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Complete abstractions through extensions of disjunctive modal transition systems

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Abstract. Modal transition systems and their many variants are established models for abstraction and specification as they explicitly specify necessary and possible state or behavior. Disjunctive modal transition systems can be more precise abstractions as they allow a disjunctive split on necessary transitions. We show that these abstractions are compact, sound, and complete for — and as expressive as — the modal mu-calculus if models are enriched with fairness constraints and conjunctive splitting abilities for possible transitions. We point out the potential benefits of our approach over other complete abstraction frameworks.

1 Introduction

Transition systems are often employed as operational models of programs, protocols, specification standards, and other dynamical systems such as metabolic networks in a cell. The branching and nondeterministic nature of transition systems is ideally captured through properties expressible in the modal mu-calculus, which has equivalent formulations via parity games or tree automaton. The ability of this calculus to nest path quantifiers and least and greatest fixed points makes it an ideal property language in the observed merging of testing, model checking, and simulation activities within the context of system validation. All properties stated in this paper are expressible in this logic.

Example 1. Consider the transition system T_1 depicted in Figure 1. It describes an infinite reactive system that can generate any value in $\{1, 2, 3, \ldots\}$ and then engage in as many send activities as specified by that value. Thereafter, the same behavior starts anew. Predicate

p_r states that the system is ready to generate a value, and predicate p_o (p_e) states that an odd (even) number of sends remain to be completed (respectively).

The complexity and size of transition systems suggests to seek a formalism in which one can reason about abstractions of such systems directly. Figure 1 also shows a transition system T_2 that abstracts T_1 in that all paths present in T_1 have matching paths in T_2. (Abstraction and refinement will be defined formally in the paper.) Such *safe* simulations enable us to conclude that universal path properties that hold in T_2 also hold in T_1, e.g. "On all paths, p_r holds only if both p_o and p_e don't hold." The property "After any *gen* event, we reach a state from which we can reach, with one or two *send* events, a state satisfying p_r," which is not a universal path property, holds in T_2 but not in T_1. So transferring positive property checks from abstract to concrete models only works for universal path properties for "safe simulations" [6] such as the one shown in Figure 1.

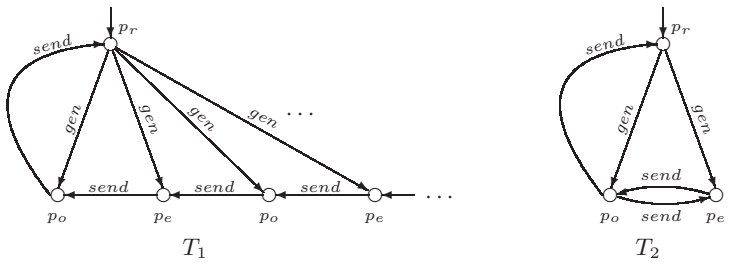


Fig. 1. Two transition systems with labels *gen* and *send* and propositions p_e and p_o, where T_2 is a safe simulation of T_1: all paths in T_1 have abstract counterparts in T_2

Modal transition systems [9] and their slight extensions, *mixed transition systems* [2], are abstract models with two kinds of behavior: *may*-transitions, which may be but don't have to be present in a refining transition system; and *must*-transitions, which must be preserved in all refining transition systems. Figure 2 shows a modal transition system M that abstracts T_1 from Figure 1. Now all paths of T_1 have matching *may*-transition paths (dashed arrows in Figure 2) in M, and all *must*-transition paths (solid arrows in Figure 2)

in M have matching paths in T_1. This split into *may* and *must* and its sense of direction enable the sound transfer of positive property checks from M to T_1 for all properties expressible in the modal mu-calculus [9, 6], as illustrated in that figure.

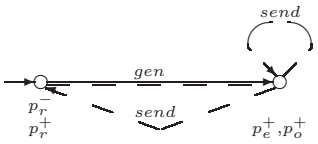


Fig. 2. A modal transition system M, obtained by predicate abstraction from the transition system T_1 in Figure 1 with sole predicate p_r. At all states, we use p^- (p^+) to denote that p *must* (resp., *may*) be true at s and solid (dashed) arrows denote *must* (resp., *may*) transitions. Model M satisfies "there is a *gen* event after which all *send* events reach states satisfying $\neg p_r \vee \neg p_o$" and so this property also holds for T_1

Disjunctive modal transition systems [10] generalize modal and mixed transition systems so that *must*-transitions (s, α, D) can have a *set* of states D as a target, specifying that a refining transition system L must refine at least one state of D, say to t', and refine s to some t such that (t, α, t') is a transition in L. This disjunctive ability can render abstractions that are more precise than those constructed as modal transition systems. For example, the disjunctive modal transition system of Figure 3, an abstraction of T_1, satisfies "after every *gen* event there are either infinitely many *send* events or p_r holds after finitely many *send* events". But one can show that no finite modal transition system that abstracts T_1 satisfies this property.

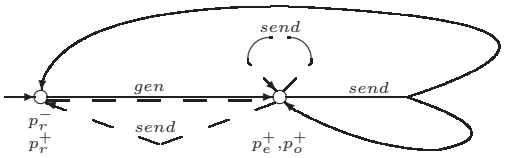


Fig. 3. A disjunctive modal transition system abstracting the transition system T_1 in Figure 1. Branching solid transitions model *must*-transitions (s, α, D)

As seen above, disjunctive modal transition systems are more expressive abstractions than modal transition systems. Unfortunately, disjunctive modal transition systems don't allow the expression of fairness constraints with which one could describe unbounded reachability with only finitely many states, an essential requirement for automatic verification. For example, one can show that there exists no finite disjunctive modal transition systems that abstracts T_1 and that satisfies the property ψ_f, recursively stating "p_r holds and after every *gen* event there are finitely many *send* events until ψ_f holds again."

Dams & Namjoshi proposed the first framework in the literature whose models yield sound and complete abstractions with respect to the modal mu-calculus, the *focussed transition systems* in [3]. Soundness means that a satisfaction relation $M \models A$ between models M and properties (encoded as tree automaton A) is closed under refinement of models: $M \models A$ and M' refines M implies $M' \models A$, where abstraction is the relational inverse of refinement. Completeness means that the truth of any satisfaction instance $M \models A$ has a finite abstraction N of M such that $N \models A$. As seen above, disjunctive modal transition systems are incomplete. Dams & Namjoshi also considered μ-automaton [7], and demonstrated in [4] how they yield sound and complete models for abstraction with respect to the modal mu-calculus. That paper also defined modal automaton as 3-valued versions of μ-automaton. Both papers [3, 4] proposed to approximate the EXPTIME-hard language inclusion problem "Is every tree that refines M accepted by A?" within NP with the outcome of a refinement game $M \sqsubseteq A$ endowed with a Rabin acceptance condition for finite M.

The research programme put forward and developed in [3, 4] is attractive in its simplicity and blend of well established techniques in operational semantics and automaton theory. With more than one sound and complete abstraction framework at hand, the question then emerges as to what *additional* properties such frameworks should enjoy. We list some desired properties that motivated the work reported here: efficient and transparent translations from the modal mu-calculus to automaton and models alike; the ability to cast desired abstractions as used in tools, e.g. (disjunctive) modal transition systems, directly as models into the complete framework;

and the preservation of "equational theories" in the approximating refinement games, e.g. that $M \models A_{\phi \wedge \psi}$ ought to equal ($M \models A_\phi$ & $M \models A_\psi$) since $M \models A$ reasons about *all* refinements of M.

Contributions of this paper. In this paper, we therefore

- enrich disjunctive modal transition systems with two ingredients: fairness constraints, similar to those used in alternating tree automaton [11]; and *may*-transitions of the form (s, α, C) where C is interpreted *conjunctively*,
- define abstraction and refinement on these fair disjunctive-conjunctive modal transition systems through games,
- define a satisfaction relation between such models and alternating tree automaton, which (under)approximates the underlying EXPTIME-hard language inclusion problems in NP for finite models,
- show that this satisfaction relation is closed under refinement, i.e. sound,
- prove that finite fair disjunctive-conjunctive modal transition systems have the same expressiveness as the modal mu-calculus, yielding a sound and complete abstraction framework for that temporal logic, and
- discuss the desirable properties that our sound and complete abstraction framework enjoys.

Outline of paper. Fair disjunctive-conjunctive modal transition systems and their refinement games are introduced in Section 2 and a satisfaction relation between such systems and alternating tree automaton is given and proved to be sound in Section 3. The fair disjunctive-conjunctive modal transition system corresponding to an alternating tree automaton is constructed in Section 4 and the conversion of a *finite* fair disjunctive-conjunctive modal transition system into an alternating tree automaton is given in Section 5, rendering completeness and equal expressiveness with the modal mu-calculus. Finally, we discuss further properties of our approach and its connections to related work in Section 6 and conclude in Section 7.

2 Fair disjunctive-conjunctive modal transition systems

We first define fair disjunctive-conjunctive modal transition systems. Throughout, $|S|$ denotes the cardinality of a set S and $\mathbb{P}(S)$ denotes its power set.

Definition 2. *A* fair disjunctive-conjunctive modal transition system *over a set of propositions,* AP*, and transition labels, $\mathcal{L}$, is a tuple $(S, S^{\mathrm{i}}, R^-, R^+, L^-, L^+, \eta)$ such that*

- *$(s \in) S$ is a set of states,*
- *$S^{\mathrm{i}} \subseteq S$ a set of initial states,*
- *$R^- \subseteq S \times \mathcal{L} \times \mathbb{P}(S)$ the set of* must*-transitions,*
- *$R^+ \subseteq S \times \mathcal{L} \times \mathbb{P}(S)$ the set of* may*-transitions,*
- *$L^-, L^+ : S \to \mathbb{P}(\mathrm{AP})$ the* must- *and* may*-labelings of atomic propositions,*
- *$\eta \colon S \to \mathbb{N}$ an acceptance condition with finite image*

Furthermore, such a system is finite *if $|S| + |R^-| + |R^+| + |\mathrm{AP}|$ is finite. We often refer to fair disjunctive-conjunctive modal transition systems as 'models'.*

The interpretation of the labelings L^- and L^+ is standard [1]: $L^-(s)$ lists those atomic propositions that must hold in any refining states of s whereas $L^+(s)$ lists those propositions that may hold in some refinement of s. The set of initial states S^{i} represents states that must have refining initial states. The acceptance condition is used to model fairness, as made precise later.

A *must*-transition $(s, \alpha, D) \in R^-$ specifies that all refining states t of s in a transition system L must have a transition (t, α, t') in L such that t' refines some state in D. Dually, a *may*-transition $(s, \alpha, C) \in R^+$ specifies that all refining states t of s in a transition system L may have transitions in L of form (t, α, t') such that t' refines all states in C.

Example 3. May-transitions can enable further state-space reductions, as illustrated in Figure 4 where $p_0, \ldots, p_n, p$ are $n + 2$ atomic propositions. Model N_1 has $n + 4$ states and abstracts T_3, which has $3 \cdot (n + 2)$ states. Furthermore, N_1 satisfies property ψ_m stating "(i)

all successor states of the initial state have, for at least one i, a path on which p_i holds globally, (ii) all successor states of the initial state satisfy p and (iii) have in turn two successor states which have a path on which p ($\neg p$) holds globally (respectively).[3] A disjunctive modal transition system that abstract T_3 and satisfy formula ψ_m has a least $2n$ states, since n states are needed for satisfying (i) and abstracting T_3, p must hold at those n states in order to satisfy (ii), and so additional n states are needed in order to satisfy (iii) and abstract T_3.

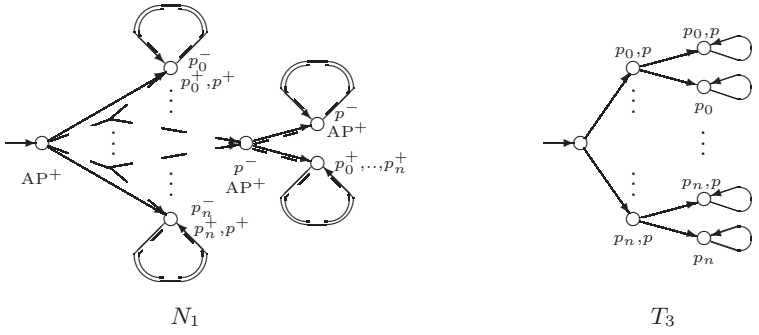


Fig. 4. An illustration of the fact that *may*-transitions of the form (s, α, C) can further reduce state spaces through abstraction: N_1 abstracts T_1 with about one third of the latter's state space and no disjunctive modal transition system can achieve this. We write AP^+ to indicate p^+ for all $p \in \mathrm{AP}$

Since our models are sought to be as expressive as the modal mu-calculus, it is reassuring that abstract states s may be inconsistent in that no refining state in transition systems exists, for example, when $(s, \alpha, \{\}) \in R^-$.

It is apparent that transition systems, modal and mixed transition systems, and disjunctive modal transition systems all have natural representations as fair disjunctive-conjunctive modal transition systems. For example, if $R^- = R^+$, $P^- = P^+$, $\eta\colon S \to \mathbb{N}$ has image $\{0\}$, and if $(s, \alpha, D) \in R^-$ implies that D is a singleton, we have a representation of a transition system. Subsequently, the components of a fair disjunctive-conjunctive modal transition system M

[3] In CTL* this can be written as $\mathbf{AX}(\bigvee_i \mathbf{EG}(p_i)) \wedge \mathbf{AX}(p \wedge \mathbf{EX}\,\mathbf{EG}(p) \wedge \mathbf{EX}\,\mathbf{EG}(\neg p))$.

are denoted by S, S^{i}, R^-, R^+, L^-, L^+, and η, tagged with indices if needed.

Abstract models represent sets of concrete models, i.e. transition systems. We provide meaning for this interpretation through a refinement notion, which we define using the basic machinery of parity games, see e.g. [12] for a reference to the notions and basic theory of parity games. We set

$$\mathrm{succ}^-_\alpha(s) = \{D \mid (s, \alpha, D) \in R^-\} \qquad \mathrm{succ}^+_\alpha(s) = \{C \mid (s, \alpha, C) \in R^+\}\,.$$

Below we write π_i for the projection into the ith component of an ordered pair, $\circ$ for function composition, and represent infinite sequences over a set X as maps from $\mathbb{N}$ to X. Also, when X is $\mathbb{N}$ and $\boldsymbol{n}$ such a sequence with finite image we write

$$\sup(\boldsymbol{n}) = \sup\{m \in \mathbb{N} \mid \forall i\, \exists j \geq i : \boldsymbol{n}(j) = m\}\,.$$

Definition 4. *1. Finite refinement plays for models M_1 and M_2 have the rules and winning conditions as stated in Table 1. An infinite play Φ is a win for Player I iff* [$\sup(\eta_1 \circ \pi_1 \circ \Phi)$ is even $\Rightarrow$ $\sup(\eta_2 \circ \pi_2 \circ \Phi)$ is even] *holds; otherwise it is won by Player II.*

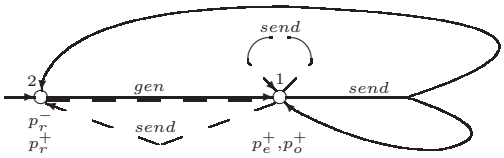
2. *The model M_1* refines (is abstracted by) *M_2 iff Player I has a strategy for the corresponding refinement game between M_1 and M_2 such that for any $s^{\mathrm{i}}_1 \in S^{\mathrm{i}}_1$ there is $s^{\mathrm{i}}_2 \in S^{\mathrm{i}}_2$ such that Player I always wins with his or her strategy in a refinement play started at $(s^{\mathrm{i}}_1, s^{\mathrm{i}}_2)$.*
3. *Let $\mathcal{T}(M)$ be the set*[4] *of transition systems that refine M.*

Example 5. Figure 5 depicts a model that abstracts the model T_1 of Figure 1 and satisfies ψ_f, recursively stating "p_r holds and after every *gen* event there are finitely many *send* events until ψ_f holds again."[5] This abstraction is witnessed by any strategy of Player I that always yields configurations consisting of tuples of corresponding initial states and tuples of non-initial states.

[4] Strictly speaking, this is a class but a skeletal set may be chosen.
[5] Described by the modal mu-calculus formula $\nu Y.(p_r \wedge [gen](\mu X.Y \vee \langle send\rangle X)))$.

L^- labeling: Player II chooses p from $L^-(s_2)$; Player I wins iff p is in $L^-(s_1)$
L^+ labeling: Player II chooses p from $\mathrm{AP} \setminus L^+(s_2)$; Player I wins iff p is not in $L^+(s_1)$
R^- transition: Player II chooses a set of states $D_2' \in \mathrm{succ}_\alpha^-(s_2)$; Player II wins if $\mathrm{succ}_\alpha^-(s_1) = \{\}$, otherwise Player I responds with $D_1' \in \mathrm{succ}_\alpha^-(s_1)$; Player I wins if $D_1' = \{\}$, otherwise Player II chooses $s_1' \in D_1'$; Player II wins if $D_2' = \{\}$, otherwise Player I responds with $s_2' \in D_2'$; the next configuration is (s_1', s_2')
R^+ transition: Player II chooses a set of states $C_1' \in \mathrm{succ}_\alpha^+(s_1)$; Player II wins if $\mathrm{succ}_\alpha^+(s_2) = \{\}$, otherwise Player I responds with $C_2' \in \mathrm{succ}_\alpha^+(s_2)$; Player I wins if $C_2' = \{\}$, otherwise Player II chooses $s_2' \in C_2'$; Player II wins if $C_1' = \{\}$, otherwise Player I responds with $s_1' \in C_1'$; the next configuration is (s_1', s_2').

Table 1. List of possible moves in the refinement game at configuration (s_1, s_2)



N_2

Fig. 5. A fair disjunctive-conjunctive modal transition system that abstracts T_1 of Figure 1 and satisfies ψ_f as defined in Example 5. In figures, we write the parity values $\eta(s)$ next to states s

Remark 6. The winning conditions above are Rabin conditions so history independent strategies for Player I suffice. Consequently, deciding refinement is in NP for finite models. For the parity game illustrated in Figure 6, Player II has a winning strategy by choosing the transitions t_1 and t_2 alternately, but Player II does not have a history independent winning strategy.

The abstraction of an abstracted transition systems yields again an abstraction of this transition system. We state this dually via refinement:

Proposition 7 (Transitivity). *Suppose M_1, M_2, and M_3 are models such that M_1 refines M_2 and M_2 refines M_3. Then M_1 refines M_3.*

3 Sound satisfaction relation

In this section we define a satisfaction relation between our models and tree automaton using the formulation of tree automaton in [11],

Fig. 6. Player II has only a history dependent winning strategy for showing that the model N_3 does not refine N_4. We use $\{\}$ to indicate the "target" of a transition $(s, \alpha, \{\})$.

which allows linear-time translations between tree automaton and modal mu-calculus formulas.

Definition 8. *An* alternating tree automaton *is a tuple* $(\mathcal{Q}, \delta, \Theta)$, *where*

- $(q \in)\mathcal{Q}$ *is a finite, nonempty set of states*
- δ *is a transition relation, which maps an automaton state to one of the following forms, where* q, q_1, q_2 *are automaton states,* $p \in \mathrm{AP}$, *and* $\alpha \in \mathcal{L}$: $p \mid \neg p \mid q \mid q_1 \tilde{\wedge} q_2 \mid q_1 \tilde{\vee} q_2 \mid \mathbf{EX}^\alpha q \mid \mathbf{AX}^\alpha q$
- $\Theta\colon \mathcal{Q} \to \mathbb{N}$ *is an acceptance condition with finite image.*

Subsequently, the components of an alternating tree automaton A are denoted by $\mathcal{Q}$, δ, and Θ, tagged with indices if needed. Furthermore, m_Θ denotes the maximal value in the image of Θ and K_A denotes the set of those states that point to states directly, via $\tilde{\wedge}$, or via $\tilde{\vee}$:

$$m_\Theta = \max\{\Theta(q) \mid q \in \mathcal{Q}\}$$
$$K_A = \{q \in \mathcal{Q} \mid \exists q_1, q_2 \in \mathcal{Q} : \delta(q) \in \{q_1, q_1 \tilde{\wedge} q_2, q_1 \tilde{\vee} q_2\}\}.$$

Definition 9. *1. Finite satisfaction plays for model* M *and alternating tree automaton* A *have the rules and winning conditions as stated in Table 2. An infinite play* C *is a win for Player I iff* $[\sup(\eta \circ \pi_1 \circ C)$ is even or $\exists i \forall j \geq i : \pi_2(C(j)) \in K_A] \Rightarrow \sup(\Theta \circ \pi_2 \circ C)$ is even*; otherwise it is won by Player II.*

2. The model M satisfies *the automaton* A *in state* $q \in \mathcal{Q}$, *written* $M \models (A, q)$ *iff Player I has a strategy for the corresponding satisfaction game between* M *and* A *such that for any* $s^{\mathrm{i}} \in S^{\mathrm{i}}$,

Player I wins all satisfaction plays started at (s^{i}, q) *with his or her strategy.*

3. *Let* $\mathcal{T}(A, q)$ *be the set of transition systems that satisfies* A *in* q*.*

p:	Player I wins iff $p \in L^-(s)$
$\neg p$:	Player I wins iff $p \notin L^+(s)$
q':	the next configuration is (s, q')
$q_1 \tilde{\wedge} q_2$:	Player II picks a q' from $\{q_1, q_2\}$; the next configuration is (s, q')
$q_1 \tilde{\vee} q_2$:	Player I picks a q' from $\{q_1, q_2\}$; the next configuration is (s, q')
$\mathbf{EX}^\alpha q'$:	Player II wins if $\mathrm{succ}^-_\alpha(s) = \{\}$, otherwise Player I picks $D' \in \mathrm{succ}^-_\alpha(s)$; Player I wins if $D' = \{\}$, otherwise Player II picks $s' \in D'$; the next configuration is (s', q')
$\mathbf{AX}^\alpha q'$:	Player I wins if $\mathrm{succ}^+_\alpha(s) = \{\}$, otherwise Player II picks $C' \in \mathrm{succ}^+_\alpha(s)$; Player II wins if $C' = \{\}$, otherwise Player I picks $s' \in C'$; the next configuration is (s', q').

Table 2. Rules for the satisfaction game $M \models A$ at current configuration (s, q)

As usual, such a satisfaction relation is inherently 3-valued since any instance $M \models (A, q)$ attempts to establish whether all refinements of M satisfy (A, q) and since some refinements of M may satisfy (A, q), some the "negation" of (A, q).

For the same reason as put forward in Remark 6, deciding $M \models (A, q)$ is in NP for guarded formulas, i.e., $\exists i \, \forall j \geq i : \pi_2(C(j)) \in K_A$ cannot hold. Note that every automaton has an equivalent guarded one [8]. Next we show that $M \models (A, q)$ soundly approximates the EXPTIME-hard relation which asks whether all transition systems T that refine M satisfy (A, q), i.e. whether $\mathcal{T}(M) \subseteq \mathcal{T}(A, q)$, where our $L \models (A, q)$ corresponds to the usual satisfaction relation between transition systems and alternating tree automaton [11]. That $M \models (A, q)$ only (under)approximates the latter language inclusion rests on a weak interpretation of disjunctions, e.g. every transition system refining the model N_2 in Figure 5 satisfies "after any *gen* event $p_e \vee \neg p_e$ holds." But if (A, q) is the automaton corresponding to that property, we have neither $N_2 \models (A, q)$ nor $N_2 \models (\text{"not"} A, q)$.

Theorem 10 (Soundness). *Let* M_1 *refine* M_2 *and let* A *be an alternating tree automaton with state* $q \in \mathcal{Q}$*. Then* $M_2 \models (A, q)$ *implies* $M_1 \models (A, q)$*.*

4 Completeness of satisfaction relation

In this section we describe how a *finite* model $M_{(A,q)}$ can be obtained from an alternating tree automaton (A, q) such that both characterize the same sets of transition systems via refinement and satisfaction, respectively:

$$\exists \text{finite } M_{(A,q)}\, \forall M\colon \quad M \models (A,q) \iff M \text{ refines } M_{(A,q)}\,. \tag{1}$$

The idea for the construction of $M_{(A,q)}$ is to take the automaton states q paired with any resolution c_A (a choice function) of disjunctions in A as states of $M_{(A,q)}$, more precisely, the set of states determined by the automaton states reachable from q, where disjunctions are resolved as specified in c_A. The predicate labelings and *may*- and *must*-transitions are determined by the labels of these collected states. For example, if such a collected state q' is labeled $\mathbf{EX}^\alpha q''$, we generate a *must*-transition, where the acceptance condition of the target is determined by the maximal acceptance condition of the states encountered in reaching q'.

We begin the formalization of these intuitions with defining the set of *or*-states O_A of A and its set T_A of states that are targets of **EX**- or **AX**-states:

$$\begin{aligned} O_A &= \{q \in \mathcal{Q} \mid \exists q_1, q_2 \in S : \delta(q) = q_1 \tilde{\vee} q_2\} \\ T_A &= \{q \in \mathcal{Q} \mid \exists q', \alpha\colon \delta(q') \in \{\mathbf{EX}^\alpha q, \mathbf{AX}^\alpha q\}\}\,. \end{aligned}$$

A *choice function* for A is a function $c_A\colon O_A \to \{1,2\}$. Let $\mathtt{Ch}_A$ be the set of all choice functions for A, let $\overline{m}$ be the set $\{n \in \mathbb{N} \mid n \le m\}$, and let $\mathcal{U} = \overline{m_\Theta} \times \mathcal{Q}$. We determine the set of reachable states, together with their maximal parity value encountered en route, for a given choice function through a least fixed-point equation. We define $\mathcal{T}_{c_A} : (\mathcal{U} \to \mathbb{P}(\mathcal{U})) \to (\mathcal{U} \to \mathbb{P}(\mathcal{U}))$ and $\mathbf{rea}_{c_A}\colon \mathcal{U} \to \mathbb{P}(\mathcal{U})$ by

$$\mathcal{T}_{c_A}(f)_{(n,q)} = \begin{cases} \{\} & \text{if } \exists p, \alpha, q' : \delta(q) \in \{p, \neg p, \mathbf{EX}^\alpha q', \mathbf{AX}^\alpha q'\} \\ \{(n', q')\} \cup f(n', q') & \\ \qquad \text{if } \delta(q) = q' \;\&\; n' = \max\{n, \Theta(q')\} & \\ \{(n_1, q_1), (n_2, q_2)\} \cup f(n_1, q_1) \cup f(n_2, q_2) & \\ \qquad \text{if } \delta(q) = q_1 \tilde{\wedge} q_2 \;\&\; n_i = \max\{n, \Theta(q_i)\} & \\ \{(n', q_{c_A(q)})\} \cup f(n', q_{c_A(q)}) & \\ \qquad \text{if } \delta(q) = q_1 \tilde{\vee} q_2 \;\&\; n' = \max\{n, \Theta(q_{c_A(q)})\} & \end{cases} \tag{2}$$

$$\mathbf{rea}_{c_A} = \bigsqcup_{i\in\mathbb{N}} \mathcal{T}^i_{c_A}(\bot) \tag{3}$$

where $\bot(n,q) = \{\}$ and $\bigsqcup_{i\in\mathbb{N}} f_i$ denotes the point-wise union of functions. It is straightforward to check that $\mathcal{T}_{c_A}$ is monotone with respect to point-wise inclusion. Subsequently, we may abbreviate $\mathbf{rea}_{c_A}(\Theta(q),q)$ by $\mathbf{rea}_{c_A}(q)$. The set of reachable states, including those that are trivially reachable, are defined by

$$\mathbf{BS}_{c_A}\colon \mathcal{Q} \to \mathbb{P}(\mathcal{U}), \qquad \mathbf{BS}_{c_A}(q) = \{(\Theta(q),q)\} \cup \mathbf{rea}_{c_A}(q)\,.$$

Unfortunately, not all (q, c_A) are appropriate to generate states for $M_{(A,q)}$, since they may be inconsistent if there exists an infinite path from q that follows c_A, has an odd maximal parity value, and remains in states from K_A forever. The following definition of $M_{(A,q)}$ takes this into account.

Definition 11. *1. A state $q \in \mathcal{Q}$ is* odd-circle free *with respect to $c_A \in \mathtt{Ch}_A$, denoted $\mathbf{ocf}_{c_A}(q)$, if there is no path over states from K_A from q to q whose maximal parity value is odd: $\mathbf{ocf}_{c_A}(q) \Leftrightarrow [\forall n\colon (n,q) \in \mathbf{rea}_{c_A}(q) \Rightarrow n$ is even$]$.*

2. A state $q \in \mathcal{Q}$ is odd-loop free *with respect to $c_A \in \mathtt{Ch}_A$, denoted $\mathbf{olf}_{c_A}(q)$, if q cannot reach a circle over states from K_A whose maximal parity value is odd: $\mathbf{olf}_{c_A}(q) \Leftrightarrow [\forall (n,q') \in \mathbf{BS}_{c_A}(q)\colon \mathbf{ocf}_{c_A}(q')]$.*

3. The model $M_{(A,q)} = (S_{A,q}, S^{\mathrm{i}}_{A,q}, R^-_{A,q}, R^+_{A,q}, L^-_{A,q}, L^+_{A,q}, \eta_{A,q})$ is defined by

$$\begin{aligned}
S_{A,q} &= \overline{m_\Theta} \times \{\mathbf{BS}_{c_A}(q') \mid c_A \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c_A}(q') \,\&\, q' \in T_A \cup \{q\}\}\\
S^{\mathrm{i}}_{A,q} &= \{(0, \mathbf{BS}_{c_A}(q)) \mid c_A \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c_A}(q)\}\\
R^-_{A,q} &= \{((n,U),\alpha,D) \mid \exists q'' \in \mathcal{Q}, (n',q') \in U\colon \delta(q') = \mathbf{EX}^\alpha q'' \,\&\\
&\qquad D = \{(n', \mathbf{BS}_{c_A}(q'')) \mid c_A \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c_A}(q'')\}\}\\
R^+_{A,q} &= \{((n,U),\alpha,C) \mid \exists f : \mathcal{Q} \to \mathtt{Ch}_A :\\
&\qquad (\forall q'' \in \mathcal{Q} : (\exists (n',q') \in U\colon \delta(q') = \mathbf{AX}^\alpha q'') \Rightarrow \mathbf{olf}_{f(q'')}(q'')) \,\&\\
&\qquad C = \{(n', \mathbf{BS}_{f(q'')}(q'')) \mid \exists q'\colon (n',q') \in U \,\&\, \delta(q') = \mathbf{AX}^\alpha q''\}\}\\
L^-_{A,q}(n,U) &= \{p \mid \exists (n',q') \in U\colon \delta(q') = p\}\\
L^+_{A,q}(n,U) &= \{p \mid \forall (n',q') \in U\colon \delta(q') \neq \neg p\}\\
\eta_{A,q}(n,U) &= n\,.
\end{aligned}$$

We provide some intuitions for the definition of *must*- and *may*-transitions in $M_{(A,q)}$. When $(n', q') \in U$ is 'labeled' with $\mathbf{EX}^{\alpha} q''$, all targets of the corresponding *must*-transition in $M_{(A,q)}$ get assigned the maximal value n'. Furthermore, D contains all sets determined by choice functions for which no inconsistent state is obtained. *May*-transitions are formed by taking a consistent choice function for every state 'in' U labeled with $\mathbf{AX}^{\alpha} q''$. Then C is determined by pairing, for all $(n', q') \in U$ 'labeled' with $\mathbf{AX}^{\alpha} q''$, the value n' with the set determined by the given choice function.

Example 12. The alternating tree automaton in Figure 7 corresponds to ψ_f as defined in Example 5. The model in Figure 7 abstracts the one in Figure 5.

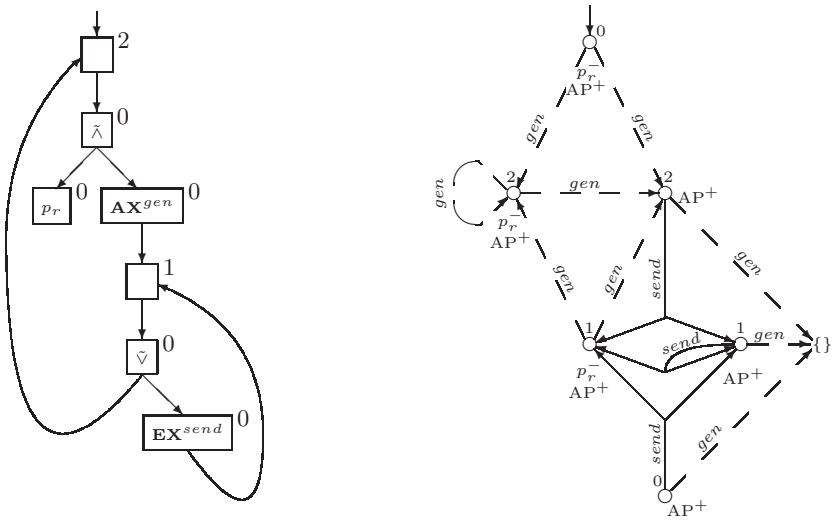


Fig. 7. Left: an alternation tree automaton A with initial state q, corresponding to the property ψ_f as defined in Example 5. Right: the disjunctive-conjunctive modal transition system $M_{(A,q)}$ for which we don't show that every state has an additional *may*-transition labeled with *send* to {}.

Theorem 13 (Completeness). *Let A be an alternating tree automaton with state q, $M_{(A,q)}$ the model constructed from (A, q) above, and ℓ the maximal number of states of $M_{A,q}$ that are reached from*

a state via a path containing only elements of K_A. Then (A, q) and $M_{(A,q)}$ are equivalent (that is, (1) holds), $|S_{A,q}| \leq (m_\Theta \cdot 2^\ell \cdot |T_A \cup \{q\}|)$, $|R^-_{A,q}| \leq |S_{A,q}| \cdot m_\Theta \cdot \ell$, and $|R^+_{A,q}| \leq |S_{A,q}| \cdot |\mathcal{L}| \cdot 2^{\ell^2}$.

5 Models as alternating tree automaton

We show that disjunctive-conjunctive modal transition systems have the same expressive power as alternating tree automaton, and are therefore exactly as expressive as the modal mu-calculus. Given $\mathcal{D} \subseteq \mathbb{P}(S)$, let $\mathrm{SF}_\mathcal{D} = \{f : \mathcal{D} \to S \mid \forall C \in \mathcal{D} : f(C) \in C\}$ be a set of selection functions. Appealing to the Axiom of Choice or letting S be finite, $\mathrm{SF}_\mathcal{D}$ is non-empty whenever $\{\} \notin \mathcal{D}$.

We now assume that $\mathcal{L}$, AP, and M are finite. Then we construct a corresponding alternating tree automaton A_M whose set of states is a superset of S. First we encode every state $s \in S$ via the formula

$$\begin{aligned} &\left(\textstyle\bigwedge_{p \in L^-(s)} p\right) \wedge \left(\textstyle\bigwedge_{p \in \mathrm{AP} \setminus L^+(s)} \neg p\right) \wedge \\ &\textstyle\bigwedge_{\alpha \in \mathcal{L}} \Big(\left(\textstyle\bigwedge_{D \in \mathrm{succ}^-_\alpha(s)} \mathbf{EX}^\alpha \left(\textstyle\bigvee_{s' \in D} s'\right)\right) \wedge \\ &\qquad \left(\textstyle\bigwedge_{f \in \mathrm{SF}_{\mathrm{succ}^+_\alpha(s)}} \mathbf{AX}^\alpha \left(\textstyle\bigvee_{C \in \mathrm{succ}^+_\alpha(s)} f(C)\right)\right)\Big). \end{aligned} \tag{4}$$

Second the overall alternating tree automaton A_M has a state set consisting of S, all sub-formulas of all instances of (4), `true` and `false`, and a state (together with its sub-formulas) representing the disjunction of the initial states in M. We write $\widehat{q}_M$ for that disjunction.

The transition relation δ_M of A_M maps each $s \in S$ to the formula in (4), and sub-formulas of (4) to sub-formulas via the corresponding top-level connective. If an empty conjunction (disjunction) occurs in a sub-formula of (4), δ_M maps to state `true` (`false`, respectively). States `true` and `false` are mapped to the predicate that is true at all (no) states (respectively). Finally, the acceptance condition of A_M maps $s \in S$ to $\eta(s)$ and all other states to 0. Note that the finiteness of $\mathcal{L}$, AP, and M is needed to ensure that the state set $\mathcal{Q}_M$ is finite. Figure 8 illustrates this construction.

Theorem 14. *The alternating tree automaton $(A_M, \widehat{q}_M)$ constructed from a model M as described above has $\mathcal{O}(|S| \cdot (|\mathrm{AP}| + |\mathcal{L}| \cdot (\zeta^- \cdot \chi^- + (\chi^+)^{\zeta^+} \cdot \zeta^+))$ many states, where $\zeta^* = \max_{s,\alpha} |\mathrm{succ}^*_\alpha(s)|$ and*

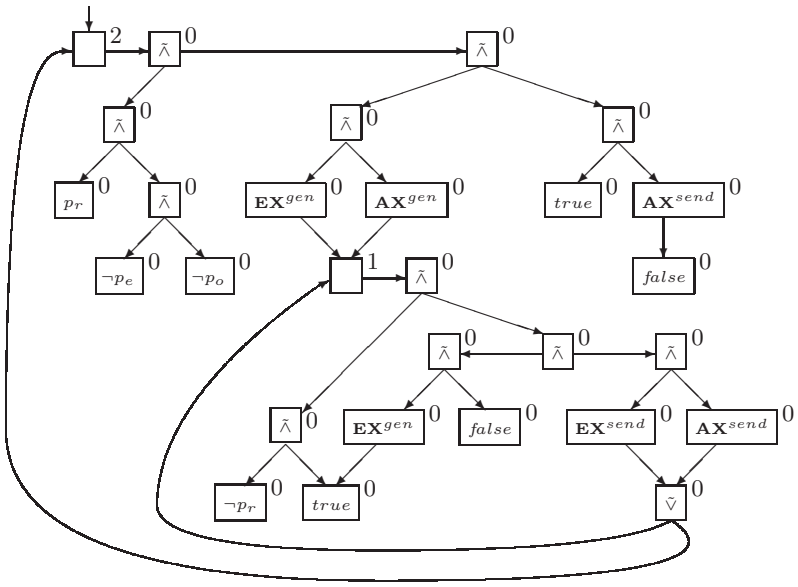


Fig. 8. The alternating tree automaton A_M constructed from the model M in Figure 5, where copies of *true* and *false* states are used

$\chi^* = \max_{s,\alpha}\{|D| \mid D \in \mathrm{succ}^*_\alpha(s)\}$ *with* $* \in \{-,+\}$. *Furthermore,* M *is equivalent to* $(A_M, \widehat{q}_M)$, *i.e.,*

$$\forall M' \colon\ M' \models (A_M, \widehat{q}_M) \iff M' \textit{ refines } M\,.$$

6 Discussion

Our framework allows trivial embeddings of transition systems and their modal and disjunctive modal generalizations into our notion of model. This is not the case for μ-automaton but is the case for the modal automaton in [4]. Our models have implicit "and" nodes and not just "or" nodes as for modal automaton; this should allow for more compact abstractions, as illustrated in Example 3. Our framework directly supports predicate abstraction since disjunctive modal transition systems are a special class of our models. We use a notion of automaton for which modal mu-calculus formulas have linear-time computable representations, no direct transformation of formulas into modal automaton is given in [4].

Our refinement and satisfaction games $M \sqsubseteq M'$ and $M \models (A, q)$ respectively, preserve important equational reasoning, e.g. "conjunction elimination" and "disjunction introduction." This is likely to be essential in identifying semantically self-minimizing patterns, as studied for modal transition systems in [5], for the models considered in this paper. We plan to classify such patterns of A for which, for *all* models M, $(M \models (A, q) \Leftrightarrow \mathcal{T}(M) \subseteq \mathcal{T}(A, q))$. Such reasoning does not appear to be possible for focussed transition systems.

Example 15. Focussed transition systems [3] $\mathcal{F}$ are tuples

$$(S, S^{\mathrm{i}}, R^{-}_{\mathcal{F}}, R^{+}_{\mathcal{F}}, L^{-}, L^{+}, F^{-}, F^{+}, \eta)$$

where $\mathcal{L}$ is a singleton, the transition relations $R^{-}_{\mathcal{F}}$ and $R^{+}_{\mathcal{F}}$ are subsets of $S \times S$, and the *focus* and *de-focus* relations F^{-} and F^{+} are subsets of $S \times \mathbb{P}(S)$. All other components, including AP, are as defined for our models. Consider the focussed transition system $\mathcal{F}_1$ depicted in Figure 9, where η is not shown as it is irrelevant for this example. Following the satisfaction definition of [3], we have $\mathcal{F}_1 \models p_1 \wedge p_2$ since Player I can choose $(s^{\mathrm{i}}, \{s_1, s_2\}) \in F^{+}$. Then Player II can choose any p_j with $j \in \{1, 2\}$. Now Player I will choose s_j to let the play continue at (s_j, p_j). But then this is won by Player I as $p_j \in L^{-}(s_j)$. Therefore $\mathcal{F}_1 \models p \wedge q$ holds. But neither $\mathcal{F}_1 \models p_1$ nor $\mathcal{F}_1 \models p_2$ hold: at (s^{i}, p_j), Player II wins since $p_j \notin L^{-}(s^{\mathrm{i}})$ for any $j \in \{1, 2\}$.

Dually, we have $\mathcal{F}_2 \models \mathbf{EX}p_1 \vee \mathbf{EX}p_1$ since Player I can choose $(s^{\mathrm{i}}, \{s'_1, s'_2\}) \in F^{-}$. Then Player II can choose s'_j for any $j \in \{1, 2\}$. Now Player I must choose $\mathbf{EX}p_1$ to let the play continue at $(s'_j, \mathbf{EX}p_1)$. But then this is won by Player I. Therefore $\mathcal{F}_2 \models \mathbf{EX}p_1 \vee \mathbf{EX}p_1$ holds. But $\mathcal{F}_2 \models \mathbf{EX}p_1$ does not hold: there is no outgoing $R^{-}_{\mathcal{F}}$ transition from the initial state.

Note that this problem for $\mathcal{F}_1$ can be dissolved by enforcing that p_1 and p_2 also hold at the initial state. Similarly, one could fix the problem for $\mathcal{F}_2$ by adding a transition from the initial state to s''_1 or to s''_2. However, the second of these dissolutions restricts the intended set of refining transition systems.

It is also of interest to note that translating our models into alternating tree automaton and back results in a state space increased by a factor of "only" m_{Θ}. We believe that the structural aspects of

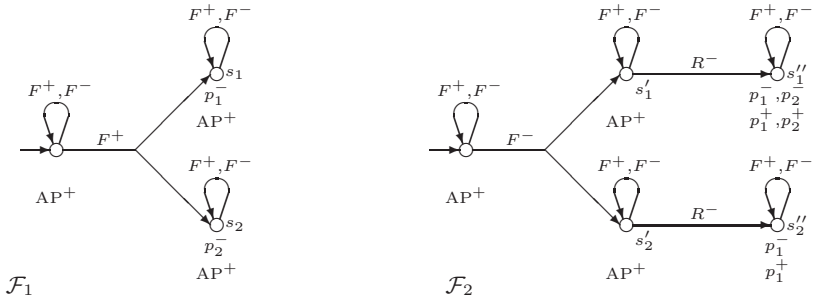


Fig. 9. Focussed transition systems $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_1 \models p_1 \wedge p_2$, $\mathcal{F}_1 \not\models p_1$, and $\mathcal{F}_2 \models \mathbf{EX}p_1 \vee \mathbf{EX}p_1$, $\mathcal{F}_2 \not\models \mathbf{EX}p_1$, illustrating that the relation $\models$ does not satisfy "conjunction elimination" or "disjunction introduction" for these models

our models will appeal to modelers and tool users as those aspects closely resemble the ones present in transition systems.

Finally, for an automaton A with 0 as only parity value[6] one can apply predicate abstraction to any transition system T to yield a finite disjunctive modal transition system M such that

$$T \text{ satisfies } A \text{ in } q \;\Leftrightarrow\; M \text{ satisfies } A \text{ in } q\,. \tag{5}$$

Future work will aim at extending such predicate abstraction techniques to automaton A with general parity acceptance conditions by constructing finite disjunctive-conjunctive modal transition systems M that satisfy (5).

7 Conclusion

We enriched disjunctive modal transition systems with two ingredients: fairness constraints and *may*-transitions of the form (s, α, C) where C is interpreted *conjunctively*. We defined abstraction and refinement on such models through games and defined a satisfaction relation between such models and alternating tree automaton, which approximates the underlying EXPTIME-hard language inclusion problems in NP. This satisfaction relation was shown to be sound, i.e. closed under refinement We proved that finite fair disjunctive-conjunctive modal transition systems have the same expressiveness as the modal mu-calculus, yielding completeness for that

[6] Such automaton represent modal mu-calculus formulas without least fixed-points.

abstraction framework. Finally, we discussed the desirable properties that our sound and complete abstraction framework enjoys.

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A Proof Sketch of Theorem 13

For technical reasons we construct an automata $\tilde{M}_{(A,q)}$ with state space $\overline{m_\Theta} \times \mathtt{Ch}_A \times \mathcal{Q}$, show that $\tilde{M}_{(A,q)}$ refines and is refined by $M_{(A,q)}$, and then prove the theorem for $\tilde{M}_{(A,q)}$ instead of $M_{(A,q)}$. This is sound as these models are refinement equivalent. Formally:

$$
\begin{aligned}
\tilde{S}_{A,q} &= \overline{m_\Theta} \times \mathtt{Ch}_A \times \mathcal{Q} \\
\tilde{S^{\mathrm{i}}}_{A,q} &= \{(0, c_A'', q) \mid c_A'' \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c_A''}(q)\} \\
\tilde{R}^-_{A,q} &= \{((n, c_A, q'''), \alpha, D) \mid \exists (n', q') \in \mathbf{BS}_{c_A}(q'''), q'' \in \mathcal{Q} : \delta(q') = \mathbf{EX}^\alpha q'' \,\& \\
&\qquad D = \{(n', c_A'', q'') \mid c_A'' \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c_A''}(q'')\}\} \\
\tilde{R}^+_{A,q} &= \{((n, c_A, q'''), \alpha, D) \mid \exists f : \mathcal{Q} \to \mathtt{Ch}_A : \\
&\qquad [\forall q'' \in \mathcal{Q} : (\exists (n', q') \in \mathbf{BS}_{c_A}(q''') : \delta(q') = \mathbf{AX}^\alpha q'') \Rightarrow \mathbf{olf}_{f(q'')}(q'')] \,\& \\
&\qquad D = \{(n', f(q''), q'') \mid \exists q' : (n', q') \in \mathbf{BS}_{c_A}(q''') \,\&\, \delta(q') = \mathbf{AX}^\alpha q''\}\} \\
\tilde{L}^-_{A,q}(n, c_A, q''') &= \{p \mid \exists (n', q') \in \mathbf{BS}_{c_A}(q''') : \delta(q') = p\} \\
\tilde{L}^+_{A,q}(n, c_A, q''') &= \{p \mid \forall (n', q') \in \mathbf{BS}_{c_A}(q''') : \delta(q') \neq \neg p\} \\
\tilde{\eta}_{A,q}(n, c_A, q''') &= n\,.
\end{aligned}
$$

Furthermore, we set

$$\widetilde{\mathrm{succ}}^-_\alpha(s) = \{D \mid (s, \alpha, D) \in \tilde{R}^-_{A,q}\} \qquad \widetilde{\mathrm{succ}}^+_\alpha(s) = \{C \mid (s, \alpha, C) \in \tilde{R}^+_{A,q}\}\,.$$

The following lemmas and Theorem 10 are used in order to prove Theorem 13.

Lemma 16. *Model $\tilde{M}_{(A,q)}$ refines $M_{(A,q)}$ and vice versa.*

Lemma 17. *Let $i, n, m \in \mathbb{N}$, $c_A \in \mathtt{Ch}_A$, and $q \in \mathcal{Q}$ with $(n, q) \in \mathcal{U}$. Then*

$$\mathcal{T}^i_{c_A}(\bot)_{(n+m,q)} = \{(\max\{n + m, n'\}, q') \mid (n', q') \in \mathcal{T}^i_{c_A}(\bot)_{(n,q)}\}\,.$$

Lemma 18. *Let $(s^{(k)}, q^{(k)})_{k\in\mathbb{N}}$ be a satisfaction play for M and A where Player I makes his or her decisions on O_A-states via $c_A \in \mathtt{Ch}_A$. Furthermore, let $k > i$ be such that for all j with $i \leq j < k$ we have $q^{(j)} \in K_A$. Then*

$$(\max\{\Theta(q^{(j)}) \mid i \leq j \leq k\}, q^{(k)}) \in \mathbf{rea}_{c_A}(q^{(i)})\,.$$

Let $\mathbb{P}_{fin}(\mathbb{N})$ denote the set of all finite subsets of $\mathbb{N}$. Define $\xi : (\mathbb{P}_{fin}(\mathbb{N}) \setminus \{\{\}\}) \to \mathbb{N}$ by

$$\xi(N) = \begin{cases} \max\{n \in N \mid n \text{ is odd}\} & \text{if } \{n \in N \mid n \text{ is odd}\} \neq \{\}, \\ \min\{n \in N \mid n \text{ is even}\} & \text{otherwise}. \end{cases}$$

Lemma 19. *For any alternating tree automata A and any of its states $q \in \mathcal{Q}$ we have*

$$\tilde{M}_{(A,q)} \models (A, q)\,.$$

Proof. Let $H_{(c_A,q',q'')} = \{n' \in \mathbb{N} \mid (n', q'') \in \mathbf{BS}_{c_A}(q')\}$. Furthermore, we say that $((n, c_A, q'), q'')$ is an *allowed satisfaction configuration* if $H_{(c_A,q',q'')} \neq \{\}$ and $\mathbf{olf}_{c_A}(q')$ holds.

Now suppose $((n, c_A, q'), q'')$ is an allowed satisfaction configuration. Then the strategy $\theta_{\tilde{M}_{(A,q)}}$ of Player I is defined as follows:

Case #1: $\delta(q'') = q_1 \tilde{\vee} q_2$. Player I picks $q_{c_A(q'')}$.
Case #2: $\delta(q'') = \mathbf{EX}^\alpha q'''$. Player I picks

$$D = \{(\xi(H_{(c_A,q',q'')}), c''_A, q''') \mid c''_A \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c''_A}(q''')\}\,.$$

Case #3: $\delta(q'') = \mathbf{AX}^\alpha q'''$. Suppose Player II picks $D \in \widetilde{\mathrm{succ}}^+_\alpha(s)$. By definition of $\tilde{R}^+_{A,q}$, there then is c''_A with $(\xi(H_{(c_A,q',q'')}), c''_A, q''') \in D$ since the satisfaction configuration $((n, c_A, q'), q'')$ is allowed. Player I then picks this very $(\xi(H_{(c_A,q',q'')}), c''_A, q''')$.

It is easily seen that $\theta_{\tilde{M}_{(A,q)}}$ is indeed a strategy for allowed satisfaction configurations. Using Lemma 18, one can also show that the set of all allowed satisfaction configurations is a "trap" for Player II with $\theta_{\tilde{M}_{(A,q)}}$ as a witnessing strategy: no matter how Player II plays, plays will not leave the set of allowed satisfaction configurations if played according to $\theta_{\tilde{M}_{(A,q)}}$.

Since (s^{i}, q) is an allowed satisfaction configuration for all $s^{\mathrm{i}} \in \tilde{S}^{\mathrm{i}}{}_{A,q}$ it remains to show that $\theta_{\tilde{M}_{(A,q)}}$ is a winning strategy for Player I:

By induction on the length of a finite play, one can easily show that Player I wins every finite play if played according to $\theta_{\tilde{M}_{(A,q)}}$. So it remains to show that Player I also wins every infinite play.

Let $C = ((n^{(k)}, c_A^{(k)}, q^{(k)}), \hat{q}^{(k)})_{k\in\mathbb{N}}$ be an infinite satisfaction play consistent with strategy $\theta_{\tilde{M}_{(A,q)}}$. We proceed with the following case analysis:

Case #1: $\exists i : \forall j \geq i : \hat{q}^{(j)} \in K_A$. It suffices to show that $\sup(\Theta \circ \pi_2 \circ C)$ is even. We use proof by contradiction. Let $\sup(\Theta \circ \pi_2 \circ C) = 2m+1$ with $m \in \mathbb{N}$. Then there exists $\hat{q}, j, k$ with $k > j \geq i$, $\hat{q} = \hat{q}^{(j)} = \hat{q}^{(k)}$, and $\delta(\hat{q}) = 2m+1$. Then by Lemma 18, we get $(2m+1, \hat{q}) \in \mathbf{rea}_{c_A^{(i)}}(\hat{q})$. Furthermore, $((n^{(i)}, c_A^{(i)}, q^{(i)}), \hat{q})$ is an allowed satisfaction configuration as plays consistent with strategy $\theta_{\tilde{M}_{(A,q)}}$ cannot leave the set of allowed satisfaction configurations. Hence, $((n^{(i)}, c_A^{(i)}, q^{(i)}), \hat{q}) \in \mathbf{BS}_{c_A^{(i)}}(q^{(i)})$ and $\mathbf{olf}_{c_A^{(i)}}(q^{(i)})$ holds. This is a contradiction to $(2m+1, \hat{q}) \in \mathbf{rea}_{c_A^{(i)}}(\hat{q})$.

Case #2: otherwise. Let $I = \{i+1 \mid \hat{q}^{(i)} \notin K_A\}$. Note that I is infinite in this case. It is easily seen that the calculation of $\sup(\tilde{\eta}_{A,q} \circ \pi_1 \circ C)$ only depends on the positions of I, since $\pi_1 \circ C$ is constant outside I. Now let $i, j \in I$ be such that $i < j$ and $I \cap \{k \in \mathbb{N} \mid i < k < j\} = \{\}$. By the definition of $\theta_{\tilde{M}_{(A,q)}}$ we get $\tilde{\eta}_{A,q}(n^{(j)}) = \xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})})$. Define $\hat{m} = \max\{\Theta(\hat{q}^{(k)}) \mid i \leq k \leq j-1\}$. By Lemma 18

$$(\hat{m}, \hat{q}^{(j-1)}) \in \mathbf{BS}_{c_A^{(i)}}(q^{(i)}). \tag{6}$$

We proceed with the following case analysis:

Case #2.1: $\hat{m}$ is odd. Then $\xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})})$ is odd and we also know $\hat{m} \leq \xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})})$ by (6) and by the definition of ξ. Hence,

$$\hat{m} \text{ is odd } \Rightarrow (\tilde{\eta}_{A,q}(n^{(j)}) \text{ is odd} \,\&\, \tilde{\eta}_{A,q}(n^{(j)}) \geq \hat{m}), \tag{7}$$

since $\xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})}) = \tilde{\eta}_{A,q}(n^{(j)})$.

Case #2.2: $\hat{m}$ is even. Then $\xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})})$ is odd or we know that $\hat{m} \geq \xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})})$ by (6) and by the definition of ξ. Hence,

$$\hat{m} \text{ is even } \Rightarrow (\tilde{\eta}_{A,q}(n^{(j)}) \text{ is odd} \,\&\, \tilde{\eta}_{A,q}(n^{(j)}) \leq \hat{m}), \tag{8}$$

since $\xi(H_{(c_A^{(i)}, q^{(i)}, \hat{q}^{(j-1)})}) = \tilde{\eta}_{A,q}(n^{(j)})$.

Now suppose $\sup(\Theta \circ \pi_2 \circ C)$ is odd. Then by (7) and (8), $\sup(\tilde{\eta}_{A,q} \circ \pi_1 \circ C)$ is odd as well. Thus $\theta_{\tilde{M}_{(A,q)}}$ is a winning strategy for Player I. □

Lemma 20. *Suppose* $c_A \in \mathtt{Ch}_A$ *and* $(n', q') \in \mathcal{T}^i_{c_A}(\bot)_{(n,q)}$. *Then there exists* $(q^{(k)})_{k \leq m}$ *such that*

- $q^{(0)} = q$,
- $q^{(m)} = q'$,
- *for all* $k < m$ *we have* $q^{(k)} \in K_A$,
- $\max(\{n\} \cup \{\Theta(q^{(j)}) \mid 1 < j \leq m\}) = n'$, *and*
- *the steps between the* $q^{(k)}$ *are made as for the satisfaction game via strategy* c_A, *i.e. for all* $k < m$ *we have*

$$\begin{aligned} \delta(q^{(k)}) = q'' &\Rightarrow q^{(k+1)} = q'' \\ \delta(q^{(k)}) = q_1 \tilde{\wedge} q_2 &\Rightarrow q^{(k+1)} \in \{q_1, q_2\} \\ \delta(q^{(k)}) = q_1 \tilde{\vee} q_2 &\Rightarrow q^{(k+1)} = q_{c_A(q^{(k)})} \,. \end{aligned}$$

Let θ be a strategy for Player I for the satisfaction game between M and A. Then $c_A^{\theta,s}$ is defined to be a choice function whose choices on $q \in O_A$ agree with those of θ on (s, q).

Lemma 21. *Suppose* θ *is a winning strategy for Player I for the satisfaction game between* M *and* A *for initial configuration* (s, q). *Then* $\mathbf{olf}_{c_A^{\theta,s}}(q)$ *holds.*

Lemma 22. *Suppose* A *is an alternating tree automaton,* $q \in \mathcal{Q}$, *and* M *is a fair disjunctive-conjunctive modal transition system. Then*

$$M \models (A, q) \Rightarrow M \text{ refines } \tilde{M}_{(A,q)} \,.$$

Proof. Let θ be a winning strategy for Player I for the satisfaction game $M \models (A, q)$. Then we call $(s, (n, c_A, q))$ an *allowed refinement configuration with respect to* θ if $c_A = c_A^{\theta,s}$ and θ is a winning strategy for (s, q).

Now suppose $(s, (n, c_A^{\theta,s}, q))$ is an allowed refinement configuration with respect to θ. Then the strategy $\vartheta_{\tilde{M}_{(A,q)}}$ of Player I is defined as follows:

Case #1: R^- transition. Suppose Player II picks $D \in \widetilde{\mathrm{succ}}^-_\alpha((n, c_A^{\theta,s}, q))$. By definition of $\tilde{M}_{(A,q)}$ there exists $(n', q') \in \mathbf{BS}_{c_A}(q)$ and q'' with $\delta(q') = \mathbf{EX}^\alpha q''$ and $D = \{(n', c''_A, q'') \mid c''_A \in \mathtt{Ch}_A \,\&\, \mathbf{olf}_{c''_A}(q'')\}$. Player I picks the D' that is chosen by θ at (s, q'), noting that $D' \in \mathrm{succ}^-_\alpha(s)$ holds.

Now suppose Player II picks $s' \in D'$. Since θ is a winning strategy for (s, q), we obviously obtain that θ is a winning strategy for (s, q'), and therefore also for (s', q''). Thus by Lemma 21 $\mathbf{olf}_{c_A^{\theta,s'}}(q'')$ holds. Hence, $(n', c_A^{\theta,s'}, q'') \in D$. Player I chooses this element.

Case #2: R^- transition. Suppose Player II picks $D' \in \mathrm{succ}^+_\alpha(s)$. Let $G = \{q'' \mid \exists n', q' : (n', q') \in \mathbf{BS}_{c_A}(q) \,\&\, \delta(q') = \mathbf{AX}^\alpha q''\}$. For $q'' \in G$ let $n_{q''} \in \mathbb{N}$ and $q_{q''} \in \mathcal{Q}$ be such that $(n_{q''}, q_{q''}) \in \mathbf{BS}_{c_A}(q)$ and $\delta(q_{q''}) = \mathbf{AX}^\alpha q''$. From the fact that θ is a winning strategy for (s, q) and $q'' \in G$, we again obtain that θ is a winning strategy for $(s, q_{q''})$. Hence, for any $q'' \in G$, there exists $s_{q''} \in D'$ (the one chosen by θ at $(s, q_{q''})$ after Player II had picked D') such that θ is a winning strategy for $(s_{q''}, q'')$. We choose any function $f\colon \mathcal{Q} \to \mathtt{Ch}_A$ satisfying that for any $q'' \in G$ we have $f(q'') = c_A^{\theta,s_{q''}}$. Note that $\mathbf{olf}_{f(q'')}(q'')$ holds for any $q'' \in G$. Hence, $D \in \widetilde{\mathrm{succ}}^+_\alpha((n, c_A^{\theta,s}, q))$ where

$$D = \{(n', f(q''), q'') \mid \exists q' : (n', q') \in \mathbf{BS}_{c_A}(q) \,\&\, \delta(q') = \mathbf{AX}^\alpha q''\}\,.$$

Player I picks this D. Now suppose Player II picks $(n', f(q''), q'') \in D$. Then $q'' \in G$. Player I chooses $s_{q''}$.

By definition, $\vartheta_{\tilde{M}_{(A,q)}}$ is indeed a strategy for allowed refinement configurations with respect to θ, and — regardless of how Player II plays — the strategy $\vartheta_{\tilde{M}_{(A,q)}}$ will not leave the set of allowed refinement configurations with respect to θ.

For $s^{\mathrm{i}} \in S^{\mathrm{i}}$ we have $(0, c_A^{\theta,s^{\mathrm{i}}}, q) \in \tilde{S}^{\mathrm{i}}{}_{A,q}$ by Lemma 21. Furthermore, we have that $(s^{\mathrm{i}}, (0, c_A^{\theta,s^{\mathrm{i}}}, q)$ is an allowed refinement configuration with respect to θ. Therefore, it only remains to show that $\vartheta_{\tilde{M}_{(A,q)}}$ is a winning strategy for Player I for any allowed refinement configuration $(s, (n, c_A^{\theta,s}, q))$ with respect to θ:

– Suppose Player II picks p from $\tilde{L}^-_{A,q}(c_A^{\theta,s}, q))$. Then, by definition, there exists $(n', q') \in \mathbf{BS}_{c_A}(q''')$ with $\delta(q') = p$. From Lemma 20

we obtain that Player II can reach the satisfaction configuration (s, q') from (s, q) if Player I uses strategy θ. Thus, $p \in L^-(s)$ as θ is a winning strategy for (s, q). Hence, Player I wins this refinement play, as required.
- The case when Player II picks p from $\mathrm{AP} \setminus \tilde{L}^+_{A,q}(c_A^{\theta,s}, q))$ follows analogously.
- Furthermore, we already argued in the definition of $\vartheta_{\tilde{M}_{(A,q)}}$ that Player II cannot win during a R^- transition or a R^+ transition step.

Now let $\Phi = (s^{(k)}, (n^{(k)}, c_A^{(k)}, q^{(k)}))_{k\in\mathbb{N}}$ be an infinite satisfaction play consistent with strategy $\vartheta_{\tilde{M}_{(A,q)}}$ such that $(s^{(0)}, (n^{(0)}, c_A^{(0)}, q^{(0)}))$ is an allowed refinement configuration with respect to θ. Therefore all configurations of that infinite sequence are allowed refinement configurations with respect to θ. By the definition of $\tilde{M}_{(A,q)}$ there exists $\tilde{q}^{(k)}$ such that $(n^{(k+1)}, \tilde{q}^{(k)}) \in \mathbf{BS}_{c_A^{(k)}}(q^{(k)})$ and $q^{(k+1)} \in \{\mathbf{EX}^\alpha \tilde{q}^{(k)}, \mathbf{AX}^\alpha \tilde{q}^{(k)}\}$. Then by Lemma 20, for all $k \in \mathbb{N}$ there exists $(q_k^{(j)})_{j\le m_k}$ such that

- $q_k^{(0)} = q^{(k)}$,
- $q_k^{(m_k)} = \tilde{q}^{(k)}$,
- for all $j < m_k$ we have $q_k^{(j)} \in K_A$ and $\max\{\Theta(q_k^{(j)}) \mid j \le m\} = n^{(k+1)}$, and
- the steps between the $q^{(k)}$ are made as for the satisfaction game according to strategy $c_A^{(k)}$.

Define $(C)_{i\in\mathbb{N}}$ by $C_i = (q^{(k)}, q_k^{(j)})$ if $i = j + \sum_{\ell=0}^{k-1}(m_\ell+1)$ and $j \le m_k$. Then it is straightforward to check that C is an infinite satisfaction play between M and A at initial satisfaction configuration (s, q). Furthermore, said satisfaction play is consistent with strategy θ in C as $c_A^{(k)} = c_A^{\theta,s^{(k)}}$. Now suppose $\sup(\eta \circ \pi_1 \circ \Phi)$ is even. Then by construction $\sup(\eta \circ \pi_1 \circ C)$ is even. Thus $\sup(\Theta \circ \pi_2 \circ C)$ is even since θ is a winning strategy in $(s^{(0)}, q^{(0)})$ for Player I, noting that $(s^{(0)}, (n^{(0)}, c_A^{(0)}, q^{(0)}))$ is an allowed refinement configuration. From $\max\{\Theta(q_k^{(j)}) \mid j \le m\} = n^{(k+1)}$ we get $\sup(\Theta \circ \pi_2 \circ C) = \sup(\tilde{\eta}_{A,q} \circ \pi_2 \circ \Phi)$ and, therefore, $\sup(\tilde{\eta}_{A,q} \circ \pi_2 \circ \Phi)$ is even. □

Remark 23. The proofs above suggest that the target sets of transitions in $\tilde{M}_{(A,q)}$ and $M_{(A,q)}$ can be further restricted to those states reflecting function ξ, i.e. only one element per reachable state would appear in the target set of a transition. This optimization was dropped in order to increase readability and accessibility of proofs.

B Proof Sketch of Theorem 14

The following lemmas and Theorem 10 are used in order to prove Theorem 14.

Lemma 24. *Suppose M is a fair disjunctive-conjunctive modal transition system. Then*

$$M \models (A_M, \widehat{q}_M)\,.$$

Proof. Let M be a model. Then (s, q_M) is defined to be an *allowed A_M satisfaction configuration* if one of the following is true:

- $q_M = s$
- $q_M = \text{true}$
- $q_M = p$ and $p \in L^-(s)$
- $q_M = \neg p$ and $p \in \text{AP} \setminus L^+(s)$
- $q_M = (q_M^{(1)} \wedge q_M^{(2)})$ and for all $j \in \{1,2\}$: $(s, q_M^{(j)})$ is an allowed A_M satisfaction configuration
- $q_M = (q_M^{(1)} \vee q_M^{(2)})$ and there is some $j \in \{1,2\}$ such that $(s, q_M^{(j)})$ is an allowed A_M satisfaction configuration
- $q_M = (\mathbf{AX}^\alpha(\bigvee_{s' \in D'} s'))$ and there is some $f \in \text{SF}_{\text{succ}_\alpha^+(s)}$ such that $D' = \{f(D'') \mid D'' \in \text{succ}_\alpha^+(s)\}$)
- $q_M = (\mathbf{EX}^\alpha(\bigvee_{s' \in D'} s'))$ and $D' \in \text{succ}_\alpha^-(s)$).

Obviously, such a notion is well defined.

Now suppose (s, q_M) is an allowed A_M satisfaction configuration. Then the strategy θ_{A_M} of Player I is straightforwardly defined as follows:

- If $\delta_M(q_M) = q_M^{(1)} \tilde{\vee} q_M^{(2)}$, choose one disjunct that yields an allowed A_M satisfaction configuration.
- If $\delta_M(q_M) = \mathbf{AX}^\alpha(\bigvee_{s' \in D'} s')$ and Player II picks $D'' \in \text{succ}_\alpha^+(s)$, then choose $f(D'')$ for the $f \in \text{SF}_{\text{succ}_\alpha^+(s)}$ that exists according to the definition above.

– If $\delta_M(q_M) = \mathbf{EX}^\alpha(\bigvee_{s' \in D'} s')$ then Player I chooses D'.

It is easily seen that θ_{A_M} is indeed a strategy for allowed A_M satisfaction configurations. Furthermore, it is straightforwardly checked that – independently of how Player II plays – allowed A_M satisfaction configurations are always obtained if the game starts in an allowed A_M satisfaction configuration and is played according to θ_{A_M}. Obviously, for all $s^{\mathrm{i}} \in S^{\mathrm{i}}$ the configuration $(s^{\mathrm{i}}, \widehat{q}_M)$ is allowed in that sense. Thus it remains to show that θ_{A_M} is a winning strategy for Player I.

For any allowed A_M satisfaction configuration $(s, \mathbf{AX}^\alpha(\bigvee_{s' \in D'} s'))$ we have $\{\} \notin \mathrm{succ}^+_\alpha(s)$. Therefore induction on the length of the play shows that Player I wins every finite play using strategy θ_{A_M}.

Now let $C = (s^{(k)}, q_M^{(k)})_{k \in \mathbb{N}}$ be an infinite satisfaction play consistent with strategy θ_{A_M}. Let $I = \{k \mid q_M^{(k)} \in S\}$. It is straightforwardly checked that I is infinite, since only sub-formulas of q_M can be reached, except when $q_M \in S$ or q_M is a predicate. By definition $\sup(\Theta_M \circ \pi_2 \circ C) = \sup((\eta(q_M^{(k)})_{k \in I})$. Since *allowed A_M satisfaction configuration* are preserved during such plays, we have $s^{(k)} = q_M^{(k)}$ for all $k \in I$. Next, it is easily checked that in between two successive elements k and k' from I the value of $s^{(n)}$ changes at most once inside the sub-sequence $(s^{(n)})_{k \leq n \leq k'}$. Hence, $\sup(\eta \circ \pi_1 \circ C) = \sup((\eta(s^{(k)})_{k \in I})$. Thus $\sup(\Theta_M \circ \pi_2 \circ C) = \sup(\eta \circ \pi_1 \circ C)$ and therefore [$\sup(\Theta_M \circ \pi_2 \circ C)$ is even $\Rightarrow \sup(\eta \circ \pi_1 \circ C)$ is even] holds as required. □

Lemma 25. *Suppose M and M' are fair disjunctive-conjunctive modal transition systems such that $M' \models (A_M, \widehat{q}_M)$. Then*

$$M' \text{ refines } M.$$

Proof. Let M and M' be models and θ a winning strategy for Player I for the satisfaction game $M' \models (A_M, \widehat{q}_M)$. Then $(s', s) \in S' \times S$ is defined to be an *allowed A_M refinement configuration* if θ is a winning strategy for Player I at configuration (s', s).

Now suppose (s', s) is an allowed A_M refinement configuration. Then the strategy ϑ_{A_M} of Player I is defined as follows:

Case #1: R^- transition. Suppose Player II picks $D \in \mathrm{succ}^-_\alpha(s)$. Then the configuration $(s', \mathbf{EX}^\alpha(\bigvee_{\ddot{s}\in D} \ddot{s}))$ is reachable from (s', s) via θ. Player I responds with the D' chosen by θ on $(s', \mathbf{EX}^\alpha(\bigvee_{\ddot{s}\in D} \ddot{s}))$, noting that $D' \in \mathrm{succ}'^-_\alpha(s')$.

Now suppose Player II picks $\ddot{s}' \in D'$. Then θ is winning for $(\ddot{s}', \bigvee_{\ddot{s}\in D} \ddot{s})$. Now apply strategy θ on $(\ddot{s}', \bigvee_{\ddot{s}\in D} \ddot{s})$ until $(\ddot{s}', \ddot{s})$ with $\ddot{s} \in D$ is *directly obtained*, i.e. stop when a state of the model is reached at the right hand side. Player I responds with $\ddot{s}$.

Case #2: R^+ transition. Suppose Player II picks $D' \in \mathrm{succ}'^+_\alpha(s')$. Then Player I responds with a $D \in \mathrm{succ}^+_\alpha(s)$ satisfying

$$\begin{aligned}&\forall \ddot{s} \in D\colon \exists s'_{\ddot{s}} \in D'\colon \exists f_{\ddot{s}} \in \mathrm{SF}_{\mathrm{succ}^+_\alpha(s)}\colon\\ &\quad (s'_{\ddot{s}}, \ddot{s}) \text{ is directly obtained from } (s', \mathbf{AX}^\alpha(\textstyle\bigvee_{\ddot{D}\in\mathrm{succ}^+_\alpha(s)} f_{\ddot{s}}(\ddot{D})))\\ &\quad \text{according to } \theta \text{ after Player II had picked } D'\end{aligned} \tag{9}$$

- First we prove that such a D exists. We use proof by contradiction. If no such D exists, then for all $D \in \mathrm{succ}^+_\alpha(s)$ equation (9) is false and so there is a witness $\ddot{s}_D \in D$ for the falsity of that equation.

  Let $\tilde{f} \in \mathrm{SF}_{\mathrm{succ}^+_\alpha(s)}$ with $\tilde{f}(D) = \ddot{s}_D$. Note that θ is winning for the configuration $(s', \mathbf{AX}^\alpha(\bigvee_{\ddot{D}\in\mathrm{succ}^+_\alpha(s)} \tilde{f}(\ddot{D})))$. Hence, there exists $\ddot{s}' \in D'$ where Player II can pick $D' \in \mathrm{succ}'^+_\alpha(s')$ such that θ is winning for $(\ddot{s}', (\bigvee_{\ddot{D}\in\mathrm{succ}^+_\alpha(s)} \tilde{f}(\ddot{D})))$. Then there is $\ddot{D} \in \mathrm{succ}^+_\alpha(s)$ such that $(\ddot{s}', \tilde{f}(\ddot{D}))$ is directly obtained from $(\ddot{s}', \mathbf{AX}^\alpha(\bigvee_{\ddot{D}\in\mathrm{succ}^+_\alpha(s)} f(\ddot{D})))$ via θ. This is a contradiction as $\tilde{f}(\ddot{D}') = \ddot{s}_D$.
- Second, suppose Player II picks $\ddot{s} \in D$. Then Player I responds with $s'_{\ddot{s}}$ which exists according to (9).

It is easily seen that ϑ_{A_M} is a strategy for allowed A_M refinement configurations. Also, it is straightforwardly checked that, no matter how Player II plays, allowed A_M refinement configurations are always obtained if the game starts in an allowed A_M refinement configuration. It is easily seen that for any $s^{\mathrm{i}\prime} \in S^{\mathrm{i}\prime}$ there exists $s^{\mathrm{i}} \in S^{\mathrm{i}}$ such that $(s^{\mathrm{i}\prime}, s^{\mathrm{i}})$ is an allowed A_M refinement configuration. Thus it remains to be shown that ϑ_{A_M} is a winning strategy for Player I:

- By induction on the length of the play we can easily show that Player I wins every finite play using strategy ϑ_{A_M}.

– Now let $\Phi = (s'_k, s_k)_{k \in \mathbb{N}}$ be an infinite refinement play where Player I uses strategy ϑ_{A_M}. One quickly sees that it is possible to extend the sequence of that play by adding automata states on the right hand side such that a corresponding play C using θ for the satisfaction game for M' and A_M in $\widehat{q}_M$ is obtained. Furthermore, no states from S are introduced on the right hand side in C. Hence, $\sup(\eta \circ \pi_2 \circ \Phi) = \sup(\Theta_M \circ \pi_2 \circ C)$ and $\sup(\eta' \circ \pi_1 \circ \Phi) = \sup(\eta' \circ \pi_1 \circ C)$. Thus, the truth of [$\sup(\eta' \circ \pi_1 \circ \Phi)$ is even $\Rightarrow$ $\sup(\eta \circ \pi_2 \circ \Phi)$ is even] follows from the fact that θ is a winning strategy. □