

Modules of generalized Kronecker quivers

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Abstract

Let k be an algebraically closed field, and let (0) denote 0-dimensional vector space over k . Recall that n -Kronecker quiver $K(n)$ is just the quiver with two vertices 1, 2, and n arrows $\gamma_1, \dots, \gamma_n : 1 \rightarrow 2$. We use $\text{rep}_k(K(n))$ to denote finite-dimensional representations of $K(n)$. If we use $\text{mod } \mathcal{K}_n$ to denote finite-dimensional modules of \mathcal{K}_n , where $\mathcal{K}_n = kK(n)$, then there is an equivalence between categories $\text{rep}_k(K(n))$ and $\text{mod } \mathcal{K}_n$. We will frequently identify these two categories.

Let $(T(n), \Omega)$ be the universal covering of $K(n)$, that is, $T(n)$ is an n -regular tree with a bipartite orientation Ω . In general, we can take k -modules of the quiver $T(n)$ as graded Kronecker modules. We use $\text{mod}(T(n), \Omega)$ to denote finite-dimensional graded Kronecker modules (or simply modules). In the end, we introduce two reflection functors $\sigma, \sigma^- : \text{mod}(T(n), \Omega) \rightarrow \text{mod}(T(n), \sigma\Omega)$ concerning all sink points or all source points of $T(n)_0$, respectively, where $\sigma\Omega$ is the orientation under action of σ . If $\sigma^t(M) \neq (0)$ for any $t \in \mathbb{Z}$, we call that M is *regular*, where $M \in \text{mod}(T(n), \Omega)$ is an indecomposable module.

Category $\text{mod}(T(n), \Omega)$ admits AR-sequences and its regular components are type of $\mathbb{Z}A_\infty$. Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. We call M a sink (source, or flow) module if there exists a path of maximal length in M connecting two sink points (or source points, or one sink and one source points). Hence we can define center $C(M)$, radius $r(M)$ of M . Moreover, a module $Y \in \text{mod}(T(n), \Omega)$ is said to be a sink (flow or source) module if $Y = \bigoplus_{i \in I} Y^i$, and all Y^i have the same center and are indecomposable sink (flow or source) modules. Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$, and let $\sigma\mathcal{D}$ be the corresponding component under the action of reflection functor σ . Let $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ be an AR-sequence in \mathcal{D} , and let $b(Z)$ be the number of flow modules in the σ -orbit of Z , then

- (a) Y is a sink (source or flow) module if and only if σZ is a sink (source or flow) module. Moreover, their radii and centers satisfy $r(Y) = r(\sigma Z) + 1$ and $C(Y) = C(\sigma Z)$.
- (b) $|\mathcal{W}(\mathcal{D}) - \mathcal{W}(\sigma\mathcal{D})| \leq 1$, $\mathcal{W}(\mathcal{D}) \geq \frac{b(\mathcal{D})+1}{2}$, where $\mathcal{W}(\mathcal{D})$ is the width of \mathcal{D} and $b(\mathcal{D}) = b(Z)$.

We define full subcategories $\text{Inj}_{n,\Omega}$, $\text{Surj}_{n,\Omega}$ and $\text{CR}_{n,\Omega}$ on $\text{mod}(T(n), \Omega)$. There exists a push-down functor $\pi_\lambda : \text{mod}(T(n), \Omega) \rightarrow \text{mod } \mathcal{K}_n$, and it is exact. Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. Let $N = \pi_\lambda(M)$, and let N be not isomorphic to a simple module. We say that M is of type $A_{1,2}$ if there exists a path in M consisting of all the direct summands $M(\gamma_i^j), M(\gamma_{i'}^j)$ for some two indexes i, i' , where $N(\gamma_x) = \bigoplus_{\pi(\gamma_x^j) = \gamma_x} M(\gamma_x^j), x \in \{i, i'\}$. J. Carlson, E. Friedlander and J. Pevtsova once introduced the subclass of modules of constant rank in 2008. We say that M is of constant rank, provided $\pi_\lambda(M)$ is of constant rank. We show that $M \in \text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$ when M is of type $A_{1,2}$ and is of constant rank. If there exists $\mathcal{W}(\sigma\mathcal{D}) = 1$ for a regular component \mathcal{D} of $\text{mod}(T(n), \Omega)$. Then we have

$$\text{CR}_{n,\Omega} \cap \mathcal{D} = (\text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}) \cap \mathcal{D}.$$

Let M be a finite-dimensional module of \mathcal{K}_n , we consider its dimension vector $\underline{\dim}M = (\dim_k M_1, \dim_k M_2)$. Let $\mathbf{F} = \{(x, y) \mid \frac{2}{n}x \leq y \leq x\}$. Suppose that $(x, y) \in \mathbf{F}$. We construct a module $X(x, y)$ of \mathcal{K}_n , and we prove it being elementary. Suppose that $\underline{\dim}M = (x, y)$. We show that:

- (a) The module M is an elementary module, then $x < 2n$.
- (b) When $x + y = n + 1$, the module M is elementary if and only if M is of the form $X(x, y)$.

Zusammenfassung

Sei k ein algebraisch abgeschlossener Körper und (0) der 0-dimensionale k -Vektorraum. Der n -Kronecker-Köcher $K(n)$ ist der Köcher mit zwei Ecken $1, 2$ und n Pfeilen $\gamma_1, \dots, \gamma_n : 1 \rightarrow 2$. Wir bezeichnen mit $\text{rep}_k(K(n))$ die Kategorie der endlich-dimensionalen Darstellungen des Köchers $K(n)$. Diese ist äquivalent zur Kategorie $\text{mod } \mathcal{K}_n$ endlich-dimensionaler Moduln der Wege-Algebra $\mathcal{K}_n := kK(n)$. Wir werden diese Äquivalenz durchgängig nutzen.

Sei $(T(n), \Omega)$ die universelle Überdeckung von $K(n)$, d.h. $T(n)$ ist ein n -regulärer Baum mit einer bipartiten Orientierung Ω . Im Allgemeinen werden wir k -Moduln des Köchers $T(n)$ als graduierte Kronecker-Moduln ansehen. Wir bezeichnen mit $\text{mod}(T(n), \Omega)$ die Kategorie endlich-dimensionaler, graduierter Kronecker-Moduln (kürzer: Moduln). Wir werden zwei Spiegelungs-Funktoren $\sigma, \sigma^- : \text{mod}(T(n), \Omega) \rightarrow \text{mod}(T(n), \sigma\Omega)$, relativ allen Senken- bzw. allen Quellen-Punkten von $T(n)_0$, definieren.

Die Kategorie $\text{mod}(T(n), \Omega)$ erlaubt das Studium von AR-Folgen, seine regulären Komponenten sind vom Typ $\mathbb{Z}A_\infty$. Sei $M \in \text{mod}(T(n), \Omega)$ ein unzerlegbarer Modul. Wir nennen M einen Senken- (Quellen- oder Fluss) Modul, falls ein Weg maximaler Länge in M existiert, welcher zwei Senken-Punkte (zwei Quell-Punkte, oder einen Senken-Punkt und einen Quell-Punkt) verbindet. Dies ermöglicht es uns, das Zentrum $C(M)$ und den Radius $r(M)$ von M zu definieren. Wir sagen ferner, dass ein Modul $Y \in \text{mod}(T(n), \Omega)$ ein Senken- (Fluss- oder Quell-) Modul ist, falls $Y = \bigoplus_{i \in I} Y^i$, wobei alle Y^i unzerlegbare Senken- (Fluss- oder Quell-) Moduln mit gleichem Zentrum sind. Sei \mathcal{D} eine reguläre Komponente von $\text{mod}(T(n), \Omega)$, und sei $\sigma\mathcal{D}$ die Komponente, welche nach Anwendung des Spiegelungs-Funktors σ entsteht. Sei $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ eine AR-Folge in \mathcal{D} , und sei $b(Z)$ die Anzahl von Fluss-Moduln in der σ -Bahn von Z , dann gelten

(a) Y ist ein Senken- (Quell- oder Fluss-) Modul genau dann, wenn σZ ein Senken- (Quell- oder Fluss-) Modul ist. Des Weiteren genügen ihre Radien und Zentren den Gleichungen $r(Y) = r(\sigma Z) + 1$ und $C(Y) = C(\sigma Z)$.

(b) $|\mathcal{W}(\mathcal{D}) - \mathcal{W}(\sigma\mathcal{D})| \leq 1, \mathcal{W}(\mathcal{D}) \geq \frac{b(\mathcal{D})+1}{2}$, wobei $\mathcal{W}(\mathcal{D})$ die Breite von \mathcal{D} ist und $b(\mathcal{D}) = b(Z)$.

Wir definieren die vollen Unterkategorien $\text{Inj}_{n,\Omega}$, $\text{Surj}_{n,\Omega}$ und $\text{CR}_{n,\Omega}$ von $\text{mod}(T(n), \Omega)$. Es existiert ein push-down Funktor $\pi_\lambda : \text{mod}(T(n), \Omega) \rightarrow \text{mod } \mathcal{K}_n$, welcher exakt ist. Sei $M \in \text{mod}(T(n), \Omega)$ ein unzerlegbarer Modul. Sei $N = \pi_\lambda(M)$, so dass N nicht einfach ist. Wir sagen, dass M vom Typ $A_{1,2}$ ist, falls ein Weg in M existiert, der aus allen direkten Summanden von $M(\gamma_i^j), M(\gamma_{i'}^j)$ für zwei Indizes i, i' existiert, wobei $N(\gamma_x) = \bigoplus_{\pi(\gamma_x^j) = \gamma_x} M(\gamma_x^j), x \in \{i, i'\}$. J. Carlson, E. Friedlander und J. Pevtsova führten 2008 die Unterklasse aller Moduln von konstantem Rang ein. Wir sagen, dass ein Modul M von konstantem Rang ist, falls $\pi_\lambda(M)$ von konstantem Rang ist. Wir zeigen, dass $M \in \text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$ gilt, falls M vom Typ $A_{1,2}$ und vom konstanten Rang ist. Falls $\mathcal{W}(\sigma\mathcal{D}) = 1$ für eine reguläre Komponente \mathcal{D} von $\text{mod}(T(n), \Omega)$, dann gilt

$$\text{CR}_{n,\Omega} \cap \mathcal{D} = (\text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}) \cap \mathcal{D}.$$

Sei M ein endlich-dimensionaler \mathcal{K}_n -Modul mit Dimensionsvektor $\underline{\dim} M = (\dim_k M_1, \dim_k M_2)$. Unter der Annahme, dass $(x, y) \in \mathbf{F} := \{(x, y) \mid \frac{2}{n}x \leq y \leq x\}$ gilt, konstruieren wir einen \mathcal{K}_n -Modul $X(x, y)$ und zeigen, dass dieser elementar ist. Gilt $\underline{\dim} M = (x, y)$, so zeigen wir ferner:

(a) Wenn M ein elementarer Modul ist, dann gilt $x < 2n$.

(b) Falls $x + y = n + 1$, M ist elementar, wenn und nur wenn M in der Form von $X(x, y)$ ist.

Declaration

I hereby declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a doctoral degree, or for publication. Except my supervisor's guidance, and where states otherwise by reference, the work presented is entirely my own. This thesis has been prepared with regard to the Rules of Good Scientific Practice of the German Research Foundation, and I have no academic degree that has ever been withdrawn.

Signature: Jie Liu

Date: 20.08.2021

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1 Introduction

Let k be an algebraically closed field. Let $T(n)$ be the covering of generalized Kronecker quiver $K(n)$. We call a finite-dimensional representation of $T(n)$ a graded module of $K(n)$, $n \geq 3$.

Let Q be a quiver. We use $\text{rep}_k(Q)$ to denote the finite-dimensional representations of Q . There is a reflection (or called shift) functor $\sigma : \text{rep}_k(Q) \rightarrow \text{rep}_k(Q)$. Working with push-down functor $\pi_\lambda : \text{mod}(T(n), \Omega) \rightarrow \text{mod } \mathcal{K}_n$ [7, 7.1], we get $\pi_\lambda \circ \sigma(M) = \sigma \circ \pi_\lambda(M)$ for all $M \in \text{mod}(T(n), \Omega)$. Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. We call M regular if $\sigma^t(M) \neq (0)$ for any $t \in \mathbb{Z}$. Functors σ, σ^- induces a quasi-inverse equivalence between k -linear full subcategories of $\text{mod}(T(n), \Omega)$ and $\text{mod}(T(n), \sigma\Omega)$ consisting of regular modules (section 3).

In section 4, we focus on a regular indecomposable module M of $\text{mod}(T(n), \Omega)$. We use $T(M)$ to denote its support. Recall that any path in $T(M)$ of maximal length will be called a diameter path [5, section 1]. All diameter paths of M have the same length $d(M)$ and the same center $C(M)$ [5, section 2]. Then we can get three different modules; sink module, source module, and flow module. More generally, we call a module $N = \bigoplus_{i \in I} N^i$ a sink (flow, or source) module, provided all its indecomposable direct summands N^i are indecomposable sink (flow, or source) modules with same center and $d(N) = \max\{d(N^i) \mid i \in I\}$, $N^i \in \text{mod}(T(n), \Omega)$. It is also well-known that regular components (sometimes called AR-components) of generalized Kronecker quivers $K(n)$ are components of type $\mathbb{Z}A_\infty$ (cf.[8]). It is the same for the covering $T(n)$. Suppose that \mathcal{D} is a regular component of $\text{mod}(T(n), \Omega)$. Then for an AR-sequence $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ in \mathcal{D} , module Y is a sink (source, or flow) module if and only if σZ is a sink (source, or flow) module.

There exist two full subcategories EIP_n , EKP_n and a width $\mathcal{W}(\mathcal{C})$ of regular component \mathcal{C} in $\text{mod } \mathcal{K}_n$ [11, Section 2]. Correspondingly, we have $\text{Inj}_{n, \Omega}$, $\text{Surj}_{n, \Omega}$ and $\mathcal{W}(\mathcal{D})$ of regular component \mathcal{D} in $\text{mod}(T(n), \Omega)$. Let $\sigma\mathcal{D}$ denote the corresponding component of \mathcal{D} under the action of reflection functor σ . Claus Michael Ringel defines an invariant $b(\mathcal{D})$ for such component \mathcal{D} in 2018 (cf.[5]). In the section 5, we combine those things and get the inequalities:

Theorem A. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Then we have*

$$|\mathcal{W}(\mathcal{D}) - \mathcal{W}(\sigma\mathcal{D})| \leq 1 \text{ and } \frac{b(\mathcal{D})+1}{2} \leq \mathcal{W}(\mathcal{D}).$$

In section 6, we talk about modules of constant rank. Let $M \in \text{mod}(T(n), \Omega)$ be indecomposable, and let $N = \pi_\lambda(M)$. Let $\gamma = \sum_{i=1}^n N(\gamma_i)$. In section 6.1, we write down the matrix γ , and find $\text{rk } \gamma \geq \text{rk } \sum_{i=1}^{n-1} N(\gamma_i) \geq \dots \geq \text{rk } N(\gamma_1)$. We say that the M is of constant rank, provided N is of constant rank. We say that the M is of type $A_{1,2}$, provided there exists a path in $T(M)$ consisting of $M(\gamma_i^p), M(\gamma_j^q)$ for some two i, j , where $N(\gamma_i) := \bigoplus_{\pi(\gamma_i^p)=\gamma_i} M(\gamma_i^p), N(\gamma_j) := \bigoplus_{\pi(\gamma_j^q)=\gamma_j} M(\gamma_j^q)$. Then we get the following result:

Theorem B. *Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module, and be of constant rank. Suppose that the module M is of type $A_{1,2}$ for some two indexes i, j . Then there exists $M \in \text{Inj}_{n, \Omega} \cup \text{Surj}_{n, \Omega}$.*

Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Then we have

Theorem C. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Suppose that $\mathcal{W}(\mathcal{D}) = 1$. Then there exists*

$$\text{CR}_{n, \sigma\Omega} \cap \sigma\mathcal{D} = (\text{Inj}_{n, \sigma\Omega} \cup \text{Surj}_{n, \sigma\Omega}) \cap \sigma\mathcal{D}$$

for the regular component $\sigma\mathcal{D}$.

Now suppose that $M \in \text{rep}_k(K(n))$ is a regular indecomposable module, and let (x, y) be its dimension vector, we denote by $\underline{\dim} M = (\dim_k M_1, \dim_k M_2) = (x, y)$. Associating to $K(n)$, there is a quadratic form defined by $q(x, y) = x^2 + y^2 - nxy$. We say that the dimension vector (x, y) is regular, provided $q(x, y) < 0$. For n -Kronecker quiver $K(n)$, $n \geq 4$,

$$1 \circ \begin{array}{c} \xrightarrow{\gamma_1} \\ \vdots \\ \xrightarrow{\gamma_n} \end{array} \circ 2,$$

it is barely known about its elementary modules. Let \mathbf{R} be the set of regular dimension vectors. By abusing notations, we introduce two maps σ, δ on the set \mathbf{R} , where $\sigma(x, y) = (nx - y, x), \delta(x, y) = (y, x), (x, y) \in \mathbf{R}$. Let $\mathbf{F} = \{(x, y) \mid \frac{2}{n}x \leq y \leq x\}$. We have seen that set \mathbf{F} is a fundamental domain for the action of the group generated by δ and σ on the set \mathbf{R} (cf.[17]). Hence we only need to focus on the set \mathbf{F} . We construct a module $X(x, y)$ for the dimension vector (x, y) , where $x + y = n + 1$. We prove

Theorem D. *Let $(x, y) \in \mathbf{F}$ with $x + y = n + 1$. Then a module $M \in \text{mod } \mathcal{K}_n$ with dimension vector (x, y) is elementary if and only if M is of the form $X(x, y)$.*

- (2) N is called *preprojective*, provided there exists some $t \in \mathbb{N}_0$ such that $\tau^t N = (0)$.
(3) N is called *regular*, provided N is not preinjective or preprojective.

It is well-known that there is no nonzero morphism from preinjective modules to preprojective or regular modules [7, Proposition 2.13]. When N is decomposable, we sometimes say that N is still regular, provided there does not exist indecomposable direct summand which is preinjective or preprojective in N .

Let $X, Y \in \text{mod } \mathcal{A}$. We say that homomorphism $f : X \rightarrow Y$ is *irreducible*, provided f is neither a split monomorphism nor a split epimorphism, and if $f = f_1 \circ f_2$, then either f_1 is a split epimorphism or f_2 is a split monomorphism. Let $\text{rad}_{\mathcal{A}}$ be the radical of $\text{mod } \mathcal{A}$. We can define the set

$$\text{Irr}(X, Y) := \text{rad}_{\mathcal{A}}(X, Y) / \text{rad}_{\mathcal{A}}^2(X, Y)$$

and call it the *space* of irreducible morphisms [1, IV.1.6].

Definition. The *Auslander-Reiten quiver* $\Gamma(\mathcal{A})$ of \mathcal{A} is defined as follows:

- (a) The points $\Gamma(\mathcal{A})_0$ correspond to the isomorphism classes $[X]$ of indecomposable modules X in $\text{mod } \mathcal{A}$.
(b) Let $[X], [Y]$ be two points in $\Gamma(\mathcal{A})_0$. Then there are $\dim_k \text{Irr}(X, Y)$ arrows from $[X]$ to $[Y]$ in $\Gamma(\mathcal{A})_1$.

Let $X, Y \in \text{mod } \mathcal{A}$ be indecomposable modules. We say that X is a *predecessor* of Y , provided there exists a directed path from $[X]$ to $[Y]$ in $\Gamma(\mathcal{A})$, i.e. a chain of irreducible maps from X to Y . We say that X is a *successor* of Y , provided there exists a directed path from $[Y]$ to $[X]$ in $\Gamma(\mathcal{A})$, i.e. a chain of irreducible maps from Y to X .

Definition. A connected component \mathcal{D} of $\Gamma(\mathcal{A})$ is called *preprojective*, *preinjective*, or *regular*, provided modules contained in it are all preprojective, preinjective, or regular.

We use $(\rightarrow Y)$ to denote the subset of $\Gamma(\mathcal{A})_0$ consisting of Y and all its predecessors, and $(Y \rightarrow)$ to denote the subset of $\Gamma(\mathcal{A})_0$ consisting of Y and all its successors, respectively.

Let X be a non-projective indecomposable module (or let Y be a non-injective indecomposable module). Then there exists a uniquely determined AR-sequence,

$$(0) \rightarrow Y \xrightarrow{f} \bigoplus_{i=1}^t M_i^{n_i} \xrightarrow{g} X \rightarrow (0),$$

where Y (or X) is indecomposable, modules M_i are pairwise non-isomorphic and indecomposable.

Suppose that $f = \begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix}$, $g = [g_1, \dots, g_t]$, where $f_i = \begin{bmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{bmatrix}$, $g_i = [g_{i1}, \dots, g_{in_i}]$. Then the maps $f_{i1}, f_{i2}, \dots, f_{in_i} : Y \rightarrow M_i$ and $g_{i1}, \dots, g_{in_i} : M_i \rightarrow X$ correspond to bases of the $\text{Irr}(Y, M_i)$ and $\text{Irr}(M_i, X)$, respectively. We write $Y = \tau X$ (or $X = \tau^- Y$) and we denote this in $\Gamma(\mathcal{A})$ by $[Y] \leftarrow [X]$. Note that we sometimes don't distinguish X (or Y) and its isomorphism class $[X]$ (or $[Y]$) in $\Gamma(\mathcal{A})$.

2.2 Graph

Definition. A graph is a pair $G = (G_0, G_1) : G_0$ (whose elements called the vertices) and G_1 (whose elements called the edges). If $\{a, b\}$ is an edge, then a, b are also called neighbours.

For a graph G , there is a map Ω , called *orientation* $: G \rightarrow G_0 \times G_0$, such that $\Omega(\{a, a'\})$ is either (a, a') or (a', a) . We call (G, Ω) an *oriented graph*. We write $a \rightarrow a'$ if $\Omega(\{a, a'\}) = (a, a')$ and call a the *start* and a' the *target*. Meanwhile we call a a *sink* (or a *source*) if it is not a start (or the target, respectively) of any arrow. If any vertex is a sink or a source, then the orientation Ω will be called *bipartite*. A *subgraph* of a graph $G = (G_0, G_1)$ is a graph $G' = (G'_0, G'_1)$ such that $G'_0 \subseteq G_0, G'_1 \subseteq G_1$.

A *path* of length $t \geq 0$ in graph G is a finite sequence (a_0, a_1, \dots, a_t) of vertices satisfying a_{i-1}, a_i are neighbours for $1 \leq i \leq t$, and $a_{j-1} \neq a_{j+1}$ for $1 \leq j < t$. If $t = 2r$, then a_r is called its *center* and r its *radius* [5, 2.1]. If $t = 2r + 1$, then $\{a_r, a_{r+1}\}$ is called its *center* and r its *radius*. Suppose that for any pair a, b of vertices in G , there always exists a path connecting a and b , then we say that G is *connected*. We call a path of length $t \geq 0$ being a *cycle* whenever its source and target coincide. A cycle of length 1 is said to be a *loop*. Graph G is called *acyclic* if it contains on cycles, and it is called *finite* if G_0 and G_1 are finite set.

Graph G is called a *tree* if it is connected and paths connecting a vertex with itself are always length 0. Let G be a tree, we define

$$d(a, b) := \text{the length of path connecting } a \text{ and } b.$$

This is well-defined since there is only one path from a to b in the tree G . If G' is a subset of tree G and $x \in G'$, we define

$$d(x, G') := \min\{d(x, a) \mid a \in G'\}.$$

For every finite tree G , the paths of maximal possible length $d(G)$ are called the *diameter* of G . Let $r(G)$ denote the radius of G . It can be shown that for a finite tree G , all diameter paths of G have the same center $C(G)$ [5, 2.1]. Suppose that (a_0, a_1, \dots, a_t) is a diameter path of G . We define,

$$C(G) = \begin{cases} a_r & t = 2r, \\ \{a_r, a_{r+1}\} & t = 2r + 1. \end{cases}$$

2.3 Quivers

A *quiver* is just an oriented graph (loops and multiple arrows are allowed), normally denoted by $Q = (Q_0, Q_1, s, t)$, where $s, t : Q_1 \rightarrow Q_0$. For $x \in Q_0$, let

$$\begin{aligned} x^+ &:= \{y \in Q_0 \mid \exists \alpha \in Q_1 : t(\alpha) = y, \text{ where } s(\alpha) = x\} \\ x^- &:= \{y \in Q_0 \mid \exists \alpha \in Q_1 : s(\alpha) = y, \text{ where } t(\alpha) = x\}. \end{aligned}$$

The *neighbourhood* $\mathcal{N}(x)$ of x is $x^+ \cup x^-$. We say that x is a *leaf*, provided x has at most one neighbour. We say that quiver $Q' = (Q'_0, Q'_1, s', t')$ is a *subquiver* of quiver $Q = (Q_0, Q_1, s, t)$, provided $Q'_0 \subseteq Q_0, Q'_1 \subseteq Q_1$ and the restrictions $s|_{Q'_1}, t|_{Q'_1}$ of s, t to Q'_1 satisfy $s|_{Q'_1} = s', t|_{Q'_1} = t'$. Subquiver Q' is called *full* if $Q'_1 = \{\alpha \in Q_1 \mid s(\alpha) \in Q'_0, t(\alpha) \in Q'_0\}$. We get quiver Q its

underlying graph \overline{Q} by forgetting the orientation of the arrows. We say that Q is connected if \overline{Q} is a connected graph. Without loss of generality, we always assume that it is acyclic and connected when we talk about a quiver Q .

A finite-dimensional representation $M = ((M_x)_{x \in Q_0}, (M(\alpha))_{\alpha \in Q_1})$ over Q consists of vector spaces M_x and k -linear maps $M(\alpha) : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$ such that $\dim_k M := \sum_{x \in Q_0} \dim_k M_x$ is finite.

A *morphism* $f : M \rightarrow N$ between representations is a collection of k -linear maps $(f_z)_{z \in Q_0}$, such that for each arrow $\alpha : x \rightarrow y$ there is a commutative diagram

$$\begin{array}{ccc} M_x & \xrightarrow{M(\alpha)} & M_y \\ \downarrow f_x & & \downarrow f_y \\ N_x & \xrightarrow{N(\alpha)} & N_y. \end{array}$$

We sometimes use $S(i), P(i), I(i)$ to denote the correspondingly simple, projective and injective representations at the point $i, i \in Q_0$. We thus define a category $\text{Rep}_k(Q)$ of all k -linear representations of Q . We denote by $\text{rep}_k(Q)$ the subcategory of $\text{Rep}_k(Q)$ consisting of finite-dimensional representations.

Definition. Let $Q = (Q_0, Q_1, s, t)$ be a finite, connected, and acyclic quiver [1, VII.5]. For every point $a \in Q_0$, we define a new quiver

$$\sigma_a Q = (Q'_0, Q'_1, s', t'),$$

as follows: all the arrows of Q having a as source or as target are reversed, all other arrows remain unchanged. More precisely, $Q'_0 = Q_0$ and there exists a bijection $Q_1 \rightarrow Q'_1$ such that if $\alpha' \in Q'_1$ denotes the arrow corresponding to $\alpha \in Q_1$ under this bijection, then:

- (1) if $s(\alpha) \neq a$ and $t(\alpha) \neq a$, then $s'(\alpha') = s(\alpha)$ and $t'(\alpha') = t(\alpha)$; whereas
- (2) if $s(\alpha) = a$ or $t(\alpha) = a$, then $s'(\alpha') = t(\alpha)$ and $t'(\alpha') = s(\alpha)$.

Definition. Let Q be a finite quiver, a a sink in Q , and let $Q' = \sigma_a Q$. We define a reflection functor

$$\mathcal{S}_a^+ : \text{rep}_k(Q) \rightarrow \text{rep}_k(Q')$$

between the categories $\text{rep}_k(Q)$ and $\text{rep}_k(Q')$ as follows [1, VII.5.5].

Let $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be an object in $\text{rep}_k(Q)$. We define $\mathcal{S}_a^+ M = (M'_i, \varphi'_\alpha)_{i \in Q'_0, \alpha \in Q'_1}$ in $\text{rep}_k(Q')$ as follows:

- (a) When $i \neq a$, we define $M'_i = M_i$. However, space M'_a is the kernel of k -linear map $(\varphi_\alpha)_\alpha : \bigoplus_{\alpha: s(\alpha) \rightarrow a} M_{s(\alpha)} \rightarrow M_a$;
- (b) when $j \neq a$, we define $\varphi'_\alpha = \varphi_\alpha$ for all arrows $\alpha : i \rightarrow j$ in Q_1 . When $\alpha : i \rightarrow a$ is an arrow in Q_1 , then $\varphi'_\alpha : M'_a \rightarrow M_i$ is the composition of the inclusion of M'_a into $\bigoplus_{\alpha: s(\alpha) \rightarrow a} M_{s(\alpha)}$ with the projection onto the direct summand M_i .

Similarly, if Q' is a finite connected quiver, a a source in Q' , and $Q = \sigma_a Q'$. There exists a reflection functor

$$\mathcal{S}_a^- : \text{rep}_k(Q') \rightarrow \text{rep}_k(Q)$$

between the categories $\text{rep}_k(Q')$ and $\text{rep}_k(Q)$ as follows. Let $M' = (M'_i, \varphi'_\alpha)_{i \in Q'_0, \alpha \in Q'_1}$ be an object in $\text{rep}_k(Q')$. We define the object $\mathcal{S}_a^- M' = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ in $\text{rep}_k(Q)$ as follows:

- (a') When $i \neq a$, we define $M'_i = M_i$. However, space M_a is the cokernel of the k -linear map $(\varphi'_\alpha)_\alpha : M'_a \rightarrow \bigoplus_{\alpha: a \rightarrow t(\alpha)} M'_{t(\alpha)}$;
- (b') when $i \neq a$, we define $\varphi_\alpha = \varphi'_\alpha$ for all arrows $\alpha : i \rightarrow j$ in Q'_1 . When $\alpha : a \rightarrow j$ is an arrow in Q_1 , then $\varphi_\alpha : M_j = M'_j \rightarrow M_a$ is the natural embedding of M'_j into $\bigoplus_{\alpha: a \rightarrow t(\alpha)} M'_{t(\alpha)}$ with the projection of this direct summand onto M_a .

Corollary 2.3.1. [1, VII. Corollary 5.7] *Let Q be a finite, connected, and acyclic quiver with at least two points; a sink in Q ; and $Q' = \sigma_a Q$. The reflection functors $\mathcal{S}_a^+ : \text{rep}_k(Q) \rightarrow \text{rep}_k(Q')$ and $\mathcal{S}_a^- : \text{rep}_k(Q') \rightarrow \text{rep}_k(Q)$ satisfy the following properties:*

- (a) *The functor \mathcal{S}_a^- is left adjoint to \mathcal{S}_a^+ .*
- (b) *If M is indecomposable in $\text{rep}_k(Q)$, then the following two conditions are equivalent;*
- (i) $\mathcal{S}_a^+ M \neq 0$,
 - (ii) $M \not\cong S(a)$.
- Moreover, if this is the case, then $\mathcal{S}_a^- \mathcal{S}_a^+ M \cong M$ and \mathcal{S}_a^+ induces an algebra isomorphism $\text{End } M \cong \text{End}(\mathcal{S}_a^+ M)$.*
- (c) *If M' is indecomposable in $\text{rep}_k(Q')$, then the following two conditions are equivalent;*
- (i) $\mathcal{S}_a^- M' \neq 0$,
 - (ii) $M' \not\cong S(a)$.
- Moreover, if this is the case, then $\mathcal{S}_a^+ \mathcal{S}_a^- M' \cong M'$ and \mathcal{S}_a^- induces an algebra isomorphism $\text{End } M' \cong \text{End}(\mathcal{S}_a^- M')$.*
- (d) *The functors \mathcal{S}_a^+ and \mathcal{S}_a^- induce quasi-inverse equivalences between the k -linear full subcategory of $\text{rep}_k(Q)$ of the representations having no direct summand isomorphic to the simple projective representation $S(a)$ and the k -linear full subcategory of $\text{rep}_k(Q')$ of the representations having no direct summand isomorphic to the simple injective representation $S(a)$.*

2.4 Covering Theory

We now focus on the generalized Kronecker quiver $K(n)$, $n \geq 3$,

$$1 \circ \begin{array}{c} \xrightarrow{\gamma_1} \\ \vdots \\ \xrightarrow{\gamma_n} \end{array} \circ 2.$$

Let $\mathcal{K}_n = kK(n)$. Clearly, algebra \mathcal{K}_n is Artin [11, Section 2.1]. Let ϵ_i correspond to the trivial path point i , $i \in \{1, 2\}$. In the following, we give a brief construction of the covering $T(n)$ of Kronecker quiver $K(n)$ [7, 7.1]. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. We define formal inverse $(Q_1)^{-1}$ on Q_1 , where

- (a) $(Q_1)^{-1} := \{\alpha^{-1} \mid \alpha \in Q_1\}$;
- (b) $s(\alpha^{-1}) := t(\alpha)$ and $t(\alpha^{-1}) := s(\alpha)$ for any $\alpha^{-1} \in (Q_1)^{-1}$;
- (c) we say that ω is a *walk* of Q_1 , provided there exists $\omega = \alpha_n^{\epsilon_n} \cdots \alpha_1^{\epsilon_1}$, where $\alpha_i \in Q_1$, $\epsilon_i \in \{1, -1\}$ and $s(\alpha_{i+1}^{\epsilon_{i+1}}) = t(\alpha_i^{\epsilon_i})$ for all $i < n$;
- (d) $s(\omega) := s(\alpha_1^{\epsilon_1})$ and $t(\omega) := t(\alpha_n^{\epsilon_n})$.

Let $W := \{\text{walks of } K(n)\}$. We introduce an equivalence relation \sim on W , which is generated by $\gamma_i^{-1} \gamma_i \sim \epsilon_1$ and $\gamma_i \gamma_i^{-1} \sim \epsilon_2$. We define an involution $(-)^{-1}$ on W , where

$$(-)^{-1}: W \rightarrow W; (\alpha_n^{\epsilon_n} \cdots \alpha_1^{\epsilon_1})^{-1} \mapsto \alpha_1^{-\epsilon_1} \cdots \alpha_n^{-\epsilon_n}.$$

Let $\pi(K(n))$ be the fundamental group of $K(n)$ in the point 1, i.e. $\pi(K(n)) = \{[\alpha] \mid s(\alpha) = t(\alpha) = 1\}$, where $[\alpha]$ is the equivalence classe of unoriented path α . The multiplication is the concatenation of paths. We define $[\omega]^{-1} := [\omega^{-1}]$ and the identity element is $[\epsilon_1]$. Then quiver $T(n)$ is given by the following data:

(a) $(T(n))_0$ is the set of equivalence classes of walks starting in 1.

(b) When $\omega' \sim \gamma_i \omega$ for some $i \in \{1, \dots, r\}$, we say that there exists an arrow from $[\omega]$ to $[\omega']$.

Let $\pi : T(n) \rightarrow K(n); [\omega] \mapsto t(\omega), ([\omega] \rightarrow [\gamma_i \omega]) \mapsto \gamma_i$. Action of group $G = \pi(K(n))$ on $T(n)$ is the concatenation of paths:

$$g.[v] = [v\omega^{-1}] \text{ and } g.([u] \rightarrow [\gamma_i u]) = ([u\omega^{-1}] \rightarrow [\gamma_i u\omega^{-1}]),$$

where $g = [\omega] \in \pi(K(n))$ and $[v], [u] \in T(n)_0$ with arrow $[u] \rightarrow [\gamma_i u] = [v]$. Finally, we define $T_n^+ := \pi^{-1}(\{1\})$, $T_n^- := \pi^{-1}(\{2\})$. Let Ω be the bipartite on $T(n)$. Then for any $M \in \text{rep}_k(T(n), \Omega)$, group G acts on M via:

$$M^g := (((M^g)_x)_{x \in T(n)_0}, (M^g(\alpha))_{\alpha \in T(n)_1}),$$

where $(M^g)_x := M_{g.x}$ and $M^g(\alpha) := M(g.\alpha)$.

Definition. Let $\emptyset \neq S \subseteq T(n)_0$ be a set of vertices.

(a) The unique minimal tree in $T(n)$ containing S is denoted by $T(S)$.

(b) A vertex $x \in T(S)_0$ is called a *leaf* of $T(S)$, provided $|\mathcal{N}(x) \cap T(S)_0| \leq 1$.

Definition. Let $x \in T(n)_0$ and $M \in \text{mod}(T(n), \Omega)$ be a module.

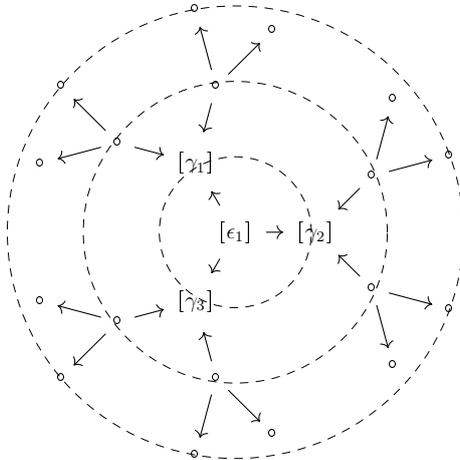
(a) The set $\text{supp}(M) := \{y \in T(n)_0 \mid M_y \neq 0\}$ is called the *support* of M .

(b) The vertex x is said to be a *leaf* M , provided x is a leaf of $T(M) := T(\text{supp}(M))$.

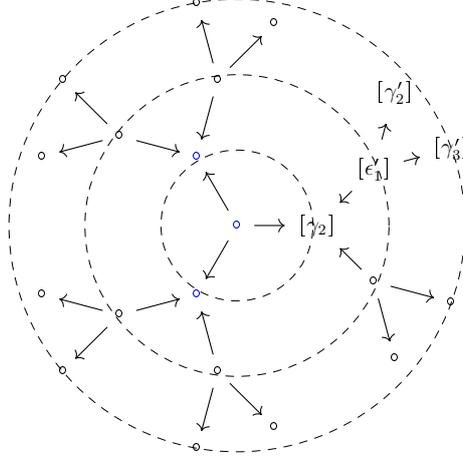
Definition. Let $M \in \text{mod}(T(n), \Omega)$. We call M a thin module, provided $\dim_k M_a \leq 1$ for every $a \in T(n)_0$.

Let $M \in \text{mod}(T(n), \Omega)$ be a thin module. According to [9, Proposition 1], we know that M is indecomposable if $\text{supp}(M)$ is connected. Now we look at an example.

Example. Let $M \in \text{rep}_k(T(3), \Omega)$, and let $\text{supp}(M)$ be the following:



where $\dim_k M_a = 1$ for all $a \in T(M)_0$, $M(\gamma_i) = \lambda_i, \lambda_i \in k \setminus \{0\}$. Let $g = [\gamma_2^{-1}\gamma_1] \in G$. Then M is indecomposable and $\text{supp}(M^g)$ is the following:



where $[\epsilon'_1] = [\gamma_1^{-1}\gamma_2], [\gamma'_2] = [\gamma_2\gamma_1^{-1}\gamma_2], [\gamma'_3] = [\gamma_3\gamma_1^{-1}\gamma_2]$. There exist $\dim_k M_a^g = 1$ for all $a \in T(M^g)_0$ and $M^g(\gamma_i) = \lambda_i$.

Let Γ be a quiver. Then Γ is said to be *locally finite*, provided every point $x \in \Gamma_0$ has only finitely many neighbours.

Definition. Let Γ be a locally finite quiver containing no loops. Let τ be a bijection between two subsets of Γ_0 . Then (Γ, τ) (or simply, Γ) is called to be a *translation quiver*, provided for any $x \in \Gamma_0$ such that τx exists, any every $y \in x^-$, the number of arrows from y to x is equal to the number of arrows from τx to y . Moreover, we say that (Γ', τ') is a *full translation subquiver* of (Γ, τ) , provided Γ' is a full subquiver of Γ and $\tau'x = \tau x$, whenever x is a vertex of Γ' such that τx belongs to Γ' .

We define a push-down functor $\pi_\lambda : \text{rep}_k(T(n), \Omega) \rightarrow \text{rep}_k(K(n)); M \mapsto (\pi_\lambda(M)_j, \pi_\lambda(M)(\gamma_i))$ factoring through $T(n)/G$, that is, $\pi_\lambda(M^g) = \pi_\lambda(M), \forall g \in G$, where

$$\pi_\lambda(M)_j := \bigoplus_{\pi(y)=j} M_y,$$

$$\pi_\lambda(M)(\gamma_i) := \bigoplus_{\pi(\beta)=\gamma_i} M(\beta) : \pi_\lambda(M)_1 \rightarrow \pi_\lambda(M)_2, j \in \{1, 2\}, 1 \leq i \leq n.$$

The functor π_λ is exact [3, 3.2]. Since the category $\text{rep}_k(K(n))$ is equivalent to the category $\text{mod } \mathcal{K}_n$ of finite-dimensional modules for path algebra \mathcal{K}_n of generalized Kronecker quiver $K(n)$, we sometimes say that a representation $M \in \text{rep}_k(K(n))$ is a module of \mathcal{K}_n .

For any two sinks $x, y \in T(n)_0$, we can see that $\sigma_x\sigma_y = \sigma_y\sigma_x$ using the definition. If we fix a bipartite orientation Ω of $T(n)$, then we can define a reflection functor

$$\sigma : \text{rep}_k(T(n), \Omega) \rightarrow \text{rep}_k(T(n), \sigma\Omega)$$

at all sinks of quiver $(T(n), \Omega)$, where $\sigma\Omega$ is the orientation of $T(n)$ under the action of functor σ . Then σ is well-defined and independent of the order. We sometimes call such functor σ the *shift functor*. Similarly, we can get another functor σ^- at all source points of $T(n)_0$. Note that

the $\text{rep}_k(T(n), \Omega)$ also admits AR-sequences (cf.[3, 2.2]). By [5, 2.6], the composition σ^2 is the translation τ .

We call modules in $\text{rep}_k(T(n), \Omega)$ the *graded Kronecker modules* (or *graded modules*, or simply *modules*). We often use $\text{mod}(T(n), \Omega)$ to denote those modules. Without specific emphasis, we always mean quiver $(T(n), \Omega)$ when we talk about covering $T(n)$.

Definition. Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. We say that M is *regular*, provided $\pi_\lambda(M)$ is regular.

We will show that functors σ and σ^- send regular indecomposable modules to regular indecomposable modules in section 3.

Theorem 2.4.1. [7, Theorem 7.1] *The following statements hold.*

- (a) π_λ sends indecomposable representations in $\text{rep}_k(T(n), \Omega)$ to indecomposable representations in $\text{rep}_k(K(n))$.
- (b) If $M \in \text{rep}_k(T(n), \Omega)$ is indecomposable, then $\pi_\lambda(M) \cong \pi_\lambda(N)$ if and only if $M^g \cong N$ for some $g \in G$.
- (c) π_λ sends AR-sequences to AR-sequences and π_λ commutes with the Auslander-Reiten translations.
- (d) If $M \in \text{rep}_k(T(n), \Omega)$ is indecomposable in a component \mathcal{D} with $\pi_\lambda(M)$ in a component \mathcal{C} , then π_λ induces a covering $\mathcal{D} \rightarrow \mathcal{C}$ of translation quivers.

Corollary 2.4.2. [7, Corollary 7.2] *Let \mathcal{D} be regular component, then the covering $\mathcal{D} \rightarrow \pi_\lambda(\mathcal{D})$ is an isomorphism of translation quivers. In particular, \mathcal{D} is of type $\mathbb{Z}A_\infty$. Moreover, a component \mathcal{C} is regular if and only if \mathcal{C} is of type $\mathbb{Z}A_\infty$.*

2.5 Graded Kronecker modules

Let $M \in \text{mod}(T(n), \Omega)$ be indecomposable. Then $T(M) = \text{supp}(M)$ is a finite tree, we can define

$$d(M) = d(T(M)), C(M) = C(T(M)) \text{ and } r(M) = r(T(M)).$$

We call them *diameter*, *center*, and *radius* of M , respectively [5, 2.4]. We say that

- (a) M is said to be a *sink* module, provided diameter paths of $T(M)$ start and end in sinks;
- (b) M is said to be a *source* module, provided diameter paths of $T(M)$ start and end in sources;
- (c) M is said to be a *flow* module, provided length of diameter path of $T(M)$ is odd.

Theorem 2.5.1. [5, Theorem 1] *Let M be a regular indecomposable module of $\text{mod}(T(n), \Omega)$. Then the shift orbit of M contains a unique sink module M_0 with smallest possible radius, saying with radius $r_0 = r_0(M)$. Let $M_i = \sigma^i M_0$ for all $i \in \mathbb{Z}$. Then there is an integer $0 \leq b(M) \leq r_0(M)$ and a path (a_0, \dots, a_b) in $T(n)$ with the following properties.*

- (a) For $i \geq 0$, the module M_{-i} is a sink module with radius $r_0 + i$ and center a_0 .
- (b) For $1 \leq i \leq b(M)$, the module M_i is a flow module with radius $r_0 - 1$ and center $\{a_{i-1}, a_i\}$.
- (c) For $i \geq 0$, the module M_{b+1+i} is a source module with radius $r_0 + i$ and center a_b .

The integer $r_0(M)$ is positive. If $r_0(M)$ is even, then a_0 is a sink, otherwise a source.

2.6 Index and Balls

Let M be a regular indecomposable module of $\text{mod}(T(n), \Omega)$. Theorem 2.5.1 tells us that there exists some $i \geq 0$ such that $\sigma^{i+1}M$ is a flow or source module. Let i be the minimal with this property, we define

$$M_0 := \sigma^{-i}M \text{ and } M_i = \sigma^i M_0, \text{ for each } i \in \mathbb{Z}.$$

Then all the module M_i with $i \leq 0$ are sink modules.

Definition. Given an indecomposable regular module $M \in \text{mod}(T(n), \Omega)$, we define its *index* $\iota(M) := t$ when $M = \sigma^t M_0, t \in \mathbb{Z}$.

For any $x, c, c_1, c_2 \in T(n)_0$ and $r \in \mathbb{N}$ [5, 2.3], let

$$\begin{aligned} B_r(c) &:= \{x \in T(n)_0 \mid d(x, c) \leq r\}, \\ B_r(c_1, c_2) &:= \{x \in T(n)_0 \mid d(x, c_i) \leq r\}, i \in \{1, 2\}. \end{aligned}$$

We call such sets $B_r(c)$ and $B_r(c_1, c_2)$ the *balls* with radius r and with center c or $\{c_1, c_2\}$, respectively. Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. Suppose that c , or $\{c_1, c_2\}$, and r are the center and radius of M , respectively. We define

$$B(M) := B_r(c), \text{ or } B(M) := B_r(c_1, c_2).$$

Clearly, $T(M)$ and $B(M)$ have the same radius and same center.

2.7 Regular component

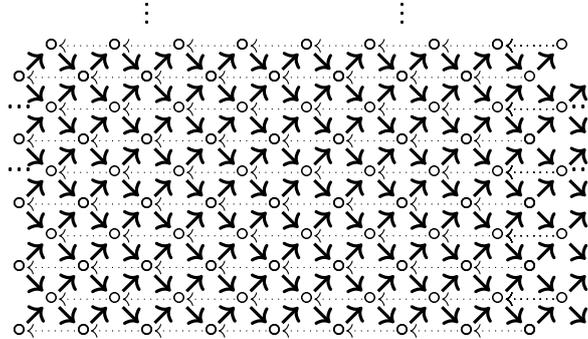


Figure 1: Regular component $\mathbb{Z}A_\infty$.

It is well-known that the regular components of $\Gamma(\mathcal{K}_n)$ are the type of $\mathbb{Z}A_\infty$ (cf.[8]). A module M is called *quasi-simple* if it is located in the bottom row of a $\mathbb{Z}A_\infty$ component. Let M be a module in a regular $\mathbb{Z}A_\infty$ -component \mathcal{C} , then there exists a quasi-simple module $X \in \mathcal{C}$ (or $Y \in \mathcal{C}$) such that we can find a chain of irreducible monomorphisms $X = X_1 \rightarrow \dots \rightarrow X_{s-1} \rightarrow X_s = M$ (or irreducible epimorphisms $M = Y_s \rightarrow Y_{s-1} \rightarrow \dots \rightarrow Y_1 = Y$). We call s the *quasi-length* of M and the module X (or Y) *quasi-socle* (or *quasi-top*) of M . We call the factors X_i/X_{i-1} or Y_i/Y_{i-1} the *quasi-composition factors* of M . It can be shown that M is uniquely determined by its quasi-length and quasi-socle (or quasi-top), whence we can define

$ql(M) := s = \text{quasi-length of } M.$

Moreover, if we let $X_0 = (0)$, we have sequences

$$(0) \rightarrow X_i/X_{j-1} \xrightarrow{\begin{pmatrix} p \\ \iota \end{pmatrix}} X_i/X_j \oplus X_{i+1}/X_{j-1} \xrightarrow{(\iota \quad -p)} X_{i+1}/X_j \rightarrow (0)$$

being AR-sequences for all $1 \leq j \leq i \leq s-1$, where p is projection, and ι is embedding [8, 3.2].

Note that for a quiver Q , we usually use $\mathcal{C}, \mathcal{D}, \dots$, to denote the regular components of $\text{rep}_k(Q)$, and $\sigma\mathcal{C}, \sigma\mathcal{D}, \dots$, its corresponding components under the action of σ in $\text{rep}_k(Q)$.

Definition. For any $\alpha \in k^n \setminus (0)$ and any $M \in \text{mod } \mathcal{K}_n$, we define

$$x_\alpha := \alpha_1\gamma_1 + \dots + \alpha_n\gamma_n.$$

We denote by

$$x_\alpha^M : M \rightarrow M$$

the linear operator associated to x_α .

Definition. Let $M \in \text{mod } \mathcal{K}_n$ be indecomposable. Module M is said to have the *equal kernels property*, provided $\ker x_\alpha^M$ is independent of α for all $\alpha \in k^n \setminus (0)$. Further, M is said to have the *equal images property*, provided $\text{im } x_\alpha^M$ is independent of α .

Let $M \in \text{mod } \mathcal{K}_n$ be indecomposable and non-simple. General theory tells us that

$$\bigcap_{i=1}^n \ker(x_{e_i}^M) = M_2 \text{ and } \sum_{i=1}^n \text{im}(x_{e_i}^M) = M_2,$$

where e_i is the i -th canonical coordinate. In particular, the module M has the equal kernels property if and only if $M_2 = \ker(x_\alpha^M)$ for all $\alpha \in k^n \setminus \{0\}$, and the equal images property if and only if $M_2 = \text{im}(x_\alpha^M)$ for all nonzero α [7, Lemma 2.20]. Hence for an indecomposable module M , we can define two sets

$$\text{EKP}_n := \{M \in \text{mod } \mathcal{K}_n \text{ does not have direct summands of type } S(1) \mid \forall \alpha \in k^n \setminus (0) : \ker(x_\alpha^M) = M_2\},$$

$$\text{EIP}_n := \{M \in \text{mod } \mathcal{K}_n \text{ does not have direct summands of type } S(2) \mid \forall \alpha \in k^n \setminus (0) : \text{im}(x_\alpha^M) = M_2\}.$$

Definition. Let $M \in \text{mod } \mathcal{K}_n$. We say that the module M is of *constant rank* if the rank of x_α^M is independent of $\alpha \in k^n \setminus (0)$.

Hence we get the set

$$\text{CR}_n := \{M \in \text{mod } \mathcal{K}_n \mid \text{there is } r_M \in \mathbb{N}_0 \text{ such that } \text{rk } x_\alpha^M = r_M \text{ for all } \alpha \in k^n \setminus (0)\}.$$

There exists $\text{EKP}_n \cap \text{EIP}_n = (0)$ [11, 2.4]. Clearly, there is $\text{EIP}_n \cup \text{EKP}_n \subseteq \text{CR}_n$ [11, Theorem 2.5].

It is shown that there are uniquely determined quasi-simple modules $M_{\mathcal{C}}$ and $W_{\mathcal{C}}$ in \mathcal{C} such that the cone $(M_{\mathcal{C}} \rightarrow)$ of all successors of $M_{\mathcal{C}}$ satisfies $(M_{\mathcal{C}} \rightarrow) = \text{EKP}_n \cap \mathcal{C}$ and the cone $(\rightarrow W_{\mathcal{C}})$ of all predecessors of $W_{\mathcal{C}}$ satisfies $(\rightarrow W_{\mathcal{C}}) = \text{EIP}_n \cap \mathcal{C}$ [11, Theorem 3.3]. Let

$$\begin{aligned} \text{Inj}_{n,\Omega} &:= \{M \in \text{mod}(T(n), \Omega) \mid \forall \delta \in (T(n))_1 : M(\delta) \text{ is injective} \}, \\ \text{Surj}_{n,\Omega} &:= \{M \in \text{mod}(T(n), \Omega) \mid \forall \delta \in (T(n))_1 : M(\delta) \text{ is surjective} \}, \\ \text{CR}_{n,\Omega} &:= \{M \in \text{mod}(T(n), \Omega) \mid \pi_\lambda(M) \text{ is of constant rank} \}. \end{aligned}$$

Let $M \in \text{mod}(T(n), \Omega)$ be regular indecomposable. According to [7, Theorem 8.1], we know that $M \in \text{Inj}_{n,\Omega}$ if and only if $M \in \text{EKP}_n$.

Definition. Let \mathcal{C} be a regular component of $\text{mod } \mathcal{K}_n$. Define $\mathcal{W}(\mathcal{C})$ an integer satisfying $\tau^{\mathcal{W}(\mathcal{C})+1}M_{\mathcal{C}} = W_{\mathcal{C}}$. We call $\mathcal{W}(\mathcal{C})$ the *width* of \mathcal{C} .

Definition. Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. We define an integer $\mathcal{W}(\mathcal{D})$ satisfying $\tau^{\mathcal{W}(\mathcal{D})+1}M_{\mathcal{D}} = W_{\mathcal{D}}$. We call $\mathcal{W}(\mathcal{D})$ the *width* of \mathcal{D} .

In fact, we want to talk about the invariants in the regular component \mathcal{D} of $\text{mod}(T(n), \Omega)$ in the following sections. Ringel shows that $b(M) = b(M')$ for any regular indecomposable modules $M, M' \in \mathcal{D}$ [5, section 7]. We combine \mathcal{D} with $\sigma\mathcal{D}$ using reflection functor σ . By discussing their widths $\mathcal{W}(\mathcal{D}), \mathcal{W}(\sigma\mathcal{D})$ and radii $r(\mathcal{D}), r(\sigma\mathcal{D})$, we get these two components $\mathcal{D}, \sigma\mathcal{D}$ being almost the same things. That is why we hardly get difference when we use push-down functor π_λ to act on them.

2.8 Duality

Recall that the duality $D : \text{mod } \mathcal{K}_n \rightarrow \text{mod } \mathcal{K}_n; (M_i, M(\gamma_j)) \mapsto (M_i^*, M^*(\gamma_j))$, where M_i^* is the k -dual of M_{3-i} , and $M^*(\gamma_j)$ is the k -dual of $M(\gamma_j)$, where $\xi : \{1, 2\} \rightarrow \{1, 2\}$ is the involution with $\xi(1) = 2, i = 1, 2, j \in \{1, \dots, n\}$.

We now introduce an involution $\zeta_0 : T(n)_0 \rightarrow T(n)_0; [w] \mapsto [\bar{w}\gamma_1]$, where $\bar{\epsilon}_1 := \epsilon_2, \bar{\epsilon}_2 := \epsilon_1$ and $\bar{\alpha}_n^{\epsilon_n} \cdots \bar{\alpha}_1^{\epsilon_1} := \alpha_n^{-\epsilon_n} \cdots \alpha_1^{-\epsilon_1}$. Then we can get a quiver anti-morphism $\zeta : T(n) \rightarrow T(n)$; for every arrow $[w] \rightarrow [\gamma_i w]$ in $T(n)$, there is a unique arrow $\zeta([w] \rightarrow [\gamma_i w])$ with start $\zeta_0([\gamma_i w]) = [\gamma_i^{-1} \bar{w}\gamma_1]$ and target $\zeta_0([w]) = [\bar{w}\gamma_1]$. Then we can define duality

$$D_{T(n)} : \text{mod}(T(n), \Omega) \rightarrow \text{mod}(T(n), \Omega); (N_i, N(\gamma_j)) \mapsto (N_i^*, N^*(\gamma_j)).$$

We have $\pi_\lambda \circ D_{T(n)}(M) \cong D \circ \pi_\lambda(M)$ for all $M \in \text{mod}(T(n))$ [7, Section 7.2]. We also have $D(\text{EKP}_n) = \text{EIP}_n$ [11, Section 2.2]. Then we have $D_{T(n)}(\text{Inj}_{n,\Omega}) = \text{Surj}_{n,\Omega}$ by [7, Theorem 8.1]. Using the definition of $D_{T(n)}$, we can see that functor $D_{T(n)}^2$ is equivalent to the identity functor on category $\text{mod}(T(n), \Omega)$.

3 Reflection functor of converging quiver

3.1 Indecomposable module and reflection functor

We want to show that reflection functor sends regular indecomposable module to regular indecomposable module. We abuse notations, and let $\sigma = \mathcal{S}_2^+, \sigma^- = \mathcal{S}_1^-$ for \mathcal{K}_n .

Lemma 3.1.1. *Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. Then $\pi_\lambda(\sigma M) = \sigma(\pi_\lambda(M))$ and $\pi_\lambda(\sigma^- M) = \sigma^-(\pi_\lambda(M))$.*

Proof. We only prove $\pi_\lambda(\sigma M) = \sigma(\pi_\lambda(M))$, and it is similar for the other one. If $\sigma M = (0)$, then $\pi_\lambda(M) \cong S(1)$, there is nothing to prove. Suppose that $\sigma M \neq (0)$. We need to show that the following diagram commutes.

$$\begin{array}{ccc} \text{mod}(T(n), \Omega) & \xrightarrow{\pi_\lambda} & \text{mod } \mathcal{K}_n \\ \downarrow \sigma & & \downarrow \sigma \\ \text{mod}(T(n), \sigma\Omega) & \xrightarrow{\pi_\lambda} & \text{mod } \mathcal{K}_n \end{array}$$

We have

$$\begin{aligned} \pi_\lambda(M)_j &= \bigoplus_{\pi(y)=j} M_y \\ \pi_\lambda(M)(\gamma_i) &= \bigoplus_{\pi(\beta)=\gamma_i} M(\beta) : \pi_\lambda(M)_1 \rightarrow \pi_\lambda(M)_2, \end{aligned}$$

for $j \in \{1, 2\}$ and $1 \leq i \leq n$.

Let $N = \sigma\pi_\lambda(M)$, where

$$N_2 = \pi_\lambda(M)_1 = \bigoplus_{\pi(y)=1} M_y,$$

and

$$\begin{aligned} N_1 &= (\sigma\pi_\lambda(M))_1 \\ &= \ker \left(\bigoplus_{\gamma_j} \pi_\lambda(M)_{s(\gamma_j)} \rightarrow \pi_\lambda(M)_2 \right) \\ &= \ker \left(\bigoplus_{\gamma_j} \bigoplus_{\pi(\beta)=\gamma_j} M_{s(\beta)} \rightarrow \bigoplus_{\pi(z)=2} M_z \right). \end{aligned}$$

By the definition of σ , each $N(\gamma_i)$ is the composition of the inclusion of N_1 into $\bigoplus_{\gamma_j} \pi_\lambda(M)_{s(\gamma_j)}$ with the projection onto the direct summand $\pi_\lambda(M)_{s(\gamma_i)}$.

On the other hand, let $N' = \sigma(M)$ and $N'' = \pi_\lambda(N')$. Note that,

$$N''_2 = \bigoplus_{\pi(y')=2} N'_{y'} = \bigoplus_{\pi(y)=1} M_y = N_2,$$

and

$$\begin{aligned}
N_1'' &= \bigoplus_{\pi(y')=1} N_{y'}' \\
&= \bigoplus_{\pi(y')=1} \sigma(M)_{y'} \\
&= \bigoplus_{\pi(z)=2} \ker \left(\bigoplus_{\beta:s(\beta)\rightarrow z} M_{s(\beta)} \rightarrow M_z \right) \\
&= \ker \left(\bigoplus_{\gamma_j} \bigoplus_{\pi(\beta)=\gamma_j} M_{s(\beta)} \rightarrow \bigoplus_{\pi(z)=2} M_z \right) \\
&= \ker \left(\bigoplus_{\gamma_j} \pi_\lambda(M)_{s(\gamma_j)} \rightarrow \pi_\lambda(M)_2 \right) \\
&= N_1.
\end{aligned}$$

By the definition of σ , each $N''(\gamma_i)$ is the composition of inclusion of $N_1'' = N_1$ into $\bigoplus_{\gamma_j} \pi_\lambda(M)_{s(\gamma_j)}$ with the projection onto the direct summand $\pi_\lambda(M)_{s(\gamma_i)}$. Then we know that

$$N(\gamma_i) = N''(\gamma_i), \quad 1 \leq i \leq n.$$

Then $N = N''$. □

Lemma 3.1.2. *Functors σ and σ^- send regular indecomposable modules to regular indecomposable modules.*

Proof. Let $M \in \text{mod}(T(n), \Omega)$ be regular indecomposable. Using Corollary 2.3.1(d) and Corollary 2.4.2, we can see that $\sigma(\pi_\lambda(M))$ is regular indecomposable. However, Lemma 3.1.1 tells us that $\pi_\lambda(\sigma M) = \sigma(\pi_\lambda(M))$, whence σM is indecomposable. For functor σ^- , the proof is similar. □

3.2 Balls and reflection functor

Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. We define

$$B_{+1}(M) := \{x \in (T(n), \Omega)_0 \mid d(x, B(M)) \leq 1\}.$$

Then $B_{+1}(M)$ is the ball having the same center with $B(M)$, and its radius $r_{+1} = r + 1$, where r is the radius of $B(M)$ (see 2.6). In particular, these balls $B(M), B_{+1}(M)$ can be taken as subquivers of $(T(n), \Omega)$ naturally, whence we sometimes call them subquivers $B(M), B_{+1}(M)$. Such subquivers are finite. From now on, we always assume that M is regular indecomposable when we talk about a module $M \in \text{mod}(T(n), \Omega)$ in this section. Let $1, \dots, n_M$ be the all sink points in $B_{+1}(M)$, and let $1', \dots, n_M'$ be the all source points in $B_{+1}(M)$, respectively, $n_M, n_M' \in \mathbb{N}$.

Lemma 3.2.1. *Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. Then*

$$\sigma(M) = \mathcal{S}_1^+ \cdots \mathcal{S}_{n_M}^+(M).$$

Proof. Let $x \in T(n)_0 \setminus B_{+1}(M)$ be a sink point. Then there exists $d(x, B(M)) \geq 2$, whence we have $y \notin \text{supp}(M)$ for any arrow $\alpha : y \rightarrow x$. Then $\mathcal{N}(x) \cap T(M)_0 = \emptyset$, whence $\mathcal{S}_x^+(M) = M$. On the other hand, reflection functors σ_x and $\sigma_{x'}$ commute for any two sink points $x, x' \in T(n)_0$. Hence we have $\sigma(M) = \mathcal{S}_1^+ \cdots \mathcal{S}_{n_M}^+(M)$. □

Lemma 3.2.1 tells us that we can take regular indecomposable module M as a representation of subquiver $B_{+1}(M)$. By duality, we have $\sigma^-(D_{T(n)}M) = \mathcal{S}_1^- \cdots \mathcal{S}_{n_M}^- (D_{T(n)}M)$. Similarly, we have the following result:

Corollary 3.2.2. *Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. Then*

$$\sigma^-(M) = \mathcal{S}_1^- \cdots \mathcal{S}_{n'_M}^- (M).$$

Proof. This is a dual version of Lemma 3.2.1. □

Lemma 3.2.3. *Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. Then*

$$\sigma^- \sigma(M) = \mathcal{S}_{n_M}^- \cdots \mathcal{S}_1^- \mathcal{S}_1^+ \cdots \mathcal{S}_{n_M}^+ (M) \cong M.$$

Proof. Let $x \in (T(n), \Omega)_0 \setminus B_{+1}(M)$ be a sink point, that is, x is a source point for $\sigma\Omega$. Let $y \in \mathcal{N}(x)$. Then y is a sink point for $\sigma\Omega$, that is, y is a source point for Ω . Moreover, point y either satisfies $\{y\} \cap B_{+1}(M) = \emptyset$ or y is a leaf of $B_{+1}(M)$ by the choice of x . Then we have $\{y\} \cap \text{supp}(M) = \emptyset$. Hence $(\sigma M)_y = M_y = (0)$. It indicates that $y \notin \text{supp}(\sigma M)$. We have

$$\mathcal{N}(x) \cap \text{supp}(\sigma M) = \emptyset.$$

Since σM is indecomposable by Lemma 3.1.2, we have $x \notin \text{supp}(\sigma M)$. It is obvious that $x \notin \text{supp}(\sigma M)$ when x is source point for Ω . Then we have $\text{supp}(\sigma M) \subseteq B_{+1}(M)$ as sets of points, and $\mathcal{S}_x^-(\sigma M) = \sigma M$ for any $x \in (T(n), \Omega)_0 \setminus B_{+1}(M)$ being a sink point. Finally, we can see that σ^- is the composition of those source points that contained in $B_{+1}(M)$ for $\sigma\Omega$, and they are exactly the points $1, \dots, n_M$. By Corollary 2.3.1(b), we get our claim. □

Note that we can also get $B(\sigma M) \subseteq B_{+1}(M)$ as unoriented graphs from $\text{supp}(\sigma M) \subseteq B_{+1}(M)$ in the proof of Lemma 3.2.3. Similarly, we can get the following result:

Corollary 3.2.4. *Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. Then*

$$\sigma \sigma^-(M) = \mathcal{S}_{n'_M}^+ \cdots \mathcal{S}_1^+ \mathcal{S}_1^- \cdots \mathcal{S}_{n'_M}^- (M) \cong M.$$

Proof. This is a dual version of Lemma 3.2.3. That is, we can change σM into $\sigma^- M$ in the proof of Lemma 3.2.3 and consider $x \in T(n)_0 \setminus B_{+1}(M)$ being a source point for Ω , then x is a sink point for $\sigma\Omega$. Comparing $\text{supp}(\sigma^- M)$ with $B_{+1}(M)$, we can finally get $\mathcal{N}(x) \cap \text{supp}(\sigma^- M) = \emptyset$ and $x \notin \text{supp}(\sigma^- M)$. Then $\text{supp}(\sigma^- M) \subseteq B_{+1}(M)$ and $\mathcal{S}_x^+(\sigma^- M) = \sigma^- M$. Hence σ is the composition of those sink points that contained in $B_{+1}(M)$ for $\sigma\Omega$, and they are exactly the points $1', \dots, n'_M$. By Corollary 2.3.1(c), we get this. □

Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. We also have $B(\sigma^- M) \subseteq B_{+1}(M)$ from the proof of Corollary 3.2.4. By Lemma 3.2.3 and Corollary 3.2.4, we can take

$$M, \sigma M, \sigma^- M, \sigma^- \sigma(M) \text{ and } \sigma \sigma^-(M)$$

as representations of subquivers $(B_{+1}(M), \Omega)$ and $(B_{+1}(M), \sigma\Omega)$, respectively.

Lemma 3.2.5. *Functor $\sigma^- \circ \sigma$ is equivalent to identity functor on regular modules.*

Proof. Let $1_{T(n)}$ denote the identity functor on full subcategory of $\text{mod}(T(n), \Omega)$ consisting of regular modules. By Lemma 3.1.2, we only need to talk about regular indecomposable modules of $\text{mod}(T(n), \Omega)$.

Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. We now consider M as a representation of subquiver $B_{+1}(M)$. According to Lemma 3.2.3, there exists an isomorphism $\eta_M : M \rightarrow \sigma^- \sigma(M)$. Let $N \in \text{mod}(T(n), \Omega)$ be regular indecomposable, and let $f : M \rightarrow N$ be a morphism. Let $B_{+1}(M, N)$ be the minimal ball containing $B_{+1}(M)$ and $B_{+1}(N)$. Then $B_{+1}(M, N) \subseteq T(n)_0$ is finite. It allows us to take M, N as representations of subquiver $B_{+1}(M, N)$. Let $F = \sigma^- \circ \sigma$, and let $1, \dots, n_{M, N}$ be the sink points of subquiver $B_{+1}(M, N)$. Consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & F(M) \\ \downarrow f & & \downarrow F(f) \\ N & \xrightarrow{\eta_N} & F(N) \end{array}$$

Note that $B_{+1}(M), B_{+1}(N) \subseteq B_{+1}(M, N)$. By the proof of Lemma 3.2.3 and Corollary 2.3.1, it is not hard to see that

$$F(X) \cong \mathcal{S}_{n_{M, N}}^- \cdots \mathcal{S}_1^- \mathcal{S}_1^+ \cdots \mathcal{S}_{n_{M, N}}^+(X) \cong X,$$

where $X \in \{M, N\}$. Moreover, we have

$$\text{Hom}_k(M, N) \cong \text{Hom}_k(F(M), F(N))$$

by Corollary 2.3.1(d). Then there exists such $F(f) \in \text{Hom}_k(F(M), F(N))$ so that the above diagram commutes. Hence there exists a natural isomorphism $\eta : 1_{T(n)} \rightarrow F$. Then functor $\sigma^- \circ \sigma$ is equivalent to $1_{T(n)}$. \square

Corollary 3.2.6. *Functor $\sigma \circ \sigma^-$ is equivalent to identity functor on regular modules.*

Proof. By Lemma 3.1.2, we only need to talk about regular indecomposable modules of category $\text{mod}(T(n), \Omega)$. Using Corollary 3.2.4 and Corollary 2.3.1(c), (d), we can finally get a dual proof with that of Lemma 3.2.5. \square

Lemma 3.2.7. *The functors σ and σ^- induce quasi-inverse equivalences on k -linear full subcategories of $\text{mod}(T(n), \Omega)$ and $\text{mod}(T(n), \sigma\Omega)$ consisting of regular modules.*

Proof. Combining Lemma 3.2.5 and Corollary 3.2.6, functors $\sigma \circ \sigma^-$ and $\sigma^- \circ \sigma$ are equivalent to the identity functor on regular modules of $\text{mod}(T(n), \Omega)$, that is, σ^-, σ are the inverse functors of σ, σ^- , respectively. Hence we have the claim. \square

Lemma 3.2.8. *Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. Then $D_{T(n)} \circ \sigma(M) \cong \sigma^{-1} \circ D_{T(n)}(M)$ and $D_{T(n)} \circ \sigma^-(M) \cong \sigma \circ D_{T(n)}(M)$.*

Proof. Using Lemma 3.2.7, we have these isomorphisms. \square

4 Middle terms of AR-sequences of covering quiver

In this part, we discuss middle terms of AR-sequences in regular components of $\text{mod}(T(n), \Omega)$. Now let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. By Theorem 2.5.1, and sticking to the abuse, we define

- (a) $\theta(M) := (a_0, \dots, a_b)$ the *center path* of M ,
- (b) $p(M) := a_0, q(M) := a_b$.

Note that $p(M)$ is the center of all sink modules of the σ -orbit of M and $q(M)$ is the center of its all source modules, respectively. Moreover, the number $b(M)$ is just the number of flow modules in the σ -orbit of M and $b(M) = d(p(M), q(M))$. Recall that $i = \iota(M)$. According to Theorem 2.5.1, we can get the following corollary.

Corollary 4.1. *Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. Then*

- (a) M is a sink module if and only if $i \leq 0$,
- (b) M is a flow module if and only if $0 < i \leq b(M)$,
- (c) M is a source module if and only if $i > b(M)$.

Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$, and let $M, M' \in \mathcal{D}$ be two regular indecomposable modules. Suppose that σ -orbit of M (or M') contains b (or b') flow modules. According to [5, section 7], we have $p(M) = p(M'), q(M) = q(M')$ and $b = b'$. This is saying that for the σ -orbits of M and M' , the centres of all the sink modules or source modules are the same. We also have $r_0(M) - ql(M) = r_0(M') - ql(M')$, where $r_0(M) = r(M_0)$. Then we define

$$\begin{aligned} p(\mathcal{D}) &= p(M), \\ q(\mathcal{D}) &= q(M), \\ b(\mathcal{D}) &= d(p(M), q(M)), \\ r(\mathcal{D}) &= r_0(M) - ql(M). \end{aligned}$$

There exist $p(M) = p(\sigma^t M) = p(M') = p(\sigma^{t'} M')$ and $q(M) = q(\sigma^t M) = q(M') = q(\sigma^{t'} M')$ for any $t, t' \in \mathbb{Z}$. That is, $\theta(\sigma^t M) = \theta(\sigma^{t'} M')$ and $b(M) = b(\sigma^t M) = b(M') = b(\sigma^{t'} M') = b$.

We say that a module $Y \in \text{mod}(T(n), \Omega)$ is a *sink* (*source*, or *flow*) module, provided all its indecomposable direct summands are sink (source, or flow) modules with same center. Suppose that $Y = \bigoplus_{i \in I} Y_i$, where Y_i are indecomposable sink (source, or flow) modules for all i , and $C(Y_i) = C(Y_j), i, j \in I$. Then we define $r(Y) = \max\{r(Y_i) \mid i \in I\}, C(Y) = C(Y_i)$. Let $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ be an AR-sequence in regular component \mathcal{D} . Suppose that $Y = Y_1 \oplus Y_2$, where $Y_1, Y_2 \in \text{mod}(T(n), \Omega)$ are indecomposable and $ql(Y_1) < ql(Y_2)$. Then we have $b(Y_1) = b(Y_2)$ and $\iota(Y_1) = \iota(Y_2)$ [5, Theorem 3]. In addition, $C(Y_1) = C(Y_2)$ and Y_1, Y_2 are of the same type modules [5, Theorem 4]. Hence we get $C(Y) = C(Y_2), r(Y) = r(Y_2)$. We define

$$\iota(Y) := \iota(Y_2) \text{ and } b(Y) := b(Y_1) = b(Y_2).$$

In the following of this section, we focus on such middle term Y and let $Y = Y_2$ when Y is indecomposable.

Lemma 4.2. *Let $X, Y, Z \in \text{mod}(T(n), \Omega)$ be regular modules. Assume that $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ is an AR-sequence. Then $\iota(Y) = \iota(\sigma Z)$. Moreover, Y is a sink (flow, or source) module if and only if σZ is a sink (flow, or source) module.*

Proof. We already know that $\iota(Y_1) = \iota(Y_2)$ by [5, Theorem 4]. We just need to prove $\iota(Y_2) = \iota(\sigma Z)$. By [5, Theorem 3], we get $\iota(Z) = \iota(Y_2) - 1$. Then $\iota(\sigma Z) = \iota(Y_2)$. This also tells us that Y has the same type with σZ [Corollary 4.1]. □

Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. We define:

$$\bar{\iota}(M) := \frac{1}{ql(M)} (\sum_{M^i \in I} \iota(M^i)),$$

where I is the set of all quasi-composition factors of M . We call $\bar{\iota}(M)$ the *average index* of M . We have $\iota(M) = \bar{\iota}(M)$ [5, Proposition 4]. Moreover, we have the following:

Lemma 4.3. [5, Lemma 3] *Let $X \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module with $\iota(X) \leq 0$. Then X is a sink module and $B(X) = B(X')$, X' is the quasi-top of X .*

Lemma 4.4. [5, Lemma 3*] *Let $X \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module with $\iota(X) \geq b(X) + 1$. Then X is a source module and $B(X) = B(X'')$, X'' is the quasi-sockle of X .*

Now we get our result:

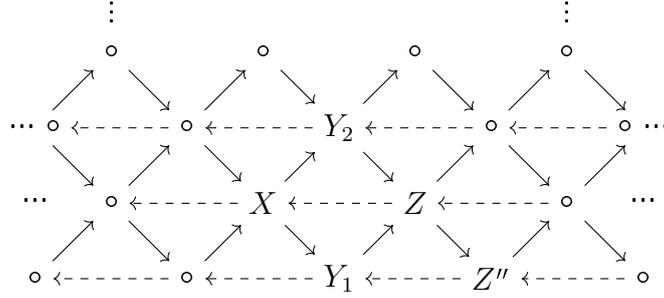
Theorem 4.5. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Suppose that $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ is an AR-sequence in \mathcal{D} . Then there exist $C(Y) = C(\sigma Z)$ and $r(Y) = r(\sigma Z) + 1$.*

Proof. We firstly assume that σZ is a sink module, then Y and Z would be sink modules using Lemma 4.2 and [5, 3.1(a)]. Then $C(Y) = p(Y) = p(Z) = C(Z) = C(\sigma Z)$. Module $\sigma^2 Z = X$ would be a source module whenever modules Y and σZ are both sources [5, 3.3], whence we still have $C(Y) = q(Y) = q(Z) = C(\sigma^2 Z) = C(\sigma Z)$ in this case.

By [5, Proposition 1.(b)], the radii of all its flow modules are all the same in the σ -orbit of a regular indecomposable module. Moreover, centres of those flow modules just follow the path from $p(M)$ to $q(M)$, that is, $\theta(M)$ [5, 3.2.(a)]. However, there exist $\theta(Y) = \theta(Z) = \theta(\sigma Z)$ and $b(Y) = b(Z) = b(\sigma Z)$. Hence we must have $C(Y) = C(\sigma Z)$ when both of them are flow modules.

For the second assertion, we use the induction on the quasi-length of Z . Suppose that $ql(Z) = 1$. It indicates that Z is quasi-simple and $Y = Y_2$. In addition, Z is the quasi-top of Y . We can see that Y is a sink module, provided σZ is a sink module [Lemma 4.2]. By Lemma 4.3, we get $B(Y) = B(Z)$, whence Z is a sink and $r(Z) = r(\sigma Z) + 1$ [5, 3.1.(a)]. Then $r(Y) = r(Z) = r(\sigma Z) + 1$. Suppose that σZ is a source module. Then Y is a source module and $X = \sigma^2 Z$ is the quasi-sockle of Y . Then $B(Y) = B(\sigma^2 Z)$ [Lemma 4.4]. On the other hand, there exists $r(\sigma^2 Z) = r(\sigma Z) + 1$, whence $r(Y) = r(\sigma Z) + 1$ in this case [5, 3.3]. Suppose that σZ is a flow module. We get Y being a flow module [Lemma 4.2]. But we have seen that all flow modules have the same radius in the σ -orbit of a regular indecomposable module. Moreover, there exist $r(\sigma Z) = r(Z_0) - 1$ and $r(Y) = r(Y_0) - 1$. Note that there exist $r_0(Y) - ql(Y) = r_0(Z) - ql(Z)$, $ql(Y_0) = ql(Y)$ and $ql(Z) = ql(Z_0)$ [Lemma 3.2.7]. Since $ql(Y) = ql(Z) + 1$, then $r_0(Y) = r(Y_0) = r_0(Z) + 1 = r(Z_0) + 1$. Finally, we have $r(Y) = r(\sigma Z) + 1$.

Suppose that $ql(Z) = 2$. There exists a module $Z'' \in \mathcal{D}$ such that



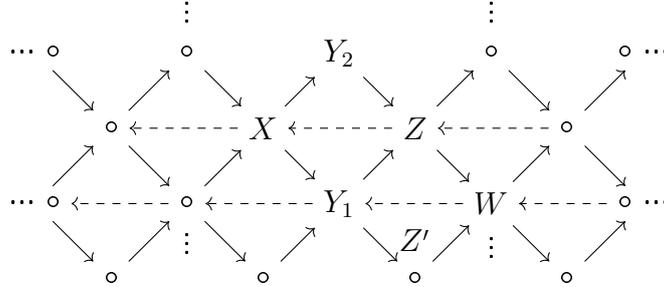
We have $ql(Y_1) = ql(Z'') = 1$ and $ql(Z) = 2$. The sequences $(0) \rightarrow Y_1 \rightarrow Z \rightarrow Z'' \rightarrow (0)$ and $(0) \rightarrow \sigma Y_1 \rightarrow \sigma Z \rightarrow \sigma Z'' \rightarrow (0)$ are AR-sequences [Lemma 3.2.7]. There exist

$$r(Z) = r(\sigma Z'') + 1, \text{ and } r(\sigma Z) = r(\sigma^2 Z'') + 1.$$

Hence there has

$$r(Y) = r(Y_2) = r(Y_1) + 2 = r(\sigma^2 Z'') + 2 = r(\sigma Z) + 1$$

by [5, Theorem 4]. When $ql(Z) > 2$, there exist modules $Z', W \in \mathcal{D}$ such that



In this case, there always exists $r(Y_1) = r(\sigma Z') + 1$ by induction. However, there exists $r(Y_2) = r(Y_1) + 2$ [5, Theorem 4]. Hence $r(Y) = r(\sigma Z') + 3 = r(\sigma Z) + 1$ since $r(Z) = r(\sigma Z') + 2$. \square

Corollary 4.6. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Then there exists $r(\mathcal{D}) = r(\sigma \mathcal{D})$.*

Proof. Let $M \in \mathcal{D}$ be a regular indecomposable module. By the definition, we have $r(\mathcal{D}) = r_0(M) - ql(M)$. However, $r_0(M) = r(M_0)$ and $ql(M) = ql(\sigma M)$ [Lemma 3.2.7]. Hence $r(\sigma \mathcal{D}) = r_0(\sigma M) - ql(\sigma M) = r(M_0) - ql(\sigma M) = r(\mathcal{D})$. \square

Lemma 4.7. *Let $X, Y, Z \in \text{mod}(T(n), \Omega)$ be regular modules. Suppose that*

$$(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$$

is an AR-sequence with $T(X)_0 \cap T(Z)_0 = \emptyset$, then Y is a flow module.

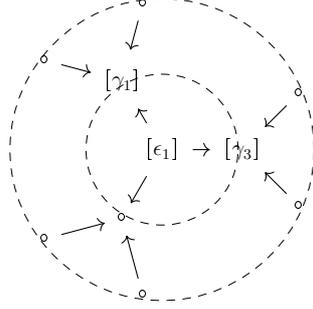
Proof. This means that σZ must be a flow module [Theorem 4.5].

(a) Suppose that σZ is a sink module with $C(\sigma Z) = x_0$, where $x_0 \in T(\sigma Z)_0$. According to [5, 3.1(a)], Z is a sink module and $C(Z) = x_0$. For $\sigma^2 Z = X$, we have $C(X) = C(\sigma Z) = x_0$ when X is a source module [5, 3.1.(c)], and $C(X) \cap C(Z) = x_0$ when X is a flow module [5, 3.1.(b)]. Hence we always have $C(Z) \cap C(X) = C(\sigma Z) \cap C(\sigma^2 Z) \neq \emptyset$ in this case. This contradicts our assumption $T(X)_0 \cap T(Z)_0 = \emptyset$.

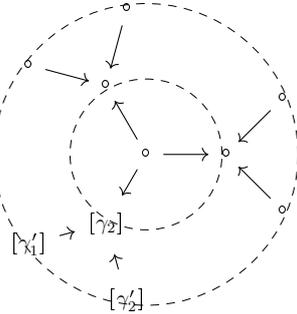
(b) Suppose that σZ is a source module with $C(\sigma Z) = y_0$, $y_0 \in T(\sigma Z)_0$. Then $\sigma^2 Z = X$ is a source module and $C(X) = C(\sigma Z)$ [5, 3.3]. Module Z has 3 possibilities: sink, flow or source. Using [5, 3.1, 3.2, 3.3], we know that $C(Z) = C(\sigma Z) = y_0$ when Z is a sink or source module. When Z is a flow module, there exists $C(Z) \cap C(\sigma Z) = y_0$, where $C(Z) = \{y'_0, y_0\}$, $y'_0 \in T(n)_0$. Since $T(X)_0 \cap T(Z)_0 = \emptyset$, but $C(X) \cap C(Z) = C(\sigma Z) \cap C(Z) \neq \emptyset$, this case also cannot happen.

Together with (a) and (b), we can see that σZ is a flow module. □

Example. Let $Z \in \text{mod}(T(3), \Omega)$, and let $\text{supp}(Z)$ be the following:



where $\dim_k Z_a = 1$, $a \in T(Z)_0$ and $Z(\gamma_i) = \lambda_i$, $\lambda_i \in k \setminus \{0\}$, $i \in \{1, 3\}$. Then Z is regular indecomposable [7, Theorem 2.18]. Suppose that $(0) \rightarrow X \rightarrow Y \rightarrow Z \rightarrow (0)$ is an AR-sequence. Then support of $X = \sigma^2 Z$ would look like the following form:



where $[\gamma'_1] = [\gamma_3^{-1}\gamma_2]$, $[\gamma'_2] = [\gamma_3^{-1}\gamma_2]$. Then Z is a sink module, and X is a source module. Moreover, there exists $T(X) \cap T(Z) = \emptyset$. Let (a_0, a_1, a_2) be the diameter path of $T(Z)$. Using direct computation, we get module $\sigma Z : k \xrightarrow{\gamma_2} k$ being a flow module. Then $C(\sigma Z) = \{a_0, a'\}$ and $r(\sigma Z) = 0$, $a' \in T(\sigma Z)_0$. According to Lemma 4.7, we get Y being a flow module. By Theorem 4.5, $r(Y) = r(\sigma Z) + 1 = 1$, $C(Y) = C(\sigma Z) = \{a_0, a'\}$.

5 Widths of regular components

In this section, we want to talk about graded modules and width in a regular component \mathcal{D} of $\text{mod}(T(n), \Omega)$. Let $M \in \text{mod}(T(n), \Omega)$ be a regular indecomposable module. We define $\mathcal{N}_M(b) := \mathcal{N}(b) \cap T(M)_0, \forall b \in T(M)_0$.

Lemma 5.1. *Let $M \in \text{Inj}_{n, \Omega}$ be a regular indecomposable module. Then*

- (a) *M is a sink module and $d(M) \geq 4$.*
- (b) *Any leaf of $T(\sigma M)$ is a sink. In particular, σM is a sink module.*

Proof. According to [7, Theorem 8.1(b)], we know that M must be a sink module. Note that σM is also indecomposable [Lemma 3.2.7]. We now suppose that there has a source vertex b_0 being a leaf in $T(\sigma M)$. Let $\alpha : b_0 \rightarrow b_1$ be the arrow in $T(\sigma M)_0$. Then k -linear map $\varphi = (\sigma M)(\alpha) : (\sigma M)_{b_0} \rightarrow (\sigma M)_{b_1}$ is non-trivial, $b_1 \in T(\sigma M)_0$. Since $\sigma^-(\sigma M) \cong M$, then there exists a map $\varphi' = M(\alpha') : M_{b_1} = (\sigma M)_{b_1} \rightarrow M_{b_0} = \text{coker } \varphi$ that is not injective in M by the definition of σ^- , this is a contradiction. Hence all leaves of $T(\sigma M)_0$ are sinks and σM is a sink module.

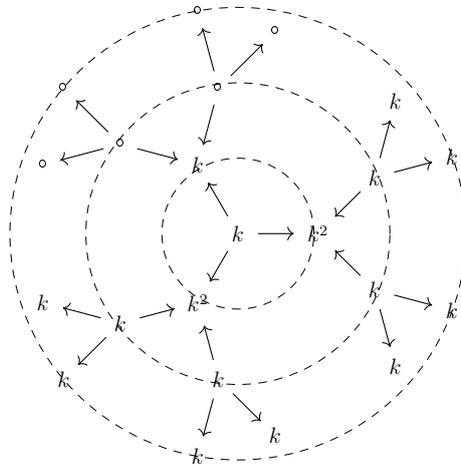
Actually, $d(M) \geq 4$ since M is a sink and regular module. Otherwise, assume that $d(M) = 2$ since $d(M)$ is even. We will get $d(\sigma M) = 0$, that is, $|T(\sigma M)_0| = 1$, so that σM is not regular, a contradiction. □

Corollary 5.2. *Let $N \in \text{Surj}_{n, \Omega}$ be a regular indecomposable module. Then*

- (a) *N is a source module and $d(N) \geq 4$.*
- (b) *Any leaf of $T(\sigma^- N)$ is a source. In particular, $\sigma^- N$ is a source module.*

Proof. Since $N \in \text{Surj}_{n, \Omega}$, then we have $D_{T(n)}(N) \in \text{Inj}_{n, \Omega}$. By Lemma 5.1, we know that N is a source module and $d(N) \geq 4$. According to Lemma 3.2.8, we can see that $\sigma \circ D_{T(n)}(N) \cong D_{T(n)} \circ \sigma^-(N)$ is a sink module. Then (b) is clear. □

Example. Let $N = k \xleftarrow{\gamma_1} k \xrightarrow{\gamma_2} k \in \text{mod}(T(3), \Omega)$, where $\gamma_1, \gamma_2 \in k \setminus \{0\}$. Then N is regular indecomposable. Suppose that $N \in \mathcal{D}$ for some regular component \mathcal{D} of. Let $M = \sigma^{-2}N$, then $T(M)$ would be the following:



Using the definition of σ^- , there exists $M \in \text{Inj}_{n,\Omega} \cap \mathcal{D}$. Note that $N \notin \text{Inj}_{n,\Omega} \cap \mathcal{D}$ and N is quasi-simple [10, Proposition 3.4]. Then $M = \sigma^{-2}N$ is quasi-simple and $M = M_{\mathcal{D}}$ [Lemma 3.2.7]. One can check that σ^-M is a sink module.

Lemma 5.3. *Let $M \in \text{Inj}_{n,\Omega} \cap \mathcal{D}$ be a regular indecomposable module. Then $\sigma^-M \in \text{Inj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$.*

Proof. Let $M' = \sigma^-M$. We want to prove that $M' \in \text{Inj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$. That is to say, every map is injective in M' [7, Theorem 8.1].

We consider $M \in \text{Inj}_{n,\Omega}$ and want to reflect at a source point $x \in T(M)_0$. In fact, all neighbours of x belong to $T(M)_0$. We have injective maps $f_i : M_x \rightarrow M_{y_i}$ for $1 \leq i \leq n, y_i \in \mathcal{N}_M(x)$. Put $M'_x := \bigoplus M_{y_i} / \text{im} f$, where $f(m) = \begin{bmatrix} f_1(m) \\ \vdots \\ f_n(m) \end{bmatrix}, m \in M_x$. For each i , we have to show that the map $g_i : M'_{y_i} = M_{y_i} \rightarrow M'_x; m_i \mapsto (0, \dots, 0, m_i, 0, \dots, 0) + \text{im} f$ is injective, where g_i maps m_i to the i -th position in M'_x . Let $m_i \in \ker g_i$. Then $(0, \dots, 0, m_i, 0, \dots, 0) \in \text{im} f$. Hence there is $m \in M_x$ such that $m_i = f_i(m)$, and $f_j(m) = 0$ for all $j \neq i$. Since each f_j is injective (actually, $\bigcap_{j \neq i} \ker f_j = (0)$ suffices), we conclude that $m = 0$, that is, $m_i = 0$. Consequently, g_i is injective. \square

Lemma 5.4. *Let $N \in \text{Surj}_{n,\Omega} \cap \mathcal{D}$ be a regular indecomposable module. Then $\sigma N \in \text{Surj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$.*

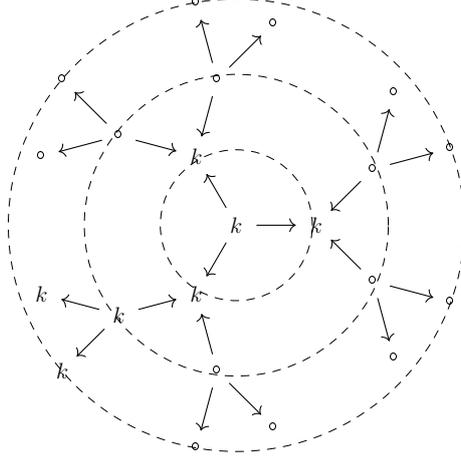
Proof. Since $D_{T(n)}(N) \in \text{Inj}_{n,\Omega}$, then we get $\sigma^- \circ D_{T(n)}(N) \stackrel{3.2.8}{\cong} D_{T(n)} \circ \sigma(N) \in \text{Inj}_{n,\sigma\Omega}$ using Lemma 5.3, that is, $\sigma N \in \text{Surj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$. \square

Lemma 5.5. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Then*

$$\iota(M_{\sigma\mathcal{D}}) \in \{\iota(M_{\mathcal{D}}) - 1, \iota(M_{\mathcal{D}}) + 1\}.$$

Proof. Now let $M = M_{\mathcal{D}}$. Since \mathcal{D} is regular and $ql(M) = 1$, then for any $N \in \text{Inj}_{n,\Omega} \cap \mathcal{D}$ with $ql(N) = 1$, it is well-known that $N = \tau^{-t}M = \sigma^{-2t}M$ and $\iota(N) = \iota(M) - 2t$ for some $t \in \mathbb{N}_0$. Hence M is the one with the biggest index $\iota(M)$ in its σ -orbit located in $\text{Inj}_{n,\Omega} \cap \mathcal{D}$. Then $\sigma^-M \in \text{Inj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$ by Lemma 5.3. Moreover, $ql(\sigma^{-1}M) = ql(M_{\sigma\mathcal{D}}) = ql(M)$ [Lemma 3.2.7], and $\iota(\sigma^-M) \leq \iota(M_{\sigma\mathcal{D}})$. Note that $M_{\sigma\mathcal{D}}$ is located in the σ -orbit of M with odd index. We have $\iota(M_{\sigma\mathcal{D}}) \geq \iota(M_{\mathcal{D}}) - 1$ and $\iota(M_{\sigma\mathcal{D}}) \neq \iota(M_{\mathcal{D}})$. Suppose that $\iota(M_{\sigma\mathcal{D}}) \leq \iota(M) - 3$. Then $\iota(\sigma^-M) = \iota(M) - 1 > \iota(M_{\sigma\mathcal{D}})$, a contradiction. Similarly, suppose that $\iota(M_{\sigma\mathcal{D}}) \geq \iota(M) + 3$. Then $\iota(\sigma^-M_{\sigma\mathcal{D}}) = \iota(M_{\sigma\mathcal{D}}) - 1 > \iota(M)$ and $\sigma^-M_{\sigma\mathcal{D}} \in \text{Inj}_{n,\Omega} \cap \mathcal{D}$ by Lemma 5.3, a contradiction. Hence $\iota(M_{\sigma\mathcal{D}}) \in \{\iota(M) - 1, \iota(M) + 1\}$. \square

Example. Continuing the example after Corollary 5.2, we now let $M = \sigma^-N$. Then $\text{supp}(M)$ would be the following:



We can see that $M \in \text{Inj}_{3,\sigma\Omega}$, and M is indecomposable. Moreover, M is a quasi-simple module [10, Proposition 3.4]. Since $N = \sigma M \notin \text{Inj}_{3,\Omega}$ and σN is a flow module, then $M = M_{\sigma\mathcal{D}}$. In fact, we already get $M_{\mathcal{D}} = \sigma^{-2}N$. Hence $\iota(M_{\sigma\mathcal{D}}) = \iota(M_{\mathcal{D}}) + 1$. Note that $\pi_{\lambda}(M_{\mathcal{D}}) \in \pi_{\lambda}(\mathcal{D})$ and $\pi_{\lambda}(M_{\sigma\mathcal{D}}) \in \pi_{\lambda}(\sigma\mathcal{D})$. It says that functor π_{λ} does not give us much information concerning such component \mathcal{D} .

Lemma 5.6. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Then*

$$\iota(W_{\sigma\mathcal{D}}) \in \{\iota(W_{\mathcal{D}}) - 1, \iota(W_{\mathcal{D}}) + 1\}.$$

Proof. Let $U \in \text{Surj}_{n,\Omega} \cap \mathcal{D}$ with $ql(U) = 1$, then it is already known that $U = \tau^s W_{\mathcal{D}} = \sigma^{2s} W_{\mathcal{D}}$ and $\iota(U) = \iota(W_{\mathcal{D}}) + 2s$ for some $s \in \mathbb{N}_0$. That is, $\iota(U) \geq \iota(W_{\mathcal{D}})$. On the other hand, we have $\sigma W_{\sigma\mathcal{D}} \in \text{Surj}_{n,\Omega} \cap \mathcal{D}$ by Lemma 5.4. Suppose that $\iota(W_{\sigma\mathcal{D}}) \leq \iota(W_{\mathcal{D}}) - 3$. Then $\iota(\sigma W_{\sigma\mathcal{D}}) = \iota(W_{\sigma\mathcal{D}}) + 1 < \iota(W_{\mathcal{D}})$, a contradiction. Similarly, suppose that $\iota(W_{\sigma\mathcal{D}}) \geq \iota(W_{\mathcal{D}}) + 3$. Then $\iota(\sigma W_{\mathcal{D}}) = \iota(W_{\mathcal{D}}) + 1 < \iota(W_{\sigma\mathcal{D}})$ and $\sigma W_{\mathcal{D}} \in \text{Surj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$ by Lemma 5.4, a contradiction. Hence $\iota(W_{\sigma\mathcal{D}}) < \iota(W_{\mathcal{D}}) + 3$. Finally, there exists $\iota(W_{\sigma\mathcal{D}}) = \iota(W_{\mathcal{D}}) + 2s' + 1$ for some $s' \in \mathbb{Z}$. \square

Theorem 5.7. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Then we have*

$$|\mathcal{W}(\mathcal{D}) - \mathcal{W}(\sigma\mathcal{D})| \leq 1 \text{ and } \frac{b(\mathcal{D})+1}{2} \leq \mathcal{W}(\mathcal{D}).$$

Proof. Since there exists $\tau = \sigma^2$ in our case, then there is $(\sigma^2)^{(\mathcal{W}(\mathcal{D})+1)} M_{\mathcal{D}} = W_{\mathcal{D}}$. It says that module $W_{\mathcal{D}}$ is located in the σ -orbit of module $M_{\mathcal{D}}$. Consequently, there is $2(\mathcal{W}(\mathcal{D}) + 1) + \iota(M_{\mathcal{D}}) = \iota(W_{\mathcal{D}})$. Then $\mathcal{W}(\mathcal{D}) = \frac{\iota(W_{\mathcal{D}}) - \iota(M_{\mathcal{D}})}{2} - 1$. According to Lemma 5.5 and Lemma 5.6, there exist inequalities

$$\begin{aligned} \frac{\iota(W_{\mathcal{D}}) - 1 - (\iota(M_{\mathcal{D}}) + 1)}{2} - 1 &= \mathcal{W}(\mathcal{D}) - 1 \\ &\leq \mathcal{W}(\sigma\mathcal{D}) = \frac{\iota(W_{\sigma\mathcal{D}}) - \iota(M_{\sigma\mathcal{D}})}{2} - 1 \\ &\leq \frac{\iota(W_{\mathcal{D}}) + 1 - (\iota(M_{\mathcal{D}}) - 1)}{2} - 1 = \mathcal{W}(\mathcal{D}) + 1. \end{aligned} \tag{1}$$

Hence

$$|\mathcal{W}(\mathcal{D}) - \mathcal{W}(\sigma\mathcal{D})| \leq 1. \tag{2}$$

There are only sink modules and source modules contained in $\text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$. Now let N be a regular indecomposable module located in the σ -orbit of module $M_{\mathcal{D}}$. Then there is $N = \sigma^q M_{\mathcal{D}}$ for some $q \in \mathbb{N}$. Suppose that N is a flow module so that $q \geq 2$. Using Lemma 5.1 and Corollary 5.2, it is shown that $\sigma M_{\mathcal{D}}$ is a sink and $\sigma^{-1} W_{\mathcal{D}}$ is a source. Then

$$\iota(\sigma M_{\mathcal{D}}) = \iota(M_{\mathcal{D}}) + 1 < \iota(N) = q + \iota(M_{\mathcal{D}}) < \iota(\sigma^{-1} W_{\mathcal{D}}) = \iota(W_{\mathcal{D}}) - 1.$$

Then $1 < q < \iota(W_{\mathcal{D}}) - 1 - \iota(M_{\mathcal{D}}) = 2\mathcal{W}(\mathcal{D}) + 1$. It indicates that the maximal length of the range of q is $2\mathcal{W}(\mathcal{D}) + 1 - 1 - 1 = 2\mathcal{W}(\mathcal{D}) - 1$, whence the number of flow modules $b(N) \leq 2\mathcal{W}(\mathcal{D}) - 1$. Moreover, there exists $b(\mathcal{D}) = d(p(N), q(N)) = b(N)$. Hence $\frac{b(\mathcal{D})+1}{2} \leq \mathcal{W}(\mathcal{D})$. \square

Corollary 5.8. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Suppose that $\mathcal{W}(\mathcal{D}) = \mathcal{W}(\sigma\mathcal{D}) = 1$, then $b(\mathcal{D}) = 0$.*

Proof. Suppose that there exists such a component \mathcal{D} so that $b(\mathcal{D}) = b(\sigma\mathcal{D}) \geq 1$. Hence there at least exists a regular indecomposable and flow module $N \in \mathcal{D}$ or $N \in \sigma\mathcal{D}$ with $ql(N) = 1$. Suppose that $N \in \mathcal{D}$. By Lemma 3.2.5, we know that $\sigma^{-1}N \notin \text{Inj}_{n, \sigma\Omega} \cap \sigma\mathcal{D}$ and $\sigma N \notin \text{Surj}_{n, \Omega} \cap \sigma\mathcal{D}$ [Lemma 5.1, Corollary 5.2]. Then $\iota(M_{\sigma\mathcal{D}}) \leq \iota(\tau^{-1}\sigma^{-1}N) = \iota(\sigma^{-1}N) - 2$ and $\iota(W_{\sigma\mathcal{D}}) \geq \iota(\tau\sigma N) = \iota(\sigma N) + 2$. Hence

$$\mathcal{W}(\sigma\mathcal{D}) = \frac{\iota(W_{\sigma\mathcal{D}}) - \iota(M_{\sigma\mathcal{D}})}{2} - 1 \geq \frac{\iota(\sigma N) + 2 - (\iota(\sigma^{-1}N) - 2)}{2} - 1 = \frac{\iota(\sigma N) - \iota(\sigma^{-1}N)}{2} - 1 + 2 = 2,$$

a contradiction. When $N \in \sigma\mathcal{D}$, we can similarly prove that $\mathcal{W}(\mathcal{D}) \geq 2$, and this is also a contradiction. \square

Example. Let $M \in \text{mod}(T(3), \Omega)$ be the following

$$\begin{array}{ccccccccc} (0) & \rightarrow & k & \xleftarrow{\lambda_1} & k & \xrightarrow{\lambda_2} & k & \xleftarrow{\lambda_3} & k & \xrightarrow{\lambda_4} & k & \leftarrow & (0) \\ & & \uparrow & & \downarrow \lambda_5 & & \uparrow \lambda_6 & & \downarrow \lambda_7 & & \uparrow & & \\ & & (0) & & k & & k & & k & & (0) & & \\ & & & & \swarrow \lambda_8 & & \searrow \lambda_9 & & & & & & \\ & & & & k & & k & & & & & & \end{array}$$

where $\lambda_i \in k \setminus \{0\}$, $i \in \{1, 2, \dots, 9\}$. Then M is indecomposable and $M \in \text{Inj}_{3, \Omega}$ is a sink module. In fact, M is a quasi-simple module [10, Proposition 3.4]. Moreover, we can show that M is regular [7, Theorem 2.18].

In fact, we have $b(M) = 0$. In particular, $M = M_{\mathcal{D}}$ for some regular component \mathcal{D} of $\text{mod}(T(3), \Omega)$ as $\sigma M \notin \text{Inj}_{3, \sigma\Omega}$. By direct computation, we know that $\sigma^2 M$ and $\sigma^3 M$ are source modules. However, $\sigma^3 M, \sigma^4 M$ are also source modules and all maps are surjective in them. Since $\sigma^2 M \notin \text{Surj}_{3, \Omega}$, then $\sigma^4 M = \tau^2 M = W_{\mathcal{D}}$ and $\mathcal{W}(\mathcal{D}) = 1$. Moreover, there exists $\sigma M \notin \text{Inj}_{3, \sigma\Omega} \cup \text{Surj}_{3, \sigma\Omega}$. Hence $\mathcal{W}(\sigma\mathcal{D}) = 1$ by Lemma 5.3.

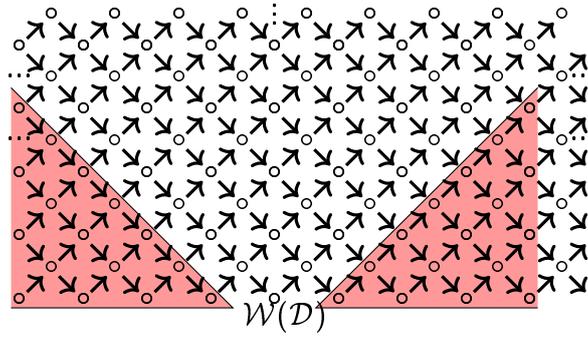


Figure 2: Regular component $\mathcal{W}(\mathcal{D}) = 1$.

6 Graded modules of constant rank

6.1 Matrices of graded module of constant rank

Let $M \in \text{mod}(T(n), \Omega)$ and $N = \pi_\lambda(M)$. We have

$$N_1 = \pi_\lambda(M)_1 = \bigoplus_{\pi(y)=1} M_y, N_2 = \pi_\lambda(M)_2 = \bigoplus_{\pi(y)=2} M_y,$$

$$N(\gamma_i) := \bigoplus_{\pi(\gamma_i^j)=\gamma_i} M(\gamma_i^j) : \pi_\lambda(M)_1 \rightarrow \pi_\lambda(M)_2,$$

for some $y \in T(n)_0$, $j \in \mathbb{N}$ and $i \in \{1, 2, \dots, n\}$. For convenience, we let $\gamma_i^j = M(\gamma_i^j)$, and let $\gamma_i = N(\gamma_i)$ in this section, where $M(\gamma_i^j) : M_{s(\gamma_i^j)} \rightarrow M_{t(\gamma_i^j)}$, $\pi(\gamma_i^j) = \gamma_i$, $1 \leq i \leq n$. By abusing notations, we still use γ_i^j and γ_i to denote their corresponding arrows in M and N , respectively.

Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. Since simple modules are trivial cases in our discussions, we assume that $\pi_\lambda(M)$ is not isomorphic to a simple module in general. We want to discuss the rank of matrix $\gamma = \sum_{i=1}^n N(\gamma_i) = \sum_{i=1}^n \gamma_i$. We use induction on the diameter $l = d(M)$ of M . Clearly, we have $l \geq 1$. Suppose that $l = 1$. Then $T(M)$ would look like $\alpha : a \rightarrow b$, $a, b \in T(M)_0$, $\alpha \in T(M)_1$. Hence

$$\Gamma_1 = \gamma = \sum_{i=1}^n \gamma_i = M(\alpha) = \gamma_j^1 = \gamma_j,$$

for some $j \in \{1, \dots, n\}$. Suppose that $l = 2$. Then M is either a sink or a source module. Moreover, there only exists a source or a sink in $T(M)_0$, that is, $N_1 = M_{s(\gamma_i^1)}$ or $N_2 = M_{t(\gamma_j^1)}$ for some $i, j \in \{1, \dots, n\}$. We can get

$$\Gamma_2 = \gamma = \sum_{i=1}^n \gamma_i = \begin{bmatrix} \gamma_{a_{11}}^1 \\ \gamma_{a_{12}}^1 \\ \vdots \\ \gamma_{a_{1n_1}}^1 \end{bmatrix}, \text{ or } \Gamma'_2 = \gamma = \sum_{i=1}^n \gamma_i = [\gamma_{a_{11}}^1 \quad \gamma_{a_{12}}^1 \quad \cdots \quad \gamma_{a_{1n_1}}^1], \quad (3)$$

for some subset $I_2 = \{a_{11}, a_{12}, \dots, a_{1n_1}\} \subseteq \{1, 2, \dots, n\}$. Since matrices Γ_2, Γ'_2 represent different trees, we sometimes call them tree Γ_2 or Γ'_2 . Note that all leaves of Γ_2 are sinks, all leaves of Γ'_2 are sources. Let $E_2 = \{\gamma_{a_{12}}^1, \gamma_{a_{13}}^1, \dots, \gamma_{a_{1n_1}}^1\}$.

Suppose that $l = 3$. Then M is a flow module. It indicates that there exist n_2 arrows $\gamma_{a_{2i}}^2$ connecting tree Γ_2 or Γ'_2 for some $n_2 \in \{1, \dots, n-1\}$, where $I_3 = \{a_{21}, \dots, a_{2n_2}\} \subseteq \{1, \dots, n\}$. Let $E_3 = \{\gamma_i^2 \mid i \in I_3\}$. Suppose that all arrows $\gamma_{a_{2i}}^2$ connect tree Γ_2 . Since $l = 3$, there exists some $a_{1j} \in I_2$ such that

$$t(\gamma_{a_{2i}}^2) = t(\gamma_{a_{1j}}^1)$$

for all arrows $\gamma_{a_{2i}}^2 \in E_3$. We reorder $\gamma_{a_{1i}}^1$ such that $t(\gamma_{a_{2i}}^2) = t(\gamma_{a_{1n_1}}^1)$. Then

$$\Gamma_3 = \sum_{i=1}^n \gamma_i = \begin{bmatrix} \gamma_{a_{11}}^1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{a_{1n_1-1}}^1 & 0 & \cdots & 0 \\ \gamma_{a_{1n_1}}^1 & \gamma_{a_{21}}^2 & \cdots & \gamma_{a_{2n_2}}^2 \end{bmatrix} = [\Gamma_2 \quad E(3)] = \Gamma_2 + \sum_{\gamma_{a_{2i}}^2 \in E_3} \gamma_{a_{2i}}^2, \quad (4)$$

where $E(3) = \sum_{\gamma_{a_{2i}}^2 \in E_3} \gamma_{a_{2i}}^2$. Similarly, we can get a matrix Γ'_3 when all arrows of E_3 connect tree Γ'_2 , that is, there exists some $a_{1j} \in I_2$ such that $s(\gamma_{a_{2i}}^2) = s(\gamma_{a_{1j}}^1)$ for all $\gamma_i^2 \in E_3$.

Suppose that $l = 4$. Then M is either a sink or a source module. Suppose that M is a sink module. Then there exist n_3 arrows $\gamma_{a_{3i}}^3$ connecting tree Γ_3 in M , $1 \leq i \leq n_3 \leq n-1$. Let $I_4 = \{a_{31}, \dots, a_{3a_{n_3}}\}$, and let $E_4 = \{\gamma_i^3 \mid i \in I_4\}$. It indicates that there exist some $\gamma_{a_{2j}}^2 \in E_3$ such that

$$s(\gamma_{a_{2j}}^2) = s(\gamma_{3i}^3), \text{ for some } \gamma_{3i}^3 \in E_4.$$

For convenience, we let $i = j$ when $a_{2j} = a_{3i}$, and we reorder a_{2j} by starting from a_{21} such that $a_{21} = a_{31}$. Then we get

$$\Gamma_4 = \begin{bmatrix} \gamma_{a_{11}}^1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{a_{n_1}}^1 & \gamma_{a_{21}}^2 & \gamma_{a_{22}}^2 & \cdots & \gamma_{a_{2n_2}}^2 \\ 0 & \gamma_{a_{31}}^3 & 0 & \cdots & 0 \\ 0 & \gamma_{a_{32}}^3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_{a_{3m_1}}^3 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_{a_{3m_2}}^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \gamma_{a_{3m_3}}^3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{a_{3m_{n_2}}}^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{a_{3m_{n_3}}}^3 \end{bmatrix} = \begin{bmatrix} \Gamma_3 \\ E(4) \end{bmatrix} = \Gamma_3 + \sum_{\gamma_{a_{3i}}^3 \in E_4} \gamma_{a_{3i}}^3, \quad (5)$$

where $E(4) = \sum_{\gamma_{a_{3i}}^3 \in E_4} \gamma_{a_{3i}}^3$. Suppose that M is a source module, that is, all arrows $\gamma_{a_{3i}}^3$ connect tree Γ'_3 , we can get a tree Γ'_4 . We can continue doing this, and we will finally get matrices $\Gamma_l = \sum_{i=1}^n \gamma_i$ when M is a sink or flow module, or $\Gamma'_l = \sum_{i=1}^n \gamma_i$ when M is a source module, and set E_l , where $l \in \mathbb{N}_{\geq 4}$. Note that E_l is the set consisting of those γ_i^j such that $t(\gamma_i^j)$ (or $s(\gamma_i^j)$) are leaves, and are located at the endings of diameter paths of $T(M)$; if $t(\gamma_i^j) = a_l$ (or $s(\gamma_i^j) = a_l$) is a leaf for some $a_l \in T(M)_0$, then there exists a diameter path (a_0, \dots, a_l) in $T(M)$. Let $E(l) = \sum_{\gamma_i^j \in E_l} \gamma_i^j$.

Lemma 6.1.1. *Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. Then there exists an equation*

$$\Gamma_l = \sum_{i=1}^n \gamma_i = \begin{cases} \begin{bmatrix} \Gamma_{l-1} & E(l) \end{bmatrix}, l = 2r - 1, \\ \begin{bmatrix} \Gamma_{l-1} \\ E(l) \end{bmatrix}, l = 2r, r \in \mathbb{N}, \end{cases} \quad (6)$$

when M is a sink or flow module, and there exists an equation

$$\Gamma'_l = \sum_{i=1}^n \gamma_i = \begin{cases} \begin{bmatrix} \Gamma'_{l-1} \\ E(l) \end{bmatrix}, l = 2r - 1, \\ \begin{bmatrix} \Gamma'_{l-1} & E(l) \end{bmatrix}, l = 2r, r \in \mathbb{N}, \end{cases} \quad (7)$$

when M is a source module, where $E(l) = \sum_{\gamma_i^j \in E_l} \gamma_i^j$.

Proof. There is nothing to prove when $\pi_\lambda(M)$ is isomorphic to simple module. Suppose that $\pi_\lambda(M)$ is not isomorphic to a simple module. We only prove the first equation, that is, M is a sink or flow module, and the second one is similar.

We use induction on the length of diameter l . Suppose that it is true when $l = 2r - 1$, or $2r$. When $l = 2r + 1$, that is, M is a flow module, we consider the set E_{2r+1} . By our construction, all $s(\gamma_i^j) \in E_l$ are source leaves and there exist some $\gamma_{i'}^{j'} \in E_{l-1}$ such that $t(\gamma_i^j) = t(\gamma_{i'}^{j'})$. Moreover, there exists $\Gamma_{l-1} = \begin{bmatrix} \Gamma_{l-2} \\ E(l-1) \end{bmatrix}$ by induction. Hence we have

$$\Gamma_{2r+1} = \begin{bmatrix} \Gamma_{l-2} & 0 \\ E(l-1) & E(l) \end{bmatrix} = [\Gamma_{l-1} \quad E(l)]$$

When $l = 2r + 2$, we consider set E_{2r+2} . By our construction, module M is a sink module. Then any $t(\gamma_i^j) \in E_l$ is a sink leaf and there exist some $\gamma_{i'}^{j'} \in E_{l-1}$ such that $s(\gamma_i^j) = s(\gamma_{i'}^{j'})$. Then we have

$$\Gamma_{2r+2} = \begin{bmatrix} \Gamma_{2r+1} \\ E(l) \end{bmatrix}.$$

Combining these two matrices $\Gamma_{2r+1}, \Gamma_{2r+2}$, we get our first equation. \square

Lemma 6.1.2. *Let $(a_0, a_1, a_2, \dots, a_l)$ be a diameter path of a finite tree Q , where $l \in \mathbb{N}_{\geq 2}$. Then the set $\mathcal{N}'(a_1) = \mathcal{N}(a_1) \setminus \{a_2\}$ consists of all leaves.*

Proof. Suppose that there exists a vertex $a' \in \mathcal{N}'(a_1)$ being not a leaf. It means that there at least exists a vertex $a'' \neq a' \in \mathcal{N}(a')$ such that the length of path $(a'', a', a_1, a_2, \dots, a_m)$ is greater than that of diameter path $(a_0, a_1, a_2, \dots, a_l)$ in Q , this is a contradiction. \square

In fact, we have inequalities as follow:

Lemma 6.1.3. *Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module with diameter $l \geq 2$. Then there exists an inequality*

$$|E_l| \leq (n-1)^r, l = 2r, \text{ or } 2r+1, r \in \mathbb{N}.$$

Proof. We use induction on the length l . When $l = 2$ or 3 , there is nothing to prove. Suppose that $|E_l| \leq (n-1)^r$ when $l = 2r$ or $2r+1$, $r \geq 2$. Suppose that M is a sink module when $l = 2r$. By our construction, each $s(\gamma_i^j)$ is a leaf for every $\gamma_i^j \in E_{2r+1}$ [Lemma 6.1.2]. Let $(a_0, a_1, \dots, a_{2r+1})$ be a diameter path of $T(M)$ when $l = 2r + 1$, and let a_{2r+1} be a source. Then $a_{2r+1} = s(\gamma_i^j)$ for some $\gamma_i^j \in E_{2r+1}$ and point a_{2r+1} has at most $n-1$ neighbours being leaves when $l = 2r + 2$. Moreover, all $\gamma_i^j \in E_{2r+1}$ have such property. Hence

$$|E_{2(r+1)}| \leq (n-1)E_{2r+1} \leq (n-1)^{(r+1)}.$$

When $l = 2(r + 1) + 1$, we assume that $(a_0, \dots, a_{2(r+1)}, a_{2(r+1)+1})$ is a diameter path of $T(M)$. It indicates that $a_{2(r+1)}$ has at most $(n - 1)$ neighbours being leaves and $a_{2(r+1)} = t(\gamma_i^j)$ for some $\gamma_i^j \in E_{2r+3}$. Then

$$|E_{2(r+1)+1}| \leq (n - 1) |E_{2(r+1)}| \leq (n - 1)^{(r+2)}.$$

Note that it is not every arrow of $E_{2(r+1)}$ connects an arrow of $E_{2(r+1)+1}$ for $l = 2(r + 1) + 1$. Otherwise, suppose that a_{r+1} is the center of $T(M)$ when $l = 2(r + 1)$. By our construction, it is not hard to see that $d(a_{r+1}, b) = d(a_{r+1}, b') = r + 2$ for any two source leaves $b = s(\gamma_i^j), b' = s(\gamma_{i'}^{j'})$ when $l = 2(r + 1) + 1$, where $\gamma_i^j, \gamma_{i'}^{j'} \in E_{2(r+1)+1}$. Then there exists an indecomposable module M (for example, a thin module) such that its leaves b, b' satisfy $(a_0, \dots, a_{r+1}, b_{r+2}, b_{r+3}, \dots, b_{2(r+1)+1} = b')$ and $(a_0, \dots, a_{r+1}, a_{r+2}, a_{r+3}, \dots, a_{2(r+1)+1} = b)$ are paths of $T(M)$ and

$$\{a_i \mid i = r + 2, r + 3, \dots, 2(r + 1) + 1\} \cap \{b_i \mid i = r + 2, r + 3, \dots, 2(r + 1) + 1\} = \emptyset.$$

That is, leaves b and b' are connected by a path in $T(M)$ passing the point a_{r+1} . Then there exists a path w of $T(M)$ such that

$$w = (b_{2(r+1)+1}, b_{2(r+1)}, \dots, b_{r+2}, a_{r+1}, a_{r+2}, a_{r+3}, \dots, a_{2(r+1)+1}),$$

and length of w is $2(r + 2)$ being even, this is a contradiction. Moreover, one can check that there are at most $n - 1$ neighbours of a_{r+1} located on the diameter paths of $T(M)$ when every arrow of $E_{2(r+1)}$ connects an arrow of $E_{2(r+1)+1}$ according to our discussion, and there only exists one neighbour of a_{r+1} consisting of the diameter path of $T(M)$ when $l = 2(r + 1) + 1$. Hence

$$|E_{2(r+1)+1}| \leq \frac{(n-1)^{r+2}}{n-1} \leq (n - 1)^{(r+1)}.$$

Similarly, one can show that the above inequality holds when M is a source module for $l = 2r$. \square

We recall an old result:

Lemma 6.1.4. [12, Theorem 19, (8.1), (8.2), (8.7)] *Ranks of Partitioned Matrices.* For matrices over an arbitrary field:

$$\begin{aligned} rk \begin{bmatrix} A & B \end{bmatrix} &\geq rk A, \quad rk \begin{bmatrix} A \\ B \end{bmatrix} \geq rk A, \\ rk \begin{bmatrix} X & 0 \\ S & Y \end{bmatrix} &\geq rk X + rk Y. \end{aligned}$$

Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module, and let $N = \pi_\lambda(M)$. We define sets $S_i := \{M(\gamma_i^j) \mid \pi(M(\gamma_i^j)) = N(\gamma_i)\}$, and $S_{i'} \subseteq S_i$ is a subset, $1 \leq i \leq n$. We define

$$N(\gamma_{i'}) := \gamma_{i'} = \bigoplus_{M(\gamma_{i'}^j) \in S_{i'}} M(\gamma_{i'}^j).$$

Lemma 6.1.5. *Let $M \in \text{mod}(T(n), \Omega)$ be a module. There exist inequalities*

$$rk \Gamma_l = rk \left(\sum_{i=1}^n \gamma_i \right) \geq rk \left(\sum_{i=1}^n \gamma_{i'} \right) \geq rk \left(\sum_{i=1}^{n-1} \gamma_{i'} \right) \geq \dots \geq rk \gamma_{1'}.$$

Proof. Without loss of generality, we assume that M is indecomposable, and $\pi_\lambda(M)$ is not isomorphic to a simple module. Let $\tilde{\Gamma}_l = \sum_{i=1}^n \gamma_{i'}$. Then matrix $\tilde{\Gamma}_l$ is gotten from matrix Γ_l by letting those γ_i^j , where $\gamma_i^j \in S_i \setminus S_{i'}$. By Lemma 6.1.1 and Lemma 6.1.4, we get the first inequality $rk \Gamma_l \geq rk \tilde{\Gamma}_l$. For the rest inequalities, proofs are similar. \square

Definition. Let $\alpha \in k^n \setminus (0)$, the map $\bar{\alpha} : P(2) \rightarrow P(1), m \mapsto (\alpha_1\gamma_1 + \alpha_2\gamma_2 + \cdots + \alpha_n\gamma_n)m$ induces an embedding of \mathcal{K}_n -modules. We set $X_\alpha := \text{coker } \bar{\alpha}$.

Definition. We call that $M \in \text{mod}(T(n), \Omega)$ is of constant rank provided $\pi_\lambda(M)$ is of constant rank.

Theorem 6.1.6. [15, Theorem 3.7] *Let $M \in \text{mod}(T(n), \Omega)$ be of constant rank. Then any direct summand of M is also of constant rank.*

Lemma 6.1.7. [11, Lemma 3.5] *Let $M \in \text{mod } \mathcal{K}_n$ be regular and not isomorphic to an X_α . Let $0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$ be the AR-sequence ending in M . If two out of the three modules are of constant rank, then so is the third.*

Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module. Recall that $\mathcal{N}_M(a) = \mathcal{N}(a) \cap T(M)_0$ for any $a \in T(M)_0$. Let $N = \pi_\lambda(M)$. In the following proofs, we always assume that module N is the push-down of the module M . We now consider matrix Γ_l . Let

$$B_i = \{s(\gamma_i^j) \mid \pi(\gamma_i^j) = \gamma_i\}, \text{ and } C_i = \{t(\gamma_i^j) \mid \pi(\gamma_i^j) = \gamma_i\}, 1 \leq i \leq n.$$

If there exists $s(\gamma_y^p) \in B_i$ for some $\gamma_y^p \in S_y, y \neq i$, it indicates that matrix γ_y^p locates in the same columns with some matrix γ_i^j in Γ_l , that is, $s(\gamma_y^p) = s(\gamma_i^j)$, where $\gamma_i^j \in B_i$. Similarly, γ_y^p locates in the same rows with γ_i^j in Γ_l if $t(\gamma_y^p) = t(\gamma_i^j) \in C_i$.

Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module, and let $N = \pi_\lambda(M)$ be not isomorphic to a simple module. Suppose that M is of constant rank. Then $l = d(M) \geq 2$, and we have the following conclusions:

Lemma 6.1.8. *Let $i \in \{1, 2, \dots, n\}$. Then for each arrow $\gamma_i^p : M_{s(\gamma_i^p)} \rightarrow M_{t(\gamma_i^p)}, \gamma_i^p \in S_i$, and every set $S_j, i \neq j$, there exist some $\gamma_j^{q_1} \in S_j$, or $\gamma_j^{q_2} \in S_j$ such that*

$$s(\gamma_j^{q_1}) = s(\gamma_i^p) \in B_i, \text{ or } t(\gamma_j^{q_2}) = t(\gamma_i^p) \in C_i.$$

Proof. Let $N = \pi_\lambda(M)$. It is clear that $\text{rk } x_\alpha^N \geq 1, \alpha \in k^n \setminus (0)$. Assume that there exists a non-trivial map $\gamma_j^u : M_{s(\gamma_j^u)} \rightarrow M_{t(\gamma_j^u)}$ in S_j for some $j \neq i$ such that $s(\gamma_j^u) \notin B_i$ and $t(\gamma_j^u) \notin C_i$.

Hence we have $\gamma_i^p + \gamma_j^u = \begin{bmatrix} \gamma_i^p & 0 \\ 0 & \gamma_j^u \end{bmatrix} = \gamma_i^p \oplus \gamma_j^u$ for all $\gamma_i^p \in S_i$. Now let $A = N(\gamma_i) + N(\gamma_j) = \gamma_i + \gamma_j$. According to Lemma 6.1.5, there exist inequalities

$$\text{rk } A = \text{rk}(\gamma_i + \gamma_j) \geq \text{rk}(\gamma_i + \gamma_j^u) = \text{rk} \begin{bmatrix} \gamma_i & 0 \\ 0 & \gamma_j^u \end{bmatrix} > \text{rk } \gamma_i. \quad (8)$$

It is a contradiction since module $N = \pi_\lambda(M)$ is of constant rank and $\text{rk } A = \text{rk } \gamma_i$. \square

Lemma 6.1.9. *Let (a_0, a_1, \dots, a_m) be a path of $T(M), m \geq 2$. Suppose that a_0 is a leaf. Then we have $|\mathcal{N}_M(a_1)| = n$.*

Proof. Since M is of constant rank, we have $\text{rk } \gamma_1 = \text{rk } \gamma_2 = \cdots = \text{rk } \gamma_n$ in N . Suppose that $|\mathcal{N}_M(a_1)| < n$. We first assume that a_0 is a sink. It indicates that there exists some γ_i such that there does not exist non-trivial direct summand $\gamma_i^p : M_{a_1} \rightarrow M_{a'}$ in $T(M)$ for all $\gamma_i^p \in S_i$, that is, $a_1 \notin B_i$, where $a' \in \mathcal{N}_M(a_1)$. As a_0 is a leaf, it also says that there does not exist direct summand $\gamma_i^q \in S_i$ such that $t(\gamma_i^q) = a_0$, that is, $a_0 \notin C_i$. On the other hand, there at least exists one map $\gamma_j^w \in E_j : M_{a_1} \rightarrow M_{a_0}$ that is non-trivial for some $\gamma_j^w \in S_j, j \neq i$. Then we have $s(\gamma_j^w) = a_1 \notin B_i$ and $t(\gamma_j^w) = a_0 \notin C_i$. According to Lemma 6.1.8, we know that this cannot happen. When a_0 is a source, we can consider non-trivial map $\gamma_{j'}^{w'} : M_{a_0} \rightarrow M_{a_1}$ for some $\gamma_{j'}^{w'} \in S_{j'}, j' \neq i$. Finally, we have $|\mathcal{N}_M(a_1)| = n$. \square

When $n \geq 3$, we have the following:

Lemma 6.1.10. *Let (a_0, a_1, \dots, a_m) be a path of $T(M)$, $m \geq 2$. Suppose that $b_i \in \mathcal{N}_M(a_1) \setminus \{a_2\}$ are leaves for $1 \leq i \leq n-1$. Then*

(a) *When a_0 is a sink, kernels of the maps $\gamma_i^1 : M_{a_1} \rightarrow M_{b_i}$ satisfy*

$$\ker \gamma_1^1 \cong \dots \cong \ker \gamma_{n-1}^1$$

and they are all surjective;

(b) *when a_0 is a source, maps $\gamma_i^1 : M_{b_i} \rightarrow M_{a_1}$ satisfy $\text{im } \gamma_1^1 \cong \dots \cong \text{im } \gamma_{n-1}^1$ and they are all injective.*

Moreover, there always exists $\text{rk } \gamma_n^1 \geq \text{rk } \gamma_i^1$, where γ_n^1 is the map between M_{a_0} and M_{a_1} .

Proof. Now we consider the module $N = \pi_\lambda(M)$. Suppose that a_0 is a sink. By Lemma 6.1.1, we have

$$\Gamma_l = \sum_{i=1}^n \gamma_i = \begin{bmatrix} \gamma_1^1 & 0 \\ \vdots & \vdots \\ \gamma_{n-1}^1 & 0 \\ \gamma_n^1 & B \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix},$$

where $\begin{bmatrix} B \\ A \end{bmatrix} = \sum_{i=1}^n (\oplus_{\gamma_i^j \in S'_i} \gamma_i^j)$, $S'_i = S_i \setminus \{\gamma_i^1\}$. We have $\text{rk } \gamma_n^1 \geq \text{rk } \gamma_i^1$ from Γ_l . Otherwise, suppose there that exists some j such that $\text{rk } \gamma_n^1 < \text{rk } \gamma_j^1, j \in \{1, \dots, n-1\}$. Then

$$\text{rk } (\gamma_n + \gamma_j) = \begin{bmatrix} \gamma_j^1 & 0 \\ \gamma_n^1 & \hat{B} \\ 0 & \hat{A} \end{bmatrix} \geq \text{rk } \begin{bmatrix} \gamma_j^1 & 0 \\ 0 & \hat{B} \\ 0 & \hat{A} \end{bmatrix} > \text{rk } \gamma_n,$$

where $\begin{bmatrix} \hat{B} \\ \hat{A} \end{bmatrix} = \oplus_{\gamma_n^i \in S'_n} \gamma_n^i + \oplus_{\gamma_j^i \in S'_j} \gamma_j^i$.

For the map $\gamma_i^1 : M_{a_1} \rightarrow M_{b_i}, 1 \leq i \leq n-1$, there exists an equation

$$\text{rk } \gamma_1^1 = \dots = \text{rk } \gamma_{n-1}^1 = \text{rk } \begin{bmatrix} \gamma_1^1 \\ \vdots \\ \gamma_{n-1}^1 \end{bmatrix}.$$

Otherwise, suppose that there exists $\text{rk } \gamma_i^1 > \text{rk } \gamma_j^1$ for some $1 \leq i \neq j \leq n-1, n \geq 3$. Then by Lemma 6.1.4 and Lemma 6.1.5, there exist inequalities

$$\text{rk } (\gamma_i + \gamma_j + \gamma_n) = \text{rk } \begin{bmatrix} \gamma_i^1 & 0 \\ \gamma_j^1 & 0 \\ \gamma_n^1 & B' \\ 0 & \\ \vdots & A' \\ 0 & \end{bmatrix} \geq \text{rk } \gamma_i^1 + \text{rk } \begin{bmatrix} B' \\ A' \end{bmatrix} > \text{rk } \begin{bmatrix} 0 & 0 \\ \gamma_j^1 & 0 \\ 0 & B'' \\ 0 & \\ \vdots & A'' \\ 0 & \end{bmatrix} = \text{rk } \gamma_j^1 + \text{rk } \begin{bmatrix} B'' \\ A'' \end{bmatrix} = \text{rk } \gamma_j,$$

where $\begin{bmatrix} B' \\ A' \end{bmatrix} = \oplus_{\gamma_i^v \in S'_i} \gamma_i^v + \oplus_{\gamma_j^v \in S'_j} \gamma_j^v + \oplus_{\gamma_n^v \in S'_n} \gamma_n^v$, $\begin{bmatrix} B'' \\ A'' \end{bmatrix} = \oplus_{\gamma_j^v \in S'_j} \gamma_j^v$. It is not hard to see that $\text{rk } \gamma_i^1 = \text{rk } \begin{bmatrix} \gamma_1^1 \\ \vdots \\ \gamma_{n-1}^1 \end{bmatrix}$, $1 \leq i \leq n-1$. In the end, $\ker \gamma_1^1 \cong \cdots \cong \ker \gamma_{n-1}^1 \cong \ker \begin{bmatrix} \gamma_1^1 \\ \vdots \\ \gamma_{n-1}^1 \end{bmatrix}$. Moreover, each γ_i^1 is surjective since M is indecomposable and $t(\gamma_i^1)$ is a leaf, $1 \leq i \leq n-1$. Suppose that a_0 is a source vertex. For maps $\gamma_i^1 : M_{b_i} \rightarrow M_{a_1}$, there is

$$\Gamma_l = \sum_{i=1}^n \gamma_i = \begin{bmatrix} \gamma_1^1 & \gamma_2^1 & \cdots & \gamma_{n-1}^1 & \gamma_n^1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & B & A & & \end{bmatrix},$$

where $\begin{bmatrix} B & A \end{bmatrix} = \sum_{i=1}^n (\oplus_{\gamma_i^j \in S'_i} \gamma_i^j)$, $S'_i = S_i \setminus \{\gamma_i^1\}$. Similarly, we can get

$$\text{rk } \gamma_1^1 = \cdots = \text{rk } \gamma_{n-1}^1 = \text{rk } \begin{bmatrix} \gamma_1^1 & \cdots & \gamma_{n-1}^1 \end{bmatrix}, \text{ and } \text{rk } \gamma_n^1 \geq \text{rk } \gamma_i^1, 1 \leq i \leq n-1.$$

Then we have $\text{im } \gamma_1^1 \cong \cdots \cong \text{im } \gamma_{n-1}^1 \cong \text{im } \begin{bmatrix} \gamma_1^1 & \cdots & \gamma_{n-1}^1 \end{bmatrix}$ by the theory of matrix. Moreover, each γ_i^1 is injective since M is indecomposable and $s(\gamma_i^1)$ is a leaf, $1 \leq i \leq n-1$. \square

Corollary 6.1.11. *If M is a thin module, then we have $M \in \text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$ and M is quasi-simple.*

Proof. Let (a_0, \dots, a_m) be path of $T(M)$ such that a_0 is a leaf, $m \geq 2$. When a_0 is a sink, we consider matrix Γ_l :

$$\Gamma_l = \sum_{i=1}^n \gamma_i = \begin{bmatrix} \gamma_1^1 & 0 \\ \vdots & \vdots \\ \gamma_{n-1}^1 & 0 \\ \gamma_n^1 & B \\ 0 & A \\ \vdots & \\ 0 & \end{bmatrix} = \begin{bmatrix} \gamma_1^1 & 0 \\ \vdots & \vdots \\ \gamma_{n-1}^1 & 0 \\ \gamma_n^1 & \Gamma_l^{(1)} \end{bmatrix},$$

where $\gamma_i^1 : M_{a_1} \rightarrow M_{a'}$, $a' \in \mathcal{N}_M(a_1)$, $1 \leq i \leq n$, and $\Gamma_l^{(1)} = \begin{bmatrix} B \\ A \end{bmatrix} = \sum_{i=1}^n (\oplus_{\gamma_i^j \in S'_i} \gamma_i^j)$, $S'_i = S_i \setminus \{\gamma_i^1\}$. Since M is a thin module, then we have

$$\ker \gamma_1^1 = \cdots = \ker \gamma_n^1 = (0).$$

If we delete set $\mathcal{N}_M(a_1) \setminus \{a_2\}$ in $T(M)_0$, we will get a new module $M^{(1)}$ and a matrix $\Gamma_l^{(1)}$. It is not hard to see that $M^{(1)}$ is still a thin module and indecomposable. Moreover, we can see that $M^{(1)}$ is also of constant rank and $\text{rk } \Gamma_l^{(1)} = \text{rk } \Gamma_l - 1$. Hence we can repeat our process, and delete such sets $\mathcal{N}_M(b) \setminus \{a_2\}$ in $T(M)_0$. Note that points of $\mathcal{N}_M(b) \setminus \{a_2\}$ are leaves. Finally, we get a new module $M^{(p)}$ and point a_2 is a sink leaf of $M^{(p)}$. Using Lemma 6.1.9, we replace M by module $M^{(p)}$ at the beginning, and discuss $M^{(p)}$ again.

After finite steps, we will get a module \bar{M} , and $\pi_\lambda(\bar{M}) \cong P(1)$, that is to say, every leaf of $T(M)$ is a sink, that is, $M \in \text{Inj}_{n,\Omega}$. Similarly, we can consider matrix Γ'_l when a_0 is a source, and get $M \in \text{Surj}_{n,\Omega}$. We get M being quasi-simple by [7, Corollary 10.16]. \square

Let $n = 2$, and let $M \in \text{mod}(T(2), \Omega)$ be an indecomposable module. Suppose that $\pi_\lambda(M)$ is not isomorphic to a simple module. Let $l = d(M)$. Then $l \geq 1$. We want to look at matrix Γ_l when $n = 2$. We choose an order for γ_1^p and γ_2^q , and we define

$$A_{1,2} := \Gamma_l = \gamma_1 + \gamma_2 = \begin{bmatrix} \gamma_2^1 & \gamma_1^1 & 0 & \cdots & 0 \\ 0 & \gamma_2^2 & \gamma_1^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \gamma_2^u & \gamma_1^u \\ 0 & 0 & 0 & 0 & \gamma_2^{u+1} \end{bmatrix}$$

where $l \geq u$. In fact, matrix $A_{1,2}$ represents a path in M starting from $s(\gamma_2^1)$ and ending at $t(\gamma_2^{u+1})$. Hence we sometimes call it path $A_{1,2}$. We have $s(\gamma_2^1)$ and $t(\gamma_2^{u+1})$ being leaves.

Let $n \geq 3$. Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module, and let $N = \pi_\lambda(M)$ be not isomorphic to a simple module. If there exists a path in $T(M)$ consisting of all direct summands γ_i^p, γ_j^q for some two arrows γ_i, γ_j in N , then module M is said to be of *type* $A_{1,2}$. Note that it may be decomposable when we restrict M to these direct summands γ_i^p, γ_j^q .

Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module, $n \geq 3$. Suppose that M is of constant rank, and $\pi_\lambda(M)$ is not isomorphic to a simple module. We have the following:

Lemma 6.1.12. *Suppose that M is of type $A_{1,2}$ for some two arrows γ_i, γ_j . Then for any direct summand γ_l^p , either $s(\gamma_l^p)$ is a leaf and γ_l^p is injective, or $t(\gamma_l^p)$ is a leaf and γ_l^p is surjective, where $\gamma_l^p \in S_l, l \neq i, j$.*

Proof. By Lemma 6.1.10, we only need to show that $s(\gamma_l^p)$ or $t(\gamma_l^p)$ is a leaf. Now we consider matrix $A_{1,2,l} = \gamma_i + \gamma_j + \gamma_l = A_{1,2} + \gamma_l$ in N , where $l \neq i, j$. By Lemma 6.1.8, we know that $s(\gamma_l^p) \in B_i$, or $t(\gamma_l^p) \in C_i$ for any $\gamma_l^p \in S_l$. Hence matrix $A_{1,2,l}$ would be the following:

$$A_{1,2,l} = \gamma_i + \gamma_j + \gamma_l = \begin{bmatrix} \gamma_l^1 & 0 & \cdots & 0 & \gamma_j^1 & \gamma_i^1 & 0 & \cdots & 0 \\ 0 & \gamma_l^2 & \cdots & 0 & 0 & \gamma_j^2 & \gamma_i^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_l^u & 0 & 0 & \cdots & \gamma_j^u & \gamma_i^u \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \gamma_j^{u+1} \\ 0 & 0 & \cdots & 0 & 0 & \gamma_l^{1'} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_l^{2'} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \gamma_l^{u'} \end{bmatrix} = \begin{bmatrix} A_l & A_{1,2} \\ \mathbf{0} & A_{l'} \end{bmatrix},$$

where $A_l = \sum_{p=1}^u \gamma_l^p, A_{l'} = \sum_{p'=1}^{u'} \gamma_l^{p'}$, and $t(\gamma_l^p) = t(\gamma_i^p), s(\gamma_l^{p'}) = s(\gamma_i^{p'}), p \in \{1, \dots, u\}$, and set $S_l = \{\gamma_l^p, \gamma_l^{p'} \mid p = 1, \dots, u\}, u = |S_l|$.

Suppose that $s(\gamma_l^q)$ is not a leaf for some $\gamma_l^q \in S_l$. It indicates that there exists some direct summand $\gamma_{l'}^w \in S_{l'}$ such that $s(\gamma_{l'}^w) = s(\gamma_l^q) \notin B_i$, and $t(\gamma_{l'}^w) \notin C_i, l' \neq i, j$, since arrow $\gamma_{l'}^w$ is not located on the path $A_{1,2}$. By Lemma 6.1.8, this is a contradiction. Using similar discussion, we can get all $t(\gamma_{l'}^{p'})$ being leaves. Moreover, $s(\gamma_j^1)$ and $t(\gamma_j^{u+1})$ are leaves because these two points are located at the two endings of path $A_{1,2}$. □

Let U_1, U_2 be two finite-dimensional vector spaces over field k , and let $\mathcal{L} \subseteq \text{Hom}_k(U_1, U_2)$ be a subspace. We say that \mathcal{L} is a *r-subspace* provided for any $0 \neq f \in \mathcal{L}$, there exists $\text{rk } f = r$. Let $U_1' \subseteq U_1$ be a subspace, let $f|_{U_1'}$ denote map f restricted to U_1' .

Lemma 6.1.13. [13, Lemma 3.1] *Let $f, g \in \mathcal{L}$ be two nonzero elements. Then there exists $f(\ker g) \subseteq \text{im } g$.*

Let $M \in \text{mod}(T(n), \Omega)$ be an indecomposable module, $n \geq 3$, and let M be of constant rank. Suppose that $N = \pi_\lambda(M)$ is not isomorphic to a simple module. Then we can get:

Lemma 6.1.14. *Suppose that M is of type $A_{1,2}$ for some two arrows γ_i, γ_j . Then there exists $M \in \text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$.*

Proof. We consider matrix $A_{1,2,l} = \gamma_i + \gamma_j + \gamma_l = A_{1,2} + \gamma_l$ in N , where $l \neq i, j$. By the proof of Lemma 6.1.12, matrix $A_{1,2,l}$ would be the following,

$$A_{1,2,l} = \gamma_i + \gamma_j + \gamma_l = \begin{bmatrix} \gamma_l^1 & 0 & \cdots & 0 & \gamma_j^1 & \gamma_i^1 & 0 & 0 \\ 0 & \gamma_l^2 & \cdots & 0 & 0 & \gamma_j^2 & \gamma_i^2 & 0 \\ 0 & 0 & \ddots & 0 & 0 & \vdots & \vdots & \\ 0 & 0 & \cdots & \gamma_l^u & 0 & \cdots & \gamma_j^u & \gamma_i^u \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \gamma_j^{u+1} \\ 0 & 0 & \cdots & 0 & 0 & \gamma_l^{1'} & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \gamma_l^{u'} \end{bmatrix},$$

where $\gamma_l^p, \gamma_l^{p'} \in S_l, u = |S_l|$, and $t(\gamma_l^p) = t(\gamma_i^p), s(\gamma_l^{p'}) = s(\gamma_i^p), p \in \{1, \dots, u\}$.

Since M is of constant rank, we have $\text{rk } A_{i,j,l} = \text{rk } \gamma_i = \text{rk } \gamma_j = \text{rk } \gamma_l$. Then we have

$$\text{rk } \gamma_l^p + \text{rk } \gamma_l^{p'} = \text{rk } \gamma_i^p, \text{ for all } p \in \{1, \dots, u\}. \quad (9)$$

Otherwise, suppose that there exists some $q \in \{1, \dots, u\}$ such that $\text{rk } \gamma_l^q + \text{rk } \gamma_l^{q'} > \text{rk } \gamma_i^q$. Then we have $\gamma_l^q \oplus \gamma_l^{q'} + \gamma_i = D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$, where $D_1 = \begin{bmatrix} \gamma_l^q & \gamma_i^q \\ 0 & \gamma_l^{q'} \end{bmatrix}$ and $D_2 = \bigoplus_{\gamma_i^j \in S_i'} \gamma_i^j, S_i' = S_i \setminus \{\gamma_i^q\}$. Hence

$$\text{rk } D = \text{rk } D_1 + \sum_{p \neq q}^u \text{rk } \gamma_i^p > \text{rk } \sum_{p=1}^u \gamma_i^p = \text{rk } \gamma_i,$$

using $\text{rk } \begin{bmatrix} \gamma_l^q & \gamma_i^q \\ 0 & \gamma_l^{q'} \end{bmatrix} \geq \text{rk } \gamma_l^q + \text{rk } \gamma_l^{q'} > \text{rk } \gamma_i^q$ by Lemma 6.1.4. Finally

$$\text{rk } (\gamma_l + \gamma_i) \geq \text{rk } (\gamma_l^q \oplus \gamma_l^{q'} + \gamma_i) > \text{rk } \gamma_i$$

by Lemma 6.1.5, this is a contradiction. Similarly, suppose that there exists $\text{rk } \gamma_l^q + \text{rk } \gamma_l^{q'} < \text{rk } \gamma_i^q$. We also have

$$\text{rk } \begin{bmatrix} \gamma_l^q & \gamma_i^q \\ 0 & \gamma_l^{q'} \end{bmatrix} \geq \text{rk } \begin{bmatrix} \gamma_l^q & \gamma_i^q \\ \gamma_l^{q'} & \end{bmatrix} \geq \text{rk } \gamma_i^q > \text{rk } \gamma_l^q + \text{rk } \gamma_l^{q'}.$$

Using Lemma 6.1.5, $\text{rk } (\gamma_l + \gamma_i) \geq \text{rk } \begin{bmatrix} \gamma_l^q & \gamma_i^q \\ 0 & \gamma_l^{q'} \end{bmatrix} + \sum_{p \neq q, \gamma_i^p \in S_i} \text{rk } \gamma_i^p > \text{rk } \gamma_l$, this is a contradiction.

By Lemma 6.1.13, we have $\gamma_l(\ker \gamma_i) \subseteq \text{im } \gamma_i$. Since $\ker \gamma_i = \bigoplus_{\gamma_i^j \in S_i} \ker \gamma_i^j$, there exists $\gamma_l^{p'}(\ker \gamma_i^p) \subseteq \text{im } \gamma_i^p$. However, $\text{im } \gamma_l^{p'} \cap \text{im } \gamma_i^p = (0)$. Then $\gamma_l^{p'}(\ker \gamma_i^p) = (0)$. Hence there exists

$$\ker \gamma_i^p \subseteq \ker \gamma_l^{p'}, p \in \{1, \dots, u\}.$$

On the other hand, there is also $\gamma_i(\ker \gamma_l) \subseteq \text{im } \gamma_l$. Similarly, we have $\gamma_i^p(\ker \gamma_l^{p'}) \subseteq \text{im } \gamma_l^p$. Suppose that $\gamma_i^p(\ker \gamma_l^{p'}) \not\subseteq \text{im } \gamma_l^p$. Since $\ker \gamma_i^p \subseteq \ker \gamma_l^{p'}$, then $\gamma_i^p|_{U^{p'}}$ is injective, where $U^{p'}$ is the complement of $\ker \gamma_l^{p'} = U''$ in vector space $U = M_{s(\gamma_i^p)}$. Then

$$\gamma_i^p(U) = \gamma_i^p(U^{p'} \oplus U'') \subseteq \gamma_i^p|_{U^{p'}} \oplus \gamma_i^p|_{U''} \not\subseteq U^{p'} \oplus \text{im } \gamma_l^p,$$

under isomorphisms of vector spaces. However, there exists

$$\dim_k U^{p'} + \text{rk } \gamma_l^p = \text{rk } \gamma_l^{p'} + \text{rk } \gamma_l^p > \text{rk } \gamma_i^p,$$

this is a contradiction with equation (9). Hence there exist isomorphisms

$$\gamma_i^p(\ker \gamma_l^{p'}) \cong \text{im } \gamma_l^p, \quad (10)$$

for all $p \in \{1, \dots, u\}$. Actually, we can also get

$$\text{rk } \gamma_l^{(p-1)'} + \text{rk } \gamma_l^p = \text{rk } \gamma_j^p. \quad (11)$$

through changing i into j . Then $\gamma_j^p(\ker \gamma_l^{(p-1)'}) \cong \text{im } \gamma_l^p, p \in \{2, \dots, u\}$.

Hence we can see that there exists decomposition of γ_i^p and γ_j^p . That is,

$$\begin{aligned} \gamma_i^p &= \gamma_i^p|_{\ker \gamma_l^{p'}} \oplus \gamma_i^p|_{U^{p'}}, \\ \gamma_j^p &= \gamma_j^p|_{\ker \gamma_l^{(p-1)'}} \oplus \gamma_j^p|_{U^{(p-1)'}}, \end{aligned}$$

where $\ker \gamma_l^{p'} \oplus U^{p'} \cong M_{s(\gamma_i^p)} = M_{s(\gamma_l^{p'})}$, $\ker \gamma_l^{(p-1)'} \oplus U^{(p-1)'} \cong M_{s(\gamma_j^p)} = M_{s(\gamma_l^{(p-1)'})}$. Hence we get

$$A_{1,2,l} = \gamma_i + \gamma_j + \gamma_l = \begin{bmatrix} \gamma_l^1 & 0 & \dots & 0 & \gamma_j^{1'} & \gamma_i^{1'} & 0 & 0 \\ 0 & \gamma_l^2 & \dots & 0 & 0 & \gamma_j^{2'} & \gamma_i^{2'} & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \gamma_l^u & 0 & \dots & \gamma_j^{u'} & \gamma_i^{u'} \end{bmatrix} \oplus \begin{bmatrix} \gamma_i^{1''} & 0 & \mathbf{0} \\ \gamma_j^{2''} & \gamma_i^{2''} & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} \\ \mathbf{0} & \gamma_j^{u''} & \gamma_i^{u''} \\ 0 & \dots & \gamma_j^{(u+1)''} \\ \gamma_l^{1'} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_l^{u'} \end{bmatrix} = A' \oplus A'',$$

where $\gamma_i^{p'} = \gamma_i^p|_{\ker \gamma_l^{p'}}$, $\gamma_j^{p'} = \gamma_j^p|_{\ker \gamma_l^{(p-1)'}}$, $\gamma_i^{p''} = \gamma_i^p|_{U^{p'}}$, $\gamma_j^{p''} = \gamma_j^p|_{U^{(p-1)'}}$, and

$$A' = \begin{bmatrix} \gamma_l^1 & 0 & \dots & 0 & \gamma_j^{1'} & \gamma_i^{1'} & 0 & 0 \\ 0 & \gamma_l^2 & \dots & 0 & 0 & \gamma_j^{2'} & \gamma_i^{2'} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \gamma_l^u & 0 & \dots & \gamma_j^{u'} & \gamma_i^{u'} \end{bmatrix}, A'' = \begin{bmatrix} \gamma_i^{1''} & 0 & \mathbf{0} \\ \gamma_j^{2''} & \gamma_i^{2''} & \mathbf{0} \\ \vdots & \vdots & \mathbf{0} \\ \mathbf{0} & \gamma_j^{u''} & \gamma_i^{u''} \\ 0 & \dots & \gamma_j^{(u+1)''} \\ \gamma_l^{1'} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_l^{u'} \end{bmatrix}.$$

Note that we only need $l \neq i, j$ and $n \geq 3$. On the other hand, M is indecomposable. Then either $A' = 0$, or $A'' = 0$. Suppose that $A' = 0$, then

$$\text{rk } \gamma_i^{p''} = \text{rk } \gamma_i^p = \text{rk } \gamma_i^{p'} = \text{rk } \gamma_j^{(p+1)''} = \text{rk } \gamma_j^{(p+1)}, 1 \leq q \leq u.$$

Hence $\ker \gamma_i^{p'} = (0)$, that is, every $\gamma_i^{p'}$ is injective, $1 \leq p \leq u$. Finally, we can see that $\gamma_i^{p''} = \gamma_i^p, \gamma_j^{(p+1)''} = \gamma_j^{(p+1)}$ are all injective in M , $1 \leq p \leq u$. Hence $M \in \text{Inj}_{n,\Omega}$. Similarly, we get $M \in \text{Surj}_{n,\Omega}$ when $A'' = 0$. □

6.2 Reflection functor and graded modules of constant rank

In this section, we want to see what would happen when we use reflection functor σ to act on $\text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$ in $\text{mod}(T(n), \Omega)$, $n \geq 3$.

Lemma 6.2.1. *Let $M \in \text{Inj}_{n,\Omega}$ be a regular indecomposable module. Suppose that σM is of constant rank. Then $\sigma M \in \text{Inj}_{n,\sigma\Omega}$.*

Proof. Since $M \in \text{Inj}_{n,\Omega}$, M is a sink module and its diameter l is even. Let $l_0 = (a_0, a_1, \dots, a_m)$ be a diameter path of $T(M)$. By [7, Theorem 8.1], all maps $\gamma_i^1 : M_{c_i} \rightarrow M_{a_2}$ are injective, $c_i \in \mathcal{N}_M(a_2)$, $1 \leq i \leq n$. Let $M' = \sigma M$. By Lemma 5.1, all leaves of M' are sinks and $m \geq 4$. By [5, Proposition 1.(a)], path $l_1 = (a_1, \dots, a_{m-1})$ is a diameter path of $T(M')$. Actually, all diameter paths of $T(M')$ are gotten by this process. By Lemma 6.1.10(a), kernels of maps $\gamma_i^1 : M'_{a_2} \rightarrow M'_{c_i}$ are isomorphic, where points $c_i \in \mathcal{N}_{M'}(a_2) \setminus \{a_3\}$ are leaves and $1 \leq i \leq n-1$ [Lemma 6.1.9]. Note that $M'_{a_2} = \ker \mu$, where $\mu = (\gamma_1^1, \dots, \gamma_n^1) : \oplus M_{c_i} \rightarrow M_{a_2}$, $c_i \in \mathcal{N}_M(a_2)$. On the other hand, we know that $\gamma_i^1 : M'_{a_2} \rightarrow M'_{c_i} = M_{c_i}; m = (m_1, \dots, m_n) \mapsto m_i, 1 \leq i \leq n-1$. Moreover

$$\ker \gamma_i^1 = \{m' = (m_1, \dots, m_{i-1}, 0, m_{i+1}, \dots, m_n) \mid \sum_{v \neq i} \gamma_v^1(m_v) = 0\}.$$

We pick an element $e_i = (m_1, \dots, m_{i-1}, 0, m_{i+1}, \dots, m_n) \in \ker \gamma_i^1$. Consider $\ker \gamma_j^1, 1 \leq j \neq i \leq n-1$. By Lemma 6.1.10(a), there exists $\ker \gamma_i^1 \cong \ker \gamma_j^1$. Then the element $e_i \in \ker \gamma_i^1 \cong \ker \gamma_j^1$ indicates that $m_j = 0$ in e_i for all $1 \leq j \neq i \leq n-1$. Finally, we get

$$e_i = (0, 0, \dots, 0, m_n) \in \ker \gamma_i^1.$$

Hence $\ker \gamma_i^1 = \{m' = (0, 0, \dots, 0, m_n) \mid \gamma_n^1(m_n) = 0\} = (0)$ because γ_n^1 is injective in M , that is, γ_i^1 is injective, $1 \leq i \leq n-1$. Note that $\text{rk } \gamma_n^1 \leq \dim_k M'_{a_2} = \text{rk } \gamma_i^1$, where $\gamma_n^1 : M'_{a_2} \rightarrow M'_{a_3}$. However, there exists $\text{rk } \gamma_n^1 \geq \text{rk } \gamma_i^1$ by Lemma 6.1.10. Hence $\text{rk } \gamma_n^1 = \text{rk } \gamma_i^1$, and γ_n^1 is also injective. Let $N' = \pi_\lambda(M')$ and $\gamma'_i = N'(\gamma_i), 1 \leq i \leq n$. According to section 6.1, we get

$$\Gamma_{l_1} = \sum_{i=1}^n (\oplus_{\pi(\gamma_i^j) = \gamma'_i} \gamma_i^j) = \begin{bmatrix} \gamma_1^1 & 0 \\ \vdots & \vdots \\ \gamma_{n-1}^1 & 0 \\ \gamma_n^1 & B \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix},$$

for M' . Since all γ'_i are injective for $1 \leq i \leq n$, we get a module M^2 , and matrix

$$\Gamma_{(l_2)} = \begin{bmatrix} B \\ A \end{bmatrix},$$

satisfying $\text{rk } \Gamma_{(l_2)} = \text{rk } \Gamma_{l_1} - \text{rk } \gamma_1^1$, where module M^2 is gotten from M' by deleting set $\{a_2\} \cup \mathcal{N}_{M'}(a_2) \setminus \{a_3\}$ in $\text{supp}(M')$, l_2 is a path of M^2 . We can see that M^2 is still of constant rank. Module M^2 may be decomposable, however, $T(M^2)$ is connected. Note that the maps in M^2 are the same with that of M' .

If $d(M') = 2$, then we are done. Suppose that $d(M') \geq 4$. In fact, we already see that all leaves of M' are sinks [Lemma 5.1(b)]. If a_3 is not a leaf of $\text{supp}(M^2)$, then we can continue to delete the set $\{b_2\} \cup \mathcal{N}_{M'}(b_2) \setminus \{a_3\}$ in $\text{supp}(M')$ when $(b_1, b_2, a_3, \dots, b_{m-1})$ is another diameter path of M' . After finite steps, we will get a module M^p such that a_3 is a sink leaf of M^p . Although M^p may be decomposable, $\text{supp}(M^p)$ is still a tree by our construction, and each direct summand of M^p is of constant rank [Theorem 6.1.6]. Suppose that $M^p = \bigoplus_{i \in I} M^{p,i}$, where every $M^{p,i}$ is indecomposable and is of constant rank. Since M' is indecomposable, there exists

$$a_3 \in \text{supp}(M^{p,i}) \cap \text{supp}(M^{p,j})$$

for any $i \neq j$. Otherwise, suppose that there exists a direct summand $M^{p,i}$ of M^p such that

$$a_3 \notin \text{supp}(M^{p,i}).$$

It indicates that $M^{p,i}$ is also a direct summand of M' by our construction, this is a contradiction. Let $l_{p,i} = (a_{i,0} = a_3, a_{i,1}, \dots, a_{i,p_i})$ be a path of direct summand $M^{p,i}$, $i \in I$. Then path $l_{p,i}$ satisfies condition of Lemma 6.1.10(a). Hence M^p satisfies condition of Lemma 6.1.10(a).

We now change M' into M^p at the beginning, and discuss one of its paths $l_p = (a_3, \dots, a_{l'})$. Then we can similarly get map $\psi_i^1 : M_{a_4}^p = M_{a_4}^i \rightarrow M_{b_i}^p = M_{b_i}^i$ being injective for every $b_i \in \mathcal{N}_{M'}(a_4)$. Then we delete set $\{a_4\} \cup \mathcal{N}_{M'}(a_4) \setminus \{a_5\}$ and get a module M^q . We repeat this process, and we can finally see that every $M'(\gamma_i^j)$ is injective in M' , whence $M' = \sigma M \in \text{Inj}_{n,\sigma\Omega}$. \square

Lemma 6.2.2. *Let $M \in \text{Surj}_{n,\Omega}$ be a regular indecomposable module. Suppose that $\sigma^- M$ is of constant rank, then $\sigma^- M \in \text{Surj}_{n,\sigma\Omega}$.*

Proof. According to Lemma 3.2.8 and [15, Proposition 5.2], we know that $D_{T(n)} \circ (\sigma^- M)$ is of constant rank and

$$D_{T(n)} \circ (\sigma^- M) \cong \sigma \circ (D_{T(n)}(M)).$$

Since $M \in \text{Surj}_{n,\Omega}$, then $D_{T(n)}(M) \in \text{Inj}_{n,\Omega}$. By Lemma 6.2.1, we know that

$$\sigma(D_{T(n)}(M)) \in \text{Inj}_{n,\sigma\Omega}.$$

Hence $\sigma^- M \in \text{Surj}_{n,\sigma\Omega}$. \square

Theorem 6.2.3. *Let \mathcal{D} be a regular component of $\text{mod}(T(n), \Omega)$. Suppose that $\mathcal{W}(\mathcal{D}) = 1$. Then there exists*

$$\text{CR}_{n,\sigma\Omega} \cap \sigma\mathcal{D} = (\text{Inj}_{n,\sigma\Omega} \cup \text{Surj}_{n,\sigma\Omega}) \cap \sigma\mathcal{D}$$

for the regular component $\sigma\mathcal{D}$.

Proof. By Theorem 5.7, we get that $\mathcal{W}(\sigma\mathcal{D}) = 1$, or 2. Now let $M \in \mathcal{D}$, and let $M \notin \text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$. Then there have $\sigma^{-2}M \in \text{Inj}_{n,\Omega}, \sigma^2M \in \text{Surj}_{n,\Omega}$ since $\mathcal{W}(\mathcal{D}) = 1$.

Suppose that $\mathcal{W}(\sigma\mathcal{D}) = 2$. We have $\sigma^{-1}M, \sigma M \notin \text{Inj}_{n,\sigma\Omega} \cup \text{Surj}_{n,\sigma\Omega}$ for component $\sigma\mathcal{D}$. Otherwise, suppose that $\sigma M \in \text{Surj}_{n,\sigma\Omega}$. Since $\mathcal{W}(\sigma\mathcal{D}) = 2$, there exist $\sigma^{-1}M, \sigma^{-3}M \notin \text{Inj}_{n,\sigma\Omega}$. Then $\sigma^{-2}M = \tau^{-1}M \notin \text{Inj}_{n,\Omega}$ by Lemma 5.3. This cannot happen since $\mathcal{W}(\mathcal{D}) = 1$. Similarly, we can show that $\sigma^{-1}M \notin \text{Inj}_{n,\sigma\Omega}$.

By Lemma 3.2.4, Lemma 3.2.7, Lemma 6.2.1 and Lemma 6.2.2, we have

$$\sigma^{-1}M \cong \sigma(\sigma^{-2}M) \notin \text{CR}_{n,\sigma\Omega} \cap \sigma\mathcal{D}, \sigma M \cong \sigma^{-1}(\sigma^2M) \notin \text{CR}_{n,\sigma\Omega} \cap \sigma\mathcal{D}.$$

Since $\mathcal{W}(\sigma\mathcal{D}) = 2$, there is

$$\text{CR}_{n,\sigma\Omega} \cap \sigma\mathcal{D} = (\text{Inj}_{n,\sigma\Omega} \cup \text{Surj}_{n,\sigma\Omega}) \cap \sigma\mathcal{D}. \quad (12)$$

Suppose that $\mathcal{W}(\sigma\mathcal{D}) = 1$. We can see that $\sigma M = W_{\sigma\mathcal{D}} \in \text{Surj}_{n,\sigma\Omega}$, or $\sigma^{-1}M = M_{\sigma\mathcal{D}} \in \text{Inj}_{n,\sigma\Omega}$. Hence there exists $M \cong \sigma^{-1}W_{\sigma\mathcal{D}}$, or $M \cong \sigma M_{\sigma\mathcal{D}}$ by Lemma 3.2.7. Since $M \notin \text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}$ and $\mathcal{W}(\mathcal{D}) = \mathcal{W}(\sigma\mathcal{D}) = 1$, then M is not of constant rank [Lemma 6.2.1, Lemma 6.2.2]. Moreover, $\text{CR}_{n,\Omega} \cap \mathcal{D} = (\text{Inj}_{n,\Omega} \cup \text{Surj}_{n,\Omega}) \cap \mathcal{D}$. Suppose that $M \cong \sigma^{-1}W_{\sigma\mathcal{D}}$. Then $\sigma^{-1}M \notin \text{Inj}_{n,\sigma\Omega}$ since $\mathcal{W}(\sigma\mathcal{D}) = 1$. However, $\sigma^{-2}M \in \text{Inj}_{n,\Omega}$ since $\mathcal{W}(\mathcal{D}) = 1$. Then $\sigma^{-1}M \cong \sigma(\sigma^{-2}M)$ is not of constant rank by Lemma 6.2.1. Suppose that $M \cong \sigma M_{\sigma\mathcal{D}}$. Then $\sigma M \notin \text{Surj}_{n,\sigma\Omega}$ since $\mathcal{W}(\sigma\mathcal{D}) = 1$. However, there is $\sigma^2M \in \text{Surj}_{n,\Omega}$ since $\mathcal{W}(\mathcal{D}) = 1$. Then $\sigma M \cong \sigma^{-1}(\sigma^2M)$ is not of constant rank according to Lemma 6.2.2. Hence equation (12) is also true when $\mathcal{W}(\sigma\mathcal{D}) = 1$. □

7 Elementary modules of \mathcal{K}_n

7.1 Dimension vectors of elementary modules

Let $\mathcal{K}_n = kK(n)$, $n \geq 3$. There exists a function called dimension vector on $\text{mod } \mathcal{K}_n$

$$\underline{\dim}: \text{mod } \mathcal{K}_n \rightarrow \mathbb{Z}^2, M \mapsto (\dim_k M_1, \dim_k M_2).$$

if $(0) \rightarrow L \rightarrow M \rightarrow N \rightarrow (0)$ is an exact sequence in $\text{mod } \mathcal{K}_n$, then $\underline{\dim} L + \underline{\dim} N = \underline{\dim} M$. There exists a bilinear form

$$\langle -, - \rangle: \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}, ((x_1, x_2), (y_1, y_2)) \mapsto (x_1 y_1 + x_2 y_2) - n x_1 y_2.$$

This bilinear form is exactly same with the Euler-Ringel form on the Grothendieck group $K_0(\mathcal{K}_n) \cong \mathbb{Z}^2$. Then we get a quadratic form

$$q: \mathbb{Z}^2 \rightarrow \mathbb{Z}, x \mapsto \langle x, x \rangle.$$

Definition. A dimension vector (x, y) is said to be regular, provided $q(x, y) < 0$.

Let σ, σ^- be the Bernstein-Gelfand-Ponomarev reflections (or BGP-functors) of $K_0(\mathcal{K}_n) = \mathbb{Z}^2$ given by $\sigma(x, y) = (nx - y, x)$, $\sigma^-(x, y) = (y, ny - x)$. By abusing notations, we take σ, σ^- as the BGP functors of $\text{mod } \mathcal{K}_n$ (we take the opposite of the n -Kronecker quiver to be again the n -Kronecker quiver).

Let $M \in \text{mod } \mathcal{K}_n$ be an indecomposable module different from $S(2)$, then there exists

$$\underline{\dim} \sigma M = \sigma \underline{\dim} M.$$

Similarly, if M is indecomposable and different from $S(1)$, then $\underline{\dim} \sigma^- M = \sigma^- \underline{\dim} M$.

Definition. Let $M \in \text{mod } \mathcal{K}_n$ be regular. We say that M is *elementary*, provided there is no short exact sequence $(0) \rightarrow L \rightarrow M \rightarrow N \rightarrow (0)$ with $L, N \in \text{mod } \mathcal{K}_n$ being nonzero regular modules.

If M is elementary, then all module $\sigma^t M$ are elementary, $t \in \mathbb{Z}$ [1, VII. Corollary 5.7(d)].

Definition. The dimension vector (x, y) is said to be *elementary*, *preprojective*, or *preinjective*, provided there exists an elementary, preprojective, or preinjective module M with $\underline{\dim} M = (x, y)$.

Let \mathbf{R} be the set of the regular dimension vectors. We have seen that σ maps \mathbf{R} onto \mathbf{R} . In fact, there is another transformation δ on $K_0(\mathcal{K}_n)$ defined by $\delta(x, y) = (y, x)$. Moreover, it sends \mathbf{R} onto \mathbf{R} . Let $M \in \text{mod } \mathcal{K}_n$, then $\delta \underline{\dim} M = \underline{\dim} M^*$, where M^* is the dual representation of M , which is, $M^* = (M_1^*, M_2^*, (M^*(\gamma_i))_{1 \leq i \leq n})$, M_1^* is the k -dual of M_2 and M_2^* is the k -dual of M_1 , the map $M^*(\gamma_i)$ is the k -dual of $M(\gamma_i)$.

We give the set

$$\mathbf{F} = \{(x, y) \mid \frac{2}{n}x \leq y \leq x\}.$$

Lemma 7.1.1. [17, Section 2. Lemma] *The set \mathbf{F} is a fundamental domain for the action of the group generated by δ and σ on the set \mathbf{R} .*

Lemma 7.1.2. [17, Lemma 3.1] *Assume that $M \in \text{mod } \mathcal{K}_n$ is a regular module with a proper nonzero submodule U such that both dimension vectors $\underline{\dim} U$ and $\underline{\dim} M/U$ are regular. Then M is not elementary.*

Proposition 7.1.3. [17, Appendix 1. Proposition] *Let $M \in \text{mod } \mathcal{K}_n$ be nonzero regular module. Then M is elementary if and only if for any submodule U of M , the submodule U is preprojective or the factor module M/U is preinjective.*

By duality and Lemma 7.1.2, a module $M \in \text{mod } \mathcal{K}_n$ is elementary if and only if its dual M^* is elementary. That is to say, if $\underline{\dim} M = (x, y)$, then (x, y) is elementary if and only if (y, x) is elementary. Hence we only need to study one of these two dimension vectors.

Lemma 7.1.4. [7, Lemma 14.11] *Let $M \in \text{mod } \mathcal{K}_n$ be an elementary module with $\underline{\dim} M = (x, y)$ and $y \leq x \leq y + n - 2$. Then $x < n$.*

Let V^i denote the i -dimensional vector space over field k , $i \in \mathbb{N}$. Let $L(V^j, V^i)$ be the linear space of all k -linear transformations from V^j to V^i . Let $l(r, j, i) = \dim_k L'$, where $L' \subseteq L(V^j, V^i)$ is the largest linear subspace such that $\text{rk } v = r$ for all $0 \neq v \in L'$.

Theorem 7.1.5. [14, Theorem] *Let $2 \leq r \leq j \leq i$ be integers. Then*

$$i - r + 1 \leq l(r, j, i) \leq i + j - 2r + 1.$$

We define a quadratic function $f(w) := w^2 - nw + 1 = (w - \frac{n}{2})^2 - \frac{n^2}{4} + 1$, $w \in k, n \in \mathbb{N}$. Since w^2 has positive coefficient, we have $\max_{a \leq w \leq b} f(w) = \max\{f(a), f(b)\}$ for all $a \leq b \in \mathbb{R}$.

Let $(x, y) \in \mathbf{F}, y \neq n - 1$. Then

$$\begin{aligned} q(x-1, y-(n-1)) &= (x-1)^2 + (y-(n-1))^2 - n(x-1)(y-(n-1)) \\ &= (y-(n-1))^2 \left(\left(\frac{x-1}{y-(n-1)} \right)^2 + 1 - n \frac{x-1}{y-(n-1)} \right) \\ &= (y-(n-1))^2 (t^2 + 1 - nt) \\ &= (y-(n-1))^2 f(t), \end{aligned} \tag{13}$$

where $t = \frac{x-1}{y-(n-1)}$. Then $q(x-1, y-(n-1)) < 0$ if and only if $f(t) < 0$.

Let Λ_n be the arrow space of \mathcal{K}_n . It is an n -dimensional vector space with basis $\{\gamma_i \mid 1 \leq i \leq n\}$. Let $M \in \text{mod } \mathcal{K}_n$. We use $\gamma_i.m$ to denote arrow γ_i acting on $m, m \in M$.

Lemma 7.1.6. *Suppose that $(x, y) \in \mathbf{F}$ and $y \geq 2(n-1)$. Then $(x-1, y-(n-1))$ is a regular dimension vector.*

Proof. Since $y \leq x$ and $n \geq 3$, we have $y - (n-1) \leq x - 1$. On the other hand, the inequalities $y \geq 2(n-1)$ and $y \geq \frac{2}{n}x$ imply $y - (n-1) \geq \frac{n-1}{n(n-1)-1}(x-1)$. This is because $\frac{2}{n} > \frac{n-1}{n(n-1)-1}$ and these two lines $y_1 = \frac{2}{n}x$ and $y_2 = \frac{n-1}{n(n-1)-1}(x-1) + (n-1)$ intersect at the point $(n(n-1), 2(n-1))$. That is, $\frac{n-1}{n(n-1)-1}(x-1) \leq y - (n-1) \leq x - 1$ and $\frac{x-1}{y-(n-1)} \in [1, \frac{n(n-1)-1}{n-1}]$. Let $t = \frac{x-1}{y-(n-1)}$. Since $f(1) \leq f(\frac{n(n-1)-1}{n-1})$, we obtain

$$\begin{aligned}
f(t) = t^2 + 1 - nt &\leq \max_{1 \leq t \leq \frac{n(n-1)-1}{n-1}} f(t) = f\left(\frac{n(n-1)-1}{n-1}\right) \\
&= \left(\frac{n(n-1)-1}{n-1}\right)^2 + 1 - n \frac{n(n-1)-1}{n-1} \\
&= \left(n - \frac{1}{n-1}\right)^2 + 1 - n^2 + \frac{n}{n-1} \\
&= n^2 - 2 - \frac{2}{n-1} + \frac{1}{(n-1)^2} + 1 - n^2 + 1 + \frac{1}{n-1} \\
&= \frac{2-n}{(n-1)^2} < 0.
\end{aligned} \tag{14}$$

Hence $(x-1, y-(n-1))$ is a regular dimension vector. \square

Lemma 7.1.7. *Let $M \in \text{mod } \mathcal{K}_n$ be a module with $\underline{\dim} M = (x, y)$ such that $n-1 \leq y \leq x+n-2$. Then M has a submodule U with dimension vector $(1, n-1)$.*

Proof. Actually, we can obtain a proof by following the arguments of Ringel mutatis mutandis (cf.[17, Lemma 3.2]). \square

Lemma 7.1.8. *Suppose that $(x, y) \in \mathbf{F}$ is an elementary dimension vector. Then $y < 2(n-1)$.*

Proof. By Lemma 7.1.4, we only need to consider the case: $x-y \geq n-1$. Suppose that there exists some elementary module M with $\underline{\dim} M = (y, x)$ such that $y \geq 2(n-1)$ and $x-y \geq n-1, (x, y) \in \mathbf{F}$. Then we have

$$\begin{cases} y \geq 2(n-1), \\ x-y \geq n-1, \\ y \geq \frac{2}{n}x. \end{cases} \tag{15}$$

That is

$$\begin{cases} y \geq 2(n-1), \\ y+n-1 \leq x \leq \frac{n}{2}y, \\ x \geq y+n-1 \geq 3(n-1). \end{cases} \tag{16}$$

Then there exist inequalities $y = y + (n-1) - (n-1) \leq x - (n-1) \leq \frac{n}{2}y - (n-1)$. Moreover, we have

$$\frac{y}{y-1} \leq \frac{x-(n-1)}{y-1} \leq \frac{\frac{n}{2}y-(n-1)}{y-1}.$$

Since $f\left(\frac{y}{y-1}\right) = -\frac{(n-2)y^2-(n-2)y-1}{(y-1)^2} < 0$ and $f\left(\frac{\frac{n}{2}y-(n-1)}{y-1}\right) = f\left(\frac{n}{2} - \frac{n-2}{2y-2}\right) = \left(\frac{n-2}{2y-2}\right)^2 - \frac{n^2}{4} + 1 < 0, y \geq 2(n-1), (y-1, x-(n-1))$ is regular. Now we consider $(y-2, x-x'), n+1 \leq x' \leq 2n-1$. Note that $y+n-1-x' \leq x-x' \leq \frac{n}{2}y-x'$ and $x'-(n-1) \leq n$. Then

$$\frac{y+(n-1)-(2n-1)}{y-2} = \frac{y-n}{y-2} \leq \frac{y+n-1-x'}{y-2} \leq \frac{x-x'}{y-2} \leq \frac{\frac{n}{2}y-x'}{y-2} \leq \frac{\frac{n}{2}y-(n+1)}{y-2} = \frac{n}{2} - \frac{1}{y-2}.$$

We have $\frac{y-n}{y-2} = 1 - \frac{n-2}{y-2} \geq 1 - \frac{n-2}{2(n-1)-2} = \frac{1}{2}$, and $f(\frac{1}{2}) = \frac{5-2n}{4} < 0, n \geq 3$. As $\frac{n}{2} - \frac{y-n}{y-2} \leq \frac{n}{2} - \frac{1}{2}$, then $f(\frac{y-n}{y-2}) \leq f(\frac{1}{2}) < 0$. On the other hand, $f(\frac{n}{2} - \frac{1}{y-2}) = (\frac{1}{y-2})^2 - \frac{n^2}{4} + 1 < 0, y \geq 2(n-1)$. Hence $(y-2, x-x')$ is regular.

Let $U \subseteq M$ be a submodule generated by an arbitrary element $0 \neq m \in M_1$. Then dimension vector of U is $(1, n)$. Otherwise, suppose that $\underline{\dim} U = (1, x_U)$ and $x_U \leq n-1$. Since M is indecomposable, we have $x_U \geq 1$ as $x_U = 0$ will indicate that M has a direct summand $S(1)$ and M is decomposable. We now let $U'' = U \oplus U'$, where U' is a semi-simple module with dimension vector $(0, n-1-x_U)$. We can see that $U'' \subseteq M$ is a submodule with dimension vector $(1, n-1)$. Since $(y-1, x-(n-1))$ is regular and $\underline{\dim} U''$ is regular, M is not elementary, this is a contradiction by Lemma 7.1.2 and Lemma 7.1.6.

We claim that it generates a submodule $E \subseteq M$ with $\underline{\dim} E = (2, x_E)$ and $x_E \geq n+1$ for any two nonzero and linearly independent elements $m_1, m_2 \in M_1$. Otherwise, suppose that there exist some two non-linear elements $m_1, m_2 \in M_1$ such that submodule \tilde{E} generated by these two elements has dimension vector $(2, x_{\tilde{E}})$ and $x_{\tilde{E}} \leq n$. We already know that any submodule $U \subseteq M$ with $\dim_k U_1 \neq 0$ has the property $\dim_k U_2 \geq n$. This implies $x_{\tilde{E}} = n$. Lemma 7.1.7 provides a submodule $E' \subseteq \tilde{E} \subseteq M$ of dimension vector $(1, n-1)$, a contradiction. Hence we have $x_E > n$. We have seen that $(y-2, x-x')$ is regular for all $n+1 \leq x' < 2n$. Then $x_E = 2n$. Note that m_1, m_2 are random elements. Now let set $\{e_1, \dots, e_y\}$ be a basis of M_1 . Let U^i be the submodule generated by single element $e_i, i = 1, \dots, y$, where $y \geq 2(n-1) \geq 4$. Comparing to submodule E , we have $U^i \cap U^j = (0)$ for any $i \neq j \in \{1, \dots, y\}$. Then we can get $x \geq ny > \frac{n}{2}y$, this is a contradiction with (16). Finally, we can see that there does not exist such elementary dimension (y, x) with $y \geq 2(n-1)$. Then $y < 2(n-1)$. □

Lemma 7.1.9. *Suppose that $(x, y) \in \mathbf{F}$ is an elementary dimension vector. Then $x < 2n$.*

Proof. By Lemma 7.1.4, we only need to show that (x, y) is not elementary when $x-y \geq n-1$, that is, $y \leq x-(n-1)$, where $x \geq 2n$. According to Lemma 7.1.8, we know that $\frac{2}{n}x \leq y < 2(n-1)$. Now we consider dimension vector (y, x) . Suppose that there exists an elementary module M with dimension vector (y, x) , where $4 = \frac{2}{n} \times 2n \leq y \leq x-(n-1), x \geq 2n$. We get

$$\begin{cases} 2n \leq x \leq \frac{n}{2}y, \\ \frac{2}{n}x \leq y \leq x-(n-1), \\ y < 2(n-1). \end{cases} \quad (17)$$

Then $2n \leq x < n(n-1)$ and

$$\begin{cases} n+1 \leq x-(n-1) < (n-1)^2, \\ \frac{2}{n}x-1 \leq y-1, \\ y < 2(n-1), \\ y \leq x-(n-1). \end{cases} \quad (18)$$

We have

$$\frac{y}{y-1} \leq \frac{x-(n-1)}{y-1} = t \leq \frac{x-(n-1)}{\frac{2}{n}x-1} = \frac{n}{2} \left(\frac{x-(n-1)}{x-\frac{n}{2}} \right) = \frac{n}{2} \left(1 - \frac{n-2}{2x-n} \right) < \frac{n}{2} \left(1 - \frac{n-2}{2n(n-1)-n} \right) = \frac{n}{2} - \frac{n-2}{4n-6}.$$

Note that $f(\frac{y}{y-1}) = -\frac{(n-2)y^2-(n-2)y-1}{(y-1)^2} < 0, 4 \leq y < 2(n-1)$, and $f(\frac{n}{2} - \frac{n-2}{4n-6}) = (\frac{n-2}{4n-6})^2 - \frac{n^2}{4} + 1 < 1 - \frac{n^2}{4} + 1 < 0, n \geq 4$. Hence $f(t) < 0, t \in [\frac{y}{y-1}, \frac{n}{2} - \frac{n-2}{4n-6}]$. Then $(y-1, x-(n-1))$ is regular.

Let $n+1 \leq x' \leq 2n-1 < x$. Consider $(y-2, x-x')$. Then

$$\frac{1}{n-1} = 1 - \frac{n-2}{2n-(n+1)} \leq 1 - \frac{n-2}{x-(n+1)} = \frac{x-(2n-1)}{x-(n-1)-2} \leq \frac{x-(2n-1)}{y-2} \leq \frac{x-x'}{y-2} \leq \frac{x-(n+1)}{y-2} \leq \frac{x-(n+1)}{\frac{n}{2}x-2}.$$

Moreover, $\frac{x-(n+1)}{\frac{n}{2}x-2} = \frac{n}{2} \left(\frac{x-(n+1)}{x-n} \right) = \frac{n}{2} \left(1 - \frac{1}{x-n} \right) \leq \frac{n}{2} \left(1 - \frac{1}{n(n-1)-n} \right) = \frac{n}{2} - \frac{1}{2(n-2)}$, $2n \leq x < n(n-1)$.

On the other hand, $f\left(\frac{1}{n-1}\right) = \frac{2-n}{(n-1)^2} < 0$ and $f\left(\frac{n}{2} - \frac{1}{2(n-2)}\right) = \left(\frac{1}{2(n-2)}\right)^2 - \frac{n^2}{4} + 1 < 0$, $n \geq 4$. Hence $(y-2, x-x')$ is regular.

Let $U \subseteq M$ be a submodule with $\underline{\dim} U = (y_U, x_U)$. When $y_U = 1$, we get $x_U = n$ since $(y-1, x-(n-1))$ is regular. Otherwise, suppose that $x_U \leq n-1$. We can let $U' = U \oplus U''$, where U'' is a semi-simple module with $\underline{\dim} U'' = (0, n-1-x_U)$. Then $\underline{\dim} U' = (1, n-1)$ is regular and $\underline{\dim} M/U' = (y-1, x-(n-1))$ is regular. Then M is not elementary [Lemma 7.1.2]. Suppose that $y_U = 2$. We claim that it generates a submodule $U^{(1)} \subseteq U$ with $\underline{\dim} U^{(1)} = (2, x_{U^{(1)}})$ and $x_{U^{(1)}} \geq n+1$ for any two nonzero and linearly independent elements $m_1, m_2 \in U_1$. Otherwise, suppose that there exist two elements $m_1, m_2 \in U_1$ such that submodule \tilde{U} generated by m_1, m_2 has dimension vector $(2, x_{\tilde{U}})$ with $x_{\tilde{U}} \leq n$. We know that any submodule $U \subseteq M$ with $\dim_k U_1 \neq 0$ has the property $\dim_k U_2 \geq n$. Then $x_{\tilde{U}} = n$. Now Lemma 7.1.7 provides a submodule $E' \subseteq \tilde{E} \subseteq M$ of dimension vector $(1, n-1)$, a contradiction. Since $(y-2, x-x_U)$ is regular when $n+1 \leq x_U < x$, and $(2, x_U)$ is regular when $n+1 \leq x_U < 2n$, we get $x_U = 2n$. This indicates that it generates a submodule $U^{(2)} \subseteq M$ with $\underline{\dim} U^{(2)} = (2, 2n)$ for any two nonzero and linearly independent elements $m_1, m_2 \in M_1$. Let $\{e_1, \dots, e_y\}$ be a basis of M_1 . Let U^i be the submodule generated by single element e_i , $i \in \{1, \dots, y\}$. Comparing to submodule U , we have $U^i \cap U^j = (0)$. Hence $x \geq ny > \frac{n}{2}y$, this is a contradiction with (17). Hence such elementary module M does not exist, that is, $x < 2n$.

Suppose that we can find an elementary module M with $\underline{\dim} M = (y, x)$ when $x < 2n$, where $(x, y) \in \mathbf{F}$. If we let U be a submodule of M generated by nonzero and linearly independent elements m_1, m_2 , then we have $\underline{\dim} U = (2, x)$ according to the above discussion, where $m_1, m_2 \in M_1$. □

Corollary 7.1.10. *Let $(x, y) \in \mathbf{F}$, $n < x < 2n$. Let $M \in \text{mod } \mathcal{K}_n$ be a module with dimension vector (y, x) . Then M is an elementary module if and only if elements of M_1 satisfy the following two conditions:*

- (a) *any nonzero element $m_1 \in M_1$ generates a submodule U' with dimension vector $(1, n)$, i.e. $U' \cong P(1)$;*
- (b) *any two nonzero and linearly independent elements $m_1, m_2 \in M_1$ generate a submodule U'' with dimension vector $(2, x)$.*

Proof. Suppose that M is an elementary module. According to the proof of Lemma 7.1.9, we get (a), (b).

Suppose that M_1 satisfies conditions (a), (b). We can first see that M is indecomposable. Otherwise, suppose that $M = M' \oplus M''$. Since $n < x < 2n$, there exists some submodule $X \in \{M', M''\}$ such that for any $0 \neq m \in X_1$, it generates a submodule U with dimension vector $(1, x_U)$ and $x_U < n$. This is a contradiction with condition (a). Since $(x, y) \in \mathbf{F}$ and dimension vector of M is (y, x) , $y \geq \frac{2}{n}x > 2$, M is regular. Finally, we can see that module M satisfies the conditions of Proposition 7.1.3. Then M is elementary. □

Lemma 7.1.11. *Suppose that $(x, y) \in \mathbf{F}$ is an elementary dimension vector, where $n < x < 2n$. Then $y^2 - y + 2x \leq 4n + 2$.*

Proof. Suppose that M is an elementary module with $\underline{\dim} M = (y, x), 2 = \frac{2}{n} \times n < y \leq x$. Let $I = \{m_1, \dots, m_y\}$ be a basis of M_1 , and let $I' = \{m'_1, \dots, m'_x\}$ be a basis of M_2 , respectively. Let $m_i, m_j \in I, i \neq j$. Assume that for each $\gamma_i \in \Lambda_n$, there exists

$$\gamma_i \cdot m_j = (a_{i1}^j, \dots, a_{ix}^j),$$

$a_{is}^j \in k, 1 \leq i \leq n, 1 \leq s \leq x, j \in \{1, 2\}$. Now consider the matrix

$$A_{(i,j)} = \begin{bmatrix} \gamma_1 \cdot m_i \\ \vdots \\ \gamma_n \cdot m_i \\ \gamma_1 \cdot m_j \\ \vdots \\ \gamma_n \cdot m_j \end{bmatrix} = \begin{bmatrix} a_{11}^1 & \cdots & a_{1x}^1 \\ \vdots & \ddots & \vdots \\ a_{n1}^1 & \cdots & a_{nx}^1 \\ a_{11}^2 & \cdots & a_{1x}^2 \\ \vdots & \ddots & \vdots \\ a_{n1}^2 & \cdots & a_{nx}^2 \end{bmatrix}.$$

It says that the rank of the matrix $A_{(i,j)}$ is independent of the choices of m_i, m_j by Corollary 7.1.10. Hence there exists $\text{rk } A_{(i,j)} \equiv x$. On the other hand, the matrix $A_{(i,j)}$ can also be treated as a linear transformation from vector space V^x to vector space V^{2n} . Since the number of set $\{(m_i, m_j) \mid i \neq j\}$ is $\binom{y}{2} = \frac{y(y-1)}{2}$, the dimension of linear space W generated by all such matrices $\{A_{(i,j)} \mid i \neq j\}$ is $\frac{y(y-1)}{2}$. Otherwise, suppose that $\dim_k W < \frac{y(y-1)}{2}$. Then there exist some $b_{i,j} \in k \setminus \{0\}$ such that $\sum_{i,j} b_{i,j} A_{(i,j)} = 0$. That is,

$$\sum_{i,j} b_{i,j} A_{(i,j)} = \begin{bmatrix} \gamma_1 \cdot (\sum b_{i,j} m_i) \\ \vdots \\ \gamma_n \cdot (\sum b_{i,j} m_i) \\ \gamma_1 \cdot (\sum b_{i,j} m_j) \\ \vdots \\ \gamma_n \cdot (\sum b_{i,j} m_j) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we can find at least an element $0 \neq m = \sum b_{i,j} m_i$ such that $\gamma_i \cdot m \equiv 0$ for all $1 \leq i \leq n$, this is a contradiction with Corollary 7.1.10(a).

On the other hand, we have $2n - x + 1 \leq l(x, x, 2n) \leq 2n + x - 2x + 1 = 2n - x + 1$, which means $l(x, x, 2n) = 2n + 1 - x$. By Theorem 7.1.5, we know that $\frac{y(y-1)}{2} \leq 2n + 1 - x$. That is, $y^2 - y + 2x \leq 4n + 2$. □

Lemma 7.1.12. *Let $M \in \text{mod } \mathcal{K}_n$ with dimension vector $(x, y), x \geq 1$. Suppose that any nonzero element $m \in M_1$ generates a submodule U with dimension vector $(1, y)$. Then M is indecomposable and $y \leq n$.*

Proof. When $x = 1$, we have done. Now let $x \geq 1$. Suppose that there exists a module M being decomposable and $M = M' \oplus M''$. Assume that $\underline{\dim} M' = (x', y')$. For any $0 \neq m' \in M'_1$, it generates a submodule U' with dimension vector $(1, y)$. Then $y' = y$. Hence $x' < x$, and we have $\underline{\dim} M'' = (x - x', 0)$. This cannot happen since random element m'' also generates a submodule U'' with dimension vector $(1, y), 0 \neq m'' \in M''_1$. Since $\dim_k \Lambda_n = n$, we get $y \leq n$. □

Lemma 7.1.13. *Let $M \in \text{mod } \mathcal{K}_n$ be a module with $\underline{\dim} M = (x, y) \in \mathbf{F}$ and $x \leq n - 1$. Let $0 \neq m \in M_1$ be an arbitrary element. Then M is elementary if and only if the submodule U generated by m has dimension vector $(1, y)$.*

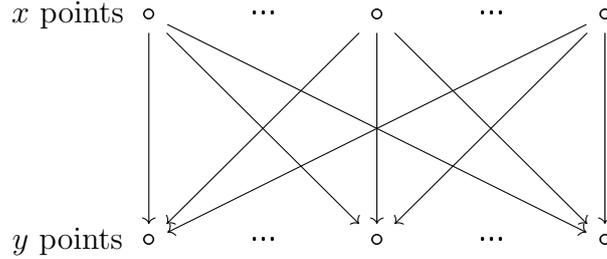
Proof. Suppose that $\underline{\dim} U = (1, y)$ for any $0 \neq m \in M_1$. Then the module U is a regular module [7, Lemma 3.5]. By Lemma 7.1.12 and $1 \leq y \leq x \leq n-1$, M is regular indecomposable. Let U' be a random proper submodule of M with dimension vector (x', y') . Then $x' \geq 1$. Note that $x' = 0$ will indicate that M is decomposable. Hence $y' = y$ and $\underline{\dim} U'$ is regular. Moreover, $\underline{\dim} M/U' = (x-x', 0)$ is preinjective. Hence M is elementary by Proposition 7.1.3.

Conversely, let M be an elementary module. Suppose that there exists some $0 \neq m \in M_1$ generating a submodule U with $\underline{\dim} U = (1, y')$ and $1 \leq y' < y$. Note that $y' = 0$ will indicate that U is isomorphic to the simple module $S(1)$ and M is decomposable. Then U is regular. Now consider factor module M/U and its dimension vector $(x-1, y-y')$

$$\begin{aligned} q(x-1, y-y') &= (x-1)^2 + (y-y')^2 - n(x-1)(y-y') \\ &= (x-1)[(x-1) - n(y-y')] + (y-y')^2 < 0. \end{aligned} \tag{19}$$

Since $x-1 \geq y-y'$ and $n(y-y') - (x-1) > y-y'$ when $y \leq x \leq n-1$, $(x-1, y-y')$ is a regular dimension vector which contradicts Lemma 7.1.2. \square

Let $(x, y) \in \mathbf{F}$, where $x < n$. Suppose that $M \in \text{mod } \mathcal{K}_n$ is an elementary module with dimension vector (x, y) . Lemma 7.1.13 tells us that M can be seen as a representation of the quiver



that is, each point of upper row has y arrows. For the elementary module M , we just put 1-dimensional vector space k on each point. We will give a precisely construction later.

Let $M \in \text{mod } \mathcal{K}_n$ be a module. We introduce an arrow γ_{n+1} and define $\gamma_{n+1}.m := 0$ for each $m \in M$. Let $\Lambda_{n+1} = \Lambda_n \oplus k\gamma_{n+1}$, where $k\gamma_{n+1}$ is the 1-dimensional vector space generated by arrow γ_{n+1} . Hence we have an embedding $\iota : \text{mod } \mathcal{K}_n \rightarrow \text{mod } \mathcal{K}_{n+1}$ in this way. Then module M can be seen as a module of $\text{mod } \mathcal{K}_{n+1}$. In the following, we always mean $\gamma_{n+1}.M = 0$ when we say $M \in \text{mod } \mathcal{K}_n \subseteq \text{mod } \mathcal{K}_{n+1}$.

Let \mathbf{F} be the fundamental domain of \mathcal{K}_n , and let $(x, y) \in \mathbf{F}$ with $x < n$. We have $\frac{2}{n}x \leq y \leq x$. Hence $\frac{2}{n+1}x < \frac{2}{n}x \leq y \leq x$. Then (x, y) is located in the fundamental domain of \mathcal{K}_{n+1} .

Corollary 7.1.14. *Let $(x, y) \in \mathbf{F}$ with $x < n$. Suppose that (x, y) is elementary for \mathcal{K}_n . Then (x, y) is elementary for \mathcal{K}_{n+1} .*

Proof. Now let $M \in \text{mod } \mathcal{K}_n$ be an elementary module with dimension vector (x, y) . We want to prove that M is also elementary for \mathcal{K}_{n+1} . Let U be a submodule of M in $\text{mod } \mathcal{K}_{n+1}$. By Lemma 7.1.13, it generates a submodule U' with dimension vector $(1, y)$ for any $0 \neq m_1 \in M_1$ for \mathcal{K}_n , whence this is also true for \mathcal{K}_{n+1} . Then $\underline{\dim} U = (x_U, y)$ for \mathcal{K}_{n+1} , where $x_U \geq 1$. Then M/U is preinjective with $\underline{\dim} M/U = (x-x_U, 0)$. Hence we can see that M is also elementary for \mathcal{K}_{n+1} [Lemma 7.1.13]. \square

Let $(x, y) \in \mathbf{F}$ with $x < n$. Suppose that $M \in \text{mod } \mathcal{K}_n$ is an elementary module with dimension vector (x, y) . Let A be a matrix over k . We use A^t to denote its transpose. Let $\{m_1, \dots, m_x\}$ be a basis of M_1 . We define $A_i := [\gamma_1 \cdot m_i \ \dots \ \gamma_n \cdot m_i]^t, \gamma_i \in \Lambda_n$. We focus on the matrix A_i . Lemma 7.1.13 tells us $\text{rk } A_i = y$. Each A_i can be seen as a linear transformation from the vector space V^y to the vector space V^n . We define $V_M := \langle A_i \mid 1 \leq i \leq x \rangle$, that is, the linear space V_M is spanned by all such A_i over k . Then we have:

Lemma 7.1.15. *Let $(x, y) \in \mathbf{F}$ with $x < n$. Suppose that $M \in \text{mod } \mathcal{K}_n$ is an elementary module with $\underline{\dim} M = (x, y)$. Then $\dim_k V_M = x$.*

Proof. Suppose that $\dim_k V_M < x$. It indicates that there exist some $b_i \in k \setminus \{0\}$ such that $\sum_{i=1}^x b_i A_i = 0$. That is

$$\sum_{i=1}^x b_i A_i = \begin{bmatrix} \gamma_1 \cdot (\sum_{i=1}^x b_i m_i) \\ \vdots \\ \gamma_x \cdot (\sum_{i=1}^x b_i m_i) \\ \vdots \\ \gamma_n \cdot (\sum_{i=1}^x b_i m_i) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

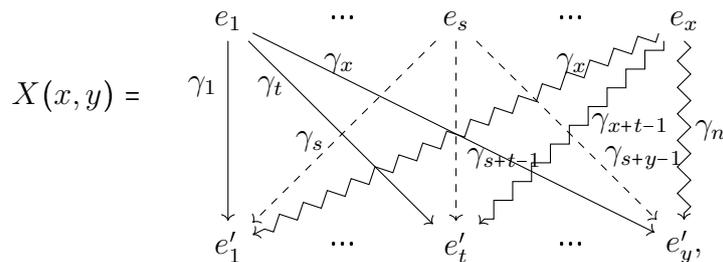
Hence we can find an element $0 \neq m' = \sum_{i=1}^x b_i m_i \in M_1$ such that $\gamma_j \cdot m' = 0$ for all $j \in \Lambda_n$. According to Lemma 7.1.13, this cannot happen. □

Lemma 7.1.16. *Suppose that $(x, y) \in \mathbf{F}$ is an elementary dimension vector with $x < n$. Then there exists an inequality $x + y \leq n + 1$.*

Proof. Suppose that $M \in \text{mod } \mathcal{K}_n$ is an elementary module with $\underline{\dim} M = (x, y)$. We now consider the linear space V_M . Note that the A_i can be treated as a linear transformation from the vector space V^y to the vector space V^n and $\text{rk } A_i = y, A_i \in V_M$. On the other hand, $n - y + 1 \leq l(y, y, n) \leq n + y - 2y + 1 = n - y + 1$ by Theorem 7.1.5. Hence $\dim_k V_M = x \leq n - y + 1$, that is, $x + y \leq n + 1$. □

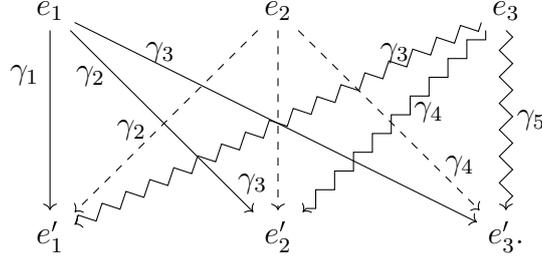
7.2 Construction of elementary modules

Let $(x, y) \in \mathbf{F}$ with $x + y = n + 1$. We construct a module $X = X(x, y) = (X_1, X_2, X(\gamma_i)_{1 \leq i \leq n})$ of \mathcal{K}_n : let $\{e_1, \dots, e_x\}$ be a standard basis of X_1 , i.e. the i -position is 1 and others are all 0 for each e_i . Let $\{e'_1, \dots, e'_y\}$ be a standard basis of X_2 . We use $[s; a_1, a_2, \dots, a_y]$ to denote the arrow γ_{a_i} mapping e_s to e'_i in $X(x, y)$, and we call it *arrow basis* of e_s , where $1 \leq s \leq x, 1 \leq i \leq y, 1 \leq a_i \leq n, \gamma_{a_i} \in \Lambda_n$. For e_1 , we define its arrow basis being $[1; 1, 2, \dots, y]$. For the second e_2 , we start from 2 to $y + 1$, that is, the arrow basis of e_2 is $[2; 2, 3, \dots, y + 1]$. For e_3 , we repeat this process by starting from 3 to $y + 2$. We keep doing it. Finally, we can get x arrow bases of $X(x, y) : [1; 1, 2, \dots, y], [2; 2, 3, \dots, y + 1], \dots, [s; s, s + 1, \dots, y + s - 1], \dots, [x; x, x + 1, \dots, n], 1 \leq s \leq x$. That is,



Let $e'_j = 0$ when $j \notin \{1, \dots, y\}$. Then $\gamma_i \cdot e_j = e'_{i-j+1}$, $\gamma_i \in \Lambda_n$, $j \in \{1, \dots, x\}$. Now we give an example.

Example. Let $X(3, 3) \in \text{mod } \mathcal{K}_5$. Then we can get its arrow bases: $[1; 1, 2, 3]$, $[2; 2, 3, 4]$, $[3; 3, 4, 5]$. The structure of $X(3, 3)$ would be the following



$$\text{Then } X(3, 3) = k^3 \begin{array}{c} \xrightarrow{X(\gamma_1)} \\ \vdots \\ \xrightarrow{X(\gamma_5)} \end{array} k^3, \text{ where } X(\gamma_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X(\gamma_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X(\gamma_3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$X(\gamma_4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, X(\gamma_5) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lemma 7.2.1. *The module $X(x, y)$ is elementary.*

Proof. Let $0 \neq m = (a_1, \dots, a_x) \in X_1$. Then we can get a matrix

$$A_m = \begin{bmatrix} \gamma_1 \cdot m \\ \vdots \\ \gamma_x \cdot m \\ \vdots \\ \gamma_n \cdot m \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_y & a_{y-1} & \cdots & a_2 & a_1 \\ a_{y+1} & a_y & \cdots & a_3 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_x & a_{x-1} & \cdots & a_{x-y+2} & a_{x-y+1} \\ 0 & a_x & \cdots & a_{x-y+3} & a_{x-y+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_x & a_{x-1} \\ 0 & 0 & \cdots & 0 & a_x \end{bmatrix}.$$

Since A_m is an $n \times y$ matrix, we get $\text{rk } A_m \leq y$. Let $j = \min \{i \mid a_i \neq 0, 1 \leq i \leq x\}$. We get a $y \times y$

$$\text{matrix } A_j = \begin{bmatrix} a_j & 0 & \cdots & 0 \\ a_{j+1} & a_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{j+y-1} & a_{j+y-2} & \cdots & a_j \end{bmatrix}. \text{ Then } \text{rk } A_j = y. \text{ Moreover, we have a partition}$$

$$A_m = \begin{bmatrix} A_j \\ A_0 \end{bmatrix},$$

$$\text{where } A_0 = \begin{bmatrix} a_{j+y} & a_{j+y-1} & \cdots & a_{j+1} \\ a_{j+y+1} & a_{j+1} & \cdots & a_{j+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_x \end{bmatrix}. \text{ Then } \text{rk } A_m \geq y \text{ [12, 8.2]}. \text{ That is to say, we have}$$

$$\text{rk } A_m \equiv y.$$

By Lemma 7.1.13, $X(x, y)$ is elementary. □

Theorem 7.2.2. *Let $(x, y) \in \mathbf{F}$ with $x + y = n + 1$. Then a module $M \in \text{mod } \mathcal{K}_n$ with dimension vector (x, y) is elementary if and only if M is of the form $X(x, y)$.*

Proof. By Lemma 7.2.1, we already know that $X(x, y)$ is elementary. Now let $M \in \text{mod } \mathcal{K}_n$ be an elementary module with dimension vector (x, y) . By Lemma 7.1.15, we get $\dim_k V_M = x$. According to Theorem 7.1.5, there exist inequalities

$$n - y + 1 \leq l(y, y, n) \leq n + y - 2y + 1 = n + 1 - y.$$

That is, $x = n + 1 - y \leq l(y, y, n) = n + 1 - y$, which is $x = l(y, y, n)$. By Lemma 7.2.1, the module $X(x, y)$ can also provide a linear space V_X satisfying $\dim_k V_X = x$ with $\text{rk } v = y$ for all $0 \neq v \in V_X$. Then we have $V_M \cong V_X$. Suppose that $\{A_1, \dots, A_x\}$ and $\{B_1, \dots, B_x\}$ are the bases of the linear spaces V_X and V_M , respectively. Then there exists an invertible matrix T such that

$$(B_1, \dots, B_x)^t = (A_1, \dots, A_x)T, \tag{20}$$

We now reconstruct a new module $X'(x, y)$ such that $(A'_1, \dots, A'_x)^t = (A_1, \dots, A_x)T$. According to Lemma 7.1.13, $X'(x, y)$ is elementary. Then we can see that $X'(x, y) \cong M$. □

Remark. Unfortunately, we couldn't give a similar result for dimension vector (x, y) when $n < x < 2n$, where $(x, y) \in \mathbf{F}$. It is still a question to describe their elementary modules.

References

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation Theory of Associative Algebras, I: Techniques of Representation Theory*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2006.
- [2] D.J. Benson, *Representation and cohomology ,I: Basic representation theory of finite groups and associative algebras*, Vol. 1, Cambridge University Press, 1998.
- [3] K.Bongartz and P.Gabriel, *Covering spaces in representation theory*, *Inventiones mathematicae* **65** (1981/82), 331-378.
- [4] Peter Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, *Representation theory I*, 1980, pp. 1–71.
- [5] Claus Michael Ringel, *The shift orbits of the graded Kronecker modules*, *Mathematische Zeitschrift* **290** (2018), no. 3, 1199-1222.
- [6] Claus Michael Ringel, *Covering theory* (2016), <https://www.math.uni-bielefeld.de/~ringel/lectures/izmir/izmir-6.pdf>.
- [7] Daniel Bissinger, *Representations of Regular Trees and Invariants of AR-Components for Generalized Kronecker Quivers* (2018), https://macau.uni-kiel.de/servlets/MCRFileNodeServlet/dissertation_derivate_00007342/DissertationDanielB.pdf.
- [8] Claus Michael Ringel, *Finite-dimensional hereditary algebras of wild representation type*, *Mathematische Zeitschrift* **161** (1978), 235-255.
- [9] Claus Michael Ringel, *Simple representations, thin representations*, <https://www.math.uni-bielefeld.de/~sek/kau/leit2v2.pdf>.
- [10] Chen Bo, *Dimension vectors in regular components over wild Kronecker quivers*, *Bulletin des Sciences Mathématiques* **137** (2013), no. 6, 730-745.
- [11] Julia Worch, *Categories of modules for elementary abelian p -groups and generalized Beilinson algebras*, *J.London Math.Soc* **88** (2013), 649-688.
- [12] George Matsaglia and George P. H. Styan, *Equalities and Inequalities for Ranks of Matrices*, *Linear and Multilinear Algebra* **2** (1974), no. 3, 269-292.
- [13] R Westwick, *Spaces of linear transformations of equal rank*, *Linear algebra and its applications* **5** (1972), no. 1, 49-64.
- [14] R.Westwick, *Spaces of matrices of fixed rank*, *Linear and Multilinear Algebra* **20** (1987), no. 2, 171–174.
- [15] J. Carlson E. Friedlander and J. Pevtsova, *Modules of constant Jordan type*, *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2008** (2008), no. 614, 191-234.
- [16] Otto Kerner and Lukas Frank, *Elementary modules*, *Mathematische Zeitschrift* **223** (1996), 421–434.
- [17] Claus Michael Ringel, *The elementary 3–Kronecker representations* (2016), <https://www.math.uni-bielefeld.de/~ringel/opus/elementary.pdf>.
- [18] Claus Michael Ringel, *Representations of K -species and bimodules*, *Journal of algebra* **41** (1976), no. 2.