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**Experimental Studies of
the Price of Anarchy in
Non-atomic Consumption-Relevance
Congestion Games**

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Experimental Studies of the Price of Anarchy in Non-atomic Consumption-Relevance Congestion Games

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Abstract

The model of non-atomic consumption-relevance congestion games (NCRCG) is an extension of the well-known non-atomic congestion games (NCG). We introduced the NCRCG model in [4] and proved worst-case lower and upper bounds on the price of anarchy. These bounds are tight up to a factor of γ , where γ is a new structural parameter of the game; we have $\gamma = 1$ for NCGs. This experimental work substantiates our conjecture that our worst-case lower bound is the best possible, i.e., that it actually also is an upper bound.

Outline. This article is organized as follows. First, we briefly review the NCRCG model and known theoretical results (Sec. 1). The reader is encouraged to also consult the more detailed presentation in [4]. Then we state our objectives and the conjecture (Sec. 2). We fix some terminology, describe our computational procedure and give some technical details (Sec. 3 and Sec. 4). Then we are ready to describe the random model used to generate the test instances (Sec. 5). In the sections starting with Sec. 6, we present the results of the experiments. First some qualitative observations are stated (Sec. 6) and then we give tables and plots for affine element latency functions (Sec. 7 and Sec. 8). More comprehensive tables with numerical data, also for element latency functions of degrees 2 and 3, can be found in the appendix. We compare our theoretical and conjectured bound to Perakis' bound (Sec. 9). The article concludes with remarks on open problems (Sec. 10).

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1. The Theory of NCRCG

1.1. Instances

Definition 1. A continuous and non-decreasing function $\ell : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is called an *element latency function*. We only consider element latency functions that are polynomials with non-negative coefficients in this article. The class of all such polynomials of degree at most p is denoted $\text{Poly}^+(p)$.

The following definition lists all items that characterize a particular instance in the NCRCG model. It is not a definition of game-theoretic concepts; these will be introduced at the end of this section.

Definition 2. An instance in the non-atomic consumption-relevance congestion game model (called shortly “a non-atomic consumption-relevance congestions game”, or “an NCRCG”) is defined by the following items.

- (i) A number $m \in \mathbb{N}$. We denote $E := [m]$ ($= \{1, \dots, m\}$), and each number in E stands for an *element*.
- (ii) For each element $e \in E$ an element latency function ℓ_e .
- (iii) A number $N \in \mathbb{N}$. Each number in $[N]$ stands for a *player class*.
- (iv) For each $i \in [N]$ a number $d_i \in \mathbb{R}_{>0}$, called the *demand* of player class i . The demand is to represent a large number of non-cooperative players. The game-theoretic concept behind this will be explained in Sec. 1.4.
- (v) For each $i \in [N]$ a finite set \mathfrak{S}_i , chosen such that $\mathfrak{S}_1, \dots, \mathfrak{S}_N$ are pairwise disjoint and each \mathfrak{S}_i has at least cardinality 2. We say that each element in \mathfrak{S}_i stands for a *strategy* for player class i . Denote $\mathfrak{S} := \bigcup_{i \in [N]} \mathfrak{S}_i$ and $n := |\mathfrak{S}|$. We will often identify $\mathfrak{S} = [n]$ to simplify notation.
- (vi) A matrix $C \in \mathbb{R}_{\geq 0}^{m \times n}$ – its entries are called *consumption numbers* – and a matrix $R \in \mathbb{R}_{\geq 0}^{m \times n}$ – its entries are called *relevance numbers*. We say that for $e \in E$ and $S \in \mathfrak{S}$ the number C_{eS} is the consumption of element e under strategy S , and the number R_{eS} is the relevance of element e for strategy S . We further demand that neither of the two matrices has a row or a column of only zeros, and that

$$C_{eS} = 0 \text{ if and only if } R_{eS} = 0$$

for all $e \in E, S \in \mathfrak{S}$.

1.2. Notational Remarks

The model is feasibly defined and interesting even with one player class, i.e., $N = 1$. That case is notationally simpler and the reader is encouraged to think of $N = 1$ on a

first read. In fact, we will only work with one player class in the experiments and also adopt the corresponding simplified notation starting with Sec. 4.

Let $S \in \mathfrak{S}$. The set of those elements that have non-zero entries in column S of matrix C (or, equivalently, R), will – abusing notation – sometimes also be denoted by S , and we also speak of an element being ‘contained’ in S . Keep in mind that two strategies with different consumption or relevance numbers can constitute the same set of elements in this sense. This will be no problem; we will always treat such strategies as distinct. Moreover, it is sometimes convenient to consider the set of all strategies which in that sense ‘contain’ a particular element e . We denote that as $\mathfrak{S}(e) := \{S \in \mathfrak{S}; C_{e,S} \neq 0\}$, which is the same as $\{S \in \mathfrak{S}; R_{e,S} \neq 0\}$.

1.3. Action Distributions, Congestions, Latencies, Social Cost

Define the set of *action distributions* by

$$\mathcal{A} := \{a \in \mathbb{R}_{\geq 0}^n; \sum_{S \in \mathfrak{S}_i} a_S = d_i \quad \forall i \in [N]\} .$$

An action distribution describes a way of distributing the demand d_i of each player class i across the strategies in \mathfrak{S}_i , i.e., the strategies available for that player class.

For the rest of this section, we present local and global performance measures and other quantities relevant to the performance of the system under a given action distribution a . We define the *vector of congestions* as $\vec{g}(a) := C \cdot a$, where the dot denotes the usual matrix-vector product. Hence for each $e \in E$ we have its congestion

$$g_e(a) = \sum_{S \in \mathfrak{S}} C_{eS} a_S = \sum_{S \in \mathfrak{S}(e)} C_{eS} a_S .$$

The congestion $g_e(a)$ of an element e is a measure of how heavily that element e is stressed under a . As can be noticed from the definition, the model allows that an element may be stressed differently by the same amount of players *depending on the strategy through which these players occupy the element*.

Recall that each element e has an element latency function ℓ_e . This function tells us how the element reacts to congestion. The quantity $\ell_e(g_e(a))$ is called the *element latency* of e under a .

Finally we have to define how these element latencies are experienced by the players. The total of players in a player class i can be understood as the interval $[0, d_i]$. Each action distribution describes a possible way of how this interval can be partitioned and the parts assigned to strategies. All the players which by this are assigned to a strategy S experience the latency

$$L_S(a) := \sum_{e \in E} R_{eS} \ell_e(g_e(a)) ,$$

which, thinking of S as a subset of E , can be written as $\sum_{e \in S} R_{eS} \ell_e(g_e(a))$. We call

this the *strategy latency* of S under a . The vector of all strategy latencies is denoted¹ $\vec{L}(a) := (L_S(a))_{S \in \mathfrak{S}}$. Since R is a $(m \times n)$ -matrix and $\vec{\ell}(g(a)) := (\ell_e(g_e(a)))_{e \in E}$ is a vector of length m , we can write $\vec{L}(a)$ in compact form using matrix-vector multiplication: $\vec{L}(a) = (\ell(g(a))^\perp \cdot R)^\perp$, which is a vector of length n .

The following quantity is called the *social cost*:

$$\text{SC}(a) := \sum_{S \in \mathfrak{S}} L_S(a) a_S .$$

It captures the overall performance of the system under a given action distribution a .

1.4. Game Theory

We adopt the standard definition for Nash equilibria in non-atomic games, also known as *Wardrop equilibria* [9].

Definition 3. Let an NCRCG be given. An action distribution a with the following property is called a *Nash equilibrium* (abbreviated “N.E.”) for the game:

$$\forall i \in [N] \quad \forall S, T \in \mathfrak{S}_i : \quad (a_S > 0 \implies L_S(a) \leq L_T(a)) . \quad (1)$$

The definition says that in a Nash equilibrium, only minimum-latency strategies are used. Existence of Nash equilibria is guaranteed under our assumption on the element latency functions being continuous. Since then SC is continuous and \mathcal{A} is compact, optimal action distributions, i.e., with minimal social cost, also exist. Their value is denoted by OPT . We always assume that $\text{OPT} > 0$, which is given, e.g., if $\ell(x) > 0$ for all element latency functions ℓ and $x > 0$. To see that, recall that there are no empty strategies, since by definition no column in C (or equivalently in R) consists of only zeros. We can now define the price of anarchy.

Definition 4. Given an NCRCG, its *price of anarchy* or *coordination ratio* is defined by $\sup_{a \text{ is N.E.}} \text{SC}(a)/\text{OPT}$. It is sometimes denoted by ρ . By continuity of $L_S(\cdot)$, $S \in \mathfrak{S}$, we can replace the supremum by a maximum.

We have now defined the NCRCG model. The case $C = R$ coincides with the previously studied NCG model [7].

¹Using simply $L(a)$ instead of $\vec{L}(a)$ and referring to its S th component by $L_S(a)$ would have been more systematic, but carries the risk for confusion that L might refer to the latency function of a single strategy – just as we sometimes denote a single element latency function by ℓ and not by ℓ_e for some $e \in E$. For the same reason, we write $\vec{\ell}(a)$ for the vector of all element latencies. We also write $\vec{g}(a)$ for the vector of all congestions.

1.5. The New Parameters γ_1 , γ_2 , and γ

We introduce new parameters for instances of the NCRCG model. They will play an important role in bounding the price of anarchy.

$$\begin{aligned}\gamma_1 &:= \max\left\{\frac{C_{eS}}{R_{eS}}; S \in \mathfrak{S}, e \in S\right\}, \\ \gamma_2 &:= \max\left\{\frac{R_{eS}}{C_{eS}}; S \in \mathfrak{S}, e \in S\right\}.\end{aligned}$$

Their product is denoted $\gamma := \gamma_1\gamma_2$.

Recall that we interpret $S \in \mathfrak{S}$ as the set of elements which have non-zero consumption number in strategy S , or, equivalently, which have non-zero relevance number in strategy S . Hence there is no division by zero and so these parameters are well-defined and finite. We have $\gamma_1 = \gamma_2 = 1$ if $C = R$, i.e., in the case of an NCG. We always have the product $\gamma = \gamma_1\gamma_2 \geq 1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ and $e \in E$. If we scale the row of R corresponding to e with ε and then replace ℓ_e by $x \mapsto \frac{1}{\varepsilon}\ell_e(x)$, essentially nothing changes for the game. The functions \vec{L} and SC are unchanged, and hence so is the set of equilibria, optima, and their social costs. Our upper bounds on the price of anarchy (to be presented in the next section) are non-decreasing in the term $\gamma = \gamma_1\gamma_2$ and hence it is desirable to reduce that quantity, while preserving other important properties of the game. An optimal way of scaling the matrix R can be found in polynomial time [4].

1.6. Bounds on the Price of Anarchy

We have lower and upper bounds on the price of anarchy for polynomial element latency functions, which are apart by a factor of γ .

Theorem 1 ([4]). *(i) Let $p \in \mathbb{N}_{\geq 1}$ and $c, r \in \mathbb{R}_{\geq 1}$. There exist NCRCGs with element latency functions only from $\text{Poly}^+(p)$ with $\gamma_1 = c$ and $\gamma_2 = r$ such that the price of anarchy is at least $\gamma^p = (\gamma_1\gamma_2)^p$.*

(ii) Let $p \in \mathbb{N}_{\geq 1}$ and $c, r \in \mathbb{R}_{\geq 1}$ such that $cr < (p+1)^{\frac{1}{p}}$. There exist NCRCGs with element latency functions only from $\text{Poly}^+(p)$ with $\gamma_1 = c$ and $\gamma_2 = r$ such that the price of anarchy is at least

$$\frac{1}{1 - \gamma\beta}, \quad \text{where } \beta = p(p+1)^{-1-\frac{1}{p}}.$$

Theorem 2 ([4]). *The price of anarchy in an NCRCG with element latency functions drawn from $\text{Poly}^+(p)$ is no more than*

$$\begin{cases} \gamma^{p+1} & \text{if } \gamma \geq (1+p)^{\frac{1}{p}} \\ \frac{1}{1-\gamma\beta}\gamma & \text{if } \gamma \leq (1+p)^{\frac{1}{p}} \end{cases}, \quad \text{where } \beta = p(p+1)^{-1-\frac{1}{p}}.$$

The parameter β is a generalization to the concept of anarchy value [7] and was introduced in [2]. It depends only on the class of eligible element latency functions and is $p(p+1)^{-1-\frac{1}{p}}$ if element latency functions are drawn from $\text{Poly}^+(p)$. In fact, there is essentially no difference between the concept of anarchy value and that of β for the classes $\text{Poly}^+(p)$. We use β here for notational convenience and for staying consistent with [4].

Another bound on the price of anarchy in certain cases was obtained by Perakis [5] using a different approach than ours. We only state the result for affine element latency functions, say, $\ell_e(x) = \theta_e x + \tau_e$ with $\theta_e, \tau_e \in \mathbb{R}_{\geq 0}$. Let $\theta := \text{diag}(\theta_e)_{e \in E}$ and $\tau := (\tau_e)_{e \in E}$. Then $G := J\vec{L}(a) = (\theta R)^{\perp} C$, independently of a . This matrix may be positive definite or not. If it is, we can apply one of Perakis' bounds. So, let G be positive definite. Then $S := (G + G^{\perp})/2$ is also positive definite (hence invertible) and symmetric. Let $\|v\|_S := \sqrt{v^{\perp} S v}$ for all $v \in \mathbb{R}^n$, then $\|\cdot\|_S$ is a norm. Define

$$c := \|S^{-1}G\|_S = \sup_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|S^{-1}Gv\|_S}{\|v\|_S},$$

which is the operator norm of $S^{-1}G$ using $\|\cdot\|_S$ as the vector norm. Then, Perakis [5] shows that the price of anarchy is bounded by

$$\begin{cases} c^2 & \text{if } c^2 > 2 \text{ or } \tau = 0 \\ \frac{4}{4-c^2} & \text{else} \end{cases}.$$

If G is only positive *semidefinite*, Perakis proves another bound, which coincides with the former if G is positive definite. We will only consider the positive definite case in our experiments, since we found only a very few instances with positive semidefinite but not positive definite G .

1.7. Computation

Optimal action distributions are characterized by the following mathematical program. The constraints are linear. The objective function, however, can be non-linear and even non-convex.

$$\begin{aligned} \min \quad & \text{SC}(a) \\ \text{s.t.} \quad & a \in \mathcal{A} \end{aligned} \tag{OPT NLP}$$

The function SC is convex on \mathcal{A} if and only if its projected Hessian is positive semidefinite. In case of affine element latency functions, i.e., from $\text{Poly}^+(1)$, the Hessian and so also the projected Hessian is the same in every point of \mathbb{R}^n . A test for convexity of SC is hence practical for affine element latency functions, up to numerical inaccuracies involved when checking a matrix for being positive semidefinite.

Nash equilibria with social cost at least c_0 , provided that they exist, are characterized by the following mathematical program.

$$\begin{aligned}
& \min \text{SC}(a) - \lambda \cdot d \\
& \text{s.t. } a \in \mathcal{A} \\
& \lambda \in \mathbb{R}^N \\
& \lambda_i \leq L_S(a) \quad \forall S \in \mathfrak{S}_i \quad \forall i \in [N] \\
& c_0 \leq \lambda \cdot d
\end{aligned} \tag{Nash NLP}$$

The constraints are linear, save possibly for ‘ $\lambda_i \leq L_S(a)$ ’, which may be non-linear and even non-convex. This constraint is linear for affine element latency functions.

2. Objectives and Conjecture

The main intention for experiments was to gather evidence as to which of the two bounds, the worst-case lower bound or the upper bound, is closer to the truth. To this end, we tried to find an instance for which we could show a price of anarchy exceeding that of Thm. 1. No instance for which we could prove a price of anarchy exceeding that bound by more than 1% was found for $p \in \{1, 2\}$, and none exceeding it by more than 4% was found for $p = 3$. We attribute those 1% and 4% to numerical errors. Hence the following conjecture, already stated in [4], stands.

Conjecture. *The price of anarchy in an NCRCG with element latency functions drawn from $\text{Poly}^+(p)$ is no more than*

$$\begin{cases} \gamma^p & \text{if } \gamma \geq (1+p)^{\frac{1}{p}} \\ \frac{1}{1-\gamma\beta} & \text{if } \gamma \leq (1+p)^{\frac{1}{p}} \end{cases}, \quad \text{where } \beta = p(p+1)^{-1-\frac{1}{p}}.$$

3. Computational Procedure

For each randomly generated instance, we do the following:

- 1.) If element latency functions are affine, check whether SC is convex or not. This is done by checking whether the projected Hessian is positive semidefinite.
- 2.) If element latency functions are affine, check whether Perakis’ bound for the positive definite case is applicable. If so, then compute it.
- 3.) Try to find an optimum a^* of (OPT NLP), using an NLP solver back-end [1, 8].
- 4.) Try to solve (Nash NLP) using the NLP solver back-end with different c_0 , following a binary search scheme. The Nash equilibrium a with the highest social cost obtained by this procedure is returned as the result.

We describe the last step in more detail in the following. Recall the mathematical program for Nash equilibria from Sec. 1.7.

$$\begin{aligned}
& \min \text{SC}(a) - \lambda \cdot d \\
& \text{s.t. } a \in \mathcal{A} \\
& \lambda \in \mathbb{R}^N \\
& \lambda_i \leq L_S(a) \quad \forall S \in \mathfrak{S}_i \quad \forall i \in [N] \\
& c_0 \leq \lambda \cdot d
\end{aligned} \tag{Nash NLP}$$

Let $\tilde{\rho}$ be an upper bound on the price of anarchy. Then all Nash equilibria have their social cost in the interval $[\text{OPT}, \tilde{\rho}\text{OPT}]$. Assuming that we can determine the feasibility of (Nash NLP) and solve it optimally if it is feasible, the following algorithm computes the price of anarchy up to an accuracy of ε . We call a vector $a \in \mathbb{R}^n$ *acceptable* roughly when it is a Nash equilibrium; this will be refined later in Sec. 4.

```

left ← OPT;
right ←  $\tilde{\rho}\text{OPT}$ ;
I ←  $\lceil \log_2 \frac{1}{\varepsilon\text{OPT}}(\text{right} - \text{left}) \rceil$  (=  $\lceil \log_2 \frac{\tilde{\rho}-1}{\varepsilon} \rceil$ );
i ← 0;
c0 ← left;
while i < I do
  if (Nash NLP) with c0 is feasible then
     $\tilde{a}$  ← optimal solution to (Nash NLP) using c0;
  else
     $\tilde{a}$  ← NIL;
  end
  if  $\tilde{a}$  is acceptable then
    a ←  $\tilde{a}$ ;
    left ← c0;
  else
    right ← c0;
  end
  c0 ← left +  $\frac{\text{right}-\text{left}}{2}$  (=  $\frac{\text{left}+\text{right}}{2}$ );
  i ← i + 1;
end
return a;

```

This algorithm always finds a solution,² since there is a Nash equilibrium of social cost at least OPT . Let a_w be a worst Nash equilibrium, so the price of anarchy is $\rho = \frac{\text{SC}(a_w)}{\text{OPT}}$. Then the result a of the algorithm fulfills $\text{SC}(a_w) - \text{SC}(a) \leq \varepsilon\text{OPT}$ and so the error is $\rho - \frac{\text{SC}(a)}{\text{OPT}} \leq \varepsilon$.

²Recall that we assumed that we can solve the involved NLP optimally. If this is not the case, it may happen that the algorithm does not find any acceptable solution. We encountered that for a small fraction of our random instances.

Definition 5. By *observed price of anarchy* we refer to the value $SC(a)/SC(a^*)$, where a^* and a are as in 3.) and 4.) above.

If we can correctly determine feasibility of the involved NLPs and solve them optimally if feasible, then the observed price of anarchy equals the price of anarchy, up to the error ε introduced by binary search. In general, the observed price of anarchy is a lower bound on the price of anarchy.

To better quantify how close the observed price of anarchy is to the bound of the conjecture, we introduce the *target ratio* of an instance. It is the ratio of observed price of anarchy to the conjectured bound. A target ratio of 1 means that the conjectured bound was hit, and a target ratio of more than 1 would disprove the conjecture. We say that an instance has TR x if its target ratio is $x\%$ or more, e.g., an instance with TR90 has a target ratio of at least 0.9. Since the conjecture is known to be true for $\gamma = 1$, we are mostly interested in the target ratios of instances with $\gamma > 1$. So later we will give figures stating what percentage of the randomly generated instances with $\gamma > 1$ had a certain TR x , and we will also state how many instances had $\gamma > 1$.

For some instances in our experiments, the binary search could not find *any* acceptable solution. This happens rarely for $p = 1$ and more frequently than that for $p > 1$, as can be seen in the tables found in the appendix. If it happens, the most common cause is that the vector returned by the NLP solver fails the test for being a Nash equilibrium. As a possible explanation, it was considered that the solver was not adjusted to produce sufficiently accurate solutions. Additional tests with the NLP solver configured for higher accuracy, however, suggest that this is not likely to be the cause. Another explanation, for the cases where only the Nash condition is not met, is that the solver simply failed to find a global optimum. This is substantiated by the observation that no instance with convex SC showed that kind of problem.

4. Experiment Setup

We wrote an implementation in C++, using various libraries. Most importantly, we use IP-OPT [8] and LANCELOT [1] to solve the mathematical programs involved. For eigenvalue computations, needed for the Perakis bound, we use the GNU Scientific Library [3]. Post-processing and plotting of gained data is done using the R System [6]. Experiments are run in parallel on several Barcelona[®] and Opteron[®] multicore machines running the Linux operating system at the Rechenzentrum Universität Kiel.

We apply the following counter-measures and workarounds for numerical inaccuracies:

- As noted earlier, all γ values are with respect to an optimal scaling, in order to reduce the bounds. That is, we use

$$\gamma_{\text{scaled}} = \min_{\varepsilon_1, \dots, \varepsilon_m > 0} \left\{ \max_{e \in E, S \in \mathfrak{S}(e)} \frac{C_{eS}}{\varepsilon_e R_{eS}} \quad \max_{e \in E, S \in \mathfrak{S}(e)} \frac{\varepsilon_e R_{eS}}{C_{eS}} \right\} .$$

We do, however, not actually scale the instances to not risk numerical instabilities. Since a scaled instance has the same optima and Nash equilibria as the original one, this procedure is correct.

- Instead of checking $\gamma > 1$ for determining the target ratios, we check $\gamma > 1.01$.
- We initialize the binary search with $left \leftarrow 0.95 \text{ OPT}$. The accuracy used in the binary search is $\varepsilon = 10^{-4}$.
- A vector $a \in \mathbb{R}^n$ is considered *acceptable* if
 - (i) the NLP solver does not signal an error,
 - (ii) $|d - \sum_{S \in \mathfrak{S}} a_S| \leq 10^{-2}d$ and $a_S \geq 0$ for all $S \in \mathfrak{S}$,
 - (iii) $c_0 - \text{SC}(a) \leq c_0 10^{-2}$,
 - (iv) $a_S(L_S(a) - \Lambda(a)) \leq d\Lambda(a)10^{-3}$ for all $S \in \mathfrak{S}$, where $\Lambda(a) := \min_{S \in \mathfrak{S}} L_S(a)$.

We generally rely on the NLP solver to signal errors and infeasibility correctly, so (ii) and (iii) exist mostly as a precaution. Test (iv) is to determine whether we have a Nash equilibrium or not. With these settings, we obtain accurate solutions up to two decimal digits for a set of 10 known instances.
- Tests for matrix definiteness are done comparing eigenvalues not against 0, but against 10^{-13} . This had turned out to be necessary in order to prevent problems due to numerical errors, e.g., the Hessian being recognized as positive semidefinite but the projected Hessian not, or Cholesky decomposition failing on an allegedly positive definite matrix.

5. Random Model and Data Sets

Since the NCRCG model comprises quite a few different parameters, which have several constraints to fulfill, it is not directly obvious how to generate random instances. We chose the following way. Recall that we restrict ourselves to one player class.

- 1.) Fix the number of elements m , number of strategies n , the demand d , and a maximal degree p .
- 2.) For each $k \in \{0, \dots, p\}$ fix $0 \leq \theta_{k,\min} \leq \theta_{k,\max}$ and $P_k \in [0, 1]$.
- 3.) Fix numbers $0 < c_{\min} \leq c_{\max}$ and $0 < r_{\min} \leq r_{\max}$.
- 4.) For each strategy S , fix $P_{S,\min}, P_{S,\max} \in [0.0, 1.0]$.
- 5.) For each strategy S , choose a number $P_S \in [P_{S,\min}, P_{S,\max}]$ uniformly at random.
- 6.) For each $S \in \mathfrak{S}$:

Initialize $P := P_S$. Then for $e = 1, \dots, m$ do: with probability $1 - P$ set C_{eS} and R_{eS} to 0, and with probability P choose C_{eS} from $[c_{\min}, c_{\max}]$ uniformly at random and R_{eS} from $[r_{\min}, r_{\max}]$. Until the strategy has at least one non-zero entry, increase P after each step in such a way that if $e = m$ and still no non-zero entry exists, we will have $P = 1$. After a non-zero entry was chosen, reset $P := P_S$ and never touch it again until the next strategy is treated. This is just one way to ensure our requirement that there must be no empty strategy.

7.) For each $e \in E$:

For each $k \in [p]$ with probability $1 - P_k$ set $\theta_{e,k}$ to 0, and with probability P_k choose $\theta_{e,k}$ from $[\theta_{k,\min}, \theta_{k,\max}]$ uniformly at random. If all so chosen coefficients are zero, start over again for that element. Finally define $\ell_e(x) := \sum_{k=0}^p \theta_{e,k} x^k$. This procedure ensures that $\ell_e(x) > 0$ for all $e \in E$ and $x > 0$, and so that $\text{OPT} > 0$.

We do experiments in series. For each series, steps 1.) to 4.) are executed once, and then steps 5.) to 7.) are executed many times, and computations are done for each instance. We now describe the scheme by which we fix the parameters in steps 1.) to 4.). We fix the following parameters for all experiments: $m = 4$ and $[P_{S,\min}, P_{S,\max}] = [0.1, 1.0]$ for all $S \in \mathfrak{S}$. We also fix $c_{\min} = r_{\min} = 1$, which is without loss of generality by a scaling argument.

The number of strategies ranges in $n \in \{2, \dots, 6, 9\}$, which was chosen to cover the range where n is not far from m , and also to show the effects of n being larger than m . Regarding demand, we run each experiment once with $d = 1$ and once with $d = 10$, in order to cover two different scales of congestions, latencies, and social costs.

Latency Functions. The maximum degree of polynomials in element latency functions ranges in $p \in \{1, 2, 3\}$. We did not consider larger degrees since the computational overhead for solving the NLPs increases substantially with the degree. Fix one p for now. We use two different sets of parameters for the element latency functions. Instances using the first set of parameters are called *single*: $P_p = 1.0$ and $P_k = 0$ for $k \neq p$, $[\theta_{k,\min}, \theta_{k,\max}] = [1.0, 1.0]$ for all k . That is, we have $\ell_e(x) = x^p$ for all $e \in E$. We denote such a vector of parameters by $single := (\theta_{k,\min}, \theta_{k,\max}, P_k)_{k \in \{0, \dots, p\}}$.

Instances using the second set of parameters are called *mixed*: we set $P_k = 0.5$ and $[\theta_{k,\min}, \theta_{k,\max}] = [0.01, 1.0]$ for all $k > 0$, and $P_0 = 0.5$ and $[\theta_{0,\min}, \theta_{0,\max}] = [0.01, c_{\max} r_{\max}]$. That is, for each element latency function, each of the coefficients has a probability of 0.5 to be non-zero, and if so, it is chosen uniformly at random from the interval $[0.01, 1.0]$ in case of a non-constant term and $[0.01, c_{\max} r_{\max}]$ for the constant term. The 0.01 is to prevent too small numbers and the $c_{\max} r_{\max}$ is inspired by the example in the proof of Thm. 1(ii) in [4]. This will lead to a variety of element latency functions: constant ones and non-constant ones, and ones with or without constant term. We denote such a vector of parameters by $mixed := (\theta_{k,\min}, \theta_{k,\max}, P_k)_{k \in \{0, \dots, p\}}$.

Consumption and Relevance Numbers. For one group of series, we fix the range for consumption numbers to $[1.0, 9.0]$ and then go through three different intervals for the relevance numbers: $[1.0, 1.0]$, $[1.0, 5.0]$, and $[1.0, 9.0]$. For the other group of series, we fix the range for relevance numbers to $[1.0, 9.0]$ and then go through three different intervals for the consumption numbers, just as above for relevance numbers.

Summary. We run the following experiments: Fix $m := 4$. For each $p \in \{1, 2, 3\}$, each $d \in \{1, 10\}$, each $(\theta_{k,\min}, \theta_{k,\max}, P_k)_{k \in \{0, \dots, p\}} \in \{single, mixed\}$, and each $n \in \{2, 3, 4, 5, 6, 9\}$ do:

1. (Fixed consumption number range.) Fix $[c_{\min}, c_{\max}] = [1.0, 9.0]$ and $r_{\min} = 1.0$. For each $r_{\max} \in \{1.0, 5.0, 9.0\}$ generate N_p random instances and compute their observed price of anarchy, bounds, and parameters.
2. (Fixed relevance number range.) Fix $[r_{\min}, r_{\max}] = [1.0, 9.0]$ and $c_{\min} = 1.0$. For each $c_{\max} \in \{1.0, 5.0, 9.0\}$ generate N_p random instances and compute their observed price of anarchy, bounds, and parameters.

The numbers N_p are $N_1 = 32,000$, $N_2 = 16,000$, and $N_3 = 16,000$. So, we treat fewer instances of higher degrees due to their higher computational overhead.

6. Qualitative Observations

We first state some qualitative observations and in the next sections substantiate them by quantitative results. All statements concerning convexity and Perakis' bound are only based on the data gained for $p = 1$.

- Provoking an observed price of anarchy that is close to the conjectured bound requires the choice of specific parameters. For many settings, the observed price of anarchy usually is far away from that bound, as can be seen by the TR values, especially for $n \geq m$.
- We observed the more instances with an observed price of anarchy close or on the conjectured bound the smaller $r_{\max} - r_{\min}$ was compared to $c_{\max} - c_{\min}$. This can be seen by comparing the TR90 values from series with fixed consumption number range, which is $[1.0, 9.0]$, and small relevance number range against values from series with fixed relevance number range. The TR values are even higher when we restrict the class of latency functions to single.
- For fixed m , the fraction of instances with convex SC decreases with increasing n , and that decrease is rapid once $n \geq m$.
- It makes a substantial difference using the projected Hessian instead of the Hessian itself for determining the convexity of SC on \mathcal{A} . There are many instances with SC convex on \mathcal{A} but not globally convex, i.e., not convex on \mathbb{R}^n .

A globally convex SC is equivalent to $J\vec{L}$ being positive semidefinite, and hence is equivalent to one of Perakis' bounds being applicable. Since so many instances have no globally convex SC, Perakis' bounds are also not applicable for many instances.

- We found only a very few instances where Perakis' semidefinite bound would have been applicable but the positive definite bound not; see the differences between columns number four and five of each table.

Table 1: Single element latency functions, $d = 1$, fixed consumption number range.

n	r_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	90.41	89.41	89.41	100.00	1.09	26384	52.10	1.92	0.1895
3	1.0	71.84	61.21	61.20	100.00	1.13	31553	28.62	1.67	0.1204
4	1.0	43.26	22.19	22.19	100.00	1.17	31978	15.06	0.93	0.1032
5	1.0	13.14	0.00	0.00	100.00	1.21	32000	9.37	0.58	0.0562
6	1.0	0.00	0.00	0.00	100.00	1.24	32000	6.59	0.36	0.0281
9	1.0	0.00	0.00	0.00	100.00	1.35	32000	3.76	0.11	0.0156
2	5.0	92.93	88.78	88.78	99.70	1.07	26221	37.45	0.05	0.0000
3	5.0	73.78	56.41	56.39	99.53	1.09	31395	16.04	0.04	0.0000
4	5.0	40.70	16.60	16.58	99.60	1.11	31860	5.20	0.00	0.0000
5	5.0	10.97	0.00	0.00	99.56	1.12	31860	1.49	0.00	0.0000
6	5.0	0.00	0.00	0.00	99.55	1.14	31857	0.51	0.00	0.0000
9	5.0	0.00	0.00	0.00	99.53	1.16	31849	0.04	0.00	0.0000
2	9.0	91.80	86.75	86.75	99.67	1.07	26375	33.55	0.06	0.0000
3	9.0	70.99	52.85	52.84	99.43	1.10	31351	12.98	0.03	0.0000
4	9.0	37.23	14.42	14.41	99.38	1.12	31784	3.84	0.00	0.0000
5	9.0	9.27	0.00	0.00	99.31	1.13	31779	1.05	0.00	0.0000
6	9.0	0.00	0.00	0.00	99.28	1.14	31768	0.32	0.00	0.0000
9	9.0	0.00	0.00	0.00	99.32	1.16	31783	0.02	0.00	0.0000

7. Tables for $p = 1$

The eight tables starting with Tab.1 show results for several parameters and 32,000 randomly generated instances in each row. Numbers are rounded to the shown number of decimal digits. The first column gives the number of strategies. The second column gives the upper bound of the range for consumption numbers or relevance numbers, respectively. The next two columns give the percentage of instances with convex SC (abbreviated “C”) and globally convex SC (abbreviated “GC”), respectively. The next column shows the percentage of instances for which Perakis’ positive definite bound is applicable.

The sixth column (titled “%OK”) gives the percentage of instances where the binary search found at least one acceptable solution. Only these instances form the basis for the values in the following columns. The next column gives the average price of anarchy. The eighth column (titled “ $\#(\gamma > 1)$ ”) gives the number of instances with $\gamma > 1$ (recall that we in fact test for $\gamma > 1.01$). In the following three columns we have the percentages of instances with the specified TR x property (relative to the number of instances with $\gamma > 1$) with $x \in \{50, 90, 100\}$. We give more TR x values, namely for $x \in \{30, 40, \dots, 100\}$, in the appendix. The reason we give column number eight is to show how many instances count for the TR x statistics.

Table 2: Single element latency functions, $d = 1$, fixed relevance number range.

n	c_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	90.58	89.52	89.52	100.00	1.01	26312	47.41	0.00	0.0000
3	1.0	71.83	60.89	60.89	99.99	1.02	31483	24.19	0.00	0.0000
4	1.0	42.90	21.93	21.93	99.98	1.03	31972	9.75	0.00	0.0000
5	1.0	13.42	0.00	0.00	99.97	1.04	31989	3.63	0.00	0.0000
6	1.0	0.00	0.00	0.00	99.98	1.04	31994	1.21	0.00	0.0000
9	1.0	0.00	0.00	0.00	99.95	1.04	31985	0.05	0.00	0.0000
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2	5.0	92.77	88.38	88.38	99.85	1.05	26426	37.85	0.05	0.0000
3	5.0	73.41	56.46	56.45	99.67	1.07	31394	15.93	0.00	0.0000
4	5.0	41.06	17.21	17.20	99.61	1.08	31861	4.90	0.01	0.0000
5	5.0	10.91	0.00	0.00	99.67	1.09	31893	1.51	0.00	0.0000
6	5.0	0.00	0.00	0.00	99.63	1.10	31882	0.40	0.00	0.0000
9	5.0	0.00	0.00	0.00	99.67	1.11	31895	0.02	0.00	0.0000
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2	9.0	91.67	86.50	86.50	99.64	1.07	26214	33.42	0.05	0.0000
3	9.0	70.73	52.17	52.17	99.45	1.10	31375	12.82	0.02	0.0000
4	9.0	37.20	14.55	14.55	99.41	1.12	31799	3.99	0.00	0.0000
5	9.0	9.32	0.00	0.00	99.36	1.13	31795	1.03	0.00	0.0000
6	9.0	0.00	0.00	0.00	99.38	1.15	31801	0.30	0.00	0.0000
9	9.0	0.00	0.00	0.00	99.37	1.16	31798	0.02	0.00	0.0000

Table 3: Mixed element latency functions, $d = 1$, fixed consumption number range.

n	r_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	77.75	55.85	55.85	100.00	1.04	26310	48.85	0.72	0.0950
3	1.0	43.65	21.51	21.51	100.00	1.07	31506	25.93	1.09	0.1206
4	1.0	14.57	3.42	3.42	100.00	1.09	31985	12.84	0.87	0.1126
5	1.0	1.84	0.00	0.00	100.00	1.12	31999	6.75	0.67	0.1156
6	1.0	0.00	0.00	0.00	100.00	1.15	31999	4.63	0.64	0.1281
9	1.0	0.00	0.00	0.00	100.00	1.24	32000	3.57	0.46	0.0406
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2	5.0	84.29	55.71	55.71	99.90	1.01	26413	35.85	0.02	0.0000
3	5.0	49.00	20.21	20.20	99.92	1.02	31525	15.14	0.04	0.0000
4	5.0	15.08	2.87	2.87	99.95	1.02	31971	4.90	0.01	0.0000
5	5.0	1.55	0.00	0.00	99.95	1.03	31983	1.36	0.01	0.0000
6	5.0	0.00	0.00	0.00	99.93	1.04	31979	0.49	0.01	0.0000
9	5.0	0.00	0.00	0.00	99.89	1.07	31965	0.12	0.00	0.0000
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2	9.0	83.29	54.21	54.21	99.95	1.01	26337	31.59	0.01	0.0000
3	9.0	47.06	19.02	19.01	99.95	1.01	31490	11.99	0.03	0.0000
4	9.0	13.62	2.34	2.34	99.95	1.02	31966	3.59	0.03	0.0000
5	9.0	1.30	0.00	0.00	99.94	1.03	31981	0.96	0.01	0.0000
6	9.0	0.00	0.00	0.00	99.94	1.03	31980	0.28	0.01	0.0000
9	9.0	0.00	0.00	0.00	99.92	1.06	31975	0.08	0.00	0.0000

Table 4: Mixed element latency functions, $d = 1$, fixed relevance number range.

n	c_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	77.56	55.82	55.82	100.00	1.00	26311	46.49	0.00	0.0000
3	1.0	44.33	22.15	22.14	100.00	1.00	31531	23.83	0.01	0.0000
4	1.0	14.60	3.40	3.39	100.00	1.01	31982	9.32	0.01	0.0000
5	1.0	1.87	0.00	0.00	100.00	1.01	31999	3.42	0.00	0.0000
6	1.0	0.00	0.00	0.00	100.00	1.01	31999	1.13	0.00	0.0000
9	1.0	0.00	0.00	0.00	99.99	1.01	31996	0.04	0.00	0.0000
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2	5.0	84.24	55.78	55.77	99.95	1.00	26301	35.61	0.03	0.0000
3	5.0	49.05	20.34	20.34	99.98	1.01	31529	14.40	0.02	0.0000
4	5.0	15.35	2.88	2.88	99.98	1.01	31973	4.72	0.02	0.0031
5	5.0	1.67	0.00	0.00	99.97	1.02	31989	1.37	0.01	0.0000
6	5.0	0.00	0.00	0.00	99.95	1.02	31985	0.43	0.01	0.0000
9	5.0	0.00	0.00	0.00	99.94	1.04	31982	0.05	0.00	0.0000
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2	9.0	83.65	54.27	54.26	99.96	1.01	26570	30.92	0.02	0.0000
3	9.0	47.02	18.80	18.80	99.96	1.01	31514	11.93	0.03	0.0000
4	9.0	13.40	2.29	2.29	99.97	1.02	31973	3.66	0.01	0.0000
5	9.0	1.25	0.00	0.00	99.93	1.03	31978	0.93	0.00	0.0000
6	9.0	0.00	0.00	0.00	99.94	1.04	31982	0.33	0.01	0.0000
9	9.0	0.00	0.00	0.00	99.90	1.06	31969	0.06	0.00	0.0000

Table 5: Single element latency functions, $d = 10$, fixed consumption number range.

n	r_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	90.22	89.21	89.21	100.00	1.09	26251	52.83	1.86	0.0114
3	1.0	71.92	60.99	60.99	100.00	1.13	31517	28.57	1.56	0.0159
4	1.0	43.01	22.31	22.30	99.99	1.17	31979	15.55	0.98	0.0063
5	1.0	12.93	0.00	0.00	99.98	1.20	31993	9.25	0.57	0.0031
6	1.0	0.00	0.00	0.00	99.99	1.24	31996	6.75	0.30	0.0031
9	1.0	0.00	0.00	0.00	99.99	1.35	31997	3.73	0.12	0.0000
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2	5.0	92.65	88.26	88.25	99.35	1.06	26214	37.63	0.01	0.0038
3	5.0	73.97	56.80	56.78	98.79	1.09	31095	15.78	0.01	0.0000
4	5.0	41.29	17.50	17.48	98.42	1.10	31487	5.11	0.00	0.0000
5	5.0	11.55	0.00	0.00	98.23	1.11	31433	1.57	0.00	0.0000
6	5.0	0.00	0.00	0.00	98.11	1.12	31396	0.42	0.00	0.0000
9	5.0	0.00	0.00	0.00	98.10	1.14	31393	0.03	0.00	0.0000
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2	9.0	91.87	86.35	86.34	99.37	1.07	26193	33.17	0.02	0.0000
3	9.0	70.46	52.11	52.11	98.71	1.09	31142	12.62	0.01	0.0000
4	9.0	37.55	14.95	14.93	98.19	1.10	31406	3.83	0.00	0.0000
5	9.0	9.41	0.00	0.00	98.08	1.11	31387	0.99	0.00	0.0000
6	9.0	0.00	0.00	0.00	97.81	1.12	31299	0.24	0.00	0.0000
9	9.0	0.00	0.00	0.00	97.80	1.15	31295	0.01	0.00	0.0000

Table 6: Single element latency functions, $d = 10$, fixed relevance number range.

n	c_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	90.12	89.05	89.05	100.00	1.01	26393	47.49	0.00	0.0000
3	1.0	72.00	61.40	61.39	99.98	1.02	31510	24.07	0.00	0.0000
4	1.0	43.43	22.74	22.73	99.94	1.03	31961	9.88	0.00	0.0000
5	1.0	13.32	0.00	0.00	99.91	1.03	31970	3.43	0.00	0.0000
6	1.0	0.00	0.00	0.00	99.87	1.04	31959	1.17	0.00	0.0000
9	1.0	0.00	0.00	0.00	99.90	1.04	31967	0.04	0.00	0.0000
2	5.0	92.86	88.33	88.32	99.58	1.05	26346	37.24	0.02	0.0000
3	5.0	73.37	56.32	56.30	99.17	1.07	31288	15.76	0.01	0.0000
4	5.0	41.55	17.59	17.57	98.95	1.08	31645	5.04	0.00	0.0000
5	5.0	11.12	0.00	0.00	98.92	1.09	31653	1.37	0.00	0.0000
6	5.0	0.00	0.00	0.00	98.79	1.09	31612	0.48	0.00	0.0000
9	5.0	0.00	0.00	0.00	98.87	1.10	31639	0.02	0.00	0.0000
2	9.0	91.87	86.64	86.64	99.39	1.06	26212	33.55	0.04	0.0000
3	9.0	70.44	52.27	52.26	98.69	1.09	31116	12.91	0.02	0.0000
4	9.0	37.44	14.74	14.74	98.24	1.10	31424	3.79	0.00	0.0000
5	9.0	9.10	0.00	0.00	98.01	1.11	31363	0.99	0.00	0.0000
6	9.0	0.00	0.00	0.00	97.72	1.12	31272	0.23	0.00	0.0000
9	9.0	0.00	0.00	0.00	97.62	1.15	31239	0.02	0.00	0.0000

Table 7: Mixed element latency functions, $d = 10$, fixed consumption number range.

n	r_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	77.69	55.66	55.65	100.00	1.10	26357	53.25	0.90	0.0038
3	1.0	43.86	21.26	21.26	100.00	1.12	31541	28.33	0.86	0.0063
4	1.0	14.49	3.52	3.52	100.00	1.13	31980	13.88	0.57	0.0000
5	1.0	1.88	0.00	0.00	100.00	1.15	31999	7.26	0.33	0.0000
6	1.0	0.00	0.00	0.00	100.00	1.15	31999	4.39	0.20	0.0031
9	1.0	0.00	0.00	0.00	100.00	1.18	31999	1.83	0.09	0.0000
2	5.0	84.55	56.15	56.14	99.46	1.03	26195	36.68	0.04	0.0000
3	5.0	49.43	20.44	20.43	99.26	1.04	31301	15.57	0.09	0.0000
4	5.0	15.17	2.93	2.93	99.27	1.05	31751	5.29	0.01	0.0000
5	5.0	1.37	0.00	0.00	99.16	1.06	31729	1.67	0.01	0.0000
6	5.0	0.00	0.00	0.00	99.10	1.06	31711	0.42	0.00	0.0000
9	5.0	0.00	0.00	0.00	99.12	1.08	31717	0.03	0.00	0.0000
2	9.0	83.43	54.24	54.24	99.56	1.02	26297	31.50	0.04	0.0000
3	9.0	46.63	18.77	18.77	99.40	1.03	31349	13.04	0.04	0.0032
4	9.0	13.49	2.48	2.48	99.34	1.04	31777	3.88	0.01	0.0000
5	9.0	1.29	0.00	0.00	99.30	1.05	31775	1.02	0.00	0.0000
6	9.0	0.00	0.00	0.00	99.28	1.06	31769	0.26	0.00	0.0000
9	9.0	0.00	0.00	0.00	99.08	1.07	31706	0.03	0.00	0.0000

Table 8: Mixed element latency functions, $d = 10$, fixed relevance number range.

n	c_{\max}	%C	%GC	%Perakis	%OK	avg.	$\#(\gamma > 1)$	%TR50	%TR90	%TR100
2	1.0	77.52	55.58	55.58	100.00	1.01	26382	47.07	0.00	0.0000
3	1.0	43.66	21.29	21.28	99.98	1.02	31489	24.08	0.07	0.0000
4	1.0	14.61	3.45	3.45	99.93	1.03	31962	10.12	0.03	0.0000
5	1.0	2.02	0.00	0.00	99.97	1.03	31988	3.74	0.00	0.0000
6	1.0	0.00	0.00	0.00	99.95	1.03	31984	1.32	0.00	0.0000
9	1.0	0.00	0.00	0.00	99.96	1.04	31986	0.08	0.00	0.0000
2	5.0	84.24	55.82	55.81	99.65	1.02	26380	35.74	0.02	0.0000
3	5.0	49.10	20.49	20.48	99.59	1.03	31388	15.39	0.03	0.0000
4	5.0	14.69	2.60	2.60	99.58	1.04	31854	5.13	0.01	0.0000
5	5.0	1.50	0.00	0.00	99.52	1.04	31846	1.42	0.01	0.0000
6	5.0	0.00	0.00	0.00	99.54	1.05	31853	0.38	0.00	0.0000
9	5.0	0.00	0.00	0.00	99.48	1.06	31834	0.04	0.00	0.0000
2	9.0	83.66	54.72	54.72	99.64	1.02	26365	31.92	0.06	0.0000
3	9.0	46.92	18.31	18.31	99.34	1.03	31337	12.66	0.04	0.0000
4	9.0	13.69	2.32	2.32	99.34	1.04	31768	3.86	0.00	0.0000
5	9.0	1.27	0.00	0.00	99.33	1.05	31785	1.04	0.01	0.0000
6	9.0	0.00	0.00	0.00	99.24	1.06	31758	0.27	0.00	0.0000
9	9.0	0.00	0.00	0.00	99.10	1.07	31712	0.02	0.00	0.0000

8. Scatter Plots for $p = 1$

We use scatter plots to give an additional impression of the distribution of observed price of anarchy in relation to the conjectured bound. A dot is drawn for every single instance. The horizontal position of the dot is determined by the conjectured bound, and the vertical position is determined by the observed price of anarchy. A red line marks where the observed price of anarchy equals conjectured bound. Dots positioned close to that line have TR x for high x , and a dot positioned to the upper left of that line would have been a counter-examples to the conjecture. A few dots are positioned vertically below 1, which is an impossible value for the price of anarchy. These dropout cases disappear when we remove the non-convex cases from the data set.

There is a peculiarity best visible in Fig. 3: a curve emerges that starts at the lower left corner and shortly before $\gamma = 2 = (p + 1)^{1/p}$, it follows the conjectured bound. A possible explanation is that our scaling technique leads to such a concentration of instances along that curve. We have no further understanding of this phenomenon yet.

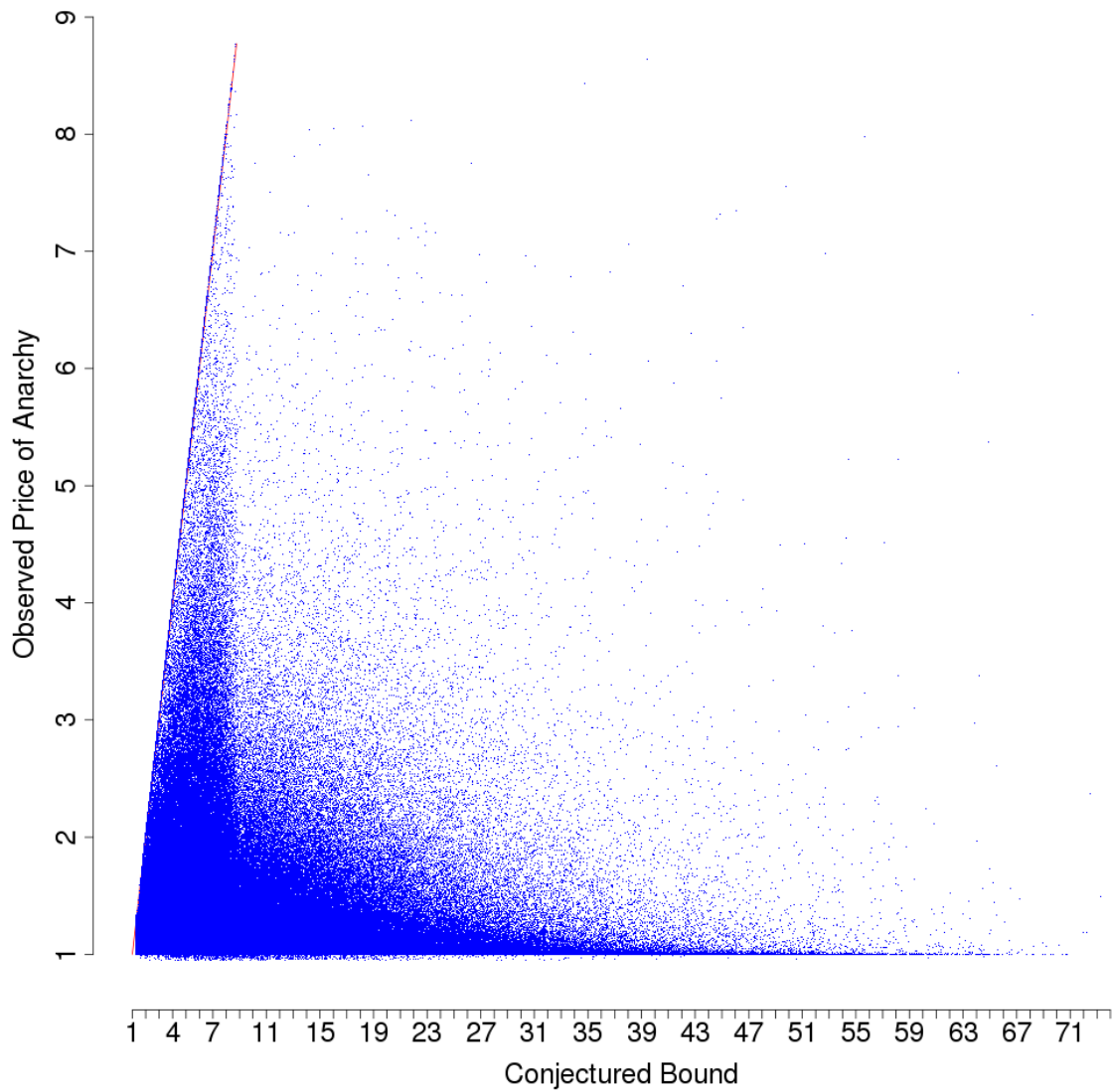


Figure 1: Scatter plot for random instances with affine element latency functions. All 2,292,886 instances from the series with *fixed consumption number range* are shown that yielded an acceptable solution in the binary search.

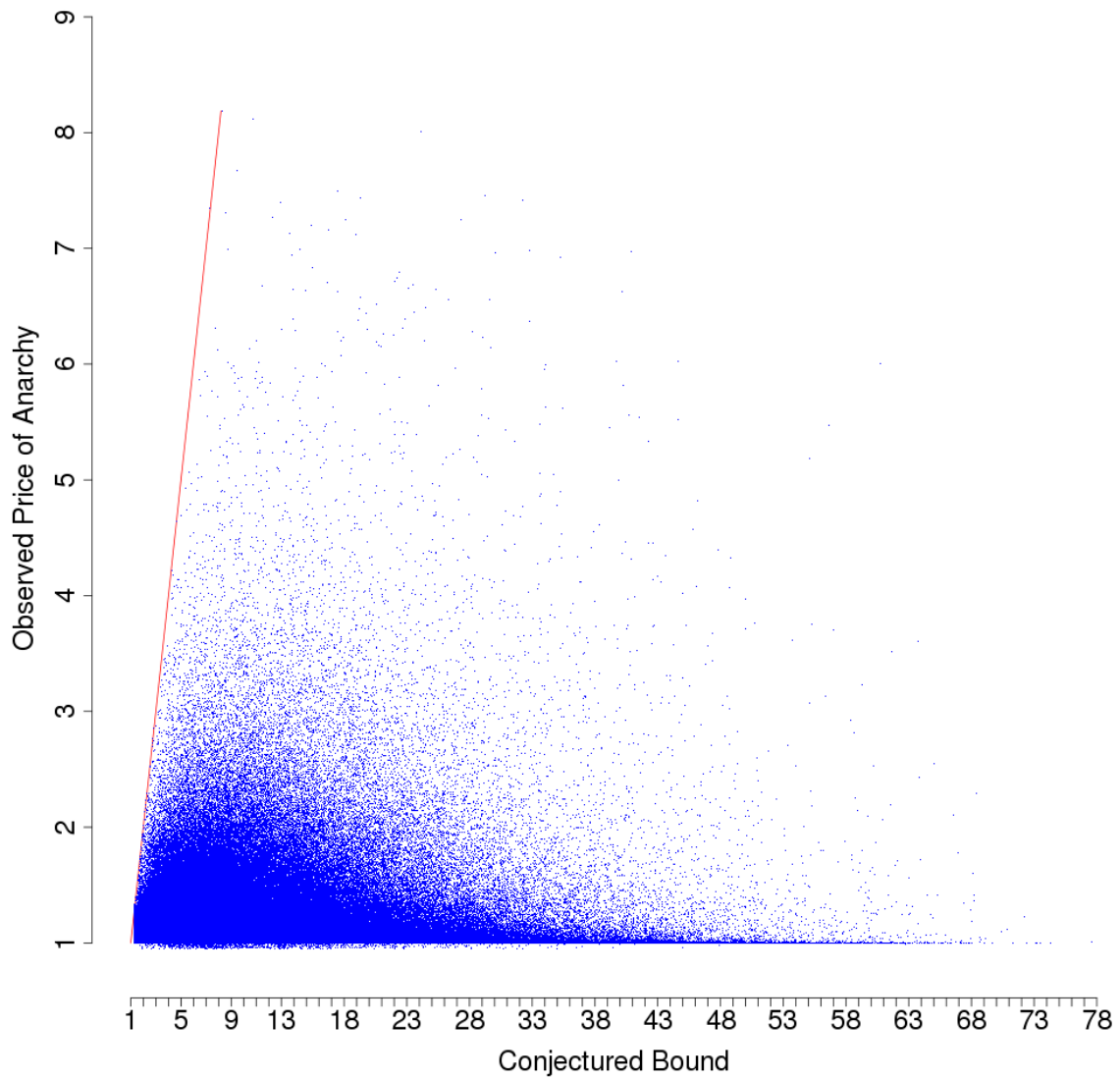


Figure 2: Scatter plot for random instances with affine element latency functions. All 2,294,621 instances from the series with *fixed relevance number range* are shown that yielded an acceptable solution in the binary search.

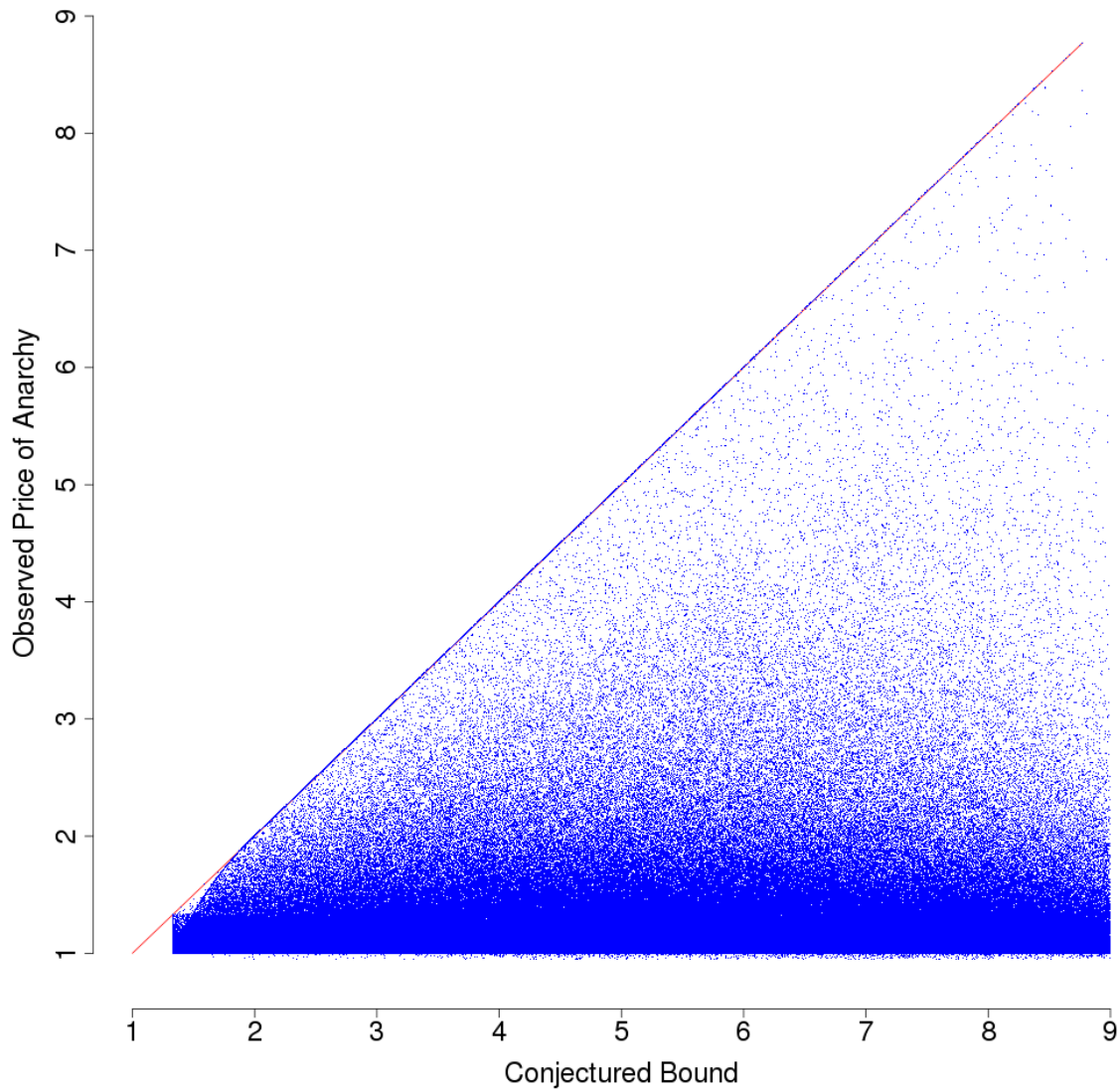


Figure 3: Scatter plot for random instances with affine element latency functions. The altogether 4,587,507 instances from Fig. 1 and Fig. 2 were considered. However, for the sake of a better scale, only those are shown which have γ smaller than the maximum observed price of anarchy, resulting in 3,557,153 instances shown.

9. Comparison with Perakis' Bound

We compare Perakis' bound for the positive definite case to our proven and our conjectured bound. Data from all 4,608,000 experiments with affine element latency functions is used in which Perakis' bound was applicable (and which yielded an acceptable solution in the binary search). This is approximately 20% of the instances, namely 926,185 instances in total. In Fig. 4 we show a comparison with our proven bound by the means of a scatter plot. Perakis' bound looks better than ours. However, the computation of Perakis' bound also yielded some extraordinary high values, up to the value of 8,246, which were removed from the data set before creating the plot for the sake of a reasonable scale. It is currently not understood whether these high values are to be attributed to numerical errors, or reflect the true values. In Fig. 5 a comparison with our conjectured bound is given. The plot promises a good relative performance of our bound – note in particular that 1,552 values of Perakis' bound are off-scale. However, in Fig 6, which shows a magnification of the left part, Perakis' bound still looks more concentrated to the left, except the tail horizontally positioned around 1 and vertically reaching out to the right.

For a more quantitative evaluation, we give two histograms in Fig. 7. They show the distribution of the ratios of Perakis' bound to our proven and conjectured bound, respectively. A ratio lesser than 1 means that Perakis' bound is smaller (hence better), and a ratio of more than 1 means that our bound is smaller.

Even our conjectured bound appears to be outperformed by Perakis' most of the time in terms of closely bounding the price of anarchy. However, our bounds have the strengths of being always applicable (not only under Jacobian definiteness conditions) and computationally much simpler to obtain.

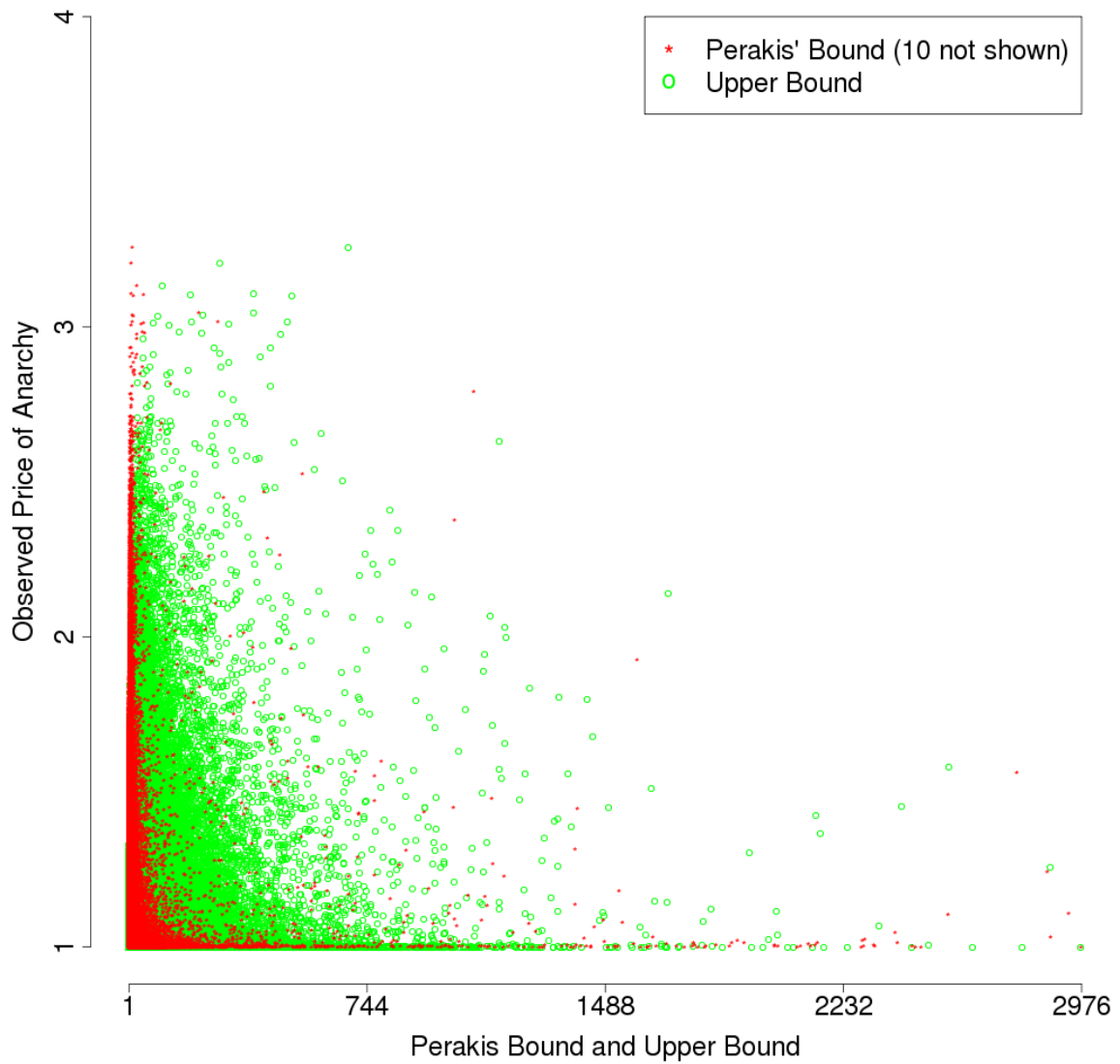


Figure 4: Scatter plot with Perakis bound (red) and our proven bound (green). Values of 926,175 instances are shown. For those not shown, the values of Perakis' bound range up to 8,246.

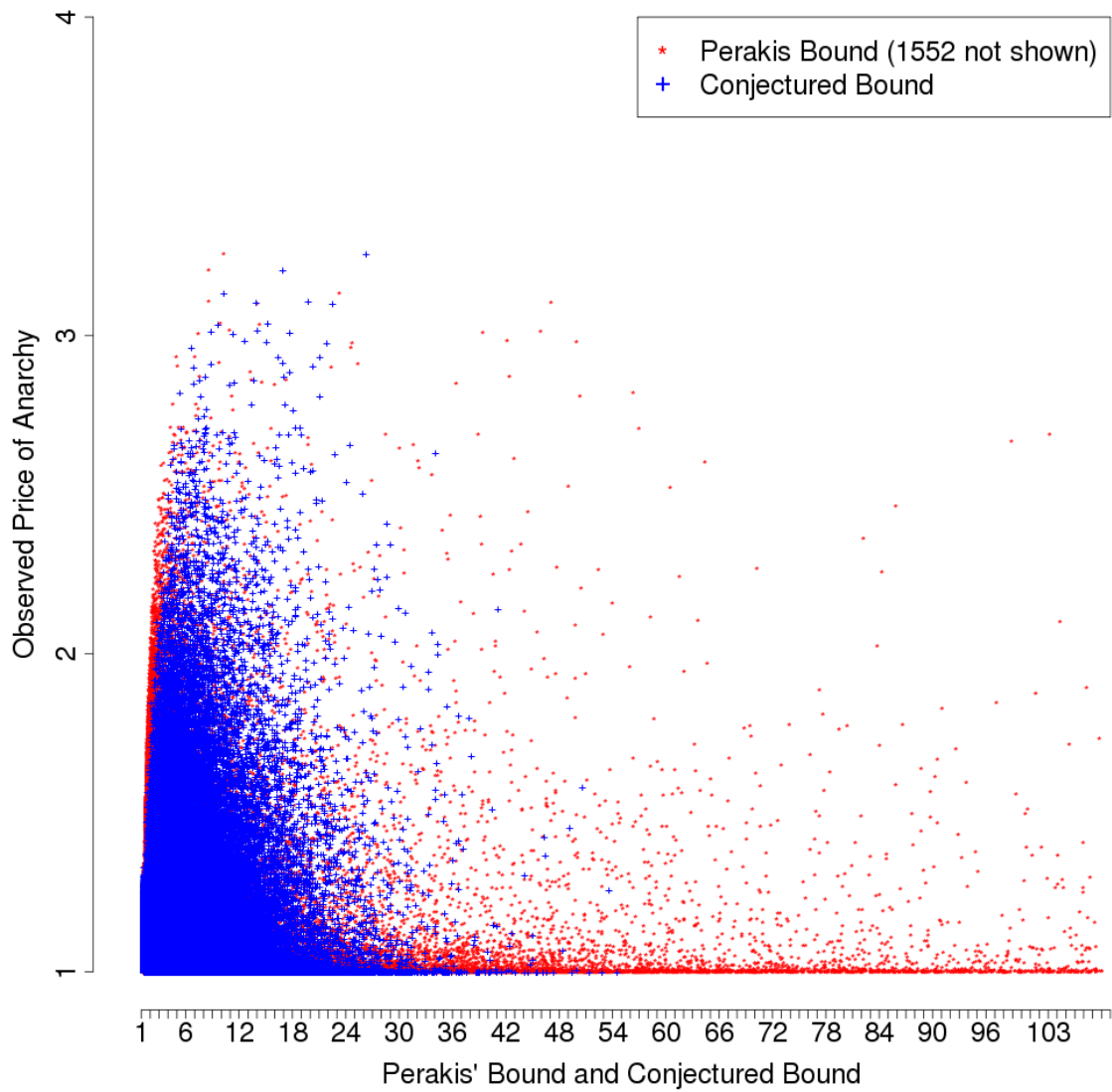


Figure 5: Scatter plot comparing the Perakis bound (red) to our conjectured bound (blue). Only those cases with Perakis' bound no more than two times the maximum conjectured bound were chosen, resulting in 924,633 instances shown.

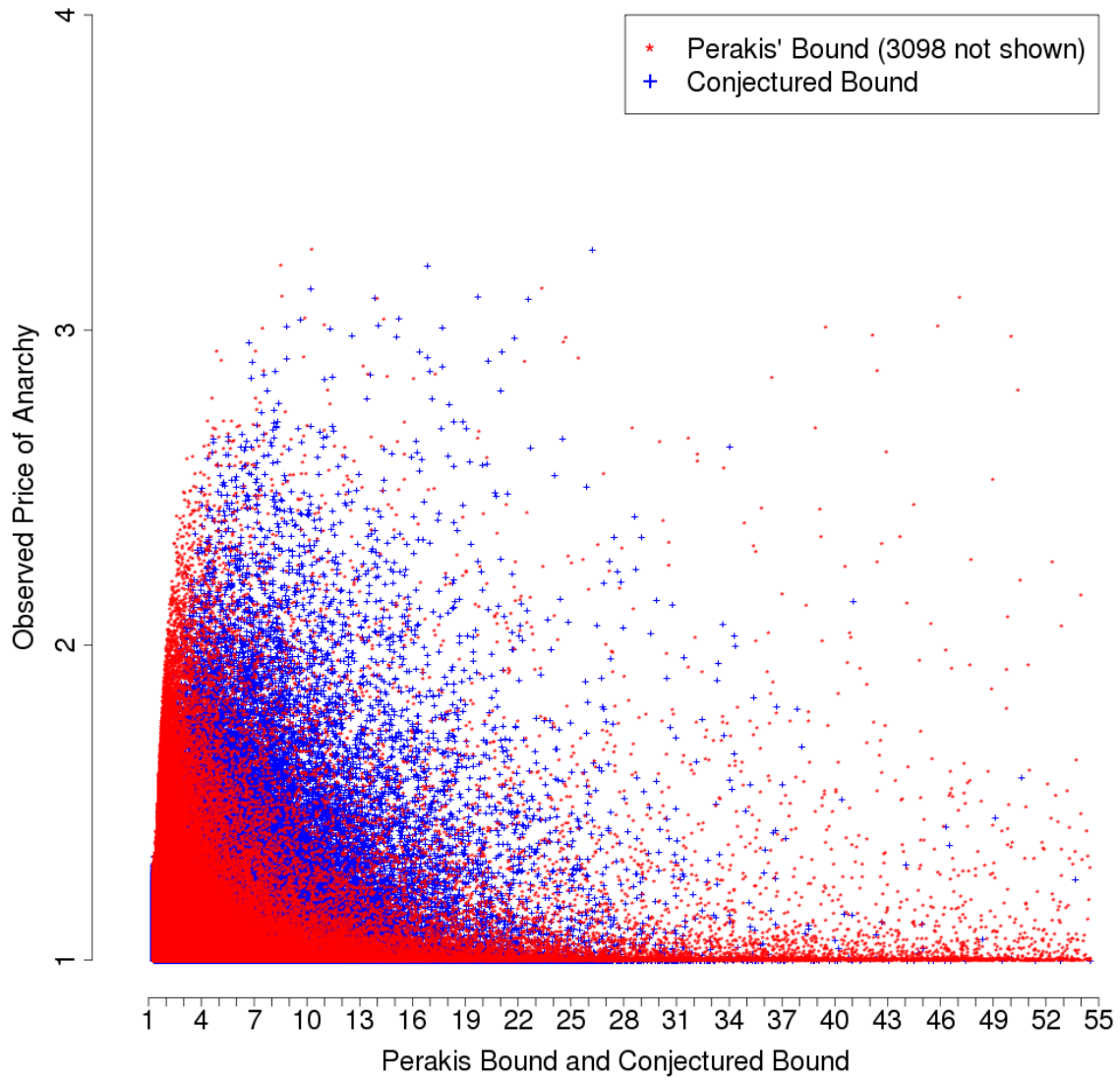


Figure 6: A magnification of the left part of Fig. 5, with Perakis' bound (red) plotted over ours (blue). The plot comprises 923,087 instances.

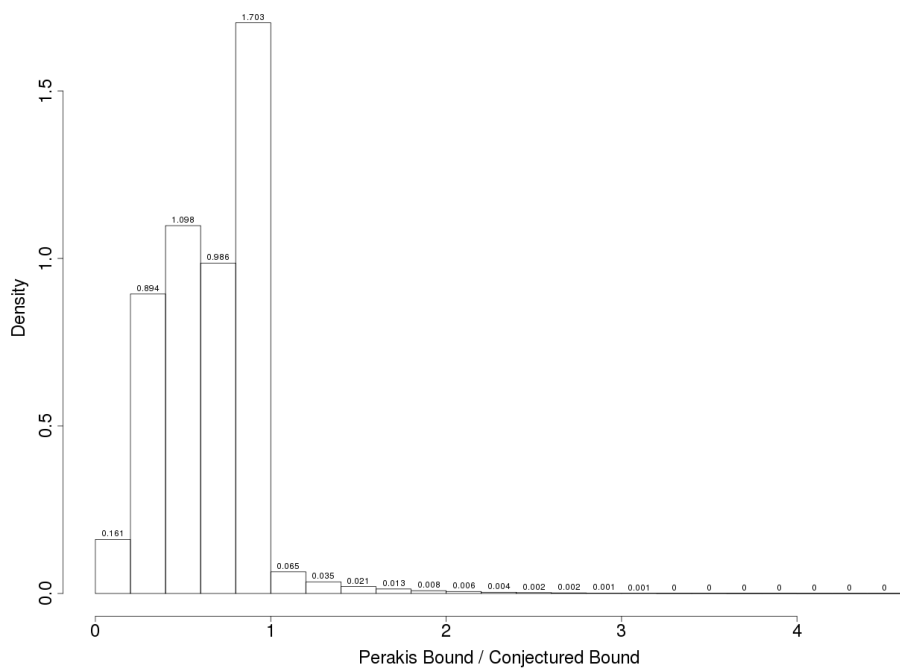
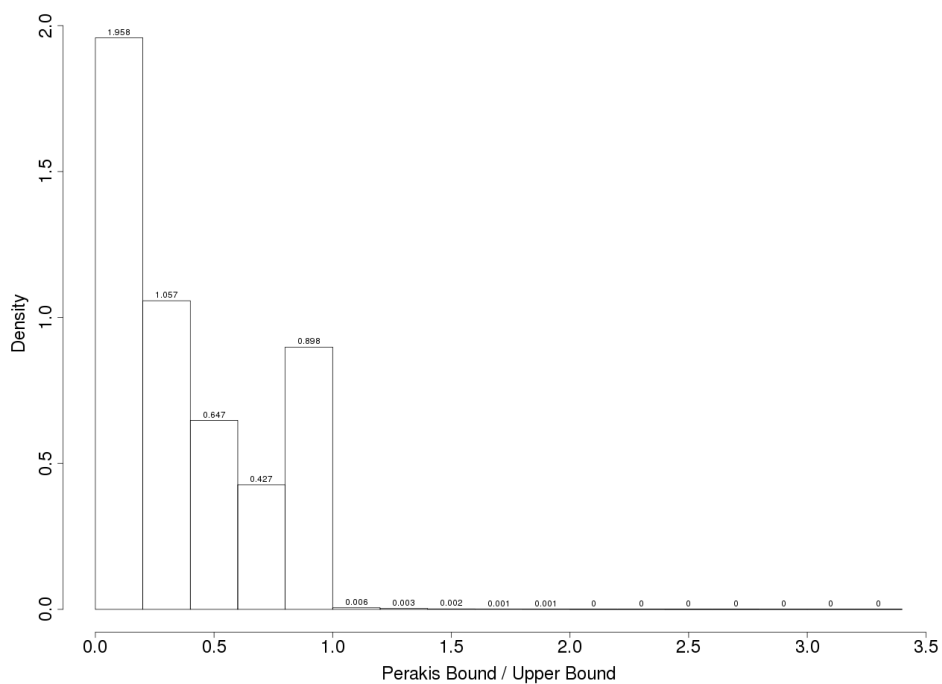


Figure 7: Histograms of the ratio between Perakis' bound and our proven bound at the top, and our conjectured bound below. Only the 97% smallest values of Perakis' bound were considered.

10. Open Problems

Challenges for future work mainly point in three directions from here.

1. Prove or disprove the conjectured upper bound on the price of anarchy.
2. Develop algorithms to compute optima and Nash equilibria in certain non-convex cases. Our results can serve as a benchmark for future experimental studies: will a new algorithm be able to deliver substantially higher TR_x values for the same random model? Will a new algorithm succeed in disproving the conjecture by the discovery of a counter-example?
3. Analyze the causalities between the random model from Sec. 5 and the observations and results in Sec. 6.

Acknowledgments

I thank Prof. Srivastav for introducing me to the subject, his support, and for giving me the opportunity to participate in the DFG Priority Program 1126. I thank our student Ole Kliemann for his work on the implementation, his support while conducting the experiments, and helpful discussions. I thank Andreas Baltz and our student Sandro Esquivel for contributions in early stages of the project. And I thank the Deutsche Forschungsgemeinschaft for their financial support through Grant Sr7-3 in the Priority Program 1126 “Algorithmics of Large and Complex Networks”.

Table 9: TR x values for degree $p = 1$, demand $d = 1$, single element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26384	81.42	66.65	52.10	37.10	16.07	2.90	1.92	0.1895
3	1.0	100.00	31553	62.85	43.03	28.62	17.41	7.61	2.80	1.67	0.1204
4	1.0	100.00	31978	46.16	26.21	15.06	8.14	3.88	1.60	0.93	0.1032
5	1.0	100.00	32000	35.29	17.72	9.37	4.87	2.39	1.00	0.58	0.0562
6	1.0	100.00	32000	28.48	13.44	6.59	3.08	1.42	0.62	0.36	0.0281
9	1.0	100.00	32000	20.11	8.81	3.76	1.57	0.57	0.23	0.11	0.0156
2	5.0	99.70	26221	68.07	51.32	37.45	24.12	8.43	0.18	0.05	0.0000
3	5.0	99.53	31395	44.47	27.00	16.04	8.15	2.20	0.08	0.04	0.0000
4	5.0	99.60	31860	25.12	11.70	5.20	2.01	0.35	0.04	0.00	0.0000
5	5.0	99.56	31860	13.38	4.65	1.49	0.44	0.09	0.01	0.00	0.0000
6	5.0	99.55	31857	7.10	1.86	0.51	0.16	0.03	0.00	0.00	0.0000
9	5.0	99.53	31849	1.37	0.23	0.04	0.01	0.00	0.00	0.00	0.0000
2	9.0	99.67	26375	61.35	46.14	33.55	21.57	7.52	0.16	0.06	0.0000
3	9.0	99.43	31351	36.44	21.56	12.98	6.72	1.78	0.09	0.03	0.0000
4	9.0	99.38	31784	18.42	8.38	3.84	1.51	0.31	0.02	0.00	0.0000
5	9.0	99.31	31779	8.88	2.99	1.05	0.26	0.03	0.00	0.00	0.0000
6	9.0	99.28	31768	4.09	1.07	0.32	0.07	0.01	0.01	0.00	0.0000
9	9.0	99.32	31783	0.50	0.10	0.02	0.00	0.00	0.00	0.00	0.0000

Table 10: TR x values for degree $p = 1$, demand $d = 1$, single element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26312	76.73	61.35	47.41	32.56	12.27	0.00	0.00	0.0000
3	1.0	99.99	31483	57.97	38.30	24.19	12.93	3.19	0.00	0.00	0.0000
4	1.0	99.98	31972	39.81	20.41	9.75	3.75	0.55	0.00	0.00	0.0000
5	1.0	99.97	31989	26.32	10.03	3.63	0.94	0.09	0.00	0.00	0.0000
6	1.0	99.98	31994	17.17	4.79	1.21	0.18	0.01	0.00	0.00	0.0000
9	1.0	99.95	31985	4.09	0.50	0.05	0.00	0.00	0.00	0.00	0.0000
2	5.0	99.85	26426	67.38	51.44	37.85	24.34	8.76	0.16	0.05	0.0000
3	5.0	99.67	31394	43.90	26.77	15.93	8.24	2.24	0.06	0.00	0.0000
4	5.0	99.61	31861	24.37	11.19	4.90	1.80	0.35	0.02	0.01	0.0000
5	5.0	99.67	31893	12.64	4.36	1.51	0.43	0.05	0.01	0.00	0.0000
6	5.0	99.63	31882	6.56	1.70	0.40	0.10	0.01	0.00	0.00	0.0000
9	5.0	99.67	31895	0.94	0.11	0.02	0.00	0.00	0.00	0.00	0.0000
2	9.0	99.64	26214	60.77	45.80	33.42	21.61	7.72	0.14	0.05	0.0000
3	9.0	99.45	31375	36.43	21.69	12.82	6.82	1.84	0.07	0.02	0.0000
4	9.0	99.41	31799	18.32	8.44	3.99	1.56	0.27	0.02	0.00	0.0000
5	9.0	99.36	31795	8.77	2.97	1.03	0.30	0.06	0.02	0.00	0.0000
6	9.0	99.38	31801	4.17	1.14	0.30	0.07	0.01	0.00	0.00	0.0000
9	9.0	99.37	31798	0.58	0.12	0.02	0.00	0.00	0.00	0.00	0.0000

Table 11: TR x values for degree $p = 1$, demand $d = 1$, mixed element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26310	78.69	63.24	48.85	34.04	13.68	1.17	0.72	0.0950
3	1.0	100.00	31506	59.50	39.75	25.93	15.17	6.18	2.00	1.09	0.1206
4	1.0	100.00	31985	42.36	23.54	12.84	6.50	2.81	1.48	0.87	0.1126
5	1.0	100.00	31999	29.67	13.91	6.75	3.51	1.93	1.19	0.67	0.1156
6	1.0	100.00	31999	22.08	9.38	4.63	2.59	1.63	0.99	0.64	0.1281
9	1.0	100.00	32000	12.94	6.18	3.57	2.16	1.34	0.82	0.46	0.0406
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2	5.0	99.90	26413	64.98	48.66	35.85	23.46	8.25	0.08	0.02	0.0000
3	5.0	99.92	31525	40.88	24.82	15.14	7.86	2.13	0.14	0.04	0.0000
4	5.0	99.95	31971	22.45	10.48	4.90	1.78	0.44	0.08	0.01	0.0000
5	5.0	99.95	31983	11.52	3.95	1.36	0.40	0.07	0.03	0.01	0.0000
6	5.0	99.93	31979	5.58	1.54	0.49	0.20	0.08	0.04	0.01	0.0000
9	5.0	99.89	31965	1.27	0.34	0.12	0.06	0.03	0.01	0.00	0.0000
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2	9.0	99.95	26337	57.18	42.95	31.59	20.34	7.01	0.05	0.01	0.0000
3	9.0	99.95	31490	32.81	19.63	11.99	6.26	1.60	0.10	0.03	0.0000
4	9.0	99.95	31966	16.02	7.55	3.59	1.48	0.33	0.07	0.03	0.0000
5	9.0	99.94	31981	7.17	2.58	0.96	0.33	0.08	0.03	0.01	0.0000
6	9.0	99.94	31980	3.39	0.91	0.28	0.10	0.04	0.02	0.01	0.0000
9	9.0	99.92	31975	0.53	0.17	0.08	0.04	0.01	0.00	0.00	0.0000

Table 12: TR x values for degree $p = 1$, demand $d = 1$, mixed element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26311	75.98	60.44	46.49	31.58	11.67	0.02	0.00	0.0000
3	1.0	100.00	31531	56.98	37.45	23.83	12.62	3.30	0.08	0.01	0.0000
4	1.0	100.00	31982	37.62	19.26	9.32	3.62	0.58	0.03	0.01	0.0000
5	1.0	100.00	31999	24.68	9.65	3.42	0.93	0.07	0.01	0.00	0.0000
6	1.0	100.00	31999	15.66	4.45	1.13	0.19	0.01	0.00	0.00	0.0000
9	1.0	99.99	31996	3.53	0.41	0.04	0.00	0.00	0.00	0.00	0.0000
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2	5.0	99.95	26301	64.29	48.49	35.61	22.81	8.07	0.06	0.03	0.0000
3	5.0	99.98	31529	40.57	24.28	14.40	7.67	2.09	0.06	0.02	0.0000
4	5.0	99.98	31973	21.78	10.12	4.72	1.85	0.44	0.06	0.02	0.0031
5	5.0	99.97	31989	11.29	3.85	1.37	0.45	0.07	0.01	0.01	0.0000
6	5.0	99.95	31985	5.50	1.44	0.43	0.11	0.03	0.02	0.01	0.0000
9	5.0	99.94	31982	0.78	0.18	0.05	0.02	0.01	0.00	0.00	0.0000
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2	9.0	99.96	26570	57.32	42.57	30.92	20.05	7.14	0.07	0.02	0.0000
3	9.0	99.96	31514	33.06	19.93	11.93	6.37	1.79	0.11	0.03	0.0000
4	9.0	99.97	31973	16.26	7.43	3.66	1.48	0.29	0.03	0.01	0.0000
5	9.0	99.93	31978	7.42	2.63	0.93	0.25	0.04	0.02	0.00	0.0000
6	9.0	99.94	31982	3.35	0.95	0.33	0.13	0.05	0.01	0.01	0.0000
9	9.0	99.90	31969	0.50	0.11	0.06	0.02	0.01	0.00	0.00	0.0000

Table 13: TR x values for degree $p = 1$, demand $d = 10$, single element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26251	82.00	67.21	52.83	37.37	16.01	2.85	1.86	0.0114
3	1.0	100.00	31517	63.11	43.00	28.57	17.29	7.42	2.68	1.56	0.0159
4	1.0	99.99	31979	46.58	26.77	15.55	8.45	4.05	1.78	0.98	0.0063
5	1.0	99.98	31993	34.60	17.33	9.25	4.93	2.33	1.05	0.57	0.0031
6	1.0	99.99	31996	28.22	13.35	6.75	3.36	1.47	0.62	0.30	0.0031
9	1.0	99.99	31997	20.13	8.84	3.73	1.50	0.57	0.20	0.12	0.0000
2	5.0	99.35	26214	67.89	51.52	37.63	24.27	8.45	0.08	0.01	0.0038
3	5.0	98.79	31095	44.33	26.79	15.78	7.86	2.12	0.06	0.01	0.0000
4	5.0	98.42	31487	24.87	11.45	5.11	1.96	0.33	0.02	0.00	0.0000
5	5.0	98.23	31433	13.47	4.62	1.57	0.45	0.05	0.01	0.00	0.0000
6	5.0	98.11	31396	6.74	1.73	0.42	0.09	0.01	0.01	0.00	0.0000
9	5.0	98.10	31393	1.26	0.18	0.03	0.01	0.00	0.00	0.00	0.0000
2	9.0	99.37	26193	60.99	45.69	33.17	21.28	7.31	0.11	0.02	0.0000
3	9.0	98.71	31142	36.21	21.61	12.62	6.49	1.74	0.03	0.01	0.0000
4	9.0	98.19	31406	18.39	8.38	3.83	1.52	0.28	0.01	0.00	0.0000
5	9.0	98.08	31387	8.69	2.79	0.99	0.31	0.04	0.00	0.00	0.0000
6	9.0	97.81	31299	3.93	1.04	0.24	0.04	0.00	0.00	0.00	0.0000
9	9.0	97.80	31295	0.48	0.07	0.01	0.00	0.00	0.00	0.00	0.0000

Table 14: TR x values for degree $p = 1$, demand $d = 10$, single element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26393	76.66	61.62	47.49	32.57	12.10	0.00	0.00	0.0000
3	1.0	99.98	31510	57.91	38.14	24.07	12.71	2.93	0.01	0.00	0.0000
4	1.0	99.94	31961	40.46	20.66	9.88	3.84	0.48	0.00	0.00	0.0000
5	1.0	99.91	31970	25.96	9.76	3.43	0.93	0.06	0.00	0.00	0.0000
6	1.0	99.87	31959	16.58	4.67	1.17	0.20	0.01	0.00	0.00	0.0000
9	1.0	99.90	31967	3.84	0.48	0.04	0.00	0.00	0.00	0.00	0.0000
2	5.0	99.58	26346	66.94	50.86	37.24	24.12	8.67	0.16	0.02	0.0000
3	5.0	99.17	31288	43.88	26.71	15.76	8.10	2.11	0.05	0.01	0.0000
4	5.0	98.95	31645	24.38	11.15	5.04	1.83	0.31	0.02	0.00	0.0000
5	5.0	98.92	31653	12.21	4.15	1.37	0.35	0.03	0.00	0.00	0.0000
6	5.0	98.79	31612	6.45	1.65	0.48	0.10	0.03	0.00	0.00	0.0000
9	5.0	98.87	31639	0.88	0.14	0.02	0.00	0.00	0.00	0.00	0.0000
2	9.0	99.39	26212	61.40	45.81	33.55	21.62	7.59	0.11	0.04	0.0000
3	9.0	98.69	31116	36.25	21.79	12.91	6.79	1.77	0.05	0.02	0.0000
4	9.0	98.24	31424	17.52	8.16	3.79	1.56	0.28	0.01	0.00	0.0000
5	9.0	98.01	31363	8.61	2.93	0.99	0.27	0.02	0.00	0.00	0.0000
6	9.0	97.72	31272	4.09	1.03	0.23	0.05	0.01	0.00	0.00	0.0000
9	9.0	97.62	31239	0.52	0.08	0.02	0.00	0.00	0.00	0.00	0.0000

Table 15: TR x values for degree $p = 1$, demand $d = 10$, mixed element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26357	81.31	67.16	53.25	38.04	17.00	2.58	0.90	0.0038
3	1.0	100.00	31541	62.95	43.23	28.33	16.64	6.79	1.85	0.86	0.0063
4	1.0	100.00	31980	45.42	25.40	13.88	6.86	2.77	1.11	0.57	0.0000
5	1.0	100.00	31999	32.77	15.37	7.26	3.50	1.43	0.71	0.33	0.0000
6	1.0	100.00	31999	23.72	9.69	4.39	1.94	0.88	0.39	0.20	0.0031
9	1.0	100.00	31999	11.34	4.04	1.83	0.91	0.46	0.20	0.09	0.0000
2	5.0	99.46	26195	65.52	49.79	36.68	23.81	8.91	0.23	0.04	0.0000
3	5.0	99.26	31301	42.29	25.71	15.57	8.41	2.88	0.39	0.09	0.0000
4	5.0	99.27	31751	24.40	11.48	5.29	2.09	0.58	0.09	0.01	0.0000
5	5.0	99.16	31729	12.25	4.39	1.67	0.55	0.09	0.01	0.01	0.0000
6	5.0	99.10	31711	6.24	1.61	0.42	0.07	0.02	0.01	0.00	0.0000
9	5.0	99.12	31717	0.86	0.16	0.03	0.01	0.01	0.01	0.00	0.0000
2	9.0	99.56	26297	57.73	43.20	31.50	20.52	7.34	0.25	0.04	0.0000
3	9.0	99.40	31349	34.62	21.25	13.04	7.09	2.37	0.29	0.04	0.0032
4	9.0	99.34	31777	17.04	8.06	3.88	1.69	0.46	0.07	0.01	0.0000
5	9.0	99.30	31775	7.88	2.85	1.02	0.34	0.06	0.02	0.00	0.0000
6	9.0	99.28	31769	3.38	0.94	0.26	0.09	0.02	0.01	0.00	0.0000
9	9.0	99.08	31706	0.38	0.07	0.03	0.02	0.00	0.00	0.00	0.0000

Table 16: TR x values for degree $p = 1$, demand $d = 10$, mixed element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	26382	76.38	61.18	47.07	31.98	12.15	0.07	0.00	0.0000
3	1.0	99.98	31489	57.27	37.81	24.08	13.47	4.46	0.51	0.07	0.0000
4	1.0	99.93	31962	39.72	20.61	10.12	4.22	0.94	0.17	0.03	0.0000
5	1.0	99.97	31988	26.00	10.24	3.74	1.20	0.22	0.02	0.00	0.0000
6	1.0	99.95	31984	16.85	4.89	1.32	0.26	0.03	0.00	0.00	0.0000
9	1.0	99.96	31986	4.55	0.61	0.08	0.01	0.00	0.00	0.00	0.0000
2	5.0	99.65	26380	64.71	49.13	35.74	22.97	8.24	0.16	0.02	0.0000
3	5.0	99.59	31388	41.68	25.64	15.39	8.41	2.56	0.30	0.03	0.0000
4	5.0	99.58	31854	23.01	10.93	5.13	2.28	0.63	0.08	0.01	0.0000
5	5.0	99.52	31846	12.06	4.11	1.42	0.46	0.11	0.03	0.01	0.0000
6	5.0	99.54	31853	6.07	1.61	0.38	0.10	0.01	0.00	0.00	0.0000
9	5.0	99.48	31834	0.68	0.10	0.04	0.01	0.00	0.00	0.00	0.0000
2	9.0	99.64	26365	58.39	43.68	31.92	20.51	7.59	0.24	0.06	0.0000
3	9.0	99.34	31337	34.05	20.58	12.66	7.21	2.40	0.23	0.04	0.0000
4	9.0	99.34	31768	17.30	8.25	3.86	1.60	0.43	0.07	0.00	0.0000
5	9.0	99.33	31785	7.84	2.81	1.04	0.35	0.08	0.02	0.01	0.0000
6	9.0	99.24	31758	3.51	0.87	0.27	0.08	0.03	0.01	0.00	0.0000
9	9.0	99.10	31712	0.38	0.07	0.02	0.01	0.00	0.00	0.00	0.0000

Table 17: TR x values for degree $p = 2$, demand $d = 1$, single element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	98.62	12952	49.72	39.03	27.10	8.35	3.51	2.54	1.79	0.4401
3	1.0	98.85	15584	26.21	17.78	10.68	4.92	2.35	1.73	1.28	0.3850
4	1.0	99.38	15893	13.31	8.17	4.71	2.43	1.38	0.98	0.77	0.2265
5	1.0	99.37	15899	7.91	4.23	2.33	1.31	0.82	0.57	0.42	0.1132
6	1.0	99.41	15906	4.99	2.46	1.37	0.84	0.58	0.43	0.35	0.1509
9	1.0	99.07	15851	2.53	1.08	0.48	0.31	0.18	0.13	0.09	0.0252
2	5.0	94.58	12384	33.96	24.22	13.37	1.90	0.21	0.12	0.02	0.0000
3	5.0	95.14	14985	13.53	8.05	3.70	0.45	0.10	0.03	0.01	0.0000
4	5.0	96.44	15423	4.07	1.88	0.72	0.09	0.01	0.01	0.01	0.0065
5	5.0	96.96	15514	1.10	0.32	0.06	0.01	0.01	0.00	0.00	0.0000
6	5.0	97.36	15577	0.30	0.12	0.01	0.00	0.00	0.00	0.00	0.0000
9	5.0	98.15	15704	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
2	9.0	94.22	12253	29.26	20.65	11.66	1.57	0.14	0.06	0.02	0.0000
3	9.0	94.59	14920	10.79	6.49	3.07	0.44	0.03	0.01	0.01	0.0000
4	9.0	95.51	15274	2.99	1.39	0.51	0.09	0.02	0.00	0.00	0.0000
5	9.0	96.01	15362	0.68	0.21	0.10	0.02	0.01	0.01	0.00	0.0000
6	9.0	96.19	15390	0.20	0.06	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	97.08	15533	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 18: TR x values for degree $p = 2$, demand $d = 1$, single element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13129	41.12	31.08	18.79	2.22	0.00	0.00	0.00	0.0000
3	1.0	99.51	15695	19.39	11.92	5.43	0.38	0.00	0.00	0.00	0.0000
4	1.0	99.59	15927	7.01	3.32	0.93	0.05	0.00	0.00	0.00	0.0000
5	1.0	99.78	15964	2.32	0.86	0.16	0.02	0.00	0.00	0.00	0.0000
6	1.0	99.90	15984	0.68	0.16	0.01	0.00	0.00	0.00	0.00	0.0000
9	1.0	99.95	15992	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
2	5.0	95.36	12296	32.84	22.95	12.91	1.67	0.11	0.04	0.02	0.0000
3	5.0	95.91	15096	13.03	7.50	3.27	0.38	0.02	0.01	0.00	0.0000
4	5.0	97.08	15526	3.66	1.75	0.61	0.05	0.00	0.00	0.00	0.0000
5	5.0	97.36	15577	1.08	0.39	0.06	0.01	0.00	0.00	0.00	0.0000
6	5.0	98.02	15683	0.24	0.04	0.01	0.01	0.00	0.00	0.00	0.0000
9	5.0	98.59	15775	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
2	9.0	94.43	12293	29.31	20.82	11.62	1.77	0.20	0.08	0.02	0.0000
3	9.0	94.61	14889	10.65	6.67	3.24	0.39	0.02	0.01	0.00	0.0000
4	9.0	95.40	15260	3.05	1.40	0.65	0.09	0.01	0.00	0.00	0.0000
5	9.0	95.99	15359	0.74	0.23	0.06	0.01	0.00	0.00	0.00	0.0000
6	9.0	96.12	15380	0.16	0.04	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	96.96	15514	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 19: TR x values for degree $p = 2$, demand $d = 1$, mixed element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	99.56	13070	48.01	36.89	24.16	7.34	2.18	0.78	0.39	0.0612
3	1.0	99.81	15721	24.09	15.37	8.91	3.97	1.47	0.73	0.32	0.0700
4	1.0	99.71	15947	11.25	6.28	3.27	1.48	0.61	0.35	0.21	0.0564
5	1.0	99.73	15957	5.46	2.63	1.35	0.61	0.34	0.22	0.17	0.0376
6	1.0	99.59	15934	2.59	1.12	0.54	0.28	0.19	0.11	0.09	0.0314
9	1.0	99.22	15875	1.00	0.50	0.31	0.16	0.11	0.08	0.05	0.0252
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2	5.0	98.89	13012	32.26	23.34	13.64	2.87	0.54	0.08	0.02	0.0000
3	5.0	98.70	15537	13.73	8.35	4.26	1.20	0.33	0.06	0.02	0.0000
4	5.0	98.90	15814	4.35	2.03	0.96	0.27	0.08	0.02	0.01	0.0000
5	5.0	98.98	15836	1.26	0.51	0.19	0.08	0.03	0.01	0.01	0.0000
6	5.0	98.98	15836	0.49	0.15	0.04	0.03	0.01	0.00	0.00	0.0000
9	5.0	98.76	15802	0.05	0.01	0.01	0.00	0.00	0.00	0.00	0.0000
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2	9.0	98.88	13095	28.51	20.44	11.88	2.34	0.36	0.05	0.00	0.0000
3	9.0	98.94	15563	10.78	6.80	3.53	0.93	0.20	0.03	0.01	0.0000
4	9.0	98.91	15820	2.98	1.49	0.62	0.20	0.08	0.02	0.01	0.0000
5	9.0	98.92	15827	0.82	0.33	0.12	0.01	0.01	0.00	0.00	0.0000
6	9.0	98.94	15830	0.30	0.11	0.03	0.01	0.00	0.00	0.00	0.0000
9	9.0	98.56	15770	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.0000

Table 20: TR x values for degree $p = 2$, demand $d = 1$, mixed element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13138	41.75	30.41	17.79	2.95	0.37	0.00	0.00	0.0000
3	1.0	99.99	15769	19.98	12.30	6.02	1.27	0.24	0.03	0.01	0.0000
4	1.0	99.99	15993	7.52	3.76	1.39	0.28	0.07	0.03	0.01	0.0000
5	1.0	99.99	15999	2.88	1.04	0.30	0.06	0.01	0.00	0.00	0.0000
6	1.0	99.98	15997	1.10	0.36	0.06	0.01	0.00	0.00	0.00	0.0000
9	1.0	99.99	15999	0.14	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
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2	5.0	99.28	13181	32.52	23.33	13.36	2.57	0.39	0.02	0.02	0.0000
3	5.0	99.32	15654	12.64	7.66	3.95	0.94	0.22	0.03	0.01	0.0000
4	5.0	99.28	15871	3.98	1.92	0.88	0.19	0.06	0.03	0.01	0.0000
5	5.0	99.30	15888	1.13	0.48	0.17	0.04	0.01	0.01	0.01	0.0000
6	5.0	99.35	15896	0.32	0.09	0.03	0.02	0.00	0.00	0.00	0.0000
9	5.0	99.29	15887	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	99.02	13019	28.54	20.27	11.74	2.58	0.31	0.04	0.02	0.0000
3	9.0	98.79	15586	10.62	6.58	3.56	1.12	0.32	0.10	0.03	0.0000
4	9.0	98.79	15791	3.17	1.56	0.56	0.16	0.03	0.02	0.00	0.0000
5	9.0	98.84	15815	0.79	0.32	0.11	0.03	0.01	0.00	0.00	0.0000
6	9.0	98.87	15819	0.20	0.06	0.02	0.01	0.01	0.01	0.00	0.0000
9	9.0	98.62	15780	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.0000

Table 21: TR x values for degree $p = 2$, demand $d = 10$, single element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	98.47	12935	50.69	39.34	27.04	8.41	3.56	2.53	1.84	0.2474
3	1.0	98.62	15530	26.14	17.75	10.78	4.73	2.38	1.64	1.31	0.1674
4	1.0	98.86	15808	13.56	8.43	4.78	2.64	1.46	0.97	0.77	0.0886
5	1.0	98.64	15782	7.43	4.05	2.34	1.25	0.79	0.61	0.42	0.0570
6	1.0	98.23	15717	5.01	2.61	1.22	0.69	0.44	0.30	0.22	0.0318
9	1.0	96.78	15485	2.25	0.93	0.48	0.26	0.20	0.10	0.06	0.0065
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2	5.0	93.66	12199	34.10	24.55	14.00	1.72	0.13	0.04	0.01	0.0082
3	5.0	92.73	14595	13.03	7.76	3.85	0.43	0.04	0.03	0.01	0.0000
4	5.0	92.70	14829	3.78	1.73	0.62	0.05	0.01	0.01	0.00	0.0000
5	5.0	92.59	14814	0.98	0.30	0.05	0.00	0.00	0.00	0.00	0.0000
6	5.0	92.81	14849	0.27	0.09	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	92.99	14879	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	93.66	12197	30.22	21.22	11.91	1.71	0.17	0.06	0.01	0.0000
3	9.0	91.76	14426	10.58	6.21	2.91	0.31	0.01	0.00	0.00	0.0000
4	9.0	91.91	14695	3.28	1.71	0.54	0.07	0.01	0.00	0.00	0.0000
5	9.0	91.88	14701	0.57	0.20	0.07	0.01	0.01	0.01	0.01	0.0000
6	9.0	91.72	14676	0.10	0.04	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	91.31	14609	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 22: TR x values for degree $p = 2$, demand $d = 10$, single element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13188	42.53	31.94	18.96	2.18	0.00	0.00	0.00	0.0000
3	1.0	99.25	15637	19.15	11.51	5.02	0.38	0.00	0.00	0.00	0.0000
4	1.0	99.34	15879	6.97	3.22	0.94	0.02	0.00	0.00	0.00	0.0000
5	1.0	99.31	15889	2.09	0.71	0.17	0.01	0.00	0.00	0.00	0.0000
6	1.0	99.26	15882	0.56	0.14	0.02	0.00	0.00	0.00	0.00	0.0000
9	1.0	99.34	15894	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	5.0	95.00	12417	32.87	23.22	13.24	1.61	0.05	0.02	0.00	0.0000
3	5.0	93.73	14744	13.05	7.87	3.64	0.41	0.03	0.01	0.01	0.0000
4	5.0	94.26	15073	4.01	1.86	0.66	0.02	0.00	0.00	0.00	0.0000
5	5.0	94.13	15061	0.80	0.28	0.04	0.01	0.01	0.00	0.00	0.0000
6	5.0	94.33	15092	0.17	0.03	0.01	0.00	0.00	0.00	0.00	0.0000
9	5.0	94.76	15161	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	93.83	12226	29.63	21.33	11.72	1.59	0.11	0.02	0.01	0.0000
3	9.0	92.22	14512	11.02	6.74	3.18	0.43	0.03	0.01	0.00	0.0000
4	9.0	91.97	14702	2.95	1.47	0.58	0.05	0.01	0.01	0.00	0.0000
5	9.0	91.56	14650	0.69	0.28	0.06	0.00	0.00	0.00	0.00	0.0000
6	9.0	91.46	14633	0.14	0.03	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	91.66	14665	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 23: TR x values for degree $p = 2$, demand $d = 10$, mixed element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	99.55	13183	53.54	42.63	31.06	14.40	7.38	4.71	2.50	0.0759
3	1.0	99.57	15688	29.42	20.25	12.83	6.55	3.64	2.40	1.49	0.0382
4	1.0	99.48	15907	14.80	8.96	5.17	2.84	1.82	1.12	0.64	0.0251
5	1.0	99.27	15883	8.40	4.45	2.66	1.57	1.02	0.60	0.37	0.0126
6	1.0	99.04	15847	4.82	2.56	1.49	0.87	0.62	0.44	0.25	0.0126
9	1.0	98.15	15704	1.56	0.78	0.38	0.22	0.13	0.08	0.05	0.0000
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2	5.0	94.86	12378	35.11	25.17	15.05	3.42	0.69	0.11	0.02	0.0000
3	5.0	93.68	14739	14.32	8.68	4.35	1.00	0.18	0.03	0.01	0.0000
4	5.0	94.47	15110	4.42	2.15	0.90	0.21	0.05	0.01	0.01	0.0000
5	5.0	95.18	15229	1.46	0.62	0.20	0.04	0.01	0.00	0.00	0.0000
6	5.0	95.94	15351	0.38	0.08	0.01	0.00	0.00	0.00	0.00	0.0000
9	5.0	97.19	15550	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	95.12	12510	30.66	21.76	12.86	2.81	0.48	0.11	0.03	0.0000
3	9.0	94.61	14898	11.63	7.44	3.79	1.01	0.17	0.03	0.01	0.0000
4	9.0	94.67	15135	3.44	1.68	0.67	0.09	0.02	0.01	0.00	0.0000
5	9.0	95.24	15238	0.80	0.29	0.13	0.06	0.01	0.00	0.00	0.0000
6	9.0	96.12	15379	0.23	0.05	0.02	0.01	0.00	0.00	0.00	0.0000
9	9.0	97.02	15523	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 24: TR x values for degree $p = 2$, demand $d = 10$, mixed element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13142	43.21	31.97	18.66	3.37	0.63	0.00	0.00	0.0000
3	1.0	99.76	15716	20.60	12.80	6.44	1.56	0.25	0.02	0.01	0.0000
4	1.0	99.72	15943	7.98	3.91	1.57	0.37	0.09	0.01	0.00	0.0000
5	1.0	99.72	15955	3.22	1.22	0.34	0.09	0.01	0.00	0.00	0.0000
6	1.0	99.79	15966	1.28	0.36	0.07	0.00	0.00	0.00	0.00	0.0000
9	1.0	99.88	15981	0.21	0.05	0.01	0.00	0.00	0.00	0.00	0.0000
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2	5.0	96.45	12624	34.50	24.48	14.60	3.20	0.51	0.07	0.01	0.0000
3	5.0	96.13	15135	13.62	8.22	3.98	1.00	0.18	0.01	0.00	0.0000
4	5.0	96.42	15418	4.28	1.97	0.79	0.21	0.01	0.00	0.00	0.0000
5	5.0	97.04	15527	1.15	0.40	0.10	0.01	0.00	0.00	0.00	0.0000
6	5.0	97.43	15589	0.37	0.08	0.03	0.01	0.00	0.00	0.00	0.0000
9	5.0	98.22	15716	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	95.44	12480	31.78	22.73	13.58	3.00	0.54	0.10	0.04	0.0000
3	9.0	94.03	14812	11.79	7.21	3.46	0.75	0.12	0.02	0.00	0.0000
4	9.0	94.67	15139	3.33	1.70	0.63	0.18	0.05	0.01	0.00	0.0000
5	9.0	95.47	15276	0.73	0.22	0.08	0.03	0.00	0.00	0.00	0.0000
6	9.0	95.69	15311	0.24	0.12	0.03	0.01	0.00	0.00	0.00	0.0000
9	9.0	96.77	15483	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 25: TR x values for degree $p = 3$, demand $d = 1$, single element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	71.88	8717	48.31	36.00	18.55	5.16	3.63	2.98	2.47	1.0210
3	1.0	88.98	14002	16.80	10.98	5.83	2.89	2.11	1.69	1.40	0.7213
4	1.0	94.51	15117	6.89	4.39	2.67	1.59	1.14	0.91	0.75	0.3308
5	1.0	93.63	14981	3.45	2.00	1.19	0.73	0.50	0.38	0.28	0.1802
6	1.0	91.61	14657	1.66	0.99	0.65	0.44	0.29	0.22	0.14	0.0750
9	1.0	90.27	14443	0.67	0.31	0.21	0.12	0.10	0.07	0.05	0.0277
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2	5.0	88.59	11376	20.59	12.46	3.08	0.36	0.16	0.03	0.01	0.0000
3	5.0	90.24	14211	6.95	3.45	0.67	0.11	0.04	0.01	0.01	0.0000
4	5.0	92.55	14799	1.59	0.62	0.09	0.03	0.01	0.01	0.01	0.0000
5	5.0	94.13	15061	0.30	0.09	0.03	0.02	0.00	0.00	0.00	0.0000
6	5.0	95.26	15242	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	96.97	15516	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	88.69	11386	18.56	11.09	2.57	0.17	0.06	0.03	0.00	0.0000
3	9.0	89.34	14050	5.67	2.81	0.56	0.06	0.05	0.03	0.01	0.0000
4	9.0	91.47	14632	1.05	0.44	0.10	0.04	0.01	0.00	0.00	0.0000
5	9.0	92.89	14863	0.19	0.06	0.01	0.00	0.00	0.00	0.00	0.0000
6	9.0	93.94	15030	0.03	0.01	0.01	0.01	0.00	0.00	0.00	0.0000
9	9.0	95.46	15273	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 26: TR x values for degree $p = 3$, demand $d = 1$, single element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13147	28.31	17.22	3.29	0.00	0.00	0.00	0.00	0.0000
3	1.0	98.81	15585	9.67	4.74	0.83	0.00	0.00	0.00	0.00	0.0000
4	1.0	99.17	15864	2.46	0.93	0.10	0.00	0.00	0.00	0.00	0.0000
5	1.0	99.42	15907	0.49	0.14	0.01	0.00	0.00	0.00	0.00	0.0000
6	1.0	99.52	15923	0.09	0.02	0.00	0.00	0.00	0.00	0.00	0.0000
9	1.0	99.76	15962	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	5.0	90.14	11588	20.75	12.36	2.93	0.34	0.15	0.06	0.03	0.0000
3	5.0	91.37	14396	6.95	3.47	0.67	0.05	0.01	0.01	0.01	0.0000
4	5.0	93.85	15005	1.46	0.48	0.09	0.01	0.00	0.00	0.00	0.0000
5	5.0	95.43	15268	0.27	0.05	0.01	0.00	0.00	0.00	0.00	0.0000
6	5.0	95.89	15342	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	97.15	15544	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	89.02	11538	19.18	11.37	2.70	0.23	0.13	0.05	0.02	0.0000
3	9.0	89.42	14060	5.67	2.85	0.55	0.05	0.02	0.00	0.00	0.0000
4	9.0	91.25	14593	1.16	0.49	0.10	0.02	0.00	0.00	0.00	0.0000
5	9.0	92.84	14854	0.20	0.07	0.01	0.01	0.00	0.00	0.00	0.0000
6	9.0	94.06	15049	0.02	0.01	0.01	0.01	0.00	0.00	0.00	0.0000
9	9.0	95.85	15336	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 27: TR x values for degree $p = 3$, demand $d = 1$, mixed element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	97.54	12828	35.30	24.54	11.67	3.93	2.03	1.01	0.42	0.0702
3	1.0	98.04	15440	14.34	8.51	3.94	1.58	0.75	0.45	0.20	0.0194
4	1.0	97.62	15616	4.76	2.50	1.18	0.60	0.36	0.19	0.13	0.0384
5	1.0	97.19	15549	1.55	0.80	0.37	0.22	0.13	0.06	0.03	0.0064
6	1.0	96.45	15432	0.67	0.30	0.14	0.10	0.06	0.04	0.03	0.0194
9	1.0	94.53	15124	0.13	0.03	0.02	0.01	0.01	0.01	0.00	0.0000
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2	5.0	95.57	12450	21.08	12.76	3.94	0.60	0.17	0.04	0.02	0.0000
3	5.0	96.17	15155	6.58	3.57	1.32	0.31	0.09	0.01	0.01	0.0000
4	5.0	97.04	15520	1.42	0.61	0.26	0.10	0.01	0.01	0.00	0.0000
5	5.0	97.19	15550	0.34	0.12	0.03	0.01	0.00	0.00	0.00	0.0000
6	5.0	97.82	15651	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	98.28	15724	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	95.89	12553	18.41	11.18	3.54	0.49	0.14	0.05	0.00	0.0000
3	9.0	96.60	15221	5.83	3.10	1.26	0.40	0.12	0.03	0.00	0.0000
4	9.0	96.63	15455	1.21	0.54	0.17	0.05	0.01	0.00	0.00	0.0000
5	9.0	96.92	15506	0.25	0.06	0.00	0.00	0.00	0.00	0.00	0.0000
6	9.0	97.27	15563	0.04	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	97.58	15612	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 28: TR x values for degree $p = 3$, demand $d = 1$, mixed element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13135	26.97	16.83	4.23	0.24	0.02	0.00	0.00	0.0000
3	1.0	99.98	15746	10.36	5.40	1.16	0.16	0.04	0.01	0.00	0.0000
4	1.0	99.98	15989	2.61	1.05	0.27	0.05	0.02	0.01	0.01	0.0000
5	1.0	99.99	15999	0.79	0.19	0.03	0.00	0.00	0.00	0.00	0.0000
6	1.0	99.98	15996	0.19	0.04	0.02	0.01	0.00	0.00	0.00	0.0000
9	1.0	99.99	15999	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	5.0	97.39	12830	20.77	12.25	3.77	0.66	0.17	0.05	0.00	0.0000
3	5.0	97.56	15398	6.46	3.40	1.29	0.34	0.10	0.03	0.00	0.0000
4	5.0	98.08	15684	1.61	0.72	0.31	0.11	0.05	0.02	0.01	0.0000
5	5.0	98.13	15701	0.24	0.07	0.04	0.00	0.00	0.00	0.00	0.0000
6	5.0	98.09	15694	0.05	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	98.51	15761	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	96.16	12571	19.03	11.94	3.91	0.60	0.17	0.05	0.01	0.0000
3	9.0	96.47	15185	6.16	3.47	1.26	0.36	0.11	0.01	0.00	0.0000
4	9.0	96.81	15479	1.14	0.50	0.19	0.05	0.01	0.01	0.00	0.0000
5	9.0	96.97	15515	0.20	0.10	0.02	0.00	0.00	0.00	0.00	0.0000
6	9.0	97.47	15596	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	97.69	15631	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 29: TR x values for degree $p = 3$, demand $d = 10$, single element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	88.02	11195	40.92	31.08	15.71	4.56	3.05	2.32	1.69	0.3841
3	1.0	88.34	13894	16.40	10.72	5.60	2.62	1.86	1.52	1.17	0.3527
4	1.0	91.53	14637	6.62	3.92	2.16	1.27	0.90	0.66	0.52	0.1913
5	1.0	90.66	14505	3.32	1.81	1.19	0.86	0.61	0.45	0.34	0.0965
6	1.0	88.83	14212	1.85	1.03	0.65	0.52	0.39	0.27	0.21	0.0704
9	1.0	86.90	13904	0.65	0.39	0.27	0.19	0.13	0.07	0.06	0.0360
2	5.0	86.98	11069	21.91	13.18	3.13	0.24	0.05	0.01	0.01	0.0000
3	5.0	86.28	13552	6.92	3.46	0.63	0.04	0.02	0.01	0.00	0.0000
4	5.0	86.89	13891	1.40	0.54	0.09	0.00	0.00	0.00	0.00	0.0000
5	5.0	86.74	13879	0.22	0.04	0.01	0.00	0.00	0.00	0.00	0.0000
6	5.0	87.26	13962	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	87.95	14072	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
2	9.0	87.39	11181	18.99	10.99	2.51	0.17	0.05	0.04	0.00	0.0000
3	9.0	85.87	13510	5.63	2.81	0.49	0.04	0.01	0.01	0.00	0.0000
4	9.0	85.76	13717	1.20	0.49	0.03	0.00	0.00	0.00	0.00	0.0000
5	9.0	85.21	13634	0.16	0.04	0.00	0.00	0.00	0.00	0.00	0.0000
6	9.0	85.47	13676	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	85.89	13743	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 30: TR x values for degree $p = 3$, demand $d = 10$, single element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13126	26.97	16.61	3.42	0.00	0.00	0.00	0.00	0.0000
3	1.0	98.35	15494	10.03	4.80	0.77	0.00	0.00	0.00	0.00	0.0000
4	1.0	98.02	15676	2.21	0.83	0.09	0.01	0.00	0.00	0.00	0.0000
5	1.0	97.67	15628	0.52	0.09	0.01	0.00	0.00	0.00	0.00	0.0000
6	1.0	97.72	15635	0.10	0.02	0.00	0.00	0.00	0.00	0.00	0.0000
9	1.0	97.90	15664	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
2	5.0	88.62	11384	21.35	12.54	3.07	0.23	0.04	0.01	0.00	0.0000
3	5.0	88.24	13875	6.62	3.22	0.56	0.01	0.00	0.00	0.00	0.0000
4	5.0	88.71	14188	1.31	0.48	0.07	0.00	0.00	0.00	0.00	0.0000
5	5.0	88.99	14238	0.18	0.05	0.01	0.00	0.00	0.00	0.00	0.0000
6	5.0	89.33	14293	0.05	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	90.72	14516	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
2	9.0	87.17	11213	18.99	11.17	2.53	0.17	0.08	0.03	0.01	0.0089
3	9.0	85.16	13389	5.74	3.02	0.60	0.02	0.00	0.00	0.00	0.0000
4	9.0	85.22	13624	1.04	0.42	0.08	0.00	0.00	0.00	0.00	0.0000
5	9.0	85.31	13649	0.14	0.04	0.00	0.00	0.00	0.00	0.00	0.0000
6	9.0	85.16	13626	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	85.29	6823 ^a	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

^aThis series did not complete on time.

Table 31: TR x values for degree $p = 3$, demand $d = 10$, mixed element latency functions, and fixed consumption number range.

n	r_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	97.31	12683	42.36	32.47	19.58	10.55	7.60	5.72	3.87	0.3075
3	1.0	97.11	15281	19.64	13.15	7.92	4.70	3.36	2.47	1.83	0.1636
4	1.0	96.38	15410	8.29	5.18	3.26	2.25	1.65	1.27	0.90	0.0844
5	1.0	94.82	15171	3.94	2.43	1.65	1.18	0.89	0.69	0.45	0.0330
6	1.0	93.77	15003	2.30	1.37	0.93	0.70	0.45	0.29	0.17	0.0200
9	1.0	91.05	14568	0.86	0.48	0.32	0.20	0.15	0.08	0.05	0.0000
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2	5.0	89.20	11454	23.53	14.55	3.97	0.55	0.26	0.10	0.03	0.0000
3	5.0	88.24	13885	7.04	3.58	0.87	0.09	0.04	0.01	0.00	0.0000
4	5.0	88.84	14206	1.57	0.61	0.13	0.01	0.01	0.00	0.00	0.0000
5	5.0	90.01	14402	0.25	0.05	0.00	0.00	0.00	0.00	0.00	0.0000
6	5.0	91.42	14627	0.04	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	94.22	15075	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	89.76	11599	20.80	12.90	3.72	0.48	0.15	0.04	0.00	0.0000
3	9.0	88.17	13852	6.00	3.12	0.72	0.09	0.01	0.00	0.00	0.0000
4	9.0	88.66	14177	1.27	0.49	0.15	0.01	0.01	0.00	0.00	0.0000
5	9.0	89.77	14363	0.15	0.06	0.01	0.00	0.00	0.00	0.00	0.0000
6	9.0	90.57	14491	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
9	9.0	93.02	14883	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

Table 32: TR x values for degree $p = 3$, demand $d = 10$, mixed element latency functions, and fixed relevance number range.

n	c_{\max}	%OK	$\#(\gamma > 1)$	%TR30	%TR40	%TR50	%TR60	%TR70	%TR80	%TR90	%TR100
2	1.0	100.00	13189	27.93	17.27	4.44	0.25	0.04	0.00	0.00	0.0000
3	1.0	99.41	15658	9.95	5.15	1.14	0.13	0.01	0.00	0.00	0.0000
4	1.0	98.97	15830	2.67	1.12	0.32	0.04	0.02	0.00	0.00	0.0000
5	1.0	98.87	15819	0.59	0.18	0.04	0.00	0.00	0.00	0.00	0.0000
6	1.0	98.85	15816	0.14	0.03	0.00	0.00	0.00	0.00	0.00	0.0000
9	1.0	99.27	15883	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
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2	5.0	91.03	11745	22.46	13.85	3.97	0.52	0.21	0.08	0.01	0.0000
3	5.0	90.56	14245	6.75	3.50	0.92	0.14	0.02	0.01	0.00	0.0000
4	5.0	91.21	14589	1.44	0.58	0.15	0.01	0.01	0.00	0.00	0.0000
5	5.0	92.49	14799	0.20	0.05	0.00	0.00	0.00	0.00	0.00	0.0000
6	5.0	93.54	14966	0.06	0.01	0.00	0.00	0.00	0.00	0.00	0.0000
9	5.0	95.40	15264	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.0000
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2	9.0	89.30	11523	20.35	12.67	3.45	0.48	0.13	0.09	0.03	0.0000
3	9.0	87.92	13840	6.19	3.22	0.86	0.12	0.04	0.01	0.00	0.0000
4	9.0	88.78	14200	1.15	0.47	0.08	0.01	0.00	0.00	0.00	0.0000
5	9.0	89.99	14399	0.17	0.05	0.01	0.00	0.00	0.00	0.00	0.0000
6	9.0	90.85	14536	0.03	0.01	0.01	0.01	0.00	0.00	0.00	0.0000
9	9.0	93.31	14929	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0000

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