[V^\pm]$. Consequently, the weights on the dual kV^+ (kV^-) are all positive (negative). As our total order $\leq$ extends $\preccurlyeq$, it also follows that all weights of T on kU_r^+ (kU_r^-) are positive (negative), see [41, II.4.8(2)]. Combining these observations with $U_{V,r}^\pm = V_{a(r)}^\pm \rtimes U_r^\pm$, this finishes our proof. □

Remark 4.4.5. In view of [41, II.1.9], one can adopt the proof of [41, Lemma II.3.2] to get an isomorphism $G_r \cong U_r^- \times T_r \times U_r^+$ of schemes given by multiplication. Our above arguments then also give rise to an isomorphism $kG_{V,r} \cong kU_{V,r}^- \otimes_k kT_r \otimes_k kU_{V,r}^+$.

We have for all $r \geq 1$ an exact sequence

$$0 \longrightarrow p^r X(T) \longrightarrow X(T) \longrightarrow X(T_r) \longrightarrow 0$$

of abelian groups. Denote by Λ_r a full set of representatives for elements of the factor group $X(T)/p^r X(T)$. Since $B_{V,r}^\pm \cong U_{V,r}^\pm \rtimes T_r$, every simple $B_{V,r}^\pm$-module corresponds to some $\lambda \in \Lambda_r$. We put

$$Z_r(\lambda) := kG_{V,r} \otimes_{kB_{V,r}^+} \lambda, \qquad Z_r^-(\lambda) = \mathrm{Hom}_{kB_{V,r}^-}(kG_{V,r}, \lambda).$$

The module $Z_r(\lambda)$ is referred to as a *(baby) Verma module with highest weight* λ. We have

$$\dim_k Z_r(\lambda) = p^{r(d^- + |R^+|)} \qquad \dim_k Z_r^-(\lambda) = p^{r(d^+ + |R^+|)}.$$

Given a character $\lambda' \in X(T)$, we can (in the obvious way) turn the simple T_r-module defined by λ into a $T_r \rtimes T = (T_r \times T)$-module $\hat{\lambda}$ which extends the given T_r-structure. We can define a T-module structure on $Z_r(\lambda)$ and $Z_r^-(\lambda)$ as follows:

$$\begin{aligned} t.(u \otimes \alpha) &:= \mathrm{Ad}(t)(u) \otimes \lambda'(t) \cdot \alpha & \forall t \in T, u \otimes \alpha \in Z_r(\lambda) \\ (t.\varphi)(u) &:= \lambda'(t) \cdot \varphi(\mathrm{Ad}(t^{-1})(u)) & \forall t \in T, \varphi \in Z_r^-(\lambda), u \in kG_{V,r}. \end{aligned}$$

These T-structures are clearly induced by the T-structures on $kG_r \otimes_k \lambda'$ and $\mathrm{Hom}_k(kG_r, \lambda')$, $Z_r(\lambda)$ is a quotient of the first and $Z_r^-(\lambda)$ a submodule of the latter. Using that T acts on kG_r via automorphisms, a direct computation shows that this in fact renders $Z_r(\lambda)$ as well as $Z_r^-(\lambda)$ being $G_{V,r} \rtimes T$-modules, denoted by $\hat{Z}_r(\hat{\lambda})$ and $\hat{Z}_r^-(\hat{\lambda})$, respectively. Then

$$\hat{Z}_r(\hat{\lambda}) \in \mathbf{mod}(G_{V,r}T) \Longleftrightarrow \hat{\lambda} \in \mathbf{mod}(T_r T) \Longleftrightarrow \lambda'|_{T_r} = \lambda$$

The second equivalence is clear, while the non-trivial direction of the first follows from [41, Proposition I.8.20], which then even implies that $Z_r(\lambda)$ has the structure of a $(G_{V,r}B_V^+)$-module and $Z_r^-(\lambda)$ that of a $(G_{V,r}B_V^-)$-module. So, given $\lambda \in X(T)$, we denote the $G_{V,r}T$-modules introduced above by $\hat{Z}_r(\lambda)$ and $\hat{Z}_r^-(\lambda)$, respectively. By definition, we have

$$\mathfrak{F}(\hat{Z}_r(\lambda)) = Z_r(\lambda), \ \ \mathfrak{F}(\hat{Z}_r^-(\lambda)) = Z_r^-(\lambda),$$

where $\mathfrak{F}$ denotes the restriction to $G_{V,r}$. We say that $M \in \mathbf{mod}(G_{V,r}T)$ ($\mathbf{mod}(G_{V,r})$) has a *$\hat{Z}$-filtration (Z-filtration)*, provided there exists a filtration of M whose filtration factors are isomorphic to various $\hat{Z}_r(\lambda)$ ($Z_r(\lambda)$). Below, we will see that the number of times a Verma module $\hat{Z}_r(\lambda)$ ($Z_r(\lambda)$) occurs up to isomorphism as a filtration factor does not depend on the choice of the filtration.

Theorem 4.4.6. *Let V be a rational representation of a reductive algebraic group G and $T \subseteq G$ a maximal torus. Let $\Lambda_r \subseteq X(T)$ be a full set of representatives for elements of the factor group $X(T)/p^rX(T)$ and denote by $\Lambda_V \subseteq X(T)$ the set of weights of T on V. Assume that $0 \notin \Lambda_V$ and that the Borel subgroup B^+ (B^-) stabilizes the subspace $V^+ = \bigoplus_{\lambda\in\Lambda_V,\lambda>0} V_\lambda$ ($V^- = \bigoplus_{\lambda\in\Lambda_V,\lambda>0} V_\lambda$) of V.*

(1) The following statements hold:

- *$\{\hat{S}_r(\lambda) := \mathrm{Top}(\hat{Z}_r(\lambda)) : \lambda \in X(T)\}$ is a full set of representatives for the iso-classes of simple $(G_{V,r}T)$-modules.*
- *$M \in \mathbf{mod}(G_{V,r}T)$ has a $\hat{Z}$-filtration if and only if the restriction of M to $B_{V,r}^-$ is projective.*
- *If M has a $\hat{Z}$-filtration, then*

$$[M : \hat{Z}_r(\lambda)] = \dim_k \mathrm{Hom}_{G_{V,r}T}(M, \hat{Z}_r^-(\lambda)).$$

(2) The following statements hold:

- *$\{S_r(\lambda) := \mathrm{Top}(Z_r(\lambda)) : \lambda \in \Lambda_r\}$ is a full set of representatives for the iso-classes of simple $G_{V,r}$-modules.*
- *If $M \in \mathbf{mod}(G_{V,r})$ has a Z-filtration, then the restriction of M to $B_{V,r}^-$ is projective. If there exists a $(G_{V,r}T)$-module $\hat{M}$ such that $\mathfrak{F}(\hat{M}) \cong M$, then the converse holds.*
- *If M has a Z-filtration, then*

$$[M : Z_r(\lambda)] = \dim_k \mathrm{Hom}_{G_{V,r}}(M, Z_r^-(\lambda)).$$

(3) Every projective indecomposable $G_{V,r}T$-module $\hat{Q}$ with top $\hat{S}$ ($G_{V,r}$-module Q with

top S) admits a $\hat{Z}$-filtration (Z-filtration) and we have $[\hat{Q} : \hat{Z}_r(\lambda)] = [\hat{Z}_r^-(\lambda) : \hat{S}]$ ($[Q : Z_r(\lambda)] = [Z_r^-(\lambda) : S]$).

Proof. As $kU^{\pm}_{V,r} \cdot kT_r = k(U^{\pm}_{V,r} \rtimes T_r) = kT_r \cdot kU^{\pm}_r$ (also the multiplication $T_r \times U^{\pm}_r \to U^{\pm}_r \rtimes T_r,\ (t,u) \mapsto (t,1) * (1,u) = (t.u,t)$ is an isomorphism of schemes), it follows from [41, I.9.6(2)] that the triangular decomposition 4.4.4 satisfies the requirements of [37, 2.1]. Recalling that $\mathbf{mod}(G_{V,r}T)$ is a sum of blocks of $\mathbf{mod}(G_{V,r} \rtimes T)$, the assertion thus follows from our above discussion and the results [37, Lemma 3.2, Theorem 4.4/5] (they are formulated for $\mathbb{Z}$-graded algebras, but also generalize to $\mathbb{Z}^n$-graded algebras). □

The Verma modules of G_r are defined analogously. Given $\lambda \in X(T)$, we put

$$Z_{G_r}(\lambda) := kG_r \otimes_{kB_r^+} \lambda,\ \ Z^-_{G_r}(\lambda) = \mathrm{Hom}_{B_r^-}(kG_r, \lambda).$$

Both modules also have structures of G_rT-modules, which we indicate by using the symbol $\hat{Z}$ and in analogy to the above, one defines simple G_rT-modules $\hat{L}_r(\lambda) := \mathrm{Top}(\hat{Z}_{G_r}(\lambda))$. Then we have $\hat{L}_r(\lambda) \cong \hat{S}_r(\lambda)|_{G_rT}$ since $V_{a(r)}$ is unipotent. The meaning of a G_rT-module having a $\hat{Z}_{G_r}$-filtration or a G_r-module having a Z_{G_r}-filtration is then the obvious one and we can use the results [41, Proposition II.11.2/4] by Jantzen.

Lemma 4.4.7. *Let $\lambda \in X(T)$, then the restriction of $\hat{Z}_r(\lambda)$ to G_rT admits a $\hat{Z}_{G_r}$-filtration and $[\hat{Z}_r(\lambda) : \hat{Z}_{G_r}(\mu)]$ equals the multiplicity $[kV^-_{a(r)} : \mu - \lambda]$ of $\mu - \lambda$ inside the T-module $kV^-_{a(r)}$ for all $\mu \in X(T)$.*

Proof. By [37, Lemma 4.1], the restriction of $\hat{Z}_r(\lambda)$ to $B^-_{V,r}T$ is the projective cover of the simple $(B^-_{V,r}T)$-module λ. Thus, the restriction of $\hat{Z}_r(\lambda)$ to the closed subgroup B_r^-T of $B^-_{V,r}T$ is projective as well and [41, Proposition II.11.2] implies that $\hat{Z}_r(\lambda)$ admits a $\hat{Z}_{G_r}$-filtration such that

$$[\hat{Z}_r(\lambda) : \hat{Z}_{G_r}(\mu)] = \dim_k \mathrm{Hom}_{G_rT}(\hat{Z}_r(\lambda), \hat{Z}^-_{G_r}(\mu)) = \dim_k \mathrm{Hom}_{B_r^-T}(\hat{Z}_r(\lambda), \mu)$$

for all $\mu \in X(T)$. Next, we note that $\hat{Z}_r(\lambda) \cong k(U^-_{V,r}) \otimes_k \lambda$ with T acting via the tensor product and the action of $B^-_{V,r} = U^-_{V,r} \rtimes T_r$ is the one given by 2.4.29 (see also 4.1.4(2b)). Since the multiplication $U_r^- \times V^-_{a(r)} \to U^-_{V,r}$ is an isomorphism of schemes, we get an isomorphism of vector spaces $kU^-_{V,r} \cong kU_r^- \otimes_k kV^-_{a(r)}$ given by multiplication. Since T acts via automorphisms of Hopf algebras on $kU^-_{V,r}$, it follows that

$$kU^-_{V,r} \otimes_k \lambda \cong kU_r^- \otimes_k (kV^-_{a(r)} \otimes_k \lambda)$$

as $(B^-_{V,r}T)$-modules, with T acting via tensor product and the action of $B_r^- = U_r^- \rtimes T_r$ given by 2.4.29. Since the projective cover of the simple B_r^-T-module $\mu \in X(T)$ is isomorphic to $kU_r^- \otimes_k \mu$ (follows by using similar arguments as above), the assertion follows by combining all our observations. □

Let W be the *Weyl group* associated to R and denote by $w_0 \in W$ the unique element that sends R^+ to $-R^+$. The dot action of W on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, defined by means of $w \bullet \lambda := w(\lambda + \rho) - \rho$, maps (the image of) $X(T)$ onto itself.

Lemma 4.4.8. *Put* $\rho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ *and* $\rho^{\pm} := \sum_{\mu \in \Lambda^{\pm}} \mu \in X(T)$, *then the following statements hold for all* $\lambda \in X(T)$*:*

(1) We have isomorphisms of $G_{V,r}T$*-modules*

$$\hat{Z}_r(\lambda) \cong \operatorname{Hom}_{kB^+_{V,r}}(kG_{V,r}, \lambda + (p^r - 1)(\rho^- - 2\rho))$$
$$\hat{Z}_r^-(\lambda) \cong kG_{V,r} \otimes_{kB^-_{V,r}} (\lambda + (p^r - 1)(\rho^+ - 2\rho)).$$

(2) $Z_r(\lambda)$ *has simple socle* $S_r(\lambda')$*, where* $\lambda' \in \Lambda_r$ *represents* $w_0 \bullet (\lambda + (p^r - 1)\rho^-)$.

(3) $\hat{Z}_r^-(\lambda)$ *has simple socle* $\hat{S}_r(\lambda)$ *and* $Z_r^-(\lambda)$ *has simple top* $S_r(\lambda')$*, where* $\lambda' \in \Lambda_r$ *represents* $w_0 \bullet (\lambda + (p^r - 1)\rho^+)$.

Proof. (1) In view of [41, Proposition I.8.20], we have isomorphisms of $G_{V,r}T$-modules

$$\hat{Z}_r(\lambda) \cong \operatorname{Hom}_{kB^+_{V,r}}(kG_{V,r}, \lambda + \chi|_T - \chi')$$

where $\chi \in X(G_V)$ resp. χ' is the character through which G_V resp. T acts on the space of left integrals of the Hopf algebra $kG_{V,r}$ resp. $kB^+_{V,r}$. Combining [41, Proposition I.8.19, II.3.4] and Lemma 2.5.29, we have $\chi|_T = (p^r - 1)\sum_{\mu \in \Lambda_V} \mu$ and $\chi' = (p^r - 1)(\sum_{\mu \in \Lambda^+} \mu + 2\rho)$, as desired. The assertion concerning $\hat{Z}_r^-(\lambda)$ follows similarly.

(2) Let $\mu \in \Lambda_r$ be another weight. Our above alternative description of $Z_r(\lambda) = \mathfrak{F}(\hat{Z}_r(\lambda))$ yields

$$\begin{aligned}
\operatorname{Hom}_{G_{V,r}}(S_r(\mu), Z_r(\lambda)) &\cong \operatorname{Hom}_{G_{V,r}}(S_r(\mu), \operatorname{Hom}_{kB^+_{V,r}}(kG_{V,r}, \lambda + (p^r - 1)(\rho^- - 2\rho))) \\
&\cong \operatorname{Hom}_{B^+_{V,r}}(S_r(\mu), \lambda + (p^r - 1)(\rho^- - 2\rho)) \\
&\cong \operatorname{Hom}_{B^+_r}(S_r(\mu), \lambda + (p^r - 1)(\rho^- - 2\rho)) \;\; (V^+_{a(r)} \text{ acts trivially}) \\
&\cong \operatorname{Hom}_{G_r}(L_r(\mu), \operatorname{Hom}_{kB^+_r}(kG_r, \lambda + (p^r - 1)(\rho^- - 2\rho)) \\
&\cong \operatorname{Hom}_{G_r}(L_r(\mu), Z_{G_r}(\lambda + (p^r - 1)\rho^-)) \;\; \text{by [41, II.3.5(2)]}.
\end{aligned}$$

As $Z_{G_r}(\lambda)$ is known to have simple socle $L_r(w_0 \bullet \lambda)$ (see [41, Corollary II.3.12]), we get (2).

(3) Graded Frobenius reciprocity (see also [40, 1.11(3)(4)]) implies:

$$\begin{aligned}\mathrm{Hom}_{G_{V,r}T}(\hat{S}_r(\mu), \hat{Z}_r^-(\lambda)) &\cong \mathrm{Hom}_{B_{V,r}^-T}(\hat{S}_r(\mu), \lambda)\\ &\cong \mathrm{Hom}_{B_r^-T}(\hat{L}_r(\mu), \lambda) \qquad (V_{a(r)}^- \text{ acts trivially})\\ &\cong \mathrm{Hom}_{G_rT}(\hat{L}_r(\mu), \hat{Z}_{G_r}^-(\lambda)).\end{aligned}$$

As $\hat{Z}_{G_r}^-(\lambda)$ has simple socle $\hat{L}_r(\lambda)$ (see [41, Proposition II.9.6]), the first assertion follows. For the second, our above alternative description of $Z_r^-(\lambda) = \mathfrak{F}(\hat{Z}_r^-(\lambda))$ yields for all $\mu \in \Lambda_r$

$$\begin{aligned}\mathrm{Hom}_{G_{V,r}}(Z_r^-(\lambda), S_r(\mu)) &\cong \mathrm{Hom}_{G_{V,r}}(kG_{V,r} \otimes_{kB_{V,r}^-} (\lambda + (p^r-1)(\rho^+ - 2\rho)), S_r(\mu))\\ &\cong \mathrm{Hom}_{B_{V,r}^-}(\lambda + (p^r-1)(\rho^+ - 2\rho), S_r(\mu))\\ &\cong \mathrm{Hom}_{B_r^-}(\lambda + (p^r-1)(\rho^+ - 2\rho)), L_r(\mu)) \;\; (V_{a(r)}^- \text{ acts trivially})\\ &\cong \mathrm{Hom}_{G_r}(kG_r \otimes_{kB_r^-} (\lambda + (p^r-1)(\rho^+ - 2\rho)), L_r(\mu))\\ &\cong \mathrm{Hom}_{G_r}(Z_{G_r}(\lambda + (p^r-1)\rho^+), L_r(\mu)) \text{ by [41, II.3.5(2)]}.\end{aligned}$$

As $Z_{G_r}^-(\lambda)$ is known to have simple top $L_r(w_0 \bullet \lambda)$ (see [41, II.3.12]), we get (3).

□

Remark 4.4.9. In view of [41, II.9.7], we could also describe the socle of $\hat{Z}_r(\lambda)$ (top of $\hat{Z}_r^-(\lambda)$) with the above methods. Since we do not really need this later and this involves further notation, we won't go into this.

As an example, we consider the reductive group $\mathrm{GL}(n)$ along with its maximal torus T of diagonal matrices whose character group $X(T)$ is identified with the free abelian group $\mathbb{Z}^n$. Denoting by $e_i \in \mathbb{Z}^n$ the i-th unit vector, the corresponding set of simple roots is $S = \{e_i - e_{i+1} : 1 \le i \le n-1\}$. The set of weights of T on the natural representation $V = k^n$ is $\{e_i : 1 \le i \le n\}$. If we consider the group $\mathrm{SL}(n)$ along with its standard torus $T' := T \cap \mathrm{SL}(n)$, whose character group $X(T')$ we shall identifiy with $\mathbb{Z}^{n-1}$, the restriction map corresponding to the inclusion $T' \subseteq T$ is given by

$$\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1},\ (x_1, \dots, x_n) \mapsto (x_1 - x_n, \dots, x_{n-1} - x_n)$$

Then $e_i' := \pi(e_i)$ for $1 \le i \le n-1$ is a basis of $X(T')$, $\pi(R)$ is the set of roots and $\pi(S) = \{e_i' - e_{i+1}' : 1 \le i \le n-2\} \cup \{v := (1, \dots, 1, 2)\}$ is a set of simple roots, and e_i' for $1 \le i \le n-1$ as well as $\pi(e_n) = (-1, \dots, -1) \in \mathbb{Z}^{n-1}$ are the weights of T' on k^n (see [41, II.1.21] for more details).

Lemma 4.4.10. *Let $n \ge 2$. With the notation of the previous paragraph, the following statements hold:*

(a) *Given $k \in \{1, \dots, n-1\}$, there exists a total $\leq$ order on $X(T)$, that extends the partial order $\preccurlyeq$ and $e_i \begin{cases} > 0, & 1 \leq i \leq k \\ < 0, & k+1 \leq i \leq n \end{cases}$. Moreover, $(X(T), \leq)$ is a totally ordered group.*

(b) *Given $k \in \{1, \dots, n-1\}$, there exists a total $\leq$ order on $X(T')$, that extends the partial order $\preccurlyeq$ such that $e_i' \begin{cases} > 0, & 1 \leq i \leq k \\ < 0, & k+1 \leq i \leq n \end{cases}$ and $v < 0$. Moreover, $(X(T'), \leq)$ is a totally ordered group.*

Proof. (a) If we can show that the submonoid

$$P := \sum_{i=1}^{k} \mathbb{N}_0 e_i + \sum_{i=k+1}^{n} \mathbb{N}_0(-e_i) + \sum_{i=1}^{n-1} \mathbb{N}_0(e_i - e_{i+1})$$

is pointed, our claim will follow from the discussion before by taking a maximal pointed submonoid containing P. As $e_i = (e_i - e_{i+1}) + e_{i+1}$ for $1 \leq i \leq k-1$ and $-e_{i+1} = (e_i - e_{i+1}) - e_i$ for $k+1 \leq i \leq n-1$, it follows that

$$P = \mathbb{N}_0 e_k + \mathbb{N}_0(-e_{k+1}) + \sum_{i=1, i \neq k}^{n-1} \mathbb{N}_0(e_i - e_{i+1})$$

Let $x \in P$, then there exist integral coefficients $a, b, \alpha_i \in \mathbb{N}_0$ such that

$$x = (\alpha_1, \alpha_2 - \alpha_1, \dots, a - \alpha_{k-1}, \alpha_{k+1} - b, \alpha_{k+2} - \alpha_{k+1}, \dots, -\alpha_{n-1}).$$

If $-x \in P$, then there are integral coefficients $c, d, \beta_i \in \mathbb{N}_0$ such that

$$\begin{aligned} -x &= (\beta_1, \beta_2 - \beta_1, \dots, c - \beta_{k-1}, \beta_{k+1} - d, \beta_{k+2} - \beta_{k+1}, \dots, -\beta_{n-1}) \\ &= (-\alpha_1, -\alpha_2 + \alpha_1, \dots, -a + \alpha_{k-1}, -\alpha_{k+1} + b, -\alpha_{k+2} + \alpha_{k+1}, \dots, \alpha_{n-1}). \end{aligned}$$

As β_{n-1} and α_{n-1} are natural numbers, it follows that they must be zero. Then

$$\alpha_{n-2} = -\alpha_{n-1} + \alpha_{n-2} = \beta_{n-1} - \beta_{n-2} = -\beta_{n-2},$$

so that the same argument gives $0 = \alpha_{n-2} = \beta_{n-2}$. Continuing in this fashion, we get $0 = \alpha_i = \beta_i$ for $k+1 \leq i \leq n-1$ and then also $0 = b = d$. If now $k = 1$, we get $0 = a = c$ with the same argument again. Otherwise, we get $0 = \alpha_1 = \beta_1$ and successively $0 = \alpha_i = \beta_i$ for $1 \leq i \leq k-1$, $0 = a = c$. Thus, $x = 0$ in any case. It follows that P is pointed.

(b) If $n = 2$, then $X(T') = \mathbb{Z}$ and the natural order on $\mathbb{Z}$ is the desired total order. Let

$n \geq 3$. As above, it suffices to show that the submonoid

$$P' := \mathbb{N}_0 e'_k + \mathbb{N}_0(-e'_{k+1}) + \sum_{i=1, i \neq k}^{n-2} \mathbb{N}_0(e'_i - e'_{i+1}) + \mathbb{N}_0 v$$

of $X(T')$ is pointed. If $x \in P'$ and $-x \in P'$, it follows that integral coefficients corresponding to v in expressions to form x and $-x$ (as in a)) must be zero. But then it follows as in a), that $x = 0$. Thus, P' is pointed.

□

For both $G = \mathrm{GL}(n), \mathrm{SL}(n)$ and any choice of total order on $X(T), X(T')$ as above, the Borel subgroup of upper (lower) triangular matrices clearly stabilizes the subspace V^+ (V^-) and 0 is not a weight of T, T' on $V = k^n$.

5 Applications

Throughout this chapter, we assume that k is an algebraically closed field of characteristic $\mathrm{char}(k) = p \geq 3$.

5.1 Some semidirect products with the (three-dimensional) Heisenberg group

We begin with determining the restricted derivations of the three-dimensional p-trivial Heisenberg algebra $\mathfrak{h}_3$. Considering its standard basis r, s, z, the bracket is determined by $[r, s] = z,\ [z, \mathfrak{h}_3] = (0)$ and the p-map is identically zero. This implies that every ordinary derivation of the Lie algebra $\mathfrak{h}_3$ is already restricted and we shall be concerned with the determination of the former. Using the standard basis, we shall interpret every linear map $\mathfrak{h}_3 \to \mathfrak{h}_3$ as a (3×3)-matrix and give a criterion depending on that matrix, to decide whether such a map is a derivation.

Lemma 5.1.1. *Let $D : \mathfrak{h}_3 \to \mathfrak{h}_3$ be a linear map. Then D is a derivation if and only if D is of the form*

$$\begin{pmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & d_{11} + d_{22} \end{pmatrix}$$

for coefficients $d_{ij} \in k$ $(1 \leq i \leq 3, 1 \leq j \leq 2)$. In particular, $\dim_k \mathrm{Der}(\mathfrak{h}_3) = 6$ *and if $\mathfrak{g} \subseteq \mathfrak{gl}(2)$ is a restricted subalgebra, then*

$$\mathfrak{g} \to \mathrm{Der}_p(\mathfrak{h}_3),\ x \mapsto \begin{pmatrix} x & 0 \\ 0 & \mathrm{tr}(x) \end{pmatrix}$$

is a homomorphism of restricted Lie algebras.

Proof. If $D = (d_{ij})$ is a derivation, then

$$\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \end{pmatrix} = D(z) = D([r, s]) = [D(r), s] + [r, D(s)] = \begin{pmatrix} 0 \\ 0 \\ d_{11} + d_{22} \end{pmatrix}$$

Thus, D is of the desired form. We finish the proof by showing that any linear map $D : \mathfrak{h}_3 \to \mathfrak{h}_3$ of the mentioned form is in fact a derivation. Therefore, we let $v =$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}, w = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathfrak{h}_3$ be arbitrary. Then

$$\begin{aligned}
D([v,w]) = D \begin{pmatrix} 0 \\ 0 \\ a\beta - b\alpha \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ (d_{22}+d_{11})(a\beta - b\alpha) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ d_{22}a\beta - d_{22}b\alpha + d_{11}a\beta - d_{11}b\alpha \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ a\beta d_{11} + b\beta d_{12} - a\alpha d_{21} - b\alpha d_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a\alpha d_{21} + a\beta d_{22} - b\alpha d_{11} - b\beta d_{12} \end{pmatrix} \\
&= [D(v), w] + [D(w), v].
\end{aligned}$$

The additional statement can now be obtained from the fact that the natural representation of $\mathfrak{gl}(2)$ on $V = k^2$ as well as the one-dimensional defined by the trace, are restricted representations of $\mathfrak{gl}(2)$ (and hence for any restricted subalgebra). $\square$

The lemma shows that we can form semidirect products such as $\mathfrak{h}_3 \rtimes \mathfrak{gl}(2)$ or $\mathfrak{h}_3 \rtimes \mathfrak{sl}(2)$.

We now consider the case where some arbitrary restricted Lie algebra $\mathfrak{g}$ acts on the three-dimensional p-trivial Heisenberg algebra $\mathfrak{h}_3$ by restricted derivations and want to determine the p-map on the semidirect product $\mathfrak{h}_3 \rtimes \mathfrak{g}$. Clearly $\mathfrak{g}$ stabilizes the center $[\mathfrak{h}_3, \mathfrak{h}_3] = kz$ of $\mathfrak{h}_3$, we let $\lambda : \mathfrak{g} \to k$ be the corresponding linear form such that $x.z = \lambda(x)z$ for all $x \in \mathfrak{g}$.

Lemma 5.1.2. *Assume that a restricted Lie algebra $\mathfrak{g}$ acts on the Heisenberg algebra $\mathfrak{h}_3$ via restricted derivations. The unique p-map on $\mathfrak{h}_3 \rtimes \mathfrak{g}$, such that (the images of) $\mathfrak{h}_3$ and $\mathfrak{g}$ are restricted subalgebras, is given by*

$$(x,y)^{[p]} = \left(\frac{1}{2} \sum_{j=0}^{p-3} \lambda(y)^j [x, y^{p-2-j}.x] + y^{p-1}.x \ , \ y^{[p]_{\mathfrak{g}}} \right) \quad \forall\, x \in \mathfrak{h}_3, y \in \mathfrak{g}$$

Proof. Let $x \in \mathfrak{h}_3$ and $y \in \mathfrak{g}$. In view of 2.5.18(1), we need to determine the coefficients $s_i(x,y)$ for $1 \le i \le p-1$, which are defined by the equation

$$\mathrm{ad}(x \otimes T + y \otimes 1)^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i \cdot s_i(x,y) \otimes T^{i-1}$$

inside the Lie algebra $(\mathfrak{h}_3 \rtimes \mathfrak{g}) \otimes_k k[T]$. Therefore we will show inductively that

$$\mathrm{ad}(x \otimes T + y \otimes 1)^\ell (x \otimes 1) = \sum_{j=0}^{\ell-2} \lambda(y)^j [x, y^{\ell-1-j}.x] \otimes T + y^\ell.x \otimes 1 \ \ \forall\, \ell \in \mathbb{N}.$$

Our claim then follows by substituting $\ell = p-1$. The case $\ell = 1$ is trivial. Let $\ell \in \mathbb{N}$ such that the above formula holds. Then

$$\begin{aligned}
\mathrm{ad}(x \otimes T + y \otimes 1)^{\ell+1}(x \otimes 1) &= \mathrm{ad}(x \otimes T + y \otimes 1)(\mathrm{ad}(x \otimes T + y \otimes 1)^\ell (x \otimes 1)) \\
&= \mathrm{ad}(x \otimes T + y \otimes 1)(\sum_{j=0}^{\ell-2} \lambda(y)^j \underbrace{[x, y^{\ell-1-j}.x]}_{\in kz} \otimes T + y^\ell.x \otimes 1) \\
&= [x, y^\ell.x] \otimes T + \sum_{j=0}^{\ell-2} \lambda(y)^{j+1} [x, y^{\ell-1-j}.x] \otimes T + y^{\ell+1}.x \otimes 1 \\
&= [x, y^\ell.x] \otimes T + \sum_{j=1}^{\ell-1} \lambda(y)^{j} [x, y^{\ell-j}.x] \otimes T + y^{\ell+1}.x \otimes 1 \\
&= \sum_{j=0}^{\ell-1} \lambda(y)^{j} [x, y^{\ell-j}.x] \otimes T + y^{\ell}.x \otimes 1 + y^{\ell+1}.x \otimes 1,
\end{aligned}$$

as desired. □

It is always useful to know whether a restricted Lie algebra is *algebraic*, i.e. whether it is the Lie algebra of an algebraic group. We let $H \subseteq \mathrm{SL}(3)$ be the group of unitriangular (3×3)-matrices, the *Heisenberg group*. Its Lie algebra, consisting of strictly upper triangular (3×3)-matrices, is isomorphic to the Heisenberg algebra $\mathfrak{h}_3$. For our purposes, it will be convenient to use another realization of the algebraic group H: The affine variety $\mathbb{A}^2 \times \mathbb{A}$ obtains the structure of an algebraic group by setting

$$(*) \ \ (v, c) \cdot (w, d) := (v + w, c + d + \det(v, w)), \ \ \forall (v, c), (w, d) \in \mathbb{A}^2 \times \mathbb{A}.$$

Remark 5.1.3. We note at this point, that the multiplication used in the literature is $(v, c) \cdot (w, d) := (v + w, c + d + \frac{1}{2} \cdot \det(v, w))$, but the mapping $(v, c) \mapsto (v, 2c)$ defines an isomorphism between these structures (here we use $p = \mathrm{char}(k) \neq 2$).

Then the following map defines an isomorphism between these structures.

$$(v, c) \mapsto \begin{pmatrix} 1 & v_1 & \frac{1}{2}(c + v_1 v_2) \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix}$$

As noted above, we will from now on let H be the group defined by $(*)$.

Lemma 5.1.4. *The group* $\mathrm{GL}(2)$ *acts on the Heisenberg group* H *via*

$$\mathrm{GL}(2) \times H \to H, (g, (v, c)) \mapsto (gv, \det(g).c).$$

If $\mathfrak{g} \subseteq \mathfrak{gl}(2)$ *is the Lie algebra of some closed subgroup* $G \subseteq \mathrm{GL}(2)$*, then the restricted Lie algebra* $\mathfrak{h}_3 \rtimes \mathfrak{g}$ *defined via the action of 5.1.1 is the Lie algebra of the semidirect product* $H \rtimes G$*, formed with respect to the above action of* G *on* H*.*

Proof. This follows from direct computation and 2.5.23, 2.4.32. □

We can also form semidirect products of the form $\mathbb{G}_a^2 \rtimes G$ for closed subgroups $G \subseteq \mathrm{GL}(2)$, where we let G act on $\mathbb{G}_a^2$ naturally. Throughout this chapter, we put

$$\begin{aligned} \mathcal{G} &:= H \rtimes \mathrm{GL}(2),\ \overline{\mathcal{G}} := \mathbb{G}_a^2 \rtimes \mathrm{GL}(2), & \mathcal{S} &:= H \rtimes \mathrm{SL}(2),\ \overline{\mathcal{S}} := \mathbb{G}_a^2 \rtimes \mathrm{SL}(2) \\ \mathfrak{g} &:= \mathrm{Lie}(\mathcal{G}),\ \overline{\mathfrak{g}} := \mathrm{Lie}(\overline{\mathcal{G}}), & \mathfrak{s} &:= \mathrm{Lie}(\mathcal{S}),\ \overline{\mathfrak{s}} := \mathrm{Lie}(\overline{\mathcal{S}}). \end{aligned}$$

The group $\mathcal{S}$ (considered over the field of complex numbers $\mathbb{C}$) originates from physics. It is called the *Schrödinger group*, its Lie algebra $\mathfrak{s}$ is called the *Schrödinger algebra.* In number theory, this group (considered over the reals $\mathbb{R}$) is also called the *Jacobi group* (see [3, Abschnitt 8.5]).

Lemma 5.1.5. *Assume that a restricted Lie algebra* $\mathfrak{g}$ *acts on the Heisenberg algebra* $\mathfrak{h}_3$ *via restricted derivations. Then the following statements hold:*

(1) We have

$$V(\mathfrak{h}_3 \rtimes \mathfrak{g}) = \begin{cases} \{(x, y) \in \mathfrak{h}_3 \rtimes \mathfrak{g} \mid y \in V(\mathfrak{g}) \text{ and } 0 = \frac{1}{2}[x, y.x] + y^2.x\} & p = 3 \\ \mathfrak{h}_3 \times V(\mathfrak{g}). & p > 3 \end{cases}$$

(2) If $\mathfrak{g} \subseteq \mathfrak{gl}(2)$ *is a restricted subalgebra, then either* $V(\mathfrak{g}) = V(\mathfrak{sl}(2))$ *is an irreducible two-dimensional variety or there exists* $x \in V(\mathfrak{sl}(2))$ *such that* $V(\mathfrak{g}) = kx$*.*

(3) Let $p = 3$*. If* $\mathfrak{g} \subseteq \mathfrak{gl}(2)$ *acts on* $\mathfrak{h}_3$ *as in 5.1.1, then*

$$V(\mathfrak{h}_3 \rtimes \mathfrak{g}) = \{(x, y) \in \mathfrak{h}_3 \rtimes \mathfrak{g} \mid y \in V(\mathfrak{g}) \text{ and } 0 = [x, y.x]\}$$

equals $\mathfrak{h}_3$ *if and only if* $\mathfrak{g}$ *is a torus and otherwise it is an equidimensional variety of dimension* $\dim V(\mathfrak{g}) + 2$*.*

Proof. (1) By the foregoing lemma, we have

$$\mathcal{V} := V(\mathfrak{h}_3 \rtimes \mathfrak{g}) = \{(x, y) \in \mathfrak{h}_3 \rtimes \mathfrak{g} \mid y \in V(\mathfrak{g}) \text{ and } 0 = \frac{1}{2} \sum_{j=0}^{p-3} \lambda(y)^j [x, y^{p-2-j}.x] + y^{p-1}.x\}.$$

As $\mathfrak{h}_3$ is a restricted $\mathfrak{g}$-module, we have $\lambda|_{\mathcal{V}(\mathfrak{g})} \equiv 0$. By the same token, $y \in V(\mathfrak{g})$ acts p-nilpotently on the three-dimensional space $\mathfrak{h}_3$. Consequently, $y^n.x = 0$ for all $n \geq 3$ and all $x \in \mathfrak{h}_3$ and hence 1).

(2) The variety $V(\mathfrak{gl}(2)) = V(\mathfrak{sl}(2)) = \{(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}) \mid a^2+bc = 0\}$ is a hypersurface inside the three-dimensional variety $\mathfrak{sl}(2)$. As the defining polynomial is irreducible, it follows that $V(\mathfrak{sl}(2))$ is an irreducible, two-dimensional variety. As $V(\mathfrak{g}) \subseteq V(\mathfrak{gl}(2))$, the assumption $\dim V(\mathfrak{g}) = 2$ forces $V(\mathfrak{g}) = V(\mathfrak{sl}(2))$. If $\dim V(\mathfrak{g}) = 1$ then there exists $x \in V(\mathfrak{sl}(2))$ such that $V(\mathfrak{g}) = kx$ is a line (2.5.7(4)). Finally, $\dim V(\mathfrak{g}) = 0$ forces $x = 0$.

(3) As the $\mathfrak{gl}(2)$-module $\mathfrak{h}_3 = k^2 \oplus k_{\mathrm{tr}}$ is a direct sum of the natural representation k^2 and the one-dimensional defined by the trace, the same argument as above shows $y^n.x = 0$ for all $n \geq 2$ and all $x \in \mathfrak{h}_3$ and $y \in V(\mathfrak{g})$. Hence $\mathcal{V} := V(\mathfrak{h}_3 \rtimes \mathfrak{g})$ is precisely the fiber at 0 of the morphism of (in view of 2)) irreducible varieties

$$\psi : \mathfrak{h}_3 \times V(\mathfrak{g}) \to kz,\ (x, y) \mapsto [x, y.x].$$

We want to show that ψ is dominant if and only if $\mathfrak{g}$ is not a torus. If $\mathfrak{g}$ is a torus, then $V(\mathfrak{g}) = \{0\}$ and ψ is the zero map, clearly not dominant. Assume that $\mathfrak{g}$ is not a torus. In view of (2), we have to look at the following two cases

- If $V(\mathfrak{g}) = V(\mathfrak{sl}(2))$, then ψ is surjective; we have $\alpha z = \psi(y, -\alpha e)$ for any $\alpha \in k$.
- If $V(\mathfrak{g}) = kx$ for some $x \in V(\mathfrak{sl}(2)) \setminus \{0\}$, then there exists $g \in \mathrm{GL}(2)$ such that $x = g.e = geg^{-1}$ (Jordan canonical form). It follows that

$$\begin{aligned} \psi(g.s, x) &= [g.s, (g.e).(g.s)] = [g.s, g.(e.s)] && 2.4.32(2) \\ &= [g.s, g.r] = (g.[s, r]) = -g.z = -\det(g).z \neq 0. \end{aligned}$$

  It is now easy to conclude the surjectivity of ψ.

Our claim follows, since surjective morphisms are (obviously) dominant. Let $\mathfrak{g}$ not be a torus. By the above, standard results on fiber dimensions (see the lemma below) yield for an arbitrary irreducible component $Z \subseteq \mathcal{V}$:

$$\dim Z \geq \dim\,(\mathfrak{h}_3 \times \mathcal{V}(\mathfrak{g})) - \dim\ kz = \dim \mathcal{V}(\mathfrak{g}) + 2$$

As ψ is not identically zero, we have $\mathcal{V} \neq \mathfrak{h}_3 \times \mathcal{V}(\mathfrak{g})$. Hence Z is a proper, closed subset of the $(\dim V(\mathfrak{g}) + 3)$-dimensional irreducible variety $\mathfrak{h}_3 \times V(\mathfrak{g})$. It follows that $\dim Z = \dim V(\mathfrak{g}) + 2$.

□

Lemma 5.1.6. *Let $\varphi : V \to W$ be a dominant morphism of irreducible affine varieties* *and put* $r := \dim V - \dim W$. *Then we have* $\dim Z \geq r$ *for every irreducible component* $Z \subseteq \varphi^{-1}(\{y\})$ *and all* $y \in \varphi(V)$.

For the following, one may recall 2.5.13.

Lemma 5.1.7. *The variety* $\mathbb{E}(2, \bar{\mathfrak{s}})$ *is irreducible.*

Proof. Consider the Borel subalgebra $\mathfrak{b} := L(1) \rtimes \mathfrak{b}_{\mathfrak{sl}(2)}$ of $\bar{\mathfrak{s}}$, where $\mathfrak{b}_{\mathfrak{sl}(2)} \subseteq \mathfrak{sl}(2)$ denotes the Borel subalgebra of upper triangular matrices (see 2.4.36) and $L(1)$ denotes the natural representation of the Lie algebra $\mathfrak{sl}(2)$. We compute $\mathbb{E}(2, \mathfrak{b})$: The subalgebra $\mathfrak{u} = ke \oplus ks \oplus kr$ is clearly isomorphic to the three-dimensional Heisenberg algebra $\mathfrak{h}_3$. By the equality $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$, where $\mathfrak{t} = k\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is a torus, we deduce

$$\mathbb{E}(2, \mathfrak{b}) = \mathbb{E}(2, \mathfrak{u}) \cong \mathbb{P}^1.$$

Now [5, Theorem 2.2.2] shows $\overline{\mathcal{S}}.\mathbb{E}(2, \mathfrak{b}) = \mathbb{E}(2, \bar{\mathfrak{s}})$, i.e. $\mathbb{E}(2, \bar{\mathfrak{s}})$ is the image of the morphism

$$\overline{\mathcal{S}} \times \mathbb{E}(2, \mathfrak{b}) \to \mathbb{E}(2, \bar{\mathfrak{s}}), \;\; (g, \mathfrak{e}) \mapsto g.\mathfrak{e}$$

which originates in an irreducible variety. Thus, $\mathbb{E}(2, \bar{\mathfrak{s}})$ is irreducible as well. □

Next, we compute the actions of the group $\mathcal{G}(k) = H \rtimes \mathrm{GL}(2)(k)$ on the two Lie algebras $\mathfrak{s} := \mathfrak{h}_3 \rtimes \mathfrak{sl}(2)$ and $\mathfrak{g} := \mathfrak{h}_3 \rtimes \mathfrak{gl}(2)$ via the formulas of 2.5.24. Let $h = (v, c) \in H$ be arbitrary. The differential of the left translation by $h^{-1} = (-v, -c)$ at h is given by

$$\mathrm{d}_h(l_{h^{-1}}) : \mathfrak{h} \longrightarrow \mathfrak{h}, \; (w, d) \mapsto (w, d - \det(v, w))$$

and the differential of $\eta_h : \mathrm{GL}(2) \longrightarrow H, g \mapsto g.h$ at e is given by

$$\mathfrak{gl}(2) \longrightarrow \mathfrak{h}, \; x \mapsto (x.v, \mathrm{tr}(x)).$$

Now consider the conjugation by h inside H

$$\kappa_h : H \longrightarrow H, \; (w, d) \mapsto h(w, d)h^{-1} = (w, d + 2\det(v, w)).$$

The differential at e can be identified with κ_h for reasons of linearity. As a result, Lemma 2.5.24(b) yields:

Lemma 5.1.8. *Let* V *be the natural representation of* $\mathrm{GL}(2)$.

(1) The $\mathrm{GL}(2)$*-module* $\mathfrak{g}$ *is the direct sum* $\mathfrak{h} \oplus \mathfrak{gl}(2) \cong V \oplus \det \oplus \mathfrak{gl}(2)$. *Let* $h = (v, c) \in$

H, $k = (w, d) \in \mathfrak{h}_3$ and $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{gl}(2)$. Then we have

$$\begin{aligned} h.k &= (w, d + 2\det(v, w)) = (w, d + 2(v_1 w_2 - v_2 w_1)) \\ h.x &= ((-x.v, -\det(v, x.v) - \mathrm{tr}(x)), x) \\ &= ((-x.v, -cv_1^2 + bv_2^2 + (a-d)v_1 v_2 - a - d), x). \end{aligned}$$

(2) The $\mathrm{GL}(2)$-module $\mathfrak{s}$ is the direct sum $\mathfrak{h} \oplus \mathfrak{sl}(2) \cong V \oplus \det \oplus \mathfrak{sl}(2)$ and the action of H is induced by that of (1).

Lemma 5.1.9. *Let (r, s, z) $((r^*, s^*, z^*))$ and (h, e, f) $((h^*, e^*, f^*))$ denote the standard bases of $\mathfrak{h}_3$ and $\mathfrak{sl}(2)$ $(\mathfrak{h}_3^*$ and $\mathfrak{sl}(2)^*)$, respectively. The $\mathrm{GL}(2)$-module $\mathfrak{s}^*$ is isomorphic to $V^* \oplus \det^{-1} \oplus \mathfrak{sl}(2)^*$. Let $\chi = (\chi_{\mathfrak{h}}, \chi_{\mathfrak{sl}(2)}) \in \mathfrak{s}^*$ and write $\chi_{\mathfrak{h}} = \alpha r^* + \beta s^* + \gamma z^*$ with $\alpha, \beta, \gamma \in k$. Then any $g \in \mathrm{GL}(2)(k)$ will take $\chi_{\mathfrak{h}}$ to $\alpha' r^* + \beta' s^* + (\det(g)^{-1}\gamma) z^*$, where*

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = (g^{-1})^t \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Moreover, given $h = (v, c) \in H$, we have $h^{-1}.\chi = h^{-1}.\chi_{\mathfrak{h}} + (\chi_{\mathfrak{sl}(2)} - \psi(h, \chi_{\mathfrak{h}}))$, where

$$\begin{aligned} h^{-1}.\chi_{\mathfrak{h}} &= (\alpha - 2\gamma v_2) r^* + (\beta + 2\gamma v_1) s^* + \gamma z^* \\ \psi(h, \chi_{\mathfrak{h}}) &= (-v_2 \beta + v_1 \alpha - 2\gamma v_1 v_2) h^* + (v_2^2 \gamma + v_2 \alpha) e^* + (v_1 \beta - \gamma v_1^2) f^*. \end{aligned}$$

Remark 5.1.10. We will see later, that the $\mathrm{GL}(2)$-module $\mathfrak{sl}(2)$ is self-dual.

Let now $G \subseteq \mathrm{GL}(2)$ be a closed subgroup. We denote by $Z = \{(0, c) \mid c \in k\}$ the center of H. The canonical projection $H \to \mathbb{G}_a^2$, $(v, c) \mapsto v$ induces an exact sequence

$$\delta : e_k \longrightarrow Z \longrightarrow H \longrightarrow \mathbb{G}_a^2 \longrightarrow e_k$$

of algebraic groups, which splits as a sequence of schemes. In view of the left-exactness of $G \mapsto G_r$, we obtain exact sequences

$$\delta_r : e_k \longrightarrow Z_r \longrightarrow H_r \longrightarrow \mathbb{G}_{a(r)}^2 \longrightarrow e_k$$

of rth Frobenius kernels for all $r \geq 1$. We will now see that the Gabriel quiver of $k(H_r \rtimes G_r) = k(H \rtimes G)_r$ is the same as that of $k(\mathbb{G}_{a(r)}^2 \rtimes G_r)$.

Lemma 5.1.11. *Let $r \geq 1$ and $G \subseteq \mathrm{GL}(2)$ be a closed subgroup. Then the following statements hold:*

(1) We have $\mathcal{D}H = Z$ and $\mathcal{D}H_r = Z_r$.

(2) *The Gabriel quiver of $k(H_r \rtimes G_r)$ is canonically isomorphic to that of $k(\mathbb{G}_{a(r)}^2 \rtimes G_r)$.*

Proof. (1) As $\mathbb{G}_a^2$ is abelian, we get $\mathcal{D}H \subseteq Z$ by the abovementioned exact sequence. Let $A \in \mathbf{Comm}_k$ and $a \in A$, then a direct computation shows that the commutator $[((a,0),0),((0,\frac{1}{2}),0)]$ equals $(0,a)$. This shows $Z \subseteq \mathcal{D}H$. As δ also induces an exact sequence on Frobenius kernels, the missing assertion follows analogously.

(2) This follows from (1), the remark below 2.4.22 and 4.1.8.

□

We note that the projection $H \to \mathbb{G}_a^2,\ (v,c) \mapsto v$ also induces an exact sequence

$$e_k \longrightarrow Z \longrightarrow H \rtimes G \longrightarrow \mathbb{G}_a^2 \rtimes G \longrightarrow e_k$$

of algebraic groups, which (by the same arguments above Lemma 5.1.11) also induces exact sequences of rth Frobenius kernels

$$e_k \longrightarrow Z_r \longrightarrow H_r \rtimes G_r \longrightarrow \mathbb{G}_{a(r)}^2 \rtimes G_r \longrightarrow e_k$$

for all $r \geq 1$.

5.2 The Gabriel quiver of the Frobenius kernels of the Schrödinger group

We consider the group $\overline{\mathcal{S}} = \mathbb{G}_a^2 \rtimes \mathrm{SL}(2)$, it is isomorphic to the quotient of the Schrödinger group $\mathcal{S}$ by its center. As a first step, we observe that the Hopf algebras of the Frobenius kernels of the group $\overline{\mathcal{S}}$ are symmetric and of wild representation type.

Lemma 5.2.1. *The algebra $k\overline{\mathcal{S}}_r$ is symmetric and of wild representation type for all $r \geq 1$.*

Proof. The symmetry follows from Lemma 2.5.29(c). If $r \geq 2$, the algebra $k\mathrm{SL}(2)_r$ is of wild representation type (cf. [51, Satz 6 in Abschnitt 4.4]). As the pullback along the projection $\overline{\mathcal{S}}_r \to \mathrm{SL}(2)_r$ provides a full embedding of $\mathbf{mod}(\mathrm{SL}(2)_r)$ into $\mathbf{mod}(\overline{\mathcal{S}}_r)$, it follows from the definition that $k\overline{\mathcal{S}}_r$ enjoys the same property. If $r = 1$, then we can apply 4.2.8(2). □

We consider the maximal torus $T \subseteq \mathrm{SL}(2)$ of diagonal matrices and the Borel subgroup B of lower triangular matrices. We recall several notions defined earlier for $\mathrm{SL}(n)$ and arbitrary $n \in \mathbb{N}$. The character group $X(T)$ is identified with the group $\mathbb{Z}$ of integers. The roots relative to T are given by $R = \{\pm 2\}$, the only positive root is 2. This induces a partial order on $X(T)$: if $\lambda, \mu \in \mathbb{Z}$, then

$$\lambda \leq \mu :\Longleftrightarrow \mu - \lambda \in 2\mathbb{N}_0.$$

As $B = U \rtimes T$ is a semidirect product of unitriangular matrices and diagonal matrices, respectively, every $\lambda \in X(T)$ gives rise to a one-dimensional B-module k_λ (also denoted λ when no confusion is possible). Let $\lambda \in X(T)_+ = \mathbb{N}_0$ be a *dominant weight.* Following [41, II.2.1(5)/2.13(1)], we define

$$H^0(\lambda) := \mathrm{Ind}_B^{\mathrm{SL}(2)}(\lambda) = \{f \in k[\mathrm{SL}(2)] : f(gb) = \lambda(b)^{-1} f(g)\ \forall g \in \mathrm{SL}(2,k), b \in B(k)\}$$

and put $V(\lambda) := H^0(\lambda)^*$, the *Weyl module with highest weight* λ. Then [41, II.2.16(4)] yields that $H^0(\lambda) \cong k[X,Y]_\lambda$ is the space of homogeneous polynomials of degree λ in the polynomial ring $k[X,Y]$ with two variables X, Y. The (simple) socle $L(\lambda)$ of this module is called *the simple* SL(2)*-module with highest weight* λ. All simple SL(2)-modules arise in this fashion (cf. [41, Proposition II.2.4 a)]) and they are all self-dual (see [41, Corollary II.2.5]). Moreover, we have $H^0(\lambda) \cong V(\lambda) \cong L(\lambda)$ for $0 \leq \lambda \leq p-1$ (see [41, II.2.10(2)/2.16(7)]). In particular, $L(0) = k$ is the trivial module $L(1) = k^2$ is the natural one.

The group SL(2) acts on the restricted (elementary abelian) Lie algebra $\mathrm{Lie}(\mathbb{G}_a^2) = L(1) = k^2$. This action lifts to an action on the restricted enveloping algebra $\mathrm{U}_0(L(1))$.

Lemma 5.2.2. *The* SL(2)*-module* $\mathrm{U}_0(L(1))$ *is completely reducible, we have*

$$\mathrm{U}_0(L(1)) \cong \bigoplus_{i=0}^{p-2} 2L(i) \oplus L(p-1).$$

Proof. Since $S := \mathrm{U}_0(L(1)) \cong k[X_1, X_2]/(X_1^p, X_2^p)$, we clearly have a decomposition

$$S = \bigoplus_{k=0}^{2p-2} S_k$$

into homogeneous components. As in the characteristic zero case, the above mentioned decomposition of S is a decomposition into a direct sum of SL(2)-modules. Moreover note that $S_{2p-2} = \langle x_1^{p-1} x_2^{p-1} \rangle_k = L(0)$ ($x_i := X_i + (X_1^p, X_2^p)$) is the trivial SL(2)-module and if $i \in \{0, \ldots, p-1\}$, then $S_i = L(i)$ is the unique simple SL(2)-module of dimension $i+1$. Using multi-index notation, any $f \in S$ can be written as $\sum_{j \in \mathbb{N}_0^2} \alpha_j x^j$. Now consider the associative bilinear form

$$\lambda : S \times S \to k \ \ ; \ \ (a,b) \mapsto \alpha_{(p-1,p-1)}(f \cdot g).$$

Fix $i \in \{0, \ldots, p-1\}$ and consider the restriction $\lambda_i : S_i \times S_{2p-2-i} \to k$. Let $g \in \mathrm{SL}(2)$ and $a \in S_i, b \in S_{2p-2-i}$, then

$$\lambda_i(g.a, g.b) = \alpha_{(p-1,p-1)}(g.a \cdot g.b) = \alpha_{(p-1,p-1)}(g. \underbrace{a \cdot b}_{\in S_{2p-2}}) = \alpha_{(p-1,p-1)}(a \cdot b) = \lambda_i(a,b).$$

Hence the form λ_i is SL(2)-stable. Moreover it is non-degenerate and therefore induces an isomorphism $S_i \to S^*_{2p-2-i}$ of SL(2)-modules. As S_i is selfdual, the result follows. $\square$

Lemma 5.2.3. *Given* $1 \leq \lambda \leq p-2$, *the* SL(2)*-module* $L(1) \otimes_k L(\lambda)$ *is isomorphic to* $L(\lambda-1) \oplus L(\lambda+1)$.

Proof. We have an exact sequence of B-modules

$$0 \longrightarrow k_{-1} \longrightarrow L(1) \longrightarrow k_1 \longrightarrow 0.$$

Tensoring with k_λ and application of the (by [41, Proposition I.3.3(a)] left exact) induction functor $\mathrm{Ind}_B^{\mathrm{SL}(2)}(-)$ in conjunction with the tensor identity [41, Proposition I.3.6] yields an exact sequence

$$0 \longrightarrow L(\lambda-1) \longrightarrow L(\lambda) \otimes_k L(1) \longrightarrow L(\lambda+1)$$

A direct computation shows that the map on the right-hand side takes the element $X^\lambda \otimes (1,0) \in L(\lambda) \otimes_k L(1)$ to $X^\lambda Y \in L(\lambda+1)$. As $L(\lambda-1)$ is simple, the above sequence is in fact a short exact seqence. In view of [41, Proposition II.2.14], we get

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{SL}(2)}(L(\lambda+1), L(\lambda-1)) &\cong \mathrm{Hom}_{\mathrm{SL}(2)}(\mathrm{rad}(V(\lambda+1)), L(\lambda-1)) \\ &\cong \mathrm{Hom}_{\mathrm{SL}(2)}(0, L(\lambda-1)) \cong (0).\end{aligned}$$

Consequently, the above sequence splits. $\square$

Given $r \geq 1$, we put $\Lambda_r := \{0, \dots, p^r-1\}$ and present elements $\lambda \in \Lambda_r$ by expanding them p-adically: $\lambda = \sum_{j=0}^{r-1} \lambda_j p^j$; sometimes we also write $\lambda = (\lambda_0, \dots, \lambda_{r-1})$. We recall from [41, Proposition II.3.10/3.15], that

$$\{L_r(\lambda) := L(\lambda)|_{\mathrm{SL}(2)_r} \mid \lambda \in \Lambda_r\}$$

is a complete set of representatives for the iso-classes of simple $\mathrm{SL}(2)_r$-modules. For $r=1$, we will occasionally also write $L(\lambda)$ instead of $L_1(\lambda)$, when no confusion is possible.

The above implies that every simple $\mathrm{SL}(2)_1$-module $L_1(\lambda)$ ($0 \leq \lambda \leq p-1$) can be lifted to an $\mathrm{SL}(2)_r$-module isomorphic to $L_r(\lambda)$. Also the $\mathrm{SL}(2)_1$-PIM's $P_1(\lambda)$ can be lifted to $\mathrm{SL}(2)_r$ and, in both cases, this lifting process is unique up to isomorphism (see [51, Bemerkung 3, Satz 3]).

Given $i \geq 0$, we denote by $(-)^{[i]}$ the pullback along the ith power of a Frobenius endomorphism $F : \mathrm{SL}(2) \to \mathrm{SL}(2)$. Note that F^i maps $\mathrm{SL}(2)_r$ to $\mathrm{SL}(2)_{r-i}$. We then have *Steinberg's tensor product theorem* available for simples and an analogous result for

PIM's (see [51, Korollar zu Satz 2]):

$$L_r(\lambda) \cong \bigotimes_{i=0}^{r-1} L_1(\lambda_i)^{[i]}, \qquad P_r(\lambda) \cong \bigotimes_{i=0}^{r-1} P_1(\lambda_i)^{[i]}.$$

For the following, we introduce some additional notation. Given $\lambda \in \Lambda_r$ and $0 \leq s \leq r-1$, we put

$$\begin{aligned} \lambda^{(s)} &:= \sum_{j=0, j\neq s}^{r-1} \lambda_j p^j = (\lambda_0, \dots, \lambda_{s-1}, 0, \lambda_{s+1}, \dots, \lambda_{r-1}) \\ \lambda^{(s),\pm} &:= \lambda^{(s)} + (\lambda_s \pm 1)p^s = (\lambda_0, \dots, \lambda_{s-1}, \lambda_s \pm 1, \lambda_{s+1}, \dots, \lambda_{r-1}). \end{aligned}$$

Lemma 5.2.4. *Let $r \geq 1, s \in \{0, \dots, r-1\}$ and $\lambda \in \Lambda_r$. Then we have isomorphisms of $\mathrm{SL}(2)_r$-modules:*

(a)

$$L_r(1) \otimes_k L_r(\mu) \cong \begin{cases} L_r(\mu-1) \oplus L_r(\mu+1), & \mu \neq p-1 \\ P_1(p-2), & \mu = p-1 \end{cases}$$

for all $1 \leq \mu \leq p-1$.

(b)

$$L_r(p^s) \otimes_k L_r(\lambda) \cong \begin{cases} L_r(p^s), & \lambda = 0 \\ L_r(\lambda^{(s),+}) \oplus L_r(\lambda^{(s),-}), & \lambda_s \neq p-1 \\ X(s,\lambda) := P_1(p-2)^{[s]} \otimes_k L_r(\lambda^{(s)}), & \lambda_s = p-1. \end{cases}$$

Proof. (a) First consider the case $\mu \neq p-1$. By 5.2.3, we have isomorphisms $L(1) \otimes_k L(\mu) \cong L(\mu-1) \oplus L(\mu+1)$ of SL(2)-modules. We thus obtain isomorphisms of $\mathrm{SL}(2)_r$-modules

$$\begin{aligned} L_r(1) \otimes_k L_r(\mu) &\cong (L(1) \otimes_k L(\mu))|_{\mathrm{SL}(2)_r} \cong (L(\mu-1) \oplus L(\mu+1))|_{\mathrm{SL}(2)_r} \\ &\cong L_r(\mu-1) \oplus L_r(\mu+1). \end{aligned}$$

In the remaining case, the *modular Clebsch-Gordan rule* (cf. [26, Kapitel 5]) implies that $L_1(1) \otimes_k L_1(p-1) \cong P_1(p-2)$ as $\mathrm{SL}(2)_1$-modules. Hence $L_1(1) \otimes_k L_1(\mu)$ is a lift of the principal indecomposable $\mathrm{SL}(2)_1$-module $P_1(p-2)$, so that our first claim follows from the uniqueness.

(b) Application of Steinberg's tensor product theorem yields

$$\begin{aligned} L_r(p^s)\otimes_k L_r(\lambda) &\cong L_1(1)^{[s]}\otimes_k \bigotimes_{i=0}^{r-1} L_1(\lambda_i)^{[i]} \\ &\cong L_1(\lambda_0)\otimes_k L_1(\lambda_1)^{[1]}\otimes_k\cdots\otimes_k (L_1(\lambda_s)^{[s]}\otimes_k L_1(1)^{[s]})\otimes_k\cdots\otimes_k L_1(\lambda_{r-1})^{[r-1]} \\ &\cong L_1(\lambda_0)\otimes_k L_1(\lambda_1)^{[1]}\otimes_k\cdots\otimes_k (L_1(\lambda_s)\otimes_k L_1(1))^{[s]}\otimes_k\cdots\otimes_k L_1(\lambda_{r-1})^{[r-1]}. \end{aligned}$$

Now apply (a) and the tensor product theorem again.

□

The block structure of the algebra $k\mathrm{SL}(2)_r$ has been determined by Pfautsch [51, Abschnitt 4.2]. We adopt the notation of [22, p.1503] and put $\mathcal{B}_r^{(r)} := \{p^r-1\}$, the block corresponding to the (projective) *Steinberg module* $St_r := L_r(p^r-1)$ as well as

$$\mathcal{B}_{i,s}^{(r)} := \left\{ \lambda = \sum_{j=0}^{r-1}\lambda_j p^j \in \Lambda_r \mid \lambda_0=\lambda_1=\cdots=\lambda_{s-1}=p-1, \lambda_s\in\{i,p-2-i\}\right\}$$

for given elements $0\le i\le \frac{p-3}{2}$ and $0\le s\le r-1$. The corresponding block consists of modules having composition factors of the form $L_r(\lambda)$ with $\lambda\in\mathcal{B}_{i,s}^{(r)}$.

We recall from Theorem 4.1.8 and Lemma 4.1.10, that the Gabriel quiver of the algebra $k\overline{\mathcal{S}}_r$ is given by the generalized McKay quiver $\Gamma_{\mathrm{SL}(2)_r}(V)$, where

$$V := H^1(\mathbb{G}_{a(r)}^2, k)^* \cong \bigoplus_{j=0}^{r-1} L_r(p^j)\in \mathbf{mod}(\mathrm{SL}(2)_r).$$

Hence, if $S_r(\lambda) := \mathrm{Inf}(L_r(\lambda))$, then there are precisely $d_r(\lambda,\mu)$ arrows $S_r(\lambda)\to S_r(\mu)$ inside $Q := Q_{k\overline{\mathcal{S}}_r}$ where

$$d_r(\lambda,\mu) := \dim_k \mathrm{Hom}_{\mathrm{SL}(2)_r}(V\otimes_k L_r(\lambda), L_r(\mu)) \;+\; \dim_k \mathrm{Ext}^1_{\mathrm{SL}(2)_r}(L_r(\lambda), L_r(\mu))$$

for all $\lambda,\mu\in\Lambda_r$. Given two blocks $\mathcal{B},\mathcal{C}$ of $k\mathrm{SL}(2)_r$, we say $\mathcal{B}$ *is connected with* $\mathcal{C}$ *inside* Q, provided there exist simple modules $S\in\mathbf{mod}(\mathcal{B}), T\in\mathbf{mod}(\mathcal{C})$ and an unoriented path between S and T inside Q. Note that this clearly defines an equivalence relation on the set of blocks of $k\mathrm{SL}(2)_r$.

Theorem 5.2.5. *The Gabriel quiver $Q := Q_{k\overline{\mathcal{S}}_r}$ of the algebra $k\overline{\mathcal{S}}_r$ is connected for all $r\ge 1$.*

Proof. We proceed in several steps.

(a) The block $\mathcal{B}_{i,s}$ is connected with $\mathcal{B}_{i+1,s}$ inside Q for all $0\le i\le \frac{p-3}{2}-1,\ 0\le s\le r-1$:

Indeed, let i, s be as above and $\lambda \in \mathcal{B}_{i,s}$ non-zero with the property that $\lambda_s = i$. Then 5.2.4 implies that

$$L_r(p^s) \otimes_k L_r(\lambda) \cong L_r(\lambda^{(s),+}) \oplus L_r(\lambda^{(s),-})$$

is a direct summand of $V \otimes_k L_r(\lambda)$, so that we get $d_r(\lambda, \lambda^{(s),+}) \neq 0$. Recalling that

$$\lambda^{(s),\pm} = (p-1, \ldots, p-1, i \pm 1, \lambda_{s+1}, \ldots, \lambda_{r-1}),$$

we conclude $\lambda^{(s),+} \in \mathcal{B}_{i+1,s}$ and therefore the desired statement follows.

(b) $\mathcal{B}_{0,s}$ is connected with $\mathcal{B}_{0,s+1}$ for all $0 \leq s \leq r-2$:

Let $0 \leq s \leq r-2$ be arbitrary and $\lambda \in \mathcal{B}_{0,s}$ such that $\lambda_s = p-2$ and $\lambda_{s+1} \in \{0, p-2\}$. Then

$$L_r(p^s) \otimes_k L_r(\lambda) \cong L_r(\lambda^{(s),+}) \oplus L_r(\lambda^{(s),-})$$

is a direct summand of $V \otimes_k L_r(\lambda)$, so that $d_r(\lambda, \lambda^{(s),+}) \neq 0$. Since

$$\lambda^{(s),+} = (p-1, \ldots, p-1, p-1, \lambda_{s+1}, \ldots, \lambda_{r-1}) \in \mathcal{B}_{0,s+1},$$

we observe that $\mathcal{B}_{0,s}$ is connected with $\mathcal{B}_{0,s+1}$.

(c) $\mathcal{B}_{i,s}$ is connected with $\mathcal{B}_{j,t}$ for all possible $0 \leq i, j \leq \frac{p-3}{2}$, $0 \leq s, t \leq r-1$:

W.l.o.g. we assume $s \leq t$ and recall that our relation on the set of blocks of $\mathrm{SL}(2)_r$ is an equivalence relation. Repeated applications of (a) show that $\mathcal{B}_{i,s}$ is connected with $\mathcal{B}_{0,s}$ and $\mathcal{B}_{j,t}$ with $\mathcal{B}_{0,t}$. But then repeated applications of (b) show that $\mathcal{B}_{0,t}$ is connected with $\mathcal{B}_{0,s}$ inside Q. Hence $\mathcal{B}_{i,s}$ is connected with $\mathcal{B}_{j,t}$.

Finally, the block $\mathcal{B}_r^{(r)}$ corresponding to the Steinberg module St_r is connected with $\mathcal{B}_{0,r-1}$ since St_r is, again by 5.2.4, a direct summand of $L_r(p^{r-1}) \otimes_k L_r(\lambda)$ for $\lambda = (p-1, \ldots, p-1, p-2) \in \mathcal{B}_{0,r-1}$. This in conjunction with (c) shows that Q is connected. □

In case $r = 1$, we can determine the Gabriel quiver completely.

Theorem 5.2.6. *We have*

$$\dim_k \mathrm{Ext}^1_{\mathrm{U}_0(\bar{\mathfrak{s}})}(S(i), S(j)) = 2\delta_{i+j,p-2} + \begin{cases} \delta_{j,1}, & i = 0 \\ \delta_{|i-j|,1} & 1 \leq i \leq p-1 \\ \delta_{j,p-2}, & i = p-1 \end{cases}$$

for all $0 \leq i, j \leq p-1$.

Proof. Recall that

$$\operatorname{Ext}^1_{\mathrm{U}_0(\bar{\mathfrak{s}})}(S(i),S(j)) \cong \operatorname{Hom}_{\mathfrak{sl}(2)}(L(1)\otimes_k L(i),L(j)) \;\oplus\; \operatorname{Ext}^1_{\mathrm{U}_0(\mathfrak{sl}(2))}(L(i),L(j)).$$

By 5.2.4, we have isomorphisms of $\mathrm{U}_0(\mathfrak{sl}(2))$-modules

$$L(1)\otimes_k L(i) \cong \begin{cases} L(1), & i=0 \\ L(i-1)\oplus L(i+1), & 1\leq i\leq p-2 \\ P(p-2), & i=p-1 \end{cases}$$

Combining this with $\dim_k \operatorname{Ext}^1_{\mathrm{U}_0(\mathfrak{sl}(2))}(L(i),L(j)) = 2\delta_{i+j,p-2}$ (see for instance [26, Kapitel 1]), we get our assertion. □

Remark 5.2.7. With the modular Clebsch-Gordan rule in hand ([26, Kap.5]), one can compute in the same fashion the Gabriel quivers of restricted enveloping algebras of semidirect products $L(i)\rtimes\mathfrak{sl}(2)$ for $i=0,\ldots,p-1$.

We recall some notations introduced in Section 4.4. Denote by $B^+\subseteq \mathrm{SL}(2)$ (B^-) the Borel subgroup of upper (lower) triangular matrices with unipotent radical U^+ (U^-) consisting of unitriangular (2×2)-matrices. Consider the closed subgroups $V^+ := \mathbb{G}_a\times\{0\}$, $V^- := \{0\}\times\mathbb{G}_a \subseteq \mathbb{G}_a^2$ and put $U_V^\pm := V^\pm \rtimes U^\pm$ as well as $B_V^\pm := V^\pm\rtimes B^\pm = U_V^\pm\rtimes T$. Then, we put for all $r\geq 1$ and $\lambda\in X(T)$

$$Z_r(\lambda) := k\overline{\mathcal{S}}_r \otimes_{kB^+_{V,r}} \lambda \in \mathbf{mod}(k\overline{\mathcal{S}}_r)$$

and denote by $\hat{Z}_r(\lambda)$ its (canonical) structure as a $\overline{\mathcal{S}}_r T$-module. We can enhance 4.4.7 and thereby obtain another proof for the fact that $k\overline{\mathcal{S}}_r$ is connected:

Lemma 5.2.8. *The restriction of $\hat{Z}_r(\lambda)$ ($Z_r(\lambda)$) to $\mathrm{SL}(2)_r T$ ($\mathrm{SL}(2)_r$) decomposes as*

$$\bigoplus_{\mu\in\Lambda_r} \hat{Z}_{\mathrm{SL}(2)_r}(\lambda-\mu) \quad \left(\bigoplus_{\mu\in\Lambda_r} Z_{\mathrm{SL}(2)_r}(\mu)\right)$$

In particular, we have $[Z_r(\lambda):S_r(p^r-1)]=1$ and $k\overline{\mathcal{S}}_r$ is a connected algebra.

Proof. The group $U_r^-\subseteq\overline{\mathcal{S}}_r$ is isomorphic to the r-th Frobenius kernel of the additive group, its coordinate ring is given by $k[T]/(T^{p^r})$ and there is an isomorphism of k-algebras

$$kU_r^- \longrightarrow k[X_0,\ldots,X_{r-1}]/(X_i^p : i\in\{0,\ldots,r-1\}) \qquad (T^{p^i})^* \mapsto X_i.$$

As in the proof of 4.4.4, one observes that the weight of X_i relative to T is given by $-p^i$. We denote the Hopf algebra of the group V_r^- with the same reason by $k[Y_0, \dots, Y_{r-1}]/(Y_i^p : i \in \{0, \dots, r-1\})$ and in what follows, we denote by x_i and y_i the images of X_i and Y_i.

By 4.4.5, we have an isomorphism

$$\hat{Z}_r(\lambda) \cong kU_{V,r}^- \otimes_k kB_{V,r}^+ \otimes_{kB_{V,r}^+} \lambda \cong kU_{V,r}^- \otimes_k \lambda$$

of vector spaces. It follows that $\hat{Z}_r(\lambda)$ has a k-basis given by

$$\mathcal{B} := \left\{ v_{n,m} := \prod_{i=0}^{r-1} x_i^{n_i} \cdot \prod_{j=0}^{r-1} y_j^{m_j} \otimes 1 \mid n, m \in \mathbb{N}^r_{0 \le \dots \le p-1} \right\}.$$

Given $m \in \mathbb{N}^r_{0 \le \dots \le p-1}$, we now consider the vector $v_m := v_{0,m}$. As U_r^- acts trivially on $V_{a(r)}^-$, the elements of $kV_{a(r)}^-$ commute with the elements of kU_r^- inside $k\overline{\mathcal{S}}_r$, so that U_r^- acts trivially on v_m. Moreover, the group T acts on v_m via $\lambda - m' \in X(T)$, where $m' := \sum_{l=0}^{r-1} m_i p^i \in \mathbb{Z} = X(T)$. As $\hat{Z}_r(\lambda)$ is an $\overline{\mathcal{S}}_r T$-module, T_r operates via the restriction of that character. (Graded) Frobenius reciprocity thus provides a homomorphism of $\mathrm{SL}(2)_r T$-modules

$$\varphi_m : \hat{Z}_{\mathrm{SL}(2)_r}(\lambda - m') \to \hat{Z}_r(\lambda)$$

such that $\mathrm{im}(\varphi_m) := \langle v_{n,m} \mid n \in \mathbb{N}^r_{0 \le \dots \le p-1} \rangle$. Consequently, the direct sum of all such φ_m provides a homomorphism

$$\varphi : \bigoplus_{\mu \in \Lambda_r} \hat{Z}_{\mathrm{SL}(2)_r}(\lambda - \mu) \to \hat{Z}_r(\lambda)$$

which is surjective since $\mathcal{B}$ is in its image. As the relevant spaces have the same dimension, we conclude that φ is an isomorphism. Application of the restriction functor $\mathrm{Res}_{\mathrm{SL}(2)_r}^{\mathrm{SL}(2)_r T}$ yields the second assertion. Finally, since $Z_r(\lambda)$ is an indecomposable $\overline{\mathcal{S}}_r$-module which has each simple object at least once as a composition factor (see 4.1.4(2c)), the algebra $k\overline{\mathcal{S}}_r$ must be connected. $\square$

Remark 5.2.9. Modules (such as $Z_r(\lambda)$), which have any simple module at least once as a composition factor, are also called *sincere*.

Since all Verma modules have the same restriction to $\mathrm{SL}(2)_r$, it follows from 4.1.4(2c), that they all have the same composition factors (along with multiplicities). Moreover, the knowledge of the composition factors with multiplicities of $\mathrm{SL}(2)_r$ Verma modules will yield the ones of $\overline{\mathcal{S}}_r$ Verma modules. However, the first mentioned seems not to be known yet except for the case $r = 1$: the $\mathrm{U}_0(\mathfrak{sl}(2))$-module Verma module $\mathrm{Z}_{\mathfrak{sl}(2)}(i)$ for $1 \le i \le p-2$ is uniserial of length two with composition factors $L(i), L(p-2-i)$, while

$Z_{\mathfrak{sl}(2)}(p-1) = L(p-1)$ is projective simple (cf. [26]).

Recall that the first Frobenius kernel $\overline{\mathcal{S}}_1$ corresponds to the restricted Lie algebra $\mathrm{Lie}(\overline{\mathcal{S}}) = \bar{\mathfrak{s}} = L(1) \rtimes \mathfrak{sl}(2)$. We now take a closer look at the Verma modules $Z(i) := Z_1(i)$ for $i \in \Lambda_1$. Let (h, e, f) and (r, s) denote the standard bases of $\mathfrak{sl}(2)$ and $L(1) = k^2$, respectively. Moreover, denote by $(v_{n,m} = f^n s^m \otimes 1 : 0 \leq n, m \leq p-1)$ the standard basis of $Z(i)$. Then the action of r is given by

$$r.v_{n,m} = \begin{cases} 0 & n = 0 \text{ or } m = p-1 \\ -n v_{n-1,m+1} & \text{otherwise.} \end{cases}$$

We denote by $S(i)$ the inflation of the simple $\mathrm{U}_0(\mathfrak{sl}(2))$-module $L(i)$ to $\mathrm{U}_0(\bar{\mathfrak{s}})$.

Lemma 5.2.10. *Let $i \in \Lambda_0$, then the following statements hold:*

(a) We have $[Z(i) : S(p-1)] = 1$ and $[\mathrm{Z}(i) : S(j)] = 2$ for all $0 \leq j \leq p-2$. In particular, the $\mathrm{U}_0(\bar{\mathfrak{s}})$-module $Z(i)$ has length $\ell(Z(i)) = 2p-1$.

(b) We have $V(\bar{\mathfrak{s}})_{Z(i)} = ke \oplus kr$.

Proof. (a) This follows from 5.2.8 and the abovementioned structure of $\mathrm{U}_0(\mathfrak{sl}(2))$ Verma modules.

(b) According to [13, Proposition 3.4] we have $V(\bar{\mathfrak{s}})_{Z(i)} \subseteq V(kh \oplus ke \oplus kr) = ke \oplus kr$. Since $ke = V(\mathfrak{sl}(2))_{Z_{\mathfrak{sl}(2)}(0)}$, we conclude that $ke \subseteq V(\bar{\mathfrak{s}})_{Z(i)}$ by 5.2.8. Denote by

$$x_r : Z(i) \to Z(i), \quad z \mapsto r.z$$

the left multiplication effected by r. If $Z(i)|_{\mathrm{U}_0(kr)}$ was projective, then $\mathrm{rk}(x_r) = p(p-1) = p^2 - p$. Since the $2p$ linearly independent vectors

$$\{v_{0,0}, \ldots, v_{0,p-1}, v_{p-1,0}, \ldots, v_{p-1,p-1}\}$$

lie in the kernel of x_r, it follows that $\mathrm{rk}(x_r) \leq p^2 - 2p < p^2 - p$. Hence $Z(i)|_{\mathrm{U}_0(kr)}$ is not projective, so that $kr \subseteq V(\bar{\mathfrak{s}})_{Z(i)}$. By indecomposability, the assumption $\dim V(\bar{\mathfrak{s}})_{Z(i)} = 1$ would render $V(\bar{\mathfrak{s}})_{Z(i)}$ being a line, which is not possible since the vectors $r, e \in \bar{\mathfrak{s}}$ are linearly independent. Hence $\dim V(\bar{\mathfrak{s}})_{Z(i)} = 2$, so that the assertion follows by irreducibility of the linear variety $ke \oplus kr$.

□

We can determine the Cartan matrix of the algebra $\mathrm{U}_0(\bar{\mathfrak{s}})$:

Lemma 5.2.11. *Let M be a d-dimensional $\mathrm{U}_0(\bar{\mathfrak{s}})$-module.*

(a) *If M admits a Z-filtration, then d must be divisible by p^2 and the length of this filtration is given by $\frac{d}{p^2}$. Moreover, we have $[M : S(j)] = \frac{2d}{p^2}$ for all $1 \leq j \leq p-1$ and $[M : S(p-1)] = \frac{d}{p^2}$.*

(b) *Let $Q(i) \in \mathbf{mod}(\mathrm{U}_0(\bar{\mathfrak{s}}))$ be the projective cover of $S(i)$, then we have*

$$[Q(i) : S(j)] = \begin{cases} 4p & i, j \leq p-2 \\ p & i = j = p-1 \\ 2p & \textit{otherwise.} \end{cases}$$

In particular, $Q(i)$ has length $4p^2 - 2p$ for all $1 \leq i \leq p-2$ and $Q(p-1)$ has length $2p^2 - p$.

Proof. (a) Let $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a Z-filtration. Then there exist $\lambda_0, \ldots, \lambda_{n-1} \in \Lambda_0$ such that

$$d = \sum_{i=0}^{n-1} \dim_k M_{i+1}/M_i = \sum_{i=0}^{n-1} \dim_k Z(\lambda_i) = np^2.$$

Thus, d is divisible by p^2 and $n = \frac{d}{p^2}$. An application of 5.2.10(a) then yields

$$[M : S(j)] = \sum_{i=0}^{n-1} [M_{i+1}/M_i : S(j)] = \sum_{i=0}^{n-1} [Z(\lambda_i) : S(j)] = \begin{cases} 2n & j \leq p-2 \\ n & j = p-1 \end{cases},$$

as desired.

(b) Consider first $0 \leq i \leq p-2$. Since the projective cover $P(i)$ of $L(i)$ over $\mathrm{U}_0(\mathfrak{sl}(2))$ is $2p$-dimensional (see [26, p.15 Satz]), it follows that $Q(i) \cong \mathrm{U}_0(L(1)) \otimes_k P(i)$ is $p^2 \cdot 2p = 2p^3$-dimensional (see 4.2.4(2b)). Since $L(p-1)$ is projective, we get $\dim_k Q(p-1) = p^3$ by the same token. Now apply (a).

□

We can also enhance the reciprocity formula 4.4.6 for principal indecomposables. It will have the same type as the one [41, II.11.4] in the classical context of G_rT-modules.

Theorem 5.2.12. *Put $\mathfrak{b}^+ := kh \oplus ke \oplus kr$ as well as $\mathfrak{b}^- := kh \oplus kf \oplus ks$.*

(a) *The k-linear map $\tau : \bar{\mathfrak{s}} \to \bar{\mathfrak{s}}$ induced by*

$$e \mapsto -f \qquad f \mapsto -e \qquad h \mapsto h$$
$$r \mapsto s \qquad s \mapsto r$$

is a restricted, involutive, antigraded antiautomorphism of $\bar{\mathfrak{s}}$ *such that* $\tau(\mathfrak{b}^+) = \mathfrak{b}^-$ *and* $\lambda^\tau \cong \lambda$ *for each simple* $\mathrm{U}_0(kh)$*-module* λ*.*

(b) Every projective indecomposable $\overline{\mathcal{S}}_1 T$*-module* $\hat{Q}$ *with top* $\hat{S}$ *(*$\overline{\mathcal{S}}_1$*-module* Q *with top* S*) admits a* $\hat{Z}$*-filtration (*Z*-filtration) and we have* $[\hat{Q} : \hat{Z}(\lambda)] = [\hat{Z}(\lambda) : \hat{S}]$ *(*$[Q : Z(\lambda)] = [Z(\lambda) : S]$*) for all* $\lambda \in \mathbb{Z} = X(T)$*.*

(c) We have $[Q(i) : Z(j)] = \begin{cases} 2 & 1 \leq i \leq p-2 \\ 1 & i = p-1 \end{cases}$ *for all* $1 \leq j \leq p-1$*.*

Proof. (a) Clearly τ is an antiautomorphism of the Lie algebra $\bar{\mathfrak{s}}$. We need to show that τ is restricted. This follows from 2.5.21(1),(2) applied to $\sigma := -\tau$, but we can also compute it directly: Denote by E_2 the unit-matrix. Inductively, one can show the following equation

$$x^{2n} = (-1)^n \cdot \det(x)^n \cdot E_2 \qquad \forall x \in \mathfrak{sl}(2), n \in \mathbb{N} \tag{1}$$

which holds inside the algebra of (2×2)-matrices. Next, note that τ is $\mathfrak{sl}(2)$-invariant and $\det(\tau(x)) = \det(x)$ for all $x \in \mathfrak{sl}(2)$. Using that $p \geq 3$ and (1), we therefore conclude, denoting by $\cdot$ matrix-vector multiplication,

$$\tau(x^p) = \tau(x)^p \qquad \tau(x)^{p-1} \cdot \tau(v) = \tau(x^{p-1} \cdot v) \qquad \forall x \in \mathfrak{sl}(2), v \in L(1) \tag{2}$$

Recall from 2.5.20 that the p-map on $\bar{\mathfrak{s}} = L(1) \rtimes \mathfrak{sl}(2)$ is given by $(v,x)^{[p]} = (x^{p-1}.v, x^p)$ for all $x \in \mathfrak{sl}(2), v \in L(1)$. Let now $y = (v,x) \in \bar{\mathfrak{s}}$. Then we have

$$\begin{aligned} \tau(y^{[p]}) &= (\tau(x^{p-1} \cdot v), \tau(x^p)) = (\tau(x)^{p-1} \cdot \tau(v), \tau(x)^p) && \text{by (2)} \\ &= \tau(y)^{[p]}. \end{aligned}$$

(b) The map τ of (a) induces an antiautomorphism $\mathrm{U}_0(\bar{\mathfrak{s}}) \to \mathrm{U}_0(\bar{\mathfrak{s}})$, which enables us to apply [37, Theorem 5.1].

(c) Apply (b) and our above lemma.

□

5.3 Reduced enveloping algebras where the linear form does not vanish on the unipotent radical

Recall that k is algebraically closed of characteristic $p \geq 3$. Our aim is to investigate the structure of reduced enveloping algebras of the restricted Lie algebras $\bar{\mathfrak{g}} = V \rtimes \mathfrak{gl}(2)$ and $\bar{\mathfrak{s}} = V \rtimes \mathfrak{sl}(2)$, where V denotes the natural representation. It turns out that the structure of algebras, whose defining linear form does not vanish on V, will provide useful information for the block structure in the arbitrary case: all algebras will be connected. We first collect the following lemma.

Lemma 5.3.1. *Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ be a direct product of restricted Lie algebras (that is, a semidirect product $n \rtimes_\tau \mathfrak{h}$ such that $\tau \equiv 0$) and $\chi \in \mathfrak{g}^*$ a linear form.*

(1) The algebras $\mathrm{U}_\chi(\mathfrak{g})$ and $\mathrm{U}_\chi(\mathfrak{n}) \otimes_k \mathrm{U}_\chi(\mathfrak{h})$ are isomorphic.

(2) Given $M \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{n})), N \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{h}))$, the rank variety of the $\mathrm{U}_\chi(\mathfrak{g})$-module $T := M \otimes_k N$ (see (1) and 2.1.30(1)) is given by $V(\mathfrak{g})_T = V(\mathfrak{n})_M \times V(\mathfrak{h})_N$. In particular, T is $\mathrm{U}_\chi(\mathfrak{g})$-projective if and only if M is $\mathrm{U}_\chi(\mathfrak{n})$-projective and N is $\mathrm{U}_\chi(\mathfrak{h})$-projective.

Proof. (1) Denote by $\iota : \mathrm{U}_\chi(\mathfrak{n}) \hookrightarrow \mathrm{U}_\chi(\mathfrak{g})$ and $i : \mathrm{U}_\chi(\mathfrak{h}) \hookrightarrow \mathrm{U}_\chi(\mathfrak{g})$ the canonical embeddings. Since $\mathfrak{h}$ acts trivially on $\mathfrak{n}$, the elements of the images of i and ι commute. Thus, the universal property of the tensor product yields a homomorphism

$$\psi := \iota \hat{\otimes} i : \mathrm{U}_\chi(\mathfrak{n}) \otimes_k \mathrm{U}_\chi(\mathfrak{h}) \to \mathrm{U}_\chi(\mathfrak{g}), u \otimes v \mapsto uv$$

of algebras. Since the generating set $\mathfrak{n} \oplus \mathfrak{h}$ of the algebra $\mathrm{U}_\chi(\mathfrak{g})$ is contained in the image of ψ, it follows that ψ is surjective. By the PBW-theorem for reduced enveloping algebras (cf. 2.5.4(1)), the relevant algebras have the same dimension. Thus, ψ is bijective and hence an isomorphism (One can also show that $\Theta : \mathrm{U}_\chi(\mathfrak{g}) \to \mathrm{U}_\chi(\mathfrak{n}) \otimes_k \mathrm{U}_\chi(\mathfrak{h})$, $\mathfrak{n} \oplus \mathfrak{h} \ni (x, y) \mapsto x \otimes 1 + 1 \otimes y$ is the inverse of ψ).

(2) It is not hard to see, that the $\mathrm{U}_\chi(\mathfrak{g})$-modules $T := M \otimes_k N$ (constructed via 2.1.30(1)) and $\mathrm{Inf}_{\mathfrak{n}}^{\mathfrak{g}}(M) \otimes_k \mathrm{Inf}_{\mathfrak{h}}^{\mathfrak{g}}(N)$ (the tensor product of two inflated $\mathfrak{g}$-modules, see 4.2.2) are isomorphic. Recall that the rank variety of $\mathfrak{g}$ is given by $V(\mathfrak{g}) = V(\mathfrak{n}) \times V(\mathfrak{h})$ (see 2.5.20(1)). Thus, we have (observing 2.5.7(3))

$$\begin{aligned} V(\mathfrak{g})_T &= V(\mathfrak{g})_{\mathrm{Inf}_{\mathfrak{n}}^{\mathfrak{g}}(M)} \cap V(\mathfrak{g})_{\mathrm{Inf}_{\mathfrak{h}}^{\mathfrak{g}}(N)} \\ &= (V(\mathfrak{n})_M \times V(\mathfrak{h})) \cap (V(\mathfrak{n}) \times V(\mathfrak{h})_N) \\ &= V(\mathfrak{n})_M \times V(\mathfrak{h})_N. \end{aligned}$$

□

We proceed by recalling several facts about the structure of reduced enveloping algebras of $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$. Denote by

$$h_1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),\ e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),\ f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right),\ h_2 := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

and

$$h := h_1 - h_2,\ e,\ f$$

the standard basis of $\mathfrak{gl}(2)$ and $\mathfrak{sl}(2)$, respectively. We will also consider the corresponding dual basis $(h_1^*, e^*, f^*, h_2^*), (h^*, e^*, f^*)$ of the dual spaces $\mathfrak{gl}(2)^*$ and $\mathfrak{sl}(2)^*$. The group $\mathrm{GL}(2)$ acts on $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$ by conjugation.

(i) The standard trace form of the $\mathfrak{gl}(2)$-module V

$$\gamma : \mathfrak{gl}(2) \times \mathfrak{gl}(2) \to k,\ (x, y) \mapsto \mathrm{tr}(xy)$$

is non-degenerate and $\mathrm{GL}(2)$-invariant. Thus, we identify the $\mathrm{GL}(2)$-modules $\mathfrak{gl}(2)$ and $\mathfrak{gl}(2)^*$. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{gl}(2)$ then corresponds to the linear form $ah_1^* + ce^* + bf^* + dh_2^* \in \mathfrak{gl}(2)^*$.

(ii) Since $\gamma|_{\mathfrak{sl}(2)\times\mathfrak{sl}(2)}$ is still non-degenerate, we can also identify the $\mathrm{GL}(2)$-modules $\mathfrak{sl}(2)$ and $\mathfrak{sl}(2)^*$. A matrix $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2)$ then corresponds to the linear form $2ah^* + ce^* + bf^* \in \mathfrak{sl}(2)^*$.

The $\mathrm{GL}(2)$-orbits (see 2.5.5) of its action on the dual spaces $\mathfrak{sl}(2)^*$ and $\mathfrak{gl}(2)^*$ are thus given by Jordan canonical forms. In case of $\mathfrak{sl}(2)$, the structure of the relevant algebras with a non-zero linear form has been determined in [28, Proposition 2.3] while that of $\mathrm{U}_0(\mathfrak{sl}(2))$ has been analyzed in [26]. Below, we will give for each non-zero normal form an overview. To that end, we denote by $\mathfrak{b}_{\mathfrak{sl}(2)} = kh \oplus ke$ the borel subalgebra of $\mathfrak{sl}(2)$ consisting of upper triangular matrices. Let $\chi \in \mathfrak{sl}(2)^*$ be a linear form such that $\chi(e) = 0$ and put $\Lambda_\chi := \{\lambda \in k : \lambda^p - \chi(h)^p - \chi(h) = 0\}$ as well as

$$Z_{\chi,\mathfrak{sl}(2)}(\lambda) := \mathrm{U}_\chi(\mathfrak{sl}(2)) \otimes_{\mathrm{U}_\chi(\mathfrak{b}_{\mathfrak{sl}(2)})} k_\lambda \ \ \forall \lambda \in \Lambda_\chi,$$

the so-called *baby Verma module with highest weight* λ. Note that if $\lambda(h) = 0$, then $\Lambda_\chi = \mathbb{F}_p$.

Lemma 5.3.2. *Let* $0 \neq \chi \in \mathfrak{sl}(2)^*$ *be in its normal form relative to the action of* $\mathrm{GL}(2)$.

(a) If $\chi = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, *then* $\mathrm{U}_\chi(\mathfrak{sl}(2))$ *has* $\frac{p+1}{2}$ *blocks, each consisting of a single simple module, represented by* $Z_{\chi,\mathfrak{sl}(2)}(\lambda)$ *for some suitable* $\lambda \in \Lambda_\chi$. *The block corresponding to the* ***Steinberg module*** $Z_{\chi,\mathfrak{sl}(2)}(p-1)$ *is a simple algebra. All other blocks are isomorphic to* $\mathrm{Mat}_p(k[X]/(X^2))$ *(see also [12, 3.3(2)]).*

(b) If $\chi = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$ *for some* $a \in k \setminus \{0\}$, *then* $\mathrm{U}_\chi(\mathfrak{sl}(2))$ *is semi-simple and* $\{Z_{\chi,\mathfrak{sl}(2)}(\lambda) : \lambda \in \Lambda_\chi\}$ *is a complete set of simple modules.*

We will also be able to give corresponding types of reduced enveloping algebras of $\mathfrak{gl}(2)$.

Lemma 5.3.3. *The following statements hold:*

(1) $\mathfrak{gl}(2) = \mathfrak{t} \oplus \mathfrak{sl}(2)$ *is a direct product of restricted Lie algebras, where* $\mathfrak{t} := k(h_1 + h_2) \subseteq \mathfrak{gl}(2)$ *denotes the torus of scalar matrices. In particular, the algebra* $\mathrm{U}_\chi(\mathfrak{gl}(2))$ *is isomorphic to* p *copies of the algebra* $\mathrm{U}_\chi(\mathfrak{sl}(2))$ *for all* $\chi \in \mathfrak{gl}(2)^*$*.*

(2) Let $0 \neq \chi \in \mathfrak{gl}(2)^*$ *be in its normal form relative to the action of* $\mathrm{GL}(2)$*.*

(a) If $\chi = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ *for* $a \in k$*, then* $\mathrm{U}_\chi(\mathfrak{gl}(2))$ *is isomorphic to* p *copies of the algebra described in 5.3.2(a).*

(b) If $\chi = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ *for some* $(a, b) \in k^2 \setminus \{0\}$*, then* $\mathrm{U}_\chi(\mathfrak{gl}(2))$ *is semi-simple with* p^2 *simple modules of dimension* p*.*

Proof. Since (2) follows from (1) and 5.3.2, we need to prove (1). The direct sum decomposition $\mathfrak{gl}(2) = \mathfrak{t} \oplus \mathfrak{sl}(2)$ is clear (the point is that $\mathfrak{t} \cap \mathfrak{sl}(2) = (0)$ since $\mathrm{char}(k) \neq 2$). Since the p-dimensional algebra $\mathrm{U}_\chi(\mathfrak{t})$ admits p one-dimensional modules parametrized by the set $\Lambda_{\mathfrak{t},\chi} := \{\lambda \in k : \lambda^p - \lambda - \chi(h_1 + h_2)^p = 0\}$, it follows that $\mathrm{U}_\chi(\mathfrak{t})$ is a product of p copies of the ground field. Now apply 5.3.1(1). $\square$

Let now $\chi \in \overline{\mathfrak{g}}^*$ $(\overline{\mathfrak{s}}^*)$. Recall that we write $\chi = (\chi_V, \chi_{\mathfrak{gl}(2)})$ $((\chi_V, \chi_{\mathfrak{sl}(2)}))$ with $\chi_V := \chi|_V \in V^*, \chi_{\mathfrak{gl}(2)} = \chi|_{\mathfrak{gl}(2)} \in \mathfrak{gl}(2)^*$ $(\chi_{\mathfrak{sl}(2)} = \chi|_{\mathfrak{sl}(2)} \in \mathfrak{sl}(2)^*)$. In addition to (i), (ii) from the sequel above 5.3.2 we make the following identification:

(iii) The $\mathrm{GL}(2)$-module V^* is isomorphic to the twist V^τ where $\tau(g) = (g^{-1})^t$ for all $g \in \mathrm{GL}(2)$. Here a vector $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) \in V^\tau$ corresponds to the linear form $\alpha r^* + \beta s^* \in V^*$, where $r = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), s = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is the standard basis of V and (r^*, s^*) the corresponding dual basis of V^*.

Recall from 2.5.27, that the group $(\mathbb{G}_a^2 \rtimes \mathrm{GL}(2))(k)$ acts on the dual spaces $\overline{\mathfrak{g}}^*$ and $\overline{\mathfrak{s}}^*$. We recall that, given $\mathfrak{h} \in \{\mathfrak{gl}(2), \mathfrak{sl}(2)\}$, the map $\psi = \psi_V$ is given by

$$\psi : V \times V^* \to \mathfrak{h}^*,\ (v, \lambda) \mapsto (x \mapsto (x.\lambda)(v) = -\lambda(x.v)).$$

A direct computation shows:

Lemma 5.3.4. *Let* $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in \mathbb{G}_a^2(k)$*. Then the following statements hold:*

(1) Let $\chi = (\chi_V, \chi_{\mathfrak{sl}(2)}) \in \overline{\mathfrak{s}}^*$ *be a linear form. Writing* $\chi_V = \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) \in V^\tau \cong V^*$*, we have*

$$v.\chi = (\chi_V,\ \chi_{\mathfrak{sl}(2)} - \psi(v, \chi_V)) = \left(\chi_V,\ \chi_{\mathfrak{sl}(2)} + \begin{pmatrix} \frac{\alpha v_1 - \beta v_2}{2} & \beta v_1 \\ \alpha v_2 & -\frac{\alpha v_1 - \beta v_2}{2} \end{pmatrix}\right).$$

(2) Let $\chi = (\chi_V, \chi_{\mathfrak{gl}(2)}) \in \overline{\mathfrak{g}}^*$ *be a linear form. Writing* $\chi_V = \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) \in V^\tau \cong V^*$*, we have*

$$v.\chi = (\chi_V,\ \chi_{\mathfrak{gl}(2)} - \psi(v, \chi_V)) = \left(\chi_V,\ \chi_{\mathfrak{gl}(2)} + \begin{pmatrix} \alpha v_1 & \beta v_1 \\ \alpha v_2 & \beta v_2 \end{pmatrix}\right).$$

Lemma 5.3.5. *Let* $v := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \in V^\tau$.

(1) The stabilizers of $v \in V^\tau$ *relative to* $\mathrm{GL}(2)$ *(its Lie algebra; here the twist is given by* $\tau(x) = -x^t$ *for* $x \in \mathfrak{gl}(2)$*) are given by*

$$\mathrm{Stab}_{\mathrm{GL}(2)}(v) = \{\left(\begin{smallmatrix}x & y\\0 & 1\end{smallmatrix}\right) \mid x \in k^\times, y \in k\}, \qquad \mathrm{Stab}_{\mathfrak{gl}(2)}(v) = \{\left(\begin{smallmatrix}a & b\\0 & 0\end{smallmatrix}\right) \mid a, b \in k\}$$

(In particular, $\mathrm{Stab}_{\mathfrak{gl}(2)}(v) = \mathrm{Lie}(\mathrm{Stab}_{\mathrm{GL}(2)}(v))$*).*

(2) Given an invertible matrix $g = \left(\begin{smallmatrix}x & y\\0 & 1\end{smallmatrix}\right) \in \mathrm{Stab}_{\mathrm{GL}(2)}(v)$ *and* $A = \left(\begin{smallmatrix}a & 0\\c & 0\end{smallmatrix}\right) \in \mathfrak{gl}(2)$*, we have*

$$gAg^{-1} = \frac{1}{x}\begin{pmatrix} ax + cy & -y(ax + cy) \\ c & -cy \end{pmatrix}.$$

Given $\mathfrak{h} \in \{\mathfrak{gl}(2), \mathfrak{sl}(2)\}$, we let $\mathcal{O}_V = \mathcal{O}_{V,\mathfrak{h}}$ be the open subset of the affine variety $\overline{\mathfrak{h}}^* = (V \rtimes \mathfrak{h})^*$ consisting of linear forms, which do not vanish on V. The group $G := \mathbb{G}_a^2 \rtimes \mathrm{GL}(2)$ stabilizes $\mathcal{O}_V$. In view of 2.5.5, it is useful to compute a full set of orbit representatives relative to this action.

Lemma 5.3.6. *Let* $G := \mathbb{G}_a^2 \rtimes \mathrm{GL}(2)$*. Then the following statements hold:*

(1) The group $G(k)$ *acts on* $\mathcal{O}_{V,\mathfrak{sl}(2)}$ *with two orbits, which are represented by*

$$\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{2} & 0\\ 1 & -\frac{1}{2}\end{pmatrix}\right), \quad \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\right).$$

Thus, for every $\chi \in \mathcal{O}_{V,\mathfrak{sl}(2)}$ *there is exactly one such representative, which we will call the* ***normal form of*** χ*.*

(2) The group $G(k)$ *acts on* $\mathcal{O}_{V,\mathfrak{gl}(2)}$ *with infinitely many orbits. A complete set of orbit-representatives is given by*

$$\chi_a := \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\right) \ (a \in k), \qquad \chi_s := \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right).$$

Thus, for every $\chi \in \mathcal{O}_{V,\mathfrak{gl}(2)}$ *there is exactly one such representative, which we will call the* ***normal form of*** χ*.*

Proof. (1) Let $\chi = (\chi_V, \left(\begin{smallmatrix}a & b\\c & -a\end{smallmatrix}\right)) \in \mathcal{O}_{V,\mathfrak{sl}(2)}$ (see 5.3.4). As $\mathrm{GL}(2)$ acts transitively on $V^\tau \setminus \{0\}$, we can assume that $\chi_V = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Then 5.3.4 implies, that the element $(-b, a-1) \in \mathbb{G}_a^2(k)$ takes χ to

$$\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{2} & 0\\ c & -\frac{1}{2}\end{pmatrix}\right).$$

If $c \neq 0$, then the diagonal matrix $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Stab}_{\mathrm{GL}(2)}(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$ takes χ to

$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}\right).$$

We are left to show, that the two linear forms

$$\chi' := \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}\right), \quad \chi := \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right)$$

are not equivalent under the action of the group $G(k)$. Assume that there is $vg \in G(k)$ such that $vg.\chi = \chi'$. Then (see 5.3.4, 5.3.5), we must have

$$g \in \mathrm{Stab}_{\mathrm{GL}(2)}(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a \in k^\times, b \in k\}.$$

Thus, there are $a \in k^\times, b \in k$ such that $g = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$. It follows, that

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} = \chi'_{\mathfrak{sl}(2)} = g.\chi_{\mathfrak{sl}(2)} + \begin{pmatrix} -\frac{v_2}{2} & v_1 \\ 0 & \frac{v_2}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -b \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{v_2}{2} & v_1 \\ 0 & \frac{v_2}{2} \end{pmatrix}.$$

Hence $0 = 1$, which is absurd. Thus, χ and χ' are not equivalent.

(2) Let $\chi \in \mathcal{O}_{V,\mathfrak{gl}(2)}$. As $\mathrm{GL}(2)$ acts transitively on $V^\tau \setminus \{0\}$, we can assume that $\chi_V = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Now write $\chi_{\mathfrak{gl}(2)} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. After acting by $\left(\begin{smallmatrix} -b \\ -d \end{smallmatrix}\right) \in \mathbb{G}_a^2(k)$, we can assume $\chi_{\mathfrak{gl}(2)} = \left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right)$. If $c \neq 0$, we put $g := \left(\begin{smallmatrix} c & -a \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Stab}_{\mathrm{GL}(2)}(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$ and $v := \left(\begin{smallmatrix} 0 \\ -a \end{smallmatrix}\right) \in \mathbb{G}_a^2$, then (see 5.3.5)

$$(vg).\chi = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} a + c\frac{-a}{c} & a(a-a)+0 \\ \frac{c}{c} & a-a \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right).$$

We are left to show, that for $a \neq b \in k$, we have $\chi_a \nsim \chi_b$, as well as $\chi_a \nsim \chi_s$. Let $vg \in G(k)$, then we have $(vg.\chi_a)|_V = g.\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Thus, $vg.\chi_a = \chi_b$ or $vg.\chi_a = \chi_s$ implies $g \in \mathrm{Stab}_{\mathrm{GL}(2)}(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$. Thus, in any case, there is $x \in k^\times, y \in k$ such that $g = \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$. Then (see again 5.3.5)

$$vg.\chi_a = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} a & -ya+v_1 \\ 0 & v_2 \end{pmatrix}\right).$$

As $a \neq b$, the right-hand side cannot equal χ_b and, obviously, it does not equal χ_s. $\square$

We can now determine the reduced enveloping algebras $\mathrm{U}_\chi(V \rtimes \mathfrak{h})$ up to Morita equivalence for $\mathfrak{h} \in \{\mathfrak{gl}(2), \mathfrak{sl}(2)\}$ and $\chi \in \mathcal{O}_{V,\mathfrak{h}}$. Therefore we need the following lemmas.

Lemma 5.3.7. *Let $\mathfrak{b} = kh \oplus ke$ be the unique non-abelian two-dimensional restricted Lie algebra with bracket and p-map determined by $[h, e] = e$, $h^{[p]} = h, e^{[p]} = 0$. Let $\chi \in \mathfrak{b}^*$ be a linear form. If $\chi(e) \neq 0$, then $\mathrm{U}_\chi(\mathfrak{b}) \cong \mathrm{Mat}_p(k)$ is a simple algebra.*

Proof. Put $\mathfrak{u} := ke \trianglelefteq \mathfrak{b}$. In view of 2.5.6(2), the algebra $\mathrm{U}_\chi(\mathfrak{u}) \cong \mathrm{U}_0(\mathfrak{u})$ is local with unique simple module being k_χ with action given by $e.1 = \chi(e).1$. We consider the p-dimensional $\mathrm{U}_\chi(\mathfrak{b})$-module $S := \mathrm{Ind}_{\mathfrak{u}}^{\mathfrak{b}}(k_\chi, \chi)$. As $\mathrm{U}_\chi(\mathfrak{b})$ is a p^2-dimensional algebra, the assertion will follow from Wedderburn's theorem if S is simple. We give two arguments for that, the first one is direct, the second general theory:

(a) Let $\mathcal{B} := (v_m := h^m \otimes 1 \mid 0 \leq m \leq p-1)$ be the standard basis of S. By the Cartan Weyl formula [25, I.1.3(4)], we have $eh^m = (\sum_{j=0}^{m} \alpha_{m,j} h^j)e$, where $\alpha_{m,j} = (-1)^{m-j}\binom{m}{j} \neq 0$. Hence, we get

$$e.v_m = \chi(e) \sum_{j=0}^{m} \alpha_{m,j} v_j.$$

Since $\alpha_{m,m} = 1$, it follows that the matrix representating the endomorphism $(e - \chi(e).\mathrm{id}_S) : S \to S$ relative $\mathcal{B}$ is strictly upper triangular, hence has a one-dimensional kernel. Since $v_0 = 1 \otimes 1$ lies in this kernel, we get that

$$kv_0 = \{s \in S \mid e.s = \chi(e).s\} = \mathrm{Soc}_{\mathrm{U}_\chi(\mathfrak{u})}(S).$$

Let now $X \subseteq S$ be a non-zero $\mathrm{U}_\chi(\mathfrak{b})$-submodule, then X must contain a simple $\mathrm{U}_\chi(\mathfrak{u})$-submodule. By the above, we get $1 \otimes 1 \in X$. Hence $S = \mathrm{U}_\chi(\mathfrak{b}).1 \otimes 1 \subseteq X$.

(b) It is well-known that the dimension of a simple modules for a solvable Lie algebra in positive characteristic p is always of the form p^n for $n \in \mathbb{N}_0$ (see [25, Corollary V.8.5]). For dimension reasons, only the cases p or $p^0 = 1$ can occur in our context. But in the latter case, 2.5.6(1) would imply the existence of a linear form $\lambda \in \mathfrak{b}^*$ with the property $0 = \lambda([\mathfrak{b}, \mathfrak{b}]) = \lambda(\mathfrak{u})$ and

$$0 = 0 - 0 = \lambda(e)^p - \lambda(e^{[p]}) = \chi(e)^p.$$

Since $\chi(e) \neq 0$, we obtain a contradiction.

□

Theorem 5.3.8. *Consider the restricted Lie algebras $\overline{\mathfrak{g}} := V \rtimes \mathfrak{gl}(2), \overline{\mathfrak{s}} := V \rtimes \mathfrak{sl}(2)$, where V denotes the natural representation. Then the following statements hold:*

(1) Let $\chi \in \mathcal{O}_{V,\mathfrak{sl}(2)} = \{\chi \in \overline{\mathfrak{s}}^* : \chi(V) \neq 0\}$ be a linear form.

(a) The algebra $\mathrm{U}_\chi(\overline{\mathfrak{s}})$ admits exactly one simple module S_χ of dimension p^2.

(b) *The module* S_χ *is not projective, its projective cover* P_χ *of* S_χ *is* p^3*-dimensional and* $[P_\chi : S_\chi] = p$*. We have* $P_\chi \cong \mathrm{Ind}_{\mathfrak{sl}(2)}^{\bar{\mathfrak{s}}}(L, \chi)$ *where* L *is any simple* $\mathrm{U}_\chi(\mathfrak{sl}(2))$ *module.*

(c) $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ *is isomorphic to the Nakayama algebra* $\mathrm{Mat}_{p^2}(k[X]/(X^p))$*.*

(d) $\{\chi \in \bar{\mathfrak{s}}^* : \mathrm{U}_\chi(\bar{\mathfrak{s}})$ *is representation-finite*$\} = \mathcal{O}_{V,\mathfrak{sl}(2)}$

(2) *Let* $\chi \in \mathcal{O}_{V,\mathfrak{gl}(2)}$ *be a linear form.*

(a) *If* $\chi \sim \chi_s$*, then* $\mathrm{U}_\chi(\bar{\mathfrak{g}})$ *admits exactly one simple module of dimension* p^3 *up to isomorphism. In particular,* $\mathrm{U}_\chi(\bar{\mathfrak{g}})$ *is a simple algebra.*

(b) *Let* $\chi \sim \chi_a$ *for some* $a \in k$*, then the algebra* $\mathrm{U}_\chi(\bar{\mathfrak{g}})$ *is Morita equivalent to the Nakayama algebra* $k\tilde{A}_{p,0}/(k\tilde{A}_{p,0})_{\geq p}$*, where* $\tilde{A}_{p,0}$ *is the oriented circle with* p *vertices and* $(k\tilde{A}_{p,0})_{\geq p}$ *denotes the subspace generated by all paths of length* $\geq p$*. The simple* $\mathrm{U}_\chi(\bar{\mathfrak{g}})$*-modules and their projective covers are given as follows: Consider the* p*-subalgebra* $\mathfrak{b} := kh_1 \oplus ke$ *of* $\mathfrak{gl}(2)$*, as well as the subset*

$$\Lambda_\chi = \Lambda_\chi(\mathfrak{b}, V) := \{\lambda \in \mathfrak{b}_V^* : 0 = \lambda(e) = \lambda(r), 1 = \lambda(s), \lambda(h_1)^p - \lambda(h_1) = \chi(h_1)^p\}$$

of the dual space $(V \rtimes \mathfrak{b})^* = \mathfrak{b}_V^*$*, which has cardinality* p*. Then the* p^2*-dimensional modules* $S_\lambda := \mathrm{Ind}_{\mathfrak{b}_V}^{\bar{\mathfrak{g}}}(\lambda, \chi)$ *for* $\lambda \in \Lambda_\chi$ *give a complete list of simple* $\mathrm{U}_\chi(\bar{\mathfrak{g}})$*-modules up to isomorphism. The projective cover* P_λ *of* S_λ *is a uniserial* p^3*-dimensional* $\mathrm{U}_\chi(\bar{\mathfrak{g}})$*-module of length* p*; we have* $\mathrm{Rad}^i(P)/\mathrm{Rad}^{i+1}(P) \cong S_{\lambda(i)}$*, where* $\lambda(i)(h_1) = \lambda(h_1) + i$ *for all* $0 \leq i \leq p-1$*.*

(c) $\{\chi \in \bar{\mathfrak{g}}^* : \mathrm{U}_\chi(\mathfrak{gl}(2)_V)$ *is representation-finite*$\} = \mathcal{O}_{V,\mathfrak{gl}(2)}$*.*

Proof. (1) In view of 2.5.5, we may throughout assume that there exists $\beta \in \{0, 1\}$ such that $\chi = \beta e^* + h^* + s^* \in \mathcal{O}_V$ is in its normal form in the sense of 5.3.6. In particular, we have $\chi(r) = 0$ and $\chi(s) = 1$.

(a) We wish to give an application of Theorem 4.3.1. Direct computation shows

$$\mathfrak{h} := \bar{\mathfrak{s}}^{\chi} = ke \oplus ks \oplus kr$$

is a p-trivial Heisenberg algebra. As χ vanishes on its central element r, it admits exactly one simple module with character χ; namely k_χ with underlying vector space k and operation given by

$$x.\alpha = \chi(x).\alpha \ \ \forall x \in \mathfrak{h}, \alpha \in k_\chi.$$

Now 4.3.1 implies that the p^2-dimensional module $S_\chi := \mathrm{Ind}_{\mathfrak{h}}^{\bar{\mathfrak{s}}}(k_\chi, \chi)$ is the unique simple $\mathrm{U}_\chi(\bar{\mathfrak{s}})$-module.

(b) The first two statements follow for dimension reasons. Let L be a simple $\mathrm{U}_\chi(\mathfrak{sl}(2))$-module. Then L is projective and p-dimensional (see 5.3.2(a)). Hence $\mathrm{Ind}_{\mathfrak{sl}(2)}^{\bar{\mathfrak{s}}}(L, \chi)$ is projective and its dimension coincides with the dimension of P_χ. Hence they must be isomorphic.

(c) A basis of $S_\chi = \mathrm{Ind}_{\mathfrak{h}}^{\bar{\mathfrak{s}}}(k_\chi)$ is given by $\{v_{n,m} := f^n h^m \otimes 1 \mid 0 \leq n, m \leq p-1\}$. Using the Cartan Weyl-formula (see [25, Proposition I.1.3]), direct computation shows that the operation of $V = kr \oplus ks$ is determined by

- $r.v_{n,m} = \begin{cases} -n\chi(s) \sum_{j=0}^{m} \binom{m}{j} v_{n-1,j} & n \neq 0 \\ 0 & n = 0 \end{cases}$
- $s.v_{n,m} = \chi(s) \sum_{j=0}^{m} \binom{m}{j} v_{n,j}$.

We will use these formulas below.

Consider the $\mathrm{U}_0(\mathfrak{h})$-module $M(\chi) := k_{-\chi} \otimes_k S_\chi$. By [11, Lemma 2.1(1), Corollary 2.5], we have natural isomorphisms

$$\mathrm{Ext}^n_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S_\chi, S_\chi) \cong \mathrm{Ext}^n_{\mathrm{U}_\chi(\mathfrak{h})}(k_\chi, S_\chi) \cong H^n(\mathrm{U}_0(\mathfrak{h}), M(\chi)), \quad \forall n \geq 0 \quad (*)$$

We want to show the $\mathrm{U}_0(V)$-projectivity of $M(\chi)$. Taking a look at the chain $(0) \subset V \subset \mathfrak{h} \subset \bar{\mathfrak{s}}$ of subalgebras, we see that $\mathfrak{h} \in \mathcal{P}_{S_\chi}(\chi, \chi)$ is a polarization in the sense of [13, Def. 2.1]. The central element $r \in \mathfrak{h}$ satisfies the condition of [13, Proposition 3.6(1)] (direct computation shows $\mathrm{Rad}(\beta_\chi) = k(e + r + 2\beta s)$). Hence S_χ is $\mathrm{U}_\chi(kr)$-projective, so that $M(\chi)$ is $\mathrm{U}_0(kr)$-projective (see 2.5.4(6)). By [13, Lemma 3.1], $M(\chi)^{kr}$ is p-dimensional. As the p linearly independent vectors $(1 \otimes v_{0,m} \mid 0 \leq m \leq p-1)$ get annihilated by r, we conclude

$$M_r(\chi) := M(\chi)^{kr} = \langle (1 \otimes v_{0,m} \mid 0 \leq m \leq p-1) \rangle_k.$$

Moreover, we have

$$\begin{aligned} s.1 \otimes v_{0,m} &= 1 \otimes s.v_{0,m} + s.1 \otimes v_{0,m} = 1 \otimes \sum_{j=0}^{m} \binom{m}{j} v_{0,j} - 1 \otimes v_{0,m} \\ &= \sum_{j=0}^{m-1} \underbrace{\binom{m}{j}}_{\neq 0} 1 \otimes v_{0,j}. \end{aligned}$$

Therefore the corresponding matrix representing the action of s on $M_r(\chi)$ is strictly upper triangular with each element on the off-diagonal being non-zero. Consequently the matrix is of rank $p-1$ and $M_r(\chi)$ is the unique principal indecomposable $\mathrm{U}_0(ks)$-module, hence free of rank 1. Moreover, again by [13, Lemma 3.1], the space $M_r(\chi)^{ks}$ is one-dimensional (it is given by $k(1 \otimes v_{0,0})$). But since $M_r(\chi)^{ks} = (M(\chi)^{kr})^{ks} = M(\chi)^V$ coincides with the space of V-invariants of $M(\chi)$, [13, Lemma 3.1] now yields the $\mathrm{U}_0(V)$-projectivity of $M(\chi)$. Therefore the spectral sequence

$$H^r(\mathrm{U}_0(\mathfrak{h}/V), H^s(\mathrm{U}_0(V), M(\chi)) \Rightarrow H^n(\mathrm{U}_0(\mathfrak{h}), M(\gamma))$$

collapses to isomorphisms

$$H^n(\mathrm{U}_0(\mathfrak{h}/V), k(1 \otimes v_{0,0}) = M(\chi)^V) \cong H^n(\mathrm{U}_0(\mathfrak{h}), M(\chi)) \;\; \forall n \geq 0.$$

Since $\mathfrak{h}/V = k(e+V) \cong \mathfrak{e}_1$ is isomorphic to the elementary abelian Lie algebra of dimension 1, the restricted enveloping algebra $\mathrm{U}_0(\mathfrak{h}/V)$ is isomorphic to the truncated polynomial ring $k[X]/(X^p)$. Now 2.1.40 in conjunction with (∗) and [12, Lemma 2.1] show that $\mathrm{cx}_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S_\chi) \leq 1$. Since S_χ is not projective, we get $\mathrm{cx}_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S_\chi) = 1$. The assertion now follows from (b) and [12, Theorem 3.3(2)].

(d) We have seen, that every algebra $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ for $\chi \in \mathcal{O}_V$ is representation-finite. If $\chi(V) = 0$, then the two-dimensional variety V is contained in the rank variety $V(\bar{\mathfrak{s}})_S$ of every simple $\mathrm{U}_\chi(\bar{\mathfrak{s}})$-module S (see 4.2.4(1)). Thus, S is not periodic (see 2.5.7(5)), so that $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ cannot be representation-finite.

(2) (a) We can assume that $\chi = \chi_s$. We already computed that $\mathrm{Stab}_{\mathfrak{gl}(2)}((\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})) = \mathfrak{b} = kh_1 \oplus ke$. Thus, 4.3.1 shows that simple $\mathrm{U}_\chi(\mathfrak{gl}(2)_V)$-modules are induced by simple $\mathrm{U}_\chi(\mathfrak{b}_V)$-modules and that this correspondence is one-one. As $kr \trianglelefteq \mathfrak{b}_V$ is a unipotent ideal and $\chi(r) = 0$, simple $\mathrm{U}_\chi(\mathfrak{b}_V)$-modules can be viewed as simple $\mathrm{U}_\chi(\mathfrak{g})$-modules (see 4.2.2), where $\mathfrak{g} := \mathfrak{b}_V/kr$ is a three-dimensional Lie algebra with basis (h_1, e, s) (by abuse of notation), bracket and p-map determined by

$$[h_1, e] = e,\; [e, s] = 0,\; [h_1, s] = 0,\; h_1^{[p]} = h_1,\; 0 = e^{[p]} = s^{[p]}.$$

Thus, we write

$$\mathfrak{g} = \mathfrak{b} \oplus ks \cong \mathfrak{b} \oplus \mathfrak{e}_1$$

as a direct sum of restricted Lie algebras. Hence, simple $\mathrm{U}_\chi(\mathfrak{g})$-modules are simple $\mathrm{U}_\chi(\mathfrak{b}) \otimes_k \mathrm{U}_\chi(ks)$-modules (see 5.3.1(1)). As $\chi(e) \neq 0$, the algebra $\mathrm{U}_\chi(\mathfrak{b})$ has exactly one simple module, which is of dimension p (see 5.3.7) and $\mathrm{U}_\chi(ks)$ admits exactly one of dimension 1. In view of 2.1.30(2), the algebra $\mathrm{U}_\chi(\mathfrak{b}_V)$ also has exactly one simple module T, which is of dimension p. As an upshot of the above, it follows that the p^3-dimensional module $S := \mathrm{Ind}_{\mathfrak{b}_V}^{\bar{\mathfrak{g}}}(T, \chi)$ is the unique simple $\mathrm{U}_\chi(\bar{\mathfrak{g}})$-module. As $\mathrm{U}_\chi(\bar{\mathfrak{g}})$ is $p^3 \cdot p^3 = p^6$-dimensional, the assertion follows.

(b) Let $\chi = \chi_a$ for some $a \in k$. It again follows from Theorem 4.3.1, that all simple $\mathrm{U}_\chi(\mathfrak{gl}(2)_V)$-modules are induced by simple $\mathrm{U}_\chi(\mathfrak{b}_V)$-modules and that this correspondence is one-one. The first derived subalgebra $[\mathfrak{b}_V, \mathfrak{b}_V] = kr \oplus ke \trianglelefteq \mathfrak{b}_V$ is unipotent. As $0 = \chi(r) = \chi(e)$, it follows that simple $\mathrm{U}_\chi(\mathfrak{b}_V)$-modules can be viewed as simple $\mathrm{U}_\chi(\mathfrak{l})$-modules (see 4.2.2), where $\mathfrak{l} := \mathfrak{b}_V/(kr \oplus ke)$ is a two-dimensional abelian Lie algebra with basis (h_1, s) and p-map determined by $h_1^{[p]} = h$ and $s^{[p]} = 0$. Thus, $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{e}_1$ is a direct sum of the torus $\mathfrak{t} := kh_1$ and the one-dimensional elementary abelian Lie algebra $\mathfrak{e}_1 = ks$. Hence, every simple $\mathrm{U}_\chi(\mathfrak{l})$-module is one-dimensional and corresponds to some linear form $\lambda : \mathfrak{l} \to k$ such that $\lambda(h_1)^p - \lambda(h_1) = \chi(h_1)^p$ and $\lambda(s) = \chi(s) = 1$ (see 2.5.6(1)). We have shown that $\{S_\lambda := \mathrm{Ind}_{\mathfrak{b}_V}^{\bar{\mathfrak{g}}}(k_\lambda, \chi) \mid \lambda \in \Lambda_\chi\}$ is a complete set of simple $\mathrm{U}_\chi(\bar{\mathfrak{g}})$-modules up to isomorphism.

Note that the restriction map $\bar{\mathfrak{g}}^* \to \bar{\mathfrak{s}}^*$ maps $\mathcal{O}_{V,\mathfrak{gl}(2)}$ to $\mathcal{O}_{V,\mathfrak{sl}(2)}$. Let $\lambda \in \Lambda_\chi$. Since $V(\bar{\mathfrak{g}}) = V(\bar{\mathfrak{s}}) = V \times V(\mathfrak{sl}(2))$ (see 5.1.5(2), 4.2.8(2)), (1) shows that $V(\bar{\mathfrak{g}})_{S_\lambda} = V(\bar{\mathfrak{s}})_T$ is one-dimensional (here T denotes the unique simple $\mathrm{U}_\chi(\bar{\mathfrak{s}})$-module). Consequently, $\mathrm{cx}_{\mathrm{U}_\chi(\bar{\mathfrak{g}})}(S_\lambda) = 1$ (see 2.5.7(5)). We can now collect all missing assertions from [12, Theorem 3.2], 2.1.39 and the following: Let

$$\mu : \mathrm{U}_\chi(\bar{\mathfrak{g}}) \to \mathrm{U}_\chi(\bar{\mathfrak{g}}),\ \bar{\mathfrak{g}} \ni x \mapsto x + \mathrm{tr}(\mathrm{ad}(x)).1$$

be the Nakayama automorphism of $\mathrm{U}_\chi(\bar{\mathfrak{g}})$. As $\mu(e) = e, \mu(h_1) = h_1 + 1$, we see that $1 \otimes 1 \mapsto 1 \otimes 1$ defines via Frobenius reciprocity a homomorphism $S_\lambda^{\mu^i} \to S_{\lambda(-i)}$ which is surjective by simplicity and hence bijective for dimension reasons.

(c) Proceed as in (1)(d).

□

We can now apply Lemma 4.3.3 and get, as stated in the beginning, the following

Corollary 5.3.9. *Let $\bar{\mathfrak{g}} = V \rtimes \mathfrak{gl}(2), \bar{\mathfrak{s}} = V \rtimes \mathfrak{sl}(2)$, where V denotes the natural representation. Then the following statements hold:*

(1) The algebra $\mathrm{U}_\chi(\bar{\mathfrak{g}})$ is connected for every linear form $\chi \in \bar{\mathfrak{g}}^*$.

(2) The algebra $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ is connected for every linear form $\chi \in \bar{\mathfrak{s}}^*$.

5.4 Gabriel quivers in the remaining cases

We wish to compute the Gabriel quivers of reduced enveloping algebras of $\bar{\mathfrak{s}}, \bar{\mathfrak{g}}$ explicitly where the defining linear form vanishes on $V = k^2$. We already know from Theorem 4.2.7 that these quivers are precisely generalized McKay quivers relative to the natural representation V of $\mathfrak{sl}(2), \mathfrak{gl}(2)$. We begin with the case of $\bar{\mathfrak{s}} = V \rtimes \mathfrak{sl}(2)$.

Put $\mathfrak{u}^+ := kr \oplus ke, \mathfrak{u}^- := ks \oplus kf$ and observe that this defines a *triangular decomposition* $\bar{\mathfrak{s}} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}^+$ with torus $\mathfrak{t} = kh$ and unipotent constitutients $\mathfrak{u}^\pm$. This allows us to define a standard class of modules having a character vanishing on $\mathfrak{u}^+$: Let $\chi \in \bar{\mathfrak{s}}^*$ a linear form such that $\chi(\mathfrak{u}^+) = 0$, put $\mathfrak{b} := \mathfrak{t} \oplus \mathfrak{u}^+$ and denote for $\lambda \in \Lambda_\chi := \{\lambda \in k \mid \lambda^p - \lambda - \chi^p(h) = 0\}$ by k_λ the one-dimensional $\mathfrak{b} = \mathfrak{u}^+ \rtimes \mathfrak{t}$-module with action given by

$$\mathfrak{u}^+.1 = (0),\ h.1 = \lambda.1.$$

The induced module $\overline{Z}_\chi(\lambda) := \mathrm{Ind}_{\mathfrak{b}}^{\bar{\mathfrak{s}}}(k_\lambda, \chi)$ is then referred to as a *reduced Verma module with highest weight* λ. Recall that we have introduced the corresponding construction for $\mathfrak{sl}(2)$ leading to Verma modules $Z_\chi(\lambda) := \mathrm{Ind}_{kh\oplus ke}^{\mathfrak{sl}(2)}(k_\lambda, \chi)$ with highest weight $\lambda \in \Lambda_\chi$ where $\chi \in \mathfrak{sl}(2)^*$ with $\chi(e) = 0$. Note that if $\lambda \in \Lambda_\chi$, then $\Lambda_\chi = \lambda + \mathbb{F}_p$.

In analogy to 5.2.8, one can show

Lemma 5.4.1. *Let $\chi \in \bar{\mathfrak{s}}^*$ such that $\chi(\mathfrak{u}^+) = 0$. The restriction of $\overline{Z}_\chi(\lambda)$ to $\mathrm{U}_\chi(\mathfrak{sl}(2))$ is isomorphic to $\bigoplus_{\lambda\in\Lambda_\chi} Z_\chi(\lambda)$.*

Let now $\chi \in \bar{\mathfrak{s}}^*$ such that $\chi(V) = (0)$. Owing to 2.5.27, the coadjoint orbits under our group $\overline{\mathcal{G}}(k)$ are then given by the Jordan canonical form of $\chi_{\mathfrak{sl}(2)}$. We therefore consider the following three types of linear forms:

- The form $\chi = 0$. This case has already been treated.
- The forms $\chi = \chi_s$ with $\chi_s(h) \neq 0$ and $0 = \chi_s(kf \oplus ke \oplus V)$.
- The form χ_n with $\chi_s(f) = 1$ and $0 = \chi_s(kh \oplus ke \oplus V)$.

Lemma 5.4.2. *Let $\chi \in \mathfrak{sl}(2)^*$ be a linear form such that $\chi(e) = 0$. Then the $\mathrm{U}_\chi(\mathfrak{sl}(2))$-module $L(i) \otimes_k Z_\chi(\lambda)$ possesses a filtration with factors*

$$\{Z_\chi(\mu) : \mu \in \{\lambda + i - 2s \mid 0 \leq s \leq i\}\}.$$

In particular, if $0 = \chi(ke \oplus kf)$, then $L(i) \otimes_k Z_\chi(\lambda) \cong \bigoplus_{\mu\in\{\lambda+i-2s|0\leq s\leq i\}} Z_\chi(\mu)$.

Proof. The h-weights of $L(i) \otimes_k k_\lambda$ are given by $\{\lambda + i - 2s \mid 0 \leq s \leq i\}$ (cf. [26, p.12 above]). Considering the Borel subalgebra $\mathfrak{b}_{\mathfrak{sl}(2)} = ke \rtimes kh$, 4.2.4(2c) shows that the composition factors of the $\mathfrak{b}_{\mathfrak{sl}(2)}$-module $L(i) \otimes_k k_\lambda$ are given by

$$\{k_\mu : \mu \in \{\lambda + i - 2s \mid 0 \leq s \leq i\}\}.$$

In view of the tensor identity $\mathrm{Ind}_{\mathfrak{b}_{\mathfrak{sl}(2)}}^{\mathfrak{sl}(2)}(L(i)\otimes_k k_\lambda, \chi) \cong L(i)\otimes_k Z_\chi(\lambda)$ (see [40, 1.12(1)]), the assertion follows from the exactness of the induction functor. The additional statement is clear, since $\mathrm{U}_\chi(\mathfrak{sl}(2))$ is semi-simple in case $\chi(ke \oplus kf) = 0$ (see 5.3.2). $\square$

For every linear form $\chi \in \bar{\mathfrak{s}}^*$ such that $\chi(V) = 0$, simple $\mathrm{U}_\chi(\bar{\mathfrak{s}})$-modules correspond to $\mathrm{U}_\chi(\mathfrak{sl}(2))$-modules in the sense that they are inflations of the latter (see 4.2.4). First, we take a look at a form of type χ_s. Here $\mathrm{U}_\chi(\mathfrak{sl}(2))$ is semi-simple with p simple modules given by $\{Z_\chi(\lambda) : \lambda \in \Lambda_\chi\}$ (see 5.3.2). We put $S(\lambda) := \mathrm{Inf}(Z_\chi(\lambda))$.

Theorem 5.4.3. *Let $\chi \in \bar{\mathfrak{s}}^*$ be of type χ_s.*

(a) The modules $S(\lambda)$ ($\lambda \in \Lambda_\chi$) form a complete set of representatives for the isomorphism classes of simple $\mathrm{U}_\chi(\bar{\mathfrak{s}})$-modules. The projective cover of $S(\lambda)$ is given by $Q(\lambda) \cong \mathrm{U}_0(L(1)) \otimes_k Z_\chi(\lambda)$. We have $[Q(\lambda) : S(\mu)] = p$ for all $\mu \in \Lambda_\chi$.

(b) The Jacobson radical of $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ is given by the ideal $I = \mathrm{U}_\chi(\bar{\mathfrak{s}})\mathrm{U}_0(V)^\dagger$ generated by V.

(c) We have $\dim_k \mathrm{Ext}^1_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S(\lambda), S(\mu)) = \begin{cases} 1 & \mu \in \{\lambda \pm 1\} \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (a) The first two statements follow from 4.2.4. Since the torus $T \subseteq \mathrm{SL}(2)$ stabilizes the form χ_s, it follows that the algebra $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ inherits a $X(T) = \mathbb{Z}$-grading, which by the PBW theorem satisfies the requirements of [37, 2.1]. Using 5.4.1, the Cartan matrix of $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ can now be obtained as in 5.2.11(b).

(b) See 4.2.4(2e).

(c) In view of 4.2.7, we have

$$\dim_k \mathrm{Ext}^1_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S(\lambda), S(\mu)) = \dim_k \mathrm{Hom}_{\mathfrak{sl}(2)}(L(1) \otimes_k Z_\chi(\lambda), Z_\chi(\mu)).$$

The statement now follows from the isomorphism $L(1) \otimes_k Z_\chi(\lambda) \cong Z_\chi(\lambda - 1) \oplus Z_\chi(\lambda + 1)$ (see 5.4.2).

□

Now we consider the form $\chi = \chi_n$. The Verma modules are also simple in this case, but

$$Z_\chi(\lambda) \cong Z_\chi(\mu) \Longleftrightarrow \lambda = p - 2 - \mu$$

The modules $Z_\chi(i)$ with $0 \leq i \leq \frac{p-3}{2}$ and $Z_\chi(p-1)$ form a complete set of simple $\mathrm{U}_\chi(\mathfrak{sl}(2))$-modules. The *Steinberg module* $Z_\chi(p-1)$ is projective. Moreover, we have $\dim_k \mathrm{Ext}^1(Z_\chi(i), Z_\chi(j)) = \delta_{i,j}$ for $0 \leq i, j \leq \frac{p-3}{2}$ and the (up to some scalar) unique self extension of $Z_\chi(i)$ is given by its $2p$-dimensional projective cover $P(i)$.

Lemma 5.4.4. *Let $\chi \in \mathfrak{sl}(2)^*$ be the unique linear form such that $\chi(f) = 1$ and $\chi(kh \oplus ke) = 0$. Then we have the following isomorphism of $\mathrm{U}_\chi(\mathfrak{sl}(2))$-modules:*

$$L(1) \otimes_k Z_\chi(i) \cong \begin{cases} Z_\chi(\frac{p-3}{2}) \oplus Z_\chi(\frac{p-3}{2} - 1), & i = \frac{p-3}{2} \\ P(0), & i = p-1 \\ Z_\chi(p-1) \oplus Z_\chi(1), & i = 0 \\ Z_\chi(i-1) \oplus Z_\chi(i+1), & 1 \leq i \leq \frac{p-3}{2} - 1. \end{cases}$$

Proof. Put $T(i) := L(1) \otimes_k Z_\chi(i) \in \mathbf{mod}(\mathrm{U}_\chi(\mathfrak{sl}(2)))$. Starting with the obvious exact sequence

$$0 \longrightarrow k_1 \longrightarrow L(1) \longrightarrow k_{-1} \longrightarrow 0$$

in $\mathrm{U}_0(\mathfrak{b}_{\mathfrak{sl}(2)})$-modules, the arguments in the proof of 5.4.2 yield an exact sequence

$$0 \longrightarrow Z_\chi(i+1) \longrightarrow T(i) \longrightarrow Z_\chi(i-1) \longrightarrow 0$$

of $\mathrm{U}_\chi(\mathfrak{sl}(2))$-modules. Assume first that $0 \leq i \leq \frac{p-3}{2}$. Then $Z_\chi(i-1) \ncong Z_\chi(i+1)$ as

$$p - 2 - (i-1) = i + 1 \Longleftrightarrow p - 2 = 2i$$

which can not be the case (as $p \geq 3$ is odd). Consequently, the above sequence splits. Observing

- $Z_\chi(\frac{p-3}{2} + 1) \cong Z_\chi(\frac{p-3}{2})$,
- $Z_\chi(-1) = Z_\chi(p-1)$,

we get the assertion for all those i. Now let $i = p-1$. As $Z_\chi(p-1)$ is projective, we conclude that the $2p$-dimensional module $T(p-1)$ is projective and the above sequence shows that $Z_\chi(p) = Z_\chi(0)$ lies in its socle. Hence it must be $P(0)$. □

We put $S(i) := \mathrm{Inf}(Z_\chi(i))$ for all $0 \leq i \leq \frac{p-3}{2}$ as well as $S(p-1) := \mathrm{Inf}(Z_\chi(p-1))$.

Theorem 5.4.5. *Let $\chi = \chi_n \in \bar{\mathfrak{s}}^*$.*

(a) The algebra $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ has exactly $\frac{p+1}{2}$ simple modules $S(0), \ldots, S(\frac{p-3}{2}), S(p-1)$ up to isomorphism. The projective cover of $S(i)$ is given by $Q(i) = \mathrm{U}_0(L(1)) \otimes_k P(i)$.

(b) We have $\dim_k \mathrm{Ext}^1_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S(i), S(j)) = \begin{cases} \delta_{j,0} & i = p-1 \\ \delta_{j,p-1} + \delta_{j,1} & i = 0 \\ \delta_{j,i-1} + \delta_{j,i+1} & 1 \leq i \leq \frac{p-3}{2} - 1 \\ \delta_{j,i-1} & i = \frac{p-3}{2} \end{cases}$ *for $i \neq j$ and*

$$\dim_k \operatorname{Ext}^1_{\mathrm{U}_\chi(\bar{\mathfrak{s}})}(S(i),S(i)) = \begin{cases} 0 & i=p-1 \\ 1 & 0 \le i \le \frac{p-3}{2}-1 \\ 2 & i=\frac{p-3}{2}. \end{cases}$$

(c) $\mathrm{U}_\chi(\bar{\mathfrak{s}})$ *is of wild representation type.*

Proof. (a) See 4.2.4(1),(2b).

(b) This follows from 4.2.7 and our above lemma.

(c) Since the rank variety $V(\bar{\mathfrak{s}})_{S(0)} = V \times ke$ of $S(0)$ is three-dimensional (see 4.2.8(2)), this is a consequence of [19, Theorem 4.1].

□

The Gabriel quivers of reduced enveloping algebra $\mathrm{U}_\chi(\bar{\mathfrak{g}})$ such that $\chi(V)=0$ can be obtained in a similar fashion from our above computations. One reduces again to linear forms of three types, namely 0, the 'nilpotent' linear forms and the 'semi-simple' ones (see 5.3.2). By way of example, we do this for the form $\chi=0$, it should be clear afterwards how one can proceed in the other cases.

Since we have a direct sum decomposition $\mathfrak{gl}(2) = \mathfrak{t} \oplus \mathfrak{sl}(2)$, where $\mathfrak{t} := k(h_1+h_2)$, we obtain $\mathrm{U}_0(\mathfrak{gl}(2)) \cong \mathrm{U}_0(\mathfrak{sl}(2)) \otimes_k \mathrm{U}_0(\mathfrak{t})$. The simple $\mathrm{U}_0(\mathfrak{t})$ are the k_i for $0 \le i \le p-1$, where the action of $\mathfrak{t}$ on $k_i = k$ is given by $(h_1+h_2).1 = i.1$. Thus, by 2.1.30, we get that the modules $L(i,j) := L(i) \otimes_k k_j$ for $0 \le i,j \le p-1$ form a complete set of simple $\mathrm{U}_0(\mathfrak{gl}(2))$-modules up to isomorphism. The projective cover of $L(i,j)$ is then given by $P(i,j) := P(i) \otimes_k k_j$, where $P(i)$ is the projective cover of $L(i)$ over $\mathrm{U}_0(\mathfrak{sl}(2))$. In this notation, the natural representation V is exactly $L(1,1)$. We denote by $S(i,j)$ the inflation of $L(i,j)$ to $\mathrm{U}_0(\bar{\mathfrak{g}})$.

Theorem 5.4.6. *We have*

$$\dim_k \operatorname{Ext}^1_{\mathrm{U}_0(\bar{\mathfrak{g}})}(S(i,j),S(k,l)) = 2\delta_{j,l}\delta_{i+k,p-2} + \delta_{l,j+1} \cdot \begin{cases} \delta_{k,1}, & i=0 \\ \delta_{|i-k|,1}, & 1 \le i \le p-2 \\ \delta_{k,p-2}, & i=p-1 \end{cases}$$

for all $0 \le i,j,k,l \le p-1$.

Proof. By 4.2.7, we have

$$\operatorname{Ext}^1_{\mathrm{U}_0(\bar{\mathfrak{g}})}(S(i,j),S(k,l)) \cong \operatorname{Hom}_{\mathfrak{gl}(2)}(L(1,1)\otimes_k L(i,j),L(k,l)) \oplus \operatorname{Ext}^1_{\mathrm{U}_0(\mathfrak{gl}(2))}((L(i,j),L(k,l)).$$

Now 5.2.4 implies that

$$\begin{aligned} L(1,1) \otimes_k L(i,j) &\cong (L(1) \otimes_k L(i)) \otimes_k (k_1 \otimes_k k_j) \cong (L(1) \otimes_k L(i)) \otimes_k k_{j+1} \\ &\cong \begin{cases} L(1, j+1) & i = 0 \\ L(i-1, j+1) \oplus L(i+1, j+1) & 1 \leq i \leq p-2 \ , \\ P(p-2, j+1) & i = p-1. \end{cases} \end{aligned}$$

Combining $\dim_k \operatorname{Ext}^1_{\mathrm{U}_0(\mathfrak{gl}(2))}((L(i,j), L(k,l)) = 2\delta_{j,l}\delta_{i+k,p-2}$ with the above, we get the claim. $\square$

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch meinen Betreuer unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe, keine anderen als die angegebenen Hilfsmittel verwendet habe und dass kein akademischer Grad entzogen wurde.

Kiel, den 20. Juli 2021

(Jan-Niclas Thiel)