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Generalised Lyndon-Schützenberger Equations

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Abstract. We fully characterise the solutions of the generalised Lyndon-Schützenberger word equations $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$, where $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq \ell$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq m$, $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$, and θ is an antimorphic involution. More precisely, we show for which ℓ , m , and n such an equation has only θ -periodic solutions, i.e., u , v , and w are in $\{t, \theta(t)\}^*$ for some word t , closing an open problem by Czeizler et al. (2011).

1 Introduction

A word is a *power* (or *repetition*) if it can be written as a repeated concatenation of one of its prefixes. A word w is a *pseudo-power* if it can be written as a repeated concatenation of one of its prefixes t and its image $f(t)$ under some morphic or antimorphic function f , thus $w \in t\{t, f(t)\}^+$, also called f -power. Introduced in [1], the latter notion seems to be a natural generalisation of the former: when f is the identity morphism, pseudo-powers are, indeed, classical powers. More interestingly, when f is the reversal (identity antimorphism), pseudo-powers are repeated concatenations of a word and its mirror image, so, in a sense, generalised palindromic structures. To this end, the study of combinatorial properties of pseudo-powers supports the development of a generalised periodicity theory.

The initial motivation of studying pseudo-repetitions (according to [1]) came from two important biological concepts: tandem repeat, i.e., the consecutive repetition of the same sequence of nucleotides in a DNA strand, and the inverted repeat, i.e., a sequence of nucleotides whose reversed Watson-Crick complement occurred before in the strand, both occurrences encoding, essentially, the same genetic information. Noting that the Watson-Crick complement can be abstracted as an antimorphic involution on the DNA-alphabet, pseudo-powers formalise generalised tandem repeats, in which one sequence is followed by several consecutive occurrences of either its copy or of its reversed complement. However, repetitions of a fragment in its original form, or slightly modified appear in other domains as well. In music or visual arts such repetitions are used to highlight important elements of that musical or artistic piece, as well as to provide unity to it.

The study of combinatorial properties of pseudo-powers was mostly concerned with the case when f is an involution; this case seems the most motivated,

modelling both the original repetitions and palindromic structures, the Watson-Crick complement, and other practical situations. In this context, generalisations of the Fine and Wilf theorem as well as avoidability results were derived ([1], [2], respectively). The combinatorial results were complemented by algorithmic results: efficient methods of testing whether a word is a pseudo-power or finding which pseudo-powers it contains were recently developed (see [3, 4]).

Naturally, the study of pseudo-powers was focused on understanding the way classical periodicity results can be translated into this new and more general setting. Accordingly, Czeizler et al. [5] investigated a generalisation of Lyndon and Schützenberger's equations. Lyndon and Schützenberger [6] showed that in all solutions of an equation $u^\ell = v^m w^n$, with $l, m, n \geq 2$, in a free group, u , v , and w are necessarily powers of a common element. Their result also holds, if u, v and w are elements of a free semigroup [7]. The generalised form of these equations is $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$, where $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq \ell$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq m$, $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$, and θ is an antimorphic involution. Following the classical case, it was studied under which conditions $u, v, w \in \{t, \theta(t)\}^+$ for some word t . In other words, one is interested in the case when the solutions u, v, w of the equation are pseudo-powers, or more precisely, θ -powers of the same word; we call such solutions θ -periodic.

The results obtained on these generalised equations in [5, 8, 9] are summarised in Table 1. One can note from this table that the more problematic equations are those with $\ell = 3$. More precisely, equations which allow non- θ -periodic solutions and equations having only θ -periodic solutions were identified; however, no precise characterisation was obtained. Hence, this case seems to be especially intricate and interesting, as the separating border between the cases when the equation has only θ -periodic solutions and the cases when it may also have non- θ -periodic solutions is drawn here. Our work closes the gap providing a full characterisation of the generalised Lyndon-Schützenberger equations having only θ -periodic solutions. This seems to us a relevant step towards the aforementioned development of a generalised periodicity theory.

Table 1. Results on the equation $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$

ℓ	m	n	$u, v, w \in \{t, \theta(t)\}^+?$	
			Known	This paper
≥ 4	≥ 3	≥ 3	Yes	
3	≥ 12	≥ 12	Yes	
3	$5 \leq \min\{m, n\}$, m or n odd		Yes	
3	$5 \leq \min\{m, n\} < 12$, m and n even		Open (1)	Yes
3	4	≥ 5 and odd	Open (2)	Yes
3	4	≥ 4 and even	No	
3	3	≥ 3	No	
	one of $\{\ell, m, n\}$ equals 2		No	

Our Results. As mentioned above, we are interested in solutions of the equation

$$u_1 u_2 u_3 = v_1 \cdots v_m w_1 \cdots w_n, \quad (1)$$

where $5 \leq \min\{m, n\} < 12$ and m and n are even, or $m = 4$ and $n \geq 5$ is odd, $u_1, u_2, u_3 \in \{u, \theta(u)\}$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq m$ and $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$, and θ an antimorphic involution.

In [9] it is already shown that if $m, n \geq 5$ and not all of u_1, u_2 and u_3 are the same or $m|v| \geq 2|u|$ then Equation (1) has only θ -periodic solutions. Thus, the part of case (1) left open, and answered positively here, regards the θ -periodicity of the solutions of Equation (1) for $u_1 = u_2 = u_3$ and $m|v| < 2|u|$. Due to space restrictions, we do not present in this extended abstract the complete proofs for this case; they are given in the Appendix. However, note that, in the light of the results from [9], the positive answer we give in this case is not surprising; essentially, a very careful exploration of the alignments occurring between the v or $\theta(v)$ and w or $\theta(w)$ factors leads to this result. The following theorem is obtained.

Theorem 1. *If $\ell = 3$, $5 \leq \min\{m, n\} < 12$, and both m and n even, then Equation (1) implies that $u, v, w \in \{t, \theta(t)\}^*$ for some word t .*

In the case of question (2), when $\ell = 3, m = 4, n \geq 5$ and odd, we give here the main proofs. This case seems very interesting to us for two reasons. Firstly, the fact that Equation (1) has only θ -periodic solutions under these restrictions seems surprising to us, as it shows a different behaviour from the case when the same bounds apply to ℓ, m , and n , but n is even; this shows exactly where the equations having only θ -periodic solutions are separated from the ones that may also have other solutions. Secondly, due to the small number of factors v or $\theta(v)$, it seems that a different (at least partly) approach is needed in this case. Indeed, the common approach in the proofs of the results of [5, 8, 9] or in those supporting Theorem 1, in the Appendix, was to find a long enough factor of $u_1 u_2 u_3$ that reflects an alignment between some of the factors v_1, \dots, v_m and some of the factors w_1, \dots, w_n , and then to apply periodicity results like Theorems 4 or 5 to get that u, v , and w are all θ -powers of the same word. Making such an approach work seems more difficult when we only have few occurrences of v or $\theta(v)$ (or, alternatively, of w or $\theta(w)$). Our proofs show how this can be done. While the basic techniques we rely on are the usual ones of combinatorics on words, including detailed case analyses, we also make use of novel arguments regarding θ -periodic structures and exhibit more general ways of applying known θ -periodicity arguments. We show the following.

Theorem 2. *If $\ell = 3, m = 4$ and $n \geq 5$ odd, then Equation (1) implies that $u, v, w \in \{t, \theta(t)\}^*$ for some word t .*

Theorems 1 and 2 and Table 1 fully characterise the solutions of Equation (1).

Theorem 3. *Equation (1) implies that $u, v, w \in \{t, \theta(t)\}^*$ for some word t if and only if (1) $\ell \geq 4$ and $m, n \geq 3$, or (2) $\ell \geq 3$ and $m, n \geq 5$, or (3) $\ell = 3, m = 4$, and n is an odd number at least 5.*

2 Preliminaries

For more detailed definitions we refer to [7]. For a finite alphabet Σ , we denote by Σ^* and Σ^+ the set of all words and the set of all non-empty words over Σ ,

respectively. The empty word is denoted by ε and the length of a word w is denoted by $|w|$. For a word $w = uvz$ we say that u is a *prefix* of w , v is a *factor* of w , and z is a *suffix* of w . We denote that by $u \leq_p w$, $v \leq_f w$, and $v \leq_s w$, respectively. If $vz \neq \varepsilon$ we call u a *proper prefix*, and we denote that by $u <_p w$, and symmetrically for suffixes. Similarly, v is called a *proper factor* of w , denoted by $v <_f w$, if $u \neq \varepsilon, z \neq \varepsilon$. A word w is called *primitive*, if $w = u^k$ implies $k = 1$ and $u = w$; otherwise, w is called *power* or *repetition*. For a word w , we define the word w^ω as the infinite word whose prefix of length $n|w|$ is w^n , for all $n \in \mathbb{N}$.

Primitive words are characterised as follows:

Proposition 1. *If w is primitive and $w = xwy$, then either $x = \varepsilon$ or $y = \varepsilon$.*

Let θ be an antimorphic involution. A word w is called *θ -primitive* if $w = u_1 \cdots u_k$ with $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq k$ implies $k = 1$ and $u = w$; otherwise, w is a *θ -power*. A θ -primitive word is primitive, but the converse does not hold. For instance, the word $w = abba$ is primitive but $w = ab\theta(ab)$, for θ being the mirror image. Any nonempty word w admits a *unique* θ -primitive word t such that $w \in t\{t, \theta(t)\}^*$, and the t is called the *θ -primitive root* of w . A word w is a *θ -palindrome* if $w = \theta(w)$.

Kari et al. [8] characterised θ -primitive words similarly to Proposition 1:

Lemma 1. *For a θ -primitive word $x \in \Sigma^+$, neither $x\theta(x)$ nor $\theta(x)x$ can be a proper factor of a word in $\{x, \theta(x)\}^3$.*

Furthermore, Czeizler et al. [1] showed the following:

Lemma 2. *Let $x \in \Sigma^+$ be a θ -primitive word, and $x_1, x_2, x_3, x_4 \in \{x, \theta(x)\}$. If $x_1x_2y = zx_3x_4$ for some words $y, z \in \Sigma^+$ with $|y|, |z| < |x|$, then $x_2 \neq x_3$.*

The results of Proposition 2 and Theorem 4 are well known (see, e.g., [7]): The words x, y from Proposition 2 are called *conjugates*, denoted by $x \sim y$.

Proposition 2. *If $xz = zy$ for some words $x, y, z \in \Sigma^*$, then there exist $p, q \in \Sigma^*$, such that $x = pq, y = qp$, and $z = (pq)^i p$ for some $i \geq 0$.*

Theorem 4. *If $\alpha \in u\{u, v\}^*$ and $\beta \in v\{u, v\}^*$ have a common prefix of length at least $|u| + |v| - \gcd(|u|, |v|)$, then $u, v \in \{t\}^+$ for a word t .*

Czeizler et al. [1] proved the following generalisation of Theorem 4:

Theorem 5. *Let $u, v \in \Sigma^+$ with $|u| \geq |v|$. If $\alpha \in \{u, \theta(u)\}^+$ and $\beta \in \{v, \theta(v)\}^+$ have a common prefix of length at least $\min\{2|u| + |v| - \gcd(|u|, |v|), \text{lcm}(|u|, |v|)\}$, then $u, v \in t\{t, \theta(t)\}^+$ for some θ -primitive word $t \in \Sigma^+$.*

Lentin [10] investigated equations similar to the ones by Lyndon and Schützenberger with the following result, which we use in our proofs:

Theorem 6. *If $u^\ell = v^m w^n x^p$ for some words $u, v, w, x \in \Sigma^*$ and $\ell, m, n, p \geq 3$, then $u, v, w, x \in \{t\}^*$ for some word $t \in \Sigma^*$.*

We will also make frequent use of the following results from [5]:

Proposition 3 (Prop. 20 and 21 in [5]). *Let $u, v \in \Sigma^+$ so that v is θ -primitive, $u_1, u_2, u_3 \in \{u, \theta(u)\}$ and $v_1, \dots, v_m \in \{v, \theta(v)\}$ for some $m \geq 3$. Assume that $v_1 \cdots v_m <_p u_1 u_2 u_3$ and $2|u| < m|v| < 2|u| + |v|$. Then:*

- *If m is odd, then $u_2 \neq u_1$ and $v_1 = \dots = v_m = z\theta(z)p$, where $p = \theta(p)$.*
- *If m is even, then one of the following holds:*
 1. *$u_1 \neq u_2$ and $v_1 = \dots = v_m = xz\theta(z)$, where $x = \theta(x)$, or*
 2. *$u_1 = u_2$, $v_1 = \dots = v_{\frac{m}{2}} = v_{\frac{m}{2}+1} = \dots = v_m = \theta(v_1)$.*

A symmetrical result (i.e., $v_1 \dots v_m <_s u_1 u_2 u_3$ in the hypothesis and u_1 and u_2 replaced by u_2 and u_3 , respectively, in the conclusions) can be easily derived.

3 Proof of Theorem 2

The case (2) from Table 1, left open in [8, 9], was that of the equations

$$u_1 u_2 u_3 = v_1 v_2 v_3 v_4 w_1 \cdots w_n, \tag{2}$$

where $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq 3$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq 4$ and $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$ and $n \geq 5$ is odd. Note that there are instances of Equation (2) with $n = 3$ that also have non- θ -periodic solutions, by symmetry to the case $\ell = 3, m = 3, n = 4$ (see Example 50 in [8]).

We begin with a series of general remarks. Firstly, it is not hard to see that we can assume v to be θ -primitive. Otherwise, replacing v with its θ -primitive root, we end up with an equation with a greater number of v or $\theta(v)$ factors, and these have only θ -periodic solutions, by Theorem 1. Secondly, if we show that two of the words u, v , and w are θ -powers of a word t , then so is the third. Finally, as θ is an involution, it is safe to assume $v_1 = v$ and $w_1 = w$; other cases can be reduced to this one by replacing v by $\theta(v')$ or w by $\theta(w')$.

To solve Equation (2), we analyse all possible values of u_1, u_2 , and u_3 , and for all these we look at the different relations between $|v_1 \cdots v_4|$ and $|u|$ separately.

The following two lemmas are straightforward applications of Theorem 5:

Lemma 3. *If $4|v| \geq 2|u| + |v|$ and Equation (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Lemma 4. *If $4|v| \leq |u| - |w|$ (so, $n|w| \geq 2|u| + |w|$) and Equation (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

■ *The case $u_1 u_2 u_3 = u\theta(u)u$. This is covered by the following general result.*

Theorem 7. *For any $n, m \geq 3$, if $u\theta(u)u = v_1 \cdots v_m w_1 \cdots w_n$, then $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. Applying θ to the equation gives $\theta(u)u\theta(u) = \theta(w_n) \cdots \theta(w_1)\theta(v_m) \cdots \theta(v_1)$, and we catenate this to the original equation to yield:

$$(u\theta(u))^3 = v_1 \cdots v_m w_1 \cdots w_n \theta(w_n) \cdots \theta(w_1)\theta(v_m) \cdots \theta(v_1).$$

Cyclic shift converts this into

$$x^3 = \theta(v_m) \cdots \theta(v_1)v_1 \cdots v_m w_1 \cdots w_n \theta(w_n) \cdots \theta(w_1),$$

where x is a conjugate of $u\theta(u)$. The known results displayed in Table 1 and Theorem 1 cover the case of such equations, showing that $v, w \in \{t, \theta(t)\}^+$. □

■ *The case $u_1u_2u_3 = uuu$.* The analysis of this case is split in three subcases depending on whether $4|v| < |u|$, $|u| < 4|v| < 2|u|$, or $4|v| > 2|u|$ holds.

Lemma 5. *If $u_1u_2u_3 = uuu$, $4|v| < |u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some t .*

Proof. According to Proposition 3, w cannot be θ -primitive in this case. Therefore, $w \in \{w', \theta(w')\}^+$ for some θ -primitive word w' , and we analyse the equation $uuu = v_1 \cdots v_4 w'_1 \cdots w'_{n'}$, where $w'_i \in \{w', \theta(w')\}$ for all $1 \leq i \leq n'$ and some even $n' \geq 10$ instead of Equation (2).

Furthermore, we get that $w' = rpr$ for some words r and p , with $r = \theta(r)$ and $rp = \theta(p)r$. Also $w'_1 = \cdots = w'_{\frac{n'}{2}} = w'_{\frac{n'}{2}+1} = \cdots = w'_{n'} = \theta(w'_1)$. If $v = \theta(v)$, then we have the equation $u^3 = v^4 w_1^{\frac{n'}{2}} \theta(w'_1)^{\frac{n'}{2}}$, and Theorem 6 verifies our claim. Therefore we also assume $v \neq \theta(v)$ in the following.

Thus, $u = prw'^{\frac{n'}{2}-1} = prw'^{\frac{n'}{2}-2}rpr = prw'^{\frac{n'}{2}-2}\theta(p)rr$. Further, $|v_1 \cdots v_4| = |u| - 2|r|$ and so $v_1 \cdots v_4 = prw'^{\frac{n'}{2}-2}\theta(p) = (prr)^{\frac{n'}{2}-2}pr\theta(p)$.

Let $\tilde{w}' = prr$. We will show that \tilde{w}'^{ω} and $v_1 \cdots v_4$ have a common prefix long enough to apply Theorem 5:

If $|w'| \geq |v|$, then $|\tilde{w}'^{\frac{n'}{2}-2}pr| \geq |\tilde{w}'^3pr| > 2|w'| + |v|$. If $|v| > |w'|$ and $|v| \geq |pr\theta(p)|$, then $v^3 \leq_p \tilde{w}'^{\frac{n'}{2}-2}pr$ and $3|v| > 2|v| + |w'|$. On the other hand if $|v| > |w'|$ and $|v| < |pr\theta(p)|$, then $|p| + 2|r| = |w'| < |v| < 2|p| + |r|$, so $|r| < |p|$. As $rp = \theta(p)r$, we have that $r \leq_p \theta(p)$, and thus $v_1 \cdots v_4 = \tilde{w}'^{\frac{n'}{2}-1}\theta(p)''$, where $\theta(p)''$ is a suffix of $\theta(p)$. Since $|\theta(p)''| < |p| < |w'| < |v|$, we have that $|\tilde{w}'^{\frac{n'}{2}-1}| > 3|v|$, so again $v_1 \cdots v_4$ and \tilde{w}'^{ω} have a common prefix of length at least $3|v| > 2|v| + |w'|$ and we can apply Theorem 5.

In all cases, we get that $v, \tilde{w}' \in \{t, \theta(t)\}^+$ for some word t . However, as v is assumed to be θ -primitive, we must have $\tilde{w}' \in \{v, \theta(v)\}^+$.

Now, if $|r| < |p|$, as $rp = \theta(p)r$, we can write $\theta(p) = rs$ for some word s . Then, since $prr \in \{v, \theta(v)\}^+$ and $v_1 \cdots v_4 = (prr)^{\frac{n'}{2}-2}pr\theta(p)$, also $prrs = pr\theta(p) \in \{v, \theta(v)\}^+$ holds. From these last two results, we see that $s \in \{v, \theta(v)\}^+$ and thus also $\theta(s) \in \{v, \theta(v)\}^+$. However, as $prr = \theta(s)rpr \in \{v, \theta(v)\}^+$, by Theorem 5, also $r \in \{v, \theta(v)\}^+$. Hence, $\theta(p) = rs \in \{v, \theta(v)\}^+$, and the same holds for p . Thus, $w' = rpr \in \{v, \theta(v)\}^+$, and we are done.

If $|r| > |p|$, we write $r = \theta(p)s'$. As $prr = pr\theta(p)s'$ and $pr\theta(p)$ are both in $\{v, \theta(v)\}^+$, so is s' . Furthermore, as $pr\theta(p) = p\theta(p)s'\theta(p) \in \{v, \theta(v)\}^+$, if $p\theta(p) \in \{v, \theta(v)\}^+$, then by Theorem 5, also $p \in \{v, \theta(v)\}^+$, and so $r = \theta(p)s' \in \{v, \theta(v)\}^+$. This is a contradiction, since $w' = rpr$ is θ -primitive. Therefore, $p\theta(p) \notin \{v, \theta(v)\}^+$, which means that $s' \in \{v, \theta(v)\}^+$ is a proper factor of some word in $\{v, \theta(v)\}^+$. By Lemma 1, we must have that $s' \in \{v\}^+$ or $s' \in \{\theta(v)\}^+$, as v is θ -primitive. However, $\theta(p)\theta(p)s' = \theta(p)r = rp = \theta(p)s'p$, so $\theta(p)s' = s'p$, and we saw before that this means that s' is a θ -palindrome. In conclusion, $v = \theta(v)$ in both cases, and we get a contradiction. \square

Lemma 6. *If $u_1u_2u_3 = uuu$, $|u| < 4|v| < 2|u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. The two possible situations in this case are depicted in Figure 1 (with $v_j = v'_j v''_j$ for $j \in \{3, 4\}$ and $w_k = w'_k w''_k$).

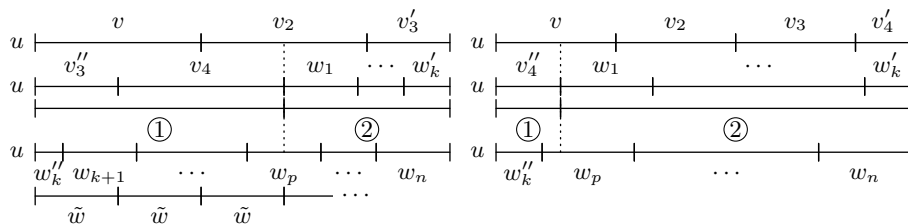


Fig. 1. The two possible cases for $uuu = v_1 \cdots v_4 w_1 \cdots w_n$.

We either have $u_2 = v''_3 v_4 w_1 \cdots w'_k$ for some k (left half of Figure 1) or $u_2 = v''_4 w_1 \cdots w'_k$ for some k (right half of Figure 1). We show that the factor ① in Figure 1 is a θ -palindrome. To streamline the presentation we assume v to be θ -primitive. Otherwise $v \in \{v', \theta(v')\}^+$ for some θ -primitive word v' , and we apply the reasoning below to v' , reaching the same conclusion.

In the case $v''_3 v_4 \leq_p u_2$, we apply Lemma 2 to $xvv_2 = v_3 v_4 y$, with x and y chosen accordingly, to get that $v_4 = \theta(v)$. If $v_3 = \theta(v_2)$, then ① is clearly a θ -palindrome. If $v_2 = v_3 = v$, then v''_3 is a prefix of v , so it also is a suffix of $v_4 = \theta(v)$. But $\theta(v''_3)$ is also a suffix of v_4 , so $v''_3 = \theta(v''_3)$, and thus $\theta(\textcircled{1}) = \theta(v''_3 v_4) = v v''_3 = \textcircled{1}$. A similar reasoning applies if $v_2 = v_3 = \theta(v)$. Furthermore, the factor ② is a θ -palindrome by the same arguments.

In the case $v''_4 \leq_p u_2$, we first show that ② is a θ -palindrome using the methods as above. Now, if $v_4 = \theta(v)$, then ① is obviously a θ -palindrome as well. If $v_4 = v$ we get that $v = v''_4 y$, where ① = v''_4 and $v_4 = v''_4 v''_4$. As ② is a θ -palindrome, it holds that $y = \theta(v''_4)$. Thus $v''_4 \theta(v''_4) = v''_4 v''_4$, and the solutions of this equation are given by $v''_4 = (\alpha\beta)^i, \theta(v''_4) = (\beta\alpha)^i, v''_4 = (\alpha\beta)^j \alpha$ for some $i \geq 1, j \geq 0$ and θ -palindromes α and β . Consequently, $v''_4 = \textcircled{1}$ is a θ -palindrome.

As the factors ① and ② are θ -palindromes, so is $w_1 \cdots w_n = \textcircled{2}\textcircled{1}\textcircled{2}$. Now, as n is odd, we get that $w_{\frac{n+1}{2}} = \theta(w_{\frac{n+1}{2}})$ and therefore $w = \theta(w)$.

Thus we have the equation $u^3 = v_1 \cdots v_4 w^n$, with $n \geq 5$.

If $|w^n| \geq |u| + |w|$, then Theorem 4 implies $u, w \in \{t\}^+$ for some t and we are done. Hence, we assume $|w^n| < |u| + |w|$, that is, $|u| > (n-1)|w|$. Then $|\textcircled{1}| = 2|u| - n|w| > |u| - |w| > \frac{n-2}{n-1}|u| > \frac{1}{2}|u|$. Hence, it suffices to consider only the case when $u = v_1 v_2 v'_3$ (Figure 1, left). Then Lemma 2 implies $v_4 = \theta(v)$. Furthermore, if $v_2 = v$, we can apply Theorem 4 to v^2 and \tilde{w}^ω , where \tilde{w} is a conjugate of w , to get that v is not primitive, a contradiction. So we also assume $v_2 = \theta(v)$. Now if $v_3 = v$, then the given equation turns into a classical Lyndon-Schützenberger equation $u^3 = (v\theta(v))^2 w^5$, which is solved as $u, v\theta(v), w \in \{t\}^+$ for some t and we are done. Otherwise, since $v_1 v_2 v_3 v_4 = \textcircled{1}\textcircled{2}\textcircled{1}$ is a θ -palindrome, we get $v = \theta(v)$, and the equation once again turns into a classical Lyndon-Schützenberger equation $u^3 = v^4 w^5$ and we are done. \square

The next lemma follows easily by Proposition 3; it is proved in the Appendix.

Lemma 7. *If $u_1u_2u_3 = uuu$, $4|v| > 2|u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

■ *The case $u_1u_2u_3 = uu\theta(u)$.* Like before, we split our analysis in two subcases: $4|v| < |u|$ or $|u| < 4|v| < 2|u|$. We remind the reader the remaining subcase, when $2|u| < 4|v|$, was already solved in [8], by the following general lemma.

Lemma 8 (Proposition 51 in [8]). *For any $m, n \geq 3$, if $u_1u_2u_3 = uu\theta(u)$, Equation (1) holds, and $m|v| > 2|u|$, then $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Lemma 9. *If $u_1u_2u_3 = uu\theta(u)$, $4|v| < |u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some t .*

Proof. Let us first assume that w is not θ -primitive. Then $w \in \{w', \theta(w')\}^r$ for some θ -primitive word w' and $r \geq 2$, and we would consider the equation $uu\theta(u) = v_1v_2v_3v_4w'_1 \cdots w'_{n'}$, where $n' = rn$. In this new equation, $n' \geq 10$ holds, but n' is not necessarily odd; we can still assume $4|v| + |w'| > |u|$. As $u_2 \neq u_3$, Proposition 3 implies that $w'_1 = \dots = w'_{n'} = w'$. Now, as $u_3 \leq_s (w')^n$, we have $u_1 \leq_p \theta(w')^n$ and $v_1 \cdots v_4 \leq_p u_1$. If $|v| \geq |w'|$, then $|v_1 \cdots v_4| \geq 2|v| + |w'|$, and so by Theorem 5, we get $v, w' \in \{t, \theta(t)\}^+$ for some word t . In fact, this reasoning still applies if $3|v| \geq 2|w'|$, that is if $\frac{3}{2}|v| \geq |w'|$. As a consequence, the case remaining to be investigated is when $|w'| > \frac{3}{2}|v|$. Since $4|v| + |w'| > |u|$ and $|w'| > \frac{3}{2}|v|$, we get that $n' \leq 8$; this means that $n \leq 4$, a contradiction.

Accordingly, we only need to analyse Equation (2) for w θ -primitive. By Proposition 3 we get that $w = p\theta(z)z$ where p is a θ -palindrome, and, as above, it also follows that we only have to analyse the cases when $n \leq 8$, so $n \in \{5, 7\}$.

If $n = 7$, then $w^3 \leq_s u_3 = \theta(u)$ and so $\theta(w)^3 \leq_p u_1 = u$. Since $|w| > \frac{3}{2}|v|$ and $4|v| + |w| > |u| > 3|v|$, it follows that $\frac{3}{2}|v| < |w| < 2|v|$. Thus, in $u_1 = u$, we have the situation depicted in Figure 2.

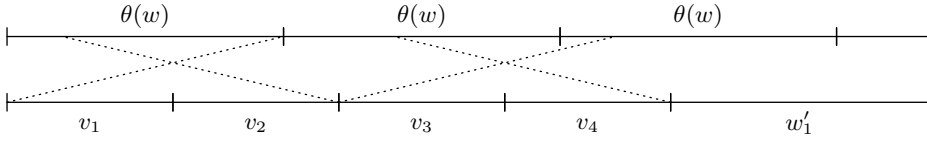


Fig. 2. The situation inside $u_1 = u$.

As v_1v_2 is a prefix of $\theta(w)\theta(w)$, it appears again as a proper factor inside $v_2v_3v_4$. In the same manner v_3v_4 appears as a proper factor inside $v_1v_2v_3$ (see the dotted lines in Figure 2). Therefore, if $v_1 \neq v_2$ or $v_3 \neq v_4$, then v is not θ -primitive by Lemma 1, a contradiction.

This leaves us with two cases only: $v_1 \cdots v_4 = v^4$, or $v_1 \cdots v_4 = vv\theta(v)\theta(v)$. In the first case v^4 is a prefix of $\theta(w)^3$ and $4|v| > 2|v| + |v| > |w| + |v|$. Thus we can apply Theorem 4, to get $v, \theta(w) \in \{t\}^+$ for some word t , and we are done. In the other case, when $v_1 \cdots v_4 = vv\theta(v)\theta(v)$, then $v_1 \cdots v_4$ is a θ -palindrome and $\theta(w)$ is a prefix of it. This means that w is a suffix of $v_1 \cdots v_4$ and so $\theta(w^{n+1})$ is a prefix of $\theta(u\theta(u)\theta(u))$ of length $(n+1)|w| > 2|u| + |w|$. Therefore we can apply Theorem 5 to get that $u, w \in \{t, \theta(t)\}^+$ for some word t in this case.

If $n = 5$, we know that $u_1 = u = \theta(w)\theta(w)\theta(z) = \theta(z)zp\theta(z)zp\theta(z)$. Thus u has a prefix $\alpha = \theta(z)zp\theta(z)z$, which is a θ -palindrome.

We observe that $|u| = 5|z| + 2|p|$ and also $4|v| = |u| - |p|$. It follows that $4|v| = 5|z| + |p| > 4|z| + |p| = |\alpha| > \frac{15}{4}|z| + \frac{3}{4}|p| = 3|v|$. So $\alpha = v_1v_2v_3v'_4 = \theta(v'_4)\theta(v_3)\theta(v_2)\theta(v_1)$ where v'_4 is a proper prefix of v_4 . Using Lemma 1 we get $v_1 = v_2 = v_3$. Without loss of generality, we assume $v_1 = v$.

If $|w| \leq 2|v|$, then $v_1v_2v_3 = v^3$ is a prefix of $\theta(w)^2$ of length at least $|w| + |v|$. Thus, by Theorem 4 we get $v, \theta(w) \in \{t\}^+$ for some word t , and we are done.

If $2|v| < |w| < 3|v|$, we have the situation in u_1 as depicted in Figure 3.

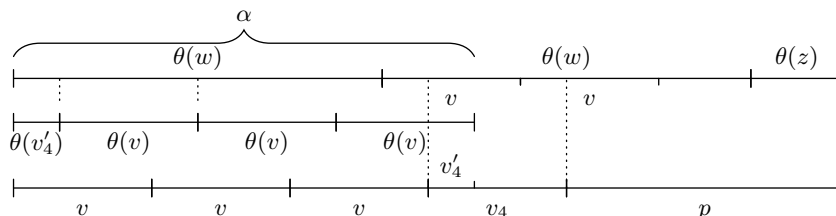


Fig. 3. The situation when $2|v| < |w| < 3|v|$.

Since v as a prefix of $\theta(w)$ appears as a proper factor inside vv_4 and it is assumed to be primitive, $v_4 = \theta(v)$ must hold. Thus $\theta(v)$ appears as a factor inside the prefix vv of $\theta(w)$ after a prefix of length $3|v| - |w|$. However, $\theta(v)$ also appears inside vv after the prefix of length $|v'_4| = |\alpha| - 3|v|$. As v is primitive, these occurrences must coincide, so $3|v| - |w| = |\alpha| - 3|v|$ must hold. Using the known values $|\alpha| = |\theta(z)zp\theta(z)z| = 4|z| + |p|$ and $|v| = \frac{5}{4}|z| + \frac{1}{4}|p|$, this equation is equivalent to $3|z| = |p|$. Furthermore, $\theta(z) \leq_s p$ and as $p \leq_s \theta(w)$, we get that $p = \theta(z)^3$. This however means that $\theta(w) = \theta(z)zp = \theta(z)z\theta(z)^3$, contradicting the θ -primitivity of w .

If $3|v| < |w| < 4|v|$, then as in the previous case $\theta(z) \leq_s p$ and $p \leq_s \theta(w)$. Therefore and since p is a θ -palindrome, $p = z^kz'$ for some $k \geq 1$ and $z' \leq_p z$. Now, as v^3 is a prefix of $\theta(w) = \theta(z)zp = \theta(z)zz^kz'$ and $|v| > |z|$ (recall that $|v| = \frac{5}{4}|z| + \frac{1}{4}|p|$), we can apply Theorem 5 here to get $v, z \in \{t, \theta(t)\}^+$ and since $|v| > |z|$, this contradicts the θ -primitivity of v . \square

The following result follows by combining arguments used in the corresponding subcase for $u_1u_2u_3 = uuu$ and in the proof of the previous Lemma. It is shown in the Appendix.

Lemma 10. *If $u_1u_2u_3 = uu\theta(u)$, $|u| < 4|v| < 2|u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some t .*

■ *The case $u_1u_2u_3 = u\theta(u)\theta(u)$.* The subcase when $4|v| < |u|$ is simple. We just have to apply θ to both sides of the equations and then use Lemma 8.

Lemma 11. *If $u_1u_2u_3 = u\theta(u)\theta(u)$, $4|v| < |u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some t .*

Lemma 12. *If $u_1u_2u_3 = u\theta(u)\theta(u)$, $|u| < 4|v| < 2|u|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some t .*

Proof. We consider two separate cases: The first is when $v_3''v_4 \leq_p u_2$, where v_3'' is a suffix of $v_3 = v_3'v_3''$. As v is assumed to be θ -primitive, $|v_3'| \neq |v_3''|$. In this case $|v| > \frac{1}{3}|u|$ and so $4|v| > \frac{4}{3}|u|$. It follows that $n|w| < \frac{5}{3}|u|$ and since $n \geq 5$, we get that $|w| < \frac{1}{3}|u| < |v|$. Furthermore we can assume that $|u| < 2|v| + |w|$, otherwise we get the claimed result by a simple application of Theorem 5. This means that $|w| > |v_3'|$.

We will now prove that $v_1 = v_3 = v_4 = v$ holds, as follows. Suppose, towards a contradiction, that $v_4 = \theta(v)$. Since $\theta(w_n) \leq_p v$, this assumption implies that v_4 ends with w_n . Then $w_nw_1 \dots w_k' \leq_s \theta(u)$, where $w_k = w_k'w_k''$ for some k , and, thus, $w_nw_1 \dots w_k' \leq_s w_1 \dots w_n$. By Lemma 1, we get that $w_n = w_1 = \dots = w_{k-1}$ and $w_{n-1} = w_n$. So, by Proposition 1, $w_n \neq w_n$ must hold, a contradiction. Thus, $v_4 = v$. On the θ -palindrome $u\theta(u)$, $\theta(v_4)\theta(v_3)$ lies inside $v_2v_3v_4$ in such a way that Lemma 1 implies $v_3 = v_4$ (recall that $|v_3'| \neq |v_3''|$). Hence, $v_1 = v_3 = v_4 = v$.

We split the discussion further, according to the relation between $|v_3'|$ and $|v_3''|$:

If $|v_3'| > |v_3''|$, then we have also $|w| > |v_3''|$, as we already established $|w| > |v_3'|$. Now, $|v| = |v_3'| + |v_3''|$ and $|u| = 3|v_3'| + 2|v_3''|$. From this we get that $n|w| = 3|u| - 4|v| = 5|v_3'| + 2|v_3''|$. However, $|w| > |v_3'|$ and $|w| > |v_3''|$, so n must be 5.

Assume $3|w| \leq 2|v|$, that is $6|w| \leq 4|v|$. Now, as $|v_3'| > |v_3''|$, we have that $\frac{5}{2}|v| < |u|$, and therefore $4|v| < \frac{8}{5}|u|$. Since $6|w| \leq 4|v|$, we get $6|w| < \frac{8}{5}|u|$, hence $5|w| < \frac{8}{6}|u|$. It follows that $4|v| + 5|w| < \frac{8}{5}|u| + \frac{8}{6}|u| = \frac{88}{30}|u| < 3|u|$, a contradiction. Thus $3|w| > 2|v|$ must hold, and as $\theta(w_5)\theta(w_4)\theta(w_3) \leq_p vv_2v$, we can apply Proposition 3 to get $v_2 = \theta(v)$ and $w_3 = w_4 = w_5$. The fact that w_1w_2' is a suffix of $w_3w_4w_5$ leads to $w_1 = w$ and $w_3 = w_4 = w_5 = \theta(w)$. Now we have that $w_5 \leq_p v_1$, so $w \leq_p v$ and we have two occurrences of w inside $u_2 = \theta(u)$: One (the prefix of v_4) is after the prefix of length $3|v| - |u|$ and the other one (w_1) is after the prefix of length $4|v| - |u|$. Both these occurrences fall inside $w_3w_4w_5 = \theta(w)^3$ inside $u_3 = \theta(u)$. In order for w to be primitive (as assumed), the length of the factor between those two occurrences must be a multiple of $|w|$. This factor is of length $|v|$ though, and since $|v| > |w|$, but $2|w| > |v|$ (as $|w| > |v_3'|$ and $|w| > |v_3''|$), this is impossible.

We reached the case when $|v_3'| < |v_3''|$; by Lemma 1 we get that $v_2 = v_3 = v_4$.

We first establish that $v_1 = v_2$ holds as well. Assume towards a contradiction, that $v_1 = \theta(v_2)$. Then, as $\theta(w_n) \leq_p v_1$, we have $w_n \leq_s v_4 = v_2 = \theta(v_1)$. Now, if w_1 is a factor of u_2 , the word $w_nw_1 \dots w_k'$ for some $k \geq 2$ is a suffix of $\theta(u)$. However, also $\theta(u) = w_k'' \dots w_n$. By Lemma 1, we get that $w_n = w_1 = \dots = w_{k-1} = w$, and also that $w_{n-1} = w_n$. But then $w_{k-1} = w_n$ appears as a proper factor inside $w_{n-1}w_n = w_nw_n$, contradicting the primitivity of w . If w_1 is not a factor of u_2 , that is, $w_1' \leq_s u_2$, where $w_1 = w_1'w_1''$, then w_nw_1' is a suffix of $w_{n-1}w_n$, so $xw_nw_1 = w_{n-1}w_ny$ where $y = w_1''$ and x is the prefix of length $|w_1''|$ of w_{n-1} . By Lemma 2 we get that $w_n \neq w_n$, a contradiction. Thus, $v_1 = \dots = v_4 = v$.

If the prefix of length $|v| + |w|$ of $\theta(w_1 \dots w_n)$ is a prefix of a word in $\{w\}^+$ or $\{\theta(w)\}^+$, then we can apply Theorem 4 to get the claimed result. Thus we assume that $w\theta(w)$ or $\theta(w)w$ appears as a factor inside vv after a prefix that is

strictly shorter than $|v|$. Without loss of generality we assume that it is $w\theta(w)$ that occurs there, and we focus on the first occurrence of this factor inside vv .

We first rule out the possibility that $w\theta(w)$ is a prefix of vv : We observe that $|u| \geq 2|w| + |v|$. To see this, assume that $|u| < 2|w| + |v|$. This means that $3|u| < 6|w| + 3|v| < 5|w| + 4|v|$, which is a contradiction. Therefore, if $w\theta(w)$ is a prefix of vv , it has another appearance inside u after a prefix of length $|v|$. However, as $|v|$ is not divisible by $|w|$ (otherwise $v \in \{w, \theta(w)\}^+$ and $|v| > |w|$, so v would not be θ -primitive), this other occurrence of $w\theta(w)$ is a proper factor of some word $w_i w_{i+1} w_{i+2}$ inside $u_3 = \theta(u)$. By Lemma 1 this is impossible if w is θ -primitive. Hence, we assume that $xw\theta(w)$ is a prefix of vv , where $x \in \{w\}^+$. As $|w| > |v'_3|$, we have an occurrence of $w\theta(w)$ in $u_2 = \theta(u)$ after a prefix of length $|x| - |v'_3|$. So in u , we have an occurrence of $w\theta(w)$ after the prefix of length $|x|$, and after the prefix of length $|u| - (|x| - |v'_3|) - 2|w|$. As $x \in \{w\}^+$, we must have $|u| - (|x| - |v'_3|) - 2|w| \geq |x|$. Now as u is prefix of $\theta(w_1 \cdots w_n)$, in order to avoid a contradiction with Lemma 1, the difference between the lengths of those two prefixes must be divisible by $|w|$. However, this difference is $|u| - (|x| - |v'_3|) - 2|w| - |x| = |u| - 2|x| + |v'_3| - 2|w|$ and as $x \in \{w\}^+$, the term $|u| + |v'_3|$ must be divisible by $|w|$. Now $|u| = 2|v| + |v'_3|$, thus $2|u| = 4|v| + 2|v'_3|$, and as $|w_1 \cdots w_n| = 3|u| - 4|v|$, we have $|w_1 \cdots w_n| = |u| + 2|v'_3|$. If now $|u| + |v'_3|$ is divisible by $|w|$, then also $|v'_3|$ must be divisible by $|w|$. This however is a contradiction, as we assumed that $|v'_3| < |w|$.

The other case to be considered is when $w''_4 \leq_p u_2$. In this case, we can apply Theorem 5 to $\theta(w_1 \cdots w_n)$ and $v_1 v_2 v_3$ if $|v| \geq |w|$ to get the claimed result. Thus we assume $|v| < |w|$ in the remainder of this proof. Theorem 5 still applies if $2|w| + |v| \leq |u|$, so we assume $2|w| + |v| > |u|$ as well. This means that $6|w| + 3|v| > 3|u|$ and hence, n must be 5. Then we can easily observe that the border between u_2 and u_3 lies inside w_3 . So, let $w_3 = w'_3 w''_3$ such that $w_1 w_2 w'_3$ is a suffix of u_2 and $u_3 = w''_3 w_4 w_5$. As also $w_1 w_2 w'_3 \leq_s \theta(u)$, by Lemma 1, we get that $w_1 = w_2 = w$ while $w_4 = w_5 = \theta(w)$.

Now, $\theta(v_1 v_2) \leq_s w_4 w_5 = \theta(w)\theta(w)$, from which we get that $v_1 v_2 \leq_p w_1 w_2 = ww$. Hence $v''_4 v_1 v_2 \leq_p \theta(u)$ and thus $\theta(v_2)\theta(v_1)\theta(v''_4) \leq_s u$. We get $v_1 = v_2 = v$.

If $3|v| > 2|w|$, since we have $v_1 v_2 v_3 \leq_p \theta(w_5)\theta(w_4)\theta(w_3)$, can apply Proposition 3 to get a contradiction with the fact that $w_4 = w_5$. Thus we assume $3|v| < 2|w|$ (if $3|v| = 2|w|$ we can apply Theorem 5 to get $v, w \in \{t, \theta(t)\}^+$ directly). Then since $ww \leq_p u$ and $v_1 = v_2 = v$, we must have $v_3 = \theta(v)$. Now there is an occurrence of vv in $\theta(u)$ after the prefix of length $4|v| - |u|$, and an occurrence of v after the prefix of length $|u| - 3|v|$. To avoid this v to appear as a proper factor inside the factor vv , we must have either $|u| - 3|v| \leq 4|v| - |u|$ or $|u| - 3|v| \geq 5|v| - |u|$. We show that both of these possibilities lead to a contradiction: If $|u| - 3|v| \leq 4|v| - |u|$, we have $2|u| \leq 7|v| = 4|v| + 3|v| < 4|v| + 2|w|$. This however means that $3|w| < |u|$ must hold, which leads to a contradiction as we have seen before. On the other hand, if $|u| - 3|v| \geq 5|v| - |u|$, we have $2|u| \geq 8|v|$, which is a contradiction straightaway, as $4|v| > |u|$. \square

Lemma 13. *If $u_1 u_2 u_3 = u\theta(u)\theta(u)$, $2|u| < 4|v| < 2|u| + |v|$, and Equation (2) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some t .*

Proof. By Proposition 3, we have $v_1 = \dots = v_4 = v$ and $v = xz\theta(z)$, for some words x and z , with $x = \theta(x)$. Furthermore, $u = vxz = xz\theta(z)xz$. Therefore, the situation in $u_3 = \theta(u)$ looks as illustrated in Figure 4. We see that $z = \theta(z)$ in this case. Now $\theta(u) = zxxzx = zzw_1 \dots w_n$, and so $xz^2x = zw_1 \dots w_n$.

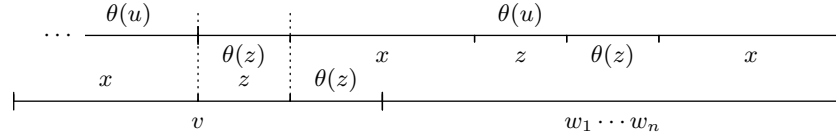


Fig. 4. The word $\theta(u)$.

Since both xz^2x and z are θ -palindromes and $|z| \leq |w_1 \dots w_n|$, the equation implies $z \leq_s w_1 \dots w_n$. Hence, there exists an integer k such that $z = w_k'' w_{k+1} \dots w_n$ where $w_k = w_k' w_k''$ for some w_k', w_k'' . If w_k'' is empty, that is, $z = w_{k+1} \dots w_n$, then $xz^2x = xz\theta(xz) = w_{k+1} \dots w_n w_1 \dots w_n$. Theorem 5 applied to this then implies $xz \in \{w, \theta(w)\}^+$ because w is θ -primitive. Combining this with $z = w_{k+1} \dots w_n$ gives $x \in \{w, \theta(w)\}^+$. Then $v = xz^2 \in \{w, \theta(w)\}^+$, but this contradicts the θ -primitivity of v , as $|v| \geq 3|w|$. If, on the other hand, w_k'' is not empty, then $xz^2x = w_k'' w_{k+1} \dots w_n w_1 \dots w_n$ implies $w_1 = \dots = w_n = w$. Now we have $xz^2x = zw^n$ and concatenating z^2 to the left on both sides results in $(z^2x)^2 = z^3w^n$. This is a conventional Lyndon-Schützenberger equation, and it implies $z^2x, z \in \{w\}^+$. Substituting this into the equation, however, implies $z^2x \in \{w\}^+$, and $|z^2x| \geq 3|w|$. Hence, z^2x is not primitive and neither is its conjugate v , a contradiction. \square

■ *Conclusion.* By the results of Lemmas 3–12 we get that Theorem 2 holds.

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Appendix: Disclaimer

This Appendix is split into two parts.

The first presents the proofs of two lemmas needed to show Theorem 2. These two lemmas, whose proofs are not very complicated and, in fact, are derived in a way similar to other results, complete the main technical section of our paper.

The second part of the Appendix contains the complete proof of Theorem 1. It was left out from the main 12 pages (an extended abstract of our results) due to the space limitations. Also, our decision to present the technicalities related to this theorem in the Appendix was influenced by the fact that Theorem 1 comes as a completion to the results from [9], while Theorem 2 solves an unrelated case, on which some of the arguments used in the other cases do not work.

Appendix: Proofs of Lemma 7, Lemma 10

Proof of Lemma 7.

Proof. By Proposition 3, we have $v_1 = v_2 = v$ and $v_3 = v_4 = \theta(v)$, and furthermore $v = rpr$ for some words r, p . From the overlap between v and $\theta(v)$ inside u_2 we see that $pr = r\theta(p)$ and $r = \theta(r)$. Furthermore, $\theta(v) = prr$ and we have that $u = rprrp = rr\theta(p)rp$, where the prefix rr is a suffix of $v_4 = \theta(v)$. Therefore $w_1 \cdots w_n = \theta(p)rp$, which is a θ -palindrome, and since n is odd, we get that $w = \theta(w)$. Thus $u = rprrp$ and $u = rrw^n$, therefore $rprrp = rrw^n$. Adding r to left on both sides of this equation gives us $(rrp)^2 = r^3w^n$, which implies $rrp, r, w \in \{t\}^+$ for some word t by Lyndon and Schützenberger’s original result. It follows that $rpr = v$ is not primitive, a contradiction. \square

Proof of Lemma 10:

Proof. As in the proof of the previous Lemma, we assume that w is θ -primitive. Recall that if it was not, i.e., $w \in \{w', \theta(w')\}^+$ for a θ -primitive word w' , then we would consider the equation $uu\theta(u) = v_1v_2v_3v_4w'_1 \cdots w'_n$, where $n' > n$. So, in general we can not assume n to be odd anymore, but for $n < 10$ we still can.

There are now two cases to be considered. The first is when $v''_3v_4 \leq_p u_2$, where v''_3 is a suffix of $v_3 = v'_3v''_3$.

In that case we see that $|v| > \frac{1}{3}|u|$ and thus $4|v| > \frac{4}{3}|u|$. It follows that $n|w| < \frac{5}{3}|u|$ and since $n \geq 5$, we get $|w| < \frac{1}{3}|u| < |v|$. Thus, if $|v'_3| \geq |w|$, we can apply Theorem 5, since u is a prefix of $\theta(w_1 \cdots w_n)$ and of vv_2v_3 , and $|u| = 2|v| + |v'_3| \geq 2|v| + |w|$ then.

Therefore we assume $|v'_3| < |w|$ in the following. An application of Lemma 2 tells us that $v_4 = \theta(v)$ must hold. Using the same methods that were used in the proof of Lemma 6 to show that the factor ① there is a θ -palindrome, we deduce that the prefix v''_3v_4 here is a θ -palindrome too.

Now, $w_1 \cdots w_n$ is a suffix of $u\theta(u)$ and $u\theta(u)$ is a θ -palindrome. Hence, the word $\theta(w_n) \cdots \theta(w_1)$ is a prefix of $u\theta(u)$. Thus $v''_3v_4 = \theta(w_n) \cdots \theta(w'_k)$ for some

k and $w_k = w'_k w_k''$, and since it is a θ -palindrome, also $v''_3 v_4 = w''_k \cdots w_n$. It follows that $u\theta(u) = w''_k \cdots w_n w_1 \cdots w_n = \theta(w_n) \cdots \theta(w_1) \theta(w_n) \cdots \theta(w''_k)$. Repeated application of Lemma 1 allows us to conclude that $w_1 = \dots = w_n = w$.

Now if $v_2 = v$, we can apply Theorem 4 to vv and the power of $\theta(w)$ that is a prefix of u to get that $u, \theta(v) \in \{t\}^+$ for some word t , which leads to $u, v \in \{t, \theta(t)\}^+$. Therefore $v_2 = \theta(v)$ must hold.

Now as $\theta(w) \leq_p v$, we get that $w \leq_s v_1 v_2$ and $w \leq_s v_4$. However, $v''_3 v_4 \leq_p v_1 v_2 \leq_p v''_3 v_4 w$, and thus w appears as a factor inside ww , contradicting its primitivity.

We move on to the other case, namely when $v''_4 \leq_p u_2$, where v''_4 is a suffix of $v_4 = v'_4 v''_4$. In this case $v_1 v_2 v_3 \leq_p u$ and $u \leq_p \theta(w_n) \cdots \theta(w_1)$. If $|w| \leq |v|$, we can thus apply Theorem 5 to get that $v, w \in \{t, \theta(t)\}^+$ for some word t . Therefore we can assume $|w| > |v|$. If $2|w| + |v| \leq |u|$ held, we could also apply Theorem 4 to get that same result. So $2|w| + |v| > |u|$ can be assumed as well. This means that $6|w| + 3|v| > 3|u|$. As a consequence, $n = 5$ must hold.

Now $|v| < \frac{1}{3}|u|$ and thus $|w| > \frac{1}{3}|u|$ must hold, and as a result, $|v| < |w|$. Moreover, $|u| < 4|v|$ implies $|w| < \frac{2}{5}|u|$, and hence, $|w| < \frac{8}{5}|v|$.

Since $u_2 u_3 = u\theta(u) = v''_4 w_1 \cdots w_5$ is a θ -palindrome and $0 < |v''_4| < |w|$, we get $w_1 = w_2 = w_3 = w_4 = w$ by Lemma 1. If $w_5 = \theta(w)$, then $v_1 = v$ is a prefix of w . Then we find as a prefix of u_2 that $v''_4 v \leq_p v v_2$, but this contradicts the θ -primitivity of v by Lemma 2. Therefore, $w_5 = w$, that is, the given equation is

$$uu\theta(u) = vv_2 v_3 v_4 w^5.$$

Now $u\theta(u) = v''_4 w^5$ implies $w = z\theta(z)p$ for some words p, z , with $p = \theta(p)$. Hence $u = (pz\theta(z))^2 pz$ and $v''_4 = p$. If $|u| \geq 2|w| + |v|$, that is, $|pz| \geq |v|$, then Theorem 5 is applied to $u = vv_2 v_3 v''_4 = (pz\theta(z))^2 pz$ to obtain that $v, \theta(w) \in \{t, \theta(t)\}^+$ for some word t , but as $|w| > |v|$, this contradicts the θ -primitivity of w . Thus, $|u| < 2|w| + |v|$, which is equivalent to $|v''_4| > |z|$. With $|u| = 3|v| + |v''_4| = 3|p| + 5|z|$, this gives $3|v| < 3|p| + 4|z|$. Thus, $vv_2 v_3 \leq_p (pz\theta(z))^2 p \leq_p vv_2 v_3 v_4$ and as $(pz\theta(z))^2 p$ is a θ -palindrome, $\theta(vv_2 v_3) \leq_s (pz\theta(z))^2 p \leq_p vv_2 v_3 v_4$. As a result, $v = v_2 = v_3$ by Lemma 1. Now we have $v^3 <_p \theta(w)^3$, to which Theorem 4 is applicable (recall $|w| < \frac{8}{5}|v|$, and hence $3|v| > |v| + |w|$) which gives $v, \theta(w) \in \{t\}^+$ for some word t , and since $|w| > |v|$, this contradicts the primitivity of w . \square

Appendix: The proof of Theorem 1

In this appendix, we thoroughly close the other case, that is, we prove Theorem 1, which states that if $\ell = 3, 5 \leq \min\{m, n\} < 12$, and both m and n are even, then Eq. (1) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t . As mentioned in the main text, for this purpose it suffices to examine the equation under the condition that $u_1 = u_2 = u_3 = u$ and $|u| < m|v| < 2|u|$ (i.e., the border between v_m and w_1 is on u_2).

At the very beginning, it must be noted that if u is not θ -primitive, that is, $u = u_1 \cdots u_{\ell'}$ for some $\ell' \geq 2$ and $u_1, \dots, u_{\ell'} \in \{u', \theta(u')\}$, then the known results immediately solve Eq. (1) as we intend. That is, substituting this representation into Eq. (1) yields $u_1 \cdots u_{\ell'} u_1 \cdots u_{\ell'} u_1 \cdots u_{\ell'} = v_1 \cdots v_m w_1 \cdots w_n$.

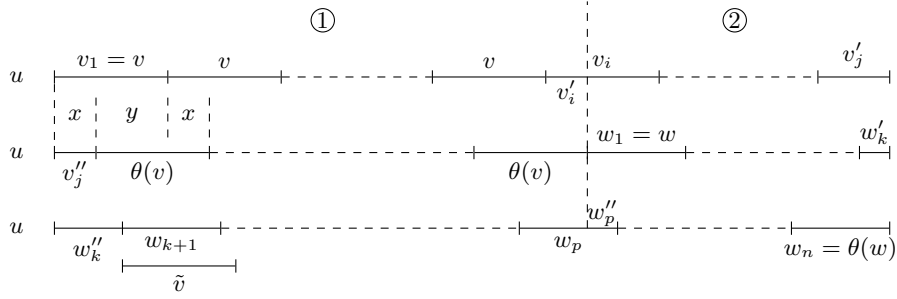


Fig. 5. Visualization of Eq. (3).

Proposition 4. *If u is not θ -primitive, then Eq. (1) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Thanks to this, throughout this appendix, we assume that u is θ -primitive and show that under this assumption, Eq. (1) does NOT hold. In proofs below, when we reach $v, w \in \{t, \theta(t)\}^+$ for some word t , which can be actually assumed to be θ -primitive, Eq. (1) implies $u_1 u_2 u_3 \in \{t, \theta(t)\}^{\geq m+n}$, that is, $u_1 \in \{t, \theta(t)\}^{\geq (m+n)/3}$. This is a clear contradiction, because $m + n$ is at least 10.

Symmetric roles of $v_1 \cdots v_m$ and $w_1 \cdots w_n$ in this equation enable us to assume $|v_1 \cdots v_m| \geq |w_1 \cdots w_n|$. Without loss of generality, we can assume $v_1 = v$ and $w_1 = w$. In summary, we only have to examine the equations of the form:

$$u^3 = v v_2 \cdots v_m w w_2 \cdots w_n, \quad (3)$$

where $v_2, \dots, v_m \in \{v, \theta(v)\}$, $w_2, \dots, w_n \in \{w, \theta(w)\}$, and both m and n are even numbers such that $6 \leq \min\{m, n\} < 12$ and $\frac{3}{2}|u| \leq m|v| < 2|u|$.

A-1 Notation and basic observations

Let us fix notation used throughout the proofs below, and also make basic observations of use.

Eq. (3) is illustrated in Fig. 5. We denote the suffix of $v_1 \cdots v_m$ that is a prefix of u_2 by ① and the rest of u_2 by ②, that is, $u_2 = u = \textcircled{1}\textcircled{2}$. The assumption $|v_1 \cdots v_m| \geq |w_1 \cdots w_n|$ is now rephrased as $|\textcircled{1}| \geq |\textcircled{2}|$. Due to $|u| < m|v| < 2|u|$, there exist indices $1 \leq j \leq m$, $1 \leq k \leq n$, and words v'_j, v''_j, w'_k, w''_k such that $v_j = v'_j v''_j$, $w_k = w'_k w''_k$, $u_1 = v_1 \cdots v_{j-1} v'_j$, $u_2 = v''_j v_{j+1} \cdots v_m w_1 \cdots w_{k-1} w'_k$, and $u_3 = w''_k w_{k+1} \cdots w_n$. Since u is assumed to be θ -primitive, none of v'_j, v''_j, w'_k, w''_k is empty. Let p be the index such that the border between ① and ② lies on w_p . The border splits w_p into the prefix w'_p and the suffix w''_p . Observe that the border between $w_{n/2}$ and $w_{n/2+1}$ is exactly at the center of ②. This means $p = n - k + 1$. As n is even, this means that the parity of p differs from that of k . That is, $p - k$ is odd. Moreover it must be at least 3 because of $|\textcircled{1}| \geq |\textcircled{2}|$ and $n \geq 6$.

Let i be the index such that the border between ① and ② lies on v_i . Since $|\textcircled{1}| \geq |\textcircled{2}|$, $m \geq 6$, and v''_j is nonempty, $i \geq 3$ must hold. The border splits

v_i into the prefix v'_i and the suffix v''_i and now we have $\textcircled{1} = v_1 v_2 \cdots v_{i-1} v'_i = v''_j v_{j+1} \cdots v_m$. Lemma 2 implies that $v_1 = v_2 = \cdots = v_{i-1} = v$ and $v_{j+1} = \cdots = v_m = \theta(v)$. Thus, $\textcircled{1}$ is a θ -palindrome. It can be represented as

$$\textcircled{1} = (xy)^{i-1}x \quad (4)$$

for some words x, y such that $v = xy$ and $v'_i = x$, and this representation implies $x = \theta(x)$ and $y = \theta(y)$. The analogous analysis works also for w_1, \dots, w_n and $\textcircled{2}$ as long as $|\textcircled{2}| \geq |w|$. Even if it is shorter, if $w_n = \theta(w_1)$, then $\textcircled{2}$ is a θ -palindrome. If $w_n = w_1$, then we can let $w_1 = \textcircled{2}y$ and $w_n = \theta(y)\textcircled{2}$; the prefix of w_n is the θ -image of the suffix of w_1 because they are a prefix and suffix of the θ -palindrome $\textcircled{1}$, respectively. Hence, we have $\textcircled{2}y = \theta(y)\textcircled{2}$, and this word equation is solved as $\textcircled{2} = (\beta\alpha)^i\beta$ and $y = \alpha\beta$ for some $i \geq 0$ and θ -palindromes α, β . As such, $\textcircled{2}$ is also a θ -palindrome. Now we can obtain useful relations $v_j = \theta(v_i)$, $w_p = \theta(w_k)$, and $w_n = \theta(w_1) = \theta(w)$ from the θ -palindromes $\textcircled{1}\textcircled{2}\textcircled{1} = v_1 \cdots v_m$ and $\textcircled{2}\textcircled{1}\textcircled{2} = w_1 \cdots w_n$. It is clear from them that both $\textcircled{1}$ and $\textcircled{2}$ are of even length. With Eq. (4), this further implies that if i is odd, then the θ -palindrome x is of even length so that we can let $x = x'\theta(x')$ for some word x' ; otherwise, the θ -palindrome y is of even length and we can let $y = y'\theta(y)$ for some word y' .

We make use of the following lemmas proved in [9].

Lemma 14 ([9]). *In the above setting, assume that v is θ -primitive and $|v|$ does not divide $|u|$. Then $v_1 = \cdots = v_{i-1} = v$ and $v_{m-i+1} = \cdots = v_m = \theta(v)$. Moreover, if $i|v| - |u| \geq \frac{|v|}{2}$, then $v_i = v$.*

Lemma 15 ([9]). *In the above setting, assume that $|u| < m|v| < 2|u|$, w is θ -primitive, $|v| > |w|$, and there exists a word $\tilde{v} \sim v$, such that \tilde{v} occurs in vv after the prefix of length $i = |u| \bmod |w|$ (see Fig. 5), and $\tilde{v} \in \{w, \theta(w)\}^+$. Then $v = \tilde{v}$.*

Let us conclude the preliminaries by presenting two conditions under which Eq. (3) can be instantly solved as $u, v, w \in \{t, \theta(t)\}^+$ for some word t : i.e.,

1. v or w is a θ -palindrome;
2. $|v| = |w|$.

In the proofs below, we will see the θ -primitivity of u cause a contradiction. This does not mean that Eq. (3) could not hold under each assumption above, but just means that θ -primitivity of u is not consistent with any of them on Eq. (3).

Lemma 16. *If v or w is a θ -palindrome, then Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. The case of $w = \theta(w)$ was already proved in [9]. As for the other case $v = \theta(v)$, let s be the primitive root of v , that is, $v = s^k$ for some s . Then $s = \theta(s)$. Now we have the equation $\textcircled{1} = v^{i-1}x$ and it implies that x is a power of s , and so is $\textcircled{1}$. From $\textcircled{1}\textcircled{2}\textcircled{1} = v^m$, this derives that $u = \textcircled{1}\textcircled{2}$ would be a power of the shorter word s , a contradiction. \square

Lemma 17. *If $|v| = |w|$, then Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. Then Eq. (3) implies $3|u| = m|v| + n|w| = (m+n)|v|$. Since $|u| < m|v| < 2|u|$ is assumed, $|u|$ must not be a multiple of $|v|$ because it is assumed to be θ -primitive. Thus, $m+n$ does not divide 3, that is, $|v|$ does divide. Let $v = x_1x_2x_3$ for some x_1, x_2, x_3 of the equal length d .

Recall $v_1 = v_2 = v$. If $|u| \bmod |v| = d$, then $v_{j+1} = \theta(v)$ and w_{k+1} occurs inside $v_1v_2 = v^2$ as $v^2 = x_1\theta(v)x_2x_3 = x_1x_2w_{k+1}x_3$. This gives $v_1 = \theta(v_1)$, $v_3 = \theta(v_2)$, and $w_{k+1} = v_3v_1v_2$, and hence, w_{k+1} is a θ -palindrome. Lemma 16 is now applicable to reach the conclusion. Otherwise, that is, if the mod is rather $2d$, we get $v^2 = x_1x_2\theta(v)x_3 = x_1w_{k+1}x_2x_3$ instead, but is to be concluded in the same manner. \square

Note 1. As done in these proofs, we will always assume the θ -primitivity of u in all the proofs below and see Eq. (3) lead us to a contradiction. This just means that in order for Eq. (3) to hold, u is required not to be θ -primitive.

A-2 Proofs for the case when $m, n \geq 10$

Eq. (3) admits only θ -periodic solutions when $m, n \geq 12$ (see Table 1). In this subsection, we aim at lowering the value from 12 to 10, that is, we prove the following theorem.

Theorem 8. *For $m, n \geq 10$, Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

The proof of this theorem is composed of the three lemmas below. Note that reductions analogous to the one introduced at the very beginning allows us to assume that both v and w are θ -primitive in this case.

Lemma 18. *If either $m = 10, n \geq 20$ or $m \geq 20, n = 10$, then Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. In any case, $m \geq 10$ and we can use this to obtain $3|v| < |\textcircled{1}|$, that is, $v^3 <_p \textcircled{1}$.

First, we consider the case when $m = 10$ and $n \geq 20$. Due to $|v_1 \cdots v_m| \geq |w_1 \cdots w_n|$, we have $10|v| \geq n|w| \geq 20|w|$. Thus, $|v| \geq 2|w|$, and hence, $w_k''w_{k+1} <_p v$. This means that $\textcircled{1}$ is long enough ($|\textcircled{1}| - |w_k''| \geq 2|v| + |w|$) for Theorem 5 to give $\tilde{v} \in \{w, \theta(w)\}^+$, where \tilde{v} is defined as stated in Lemma 15, and the lemma gives $v = \tilde{v}$. However, this contradicts the θ -primitivity of v since $|v| \geq 2|w|$.

Let us proceed to the other case: $m \geq 20$ and $n = 10$. In this case, $m|v| < 2|u|$ implies $|v| < |w|$. Recall that $p - k$ is odd and at least 3. If $p - k \geq 5$, then $w_{k+1}w_{k+2}w_{k+3}w_{k+4}$ is a factor of $\textcircled{1}$ so that Theorem 5 gives $w \in \tilde{v}\{\tilde{v}, \theta(\tilde{v})\}^+$, but this contradicts the θ -primitivity of w . Thus, $p - k$ must be 3. Then $|\textcircled{1}| = 2|w| + 2|w_k''|$, while $|\textcircled{2}| = (10|w| - |\textcircled{1}|)/2$. Substituting these into $|\textcircled{1}| > |\textcircled{2}|$ gives $|w_k''| > \frac{2}{3}|w|$. We also have $v_1 \cdots v_6 <_p \textcircled{1}$ due to $m \geq 20$. Thus, $|v| < \frac{5}{6}|w_k''|$. In summary, we have $|\textcircled{1}| - |w_k''| = 2|w| + |w_k''| > 2|w| + |v|$, and hence, Theorem 5 is applicable and leads us to the same contradiction. \square

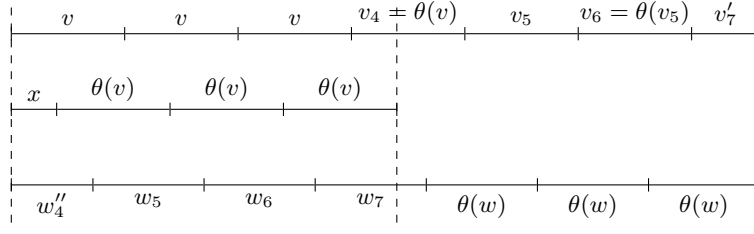


Fig. 6. Eq. (3) with $m = n = 10$.

Lemma 19. *For $m = 10$ and $n \geq 10$, Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. First of all, the assumption $|v_1 \cdots v_m| \geq |w_1 \cdots w_n|$ gives $10|v| \geq n|w| = 10|w|$, so $|v| \geq |w|$ and Lemma 17 allows us to proceed with $|v| > |w|$. Due to Lemma 16, we assume $w \neq \theta(w)$.

Lemma 18 makes it sufficient to examine the cases of $n \in \{10, 12, 14, 16, 18\}$ under the assumption that both v and w are θ -primitive. As explained previously, $v_1 = \cdots = v_{i-1} = v$ and $i \geq 4$ due to $m = 10$.

If $i \geq 5$, then due to $|v| \geq |w|$, $(w''_k)^{-1}v_1 \cdots v_4 = (w''_k)^{-1}v^4$ is of length at least $3|v| \geq 2|v| + |w|$, so Theorem 5 and Lemma 15 are applicable to it to give $v \in \{w, \theta(w)\}^{\geq 2}$ due to $|v| > |w|$, but this contradicts the θ -primitivity of v . Even for $i = 4$, if $v_4 = v$, then the contradiction arises in the same manner. Note that if $|v'_4| \geq \frac{1}{2}|v|$, then $v_4 = v$ due to Lemma 14. One more condition under which the contradiction thus arises is $p - k \geq 7$, when $w''_k w_{k+1} \leq_p v_1 x$ so that $(w''_k)^{-1}\textcircled{1}$ is long enough for the purpose.

As such, $i = 4$, $v_4 = \theta(v)$ (i.e., $3|v| \leq |\textcircled{1}| \leq \frac{7}{2}|v|$), and $p - k$ is either 3 or 5. Combining $|\textcircled{1}| < \frac{7}{2}|v|$ with $\textcircled{1}\textcircled{2}\textcircled{1} = v_1 \cdots v_{10}$ gives $|\textcircled{2}| > 3|v|$. Then $\textcircled{1} = v^3 v'_4 = (xy)^3 x$ and $y = y' \theta(y')$ (for the θ -palindromes x and y , recall the notation in Sect. A-1). First we consider the case of $n \geq 12$. With $p - k$ being 3, $|\textcircled{1}| < 4|w|$ while $|\textcircled{2}| > \frac{n-4}{2}|w| \geq 4|w|$, a contradiction. Thus $p - k = 5$. Then $\textcircled{1} = (xy)^3 x = w''_{n/2-2} w_{n/2-1} w_{n/2} w_{n/2+1} w_{n/2+2} w'_{n/2+3}$. Then the first half of $\textcircled{1}$ is written as $xy' \theta(y') xy' = w''_{n/2-2} w_{n/2-1} w_{n/2}$. We claim $|xy' \theta(y') x| \geq |w''_{n/2-2} w_{n/2-1}|$. Indeed, if not, the previous equation implies $\frac{1}{2}|y| = |y'| > |w|$, but it would lead us to a contradictory inequality $2|w| + 2|x| < |y| + 2|x| = |v| + |x| < |w''_{n/2-2}| + |w| < 2|w|$. Now we have $w''_{n/2-2} w_{n/2-1} \leq_p vx$, that is, $|(w''_{n/2-2})^{-1}\textcircled{1}| \geq 2|v| + |w|$, but we already saw this lead us to the contradiction $v \in \{w, \theta(w)\}^{\geq 2}$.

Lastly we examine the case of $n = 10$. See Fig. 6. Recall $|\textcircled{1}| < \frac{7}{2}|v|$ and $|\textcircled{2}| > 3|v|$. They imply $|\textcircled{1}| < \frac{7}{6}|\textcircled{2}|$, and hence, $p - k = 3$ in this case. Then we have $\textcircled{1} = v^3 v'_4 = w''_4 w_5 w_6 w'_7$ and $\textcircled{2} = v'_4 v_5 v_6 v'_7 = w''_7 \theta(w)^3$. We show $w_7 = \theta(w)$. The presentation of $\textcircled{1}$ means that the border between w_5 and w_6 is on v_2 . Due to $|v| \geq |w|$, the borders between the succeeding words $w_6, w_7, w_8, w_9, w_{10}$ are on v_3, v_4, v_5, v_6 , respectively. If $v_5 = v$, then $v_3 v_4 v_5 v_6 = (v \theta(v))^2$ is a θ -palindrome, and can be written as $z w_7 \theta(w)^2 z'$ for some $z \leq_s w_6$ and $z' \leq_p w_{10}$. Being a

θ -palindrome, it implies $zw_7\theta(w)^2z' = \theta(z')w^2\theta(w_7)\theta(z)$. Under the assumption $w \neq \theta(w)$, this implies $w_7 = \theta(w)$ due to Lemma 2. If $v_5 = \theta(v)$, we can rather use $v_2v_3v_4v_5$ as a θ -palindrome based on which we reach $w_7 = \theta(w)$. Now $w_7w_8w_9 = w_8w_9w_{10} = \theta(w)^3$, and the former has a factor $\theta(v)v_5$ while the latter has a factor $v_5\theta(v_5)$. Moreover, their overlap is of length at least $|v|$. The assumption $v \neq \theta(v)$ prevents them from overlapping in a trivial way, but on the other hand Lemma 2 prohibits any nontrivial overlap, a contradiction. \square

Lemma 20. *For $m \geq 10$ and $n = 10$, Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. It suffices to consider the cases of $m \in \{12, 14, 16, 18\}$. Then the assumption $|v_1 \cdots v_m| \geq |w_1 \cdots w_n|$ implies $i \geq 5$. Due to Lemma 17, we assume $|v| \neq |w|$. Recall that both v and w are assumed to be θ -primitive.

We have $|\textcircled{1}| = (i-1)|v| + |x| = (p-k-1)|w| + 2|w''_k|$. This implies $\frac{|\textcircled{1}|}{2} = \frac{p-k-1}{2}|w| + |w''_k|$ and $\frac{|\textcircled{1}|}{2} = \frac{1}{2}((i-1)|v| + |x|)$. Summing them gives $|\textcircled{1}| - |w''_k| = \frac{p-k-1}{2}|w| + \frac{i-1}{2}|v| + \frac{1}{2}|x|$, which is at least $2|w| + 2|v|$ if $p-k \geq 5$. This makes Theorem 5 applicable to $w''_k{}^{-1}\textcircled{1}$ to obtain $\tilde{v} \in \{w, \theta(w)\}^+$. Thus, $|v| > |w|$ (n.b. $|v| \neq |w|$ was assumed). Then Lemma 15 converts this into $v \in \{w, \theta(w)\}^{\geq 2}$, but this contradicts the θ -primitivity of v .

Now that we have proved $p-k = 3$. Since $n = 10$, this means $p = 7$ and $k = 4$. We have $|\textcircled{1}| = (i-1)|v| + |x| = 2|w| + 2|w''_4|$ and $|\textcircled{2}| = 4|w| - |w''_4|$. Then the inequality $|\textcircled{1}| \geq |\textcircled{2}|$ yields $|w| \leq \frac{3}{2}|w''_4|$. One more to claim is $|v| \leq |w|$. Indeed, in order for $p-k$ to be as small as 3, $|\textcircled{1}| \leq \frac{4}{7}|u|$, or equivalently, $|\textcircled{2}| \geq \frac{3}{7}|u|$, must hold. Then, $m|v| = |u| + |\textcircled{1}| \leq \frac{11}{7}|u|$ and $10|w| = |u| + |\textcircled{2}| \geq \frac{10}{7}|u|$, which give $|v| \leq |w|$ (recall that in the current investigation, m is at least 12).

If $i \geq 6$, then $5|v| \leq |\textcircled{1}| = 2|w| + 2|w''_4| \leq 5|w''_4|$, that is, $|v| \leq |w''_4| = |w'_7|$. Thus, the suffix $w_5w_6w'_7$ of $\textcircled{1}$ is of length at least $2|w| + |v|$, that is, long enough for the sake of Theorem 5 and Lemma 15. We have actually completed the proof for $m \geq 16$ as $i \geq 6$ holds then.

Below, we will prove that even with $i \geq 5$, a contradiction is inevitable. We claim that if $v_5 = v$, then Theorem 5 and Lemma 15 are applicable to $(w''_4)^{-1}v_1 \cdots v_5$ and would bring a contradiction as above. Indeed, if $|w'_7| \geq |v|$, then the word is of length at least $2|w| + |v|$ because $|(w''_4)^{-1}v_1 \cdots v_5| \geq |w_5w_6w'_7|$. Otherwise, $|(w''_4)^{-1}v_1 \cdots v_5| - (2|w| + |v|) \geq 4(|v| - |w''_4|) > 0$ using $|w| \leq \frac{3}{2}|w''_4|$. The condition $v_5 = v$ certainly holds when $m = 14$. Actually, then $|\textcircled{1}| = 4|v| + |x|$ and $|\textcircled{2}| = 6|v| - 2|x|$, and substituting these into $|\textcircled{1}| \geq |\textcircled{2}|$ results in $|x| \geq \frac{2}{3}|v|$. This gives $v_5 = v$ in light of Lemma 14.

The only remaining case is when $m = 12$, $i = 5$, and $v_5 = \theta(v)$. Let us shift our focus to suffixes of u . With $w_7 = \theta(w)$, Theorem 5 and Lemma 15 are applied to $w_7 \cdots w_{10}(v'_8)^{-1}$ to reach the contradiction. Thus, $w_7 = w$. If $v_6 = \theta(v)$, then $v_3v_4v_5v_6 = v^2\theta(v)^2$ is a θ -palindrome and also contains $w_6w_7w_8 = w_6w\theta(w)$ as its factor. No matter how they overlap, we cannot avoid a contradiction either with Lemma 1 or with the assumption $w = \theta(w)$, as seen in the proof of Lemma 19. If $v_6 = v$, then $v_7 = \theta(v)$ since $\textcircled{2}$ is a θ -palindrome. Hence, $v_4v_5v_6 = v\theta(v)v$ and contains $w_7w_8 = w\theta(w)$ as its factor. Moreover, its θ -image

$v_5v_6v_7 = \theta(v)v\theta(v)$ is a proper factor of $w_7 \cdots w_{10}$. Thus, even in this case, we have reached a contradiction either with Lemma 1 or with $w \neq \theta(w)$. \square

Now that Theorem 8 has been proved, and we conclude this subsection.

A-3 Proofs for the case when $m \leq n$

Let us continue the study of Eq. (3) under the condition $m \leq n$ in this subsection. Having already solved the equation for $m, n \geq 10$, it is enough to examine the equation first for $m = 6$ and $n \geq 6$, and then for $m = 8$ and $n \geq 8$.

Proofs for the subcase when $m = 6$ and $n \geq 6$ If $|v_1 \cdots v_6| = |w_1 \cdots w_n|$, then $u = v_1 \cdots v_4$ would not be θ -primitive. Hence we assume $|v_1 \cdots v_6| > |w_1 \cdots w_n|$. Then $2|v| < |\textcircled{1}| < 3|v|$, so we have $v_1 = v_2 = v$ and $\textcircled{1} = v^2x$. From $2|\textcircled{1}| + |\textcircled{2}| = 6|v|$ and $|\textcircled{1}| + 2|\textcircled{2}| = n|w|$, we get $n|w| = 3|x| + 6|y|$.

Let us handle the least repetitive case first.

Lemma 21. *If $m = n = 6$, Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. We have $\textcircled{1} = xyx'\theta(x')yx$ and it is a θ -palindrome. At its center is the border between w_3 and w_4 . Then

$$\theta(x')yx'\theta(x')\textcircled{2} = w_4w_5\theta(w). \quad (5)$$

With $n = 6$, the relation $n|w| = 3|x| + 6|y|$ implies $|w| = |x'| + |y|$. This divides Eq. (5) into $w_4 = \theta(x')y$ and $w_5w_6 = \theta(x')x'\textcircled{2}$.

Let us begin with the case of $v_3 = v$. Then $\textcircled{2} = y^2$ and $w_5w_6 = w'_5\textcircled{2} = x'\theta(x')y^2$. If $w_5 = w$, then $w_5w_6 = xyy$ becomes a θ -palindrome, that is, $xyy = yyx$. This is a well-known commutativity relation, imposing that $x, y \in s^+$ for some word s . Then $w_5w_6 \in s^{\geq 3}$, but this contradicts the θ -primitivity of w . Thus $w_5 = \theta(w)$; then $x'\theta(x')yy = \theta(w)^2$, which means $\theta(w) = x'y$. If $w_4 = \theta(w)$, then $x' = \theta(x')$, which results in an original Lyndon-Schützenberger equation $x'^2y^2 = \theta(w)^2$, giving $x', y \in s^+$ for some s . Otherwise, $w = \theta(x')y$, so we obtain the commutativity $\theta(w) = yx' = x'y$, that is, $x', y' \in s^+$. In any case, however, $w = y\theta(x')$ would not be θ -primitive, a contradiction.

Therefore, v_3 must be $\theta(v)$. Lemma 14 implies $|x| < \frac{1}{2}|v|$ (i.e., $|x| < |y|$). Let $\textcircled{2} = z\theta(z)$ for some z . Then $v_3 = \theta(v) = xz$, and hence, $xz = yx$. Moreover,

$$x'\theta(x')\textcircled{2} = x'\theta(x')z\theta(z) = w_5\theta(w). \quad (6)$$

We claim $w_5 = w$. Suppose otherwise. Then Eq. (6) implies that $xz\theta(z) = z\theta(z)x$. Thus, $x, z \in \{t, \theta(t)\}^+$ for some θ -primitive word t , but then their catenation $\theta(v)$ would not be θ -primitive, a contradiction.

Thus, $w_5 = \theta(w)$. Then we have $x'\theta(x')z\theta(z) = \theta(w)^2$, that is, $\theta(w) = x'\theta(z)$. Substituting this back to the representation of $\theta(w)^2$ yields $\theta(x')z = \theta(z)x'$. Due to $|x| < |y| = |z|$, this equation lets $z = \gamma x'$ for some θ -palindrome γ . If $w_4 = w$, then $\theta(w) = yx' = x'\theta(z)$ (recall $w_4 = \theta(x')y$). Substituting $z = \gamma x'$ into $yx = xz$

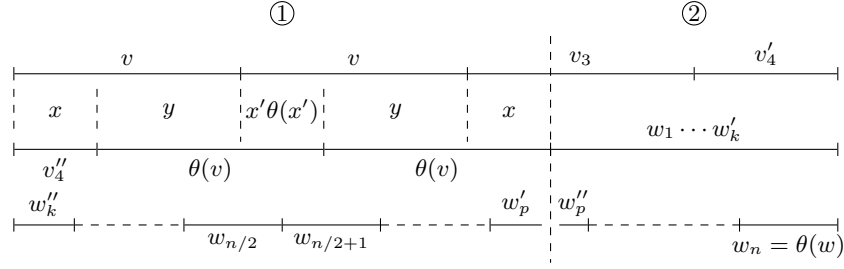


Fig. 7. The equation $u^3 = v_1 \cdots v_6 w_1 \cdots w_n$ for $n \geq 6$.

gives $yx'\theta(x') = x\gamma x'$, which shows $x' = \theta(x')$. Catenating x' from the right to both sides of $yx = xz$ gives

$$yx'^3 = x'^2 r x'^2. \quad (7)$$

This actually gives $yx'^3 = x'^3 y$ as its RHS is a θ -palindrome. Hence, $x', y \in \{t, \theta(t)\}^+$ for some t . However, then $v = x'^2 y$ would not be θ -primitive. Thus, $w_4 = \theta(w)$. Then we have $x'\theta(z) = \theta(x')y$, that is, $x' = \theta(x')$ and $y = \theta(z)$. Since y is a θ -palindrome, now we have $x'^2 y^2 = \theta(w)^2$, but we already saw this contradict the θ -primitivity of $\theta(w)$. \square

We shall strengthen this lemma by the next lemma.

Lemma 22. For $m = 6$ and $n \geq 8$, Equation (3) implies $u, v, w \in \{t, \theta(t)\}^+$.

Proof. Lemmas 16 and 17 allow us to assume $w \neq \theta(w)$ and $|v| \neq |w|$ and avoid some cumbersome case analyses. The latter can be strengthened as $|v| > |w|$ because $|\textcircled{1}\textcircled{2}\textcircled{1}| = 6|v|$, $|\textcircled{2}\textcircled{1}\textcircled{2}| \geq 6|w|$, and $|\textcircled{1}| \geq |\textcircled{2}|$. As before, $i = 3$, that is,

$$\textcircled{1} = v^2 v'_3 = x'\theta(x')\underline{yx'\theta(x')yx'\theta(x')} = w''_k w_{k+1} \cdots w_{n/2} w_{n/2+1} \cdots w_{p-1} w'_p \quad (8)$$

(the underline will be made use of later).

Should $w''_k w_{k+1}$ be shorter than x , Theorem 5 and Lemma 15 would work on $(w''_k)^{-1}\textcircled{1}$ and produce a contradiction $v \in \{w, \theta(w)\}^{\geq 2}$. Thus, $|w''_k w_{k+1}| > |x| = 2|x'|$.

Next, we will show $|w''_k| \neq |x'|$. For the sake of contradiction, suppose otherwise. Then Eq. (8) provides $x' = w''_k$ and $\theta(x')yx' = w_{k+1} \cdots w_{n/2}$. Being a θ -palindrome, the latter actually tells $\theta(x')yx' = w_{k+1} \cdots w_{n/2} = \theta(w_{n/2}) \cdots \theta(w_{k+1})$. The assumption $w \neq \theta(w)$ makes $\frac{n}{2} - (k+1) + 1$ even. Hence, $w_{k+1} \leq_p \theta(x')y$. Therefore, with $v_3 = v$, we would get the same contradiction as above based on Theorem 5 and Lemma 15. Thus, $v_3 = \theta(v)$. This means $v'_3 = x <_p yx = \theta(v)$ and $|x| < |y|$ due to Lemma 14. See Fig. 7. Note that $\theta(x')\textcircled{2} = \theta(w''_k)w''_p w_{p+1} \cdots w_n = w'_p w''_p w_{p+1} \cdots w_n = w_p \cdots w_n$. This implies $|x'| + |\textcircled{2}| = |x'| + 2|y| = (n-p)|w| = k|w|$. Substituting this into $3|x| + 4|y| = \frac{1}{2}|x| + (n-k)|w|$, which derives from $u = (xy)^2 x \textcircled{2} = w''_k w_{k+1} \cdots w_n$, yields $\frac{3}{2}|x| = (n-3k)|w|$, and this means $|w| = \frac{3}{n-3k}|x|$ when $|x'| = |w''_k|$. Thus, the length of w is either $3|w''_k|$ or $\frac{3}{2}|w''_k|$.

See Eq. (8) and focus on the prefix $w_k''w_{k+1}w_{k+2}$ with $x <_p yx$ in mind. If $|w| = 3|w_k''|$, then the prefix $w_k''w_{k+1}$ is equal to $x'\theta(x')x'\theta(x')$, that is, $w_{k+1} = \theta(x')x'\theta(x')$, but this contradicts the θ -primitivity of v . Hence $|w| = \frac{3}{2}|w_k''|$. Then $w_k'' = x'$ must be of even length, that is, $x' = s_1s_2$ for some words s_1, s_2 of the same length. Then $w_{k+1} = \theta(s_2)\theta(s_1)s_1$ and $w_{k+2} = s_2\theta(s_2)\theta(s_1)$. Since w_{k+2} is certainly equal to either w_{k+1} or $\theta(w_{k+1})$, these equations would imply $s_2 = s_1$ or $s_2 = \theta(s_1)$, but in any case, w_{k+1} would not be θ -primitive, a contradiction.

Having proved $|w_k''| \neq |x'|$, now we have either $|w_k''| < |x'|$ or $|w_k''| < |x'| < |w_k''w_{k+1}|$. The first half of Eq. (8), that is, $x'\theta(x')yx' = w_k''w_{k+1} \cdots w_{n/2}$, occurs again to the right as underlined. This implies $w_{k+1} \cdots w_{n/2}$ overlaps with $w_k \cdots w_p$ and does so in such a nontrivial way that Lemma 1 enforces $w_{k+1} = \cdots = w_{n/2}$ and $w_{n/2+1} = \cdots = w_{k+1} = \theta(w_{n/2})$.

Let us examine the two cases depending on the value of v_3 one after another. The first case is $v_3 = \theta(v)$. Then $v_3 = xz$ for some word z such that $\textcircled{2} = z\theta(z)$. We claim $w_p = w_{p-1}$. Suppose otherwise for the sake of contradiction. Then $w_{p-1}w_p$ is a θ -palindrome. We have

$$v_2v_3'\textcircled{1} = vxz\theta(z) = x'\theta(w_p)^{n/2-k}w_p\theta(w)^{k-1}. \quad (9)$$

The θ -palindrome v_2v_3 contains the θ -palindrome $w_{p-1}w_p$ as its factor because of $k-1 \geq 2$ (see Fig. 7). That is, $v_2v_3 = \alpha w_{p-1}w_p\beta$ for some words α, β . It reflects $w_{p-1}\theta(w_{p-1})$ to the symmetric position as $v_2v_3 = \alpha w_{p-1}w_p\beta = \theta(\beta)\theta(w_{p-1}w_p)\theta(\alpha)$. Note that $w_{k+1} \cdots w_n = w_p^{(n-k)/2}\theta(w_p)^{(n-k)/2}w_p\theta(w)^{k-1}$, and v_2v_3 is its factor. Thus, the θ -image $\theta(w_{p-1}w_p) = \theta(w_p)w_p$ is also a factor of the θ -power of w . In order to avoid the contradiction with Lemma 1, the image must match $w_s w_{s+1}$ for some index s such that $k+1 \leq s < n$. Due to the assumption $w \neq \theta(w)$, the s must be chosen so that $w_s = \theta(w_p)$ and $w_{s+1} = w_p$. They mean $s = p-1$, that is, the center of $w_{p-1}w_p$ is exactly at the center of v_2v_3 , or in other words, $|\alpha| = |\beta|$. This derives from Eq. (9) that

$$(x')^{-1}v_2 = \theta(x')y = \theta(w_p)^{n/2-k}, \quad (10)$$

and $v_3'\textcircled{1} = x'\theta(x')z\theta(z) = w_p\theta(w)^{k-1}$. The latter is twice as long as the former, and hence, the latter can be halved as

$$x'\theta(x')z(\theta(x'))^{-1} = w_p\theta(w)^{k/2-1} \quad (11)$$

$$\theta(x')\theta(z) = \theta(w)^{k/2} \quad (12)$$

(n.b. $x'\theta(x')$ is a suffix of z since $\theta(v) = yx = xz$ and $|x| < |y| = |z|$). If $w_p = w$, then Eq. (10) and Eq. (12) imply $y = \theta(z)$ and this actually means $y = z$ since $y = \theta(y)$. However, then $\theta(v) = yx = xy$, and hence, $\theta(v)$ would not be θ -primitive. Thus, $w_p = \theta(w)$. Then Eq. (10)-(12) give $x'\theta(x')z(\theta(x'))^{-1} = \theta(x')\theta(z) = yx'$, that is, $x' = \theta(x')$ and $x'\theta(z) = yx'$. Substituting these into $\theta(v) = xz = yx$ results in $x'^2\theta(z) = x'\theta(z)x'$, that is, $x', \theta(z) \in s^+$ for some s . Then $v = \theta(z)x'^2$ would not be θ -primitive. The claim $w_p = w_{p-1}$ has been thus proved.

Recall that $\theta(x')yxz\theta(z) = w_{n/2+1} \cdots w_n$ and $|x| < |w_{p-1}w_p'|$. With $n/2$ being a multiple of 3, the center of the θ -palindrome v_2v_3 is at the border between w_{p-1}

and w_p , but this contradicts with $w \neq \theta(w)$ because $w_p = w_{p-1}$. Thus, $n/2$ is not a multiple of 3. The above equation implies $3|x'| + 3|y| = \frac{n}{2}|w|$, and now this means that $|w|$ is a multiple of 3. Let $w_p = s_1s_2s_3$ for some words s_1, s_2, s_3 of the equal length. If $n/2 \bmod 3 = 1$, then $\theta(x')y = (s_1s_2s_3)^{\lfloor n/6 \rfloor} s_1$ and the prefix yx' of $v_3 = \theta(v)$ has a prefix $s_2s_3w_{p+1}$. Thus, $\theta(w_{p+1})\theta(s_3)\theta(s_2) \leq_s (s_1s_2s_3)^{\lfloor n/6 \rfloor} s_1$. This implies $s_2 = \theta(s_1)$ and $s_3 = \theta(s_3)$, and with $w_{p+1} = \theta(w_p)$, we would obtain $s_3 = \theta(s_2)$, that is, $w_p = s_1\theta(s_1)s_1$, a contradiction with the θ -primitivity of w . Even when $n/2 \bmod 3 = 2$, the same argument turns $w_{p+1} = \theta(w_p)$ into the contradiction. Thus, $w_p = w_{p+1}$, that is, $w_{n/2+1} = \dots = w_n = \theta(w)$. We have

$$(x')^{-1}v_2v_3v'_4 = \theta(x')y^2x'\theta(x')v'_4 = \theta(w)^{n/2}, \quad (13)$$

and it has a prefix $\theta(x')yv'_3w_1$. Its another prefix $\theta(x')y^2x'$ is a θ -palindrome and also ends with w because $\theta(w)$ is its prefix. Moreover, $w_1 = w$ is a factor of $w_pw_{p+1} = \theta(w)^2$ and being primitive, it admits at most one way to be factor of $\theta(w)^2$. This means that the difference of the lengths of these two prefixes, which is $|y| - \frac{1}{2}|x'|$, must be a multiple of $|w|$. This also means that $\theta(x')v'_4u$ is a θ -power of w and of length $7|x'| + 5|y|$. With $3|x'| + 3|y|$ being a multiple of $|w|$, they imply that both $|x'|$ and $|y|$ are a multiple of $|w|$. Then Eq. (13) would imply that 3 divides $n/2$, a contradiction.

Now that we have seen $v_3 = v$ cause a contradiction, lastly we deny the other possibility: $v_3 = v$. We have

$$u^3 = v^3\theta(v)^3w^{k-1}\theta(w_p)\theta(w_{p-1})^{n/2-k}w_{p-1}^{n/2-k}w_p\theta(w)^{k-1}. \quad (14)$$

This case can be handled much simply.

When $p - k \geq 5$, we can easily reach the conclusion. If $w_{n/2}$ begins to the left of the border between v_1 and v_2 , then w_{k+1} occurs to the left of the border, too. Hence, we can apply Theorem 5 to $v_1v_2v_3 = v^3$ to obtain $\tilde{v} \in \{w, \theta(w)\}^+$, from which Lemma 15 derives $v \in \{w, \theta(w)\}^+$. Note that $w_{n/2}$ is a suffix of v_1x' , and hence, it begins to the right of the border between v_1 and v_2 if and only if $|w_{n/2}| < |x'| = \frac{1}{2}|v''_j|$. Thus, $w''_k w_{k+1}$ is a prefix of v''_j . Theorem 5 and Lemma 15 are applicable as above to reach the same conclusion.

The condition $p - k \geq 5$ always hold for $n \geq 12$. Hence, we consider the case when n is either 8 or 10. We have

$$\textcircled{2} = y^2 = w_1w_2 \dots w'_k, \quad (15)$$

and $n \geq 8$ implies $k \geq 3$. Recall that y is a θ -palindrome. In this case, a common prefix of length $|y| + |w|$ is long enough for the extended FW theorem (Theorems 25 and 26 in [1]). As such, $y \in \{w, \theta(w)\}^+$, and hence, w'_k is empty. However, this means $w_{k'} = w_k$ and $u_3 = w_k \dots w_n$, a contradiction with the θ -primitivity of u . \square

Combining Lemmas 21 and 22 concludes the analysis for $m = 6$ as follows.

Theorem 9. *If $m = 6$ and $n \geq 6$, Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$.*

Proofs for the subcase when $m = 8$ and $n \geq 8$ In this case, Eq. (3) becomes

$$w^3 = v_1 \cdots v_8 w_1 \cdots w_n$$

with $n \geq 8$. As before, we assume that w is θ -primitive, $v \neq \theta(v)$, and $w \neq \theta(w)$. In contrast, we cannot assume the θ -primitivity of v because now m is not at least 8 but exactly 8. As before, $|\textcircled{1}| \geq |\textcircled{2}|$ implies $|v| \geq |w|$, or more strongly, $|v| > |w|$ due to Lemma 17.

Lemma 23. *If $m = 8$ and $n \geq 8$, then Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. There are two cases to be examined; either $i = 4$ or $i = 3$.

When $i = 4$, we have $\textcircled{1} = v^3 x = (xy)^3 x$ for some θ -palindromes x, y , and $\textcircled{2} = v_4'' \theta(v_4'')$. With $v_4 = v$, $(w_k'')^{-1} v_1 \cdots v_4 \geq 2|v| + |w|$, and hence, it would be long enough for Theorem 5 and Lemma 15 to give a contradiction $v \in \{w, \theta(w)\}^+$. Then $u \in \{w, \theta(w)\}^{\geq m+n}$, but this contradicts the θ -primitivity of u . Thus, $v_4 = \theta(v)$, and $|x| < |y|$ is necessary due to Lemma 14. Then we can let $v_4 = \theta(v) = xz$ for some z . Note that

$$\textcircled{1} = xyxyxyx = w_k'' w_{k+1} \cdots w_{p-k-1/2} \theta(w_k'' w_{k+1} \cdots w_{p-k-1/2}).$$

This is a θ -palindrome, and its center matches that of the second y . If $w_k'' w_{k+1}$ were a prefix of xyx , then Theorem 5 and Lemma 15 would be applicable as above. This is the case when $p - k \geq 5$, and this condition in turn holds for $n \geq 10$. Even for $n = 8$, if $|\textcircled{1}| \geq 2|\textcircled{2}|$, then $p - k = 5$.

Thus, what we have to examine is the case when $n = 8$ and $p - k = 3$. In this case, $p = 6$. The condition $p - k = 3$ is equivalent to $|x| < \frac{1}{4}|y|$. Then the second half of $w_1 \cdots w_8$ admits the representation

$$\theta(y')xyv_4\theta(z) = \theta(y')xyxz\theta(z) = w_5w_6\theta(w)^2, \quad (16)$$

where $z\theta(z) = \textcircled{2}$. This equation gives $\theta(y')xyx = w_5w_6'$ and $z\theta(z) = w_6''\theta(w)^2 = w^2w_3'$, and hence, $w <_p z$. The following relation of use derives from this:

$$\theta(y') \leq_p w_5 \leq_p \theta(y')v \leq_p \theta(y')x\theta(v) <_p w_5w_6. \quad (17)$$

In addition, $w <_p z$ implies that w is a factor of $\theta(v) = xz$, or equivalently, $\theta(w)$ is that of $v = \theta(z)x$. Then the third and fourth inequalities in (17) and that w is primitive result in $w_5 \neq w_6$. In other words, w_5w_6 is a θ -palindrome.

Eq. (16) gives $7|y'| + 2|x| = 4|w|$. Hence, $\frac{7}{2}|y'| + |x| = 2|w| = |w_5w_6|$. This means that y' is of even length so that we let $y' = y_1y_2$ for some words y_1, y_2 of the same length. In addition, $y_1 <_p yx = \theta(v) = v_4$. Based on these, it is clear from Eq. (16) that

$$\theta(y')xyy_1 = \theta(y_2)\theta(y_1)xy_1y_2\theta(y_2)\theta(y_1)y_1 = w_5w_6.$$

Being a θ -palindrome, this gives $y_1 = \theta(y_1) = y_2$ and $xy_1y_2\theta(y_2)$ is a θ -palindrome. They imply $xy_1^3 = y_1^3x$. Thus, $x, y_1 \in \{t, \theta(t)\}^+$ for some θ -primitive word t , and

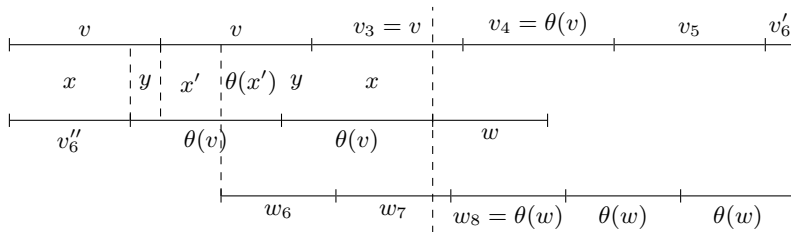


Fig. 8. $u^3 = v_1 \dots v_8 w_1 \dots w_n$ for $n = 10$.

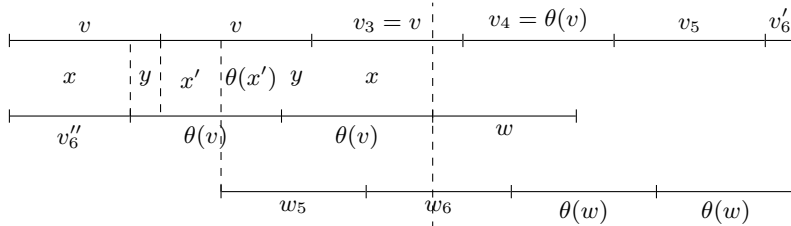


Fig. 9. $u^3 = v_1 \dots v_8 w_1 \dots w_n$ for $n = 8$.

hence, $v = xy_1^4 \in \{t, \theta(t)\}^+$. Now $w_5 w_6 = y_1^2 x y_1^5$ so that $w \in \{t, \theta(t)\}^{\geq 4}$, but this contradicts the θ -primitivity of w . Consequently, i cannot be 4.

Now it suffices to deny $i = 3$. Suppose this condition and see it certainly lead us to a contradiction. See Fig. 8, where the longest vertical dashed line splits u into ① and ②. In order for $|\textcircled{1}| \geq |\textcircled{2}|$, $|x| \geq 2|y|$ so that Lemma 14 determines $v_3 = v$ (hence, $v_6 = \theta(v)$). We already know that Theorem 5 and Lemma 15 cause a contradiction with the θ -primitivity of u if v_4 is also v or if $v_4 = \theta(v)$ and $p - k \geq 5$. Since $p - k \geq 5$ holds as long as $n \geq 12$, below we will examine the cases of $n = 10$ and $n = 8$ with $v_4 = \theta(v)$ (that is, $v_5 = v$) and $p - k = 3$.

Let us consider the case of $n = 10$ first (see Fig. 8). In this case, $|\textcircled{1}| = 3|x| + 2|y|$ and $|\textcircled{2}| = 2|x| + 4|y|$ so that $|w| = \frac{1}{10}|w_1 \dots w_{10}| = \frac{1}{10}|\textcircled{2}\textcircled{1}\textcircled{2}| = \frac{7}{10}|x| + |y|$. It makes $|\textcircled{2}| - \frac{5}{2}|w|$ positive. This means that the border between ① and ② is to the left of the center of w_8 , and hence, $w_8 = \theta(w)$ due to Lemma 14. Note that $|v_3 v_4 v_5 v'_6| = 3|x| + 4|y|$, $|w_7 \dots w_{10}| = \frac{14}{5}|x| + 4|y|$, $|w_8 w_9 w_{10}| = \frac{21}{10}|x| + 3|y|$, and $|v_4 v_5 v'_6| = 2|x| + 3|y|$. They mean that $w_7 \theta(w)$ is a factor of the θ -palindrome $v_3 v_4$, and their centers are out of alignment. Then the θ -image of $w_7 \theta(w)$ with respect to the symmetric axis of $v_3 v_4$ overlaps with $w_6 \dots w_{10}$ nontrivially, and hence, $w_7 = w_8$ must hold due to Lemma 1. Now $w_7 = w_8 = w_9 = w_{10}$ and $w_7 w_8 w_9 w_{10} (v'_6)^{-1}$ is long enough for Theorem 5 and we have that v_5 is a θ -power of a strictly shorter word as $|w| < |v|$. This converts the given equation into another equation of the form (3) with $m, n \geq 10$. Theorem 8 solves it as $u, v, w \in \{t, \theta(t)\}^+$, but this contradicts the θ -primitivity of u .

Now we proceed to the last case to be handled: $n = 8$. See Fig. 9. We have

$$w_5 w_6 \theta(w)^2 = \theta(x') y x y x' \theta(x') x' \theta(x') y y. \tag{18}$$

This gives $|w| = \frac{7}{8}|x| + \frac{5}{4}|y|$. Since $m = n$, $|\textcircled{1}| = |\textcircled{2}|$ would contradict the assumption $|v| \neq |w|$. Thus, $|\textcircled{1}| > |\textcircled{2}|$, or equivalently, $|x| > 2|y|$. Recall also that we are now working with $v_4 = \theta(w)$.

Suppose $w_6 = \theta(w)$. Note that being primitive, w admits at most one way to be a factor of $\theta(w)^2$. Length arguments based on $|w| = \frac{7}{8}|x| + \frac{5}{4}|y|$ give $\theta(x')y <_p w_5 <_p \theta(x')yv_3 <_p w_5w_6 <_p \theta(x')yv_3v_4 < w_5w_6w_7$. This means that w_6 is a factor of v_3v_4 as $v_3v_4 = \alpha w_6 \beta$ for some words α, β . Then $|\alpha| = |w| - |\theta(x')y| = \frac{3}{8}|x| + \frac{1}{4}|y|$, whereas $|\beta| = |\theta(x')yv_3v_4| - 2|w| = \frac{3}{4}|x| + \frac{1}{2}|y|$. Hence, $|\beta| = 2|\alpha| < |x|$. Since $v_3v_4 = v\theta(v) = \theta(\beta)\theta(w_6)\theta(\alpha)$, we have $\beta \leq_p \theta(\alpha)\theta(w_6) < xw_1 = xw$. This means that $\theta(w_6)$ occurs inside $w_6\theta(w)$ but strictly to the left of $w_1 = w$. We should interpret this as that if w_6 is $\theta(w)$, then w would appear on $w_6\theta(w) = \theta(w)^2$ in two different manners, which contradicts the primitivity of w as stated above.

Therefore, $w_6 = w$. Then the factor yx' (of even length) at the center of RHS of Eq. (18) is included in the θ -palindrome w_6w_7 in such a way that their centers match, and hence, we have $yx' = \theta(yx')$. With w_5 being w , the LHS of Eq. (18) becomes a θ -palindrome so that the equation also gives $\theta(x')yx'\theta(x')y = yyx'\theta(x')x'$. The relation $yx' = \theta(yx')$ transforms the second into $yx'x'yx' = y\theta(x')y\theta(x')x'$. Thus, $x' = \theta(x')$, and this turns $yx' = \theta(yx')$ into $x'y = yx'$. Therefore, $x', y \in \{t, \theta(t)\}^+$ for some t . Then $w_5w_6w_7w_8 \in \{t, \theta(t)\}^{\geq 12}$, but this would imply a contradiction $w \in \{t, \theta(t)\}^{\geq 3}$.

Now we know $w_5 = \theta(w)$. By letting θ -palindrome yx' as $yx' = \theta(x')y = z\theta(z)$, we get $w_5w_6 = \theta(x')yx'\theta(x')yz = z\theta(z)x'z\theta(z)z$. Being a θ -palindrome, it gives $z = \theta(z)$ and $x'z = z\theta(x')$. Using these, we expand $w_7w_8 = \theta(w)^2$ as

$$\theta(w)^2 = \theta(z)\theta(x')x'\theta(x')yy = z\theta(x')x'zzy = (z\theta(x'))^2zy.$$

From $yx' = \theta(x')y$ and $|y| < \frac{1}{2}|x| = |x'|$, we derive $y <_p x'$. Theorem 4 is hence applicable to the equation above, but $\theta(w)$ would turn out to be a power of the strictly shorter word $z\theta(x')$, a contradiction. \square

Consequently, we have successfully proved the next theorem in this subsection.

Theorem 10. *For m, n with $n \geq m \geq 6$, Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

A-4 Proofs for the case when $m > n$

Having concluded our investigation on the cases of $6 \leq m \leq n$ and $m, n \geq 10$, now we will proceed to the last case $m > n \geq 6$ with n being either 6 or 8. Without loss of generality, v can be assumed to be θ -primitive. In these two cases, if w is not θ -primitive, then Eq. (3) is reduced to another equation that is known to bring the conclusion. Therefore, we assume that w is θ -primitive. We also assume $v \neq \theta(v)$ and $w \neq \theta(w)$.

As before, if u is not θ -primitive, Eq. (3) is immediately solved as $u, v, w \in \{t, \theta(t)\}^+$ for some word t . Thus, in the rest of this subsection, we assume the θ -primitivity and see that under this assumption, Eq. (3) does not hold.

Let us begin with the first case.

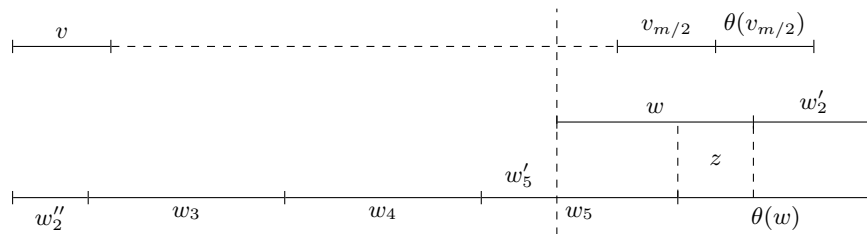


Fig. 10. The equation $u^3 = v_1 \dots v_m w_1 \dots w_6$ for $m \geq 8$ with the border between u_2 and u_3 being on w_2 .

Lemma 24. *If $m \geq 8$ and $n = 6$, then Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. First of all, let us note that the border between u_2 and u_3 is either on w_1 or on w_2 (since $|\textcircled{1}| \geq |\textcircled{2}|$, it cannot be located to the further right).

The case when the border is on w_1 can be handled easily. The border being there is equivalent to $|\textcircled{2}| \leq \frac{1}{5}|u|$. Then $|\textcircled{1}| \geq 4|w|$. If $m \geq 10$, then the length is also at least $4|v|$, and this actually means that $\textcircled{1}$ is a prefix of a power of v of length at least $4|v|$. From this, Theorem 5 and Lemma 15 can derive $v \in \{w, \theta(w)\}^+$. This would give a contradiction $u^3 \in \{w, \theta(w)\}^{\geq m+n}$. As for $m = 8$, the border between v_8 and w_1 is on v_4 . Hence, $|\textcircled{1}| = 4|x| + 3|y|$ while $|\textcircled{2}| = 2|y|$. Then the condition $|\textcircled{1}| \geq |\textcircled{2}|$ means $|x| \geq \frac{1}{8}|u|$ and $|y| \leq \frac{1}{10}|u|$, that is, $|x| > |y|$. Due to Lemma 14, $v_4 = v$ and now the same contradiction would arise.

The border must be on w_2 . See Fig. 10. The longest vertical dashed line separates $\textcircled{1}$ from $\textcircled{2}$, and is located such that $|\textcircled{2}| > \frac{1}{5}|u|$. Since $\textcircled{1}$ is a prefix of a power of v , if $|w'_2| = |w'_5| \geq |v|$, then Theorem 5 could be applied to $w_3 w_4 w'_5$ to imply that \tilde{v} is either w or $\theta(w)$ (n.b. w is assumed to be θ -primitive). This however contradicts with $|w| > |w'_2| \geq |v|$. Thus, $|v| > |w'_2| = |w'_5|$.

Before proceeding further, let us partially determine the values of w_1, \dots, w_6 . Let us denote the overlap between $w_1 = w$ and $w_6 = \theta(w)$ by z and $z = \theta(z)$ holds. Hence, let $z = z'\theta(z')$. As explained before, $w_5 = \theta(w_2)$.

Let us show that $v_{m/2}$ cannot be a factor of w_1 . For the sake of contradiction, suppose it were. See Fig. 10. Then $|v| < |w|$ and $v_{m/2} v_{m/2+1}$, which is always a θ -palindrome, is a factor of $w_1 w_2$. There are three subcases to be examined depending on the choices of w_2 and w_4 .

If $w_2 = \theta(w)$ (i.e., $w_5 = w$), then $w_1 w_2 = w\theta(w)$ contains the θ -palindrome $v_{m/2} v_{m/2+1}$ (either $v\theta(v)$ or $\theta(v)v$), as a factor, and so does $w_5 w_6 = w\theta(w)$. This occurrence of the θ -palindrome occurs to the left of the center of $\textcircled{2}$ by $|z|$. Lemma 1 enforces this to match $v_k v_{k+1}$ for some k precisely. With the fact that $v_{m/2}$ ends at the center of $\textcircled{1}$, we now know that $|v| \leq |z| = |w'_5|$, but this contradicts with the assumption $|v| > |w'_5|$.

Thus, $w_2 = w$, that is, $w_5 = \theta(w)$. If $w_4 = \theta(w)$, then $\theta(w)^2$ appears twice as $w_4 w_5$ and $w_5 w_6$. On $w_5 w_6$, the θ -palindrome $v_{m/2} v_{m/2+1}$ occurs in such a way that its center is to the right of the center of $w_5 w_6$ by $\frac{1}{2}|z|$. Due to $w_4 w_5 = w_5 w_6$,

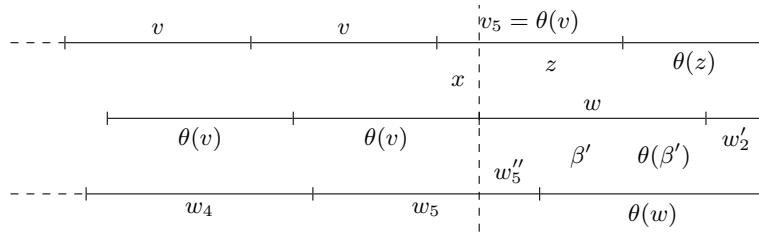


Fig. 11. The equation $u^3 = v_1 \dots v_{10} w_1 \dots w_6$.

the θ -palindrome also occurs so as for its center to be at the center of w'_5 . Lemma 1 implies that it must match $v_k v_{k+1}$ for some k , which implies that $|w|$ is a multiple of $|v|$ and $v_{m/2} = v$ as $v_k = v$. The latter means that $v\theta(z') <_s w_1 = w$. Since $w_3 = w$ is a factor of a power of v , we get $w'_2 w_3 (\theta(z'))^{-1} \in v^+$. This implies that $|w| + \frac{1}{2}|z|$ is a multiple of $|v|$, and hence, so is $\frac{1}{2}|z|$. However, this contradicts $|v| > |w'_2| = |z|$.

Only one choice is left now: $w_2 = w_4 = w$. We will make use of $v_{m/2}\theta(z') \leq_s w$. If $v_{m/2} = v$, then $v\theta(z') \leq_s w_4$. Due to the primitivity of v , this suggests that $w'_2 w_3 w_4 (\theta(z'))^{-1}$ is a power of v . However, this contradicts the θ -primitivity of v since this word is actually the first half of u . Thus, $v_{m/2} = \theta(v)$. Then $z'v \leq_p w_3$ so that $w'_2 z' = z'\theta(z')z'$ is a power of v . Since v is assumed to be θ -primitive, this means that z' is a θ -power of v , that is, $|z'| \geq |v|$, a contradiction.

We have proved that $v_{m/2}$ starts somewhere to the left of the position where w_1 starts. Note that $|v_1 \dots v_m| = 2|\textcircled{1}| + |\textcircled{2}| = 2(2|w| + 2|z|) + 2|w| - |z| = 6|w'_5 w \theta(z')^{-1}|$. Hence, if $m \geq 12$, then $|w'_5 w \theta(z')^{-1}| \leq 2|v|$. This implies that either $v_{m/2}$ is a factor of $\textcircled{2}$ or otherwise $|w'_5| \leq |v|$, but both of them were already denied.

Having examined the case of $m \geq 12$, we will work on the case of $m = 10$ and the case of $m = 8$ one after another, under the assumption that the border between u_2 and u_3 is on w_2 and $v_{m/2}$ is not included in $\textcircled{2}$. Let us begin with the former case. Eq. (3) is

$$u^3 = v_1 \dots v_{10} w_1 \dots w_6. \tag{19}$$

Due to the assumptions, $v_1 v_2 v_3 v_4$ is a prefix of $\textcircled{1}$ so that $\textcircled{1} = (xy)^4 x$.

We have $|w| > \frac{1}{5}|u|$ while $|v| < \frac{9}{50}|u|$ so that $|v| < |w|$. We claim $v_5 = \theta(v)$; otherwise Theorem 5 is applied to $w_3 w_4 w_5 w_6$ and gives $\tilde{v}, w \in \{t, \theta(t)\}^+$ for some t . With $|v| < |w|$, however, this is not consistent with the θ -primitivity of w . Set $v_5 = \theta(v)$, and see Fig. 11.

Firstly, we show $w_4 = w_5$. Suppose not. Then $w_4 w_5$ becomes a θ -palindrome. As shown in the figure, it can be represented in two ways as $w_4 w_5 = \theta(x')\theta(v)^2 w'_5 = \theta(x')y v x w'_5 = \theta(w'_5)x\theta(v)y x'$, where the last one is just the θ -image of the second. Since $|w'_5| < |z| = |y| < |y x'| < |v|$, this equation suggests that $\theta(v)$ would be a proper factor of its square, a contradiction.

We claim that $w_4 = w_5 = w$. Indeed, otherwise, that is, if $w_4 = w_5 = \theta(w)$, then $\theta(w)^2 = \theta(x')\theta(v)^2 w'_5$. It has a suffix $\theta(v)\theta(z)$ because of $w_4 w_5 = w_5 w_6$.

Since $|v| + |w_5''| > |z| > |w_5''|$, this would imply that $\theta(v)$ were a proper factor of its square, a contradiction. Therefore, $w_4 = w_5 = w$ must be the case. Then $w_5w_6 = w\theta(w) = v_s\theta(v)\theta(z)$ for some suffix v_s of $v_4 = v$. This turns into $v_s\theta(v)\theta(z) = zv_s$. Since this is of length at least $2|v|$, this means that $v\theta(v)$ is equal to its conjugate, which has two conclusions; either $v = \theta(v)$ or v is not θ -primitive, but any of them is contradictory.

Lastly, we examine the case of $m = 8$ and $n = 6$. Let us claim $v_4 = \theta(v)$. Suppose it were v ; there are two possibilities. If $|v| \geq |w|$, then $w_2'' \leq_p v_1$. Then $w_2''^{-1}v_1v_2v_3v_4$ is so long (at least of length $2|v| + |w|$) that Theorem 5 and Lemma 15 are applicable to get $v \in \{w, \theta(w)\}^+$, and this actually means that v is either w or $\theta(w)$ since it is assumed to be θ -primitive. Hence, $u^3 \in \{w, \theta(w)\}^{14}$, but this contradicts the θ -primitivity of w as 3 does not divide 14. If $|v| < |w|$, on the other hand, the border between v_3 and v_4 is located to the right of the border between w_4 and w_5 because $\theta(y')v_3v_4v_5' = w_4w_5w_6$. Thus, $|w_2''^{-1}v_1v_2v_3v_4| \geq 2|w| + |v|$, which is long enough to apply Theorem 5 to get $\tilde{v}, w \in \{t, \theta(t)\}^+$ for some t , but this would lead us to the contradiction $w \in \{t, \theta(t)\}^{\geq 2}$ due to $|\tilde{v}| = |v| < |w|$.

Therefore, $v_4 = \theta(v)$, and this requires $|x| < |y|$ due to Lemma 14. Then

$$\theta(v) = yx = xz \tag{20}$$

for some z such that $\textcircled{2} = z\theta(z)$. Then $w_4w_5w_6 = \theta(y')xyxz\theta(z)$, and this gives

$$|w| = \frac{2}{3}|x| + \frac{7}{6}|y|. \tag{21}$$

We can easily see $w_4 = w_5 = \theta(w)$ contradict the primitivity of v . Observe that $v_4\theta(z) = \theta(v)\theta(z)$ is a suffix of $w_5w_6 = \theta(w)^2$. Hence, if $w_4 = w_5 = \theta(w)$, then w_4w_5 also has this as its suffix. However, then the $\theta(v)$ would be a factor of the suffix $\theta(v)^2$ of $\textcircled{1}$. This is confirmed by letting $\textcircled{2} = w_5''w_6 = z\theta(z)$ and seeing $|w_5''| < |z|$ from it.

In order to deny the remaining three possibilities of the values of w_4, w_5 , we prove $|v| < |w|$. This is equivalent to $|x| < \frac{1}{2}|y|$, and we prove this. For the sake of contradiction, suppose $|x| \geq \frac{1}{2}|y|$. Then Eq. (20) implies $y' \leq_p x$, and with this, it further implies $y' \leq_p \theta(z)$. Then the θ -palindrome $\theta(y')v\theta(v)y'$ occurs as a prefix of $w_4w_5w_6$. More strongly,

$$\theta(y')xyx <_p w_4w_5 <_p \theta(y')v\theta(v)y' <_p w_4w_5w_6 \tag{22}$$

holds due to Eq. (21). This means that $\theta(w_5)\theta(w_4)$ is a proper factor of $w_4w_5w_6$, and Lemma 1 derives $w_4 = w_5$ from this. In fact, we already know that this means $w_4 = w_5 = w$. Using the above prefix relation (22), we will show that even this is contradictory. For that, we will prove that w_5 is a proper factor of xyx which is a factor of the θ -palindromic prefix $\theta(y')v\theta(v)y'$. The relation (22) lets $\theta(y')v\theta(v)y' = w_4w_5\gamma$ for some $\gamma \leq_p w_6$. Then $\theta(y')xyxzy' = \theta(\gamma)\theta(w_5)\theta(w_4)$. Eq. (21) gives $|\gamma| = |w_4w_5w_6| - |\theta(y')v\theta(v)y'| = |y'|$. Moreover, $|xyx| - |w| = \frac{1}{3}|x| - \frac{1}{6}|y|$ and it is nonnegative because of the assumption $|x| \geq \frac{1}{2}|y|$. Thus,

$\theta(w_5) \leq_p xyx$. Combining this with the first prefix relation in (22) would imply that $w_5 = w$ would be a proper factor of $w_4w_5 = w^2$, a contradiction.

Having proved that $|v| < |w|$ must hold, now we can prove that any of the three remaining pairs of values that w_4 and w_5 take is contradictory. Let us consider the two pairs at the same time that make w_4w_5 a θ -palindrome. Note that $v_8 = \theta(v)$ is a factor of w_4w_5 as $v_8w_5'' \leq_s w_4w_5$. Hence, $\theta(w_5'')v \leq_p w_4w_5 \leq_p \theta(y')vxx$. Since $|x| < \frac{1}{2}|y|$, we have $|w_5''| > |\theta(y')|$, and furthermore, the prefix is shorter than $\theta(y')vx$. This means that v is a proper factor of v^2 , a contradiction. The only remaining possibility is $w_4 = w_5 = w$. In this case, w_5w_6 becomes a θ -palindrome of length at least $2|v|$, and it is a factor of $v_3v_4v_5 = v\theta(v)v$. We have already seen this imply either $v = \theta(v)$ or v being not θ -primitive, but both of them contradict the assumptions. \square

Having handled the case when $m \geq 8$ and $n = 6$, now we will solve the final case: $m \geq 10$ and $n = 8$. Combining all we proved, solving Eq. (3) under the condition amounts to the proof of our main theorem; when $5 \leq \min\{m, n\} \leq 12$, Eq. (3) admits only θ -periodic solutions.

Lemma 25. *If $m \geq 10$ and $n = 8$, then Eq. (3) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t .*

Proof. As before, we can assume the θ -primitivity of u, v, w , $v \neq \theta(v)$, and $w \neq \theta(w)$ (the negation of one of them immediately leads us to the conclusion). The assumption $|\textcircled{1}| \geq |\textcircled{2}|$ is translated in this case as $|\textcircled{2}| < 3|w|$.

First of all, we prove $|w_1w_2| < |\textcircled{2}|$. Suppose not. Then $|\textcircled{1}| \geq 4|w|$. If $|w| \geq |v|$, then we can apply Theorem 5 and Lemma 15 to $\textcircled{1}$ and obtain $v \in \{w, \theta(w)\}$, but then $u^3 \in \{w, \theta(w)\}^{m+8}$, which contradicts the θ -primitivity of u . If $|w| < |v|$, on the other hand, $v_1 \cdots v_{m/2-1} \leq_p \textcircled{1}$, and the same strategy based on Theorem 5 and Lemma 15 would lead us to the contradiction. The inequality was thus proved, or more strongly, $w_1w_2 <_p \textcircled{2}$. Using $|\textcircled{2}| < 3|w|$, this is strengthened as $\textcircled{2} = w_1w_2w_3'$ for some proper nonempty prefix w_3' of w_3 . In order for this to happen, $m|v| < \frac{5}{3}|u|$ and $8|w| > \frac{4}{3}|u|$ must hold, and for $m \geq 10$, this implies $|v| < |w|$. Now if $|w_3''| = |w_6''| \geq |v|$, then $(w_3'')^{-1}\textcircled{1} = (w_3'')^{-1}w_4w_5w_6'$ would be long enough for Theorem 5 to give $w \in \{\tilde{v}, \theta(\tilde{v})\}^{\geq 2}$, but this contradicts the θ -primitivity of w . Thus, $|w_3''| < |v|$.

Now we know that the θ -palindrome $v_{m/2}v_{m/2+1}$ is a factor of $\textcircled{2}$. Note that $|\textcircled{2}| = 2|w| + |w_3'|$ and $\textcircled{1} = 4|w| - 2|w_3'|$, and substituting these into $|v_1 \cdots v_m| = 2|\textcircled{1}| + |\textcircled{2}|$ gives $m|v| = 10|w| - 3|w_3'|$, and m being even implies that $5|w| - \frac{3}{2}|w_3'|$ is a multiple of $|v|$. Based on this, we show that w_6 must be different from w_5 . Suppose $w_5 = w_6 = \theta(w)$. Then $w_5w_6w_7 = w_6w_7w_8 = \theta(w)^3$. From the two facts that the θ -palindrome $v_{m/2}v_{m/2+1}$ occurs inside $w_6w_7w_8$ and that $w_5w_6w_7$ is a factor of $v_1 \cdots v_m$, the occurrence of $v_{m/2}v_{m/2+1}$ must match $v_{s-1}v_s$ for some index $s < m/2$ due to Lemma 1. Since the occurrence and $v_{m/2}v_{m/2+1}$ are far apart by $|w|$, this means that $|w|$ is a multiple of $|v|$. The fact that $5|w| - \frac{3}{2}|w_3'|$ is a multiple of $|v|$ now implies that $|w| - \frac{3}{2}|w_3'| = |w_3''| - \frac{1}{2}|w_3'|$ is a multiple of $|v|$, but this would mean $|w_3''| > |v|$, a contradiction. An analogous argument works for $w_5 = w_6 = w$. In this case, $w_1w_2w_3 = w_5w_6w_7$ holds, and

from this we rather get that $2|w| - |w'_3|$ is a multiple of $|v|$. Combining this with $5|w| - \frac{3}{2}|w'_3|$ being a multiple of $|v|$ implies that $|w'_3|$ is so, and hence, $|\textcircled{2}| = 2|w| + |w'_3|$ is a multiple of $|v|$. Then Eq. (3) would imply $\textcircled{2} \in \{v, \theta(v)\}^+$ and $\textcircled{1}$ is also in this set, but then u would not be θ -primitive.

Therefore, $w_5 \neq w_6$ must hold, that is, w_5w_6 is a θ -palindrome. Let us make several useful observations. We can find an integer i such that $\textcircled{1} = v^{i-1}x$, where x is a prefix of v with $v = xy$ and both x and y are θ -palindromes. Then $|\textcircled{1}| = i|x| + (i-1)|y|$ and $|\textcircled{2}| = m|v| - 2|\textcircled{1}| = (m-2i)|x| + (m+2-2i)|y|$. The condition $2|w| < |\textcircled{2}| < 3|w|$ provides the i with lower and upper bounds as: $\lfloor m/3 \rfloor + 1 \leq i \leq \frac{3}{8}m$. Substituting these into $4|w| = \frac{1}{2}|\textcircled{1}| + |\textcircled{2}|$ yields $|w| = \frac{2m-3i}{8}|x| + \frac{2m+3-3i}{8}|y|$. Then from $|w''_6| = |\textcircled{2}| - 2|w|$, these equations give $|w''_6| = \frac{2m-5i}{4}|x| + \frac{2m-5-5i}{4}|y|$ and $|w'_6| = |w| - |w''_6| = \frac{-2m+7i}{8}|x| + \frac{-2m-7+7i}{8}|y|$. Using the respective lower and upperbound of i , from these we can obtain $|w''_6| \geq |y|$ and $|w'_6| > |x|$. These mean that v_i is a proper factor of w_6 .

Now we can show that v_i cannot be $\theta(v)$. If so, then $v_{i-1}v_i = v\theta(v)$ is a factor of the θ -palindrome w_5w_6 , and moreover, its center is strictly to the right of the center of w_5w_6 . Being a θ -palindrome, w_5w_6 has another factor $v\theta(v)$ to the left of $v_{i-1}v_i$. Since $v_1 = \dots = v_{i-1} = v$, Lemma 1 implies $v = \theta(v)$, but this contradicts the assumption. Thus, $v_i = v$. Since v_i is a factor of w_6 , $(w''_3)^{-1}v_1 \dots v_i$ is long enough for Theorem 5 and Lemma 15 to give $v \in \{w, \theta(w)\}$, but we already know that this causes a contradiction with the θ -primitivity of u . \square