

Hyperkähler Metrics on the Moduli Space of Weakly Parabolic Higgs Bundles

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Abstract

This thesis examines a construction of hyperkähler metrics on the moduli space of weakly parabolic $\mathrm{SL}(2, \mathbb{C})$ -Higgs bundles. Given a Hermitian vector bundle E of rank 2 over a compact Riemann surface \mathcal{C} and some $R > 0$, this space can be identified as the moduli space \mathcal{M} of solutions (A, Φ) to the R -rescaled Hitchin equations:

$$\begin{aligned} F_A + R^2 [\Phi, \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0. \end{aligned}$$

Here A is a unitary connection on E with curvature F_A , and Φ is a tracefree $\mathrm{End}(E)$ -valued 1-form on \mathcal{C} . In the weakly parabolic setting the Higgs field Φ has a finite set of simple poles p_i and its determinant defines a meromorphic quadratic differential q with double poles on \mathcal{C} . On the subset $\mathcal{M}' \subset \mathcal{M}$ of generic solutions of Hitchin's equations this differential has only simple zeroes and determines a family of triangulations T^ζ of \mathcal{C} , parametrized by complex numbers $\zeta \in \mathbb{C}^\times$, with vertices at the poles of q . Additionally a solution (A, Φ) defines a flat connection on E , given by:

$$\nabla = A + \frac{R}{\zeta} \Phi + R\zeta \Phi^*.$$

Locally the flatness equation $\nabla s_{p_i} = 0$ near each pole p_i of q defines a Volterra integral equation of the second kind on a non-compact interval. A corresponding solution theory is established and, using recent results regarding the structure of \mathcal{M} for large R , it is shown that for $R \gg 0$ there exists a one dimensional space of solutions of the flatness equation for each pole p_i , which exponentially decay along an edge of the triangulation.

The triangulation T^ζ together with the sections s_{p_i} is used to construct a set of coordinates $\mathcal{X}_j(\zeta)$ on \mathcal{M}' . It is then proven that these coordinates are dominated by a leading term which is given by the coordinates for a corresponding simpler space of limiting configurations. Furthermore it is shown that the deviation from the leading term is given by a remainder that is exponentially suppressed in R .

If one varies the parameter ζ the triangulation jumps at certain rays in \mathbb{C}^\times . This leads to the formulation of a Riemann-Hilbert problem that is shown to be solved by the coordinates $\mathcal{X}_j(\zeta)$. From this perspective precise estimates of the exponential decay rate of each \mathcal{X}_j for large R are derived. They are given in terms of period integrals $\int_{\gamma_j} q^{1/2}$ along simple closed curves γ_j on \mathcal{C} which surround two simple zeroes or one pole of q .

Finally it is explained how one can use twistor theory to construct a hyperkähler metric g_{twist} on \mathcal{M}' out of the coordinates $\mathcal{X}_j(\zeta)$. A comparison of this metric with the simpler semiflat metric g_{sf} then shows that their difference is of the form $g_{\mathrm{twist}} - g_{\mathrm{sf}} = O(e^{-\mu R})$, for some constant $\mu \in \mathbb{R}_{>0}$.

Zusammenfassung

In dieser Arbeit wird eine Konstruktion von Hyperkähler-Metriken auf dem Modulraum von schwach parabolischen $\mathrm{SL}(2, \mathbb{C})$ -Higgs Bündeln untersucht. Für ein Hermitesches Vektorbündel E von Rang 2 auf einer kompakten Riemannschen Fläche \mathcal{C} und ein $R > 0$ kann dieser Raum mit dem Modulraum \mathcal{M} von Lösungen (A, Φ) der R -reskalierten Hitchin-Gleichungen identifiziert werden:

$$F_A + R^2 [\Phi, \Phi^*] = 0, \\ \bar{\partial}_A \Phi = 0.$$

Hier ist A ein unitärer Zusammenhang auf E mit Krümmung F_A , und Φ ist eine spurfreie $\mathrm{End}(E)$ -wertige 1-Form auf \mathcal{C} . In dem schwach parabolischen Setting hat das Higgs-Feld Φ eine endliche Menge einfacher Polstellen und seine Determinante definiert ein meromorphes quadratisches Differential q mit doppelten Polstellen auf \mathcal{C} . Auf der Teilmenge $\mathcal{M}' \subset \mathcal{M}$ generischer Lösungen der Hitchin-Gleichungen hat dieses Differential nur einfache Nullstellen und bestimmt eine durch komplexe Zahlen $\zeta \in \mathbb{C}^\times$ parametrisierte Familie von Triangulierungen T^ζ von \mathcal{C} mit Ecken an den Polstellen von q . Außerdem definiert eine Lösung (A, Φ) einen flachen Zusammenhang auf E , welcher gegeben ist durch:

$$\nabla = A + \frac{R}{\zeta} \Phi + R\zeta \Phi^*.$$

Lokal definiert die Flachheits-Gleichung $\nabla s_{p_i} = 0$ nahe einer jeden Polstelle p_i von q eine Volterra-Integralgleichung zweiter Art auf einem nicht-kompakten Intervall. Eine entsprechende Lösungstheorie wird etabliert und es wird unter Benutzung jüngster Ergebnisse bezüglich der Struktur von \mathcal{M} für große R gezeigt, dass für $R \gg 0$ nahe jeder Polstelle p_i ein eindimensionaler Raum von Lösungen der Flachheitsgleichung existiert, welche entlang einer Kante der Triangulierung exponentiell abfallen.

Die Triangulierung T^ζ , zusammen mit den Schnitten s_{p_i} , wird genutzt um eine Menge von Koordinaten $\mathcal{X}_j(\zeta)$ auf \mathcal{M}' zu konstruieren. Es wird dann bewiesen, dass diese Koordinaten von einem Führungsterm dominiert werden, der durch die Koordinaten eines korrespondierenden einfacheren Raums von Limiting-Konfigurationen gegeben ist. Außerdem wird gezeigt, dass die Abweichung von dem führenden Term durch einen Rest gegeben ist, der in R exponentiell unterdrückt wird.

Wird der Parameter ζ variiert, so springt die Triangulierung bei bestimmten Strahlen in \mathbb{C}^\times . Dies führt zu der Formulierung eines Riemann-Hilbert-Problems, von dem gezeigt wird, dass es durch die Koordinaten $\mathcal{X}_j(\zeta)$ gelöst wird. Aus dieser Sicht werden präzise Abschätzungen für die exponentiellen Abklingraten für jedes \mathcal{X}_j für große R abgeleitet. Diese sind in Termen von Periodenintegralen $\int_{\gamma_j} q^{1/2}$ entlang einfach geschlossener Kurven γ_j auf \mathcal{C} gegeben, welche zwei einfache Nullstellen oder eine Polstelle von q umlaufen.

Zuletzt wird erklärt, wie Twistor-Theorie benutzt werden kann, um aus den Koordinaten $\mathcal{X}_j(\zeta)$ eine Hyperkähler-Metrik g_{twist} auf \mathcal{M}' zu konstruieren. Ein Vergleich dieser Metrik mit der einfacheren semi-flachen Metrik g_{sf} zeigt dann, dass ihre Differenz von der Form $g_{\mathrm{twist}} - g_{\mathrm{sf}} = O(e^{-\mu R})$ für eine Konstante $\mu \in \mathbb{R}_{>0}$ ist.

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Chapter 1

Introduction

In 2009 D. Gaiotto, G. Moore and A. Neitzke published a theory which they argued would lead to a very concrete way of constructing hyperkähler metrics on a large class of manifolds, using twistor theory on integrable systems. The general theory was first presented in [GMN10], and in [GMN13b] an explicit adaptation for the moduli space \mathcal{M} of weakly parabolic Higgs bundles on a compact Riemann surface \mathcal{C} was developed. This also introduced the notion of spectral networks, which was further developed in [GMN13a]. While their general theory remained conjectural a lot of progress has been made since then, proving some of their conjectures in certain special cases. In 2017 C. Garza was able to show in [Gar17] how the construction works for the Pentagon case, an integrable system with torus fibers that exhibits a nodal degeneration on a certain locus, very similar to the known behavior of the Higgs bundle moduli space. Notably he was able to show how the Ooguri-Vafa metric is a suitable model for the singular fibers. The most recent step in this direction was taken by I. Tulli, who showed in [Tul19] how the method can be applied to a certain moduli space of framed wild harmonic bundles on \mathbb{CP}^1 . Thus the progress in the general theory has mostly been made for very specific examples while the conjectures were formulated in a much broader context.

One of the most prominent conjectures of Gaiotto, Moore and Neitzke (GMN) concerns the comparison of the natural L^2 -metric g_{L^2} , which is defined on all of \mathcal{M} , and the semiflat metric g_{sf} , which is defined on the regular locus $\mathcal{M}' \subset \mathcal{M}$ where the determinants of the Higgs fields have only simple zeroes. GMN argued for the exponential decay of the difference of these two metrics on \mathcal{M}' for large Higgs fields, i. e. when considering generic Higgs bundles $(\mathcal{E}, R\Phi)$ for large R the difference of the two metrics should be $g_{L^2} - g_{\text{sf}} = O(e^{-\gamma R})$. Additionally they conjectured that γ is given by the length of the shortest saddle connection in the metric on \mathcal{C} induced by $\det\Phi$. This problem was later also tackled by R. Mazzeo, J. Swoboda, H. Weiß and F. Witt using different methods for the case of regular $\text{SU}(2)$ -Higgs bundles in [Maz+12], [Maz+16] and [Maz+19]. They were able to prove a polynomial decay by introducing the notion of limiting configurations. For large Higgs fields these almost solve Hitchin's equations, leaving only an exponentially small remainder term which allows for good approximations of the natural metric. The decaying behaviour was later on improved to an actual exponential decay by D. Dumas and A. Neitzke for a special case in [DN19], and in a general setting by L. Fredrickson in [Fre20], who also expanded the theory for the $\text{SU}(N)$ case in [Fre18].

Although it seemed reasonable to suspect that the two theories (twistorial construction on the integrable system and limiting configurations) could complement each other, they weren't put together on a general class of Higgs bundle moduli spaces before this thesis, as the twistorial construction needs to consider Higgs bundles with singularities while until recently the limiting configurations were only developed for regular Higgs bundles. But this has changed now as L. Fredrickson, R. Mazzeo, J. Swoboda and H. Weiß (FMSW)

were able to develop their theory for the case of weakly and strongly parabolic Higgs bundles in [Fre+20]. This finally allows to merge both paths and show that they fit together quite well as they give very concrete expressions and good approximations for the hyperkähler structure on \mathcal{M}' . Thus we will argue that the conjectural picture of GMN works for moduli spaces of weakly parabolic Higgs bundles of rank 2 on Riemann surfaces of arbitrary genus with an arbitrary number of parabolic points. In particular a hyperkähler metric can be constructed in a twistorial way whose difference to the semiflat metric decays exponentially.

In order to understand in which context the questions mentioned above arise, we will begin in chapter 2 with a short account of the basic ideas of hyperkähler metrics and the moduli space of Higgs bundles. We will start with the rigorous treatment after that, so let us now give a brief summary of the key steps in the construction of the hyperkähler metric which will serve as a guide for our endeavor. One starts with the moduli space of weakly parabolic Higgs bundles regarded as the moduli space \mathcal{M}_R of solutions (A, Φ) of the R -rescaled Hitchin equations:

$$\begin{aligned} F_A + R^2 [\Phi, \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0. \end{aligned}$$

Here A is a connection with curvature F_A and Φ is an endomorphism-valued 1-form, called the Higgs field, on some fixed complex Hermitian vector bundle (E, h) of rank 2 over a Riemann surface \mathcal{C} and $R \in \mathbb{R}_{>0}$. In the weakly parabolic setting the Higgs fields have simple poles at a finite number of fixed points $p_1, \dots, p_n \in \mathcal{C}$. As of the Kobayashi-Hitchin correspondence it holds $\mathcal{M}_R \cong \mathcal{M}$, and there is a corresponding regular locus $\mathcal{M}'_R \subset \mathcal{M}_R$. The basic theory of these moduli spaces will be explained in chapter 3 where we mostly follow the setup of [Fre+20], as their results are fundamental for our work here.

To each Higgs pair (A, Φ) one can then associate a decorated triangulation of \mathcal{C} . This entails a triangulation of \mathcal{C} that is given by certain curves which are geodesics when considered in the metric on \mathcal{C} induced by $\det \Phi$. On the other hand the decoration consists of a choice of a 1-dimensional subspace of E at each vertex of the triangulation. In our setting this subspace is given by a solution s to the flatness equation $\nabla(\zeta)s = 0$ (for $\zeta \in \mathbb{C}^\times$), where $\nabla(\zeta) = A + \frac{R}{\zeta}\Phi + R\zeta\Phi^*$ is a connection on E that is flat for bundles of parabolic degree 0. We will constrain ourselves to this case and describe the restriction later on in more detail. As V. Fock and A. Goncharov proved in [FG06], one can construct a coordinate system for the moduli space of flat connections out of such decorated triangulations. Following [GMN13b] chapter 4 will be concerned with constructing the triangulation and explaining how the coordinates of Fock and Goncharov relate to coordinates of the Higgs bundle moduli space.

The main difficulty in this construction comes from solving the flatness equation. Locally this is a non-autonomous linear ODE whose coefficients are given by the solution of Hitchins equations. For the construction to work one has to be able to choose certain solutions in a unique way (up to scalar multiplication), such that they have the right asymptotics for $\zeta \rightarrow 0$ and $\zeta \rightarrow \infty$. The existence of such sections, called *small flat sections*, was conjectured in [GMN13b], but had not been rigorously proven before this thesis. In chapter 5 we will use the results of [Fre+20] to obtain a local description of Higgs pairs as the sum of a diagonal leading term and a remainder that is exponentially suppressed, the

general idea being that if the connection matrix of $\nabla(\zeta)$ has this composite form, then the solutions of $\nabla(\zeta)s = 0$ should also be determined by an explicitly known leading term and some remainder that is exponentially suppressed.

GMN already provided an argument along these lines in [GMN13b], where they used the WKB approximation to conjecture the right behaviour of the solutions. Here we will not use this approach but instead develop a rigorous theory of "initial value problems at infinity" in chapter 6 which expands the existing theory of Volterra integral equations. Explicitly we show the unique existence, as well as the continuous and differentiable dependence on parameters of solutions to Volterra equations of the second kind on unbounded intervals for a general class of integral kernels. This theory hadn't been developed before and might also be useful in different contexts.

With the tools of this theory at hand we then proceed to chapter 7, where we study the solutions to the flatness equation and obtain our first main result.

Theorem 1

At each weakly parabolic point of a Higgs bundle in \mathcal{M}'_R there exists a solution of the flatness equation that is unique up to multiplication by a scalar and s. t. the Fock-Goncharov coordinates constructed out of these decorations are of the form

$$\mathcal{X} = (1 + r) \exp \left(\frac{R}{\zeta} \pi Z + i\theta + R\zeta \pi \bar{Z} \right),$$

where r is exponentially suppressed in R and Z and θ are coordinates of the Hitchin fibration.

We note here that the calculations necessary for this theorem show that the combination of our theory of initial value problems at infinity and the theory of geodesics for quadratic differentials is a very good tool for the general problem of parallel transport.

The \mathcal{X} -coordinates thus constructed are defined on \mathcal{M}'_R for generic values of $\zeta \in \mathbb{C}^\times$ and can be associated to elements of a certain lattice $\Gamma_q \subset H_1(\Sigma'_q; \mathbb{Z})$ where Σ'_q is the spectral cover of $\mathcal{C} \setminus \{p_1, \dots, p_n\}$ corresponding to the determinant q of the Higgs field. For specific rays in \mathbb{C}^\times the underlying triangulation on \mathcal{C} changes, which leads to jumps of the coordinates. On a certain generic subset $\widetilde{\mathcal{M}}'_{R,c} \subset \mathcal{M}'_R$ the jumps are completely understood and can be used to formulate a Riemann-Hilbert problem which is then solved by the \mathcal{X} -coordinates. We show how this is done in chapter 8. Our main theorem in that chapter then concerns the solvability of a certain integral equation which was proposed by GMN for the general construction of Hyperkähler metrics on integrable system, where the jumps are determined by integers Ω , which are called (generalized) Donaldson-Thomas invariants.

Theorem 2

The \mathcal{X}_γ -coordinates are solutions to the integral equation

$$\mathcal{X}_\gamma(\zeta) = c_\gamma \mathcal{X}_\gamma^{sf}(\zeta) \exp \left[-\frac{1}{4\pi i} \sum_{\beta \in \Gamma_q} \Omega(\beta; q) \langle \gamma, \beta \rangle \int_{l_\beta(q)} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_\beta(x, \zeta')) \right],$$

for some $c_\gamma \in \mathbb{C}$ with $\|c_\gamma\| = 1$ which does not depend on ζ and is exponentially close to 1 for large R .

In our setting the consideration of this integral equation has the advantage that we obtain additional analytic details regarding certain asymptotics of the \mathcal{X} -coordinates.

Coming back to the general construction, the idea proposed by GMN is that the \mathcal{X} -coordinates fit into the picture of [GMN10], i. e. one can use them to construct a holomorphic symplectic form from which one can construct the hyperkähler metric on $\widetilde{\mathcal{M}}'_{R,c} \subset \mathcal{M}'_R$ by use of the twistor theorem of N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček. We will explain this general construction of GMN in the final chapter 9. Here we just note that it is the use of the twistor theorem for which the ζ -asymptotics of the \mathcal{X} -coordinates and their derivatives have to be known. The constructed metric is then the object of our final result.

Theorem 3

There is a hyperkähler metric on $\widetilde{\mathcal{M}}'_{R,c}$ constructed by the twistorial method whose difference to the semiflat metric decays exponentially in R .

Additionally we give a criterion for the appearance of the conjectured optimal decay rate. Finally we note here that we do not yet know whether our twistorial hyperkähler metric is the natural L^2 metric on the moduli space. It is natural to conjecture this at this point and at the end of the last section we will sketch an outline of a possible proof, which was communicated to us by Andrew Neitzke.

Chapter 2

Fundamental notions

Before we start with precise definitions it might be enlightening to briefly discuss the general setting in which our theory is placed. We want to study hyperkähler metrics on the moduli space of Higgs bundles. Therefore we start with a summary of hyperkähler geometry in section 2.1, in which we explain the basic ideas of hyperkähler structures, why they are interesting and in which ways they can be constructed. It then follows section 2.2 where we summarize the origin of the Higgs bundle moduli space and its different representations, as well as the peculiarity of its hyperkähler metric.

We stress that these accounts are only meant as a helpful overview, leaving rigorous treatment to the following chapters.

2.1 Hyperkähler geometry

A hyperkähler manifold can be defined as a Riemannian manifold (M, g) equipped with three complex structures I, J and K , s. t. $IJ = K^1$ and (M, g) is Kähler for all of the three complex structures. This entails that g is preserved by all of the complex structures and that they are parallel, i. e. for ∇ the Levi-Civita connection it holds $\nabla I = \nabla K = \nabla J = 0$. Alternatively Kähler metrics can also be defined by their Kähler forms, which (for the complex structure I) is a 2-form ω_I on M given by $\omega_I(X, Y) := g(IX, Y)$. One can then show that $\nabla I = 0$ is equivalent to ω_I being closed, cf. [Nei17].

The different conditions can be relaxed by only starting with almost complex structures and in this case the integrability of, say, I follows from demanding I to be parallel. On the other hand, if we start with $d\omega_I = 0$, it does not necessarily follow that I is integrable, so we have to include this assumption to infer $\nabla I = 0$. For hyperkähler manifolds however this is different. By a result of Hitchin in [Hit87] the integrability of the almost complex structures I, J and K follows if the corresponding 2-forms ω_I, ω_J and ω_K are closed. We see in this difference a first glimpse of the additional rigidity a hyperkähler manifold has in comparison to a Kähler manifold. This rigidity is clearly captured in the relation of the Kähler forms and complex structures. In fact, after an appropriate identification one can show that $I = \omega_K^{-1}\omega_J$ holds, as well as the corresponding relations for cyclic permutations of I, J and K . So the information of the complex structures is completely encoded in the Kähler forms. This even implies that the information of the metric is encoded similarly, i. e. $g = -\omega_I\omega_K^{-1}\omega_J$. Later on we will see how the information can be condensed into a single holomorphic symplectic 2-form. Here we only note that for, say, I the 2-form $\varpi_I := \omega_J + i\omega_K$ is holomorphic symplectic, from which it follows that hyperkähler manifolds are necessarily of real dimension $4n$, cf. [Nei17].

¹Note that together with $I^2 = J^2 = K^2 = -1$ this is equivalent to I, J, K fulfilling the quaternion relations.

Another important aspect of hyperkähler manifolds concerns their holonomy. Since I, J and K are parallel and they obey the quaternion relations, it follows that the holonomy group is contained in $\mathrm{Sp}(n) := \mathrm{O}(4n) \cap \mathrm{GL}(n, \mathbb{H})$, which is one of the cases appearing in Bergers classification of holonomy groups for irreducible and nonsymmetric Riemannian manifolds from 1955 in [Ber51]. One can also consider $\mathrm{Sp}(n)$ as the subgroup of $\mathrm{U}(2n)$ preserving ϖ_I . It then also preserves the holomorphic volume form $\wedge^n \varpi_I$, so the holonomy group is contained in $\mathrm{SU}(2n)$ and M is Ricci-flat. Thus interest in hyperkähler manifolds arises as they can provide non-flat geometries, which are (nevertheless) Ricci-flat and thus a special case of Einstein manifolds, cf. [Iza+21], [Hyp99] and [Hit92].

In the smallest possible dimension it holds $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ and one obtains that compact hyperkähler manifolds are the same as Calabi-Yau manifolds, which, due to the Enriques-Kodaira classification, are either K3 surfaces or compact tori $(S^1)^4$, cf. [Bar+04]. A particular important example of a hyperkähler manifold is the Ooguri-Vafa space, a T^2 bundle that can be constructed through the use of the Gibbons-Hawking ansatz, cf. [GH78], [GH79] and [Nei17]. It entails one singular fiber (a nodal torus), that nevertheless carries a hyperkähler metric even over the singular point. This feature is relevant for the study of Higgs bundle moduli spaces as they also exhibit such nodal tori while carrying a complete hyperkähler metric. The explicitly known Ooguri-Vafa metric is therefore a likely candidate for a local model of the metric on the Higgs bundle moduli space near singular fibers.

Turning to higher dimensions we first note that in the compact case there is a relation to complex symplectic manifolds. Notably if X is a compact complex manifold that is irreducible, i. e. simply-connected and not a product of lower-dimensional manifolds, and admits Kähler metrics, then there is a unique hyperkähler metric in each Kähler class on X . Conversely, on every compact hyperkähler manifold X there exists a complex structure J , s. t. (X, J) is an irreducible complex symplectic manifold, cf. section 10.4 of [Joy07], where some concrete examples of compact hyperkähler manifolds are also studied.

In the non-compact case two methods of constructing hyperkähler manifolds are particularly important here. The first one is given by the use of hyperkähler quotients, where one starts with a hyperkähler manifold M . Similar to the symplectic quotient construction, the task is to find a moment map $\mu : M \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$ where \mathfrak{g} is the Lie algebra for a Lie group G acting on M that preserves the three Kähler forms of M . Now if ζ is a regular value for the moment map μ , then $\mu^{-1}(\zeta)/G$ is a hyperkähler manifold, cf. [Hit92]. Note that it is possible to use this construction for infinite dimensional manifolds M which can then lead to a finite dimensional quotient space. Such a quotient construction is a standard procedure in gauge theory and it is this construction from which the hyperkähler metric on the Higgs bundle moduli in [Hit87] space originates.

The other method is the twistor construction. For this we note that given a hyperkähler manifold M with three complex structures I, J and K one obtains another complex structure $aI + bJ + cK$ which is also parallel for every unit vector $(a, b, c) \in \mathbb{R}^3$. Thus a hyperkähler manifold actually carries a whole \mathbb{CP}^1 of complex structures. It turns out to be useful to study the corresponding twistor space $Z := M \times \mathbb{CP}^1$, which has a complex structure and is a holomorphic fiber bundle over \mathbb{CP}^1 . This space also carries a list of some other structures which we will consider in more detail later on. Here we just mention that the Kähler forms ω_I, ω_J and ω_K can be joined together into a single $\mathcal{O}(2)$ -twisted 2-form ϖ . The twistor theorem now allows to reverse the process. Given a complex

manifold that is given as a holomorphic fibration over \mathbb{CP}^1 with the aforementioned list of structures, the parameter space of real sections becomes a hyperkähler manifold. Recalling the considerations above the metric is mostly encoded in the 2-form ϖ . Thus this method is a particularly good candidate for obtaining information about the metric if one knows ϖ .

However, in a general setting such a ϖ still seems hard to come by and one would like to have a method of constructing such a form. This is what was proposed in the work of Gaiotto, Moore and Neitzke. In [GMN10] they started by only assuming that M is a torus fibration and proposed a list of conditions for a set of functions \mathcal{X}_i thereon. The functions are supposed to carry all the information necessary for the twistor construction, leading to a hyperkähler metric for which the logarithms of the \mathcal{X}_i should then be Darboux coordinates.² The upshot of this method is that the \mathcal{X}_i can be considered as solutions of a set of differential equations, which can be considered as a version of the Cauchy-Riemann equations.³

The construction has thus become the more concrete task of building and solving this set of differential equations. Additionally GMN proposed a way of doing just this, by formulating the problem in terms of a Riemann-Hilbert problem. For this the \mathcal{X}_i are considered as functions on the supposed twistor space $M \times \mathbb{CP}^1$ which, for a fixed element of M , should be holomorphic on \mathbb{CP}^1 , except along a (countable) set of rays, where a certain jumping behaviour is specified. In this theory the jumps are mostly determined by a set of integers, often called generalized Donaldson-Thomas-invariants, which are defined on M and jump along a real-codimension-1 locus on the base of the torus fibration. Given these DT-invariants the Riemann-Hilbert problem can be formulated, and GMN also described a way of solving it by means of a certain integral equation for the \mathcal{X}_i . Thus we have arrived at the final formulation of the construction. After specifying appropriate DT-invariants on M it suffices to solve an associated integral equation, whose solutions should then solve a Riemann-Hilbert problem and obey the list of conditions established by GMN for the twistor construction. In the most general setting for this theory one may even start without the torus fibration M and instead only specify the data which is needed to build a complex integrable system on some base manifold B . For a concise overview we refer to the article [Nei13].

While this construction is very concrete and should provide detailed knowledge of the constructed hyperkähler metric, one has to be careful as to which parts of the story work in the intended way in a given setting. If one considers, for example, the integral equation, it is not quite known, under which conditions for the DT-invariants a unique solution exists, cf. [Bri19]. In order to understand all the details of this construction it is therefore important to study examples, where each step can be thoroughly investigated. Our endeavor can be considered such an example, as we will carry out this modified twistor construction on the moduli space of Higgs bundles, which was previously obtained from the quotient construction.

²It is customary to call the functions \mathcal{X}_i themselves Darboux coordinates and we will keep referring to them in such a way.

³The twistor theorem and the adaptation of GMN can be found in section 9.1.

2.2 Higgs bundles and the self-duality equations

Higgs bundles were introduced by N. Hitchin in his seminal paper [Hit87] in 1986. Our overview in this section is mostly guided by the comprehensive survey article [Swo21] by J. Swoboda. Hitchin's starting point was the examination of the self-dual Yang-Mills equations $*F_\nabla = F_\nabla$ on \mathbb{R}^4 for the curvature F_∇ of a unitary connection ∇ . These equations are of importance in physics as their solutions are solutions of the Yang-Mills equations $d_\nabla * F_\nabla = 0$, which are the Euler-Lagrange equations for the Yang-Mills functional $\text{YM}(\nabla) = \int_{\mathbb{R}^4} \|F_\nabla\|^2 \text{dvol}$. Hitchin considered solutions of the self-duality equations which are invariant under translation in two dimensions. Combining the non-invariant parts into a unitary connection A and the invariant parts into an endomorphism field Φ the self-duality equations translate into

$$\begin{aligned} F_A + [\Phi, \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0. \end{aligned} \tag{2.1}$$

Here $\bar{\partial}_A$ is the holomorphic structure induced from A . These equations can be shown to be conformally invariant on \mathbb{R}^2 and can thus be defined on Riemann surfaces. They are usually referred to as Hitchin equations or Hitchin's self-duality equations and Hitchin considered them for a complex hermitian vector bundle (E, h) of rank 2 and odd degree on a compact Riemann surface \mathcal{C} . In this way (2.1) is a set of equations for a pair (A, Φ) consisting of a connection A on E and an endomorphism valued 1-form Φ , where a hermitian metric is fixed on the vector bundle E . Alternatively one can start with a pair $(\bar{\partial}_E, \Phi)$ on E , where $\bar{\partial}_E$ is a holomorphic structure and Φ is holomorphic in the complex structure induced by $\bar{\partial}_E$, i. e. it holds $\bar{\partial}_E \Phi = 0$. Now (2.1) can be considered as an equation for the hermitian metric, which induces the Chern connection A . The pair $(\bar{\partial}_E, \Phi)$ is called a Higgs bundle and Φ the Higgs field and we see that a solution to (2.1) gives rise to a Higgs bundle.

From a physical perspective different mathematical solutions to a system of equations such as (2.1) can refer to the same physical entity (e. g. in different coordinate bases) and one is interested in the set of solutions which define distinct solutions in the physical sense. Mathematically this is reflected through the use of gauge theory. We denote the bundle automorphisms of E by $\mathcal{G}^c = \text{GL}(E)$ and call \mathcal{G}^c the group of complex gauge transformations of E . If a hermitian metric h is given on E , we denote the h -unitary gauge transformations by \mathcal{G} . They act on all of the different objects we consider here (e. g. on $\bar{\partial}_E$ and Φ by conjugation) and one can show that (2.1) is invariant under \mathcal{G} . Therefore it is reasonable to study the space of solutions of (2.1) modulo gauge transformation of \mathcal{G} . As it turns out, in order to obtain a well behaved moduli space it is necessary to only consider irreducible pairs (A, Φ) , i. e. pairs that do not split into a direct sum of solutions on any non-trivial subbundle. One can show that with this restriction

$$\mathcal{M}^{\text{sd}} := \frac{\{(A, \Phi) \mid (A, \Phi) \text{ is an irreducible solution of eq. (2.1)}\}}{\mathcal{G}}$$

is a finite-dimensional smooth manifold, whose dimension only depends on the genus of \mathcal{C} , cf. [Hit87].

Similarly one can consider the set of all Higgs bundles. Here the action of \mathcal{G}^c is appropriate and again one has to consider some restriction to obtain a moduli space that

is Hausdorff. For E of rank r_E and degree d_E the slope μ_E of E is defined by $\mu_E := \frac{d_E}{r_E}$. A Higgs bundle is called stable if for every proper Φ -invariant subbundle F of E it holds $\mu_F < \mu_E$, cf. [WG08]. We then obtain the moduli space of stable Higgs bundles

$$\mathcal{M}^{\text{Higgs}} := \frac{\{(\bar{\partial}_E, \phi) \mid (\bar{\partial}_E, \Phi) \text{ is a stable Higgs bundle}\}}{\mathcal{G}^c}.$$

Famously, Hitchin proved in [Hit87] that the moduli spaces $\mathcal{M}^{\text{Higgs}}$ and \mathcal{M}^{sd} are homeomorphic. This is known as the Kobayashi-Hitchin correspondence for Higgs bundles.

Two more moduli spaces exist which are also homeomorphic to \mathcal{M}^{sd} and provide additional motivation to study these spaces. Given a solution (A, Φ) of Hitchin's equation (2.1) one can build the connection $A + \Phi + \Phi^*$. From (2.1) it then follows that this is a flat connection. For the space of all such flat connections on E we can again consider the action of \mathcal{G}^c and obtain the corresponding de Rham moduli space

$$\mathcal{M}^{\text{dR}} := \frac{\{D \mid D \text{ is an irreducible flat connection on } E\}}{\mathcal{G}^c}.$$

Now every flat connection determines a representation $\rho_D : \pi_1(\mathcal{C}) \rightarrow \text{GL}(r_E, \mathbb{C})$ through its monodromy. It is part of the Riemann-Hilbert correspondence that \mathcal{M}^{dR} is homeomorphic to the Betti moduli space \mathcal{M}^{B} of irreducible representations of $\pi_1(\mathcal{C})$ in $\text{GL}(r_E, \mathbb{C})$. Donaldson proved in [Don87] that there is an inverse of the map from \mathcal{M}^{sd} to \mathcal{M}^{dR} , so we obtain

$$\mathcal{M}^{\text{sd}} \cong \mathcal{M}^{\text{dR}} \cong \mathcal{M}^{\text{B}}.$$

This collection of homeomorphisms is called the non-abelian Hodge correspondence.

We note that all of the above correspondences generalize to different settings in many ways, including higher ranks of E . Another important possibility is to allow a constant central curvature part for the connection A and restrictions of the Higgs field Φ to certain subgroups of $\text{End}(E)$. This will be explained in more detail in section 3.1, as will be the inclusion of poles in the Higgs field, leading to parabolic Higgs bundles. Finally there exist generalizations to Higgs bundles on higher dimensional base space, where \mathcal{C} is replaced by a compact Kähler manifold, cf. [WG08].

These different aspects of the Higgs bundle moduli space allow for a wide range of different structures that can be studied on it. Here we will focus on the fact, that there exists a hyperkähler metric on \mathcal{M}^{sd} , which was already introduced by Hitchin in [Hit87]. At any point $[(A, \Phi)] \in \mathcal{M}^{\text{sd}}$ elements of the tangent space can be identified with pairs $(\alpha, \varphi) \in \Omega^{0,1}(\mathcal{C}, \text{End}(E)) \oplus \Omega^{1,0}(X, \text{End}(E))$. If such a pair is L^2 -orthogonal to the unitary gauge orbit through (A, Φ) it is said to be in *Coulomb gauge*, and in these terms the L^2 -metric is given as

$$\|(\alpha, \varphi)\|_{L^2}^2 := 2i \int_{\mathcal{C}} \text{Tr}(\alpha^* \wedge \alpha + \varphi \wedge \varphi^*).$$

The main feature of this metric for us is the fact, proven by Hitchin, that it is a complete hyperkähler metric. This becomes particularly interesting when considering the Hitchin fibration of $\mathcal{M}^{\text{Higgs}}$. For this one considers the map $\text{Hit} : \mathcal{M}^{\text{Higgs}} \rightarrow \mathcal{B}$, $(\bar{\partial}_E, \Phi) \rightarrow \text{char}_{\Phi}(\lambda)$ which associates the characteristic polynomial of Φ in λ to the Higgs bundle $(\bar{\partial}_E, \Phi)$.

Here \mathcal{B} is identified with the coefficients of the characteristic polynomial, so elements in \mathcal{B} are given by a collection of differential forms on \mathcal{C} . This map exhibits $\mathcal{M}^{\text{Higgs}}$ as a completely integrable system over generic points of \mathcal{B} , where the fibers are affine tori. Over non-generic elements of \mathcal{B} however (i. e. where $\det\Phi$ has non-simple zeros) the tori degenerate to nodal tori. Still the L^2 metric also extends to these points, similar to what one encounters on the Ooguri-Vafa space.

With these phenomena and the general interest in hyperkähler metrics in mind the hyperkähler geometry on the Higgs bundle moduli space presents itself as a multifaceted structure, worthy of further investigation.

Chapter 3

Parabolic Higgs bundles

In this section we start by developing the basic constructions for the moduli space we are interested in, following mostly [Fre+20]. For the general construction and further details see also [Nak96] and [BY96]. We will start by giving an overview of parabolic Higgs bundles, which can roughly be described as Higgs bundles that may have singularities at certain fixed points. But there are some different notions that will have to be separated here concerning the structure of the poles. One idea is to allow simple poles and residues of the Higgs field which are nilpotent. Those were introduced in [Kon93] and are called *strongly parabolic* (with regular singularities). at this moment our construction does not seem to work on the corresponding moduli space for these bundles. Nevertheless a thorough investigation of the possibilities of adapting the methods presented here to that case would be highly interesting. Instead we will focus on weakly parabolic bundles which also have simple poles but allow for diagonalizable residues of the Higgs field. Finally, there also is a moduli space for bundles with singularities of order > 1 . These are called *wild* or *irregular*, were presented in [BB04] and may also be strongly or weakly parabolic. For us they are of importance as an application of the theory presented here to a special case of wild bundles on the sphere was carried out in [Tul19], which shows that the theory can actually be adapted to different settings.

3.1 The moduli space of parabolic Higgs bundles.

Let \mathcal{C} be a compact Riemann surface of genus g with a metric $g_{\mathcal{C}}$ and Kähler form $\omega_{\mathcal{C}}$ and let E be a \mathcal{C}^{∞} -vector bundle over \mathcal{C} of rank r and degree d . This is the data that stays fixed throughout all of the following constructions. We also fix a holomorphic structure $\bar{\partial}_{\text{Det}E}$ on the complex line bundle $\text{Det}E = \Lambda^r E$ with associated Hermitian-Einstein metric $h_{\text{Det}E}$. This metric induces the Chern connection A_{Det} on this holomorphic Hermitian line bundle and is characterized by the curvature

$$F_{A_{\text{Det}}} = -\sqrt{-1} \deg E \frac{2\pi\omega_{\mathcal{C}}}{\text{vol}_{g_{\mathcal{C}}}(\mathcal{C})} \text{Id}_{\text{Det}E}$$

We can then equip E with a holomorphic structure $\bar{\partial}_E$ demanding that $\bar{\partial}_E$ induces $\bar{\partial}_{\text{Det}E}$ on $\text{Det}E$ and denote by $\mathcal{E} := (E, \bar{\partial}_E)$ the emerging holomorphic vector bundle.

Now we introduce the parabolic data which starts with a divisor $D = p_1 + \dots + p_n$ on \mathcal{C} , i. e. we fix n distinct points on the surface. To these points we associate data that fixes the behavior of certain objects near them.

Definition 3.1.1

A *parabolic structure* on E consists of a choice of weighted flags $\mathcal{F}(p) = F_\bullet(p)$, i. e.

$$\begin{aligned} E_p &= F_1(p) \supset \dots \supset F_{s_p}(p) = 0 \\ 0 &\leq \alpha_1(p) < \dots < \alpha_{s_p}(p) < 1 \end{aligned}$$

for each $p \in D$ with $\sum_{i=1}^n \alpha_i = 1$. A vector bundle with parabolic structure is called a *parabolic bundle*.

Now let $K_{\mathcal{C}}$ be the canonical line bundle. We are interested Higgs fields that have simple poles at the marked points, which are defined as follows:

Definition 3.1.2

A *parabolic Higgs field* is a bundle map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes K_{\mathcal{C}}(D)$ which is holomorphic w. r. t. the induced holomorphic structure of $\bar{\partial}_E$. The Higgs field is said to *preserve* the parabolic structure if for all i and p the residue preserves the flags, i. e.:

$$\text{res}_p \Phi(F_i(p)) \subseteq F_i(p).$$

A *parabolic Higgs bundle* is a holomorphic vector bundle \mathcal{E} together with a parabolic structure and a parabolic Higgs field Φ preserving the parabolic structure.

Note that the twist by D in the definition amounts to Φ having simple poles, so we will also call the Higgs field and associated objects meromorphic. At this point there are two possibilities for the residue of Φ . It may be nilpotent, in which case the Higgs field is called *strongly parabolic*, or it may be diagonalizable, in which case it is called *weakly parabolic*. In the case of weakly parabolic bundles the residue data, i. e. the eigenvalues of the residue at each parabolic point shall also be fixed.

In the following we will only consider weakly parabolic bundles of rank 2. Furthermore we restrict ourselves to the $\text{SL}(2, \mathbb{C})$ case. For the definition we state all the fixed data.

Definition 3.1.3

Let E be a complex rank 2 vector bundle over a compact Riemann surface of degree d , together with a divisor $D = p_1 + \dots + p_n$ and holomorphic structure $\bar{\partial}_{\text{Det} E}$ on $\text{Det} E$. Furthermore for each $p \in D$ let a weight vector $\vec{\alpha}(p) = (\alpha_1(p), \alpha_2(p)) \in [0, 1]^2$ with $\alpha_1(p) + \alpha_2(p) = 1$ be given, as well as $\sigma_p \in \mathbb{C}^\times$.

A weakly parabolic $\text{SL}(2, \mathbb{C})$ -Higgs bundle over (\mathcal{C}, D) is a triple $(\bar{\partial}_E, \{\mathcal{F}(p)\}_{p \in D}, \Phi)$, where $\bar{\partial}_E$ is a holomorphic structure on E inducing $\bar{\partial}_{\text{Det} E}$, $\mathcal{F}(p)$ is a complete flag for each $p \in D$ and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes K_{\mathcal{C}}(D)$ is a Higgs field which is traceless and has residue eigenvalues σ_p and $-\sigma_p$ at each $p \in D$.

It is useful to write \mathcal{E} for the holomorphic bundle $(E, \bar{\partial}_E)$ together with the flag, s. t. a parabolic Higgs bundle may be expressed as a pair (\mathcal{E}, Φ) just as in the regular case.

To obtain a well behaving moduli space a notion of stability is necessary. For this one defines the (weight depended) parabolic degree of \mathcal{E} as

$$\text{pdeg}_{\vec{\alpha}}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{p \in D} (\alpha_1(p) + \alpha_2(p)).$$

A parabolic structure on \mathcal{E} induces a parabolic structure on its holomorphic subbundles and so one defines (\mathcal{E}, Φ) to be $\vec{\alpha}$ -stable iff

$$\mu(\mathcal{E}) := \frac{\text{pdeg}_{\vec{\alpha}}(\mathcal{E})}{\text{rank}(\mathcal{E})} > \text{pdeg}_{\vec{\alpha}}\mathcal{L}.$$

for every holomorphic line subbundle \mathcal{L} preserved by Φ . Here $\mu(\mathcal{E})$ is the *parabolic slope* of \mathcal{E} . The notion of semistability follows if instead of $>$ one allows \geq , and the weights are called *generic* if any $\vec{\alpha}$ -semistable bundle is $\vec{\alpha}$ -stable.

For the definition of the moduli space we start by fixing the residues $\sigma = \{\sigma(p)\}_{p \in D}$ and the flags $\mathcal{F}(p)$ for each $p \in D$. We denote by $\text{SL}(E)$ the bundle of automorphisms of E that induce the identity on $\text{Det}(E)$ and by $\mathfrak{sl} = \text{End}_0 E$ the bundle of tracefree endomorphisms. Now denote by \mathcal{A}_0 the affine space of all holomorphic structures on E that induce the fixed holomorphic structure $\bar{\partial}_{\text{Det} E}$ on $\text{Det} E$ and by $\mathcal{P}_{0,\sigma} \subset \Omega^{1,0}(\mathcal{C}, \text{End}_0 E)$ the set of sections of $\text{End}_0 E \otimes K_{\mathcal{C}}(D)$ (i. e. tracefree endomorphisms with simple poles), which are compatible with the flag and have residues $\sigma(p)$ and $-\sigma(p)$ at each $p \in D$.

We then start with the space

$$\mathcal{H} := \{(\bar{\partial}_E, \Phi) \in \mathcal{A}_0 \times \mathcal{P}_{0,\sigma} : \Phi \text{ is meromorphic w. r. t. to } \bar{\partial}_E\}.$$

Now we fix a generic weight vector $\vec{\alpha}(p)$ at each $p \in D$ and denote by $\mathcal{H}_{\vec{\alpha}}$ the subspace of pairs $(\bar{\partial}_E, \Phi) \in \mathcal{H}$ which are $\vec{\alpha}$ -stable. For the gauge group let $\text{ParEnd}(E)$ be the bundle of automorphisms of E that preserve the flag at each $p \in D$. Our gauge group will then be

$$\mathcal{G}_{\mathbb{C}} := \Gamma(\text{SL}(E) \cap \text{ParEnd}(E)).$$

Finally we obtain the moduli space of $\text{SL}(2, \mathbb{C})$ -Higgs bundles $\mathcal{M}^{\text{Higgs}} := \mathcal{H}_{\vec{\alpha}}/\mathcal{G}_{\mathbb{C}}$, which is a Hyperkähler manifold of dimension

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}} = 6(g-1) + 2n$$

for g the genus of \mathcal{C} , cf. [BY96]. For further details see also [Yok93] and [Kon93].

3.2 Kobayashi-Hitchin correspondence

Although we usually talk about the Higgs bundle moduli space the objects we are actually working with are most of the time not the Higgs bundles (\mathcal{E}, Φ) but their associated Higgs pairs $(\Phi, D(\bar{\partial}_E))$, which we will introduce now.

Let (\mathcal{E}, Φ) be a parabolic Higgs bundle. By adding a Hermitian metric h on E the holomorphic structure $\bar{\partial}_E$ canonically induces a connection $D(\bar{\partial}_E, h)$ on E , called the Chern connection. After choosing a base connection $D(\bar{\partial}_E, h)$ can be identified with an endomorphism valued 1-form A and may also be written as d_A . Explicitly, if $\bar{\partial}_E$ is locally $\bar{\partial} + A_{\bar{z}} d\bar{z}$ in an h -unitary frame then d_A is simply given by $D(\bar{\partial}_E, h) = d_A = d + A = d + A_{\bar{z}} d\bar{z} - (A_{\bar{z}})^{*h} dz$ where d is the standard differential. Depending on the situation the connection is referred to as $D(\bar{\partial}_E, h)$, d_A or simply A .

We can now search for a hermitian metric h , such that the pair $(D(\bar{\partial}_E, h), \Phi)$ satisfies Hitchin's equation

$$F_{D(\bar{\partial}_E, h)}^\perp + [\Phi, \Phi^{*h}] = 0, \quad (3.1)$$

where

$$F_{D(\bar{\partial}_E, h)}^\perp - F_{D(\bar{\partial}_E, h)} = \sqrt{-1} \mu(E) \frac{2\pi \varpi_{\mathcal{C}}}{\text{vol}_{g_{\mathcal{C}}}(\mathcal{C})} \text{Id}_{\text{Det} E}.$$

Note that the right hand side matches the curvature of the fixed Chern connection on $\text{Det} E$. In this way $F_{D(\bar{\partial}_E, h)}^\perp$ is the trace-free part of the curvature of $D(\bar{\partial}_E, h)$.

As stated in [Fre+20] in the parabolic setting one expects the Hermitian metric to match the parabolic structure in a suitable sense. For this one has to consider filtrations of the sections of $\mathcal{E} \otimes \mathcal{O}_{\mathcal{C}}(*D)$, for $\mathcal{O}_{\mathcal{C}}(*D)$ the sheaf of algebras of rational functions with poles at D . A parabolic structure determines such a filtration, as does a Hermitian metric, and if these two filtrations coincide the Hermitian metric is said to be *adapted* to the parabolic structure. This can be expressed locally by demanding that near each $p \in D$ one has $h(s, s) \sim |z|^{2\alpha_i(p)}$ for a holomorphic basis of sections e_1, e_2 with $e_i \in F_i(p)$. For further details on this see [Moc06].

With these constraints one can now uniquely solve Hitchin's equation (3.1) in the class of Hermitian metrics h , adapted to the parabolic structure on \mathcal{E} , cf. [Sim90]. Then one obtains one part of the Kobayashi-Hitchin correspondence

$$\mathcal{M}^{\text{Higgs}} \rightarrow \mathcal{M}^{\text{sd}}, \quad (\mathcal{E}, \Phi) \mapsto (\mathcal{E}, \Phi, h).$$

Here \mathcal{M}^{sd} is the space of solutions to Hitchin's equations modulo gauge equivalence, where we consider the gauge group

$$\mathcal{G} = \Gamma(\text{SU}(E) \cap \text{ParEnd}(E)).$$

of h -unitary transformations. We note the action of the gauge group as

$$g \cdot \bar{\partial}_E = g \circ \bar{\partial}_E \circ g^{-1}, \quad g \cdot \Phi = g\Phi g^{-1} \quad \text{and} \quad (g \cdot h)(v, w) = h(gv, gw).$$

The action on $\bar{\partial}_E$ then leads to an induced action on the Chern connection A .

On the other hand one obtains a Higgs bundle from every solution of Hitchin's equation, leading to the full Kobayashi-Hitchin correspondence

$$\mathcal{M}^{\text{Higgs}} \cong \mathcal{M}^{\text{sd}}$$

for Higgs bundles.

The construction above, i. e. considering Hitchin's equation as an equation for the harmonic metric h , is called the holomorphic point of view in [Fre+20]. As we already mentioned in section 2.2, the space \mathcal{M}^{sd} can also be considered in a different but equivalent, way. For this one fixes a hermitian background metric h_0 on E and then considers the equations

$$\begin{aligned} F_A^\perp + [\Phi, \Phi^{*h_0}] &= 0 \\ \bar{\partial}_A \Phi &= 0 \end{aligned} \quad (3.2)$$

as equations for a pair consisting of a unitary connection A on E and the endomorphism valued 1-form Φ . This is called the unitary point of view and pairs (A, Φ) are then called *Higgs pairs*.

Remark 3.2.1

We will mostly work in the holomorphic picture here, but also talk about \mathcal{M}^{sd} as the space of Higgs pairs. This will be justified at the important points in the story and we start here by noting the basic equivalence, which is already explained in the work of FMSW. Suppose (\mathcal{E}, Φ) is a Higgs bundle and h a solution of eq. (3.1). Now for a fixed hermitian metric h_0 on E there is an $\text{SL}(E)$ -valued h_0 -hermitian transformation H such that $h(v, w) = h_0(Hv, w)$. It is now easy to show that the pair $(D(H^{1/2} \cdot \bar{\partial}_E), H^{1/2} \cdot \Phi)$ is a solution to the equations (3.2). Conversely any solution (A, Φ) to (3.2) directly gives us a triple (\mathcal{E}, Φ, h_0) , where the holomorphic structure on E is induced by A . Thus both pictures only differ by a gauge transformation.¹

We have thus introduced the usual notions of the (parabolic) Higgs bundle moduli space. Our construction, however, will only be valid of large Higgs pairs in the following sense. Given a Higgs bundle (\mathcal{E}, Φ) we will not consider equation (3.1) but instead the R -rescaled Hitchin equation

$$F_{D(\bar{\partial}_E, h_R)}^\perp + R^2 [\Phi, \Phi^{*h_R}] = 0. \quad (3.3)$$

for $R > 0$. FMSW obtained precise estimates for the solution h_R of this equation for large R which will be the foundation for our construction. In the regular and strongly parabolic setting such solutions can honestly be considered as large Higgs pairs, since $(\mathcal{E}, R\Phi, h_R)$ is a solution of (3.1) with large Higgs field $R\Phi$ in \mathcal{M}^{sd} . But in our weakly parabolic setting the residues of the Higgs field are fixed, so if Φ is a Higgs field in \mathcal{M}^{sd} for some fixed data, then $R\Phi$ will not be part of that same space for $R \neq 1$. Thus we have to consider the family of moduli space $\mathcal{M}_R^{\text{sd}}$, each of which consists of the solution to the corresponding R -rescaled equation. Note that, as of [Fre+20], the Kobayashi-Hitchin correspondence still holds, although $\mathcal{M}^{\text{Higgs}}$ does not involve the parameter R , i. e. for every Higgs bundle $(\mathcal{E}, \Phi) \in \mathcal{M}^{\text{Higgs}}$ there exists a harmonic metric h_R solving (3.3) and accordingly a Higgs pair (A, Φ) in $\mathcal{M}_R^{\text{sd}}$. This correspondence also justifies why we talk about the construction of the the hyperkähler metric on the Higgs bundle moduli space $\mathcal{M}^{\text{Higgs}}$, although we will actually construct it on $\mathcal{M}_R^{\text{sd}}$. For notational convenience we will from now on write $\mathcal{M} := \mathcal{M}^{\text{Higgs}}$ and $\mathcal{M}_R := \mathcal{M}_R^{\text{sd}}$.

3.3 Spectral data and integrable system

Before we can start with the general idea of the construction of the Darboux coordinates on \mathcal{M}_R we have to introduce a final feature, again following [Fre+20] where additional references are given. To every Higgs bundle (\mathcal{E}, Φ) one can associate the characteristic polynomial $\text{char}_\Phi(\lambda)$ which does not depend on the parabolic structure. This gives rise to the Hitchin map

$$\text{Hit} : \mathcal{M} \rightarrow \mathcal{B}, \quad (\mathcal{E}, \Phi) \mapsto \text{char}_\Phi(\lambda).$$

¹As [Fre+20] is our primary source we point out an important difference in notation. Here we use the Symbol Φ for a parabolic Higgs field in the holomorphic point of view, while FMSW use the symbol φ to denote the Higgs field in this picture. They, on the other hand, use the symbol Φ for the Higgs field in the unitary point of view. As we work mostly in the holomorphic picture we reserve the symbol φ for the matrix part of the Higgs field Φ .

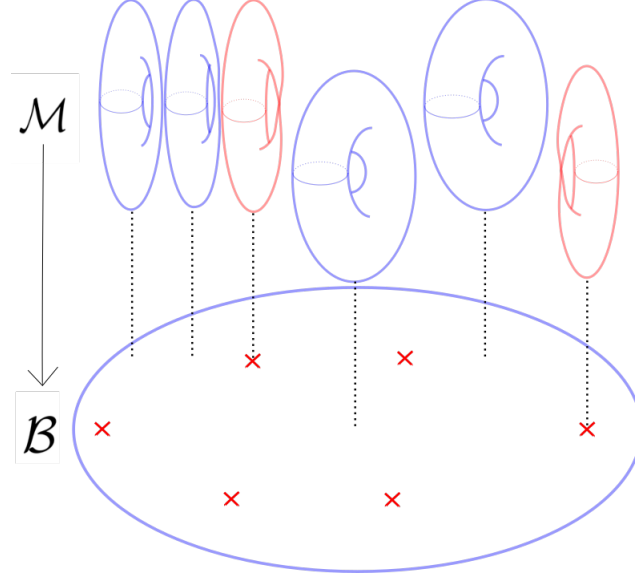


Figure 3.1: A subset of the Hitchin fibration in dimension 4. The red crosses in the Hitchin base signify the elements of the discriminant locus over which the torus fibration becomes nodal.

Here \mathcal{B} denotes the Hitchin base, identified with the affine space of coefficients of $\text{char}_\Phi(\lambda)$, so elements in \mathcal{B} are collections of differential forms. In the case of $\text{SL}(2, \mathbb{C})$ bundles it holds $\text{tr} \Phi = 0$ and thus the characteristic polynomial is just the determinant of Φ .

For weakly parabolic bundles $\text{Det} \Phi$ is a meromorphic quadratic differential on \mathcal{C} with a double pole of the form $-\sigma^2 \frac{dz^2}{z^2}$ near any weakly parabolic point p . One can associate a *spectral curve* $\Sigma \subset \text{Tot}(K_{\mathcal{C}}(D))$ to each point in \mathcal{B} which is a $2 : 1$ ramified cover of \mathcal{C} in $K_{\mathcal{C}}(D)$, where each sheet represents a different eigenvalue of the Higgs field.

The subset $\mathcal{B}' \subset \mathcal{B}$ of differentials in \mathcal{B} that are generic, i. e. have only simple zeroes, are the ones for which Σ is smooth. The following constructions will focus on the subset $\mathcal{M}' := \text{Hit}^{-1}(\mathcal{B}')$ (and the corresponding \mathcal{M}'_R) which is called the *regular locus*. Its complement is called the *discriminant locus*, cf. figure 3.1.

Lastly we want to mention the result going back to [Hit87] for the regular case that the Hitchin map equips \mathcal{M} with the structure of an algebraically completely integrable system. Notably every fiber $\text{Hit}^{-1}(q)$ for each $q \in \mathcal{B}'$ forms a torus which implies the notion of action angle-coordinates Z, θ . Recovering these coordinates will be one of the main parts in the twistorial construction of the hyperkähler metrics.

Chapter 4

Decorated triangulations and their coordinates

In this section we want to show how to build a (version of a) system of coordinates on the moduli space of flat connections on a Riemann surface \mathcal{C} that was introduced by Fock and Goncharov in [FG06]. These coordinates are constructed via the data of triangulations on \mathcal{C} together with a certain choice of decoration at each vertex. One crucial idea of GMN was that the corresponding triangulations can be obtained from quadratic differentials which appear as the determinants of the Higgs fields and that the decorations correspond to solutions of a flatness equation for a certain connection built out of a solution of Hitchin's equations. We stress that the entirety of this construction is a summary of [GMN13b] and no new results are presented in the first three sections.

The theory presented in this section can mostly be considered independently of the theory of Higgs bundles and is interesting in its own right. So ignoring their possible origin as determinants we start in section 4.1 by introducing trajectories for arbitrary quadratic differentials and angles $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, and we explain how a triangulation of \mathcal{C} is built from them. Most of this theory was developed by K. Strebel in [Str84]. We will follow [GMN13b] in showing how to construct a triangulation with these trajectories and which phenomena may occur that are important for us. It should also be mentioned that these constructions and phenomena may be seen as an introduction to the theory of *spectral networks* which were introduced by GMN in [GMN13a]. Since then they have become widely used and generalized especially in the physics literature. The theory shown here is just their simplest example which already entails quite a lot of the general theory's important structure.

In section 4.2 we continue to present the work of [GMN13b], now turning to the construction of the decorations of the triangulation and how to build the Fock-Goncharov coordinate system out of them. To obtain the full picture we will introduce the spectral curve in section 4.3. This is necessary to single out a certain subset of coordinates by use of the homology group of the spectral curve in order to obtain a set of coordinates that can later be identified as a sort of Darboux coordinate system.

Finally we have to consider what happens when we vary ϑ . At certain values of ϑ the triangulation jumps which leads to corresponding jumps in the \mathcal{X} -coordinates. For the construction to work these jumps have to be of a certain kind and accordingly GMN carried out the analysis for the most important jumps in [GMN13b]. Still a certain phenomenon may appear if \mathcal{C} has genus greater than 0 which may pose a problem. We will introduce the important notions in section 4.4 and, using the work of T. Bridgeland and I. Smith [BS15], fill in some small gaps in the general theory.

4.1 ϑ -trajectories and triangulations

We begin this section by recalling the construction of the ϑ -triangulations which were introduced in [GMN13b] where they were called WKB triangulations.¹ In the following \mathcal{C} will always denote a compact Riemann surface. The foundational work for this section is [Str84] where all relevant facts are already given. We present the relevant results in this section and only mention those details which help getting an intuitive understanding of the phenomena that may occur. So nothing new is presented here and the representation mostly follows [GMN13b].

Definition 4.1.1

Given a meromorphic quadratic differential q on (a subset of) \mathcal{C} and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ a ϑ -trajectory of q is a curve $\gamma : I \rightarrow \mathcal{C}$ for some interval $I \subset \mathbb{R}$, s. t.

$$q(\gamma', \gamma') \in e^{2i\vartheta} \mathbb{R}_{>0}.$$

Alternatively, if q has a well defined square root $q^{1/2}$ a ϑ -trajectory of q may be defined as any curve $\gamma : I \rightarrow \mathcal{C}$ for some interval $I \subset \mathbb{R}$, s. t.

$$q^{1/2}(\gamma') \in \mathbb{R}^\times e^{i\vartheta}.$$

Note that the second definition is more in line with calling the curve ϑ -trajectory and is well defined, although the square root of q is only defined up to sign. This amounts to the fact that the ϑ -trajectories are not oriented, i. e. if $\gamma(t)$ is a ϑ -trajectory for q , so is $\gamma(-t)$ going in the opposite direction.

As we will explain below ϑ -trajectories can be considered as straight lines with the inclination given by ϑ if considered in a suitable neighborhood. When talking about ϑ -trajectories for varying ϑ it is also useful to have a notion that is independent of the specific angle and just reflects that each one is of such a constant angle.

Definition 4.1.2

Let q be a meromorphic quadratic differential. A curve $\gamma : I \rightarrow \mathcal{C}$ shall be called a ca -trajectory of q if there exists some $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ s. t. γ is a ϑ -trajectory.

One important aspect of ca -trajectories is their behavior near zeroes and poles of the quadratic differential, which are also called critical points, cf. [FO08]. In the work of GMN the zeroes of q are also called turning points and we will use this notion at some places. In the following we are interested in a certain behavior that is exhibited by the ca -trajectories only if they have a specific kind of critical points, so we restrict ourselves to these differentials that are exactly the ones in the regular locus of the Hitchin base. Note however that a lot is also known about the ca -trajectories for differentials in the

¹Here and in the following we will use the notions ϑ -trajectory, triangulation, etc. instead of WKB curve, triangulation, etc. which were used in [GMN13b]. This is in part due to the fact that it is somewhat more in line with the mathematical literature concerning quadratic differentials, but also as of the fact that "WKB" is a reference to a certain approximation method used in [GMN13b]. It is a rather important part of the results presented here that we refrain from using the WKB method. So while it may be inconvenient to differ from the language introduced in the primary source for this construction it would be mathematically misleading not to do so.

discriminant locus, so even though the following structures break down at these points, one may be able to obtain at least some information as to how the structure breaks down in further studies.

Definition 4.1.3

A *generic differential* is a meromorphic quadratic differential q on \mathcal{C} , which has only poles of order 2 and simple zeros.

When talking about certain standard behaviors (e. g. as growing or decaying) along a ca -trajectory it is useful to fix a standard for the parametrization, which we can always obtain.

Lemma 4.1.4

If q is a generic differential there always exists a parametrization of its ϑ -trajectories in which

$$q^{1/2}(\gamma') = \pm e^{i\vartheta}.$$

This shall be called the *standard parametrization* of γ .

We now record the local structure of ca -trajectories near the different points we will encounter.

Lemma 4.1.5

Let q be a generic differential, γ a ϑ -trajectory and p one of the (second order) poles of q . Then γ has one of the following forms in a local neighborhood around p :

- γ is a logarithmic spiral running into (or away from) p .
- γ is a circle with center p .
- γ is a straight line running into p .

Proof. Since q is a meromorphic quadratic differential it is known from [Str84] that there exists a coordinate neighborhood z with $z(p) = 0$ and $q = (M/z^2)dz^2$ for some complex number M . By shrinking our neighborhood we may thus assume, that $q^{1/2}$ is of the form $\frac{m}{z}dz$ for some constant m . Then we obtain the following general form of the ϑ -trajectory in the local coordinate:

$$z(t) = z_0 e^{\pm \frac{e^{i\vartheta}}{m} t}.$$

Here z_0 is some complex constant. Depending on the specific values of m and ϑ one of the three asserted possibilities occurs. □

For more details on this see [GMN13b] or, more generally, the original work of Strebel [Str84]. Here we note that in such a neighborhood for any given ϑ there are infinitely many ϑ -trajectories running into the pole. This is contrasted by the ϑ -trajectories running into the other exceptional points.

Lemma 4.1.6

Let q be a generic differential and p one of its (simple) zeros, i. e. a turning point. Then for any $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ there are exactly three ϑ -trajectories that run into the turning point.

Proof. This can be shown by switching to a coordinate where $q = Mzdz^2$ and integrating the defining differential equation. \square

Finally we are interested in the behavior away from any critical point.

Lemma 4.1.7

Let q be a generic differential and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. Then in a local neighborhood around any regular point p of q the ϑ -trajectories form a foliation by straight lines.

Proof. In the coordinates $w(z) := \int_{z_0}^z q^{1/2}$ the ϑ -trajectories are a foliation by straight lines. \square

Thus the behavior near every point of \mathcal{C} is known and we obtain the following global picture.

Corollary 4.1.8

Let q be a generic differential and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. Then the ϑ -trajectories are the leaves of a (singular) foliation of \mathcal{C}

With this picture in mind we now classify the different kinds of ca -trajectories that may occur in the following list that exhausts all the possibilities.

Definition 4.1.9

Let γ be a ϑ -trajectory for generic differential q . It is called

- *generic*, iff it converges in both directions to a pole of q (which may be the same for both directions).
- *separating*, iff it converges in one direction to a zero of q and in the other direction to a pole of q .
- a *saddle*, iff it converges in both directions to a zero of q (which may be the same for both directions).
- *periodic*, iff it is diffeomorphic to a circle and does not converge to any pole or zero of q .
- *divergent*, in all other cases. In this case γ does in at least one direction not converge to a pole or zero of q .

As the name suggests almost all ca -trajectories in our setting will be generic and indeed most of our calculations will be done along such trajectories. Of particular importance are saddle trajectories, to which we will also refer as *saddle connections*. An example of a saddle connection that connects two different zeros is depicted in the middle of figure 4.4. A saddle connection may also converge towards the same zero in both directions, forming (together with the zero of q) a simple closed curve on \mathcal{C} . Additionally, a simple closed

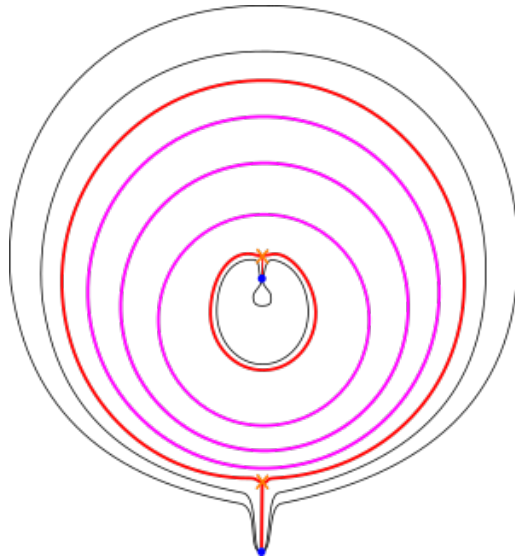


Figure 4.1: A ring domain. Orange crosses indicate a zero and blue dots a pole of q . The boundary is formed by saddle trajectories depicted in red, while the interior is filled with periodic trajectories.

curve can also be formed by two or more saddle connections $\gamma_1, \dots, \gamma_n$ together with $n - 1$ zeros of q . If two such simple closed curves form the boundary of a domain in \mathcal{C} in which no other pole or zero of q lies, then the domain is filled with periodic trajectories and is called a *ring domain*. Figure 4.1 depicts such saddle connections that converge to the same zero and form the boundary of a ring domain.

Later on we will also need the following observation which we may already infer from the local behavior near their "end points" using standard methods for ODEs.

Lemma 4.1.10

A generic ϑ -trajectory in standard parametrization can be extended to all of \mathbb{R} .

The infinite domain of the curve leads to the necessity of using integral equations with initial values at infinity later on as infinity corresponds to the poles of the quadratic differential.

Before we continue with the construction of the triangulation note the following remarks for some intuition concerning ca -trajectories:

First remember that $q^{1/2}$ was only defined up to sign, which is why the ca -trajectories are not canonically oriented. The description of the behavior around poles as spiraling into the point may still be useful as a reminder of what happens. Phenomenologically one might describe the singularities of q as some (massive) object which "captures" all curves that come near enough. Thus finite curves have to be "rare". Indeed, for one fixed ϑ their number is limited by the number of turning points and, even more, a curve coming from one turning point is more "likely" to run into a singularity than another turning point. But the parentheses here are necessary when we talk about ca -trajectories of general quadratic differentials. For example one can show that on the four-punctured sphere with a quadratic differential having only simple poles, for some angles only two finite ca -curves and one divergent curve exist which fill up the whole sphere. The angles

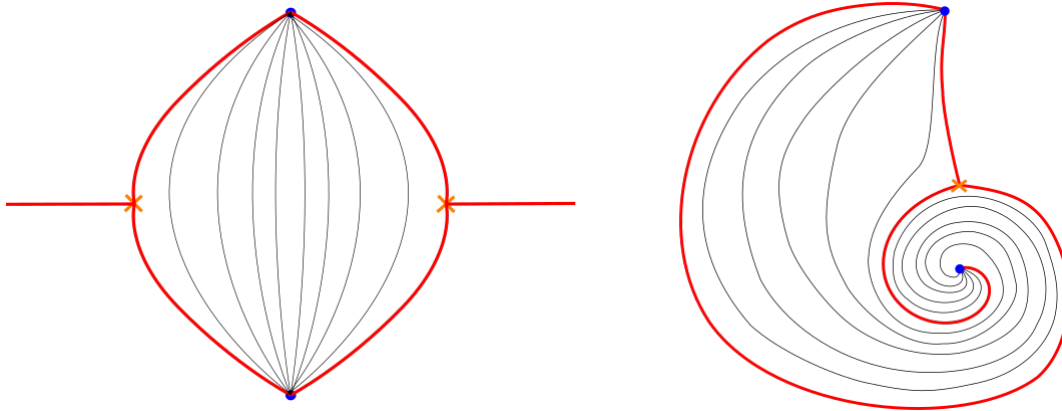


Figure 4.2: A diamond and a disc. Orange crosses indicate a zero and blue dots a pole of q . Red lines represent separating trajectories, while black lines represent generic trajectories.

for which such a behavior occurs even form a dense subset of $\mathbb{R}/2\pi\mathbb{Z}$. However, by working with generic quadratic differentials, the generic trajectories may safely be regarded as the ones which "usually" occur, while finite curves don't exist for most angles.² This is why the following fact is crucial which was shown by GMN and which specifies some of the vague formulations above. Further clarification about this will follow in section 4.4.

Proposition 4.1.11

Let q be a generic differential and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ such that there is no finite ϑ -trajectory. Then there is also no divergent ϑ -trajectory.

Proof. The proof relies on the existence of a singularity, since in that case a ca -trajectory can't fill up the whole complex curve, as it would "fall into the singularity". For more details see [GMN13b]. \square

As the case of no finite trajectory is the standard one for the following triangulations, we also denote them as such.

Definition 4.1.12

Let q be a generic differential. If $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ is such that there is no finite trajectory, then ϑ is called a *generic angle*.

It follows that in this generic case the foliation by ϑ -trajectories consists of a finite collection of separating trajectories and an infinite amount of generic ones, so the following structure emerges:

The separating trajectories form boundaries of "cells" on \mathcal{C} which are swept out by generic trajectories. The boundaries consist of at most four separating trajectories (which may be regarded as the generic case), two of which emanate from a common turning point. This is called a "diamond" in [GMN13b]. There may however be cells whose boundary consists only of three separating trajectories which all emanate from the same turning point. This may be regarded as a degenerate case, called a "disc" in [GMN13b], which

²In fact the appearance of finite curves is the defining property for the definition of the DT invariants in the general Riemann-Hilbert we will later turn to.

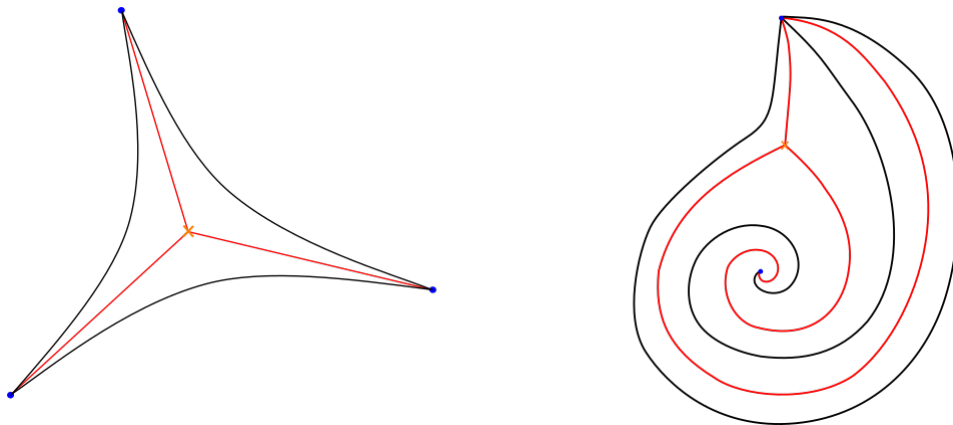


Figure 4.3: A generic triangle on the left and a degenerate triangle on the right.

will require a bit more attention later on, as the constructions need to be aware of this special case, cf. figure 4.2. For understanding (and visualizing) the theory however, it is enough for now to think of the foliation as forming those swept out diamonds on all of \mathcal{C} .

With this structure in mind it is possible to define the ϑ -triangulation.

Definition 4.1.13

Let q be a generic differential and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ a generic angle. A ϑ -triangulation is a collection of generic ϑ -trajectories, such that every two ϑ -trajectories belong to two different cells given by the separating trajectories, and such that in each cell a generic trajectory is chosen.

One can now look at the emerging structure and notice that, corresponding to the discs above, there may be degenerate triangles with only two distinct edges, cf. figure 4.3. Counting these still as triangles one obtains the following result, cf. [GMN13b].

Proposition 4.1.14

A ϑ -triangulation is a triangulation of \mathcal{C} , in which each face contains a single turning point.

Such a triangulation is of course not unique as there is family of generic ϑ -trajectories in each diamond or disc. When talking about these triangulation one uses the fact that two different ones differ only by an isotopy, which won't be relevant for the following procedure. As GMN point out, one really chooses an isotopy class of triangulations.

Remark 4.1.15

Typically our constructions in the following sections will only consider the case of generic triangles. However, the notion of degenerate triangles has to be included for the full picture as there is no way to make sure that these triangles do not occur in our cases. GMN have shown in [GMN13b] how all of the constructions that we consider here and depend on the triangulation can be adapted to the case of degenerate triangles. We will only briefly consider this case later on as no real problems arise in their treatment.

4.2 Decorations and Fock-Goncharov coordinates

Thus far we have considered a compact Riemann surface \mathcal{C} and constructed a triangulation for every generic differential and generic angle on \mathcal{C} . This is one half of the data one needs to construct the Fock-Goncharov coordinate system, the other being a decoration at each vertex, for which we now have to consider a rank-2 complex vector bundle E over \mathcal{C} . Though the moduli space we are interested in is the space of solutions of Hitchin's equations, the coordinates are constructed on the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ -connections $\mathcal{M}^{\mathrm{dR}}$. Later on we will identify the moduli spaces and thus pull back the coordinates, but for explaining how the construction works we stay on $\mathcal{M}^{\mathrm{dR}}$. The construction of [GMN13b] now proceeds as follows.

Definition 4.2.1

Let $E \rightarrow \mathcal{C}$ be a complex rank-2 vector bundle and \mathcal{A} a flat $\mathrm{SL}(2, \mathbb{C})$ -connection on E with a regular singularity at a point $p \in \mathcal{C}$ with monodromy operator M . A *decoration* at p is a choice of one of the two eigenspaces of M . The corresponding eigenvalue is denoted by μ .

A *decorated triangulation* is a triangulation T together with a decoration for each vertex of T . We denote the corresponding decorated triangulation by $T(q, \vartheta)$ where, when no ambiguity may arise, we may suppress the dependence of T on q and ϑ .

Note that this definition differs slightly from the original construction by Fock and Goncharov as they considered flags instead of decorations. Furthermore, as GMN point out, in the case at hand the conjugacy classes of the monodromy operators M are fixed for the moduli space, so they are part of the discrete datum of the triangulation, while Fock and Goncharov considered varying M for the connections.

We now have the data we need to build the coordinate system. For a given edge e of the triangulation consider the two triangles which have this edge in common. They make up a quadrilateral Q_e and their vertices p_i shall be numbered in counter-clockwise order (for \mathcal{C} canonically oriented). For each p_i there is a decoration which corresponds to a solution v_i of the flatness equation

$$\mathcal{A}v_i = 0. \quad (4.1)$$

The v_i may be regarded as eigenvectors of the monodromy M_i with eigenvalue μ_i . They are only defined up to a scalar multiple, but this ambiguity will vanish in the following step. Note that as \mathcal{A} is flat the sections exist on any simply connected subset of $\mathcal{C} \setminus \{p_1, \dots, p_n\}$.

For two of the sections v_i and v_j the \wedge -product $v_i \wedge v_j$ is an element of the determinant line bundle $\mathrm{Det}E$. If $\mathcal{A}_{\mathrm{Det}}$ denotes the induced connection on $\mathrm{Det}E$, it follows that the \wedge -product solves the induced flatness equation

$$\mathcal{A}_{\mathrm{Det}}(v_i \wedge v_j) = 0.$$

When considering a frame (b_1, b_2) on Q_e , so $v_i = s_1^i b_1 + s_2^i b_2$ and $v_j = s_1^j b_1 + s_2^j b_2$ the \wedge -product is locally given as

$$v_i \wedge v_j(z) = (s_1^i s_2^j - s_2^i s_1^j) b_1 \wedge b_2.$$

We may thus identify $v_i \wedge v_j$ with the Wronskian of the (local expression of the) two solutions of the flatness equation (4.1) evaluated at some point z in Q_e . As all of our

calculations later on are made in a specific frame we will always use this interpretation, which we will also write as

$$s^i \wedge s^j = \det [s^i(z), s^j(z)].$$

If \mathcal{A} is represented by the matrix A in the frame F , i. e. $\mathcal{A} = d + A$ for the standard differential d , the induced equation on $\text{Det } E$ is given by the trace of A :

$$d(s^i \wedge s^j) = \text{tr}(A)(s^i \wedge s^j). \quad (4.2)$$

Using this identification it is possible to consider the quotient of two \wedge -products of sections for v_i with local expressions s_i as

$$\frac{v_1 \wedge v_2}{v_3 \wedge v_4}(z) := \frac{s^1 \wedge s^2}{s^3 \wedge s^4}(z)$$

evaluated at some point $z \in Q_e$. Using the fact that both \wedge -products are solutions to the same 1-dimensional linear differential equation (4.2) one obtains that $\frac{v_1 \wedge v_2}{v_3 \wedge v_4}$ is actually independent of the evaluation point z . Furthermore, as of $Gv_i \wedge Gv_j = \det(G)v_i \wedge v_j$ for any $G \in \text{GL}(2, \mathbb{C})$ the quotient is independent of transformations $G \in \text{GL}(2, \mathbb{C})$. We are now able to define the following functions for each edge.

Definition 4.2.2

Let $E \rightarrow \mathcal{C}$ be a complex rank-2 vector bundle and \mathcal{A} a flat $\text{SL}(2, \mathbb{C})$ -connection on E with regular singularities and $T(q, \vartheta)$ a decorated triangulation. Let e be an edge of T with corresponding quadrilateral Q_e with vertices $p_1, p_2, p_3, p_4 \in \mathcal{C}$ and corresponding flat sections v_i with local expressions s_i in some frame on Q_e . Define the *canonical coordinate* for e as

$$\mathcal{X}_e^{T(q, \vartheta)} := -\frac{(v_1 \wedge v_2)(v_3 \wedge v_4)}{(v_2 \wedge v_3)(v_4 \wedge v_1)} := -\frac{s^1 \wedge s^2}{s^4 \wedge s^1} \frac{s^3 \wedge s^4}{s^2 \wedge s^3}.$$

As promised the ambiguity in the choice of the v_i vanishes in the final expression and just like the single quotients the full function is independent of transformations $G \in \text{GL}(2, \mathbb{C})$ when taken for all sections simultaneously.

We may now conclude this section with the main statement for our construction. The \wedge -product may become 0 for two adjacent vertices, which would lead to the corresponding coordinate function becoming 0 or ∞ , but, as GMN point out, this can only happen in a co-dimension-1 subvariety of \mathcal{M}^{dR} . One can see this in the local expression as the \wedge -product becomes 0 iff the sections are linearly dependent. This characterization will be our main tool in determining whether our choice of sections later on is appropriate. In this case we finally obtain the desired coordinates, though we have to select a certain set out of all the defined functions.

Proposition 4.2.3

The \mathcal{X}_e^T contain a holomorphic coordinate system on a Zariski open patch $\mathcal{U}_T \subset \mathcal{M}^{\text{dR}}$.

Proof. The original proof is found in [FG06]. In appendix A of [GMN13b] it is shown how the data of the monodromy representation of \mathcal{A} is encoded in the \mathcal{X}_e^T . \square

In the following we will often suppress the indices when considering these and related coordinates in the general picture and simply talk about the \mathcal{X} -coordinates or \mathcal{X} -functions.

Remark 4.2.4

Let us briefly remark on the word "contain" here. As said above we do need to select some coordinates as the constructed ones are too many, which can be seen as follows. As we have obtained one \mathcal{X} -function for each edge of the triangulation we can count them using Euler's formula. We then see that for a Riemann surface \mathcal{C} of genus g with n singular points we obtain $6g - 6 + 3n$ coordinate functions. The dimension of \mathcal{M}^{dR} however is $\dim \mathcal{M}^{dR} = 6g - 6 + 2n$. The difference reflects the fact that we consider the moduli space with fixed monodromy eigenvalues at the singular points. As GMN show, n combinations of the \mathcal{X}_E functions give the monodromy eigenvalues $(\mu_i)^2$. As these are fixed the information is superfluous, so that we can discard n of the functions (one for each singular point) and obtain the correct number of coordinates. At the moment however there is no way of distinguishing which functions to keep and which to discard. This is done in the next section.

Remark 4.2.5

As is also remarked upon in appendix A of [GMN13b] the coordinates of Fock and Goncharov are at first defined on the space of $\mathrm{PSL}(2, \mathbb{C})$ connections, so they have to be pulled back to the space of $\mathrm{SL}(2, \mathbb{C})$ connections. In the case of $\mathcal{C} = \mathbb{CP}^1$ this can be done in a unique way, but for larger genus the moduli space of $\mathrm{SL}(2, \mathbb{C})$ connections is a discrete cover of the moduli space of $\mathrm{PSL}(2, \mathbb{C})$ connections. This will not be of much concern to us. On the one hand there is the possibility to work directly with $\mathrm{PSL}(2, \mathbb{C})$ connections, cf. [Tes11] and references therein. But also when working with $\mathrm{SL}(2, \mathbb{C})$ connections as is done here, we can simply use the constructed coordinates as a complete set of coordinates on each sheet of the covering which is enough for our construction.

4.3 Homology and lattices

As explained in the last section, it is necessary to select a certain subset of the constructed functions to obtain the correct coordinates. Additionally, later on the hyperkähler metric will be obtained from a symplectic form that is constructed from these coordinates we just described. So our choice of coordinates also has to ensure that the emerging form is actually symplectic. Such a selection is possible via the use of certain lattices, one of which carries a symplectic pairing. We will now summarize the important facts from [GMN13b] about these lattices, how they correspond to the edges and finally how they select the right coordinates.

All of the lattices we consider are given by the spectral data, i. e. we have to consider the spectral curve. As we are dealing with differentials with singularities we have to be careful as to whether we include or exclude the poles and unfortunately the notation is not canonical throughout the literature. For our use of symbols so far the following notation seems appropriate.

Definition 4.3.1

Let q be a generic differential with n singular points at p_1, \dots, p_n on the compact Riemann surface \mathcal{C} . Denote by $\mathcal{C}' := \mathcal{C} \setminus \{p_1, \dots, p_n\}$ the punctured surface. Then the *spectral curve* is defined as

$$\Sigma'_q := \{(z \in \mathcal{C}', \lambda \in T_z^* \mathcal{C}) : \lambda^2 = q(z)\} \subset T^* \mathcal{C}.$$

The spectral cover is a noncompact complex curve which is smooth for generic q . It is a double covering $\pi' : \Sigma'_q \rightarrow \mathcal{C}'$ which is branched over the zeroes of q . With this notation Σ'_q is "punctured over the singular points p_i " and there is a corresponding compactification Σ_q of Σ'_q which is obtained by filling in these punctures. The compactification also has a corresponding projection $\pi : \Sigma_q \rightarrow \mathcal{C}$. Depending on the question at hand we might also talk about Σ_q as the spectral curve or don't distinguish Σ_q and Σ'_q at all if not necessary. However it is necessary for the next step, where we consider their homology groups.

We start by noting that Σ'_q is equipped with an involution $i : (z, \lambda) \mapsto (z, -\lambda)$ that acts accordingly on cycles in the first homology group.

Definition 4.3.2

Let q be a generic differential with n singular points at p_1, \dots, p_n on the compact Riemann surface \mathcal{C} . The *charge lattice*³ Γ_q is defined as the subgroup of $H_1(\Sigma'_q; \mathbb{Z})$ that is odd under the involution i .

When varying q one can think of Γ_q as the fiber of a local system Γ over the space of differentials. Note that Γ_q is equipped with the intersection pairing $\langle \cdot, \cdot \rangle$ which may be degenerate.

Definition 4.3.3

Let Γ_q be the charge lattice of a generic differential q with intersection pairing $\langle \cdot, \cdot \rangle$. The *flavor lattice* Γ_q^f is defined as the radical of $\langle \cdot, \cdot \rangle$. The quotient $\Gamma_q^g := \Gamma_q / \Gamma_q^f$ is called the *gauge lattice*.

Note that Γ_q^g does not change with varying q , so it can be regarded as a fixed lattice Γ^g instead of a local system of lattices. Most important for us it is that the gauge lattice Γ_q^g , which can be identified with $H_1(\Sigma; \mathbb{Z})$, also carries the intersection pairing which is skew-symmetric and nondegenerate as of the quotient construction. This is important as it allows us to obtain a symplectic basis.

Proposition 4.3.4

Let Γ_q^g be the gauge lattice of a generic differential q . Then there exists a symplectic basis $\gamma_1^e, \dots, \gamma_r^e, \gamma_1^m, \dots, \gamma_r^m$ of Γ_q^g , i. e.

$$\langle \gamma_i^e, \gamma_j^e \rangle = 0, \quad \langle \gamma_i^m, \gamma_j^m \rangle = 0, \quad \langle \gamma_i^e, \gamma_j^m \rangle = \delta_{ij}.$$

We want to use this structure, so we need to identify charges $\gamma \in \Gamma_q$ with edges e of the triangulation. We only sketch this process here and refer to [GMN13b] for the details. The idea is, that for each edge the corresponding quadrilateral Q_e contains two zeroes of q .

³The terminology in this part is mostly in accordance with the physical theory from which the indices e and m below are also taken, as they refer to so called electric and magnetic charges.

Thus we could try to lift a loop around these two points to a cycle in Σ'_q . The problem with this is that there are ambiguities in the orientation of the loop as well as the choice of the sheet of the covering. These choices need to be canonically decided which is done by GMN by using a canonical orientation for the lift of the edges and a pairing of $H_1(\Sigma'_q; \mathbb{Z})$ with the relative homology group $H_1(\Sigma'_q, \{p_i\}; \mathbb{Z})$. A corresponding construction also works for degenerate edges and in this way one obtains a canonical cycle γ_e for each edge e of the triangulation T and GMN proceed by showing that these cycles suffice.

Proposition 4.3.5

Let Γ_q be the charge lattice of a generic differential q and $T(q, \vartheta)$ a triangulation for a generic angle ϑ . Then the set of cycles $(\gamma_{e_i})_i \subset \Gamma_q$ obtained from the edges in the way described above generate Γ_q .

With these identifications at hand we may now define the coordinate functions on the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ -connections labeled by charges.

Definition 4.3.6

Let q be a generic differential, $T(q, \vartheta)$ a decorated triangulation and Γ_q the charge lattice for q with the generating cycles $(\gamma_{e_i})_i$ associated to the edges $(e_i)_i$ of the triangulation. For each such e_i we define

$$\mathcal{X}_{\gamma_{e_i}}^{(q, \vartheta)} := \mathcal{X}_{e_i}^{T(q, \vartheta)}.$$

For the sum of two charges $\gamma, \gamma' \in \Gamma_q$ we define

$$\mathcal{X}_{\gamma+\gamma'}^{(q, \vartheta)} := \mathcal{X}_{\gamma}^{(q, \vartheta)} \mathcal{X}_{\gamma'}^{(q, \vartheta)}.$$

Thus $\mathcal{X}_{\gamma}^{q, \vartheta}$ is well defined for all elements $\gamma \in \Gamma_q$. In constructing the hyperkähler metric we won't actually need the \mathcal{X}_{γ} -functions for charges γ that are not in the symplectic bases. We added the definition here as it amounts to a certain Poisson structure which is relevant in the more abstract setting of integrable systems, for which our case may be thought of as a specific example. For an overview of this setting we refer to [Nei13]. Depending on the context we will suppress the dependence on q, ϑ or γ in the notation and simply talk about the \mathcal{X} -coordinates.

Finally we turn to the matter of dimension. Remembering that we had n more coordinate functions than needed when labeled by the edges we accordingly have the same redundancy when we label them by generators of Γ_q . At this point the gauge lattice Γ_q^g comes into play. Not only does this lattice carry the symplectic pairing we will need later on. It is also of the correct rank: When we identify Γ_q^g as the first homology group $H_1(\Sigma_q; \mathbb{Z})$ of the unpunctured spectral curve Σ we notice that by filling in the punctures of Σ'_q we lose exactly those generators in $H_1(\Sigma'_q; \mathbb{Z})$ that correspond to cycles around the preimage of the singular points $\pi^{-1}(p_i)$. Thus we lose exactly those cycles which correspond to the fixed monodromy at the singular points and it remains the true coordinate system for $\mathcal{M}^{\mathrm{dR}}$.

At this point the general construction of the coordinate system is finished, although some ambiguity remains which will only be resolved when we apply it to our case. While the quadratic differential q has a natural interpretation in our setting as the determinant of the Higgs field, the angle ϑ seems arbitrary. In our application this parameter will be related to another parameter $\zeta \in \mathbb{C}^\times$ that is built into the flat connection \mathcal{A} and which

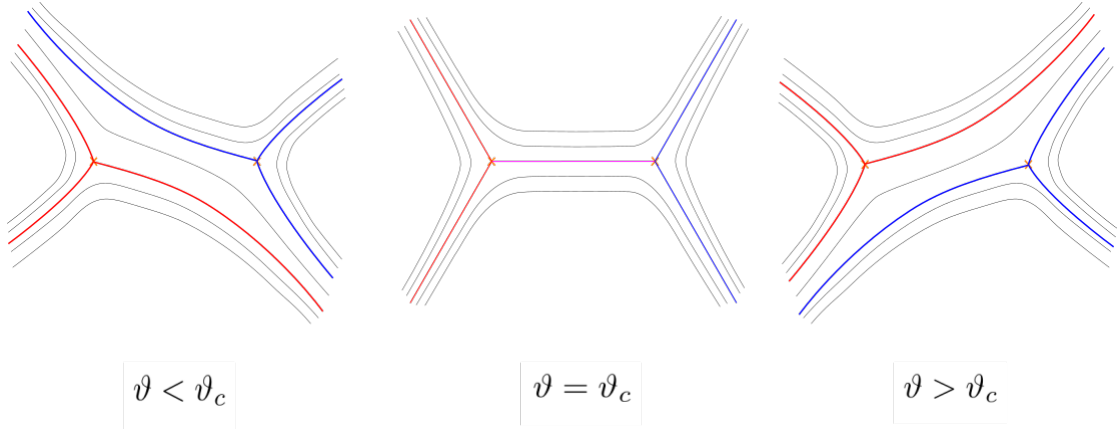


Figure 4.4: A pop in the triangulation.

will encode the complex structure in an application of Hitchin’s twistor theorem. For the most part of the following constructions we will focus on a general relation between the ϑ and ζ which allows for them both to vary in relation to another. This is done as many of our results hold for this general setting and a very large part of the theory developed by GMN concerns the relation of these two parameters. Still, when it comes to constructing the hyperkähler metric we will need to fix ϑ as it is not intrinsic to the moduli space. We will do this by setting $\vartheta = \arg(\zeta)$ which will obey all relations we demand throughout the calculations. But this implies that we can’t constrain ourselves to generic angles later on, because the twistor theorem demands to consider all elements $\zeta \in \mathbb{CP}^1$. So we now have to consider how the triangulation behaves when a finite trajectory appears.

4.4 The spectrum of a quadratic differential

In this section we consider again the general setting of the triangulation, i. e. we consider some generic quadratic differential q , but now we vary the angles ϑ to study the changes of the associated ϑ -triangulations. It turns out that the changes are mostly trivial, as for generic ϑ , i. e. such that no finite trajectories occur, the edges of the triangulation only vary by a continuous homotopy. Such variations don’t change the isotopy class of the triangulation and the coordinates will therefore be constant as functions of ϑ as long as no finite trajectories appear. If they do occur, however, there are in fact discontinuous jumps, cf. the footnote in [GMN13b], p. 312. Understanding the behavior of the triangulations and coordinates as these jumps occur is one of the most important aspects of the work in [GMN13b]. From now on, an angle ϑ at which such a jump occurs shall be denoted by ϑ_c as for critical phase, cf. figure 4.4.⁴ GMN present three different ways in which the triangulation can change at a critical phase, which are denoted as flips, pops and juggles.

⁴The appearance of finite ca -trajectories is intricately linked to the notion of generalized DT invariants which are of utmost importance for the generalization of the construction presented here to general integrable systems. We will return to these invariants in section 8.

Definition 4.4.1

Let ϑ_c be a critical phase for which a finite ϑ_c -trajectory occurs.

1. If the curve connects to two different turning points, the corresponding change in the triangulation shall be called a *pop*.
2. If the curve is closed and surrounds one singular point, the corresponding change in the triangulation shall be called a *flip*.
3. If the curve is closed and does not contract to a single point the corresponding change in the triangulation shall be called a *juggle*.

GMN show that the coordinates actually don't change when a pop occurs, so the only relevant jumps for that aspect are the flips and juggles. The property we need now is that although the coordinates jump, a certain 2-form ϖ that is built out of the derivatives of the \mathcal{X} -coordinates is well defined.⁵ This works when the jumps do not change ϖ , and GMN devote a great effort to proof that this is the case for the most important jumps in [GMN13b]. However there may be jumps for which their argument does not work, so we have to talk about what is rigorously known now. GMN showed that the jumps which correspond to the appearance of a single saddle connection, i. e. pops, are of the right nature and one kind of jumps which corresponds to the appearance of ring domains, i. e. certain juggles, are also fine.

Unfortunately, as GMN themselves note in section 7.6.2 *Symplectomorphisms from a BPS vectormultiplet* of [GMN13b], a certain phenomenon of the ca -triangulation may occur on a Riemann surface \mathcal{C} with genus $g > 0$. On such a \mathcal{C} a ring domain may cut off a torus without critical points, i. e. the complement of the ring domain is given by two disjoint subsurfaces, one of which has genus one and does not contain any zeros or poles of q . In that case their direct calculation of the jump of the coordinates does not work. Instead they refer to their work on the more general concept of spectral networks in [GMN13a] for this case. There they examine differentials which come from a degree K polynomial instead of one with degree 2 as for our generic differentials here. This leads to a lot of additional language which we can't properly introduce here, but most importantly they study the jumping behaviour of a certain *formal generating function* F of formal variables X_γ . The idea is that our \mathcal{X} -coordinates can be considered as a special case of these variables and their argument there in section 7.2 *Closed loops* thus delivers the missing case. But there are some problems with this argument, one being the quite different approach which is not easily translated into the setting here. More importantly though, even if the argument for that case works there may still be a missing case, when one of the boundaries of the ring domains consists of more than one saddle connection.

Now to make these problems precise it is useful to start with the work of T. Bridgeland and I. Smith. In section 5 of [BS15] they first show what kind of ring domains can occur. It turns out that, as stated above, most of these were shown to work in [GMN13b]. The only possibility that has not been rigorously proven is given by the occurrence of a ring domain that cuts off a torus without critical points. Thus we now want to concentrate on those differentials where such a phenomenon does not occur. To formulate this condition it is useful to first introduce the spectrum of a differential.

⁵We will introduce ϖ properly in chapter 9.

Definition 4.4.2

For q a generic quadratic differential, the *spectrum* of q is the set of all ring domains and all saddle-trajectories for any angle $\vartheta \in [0, 2\pi)$.

The elements of the spectrum that may cause problems are ring domains that cut off a torus on which there are no critical points. This case may be pictured accordingly as a torus without any relevant structure and which has a very intricate ring structure attached to one side of it. We hope that the following term captures this picture.

Definition 4.4.3

Let q be a generic differential. A ring domain that cuts of a torus without any singularities shall be called a *signet ring*. If the spectrum of q does not contain a signet ring, it shall be called *unsealed*.

Note that in particular on the sphere (with an arbitrary amount ≥ 4 of punctures) every generic differential has an unsealed spectrum, so the theory considered here is not vacuous.

For the upcoming discussion we identify the finite ca -trajectories with elements of the charge lattice. Following [Nei17] we note that the closure of the lift of a saddle connection for q to the spectral cover Σ'_q is an oriented loop, while the lift of a closed loop to the spectral cover is the union of two oriented loops.

Definition 4.4.4

Let q be a generic differential and c a finite ca -trajectory for q . The class in Γ_q of the corresponding loop (for a saddle connection) or the union of the two loops (for a ring domain) on Σ'_q is called the *associated charge*.

Note that these associated charges correspond to the elements of Γ_q that were identified with the edges of a ca -triangulation in section 4.3. This identification now allows us to see that saddle connections correspond to specific angles of the ca -triangulation. For this we consider the period map⁶ $Z_\gamma(q^{1/2}) = \int_\gamma q^{1/2}$, for $\gamma \in \Gamma_q$ and $q^{1/2}$ denoting the tautological 1-form on $T^*\mathcal{C}$.

Lemma 4.4.5

For a ϑ -saddle connection of q with associated charge γ it holds $e^{-i\vartheta} Z_\gamma(q^{1/2}) \in \mathbb{R}_+$.

Proof. This follows directly from integrating the differential along the ϑ -trajectory. \square

Thus it is clear, that proportional elements of Γ_q correspond to the same ray in the ζ -plane. Many of the general properties proven in [BS15] now only hold, when there are no non-proportional elements defining the same ζ -ray.⁷ Here we focus notationally on the complement of this set, which are elements of the *walls of marginal stability* in the work of GMN.

⁶This is sometimes also called the (central) charge function.

⁷Note that a quadratic differential with this property is called *generic* in [BS15]. But we already use *generic* as the notion for a differential with only simple zeroes and we will continue to do so in this section.

Definition 4.4.6

A generic quadratic differential with unsealed spectrum shall be called a *brick* if there exist distinct saddle connections of q such that for their associated elements γ_1 and γ_2 in Γ_q it holds $\mathbb{Z}\gamma_1 \neq \mathbb{Z}\gamma_2$, but $Z_{\gamma_1}(q^{1/2})/Z_{\gamma_2}(q^{1/2}) \in \mathbb{R}_+$.

The connected components of the complement of the set of bricks in the set of generic quadratic differentials with unsealed spectrum shall be called *chambers*.

The $Z_\gamma(q^{1/2})$ for the associated charges γ for certain saddle connections of q form local coordinates on the space of quadratic differentials, cf. section 4 in [BS15]. From this it follows that the set of bricks is a countable union of real codimension 1 submanifolds, from which the name of a wall (and thus of a brick here) arises. From this it follows directly that the chamber-elements are dense in the set of all generic quadratic differentials with unsealed spectrum. In our construction we need to vary a differential without hitting a wall, so we need the following lemma.

Lemma 4.4.7

Chambers are open in the space of all quadratic differentials.

Proof. Let q be an element of a chamber. So at any given angle ϑ at most one element of Γ_q (up to integer multiples) is the associated charge of some ϑ -trajectory.

For a contradiction suppose that in every neighborhood U_t of q there exists a brick q_t . Thus in every U_t there are distinct saddle trajectories c_t^1 and c_t^2 . They appear for angles $\vartheta_t \in [0, 2\pi)$, so we obtain an infinite sequence of angles, which accumulates at some $\vartheta \in [0, 2\pi)$ and we pass to a corresponding subsequence converging to ϑ for $t \rightarrow \infty$. As q is not a brick, it follows from Lemma 5.1 in [BS15] that either there is only one finite trajectory or there are two finite trajectories which form the boundary of a ring domain.

We first consider the case of a single trajectory and distinguish two cases. First we suppose that there is no t_0 such that for all $t > t_0$ $\vartheta_t = \vartheta$. Thus near q there are at least two ca -trajectories for angles arbitrarily close to ϑ . Now since q has only one finite ϑ -trajectory it follows from Proposition 5.5 in [BS15] that in a sufficiently small neighborhood of q there can't be a $\vartheta + \epsilon$ -trajectory for small enough $\epsilon \neq 0$. So this case cannot occur.

On the other hand, if such a t_0 exists, we obtain a sequence of differentials q_t near q which all have at least two distinct ϑ -trajectories. But, as of Lemma 5.2. in [BS15] the subset of differentials with only one ϑ -trajectory is open in the space of all quadratic differentials. So this case also cannot occur.

For the case of a ring domain the analogous statements follow from the same results in [BS15]. Thus a sequence of wall-elements can only accumulate at wall-elements, so the subset of non-wall elements is open. □

Before moving on, let us consider the spectrum in some more detail. The use of some additional technique of [BS15] will allow us to loosen the restriction of the differential in Proposition 9.1.3 quite a bit. In the definition of the spectrum we did not count the infinitely many closed ϑ -trajectories which sweep out a ring domain as separate elements, although they are finite ca -trajectories. The reason for this is the fact that the ring domain as a whole occurs at some specific angle ϑ_c and leads to a (collective) jump in the ca -triangulation at ϑ_c , as was shown in [GMN13b]. Additionally all these closed trajectories

have the same lift on the spectral cover as they are homotopic. But the existence of a ring domain still does lead to an infinite spectrum, because near ϑ_c an infinite sequence of saddle connections may occur. These saddle connections lead to distinct jumps of the triangulation and accumulate at ϑ_c . So it seems quite natural for the spectrum of a generic differential to be infinite, but the way in which a generic differential with unsealed spectrum is infinite is well understood. Indeed, infinitely many saddle connections occur only near ring domains (cf. section 5.3. in [Tah18]) and in the case of an unsealed spectrum their behaviour near such ring domains is completely described in [GMN13b]. This also implies that ring domains do not accumulate, as they would have to accumulate near a ring domain, which is not possible as of the analysis in [GMN13b]. Therefore we have the following observation:

Lemma 4.4.8

An unsealed spectrum has at most finitely many ring domains.

The analysis in [GMN13b] tells us how to calculate the jumps for ring domains and their corresponding infinite amount of saddle connections. With this lemma it is now clear, that there can only be finitely many additional jumps from single saddle connections and thus all jumps can be calculated. Otherwise, if there were an infinite sequence of ring domains, there would be technical difficulties in defining the right order for the occurring jumps. But for unsealed spectra we don't have to worry about this.

We will come back to this property of only having finitely many accumulation points later on, when considering the associated Riemann-Hilbert problem. Here the only condition which really seems to shorten the applicability of our results is the existence of signet rings, but there is a way to lessen their impact, which we will describe now.

Remark 4.4.9

Suppose q has a spectrum that is at first not unsealed. Thus Proposition 9.1.3 does not work as there is a ring domain cutting off a torus without singularities. In those cases, as was shown in [BS15], the ring domain has boundaries which are made up of some number of saddle connections and are thus *strongly non-degenerate* in the terminology of Bridgeland and Smith. It is then shown that such strongly non-degenerate ring domains can be removed without altering the genus of the surface or the differential outside of the ring domain. This process is called ring-shrinking and can roughly be considered as compressing the ring domain into a single closed curve.

After the ring shrinking the jump of the former ring domain becomes one of the jumps that was shown to be fine. If there are only finitely many signet rings it should then be possible to obtain an unsealed spectrum after finitely many ring shrinking moves, so there is no danger of shrinking the surfaces to a point or changing the topology in any other way, which could broaden the applicability of the theory presented here a lot.

However, one has to show that the ring shrinking, which eliminates a ring domain for one specific ϑ_c does not produce another ring domain at some other angle. Additionally, we are interested in properties for the moduli space for varying differentials. Therefore we would need the ring shrinking to work uniformly in at least some open subset of the Hitchin base. But varying the differential in the wrong direction might exactly reproduce the ring domain we just got rid of, cf. section 5.9 *Juggles* in [BS15]. Finally, it seems

reasonable that there are only finitely many signet rings in the spectrum of any given generic differential, but this is not rigorously proven yet.

Further explorations of these phenomenons would be very useful to obtain the most general setting for our theory. Additionally, as we explained above, the jumps corresponding to the shrunk ring domains may actually also be such that the holomorphic symplectic form ϖ is well defined, and it may be possible to obtain a proof for those jumps from the arguments given in [GMN13a]. Such a proof, written in terms of the \mathcal{X} -coordinates, would be quite useful to obtain more information about the actual limits of this construction. But for now and for the rest of this section we will be content with the demand for q to have an unsealed spectrum.

Chapter 5

Local description of solutions to Hitchin's equations

In this chapter we calculate local expressions of all the objects of the Higgs bundle moduli space we are going to need in a frame that fits the aforementioned construction of the canonical coordinates. We start with a description of the fundamental result of [Fre+20] in section 5.1 and then build a standard frame over certain subsets of the standard quadrilaterals that were introduced in the previous section. This will then allow us to calculate local forms of the harmonic metric, Higgs field and Chern connection in section 5.2 that are well suited for our construction.

5.1 Construction of the standard frame

The construction of the metric will mostly be done with local data, coming from the solutions of Hitchin's equation, for which we build on the theory of limiting configurations developed in [Fre+20]. For the rest of this section we fix a stable $\mathrm{SL}(2, \mathbb{C})$ -Higgs bundle (\mathcal{E}, Φ) in the regular locus \mathcal{M}' of \mathcal{M} s. t. $-q := \det \Phi$ has only simple zeroes. FMSW constructed solutions h_R of the R -rescaled Hitchin equation

$$F_{A_{h_R}}^\perp + R^2 [\Phi, \Phi^{*h_R}] = 0$$

for large $R \in \mathbb{R}$.¹ The idea behind this rescaling is, that for large R , corresponding to large Higgs fields, Hitchin's equation would decouple leading to a much simpler set of equations that can be solved explicitly. The decoupled equations are the *limiting equations*

$$F_{A_{h_\infty}}^\perp = 0, \quad \text{and} \quad [\Phi, \Phi^{*h_\infty}] = 0 \tag{5.1}$$

and the harmonic metric h_∞ is called the limiting metric, which corresponds to the limiting pair (A_∞, Φ_∞) . Local forms near zeroes and poles of these objects are given in [Fre+20], which were then glued together to create an approximate solution h_{app} of the true Hitchin equation.² The foundational result of [Fre+20] for our work is now that for large enough R there exists a gauge transformation that transforms the approximate solution into a true solution. We paraphrase their result as follows:

¹In [Fre+20] this parameter is t , while our notation comes from the work of GMN. As we will need t mostly as the usual parameter of the ca -trajectories, it seems reasonable to side with GMN on this question.

²From here on out we will use "true" as signifying solutions which correspond to Hitchin's equation to separate them from approximate solutions or solutions of the decoupled equations.

Theorem 5.1.1

There exists $R_0 > 1$ such that for every $R \geq R_0$ there exists an h_0 -Hermitian endomorphism γ_R , s. t. $g \cdot h_{app}$ for $g = \exp(\gamma)$ is a true solution of the R -rescaled Hitchin equation. Furthermore γ_R is unique amongst endomorphisms of small norm.

Not only did FMSW show the existence of these approximate solutions and the correction mechanism. They also showed that the transformation is exponentially suppressed in R . These results are important for the work at hand as they seem to imply that other objects that were constructed with the help of these solutions might as well inherit the structure of an explicit approximate solution plus some correctional term that is exponentially suppressed. In order to really make use of the results one would however need a way of implementing it in an appropriate construction of e. g. the hyperkähler metric of \mathcal{M} . As it turns out, the proposed construction by GMN allows for precisely such an implementation as we want to show here.

For all of this to work we will have to make heavy use of the explicitly known local forms of all the moving parts which will later be the ingredients of our flat connection for the GMN construction. So our goal here is to write the ingredients of the twistor connection as a diagonal leading term plus some remainder, or "error"-term, which behaves in a certain way when going into a parabolic point, as well as when considering the large R -asymptotics.

In the following z will always be a local coordinate on \mathcal{C} with $r := |z|$ in any coordinate centered around a parabolic point. We start by considering the local frame (e_1, e_2) in which we would like to work. We are interested in frames that are adjusted to the Higgs field $\Phi = \varphi dz$, so that φ has diagonal form

$$\varphi = \begin{pmatrix} f^{1/2} & 0 \\ 0 & -f^{1/2} \end{pmatrix}$$

for some square root $f^{1/2}$ of $f = -\det \varphi$.

Lemma 5.1.2

Let (\mathcal{E}, Φ) be a parabolic Higgs bundle with local Higgs field $\Phi = \varphi dz$ with determinant $\det \Phi = \det \varphi dz^2 = -f dz^2$. In any gauge in which φ is diagonal, the holomorphic structure splits, i. e. is also represented by a diagonal matrix.

Proof. Assume φ to be diagonal, i. e. of the form

$$\varphi = \begin{pmatrix} f^{1/2} & 0 \\ 0 & -f^{1/2} \end{pmatrix}$$

for some square root $f^{1/2}$ of f . We now write the holomorphic structure locally as

$$\bar{\partial}_E = \bar{\partial} + A_{\bar{z}} d\bar{z} = \bar{\partial} + \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{a}_3 & \bar{a}_4 \end{pmatrix} d\bar{z},$$

where $\bar{\partial}$ is the trivial holomorphic structure induced by the coordinate z . As $(\Phi, \bar{\partial}_E)$ is a Higgs bundle it holds $\bar{\partial}_E \Phi = 0$, i. e.

$$0 = \partial_{\bar{z}} \varphi + [A_{\bar{z}}, \varphi].$$

In some more detail: Let s be any section of E , then it holds:

$$\begin{aligned}
 0 &= (\bar{\partial}_E \varphi)s = \bar{\partial}_E(\varphi s) - \varphi \bar{\partial}_E(s) = \partial_{\bar{z}}(\varphi s) + A_{\bar{z}}\varphi s - \varphi \partial_{\bar{z}}s - \varphi A_{\bar{z}}s \\
 &= \begin{pmatrix} \partial_{\bar{z}}(f^{1/2}s_1) \\ -\partial_{\bar{z}}(f^{1/2}s_2) \end{pmatrix} + \begin{pmatrix} \overline{a_1} & \overline{a_2} \\ \overline{a_3} & \overline{a_4} \end{pmatrix} \begin{pmatrix} f^{1/2} & 0 \\ 0 & -f^{1/2} \end{pmatrix} s \\
 &\quad - \begin{pmatrix} f^{1/2} & 0 \\ 0 & -f^{1/2} \end{pmatrix} \begin{pmatrix} \partial_{\bar{z}}s_1 \\ \partial_{\bar{z}}s_2 \end{pmatrix} - \begin{pmatrix} f^{1/2} & 0 \\ 0 & -f^{1/2} \end{pmatrix} \begin{pmatrix} \overline{a_1} & \overline{a_2} \\ \overline{a_3} & \overline{a_4} \end{pmatrix} s \\
 &= \begin{pmatrix} (\partial_{\bar{z}}f^{1/2})s_1 \\ -(\partial_{\bar{z}}f^{1/2})s_2 \end{pmatrix} + \begin{pmatrix} -2\overline{a_2}f^{1/2}s_2 \\ 2\overline{a_3}f^{1/2}s_1 \end{pmatrix}.
 \end{aligned}$$

Plugging in $s = (1, 0)$ and $s = (0, 1)$ we obtain $\overline{a_2} = \overline{a_3} = 0$. \square

We now introduce the standard neighborhood on which most of the following local calculations take place, for which we recall the ca -triangulations of section 4.1. We take $-q = \det \Phi$ and consider any generic angle ϑ , i. e. the corresponding ϑ -triangulation has no finite ϑ -trajectories. For any edge of the triangulation we consider the two triangles that share this edge. They form a quadrilateral Q bounded by four different ϑ -trajectories inside some bigger open patch $U \subset C$. The trajectories connect four different weakly parabolic points and inside the quadrilateral are two (distinct) zeroes of $\det \phi$, cf. figure 5.1.

Now we may take a simple closed curve in U that touches all four weakly parabolic points and whose interior $O \subset U$ contains all four ϑ -trajectories that bound the quadrilateral. Inside O we may also take a simple closed curve whose interior $\tilde{O} \subset O$ contains both zeroes of $\det \Phi$ and no point of any of the edges. Then $S := O \setminus \tilde{O}$ is homeomorphic to an annulus, does not contain the zeroes or poles of $\det \Phi$, but does contain the ϑ -trajectories that form the boundary of the quadrilateral.

Definition 5.1.3

Let q be a generic differential and ϑ a generic angle. Any quadrilateral obtained from two triangles that share an edge shall be called a *standard quadrilateral* and any domain $S \in \mathcal{C}$ constructed in the way described above out of Q shall be called a *standard annulus*³ of the triangulation. In particular such a domain does not contain any of the critical points of q and any simple closed curve in S surrounds either no or two zeroes of $-q$ in Q .

Note that if we vary $-q = \det \Phi$ or the angle ϑ of the ca -triangulation the poles, zeroes and curves may vary. If we do this continuously depending on some parameter ϵ all the parts depend continuously on $-q$ and ϑ (except for the critical values for ϑ). Thus it is always possible to choose a domain that is a standard annulus for all ϵ in some interval.

Thus defined we now establish that a standard annulus is a good working environment for our case. For the rest of this section a Higgs field Φ whose determinant $-q$ is a generic differential and a generic angle ϑ shall be fixed.

³Although such a domain is homeomorphic to an annulus it has to be noted that as the ϑ -trajectories form logarithmic spirals near the parabolic points, the whole domain forms such a spiral structure there. The important fact is, that the boundary of the annulus can be taken between any such two spirals.

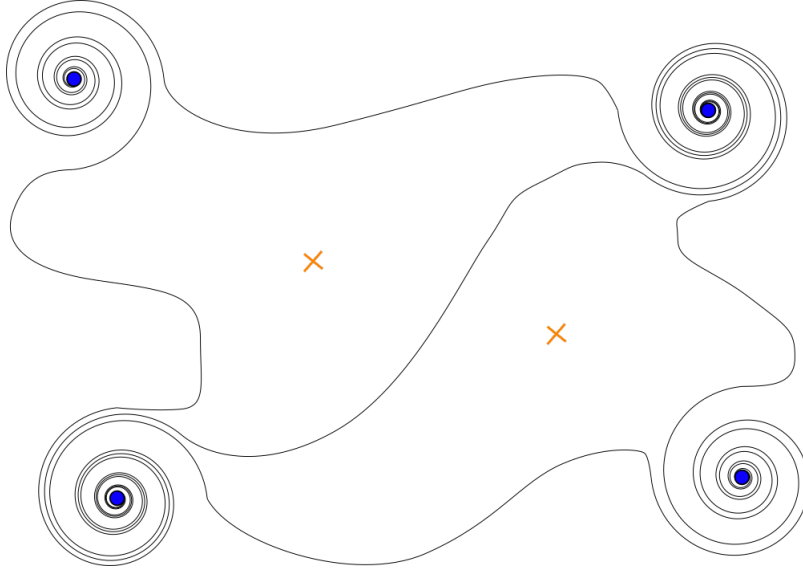


Figure 5.1: A standard quadrilateral consisting of two triangles that share a common ca -trajectory. The blue dots represent weakly parabolic points and the orange crosses represent zeroes of $-q$.

Lemma 5.1.4

Let $\Phi = \varphi dz$ be a Higgs field on a standard quadrilateral Q with standard annulus S . Then there exists a (holomorphic) frame on S in which φ is diagonal.

Proof. Let $c : [a, b] \rightarrow \mathcal{C}$ be a simple closed curve in S . Then c is homotopic to a point or c surrounds the two zeroes of $-q = \det \Phi$ in Q . As c can't surround only one zero a standard result of complex analysis shows that a well defined square root $q^{1/2}$ on the whole annulus can be chosen.⁴ In every point x in S the eigenvalues $\pm q^{1/2}(x)$ are distinct and as the square root is well defined on S they don't interchange. Thus we obtain two disjoint eigenspaces E_{x+} , E_{x-} in every point x which form two distinct line bundles E_+ , E_- on S . Locally we can thus choose a frame in which φ is diagonal. As $\bar{\partial}_E(\Phi) = 0$, the diagonality of φ implies as of Lemma 5.1.2 that the holomorphic structure $\bar{\partial}_E$ is diagonal, i. e. splits into a holomorphic structure on each line bundle. Thus E_+ and E_- are holomorphic line bundles. Since they are defined on the non-compact Riemann surface S they have to be holomorphically trivial, cf. [For81] p. 229. Thus we can choose some non-vanishing sections over S with respect to which φ will be diagonal on all of S . \square

In the following we compute the local forms of the Higgs field, its adjoint and the Chern connection of the holomorphic structure w. r. t. the Hermitian metric h_R on a standard annulus or subsets thereof. For these computations we first need to collect the information on h_R from [Fre+20] which for simplicity we will just call $h := h_R$ for the rest of the section, keeping in mind that all the constructions are considered for large enough R . So we start by choosing the approximate metric h_0 , which was defined in [Fre+20] as h_R^{app} , as our background metric on E . In [Fre+20] it is shown for $\text{pdeg}(\mathcal{E}) = 0$ that $h_0 = h_R^{app}$ is adapted to the parabolic structure for weakly parabolic bundles and induces the fixed flat

⁴This corresponds to choosing one sheet of the double cover by the spectral curve.

Hermitian structure on $\text{Det}E$.⁵ The metric is defined in such a way that away from zeroes of q the eigenspaces of Φ are orthogonal, thus Φ is represented by a diagonal matrix in any frame given by sections of the eigenspaces such as the frame constructed above. In [Fre+20] we also find local forms for h_0 near all the different exceptional points for q . Here we only note that the holomorphic frame above can be rescaled into a frame $F^* := (e_1^*, e_2^*)$ on S s. t. the matrix H_0 representing h_0 has the following form near the (double) poles p_i of q :

$$H_0 = Q \begin{pmatrix} |z|^{2\alpha_1(p)} & 0 \\ 0 & |z|^{2\alpha_2(p)} \end{pmatrix}.$$

Here α_1, α_2 are the parabolic weights and Q is some locally defined and non-vanishing smooth function, completely determined by $h_{\text{Det}\mathcal{E}}$, the choice of holomorphic section of $\text{Det}\mathcal{E}$ and coordinate z . Note that this metric coincides with h_∞ in [Fre+20]. For our calculations it is convenient to change to an h_0 -unitary frame near the p_i , so we define

$$e_1 = \frac{e_1^*}{Q^{1/2}|z|^{\alpha_1}}, \quad e_2 = \frac{e_2^*}{Q^{1/2}|z|^{\alpha_2}}.$$

Then it holds $H_0 = E_2$ in the frame $F := (e_1, e_2)$ near the p_i .

Away from poles or zeros of $\det\Phi$ the metric is still diagonal in any holomorphic frame and has the general form

$$H_0 = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$$

for some non-vanishing functions f, g . Thus via some diagonal transformation we obtain the local form $H_0 = E_2$ everywhere away from zeros of $\det\Phi$ in this frame F . Note that scaling away Q is not strictly necessary but rather simplifies a lot of the following expression.

This specific choice of frame will be useful in section 7.1 as it allows for a simple construction of the leading terms of the Fock-Goncharov-coordinates. In almost all of the following calculations we will work in this frame, so we denote it as our standard.

Definition 5.1.5

The h_0 -unitary frame F constructed above on a standard annulus S shall be called the *standard frame* on S .

5.2 Expressions in the standard frame

As before we fix a Higgs bundle $(E, \bar{\partial}_E, \Phi)$. We now want to describe the harmonic metric h in the h_0 -unitary frame F , which as of [Fre+20] we obtain for large enough R by the action of a gauge transformation g acting on h_0 via $h = g \cdot h_0(v, w) = h_0(gv, gw)$. In the chosen frame we have $h_0(v, w) = \bar{v}^t H_0 w$ and thus

$$g \cdot h_0(v, w) = \bar{g} \bar{v}^t H_0 g w = \bar{v}^t \bar{g}^t H_0 g w = \bar{v}^t \bar{g}^t g w.$$

⁵While it seems likely that their arguments can be adapted to case of $\text{pdeg}(\mathcal{E}) \neq 0$ we will constrain ourselves to case of $\text{pdeg}(\mathcal{E}) = 0$ where it is useful.

Let H be the matrix representing h in this frame, i. e. $h(v, w) = \bar{v}^t H w$. Then, as $h = g \cdot h_0$ we obtain $H = \bar{g}^t H_0 g = \bar{g}^t g$. Now from [Fre+20] we have $g = \exp(\gamma)$ for γ h_0 -Hermitian and traceless, so g is also h_0 -Hermitian and has determinant 1. We write g accordingly as

$$g =: \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$$

with a, d real-valued functions on S and $ad - |b|^2 = 1$. The behavior of the entries of g is determined by the behavior of the entries of γ in the following way:

Lemma 5.2.1

Let γ be tracefree and h_0 -hermitian with $D := \det(\gamma)$. Then

$$g = \exp(\gamma) = \cosh(\sqrt{-D})E_2 + \frac{\sinh(\sqrt{-D})}{\sqrt{-D}}\gamma.$$

Proof. This is a standard calculation: As γ is tracefree and h_0 -hermitian we may write it as

$$\gamma = \begin{pmatrix} l & o \\ \bar{o} & -l \end{pmatrix}$$

for some real valued function l and complex valued function o . As basic facts we have $D := \det(\gamma) = -l^2 - |o|^2 \leq 0$ and $\gamma^2 = -DE_2$. From this we obtain

$$\begin{aligned} \exp(\gamma) &= \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} = \sum_{k=0}^{\infty} \frac{\gamma^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\gamma^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-D)^k E_2}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-D)^k \gamma}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{\sqrt{-D}^{2k}}{(2k)!} E_2 + \sum_{k=0}^{\infty} \frac{\sqrt{-D}^{2k}}{(2k+1)!} \gamma \\ &= \sum_{k=0}^{\infty} \frac{\sqrt{-D}^{2k}}{(2k)!} E_2 + \frac{1}{\sqrt{-D}} \sum_{k=0}^{\infty} \frac{\sqrt{-D}^{2k+1}}{(2k+1)!} \gamma = \cosh(\sqrt{-D})E_2 + \frac{\sinh(\sqrt{-D})}{\sqrt{-D}}\gamma. \end{aligned}$$

□

This lemma relates the properties of γ to the properties of g , from which we obtain the necessary information about H as we have:

$$H = \bar{g}^t g = \begin{pmatrix} a^2 + |b|^2 & (a+d)b \\ (a+d)\bar{b} & d^2 + |b|^2 \end{pmatrix} =: \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix}.$$

We will use the following observation in some calculations later on:

Lemma 5.2.2

In the standard frame F on the standard annulus the matrix H that represents the Hermitian harmonic metric h has determinant 1.

Proof. This follows readily from the discussion of the local form above as g has determinant 1. □

We are interested in the decaying behavior of the entries of H near weakly parabolic points and we will also need the behavior of the entries of $H^{-1}\partial H$. Using Lemma 5.2.2 we obtain

$$H^{-1}\partial H = \begin{pmatrix} \delta\partial\alpha - \beta\partial\bar{\beta} & \partial(\beta\delta) \\ -\partial(\alpha\bar{\beta}) & -(\delta\partial\alpha - \beta\partial\bar{\beta}) \end{pmatrix} =: \begin{pmatrix} h_1 & h_2 \\ h_3 & -h_1 \end{pmatrix}.$$

We now obtain all the information from the properties of γ which were established in [Fre+20].

Lemma 5.2.3

There exists a local coordinate z centered around any weakly parabolic point, such that the following asymptotics hold for the functions defined above (with $r := |z|$) for some $\mu > 0$ and all $1 \leq i \leq 3$:

$$a, d, \alpha, \delta = O(1), \quad b, \beta = O(r^\mu), \quad h_i = O(r^{\mu-1}).$$

Furthermore, if ϵ denotes a C^2 -coordinate of \mathcal{M}'_R , then the derivatives w. r. t. ϵ exist and have the same asymptotics:

$$\begin{aligned} \partial_\epsilon a, \partial_\epsilon d, \partial_\epsilon \alpha, \partial_\epsilon \delta &= O(1), & \partial_\epsilon b, \partial_\epsilon \beta &= O(r^\mu), \\ \partial_\epsilon^2 a, \partial_\epsilon^2 d, \partial_\epsilon^2 \alpha, \partial_\epsilon^2 \delta &= O(1), & \partial_\epsilon^2 b, \partial_\epsilon^2 \beta &= O(r^\mu), \\ \partial_\epsilon h_i, \partial_\epsilon^2 h_i &= O(r^{\mu-1}). \end{aligned}$$

Additionally there exist constants $C, \nu > 0$ s. t.

$$|r^{-\mu}b|, |r^{-\mu}\beta|, |r^{1-\mu}h_i| \leq Ce^{-\nu R} \quad \text{and} \quad |r^{1-\mu}\partial_\epsilon h_i| \leq Ce^{-\nu R}$$

for large enough R .

Proof. One important result in [Fre+20] is that the gauge transformation γ that transforms the approximate solution of the Hitchin equations lies in the Friedrichs domain $\mathcal{D}_{\text{Fr}}^{0,\alpha}(\nu)$. In particular γ has a partial expansion in (non-negative) powers of r , where the first term is a constant diagonal matrix. This gives the first asymptotics for the entries of γ by definition of the domain.

The ϵ -differentiability for the entries of γ follows from an appropriate enhancement of the application of the inverse function theorem that shows the existence of γ . The asymptotics for the ϵ -derivatives follow readily as the r -dependence in $\mathcal{D}_{\text{Fr}}^{0,\alpha}(\nu)$ is independent of the ϵ -dependence.

Finally the exponential suppression is also an explicit part of the result of [Fre+20].

This shows all the conditions for the entries of γ . As $\det(\gamma)$ is well defined and non-zero on the standard annulus it follows (using a series expansion for the cosh-part) from Lemma 5.2.1 that g is of the form $g = E_2 + g_e$ where g_e inherits the decaying properties from γ . The properties for the h_i follow in the same way, using the fact, that the first two polar derivatives also lie in the corresponding domain as shown in [Fre+20]. \square

With this information we can now calculate the rest of the protagonists of our construction.

Corollary 5.2.4

In the standard frame F on the standard annulus S the adjoint Higgs field (w. r. t. the harmonic metric h) can be written as

$$\Phi^{*h} = \left(\overline{f^{1/2}} \sigma^3 + \overline{f^{1/2}} \varphi^e \right) d\bar{z}$$

for φ^{er} some traceless matrix whose entries are $O(r^\mu)$ for some $\mu > 0$ near any weakly parabolic point.

Proof. Using Lemma 5.2.2 the adjoint Higgs field $\Phi^{*h} = \varphi^{*h} d\bar{z}$ is calculated as

$$\begin{aligned} \varphi^{*h} &= H^{-1} \bar{\varphi}^t H = \begin{pmatrix} \delta & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix} \begin{pmatrix} \overline{f^{1/2}} & 0 \\ 0 & -\overline{f^{1/2}} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix} \\ &= \overline{f^{1/2}} \begin{pmatrix} \alpha\delta + |\beta|^2 & 2\delta\beta \\ -2\alpha\bar{\beta} & -\alpha\delta - |\beta|^2 \end{pmatrix} = \begin{pmatrix} \overline{f^{1/2}} & 0 \\ 0 & -\overline{f^{1/2}} \end{pmatrix} + 2\overline{f^{1/2}} \begin{pmatrix} |\beta|^2 & \delta\beta \\ -\alpha\bar{\beta} & -|\beta|^2 \end{pmatrix}. \end{aligned}$$

Thus the adjoint Higgs field consists of a diagonal leading term plus a remainder whose entries are all at least linear in $|\beta|$. As of Lemma 5.2.3 the remainder behaves in the specified way. \square

The final ingredient is the Chern connection for $\bar{\partial}_E$ for the harmonic metric h , which we also want to describe with a "simple" leading term and a "small" remainder.

Corollary 5.2.5

In the standard frame F on the standard annulus S the Chern connection A (w. r. t. the harmonic metric h) of $\bar{\partial}_E$ can be expressed as

$$A = d + \begin{pmatrix} \overline{a_1} d\bar{z} - a_1 dz & 0 \\ 0 & \overline{a_2} d\bar{z} - a_2 dz \end{pmatrix} + (a_2 - a_1) A^e dz + H^{-1} \partial H,$$

where $d = \partial + \bar{\partial}$ is the standard differential induced by the coordinate z and A^e is some traceless matrix whose entries are $O(r^\mu)$ for some $\mu > 0$ near any parabolic point.

Proof. By construction the holomorphic structure in the standard frame F has the form

$$\bar{\partial}_E = \bar{\partial} + A_{\bar{z}} = \bar{\partial} + \begin{pmatrix} \overline{a_1} & 0 \\ 0 & \overline{a_2} \end{pmatrix} d\bar{z},$$

for some function a_1, a_2 . A direct calculation using that H is h_0 -hermitian shows that the Chern connection $d_A = d + A$ with respect to h is given as

$$d + A = d + A_{\bar{z}} d\bar{z} - A_{\bar{z}}^{*h} dz + H^{-1} \partial H.$$

So we need to calculate $A_{\bar{z}}^{*h}$, which becomes

$$\begin{aligned} A_{\bar{z}}^{*h} &= H^{-1} \overline{A_{\bar{z}}} H = \begin{pmatrix} \delta & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix} dz \\ &= \begin{pmatrix} a_1 \alpha \delta - a_2 |\beta|^2 & (a_1 - a_2) \delta \beta \\ -(a_1 - a_2) \alpha \bar{\beta} & a_2 \alpha \delta - a_1 |\beta|^2 \end{pmatrix} dz = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + (a_1 - a_2) \begin{pmatrix} |\beta|^2 & \delta \beta \\ -\alpha \bar{\beta} & -|\beta|^2 \end{pmatrix} dz. \end{aligned}$$

Here we used Lemma 5.2.2. Thus the whole connection becomes

$$d_A = d + \begin{pmatrix} \overline{a_1}d\bar{z} - a_1dz & 0 \\ 0 & \overline{a_2}d\bar{z} - a_2dz \end{pmatrix} + (a_2 - a_1) \begin{pmatrix} |\beta|^2 & \delta\beta \\ -\alpha\bar{\beta} & -|\beta|^2 \end{pmatrix} dz + H^{-1}\partial H.$$

Again the structure is that of a diagonal leading term and a remainder term whose entries depend at least linearly on $|\beta|$ and thus have the specified decaying behavior. \square

Note that the factor $H^{-1}\partial H$ should also be considered as part of the remainder term as it is part of the deviation from the diagonal leading term. It consists only of data coming from the harmonic metric and thus only depends indirectly (via the solution of Hitchin's equation) on the Higgs field and holomorphic structure. Most importantly it also has the structure we need at the poles: In Lemma 5.2.3 it was shown that the h_i are $O(r^{\mu-1})$ while b and β are $O(r^\mu)$. This matches in the emerging formula in section 7, as the other remainder terms for the Higgs field and the Chern connection obtain a pole of order 1 through $f^{1/2}$ and $a_2 - a_1$.

Remark 5.2.6

We have obtained the structure of a leading term plus a "small" remainder for both the data from the Higgs field Φ and from the connection A which will together form the connection matrix for a connection ∇ in chapter 7. That is, we can construct $\nabla(\zeta) = d_A + \frac{R}{\zeta}\Phi + R\zeta\Phi^{*h}$ for $\zeta \in \mathbb{C}^\times$ and try to solve the flatness equation $\nabla(\zeta)v = 0$. We already observe the aforementioned structure of a diagonal leading term and a "small" remainder in the connection matrix of $\nabla(\zeta)$. The remainder $f^{1/2}\varphi^e d\bar{z} + (a_2 - a_1)A^e dz + H^{-1}\partial H$ of this matrix is traceless which is seen directly for the parts φ^e and A^e and follows for $H^{-1}\partial H$ by a simple calculation from the fact that H has constant determinant.

As φ is traceless by definition of our moduli space the only part that isn't comes from the leading part of the Chern connection A . One could arrange for this part to also be traceless by shifting the parabolic weights or maybe even some appropriate gauge transformation, which would simplify some of the following calculations and which is actually the form [GMN13b] use. We don't do this here as we follow the setup of [Fre+20]. Additionally this shows that no specific choice of weights is necessary. However, thinking of A as traceless leads to more convenient formulas and our treatment actually shows that one can think of these objects in this way without losing any important structure.

So far we worked in the holomorphic picture, but we will also consider \mathcal{M}_R in the unitary picture. We already explained in Remark 3.2.1 how these points of view relate to another, and it is useful to note, how the relation carries over to $\nabla(\zeta)$ and it's solutions.

Lemma 5.2.7

Let $(\mathcal{E}, \Phi) \in \mathcal{M}$ be a Higgs bundle with holomorphic structure $\bar{\partial}_E$, h_R the corresponding harmonic metric and Chern connection A (for h). Let h_0 be the approximate metric as defined above and (D, ϕ) the corresponding Higgs pair for h_0 , where D is induced by $H^{1/2} \cdot \bar{\partial}_E$ and $\phi = H^{1/2} \cdot \Phi$. In the standard frame F it then holds

$$H^{1/2} \circ \left(d_D + \frac{R}{\zeta}\phi + R\zeta\phi^{*h_0} \right) \circ H^{-1/2} = d_A + \frac{R}{\zeta}\Phi + R\zeta\Phi^{*h}.$$

Proof. This follows from a direct calculation with the same arguments we used above and the fact that h_0 is represented by the identity in the standard frame. \square

The left-hand side of the equality in this lemma can be considered as $H^{1/2} \cdot \nabla^u(\zeta)$, where $\nabla^u(\zeta)$ is the connection associated to an element in \mathcal{M}_R in the unitary picture and $H^{1/2}$ acts via conjugation, while the right-hand side corresponds to the connection $\nabla^h(\zeta)$ associated to the element in the holomorphic picture. The lemma thus shows that the associated connections in the holomorphic and unitary picture only differ by the gauge transformation $H^{1/2}$. Note that this can also be considered as a special case of the fact that the nonabelian Hodge correspondence $\mathcal{M}_R \rightarrow \mathcal{M}^{dR}$, which we will later on use, is well defined. It follows that v is a solution of $\nabla^h(\zeta)v = 0$ iff $\nabla^u(\zeta)(H^{-1/2}v) = 0$. Thus solutions are related via gauge transformations and we saw in the section 4.2 that the \mathcal{X} -coordinates on the space of flat connections are invariant under such transformations. It is therefore justified to switch between the holomorphic and unitary picture when investigating the solutions of $\nabla(\zeta)v = 0$.

Of course, to make use of this we first need to find out whether such solutions exist at all and how we can obtain them. As it turns out it is possible to solve the flatness equation along *ca*-curves but to do so in a proper way, one needs some non-standard theory for singular ODEs which we will develop in the next section.

Chapter 6

Initial value problems at infinity

Crucial ingredients in our construction are the "small flat sections" obeying the flatness equation of a connection build out of a Higgs pair. This connection, when trivialized along a ca -curve, becomes a 2×2 non-autonomous linear ODE. The small flat sections are supposed to be solutions of this ODE that decay in a certain way as $t \rightarrow \infty$. Thus we want to solve an ODE uniquely (up to multiplication by a complex scalar) by demanding a certain behaviour at ∞ . This differs substantially from the usual boundary value problems where the boundary value is given at a finite time. This becomes a problem as we want to differentiate the solutions with respect to a parameter and the standard theory requires a boundary value or initial value at some finite time. It turns out that to solve this problem the theory of Volterra integral equations is helpful, as they encode the boundary value in an explicit term of the equation. This allows for a rigorous definition of the small flat section and exploration of their differentiable behavior. Therefore we will use this section to study these types of equations. We will completely ignore the origin of the problem in the theory of Higgs bundles in this section and instead develop the theory in its own purely analytical setting, which may also be of interest in its own right and for completely different applications.

Before we start let us briefly summarize what is known for this sort of equations. Generally a linear Volterra equation of the second kind is an equation of the form

$$x(t) = a(t) - \int_b^t K(t, s)x(s)ds. \quad (6.1)$$

which has to be solved for some function x which may take values in any vector space for which the operations are defined. The "second kind" refers to the existence of the term $a(t)$, which takes the aforementioned role of the initial value as one can see by evaluating x at $t = b$. The term K is the *integral kernel*, which is the main object of study to determine whether solutions exist, are smooth, and so on.

The equations we consider, however, are different in the important detail that the integral is evaluated up to infinity:

$$x(t) = a(t) - \int_t^\infty K(t, s)x(s)ds. \quad (6.2)$$

Now in this equation $a(t)$ becomes the "boundary value at infinity". We know from the work of Levinson [Lev48] that in some cases a specific solution can be constructed, but as one might guess, the general theory becomes a bit more intricate, mostly because we can't work on compact intervals. One article studying this problem is [FO08] by D. Franco and D. O'Regan. They show the existence of solutions for specific kernels, but they mostly consider kernels with some periodic behavior that we don't assume here. Additionally they do not consider the differentiable dependence on parameters.

A better guiding light is the article [Win10] by A. Windsor. He studies equations of the finite form (6.1) and proves existence, continuity and smoothness of the solutions with respect to parameters. Most of our arguments are adaptations of his treatment to our case, as is the general structure of this section. But quite some work has to be done to adjust the theory for the unbounded intervals on which the integral lives. This becomes most apparent when it comes to the differentiable dependence, where the assumptions on the integral kernel are quite strong and in fact stronger than one might deem necessary, i. e. we have to demand that K and a are two times differentiable to obtain one time differentiability for our solution. We will also need a product structure on the kernel that exists in our special case as the ODE, from which the integral equation arises, has a leading part and a bounded remainder term. The product structure then arises as we transform the ODE into the integral equation. This is quite a special structure but it may appear in different settings and the theory we develop should be worth looking at whenever an ODE splits into a well-behaved leading part and some small perturbation.

6.1 Existence and uniqueness

We want to start with proving the unique existence of a solution of equations of the type (6.2). Our main tool for this is the Banach-Caccioppoli contraction mapping principle, which we will also need for the smooth dependence.

Theorem 6.1.1

Let (X, d) be a complete metric space and $P : X \rightarrow X$ be a contraction mapping, i. e. there exists $0 \leq \lambda < 1$ s. t.

$$d(P(x), P(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$.

Then P has a unique fixed point, which will be denoted by $\varpi(P)$. Moreover, for any $x \in X$ we have

$$d(x, \varpi(P)) < \frac{d(x, P(x))}{1 - \lambda}.$$

To use this we have to start by building our function spaces. We fix a positive number $n \in \mathbb{N}$ for the dimension of our target space (in our application this would be $n = 2$). For any $T \in \mathbb{R}$ consider functions on the interval $[T, \infty)$, so our space X will be

$$B_T := \{f \in C^0([T, \infty), \mathbb{C}^n) : \exists M \in \mathbb{R}, \forall x \in [T, \infty) : |f(x)| \leq M\},$$

the space of continuous bounded functions on the half line beginning with T with values in \mathbb{C}^n (where $|f(x)|$ denotes the usual euclidean norm on \mathbb{C}^n). It is well known that B_T is complete with respect to the metric d_∞ induced by the sup-norm

$$\|f\|_\infty := \sup_{x \in [T, \infty)} |f(x)|,$$

so from now on B_T shall be equipped with this norm. In order to obtain the contraction property we will need our kernels to decay fast enough. They will live on sectors starting

with T , i. e. we write $\Delta(T) := \{(t, s) \in \mathbb{R}^2 : T \leq t \leq s\}$. Now the 'good' kernels will be the ones in

$$\mathcal{K}_T := \left\{ K \in C^0(\Delta(T), M_n(\mathbb{C})) : \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s)\| ds < 1 \right\}.$$

The bound for the norm of the integral will be the contraction factor, so we denote it from now on by $\lambda := \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s)\| ds < 1$. The choice of matrix norm is not important at this point, so we just take the operator norm. We consider \mathcal{K}_T equipped with the metric $d_{\mathcal{K}}(K_1, K_2) := \sup_{t \in [T, \infty)} \int_t^\infty \|K_1(t, s) - K_2(t, s)\| ds$. Now we are ready to formulate our existence and uniqueness result similar to the one in [Win10]:

Theorem 6.1.2

Let $T \in \mathbb{R}$, $a \in B_T$ and $K \in \mathcal{K}_T$. Then there exists a unique solution of the integral equation

$$x(t) = a(t) - \int_t^\infty K(t, s)x(s)ds, \quad (6.3)$$

on the interval $[T, \infty)$ and this solution is an element of B_T .

Proof. For any $\phi \in B_T$ we know from the integral property of K that $\int_t^\infty K(t, s)\phi(s)ds$ exists for $t \geq T$. As a is also bounded it follows that

$$P(\phi)(t) := a(t) - \int_t^\infty K(t, s)\phi(s)ds.$$

defines an operator $P : B_T \rightarrow B_T$. For two elements $\phi_1, \phi_2 \in B_T$ we obtain

$$\begin{aligned} d_\infty(P(\phi_1), P(\phi_2)) &\leq \sup_{t \in [T, \infty)} \left| \int_t^\infty K(t, s)(\phi_1(s) - \phi_2(s))ds \right| \\ &\leq \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s)\| |\phi_1(s) - \phi_2(s)| ds \\ &\leq \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s)\| ds \sup_{t \in [T, \infty)} |\phi_1(s) - \phi_2(s)| \\ &= \lambda d_\infty(\phi_1, \phi_2), \end{aligned}$$

where $\lambda < 1$ since $K \in \mathcal{K}_T$. Thus P is a contraction mapping and we obtain a unique fixed point $x \in B_T$ as of Theorem 6.1.1. Although this solution is unique in B_T there could still exist another (non-continuous) solution of (6.3). This possibility may be excluded as follows:

A non-continuous solution would still lead to a continuous function

$$a(t) - \int_t^\infty K(t, s)x(s)ds$$

as a is continuous, K is continuous in t and the integral over a bounded function is continuous. But as this expression equals x the solution has to be continuous in the first place. \square

Remark 6.1.3

Windsor gives another argument to prove that there can't be a non-continuous bounded solution, using that the difference of two solutions also satisfies an integral equation and obtaining a contradiction for the case of the difference being non-zero. His argument would also work for our case but it requires more estimates. It may be more interesting to note that he can't use our argument as he doesn't demand his kernel to be continuous in t . Thus, although the integration over s leads to a continuity from the integral, there may still be a non-continuous part in K .

Later on we will need the aforementioned estimate for the fixed point in the following form:

Corollary 6.1.4

The following estimate holds for the solution x of (6.3):

$$\|x\|_\infty = d_\infty(0, x) \leq \frac{d_\infty(0, P(0))}{1 - \lambda} = \frac{\|a\|_\infty}{1 - \lambda}. \quad (6.4)$$

6.2 Continuous dependence on parameters

Now that we have the unique existence we consider the case where a and K may vary in order to obtain first continuous and then differentiable dependence on parameters. We start with the continuous dependence by restating Windsors theorem and proof concerning arbitrary contraction maps on metric spaces. For this denote by $Ctr(X)$ the space of contraction mappings on a complete metric space (X, d) . Every contraction map is Lipschitz continuous and therefore we can regard $Ctr(X)$ as a subspace of the space $C^0(X)$ of continuous functions on X . On $Ctr(X)$ we consider the supremum ∞ -metric d_0 , i. e. $d_0(P, Q) := \sup_{x \in X} d(P(x), Q(x))$. Note that the value of d_0 may indeed be ∞ in general settings but this has no consequences for the questions regarding continuity here. Additionally in the main Theorem 6.2.2 of this subsection the space of interest is shown to be of finite diameter, where d_0 becomes an honest metric. As every contraction map $P \in Ctr(X)$ has a unique fixed point $\varpi(P)$, we can define the map

$$\varpi : Ctr(X) \rightarrow X, \quad P \mapsto \varpi(P).$$

With this [Win10] first obtains the following result:

Lemma 6.2.1

The function ϖ is continuous.

Proof. Let $P \in Ctr(X)$. Then there is some $0 \leq \lambda < 1$, s. t.

$$d(P(x), P(y)) \leq \lambda d(x, y)$$

holds for all $x, y \in X$. Let $\epsilon > 0$ and define $\delta := (1 - \lambda)\epsilon$. Now let $Q \in Ctr(X)$ be a contraction mapping with $d_0(P, Q) < \delta$. We apply the estimate from the contraction mapping theorem 6.1.1 for P to $\varpi(Q)$ to obtain the continuity of ϖ :

$$\begin{aligned} d(\varpi(Q), \varpi(P)) &< \frac{d(\varpi(Q), P(\varpi(Q)))}{1 - \lambda} = \frac{d(Q(\varpi(Q)), P(\varpi(Q)))}{1 - \lambda} \\ &< \frac{\delta}{1 - \lambda} = \epsilon. \end{aligned}$$

□

Now we come to the continuous dependence for our solutions. For this denote the unique solution of (6.3) for any $a \in B_T$ and $K \in \mathcal{K}$ by $x_{a,K} \in B_T$. Note that $x_{a,K}$ thus does depend on T , but as T is just a parameter that is fixed from the beginning, we don't write this dependence into the notation. We now claim:

Theorem 6.2.2

Fix $T \in \mathbb{R}$. Then the map

$$X : B_T \times \mathcal{K}_T \rightarrow B_T$$

given by $X(a, K) := x_{a,K}$ is continuous (where $B_T \times \mathcal{K}_T$ is equipped with the usual product metric.)

Proof. Again the proof works similar to the one found in [Win10]. Fix $a \in B_T$ and $K \in \mathcal{K}_T$. We consider $\mathcal{B} := \{\phi \in B_T : d_\infty(\phi, x_{a,K}) < 1\}$. By the arguments from the beginning the operator

$$P_{a,K}(\phi)(t) := a(t) - \int_t^\infty K(t, s)\phi(s)ds$$

is a contraction mapping on B_T and for $\phi \in \mathcal{B}$ it holds

$$d_\infty(P_{a,K}(\phi), x_{a,K}) = d_\infty(P_{a,K}(\phi), P_{a,K}(x_{a,K})) \leq \lambda d_\infty(\phi, x_{a,K})$$

for $\lambda = \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s)\| ds < 1$. Thus $P_{a,K}(\phi)$ descends to a contraction mapping on (i. e. takes values in) \mathcal{B} with contraction factor λ .

Now take $a' \in B_T$ and $K' \in \mathcal{K}_T$. Then it holds

$$\begin{aligned} \|P_{a',K'}(\phi) - P_{a,K}(\phi)\|_\infty &\leq \|a' - a\|_\infty + \left\| \int_t^\infty (K'(t, s) - K(t, s))\phi(s)ds \right\|_\infty \\ &\leq \|a' - a\|_\infty + \gamma \|\phi\|_\infty, \end{aligned} \quad (6.5)$$

for

$$\gamma := \sup_{t \in [T, \infty)} \int_t^\infty \|K'(t, s) - K(t, s)\| ds = d_{\mathcal{K}}(K, K').$$

Note that γ is finite as K' and K are elements of \mathcal{K}_T . Again as before $P_{a',K'}$ is a contraction mapping on B_T and we now want to obtain a condition under which $P_{a',K'}$ also descends to a contraction mapping on \mathcal{B} . For this assume that a' and K' are such that

$$\|a' - a\|_\infty + \gamma(\|x_{a,K}\|_\infty + 1) < 1 - \lambda$$

holds. Then via triangle inequality and the fact that $x_{a,K}$ is the fixed point of $P_{a,K}$ we obtain

$$\begin{aligned} d_\infty(P_{a',K'}(\phi), x_{a,K}) &\leq d_\infty(P_{a',K'}(\phi), P_{a,K}(\phi)) + d_\infty(P_{a,K}(\phi), x_{a,K}) \\ &\leq \|P_{a',K'}(\phi) - P_{a,K}(\phi)\|_\infty + d_\infty(P_{a,K}(\phi), P_{a,K}(x_{a,K})) \\ &\leq \|a' - a\|_\infty + \gamma \|\phi\|_\infty + \lambda d_\infty(\phi, x_{a,K}) \\ &\leq \|a' - a\|_\infty + \gamma(\|x_{a,K}\|_\infty + 1) + \lambda \\ &< 1 - \lambda + \lambda = 1. \end{aligned}$$

Here we used the fact $\|\phi\|_\infty \leq \|x_{a,K}\|_\infty + 1$ which follows from the inverse triangle inequality and the definition of $\phi \in \mathcal{B}$. Therefore in this case $P_{a',K'}$ defines a contraction mapping on \mathcal{B} and we define the corresponding set

$$\mathcal{P} := \{(a', K') \in B_T \times \mathcal{K}_T : \|a' - a\|_\infty + \gamma(\|x_{a,K}\|_\infty + 1) < 1 - \lambda\}.$$

This set contains (a, K) and is open in $B_T \times \mathcal{K}_T$ as the defining inequality is non-strict. Now define the map $P : \mathcal{P} \rightarrow Ctr(B_T)$ by $P(a', K') := P_{a',K'}$. As of the inequality (6.5) we see that $P(\mathcal{P})$ is of finite diameter in d_0 and we obtain

$$\begin{aligned} d_0(P_{a',K'}, P_{a,K}) &= \sup_{\phi \in \mathcal{B}} d_\infty(P_{a',K'}(\phi), P_{a,K}(\phi)) \\ &= \sup_{\phi \in \mathcal{B}} \|P_{a',K'}(\phi)(t) - P_{a,K}(\phi)(t)\|_\infty \\ &\leq \sup_{\phi \in \mathcal{B}} \|a' - a\|_\infty + \gamma \|\phi\|_\infty \\ &\leq d_\infty(a, a') + d_K(K, K')(\|x_{a,K}\|_\infty + 1). \end{aligned}$$

Thus for (a', K') close enough to (a, K) the distance $d_0(P_{a',K'}, P_{a,K})$ becomes arbitrarily small and we see that P is continuous at (a, K) . From the previous lemma, the map $\varpi : Ctr(B_T)|_{P(\mathcal{P})} \rightarrow \mathcal{B}$ is continuous and thus $\varpi \circ P : \mathcal{P} \rightarrow \mathcal{B}$ is continuous at (a, K) . But on the open subset \mathcal{P} of $B_T \times \mathcal{K}_T$ we have $\varpi \circ P = X|_{\mathcal{P}}$ and thus X is continuous at (a, K) . \square

Next up, we would like to reformulate the preceding theorem in terms of an additional parameter on which the equation depends in a continuous way, i. e. the "boundary value" a and kernel K shall now depend continuously on some parameter ϵ . We expand our notation accordingly for some open subset $U \subset \mathbb{R}$:

$$\begin{aligned} B_{T,U} &:= \{f \in C^0([T, \infty) \times U, \mathbb{C}^n) : \exists M \in \mathbb{R} : \forall \epsilon \in U \quad \forall x \in [T, \infty) : |f(x)| \leq M\}, \\ \mathcal{K}_{T,U} &:= \left\{ K \in C^0(\Delta(T) \times U, M_n(\mathbb{C})) : \forall \epsilon \in U : \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s, \epsilon)\| ds < 1 \right\}. \end{aligned}$$

For $a \in B_{T,U}$ we write the corresponding element in B_T as a_ϵ (i. e. if we regard ϵ as fixed) and in the same way for $K \in \mathcal{K}_{T,U}$ the element in \mathcal{K}_T will be K_ϵ . Now let any $a \in B_{T,U}$ and $K \in \mathcal{K}_{T,U}$ be given. Then for each $\epsilon \in U$ there is a unique continuous solution $x_\epsilon \in B_T$ of

$$x_\epsilon(t) = a_\epsilon - \int_t^\infty K_\epsilon(t, s)x_\epsilon(s)ds.$$

as of Theorem 6.1.2. We can thus define the ϵ -dependent solution as

$$x : [T, \infty) \times U \rightarrow \mathbb{C}^2, \quad x(t, \epsilon) := x_\epsilon(t).$$

With this notation we can now reformulate the continuity of x by a standard argument:

Corollary 6.2.3

Let $a \in B_{T,U}$ and $K \in \mathcal{K}_{T,U}$ be such that the maps $\epsilon \mapsto a_\epsilon$ and $\epsilon \mapsto K_\epsilon$ are continuous. Let x be the solution of the parameter-dependent integral equation. Then x is continuous in $\epsilon \in U$ and $t \in [T, \infty)$.

Proof. From our assumption the map $\epsilon \mapsto a_\epsilon$ is continuous as is the map $\epsilon \mapsto K_\epsilon$. Theorem 6.2.2 then tells us that the concatenation with the map $X : B_T \times \mathcal{K}_T \rightarrow B_T$ is continuous, i. e. $\epsilon \mapsto x_\epsilon$. Thus the map

$$f : [T, \infty) \times U \rightarrow [T, \infty) \times B_T, \quad (t, \epsilon) \mapsto (t, x_\epsilon)$$

is continuous in ϵ and t . Finally the evaluation map

$$e : [T, \infty) \times B_T \rightarrow \mathbb{C}^n, \quad (t, x_\epsilon) \mapsto x_\epsilon(t)$$

is continuous. This can be derived from the fact that we are considering bounded continuous functions with the sup-norm by a standard $\epsilon - \delta$ argument. \square

Note that the continuity of the maps $\epsilon \mapsto a_\epsilon$ and $\epsilon \mapsto K_\epsilon$ does not follow simply from the assumption $a \in B_{T,U}$ and $K \in \mathcal{K}_{T,U}$ as the functions in $B_{T,U}$ and $\mathcal{K}_{T,U}$ are defined on the ray $[T, \infty)$. This is a complication compared to Windsor, since he works with functions on a compact interval.

6.3 Differentiable dependence on parameters

We now want our initial conditions to depend differentiably on the parameter ϵ . We would like to just assume that one derivative exists and is continuous and obtain a differentiable solution in the same way as it was done in [Win10]. But the non-compactness now demands some stronger conditions. So the greatest difficulty here is to formulate the right conditions which allow us to obtain the differentiable dependence, while still being loose enough so that they are fulfilled by the functions arising in our applications in section 7. This amounts to demanding the right uniform behavior of our kernels K , for which we find the following notions to be adequate:

A map $B : [T, \infty) \times U \rightarrow M_n(\mathbb{C})$ shall be called *integrally bounded* iff for every $\epsilon \in U$ there exists an open subset $V_\epsilon \subset U$ containing ϵ and a bounded function $g_\epsilon : [T, \infty) \rightarrow \mathbb{R}_{\geq 0}$, s. t.

$$\|B(s, \mu)\| \leq g_\epsilon(s) \quad \forall \mu \in V_\epsilon, \quad \forall s \geq T \quad \text{and} \quad \int_T^\infty g_\epsilon(s) ds < \infty,$$

Note that the integral bound and the boundedness of g generally don't imply one another. Now a kernel K shall be called *of product type* iff it can be written as

$$K(t, s, \epsilon) = A(t, s, \epsilon)B(s, \epsilon),$$

for two functions $A : \Delta(T) \times U \rightarrow M_n(\mathbb{C})$ and $B : [T, \infty) \times U \rightarrow M_n(\mathbb{C})$. Furthermore K shall be called *of product type of zeroth order*, iff it is of product type and A is bounded by some constant $\alpha \in \mathbb{R}$, while B is integrally bounded. It shall be called of product type of order $(1, 0)$, iff it is of product type of zeroth order and $\partial_\epsilon A$ exists and is bounded by some constant $\beta \in \mathbb{R}$ and of order $(0, 1)$, iff $\partial_\epsilon B$ exists and is integrally bounded. The general notion of product type of order (m, n) follows accordingly.

With this notion we can define the space of kernels of any order:

$$\mathcal{K}_{T,U}^{(m,n)} := \{K \in \mathcal{K}_{T,U}, \quad K \text{ is of product type of order } (m, n)\}.$$

Note that this assumption on K is not the most general one with which the following proofs work but it suffices for our set up. The important structure here is that only A depends on t and this factor is (uniformly) bounded. This will allow us to extract an upper bound of the t -dependence out of some of the following integrals. It is this uniform behaviour that we actually need for our results and the sufficient uniform bound will be provided in our applications on the Higgs bundle moduli space later on.

For our main theorem we will first need two results concerning the continuity of our ingredients, which we obtain from the following standard generalization of the mean value theorem.

Proposition 6.3.1

Let $f : (a, b) \rightarrow E$ be a differentiable function from an open interval in \mathbb{R} to a Banach space E . Then for all $x, y \in (a, b)$ it holds

$$\|f(y) - f(x)\| \leq |y - x| \sup_{0 \leq \tau \leq 1} \|f'(x + \tau(y - x))\|.$$

As a reminder for our set up the Banach spaces in question will be \mathbb{C}^n and $M_n(\mathbb{C})$, the first with the standard norm and the second with the operator norm. With this at hand we may now proceed. First up is the "boundary value":

Lemma 6.3.2

Let $a \in B_{T,U}$ be differentiable with respect to $\epsilon \in U$ and such that the derivative is still bounded, i. e.

$$\frac{\partial a}{\partial \epsilon} \in B_{T,U}$$

Then the map

$$f : U \rightarrow B_T, \quad \epsilon \mapsto a_\epsilon$$

is continuous.

Proof. We start with $a \in B_{T,U}$ and consider continuity of f at some $\epsilon_0 \in U$. Since $\frac{\partial a}{\partial \epsilon} \in B_{T,U}$ there exists an $M \in \mathbb{R}$ s. t. $|\frac{\partial a}{\partial \epsilon}(\mu, t)| < M$ for all $t \in [T, \infty)$ and $\mu \in U$. For any $\epsilon_1 \in U$ it then follows from the generalized mean value theorem:

$$|a(\epsilon_0, t) - a(\epsilon_1, t)| \leq \sup_{0 \leq \tau \leq 1} \left| \frac{\partial a}{\partial \epsilon}((\epsilon_0 + \tau(\epsilon_1 - \epsilon_0)), t) \right| |\epsilon_0 - \epsilon_1| \leq M |\epsilon_0 - \epsilon_1|.$$

Now for any $\nu > 0$ define $\delta := \frac{\nu}{M}$. Then, if we have $|\epsilon_0 - \epsilon_1| < \delta$ it follows

$$\begin{aligned} \|f(\epsilon_0) - f(\epsilon_1)\|_\infty &= \sup_{t \in [T, \infty)} |f(\epsilon_0)(t) - f(\epsilon_1)(t)| = \sup_{t \in [T, \infty)} |a(\epsilon_0, t) - a(\epsilon_1, t)| \\ &\leq \sup_{t \in [T, \infty)} M |\epsilon_0 - \epsilon_1| \\ &< M\delta \leq \nu. \end{aligned}$$

So f is continuous. □

Next up is the kernel:

Lemma 6.3.3

Let $K \in \mathcal{K}_{T,U}^{(1,1)}$. Then the map

$$f : U \rightarrow \mathcal{K}_T, \quad \epsilon \mapsto K_\epsilon$$

is continuous.

Proof. For the continuity of f we have to work with the metric d_K mentioned above. We want to show the continuity at some $\epsilon_0 \in U$. For any other $\epsilon_1 \in U$ we first use the mean value estimate

$$\begin{aligned} d_K(K(t, s, \epsilon_0), K(t, s, \epsilon_1)) &= \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s, \epsilon_0) - K(t, s, \epsilon_1)\| ds \\ &\leq \sup_{t \in [T, \infty)} \int_t^\infty \sup_{0 \leq \tau \leq 1} \left\| \frac{\partial K}{\partial \epsilon}(t, s, \epsilon_0 + \tau(\epsilon_1 - \epsilon_0)) \right\| |\epsilon_0 - \epsilon_1| ds \end{aligned}$$

Now K is of product type of order $(1, 1)$, i. e. it holds

$$\begin{aligned} K(t, s, \epsilon) &= A(t, s, \epsilon)B(s, \epsilon) \text{ and} \\ \frac{\partial K}{\partial \epsilon}(t, s, \epsilon) &= \frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon) + A(t, s, \epsilon)\frac{\partial B}{\partial \epsilon}(s, \epsilon), \end{aligned}$$

where A and $\partial_\epsilon A$ are bounded by some constants $\alpha, \beta \in \mathbb{R}$ and $B(s, \mu)$ and $\partial_\epsilon B(s, \mu)$ are integrally bounded by some functions g_0 and g_1 for all $\mu \in U$ close enough to ϵ . Therefore in the integral above, we obtain for ϵ_1 close enough to ϵ_0

$$\sup_{0 \leq \tau \leq 1} \left\| \frac{\partial K}{\partial \epsilon}(t, s, \epsilon_0 + \tau(\epsilon_1 - \epsilon_0)) \right\| \leq \alpha g_1(s) + \beta g_2(s).$$

It follows

$$\begin{aligned} d_K(K(t, s, \epsilon_0), K(t, s, \epsilon_1)) &\leq \sup_{t \in [T, \infty)} \int_t^\infty \alpha g_1(s) + \beta g_2(s) ds |\epsilon_0 - \epsilon_1| \\ &\leq \int_T^\infty \alpha g_1(s) + \beta g_2(s) ds |\epsilon_0 - \epsilon_1| \\ &\leq C |\epsilon_0 - \epsilon_1|. \end{aligned}$$

for some constant $C \in \mathbb{R}$. Therefore $d_K(K(t, s, \epsilon_0), K(t, s, \epsilon_1))$ becomes arbitrarily small and f is continuous. □

We now come to the main theorem in this section.

Theorem 6.3.4

Let $a \in B_{T,U}$ be bounded and such that the first and second derivative with respect to ϵ exist and are elements of $B_{T,U}$. Let $K \in \mathcal{K}_{T,U}^{(2,1)}$ be of product type of order $(2, 1)$. Then the solution $x(t, \epsilon)$ of the integral equation

$$x(t, \epsilon) = a(t, \epsilon) - \int_t^\infty K(t, s, \epsilon)x(s, \epsilon)ds$$

is differentiable in ϵ with $\frac{\partial x}{\partial \epsilon} \in B_{T,U}$ and the partial derivative satisfies the Volterra integral equation

$$\frac{\partial x}{\partial \epsilon}(t, \epsilon) = \left(\frac{\partial a}{\partial \epsilon}(\epsilon, t) - \int_t^\infty \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(\epsilon, s)ds \right) - \int_t^\infty K(t, s, \epsilon) \frac{\partial x}{\partial \epsilon}(s, \epsilon)ds$$

for every $\epsilon \in U$.

Proof. With the notions introduced above it is now possible for the proof to follow the general idea of [Win10]. Still the non-compact integration range demands a lot of additional estimates now. The parameter dependent solution $x(t, \epsilon)$ of the integral equation exists and depends continuously on ϵ as of Corollary 6.2.3 and the preceding two lemmas. In order to show the differentiability we first construct a candidate for the derivative and then show, that this candidate is indeed the derivative.

Formally differentiating the integral equation with respect to ϵ yields

$$\begin{aligned} \frac{\partial x}{\partial \epsilon}(t, \epsilon) &= \frac{\partial a}{\partial \epsilon}(\epsilon, t) - \int_t^\infty \left(\frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(\epsilon, s) - K(t, s, \epsilon) \frac{\partial x}{\partial \epsilon}(s, \epsilon) \right) ds \\ &= \left(\frac{\partial a}{\partial \epsilon}(\epsilon, t) - \int_t^\infty \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(\epsilon, s)ds \right) - \int_t^\infty K(t, s, \epsilon) \frac{\partial x}{\partial \epsilon}(s, \epsilon)ds. \end{aligned} \tag{6.6}$$

As of our assumptions the term

$$\frac{\partial a}{\partial \epsilon}(\epsilon, t) - \int_t^\infty \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(\epsilon, s)ds$$

is continuous in t (for every $\epsilon \in U$) and bounded. Therefore Theorem 6.1.2 tells us, that there is a unique solution of equation (6.6), which shall be denoted by $\widehat{\frac{\partial x}{\partial \epsilon}}$.

Note that $\widehat{\frac{\partial x}{\partial \epsilon}}$ only formally satisfies the equation for the derivative. In fact, in formulating equation (6.6) we pulled the differential under the integral which entails the assumption that the solution is still continuous in ϵ , which we can't know yet. So instead we have to consider only the final equation as of itself and now show, that the solution is in fact the derivative. For this consider the difference quotient

$$q(\epsilon, \Delta\epsilon, t) := \frac{x(\epsilon + \Delta\epsilon) - x(\epsilon, t)}{\Delta\epsilon}.$$

As x solves the corresponding integral equation in a neighborhood of ϵ the difference quotient satisfies the following integral equation

$$q(\epsilon, \Delta\epsilon, t) = \frac{a(t, \epsilon + \Delta\epsilon) - a(t, \epsilon)}{\Delta\epsilon} - \int_t^\infty \frac{K(t, s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)x(s, \epsilon)}{\Delta\epsilon} ds.$$

Now we reformulate the integrand to obtain an expression involving the difference quotient:

$$\begin{aligned}
 & \frac{K(t, s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)x(s, \epsilon)}{\Delta\epsilon} \\
 &= \frac{(K(t, s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon))x(s, \epsilon + \Delta\epsilon) + K(t, s, \epsilon)(x(s, \epsilon + \Delta\epsilon) - x(s, \epsilon))}{\Delta\epsilon} \\
 &= \frac{K(t, s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)}{\Delta\epsilon}x(s, \epsilon + \Delta\epsilon) + K(t, s, \epsilon)\frac{x(s, \epsilon + \Delta\epsilon) - x(s, \epsilon)}{\Delta\epsilon} \\
 &= \frac{K(t, s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)}{\Delta\epsilon}x(s, \epsilon + \Delta\epsilon) + K(t, s, \epsilon)q(\epsilon, \Delta\epsilon, s).
 \end{aligned}$$

With this, we can now look at the difference $\frac{\widehat{\partial x}}{\partial \epsilon} - q(\epsilon, \Delta\epsilon, t)$ and obtain another integral equation:

$$\begin{aligned}
 \frac{\widehat{\partial x}}{\partial \epsilon}(t, \epsilon) - q(\epsilon, \Delta\epsilon, t) &= \frac{\partial a}{\partial \epsilon}(\epsilon, t) - \frac{a(t, \epsilon + \Delta\epsilon) - a(t, \epsilon)}{\Delta\epsilon} \\
 &\quad - \int_t^\infty \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(s, \epsilon) - \frac{K(t, s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)}{\Delta\epsilon}x(s, \epsilon + \Delta\epsilon)ds \\
 &\quad - \int_t^\infty K(t, s, \epsilon) \left(\frac{\widehat{\partial x}}{\partial \epsilon}(s, \epsilon) - q(\epsilon, \Delta\epsilon, s) \right) ds \\
 &=: \widehat{a}(t, \epsilon, \Delta\epsilon) - \int_t^\infty K(t, s, \epsilon) \left(\frac{\widehat{\partial x}}{\partial \epsilon}(s, \epsilon) - q(\epsilon, \Delta\epsilon, s) \right) ds
 \end{aligned}$$

Again the first term defines a bounded continuous function on $[T, \infty)$, so we know that $\frac{\widehat{\partial x}}{\partial \epsilon} - q(\epsilon, \Delta\epsilon, t)$ is the unique solution to this integral equation. This is helpful, as we can now use the estimate from the contraction mapping principle to obtain an estimate for the difference $\frac{\widehat{\partial x}}{\partial \epsilon} - q(\epsilon, \Delta\epsilon, t)$. In fact in our situation Corollary 6.1.4 tells us

$$\left\| \frac{\widehat{\partial x}}{\partial \epsilon} - q(\epsilon, \Delta\epsilon, t) \right\|_\infty \leq \frac{\|\widehat{a}\|_\infty}{1 - \lambda}.$$

Note that $\lambda = \sup_{t \in [T, \infty)} \int_t^\infty \|K(t, s, \epsilon)\| ds$ only depends on ϵ and not on $\Delta\epsilon$, so it remains the same as we now want to consider the limit $\Delta\epsilon \rightarrow 0$.

The Taylor expansion of a at ϵ up to first order is

$$a(t, \epsilon + \Delta\epsilon) = a(t, \epsilon) + \frac{\partial a}{\partial \epsilon}(t, \epsilon)\Delta\epsilon + R(t, \Delta\epsilon),$$

for the remainder R that we may estimate in Lagrange form via

$$\|R(t, \Delta\epsilon)\| \leq \frac{1}{2} \left\| \frac{\partial^2 a}{\partial \epsilon^2}(t, \mu)(\Delta\epsilon)^2 \right\|,$$

for some $\mu \in [\epsilon, \epsilon + \Delta\epsilon]$. Thus the norm of the first part of the "boundary value" of the last integral equation can be estimated as

$$\begin{aligned}
 & \left\| \frac{\partial a}{\partial \epsilon}(t, \epsilon) - \frac{a(t, \epsilon + \Delta\epsilon) - a(t, \epsilon)}{\Delta\epsilon} \right\|_\infty \leq \left\| \frac{R(t, \Delta\epsilon)}{\Delta\epsilon} \right\| \\
 & \leq \frac{1}{2} \left\| \frac{\partial^2 a}{\partial \epsilon^2}(t, \mu) \right\| |\Delta\epsilon|.
 \end{aligned}$$

As we assumed our a to have a bounded second derivative we thus see, that the norm vanishes in the $\Delta\epsilon \rightarrow 0$ limit.

We would like to do the same for the integrand part, which basically works, but we have to put more work into this part, as the $\Delta\epsilon$ limit, the sup-norm and the integral generally don't commute. It is mainly for this reason that we introduced the product structure for K as this gives us the desired uniform behavior. So now we write our kernel K as $K(t, s, \epsilon) = A(t, s, \epsilon) \cdot B(s, \epsilon)$. This leads to

$$\begin{aligned}
 & \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(s, \epsilon) - \frac{K(t, s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)}{\Delta\epsilon}x(s, \epsilon + \Delta\epsilon) \\
 &= \left(\frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon) + A(t, s, \epsilon)\frac{\partial B}{\partial \epsilon}(s, \epsilon) \right) x(s, \epsilon) \\
 & - \left(\frac{A(t, s, \epsilon + \Delta\epsilon)B(s, \epsilon + \Delta\epsilon) - A(t, s, \epsilon)B(s, \epsilon)}{\Delta\epsilon} \right) x(s, \epsilon + \Delta\epsilon) \\
 &= \left(\frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon) + A(t, s, \epsilon)\frac{\partial B}{\partial \epsilon}(s, \epsilon) \right) x(s, \epsilon) \\
 & - \left(\frac{A(t, s, \epsilon + \Delta\epsilon) - A(t, s, \epsilon)}{\Delta\epsilon}B(s, \epsilon + \Delta\epsilon) + A(t, s, \epsilon)\frac{B(s, \epsilon + \Delta\epsilon) - B(s, \epsilon)}{\Delta\epsilon} \right) x(s, \epsilon + \Delta\epsilon) \\
 &= \frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon)x(s, \epsilon) - \frac{A(t, s, \epsilon + \Delta\epsilon) - A(t, s, \epsilon)}{\Delta\epsilon}B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon) \\
 & + A(t, s, \epsilon)\frac{\partial B}{\partial \epsilon}(s, \epsilon)x(s, \epsilon) - A(t, s, \epsilon)\frac{B(s, \epsilon + \Delta\epsilon) - B(s, \epsilon)}{\Delta\epsilon}x(s, \epsilon + \Delta\epsilon).
 \end{aligned} \tag{6.7}$$

Let us consider the first half of the final expression above. We write it as

$$\begin{aligned}
 & \frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon)x(s, \epsilon) - \frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon) \\
 & + \frac{\partial A}{\partial \epsilon}(t, s, \epsilon)B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon) - \frac{A(t, s, \epsilon + \Delta\epsilon) - A(t, s, \epsilon)}{\Delta\epsilon}B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon) \\
 &= \frac{\partial A}{\partial \epsilon}(t, s, \epsilon) (B(s, \epsilon)x(s, \epsilon) - B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon)) \\
 & + \left(\frac{\partial A}{\partial \epsilon}(t, s, \epsilon) - \frac{A(t, s, \epsilon + \Delta\epsilon) - A(t, s, \epsilon)}{\Delta\epsilon} \right) B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon)
 \end{aligned} \tag{6.8}$$

Now for the first part of this expression, we know that $B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon)$ converges point-wise to $B(s, \epsilon)x(s, \epsilon)$ as $\Delta\epsilon \rightarrow 0$. As of our assumption on B and the boundedness of x there exists some function $b : [T, \infty) \rightarrow \mathbb{R}_{\geq 0}$ with $\int_T^\infty b(s)ds < \infty$ s. t. $\|B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon)\| \leq b(s)$ for all $\Delta\epsilon$ close enough to 0. Therefore, by the Dominated convergence theorem (for Bochner integrable functions) it holds

$$\lim_{\Delta\epsilon \rightarrow 0} \int_T^\infty \|B(s, \epsilon)x(s, \epsilon) - B(s, \epsilon + \Delta\epsilon)x(s, \epsilon + \Delta\epsilon)\| ds = 0.$$

Thus, by taking into account that there is an upper bound $\beta \in \mathbb{R}$ for $\frac{\partial A}{\partial \epsilon}(t, s, \epsilon)$ we get for the sup-norm of the whole first term of (6.8):

$$\begin{aligned} & \lim_{\Delta \epsilon \rightarrow 0} \left\| \int_t^\infty \frac{\partial A}{\partial \epsilon}(t, s, \epsilon) (B(s, \epsilon)x(s, \epsilon) - B(s, \epsilon + \Delta \epsilon)x(s, \epsilon + \Delta \epsilon)) ds \right\|_\infty \\ & \leq \beta \lim_{\Delta \epsilon \rightarrow 0} \int_T^\infty \|B(s, \epsilon)x(s, \epsilon) - B(s, \epsilon + \Delta \epsilon)x(s, \epsilon + \Delta \epsilon)\| ds = 0. \end{aligned}$$

It is for this inequality (and for following ones of the same kind) that we assumed the product structure of our kernel K , as it gives us the desired uniform behavior.

Next up in (6.8) is the term

$$\int_t^\infty \left(\frac{\partial A}{\partial \epsilon}(t, s, \epsilon) - \frac{A(t, s, \epsilon + \Delta \epsilon) - A(t, s, \epsilon)}{\Delta \epsilon} \right) B(s, \epsilon + \Delta \epsilon)x(s, \epsilon + \Delta \epsilon) ds.$$

The x part is now bounded by some constant C for all $\Delta \epsilon$ small enough. For the difference quotient we consider again the Taylor approximation of $A(t, s, \epsilon)$ with respect to ϵ (i. e. t and s are fixed):

$$A(t, s, \epsilon + \Delta \epsilon) = A(t, s, \epsilon) + \frac{\partial A}{\partial \epsilon}(t, s, \epsilon)\Delta \epsilon + R(t, s, \Delta \epsilon),$$

with the remainder term R . Thus the A part in the integrand can also be regarded as

$$-\frac{R(t, s, \Delta \epsilon)}{\Delta \epsilon}.$$

For the remainder we again use the Lagrange estimate, i. e.:

$$\|R(t, s, \Delta \epsilon)\| \leq \frac{1}{2} \left\| \frac{\partial^2 A}{\partial \epsilon^2}(t, s, \mu) (\Delta \epsilon)^2 \right\|$$

for some $\mu \in [\epsilon, \epsilon + \Delta \epsilon]$. Now we use the assumption that in a neighbourhood of ϵ the second derivative of A is still bounded by some γ , so we obtain for $\Delta \epsilon$ small enough:

$$\begin{aligned} & \left\| \int_t^\infty \left(\frac{\partial A}{\partial \epsilon}(t, s, \epsilon) - \frac{A(t, s, \epsilon + \Delta \epsilon) - A(t, s, \epsilon)}{\Delta \epsilon} \right) B(s, \epsilon + \Delta \epsilon)x(s, \epsilon + \Delta \epsilon) ds \right\|_\infty \\ & \leq \sup_{t \in [T, \infty)} \int_t^\infty \left\| \left(\frac{\partial A}{\partial \epsilon}(t, s, \epsilon) - \frac{A(t, s, \epsilon + \Delta \epsilon) - A(t, s, \epsilon)}{\Delta \epsilon} \right) B(s, \epsilon + \Delta \epsilon)x(s, \epsilon + \Delta \epsilon) \right\| ds \\ & \leq \frac{C}{2} \sup_{t \in [T, \infty)} \int_t^\infty \left\| \frac{\partial^2 A}{\partial \epsilon^2}(t, s, \mu) \right\| \|B(s, \epsilon + \Delta \epsilon)\| ds \Delta \epsilon \leq \frac{C}{2} \gamma \int_T^\infty \|B(s, \epsilon + \Delta \epsilon)\| ds \Delta \epsilon \\ & \leq \frac{C\gamma}{2} \int_T^\infty \|B(s, \epsilon + \Delta \epsilon)\| ds \Delta \epsilon \leq \frac{C\gamma}{2} \int_T^\infty g(s) ds \Delta \epsilon. \end{aligned}$$

As g is integrable this term also vanishes in the $\Delta \epsilon \rightarrow 0$ limit.

Finally there is the last part in (6.7)

$$\int_t^\infty A(t, s, \epsilon) \left(\frac{\partial B}{\partial \epsilon}(s, \epsilon)x(s, \epsilon) - \frac{B(s, \epsilon + \Delta \epsilon) - B(s, \epsilon)}{\Delta \epsilon} x(s, \epsilon + \Delta \epsilon) \right) ds.$$

Here we use a combination of the arguments above. Firstly A is bounded by some $\alpha \in \mathbb{R}$ for all $(t, s) \in \Delta(T)$. From the mean value estimate we obtain

$$\|B(s, \epsilon + \Delta\epsilon) - B(s, \epsilon)\| \leq |\Delta\epsilon| \sup_{0 \leq \tau \leq 1} \left\| \frac{\partial B}{\partial \epsilon}(s, \epsilon + \tau \Delta\epsilon) \right\|.$$

We assumed $\partial_\epsilon B$ to be integrally bounded, so there is some integrable function h which gives a pointwise upper bound for all $\Delta\epsilon$ small enough. As x is also bounded the whole right side in the integrand is bounded by some general integrable function. As the right side also converges pointwise to the left side, we obtain by the dominant convergence theorem that the integral vanishes in the $\Delta\epsilon \rightarrow 0$ limit, and thus

$$\begin{aligned} & \lim_{\Delta\epsilon \rightarrow 0} \left\| \int_t^\infty A(t, s, \epsilon) \left(\frac{\partial B}{\partial \epsilon}(s, \epsilon)x(s, \epsilon) - \frac{B(s, \epsilon + \Delta\epsilon) - B(s, \epsilon)}{\Delta\epsilon} x(s, \epsilon + \Delta\epsilon) \right) ds \right\|_\infty \\ & \leq \lim_{\Delta\epsilon \rightarrow 0} \alpha \sup_{t \in [T, \infty)} \int_t^\infty \left\| \left(\frac{\partial B}{\partial \epsilon}(s, \epsilon)x(s, \epsilon) - \frac{B(s, \epsilon + \Delta\epsilon) - B(s, \epsilon)}{\Delta\epsilon} x(s, \epsilon + \Delta\epsilon) \right) \right\| ds \\ & \leq \alpha \lim_{\Delta\epsilon \rightarrow 0} \int_T^\infty \left\| \left(\frac{\partial B}{\partial \epsilon}(s, \epsilon)x(s, \epsilon) - \frac{B(s, \epsilon + \Delta\epsilon) - B(s, \epsilon)}{\Delta\epsilon} x(s, \epsilon + \Delta\epsilon) \right) \right\| ds = 0. \end{aligned}$$

We have thus shown that all terms in (6.7) vanish and we obtain

$$\lim_{\Delta\epsilon \rightarrow 0} \left\| \int_t^\infty \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(s, \epsilon) - \frac{K(t, s, \epsilon + \Delta\epsilon) - K(t, s, \epsilon)}{\Delta\epsilon} x(s, \epsilon + \Delta\epsilon) ds \right\|_\infty = 0.$$

And with this the estimate 6.1.4 tells us

$$\lim_{\Delta\epsilon \rightarrow 0} \left\| \widehat{\frac{\partial x}{\partial \epsilon}} - q(\epsilon, \Delta\epsilon, t) \right\|_\infty \leq \frac{\lim_{\Delta\epsilon \rightarrow 0} \|\widehat{a}\|_\infty}{1 - \lambda} = 0.$$

So the difference quotient converges towards our candidate for the derivative $\widehat{\frac{\partial x}{\partial \epsilon}}$ in the sup-norm, and thus also point-wise. So it is in fact the true derivative. As it is a solution of the aforementioned integral equation (6.6) we also know that it is bounded and continuous in t . As x is bounded and $\partial_\epsilon K$ is of product type and integrally bounded it is also continuous in ϵ . \square

This finishes the proof for the continuous differentiability in ϵ and one may expect that corresponding propositions for higher derivatives hold, given the corresponding differentiability of the ingredients a and K . In fact Windsor gives a proof by induction for his setting, and the arguments should translate to our set up, but as they are not needed for our applications in the second part and don't promise more important insights they are not pursued at this point.

We will, however, need the extension of the arguments for holomorphic dependence. For this we first note that the notion of kernels of product type readily translates to the notion of holomorphic product types $K \in \mathcal{K}_{T,U}^h$. We don't write them here explicitly but only note that we do not need to distinguish product types of order greater than 1. Instead by demanding the parts of the kernel to be complex differentiable we already have that they are infinitely often complex differentiable and, thanks to Cauchy's differentiation formula, the upper bound and integral bound for the kernel are also corresponding bounds for the derivatives. Thus the notion of order is not necessary in the holomorphic setting.

Theorem 6.3.5

Let $a \in B_{T,U}$ be bounded and holomorphic in $\epsilon \in U$ where U is some open subset of \mathbb{C} . Let $K \in \mathcal{K}_{T,U}^h$ be of holomorphic product type. Then the solution $x(t, \epsilon)$ of the integral equation

$$x(t, \epsilon) = a(t, \epsilon) - \int_t^\infty K(t, s, \epsilon)x(s, \epsilon)ds$$

is holomorphic in ϵ with $\frac{\partial x}{\partial \epsilon} \in B_{T,U}$ and the partial derivative satisfies the Volterra integral equation

$$\frac{\partial x}{\partial \epsilon}(t, \epsilon) = \left(\frac{\partial a}{\partial \epsilon}(\epsilon, t) - \int_t^\infty \frac{\partial K}{\partial \epsilon}(t, s, \epsilon)x(s, \epsilon)ds \right) - \int_t^\infty K(t, s, \epsilon) \frac{\partial x}{\partial \epsilon}(s, \epsilon)ds$$

for every $\epsilon \in U$.

Proof. For the proof we simply have to retrace all the steps in section 6.3. We don't do this in detail and just note that the two main estimates that were used in the proof of Theorem 6.3.4 also hold in the holomorphic setting. That is firstly the mean value inequality, for which a corresponding estimate in the holomorphic setting is standard. The other estimate concerns the use of the Lagrangian remainder in the Taylor expansion. Similar to the explanation above this is not needed now. Instead the remainder term for all relevant parts is bounded by the some multiple of the function itself thanks to Cauchy's differentiation formula. Thus all relevant estimates are given and the proof can be recast in an obvious manner. \square

Chapter 7

Construction of the coordinate system

We now come to the main part of this endeavor. While the previous section was mostly concerned with a theory that can stand on its own, we now want to use it for the construction of hyperkähler metrics on the moduli space of weakly parabolic Higgs bundles. Here most of the essential work is done as we use the theory developed in chapter 6 to obtain the correct decorations and deduce their asymptotic behaviour. This is first done for the small flat sections in section 7.1 and then for the canonical coordinates in section 7.2. Finally in section 7.3 the logarithmic derivatives of the coordinates are studied as they will provide us with the holomorphic symplectic form from which the hyperkähler metric is obtained in the last section.

7.1 Obtaining the small flat sections

As was already hinted at at the beginning of section 4.2 we actually construct coordinates on the moduli space of flat connections \mathcal{M}^{dR} which we identify with \mathcal{M} and \mathcal{M}_R in the following way. We start with a stable parabolic Higgs bundle $(E, \bar{\partial}_E, \Phi)$ of parabolic degree $\text{pdeg}(\mathcal{E}) = 0$ and a solution h of the R -rescaled Hitchin equations to obtain a pair (A, Φ) consisting of the Higgs field Φ and the Chern-connection $A = A(\bar{\partial}_E)$ on the Hermitian vector bundle (E, h) . Additionally we now take any $\zeta \in \mathbb{C}^\times$ and construct the connection

$$\nabla(\zeta) := d_A + \frac{R}{\zeta} \Phi + R\zeta \Phi^{*h}.$$

The ζ -dependence will later on lead to the objects of interest being twisted by the line bundle $\mathcal{O}(2)$ in accordance with the twistor theorem. Therefore this connection will be referred to as *twistor*-connection here. The fact that (A, Φ) is a solution to Hitchin's equations and $\text{pdeg}(\mathcal{E}) = 0$ implies that this connection is flat. Note that the flatness of $\nabla(\zeta)$ is of quite some importance in the following arguments, as we are interested in a unique way of parallel transporting sections in E . For different parabolic degree one would obtain projectively flat connections and it seems reasonable that the theory can be adapted accordingly. But as we also used this restriction on the degree before in section 5.1 when we established the background metric on the Higgs bundles, we will only consider the strictly flat case here.

In the rest of this section we want to study the decorations that belong to this family of connections. So in light of the construction in section 4.2.2 we have to study the ∇ -flat sections of E and choose one for each vertex p of any ca -triangulation. So we look at the solutions s of the flatness equation

$$\nabla(\zeta)s = 0.$$

We first briefly describe how GMN defined the decorations and which problems arise in their setting, so that we can then state the goal our of treatment of the subject. For this consider a local patch around any of the poles such that the pole has coordinate $z = 0$. GMN consider Higgs pairs (Φ, A) of the form

$$\begin{aligned}\Phi &= m\sigma^3 \frac{dz}{z} + \text{regular}, \\ A &= m^{(3)}\sigma^3 \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \text{regular},\end{aligned}\tag{7.1}$$

for $m \in \mathbb{C}$ and $m^{(3)} \in \mathbb{R}$. Note that $m^{(3)}$ corresponds to a choice of the parabolic weights as $\alpha_1 = 2m^{(3)} = -\alpha_2$, while m equals the residue of the Higgs field at the pole at $z = 0$.

Then they argue that a Frobenius analysis near a regular singular point shows that there are two flat sections of $\nabla(\zeta)$ of the form

$$\begin{aligned}s^{(1)} &= z^{-R\zeta^{-1}m+m^{(3)}} \bar{z}^{-R\zeta\bar{m}+m^{(3)}} \begin{pmatrix} 1 + O(|z|) \\ O(|z|) \end{pmatrix}, \\ s^{(2)} &= z^{R\zeta^{-1}m-m^{(3)}} \bar{z}^{R\zeta\bar{m}+m^{(3)}} \begin{pmatrix} O(|z|) \\ 1 + O(|z|) \end{pmatrix}.\end{aligned}$$

These have clockwise monodromy eigenvalues $\mu^{(1)} = e^{2\pi i\nu}$, $\mu^{(2)} = e^{-2\pi i\nu}$ with $\nu = R\zeta^{-1}m - 2m^{(3)} - R\zeta\bar{m}$. To single out one of the solutions they consider these solutions along ca -trajectories and ask whether they decay or grow when running into the pole. Thus we now fix some generic angle ϑ . It turns out that for the following constructions to work the choice of ϑ and ζ have to be tied together to gain control over how a ca -trajectory behaves. This is done by introducing the half-plane centered around $e^{i\vartheta}\mathbb{R}_+$:

$$\mathbb{H}_\vartheta := \left\{ \zeta \in \mathbb{C}^\times : \vartheta - \frac{\pi}{2} < \arg \zeta < \vartheta + \frac{\pi}{2} \right\}.$$

We will now always take $\zeta \in \mathbb{H}_\vartheta$. Then along any ϑ -trajectory with $\zeta \in \mathbb{H}_\vartheta$ one of the sections is exponentially decaying while the other one is exponentially growing. The space of exponentially decaying sections is thus a 1-dimensional subspace of solutions and seems to be well equipped to be the decoration for the parabolic point.

Following GMN one could now define the *small flat section* at a weakly parabolic point p as the (up to complex rescaling) unique $\nabla(\zeta)$ -flat section s of E that decays exponentially when following a ϑ -trajectory into p . We will not do this here, as there are some problems with this definition. A minor detail is that we do not work with a traceless connection matrix, so the argument would have to be adjusted to this. Though this might be possible another problem is that from the setup of [GMN13b] it is a priori not clear that the remainder terms in the expansion of the Higgs field and Chern connection are regular enough for the asymptotic analysis to hold rigorously for all Higgs pairs. One could try to argue for this using the results from the previous section, but the most important difficulty would remain even if such an analysis were possible, as it is a statement concerning only small neighborhoods of the parabolic points. Even if one were able to define these sections in this way it is not clear at all at this point that this amounts to a decoration for which the construction of the Fock-Goncharov coordinate system via Theorem 4.2.3 works, as we

do not know how these sections propagate throughout the rest of \mathcal{C} . It is important that such a section which is exponentially decaying when running into some weakly parabolic point p_1 along a ϑ -trajectory is not also decaying when running in the opposite direction into another weakly parabolic point p_2 . Only if this can be ruled out it is clear that the \mathcal{X} -coordinates are well defined and no linear dependence appears in the \wedge -products. Even then, to construct the \wedge -product of two such decorations we have to evaluate them at some common points on \mathcal{C} for which we also need to know how the sections develop away from weakly parabolic points where they are singled out. Finally we need to know the asymptotics of the sections for $\zeta \rightarrow 0$, $\zeta \rightarrow \infty$ and we would also like to have the asymptotics for $R \rightarrow \infty$. GMN argued that all of these aspects are as desired mainly by using an argument for the asymptotics build on the WKB approximation. The main goal here is to give a rigorous proof of these assertions via the limiting configurations of [Fre+20] and the theory of initial value problems at infinity.

With these goals in mind we now proceed to recast the problem in the formalism of section 6. For the rest of this section a generic angle ϑ for $-q = \det \Phi$ shall be fixed. We now also fix a standard annulus S for a standard quadrilateral Q and corresponding frame F as working environment for all our calculations of this section. Locally the flatness equation for s is given as

$$0 = \nabla s = \left(d + \frac{R}{\zeta} \varphi dz + A_z dz + A_{\bar{z}} d\bar{z} + R\zeta \varphi^{*h} d\bar{z} \right) s \quad (7.2)$$

The form of ∇ implies that the dz and $d\bar{z}$ part of ds have to become zero independently, so we obtain the following two PDEs:

$$\partial_z s = \left(-\frac{R}{\zeta} \varphi - A_z \right) s \quad \wedge \quad \partial_{\bar{z}} s = (-R\zeta \varphi^{*h} - A_{\bar{z}}) s.$$

Now let $\gamma(t)$ be a ϑ -trajectory for $q^{1/2}$ and some generic angle $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ that runs between two poles p_1 and p_2 of q and thus forms one of the boundary curves of the quadrilateral Q . We also fix the parametrization of the trajectory for the following calculations in such a way that $q^{1/2}(\gamma') = f^{1/2}(z(\gamma(t)))z'(\gamma(t)) = -e^{i\vartheta}$. Note that we have thus fixed an orientation for the trajectory which determines whether the trajectory runs from p_1 to p_2 for increasing t or in the other direction.

We take this curve to be extend to all of \mathbb{R} as of Proposition 4.1.10. Thus trivialized the section s can be regarded as a function $s : \mathbb{R} \rightarrow \mathbb{C}^2$. For notational simplicity let $z(t)$ denote $z(\gamma(t))$. Locally in S along the ϑ -trajectory $\gamma(t)$ the flat section $s(t) = s(z(t))$ then satisfies the following ODE-form of the flatness equation:

$$\begin{aligned} \frac{ds}{dt}(t) &= \frac{\partial s}{\partial z}(z(t)) \frac{dz}{dt}(t) + \frac{\partial s}{\partial \bar{z}}(\bar{z}(t)) \frac{d\bar{z}}{dt}(t) \\ &= \left(\left(-\frac{R}{\zeta} \varphi - A_z \right) z' + (-R\zeta \varphi^{*h} - A_{\bar{z}}) \bar{z}' \right) s =: B(t)s(t). \end{aligned} \quad (7.3)$$

Note that this trivialized s depends on the chosen gauge. This dependency vanishes in the final expression \mathcal{X} , as the gauge transformations cancel each other in the defining formula.

To solve the ODE we start by rewriting B in the acquired form of leading term plus remainder from section 5 in any contractible open subset U of S . For the leading part we define

$$\begin{aligned}\lambda_1 &:= \frac{R}{\zeta} e^{i\vartheta} - \overline{a_1} z' + a_1 z' + R\zeta e^{-i\vartheta} = \frac{R}{\zeta} e^{i\vartheta} + 2i\text{Im}(a_1 z') + R\zeta e^{-i\vartheta}, \\ \lambda_2 &:= \frac{R}{\zeta} e^{i\vartheta} + \overline{a_2} z' - a_2 z' + R\zeta e^{-i\vartheta} = \frac{R}{\zeta} e^{i\vartheta} - 2i\text{Im}(a_2 z') + R\zeta e^{-i\vartheta}.\end{aligned}\tag{7.4}$$

Note that a_1 and a_2 have simple poles at the weakly parabolic points, so by multiplying with z' the whole expression for the leading part becomes regular. More details on this follow shortly in the next proof. For the error part we define

$$\begin{aligned}e_1 &:= (a_1 - a_2)|\beta|^2 z' + R\zeta 2e^{-i\vartheta}|\beta|^2 + h_1 z' = ((a_2 - a_1)z' + R\zeta 2e^{-i\vartheta})|\beta|^2 - h_1 z', \\ e_2 &:= (a_1 - a_2)\delta\beta z' + R\zeta 2e^{-i\vartheta}\delta\beta + h_2 z' = ((a_2 - a_1)z' + R\zeta 2e^{-i\vartheta})\delta\beta - h_2 z', \\ e_3 &:= -(a_1 - a_2)\alpha\bar{\beta}z' - R\zeta 2e^{-i\vartheta}\alpha\bar{\beta} - h_3 z' = (-(a_1 - a_2)z' - R\zeta 2e^{-i\vartheta})\alpha\bar{\beta} + h_3 z'.$$

Note that the error part only has a linear dependence on ζ , while ζ^{-1} only exists in the leading term. This is a consequence of the choice of holomorphic frame in section 5.1 and important for us, as we will consider the terms for small ζ which is in this form possible without an unbounded increase of the error part.

As a corollary of Lemma 5.2.3 we obtain the asymptotics for the entries in the error-matrix:

Lemma 7.1.1

There exists a local coordinate z centered around any weakly parabolic point, such that along a generic ϑ -trajectory the following asymptotics hold for the entries of the error matrix defined above (with $r := |z|$) for some $\mu > 0$:

$$e_1, e_2, e_3 = O(r^\mu).$$

Furthermore, if ϵ denotes a C^2 -coordinate of \mathcal{M}'_R , then the derivatives w. r. t. ϵ exist and have the same asymptotics:

$$\begin{aligned}\partial_\epsilon e_1 \partial_\epsilon e_2, \partial_\epsilon e_3 &= O(r^\mu), \\ \partial_\epsilon^2 e_1 \partial_\epsilon^2 e_2, \partial_\epsilon^2 e_3 &= O(r^\mu).\end{aligned}$$

Additionally there exists $C, \delta > 0$ s. t.

$$|r^{-\mu} e_1|, |r^{-\mu} e_2|, |r^{-\mu} e_3| \leq C e^{-\delta R}$$

for large enough R .

Proof. We start with a sufficiently small neighborhood centered around a weakly parabolic point p , s. t. Lemma 5.2.3 holds, which tells us that the factors involving α, δ and β give exactly the asymptotics as stated in the lemma. So for the statement at hand to hold the terms in front have to be bounded. This is clear for the part $R\zeta 2e^{-i\vartheta}$, so we only have to check the part $|(a_1 - a_2)z'|$.

From [Fre+20] $a_1 - a_2$ has a simple pole at p , so $a_1 - a_2$ is of the form $\frac{\kappa}{z} + O(r^{\mu-1})$ for some constants $\kappa \in \mathbb{C}$ and $\mu > 0$. From Lemma 4.1.5 we know that $z(t) = z_0 e^{-\frac{e^{i\vartheta}}{m}t}$, so

$$z'(t) = -\frac{e^{i\vartheta}}{m} z_0 e^{-\frac{e^{i\vartheta}}{m}t} = -\frac{e^{i\vartheta}}{m} z(t).$$

It is thus clear that $|(a_2 - a_1)z'|$ is bounded along the ϑ -trajectory.

Finally we have to check the $h_i z'$ parts. From Lemma 5.2.3 we see that the h_i are $O(r^{\mu-1})$, so multiplying with $z' = -\frac{e^{i\vartheta}}{m} z$ results in $h_i z' = O(r^\mu)$.

This proves the first part, and the argument for the derivatives follows in the same way. Finally the exponential suppression also follows directly from Lemma 5.2.3. \square

Note that the last statement of the lemma depends crucially on the fact that all entries of the error matrix are at least linear in the off diagonal term β of the matrix H that represents the metric h . This in turn leads to the at least linear dependence on the off diagonal term b of the gauge transformation g , whose decaying behaviour is thus the source of these estimates. We now obtain the following form for our flatness ODE:

$$\frac{ds}{dt} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} s + \begin{pmatrix} e_1 & e_2 \\ e_3 & -e_1 \end{pmatrix} s.$$

The important feature is that the second matrix is integrable (which we will show below), while choosing ζ in \mathbb{H}_ϑ leads to the upper left entry of the leading term having positive real part while the lower right entry has negative real part. The small flat section (for the pole at $+\infty$) should now be given as the unique solution of the following integral equation:

$$s(t) = \begin{pmatrix} 0 \\ e^{-\Lambda_2(t)} \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{\Lambda_1(t)-\Lambda_1(\tau)} & 0 \\ 0 & e^{-\Lambda_2(t)+\Lambda_2(\tau)} \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} s(\tau) d\tau.$$

Here the Λ_i are primitives of the λ_i along the ca -curve. Note that from the structure of the λ_i and the choice of ζ it follows that $e^{-\Lambda_i(t)} \xrightarrow{t \rightarrow \infty} 0$ for both $i \in \{1, 2\}$.

This equation may already be regarded as an initial value problem at infinity. The limit for $t \rightarrow \infty$ of the initial value function corresponds to the limit for $t \rightarrow \infty$ of the solution s , i. e. the limit of the section evaluated along a ϑ -trajectory going into a parabolic point. In fact solutions of this equation can already be constructed via the method of successive approximation which was observed by N. Levinson in [Lev48]. This is also what I. Tulli did for the case of wild Higgs bundles on the sphere in [Tul19]. Although that construction works and is useful to obtain good approximations of the solution it does not seem to be adequate for questions regarding the structure of the solution space as well as uniqueness of solutions and differentiable dependence of a family of solutions. This is why we had to build the general theory for these kinds of equations in section 6 of our work.

For our theory to work in this case however we have to modify the kernel of this equation. This is done by detaching the decaying leading term, which corresponds to finding the leading term in the WKB approximation. So we write $s(t) = x(t)e^{-\Lambda_2(t)}$ and the integral equation for x takes the form

$$\begin{aligned} x(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{\Lambda_1(t)-\Lambda_1(\tau)+\Lambda_2(t)-\Lambda_2(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} x(\tau) d\tau \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} x(\tau) d\tau. \end{aligned}$$

In the last equation we introduced $\Lambda(t) := \Lambda_1(t) + \Lambda_2(t)$ for better readability. Note that the leading term corresponds to the part of the solutions of Hitchin's equation that comes from the limiting configurations. Thus it is this decoupling were the results of [Fre+20] really come into play.

In order to use our theorem we have to make sure, that the kernel is now at least for large enough R and small enough ζ a good one. Note that we will use the notion of "small enough ζ " as meaning the existence of some $\zeta_0 \in \mathbb{H}_\vartheta$ s. t. for all $\zeta \in \mathbb{H}_\vartheta$ with $|\zeta| < \zeta_0$ the assertion holds. This corresponds to the use of "large enough R ".

We start by showing the integrability locally around a parabolic point.

Lemma 7.1.2

Fix $\zeta_0 \in \mathbb{H}_\vartheta$. Locally around any weakly parabolic point p there exists a coordinate z , s. t.

$$\int_t^\infty \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau < U(t)e^{-\delta R}$$

for some constant δ and bounded function $U(t) > 0$ that is independent of R for all ζ small enough and all t for which $z(t)$ is still in the neighborhood.

Proof. Let

$$K := \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\|$$

denote the integral kernel. We are interested in the existence of $\int_t^\infty K(\tau)d\tau$. Clearly $\int_t^\infty K(\tau)d\tau$ exists if $\int_{t'}^\infty K(\tau)d\tau$ exists for some large t' , so we consider K more closely.

First note that Λ is a primitive of a function with positive real part, which implies that its real part is monotonically increasing. Thus in this integral $\exp(\Lambda(t) - \Lambda(\tau)) < 1$, so the first matrix in the integrand has norm smaller than 1 in any coordinate neighborhood of p .

We center around a weakly parabolic point p , s. t. Lemma 7.1.1 holds, so $r^{-\mu}K \leq Ce^{-\delta R}$. As considered in Lemma 4.1.5 we know that $z(\tau) = z_0 e^{-\frac{e^{i\vartheta}}{m}\tau}$. Thus the integral over K is bounded from above:

$$\int_{t'}^\infty K d\tau = \int_{t'}^\infty r^{-\mu} K r^\mu d\tau \leq \int_{t'}^\infty C e^{-\delta R} r^\mu d\tau \leq C z_0 \int_{t'}^\infty e^{-\frac{e^{i\vartheta}}{m}\mu\tau} d\tau.$$

The last integral is indeed finite, i. e. $\int_{t'}^\infty K(\tau)d\tau$ exists and is smaller than $U(t')e^{-\delta R}$ for some function U that is independent of R . It follows that $\int_t^\infty K d\tau$ exists for all t for which the curve is in this neighborhood and is bounded by some $U(t)$ where the t -dependence denotes that decreasing t increases the integration range and thus the upper bound. We will obtain a uniform bound from this shortly in Lemma 7.1.4.

As we consider the asymptotic behavior for $\zeta \rightarrow 0$ we note that decreasing $|\zeta|$ may only decrease the norm of the error matrix, so while $U(t)$ may depend on ζ_0 the bound holds for all ζ with smaller absolute value. □

Note that this is the main use of the special form of a ca -trajectory in our analysis. We use the fact that the local form of the curve is known and the distance of the curve towards the parabolic point decays fast enough. This is enough to guarantee the existence

of the solution and together with the exponential suppression of the remainder term later on also suffices to obtain the correct asymptotics and decaying behavior of the hyperkähler metric. This is how we circumvent the use of the WKB approximation.

Now that we have the desired statement locally around the parabolic points we can expand the region for which it holds along the whole ϑ -trajectory. For this we now use contractible neighborhoods.

Lemma 7.1.3

Fix $\zeta_0 \in \mathbb{H}_\vartheta$. In any contractible open set W inside the standard annulus it holds

$$\int_t^\infty \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau < U(t)e^{-\delta R}$$

for some constant δ and bounded function $U(t) > 0$ that is independent of R for all ζ small enough along the generic ϑ -trajectory inside W .

Proof. The standard neighborhood runs along the ϑ -trajectory up until p , so all of the curve (excluding the end point p) lies in the intersection of this neighborhood and the local one of the preceding Lemma 7.1.2. Inside this intersection the integrands only differ by the derivative of the change of coordinates map, i. e. if the coordinates are z and \tilde{z} we have

$$\begin{aligned} \int_{\tilde{t}}^\infty \left\| \begin{pmatrix} e^{\Lambda(\tilde{t})-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau \\ = \int_t^\infty \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| D(\tilde{z} \circ z^{-1}) d\tau. \end{aligned}$$

As $\tilde{z} \circ z^{-1}$ is a diffeomorphic map from a bounded region into a bounded region of \mathbb{C} it follows (from Cauchy's integral formula), that $|D(\tilde{z} \circ z^{-1})|$ is bounded.

Thus the integral is also finite in the standard neighborhood along the intersection. But for smaller values of t in the standard neighborhood only a finite term is added. Thus the integral exists along the whole ϑ -trajectory. As the entries of the error matrix are also exponentially suppressed everywhere, it follows the claim. \square

This would now already be enough to infer the existence of the solution on any interval $[t, \infty)$, but the upper bound involves a t -dependence in $U(t)$ which could a priori increase unbounded as $t \rightarrow -\infty$ which would then prohibit us from finding a uniform R for which the solution exists all along the curve. Fortunately this can't happen in our setting, as the generic ϑ -trajectory starts and ends at a parabolic point, at each of which the consideration of Lemma 7.1.2 applies. So we may take another step to achieve some more uniform behavior.

Lemma 7.1.4

There exist uniform constants $\delta, \tilde{C} > 0$, i. e. independent of $|\zeta| < |\zeta_0|$ and t such that

$$\int_t^\infty \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau < \tilde{C}e^{-\delta R}$$

for all R large enough and ζ small enough.

Proof. As both ends of the generic ϑ -trajectory run into a weakly parabolic point the arguments we used in the proof of Lemma 7.1.2 apply for both ends of the curve and as mentioned in Lemma 7.1.3 between the ends only a finite part is added. Thus

$$\int_{-\infty}^{\infty} \left\| \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau$$

exists. Furthermore it is exponentially suppressed by $\tilde{C}e^{-\delta R}$ for some constants \tilde{C} and δ which are now both independent of t . With the same arguments as in Lemma 7.1.2 they can also be chosen independent of $|\zeta| < |\zeta_0|$. As $\exp(\Lambda(t) - \Lambda(\tau)) \leq 1$ still holds everywhere along the ϑ -curve we have for all $t \in \mathbb{R}$ and $|\zeta| < |\zeta_0|$ the exponential suppression

$$\int_t^{\infty} \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau \leq \int_t^{\infty} \left\| \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau < \tilde{C}e^{-\delta R}.$$

□

We have thus achieved the main estimate for obtaining the solution of the Volterra equation which shall be our basic ingredient for constructing the metric.

Theorem 7.1.5

For large enough R there exists a unique solution of the integral equation

$$x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_t^{\infty} \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} x(\tau) d\tau \quad (7.5)$$

all along any generic ϑ -trajectory.

Proof. From the Lemma 7.1.4 above we infer that the integral over the kernel is < 1 for large enough R and the initial value function is bounded. Thus via Theorem 6.1.2 a unique solution to the equation exists all along the ϑ -trajectory. □

With the unique existence of a solution x to the equation we obtained by detaching the leading part we have now a good characterization of the small flat section along a generic ϑ -trajectory γ as $s(t) = x(t)e^{-\Lambda_2(t)}$.

Definition 7.1.6

The *small flat section* for a weakly parabolic point p is the solution of the corresponding initial value problem at infinity that becomes finite when going into p along a generic ca -trajectory after detaching the leading term.

In this definition the choice of $(0, 1)^t$ as leading term is arbitrary. Other complex multiples would work just as well and cancel in the final expression of \mathcal{X} . Additionally the choice for p now depends on the specific ca -curve we use to approach p . Taking the boundary of a standard quadrilateral works, as we simply need a way to choose a section. One might worry, that the section that is small along one curve may become large for a curve in the same isotopy class which was used in defining the triangulation, but as we will show in the next section, this can actually not occur.

7.2 Constructing the coordinates

Now that we have the solution of the flatness equation we can say more about its asymptotic behavior and its derivative. For this we keep the notions that we fixed before and also fix a $\zeta_0 \in \mathbb{H}_\vartheta$. We now talk about "small" $\zeta \in \mathbb{H}_\vartheta$ i. e. $|\zeta| < |\zeta_0|$, so the statements above lead to uniform constants.

Proposition 7.2.1

For small ζ the small flat section along a ϑ -trajectory γ is of the form

$$s(t) = e^{-\Lambda_2(t)} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + x^r \right),$$

with x^r some \mathbb{C}^2 -valued function with $|x^r| \leq \tilde{K}e^{-\delta R}$ for some constant \tilde{K} .

Proof. First we note, that the previous considerations lead for small enough ζ to a (uniform) constant K , s. t.

$$\int_t^\infty \left\| \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right\| d\tau < Ke^{-\delta R}.$$

The estimate from the integral equation theory 6.1.4 thus leads to

$$|x| \leq \frac{1}{1 - Ke^{-\delta R}}.$$

Plugging this back into the right hand side of the integral equation (7.5), we obtain the estimate

$$\left\| \int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} x(\tau) d\tau \right\| \leq Ke^{-\delta R} \cdot \frac{1}{1 - Ke^{-\delta R}} =: \tilde{K}e^{-\delta R}.$$

Here we use that for R large enough $\frac{K}{1 - Ke^{-\delta R}}$ is bounded by some (uniform) constant \tilde{K} . Thus the integral equation leads to

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x^r$$

where x^r is some function with $|x^r| \leq \tilde{K}e^{-\delta R}$. The form for s follows readily. \square

Thus we see that for large enough R the small flat section s is dominated by the leading part $(0, 1)^t$ of x .

Now we want to build the product $s^1 \wedge s^2$ where s^1 and s^2 are the small flat sections for two weakly parabolic points p_1 and p_2 joined by the ϑ -trajectory. We recall that we fixed the orientation of the ϑ -trajectory γ in section 7.1. So if s^1 is the small flat section constructed there, then for constructing s^2 we have to change the orientation of the ϑ -curve, i. e. consider the flatness equation (7.2) along $\tilde{\gamma}(t) := \gamma(-t)$ which leads to a change of sign in the ODE, which in turn exchanges the role of the first and second coordinate in the local expression of the solution. The calculations then work just as before and we

obtain a solution s^2 that decays in the other direction. Now evaluating this section again along γ we obtain

$$s^2(t) = e^{\Lambda_1(t)} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y^r \right),$$

for some exponentially suppressed y^r . We have thus obtained that the small flat section for one direction grows in the other direction and can thus infer the next important result.

Proposition 7.2.2

The small flat sections for two weakly parabolic points that are connect by a generic ca-trajectory are linearly independent decorations.

Note that as of the complex 2-dimensional nature of the space of solutions of the linear ODE (7.2) we have thus obtained all solutions there are.

Corollary 7.2.3

The small flat sections for two weakly parabolic points connected by a generic ca-curve form a fundamental system for the ODE (7.2).

Before proceeding, we have to take care that the definitions make sense everywhere in a quadrilateral containing two zeroes of q . The important objects to take care of are the primitive functions Λ_j of λ_j for $j \in \{1, 2\}$. These can't be chosen globally, so we take one primitive Λ_j^i for each puncture, each defined on a neighborhood of the two ϑ -trajectories running into the parabolic point p_i (and excluding p_i itself). Now we can calculate the \wedge -product which is locally evaluated as $\det[s^1, s^2]$.

Corollary 7.2.4

The wedge-product of two small flat sections s^1 and s^2 along a ϑ -trajectory with remainder terms x^r and y^r is

$$s^1 \wedge s^2 = e^{-\Lambda_2} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + x^r \right) \wedge e^{\Lambda_1} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y^r \right) = e^{\Lambda_1 - \Lambda_2} (1 + r),$$

for some function r which is exponentially suppressed in R (independently of ζ).

Until now we considered the sections evaluated along some ϑ -trajectory γ . For the formula of the \mathcal{X} -coordinates the section needs to be evaluated at points along two ϑ -trajectories (that are the two sides of the quadrilateral running into the same vertex). For this we consider that the section is well defined on the contractible neighborhood of the two ϑ -curves and obtained from the solution along γ via parallel transport. Thus we may obtain the value on the second curve by connecting the two curves along any other curve inside the neighborhood and solving the ODE along this connecting curve. But we may use the fact that the solutions along the other curve are also obtained as corresponding small flat sections, as they give us the full space of solutions of the ODE. Thus we only need to examine whether the curve may actually become unbounded along the second curve when running into the pole. Otherwise it will still be a small flat section along this second curve and thus a solution to the ODE we already solved, albeit it with maybe another initial value at infinity which will be important later as we discuss the derivatives of the sections.

To get a good control over the parallel transport we use the neighborhood around a pole as the ϑ -trajectories are well known there and they allow for good approximation of the solution as we've already seen. The following observation is needed later on and follows rather directly from the definition:

Lemma 7.2.5

Let α be a ϑ -curve and β be a $\tilde{\vartheta}$ -curve for some quadratic differential q in any local chart. Then at every intersection the (euclidean) angle between α and β is independent of the point of intersection and the specific curves α and β .

Proof. Consider α and β in standard parametrization. Let x be a point of intersection, i. e. $\alpha(t_1) = x = \beta(t_2)$. From the definition it follows

$$q(\alpha(t))\alpha'(t) = e^{i\vartheta} = e^{i(\vartheta-\tilde{\vartheta})}e^{i\tilde{\vartheta}} = e^{i(\vartheta-\tilde{\vartheta})}q(\beta(t))\beta'(t)$$

for all t . Thus at the point of intersection we obtain

$$\alpha'(t_1) = e^{i(\vartheta-\tilde{\vartheta})}\beta'(t_2).$$

Thus we see that the tangents are related by a rotation of angle $\vartheta - \tilde{\vartheta}$ which is independent of the choice of curves or intersection point. \square

Now let γ and ν be the two sides of Q that share some vertex p and s the small flat section along γ . The idea now is to use the same techniques as before to obtain an estimate along ν by estimating a corresponding integral equation along a curve that connects γ and ν . For this we first have to define a curve along which we parallel transport the section. We need the curve to lie inside the quadrilateral and it has to be of a form which allows for the same kinds of calculations as before, i. e. the growth of the section along the curve has to be known in some way. It is thus natural in our setting to look for ca -curves connecting the sides of the quadrilateral.

Proposition 7.2.6

Let U be the usual neighborhood around a weakly parabolic point of q and γ and μ the parts of two sides of the standard quadrilateral for ϑ in this neighborhood. Let V be a neighborhood of ϑ and $x = \gamma(t_x)$ any Element in the image of γ .

Then there exists $\tilde{\vartheta} \in V$, and a $\tilde{\vartheta}$ -trajectory $\tilde{\gamma}$ that intersects γ in x and μ in $y := \mu(t_y)$ that lies in the quadrilateral for all $t_x < t < t_y$.

Furthermore $t_y - t_x = d_{\tilde{\gamma}}$ for some constant $d_{\tilde{\gamma}}$ that does depend on $\tilde{\vartheta}$ and the choice of γ and μ , but not on the choice of x on γ .

Proof. We consider the usual neighborhood U around a weakly parabolic point, where $qdz = \det(\Phi) = \frac{M}{z^2}dz^2$. In this neighborhood the ϑ -trajectories are known to be logarithmic spirals (or straight lines) except for two exceptional values of ϑ_c where they become concentric circles around the parabolic point. We may parametrize all these spirals as $c(t) = c_0 e^{i\omega t}$ for differing values of c_0 and with ω only depending on ϑ with $\text{Re}(\omega) < 0$ as seen before. Let γ and μ be the parts of two sides of the standard quadrilateral in this neighborhood. Their images together with 0 separates the neighborhood into two regions,

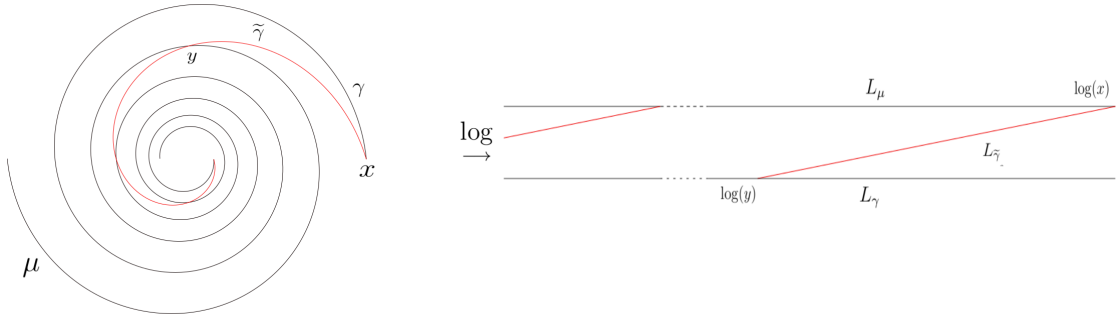


Figure 7.1: The ϑ -trajectories are mapped by the logarithm to a horizontal strip, whereas the $\tilde{\vartheta}$ -trajectory is mapped to a straight line intersecting the strip at constant angles.

one of which corresponds to the inside of the quadrilateral, which shall be called I . As the ϑ -trajectories don't intersect, there are ϑ -trajectories that lie in the complement I^c . We may take any one of them, say α , to be a branch cut on U and define a biholomorphic logarithm on $U \setminus (\text{Im}(\alpha) \cup \{0\})$. The domain of \log then includes I . The logarithm now maps all the ϑ -trajectories c (excluding α) to parallel lines L_c which we rotate to be horizontal, so L_γ and L_μ define a horizontal strip S . In the following \log shall denote the composition of the logarithm and the rotation. In $\text{Im}(\log)$ the left endpoints of the L_c correspond to the values c_0 of c at $t = 0$ in U while $0 \in U$ is mapped to $-\infty$ to the left, cf. figure 7.1.

Now let $\tilde{\gamma}$ be a ca -trajectory in for some $\tilde{\vartheta} \neq \pm\vartheta$ that intersects γ at some point x . The part of the curve starting at x is then also mapped to a straight line at least until some point, where it intersects α , which happens somewhere, as $\tilde{\vartheta} \neq \pm\vartheta$.

As the mapping is biholomorphic $L_{\tilde{\gamma}}$ intersects L_γ in the point $\log(x)$ at some non-zero angle β which, as of Lemma 7.2.5 is constant for all intersection points. We recall that the ca -trajectories are oriented as running into 0. For $\tilde{\vartheta}$ close enough to ϑ we thus have that depending on whether $\tilde{\vartheta}$ is bigger or smaller than ϑ the straight line $L_{\tilde{\gamma}}$ continues in the strip S or outside of S . We may thus take $\tilde{\vartheta}$ arbitrarily close to ϑ and such that $L_{\tilde{\gamma}}$ continues in the strip S . Being a straight line, $L_{\tilde{\gamma}}$ will intersect L_ν at some point $\log(y)$ and the thus defined segment is the image of a part of the ca -trajectory $\tilde{\gamma}$ inside I under the logarithm. Thus $\tilde{\gamma}$ intersects γ and ν in $x = \gamma(t_x)$ and $y = \mu(t_y)$ stays inside I for all $t_x < t < t_y$.

To calculate $t_x - t_y$ note that $L_\gamma = a_1 + t \text{Re}(\omega)$ and $L_\nu = a_2 + t \text{Re}(\omega)$ for $a_1 := \log(\gamma(0))$ and $a_2 := \log(\mu(0))$. Now $L_{\tilde{\gamma}}$ has the form $L_{\tilde{\gamma}}(t) = L_0 + te^{i\beta}$ and intersects L_γ in t_x and L_μ in t_y . We obtain, that

$$\begin{aligned} a_2 - a_1 + \text{Re}(\omega)(t_y - t_x) &= a_2 + t_y \text{Re}(\omega) - (a_1 + t_x \text{Re}(\omega)) = L_\mu(t_y) - L_\gamma(t_x) \\ &= L_{\tilde{\gamma}}(t_y) - L_{\tilde{\gamma}}(t_x) = t_y e^{i\beta} - t_x e^{i\beta} = (t_y - t_x) e^{i\beta} \\ \Rightarrow |t_y - t_x| &\leq \left| \frac{a_2 - a_1}{e^{i\beta} - \text{Re}(\omega)} \right| =: d_{\tilde{\vartheta}}. \end{aligned}$$

Note that a_2 and a_1 are constants that only depend on the sides of the quadrilateral, as is ω . Also β is constant for all intersections of $\tilde{\gamma}$ and any part of any ϑ -trajectory

everywhere in I as of Lemma 7.2.5 and the biholomorphicity of \log . Thus we conclude that $|t_y - t_x| = d_{\tilde{\gamma}}$ is constant for all $\tilde{\gamma}$ -trajectories connecting γ and μ in U . \square

We have thus obtained the prerequisites for transporting the small flat section that was obtained along some side γ of the quadrilateral Q to another side μ . For the proof we can now use ideas similar to the ones that helped before.

Proposition 7.2.7

Let γ and μ be two sides of the quadrilateral Q sharing a vertex p and s the small flat section along γ . If ζ is small enough then s evaluated along μ is of the form

$$s(t) = e^{-\Lambda_2(t)} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + x^r \right),$$

with x^r some \mathbb{C}^2 -valued function with $|x^r| \leq \tilde{K}e^{-\delta R}$ for some constant \tilde{K} .

Proof. To parallel transport the small flat section from one side of the quadrilateral to another we use the formulation via an integral equation as we have done before, only now we do not need to consider the integral up to infinity. We recall that in our set-up ϑ is fixed and we consider only those ζ which lie in the open half-plane \mathbb{H}_{ϑ} . Thus ζ also lies in the half-plane $\mathbb{H}_{\tilde{\vartheta}}$ for $\tilde{\vartheta}$ close enough to ϑ . Now we chose any point p_s on μ close enough to the parabolic point, s. t. it lies in the neighborhood where the ca -trajectories may be expressed as logarithmic spirals. As of the previous Proposition 7.2.6 there exists a $\tilde{\vartheta}$ -trajectory $\alpha : [a, b] \rightarrow C$ with starting point $\alpha(a) = p_s$ and first intersection with γ at $\alpha(b) = p_e$, s. t. the image of the curve is contained in the quadrilateral and $b - a \leq d$ for some constant d that does not depend on the starting point p_s .

We now consider the solution of our ODE via variation of constants along α , i. e. the parallel transport along α is the unique solution $x : [a, b] \rightarrow \mathbb{C}$ to the integral equation

$$x(t) = \begin{pmatrix} e^{\Lambda(t)-\Lambda(b)} & 0 \\ 0 & 1 \end{pmatrix} x(b) - \int_t^b \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & -e_1 \end{pmatrix} x(\tau) d\tau. \quad (7.6)$$

Note that $x(b)$ is now the "initial value" of x at the end point p_e instead of the value at infinity. Here Λ is again a primitive of λ as above but now evaluated along α . The theory developed in chapter 6 also holds for these usual Volterra equations where the integral does not go to infinity, cf. [Win10].¹

Thus we have a unique solution from which we would like to gather information about $x(a) = x(p_s)$ on μ by estimating it against $x(b) = x(p_e)$ on γ , using the techniques we already developed for the infinite case. As ζ lies in the half-plane $\mathbb{H}_{\tilde{\vartheta}}$ we still have that $\Lambda(t) - \Lambda(\tau)$ has negative real part, which holds in particular for $\tau = b$. Thus the norm of the first matrix in the integrand is smaller than 1 while the norm of the error matrix stays bounded by some $Ce^{-\delta R}$ as before for ζ small enough. As the domain of the integral is bounded by d we obtain for large enough R that the integral over the kernel is smaller than 1, so a unique solution exists. As the matrix in the initial value function is also bounded by 1, we obtain:

$$|x(t)| \leq \frac{|x(b)|}{1 - dCe^{\delta R}}.$$

¹More accurately the theory developed here tried to broaden the already existing theory for these equations to the case of an initial value at infinity.

Plugging this estimate back into the the right hand side of the equation (7.6) we obtain

$$\left| \int_t^b \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & -e_1 \end{pmatrix} x(\tau) d\tau \right| \leq \int_t^b C e^{-\delta R} \frac{|x(b)|}{1 - dC e^{-\delta R}} d\tau \quad (7.7)$$

$$\leq |x(b)| \tilde{C} e^{-\delta R}$$

for some uniform constant \tilde{C} . The full equation (7.6) thus shows us that we have:

$$x(t) = \begin{pmatrix} e^{\Lambda(t)-\Lambda(b)} & 0 \\ 0 & 1 \end{pmatrix} x(b) + v(t) \quad (7.8)$$

where $|v(t)| \leq |x(b)| \tilde{C} e^{-\delta R}$ along the whole segment. As $\Lambda(t) - \Lambda(\tau) \leq 0$ along the segment, we obtain the estimate

$$|x(t)| \leq |x(b)| (1 + \tilde{C} e^{-\delta R}). \quad (7.9)$$

In particular we have the (in)equalities (7.8) and (7.9) for $t = a$, i. e. for large enough R the endpoints $x(a)$ and $x(b)$ on the two sides of the quadrilateral differ by some (bounded) multiplicative factor in the first component (that vanishes for $\zeta \rightarrow 0$ or $R \rightarrow \infty$) and an added vector which is bounded by the norm of the function on the other side multiplied with some exponentially suppressed constant. Note that this argument is valid for all ϑ segments arbitrarily close to the weakly parabolic point p and the upper bounds can be chosen to be valid for all arbitrarily close segments. Now $x(b)$ is the value of the section on the side where the small flat section was defined which is bounded near p , so the value on the other side is also bounded. This already shows that the transported section along the second side must also be a small flat section as the big sections would have to surpass every upper bound somewhere close to the parabolic point. Thus we know that the section has a limit along the second side going into the parabolic point and the first component of this limit must vanish. As $x(b)$ has limit $(0, 1)^t$ going into p along γ the limit along the second side μ is $(0, 1 + \epsilon)$ for some constant ϵ with norm $|\epsilon| \leq \tilde{C} e^{-\delta R}$ for some uniform constant \tilde{C} for ζ small enough.

Therefore the section along μ is the solution of

$$x(t) = \begin{pmatrix} 0 \\ 1 + \epsilon \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} x(\tau) d\tau,$$

where Λ and the integral are now taken along μ . As ϵ is exponentially bounded it follows with the same arguments as in Proposition 7.2.1 that x is of the form

$$x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x^r,$$

with x^r is some \mathbb{C}^2 -valued function with $|x^r| \leq \tilde{K} e^{-\delta R}$ for some constant \tilde{K} . Again the form for s follows readily. \square

We can now proceed to build the \mathcal{X}_e coordinate (where we mostly suppress the explicit dependence on the triangulation, q and ϑ in the notation). For this consider again the standard quadrilateral Q associated to an edge e of the triangulation with vertices given

by four weakly parabolic points p_i .² We want to build the \wedge -product according to the Definition 4.2.2 of \mathcal{X}_e . But to do so we have to consider that our calculations up until now was always validated on contractible subsets of the standard annulus S in Q and we already introduced different primitives for λ in the different patches belonging to two different ϑ -curves. This now has to be done for all of the four sections of the quadrilateral. Additionally for each vertex p_i there are two sides joining p_i and thus two small flat sections of $\nabla(\zeta)$ which are given by the solution of the integral equation along these two curves, from which one has to be chosen. Our way for doing this is the following:

First we remember that we fixed the orientation for the ϑ -curves depending on $q^{1/2}$, cf. the discussion of the flatness equation (7.2). This implies that if one side of the quadrilateral Q is given by a curve oriented as running into a vertex p_i (for $t \rightarrow \infty$) the other side of Q sharing the vertex p_i is also oriented as running into p_i . Thus two vertices of Q are points into which the ϑ -curves run, and the other two vertices are points from which the curves emerge. We start with labeling one of the points into which the curves run as p_1 and label the following vertices clockwise (for a standard orientation of \mathcal{C}). Now we label the side joining p_4 and p_1 as γ_1 with is thus oriented in such a way that γ_1 converges to p_1 for $t \rightarrow \infty$. The other side joining p_1 connects p_1 and p_2 and shall be labeled as γ_2 and is also oriented in such a way that γ_2 converges to p_1 for $t \rightarrow \infty$. Now p_2 will be a vertex from which ϑ -curves emerge, while p_3 will be of the same type as p_1 . So we proceed for γ_3 and γ_4 in the same way as for γ_1 and γ_2 , just now for the vertex p_3 . In this way we obtain four oriented curves as sides of Q . For each of the vertices p_i let Λ_1^i be a primitive of $\lambda_1 := -\frac{R}{\zeta}q^{1/2} - R\zeta q^{1/2} - \overline{a_1}d\bar{z} + a_1dz$ and Λ_2^i a primitive of $\lambda_2 := -\frac{R}{\zeta}q^{1/2} - R\zeta q^{1/2} + \overline{a_2}d\bar{z} - a_2dz$ defined on a simply connected neighborhood U_i of γ_i and γ_{i+1} in S .³ Note that these primitives exist as p is holomorphic and $d(\overline{a_j} - a_j) = 0$ because these are the entries of connection of the limiting configuration which is flat. Evaluated along a ϑ -curve these primitives become the primitives that were used in the calculations above, cf. equation (7.2). Finally for each vertex we choose s^i to be the small flat section along γ_i defined using the primitives Λ_j^i and extended to sections of $\nabla(\zeta)$ on U_i (cf. figure 7.2). This extension is possible as the U_i are simply connected and $\nabla(\zeta)$ is flat. We note that this way of choosing the sections is not necessary. Other choices lead to the same \mathcal{X} coordinates as the small flat sections only differ by a (constant) complex scalar, which cancels in the final \wedge -product. This also justifies the arbitrary choice of the "initial value at infinity" as $(0, 1)^t$. Thus we may now define the \mathcal{X} -coordinate for the edge e .

Proposition 7.2.8

Let Q be a standard quadrilateral with vertices p_i and small flat sections chosen as described above. Then for $z_1 \in U_1 \cap U_4$ and $z_3 \in U_2 \cap U_3$ the product

$$-\frac{(s^1 \wedge s^2)(z_1)(s^3 \wedge s^4)(z_3)}{(s^4 \wedge s^1)(z_1)(s^2 \wedge s^3)(z_3)} =: -\frac{(s^1 \wedge s^2)(s^3 \wedge s^4)}{(s^4 \wedge s^1)(s^2 \wedge s^3)} =: \mathcal{X}_e$$

is well defined, non-zero and independent of the evaluation points z_1, z_3 .

²In the following all of the expressions involving the indices of the vertices, like p_{i+2} , are to be considered as mod 4 as we talk about a quadrilateral with four vertices.

³This neighborhood can be taken to be S minus some branch cut connecting p_{i+2} and the inner boundary of the annulus S .

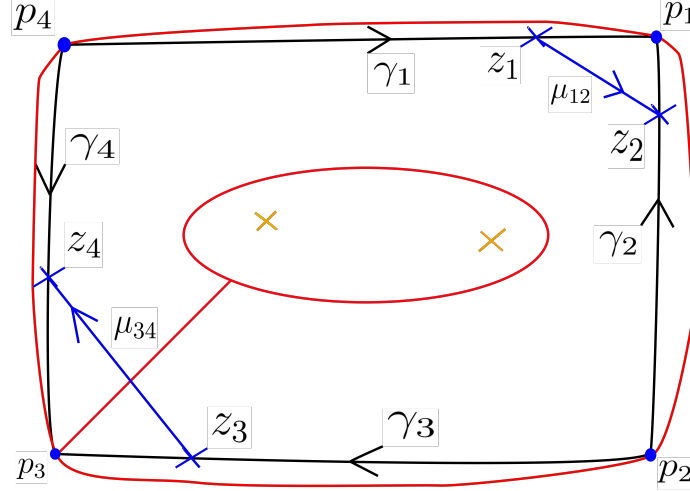


Figure 7.2: A standard annulus given by four oriented ϑ -curves γ_i . The orange crosses denote the zeroes of q and the blue circles the weakly parabolic points. The blue crosses denote the evaluation points for the small flat sections and the blue lines are curves along which two of the Λ -products are transported. The red outer and inner "circles" are the boundary of a standard annulus. Together with the red straight arc they form the boundary of the simply connected neighborhood U_1 for γ_1 .

Proof. By construction the Λ -products are defined on z_1 and z_3 . As of Propositions 7.2.2 and 7.2.7 it follows that all of the Λ -products are non-zero, so \mathcal{X} is well defined and non-zero. The independence of the evaluation points is also true by construction, cf. the discussion preceding the Definition 4.2.2. \square

So \mathcal{X}_e only depends on the parameters of the Higgs pair, ζ , ϑ and the choice of quadrilateral. Thus if we fix ζ and ϑ we obtain functions on \mathcal{M}^{dR} . As described in section 4.2 it follows from the work of Fock and Goncharov [FG06] and the treatment in [GMN13b] that the collection of these functions for all edges are actually coordinates of \mathcal{M}^{dR} , called the *canonical coordinates*.

Now consider the cycle $\gamma_e \in H_1(\Sigma'_q; \mathbb{Z})$ encircling the preimage of the two zeroes inside of Q in the (punctured) spectral curve Σ'_q . One of the key claims in [GMN13b] is that in the $\zeta \rightarrow 0$ asymptotic the period integral $\oint_{\gamma} (q^{1/2})^*$ emerges as the leading term for the \mathcal{X} -coordinates, where $(q^{1/2})^*$ denotes the Liouville (tautological) 1-form on T^*C . That is by pushing γ_e down to a cycle on C (which we will also denote by γ_e) one has $\mathcal{X}_{\gamma_e} \sim c_{\gamma_e} \exp(\frac{R}{\zeta} \oint_{\gamma_e} q^{1/2} dz)$ for some constant c_{γ_e} . Their conjecture was based on the WKB method but had not been rigorously proven until now. With the calculations done above we can now achieve this, but one technical detail has to be accounted for. As noted above [GMN13b] used the convention that the matrix representing $\nabla(\zeta)$ is traceless, which leads to a z -invariance of the Λ -products even before taking the quotient in the definition of the \mathcal{X} -functions. Thus they can evaluate the sections on each curve so that the primitives combine in just the right way to create the period integral. In our set-up the connection matrix is not traceless, so this independence does not hold. Additionally from Propositions 7.2.2 and 7.2.7 we only have the detailed description of $s^i \wedge s^k$ along the curve connecting the vertices p_i and p_k . But the definition of \mathcal{X} implies that at least two of the Λ -products

have to be evaluated in a point on some other side. We have to account for this by parallel transporting two of the \wedge -products to other sides. As the $s^i \wedge s^k$ are sections of a line bundle this is done rather easily.

Lemma 7.2.9

Let γ_i and γ_{i+1} be the sides of a standard quadrilateral Q sharing the vertex p_i , with $z_i \in \text{Im}(\gamma_i)$ and $z_{i+1} \in \text{Im}(\gamma_{i+1})$ two points on these sides of Q . Let s^i and s^{i+1} be the corresponding small flat sections defined on the neighborhoods U_i and U_{i+1} .

Then for every curve $\mu \in U_i \cap U_{i+1}$ connecting z_i and z_{i+1} it holds:

$$s^i \wedge s^{i+1}(w_2) = e^{\int_{\mu} (a_1 + a_2) dz - \overline{a_1 + a_2} d\bar{z}} s^i \wedge s^{i+1}(w_1),$$

Proof. From Proposition 7.2.7 and Corollary 7.2.3 s^i and s^j define a fundamental system for the linear ODE $s'(t) = B(t)s(t)$ given by the flatness equation (7.2) along μ . The equation is now simply an application of Liouville's formula where we note that the only part of the connection matrix of $\nabla(\zeta)$ that is not traceless comes from the leading part of the Chern connection (cf. Remark 5.2.6), which is just $(a_1 + a_2)dz - \overline{a_1 + a_2}d\bar{z}$, cf. equation (7.4). \square

Finally we can see the emergence of the period integrals.

Theorem 7.2.10

Let Q be the standard quadrilateral for an edge e of a ϑ -triangulation of C . The canonical coordinate for Q can be expressed as

$$\mathcal{X}_e = -(1 + r_q) \exp \left(\frac{R}{\zeta} \oint_{\gamma_e} q^{1/2} + R\zeta \oint_{\gamma_e} \overline{q^{1/2}} + \oint_{\gamma_E} a_1 dz - \overline{a_1} d\bar{z} \right),$$

where r_q is exponentially suppressed in R for small enough ζ , and γ_e is the projection of a cycle in $H_1(\Sigma'_q; \mathbb{Z})$ corresponding to the edge e , that is a simple closed curve in Q encircling the two zeros of q once.

Proof. With the same notations as before we choose a point $z_i \in \text{Im}(\gamma_i)$ on each of the sides of the standard quadrilateral Q and curves μ_{kl} connecting the points z_k and z_l in $U_k \cap U_l$, cf. figure 7.2. For abbreviation we write $\text{tr}(\nabla(\zeta)) := \overline{a_1 + a_2}d\bar{z} - (a_1 + a_2)dz$. Then, using Lemma 7.2.9 to parallel transport the \wedge -products to the sides where they can be evaluated as in Corollary 7.2.4, we obtain the following expression: ⁴

$$\begin{aligned} \mathcal{X}_e &= -\frac{(s^1 \wedge s^2)(z_2)(s^3 \wedge s^4)(z_4)}{(s^4 \wedge s^1)(z_2)(s^2 \wedge s^3)(z_4)} = -\frac{(s^1 \wedge s^2)(z_2)(s^3 \wedge s^4)(z_4)}{(s^4 \wedge s^1)(z_1)(s^2 \wedge s^3)(z_3)} e^{-\int_{\mu_2} \text{tr}(\nabla(\zeta)) - \int_{\mu_4} \text{tr}(\nabla(\zeta))} \\ &= -\frac{e^{-\Lambda_2^1(z_2) + \Lambda_1^2(z_2)}(1 + r_{12})e^{-\Lambda_2^3(z_4) + \Lambda_1^4(z_4)}(1 + r_{34})}{e^{\Lambda_1^4(z_1) - \Lambda_2^1(z_1)}(1 + r_{41})e^{\Lambda_1^2(z_3) - \Lambda_2^3(z_3)}(1 + r_{23})} e^{-\int_{\mu_2} \text{tr}(\nabla(\zeta)) - \int_{\mu_4} \text{tr}(\nabla(\zeta))} \\ &= -(1 + r_q) \cdot e^{\Lambda_2^1(z_1) - \Lambda_2^1(z_2) + \Lambda_1^2(z_2) - \Lambda_1^2(z_3) + \Lambda_2^3(z_3) - \Lambda_2^3(z_4) + \Lambda_1^4(z_4) - \Lambda_1^4(z_1)} e^{-\int_{\mu_2} \text{tr}(\nabla(\zeta)) - \int_{\mu_4} \text{tr}(\nabla(\zeta))}, \end{aligned}$$

⁴The right choice of lower indices of Λ_j^i can be a bit confusing here. Proposition 7.2.1 and the following discussion imply that a small flat section s^i for the vertex p_i along the curve γ_i running into p_i has the exponent $-\Lambda_2^i$ while a small flat section s^{i+1} for the vertex p_{i+1} along the curve γ_i running into p_i has the exponent Λ_1^{i+1} .

where the r_{kl} denote the corresponding remainder terms and r_q is the term emerging from all of the r_{kl} , which is thus, as of Corollary 7.2.4, exponentially suppressed in R .

The key step is now noticing that $\Lambda_j^i(z_k) - \Lambda_j^i(z_l) = \int_{\mu_{lk}} \lambda_j$ leads to the desired period integrals, which is a bit concealed here as we work with the different λ_j , emerging from $\text{tr} \nabla(\zeta) \neq 0$. So at first we obtain the following:

$$\begin{aligned} \mathcal{X}_e = & -(1 + r_q) \exp \left(\int_{\mu_{21}} \lambda_2 + \int_{\mu_{32}} \lambda_1 + \int_{\mu_{43}} \lambda_2 + \int_{\mu_{14}} \lambda_1 \right) \\ & \cdot \exp \left(- \int_{\mu_{12}} \text{tr}(\nabla(\zeta)) - \int_{\mu_{34}} \text{tr}(\nabla(\zeta)) \right). \end{aligned}$$

To put these integrals together we note that λ_1 and λ_2 differ only by containing expressions of a_1 or a_2 . So for the parts involving $q^{1/2}$ we already obtain a period integral along a simple closed curve μ encircling the two zeroes of q , while we observe the following cancellation for the rest

$$\begin{aligned} \mathcal{X}_e = & -(1 + r_q) \exp \left(-\frac{R}{\zeta} \oint_{\mu} q^{1/2} - R\zeta \oint_{\mu} \overline{q^{1/2}} \right) \\ & \exp \left(\int_{\mu_{21}} \overline{a_2} d\bar{z} - a_2 dz - \int_{\mu_{32}} \overline{a_1} d\bar{z} - a_1 dz + \int_{\mu_{43}} \overline{a_2} d\bar{z} - a_2 dz - \int_{\mu_{14}} \overline{a_1} d\bar{z} - a_1 dz \right) \\ & \exp \left(\int_{\mu_{12}} \overline{a_1} d\bar{z} + \overline{a_2} d\bar{z} - \int_{\mu_{12}} a_1 dz + a_2 dz + \int_{\mu_{34}} \overline{a_1} d\bar{z} - \overline{a_2} d\bar{z} - \int_{\mu_{34}} a_1 dz + a_2 dz \right) \\ = & -(1 + r_q) \exp \left(-\frac{R}{\zeta} \oint_{\mu} q^{1/2} - R\zeta \oint_{\mu} \overline{q^{1/2}} + \oint_{\mu} a_1 dz - \overline{a_1} d\bar{z} \right). \end{aligned} \tag{7.10}$$

As $q^{1/2}$ is holomorphic on \mathcal{C} , the integral does not depend on the specific curve but only on its homotopy class. For the term $a := a_1 dz - \overline{a_1} d\bar{z}$ we note that this corresponds to the entries of the Chern connection A_0 of $\overline{\partial}_E$ w. r. t. the limiting metric h_0 , and this solves the decoupled Hitchin equations 5.1. Thus A_0 is a flat connection and this implies $da = 0$, so this integral is also defined on the corresponding homotopy class and we obtain the period integrals on \mathcal{C} . The construction as outlined in section 4.3 and described in more detail in [GMN13b] guarantees that the signs match in the right way when comparing μ with γ_e coming from a cycle in $H_1(\Sigma'_q; \mathbb{Z})$, leading to the desired result. \square

As noted before the $\oint_{\gamma_e} q^{1/2}$ should be thought of as living on the spectral cover, where the tautological 1-form is integrated along $\gamma_e \in H_1(\Sigma'_q; \mathbb{Z})$. As [GMN13b] already noted the period integrals would vanish if $q^{1/2}$ could be well defined on the whole quadrilateral. But as we had to cut out a disc around the zeroes of q , i. e. the branch points of the spectral curve, we obtain some nonzero value. Following the notation of GMN the periods are denoted $Z_\gamma := \frac{1}{\pi} \oint_{\gamma} q^{1/2}$. They give "special coordinates" on the Hitchin Base \mathcal{B}' corresponding to its special Kähler metric. Similarly note that $\oint_{\gamma} a_1 dz - \overline{a_1} d\bar{z}$ is imaginary-valued and one can show that the functions θ with $i\theta := \oint_{\gamma} a_1 dz - \overline{a_1} d\bar{z}$ for $\gamma_e \in H_1(\Sigma'_q; \mathbb{Z})$ give coordinates of the torus fibers of the Hitchin fibration, thus completing the set of coordinates for the semiflat metric on \mathcal{M}' .

One might wonder at this point why the coordinates don't depend on a_2 anymore which was one of the two entries of the holomorphic structure from which the Chern connection of the Higgs pair was constructed. The vanishing of this term does not lead to a loss of information as the connections for two Higgs pairs only differ by a traceless matrix so all necessary information is encoded in one of the diagonal entries.

Let us now proceed to show a property of \mathcal{X}_e , which will be important for the twistorial construction later on. First we want to expand the definition of \mathcal{X}_e to large ζ , for which we first have to check whether the small flat sections can be treated in the same way as before for large ζ . We do this by considering $\nabla(-1/\bar{\zeta})$ for small ζ . It is useful in this context to switch the frame and consider the flatness equation $0 = \nabla(\zeta)s$ in an h -unitary frame. So in the following the indices h and h_0 denote the local expression in the h -unitary and h_0 -unitary frame (where the second one is the frame F we used up until now).

In this new frame $H_h = E_2$ locally and so $\varphi_h^{*h} = \overline{\varphi_h}^t$ and $A_{h\bar{z}}^{*h} = \overline{A_{h\bar{z}}}^t$. By construction the change of basis matrix from the h -unitary frame to the h_0 -unitary frame is g^{-1} for g from section 5.2. So if s_{h_0} is the local expression of a solution of the flatness equation in the h_0 -unitary frame, $s_h := gs_{h_0}$ is the local expression in the h -unitary frame. Now from

$$g \circ (\nabla_{h_0}) \circ g^{-1}s_h = g \circ (\nabla_{h_0}s_{h_0}) = 0$$

it follows that s_h is a solution of the flatness equation in the h -unitary frame.

We denote by B_h the connection matrix of the flat connection $\nabla(\zeta)$ as in equation 7.3. Taking the ζ -dependence of B_h into account and calculate $\overline{B_h}^t$ using the fact that A is h -unitary:

$$\begin{aligned} \overline{B_h}^t(\zeta, t) &= \left(-\frac{R}{\bar{\zeta}} \overline{\varphi_h}^t - \overline{A_{h\bar{z}}}^t \right) \bar{z}' + \left((-R\bar{\zeta}) \overline{\varphi_h^{*h}}^t - \overline{A_{h\bar{z}}}^t \right) z' \\ &= \left(-\frac{R}{\bar{\zeta}} \varphi_h^{*h} + A_{h\bar{z}} \right) \bar{z}' + \left((-R\bar{\zeta}) \varphi_h + A_{h\bar{z}} \right) z' \\ &= -B_h(-\frac{1}{\bar{\zeta}}, t). \end{aligned}$$

Let s and b denote the small flat sections for two parabolic points p_1 and p_2 joined by γ . From Proposition 7.2.2 we know that they are linearly independent and thus defining X as

$$X := \begin{pmatrix} s_h^1 & b_h^1 \\ s_h^2 & b_h^2 \end{pmatrix}.$$

we obtain a fundamental system for the ODE given by $B_h(\zeta)$. Combining the result above with a standard argument we obtain

$$\begin{aligned} 0 &= \frac{d(XX^{-1})}{dt} = X'X^{-1} - X(X^{-1})' = B_h(\zeta)XX^{-1} - X(X^{-1})' \\ &\Rightarrow (X^{-1})' = -X^{-1}B_h(\zeta) \\ &\Rightarrow (\overline{X^{-1}}^t)' = -\overline{B_h(\zeta)}^t \overline{X^{-1}}^t \\ &\Rightarrow (\overline{X^{-1}}^t)' = B_h(-\frac{1}{\bar{\zeta}}) \overline{X^{-1}}^t. \end{aligned}$$

So we see that a fundamental system for $-\frac{1}{\zeta}$ is given by

$$\overline{X^{-1}}^t = \frac{1}{s_h^1 b_h^2 - s_h^2 b_h^1} \begin{pmatrix} \overline{b_h^2} & -\overline{s_h^2} \\ -\overline{b_h^1} & \overline{s_h^1} \end{pmatrix}.$$

Thus we see that the sections given by these rows of this matrix are flat sections for $-\frac{1}{\zeta}$. Denote them by $\tilde{s} := \frac{1}{s_h^1 b_h^2 - s_h^2 b_h^1} (-\overline{s_h^2}, \overline{s_h^1})$ and $\tilde{b} := \frac{1}{s_h^1 b_h^2 - s_h^2 b_h^1} (\overline{b_h^2}, -\overline{b_h^1})$. We would like these sections to be the corresponding small flat sections, so we have to check how they behave as $t \rightarrow \infty$. But to be able to compare them to the sections considered before we first have to transform back to the h_0 -unitary frame. Using the fact that the transformation is given by g^{-1} and using Lemma 5.2.3 a short calculation shows that \tilde{s} decays exponentially for $t \rightarrow \infty$, while \tilde{b} grows. Thus we have the same structure as before and can expand our choice of small flat sections to $-\frac{1}{\zeta}$ for small ζ . As before there is a complex 1-dimensional subspace of small flat sections and we may chose a unique one by demanding a certain limit for $t \rightarrow \infty$ after extracting the leading term. Again this choice is of no real importance as complex factors cancel in the final expression for \mathcal{X}_e .

Now we remember that s and b are the small flat sections for the vertices p_1 and p_2 for some small ζ and a ϑ -curve running from the p_2 into p_1 . We want \tilde{s} to be the small flat section for some vertex. As explained above the behaviour is just right but we have to take care of the correct domain in the ζ -plane. By construction the decoration at any parabolic point is only defined if $\zeta \in \mathbb{H}_\vartheta$. Now $-\frac{1}{\zeta}$ is not in \mathbb{H}_ϑ as it has exactly the opposite phase. But it is thus in the half space $\mathbb{H}_{\vartheta+\pi}$. Now it remains to be checked whether \tilde{s} is small along a ca -trajectory with angle in the triangulation $T(\vartheta + \pi)$. At first ca -trajectories are only defined up to orientation, i. e. γ is also a ca -trajectory for $\vartheta + \pi$ as $e^{i(\vartheta+\pi)} = -e^{i\vartheta}$. But now the orientation is the opposite one if we demand $q^{1/2}(\gamma') = -e^{i\vartheta}$ as before. Thus \tilde{s} is now the small flat section for p_2 and \tilde{b} for p_1 , which corresponds to a change of the decoration at both vertices, which is called *pop* in [GMN13b]. In the full picture for a standard quadrilateral this amounts to switching all of the decorations, which is called an *omnipop*. For our calculations we note that the omnipop leads to switching all sections in the wedge products of \mathcal{X}_e , leading to four changes of the sign, which thus overall cancel each other. We note that GMN use the equality under the omnipop as a quite powerful tool to determine the BPS spectrum. Here we will use these observations to derive the reality condition. For this we will from now on use Definition 4.3.6 and take the cycles $\gamma \in \Gamma_q$ as labels for the \mathcal{X} -coordinates.

Theorem 7.2.11

Let q be a generic differential. For small enough ζ the \mathcal{X} -coordinates obey the reality condition

$$\mathcal{X}_\gamma^\vartheta(\zeta) = \overline{\mathcal{X}_{\gamma^{\vartheta+\pi}}(-1/\bar{\zeta})},$$

for each $\gamma \in \Gamma_q$.

Proof. Proposition 4.3.5 and Definition 4.3.6 show that it suffices to show the condition for the basis of Γ_q which is obtained from the edges of the triangulation. With the considerations from above and the notation as for Theorem 7.2.10 we obtain the following equality:

$$\begin{aligned} \mathcal{X}_e^{T(\vartheta+\pi)}(-1/\bar{\zeta}) &= \frac{\left(\frac{-\bar{s}_2^1}{s_1^1}\right) \wedge \left(\frac{-\bar{s}_2^2}{s_1^2}\right)}{s^1 \wedge s^2} (z_2) \frac{\left(\frac{-\bar{s}_2^3}{s_1^3}\right) \wedge \left(\frac{-\bar{s}_2^4}{s_1^4}\right)}{s^3 \wedge s^4} (z_4) \\ &\quad \cdot \left(\frac{\left(\frac{-\bar{s}_2^4}{s_1^4}\right) \wedge \left(\frac{-\bar{s}_2^1}{s_1^1}\right)}{s^1 \wedge s^2 s^3 \wedge s^4} (z_2) \frac{\left(\frac{-\bar{s}_2^2}{s_1^2}\right) \wedge \left(\frac{-\bar{s}_2^3}{s_1^3}\right)}{s^1 \wedge s^2 s^3 \wedge s^4} (z_4) \right)^{-1} \\ &= \frac{s^1 \wedge s^2(z_4) \overline{s^3 \wedge s^4(z_2)}}{s^4 \wedge s^1(z_2) \overline{s^2 \wedge s^3(z_4)}}. \end{aligned}$$

Now we can again use Liouville's formula to transport $s^1 \wedge s^2$ from z_4 to z_2 and $s^3 \wedge s^4$ from z_2 to z_4 . The emerging exponential factors then cancel and we obtain $\mathcal{X}_e^{\vartheta+\pi}(-1/\bar{\zeta}) = \overline{\mathcal{X}^\vartheta(\zeta)}$. By construction

$$\gamma_e^\vartheta = -\gamma_e^{\vartheta+\pi},$$

and thus combined we obtain

$$\mathcal{X}_\gamma^\vartheta(\zeta) = \overline{\mathcal{X}_\gamma^{\vartheta+\pi}(-1/\bar{\zeta})}.$$

□

Remark 7.2.12

Before concluding this section we may shortly consider the case of degenerate edges. As stated in remark 4.1.15 we do not work this out in detail here as we can mostly adapt the way this is done in [GMN13b]. The basic idea is that in the case of a degenerate edge e corresponding to a degenerate triangle the corresponding quadrilateral also degenerates and one has to consider a covering surface on which the necessary cycles can be defined. After appropriately matching the definitions one obtains cycles consisting of two loops running in opposite directions around the two lifts of a vertex on the spectral cover. The period integrals are then given by the residues at the vertex which are fixed for the moduli space.

But GMN also show that in this case $\mathcal{X}_e = \mu^2$ where μ is the clockwise monodromy around the degenerate vertex, which they have calculated from their Frobenius analysis, cf. the discussion following equation (7.1). This is the only part in which we differ, as we have not obtained the small flat section in this way. But our way works just as good, as we may also evaluate our small flat sections along a circle around the degenerate vertex. Indeed, there is always some ϑ such that the ϑ -curves are circles around the poles of q . We can thus simply use our analysis just like before and obtain the monodromy data, which matches the one from GMN. One then sees directly by comparing with the residue data that \mathcal{X}_e again becomes dominated by the period integral without any further calculations, cf. [GMN13b] section 7.

7.3 The derivatives of the coordinates

In the last section of this part we want to check the logarithmic derivatives of the \mathcal{X} -functions as the hyperkähler metric will be build out of these. After considering the possibility of differentiating at all, the important result here will be that the logarithmic derivative has only simple poles for $\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$. To show this we first take a look at the derivative ∂_ϵ of $x \wedge y$, where we differentiate with respect to some coordinate of the moduli space which is a parameter ϵ of the ODE. Generally, if the sections are differentiable we have the usual product rule

$$\partial_\epsilon(x \wedge y) = (\partial_\epsilon x) \wedge y + x \wedge (\partial_\epsilon y).$$

Thus we need to differentiate the solutions of the integral equation, but before we can carry out the explicit calculations there is one rather technical point we have to address.

Remark 7.3.1

In this chapter we will assume that the coefficients of the local expressions of our objects, like the Higgs field, depend differentiably on differentiable coordinates of the moduli space. As long as we only considered the functions \mathcal{X} at some point on the moduli space this was not important, but as we begin to describe the derivatives we are moving in the moduli space and one has to lift families of elements of the moduli spaces (i. e. the equivalence classes) to corresponding families of representatives in such a way that the differentiable structure is preserved. The possibility of such a differentiable choice of representatives is basically built directly into the construction of the moduli space as a hyperkähler quotient, which is why we won't go into the technical details here. Instead we refer to the thorough explanation of the process of building the quotient in [May16].

But this choice of representatives only addresses the part of the story where the objects as global functions depend differentiably on the coordinates. As we are going to work with the local expressions, we also have to consider how the differentiability survives the process of evaluation in a local frame. As long as the frame is fixed for a curve of Higgs fields Φ_t one obtains the differentiable dependence of the matrix coefficients as the evaluation at a point in a local frame defines a bounded linear operator. But in our construction in section 5.1 we adapted the frame to the Higgs field by taking a frame consisting of eigenvectors of ϕ . The \mathcal{X} -coordinates themselves are independent of any frame, but we have to make sure that our construction of the small flat sections from the corresponding integral equations works. Therefore it needs to be possible to chose the eigenvectors of Φ_t in such a way that they vary differentiably with the entries of the local expressions of Φ_t in the fixed frame. But this follows from a short calculation in linear algebra, so everything works out and we have the desired structure of a diagonal leading term plus some small error matrix. However, at this point we have only diagonalized the Higgs field Φ_t and not scaled the frame to a standard form, as was also done in section 5.1. This should also be possible as all the transformations are differentiable, but we actually don't have to adapt the frames in this way. Indeed, by not adapting the frame to the limiting metric and holomorphic structure, we only obtain a slightly different expression of the leading term in the flatness equation and some additional finite multiplicative factors in the entries of the error matrix. As they are finite, they don't change the analytical properties of our integral equation at

all. Therefore we can safely ignore these changes and continue to work with the form of our equation we used up until now.

Lemma 7.3.2

The small flat sections are differentiable with respect to any differentiable coordinate ϵ of \mathcal{M}_R and the derivative of the remainder-term is bounded by some uniform constant for small enough ζ .

Proof. The leading term $e^{-\Lambda_2}$ is differentiable by construction. So what we have to consider is the remainder term. We want to use Theorem 6.3.4, so we have to check that the kernel is of product type of second order w. r. t. a coordinate ϵ of the moduli space \mathcal{M}_R . The kernel is of product type as was shown above, i. e. it is of the form AB for

$$A := \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix}.$$

The derivative of A is

$$\partial_\epsilon \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \partial_\epsilon(\Lambda(t) - \Lambda(\tau))e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 0 \end{pmatrix}.$$

So we have to consider $\partial_\epsilon \Lambda(t)$ where Λ is a primitive function for $\lambda_1 + \lambda_2$, i. e. for

$$\lambda = \left(2\frac{R}{\zeta}e^{i\vartheta} + 2i(\text{Im}(a_1 z') - 2i\text{Im}(a_2 z')) + 2R\zeta e^{-i\vartheta} \right).$$

Now a_1 and a_2 are obtained directly from the connection A of the Higgs pair and are of the form $\frac{c_i(\epsilon)}{z} + f(\epsilon)$ where c_i and f_i depend differentiably on ϵ . Thus the pole structure does not change and $(\partial_\epsilon a)z'$ is finite everywhere. The other part of the derivative is $a_i \partial_\epsilon(z')$, so we need to consider $\partial_\epsilon(z')$. For this we need to recall that $z(t)$ denoting the ϑ -trajectory for $\det \Phi = -q = f dz^2$ depends on ϵ in the way q depends on ϵ . We may now consider any initial value problem for the ϑ -trajectories to obtain the differentiability of $z'(t, \epsilon)$ w. r. t. ϵ . For example when considering the derivative in ϵ_0 we can consider the initial value problem with $z'(0, \epsilon) = z'(0, \epsilon_0)$ for all ϵ near ϵ_0 and thus obtain a well defined initial value problem whose solution thus depends differentiably on ϵ . For calculations we note that by definition $f^{1/2}z' \equiv \text{const}$. Thus

$$\partial_\epsilon(-f^{1/2}z') = 0, \quad \Rightarrow \quad \partial_\epsilon z' = \frac{\partial_\epsilon f^{1/2}}{f^{1/2}} z'.$$

By a local description in a coordinate centered around a weakly parabolic point it is clear that the fraction is finite and thus $\partial_\epsilon z'$ behaves like z' . Consequently $a_i \partial_\epsilon(z')$ is finite and so is $\partial_\epsilon(\text{Im}(a_i z'))$.

For the derivative of B we look at the derivative of the e_i . We consider only

$$\partial_\epsilon e_2 = \partial_\epsilon \left((a_1 - a_2)z' + R\zeta 2e^{-i\vartheta} \right) \delta\beta + \left((a_1 - a_2)z' + R\zeta 2e^{-i\vartheta} \right) \partial_\epsilon(\delta\beta) - \partial_\epsilon(h_1 z')$$

as similar arguments hold for e_1 and e_3 . The considerations from before lead to the finiteness of the derivative of the part in the brackets for the first term, which thus remains

integrable. For the second term the bracket is still finite and we use the fact that $\partial_\epsilon(\delta\beta)$ is integrable in the same way as is $\delta\beta$, which follows from Lemma 5.2.3 in the same way as was shown for the kernel of the original (non derived) equation in section 7.1. Finally the integrability of $\partial_\epsilon(h_1 z')$ follows just as for $\partial_\epsilon(\text{Im}(a_i z'))$ and using Lemma 5.2.3. Thus the kernel is of product type of order $(1, 1)$. The same arguments now hold again for the second derivative of A and thus the kernel is of product type of order $(2, 1)$. We conclude from Theorem 6.3.4 that the derivative of x exists and is the unique solution to the following equation:

$$\begin{aligned} \partial_\epsilon x(t) = & - \int_t^\infty \partial_\epsilon \left(\begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right) x(\tau) d\tau \\ & - \int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \partial_\epsilon x(\tau) d\tau. \end{aligned} \quad (7.11)$$

Now as x is bounded by some (uniform) constant and by the properties we have shown above it follows that the first integral is bounded by some constant C . The kernel of this integral equation remains the same as before, so we obtain just like before from Corollary 6.1.4 the estimate

$$|\partial_\epsilon x(t)| \leq \frac{C}{1 - K e^{-\delta R}}.$$

Again this is bounded by a uniform constant for large enough R which finishes the proof. \square

We have thus shown that the derivative of the remainder-terms exist and are bounded. This is all we need to obtain the behaviour of the \wedge -product that we need.

Theorem 7.3.3

Let s^1 and s^2 be the small flat section for two weakly parabolic points that are connected by a generic ϑ -trajectory. Then the logarithmic derivative of $s^1 \wedge s^2$ has a simple pole at $\zeta = 0$. Furthermore this pole is precisely the one coming from the leading term.

Proof. For the derivative of the full product we consider again the different primitives of the λ_j and obtain

$$\begin{aligned} \partial_\epsilon(s^1 \wedge s^2) &= \partial_\epsilon \left(e^{\Lambda_1^2 - \Lambda_2^1} \right) (x \wedge y) + e^{\Lambda_1^2 - \Lambda_2^1} \partial_\epsilon(x \wedge y) \\ &= (\partial_\epsilon(\Lambda_1^2 - \Lambda_2^1)(x \wedge y) + \partial_\epsilon(x \wedge y)) e^{\Lambda_1^2 - \Lambda_2^1}. \end{aligned}$$

Thus the logarithmic derivative has the form

$$\frac{\partial_\epsilon(s^1 \wedge s^2)}{s^1 \wedge s^2} = \frac{\partial_\epsilon(\Lambda_1^2 - \Lambda_2^1)(x \wedge y) + \partial_\epsilon(x \wedge y)}{x \wedge y}.$$

Since $\partial_\epsilon(x \wedge y)$ is bounded and $x \wedge y = 1 + r$ for some bounded and exponentially suppressed r it follows that there is only the simple pole at $\zeta = 0$ coming from the Λ part. \square

This boundedness already suffices to prove the necessary condition for our \mathcal{X} -coordinate. Nevertheless before showing that we will proof that $\partial_\epsilon(x \wedge y)$ is even exponentially suppressed as we will need that later for the asymptotics of the metric, and the corresponding calculations are similar to the ones here.

Theorem 7.3.4

Let s^1 and s^2 be the small flat section for two weakly parabolic points that are connected by a generic ϑ -trajectory with remainder-terms x and y . Then the product $\partial_\epsilon(x \wedge y)$ is exponentially suppressed in R .

Proof. We look again at equation (7.11) for $\partial_\epsilon x(t)$. As $\partial_\epsilon x(t)$ is bounded by Lemma 7.3.2, plugging it back into the right hand side and using the exponential bound of the kernel as in the proof of Proposition 7.2.1 we see that the second integral is some vector v whose norm is bounded by some exponentially suppressed K .

Now we consider again the "initial value function" of the derivative:

$$\begin{aligned} & \int_t^\infty \partial_\epsilon \left(\begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} x(\tau) d\tau \\ & \quad + \int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \partial_\epsilon \left(\begin{pmatrix} e_1(\tau) & e_2(\tau) \\ e_3(\tau) & -e_1(\tau) \end{pmatrix} \right) x(\tau) d\tau \\ &= \int_t^\infty \partial_\epsilon (\Lambda(t) - \Lambda(\tau)) e^{\Lambda(t)-\Lambda(\tau)} \begin{pmatrix} e_1(\tau) & e_2(\tau) \\ 0 & 0 \end{pmatrix} x(\tau) d\tau \\ & \quad + \int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} \partial_\epsilon e_1(\tau) & e^{\Lambda(t)-\Lambda(\tau)} \partial_\epsilon e_2(\tau) \\ \partial_\epsilon e_3(\tau) & -\partial_\epsilon e_1(\tau) \end{pmatrix} x(\tau) d\tau. \end{aligned}$$

For the first term the e_i imply, as mentioned above, that the whole integral is given by some function $\begin{pmatrix} g \\ 0 \end{pmatrix}$, which is exponentially suppressed by some $C_1 e^{-\delta_1 R}$ for an appropriate constants C_1 and δ_1 .

For the second term we have to take a closer look at the derivative. Remember that $x = (0, 1)^t + x^r$ where x^r is exponentially suppressed. We already saw that $\partial_\epsilon e_i$ is finite (and linear in R), so the only part that may not be exponentially suppressed in R is the one coming from $(0, 1)^t$, which is

$$\int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} \partial_\epsilon e_2(\tau) \\ -\partial_\epsilon e_1(\tau) \end{pmatrix} d\tau.$$

Here we look at the second entry and remember that in

$$e_1 = 2(a_1 - a_2)|\beta|^2 z' + R\zeta 2e^{-i\vartheta}|\beta|^2 - h_1 z' = (2(a_1 - a_2)z' + R\zeta 2e^{-i\vartheta})|\beta|^2 - h_1 z'$$

the suppressing factor $|\beta|$ comes into play quadratically and so they contain the derivative

$$\partial_\epsilon |\beta|^2 = 2|\beta| \partial_\epsilon |\beta|.$$

As $\partial_\epsilon |\beta|$ is finite by the argument above the dependence on $|\beta|$ leads to the first part still being exponentially suppressed. Additionally from Lemma 5.2.3 it follows that $\partial_\epsilon (h_1 z')$ is exponentially suppressed in the right way and thus the whole term is, so

$$\int_t^\infty \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} \partial_\epsilon e_2(\tau) \\ -\partial_\epsilon e_1(\tau) \end{pmatrix} d\tau = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

for some bounded function f_1 and a function f_2 bounded by $C_2 e^{-\delta_2 R}$ for appropriate C_2 and δ_2 . So the only term not being exponentially suppressed is the first component. Luckily

for the \wedge -product the first term is undone by the leading part of y and the remainder-term y_r of y is itself exponentially suppressed. So all together we obtain

$$(\partial_\epsilon x) \wedge y = \left(\begin{pmatrix} f_1 + g \\ f_2 \end{pmatrix} + v \right) \wedge \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y^r \right) = f_2 + \begin{pmatrix} f_1 + g \\ f_2 \end{pmatrix} \wedge y_r + v \wedge \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y^r \right).$$

So with the deduced bounds we obtain

$$|(\partial_\epsilon x) \wedge y| \leq |f_2| + \left| \begin{pmatrix} f_1 + g \\ f_2 \end{pmatrix} \right| |y_r| + |v|(1 + |y^r|) \leq C e^{-\delta R}$$

for appropriate constants C, δ . The corresponding calculations also hold for $x \wedge (\partial_\epsilon y)$ and thus also for $\partial_\epsilon(x \wedge y)$. \square

As a final step we now have to transport the derivative of x to the other sides of the quadrilateral as we have done in Proposition 7.2.7 for x itself. Here we use the same techniques and estimates as before:

Proposition 7.3.5

Let γ_1 and γ_2 be two sides of the standard quadrilateral Q joining in the weakly parabolic point p of Q and let \tilde{p} be the other end of γ_2 . Let x denote the remainder term of the small flat section along γ_1 and y the remainder term of the small flat section for \tilde{p} . Then the product $\partial_\epsilon(x \wedge y)$ on γ_2 is exponentially suppressed in R .

Proof. Just as for the small flat section along the first ϑ -trajectory, we obtain the following equation for the derivative along the connecting circle segment described in Proposition 7.2.7:

$$\begin{aligned} x_\epsilon(t) = x_\epsilon(a) &+ \int_a^t \partial_\epsilon \left(\begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & -e_1 \end{pmatrix} \right) x(\tau) d\tau \\ &+ \int_a^t \begin{pmatrix} e^{\Lambda(t)-\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & -e_1 \end{pmatrix} x_\epsilon(\tau) d\tau. \end{aligned}$$

The validity of this equation follows via the methods analogous to the ones developed before. From Proposition 7.2.7 we have the form $x(\tau) = (0, 1) + x^r$ for some exponentially suppressed x^r just as for the function along the first side. Thus the same argument as in Lemma 7.3.2 may be used to obtain an uniform bound for the first integral. But then the equation for x_ϵ differs only by this uniform bound from the equation for the non derived x as we transported it to the other side in the proof of Proposition 7.2.7. Thus the same arguments as in that proof can be used to obtain the limits when going into the weakly parabolic point. It follows again that the values on the two sides of the quadrilateral only differ by an exponentially suppressed term plus the limit of the first integral, which has the same structure as the "initial value integral" in the proof of Theorem 7.3.4. When taking the wedge product with y we may then argue in the same way as in the proof of Theorem 7.3.4, the only difference being an additional exponentially suppressed term. Thus we conclude that the derivatives of the transported sections behave in the same way as the non transported ones. \square

Now we have all the information we need for our next main result concerning the \mathcal{X} -coordinates:

Theorem 7.3.6

The logarithmic derivative of any \mathcal{X} -coordinate has at most a simple pole $\zeta = 0$.

Proof. First note that all factors in the expression for \mathcal{X} are nonzero and so is \mathcal{X} itself. By the product rule we have:

$$\begin{aligned} \partial_\epsilon \mathcal{X} &= \partial_\epsilon \left(\frac{(s^1 \wedge s^2)(s^3 \wedge s^4)}{(s^4 \wedge s^1)(s^2 \wedge s^3)} \right) \\ &= \frac{(\partial_\epsilon(s^1 \wedge s^2))(s^3 \wedge s^4)(s^4 \wedge s^1)(s^2 \wedge s^3) + (s^1 \wedge s^2)(\partial_\epsilon(s^3 \wedge s^4))(s^4 \wedge s^1)(s^2 \wedge s^3)}{(s^4 \wedge s^1)^2 (s^2 \wedge s^3)^2} \\ &\quad - \frac{(s^1 \wedge s^2)(s^3 \wedge s^4)(\partial_\epsilon(s^4 \wedge s^1))(s^2 \wedge s^3) + (s^1 \wedge s^2)(s^3 \wedge s^4)(s^4 \wedge s^1)\partial_\epsilon(s^2 \wedge s^3)}{(s^4 \wedge s^1)^2 (s^2 \wedge s^3)^2} \\ &= \frac{(\partial_\epsilon(s^1 \wedge s^2))(s^3 \wedge s^4)}{(s^4 \wedge s^1)(s^2 \wedge s^3)} + \frac{(s^1 \wedge s^2)\partial_\epsilon(s^3 \wedge s^4)}{(s^4 \wedge s^1)(s^2 \wedge s^3)} \\ &\quad - \frac{(s^1 \wedge s^2)(s^3 \wedge s^4)\partial_\epsilon(s^4 \wedge s^1)}{(s^4 \wedge s^1)^2 (s^2 \wedge s^3)} - \frac{(s^1 \wedge s^2)(s^3 \wedge s^4)\partial_\epsilon(s^2 \wedge s^3)}{(s^4 \wedge s^1)(s^2 \wedge s^3)^2}. \end{aligned}$$

Thus we obtain for the logarithmic derivative

$$\partial_\epsilon \log(\mathcal{X}) = \frac{\partial_\epsilon(\mathcal{X})}{\mathcal{X}} = \frac{\partial_\epsilon(s^1 \wedge s^2)}{s^1 \wedge s^2} + \frac{\partial_\epsilon(s^3 \wedge s^4)}{s^3 \wedge s^4} - \frac{\partial_\epsilon(s^4 \wedge s^1)}{s^4 \wedge s^1} - \frac{\partial_\epsilon(s^2 \wedge s^3)}{s^2 \wedge s^3}.$$

By the calculations above all terms in the final expression have only the simple pole at $\zeta = 0$ coming from the leading term which thus is also true for $\partial_\epsilon \log(\mathcal{X})$. \square

We now also need the same structure for $\zeta = \infty$, which could naturally be guessed as of the symmetry in the construction of ∇ . The precise result could be obtained by working in an anti-holomorphic frame (i. e. a frame, where φ^* is diagonal) and repeating all the steps up until now, where one would have to consider some additional term in the ODE coming from the change of basis and its z -derivative. Though the information obtained from [Fre+20] is good enough to also make this work, it also follows directly from the reality condition, in the proof of which we already used the stated symmetry of ∇ . This is also the way in which [Tul19] derives this asymptotic, although he differs in the proof of the reality condition for his case.

Corollary 7.3.7

The logarithmic derivative of \mathcal{X} has at most a simple pole at $\zeta = \infty$.

Proof. This follows directly from the asymptotics for $\zeta \rightarrow 0$ from Theorem 7.3.6 and the reality condition Theorem 7.2.11. \square

Chapter 8

A Riemann-Hilbert problem

By now it would already be possible to construct the hyperkähler metric and even infer an exponential decay towards the semiflat metric. However we will not do this directly and instead reformulate the whole construction in terms of a Riemann-Hilbert problem in section 8.1. This approach was also already introduced in the general construction in [GMN10]. The idea in the general case is that even when there is no explicit moduli space but instead only some general integrable system it is still possible to formulate a Riemann-Hilbert problem for \mathcal{X} -functions, whose solution should naturally fit in the twistorial construction of hyperkähler metrics. The main new ingredient for this are the (generalized) Donaldson-Thomas invariants Ω which determine the jumping behaviour of \mathcal{X} . In our special case of the Higgs bundle moduli space these invariants are explicitly known, and we can use the known behaviour of our constructed \mathcal{X} -coordinates to show that they do solve the appropriate Riemann-Hilbert problem in section 8.2. We then use this to show that they solve a certain integral equation, which was also formulated in [GMN10]. The upshot of this additional work is that the integral equation allows us to obtain more detailed information about the asymptotic behaviour of the \mathcal{X} -coordinates. In Section 8.3 we will provide a criterion for submanifolds on which the additional information of the coordinates translates into corresponding information about their derivatives.

8.1 Formulation of the Riemann-Hilbert problem

In this (and the next) section we fix an element (A, Φ) of \mathcal{M}'_R such that the determinant q of the Higgs field is an element of a chamber in the Hitchin base. We consider the $\mathcal{X}_\gamma^\vartheta$ -coordinates, now for all γ in the charge lattice Γ_q , and we eliminate the ϑ -dependence of the coordinates by specializing $\mathcal{X}_\gamma^\vartheta$ to $\vartheta = \arg(\zeta)$, i. e.:

$$\mathcal{X}_\gamma(\zeta) := \mathcal{X}_\gamma^{\arg(\zeta)}(\zeta).$$

As we work with a fixed element in \mathcal{M}'_R we will suppress the dependence of \mathcal{X}_γ on the elements of \mathcal{M}'_R . So we only keep the dependence on ζ and \mathcal{X}_γ becomes a map from \mathbb{C}^\times to \mathbb{C}^\times . The important thing to note here, is that the definition above leads to discontinuous jumps of the \mathcal{X}_γ -coordinates at certain values of ζ for which the ca -triangulation jumps. As this only depends on $\arg(\zeta)$ we have jumps along rays in the ζ -plane, which are called *active BPS rays*. It is important to note that although the jumps stem from changes in the triangulations, the sides of the standard quadrilaterals which are built from the triangulation still vary continuously when the jumps occur. Thus it is possible to analytically extend each \mathcal{X}_γ across a BPS ray by simply fixing the corresponding product of small flat sections. In the following for any BPS ray l we denote the analytic continuation of any \mathcal{X}_γ in the clockwise and counter-clockwise direction across l with \mathcal{X}_γ^- and \mathcal{X}_γ^+ .

As we explained in section 4.4, in the case of an unsealed spectrum there are only two types of jumps, for which the coordinates actually change. They are characterized by the appearance of finite *ca*-trajectories which are saddle connections (connecting two zeroes) or periodic, i. e. closed loops. Crucially in both types of jumps the changes of the triangulation are known explicitly and GMN were able to obtain corresponding formulas for these jumps. In particular there is always an associated charge $\beta \in \Gamma_q$ corresponding to the jump. To formulate the jumps we now define the quadratic refinement $\sigma : \Gamma_q \rightarrow \{-1, 1\}$ and (generalized) Donaldson-Thomas invariants $\Omega : \Gamma_q \rightarrow \mathbb{Z}$, where we use Lemma 4.4.5 implicitly to obtain that the objects are well defined.

Definition 8.1.1

Let $\beta \in \Gamma_q$ be a charge such that there occurs a jump in the *ca*-triangulation corresponding to β at some angle $\arg(\zeta)$. Then

$$\Omega(\beta; q) := \begin{cases} 1, & \text{if } \beta \text{ corresponds to a saddle connection,} \\ -2, & \text{if } \beta \text{ corresponds to a closed loop.} \end{cases}$$

Additionally define $\Omega(\beta; q) := 0$ for all other $\beta \in \Gamma_q$.

Remark 8.1.2

There is a different way of introducing the specified values of Ω (cf. [Nei13]), where one counts for $\beta \in \Gamma_q$ the number of saddle connections that lift to β . Let this number be SC . Additionally one counts the set of isotopy classes of closed loops that lift to β and denotes their number by CL . Now one can define

$$\Omega(\beta; q) = SC - 2 \cdot CL.$$

Generically it should be the case that this amounts to Ω being 0, 1 or -2 in accordance to the definition above as no two different finite *ca*-trajectories may appear that lift to the same β . The results in [BS15] support this partially at least in the case of quadratic differentials with double poles and simple zeroes considered here. Still, there may be possibilities left, where Ω thus defined gives different values from before. But we are interested in the jumping behaviour of the coordinates which was checked in [GMN13b] for the values obtained from the first definition. So we keep that as the definition and postpone the question of whether this coincides with the count of finite *ca*-trajectories.

For the quadratic refinement we proceed quite similarly.

Definition 8.1.3

Let β be a generator of Γ_q defined as in section 4.3 (but now without demanding that they correspond to a jump in the triangulation as for Ω). We then define

$$\sigma(\beta) := \begin{cases} -1, & \text{if } \beta \text{ corresponds to a saddle connection at some angle } \arg(\zeta), \\ 1, & \text{if } \beta \text{ corresponds to a closed loop at some angle } \arg(\zeta). \end{cases}$$

The rest of the values is now obtained from $\sigma(0) := 1$ and the formula

$$\sigma(\gamma)\sigma(\beta) = (-1)^{\langle \gamma, \beta \rangle} \sigma(\gamma + \beta).$$

As GMN show in [GMN13b] section 7.7 the function σ is independent of q and the angle of the triangulation.

We can now define the basic building block for the following theory. We start by defining for each $\gamma, \beta \in \Gamma_q$ a function \mathcal{K}_β^γ , given by

$$\mathcal{K}_\beta^\gamma : \mathbb{C}^\times \rightarrow \mathbb{C}, \quad \zeta \mapsto (1 - \sigma(\beta)\mathcal{X}_\beta^-(\zeta))^{\langle \gamma, \beta \rangle}.$$

Note that from our analysis in the sections above for R large enough it holds $|\mathcal{X}_\beta| < 1$, so $\mathcal{K}_\beta^\gamma(\zeta) \in \mathbb{C}^\times$ for all $\zeta \in \mathbb{C}^\times$. We also regard \mathcal{K}_β^γ as acting on the space of \mathcal{X} -functions via multiplication $\mathcal{X}_\gamma^- \mapsto \mathcal{X}_\gamma^- \mathcal{K}_\beta^\gamma$. This point of view is more in line with the general setting where the \mathcal{K}_β^γ are defined as automorphisms acting on the space of twisted unitary characters of Γ_q . But for us this setting becomes very concrete as we can regard $\mathcal{K}_\beta^\gamma(\zeta)$ simply as a complex number.

Now for each $\beta \in \Gamma_q$ define the ray $l_\beta := Z_\beta(q^{1/2})\mathbb{R}_-$. These are the *BPS rays* mentioned before. Here they arise from the observation that a jump of the triangulation is accompanied by the appearance of a finite $\arg(\zeta)$ -trajectory c . They are only defined up to orientation, but their lift to the spectral cover is canonically oriented as mentioned in section 4.3, from which we obtain the corresponding charge, say δ . From Lemma 4.4.5 we know that $Z_\delta(q^{1/2}) \in \mathbb{R}_+ e^{i\arg(\zeta)}$. On the other hand as of the unoriented nature of the trajectory c , it is also a trajectory for $\arg(\zeta) + \pi$, leading to the charge $-\delta$. For this charge it holds $Z_{-\delta}(q^{1/2}) \in \mathbb{R}_- e^{i\arg(\zeta)}$, i. e. $\zeta \in Z_{-\delta}(q^{1/2})\mathbb{R}_-$. The calculations in [GMN13b] now show that it is the charge $\beta := -\delta$ which determines the jump of a coordinate \mathcal{X}_γ with $\langle \gamma, \beta \rangle \neq 0$ at l_β via the formula above. Thus we see that a jump can only occur when ζ crosses a BPS ray. We stress that this means that at the ray l_β a jump of a coordinate \mathcal{X}_γ occurs, while the coordinate \mathcal{X}_β is continuous. The choice of direction for l will also be important later on, when we need the decaying behaviour of the coordinates along the BPS ray.

Now one defines for each ray l in the ζ -plane the *jump-factors* as the products of all the corresponding factors for which $l_\beta = l$:

$$S_l^\gamma(\zeta) := \prod_{\beta: l_\beta = l} (\mathcal{K}_\beta^\gamma(\zeta))^{\Omega(\beta; q)}. \quad (8.1)$$

This definition is taken from the general theory of GMN. As we will only consider these transformations inside the chambers of the Hitchin Base (cf. Definition 4.4.6) only one factor remains for each BPS ray. Therefore we don't have to worry about a particular order in the product, which could otherwise have infinite contributions.¹

A BPS ray l_β for which $\Omega(\beta; q) \neq 0$ is called *active*. The values for σ and Ω are determined from the considerations of GMN mentioned above, so that the S_l describe the jumps at any BPS ray. From section 7.2 we recall that the \mathcal{X} -coordinates have an explicitly known leading term given by the periods Z_γ and θ_γ . We denote this leading term (i. e. the function we obtain in equation 7.10 by setting $r_q = 0$) as \mathcal{X}^{sf} . With these definitions together with the properties of the Fock-Goncharov coordinates the following theorem holds.

¹For a way of ordering the product in the general case see section 2.3 in [GMN13b].

Theorem 8.1.4

For any fixed point in a chamber of \mathcal{M}'_R and $\gamma \in \Gamma_q$ with unsealed spectrum the canonical coordinate $\mathcal{X}_\gamma : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ has the following properties:

1. \mathcal{X}_γ depends piecewise-holomorphically on $\zeta \in \mathbb{C}^\times$, with discontinuities only at the rays $l_\gamma(u)$ with $\Omega(\gamma; q) \neq 0$.
2. The limits \mathcal{X}_γ^\pm of \mathcal{X}_γ as ζ approaches any ray l from both sides exist and are related by

$$\mathcal{X}_\gamma^+ = (S_l^\gamma)^{-1} \mathcal{X}_\gamma^-.$$

3. $\mathcal{X}_\gamma(\zeta)/\mathcal{X}_\gamma^{\text{sf}}(\zeta)$ is bounded near $\zeta = 0$ and $\zeta = \infty$.

We note that property 2 was established by GMN, while the properties 1 and 3 have been rigorously proven in the earlier chapters of this thesis. These three properties together with the property that the value of \mathcal{X}_γ is never 0 are one way of describing a Riemann-Hilbert problem. Classically such a problem is given by a contour in the complex plane and a specified jumping behaviour, cf. [Its03], [Mus53] and [Fok+06]. One then wants to find a function that is holomorphic in the complement of the contour and jumps in the prescribed way. Here the problem is more complicated as the jumping behaviour is determined by the functions themselves. Additionally the classical case defines jumps by the action of (piecewise) constant matrices, so they are elements of a finite-dimensional space, whereas the jumps here live in the infinite-dimensional space of functions on \mathbb{C}^\times . Also note that the third property here differs from the usual formulation where a specific limit in 0 or ∞ is specified, which leads to the uniqueness of solutions. This will be described in more detail below.

8.2 Integral equations and solutions

Although such Riemann-Hilbert problems are interesting in themselves, here we consider it to obtain another representation for our coordinates, that allows for a better estimate of the decaying properties. To this effect GMN argued that a solution of the Riemann-Hilbert problem is given by the solution of a certain integral equation. This equation is mostly encountered in the following form:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left[-\frac{1}{4\pi i} \sum_{\beta \in \Gamma_q^g} \Omega(\beta; q) \langle \gamma, \beta \rangle \int_{l_\beta(q)} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_\beta(\zeta')) \right]. \quad (8.2)$$

Note that as of our discussion in section 8.1 the functions \mathcal{X}_γ may jump at the ray l_β , while \mathcal{X}_β is continuous there and thus along the ray of integration. In this form the equation is considered as an actual equation for all of the \mathcal{X}_γ and the approach to solving this equation in [GMN13b] is by iteration, starting with $\mathcal{X}_\gamma^{\text{sf}}$ as initial approximation. It remains to be shown that this process converges to a solution with all the desired properties like holomorphicity in the general case. In our setting the situation is different as we already constructed a solution of the Riemann-Hilbert problem and we want to

show that this solution solves the integral equation. To see this, it is useful to write the equation in the following form, which was the starting point in [GMN13b]:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left[\frac{1}{4\pi i} \sum_l \int_l \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log \left(\frac{\mathcal{X}_\gamma(\zeta')}{S_l^\gamma \mathcal{X}_\gamma(\zeta')} \right) \right]. \quad (8.3)$$

Note that here an analytic continuation is implicit, since the functions \mathcal{X}_γ jump exactly at the active BPS rays l . We will make this more precise below. Although not difficult it is enlightening to see how the two formulas relate. Consider the log-part in the second formula for some BPS ray l . From the definition (and suppressing the ζ' -dependence) it can be expanded as

$$\begin{aligned} \log \left(\frac{\mathcal{X}_\gamma}{S_l^\gamma \mathcal{X}_\gamma} \right) &= \log(\mathcal{X}_\gamma) - \log(S_l^\gamma \mathcal{X}_\beta) = \log(\mathcal{X}_\gamma) - \log \left(\prod_{\beta: l_\beta=l} (\mathcal{K}_\beta^\gamma)^{\Omega(\beta;q)} \mathcal{X}_\gamma \right) \\ &= \log(\mathcal{X}_\gamma) - \log \left(\mathcal{X}_\gamma \prod_{\beta: l_\beta=l} (1 - \sigma(\beta) \mathcal{X}_\beta)^{\langle \gamma, \beta \rangle \Omega(\beta;q)} \right) \\ &= -\log \left(\prod_{\beta: l_\beta=l} (1 - \sigma(\beta) \mathcal{X}_\beta)^{\langle \gamma, \beta \rangle \Omega(\beta;q)} \right) \\ &= - \sum_{\beta: l_\beta=l} \langle \gamma, \beta \rangle \Omega(\beta;q) \log(1 - \sigma(\beta) \mathcal{X}_\beta). \end{aligned}$$

By taking the sum over all BPS rays l we obtain the integral \int_{l_β} for every BPS ray l_β with $\Omega(\beta;q) \neq 0$. Conversely the condition $\Omega(\beta;q) \neq 0$ is only fulfilled for $\beta \in \Gamma_q^g$ which correspond to some BPS ray l_β . Thus the sum in the expression above for all BPS rays l equals exactly the sum over all $\beta \in \Gamma_q^g$ as in the first equation.

Our method of showing that our constructed set of coordinates solves the integral equation can be summarized as follows. First we define an a priori new set of functions \mathcal{Y}_γ via the integral equation, where we use the existence of the \mathcal{X}_γ -coordinates. We then show that these new functions are again a solution of the Riemann Hilbert problem. Finally a uniqueness argument ensures that the solutions have to coincide up to some scalar independent of ζ . Analyzing this scalar will then be the last task to obtain the desired decay rates.

Let us start by defining the "new" set of functions.

Definition 8.2.1

Let \mathcal{X}_γ (for all $\gamma \in \Gamma_q$) denote the constructed set of coordinates that solve the Riemann Hilbert problem 8.1.4. The corresponding *integral coordinates* on \mathbb{C}^\times are defined as

$$\mathcal{Y}_\gamma(\zeta) := \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left[\frac{1}{4\pi i} \sum_l \int_l \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log \left(\frac{\mathcal{X}_\gamma^+(\zeta')}{S_l^\gamma \mathcal{X}_\gamma^+(\zeta')} \right) \right].$$

In our proof that these function are solutions to the Riemann Hilbert problem we will already need the main estimates which will later on give us the optimal decay rate. So we have to make a lengthy calculation now.

Theorem 8.2.2

For every $\gamma \in \Gamma_q$ there exist scalars $C_{\beta,n}$ that depend on the distance of ζ to l_β and which are independent of R and the coordinates of the moduli space, such that the following estimate holds for R large enough:

$$\left\| \log \left(\frac{\mathcal{Y}_\gamma}{\mathcal{X}_\gamma^{\text{sf}}} \right) \right\| \leq \sum_{\beta} \Omega(\beta; u) \langle \gamma, \beta \rangle \sum_{n>0} C_{\beta,n} \frac{1}{\sqrt{4\pi n R |Z_\beta|}} e^{-2n\pi R |Z_\beta|}.$$

Proof. We want to use our knowledge that for any $\beta \in \Gamma_q$ the coordinates are of the form $\mathcal{X}_\beta = \mathcal{X}_\beta^{\text{sf}}(1 + \kappa)$ where κ is exponentially bounded to estimate each integral

$$I_{l_\beta} := - \int_{l_\beta} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_\beta(\zeta')).$$

Along l_β we have $\zeta' = -Z_\beta s$ for $s \in \mathbb{R}_+$, so

$$\mathcal{X}_\beta^{\text{sf}}(\zeta) = \exp \left(-\frac{R}{s} \pi + i\theta_\beta - R\pi |Z_\beta|^2 s \right) = \exp \left(-|Z_\beta| R\pi \left(\frac{1}{|Z_\beta|s} + |Z_\beta|s \right) + i\theta_\beta \right).$$

For R large enough $\mathcal{X}_\beta < 1$, so we expand $-\log(1 - \mathcal{X}_\beta)$ as $\sum_{n=1}^{\infty} \frac{\mathcal{X}_\beta^n}{n}$ and obtain for each $n \geq 1$ the integral

$$I_{l_\beta,n} := \int_0^\infty \frac{1}{s} \frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \frac{(1 + \kappa(-Z_\beta s))^n}{n} \exp \left(-n|Z_\beta| R\pi \left(\frac{1}{|Z_\beta|s} + |Z_\beta|s \right) + in\theta_\beta \right) ds$$

Following [FS17] a change of variable $s = e^x$ leads to

$$\begin{aligned} I_{l_\beta,n} &= \int_{-\infty}^\infty \frac{Z_\beta e^x - \zeta}{Z_\beta e^x + \zeta} \frac{(1 + \kappa(-Z_\beta e^x))^n}{n} \\ &\quad \cdot \exp \left(-n|Z_\beta| R\pi \left(e^{-x-\log|Z_\beta|} + e^{x+\log|Z_\beta|} \right) + in\theta_\beta \right) dx \\ &= \int_{-\infty}^\infty \frac{Z_\beta e^x - \zeta}{Z_\beta e^x + \zeta} \frac{(1 + \kappa(-Z_\beta e^x))^n}{n} \exp \left(-2n|Z_\beta| R\pi \cosh(x + \log|Z_\beta|) + in\theta_\beta \right) dx. \end{aligned}$$

Another change of variable $y = x + \log|Z_\beta|$ then leads to:

$$\begin{aligned} I_{l_\beta,n} &= \int_{-\infty}^\infty \frac{Z_\beta e^{y-\log|Z_\beta|} - \zeta}{Z_\beta e^{y-\log|Z_\beta|} + \zeta} \frac{(1 + \kappa(-Z_\beta e^{y-\log|Z_\beta|}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta| R\pi \cosh(y)) dy \\ &= \int_{-\infty}^\infty \frac{\frac{Z_\beta}{|Z_\beta|} e^y - \zeta}{\frac{Z_\beta}{|Z_\beta|} e^y + \zeta} \frac{\left(1 + \kappa\left(-\frac{Z_\beta}{|Z_\beta|} e^y\right)\right)^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta| R\pi \cosh(y)) dy. \end{aligned}$$

Now we write $\psi_\beta := \arg(Z_\beta)$ and obtain

$$\begin{aligned} I_{l_\beta,n} &= \int_{-\infty}^\infty \frac{e^{y+i\psi_\beta} - \zeta}{e^{y+i\psi_\beta} + \zeta} \frac{(1 + \kappa(-e^{y+i\psi_\beta}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta| R\pi \cosh(y)) dy \\ &= \int_0^\infty \frac{e^{y+i\psi_\beta} - \zeta}{e^{y+i\psi_\beta} + \zeta} \frac{(1 + \kappa(-e^{y+i\psi_\beta}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta| R\pi \cosh(y)) dy \\ &\quad + \int_0^\infty \frac{e^{-y+i\psi_\beta} - \zeta}{e^{-y+i\psi_\beta} + \zeta} \frac{(1 + \kappa(-e^{-y+i\psi_\beta}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta| R\pi \cosh(y)) dy, \end{aligned}$$

where we used the symmetry of \cosh in the last equality. The first fraction is a bounded continuous function that depends on the distance of ζ to l_β , and κ is exponentially bounded. Thus there exist constants $a, b \in \mathbb{R}_{>0}$ (independent of R and n) so that we have the following estimate:

$$\|I_{ln}\| \leq a \frac{b^n}{n} \int_0^\infty \exp(-2n\pi R|Z_\beta| \cosh(y)) dy = \left(a \frac{b^n}{n}\right) K_0(2n\pi R|Z_\beta|),$$

where K_0 is the modified Bessel function of the second kind. The main asymptotic estimate is now given by $K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} (1 + \mathcal{O}(1/x))$. Summing over all n we obtain the asymptotic estimate for the integral along l_β as

$$\|I_{l_\beta}\| \leq \sum_{n>0} \|I_{l_\beta, n}\| \leq \sum_{n>0} a \frac{b^n}{n} K_0(2n\pi R|Z_\beta|) \sim \sum_{n>0} C_{\beta, n} \frac{1}{\sqrt{nR|Z_\beta|}} e^{-2n\pi R|Z_\beta|}.$$

Here we wrote $C_{\beta, n}$ for an upper bound of $\frac{b^n}{n} (1 + f_n)$, where $f_n = \mathcal{O}(1/x)$ is the remainder for the asymptotic behaviour of K_0 . The estimate for the full coordinate now follows by summing over all corresponding integrals. \square

Now that we have good control over the new functions, we can continue with the plan.

Proposition 8.2.3

The set of \mathcal{Y}_γ is a solution to the Riemann Hilbert problem 8.1.4 with the jump-factors S_l^γ still determined by the \mathcal{X} -functions.

Note that this means that the \mathcal{Y} -functions have jumps which are determined by the \mathcal{X} -functions and not necessarily the \mathcal{Y} -functions. But we will show below that both sets of functions may only differ by a collection of phases, i. e. complex scalars with absolute value 1. The proof of the proposition uses that the standard approach to solving a Riemann Hilbert problem is to define a Cauchy-type integral where the integrand is given by the jumping behaviour. Here we follow in particular [FS17] for the general approach. They examine an example which is quite similar to ours, so at first we simply have to interpret the definition above in this way, but some additional complexities arise as we may have infinitely many jumps.

Proof. We start the proof with the description of a single jump, from which the result for finitely many jumps follows immediately. Finally, we have to consider the case of infinitely many jumps which is quite a bit more intricate and uses some additional information.

As \mathcal{X}_γ is a solution to the Riemann Hilbert problem for all $\gamma \in \Gamma_q$ it holds

$$\frac{\mathcal{X}_\gamma^+(\zeta')}{\mathcal{X}_\gamma^-(\zeta')} = \frac{\mathcal{X}_\gamma^+(\zeta')}{S_l^\gamma \mathcal{X}_\gamma^+(\zeta')} = (S_l^\gamma)^{-1}.$$

Thus we see that the integrand for each BPS ray l is given by

$$\frac{1}{\zeta' - \zeta} \frac{\zeta' + \zeta}{\zeta'} \frac{1}{2} \log(S_l^{-1}) =: \frac{1}{\zeta' - \zeta} \kappa_l.$$

Here we included the factor $1/2$ in κ_l , so that the integral obtains the standard coefficient $\frac{1}{2\pi i}$. In this form κ_l is continuous along l (for all ζ) and called the density, while the

factor $\frac{1}{\zeta' - \zeta}$ encodes the pole structure and is called the kernel. For ζ outside the BPS ray l standard complex analysis tells us, that $I := \frac{1}{2\pi i} \int_l \frac{1}{\zeta - \zeta'} \kappa_l d\zeta'$ defines a holomorphic function on the complement of l .

For the next step we start by observing that the continuation S_l^γ is holomorphic in ζ with bounded derivative along l , cf. section 6.3. In fact it decays exponentially for $\zeta \rightarrow 0$ and $\zeta \rightarrow \infty$, so κ_l is also holomorphic with bounded derivative along l . Note that this depends on the choice of direction for l . It follows that κ_l is Hölder continuous which is sufficient for the existence of the Cauchy principal value $\mathcal{P}I$ of the integral. Crucially the Plemelj-Sokhotski formula now holds (cf. [Mus53], in particular section 2). It guarantees the existence of the limits $I^+(\zeta_0)$ and $I^-(\zeta_0)$ of I as ζ approaches $\zeta_0 \in l$ from both sides and relates them as

$$\begin{aligned} I^+(\zeta_0) &= \frac{1}{2} \kappa(\zeta_0) + \frac{1}{2\pi i} \mathcal{P} \int_l \frac{\kappa}{\zeta' - \zeta_0} d\zeta', \\ I^-(\zeta_0) &= -\frac{1}{2} \kappa(\zeta_0) + \frac{1}{2\pi i} \mathcal{P} \int_l \frac{\kappa}{\zeta' - \zeta_0} d\zeta'. \end{aligned}$$

We don't really have to deal with the principal value and can just use the implication that the difference of both sides is simply

$$I^+(\zeta_0) - I^-(\zeta_0) = \kappa(\zeta_0) = \log(S_l^{-1}(\zeta_0)).$$

Accordingly $\exp(I^+(\zeta_0))$ and $\exp(I^-(\zeta_0))$ exist and are related by

$$\frac{\exp(I^+(\zeta_0))}{\exp(I^-(\zeta_0))} = \exp(I^+(\zeta_0) - I^-(\zeta_0)) = \exp(\log(S_l^{-1}(\zeta_0))) = S_l^{-1}(\zeta_0).$$

Thus we see that $\exp(I(\zeta))$ is a holomorphic function on the complement of l in \mathbb{C}^\times with the desired jumping behaviour at l . As the corresponding function for a different BPS ray l' is holomorphic along l they can safely be multiplied to obtain a function that jumps at l and l' and is holomorphic at their complement, which can be reiterated for all of the BPS rays in a finite sum. The same holds for multiplication with \mathcal{X}_{sf} which is holomorphic on all of \mathbb{C}^\times so that we obtain the desired behaviour for the whole \mathcal{Y}_γ if only finitely many jumps appear.

It remains to consider the case of infinitely many jumps, cf. figure 8.1. From the discussion in [GMN13b] we know that this happens precisely, when a ring domain appears and we know that the corresponding charges in Γ_q are given by $\beta + m\gamma$ for $m \in \mathbb{Z}$ where γ is the charge of the ring domain, cf. [Bri19]. We start by considering the product of jump-functions $P_k := \prod_{m=1}^k \exp(I_{\beta+m\gamma})$ for $k \in \mathbb{N}$. When the ring domain γ appears at some angle ϑ , then these are exactly the first k jumps which appear on "one side" of ϑ , say for angles smaller than ϑ . From Theorem 8.2.2 we infer

$$\|I_{\beta+m\gamma}\| \leq \frac{A}{\sqrt{R|Z_{\beta+m\gamma}|}} e^{-\pi R|Z_{\beta+m\gamma}|} \leq B e^{-mR|Z_\gamma|}$$

for some constants A, B which do not depend on m . So we obtain the following estimate.

$$\left| \sum_{m=0}^{\infty} I_{\beta+m\gamma} \right| \leq \sum_{m=0}^{\infty} |e^{-mR|Z_\gamma|}|.$$

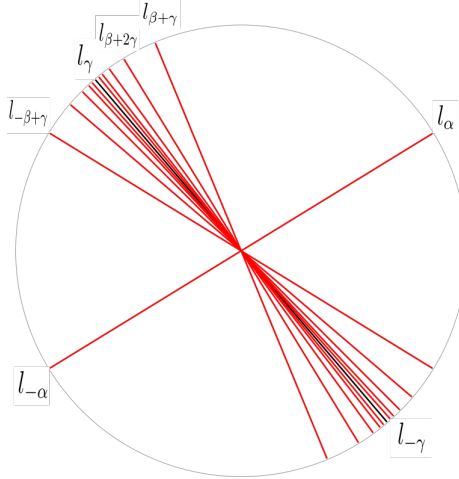


Figure 8.1: A collection of rays in the ζ -plane where jumps of the triangulation and thus of the \mathcal{X} -coordinates appear. Red rays indicate jumps from saddle connections. The black rays indicate jumps from a ring domain, where jumps from saddle connections accumulate.

From this we infer that $(P_k)_{k \in \mathbb{N}}$ is uniformly Cauchy and therefore converges uniformly towards $P = \prod_{m=1}^{\infty} \exp(I_{\beta+m\gamma})$. As all P_k are continuous away from the rays of the jumps, P is also continuous at this locus, which includes the ray at ϑ . Additionally all P_k are holomorphic away from the ray at ϑ and the rays of the jump and so is P .

The same procedure works for the infinite amounts of jumps on the right side of ϑ , so their product is a function that is holomorphic away from the BPS rays and the ray of angle ϑ , and whose right and left hand side limits for ϑ match. Finally we can build the product with the jump function for ϑ and obtain the function with correct jumping behaviour everywhere for a ring domain. As we have only finitely many ring domains we can do this for all ring domains and obtain the correct function. \square

Remark 8.2.4

The last step of the proof above uses again the fact that we only have finitely many ring domains, which, as of Lemma 4.4.8, is the case for generic differentials with unsealed spectrum. The spectrum had to be unsealed there, as otherwise another type of jump of the ca -triangulation could occur, which is not yet fully understood. So the problems there were of a rather geometrical nature. Here on the other hand the specific geometric meaning of the jumps is not as important. Instead the problem with another type of infinitely many jumps leads to problems in the definition of the supposed solution of the Riemann-Hilbert problem. But this is rather an analytical problem, as even for, say, an infinite amount of accumulations points of jumps, one might come up with a good way of ordering them and obtaining a well-behaved solution. In fact, the assumptions here are hardly the most general one for obtaining a solution to such a Riemann-Hilbert problem. For a more general treatment of these problems we refer to [Bri19]. So we note that this theory provides a link between two aspects of quadratic differentials of quite different nature (geometric and analytic).

We can now complete the link of the Riemann Hilbert problem and the geometric construction from before with the following uniqueness property.

Proposition 8.2.5

The solution of the Riemann Hilbert 8.1.4 with the jump-factors S_l^γ determined by the \mathcal{X} -functions is unique up to multiplication by a constant.

Proof. Let X and Y denote two solutions of the Riemann Hilbert problem and consider $Z := \frac{X}{Y}$. This function is holomorphic on the complement of the BPS rays and the limits when approaching any ray l from both sides are related as

$$Z^+ = \frac{X^+}{Y^+} = \frac{S_l^{-1} X^-}{S_l^{-1} Y^-} = \frac{X^-}{Y^-} = Z^-.$$

We see that Z does not jump at the rays and thus defines a continuous function on all of \mathbb{C}^\times . We also obtain that Z is holomorphic on \mathbb{C}^\times , where again some care is necessary for the case of infinitely many jumps. If a ring domain occurs at angle ϑ the holomorphicity away from the BPS rays follows directly. On the BPS rays away from ϑ the holomorphicity follows from Morera's theorem, as Z is holomorphic on a neighborhood of the ray. Thus only the ray at ϑ remains and we can again use Morera's theorem for this ray and obtain that Z is holomorphic on all of \mathbb{C}^\times . (Note that we used the fact that only finitely many ring domains may occur here. Otherwise different accumulation phenomena could occur.) From the boundedness assumption we obtain that Z has a removable singularity at 0, so as of Riemann's theorem on removable singularities Z can be continued to a holomorphic function on \mathbb{C} . The boundedness at ∞ on the other hand now guarantees as of Liouville's theorem that Z is constant, i. e. $X = AY$ for some $A \in \mathbb{C}^\times$. \square

We note that similar arguments concerning the uniqueness of solutions to Riemann-Hilbert problems can be found in [Bri19] and [Nei17]. Crucially we obtain the following corollary.

Corollary 8.2.6

For each $\gamma \in \Gamma_q$ there exists some A_γ independent of $\zeta \in \mathbb{C}^\times$, s. t. $\mathcal{X}_\gamma = A_\gamma \mathcal{Y}_\gamma$.

The value of A_γ can be regulated by demanding the appropriate behaviour for $\zeta \rightarrow \infty$ or $\zeta \rightarrow 0$. By conjecture it should be 1 in our case, but at the moment it is not clear, why this should be the case. The question is rather important, as A_γ is as of now only constant w. r. t. ζ and not the coordinates of \mathcal{M}'_R , so we need more information about this as we want to talk about asymptotic decay when varying the objects on \mathcal{M}'_R . Luckily we have such information, so that we can obtain our main theorem of this section.

Theorem 8.2.7

For every $\gamma \in \Gamma_q$ the factor A_γ is of the form $A_\gamma = \frac{1+K}{e^k}$, where K and k are exponentially bounded.

Proof. The factor A_γ is given by the quotient

$$A_\gamma = \frac{\mathcal{X}_\gamma}{\mathcal{Y}_\gamma} = \frac{\mathcal{X}_\gamma (\mathcal{X}_\gamma^{\text{sf}})^{-1}}{\mathcal{Y}_\gamma (\mathcal{X}_\gamma^{\text{sf}})^{-1}},$$

which is independent of ζ as of the preceding argument. We added the quotient by $\mathcal{X}_\gamma^{\text{sf}}$ to highlight that this expression appears in \mathcal{X}_γ as well as in \mathcal{Y}_γ and thus cancels in

the expression for A_γ . From the calculations in the proof of Theorem 8.2.2 it follows $\lim_{\zeta \rightarrow 0} (\mathcal{Y}_\gamma \mathcal{X}_\gamma^{\text{sf}})^{-1} = e^k$ for some k with the appropriate behaviour. On the other hand we have from section 7.2 that $\lim_{\zeta \rightarrow 0} (\mathcal{X}_\gamma \mathcal{X}_\gamma^{\text{sf}})^{-1}$ is of the proposed form, so the constant follows as the corresponding quotient. \square

8.3 Improvement of decay estimates

We have now obtained a lot more information about our \mathcal{X} -coordinates. They are dominated not only by the semiflat part, but also by the integrals over the BPS rays for which we already calculated the main estimate in Theorem 8.2.2. Similar calculations can now also be done for their differentials, which are needed to obtain additional information about the metric. More precisely we need to calculate $d \log \mathcal{X}_\gamma = \frac{d\mathcal{X}_\gamma}{\mathcal{X}_\gamma}$ where d is the differential w. r. t. coordinates of the moduli space. Accordingly the \mathcal{X}_γ are now considered on open subsets of \mathcal{M}'_R where we still demand that all Higgs fields have determinants in a chamber of the Hitchin base, cf. Lemma 4.4.7. Although we thus need the dependence of the \mathcal{X} -coordinates on \mathcal{M}'_R we will still suppress the dependence in the notation at times for better readability. Now start with the expression

$$\mathcal{X}_\gamma = A_\gamma \mathcal{X}_\gamma^{\text{sf}} \exp \left(\frac{1}{4\pi i} \sum_{\beta} \Omega(\beta; q) \langle \gamma, \beta \rangle I_{l_\beta} \right).$$

Away from the jump locus in the Hitchin base Ω is constant, so we obtain the following parts

$$d \log \mathcal{X}_\gamma = \frac{dA_\gamma}{A_\gamma} + \frac{R}{\zeta} \pi dZ_\gamma + id\theta_\gamma + R\zeta \pi d\overline{Z}_\gamma + \frac{1}{4\pi i} \sum_{\beta} \Omega(\beta; q) \langle \gamma, \beta \rangle dI_{l_\beta}. \quad (8.4)$$

We ignore the first part for now and first observe that the second part corresponds to the semiflat part. For the third part we consider again the definition of I_{l_β} and obtain

$$\begin{aligned} dI_{l_\beta} &= - \int_0^\infty \frac{1}{s} d \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) \log(1 - \mathcal{X}_\beta) ds \\ &\quad - \int_0^\infty \frac{1}{s} \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) d(1 + \kappa(-Z_\beta s)) \frac{\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds \\ &\quad - \int_0^\infty \frac{1}{s} \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) (1 + \kappa(-Z_\beta s)) \frac{d\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds := I_{l_\beta}^1 + I_{l_\beta}^2 + I_{l_\beta}^3. \end{aligned} \quad (8.5)$$

The first integral becomes

$$I_{l_\beta}^1 = - \int_0^\infty \frac{2\zeta}{(Z_\beta s + \zeta)^2} \log(1 - \mathcal{X}_\beta) ds dZ_\beta.$$

Expanding the logarithm and changes of variables just like in the proof of Theorem 8.2.2 now leads to

$$\begin{aligned} I_{l_\beta, n}^1 &= \int_{-\infty}^{\infty} \frac{2\zeta}{(e^{y+i\psi} + \zeta)^2} e^{y - \log |Z_\beta|} \frac{(1 + \kappa(-e^{y+i\psi_\beta}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta|R\pi \cosh(y)) dy dZ_\beta \\ &= \int_0^\infty \frac{2\zeta}{(e^{y+i\psi} + \zeta)^2} e^{y - \log |Z_\beta|} \frac{(1 + \kappa(-e^{y+i\psi_\beta}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta|R\pi \cosh(y)) dy dZ_\beta \\ &\quad + \int_0^\infty \frac{2\zeta}{(e^{y+i\psi} + \zeta)^2} e^{-y - \log |Z_\beta|} \frac{(1 + \kappa(-e^{-y+i\psi_\beta}))^n}{n} e^{in\theta_\beta} \exp(-2n|Z_\beta|R\pi \cosh(y)) dy dZ_\beta. \end{aligned}$$

The main difference to the integrals before is the additional integrating factor e^y . Still the rest in front of the exponential is bounded, so we may estimate (the function part of) $I_{l_\beta, n}^1$ as

$$\|I_{l_\beta, n}^1\| \leq \frac{1}{|Z_\beta|} a \frac{b^n}{n} \int_0^\infty \cosh(y) \exp(-2n|Z_\beta|R\pi \cosh(y)) dy = \frac{1}{|Z_\beta|} a \frac{b^n}{n} K_1(2n\pi R|Z_\beta|)$$

for some scalars a, b as in Theorem 8.2.2. We see that the structure is mostly the same as before, only now with K_1 instead of K_0 .

For the second integral in (8.5) we obtain

$$I_{l_\beta}^2 = \int_0^\infty \frac{1}{s} \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) (d\kappa)(-Z_\beta s) \frac{\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds dZ_\beta.$$

From section 6.3 we know that the derivatives of κ are exponentially bounded, and for R large enough $|1 - \mathcal{X}_\beta|^{-1}$ is also bounded by, say, 2. Thus the same arguments as in the estimates before can be used to obtain upper bounds in terms of K_0 for this integral

Finally the last part of (8.5) is

$$\begin{aligned} I_{l_\beta}^3 &= - \int_0^\infty \frac{1}{s} \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) (1 + \kappa(-Z_\beta s)) d \left(-\frac{R}{s} \pi + i\theta_\beta - R\pi|Z_\beta|^2 s \right) \frac{\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds \\ &= - \int_0^\infty \frac{1}{s} \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) (1 + \kappa(-Z_\beta s)) i \frac{\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds d\theta_\beta \\ &\quad + \int_0^\infty \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) (1 + \kappa(-Z_\beta s)) R\pi \overline{Z_\beta} \frac{\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds dZ_\beta \\ &\quad + \int_0^\infty \left(\frac{Z_\beta s - \zeta}{Z_\beta s + \zeta} \right) (1 + \kappa(-Z_\beta s)) R\pi Z_\beta \frac{\mathcal{X}_\beta^{\text{sf}}}{1 - \mathcal{X}_\beta} ds d\overline{Z_\beta}. \end{aligned}$$

Now the same arguments as before apply and we obtain bounds in terms of the Bessel functions, which thus give us bounds for all of dI_{l_β} .

The last part we now have to check in (8.4) is dA/A . For this we first need the following observation.

Lemma 8.3.1

The integral coordinates \mathcal{Y}_γ obey the reality condition $\mathcal{Y}_\gamma(\zeta) = \overline{\mathcal{Y}_{-\gamma}(-\frac{1}{\zeta})}$.

Proof. This follows from a direct calculation, using that, as of Theorem 7.2.11, \mathcal{X}_γ obeys the reality condition and the fact that for each active BPS ray l_β the ray $l_{-\beta}$ is also active. \square

From the reality condition we infer that

$$A_\gamma \mathcal{Y}_\gamma(\zeta) = \mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-\frac{1}{\zeta})} = \overline{A_{-\gamma} \mathcal{Y}_{-\gamma}(-\frac{1}{\zeta})} = \overline{A_{-\gamma}} \mathcal{Y}_\gamma(\zeta).$$

Thus it holds $A_\gamma = \overline{A_{-\gamma}}$. On the other hand it holds $1 = \mathcal{X}_\gamma \mathcal{X}_{-\gamma}$, which also implies $A_{-\gamma} = A_\gamma^{-1}$. These two facts together tell us that $\|A_\gamma\| = 1$. Note that this shows that the \mathcal{X} - and \mathcal{Y} -coordinates only differ by a phase factor. Most importantly for us however is that it allows us to compute one Wirtinger derivative if the other one is known. More explicitly it holds

$$\partial_u A_\gamma = \overline{\partial_{\bar{u}} \overline{A_\gamma}} = \overline{\partial_{\bar{u}} \frac{1}{A_\gamma}} = - \overline{\left(\frac{\partial_{\bar{u}} A_\gamma}{(A_\gamma)^2} \right)} \quad (8.6)$$

for any coordinates u of the moduli space.

Equipped with this identity we are now able to obtain the desired bounds along sections which leave one part of a Higgs pair constant.

Lemma 8.3.2

Let u be a holomorphic coordinate of \mathcal{M}'_R in the complex structure I such that the connection of the Higgs pairs is constant in u or such that the eigenline bundles of the Higgs fields are constant in u . Then $\frac{\partial_{\bar{u}} A_\gamma}{A_\gamma}$ and $\frac{\partial_u A_\gamma}{A_\gamma}$ are bounded in terms of the Bessel functions K_0 and K_1 for every $\gamma \in \Gamma_q$,

Proof. To obtain information for the holomorphic (or anti-holomorphic) dependence of A_γ we consider the flatness equation in the unitary picture. So we fix a hermitian background metric h_0 and consider \mathcal{M}_R as the space of Higgs pairs (A, Φ) solving Hitchin's equations for the fixed metric h_0 . In a corresponding unitary frame the flatness equation takes the form

$$0 = \nabla(\zeta)s = \left(d + A_{\bar{z}} - (\overline{A_z})^t + \frac{R}{\zeta} \Phi + R\zeta \bar{\Phi}^t \right) s.$$

We want to use our well-established theory, so we diagonalize $\Phi = \varphi dz$ with some change of basis matrix g , so the equation takes the form

$$\begin{aligned} 0 &= \left(d + g^{-1}(\partial_z g) + g^{-1}(\partial_{\bar{z}} g) + g^{-1} A_{\bar{z}} g - g^{-1} (\overline{A_z})^t g + \frac{R}{\zeta} g^{-1} \Phi g + R\zeta g^{-1} \bar{\Phi}^t g \right) s \\ &=: d + \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} s + Bs. \end{aligned} \quad (8.7)$$

Here $\pm\lambda$ denotes the eigenvalues of Φ and we defined B as the matrix containing all the of the other terms. Let Λ be a primitive of λ and write $s = e^{\frac{R}{\zeta} \Lambda} x$. Note that this means that we do not detach the whole leading term from before, so the remaining x will not equal the whole remainder, but instead will involve some additional part. We will later see that this poses no problem. Now x solves

$$dx = \begin{pmatrix} 2\lambda & 0 \\ 0 & 0 \end{pmatrix} x + Bx.$$

Along a ϑ -trajectory with appropriate angle ϑ the small flat section can again be obtained from the integral equation

$$x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{2\Lambda(t)-2\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} Bx(\tau) d\tau.$$

Now we can consider the limit $\zeta \rightarrow 0$ and note that $\Lambda(t)$ is of the form $\frac{1}{\zeta} Re^{i\vartheta}t + f$ for some constant f . As before, the choice of ϑ implies that for large enough R the real part of Λ is positive. The entries of B are affine-linear in ζ and thus also bounded by some constant ϵ for all ζ small enough. We first want to show that in the limit the integral equation decouples. For this we remember from section 7.2 that $\|x(t)\|$ is bounded by some constant c that holds for all ζ small enough. Therefore the equation tells us

$$\begin{aligned} |x_1(t)| &\leq \left\| e^{\frac{1}{\zeta} Re^{i\vartheta}t + f} \int_t^\infty e^{-\frac{1}{\zeta} Re^{i\vartheta}\tau - f} \epsilon x(\tau) d\tau \right\| \\ &\leq \left\| \frac{\epsilon c}{\frac{1}{\zeta} Re^{i\vartheta}} \right\|. \end{aligned}$$

Thus we see that for $\zeta \rightarrow 0$ the first component x_1 vanishes uniformly. For the second component we now write the components of B as b_{ij} and obtain

$$x_2(t) = 1 - \int_t^\infty b_{21}(\tau) x_1(\tau) + \int_t^\infty b_{22}(\tau) x_2(\tau) d\tau.$$

An application of the variation of constants leads to an appropriate expression for x_1 depending on the "inhomogeneity" $1 - \int_t^\infty b_{21}(\tau) x_1(\tau)$. As this expressions converges uniformly to 1 for $\zeta \rightarrow 0$ we obtain that x_2 converges to the solution of

$$x_2(t) = 1 + \int_t^\infty b_{22}(\tau) x_2(\tau) d\tau.$$

Thus we have obtained that in the limit $\zeta \rightarrow 0$ the small flat section converges uniformly to

$$x(t) = \begin{pmatrix} 0 \\ e^{\int_t^\infty b_{22}^l(\tau) d\tau} \end{pmatrix}$$

where b_{22}^l denotes the limit of b_{22} for $\zeta \rightarrow 0$. We thus see that in this limit the expression of x is rather simple and we can obtain the corresponding expression for all small flat sections for a standard quadrilateral. Putting them together we obtain the integral along the boundary b of the quadrilateral:

$$\mathcal{X}_\gamma^r(0) = \lim_{\zeta \rightarrow 0} \mathcal{X}_\gamma^r(\zeta) = e^{\oint_b b_{22}^l}.$$

Note that the fact that (A, Φ) solves Hitchin's equations implies that the integrand is actually closed, so we can consider this as a period integral for the standard period surrounding the two zeroes of the quadrilateral. Comparing this function with the usual \mathcal{X}^e at $\zeta = 0$ we see that $e^{i\theta} \mathcal{X}^e = \mathcal{X}^r$.

Now consider a holomorphic coordinate u of \mathcal{M}'_R in complex structure I . For such a coordinate the entries of Φ and $A_{\bar{z}}$ depend holomorphically on u , while the entries of $\bar{\Phi}$ and A_z are anti-holomorphic. Additionally the change of frame matrix g from the unitary to the diagonal frame for Φ can be chosen to vary holomorphically with u . From Theorem 6.3.5 we obtain that x depends holomorphically on every holomorphic parameter and so does \mathcal{X}^r .

Now let u be a coordinate which leaves A constant. In this case we see from equation 8.7 that the only part that is not holomorphic in u comes from $R\zeta\bar{\Phi}$. But this part vanishes for $\zeta \rightarrow 0$, so in this case b_{22}^l is holomorphic in u and therefore \mathcal{X}^r is as well.

We may now write A as

$$A = \frac{\mathcal{X}^e}{\mathcal{Y}^e} = \frac{\mathcal{X}^r}{e^{-i\theta}\mathcal{Y}^e}.$$

From the holomorphicity of \mathcal{X}^r we obtain

$$\frac{\partial_{\bar{u}}A}{A} = -\frac{\partial_{\bar{u}}e^{-i\theta}\mathcal{Y}^e}{e^{-i\theta}\mathcal{Y}^e}.$$

Since u leaves A invariant, θ is constant in u , so $\frac{\partial_{\bar{u}}A}{A}$ is simply given by the $\frac{\partial_{\bar{u}}\mathcal{Y}^e}{\mathcal{Y}^e}$. The calculations above now show that this is bounded in the stated way and with equation (8.6) we obtain the corresponding bound for $\frac{\partial_{\bar{u}}A}{A}$.

For the second part let u now be such a holomorphic coordinate of \mathcal{M}'_R that the eigenspaces of Φ do not vary with u . In this case the transformation g can be taken to be constant and we know from Lemma 5.1.2 that $A_{\bar{z}}$ is also diagonal in this frame.

Let a be the upper left entry of $A_{\bar{z}}$ and write ∇ as $\nabla + \bar{a}E_2 - \bar{a}E_2$. This allows us to write s as $s = e^{\frac{R}{\zeta}\Lambda+\alpha}x$ for α a primitive of a . Then x solves the equation

$$x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{2\Lambda(t)-2\Lambda(\tau)} & 0 \\ 0 & 1 \end{pmatrix} \tilde{B}x(\tau)d\tau.$$

where \tilde{B} now depends anti-holomorphically on u . Thus we obtain that x depends anti-holomorphically on u . Note that this time we also have detached the primitive for a . Therefore this construction leads to the full remainder \mathcal{X}^e which now depends anti-holomorphically on u . The same argument as above, only this time starting with $\frac{\partial_{\bar{u}}A}{A}$, now lead to the desired estimates for $\frac{\partial_{\bar{u}}A}{A}$ and $\frac{\partial_{\bar{u}}A}{A}$. □

Altogether we obtain the following main result in this section.

Proposition 8.3.3

Let \mathcal{N} be any submanifold of \mathcal{M}'_R given by holomorphic coordinates in complex structure I that leave the Higgs field or the connection of the Higgs pairs constant. Then for every $\gamma \in \Gamma_q$ it holds

$$d \log \mathcal{X}_\gamma = \frac{R}{\zeta} \pi dZ_\gamma + i d\theta_\gamma + R\zeta \pi d\bar{Z}_\gamma + f_\gamma^e,$$

where f_γ^e is bounded by terms of the form $\sum_n C_n(K_0(2n\pi R|Z_\beta|) + K_1(2n\pi R|Z_\beta|))$ for some constants C_n .

Chapter 9

Twistorial construction of the hyperkähler metric

In this final chapter we construct the hyperkähler metric out of the \mathcal{X} -coordinates. For this we first summarize the general idea of constructing hyperkähler metrics by twistor methods in section 9.1 and show how this works for the concrete example of the semiflat metric in section 9.2. Finally we use the construction for the true coordinate functions that we built in the last sections to obtain another hyperkähler metric which is then shown to decay exponentially towards the semiflat metric in section 9.3.

9.1 Basic twistor theory

The foundation for all of the upcoming constructions is the twistor theorem which we therefore state at this point. The original construction is found in [Hit+87] and [Hit92]. Consider any hyperkähler manifold M with complex structures I , J and K and corresponding Kähler forms ω_I , ω_J and ω_K . One of the crucial ideas of twistor theory is to make use of the fact that in this case there is actually a whole \mathbb{CP}^1 worth of complex structures on M . They may be denoted by

$$I^\zeta = \frac{(1 - |\zeta|^2)I + i(-\zeta + \bar{\zeta})J - (\zeta + \bar{\zeta})K}{1 + |\zeta|^2},$$

where ζ is a local coordinate of \mathbb{CP}^1 . The corresponding Kähler form is then given by

$$\omega_\zeta = \frac{(1 - |\zeta|^2)\omega_I + i(-\zeta + \bar{\zeta})\omega_J - (\zeta + \bar{\zeta})\omega_K}{1 + |\zeta|^2}.$$

Additionally there is the holomorphic symplectic structure

$$\varpi(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_I - \frac{i}{2}\zeta\omega_- \quad (9.1)$$

for $\omega_\pm = \omega_J \pm i\omega_K$. This structure is used to build the twistor space of \mathcal{Z} of M as the product manifold $Z := M \times \mathbb{CP}^1$ equipped with the complex structure

$$\underline{I} := (I^\zeta, I_0),$$

where I_0 is the complex structure on \mathbb{CP}^1 obtained by multiplication with i on the tangent space $T_\zeta\mathbb{CP}^1$ of $\zeta \in \mathbb{CP}^1$. There is then an obvious fibration of the twistor space \mathcal{Z} over \mathbb{CP}^1 with holomorphic sections $\zeta \mapsto (m, \zeta)$ for each $m \in M$. These are called the twistor lines and they each have a normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$. Additionally there is

a real structure $\tau : M \times \mathbb{CP}^1 \rightarrow M \times \mathbb{CP}^1$ given by $\tau(m, \zeta) = \left(m, \frac{1}{\bar{\zeta}}\right)$ that is compatible with all of the aforementioned structure in a suitable sense and induces the antipodal map on \mathbb{CP}^1 .

Crucially this collection of structures on its own is enough of to obtain a hyperkähler manifold. This is the famous twistor theorem of N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, cf. [Hit+87], [Hit92].

Theorem 9.1.1

Let \mathcal{Z}^{2r+1} be a manifold of complex dimension $2r + 1$ with the following structure:

1. \mathcal{Z} is a holomorphic fiber bundle $p : \mathcal{Z} \rightarrow \mathbb{CP}^1$ over the projective line.
2. The bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2r} \otimes \mathcal{O}(1)$.
3. There exists a holomorphic section ϖ of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$, defining a symplectic form on each fiber, where $T_F := \text{Ker } dp : T\mathcal{Z} \rightarrow T\mathbb{CP}^1$ is the tangent bundle along the fibers.
4. \mathcal{Z} has an anti-holomorphic involution $\tau : \mathcal{Z} \rightarrow \mathcal{Z}$ which covers the antipodal map on \mathbb{CP}^1 : $(m, \zeta) \mapsto (m, -1/\bar{\zeta})$, and preserves ϖ , i. e. $\tau^* \varpi = \overline{\varpi}$.

Then the parameter space of real sections is a manifold M of real dimension $4r$ with a natural hyperkähler metric for which \mathcal{Z} is the twistor space. Additionally if one starts with a hyperkähler manifold M' and uses this construction on its twistor space to obtain M , then M' is naturally included in M and the structures coincide.

Starting with [GMN10] GMN proposed a way of using this theorem on a general class of integrable systems. They stated a list of conditions for a set of coordinate functions on such an integrable system which, if fulfilled, would allow for the construction of a holomorphic structure and a holomorphic symplectic form fulfilling all of the conditions of Theorem 9.1.1. We state here the general conjecture of GMN.

Conjecture 9.1.2

Let \mathcal{B} be a r -dimensional complex manifold with a $2r$ -dimensional torus fibration $\mathcal{M} \rightarrow \mathcal{B}$. Additionally let there be a local system of lattices Γ of rank $2r$ with generators γ_i equipped with a symplectic pairing with coefficients ϵ^{ij} , such that for every choice of local patch in \mathcal{B} and local sections γ of Γ there are locally defined \mathbb{C}^\times -valued functions $\mathcal{X}_\gamma(u, \theta; \zeta)$ of $(u, \theta) \in \mathcal{M}$ and $\zeta \in \mathbb{C}^\times$. Then define

$$\varpi(\zeta) := \frac{1}{8\pi^2 R} \sum_{ij} \epsilon_{ij} \frac{d\mathcal{X}_{\gamma^i}}{\mathcal{X}_{\gamma^i}} \wedge \frac{d\mathcal{X}_{\gamma^j}}{\mathcal{X}_{\gamma^j}},$$

where d is the fiber wise differential, which ignores ζ and ϵ_{ij} is the inverse of ϵ^{ij} .

Now assume that the following properties hold:

1. The \mathcal{X}_γ are multiplicative, i. e. $\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$.
2. The \mathcal{X}_γ obey the following reality condition

$$\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}.$$

3. All \mathcal{X}_γ are solutions to a single set of differential equations of the form

$$\begin{aligned}\frac{\partial}{\partial u^i} \mathcal{X}_\gamma &= \left(\frac{1}{\zeta} \mathcal{A}_{u^i}^{(-1)} + \mathcal{A}_{u^i}^{(0)} \right) \mathcal{X}_\gamma, \\ \frac{\partial}{\partial \bar{u}^i} \mathcal{X}_\gamma &= \left(\mathcal{A}_{\bar{u}^i}^{(0)} + \zeta \mathcal{A}_{\bar{u}^i}^{(1)} \right) \mathcal{X}_\gamma,\end{aligned}$$

where the $\mathcal{A}_{u^i}^{(n)}$ and $\mathcal{A}_{\bar{u}^i}^{(n)}$ are complex vector fields on the torus fiber \mathcal{M}_u , with the $\mathcal{A}_{u^i}^{(-1)}$ linearly independent at every point and similarly $\mathcal{A}_{\bar{u}^i}^{(1)}$.

4. For each $x \in \mathcal{M}$, $\mathcal{X}_\gamma(x; \zeta)$ is holomorphic in ζ on a dense subset of \mathbb{C}^\times .

5. $\varpi(\zeta)$ is globally defined (in particular the $\varpi(\zeta)$ defined over different local patches of \mathcal{B} agree with one another) and holomorphic in $\zeta \in \mathbb{C}^\times$.

6. $\varpi(\zeta)$ is nondegenerate in the appropriate sense for a holomorphic symplectic form, i.e. $\ker \varpi(\zeta)$ is a $2r$ -dimensional subspace of the $4r$ -dimensional $T_{\mathbb{C}}\mathcal{M}$.

7. $\varpi(\zeta)$ has only simple poles as $\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$.

Then $\mathcal{Z} := \mathcal{M} \times \mathbb{CP}^1$ has all the structure necessary for the Twistor construction 9.1.1. In particular, a hyperkähler metric g on \mathcal{M} can be reconstructed from \mathcal{Z} .

The notes [Nei13] from A. Neitzke give an overview of this construction and contain some examples where the construction should be successful, one of which is the case of weakly parabolic Higgs bundles we are considering here. On this specific space it was conjectured in [GMN13b] that a variation of this construction should work, where the necessary adjustments are stated at the beginning. In particular the dependence of the triangulations on ϑ and the relationship of ϑ and ζ and the corresponding jumps have to be considered, for which the half-plane \mathbb{H}_ϑ was originally introduced.

Although our case therefore fits in the general picture we don't work with the whole construction here, as we are not interested in building a hyperkähler metric on some previously unknown integrable system, but rather on a space where a lot of structure is already given and we can simply replace the parts we want. The important part we do consider is the construction of the holomorphic symplectic form, i. e. we consider the following:

Let Γ be a lattice with generators γ_i and for each γ_i let \mathcal{X}_{γ_i} be a function defined on $M \times \mathbb{C}^\times$, s. t. for each fixed $\zeta \in \mathbb{C}^\times$ the \mathcal{X}_{γ_i} are coordinates of M . Let there also be given a pairing on Γ with coefficients ϵ^{ij} and inverse ϵ_{ij} . Then define for any parameter $R \in \mathbb{R}_+$ the following 2- form:

$$\varpi := \frac{1}{8\pi^2 R} \sum_{i,j} \epsilon_{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}. \quad (9.2)$$

As above d is the fiberwise differential that ignores ζ . In the general construction the conditions on the \mathcal{X} are supposed to ensure that ϖ is actually a holomorphic 2-form that is twisted by $O(2)$ and defines a symplectic form on each fiber, which is just what is needed

by Theorem 9.1.1. As we will show below, our work in the previous chapters gives us exactly this, as was conjectured by GMN in [GMN13b].

We use this moment to highlight the property 5 of the construction. It states that ϖ is holomorphic as a function depending on $\zeta \in \mathbb{C}^\times$. But ϖ as defined in (9.2) is built out of the \mathcal{X} -coordinates which jump at certain rays in the \mathbb{C}^\times -plane, so some cancellation has to occur. To see this we consider some part $d \log \mathcal{X}_\gamma \wedge d \log \mathcal{X}_\beta$ of ϖ . The important fact, following from the discussion in [GMN13b], is that if a jump of \mathcal{X}_γ occurs, \mathcal{X}_β is holomorphic, i. e. $\mathcal{X}_\beta^+ = \mathcal{X}_\beta^-$, and the jump for \mathcal{X}_γ is given by formula (8.1), which in the logarithm becomes $\log \mathcal{X}_\gamma^- \mapsto \log \mathcal{X}_\gamma^- + \log \mathcal{X}_\beta$. But this now cancels if we take the \wedge -product of the derivatives, so

$$(d \log \mathcal{X}_\gamma \wedge d \log \mathcal{X}_\beta)^- = (d \log \mathcal{X}_\gamma \wedge d \log \mathcal{X}_\beta)^+.$$

In the general setting this is the reason why the \mathcal{K}_β^γ , considered as automorphism on the space of unitary characters of Γ_q , are called symplectomorphisms. Here we simply denote this in the following way.

Proposition 9.1.3

Let q be a generic differential with unsealed spectrum. Then ϖ as defined in (9.2) with the coordinate functions $\mathcal{X}_\gamma : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ is holomorphic on all of \mathbb{C}^\times .

Before we go on, let us recall why we go through the trouble of this construction. After all we are about to obtain a hyperkähler metric on a space which we already know is hyperkähler. The upshot is simply, as already noted in [GMN10], that we not only obtain the existence of a hyperkähler metric but we get a very concrete expression of the metric which allows for a much better understanding of its behavior. To see this, consider the case that ϖ indeed fulfills the required conditions. Then by the twistor theorem we obtain that M is a hyperkähler space whose twistor space is equipped with the holomorphic symplectic form ϖ . But we know from the general twistor theory that the holomorphic symplectic form can be decomposed as in formula (9.1), i. e. we may write

$$\varpi(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_I - \frac{i}{2}\zeta\omega_- \quad (9.3)$$

where ω_I , ω_J and ω_K are the Kähler forms of the hyperkähler metric g . Thus we can extract the metric by specifying ζ to an appropriate value. Below we will use that the complex structure I in our case is simply given by multiplication by i , so we state the according extraction here.

Lemma 9.1.4

Assume that all the conditions of the twistor theorem are fulfilled. Using the notation from above the hyperkähler metric is given as

$$g(v, w) = \omega_I(v, Iw) = \operatorname{Re}(\varpi|_{\zeta=-1}(v, Iw)).$$

Proof. Setting $\zeta = -1$ we obtain

$$\varpi|_{\zeta=-1} = \frac{i}{2}\omega_+ + \omega_I + \frac{i}{2}\omega_- = \frac{i}{2}(\omega_J + \omega_K) + \omega_I + \frac{i}{2}(\omega_J - \omega_K) = \omega_I + i\omega_J.$$

Thus we have $\omega_I = \text{Re}(\varpi|_{\zeta=-1})$. As ω_I is the Kähler form for the action of I we obtain

$$g(v, w) = \omega_I(v, Iw) = \text{Re}(\varpi|_{\zeta=-1}(v, Iw)).$$

□

In this way knowing ϖ amounts to knowing g and the better we understand ϖ the better we understand g . In our case, as ϖ is constructed from the small flat sections in a concrete manner, we obtain that the better we understand the small flat sections, the better we understand ϖ and therefore g . This justifies the work we put in the detailed calculations concerning those sections. We conclude this introduction with a corresponding notion which will help in distinguishing different hyperkähler metrics in the following sections.

Definition 9.1.5

Any hyperkähler metric that is obtained in the way described above, i. e. from a holomorphic symplectic form that is built out of a set of coordinate functions via formula (9.2), shall be called *twistorial*.

9.2 The semiflat metric in the twistorial picture

Before we go on to use the method we just described for our true coordinate functions, it may be enlightening to describe the much simpler case of the semiflat metric and how it fits into the twistorial picture. Basically we can just do everything we've done until now, only that we do not start with the true solutions of Hitchin's equations but rather work only with the limiting configurations.

Before we do this, however, we have to point to the fact that there are two different points of view we have to consider here. We could just build all the objects via the formulas given above out of the limiting configurations and obtain a symplectic form ϖ^{sf} on the true moduli space \mathcal{M}_R of solutions of the R -rescaled Hitchin's equation. However in this way we don't know whether ϖ^{sf} is a holomorphic form on \mathcal{M}_R . Our argument for the holomorphicity won't work as the corresponding small flat sections are not solutions to the true ODE and thus won't necessarily vary holomorphically with the ODE data. We do know however that there is semiflat metric g_{sf} on \mathcal{M} from the usual theory of parabolic Higgs bundles.

On the other hand we can consider the moduli space of limiting configurations \mathcal{M}_∞ (cf. [Maz+16] for the regular case and [Fre+20] and references therein for the parabolic one) with corresponding regular locus \mathcal{M}'_∞ . In this case the construction does work and we obtain the twistorial metric $g_{\text{twist}}^{\text{lim}}$. The relevant fact here is that these two sides actually coincide, i. e. with the appropriate identifications we have $g_{\text{sf}} = g_{\text{twist}}^{\text{lim}}$. This fact can be obtained from the explicit expression of $g_{\text{twist}}^{\text{lim}}$ which we will now describe on \mathcal{M}'_∞ .

As our goal in this section is to describe how the explicitly known semiflat metric can be obtained in the twistorial picture, we refrain from giving any detailed proofs here. The fact that the finally resulting form is actually the Kähler form of the semiflat metric simply follows from the appearance of the explicitly known structure. For the rigorous argument we simply note that all calculations are simpler versions of the ones we already did and the remaining arguments concerning the twistor theory will be given for the more complicated version of the twistorial metric on \mathcal{M}_R in the next section.

In the space of limiting configurations all of the calculations become way easier as the Higgs field as well as the connection form are diagonal in a unitary frame for the corresponding Hermitian metric. We may thus write the flat connection ∇^L of the limiting case in the following way, using the third Pauli matrix σ^3 for convenience:

$$\nabla^L(\zeta) := d + A_\infty + \frac{R}{\zeta}\Phi_\infty + R\zeta\Phi_\infty^* = \partial + \frac{R}{\zeta}q^{1/2}\sigma^3 dz + A_z dz + \bar{\partial} + A_{\bar{z}} d\bar{z} + R\zeta\overline{q^{1/2}}\sigma^3 d\bar{z}.$$

Here $\Phi_\infty = q^{1/2}\sigma_3$, A_z is diagonal with entries $-a_1$ and $-a_2$, and $A_{\bar{z}} = -\overline{A_z}$.¹ We also consider the same conventions for the quadrilateral, annulus and frame as constructed in section 5. As the connection is in "diagonal gauge" and the description is in local coordinates the sections can be computed in each entry of $s = (s_1, s_2)$ independently. For s_i the differential equation becomes:

$$0 = \partial s_i - a_1 s_1 dz + \frac{R}{\zeta}q^{1/2}s_i dz + \bar{\partial} s_i + \overline{a_1} s_i d\bar{z} + R\zeta\overline{q^{1/2}}s_i d\bar{z}.$$

As before the dz and $d\bar{z}$ part have to become 0 independently, so the equation splits into the set of two PDEs:

$$\frac{\partial s_i}{\partial z} = \left(-\frac{R}{\zeta}q^{1/2} + a_1\right)s_i \quad \wedge \quad \frac{\partial s_i}{\partial \bar{z}} = (-R\zeta\overline{q^{1/2}} - \overline{a_1})s_i.$$

Denote by \overline{P} a primitive of $q^{1/2}$. From $\overline{\partial} \overline{P} = \overline{\partial} \overline{P} = \overline{q^{1/2}}$ it then follows, that \overline{P} is a primitive of $\overline{q^{1/2}}$. Furthermore let Δ_i be a primitive of $a_i - \overline{a_i}$. With these notations the solution s_1 of the set of PDEs becomes

$$s_1 = C_1 e^{-\frac{R}{\zeta}P + \Delta_1 - R\zeta\overline{P}},$$

for any $C_1 \in \mathbb{C}$. For the other component the different sign in σ^3 leads to the corresponding solution

$$s_2 = C_2 e^{\frac{R}{\zeta}P - \Delta_2 + R\zeta\overline{P}},$$

for any $C_2 \in \mathbb{C}$. So we have the general solution

$$s_L := \begin{pmatrix} C_1 e^{-\frac{R}{\zeta}P + \Delta - R\zeta\overline{P}} \\ C_2 e^{\frac{R}{\zeta}P - \Delta + R\zeta\overline{P}} \end{pmatrix},$$

where the subscript L indicates that these are the flat sections for the limiting configurations.

¹Here the signs are chosen to match the formulas in section 5.

Just as in section 7.1 we want to choose a small flat section for each vertex, but now this is easily obtained as no remainder term exists and we see directly which section to choose for each vertex. If we label the vertices of a quadrilateral by 1, 2, 3, 4 we obtain for vertex 1 the section

$$s_L^1(\zeta, z) := \begin{pmatrix} 0 \\ e^{\frac{R}{\zeta} P_1(z) - \Delta(z)_2 + R\zeta \overline{P_1}(z)} \end{pmatrix}.$$

As before we need to consider the primitives in a contractible subset of the standard annulus, but as we know the solutions completely, we don't have to transport the solutions to the ca -trajectories in order to evaluate them.

Then we may build the \mathcal{X}_γ^L coordinates and obtain the corresponding period integrals.

$$\begin{aligned} \mathcal{X}_\gamma^L &= -\frac{(s_L^1 \wedge s_L^2)(s_L^3 \wedge s_L^4)}{(s_L^2 \wedge s_L^3)(s_L^4 \wedge s_L^1)} \\ &= -\exp\left(\frac{R}{\zeta} \oint_\gamma q^{1/2} + \left(\oint_\gamma a_1 dz - \overline{a_1} d\overline{z}\right) + R\zeta \oint_\gamma \overline{q}^{1/2}\right). \end{aligned}$$

Just as before we can now define $i\theta := \left(\oint_\gamma a_1 dz - \overline{a_1} d\overline{z}\right)$. Together with the periods $Z_\gamma := \frac{\pi}{\oint_\gamma} q^{1/2}$ we obtain the action angle coordinates on the corresponding integrable system, where the Z_γ are coordinates on the Hitchin base and the θ are coordinates on the torus fibers. It follows from the theory of limiting configurations that these are in fact periodic, so they give appropriate coordinates on the torus fibers of \mathcal{M}'_∞ . With these the final form for the \mathcal{X}_γ^L -coordinates for the limiting configuration is obtained:

$$\mathcal{X}_\gamma^L = -\exp\left(\frac{R}{\zeta} \pi Z_\gamma + i\theta_\gamma + R\zeta \pi \overline{Z}_\gamma\right). \quad (9.4)$$

In light of the goal to describe the asymptotic behaviour of the true coordinate system we note here that this is exactly the term we obtained from the leading term for the true \mathcal{X} coordinates in Theorem 7.2.10. Thus the difference amounts to the addition of the remainder term r_q for the true coordinates. We will come back to this at the end.

Again these coordinate functions can be defined for each charge γ in the charge lattice Γ_q , although in the following we only use the generators of the gauge lattice Γ_q^g . We then obtain the holomorphic symplectic form on \mathcal{M}'_∞ as in the description of equation (9.2) above:

$$\varpi^{\text{sf}}(\zeta) := \frac{1}{8\pi^2 R} \sum_{i,j} \epsilon_{ij} \frac{d\mathcal{X}_{\gamma_i}^L}{\mathcal{X}_{\gamma_i}^L} \wedge \frac{d\mathcal{X}_{\gamma_j}^L}{\mathcal{X}_{\gamma_j}^L}.$$

Here the coefficients ϵ_{ij} arise as the inverse of the intersection pairing on Γ^g , i. e. we regard dZ and $d\theta$ as elements in $\Omega^1((\Gamma^g)^*)$ and use the corresponding dual pairing of the intersection pairing on $(\Gamma^g)^*$.

We then obtain the metric $g_{\text{twist}}^{\text{lim}}$ by simply plugging in the known expression for \mathcal{X}_γ^L . Here it is important to note the fact that

$$\sum_{ij} \epsilon_{ij} dZ_{\gamma_i} \wedge dZ_{\gamma_j} = 0. \quad (9.5)$$

This is clear if the dimension of the Hitchin Base is 1, which is the case for the four-punctured sphere. A proof for higher dimensions can be found in the script [Nei17] where it is referred to as "Griffiths transversality for spectral curves" and interpreted as an embedding of the Hitchin base as a complex Lagrangian submanifold of some complex symplectic vector space. The main idea of the proof is to use a "Riemann bilinear identity" to identify the intersection pairing in the homology with the integral over the wedge product of two holomorphic 1-forms on the spectral curve, which then has to vanish.

Using this cancellation and the fact that the θ coordinates are real-valued one obtains for $\zeta = -1$ the Kähler form

$$\omega_I = \operatorname{Re}(\varpi^{\text{sf}}|_{\zeta=-1}) = \sum_{i,j} \epsilon_{ij} \left(\frac{R}{4} dZ_{\gamma_i} \wedge d\bar{Z}_{\gamma_j} - \frac{1}{8\pi^2 R} d\theta_{\gamma_i} \wedge d\theta_{\gamma_j} \right).$$

Now this is in fact the Kähler form for the semiflat metric, so we obtain as stated above the equality $g_{\text{twist}}^{\lim} = g_{\text{sf}}$. This concludes the twistorial construction of the semiflat metric. It will appear again at the end of the last section, as it is our goal to compare g_{sf} and g_{twist} on (a subset of) \mathcal{M}' .

9.3 The twistorial hyperkähler metric

Finally we use the construction of section 9.1 on the space of true solutions to the R -rescaled Hitchin's equations \mathcal{M}_R . Up until now the construction of the \mathcal{X} -coordinates depended on the (almost) arbitrary angle parameter ϑ of the triangulation. But now we have to specialize the parameter so that everything is well defined and all the results from the sections before hold. This is done by setting $\vartheta := \arg(\zeta)$ and building the triangulation and coordinates accordingly. Obviously in this way $\zeta \in \mathbb{H}_{\vartheta}$ holds, which was the sufficient condition for the construction to work. But now we can't control ϑ to be generic anymore. Indeed there is now the real co-dimension-1 locus in the ζ -plane where the coordinates jump. Away from this locus the coordinates are fine, and it is one of the crucial parts of the main Theorem 9.3.5 in this section that these jumps do not pose a problem for us, so all the considerations of section 4.4 come into play.

We start by investigating the complex structures on \mathcal{M}_R . For more details on this we refer to the original work [Hit87], and also to [Swo21], [Maz+19] and [Fre+20]. In order to define them we have to consider the tangent space at $(A, \Phi) \in \mathcal{M}_R$. The space of unitary connections on E that induce the fixed connection on $\det E$ is an affine space modeled on $\Omega^1(\mathfrak{su}(E))$. With the identification $\Omega^1(\mathfrak{su}(E)) \cong \Omega^{0,1}(\operatorname{End}_0(E))$, $D \mapsto D^{0,1} := \alpha$ we may regard the deformations of the connection part of the Higgs pair (A, Φ) as an element of $\Omega^{0,1}(\operatorname{End}_0(E))$. We observe that this is just the complex conjugate of the space of traceless Higgs fields $\Omega^{1,0}(\operatorname{End}_0(E))$, so deformations of Higgs pairs can be regarded as pairs $(\alpha, \phi) \in \Omega^{0,1}(\operatorname{End}_0(E)) \times \Omega^{1,0}(\operatorname{End}_0(E))$. With this identification we can understand that the following maps are complex structures on \mathcal{M}_R :

$$I(\alpha, \phi) = (i\alpha, i\phi), \quad J(\alpha, \Phi) = (-R\phi^{*h}, \frac{1}{R}\alpha^{*h}), \quad K(\alpha, \phi) = (-iR\phi^{*h}, \frac{i}{R}\alpha^{*h}).$$

Note that the factor R appears in the definition of the complex structures J and K . This is an artifact of the the moduli space of weakly parabolic Higgs bundles, where the \mathbb{C}^\times -action

of the regular and strongly parabolic moduli spaces does not exist. In those cases the \mathbb{C}^\times action allows to regard solutions of the R -rescaled Hitchin equations as solutions of the original Hitchin equations (i. e. with $R = 1$). But in our case the fixed residues at the poles of $\det \Phi$ prevent us from using this identification. Thus each $R \in \mathbb{R}_{>0}$ corresponds to its own moduli space and the corresponding scaling has to be considered in the complex structures and hyperkähler metrics.

Remark 9.3.1

The description of the holomorphic structures above might seem odd, as they are not defined on the elements of the moduli space but on their representatives. This basically boils down to the same considerations we explained in remark 7.3.1. The fact that the holomorphic structure is preserved when taking the equivalence class is a direct consequence of the construction of the moduli space as a hyperkähler quotient. The construction actually starts with complex structures on the set of all the corresponding objects (i. e. before taking the quotient by gauge transformations) and then proceeds by building the structure on the moduli space in exactly such a way that they commute with taking representatives. Therefore it suffices to talk about the action of the complex structure on representatives and not on the elements of the moduli space as we just did above. Again we refer to [May16] for more details.

We may now consider again the (twisted) non-abelian Hodge correspondence that we used to identify the moduli space of solutions of the R -rescaled Hitchin equations \mathcal{M}_R and the moduli space of flat connections \mathcal{M}^{dR} in section 7.1. There the emphasis lay on the parameter $\zeta \in \mathbb{C}^\times$ while we now, in light of the discussion above, add R to the discussion. Concretely for every $R \in \mathbb{R}_{>0}$ and $\zeta \in \mathbb{C}^\times$ we obtain the map

$$\text{NAHC}_{R,\zeta} : \mathcal{M}_R \rightarrow \mathcal{M}^{\text{dR}}, \quad (A, \Phi) \mapsto d_A + \frac{R}{\zeta} \Phi + R\zeta \Phi^{*h} = \nabla(R, \zeta).$$

The space of flat connections is equipped with the trivial complex structure \hat{I} , that is given by multiplication with i . On the other hand the three complex structures on \mathcal{M}_R lead to the general complex structure $I^{R,\zeta}$, which we described in the beginning of this chapter and where we added R to emphasize the dependence we explained above. By a simple calculation one can now show that $\text{NAHC}_{R,\zeta}$ preserves the complex structure in the following way.

Lemma 9.3.2

Fix $R \in \mathbb{R}_{>0}$ and $\zeta \in \mathbb{C}^\times$. Then $\text{NAHC}_{R,\zeta} : (\mathcal{M}_R, I^{R,\zeta}) \rightarrow (\mathcal{M}^{\text{dR}}, \hat{I})$ is holomorphic.

In this formulation we exclude the complex structures I and $-I$ on \mathcal{M}_R which corresponds to the moduli space of Higgs bundles. They would be given by $\zeta = 0$ and $\zeta = \infty$, where $\text{NAHC}_{R,\zeta}$ has simple poles. Still, the construction below will give us a form ϖ that is also holomorphic in these complex structures as it will be twisted by the bundle $O(2)$.

We now recall the restrictions we have to take to make sure that the construction works.

Definition 9.3.3

Let \mathcal{M}_R denote the moduli space of solutions of weakly parabolic Higgs bundles as described in section 3 for Higgs bundles with parabolic degree 0. In this space \mathcal{M}'_R denotes

the regular locus, in which the quadratic differential q given by the determinant of the Higgs field has only simple zeroes and $\widetilde{\mathcal{M}}'_R$ the subset of \mathcal{M}'_R , in which the determinants q have unsealed spectrum. Finally denote by $\widetilde{\mathcal{M}}'_{R,c}$ the subset corresponding to the chambers in the Hitchin base \mathcal{B} .

Note that as explained in section 4.4 in the case of weakly parabolic Higgs bundles on the sphere with at least 4 punctures it holds $\widetilde{\mathcal{M}}'_R = \mathcal{M}'_R$ and the chambers form an open dense subset of $\widetilde{\mathcal{M}}'_R$. Thus there is a large class of moduli spaces where the following construction works without further deformations of ring domains or other adjustments.

Definition 9.3.4

For $\text{Det}\Phi = q$ let $\Gamma_q^g = H_1(\Sigma_q; \mathbb{Z})$ denote the gauge lattice with generators γ_i the symplectic basis of Proposition 4.3.4. For each generator γ_i let \mathcal{X}_{γ_i} be the corresponding true coordinate function on \mathcal{M}' constructed with triangulation angle $\vartheta := \arg(\zeta)$. With ϵ^{ij} denoting the coefficients of the symplectic form on Γ_q^g and ϵ_{ij} its inverse on $(\Gamma_q^g)^*$ define

$$\varpi := \frac{1}{8\pi^2 R} \sum_{i,j} \epsilon_{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}.$$

All of our calculations in section 7 and the arguments in [GMN10] and [GMN13b] now come together in the following theorem.

Theorem 9.3.5

Let $\mathcal{Z} := \widetilde{\mathcal{M}}'_{R,c} \times \mathbb{CP}^1$ be the twistor space for $\widetilde{\mathcal{M}}'_{R,c}$ with projection $p : \mathcal{Z} \rightarrow \mathbb{CP}^1$ and let $T_F = \ker dp$ denote the tangent bundle along the fibers of the twistor space. If R is large enough ϖ is a holomorphic section of $\Lambda^2 T_F^* \otimes O(2)$, defining a symplectic form on each fiber.

Proof. The results of Theorem 7.3.6 and Corollary 7.3.7 concerning the pole structure together with the Lagrangian property of equation (9.5) ensure that ϖ is a well-defined section of $\Lambda^2 T_F^* \otimes O(2)$ where the $O(2)$ twist matches the simple poles at $\zeta = 0$ and $\zeta = \infty$.

From [FG06] it is known that the \mathcal{X} -coordinates are holomorphic on \mathcal{M}^{dR} (in complex structure \hat{I}). As of Lemma 9.3.2 they become holomorphic coordinates on $\widetilde{\mathcal{M}}'_{R,c}$ in the complex structure $I^{R,\zeta}$, and thus the fiberwise part of ϖ is holomorphic in complex structure $I^{R,\zeta}$ for all $\zeta \in \mathbb{C}^\times$.

For the ζ -dependence we note that the entries of the flat connection $\nabla(R, \zeta)$ depend holomorphically on ζ . Theorem 6.3.5 then tells us that the small flat sections also depend holomorphically on ζ and the construction of the \mathcal{X} -functions proceeds by operations which maintain the holomorphicity. So they are ζ -holomorphic for any $\zeta \in \mathbb{C}^\times$ as long as the ϑ -triangulation with angle $\vartheta = \arg(\zeta)$ does not obtain values for which finite ϑ -trajectories appear. Thus it is piecewise holomorphic with jumps on a real co-dimension-1 domain. As of Proposition 9.1.3 the jumps in the coordinates do not lead to jumps of ϖ , so ϖ is still well defined and continuous on all of \mathbb{C}^\times . By an application of Morera's theorem it is thus holomorphic on all of \mathbb{C}^\times . Now ϖ has only simple poles for $\zeta \rightarrow 0$ and $\zeta \rightarrow \infty$ and, considered as a section of the twisted bundle $\Lambda^2 T_F^* \otimes O(2)$, it is holomorphic everywhere on \mathbb{CP}^1 by the Riemann removable singularity theorem. The continuity of the complex

structure $I^{R,\zeta}$ now implies that ϖ also defines a holomorphic form on the fiber over $\zeta = 0$ and $\zeta = \infty$.

Lastly the form on each fiber is symplectic as of the intersection pairing which extracts the Darboux coordinate system out of the set of the constructed canonical functions \mathcal{X}_γ , cf. Proposition 4.3.4. As the semiflat part leads to a nondegenerate form and the correction terms are exponentially suppressed in R it follows that ϖ is also non-degenerate for large enough R . \square

Thus the twistor theorem applies and we obtain a twistorial hyperkähler metric g_{twist} on $\widetilde{\mathcal{M}}'_c$. We use this point as a reminder that $\widetilde{\mathcal{M}}'_{R,c}$ is already known to be a hyperkähler space and thus already carries all of the structure needed in the twistor theorem. For our metric we simply forget the given existing holomorphic symplectic form and replace it with the ϖ from our construction while keeping the rest of the structure, i. e. the complex structures, the fibration of the twistor space \mathcal{Z} over \mathbb{CP}^1 , the family of holomorphic sections with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ and the compatible real structure τ . Thus we obtain the following theorem out of the twistor construction.

Theorem 9.3.6

For large enough R there exists a hyperkähler metric g_{twist} on $\widetilde{\mathcal{M}}'_{R,c}$ whose twistorial data matches the natural one except for the holomorphic symplectic form which is given by ϖ .

We can now use Lemma 9.1.4 to obtain the twistorial metric g_{twist} on $\widetilde{\mathcal{M}}'_{R,c}$ as

$$g_{\text{twist}}(v, w) = \omega_I(v, Iw) = \text{Re}(\varpi^{\zeta=-1}(v, Iw)).$$

The benefit of this construction becomes perfectly clear as we consider the asymptotics of this metric for large R . From the construction of the \mathcal{X}_{γ_i} it is clear that they can be expressed as the limiting coordinates $\mathcal{X}_{\gamma_i}^L$ modified by some term that decays exponentially in R . Thanks to the explicit formulation we can easily see that this decaying behavior descends through the construction of the metric, leading to the final result.

Theorem 9.3.7

For large enough R the coefficients of the difference $g_{\text{twist}} - g_{\text{sf}}$ on $\widetilde{\mathcal{M}}'_{R,c}$ are exponentially suppressed.

Proof. The necessary calculations have been carried out above. With them we may now write the holomorphic symplectic form as

$$\begin{aligned} \varpi(\zeta) &= \frac{1}{8\pi^2 R} \sum_{i,j} \epsilon_{ij} d \log \mathcal{X}_{\gamma_i}(\zeta) \wedge d \log \mathcal{X}_{\gamma_j}(\zeta) \\ &= \sum_{i,j} \epsilon_{ij} \frac{R}{8} dZ_{\gamma_i} \wedge d\bar{Z}_{\gamma_j} - \frac{1}{8\pi^2 R} d\theta_{\gamma_i} \wedge d\theta_{\gamma_j} \\ &\quad + \frac{i}{8\pi} \left(\frac{1}{\zeta} dZ_{\gamma_i} \wedge d\theta_{\gamma_j} + \zeta d\theta_{\gamma_i} \wedge d\bar{Z}_{\gamma_j} + \zeta d\bar{Z}_{\gamma_i} \wedge d\theta_{\gamma_j} + \frac{1}{\zeta} d\theta_{\gamma_i} \wedge dZ_{\gamma_j} \right) + \varpi_e(\zeta). \end{aligned}$$

Following the construction of the \mathcal{X} -coordinates it follows from Theorem 7.3.4 and Proposition 7.3.5 that the 2-form $\varpi_e(\zeta)$ is exponentially suppressed. The first part of this

expression corresponds to the semiflat part, while ϖ_e denotes the *instanton-corrections*. As explained above, we obtain the corresponding expression for the instanton-corrected metric

$$g_{\text{twist}} = g_{\text{sf}} + g_e,$$

where the coefficients of g_e are bounded just like the ones from $\varpi_e(\zeta)$, i. e. $\|g_e\| < Ce^{-\delta R}$ for some constants $C, \delta \in \mathbb{R}_{>0}$. \square

The assumptions of Proposition 8.3.3 now translate to the improved decay rate on a corresponding submanifold.

Corollary 9.3.8

Let \mathcal{N} be any submanifold of $\widetilde{\mathcal{M}}'_{R,c}$ given by holomorphic coordinates in complex structure I that leave the Higgs field or the connection of the Higgs pairs constant. Then the coefficients of $g_e|_{\mathcal{N}}$ are bounded by terms of the form $\sum_n C_n(K_0(2n\pi R|Z_\beta|) + K_1(2n\pi R|Z_\beta|))$ for some constants C_n .

Remark 9.3.9

We have now constructed a hyperkähler metric that decays exponentially towards the semiflat metric. This is exactly the behaviour that is expected from the natural L^2 metric on \mathcal{M} and it was accordingly conjectured in the work of GMN that the constructed twistorial metric is actually the L^2 -metric. At this point we are not able to rigorously prove this, but there is an outline of a possible proof. The author is grateful to Andrew Neitzke, who communicated the following ideas to him:

The moduli space of flat connections is equipped with a holomorphic version of the Atiyah-Bott-Goldman symplectic form. Using the twisted nonabelian Hodge correspondence one can compare this form with the holomorphic symplectic form on the moduli space of solutions to Hitchin's equations, which is guaranteed by the twistor theorem for the L^2 -metric, and these two forms should coincide. The Fock-Goncharov coordinates on the moduli space of flat connections on the other hand should be Darboux coordinates for the Atiyah-Bott-Goldman symplectic form. Combining these arguments it would follow that the coordinates we considered here are actually Darboux coordinates for the holomorphic symplectic form that corresponds to the L^2 -metric. As of the twistor theorem the holomorphic symplectic form determines the hyperkähler metric, thus inferring that the constructed metric is actually the L^2 -metric. Working out the details of this argument is subject of future research.

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Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift - abgesehen von der Beratung durch Prof. Dr. Hartmut Weiß - in Inhalt und Form eigenständig erstellt habe und nur die angegebenen Hilfsmittel verwendet habe. Die Arbeit hat weder zum Teil noch als Ganzes schon an einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Ergebnisse der Dissertation sind bereits in dem folgenden Vorabdruck auf dem Dokumentenserver arXiv erschienen:

Maximilian Holdt: Hyperkähler metrics on the moduli space of weakly parabolic Higgs bundles, <https://arxiv.org/abs/2106.16017>

Ich versichere, dass ich die Arbeit unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft angefertigt habe. Des Weiteren ist mir kein akademischer Grad entzogen worden.

A handwritten signature in black ink, appearing to read 'Maximilian Holdt', written in a cursive style.

Kiel, den 03.06.2022

