

Decidability of the first-order theory of $\langle \mathbb{N}; <, P \rangle$ for morphic predicates P ¹

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Abstract

We define a notion of morphism on multi-dimensional words on a finite alphabet Σ . We call *morphic word* a fixed point of such a morphism and we consider special kinds of morphic words (those that satisfy a notion called *shape-symmetry*). The 1-dimensional morphic words always satisfy this notion.

We show that given such a δ -dimensional shape-symmetric fixed point P , the first order theory of the structure $\langle \mathbb{N}; <, 0, \mathcal{P} \rangle$ is decidable, when \mathcal{P} is the collection of δ -ary predicates whose characteristic word is the image of P by a letter-to-letter application $\Sigma \rightarrow \{0, 1\}$.

The proof is based on the same scheme as the usual proofs of decidability by finite automata, but is slightly more general.

We give examples of non-shape-symmetric fixed points of morphisms leading to undecidable theories.

1 Introduction

In this paper, we study the problem of the decidability of the first-order theory of the structures $\langle \mathbb{N}; <, 0, P \rangle$ when P is a morphic predicate — i.e. the predicate associated to a projection on the alphabet $\{0, 1\}$ of the fixed

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point of a morphism φ on a finite alphabet Σ (definitions are given in Section 3).

So far, it is known that when φ is a fixed-length morphism or corresponds to a numeration system, the above theory is decidable by deterministic finite automata — complete in the first case (see [2]) and incomplete in the second one (see [6,1]).

Another way to answer this question is to use a theorem by Semëmov [13] stating that for any unary predicate P (morphic or not), the structure $\langle \mathbb{N}; <, 0, P \rangle$ is decidable by quantifier elimination if and only if a certain property is satisfied — namely, the characteristic word of P must be almost-periodic (see also [10]). We have recently given an effective characterization of the morphisms that generate almost-periodic fixed points [8,9].

However, none of these results answers the general case, i.e. for any morphic predicate P .

The main result of this paper is to prove the decidability of $\langle \mathbb{N}; <, 0, P \rangle$ for any unary morphic predicate P , and for special (namely *shape-symmetric*) δ -ary ‘morphic’ predicates (for a suitable notion of δ -dimensional morphism, $\delta \in \mathbb{N}$). Our proof uses the same scheme as in the classical case of ‘decidability by automata’. However, instead of using automata, we use these morphisms on words and on multi-dimensional words (or *pictures*).

The main advantage is that the key notion of *shape-symmetry* will become ‘visual’. This notion is implicitly contained in the aforementioned automata. Moreover, our morphic predicates ‘extend’ the finite automata in the following sense: sets definable by finite automata are definable by morphisms; however, there are morphic pictures/predicates that are not definable by finite automata (at least by the classical automata that correspond to numeration bases). Thus, our result allows to obtain new decidability results.

After introducing the basic definitions and notations about *pictures* in Section 2, we define the notions of multi-dimensional *morphisms*, of *fixed points* of these morphisms and the key notion of *shape-symmetric* fixed point in Section 3. We illustrate all these notions with a few examples.

Then, in Section 4, we show how to consider δ -dimensional pictures as δ -ary *predicates* on \mathbb{N} by means of *projections* and characteristic words.

Section 5 is the main section. We state and prove the main result (see above). We first give the guideline of the proof in Subsection 5.1. The arguments developed here are almost the same as for deciding theories by means of finite automata.

In Subsection 5.2, we show that both the problems of checking whether a given application on pictures is a multi-dimensional morphism and whether the fixed point of a given morphism is shape-symmetric are decidable. We give an effective (but technical) algorithm for answering these two questions.

In Subsection 5.3, we give a few basic constructions that can be operated on fixed points of morphisms and we show that most of them preserve the property of shape-symmetry.

We finally prove the main theorem in Subsection 5.4.

We give a few ways to extend the main result in Section 6. For instance we show that the theory of the structure $\langle \mathbb{N}; <, 0, \mathcal{P} \rangle$ remains decidable if we use modular quantifiers $\exists^{(p,q)} \varphi(x)$ (meaning that ‘the number of x such that $\varphi(x)$ is $p \bmod q$ ’). We also explain why deciding, with the same method, the first-order theory of the structure $\langle \mathbb{N}; +, 0, \mathcal{P} \rangle$ is generally much harder: the addition relation must be ‘defined’ as a morphic predicate compatible with P — this notion of compatibility being related to the notion of shape-symmetry.

In Section 7, we give examples of well-known decidable theories (decidable by finite automata) and show that they can also be decided by means of morphisms. We emphasize the links between automata and morphisms for numeration systems. We give an example of morphism and automaton that do not correspond to a numeration system but still lead to a decidable theory. Finally, we give an example of non-shape-symmetric morphic picture giving an undecidable theory.

2 Definitions and notations

Definition 1 Let Σ be a finite alphabet. Let $\delta \in \mathbb{N}$. A δ -dimensional finite (resp. infinite) *picture* on Σ is a finite (resp. infinite) δ -dimensional “rectangular array” of symbols of Σ . More precisely, it is a mapping $D \rightarrow \Sigma$ where the finite (resp. infinite) domain $D \subseteq \mathbb{N}^\delta$ is a cartesian product all of whose factors are either the set of natural numbers \mathbb{N} or an initial segment of it. A 1-dimensional picture is also called a *word*. See Example 2.

We allow pictures to be finite in some directions and infinite in others. The *size* of a δ -dimensional picture p is the k -tuple of its lengths, i.e. is a k -tuple of elements of $\mathbb{N} \cup \{\omega\}$. An *empty picture* is a picture whose size has a zero component. So we allow different kind of empty pictures.

We denote the set of finite δ -dimensional pictures by $\Sigma^{*\delta}$ — to be considered as $\Sigma^{(*\delta)}$ and not $(\Sigma^*)^\delta$. We denote the set of infinite pictures (infinite in every direction) by Σ^{ω^δ} . The set Σ^{*2} is sometimes written Σ^{**} in the literature.

We usually use lower case letters to denote symbols of the alphabet, UPPER CASE letters to denote finite or infinite pictures, and *C*urly letters for sets of pictures and sets of sets.

Example 2 On the alphabet $\Sigma = \{a, b, c, d\}$, the picture P_1 is 1-dimensional and the picture P_2 is 2-dimensional. Of course, P_1 could be considered as a 2-dimensional (or higher) picture as well.

$$P_1 = \begin{array}{|c|c|c|c|} \hline a & b & a & c \\ \hline \end{array} \qquad P_2 = \begin{array}{|c|c|c|c|} \hline a & b & a & d \\ \hline c & d & a & d \\ \hline \end{array}$$

Definition 3 Let $\bar{x} = (x_1, \dots, x_\delta)$ and $\bar{y} = (y_1, \dots, y_\delta)$ be two δ -tuples of natural numbers, $\delta \in \mathbb{N}$. We use the following notations:

- $\bar{y} \leq \bar{x}$ is the partial ordering satisfied if and only if $\forall i \leq \delta$ ($y_i \leq x_i$), and similarly, $\bar{y} < \bar{x}$ is satisfied if and only if $\forall i \leq \delta$ ($y_i < x_i$).
- $\bar{x} + \bar{y}$ is the δ -tuple $(x_1 + y_1, \dots, x_\delta + y_\delta)$, and similarly, when $\bar{y} \leq \bar{x}$, $\bar{x} - \bar{y}$ is the δ -tuple $(x_1 - y_1, \dots, x_\delta - y_\delta)$.
- for $j \in \mathbb{N}$, the relation $j \leq \bar{x}$ is satisfied if and only if $\forall i \leq \delta$ ($j \leq x_i$),

and similarly $j < \bar{x}$ is satisfied if and only if $\forall i \leq \delta$ ($j < x_i$).

- for $j \in \mathbb{N}$, we denote by $\bar{x} + j$ the δ -tuple $(x_1 + j, \dots, x_\delta + j)$, and similarly, if $j \leq \bar{x}$, we denote by $\bar{x} - j$ the δ -tuple $(x_1 - j, \dots, x_\delta - j)$.
- for $1 \leq i \leq \delta$, we denote by $\bar{x}_{\dot{i}}$ the $(\delta - 1)$ -tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\delta)$.

Definition 4 Let P be a δ -dimensional picture of size $\bar{d} = (d_1, \dots, d_\delta) \in \mathbb{N} \cup \{\omega\}$. We use the following notations:

- $|P| = \bar{d}$ is the size of P ,
- for $1 \leq i \leq \delta$, $|P|_i = d_i$ is the i^{th} component of $|P|$,
- for $1 \leq i \leq \delta$, $|P|_{\dot{i}}$ is the $(\delta - 1)$ -tuple $(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_\delta)$,
- for $\bar{x} = (x_1, \dots, x_\delta) \in \mathbb{N}^\delta$ such that $\bar{x} < |P|$, $P(\bar{x})$ is the element at position \bar{x} in P ,
- for $\bar{x}, \bar{y} \in \mathbb{N}^\delta$ such that $\bar{x} \leq \bar{y} < |P|$, $P[\bar{x}, \bar{y}]$ is the unique δ -dimensional picture of size $\bar{y} - \bar{x} + 1$ such that, for all $\bar{z} \in \mathbb{N}^\delta$ with $\bar{z} \leq \bar{y} - \bar{x}$:

$$P[\bar{x}, \bar{y}](\bar{z}) = P(\bar{x} + \bar{z}).$$

$P[\bar{x}, \bar{y}]$ is called a *factor* of P . An *initial factor* is a factor of the form $P[\bar{0}, \bar{y}]$.

- We usually denote the δ directions corresponding to the dimensions of the picture P as being x_1, \dots, x_δ .
- for $1 \leq i \leq \delta$ and $k \in \mathbb{N}$, $P_{\uparrow x_i = k}$ is the unique $(\delta - 1)$ -dimensional picture of size $\bar{d}_{\dot{i}}$ such that, for all $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\delta) \in \mathbb{N}^{\delta-1}$ with $x_j < |P|_j$ ($1 \leq j \leq \delta, j \neq i$):

$$P_{\uparrow x_i = k}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\delta) = P(x_1, \dots, x_{i-1}, k, x_{i+1}, \dots, x_\delta),$$

The picture $P_{\uparrow x_i = k}$ is called a *hyperplane* of P normal to the direction x_i .

- for $\bar{n} \in \mathbb{N}^\delta$, $\text{Fact}_{\bar{n}}(P)$ is the set of δ -dimensional *factors* of P of size \bar{n} . We abbreviate $\text{Fact}_{(1,1,\dots,1)}(P)$ as $\text{Fact}_1(P)$.

Definition 5 Let P and Q be two δ -dimensional pictures, and let $i \leq \delta$. If $|P|_{\dot{i}} = |Q|_{\dot{i}}$, then we can *concatenate* P and Q in the direction x_i into a new δ -dimensional picture $P \odot_i Q$. The picture $P \odot_i Q$ is the only picture of dimension $|P| + |Q|$ that satisfies

$$P = (P \odot_i Q) [(0, \dots, 0), |P| - 1],$$

$$Q = (P \odot_i Q) [(0, \dots, 0, |P|_i, 0, \dots, 0), (0, \dots, 0, |P|_i, 0, \dots, 0) + |Q| - 1].$$

Example 6 In Example 2, if we consider $P_1, P_2 \in \Sigma^{*2}$, then $|P_1| = (4, 1)$ and $|P_2| = (4, 2)$. Since $|P_1|_{\perp} = (1) \neq |P_2|_{\perp} = (2)$, we cannot concatenate P_1 and P_2 in direction x_1 (horizontally). But $|P_1|_{\sharp} = |P_2|_{\sharp} = (4)$, and so we can concatenate P_1 and P_2 in direction x_2 to get

$$P_1 \odot_2 P_2 = \begin{array}{|c|c|c|c|} \hline a & b & a & c \\ \hline a & b & a & d \\ \hline c & d & a & d \\ \hline \end{array}$$

As another example, we have that for any δ -dimensional picture P and any $i \leq \delta$ such that $|P|_i$ is finite, P can be decomposed into a product of hyperplanes:

$$P = \bigodot_{0 \leq k < |P|_i} P_{\uparrow x_i = k}$$

3 δ -dimensional morphisms

We would like to generate infinite pictures by iterating a ‘morphic-like’ application, just like in the case of morphic words (see [8,9]). The infinite picture will be the ‘limit’ of a series of growing finite pictures. We now define precisely what is a morphism and how it acts on δ -dimensional pictures.

Definition 7 Let Σ be a finite alphabet and δ be a natural number. Let φ be an application $\Sigma \rightarrow \Sigma^{*\delta}$. We extend φ as a partial function on $\Sigma^{*\delta}$ as follows:

Let P_0 be a finite δ -dimensional picture such that

$$\forall i \leq \delta, \forall k < |P_0|_i, \forall a, b \in \text{Fact}_1(P_0_{\uparrow x_i = k}) \quad |\varphi(a)|_i = |\varphi(b)|_i \quad (1)$$

(i.e. any two symbols of a hyperplane of P_0 have images of the same size in the direction normal to that hyperplane), then we define $\varphi(P_0)$ to be the picture

$$\varphi(P_0) = \bigodot_{0 \leq i_1 < |P_0|_1} \left(\bigodot_{0 \leq i_2 < |P_0|_2} \left(\cdots \left(\bigodot_{0 \leq i_\delta < |P_0|_\delta} \varphi(P_0(i_1, i_2, \dots, i_\delta)) \right) \right) \right). \quad (2)$$

Basically, Formula (1) is a necessary and sufficient condition to ensure that $\varphi(p)$ is ‘gap-free’, i.e. that all the blocks we shall concatenate in Formula (2) will have a compatible size. In this case, notice that the order of the products in Formula (2) is of no importance. For instance, if φ is the application on $\{a, b\}$ given by

$$\varphi : \left\{ \begin{array}{l} a \mapsto \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \quad b \mapsto \begin{array}{|c|} \hline b \\ \hline \end{array} \end{array} \right.$$

then the image by φ of the picture $\begin{array}{|c|c|} \hline a & b \\ \hline b & a \\ \hline \end{array}$ is undefined; it is **not** the picture $\begin{array}{|c|} \hline a & a \\ \hline b & b \\ \hline \end{array}$. On the other hand, the image of $\begin{array}{|c|c|} \hline a & a \\ \hline b & b \\ \hline \end{array}$ exists and is $\begin{array}{|c|c|} \hline a & a \\ \hline b & b \\ \hline \end{array}$.

Definition 8 We say that the application $\varphi : \Sigma \rightarrow \Sigma^{*\delta}$ extends to (or just is) a δ -dimensional morphism on Σ if for any $p_0 \in \Sigma$ and any $n \in \mathbb{N}$, $\varphi^n(p_0)$ can be defined inductively (from $\varphi^{n-1}(p_0)$) according to the condition and definition expressed by Formula (1) and Formula (2).

We shall show later that we may effectively check whether a given application φ is a morphism, i.e. we only need to check a finite number of conditions (see Proposition 21). Also, this condition is trivially satisfied for the usual notion of morphism on (1-dimensional) words: in that case, there is only one symbol on each ‘hyperplane’ $x_1 = k$.

Remark 9 We must be careful with this notion of ‘morphism’. Although 1-dimensional morphisms are exactly morphisms on words, a general δ -dimensional morphism does not exactly behave like the usual morphisms: a δ -dimensional morphism is a partial function.

So, given a δ -dimensional morphism φ , a direction x_i and two finite pictures $P, Q \in \Sigma^\delta$ with $|P|_{\dot{x}} = |Q|_{\dot{x}}$, we cannot assert in general that

$$\varphi(P \odot_i Q) = \varphi(P) \odot_i \varphi(Q). \tag{3}$$

The reason for this is that nothing prevents $|\varphi(P)|_{\dot{x}}$ and $|\varphi(Q)|_{\dot{x}}$ to be different, so that $\varphi(P)$ and $\varphi(Q)$ would not be concatenable (see Example 12). However Formula (3) becomes true if we add the condition that $|\varphi(P)|_{\dot{x}} = |\varphi(Q)|_{\dot{x}}$.

Also remark that this definition allows a morphism to send a non-empty picture on an empty one (this just means that the size of the image has a zero component).

Proposition 10 *Let φ be a δ -dimensional morphism on Σ , and let P_0 be a finite δ -dimensional picture on Σ such that P_0 is an initial factor of $\varphi(P_0)$, i.e.*

$$P_0 = \varphi(P_0)[(0, \dots, 0), |P_0| - 1].$$

Then one shows inductively that $\varphi^i(P_0)$ is an initial factor of $\varphi^j(P_0)$ for any $i, j \in \mathbb{N}$ with $i \leq j$, so that the following limit is defined

$$P = \varphi^\omega(P_0) = \lim_{k \rightarrow \omega} \varphi^k(P_0).$$

Definition 11 This picture $P = \varphi^\omega(P_0)$ is called the *fixed point generated by φ on P_0* . We also say that $\varphi^\omega(P_0)$ is a *morphic picture*.

Example 12 Let φ_1, φ_2 be the applications on $\Sigma = \{a, b, c, d\}$ defined by

$$\varphi_1 : \begin{cases} a \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} & b \mapsto \begin{array}{|c|} \hline d \\ \hline b \\ \hline \end{array} \\ c \mapsto \begin{array}{|c|c|} \hline c & b \\ \hline a & d \\ \hline \end{array} & d \mapsto \begin{array}{|c|} \hline b \\ \hline b \\ \hline \end{array} \end{cases} \quad \varphi_2 : \begin{cases} a \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} & b \mapsto \begin{array}{|c|} \hline d \\ \hline \end{array} \\ c \mapsto \begin{array}{|c|c|} \hline c & b \\ \hline a & d \\ \hline \end{array} & d \mapsto \begin{array}{|c|} \hline b \\ \hline b \\ \hline \end{array} \end{cases}$$

Then φ_1 is a 2-dimensional morphism (see Proposition 21 for the proof of this claim) and generates a fixed point on a (see Figure 1), but φ_2 is not a morphism because $\varphi_2(\varphi_2(a))$ is undefined.

Notice that although φ_1 is a morphism and $a \odot_2 b = \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}$, the value of $\varphi_1(a \odot_2 b)$ is undefined because $\varphi_1(a)$ and $\varphi_1(b)$ cannot be concatenated using \odot_2 (they do not have the same size in direction x_1). This illustrates Remark 9.

Remark 13 Although a fixed point might be finite in some directions, we shall only be interested in fixed points that belong to the set Σ^{ω^δ} . It will be obvious that the decidability results of this paper are valid in any case:

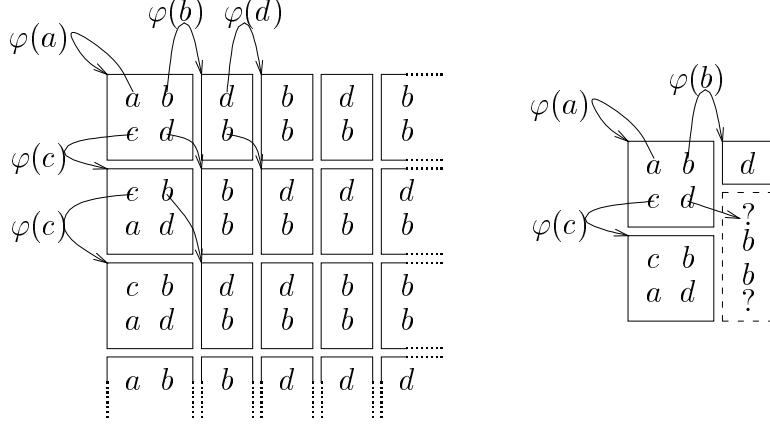


Fig. 1. φ_1 generates a fixed point on a but not φ_2 .

a finite dimension will mean a finite disjunction.

Moreover, in the remainder of this paper, we shall only consider certain type of fixed points of morphisms. These will be conditioned by the way ‘blocks’ (i.e. images of letters) are concatenated: we want a certain symmetry with respect to their sizes. We introduce the following important notion:

Definition 14 Let φ be a δ -dimensional morphism on Σ with fixed point P . We say that P is *shape-symmetric* if the following condition is satisfied:

$$\forall i, j \leq \delta, \forall k \in \mathbb{N}, \forall a \in \text{Fact}_1(P_{\uparrow x_i=k}), \forall b \in \text{Fact}_1(P_{\uparrow x_j=k}), \quad |\varphi(a)|_i = |\varphi(b)|_j \quad (4)$$

Again, we shall show later that the above conditions can be checked effectively (see Proposition 22).

Remark 15 Compared to the definition of a morphism (see Formula (1) of Definition 7), this definition states that for any two directions x_i, x_j , and for any two hyperplanes $P_{\uparrow x_i=k}$ and $P_{\uparrow x_j=k}$ at the same ‘distance’ k from the origin, the size in direction x_i of the images of the symbols of the first hyperplane is the same as that in direction x_j of the images of the symbols of the second hyperplane. See Figures 2 and 3.

Equivalently, we may require that the images of all the symbols on the diagonal of P (i.e. the elements of the form $P(n, n, \dots, n)$) are δ -dimensional cubes of variable sizes.

Definition 16 Given a δ -dimensional morphism φ with fixed points P , we

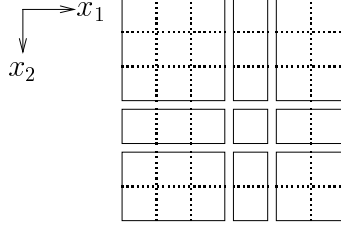


Fig. 2. The ‘blocks’ of a 2-dimensional shape-symmetric picture.

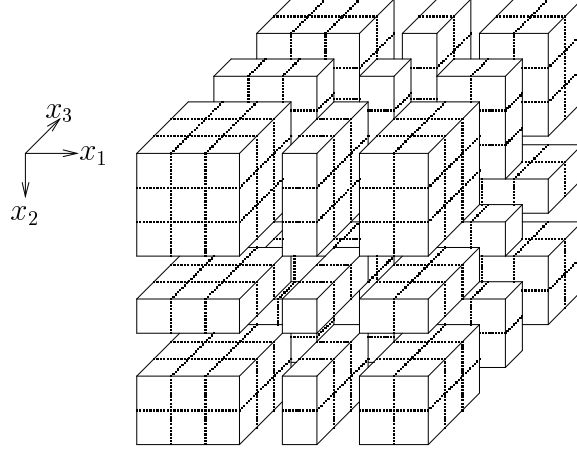


Fig. 3. The ‘blocks’ of a 3-dimensional shape-symmetric picture.

define the *shape* of P in direction x_i ($i \leq \delta$) to be the sequence

$$\text{Shape}_i(P) = (|\varphi(P_{\uparrow x_i=k})|_i)_{k \in \mathbb{N}}.$$

So P is shape-symmetric if and only if

$$\text{Shape}_1(P) = \dots = \text{Shape}_\delta(P).$$

Given two morphisms φ_1, φ_2 of dimension δ_1, δ_2 on Σ_1, Σ_2 with shape-symmetric fixed points P_1, P_2 , we say that P_1 and P_2 are *compatible* if their shapes are equal (whichever direction you consider, of course). Notice that the compatibility is an equivalence relation.

Example 17 Let φ be the 2-dimensional morphism on the alphabet $\Sigma =$

$\{a, b, c, d, e, f\}$ given by

$$\varphi : \left\{ \begin{array}{lll} a \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} & b \mapsto \begin{array}{|c|} \hline e \\ \hline c \\ \hline \end{array} & c \mapsto \begin{array}{|c|c|} \hline e & b \\ \hline \end{array} \\ \\ d \mapsto \begin{array}{|c|} \hline f \\ \hline \end{array} & e \mapsto \begin{array}{|c|c|} \hline e & b \\ \hline c & d \\ \hline \end{array} & f \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \end{array} \right.$$

Then φ generates a fixed point on a , and $\varphi^\omega(a)$ is shape-symmetric (see Figure 4).

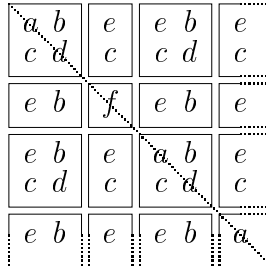


Fig. 4. $\varphi^\omega(a)$ is shape-symmetric.

Remark 18 Without real loss of generality, we shall only consider morphic pictures generated on a 1-letter picture $p_0 \in \Sigma$. In the previous example, for instance, we might have considered P as generated on a or on $\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$.

The only cases we are missing with such a restriction are those where the initial letter of the fixed point P only generates a finite factor of this fixed point. For instance, in the previous example, if we have yet another symbol g sent on itself by φ , then the fixed point, on $\begin{array}{|c|c|} \hline g & c \\ \hline b & a \\ \hline \end{array}$ is infinite, but not that on g !

However, in order to improve clarity, we shall only prove the restricted case. The results extend to the general case thanks to the fact that the ‘finitely many initial exceptions’ can be managed by a finite disjunction.

4 δ -dimensional predicates

Definition 19 Given a δ -dimensional picture P on a finite alphabet Σ , a *projection of P* is a δ -dimensional picture P' on an alphabet Σ' such that

P' obtained as the image of P under (the extension of) a letter-to-letter application $\Sigma \rightarrow \Sigma'$.

In the remainder, most of the projections we shall consider will be projection on the $\{0,1\}$ alphabet. Given a picture P on an alphabet Σ , we define \mathcal{P} as the collection of all the possible projections of P on the set $\{0,1\}$. We index the elements $P_S \in \mathcal{P}$ by the set $S \subseteq \Sigma$ of letters sent on 1. Thus $|\mathcal{P}| = 2^{|\text{Fact}_1(P)|} \leq 2^{|\Sigma|}$.

Given a δ -ary predicate $R(\bar{x})$ on \mathbb{N} , that is, a δ -ary relation, we can consider the corresponding δ -dimensional characteristic picture P_R on the alphabet $\{0,1\}$ and conversely:

$$\forall \bar{x} \in \mathbb{N}^\delta \quad R(\bar{x}) \Leftrightarrow P_R(\bar{x}) = '1'.$$

We identify these two notions and consider any projection of a morphic word on the alphabet $\{0,1\}$ as a predicate on \mathbb{N} . The set \mathcal{P} will thus be equally considered as a collection of pictures or of predicates.

5 Decidability of Theories with the Order Relation

5.1 Guideline

Let P be a shape-symmetric δ -dimensional picture obtained as fixed point of a morphism φ on a symbol $p_0 \in \Sigma$, and let $\mathcal{P} = \{P_S \text{ s.t. } S \subseteq \Sigma\}$ be the collection of projections of P on the alphabet $\{0,1\}$ as defined in the previous section.

We consider the first-order theory of the structure $\langle \mathbb{N}; <, 0, \mathcal{P} \rangle$ where ' $<$ ' is the usual ordering of the natural numbers and '0' is the zero constant.

Larger theories (with modular quantifiers or functions like addition) will be considered in Section 6.

We want to show the following result:

Theorem 20 *The first-order theory of the structure $\mathcal{A} = \langle \mathbb{N}; <, 0, \mathcal{P} \rangle$ is decidable.*

The proof uses the same method as for deciding theories using deterministic finite automata (see [7]) and consists in the following arguments (see Section 5.4 for the complete proof):

Given a projection P_S of P and given a tuple of natural numbers (m_1, \dots, m_δ) , it is very easy to compute is the element $P(m_1, \dots, m_\delta)$, and thus to check the satisfaction of the atomic formula

$$\mathcal{A} \models P_S(m_1, \dots, m_\delta)$$

(it suffices to compute the iterations of φ on the generating symbol p_0 until the element (m_1, \dots, m_δ) is known and to check whether it is in S).

Given a first-order formula $\theta(\bar{x})$ to be decided in the structure \mathcal{A} , we construct (inductively on the syntax of θ) a *new* alphabet Σ' and a *new* δ' -dimensional morphism φ' on Σ' generating a *new* δ' -dimensional shape-symmetric morphic picture P' on Σ' such that in the *new* structure $\mathcal{A}' = \langle \mathbb{N}; 0, \mathcal{P}' \rangle$, the relation on \mathbb{N} defined in \mathcal{A} by the formula $\theta(\bar{x})$ is definable by an atomic formula involving a predicate of \mathcal{P}' . Notice that we can even get rid of the order relation.

In other words, given a complicate formula $\theta(\bar{x})$ in a structure \mathcal{A} that depends on a morphic picture, we replace the problem

$$\mathcal{A} \stackrel{?}{\models} \theta(\bar{n})$$

by an equivalent problem

$$\mathcal{A}' \stackrel{?}{\models} \theta'(\bar{n})$$

where the structure \mathcal{A}' depends on a more complicate morphic picture, but where $\theta'(\bar{x})$ is just an atomic formula.

In case we want to decide a sentence, we shall end the induction process

with a sentence $P'_S(0, \dots, 0)$ for some $P'_S \in \mathcal{P}'$. Its satisfaction in \mathcal{A}' will depend on whether the initial symbol of P' is in S . In fact, this corresponds to an emptiness argument for finite automata, and this correspondence will be clear from the way we treat the quantifiers.

The property of *shape-symmetry* is the key notion for this decidability result. An example of non-shape-symmetric morphic predicate leading to an *undecidable* theory is given in Example 34.

But first we need to prove a few propositions and lemmas on morphic pictures.

5.2 Effectiveness results

Proposition 21 *Let Σ be a finite alphabet, let δ be a natural number and let φ be an application $\Sigma \rightarrow \Sigma^{*\delta}$. It is possible to check effectively whether φ extends to a δ -dimensional morphism (see Definition 8).*

Proof. For each $p_0 \in \Sigma$, we must show that we can check whether Formula (1) of Definition 7 is true for all $n \in \mathbb{N}$, so that $\varphi^{n+1}(p_0)$ is defined.

In order to simplify the writing of this proof, we suppose that p_0 satisfies $\varphi(p_0)(0, \dots, 0) = p_0$. A more complete proof would use the same arguments, but would have to combine them with the fact that the sequence $p_0(0, \dots, 0), \varphi(p_0)(0, \dots, 0), \varphi^2(p_0)(0, \dots, 0), \dots$ is a ultimately periodic sequence.

We turn the problem differently by treating each direction independently: given a direction $x_i, i \leq \delta$, we *suppose* that Formula (1) is satisfied with $P_0 = \varphi^n(p_0)$ for all other directions $x_j, j \neq i$, for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$. We show how to check this formula for x_i and globally for all n by computing the sets $S_{(i,k)}$ of letters that would occur in some hyperplane $P_{\uparrow x_i=k}$ if φ was a morphism generating P on p_0 .

We first consider the set $S_{(i,0)} = \text{Fact}_1(P_{\uparrow x_i=0})$. If P is defined, and even if we do not know it, the set $S_{(i,0)}$ can be computed using the following

procedure: we initialize a *set variable* $T = \emptyset \subset \Sigma$ and add the symbol p_0 to it (clearly, $p_0 \in S_{(i,0)}$). Now, for each element $q \in T$, we add to T the symbols of $\varphi(q)|_{x_i=0}$, i.e. we execute

$$T := \bigcup_{q \in T} \text{Fact}_1(\varphi(q)|_{x_i=0}).$$

Notice that the new value of T is a superset of the previous one since we supposed that $\varphi(p_0)(0, \dots, 0) = p_0$. We iterate this process until no new element is added. This process eventually stops since $T \subset \Sigma$. Then, clearly, $S_{(i,0)} = T$.

Since for any $n \in \mathbb{N}$, $\text{Fact}_1(\varphi^n(p_0)|_{x_i=0}) \subseteq S_{(i,0)}$, the condition

$$\forall a, b \in S_{(i,0)} \quad |\varphi(a)|_i = |\varphi(b)|_i \quad (5)$$

is clearly sufficient to ensure that Formula (1) is satisfied in direction x_i with $P_0 = \varphi^n(p_0)$ for any n and with $k = 0$ (as long as $\varphi^n(p_0)$ is defined). Conversely, this condition is also necessary for φ to be a morphism (otherwise Formula (1) would fail for some n and $k = 0$).

So we check Condition (5). Then we are able to compute what would be the sets $S_{(i,k)} = \text{Fact}_1(P|_{x_i=k})$ for $k < |\varphi(p_0)|_i$. Indeed, the word $P|_{x_i=k}$ is the k^{th} hyperplane of the image by φ of $P|_{x_i=0}$. So

$$S_{(i,k)} = \bigcup_{q \in S_{(i,0)}} \text{Fact}_1(\varphi(q)|_{x_i=k}).$$

Then, provided $\varphi^n(p_0)$ exists, Formula (1) is satisfied with $P_0 = \varphi^n(p_0)$ for any n and with $k < |\varphi(p_0)|_i$ if and only if

$$\forall a, b \in S_{(i,k)} \quad |\varphi(a)|_i = |\varphi(b)|_i \quad (6)$$

This value would actually be the k^{th} component of $\text{Shape}_i(P)$.

However, remark that we are not really interested in the value of k : we just want to be sure that the images of two letters in a hyperplane have equal sizes in the direction normal to that hyperplane, whatever its coordinate

is (just like we wanted to know $S_{(i,0)}$ without caring of the exact position of the symbols).

So we want to compute the (finite) collection $\mathcal{S}_{(i)}$ of sets of letters occurring in the some hyperplane normal to the direction x_i . We use the same method as for $S_{(i,0)}$: we initialize a ‘set of sets’ variable $\mathcal{T} = \emptyset \subset \mathcal{P}(\Sigma)$ and add the set $S_{(i,0)}$ to it (clearly, $S_{(i,0)} \in \mathcal{S}_{(i)}$). Now, for each set $Q \in \mathcal{T}$, we check that

$$\forall a, b \in Q \quad |\varphi(a)|_i = |\varphi(b)|_i. \quad (7)$$

If so, we define n_Q to be this common value; otherwise φ cannot be a morphism. Then, for each $Q \in \mathcal{T}$, we add to \mathcal{T} the n_Q sets of symbols that will occur in the successive ‘hyperplanes’ made with the images of the elements of Q by φ , i.e. we execute

$$\mathcal{T} := \bigcup_{Q \in \mathcal{T}} \left(\bigcup_{k < n_Q} \left\{ \bigcup_{q \in Q} \text{Fact}_1(\varphi(q)|_{x_i=k}) \right\} \right).$$

Again, the new value of \mathcal{T} is a superset of the previous one, and we iterate this process until no new element is added. This process eventually stops since $\mathcal{T} \subset \mathcal{P}(\Sigma)$. When it stops, then $\mathcal{S}_{(i)} = \mathcal{T}$ and we have checked that (provided $P_0 = \varphi^n(p_0)$ exists), Formula (1) is true for any n and any k in direction x_i .

Finally, if the computation of $\mathcal{S}_{(i)}$ is successful for each $i \leq \delta$, then $\varphi^n(p_0)$ does exist for all n and thus φ generates a fixed point on p_0 . \square

Proposition 22 *Let Σ be a finite alphabet, let δ be a natural number and let φ be a δ -dimensional morphism on Σ with fixed point on p_0 . It is possible to check effectively whether $\varphi^\omega(p_0)$ is shape-symmetric (see Definition 14).*

Proof. This proof uses the same ingredients as that of Proposition 21.

We can handle it in two ways: either we apply the full procedure detailed in the proof of Proposition 21, and we only check that all the diagonal

elements are sent on δ -dimensional cubes (using, again, a similar procedure that eventually stops).

Or we slightly change the previous proof: we define the sets $S_{(i,0)}$ exactly in the same way, but replace the set \mathcal{T} by a δ -tuple of sets $(\mathcal{T}_1, \dots, \mathcal{T}_\delta)$ representing the sets of symbols occurring in the different hyperplanes at the same distance from the origin (the ‘ k ’). We add a ‘compatibility’ condition in the sense of Formula (4) of Definition 14.

This allows us to check at the same time that we generate a morphic word on p_0 and that this fixed point is shape-symmetric. \square

5.3 Operations on Morphisms and Shape-symmetric Pictures

In this section, we introduce a few basic operations on morphisms and morphic pictures that will be used in Section 5.4. Most of these results are intuitively clear, but need to be stated explicitly.

We show that if a morphism φ' is ‘more detailed’ than a morphism φ — that is, φ' acts on a larger alphabet but can be projected on φ — then the projections of the fixed point of φ are projections of that of φ' too.

We define the ‘cross-product’ of two morphisms and show that if they both generate a shape-symmetric fixed point and are compatible, then this product admits a shape-symmetric fixed point too.

We show that the restriction of a shape-symmetric morphic picture P to a hyperplane $x_i = 0$ or to a ‘diagonal section’ $x_i = x_j$ is still a shape-symmetric morphic picture compatible with P .

We finally show that we can ‘merge’ two morphic pictures into a new one.

An example for each of these constructions is given in Example 29.

Lemma 23 *Let P be the shape-symmetric fixed point on p_0 of a δ -dimensional morphism φ on Σ . Let $\varphi' : \Sigma' \rightarrow \Sigma'^{\delta}$ be an application on another alphabet Σ' . If there exists a projection $\pi : \Sigma' \rightarrow \Sigma$ such that*

$$\forall b \in \Sigma' \quad \varphi(\pi(b)) = \pi(\varphi'(b))$$

and if there exists some $q_0 \in \pi^{-1}(p_0)$ such that $q_0 = \varphi'(q_0)(\bar{0})$, then φ' is a morphism and $P' = \varphi'^{\omega}(q_0)$ is shape-symmetric and compatible with P . In fact, $P = \pi(P')$ and thus any projection of P is also a projection of P' .

Proof. The proof is just an application of the definitions. □

Definition 24 Let φ_1, φ_2 be two morphisms on Σ_1, Σ_2 of dimension δ_1, δ_2 . The *cross-product* of φ_1 and φ_2 , denoted $\varphi_1 \times \varphi_2$ is the morphism on $\Sigma_1 \times \Sigma_2$ defined (in the natural way) by sending each pair $(a, b) \in \Sigma_1 \times \Sigma_2$ on the picture $\varphi_1 \times \varphi_2(a, b)$ of size $(|\varphi_1(a)|_1, \dots, |\varphi_1(a)|_{\delta_1}, |\varphi_2(b)|_1, \dots, |\varphi_2(b)|_{\delta_2})$ given by

$$\forall \bar{n} \in \mathbb{N}^{\delta_1 + \delta_2} \text{ s.t. } \bar{n} < |\varphi_1 \times \varphi_2(a, b)|, \\ \varphi_1 \times \varphi_2(a, b)(\bar{n}) = (\varphi_1(a)(n_1, \dots, n_{\delta_1}), \varphi_2(b)(n_{\delta_1+1}, \dots, n_{\delta_1+\delta_2})).$$

Lemma 25 *The application $\varphi_1 \times \varphi_2$ is a morphism. Moreover, if P_1 (resp. P_2) is a shape-symmetric fixed point on p_0 (resp. q_0) of φ_1 (resp. φ_2), and if P_1 and P_2 are compatible (see Definition 16), then $\varphi_1 \times \varphi_2$ admits a shape-symmetric fixed point $P_1 \times P_2$ on (p_0, q_0) . This fixed point is compatible with P_1 and P_2 , and actually, P_1 (resp. P_2) is a section of the projection of $P_1 \times P_2$ by the map $(a, b) \mapsto a$ (resp. $(a, b) \mapsto b$).*

Proof. Again, the proof is just an application of the definitions and of the

previous lemma. □

Lemma 26 *Let $P(x_1, \dots, x_\delta)$ be a shape-symmetric fixed point of a δ -dimensional morphism φ on Σ . Let f be a mapping from the set $\{1, \dots, \delta\}$ into the set $\{0, 1, \dots, \delta'\}$. Let P' be the δ' -dimensional picture defined by $P'(y_1, \dots, y_{\delta'}) = P(y_{f(1)}, \dots, y_{f(\delta)})$ with the convention $y_0 \equiv 0$. Then P' is the projection of a shape-symmetric morphic picture compatible with P .*

Proof.

First suppose that f is surjective on the set $\{1, \dots, \delta'\}$. Then let g be a mapping from $\{1, \dots, \delta'\}$ into $\{1, \dots, \delta\}$ such that $f \circ g = \text{id}$.

The picture P' is generated as the fixed point on p_0 of the morphism φ' defined on Σ as follows: for any symbol a of P' , $\varphi'(a)$ is the picture of size $(|\varphi(a)|_{g(1)}, \dots, |\varphi(a)|_{g(\delta)})$ given by

$$\forall \bar{n} < |\varphi'(a)| \quad \varphi'(a)(n_1, \dots, n_{\delta'}) = \varphi(a)(n_{f(1)}, \dots, n_{f(\delta)}).$$

This application is well-defined (it does not depend on the choice of g) since we supposed that P is shape-symmetric: if, say, $f(1) = f(2) = 1$ then we could choose $g(1) = 1$ or $g(1) = 2$, but for any $a \in P'(k, \text{something})$, a also occurs in $P(k, k, \text{something})$ and thus $|\varphi(a)|_1 = |\varphi(a)|_2$. So all the possible choices for g are equivalent.

It is easy to check that in this case, φ' is indeed a morphism and that P' is shape-symmetric and compatible with P .

Now, if f is not surjective on $\{1, \dots, \delta'\}$, then we proceed inductively on the amount of values that are not in the image of f .

Suppose that $\delta' \notin \text{Im}(f)$. Then $P''(y_1, \dots, y_{\delta'-1}) = P(y_{f(1)}, \dots, y_{f(\delta)})$ is the projection of a morphic shape-symmetric picture Q by the induction hypothesis. Let ψ be the $(\delta' - 1)$ -dimensional morphism that generates Q .

By definition, all the hyperplanes $P'_{|x_{\delta'}=k}$ are the same (i.e. are P''). So P' is a projection of the picture

$$\bigodot_{\substack{\delta' \\ k \in \mathbb{N}}} Q(y_1, \dots, y_{\delta'-1}).$$

However, we need to exhibit a generating δ' -dimensional morphism φ' so that P' has the same *shape* in direction $x_{\delta'}$ than in the other directions (P' must be shape-symmetric).

Let η be any 1-dimensional morphism compatible with P , for instance, the morphism $\varphi_{|x_2=\dots=x_\delta=0}$ that defines the word

$$R = P[(0, \dots, 0), (\omega, 0, \dots, 0)]$$

(this morphism exists by the proof of the surjective case).

We define $Q'(y_1, \dots, y_{\delta'})$ as the direct product $Q \times R$ generated by the morphism $\psi' = \psi \times \eta$.

Clearly, P' is a projection of Q' . Moreover, Q' is morphic, shape-symmetric and compatible with P by the induction hypothesis and Lemmas 23 and 25. \square

Definition 27 Let P_1, P_2 be two compatible shape-symmetric morphic pictures of dimension $\delta_1 > \delta_2$ on Σ_1, Σ_2 . We can *merge* P_2 into P_1 to get a picture P'_1 of dimension δ_1 defined on $\Sigma_1 \times \Sigma_2$ by

$$\forall \bar{n} \in \mathbb{N}^{\delta_1} \quad P'_1(\bar{n}) = (P_1(n_1, \dots, n_{\delta_1}), (P_2(n_1, \dots, n_{\delta_2}))).$$

Lemma 28 *The picture obtained by merging P_2 into P_1 is morphic, shape-symmetric, and both P_1 and P_2 are projections of it.*

Proof. The corresponding morphism is defined by sending a pair (a, b) on the picture obtained by merging $\varphi_2(b)$ into $\varphi_1(a)$. The proof is similar to the previous ones. \square

Example 29 Let φ be the morphism defined in example 17 on page 10, and let ψ be the morphism defined by

$$\psi : \{ a \mapsto \boxed{a \mid b} \quad b \mapsto \boxed{a} \}$$

Let P be the fixed point of φ on a , and Q be that of ψ on a . The word $P'(y_1) = P(y_1, y_1)$ is morphic (by Lemma 26) and obtained as the fixed point on a of the morphism on $\{a, d, f\}$

$$\varphi' : \{ a \mapsto \boxed{a \mid d} \quad d \mapsto \boxed{f} \quad f \mapsto \boxed{a \mid d} \}$$

The projection $\pi : \{a, d, f\} \rightarrow \{a, b\}$ defined by $\pi(a) = \pi(f) = a$ and $\pi(d) = b$ converts φ' to ψ . So P' and Q are compatible by Lemma 23 (together with P).

The cross product of Q by P' is the fixed point on (a, a) of the morphism on the alphabet $\{a, b\} \times \{a, d, f\} = \{(a, a), (a, d), (a, f), (b, a), (b, d), (b, f)\}$ given by

$$\psi \times \varphi' : \begin{cases} (a, a) \mapsto \boxed{\begin{array}{cc} (a, a) & (b, a) \\ (a, d) & (b, d) \end{array}} & (a, d) \mapsto \boxed{(a, f) \mid (b, f)} & (a, f) \mapsto \boxed{\begin{array}{cc} (a, a) & (b, a) \\ (a, d) & (b, d) \end{array}} \\ (b, a) \mapsto \boxed{\begin{array}{c} (a, a) \\ (a, d) \end{array}} & (b, d) \mapsto \boxed{(a, f)} & (b, f) \mapsto \boxed{\begin{array}{c} (a, a) \\ (a, d) \end{array}} \end{cases}$$

Finally, we can merge Q and P' and the result is the fixed point on (a, a) of the morphism

$$\rho : \{ (a, a) \mapsto \boxed{(a, a) \mid (b, d)} \quad (b, d) \mapsto \boxed{(a, f)} \quad (a, f) \mapsto \boxed{(a, a) \mid (b, d)} \}$$

5.4 Proof of Theorem 20

Let Σ be a finite alphabet. Let p_0 be a symbol of Σ and let φ be a δ -dimensional morphism with shape-symmetric fixed point P on p_0 .

Let \mathcal{P} be set of predicates/projections on $\{0,1\}$ obtained from P .

Since the ‘=’ relation is definable from ‘<’, we do not need to consider it as an atomic relation. We keep the ‘zero’ constant for simplicity.

We show the following result (without loss of generality for the predicates $P_{S_1}, P_{S_2} \in \mathcal{P}$ and the variables $\bar{y} = (y_1, \dots, y_n)$ we choose):

For any of the following formulas $\theta(\bar{y})$, it is possible (and the procedure is effective) to find a morphism φ' on an alphabet Σ' generating a shape-symmetric picture P' as fixed point of some p'_0 and with projections \mathcal{P}' such that each projection of \mathcal{P} is definable by an **atomic formula** involving a projection of \mathcal{P}' **and so is θ** :

- (1) for $\theta(\bar{y}) = \neg P_{S_1}(\bar{y})$,
- (2) for $\theta(\bar{y}) = P_{S_1}(\bar{y}) \wedge P_{S_2}(\bar{y})$,
- (3) for $\theta(\bar{y}) = y_1 < y_2$,
- (4) for $\theta(\bar{y}_{\neq}) = \exists y_n P_{S_1}(\bar{y})$.

Let us treat these cases one by one.

Case 1 $\theta(\bar{y}) = \neg P_{S_1}(\bar{y})$

This case is the easiest. The formula θ is already definable as $P_{\Sigma \setminus S_1}(\bar{y})$ which is an atomic formula. We keep the same Σ, φ and P .

Case 2 $\theta(\bar{y}) = P_{S_1}(\bar{y}) \wedge P_{S_2}(\bar{y})$

First suppose that the same variables occur at the same places in $P_{S_1}(\bar{y})$ and $P_{S_2}(\bar{y})$. Then θ is definable as $P_{S_1 \cap S_2}(\bar{y})$. Again, we keep the same Σ, φ and P .

Now, suppose that the variables occurring in the first atomic formula are not the same and/or do not occur at the same place as in the second atomic formula. However, we suppose that every variable of \bar{y} occurs at least once in θ (otherwise, just remove it from \bar{y}).

This case is the most difficult, because it is here that pictures of higher dimensions might have to be defined in order to “encode” both P and θ into a new morphic picture. It strongly uses the property of shape-symmetry and the results of Section 5.3.

Suppose $\bar{y} = (y_1, \dots, y_{\delta'})$ contains at least δ variables (i.e. $\delta \leq \delta' \dots$ otherwise, just add useless variables).

Let $\bar{\sigma}_1$ (resp. $\bar{\sigma}_2$) be the *sequence of arguments* of P_{S_1} (resp. of P_{S_2}) in θ . So these are sequences of elements from the set of symbols $\{0, y_1, \dots, y_{\delta'}\}$ where ‘0’ is the ‘zero constant’ symbol.

We define three new δ' -dimensional pictures, namely

$$\begin{aligned} R(y_1, \dots, y_{\delta'}) &= P(y_1, \dots, y_{\delta}), \\ Q_1(y_1, \dots, y_{\delta'}) &= P(\bar{\sigma}_1), \\ Q_2(y_1, \dots, y_{\delta'}) &= P(\bar{\sigma}_2). \end{aligned}$$

By the results of Section 5.3, R , Q_1 and Q_2 are projections of compatible shape-symmetric morphic pictures on some alphabets $\tilde{\Sigma}$, Γ_1 and Γ_2 . Let ρ , π_1 and π_2 be the corresponding projections.

We can merge these morphic pictures into a new shape-symmetric morphic picture P' according to Definition 27, whose generating morphism is known by Lemma 28.

Then P is a projection of a section of P' (thanks to R), and so is $\theta(\bar{y}) = P_{S_1}(\bar{y}) \wedge P_{S_2}(\bar{y})$ since it corresponds to the projection

$$P'_{\tilde{\Sigma} \times \pi_1^{-1}(S_1) \times \pi_2^{-1}(S_2)}.$$

Case 3 $\theta(\bar{y}) = y_1 < y_2$

This case is not difficult and uses the property of shape-symmetry of P .

If P is 1-dimensional, we cannot immediately ‘encode’ the relation $y_1 < y_2$ in it. So we replace P and φ by $P \times P$ and $\varphi \times \varphi$ (and of course P is definable as projection of $P \times P$).

So we suppose P is δ -dimensional with at least $\delta \geq 2$.

We shall replace P by another δ -dimensional word P' on a larger alphabet. Basically, for any δ -tuple of natural numbers (n_1, \dots, n_δ) , we replace the symbol $P(\bar{n})$ by a new symbol that also carries the information whether $n_1 < n_2$.

To see how to do this, we consider the following very simple morphism ψ on the alphabet $\{\equiv, \gg, \ll\}$

$$\psi : \left\{ \begin{array}{l} \equiv \mapsto \begin{array}{|c|c|} \hline \equiv & \gg \\ \hline \ll & \equiv \\ \hline \end{array} \quad \gg \mapsto \begin{array}{|c|c|} \hline \gg & \gg \\ \hline \gg & \gg \\ \hline \end{array} \quad \ll \mapsto \begin{array}{|c|c|} \hline \ll & \ll \\ \hline \ll & \ll \\ \hline \end{array} \end{array} \right.$$

Clearly, the picture generated by ψ on a is

\equiv	\gg	\gg	\gg	\dots
\ll	\equiv	\gg	\gg	\dots
\ll	\ll	\equiv	\gg	\dots
\ll	\ll	\ll	\equiv	\dots
\vdots	\vdots	\vdots	\vdots	\ddots

and the order relation $x_1 < x_2$ can be defined as the image of $\psi^\omega(\equiv)$ by the projection that sends \equiv and \gg on 0 and \ll on 1. We would like to merge this picture in P , but, in general, the above morphism is not compatible with φ .

We define Σ' as $\Sigma \times \{\equiv, \gg, \ll\}$ and want P' to be such that $P'(\bar{n})$ is the pair $P(\bar{n}, \star)$ where \star is \equiv (resp. \gg or \ll) if $n_1 = n_2$ (resp. $n_1 > n_2$ or $n_1 < n_2$).

The generating symbol p'_0 is the pair (p_0, \equiv) .

Thanks to the property of shape-symmetry of P , the diagonal elements of P (“diagonal” with respect to the first two components) are sent by φ on blocks whose diagonal elements are also on the diagonal of P . As a consequence, for a pair $(a, \equiv) \in \Sigma'$, we define $\varphi'(a, \equiv)$ as a picture of size $|\varphi(a)|$ whose element in position $(m_1, \dots, m_\delta) < |\varphi(a)|$ is

- the pair $(\varphi(a)(m_1, \dots, m_\delta), \equiv)$ if $m_1 = m_2$,
- the pair $(\varphi(a)(m_1, \dots, m_\delta), \gg)$ if $m_1 > m_2$,
- the pair $(\varphi(a)(m_1, \dots, m_\delta), \ll)$ if $m_1 < m_2$.

Then we define $\varphi'(a, \gg)$ as the δ -dimensional picture obtained from $\varphi(a)$ by replacing each symbol $x \in \Sigma$ by the symbol $(x, \gg) \in \Sigma'$. And similarly, we define the image of $(a, \ll) \in \Sigma'$ by replacing each symbol x of $\varphi(a)$ by $(x, \ll) \in \Sigma'$.

The conclusion follows by Lemma 23 and the fact that the predicate $y_1 < y_2$ can be defined by the atomic formula $P'_{\Sigma \times \{\ll\}}(y_1, y_2, 0, \dots, 0)$.

Case 4 $\theta(\bar{y}_n) = \exists y_n P_{S_1}(\bar{y})$

Here, we do not need the fact that P is shape-symmetric if y_n occurs only once in $P_{S_1}(\bar{y})$. Also, we can suppose that $n = \delta$, so that $\bar{y} = (y_1, \dots, y_\delta)$ and $y_n = y_\delta$.

We shall use here the fact that, given a $(\delta - 1)$ -tuple of natural numbers $\bar{n}_\delta = (n_1, \dots, n_{\delta-1})$, it is possible to determine the set of symbols that occur in the infinite 1-dimensional word $P|_{x_1=n_1, \dots, x_{\delta-1}=n_{\delta-1}}$.

Again, we shall ‘encode’ some information in a new δ -dimensional picture P' and in a new morphism φ' using a new alphabet Σ' . This time, given a tuple \bar{y} , we want to transform the element $P(\bar{y})$ into a pair $(P(\bar{y}), \mathcal{T})$ where $\mathcal{T} \subseteq \Sigma$ is the set of symbols on the line $P[(\bar{y}_\delta, 0), (\bar{y}_\delta, \omega)]$. We keep the same dimension δ for P' in order to be able to recover all the projections of P in \mathcal{P}' .

Let $\bar{\sigma}$ be the sequence of arguments of P_{S_1} (like on page 23).

We first concentrate on the word $Q(y_\delta)$ obtained from $P(\bar{\sigma})$ when $y_1 = \dots = y_{\delta-1} = 0$. This word starts with p_0 and might be a diagonal line if y_δ occurs more than once in P_{S_1} . This is a morphic word by Lemma 26, and we can easily compute the set \mathcal{T}_{p_0} of letters occurring in it (with the same kind of procedure as in Propositions 21 and 22).

We consider the $(\delta - 1)$ -dimensional picture R on the alphabet $\mathcal{P}(\Sigma)$ defined as follows:

The initial symbol of R is \mathcal{T}_{p_0} . Then we expand R by iterating the application ψ that sends a symbol \mathcal{T} of R (i.e. a subset of symbols of Σ) on the $(\delta - 1)$ -dimensional picture of size $|\varphi(a)|_\delta$ (where a is any symbol of \mathcal{T}) given by

$$\forall \bar{n}_\delta < |\varphi(a)|_\delta, \quad \psi(\mathcal{T})(\bar{n}_\delta) = \bigcup_{b \in \mathcal{T}} \bigcup_{k < |\varphi(b)|_\delta} \{\varphi(b)(\bar{n}_\delta, k)\}.$$

The application ψ is well-defined (i.e. it does not depend on the choice of a) since P is a shape-symmetric morphic picture.

Then $R = \psi^\omega(\mathcal{T}_{p_0})$ is a shape-symmetric morphic picture compatible with P , and thus we can merge it with P into a new picture P' . All the elements of \mathcal{P} are definable by atomic formula involving a predicate of \mathcal{P}' , and θ is definable as

$$\theta(\bar{y}_\delta) = \exists y_\delta P_{S_1}(\bar{y}) = P'_{\Sigma \times \mathcal{M}}(\bar{y}_\delta, 0)$$

where \mathcal{M} is the set of all the elements of $\mathcal{P}(\Sigma)$ that intersect S_1 .

This concludes the proof Theorem 20. \square

Remark 30 Although the proof of the decidability of the first-order theory of the structure $\langle \mathbb{N}; <, 0, \mathcal{P} \rangle$ is now complete, some improvements can still be done.

One of them consists in lowering the dimension of the predicates we are working with.

Suppose that for *all* the occurrences of the projections of P in the formula we want to decide, one argument (say the last one) is always the zero constant. Then we can replace P and φ by their $(\delta - 1)$ -dimensional restriction to the hyperplane $x_\delta = 0$.

Similarly, if for *all* the occurrences of the projections of P , two arguments (say the two last) are always equal, then we can replace P and φ by their $(\delta - 1)$ -dimensional restriction to the hyperplane $x_{\delta-1} = x_\delta$. Notice that, in order to define them in this way, we use Lemma 26 which, in turn, uses the fact that P is shape-symmetric !

Another improvement would be to reduce the alphabet. This should be possible using tools and methods developed for computing minimal automata (see Remark 33).

6 Extensions

One way to extend this result consists in adding special n -ary predicates or functions like the addition (of course, any function is first-order definable from its graph relation, so it suffices to consider the case of n -ary predicates). In order to use this method, it ‘suffices’ to show that the characteristic words of these predicates can be obtained as projection of some shape-symmetric morphic picture *compatible* with P . Although the order relation is easily definable from a picture compatible with P , as proved in the previous section, it is not known yet what conditions on the shape of P allow to define the addition relation in a compatible way. See Section 7 for a few examples, including an example of morphic predicate with which the addition *cannot* be compatible.

Another way to extend this method is to show that it can be used with stronger theories. For instance, we can ‘add’ *modular quantifiers* $\exists^{(p,q)}$ to the theory $\langle \mathbb{N}; <, 0, \mathcal{P} \rangle$, and the resulting theory remains decidable. A modular quantifier ‘ $\exists^{(p,q)} x \varphi(x)$ ’ means that ‘the number of x such that $\varphi(x)$ is $p \bmod q$ ’.

Let us show this. Suppose that $\theta(\bar{y}_\delta)$ is the formula $\exists^{(p,q)} y_\delta P_{S_1}(\bar{y})$.

We use the same set \mathcal{T}_{p_0} as in Case 4 of the proof of Section 5.4. For each symbol $a \in \mathcal{T}_{p_0}$, it is easy to determine whether a occurs finitely or infinitely often in the word $P_{\upharpoonright_{x_0=\dots=x_{\delta-1}=0}}$.

Indeed, a ‘pumping argument’ (see [8]) shows that any symbol (and only them) occurring infinitely often in $P_{\upharpoonright_{x_1=\dots=x_{\delta-1}=0}}$ must occur in the word

$$\bigodot_{\delta}^{|\Sigma| < m \leq 2|\Sigma|} w_m,$$

where w_m is the word defined by

$$(\varphi^m(p_0))_{\upharpoonright_{x_1=\dots=x_{\delta-1}=0}} = \varphi^{m-1}(p_0)_{\upharpoonright_{x_1=\dots=x_{\delta-1}=0}} \bigodot_{\delta} w_m.$$

We define a variation \tilde{R} of the picture R defined in Case 4. The alphabet of \tilde{R} is not $\mathcal{P}(\Sigma)$ anymore (we might have written it 2^Σ), but is the set

$\{0, \dots, q-1, \infty\}^\Sigma$, i.e. is the set of functions from Σ into the set $\{0, \dots, q-1, \infty\}$.

The initial symbol of \tilde{R} is the function f_{p_0} that sends each symbol $a \in \mathcal{T}_{p_0}$ on the remainder modulo q of the number of occurrences of a in $P_{\upharpoonright x_1=\dots=x_{\delta-1}=0}$ (∞ in case it occurs infinitely often). Notice that there is always at least one symbol occurring infinitely often.

Then we expand \tilde{R} by iterating the application $\tilde{\psi}$ that sends a ‘symbol’ (i.e. a function) f of $\{0, \dots, q-1, \infty\}^\Sigma$ on the $(\delta-1)$ -dimensional picture of size $|\varphi(x)|_\delta$ (where x is any symbol of $f^{-1}(\infty)$) whose element in position \bar{n}_δ is the function that sends each symbol $a \in \Sigma$ on the value

$$\left(\sum_{\substack{b \in \Sigma \\ f(b) \neq 0}} f(b) \cdot \sum_{k < |\varphi(b)|_\delta} \#_a(\varphi(b)(n_\delta, k)) \right) \bmod q$$

where $\#_a(w)$ is the number of symbols a in w .

We merge P and \tilde{R} into a new picture P' , and so $\theta(\bar{y}_\delta)$ is definable by the formula

$$\theta(\bar{y}_\delta) = \exists^{(p,q)} y_\delta P_{S_1}(\bar{y}) = P'_{\Sigma \times \mathcal{F}}(\bar{y}_\delta, 0)$$

where \mathcal{F} is the set of functions f that satisfy

$$\sum_{a \in S_1} f(a) = p \pmod{q}.$$

7 Examples

We give two well-known examples of decidable theories involving shape-symmetric pictures, and an undecidable theory involving a surprising non-shape-symmetric morphic picture.

Example 31 Let n be a natural number. Let P be the set of powers of n . The set P is morphic: for instance, for $n = 2$, it is the image of the fixed

point on a of the morphism

$$\varphi_{P_2} : \left\{ \begin{array}{l} a \mapsto \boxed{a \mid b} \quad b \mapsto \boxed{b \mid c} \quad c \mapsto \boxed{c \mid c} \end{array} \right.$$

under the projection $a, c \mapsto 0, b \mapsto 1$.

The set of powers of 2 is 2-recognizable, i.e. there is a deterministic finite automaton that accepts the binary writings of the powers of 2, and the above morphism is deduced from this automaton (see Figure 5). It is well-known that the addition relation is 2-recognizable too (see Figure 6). As a consequence, the addition relation can be obtained as projection of the 3-dimensional fixed point on a of φ_{+2} , which is compatible with φ_{P_2} (they are both morphisms of length 2):

$$\varphi_{+2} : \left\{ \begin{array}{l} a \mapsto \begin{array}{|c|c|} \hline a & c \\ \hline c & c \\ \hline \end{array} \odot_3 \begin{array}{|c|c|} \hline b & a \\ \hline a & c \\ \hline \end{array} \quad b \mapsto \begin{array}{|c|c|} \hline c & b \\ \hline b & a \\ \hline \end{array} \odot_3 \begin{array}{|c|c|} \hline c & c \\ \hline c & b \\ \hline \end{array} \quad c \mapsto \begin{array}{|c|c|} \hline c & c \\ \hline c & c \\ \hline \end{array} \odot_3 \begin{array}{|c|c|} \hline c & c \\ \hline c & c \\ \hline \end{array} \end{array} \right.$$

Since these two pictures are compatible, we can replace them by one shape-symmetric picture (by merging them or taking their cross product). By Theorem 20, this proves the well-known result that the first-order theory of the structure $\langle \mathbb{N}; <, +, 0, P_n \rangle$ is decidable.

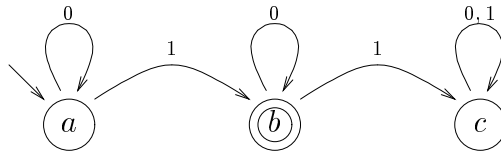


Fig. 5. Automaton accepting the powers of 2 in base 2.

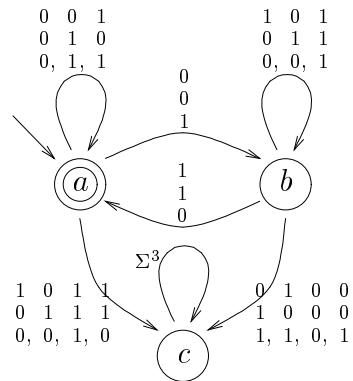


Fig. 6. Automaton accepting the sum relation in base 2.

See [2] for more on n -recognizability and definability.

Example 32 Now, we would like to consider the set $P_F = \{1, 2, 3, 5, 8, \dots\}$ of Fibonacci Numbers. Again, this set is morphic, and is obtained as the fixed point on 0 of the morphism φ_F (that corresponds to the automaton of Figure 7):

$$\varphi_{P_F} : \begin{cases} 0 \mapsto \boxed{01} & 1 \mapsto \boxed{x} \\ x \mapsto \boxed{xy} & y \mapsto \boxed{z} & z \mapsto \boxed{zy} \end{cases}$$

under the projection $0, y, z \mapsto 0, 1, x \mapsto 1$.

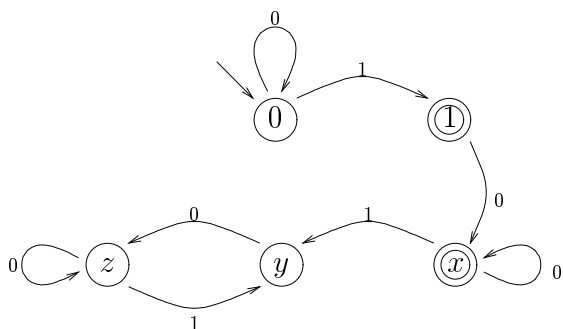


Fig. 7. Automaton accepting Fibonacci numbers in Fibonacci base.

Notice that this automaton is not complete: there is only one edge (labelled 0) leaving the states 1 and y . This is the reason for the images of 1 and y by φ_{P_F} to be of length 1.

This is due to the fact that the Fibonacci representation of a natural number is a word on the alphabet $\{0, 1\}$ that does not contain the factor 11. The words of $\{0, 1\}^*$ that correspond to the Fibonacci representation of a natural number are those that are accepted by the automaton of Figure 8.

This automaton (and its associated morphism φ_F) determine the coding of the natural numbers. Moreover, a given morphism will be compatible with φ_{P_F} if and only if it is compatible with φ_F .

Again, the addition is recognizable in base Fibonacci, i.e. is accepted by an automaton whose associated 3-dimensional morphism φ_{+F} is compatible with φ_F . However, this automaton has 37 states (and so φ_{+F} acts on an alphabet of 37 symbols). You can understand that neither the automaton, nor the morphism will be given here.

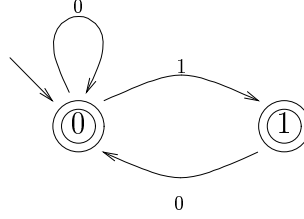


Fig. 8. The Automaton accepting Fibonacci representations.

Now, since φ_{P_F} and φ_{+F} are compatible, by Theorem 20, this proves the well-known result that the first-order theory of the structure $\langle \mathbb{N}; <, +, 0, P_F \rangle$ is decidable.

See [1,6] for more on numerations systems.

Remark 33 Lemma 23 has clearly an equivalent in terms of automata: it consists in sending the automaton that corresponds to the morphisms φ' of that lemma on the automaton of φ by identifying some of its nodes under the projection π in such a way that the edges are mapped correspondingly. This defined a notion of compatibility for automata.

Since the compatibility of morphisms is an equivalence relation, given a shape-symmetric morphic picture P , there exists a minimal 1-dimensional morphism (with respect to the alphabet it works on) whose fixed point is compatible with P . This morphism corresponds to the minimal automaton (with respect to the number of nodes) compatible with the morphism generating P .

In the case of Fibonacci numbers, it is exactly the automaton of Figure 8.

Example 34 Finally, we consider the set P_{x^2} of square numbers.

This set is obtained from the fixed point on 0 of the morphism

$$\varphi_{P_{x^2}} : \begin{cases} 0 \mapsto \boxed{0} \boxed{a} \boxed{b} \boxed{c} & a \mapsto \boxed{a} \\ b \mapsto \boxed{b} \boxed{c} \boxed{c} & c \mapsto \boxed{c} \end{cases}$$

under the projection $b, c \mapsto 0; 0, a \mapsto 1$.

$$\varphi_{P_{x^2}}^\omega(0) = 0abcabcccabccccabcccccc \dots$$

This morphism is build ‘from scratch’ using the fact that the ‘distance’ between n^2 and $(n + 1)^2$ increases by 2 when n increases by 1: this increase is managed by the letter ‘ b ’ that adds two ‘ c ’ at each iteration of the morphism.

We can associate an automaton to this morphism: the set of states is Σ , the initial state is $p_0 = 0$, the accepting states are those that are projected on 1, and we draw an edge labelled $n \in \mathbb{N}$ between two symbols $p, q \in \Sigma$ if the n^{th} symbol of $\varphi_{P_{x^2}}(p)$ is q (see Figure 9). This leads to a *coding* of the natural numbers, which consist in sending the n^{th} natural number on the n^{th} word accepted by the automaton, these words being ordered lexicographically.

However, this representation of the natural numbers is *not* a numeration system in the sense of [1] but is just a *coding* of the natural numbers. This means, for instance, that the coding of a natural number is not obtained using a ‘euclidian division’ algorithm; the value natural number coded by ‘2000’ (17 in the current case) is not twice that of the number coded by ‘1000’ (i.e. 16).

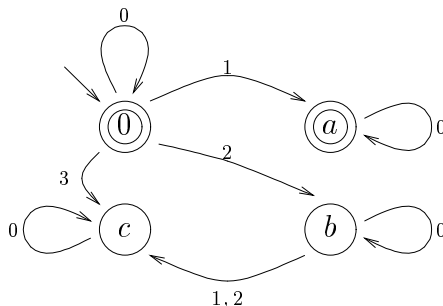


Fig. 9. Automaton accepting the square numbers.

Now, the following amazing property has to be noticed: there is **no** 3-dimensional morphism ψ generating a shape-symmetric fixed point Q **compatible with** P_{x^2} such that the addition is definable as a projection of Q .

Indeed, a result by Putnam [11] (see also Büchi [4]) shows the undecidability of the first-order theory of any structure in which both the set of square numbers and the addition are definable. If there existed a morphic picture Q from which the addition would be definable and compatible with P_{x^2} , we would have a contradiction with Theorem 20.

Of course, we may present this result on a different way: if we consider the fixed point R on $(0, 0)$ of the non-shape-symmetric morphism obtained as the cross-product of $\varphi_{P_{x,2}}$ by φ_{+_2} of Example 31, we have that the first-order theory of the structure $\langle \mathbb{N}; <, 0, \mathcal{R} \rangle$ is *undecidable*.

This shows that theorem 20 cannot be generalized to *all* δ -dimensional morphisms. However, it is valid for any 1-dimensional morphism.

We can wonder what is the lowest natural number δ for which there exists a δ -dimensional non-shape-symmetric morphic picture P leading to an undecidable theory. So far, $\delta \leq 4$ is the best upper bound known.

Remark 35 The ‘compatibility’ condition of two morphisms (as well as the notion of shape-symmetry) involves much more the *shapes* of the morphisms than their contents.

Given a predicate Q , we can wonder what are the (shapes of the) morphisms that will allow to define Q as a projection of their fixed point. We have seen that some predicates, like the order relation, are compatible (as projection) with any shape-symmetric picture. Other predicates, like the addition, are compatible only with certain shape-symmetric picture. Finally, a counting argument shows that some predicates are not morphic (projections).

So the main question is: given two non-compatible shape-symmetric morphic pictures P, Q , under which conditions is there a shape-symmetric morphic picture R such that both P and Q are projections of R ? This question seems related to Cobham’s [5] and Semënov’s [12] famous results.

The answer is known in certain cases. For instance, if V_k is the graph of the function that sends a natural n on the largest power of k dividing n , then the predicates V_k and V_l are morphic — they are obtained from fixed-length morphisms of length k and l — and can both be obtained as projections of a shape-symmetric picture R (generate by a fixed-length morphism) *if and only if* k and l are multiplicatively dependent, i.e. if and only if there exists two non-zero natural number n, m such that $k^n = l^m$ (R is then obtained by a fixed-length morphism of length $k^n = l^m$). Indeed, by a result of Villemaire [14], the first-order theory of the structure $\langle \mathbb{N}; +, V_k, V_l \rangle$ is undecidable if k and l are multiplicatively independent. See Bès [3] for

similar results related to Cobham-Semënov theorem.

However, the general case is still unsolved.

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