

# Singleton Acceptance Conditions in $\omega$ -Automata

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## Abstract

Regular  $\omega$ -languages can be defined by deterministic  $\omega$ -automata with Rabin acceptance condition, referring to a list of pairs  $(E_k, F_k)_{1 \leq k \leq m}$  of state sets of the automaton: A successful run should visit some  $E_k$  only finitely often but the corresponding  $F_k$  infinitely often. If only the non-existence, respectively existence of such visits is required, precisely the Staiger-Wagner definable  $\omega$ -languages are recognized. We study the question whether the sets  $E_k, F_k$  can be reduced to singletons, i.e. whether one can refer to individual states in place of state sets. It is shown that for the usual Rabin acceptance condition this causes no loss of expressive power, while for the Staiger-Wagner case the acceptance by singletons leads to a proper restriction.

## 1 Introduction

Acceptance conditions in  $\omega$ -automata serve to declare certain automaton runs to be “successful”, usually by a restriction regarding the states which are infinitely often visited (or just: visited) in a run. All standard acceptance conditions (like the conditions due to Büchi, Muller, Rabin, Streett) make use of designated state sets to which such an acceptance condition refers. In the

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present paper we study to which extent it is sufficient to specify individual states (or equivalently: singleton sets of states) such that acceptance can be defined in terms of their visits in runs.

The classical Büchi condition declares a run  $\rho \in Q^\omega$  of an automaton over the state set  $Q$  as successful if

$$(*) \quad \exists^\omega i \rho(i) \in F,$$

(“there are infinitely many positions  $i$  in  $\rho$  where a state in  $F$  is assumed”); here  $F \subseteq Q$  is a designated set of “final states”. Other acceptance conditions are obtained as boolean combinations of statements of the Büchi form. For example, the Rabin condition (first mentioned in [Rab69]) is a disjunction of conjunctions

$$\exists^{<\omega} i \rho(i) \in E \wedge \exists^\omega i \rho(i) \in F$$

where  $\exists^{<\omega}$  (“there are only finitely many”) is the negation of the quantifier  $\exists^\omega$ . The Streett condition (cf. [Str82]) is the dual, i.e., a conjunction of disjunctions

$$\exists^{<\omega} i \rho(i) \in E \vee \exists^\omega i \rho(i) \in F.$$

It is well-known that nondeterministic finite automata with the Büchi condition recognize precisely the regular  $\omega$ -languages, as do deterministic (and also nondeterministic) finite automata with the Rabin condition or with the Streett condition (see e.g. [Tho90]).

A more refined view of acceptance conditions is obtained when the atomic conditions refer to singletons, i.e. to individual states. Then they take the form

$$(**) \quad \exists^\omega i \rho(i) = q$$

for states  $q$  of the automaton under consideration. The Büchi condition for  $F$  is equivalent to a disjunction (over the  $q \in F$ ) of such conditions. Hence also the Rabin and Streett conditions are boolean combinations of conditions of the “singleton type” (\*\*). The Muller acceptance condition is easily formulated in this way as well: For a system  $\mathcal{F}$  of final state sets over  $Q$  it reads

$$\bigvee_{F \in \mathcal{F}} \left( \bigwedge_{q \in F} \exists^\omega i \rho(i) = q \wedge \bigwedge_{q \in Q \setminus F} \exists^{<\omega} i \rho(i) = q \right).$$

A general theory of acceptance conditions would have to classify the possible boolean combinations of conditions (\*\*) with respect to the expressive power of corresponding  $\omega$ -automata. In particular, those types of boolean combinations should be determined with which all regular  $\omega$ -languages are recognizable. It seems that this general question was considered before only by

Emerson and Lei [EL87] in their framework of “general fairness constraints” (however regarding state sets rather than states).

The analogous question can be asked on a lower level, where instead of the infinite occurrence of states in runs the mere occurrence suffices. In this case we speak of *occurrence acceptance conditions*. The analogue of the Büchi condition in this sense states for the run  $\rho$

$$\exists i \rho(i) \in F,$$

for some designated set  $F$  of states. For nondeterministic automata, this condition allows to define precisely the Staiger-Wagner recognizable  $\omega$ -languages ([SW74]). The *Staiger-Wagner acceptance condition* is a “Muller type” condition for occurrence acceptance in deterministic automata:

$$\bigvee_{F \in \mathcal{F}} \left( \bigwedge_{q \in F} \exists i \rho(i) = q \wedge \bigwedge_{q \in Q \setminus F} \neg \exists i \rho(i) = q \right).$$

The Staiger-Wagner recognizable sets form a boolean algebra and give a basic proper subclass of the regular  $\omega$ -languages. One can show (see the next section) that Staiger-Wagner recognizable sets can be defined (often more succinctly) by *occurrence acceptance of Rabin type*: Here we use a list  $(E_k, F_k)_{1 \leq k \leq m}$  of pairs of states and consider a run  $\rho$  successful if

$$\bigvee_{1 \leq k \leq m} \left( \neg \exists i \rho(i) \in E_k \wedge \exists i \rho(i) \in F_k \right).$$

In this framework of occurrence acceptance conditions, we shall again analyze the role of the singleton conditions; these are now boolean combinations of statements

$$\exists i \rho(i) = q.$$

For example, the occurrence acceptance by Rabin-singletons requires a disjunction of conjunctions  $\neg \exists i \rho(i) = p \wedge \exists i \rho(i) = q$ .

The main results of the present paper say that for deterministic automata the acceptance by “Rabin-singletons”, i.e. with Rabin conditions involving singletons  $E_k, F_k$ , is expressively complete: This singleton acceptance allows to define all regular  $\omega$ -languages. The corresponding claim for occurrence acceptance conditions is shown to fail: there are Staiger-Wagner recognizable  $\omega$ -languages which are not definable by an occurrence acceptance condition of “Rabin type” involving singletons only. Along with these results some related facts on expressive completeness are derived. A more comprehensive analysis is carried out in [Leo96].

Before turning to the proofs, let us recall some obvious facts connected with the question of acceptance by singletons. For finite automata over finite words it is an easy exercise to see that the final state sets of deterministic automata cannot be bounded in size. In nondeterministic finite automata, a single final state suffices (if the empty word is discarded). In the usual model of (nondeterministic) Büchi automata, it is again trivial to see that acceptance by singletons is a proper restriction: If the language with the two  $\omega$ -words  $a^\omega$  and  $b^\omega$  is to be recognized, a single final state  $q$  would have to be reachable via  $a$  and via  $b$  as first letter of the input word, and from  $q$  both  $b^\omega$  and  $a^\omega$  could be scanned with infinitely many revisits of  $q$ . Thus, also words in  $b\{a,b\}^*a^\omega$  and  $a\{a,b\}^*b^\omega$  could be accepted, a contradiction. A simple refinement of the argument shows that the cardinality of the final state set in Büchi automata cannot be bounded by any fixed number if arbitrary regular  $\omega$ -languages should be recognized (e.g.  $a_1^\omega + \dots + a_n^\omega$  requires  $n$  final states). Considering the Muller condition with singletons in the system  $\mathcal{F}$  of final state sets, a (deterministic or nondeterministic) automaton can only serve to define  $\omega$ -languages of the form  $UV^\omega$  where  $V$  is a set of alphabet letters. Thus, singleton Muller acceptance can only define unions of such  $\omega$ -languages, which are very special cases of regular  $\omega$ -languages. These observations lead us to focus on the Rabin condition (and the Streett condition) in deterministic automata.

The remainder of the paper is structured as follows. Section 2 summarizes terminology. In Section 3, the Rabin condition with singletons is shown to be expressively complete for regular  $\omega$ -languages. Section 4 deals with the analogous notion of occurrence acceptance; here a Staiger-Wagner recognizable set is exhibited which cannot be defined with occurrence acceptance of Rabin type, using only singletons.

## 2 Preliminaries

A (deterministic) *Rabin automaton* has the form  $\mathcal{A} = (Q, A, q_0, \delta, \Omega)$  where  $Q$  is a finite set of states,  $A$  the input alphabet,  $q_0$  the initial state,  $\delta : Q \times A \rightarrow Q$  the transition function, and  $\Omega = ((E_1, F_1), \dots, (E_m, F_m))$  a list of “accepting pairs” of state sets. Extend the transition function to  $Q \times A^*$  by setting  $\delta(q, \epsilon) = q$  and  $\delta(q, wa) = \delta(\delta(q, w), a)$ .

Any  $\omega$ -word  $\alpha \in A^\omega$  determines the *run*  $\rho \in Q^\omega$  of  $\mathcal{A}$  on  $\alpha$ , defined by  $\rho(i) = \delta(q_0, \alpha(0) \dots \alpha(i-1))$  for  $i \geq 0$ . The run is successful if

$$\bigvee_{1 \leq k \leq m} \left( \exists^{<\omega} i \rho(i) \in E_k \wedge \exists^\omega i \rho(i) \in F_k \right).$$

An  $\omega$ -word  $\alpha$  is accepted by  $\mathcal{A}$  if the unique run of  $\mathcal{A}$  on  $\alpha$  is successful. An  $\omega$ -language  $L \subseteq A^\omega$  is said to be *Rabin recognizable* if it consists of all  $\omega$ -words accepted by a Rabin automaton.

A special form of Rabin automaton will be useful in Section 3, the *Rabin chain automaton* (or: *parity automaton*) introduced in [Mos84], and applied independently in [Mos91] and [EJ91] for the complementation of tree automata. A Rabin chain automaton is a Rabin automaton whose list of designated pairs  $(E_k, F_k)$  of state sets form an increasing chain:

$$E_1 \subseteq F_1 \subseteq E_2 \subseteq F_2 \subseteq \dots \subseteq E_m \subseteq F_m.$$

It is well-known that any Rabin automaton is equivalent to a Rabin chain automaton ([Mos91],[Car94]). A simpler simulation can be based on the idea of “latest appearance record” ([GH82]) or “order vector” ([Büc83]): The desired automaton has as states all permutations of states of the given automaton (order vectors), where each vector is extended by a pointer (“hit” in Büchi’s terminology) to one position of the vector. If the given transition function transforms  $p$  to  $q$  via letter  $a$ , any vector  $(q_1, \dots, q_{j-1}, q, q_{j+1}, \dots, q_{n-1}, p)$  is transformed via  $a$  into  $(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_{n-1}, p, q)$ , and the hit (regardless of its previous value) is set to the position  $j$  in the new vector. This construction serves to keep track of the states visited last, by listing them at the end of the vectors in the order of their last visits. The sets  $E_k, F_k$  are now state sets composed of vectors with hit position  $\leq k$ ; in  $E_k$ , the vectors are collected in which states from the hit onwards to the right do not form a set satisfying the Rabin condition of the given (non-chain) Rabin automaton, while for the vectors in  $F_k$  this condition should hold (again regarding the states from the hit onwards). A more complete proof is given in [Tho].

A *Streitt automaton* is defined as a Rabin automaton, however with the acceptance condition

$$\bigwedge_{1 \leq k \leq m} \left( \exists^{<\omega} i \rho(i) \in E_k \vee \exists^{\omega} i \rho(i) \in F_k \right).$$

Streitt automata define the *Streitt recognizable*  $\omega$ -languages.

The general framework of acceptance conditions to be studied in this paper is given by boolean combinations of conditions  $\exists^{\omega} i \rho(i) \in F$  or of conditions  $\exists^{\omega} i \rho(i) = q$ , where  $\rho$  is an automaton run,  $F$  a state set, and  $q$  a state. The Muller acceptance condition (as introduced in the introduction) is most general in the sense that it amounts to the disjunctive normal form over the statements  $\exists^{\omega} i \rho(i) = q$ . In our general framework, such an atomic condition is called a positive literal and its negation a negative literal. Special forms of acceptance conditions to be considered below are, for example,

disjunctions of conjunctions of positive literals only, disjunctions of conjunctions of negative literals only, and mere conjunctions of literals (positive and negative).

A disjunction of two-literal conjunctions containing precisely one negative and one positive literal is called a *Rabin-singletons condition*. (It amounts to the Rabin condition where the sets  $E_k, F_k$  are singletons.) In this case we specify a list  $(p_k, q_k)$  of “accepting pairs” of states, and a run is accepting by the pair  $(p_k, q_k)$  if it visits  $p_k$  only finitely often but  $q_k$  infinitely often. The notion of Streett-singletons condition is introduced analogously, as a conjunction of disjunctions of one negative and positive literal each.

Let us turn to the occurrence acceptance conditions, which define Staiger-Wagner recognizable  $\omega$ -languages. We remark that any Staiger-Wagner recognizable  $\omega$ -language is also recognizable with occurrence acceptance of Rabin type, using pairs  $(E_k, F_k)$  of designated state sets. If a (deterministic) automaton  $\mathcal{A} = (Q, A, q_0, \delta, \mathcal{F})$  with  $\mathcal{F} \subseteq 2^Q$  is given, construct the equivalent automaton  $\mathcal{A}'$  with occurrence acceptance of Rabin type over the state set  $Q' \subseteq Q \times 2^Q$ ; here the first component serves to simulate  $\mathcal{A}$ , and in the second component the visited states are collected (starting with  $(q_0, \{q_0\})$ ). The accepting pairs of  $\mathcal{A}'$  are indexed by the sets  $S \in \mathcal{F}$ ; one set  $E_S = \{(p, P) \mid p \in P \subseteq Q \wedge P \not\subseteq S\}$  and  $F_S = \{(q, S) \mid q \in S\}$ .

If we allow only singletons in occurrence acceptance conditions of Rabin type, we speak of *occurrence acceptance with Rabin-singletons*, in analogy to the previous case concerning infinite occurrences of states.

### 3 Rabin Acceptance with Singletons

In the classical Rabin condition over finite state sets, a conjunction

$$\exists^{<\omega} i \rho(i) \in E_k \wedge \exists^{\omega} i \rho(i) \in F_k$$

is easily broken up such that at the place of  $F_k$  only singletons occur: Since the  $F_k$ -part says that *some*  $F_k$ -state is visited infinitely often, one replaces the displayed conjunction by a disjunction (over all states  $q \in F_k$ ) of conjunctions with  $\{q\}$  in place of  $F_k$ . This disjunction is absorbed by the outer disjunction of the Rabin condition.

It is nontrivial to achieve a corresponding reduction for the negative literals which form the  $E_k$ -part, requiring the *all* states of  $E_k$  are visited only finitely often. By splitting the set  $E_k$ , the automaton would just check the existence of an  $E_k$ -state visited only finitely often.

The idea to overcome this problem is to introduce for each transition leaving  $E_k$  a new state which signals that the set  $E_k$  has been visited previously.

A signal state will be a state of the automaton that is reached from  $E_k$  in one transition step. If the given automaton accepts with the pair  $(E_k, F_k)$ , the corresponding signal state may only be visited finitely often in an accepting run. In Figure 1 the states  $\bar{q}_1$  and  $\bar{q}_2$  are the signal states that indicate that  $E_k$  has been visited.

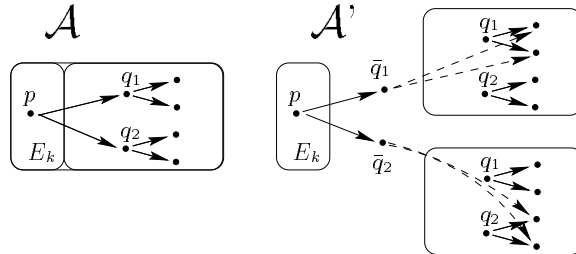


Figure 1: Introducing signal states and copies

For the formation of singleton acceptance pairs we have to deal with the situation that an  $F_k$ -state  $p$  is reachable from two signal states. Here we can save the singleton acceptance if for each transition into  $p$ , say from signal state  $s$ , an own “sub-automaton”  $\mathcal{A}_s$  is defined, consisting of an associated copy of  $p$  and all states in  $Q \setminus E_k$  reachable from  $s$ . Thus we use  $l$  copies of all states not in  $E_k$  if we have  $l$  signal states. In Figure 1 we have two copies of the states not in  $E_k$ . The (dashed) transitions into the copies from the signal states are derived in a natural way from the transitions of  $q_1$  and  $q_2$ .

But what happens if we do not use a different copy for each signal state? If a state  $s \notin E_k$  is reachable from both signal states  $\bar{q}_1$  and  $\bar{q}_2$ , then the pairs  $(\bar{q}_1, s)$  and  $(\bar{q}_2, s)$  do not guarantee a correct acceptance. E.g.  $\bar{q}_1$  and  $s$  could both be visited infinitely often (and thus a state in  $E_k$ ) but the run will be accepted with the accepting pair  $(\bar{q}_2, s)$  if  $\bar{q}_2$  is visited only finitely often contradicting the desired idea of simulating the chain automaton.

Let us explain why this idea is difficult to pursue when we start with the classical Rabin acceptance. Consider the situation where we have to handle two accepting pairs  $(E_1, F_1)$  and  $(E_2, F_2)$ . In a first step we want to replace the pair  $(E_1, F_1)$  by singleton pairs. Then the signal states for the transitions leaving  $E_1$  will lead into copies of the  $E_1$ -complement in which  $E_2$ - and  $F_2$ -states may be present, called  $E_{2,1}$  and  $F_{2,1}$  in the first copy, and  $E_{2,2}$  and  $F_{2,2}$  in a second copy. But these pairs are not suitable to capture acceptance, indeed we have to work with the pairs  $(E_{2,1} \cup E_{2,2}, F_{2,1})$  and  $(E_{2,1} \cup E_{2,2}, F_{2,2})$ .

In a next step (when replacing  $(E_{2,1} \cup E_{2,2}, F_{2,1})$  by singleton pairs) we have to introduce signal states for leaving  $E_{2,1} \cup E_{2,2}$ . Here we might get several copies of states in  $E_1$  and the corresponding signal states. Then, however, it

is not sufficient that these signal states are finally avoided individually, but it is necessary that all their copies are avoided simultaneously, contradicting the aim of acceptance by singletons.

It is not clear if this can be avoided when simulating a Rabin automaton directly. This problem is due to the different roles states play in two different Rabin pairs. Therefore we work with a kind of Rabin acceptance condition where a state plays the same role for all loops. This is the case in the Rabin chain condition. As explained in the previous section we can assume that a Rabin chain automaton is given with the accepting pairs  $(E_1, F_1), \dots, (E_m, F_m)$ .

For such an automaton a (rejecting) state in  $E_k$  is also rejecting for all pairs  $(E_j, F_j)$  where  $j \geq k$ . In Rabin automata with singleton condition we can use one step of the above described construction for the minimal accepting pair  $(E_1, F_1)$ . We replace this accepting pair by singleton pairs and get several copies of the transition structure of the automaton (leaving  $E_1$ -states uncopied) and corresponding copies of the accepting pairs  $(E_2, F_2), \dots, (E_m, F_m)$ . So in this case we do not need unions of  $E_j$ -sets since  $E_{j,s}$  is reachable from  $F_{j,s'}$  only via  $E_1$  and thus via the signal state  $s$ . As explained above,  $\mathcal{A}_s$  initiated from a signal state  $s$  is now a “sub-automaton” with Rabin chain condition given by  $(E_{2,s}, F_{2,s}), \dots, (E_{m,s}, F_{m,s})$ . Thus we can use the same construction recursively for each sub-automaton  $\mathcal{A}_s$ , leaving the parts outside of  $\mathcal{A}_s$  unchanged.

**Theorem 1** *Every regular  $\omega$ -language is recognised by a deterministic automaton with Rabin-singletons acceptance.*

*Proof.* We construct a deterministic automaton with Rabin-singletons acceptance that is equivalent to a given Rabin chain automaton. We use the idea described above, giving a global definition of the automaton instead of a recursive procedure.

So let  $\mathcal{A} = (Q, A, q_0, \delta, \Omega)$  be a Rabin automaton with acceptance component  $\Omega = ((E_1, F_1), \dots, (E_m, F_m))$  and  $E_1 \subseteq F_1 \subseteq E_2 \dots \subseteq F_m$ , i.e.,  $\mathcal{A}$  is a Rabin chain automaton.

For simplicity of notation we use the convention that  $E_0 = F_0 = \emptyset$  and  $E_{m+1} = F_{m+1} = Q$ . For a state  $p$  let  $index(p)$  the unique integer  $i$  such that  $p \in E_i \setminus E_{i-1}$ .  $Q_j$  is the set of states not in  $E_j$ , i.e. the set of states in  $Q$  with  $index \geq j$ . Besides that let  $\bar{Q} = \{\bar{q} \mid q \in Q\}$ .

So the Rabin automaton with singleton acceptance condition is given by  $\mathcal{A}' = (Q', A', q'_0, \delta', \Omega')$  where

- the state space is  $Q' \subseteq \bigcup_{j=2}^{m+1} (Q \cup \bar{Q}) \times Q_2 \times \dots \times Q_j$  where for  $(p, p_2, p_3, \dots, p_s)$  for all  $i$   $index(p_i) \leq index(p_{i+1})$  holds;



- the input alphabet remains  $A' = A$ ;
- the initial state  $q'_0 = (q_0, p_1, \dots, p_j)$  is an arbitrary state in  $Q'$  with first component  $q_0$ ;
- for states  $p \in Q$  and  $\delta(p, a) = q$  we have

$$\delta'((p, p_2, \dots, p_s), a) = ((p', p'_2, \dots, p'_{index(q)})$$

with

$$p' = \begin{cases} \bar{q} & \text{if } index(p_{j-1}) \neq index(q) \\ q & \text{else} \end{cases}$$

and

$$p'_i = \begin{cases} p_i & \text{for } i < j \\ q & \text{for } i \geq j \end{cases}$$

where  $j$  is the minimal integer  $r$  with  $index(p_r) > index(q)$  if such  $r$  exists and  $s + 1$  otherwise. For states  $\bar{p} \in \bar{Q}$  we just define  $\delta'$  as for the states  $p \in Q$ :

$$\delta'((\bar{p}, p_2, \dots, p_s), a) = \delta'((p, p_2, \dots, p_s), a)$$

- and the accepting singleton pairs are given by

$$\Omega' = \left\{ \left( \{(\bar{p}, p_2, \dots, p_j)\}, \{(q, p_2, \dots, p_j)\} \right) \mid q \in F_{j+1} \setminus E_{j+1}, j \leq m \right\}.$$

Let us shortly compare this construction with the idea explained before. The first component of a state  $q$  gives the state the automaton  $\mathcal{A}$  assumes when  $\mathcal{A}'$  assumes  $q$ . The number of other components in such a state indicates which rejecting sets  $E_i$  have previously been visited (and left) and the component  $p_i$  gives the state with minimal index visited since the set  $E_{i-1}$  was left. Obviously, if a state in  $Q'$  consists of only one component it is a state in the minimal rejecting set  $E_1$ . A state  $(q, p, p)$  indicates that  $q$  is in  $E_3 \setminus E_2$ , that since leaving  $E_1$  and  $E_2$  the state with minimal index assumed in the automaton  $\mathcal{A}$  was the state  $p$  (outside  $E_1, E_2$ ). States with an element of  $\bar{Q}$  as first component are the signal states. A state in  $(\bar{r}, p, p, r, r)$  in  $\mathcal{A}'$  indicates that  $r$  is in  $E_5 \setminus E_4$  and that  $r$  was a state with index  $\geq 4$  assumed after leaving  $E_3$  and no state with index 4 has been reached since then.

The transitions in the definition of  $\delta'$  realize the move into the copies of the states as explained before. The condition  $index(p_{j-1}) \neq index(q)$  is true

iff no state is in  $p_2, \dots, p_s$  with same index as  $q$ . Thus  $q$  is the state with minimal index in  $E_{index(q)}$  assumed after  $E_{index(q)-1}$  was left. So we enter a signal state.

The number  $j$  gives the first component in  $p_2, \dots, p_s$  with greater index than  $q$ , i.e. the first component that has to be changed. The components  $p_i$  in  $p_2, \dots, p_{j-1}$  are still the states with minimal index assumed after leaving  $E_{i-1}$ , other can have changed due to the visit of  $q$ .

For the accepting singleton pairs we used states that only differ in the first component. The rejecting state is obviously a signal state. So a run in  $\mathcal{A}'$  is accepted if finally a set  $F_j \setminus F_{j-1}$  is visited infinitely often, but  $E_j$  only finitely often.

It is obvious by the definition of the transition function  $\delta'$  that for each  $\omega$ -word  $\alpha \in A^\omega$  there exist corresponding runs  $\rho$  in  $\mathcal{A}$  and  $\rho'$  in  $\mathcal{A}'$  such that for all  $i$  we have  $\rho(i) = Pr_1(\rho'(i))$  where  $Pr_1$  is the projection to the first component of a state in  $\mathcal{A}'$ .

It remains to show that they also correspond regarding the acceptance conditions of  $\mathcal{A}$  and  $\mathcal{A}'$ , respectively, i.e., that the run  $\rho$  is accepting in  $\mathcal{A}$  iff  $\rho'$  is accepting in  $\mathcal{A}'$ .

Let  $\rho$  be an accepting run in  $\mathcal{A}$ . By definition we then have a pair  $(E_j, F_j)$  such that the states in  $E_j$  are assumed only finitely often whereas a state in  $F_j$  is assumed infinitely often in the run  $\rho$ . Let the state visited infinitely often in  $F_j$  be  $q$ . Then  $index(q) = j + 1$  and for all states  $p$  in  $\rho$  that are visited infinitely often we have  $index(p) \geq j + 1$ .

Let  $k$  be the last position in  $\rho$  where a state in  $E_j$  is assumed and  $k'$  be the first position after  $k$  where in  $\rho$  the state  $q$  occurs. Then by construction of  $\mathcal{A}'$  in  $\rho(k')$  the corresponding state  $q' = (q, p_2, \dots, p_{j+1})$  with the additional components  $p_2, \dots, p_{j+1}$  is visited. Since all states visited later than  $k$  have an index greater or equal to  $j + 1$  in  $\mathcal{A}'$  there will never be used a transition to a state without the components  $p_2, \dots, p_{j+1}$  after the position  $k$ . So there cannot occur a state in the run  $\rho'$  of the form  $p' = (\bar{p}, p_2, \dots, p_{j+1})$  in positions later than  $k$ . Obviously the state  $q' = (q, p_2, \dots, p_{j+1})$  must occur infinitely often in  $\rho'$  and hence  $\rho'$  is an accepting run in  $\mathcal{A}'$ .

For the other direction assume we have an accepting run  $\rho'$  in  $\mathcal{A}'$ . Then by definition of  $\Omega'$  there is a pair of states

$$(\{(\bar{p}, p_2, \dots, p_{j+1})\}, \{(q, p_2, \dots, p_{j+1})\}) \text{ with } q \in F_j \setminus E_j \text{ and } j \leq m$$

such that  $(\bar{p}, p_2, \dots, p_{j+1})$  occurs finitely often in  $\mathcal{A}'$ , whereas  $(q, p_2, \dots, p_{j+1})$  occurs infinitely often in  $\rho'$ . Since  $\rho$  is the run corresponding to  $\rho'$  the state  $q \in F_j$  occurs infinitely often in  $\rho$ . So we have to show that none of the states in  $E_j$  is visited infinitely often in  $\rho$ .

Now let us assume that a state  $r$  in  $E_j$  is visited infinitely often. Since  $r$  is in  $E_j$  we know that  $index(r) = k < j + 1$ . Thus in  $\mathcal{A}'$  there are only states with first component  $r$  reachable (from any other state) that are of the form  $(r, p'_2, \dots, p'_k)$ . So the  $j + 1$ -th component in these states is missing.

The  $j + 1$ -th component in  $(\bar{p}, p_2, \dots, p_{j+1})$  is  $p_{j+1} = p$ . (By the definition of  $\delta'$  only those states with first component  $\bar{p}$  are reachable.) The state  $(q, p_2, \dots, p_{j+1})$  has the same last component. So  $(q, p_2, \dots, p_{j+1})$  is reachable from any state without that component  $p_{j+1}$  only via the state  $(\bar{p}, p_2, \dots, p_{j+1})$ . States with first component  $\bar{p}$  and different  $j + 1$ -th component are not reachable since  $index(p) = j + 1$ . Thus this state is visited infinitely often when  $(q, p_2, \dots, p_{j+1})$  and a state with less components (as  $(r, p'_2, \dots, p'_k)$ ) are visited infinitely often. But this contradicts the assumption that  $\rho'$  is accepting.  $\square$

Let us analyze the complexity of the above construction, assuming that a Rabin chain automaton with  $n$  states and  $m$  accepting pairs is given.

The state space of the singletons automaton is the product of  $m + 1$  sets. The sets  $C_i$  are at most of size  $n$  and  $|Q \cup \bar{Q}| \leq 2n$ . So the Rabin-singletons automaton has at most  $O(n^{m+1})$  states.

For each accepting pair we have to combine a state with first component in  $Q$  and a state with first component in  $\bar{Q}$  where all other components are identical. So we get at most  $O(n^{m+2})$  accepting singleton pairs. Thus we have an exponential blow-up for this transformation. Now also the transformation from Rabin automata to Rabin chain automata is exponential, so we have a double exponential blow-up for the transformation from Rabin automata to Rabin-singletons automata in the present approach. We do not know whether this double exponential blow-up is really necessary.

Let us note a simple consequence of the preceding theorem:

**Corollary 2** *Any regular  $\omega$ -language is recognized by a deterministic automaton with Streett-singletons acceptance.*

*Proof.* The complements of the  $\omega$ -languages recognized with Rabin-singletons acceptance cover the whole class of regular  $\omega$ -languages (since this class is closed under complement). These complements are recognized by the given automata, now with negated Rabin acceptance conditions. But these negated acceptance conditions are clearly equivalent to conditions with Streett-singletons.  $\square$

Finally we add some remarks on more special forms of acceptance conditions with singletons, for which a more detailed treatment is given in [Leo96].

Among the many possibilities, we consider here three variants of the Rabin-singletons condition.

In the first two cases, we use  $\wedge$ - $\vee$ -combinations over positive, resp. negative literals, i.e. when considering disjunctive normal forms we replace the two literals in a conjunction of a Rabin condition by a sequence of positive literals, respectively by a sequence of negative literals.

**Proposition 3** *Deterministic  $\omega$ -automata with acceptance conditions*

$$\bigvee_{1 \leq k \leq m} \left( \bigwedge_{1 \leq j \leq l_k} \exists^{\omega} i \rho(i) = q_{k,j} \right)$$

*are expressively equivalent to deterministic Büchi automata (and hence do not cover all regular  $\omega$ -languages).*

*Proof.* Trivially each Büchi automaton has an acceptance condition of the form above. Conversely, from an automaton  $\mathcal{A}$  with such an acceptance condition (over state set  $Q$ ) construct a Büchi automaton over  $Q \times 2^Q$  where the first component simulates  $\mathcal{A}$  and the second component serves to cumulate visited states until some set  $\{q_{k,1}, \dots, q_{k,l_k}\}$  is exhausted, whence a reset to the empty set is done. The final states are those where the second component is empty.  $\square$

As we consider deterministic automata, we can proceed from a class of regular  $\omega$ -languages to the class of complement languages by complementing the acceptance type. Applied to the type of the previous proposition, we obtain the following claim (where the negation in the acceptance condition is pushed inside).

**Proposition 4** *Deterministic  $\omega$ -automata with acceptance conditions*

$$\bigwedge_{1 \leq k \leq m} \left( \bigvee_{1 \leq j \leq l_k} \exists^{<\omega} i \rho(i) = q_{k,j} \right)$$

*recognize precisely the complements of  $\omega$ -languages defined by deterministic Büchi automata (and hence do not cover all regular  $\omega$ -languages).*

Finally, we consider the option to cancel the disjunction as it appears in the Rabin condition, but allow an arbitrary number of positive and negative literals in a conjunction. Not surprisingly, the corresponding family of recognized  $\omega$ -languages is not closed under union:

**Proposition 5** *Deterministic  $\omega$ -automata with acceptance conditions of the form*

$$\bigwedge_{1 \leq k \leq m} \exists^{<\omega} i \rho(i) = p_k \wedge \bigwedge_{1 \leq j \leq l} \exists^{\omega} i \rho(i) = q_j$$

*are not expressively complete for the class of regular  $\omega$ -languages.*

*Proof.* An example language not recognizable by automata with this acceptance condition is defined by the regular expression  $(a+b)^* a^\omega + ((a+b)^* bb)^\omega$ . We skip the details.  $\square$

For the expressively complete acceptance conditions, one may ask about the hierarchy induced by the length  $m$  of the outer disjunction (as in the Rabin condition), respectively of the outer conjunction (as in the Streett condition). In the classical Rabin condition, the minimal  $m$  which a recognizing automaton can have for a given  $\omega$ -language  $L$  is called the *Rabin index* of  $L$ . By Wagner's work [Wag79], the Rabin index of a regular  $\omega$ -language (specified say by a Büchi automaton) is computable. Is the analogous index for Rabin-singletons conditions computable?

## 4 Singletons in Occurrence Acceptance Conditions

In this section we show that the main theorem of the previous section cannot be proved in the context of occurrence acceptance.

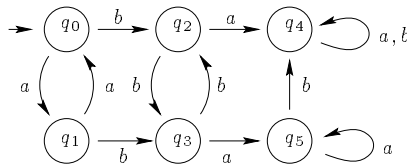


Figure 2: Automaton  $\mathcal{A}_0$

Let us first consider an instructive example which seems to indicate at first sight that occurrence acceptance with Rabin-singletons is insufficient: The  $\omega$ -language

$$L_0 = a^\omega + (aa)^+(bb)^+ a^\omega + (aa)^* a (bb)^* ba^\omega$$

is recognized by the automaton with transition structure as in Figure 2 with the following occurrence acceptance condition: Either none of  $q_2$  and  $q_3$  is

seen but  $q_0$  is (which means to accept  $a^\omega$ ), or  $q_4$  is not visited but  $q_5$  is (which means to accept an  $\omega$ -word in the latter two summands of the expression above). It seems obvious that the two states  $q_0, q_1$  are necessary for counting modulo 2, and that both exits to  $q_2, q_3$  are necessary because the result of this counting has to be remembered. So for acceptance of  $a^\omega$  it seems necessary to forbid the visit of *both*  $q_2, q_3$  (which means that acceptance with just single negative literals is insufficient). However, a closer analysis shows that a more complicated structure of the automaton allows to circumvent the problem and to accept with Rabin-singletons. The reader may verify that the automaton  $\mathcal{A}'_0$  using the transition structure given in Figure 3 with the Rabin pairs  $(q_4, q_1)$  and  $(q_3, q_8)$  recognizes the  $\omega$ -language  $L_0$ .

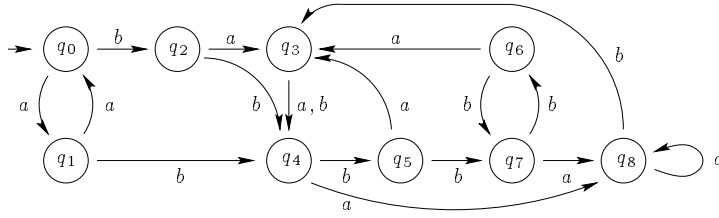


Figure 3: Automaton  $\mathcal{A}'_0$

Let us now turn to a better example language which cannot be recognized by occurrence conditions with Rabin-singletons:

**Theorem 6** *The  $\omega$ -language  $L_1 \subseteq \{1, a, b, c\}^\omega$ , defined by the regular expression  $1^*(a^* + b^* + c^*)1^\omega$ , is Staiger-Wagner recognizable, but not recognizable by occurrence acceptance with Rabin-singletons.*

*Proof.* The  $\omega$ -language  $L_1$  is recognized by the automaton  $\mathcal{A}_1$  depicted in Figure 4 with the following acceptance condition of Rabin type: “ $q_0$  but none of the states  $q_a, q_b, q_c$  is reached” (for acceptance of  $1^\omega$ ) “or  $q_1$  but not  $q_2$  is reached” (for acceptance of  $\omega$ - words in  $1^*(a^+ + b^+ + c^+)1^\omega$ ). Our aim is to show that the use of non-singletons such as  $\{q_a, q_b, q_c\}$  is unavoidable.

We proceed by contradiction and assume that the automaton  $\mathcal{A} = (Q, A, q_0, \delta, \Omega)$  recognizes  $L_1$  by occurrence acceptance with Rabin-singletons, say with  $\Omega = ((p_1, q_1), \dots, (p_m, q_m))$ . Let us further assume that  $|Q| = n$ .  $\mathcal{A}$  accepts  $1^\omega$ , say by the pair  $(p_i, q_i)$  from the list  $\Omega$ . We shall write  $p$  for  $p_i$  and  $q$  for  $q_i$ . The run on  $1^\omega$  thus does not visit  $p$  but reaches  $q$  somewhere. Pick  $k$  minimal such that  $\delta(q_0, 1^k) = q$ . Clearly the prefixes of  $1^k$  do not lead the automaton into  $p$ . If we continue  $1^k$  by  $a^\omega$ ,  $b^\omega$ , or  $c^\omega$ , we obtain  $\omega$ -words rejected by  $\mathcal{A}$ ; in all three cases somewhere the state  $p$  must be reached

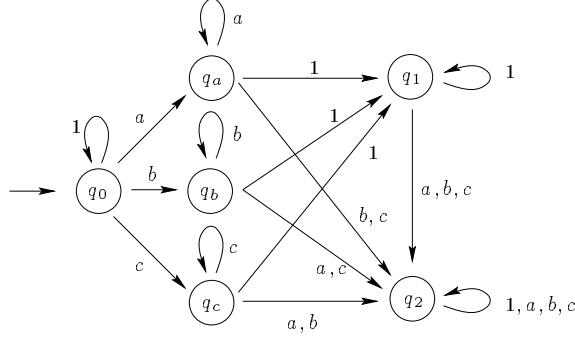


Figure 4: Automaton  $\mathcal{A}_1$

(because otherwise acceptance would hold by the pair  $(p, q)$ ). Choose  $m_a, m_b, m_c$  minimal such that  $\mathcal{A}$  reaches state  $p$  by the input  $1^k a^{m_a}, 1^k b^{m_b}, 1^k c^{m_c}$ , respectively. For later use let us note the following:

(\*) All the states reached via  $1^k, 1^k a$ , etc. up to  $1^k a^{m_a}$  are distinct.

Namely,  $q$  and  $p$  are distinct,  $p$  does not occur before  $1^k a^{m_a}$ , and a state repetition before  $1^k a^{m_a}$  would define a loop which prohibits the visit to  $p$ . Similarly we argue with the letters  $b$  and  $c$ .

Now consider the  $\omega$ -words  $1^k a^{m_a+n} 1^\omega, 1^k b^{m_b+n} 1^\omega$ , and  $1^k c^{m_c+n} 1^\omega$ , all accepted by  $\mathcal{A}$ , say with the accepting pairs  $(p_a, q_a), (p_b, q_b), (p_c, q_c)$ , respectively. The state  $q_a$  has to be reached at a later point than within the prefix  $1^k a^{m_a}$ . Otherwise, by avoidance of  $p_a$  up to the word  $1^k a^{m_a+n}$  and by the state repetition which must occur within the  $a$ -part (recall that  $n = |Q|$ ), the  $\omega$ -word  $1^k a^\omega$  would be accepted, contradicting the recognition of  $L_1$  by  $\mathcal{A}$ . Analogously, we see that the states  $q_b$  and  $q_c$  are reached on the considered  $\omega$ -words at a later point than within  $1^k b^{m_b}, 1^k c^{m_c}$ , respectively.

We shall conclude that between the prefix  $1^k$ , where state  $q$  is reached, and  $1^k a^{m_a}$ , where state  $p$  is reached, *both  $p_b$  and  $p_c$  are visited*.

We show this for  $p_b$  first. Consider the extension of  $1^k a^{m_a}$  to the  $\omega$ -word  $1^k a^{m_a} b^n 1^\omega$ : Here the run on the suffix  $b^n 1^\omega$  starts with  $p$  and thus is the same as on the same suffix in  $1^k b^{m_b} b^n 1^\omega$ . On this suffix,  $q_b$  is visited (not earlier, as shown above!), and  $p_b$  is not visited (by acceptance of the whole  $\omega$ -word with the pair  $(p_b, q_b)$ ). So on the (rejected) word  $1^k a^{m_a} b^n 1^\omega$ , where  $q_b$  is visited somewhere later than  $1^k a^{m_a}$ ,  $p_b$  has to be visited (otherwise the  $\omega$ -word would be accepted). The visit to  $p_b$  is necessarily after the prefix  $1^k$  (because  $p_b$  is not visited at all on  $1^k b^{m_b+n} 1^\omega$ ) but within the prefix  $1^k a^{m_a}$  (because we excluded above that on  $1^k a^{m_a+n} b^n 1^\omega$  the visit to  $p_b$  occurs in the suffix  $b^n 1^\omega$ ).

In a similar way, one shows that  $p_c$  is visited on  $1^k a^{m_a}$  after the prefix  $1^k$ .

By symmetry, one obtains that on  $1^k b^{m_b}$ , the states  $p_a$  and  $p_c$  are visited somewhere after the prefix  $1^k$ . Without loss of generality, suppose that on the  $a$ -part of  $1^k a^{m_a}$  the  $p_b$ -visit occurs before the  $p_c$ -visit, and that on the  $b$ -part of  $1^k b^{m_b}$ , the  $p_a$ -visit is before the  $p_c$ -visit. Then paths exist in  $\mathcal{A}$  as indicated in Figure 5.

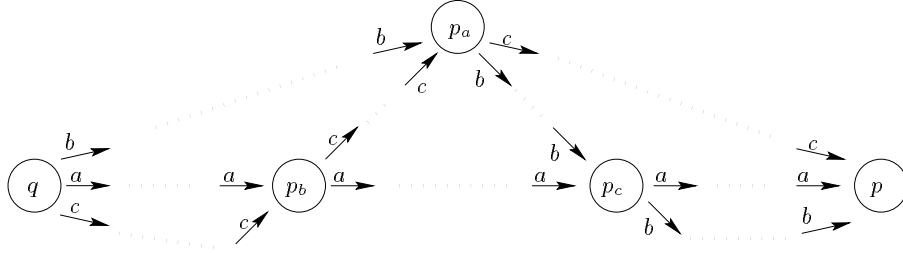


Figure 5: Paths in automaton  $\mathcal{A}$

Note that all three states  $p_a, p_b, p_c$  are distinct and not equal to either  $p$  or  $q$ : Otherwise some of  $p_a, p_b, p_c$  would be on all three paths from  $1^k$  to  $1^k a^{m_a}, 1^k b^{m_b}, 1^k c^{m_c}$ . If that happened, say with  $p_a$ , this would contradict that  $p_a$  is not reached on  $1^k a^{m_a+n} 1^\omega$ , as assumed.

In view of this situation (as in the figure), we can define a path from  $q$ , as reached via  $1^k$ , to the state  $p_c$  via some  $a^r$ , and continue to  $p$  via some  $b^s$ . On this path,  $p_a$  is not visited: not on the first part because  $p_a$  is not visited on  $1^k a^{m_a+n} 1^\omega$ , not on the second part because this would lead to a repetition of  $p_a$  on the  $b^{m_b}$ -path between  $q$  and  $p$ , which was excluded above in (\*).

Hence the  $\omega$ -word  $1^k a^r b^s a^n 1^\omega$  is accepted with the pair  $(p_a, q_a)$  (which yields the desired contradiction): As just verified,  $p_a$  is not visited within the prefix  $1^k a^r b^s$ . It is also not visited on the suffix  $a^n 1^\omega$ , because the run on this suffix, starting in  $p$ , coincides with the run on the corresponding suffix of  $1^k a^{m_a+n} 1^\omega$ , where  $p_a$  is not visited at all. On the other hand,  $q_a$  is visited, by the same coincidence of runs on the suffixes  $a^n 1^\omega$  and because on  $1^k a^{m_a+n} 1^\omega$  the state  $q_a$  is reached somewhere after the prefix  $1^k a^{m_a}$ .

□

We leave open here whether two sharpened versions of the theorem above are true: The first is concerned with the case of two-letter alphabets; we do not have a proof for the theorem over an alphabet of less than four letters. Secondly, it is not known whether in occurrence acceptance conditions with Rabin-singletons, the sizes of the (negative) sets have to be unbounded. We have shown that size 1, i.e., singleton acceptance, does not suffice, but already



in our example language it is conceivable that negative state sets with two states are as well insufficient for acceptance.

## 5 Conclusion

In this paper we considered  $\omega$ -automaton acceptance conditions of Rabin and Streett type where the designated state sets are singletons. In particular, it was clarified in which situations the avoidance of a set of states can be captured by the avoidance of an individual state. Such a reduction turned out possible for the usual Rabin and Streett conditions but not for the corresponding occurrence acceptance conditions. Some questions arising from this investigation were raised, regarding hierarchies of acceptance conditions which involve singletons.

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