

Tilting modules of tilted algebras and related topics

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Zusammenfassung

Kipp-Theorie ist seit geraumer Zeit ein zentrales Thema in der Darstellungstheorie. Ist T_0 ein Kippmodul einer erblichen Algebra Λ , so nennt man $B := \text{End}_A(T_0)$ eine gekippte Algebra vom Typ Λ . In dieser Arbeit werden Fragestellungen bezüglich dieser Algebren behandelt. Insbesondere studieren wir unter Benutzung von Spuren allgemeine Verfahren, um aus Kippmoduln von Λ solche von B zu konstruieren.

Sei A eine endlichdimensionale Algebra über einem algebraisch abgeschlossenen Körper k . Das Studium der Äquivalenzklassen von A -Kippmoduln wurde von Riedtmann und Schofield in [47] begründet. In [32] definierten Happel und Unger später den Kippköcher $\overrightarrow{\mathcal{K}}_\Lambda$. Mithilfe unseres Konstruktionsverfahrens können wir den Kippköcher von B aus dem von Λ bestimmen, sofern B eine sogenannte BB-gekippte Algebra des Typs Λ ist.

In Verallgemeinerung der klassischen Kipp-Theorie wurde von Adachi, Iyama und Reiten in [1] die sogenannte τ -Kipp-Theorie studiert. Analog zum Kippköcher $\overrightarrow{\mathcal{K}}_A$ definierten die Autoren dabei in [1] die sogenannten τ -Trägerkippköcher $Q(s\tau\text{-tilt}A)$ von A . Wir betrachten in obigem Kontext das Problem der Konstruktion des τ -Trägerkippköchers $Q(s\tau\text{-tilt}B)$ aus $Q(s\tau\text{-tilt}\Lambda)$.

Abstract

Tilting theory has been a central research topic in representation theory for a long time. Let Λ be a hereditary algebra, T_0 be a tilting Λ -module, $B := \text{End}_\Lambda(T_0)$, then B is called a tilted algebra of type Λ . In this thesis, we will study some related topics of tilted algebras.

We will give some general ways to construct tilting B -modules from tilting Λ -modules. The main tools we will use in the constructions are the traces and rejects.

Let A be a finite dimensional k algebra over an algebraically closed field k , the study of the equivalence classes of tilting A -modules was initiated by Riedtman and Schofield in [47]. Later in [32], Happel and Unger defined the tilting quiver $\overrightarrow{\mathcal{K}}_A$ for a given algebra A . With the constructions of tilting modules we have obtained, we will show how to get the tilting quiver of B from that of Λ , when B is a BB -tilted algebra of type Λ .

As a generalization of the classic tilting theory, τ -tilting theory was introduced by Adachi, Iyama and Reiten in [1]. For a given algebra A , in analogy with the tilting quiver $\overrightarrow{\mathcal{K}}_A$, the authors in [1] defined a quiver which is called the support τ -tilting quiver $Q(s\tau\text{-tilt}A)$. We will consider the reconstruction of the support τ -tilting quiver $Q(s\tau\text{-tilt}B)$ from the support τ -tilting quiver $Q(s\tau\text{-tilt}\Lambda)$.

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Introduction

Background

Tilting theory has been a central research topic in representation theory for a long time. It was initiated in [7], [11] (or even earlier in [9], through the study of the reflection functors), then axiomatised in [10], [28] and generalized in [40].

One of the most important conclusions in tilting theory is the so-called "Tilting Theorem" by Brenner and Butler in [11] and then completed by Happel and Ringel in [28]. The authors proved: When A is an algebra, $B = \text{End}_A(T)$ with T_A being a classic tilting A -module, there are equivalences $\mathcal{T}(T) \cong \mathcal{Y}(T)$ and $\mathcal{F}(T) \cong \mathcal{X}(T)$, where $\mathcal{T}(T)$, $\mathcal{F}(T)$ and $\mathcal{X}(T)$, $\mathcal{Y}(T)$ are full subcategories of $\text{mod } A$ and $\text{mod } B$, respectively.

Many different points of views about tilting theory were developed during the past. The study of the equivalence classes of tilting modules, first taken up by Riedtmann and Schofield in [47] and then by Unger in [51]. There, the authors discussed the geometric and topological aspects about this topic. Several years later, this topic was re-studied by Happel and Unger in [32]. In [32], the authors proved: For an artin algebra A , the tilting quiver $\overrightarrow{\mathcal{K}}_A$ and the Hasse quiver defined on the partially ordered set (\mathcal{T}_A, \leq) coincide. In some related articles, the tilting quiver $\overrightarrow{\mathcal{K}}_A$ turned out to be interesting and useful, because it has some good properties and provides information about A . For example it was proved in [32]: If $\overrightarrow{\mathcal{K}}_A$ has a finite component \mathcal{C} , then $\overrightarrow{\mathcal{K}}_A = \mathcal{C}$. For more details, the reader is referred to [32], [31]. Moreover, it was proved in [33]: Under some assumptions on the algebra, one can even reconstruct the algebra from the partially ordered set (\mathcal{T}_A, \leq) .

τ -tilting theory was introduced in [1], which can be considered as a generalization of the classic tilting theory from the perspective of mutations (or completions). Due to its various connections with some other topics, such as functorial finite torsion classes, 2-term silting complexes, g -vectors, it has been a fruitful topic since then. For more, we refer to [19], [35]. As a generalization of the classic tilting theory, in [1], the authors also defined a support τ -tilting quiver $Q(s\text{-tilt } A)$ for an algebra A and discussed some basic properties about it.

Motivation

There are several motivations which lead to this thesis.

First, according to [22] (or [20], [42]), hereditary algebras are well known from the view of classifications of representation types. As we know hereditary algebras are just algebras of global dimension less than or equal to one, so it is quite natural for us to take algebras of global dimension less than or equal to two into consideration. Moreover, conclusions in [28] showed that tilted algebras are of global dimension no more than two, which throw tilted algebras into our eyesight.

Secondly, let A be a finite dimensional k -algebra, T be a classic tilting A -module, $B =$

$\text{End}_A(T)$. Then the Tilting Theorem provides two equivalences $\mathcal{T}(T) \cong \mathcal{Y}(T)$ and $\mathcal{F}(T) \cong \mathcal{X}(T)$, where $\mathcal{T}(T), \mathcal{F}(T)$ are subcategories of $\text{mod } A$ and $\mathcal{X}(T), \mathcal{Y}(T)$ are subcategories of $\text{mod } B$. The two equivalences make us wonder: Can we obtain some properties of B which are related to $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ from that of A which are related to $\mathcal{T}(T)$ and $\mathcal{F}(T)$? Of course, this has always been the central topic of the tilting theory and many good results have been discovered, for more details we refer to [10], [28]. In particular, it was proved in [28]: When A is a hereditary algebra, the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting in $\text{mod } B$. Hence in this case, $\text{mod } B = \mathcal{X}(T) \uplus \mathcal{Y}(T)$, i.e., each indecomposable B -module either belongs to $\mathcal{X}(T)$ or belongs to $\mathcal{Y}(T)$. So we wonder: If we are given some information of A related to $\mathcal{T}(T)$ and $\mathcal{F}(T)$, is it possible to obtain all information about B ?

Thirdly, we all know APR-tilting modules (or reflection functors) play an important role in the research of representation theory of hereditary algebras. So BB-tilting modules, which were introduced as a generalization of APR-tilting modules, are natural to be expected to be useful. But unfortunately, until now there is no that much research about it.

Main results

The constructions of tilting modules have been a central topic with emphasis on finding complements to almost complete tilting modules (see [29], [16]). In this thesis, we will give a way of constructing tilting modules from a given one. One of our main results reads as follows:

Theorem A (=Theorem 2.3.4). *Let $B = \text{End}_\Lambda(T_0)$ be a tilted algebra of type Λ . If $T = Y \oplus X$ is a tilting Λ -module with $Y \in \mathcal{T}(T_0)$, $X \in \mathcal{F}(T_0)$ such that no indecomposable summand of Y is generated by X , then $\text{Hom}_\Lambda(T_0, Y/\text{Tr}_Y X) \oplus \text{Ext}_\Lambda^1(T_0, X)$ is a tilting B -module.*

Here Λ is a hereditary algebra, $(\mathcal{T}(T_0), \mathcal{F}(T_0))$ is the torsion pair in $\text{mod } \Lambda$ associated to T_0 , and $\text{Tr}_Y X$ denotes the trace of X in Y .

Let A be a finite dimensional k algebra over an algebraically closed field k , \mathcal{T}_A be the set of equivalence classes of tilting A -modules (tilting modules T_1 and T_2 are supposed to be in the same class, provided $\text{add}(T_1) = \text{add}(T_2)$, so in general we will just consider basic tilting modules). Here $\text{add}(T) \subset \text{mod } A$ denotes the additive closure of T . The tilting quiver $\overrightarrow{\mathcal{K}}_A$ was introduced in [32], [47]: the vertices are elements of \mathcal{T}_A , and for two basic tilting A -modules, there exists an arrow from T_1 to T_2 if and only if $T_1 = M \oplus X$, $T_2 = M \oplus Y$ with X, Y being indecomposable summands belonging to a non-split exact sequence $0 \rightarrow X \rightarrow \overline{M} \rightarrow Y \rightarrow 0$ with $\overline{M} \in \text{add}M$. Given $T_1, T_2 \in \mathcal{T}_A$, a partial order \leq on \mathcal{T}_A is defined via: $T_1 \leq T_2$ if and only if $\text{T}_1^\perp \subset \text{T}_2^\perp$. Here T_i^\perp are full subcategories of $\text{mod } A$ (see below). It was proved in [32], that the Hasse quiver of the partially ordered set (\mathcal{T}_A, \leq) coincides with the tilting quiver $\overrightarrow{\mathcal{K}}_A$.

Let T_0 be a classic tilting A -module, $B = \text{End}_A(T_0)$. Then as we have mentioned, according to the Tilting Theorem (see Theorem 1.2.7 below), there exist equivalences $\mathcal{T}(T_0) \cong \mathcal{Y}(T_0)$, and $\mathcal{F}(T_0) \cong \mathcal{X}(T_0)$. Where $\mathcal{T}(T_0)$ and $\mathcal{F}(T_0)$, $\mathcal{X}(T_0)$ and $\mathcal{Y}(T_0)$ are full subcategories given by torsion pairs in $\text{mod } A$ and $\text{mod } B$, respectively. Let $\mathcal{T}_{\mathcal{T}} \subset \mathcal{T}_A$ be the subset of \mathcal{T}_A consisting of those equivalence classes whose representatives belong to $\mathcal{T}(T_0)$; $\mathcal{T}_{\mathcal{F}} \subset \mathcal{T}_A$ be the subset of \mathcal{T}_A consisting of those equivalence classes whose representatives belong to $\mathcal{F}(T_0)$; $\mathcal{T}_{\mathcal{T} \cup \mathcal{F}}$ be the subset of \mathcal{T}_A consisting of those equivalence classes whose representatives belong to $\mathcal{T}(T_0) \cup \mathcal{F}(T_0)$ and let $\mathcal{T}_{\mathcal{T}, \mathcal{F}} \subset \mathcal{T}_A$ be the subset of \mathcal{T}_A consisting of elements in $\mathcal{T}_{\mathcal{T} \cup \mathcal{F}}$ but not in the union of $\mathcal{T}_{\mathcal{T}}$ and $\mathcal{T}_{\mathcal{F}}$. Similarly, we can define $\overrightarrow{\mathcal{T}}_{\mathcal{Y}}, \overrightarrow{\mathcal{T}}_{\mathcal{X}}, \overrightarrow{\mathcal{T}}_{\mathcal{X} \cup \mathcal{Y}}, \overrightarrow{\mathcal{T}}_{\mathcal{X}, \mathcal{Y}} \subset \overrightarrow{\mathcal{T}}_B$. It is quite reasonable to consider the relationship between $\overrightarrow{\mathcal{T}}_{\mathcal{T}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{Y}}, \overrightarrow{\mathcal{T}}_{\mathcal{F}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{X}}, \overrightarrow{\mathcal{T}}_{\mathcal{T} \cup \mathcal{F}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{X} \cup \mathcal{Y}}, \overrightarrow{\mathcal{T}}_{\mathcal{T}, \mathcal{F}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{X}, \mathcal{Y}}$, and

then the relationship between $\overrightarrow{\mathcal{K}}_A$ and $\overrightarrow{\mathcal{K}}_B$. For a subset $\mathcal{D} \subset \mathcal{T}_A$, $\overrightarrow{\mathcal{D}}$ means the full subquiver of $\overrightarrow{\mathcal{K}}_A$ consisting of elements in \mathcal{D} .

Theorem A provides a map $\Phi : \mathcal{T}_{\mathcal{T} \cup \mathcal{F}} \rightarrow \mathcal{T}_B$ (see Proposition 3.3.1 below). Restricting this map to certain subquivers of $\overrightarrow{\mathcal{K}}_A$ yields some quiver isomorphisms. More precisely, we obtain the following result.

Theorem B (=Theorem 3.3.8). *When Λ is hereditary and T_0 is a BB-tilting Λ -module, there exists isomorphisms of quivers between $\overrightarrow{\mathcal{T}}_{\mathcal{T}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{Y}}$, $\overrightarrow{\mathcal{T}}_{\mathcal{T}, \mathcal{F}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{X}, \mathcal{Y}}$, respectively.*

With this conclusion and some further observations (see Propositions 3.3.9, 3.3.10, 3.3.13), we can construct $\overrightarrow{\mathcal{K}}_B$ from $\overrightarrow{\mathcal{K}}_A$.

As we have mentioned in the background, support τ -tilting modules were introduced as a generalization of the classic tilting modules. Moreover, the support τ -tilting quivers were defined in [1] in analogy with the tilting quivers. So as an extension, we will make the similar consideration in the setting of τ -tilting theory.

Let A be a finite dimensional algebra, $s\tau\text{-tilt}A$ denotes the isomorphism classes of basic support τ -tilting A -modules. The support τ -tilting quiver $Q(s\tau\text{-tilt}A)$ is defined to be: vertices are elements of $s\tau\text{-tilt}A$, there exists an arrow from M_1 to M_2 if and only if M_2 is a left mutation of M_1 (for the definition of mutations, see Definition 4.3.1). Let $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of elements whose representatives belong to $\mathcal{T}(T) \cup \mathcal{F}(T)$, $s\tau\text{-tilt}\Lambda_{\mathcal{X} \cup \mathcal{Y}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of elements whose representatives belong to $\mathcal{X}(T) \cup \mathcal{Y}(T)$. Similarly, let $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of elements whose representatives belong to $\mathcal{T}(T)$, $s\tau\text{-tilt}\Lambda_{\mathcal{F}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of elements whose representatives belong to $\mathcal{F}(T)$, and $s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of elements contained in $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ but not in $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ nor in $s\tau\text{-tilt}\Lambda_{\mathcal{F}}$. So $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ is the disjoint union of $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$, $s\tau\text{-tilt}\Lambda_{\mathcal{F}}$ and $s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$, i.e., $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}} = s\tau\text{-tilt}\Lambda_{\mathcal{T}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{F}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$. And finally we prove the following conclusion.

Theorem C (=Theorem 4.3.9). *When $B = \text{End}_{\Lambda} T_0$ is a BB-tilted algebra of type Λ , we can define a map $F : s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}} \rightarrow s\tau\text{-tilt}B$ such that the restrictions of F induce two quiver embedding $Q(s\tau\text{-tilt}\Lambda_{\mathcal{T}}) \hookrightarrow Q(s\tau\text{-tilt}B)$ and $Q(s\tau\text{-tilt}\Lambda_{\mathcal{F}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}) \hookrightarrow Q(s\tau\text{-tilt}B)$. Moreover, F reverses the directions of arrows between elements of $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ and elements of $s\tau\text{-tilt}\Lambda_{\mathcal{F}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$.*

This thesis is organized as follows: In the introduction, we will say something about the backgrounds, our motivations and some main results. In Chapter 1, we will recall some basic definitions and well-known results, and also state some lemmas for future reference. In Chapter 2, we will provide some ways to construct tilting modules for tilted algebras and prove Theorem A. In Chapter 3, we will prove Theorem B and then use it to construct $\overrightarrow{\mathcal{K}}_B$ from $\overrightarrow{\mathcal{K}}_A$. The proof of Theorem C is contained in Chapter 4 and then we will try to use it to construct $Q(s\tau\text{-tilt}B)$ from $Q(s\tau\text{-tilt}\Lambda)$. In Chapter 5, we will propose an interesting question about stable equivalence of tilted algebras and tell something that we know.

Notation

Throughout this thesis, we suppose k is an algebraically closed field, A is a finite dimensional k algebra and assume that A admits n non-isomorphic simple modules. All modules considered here are finitely generated right modules and we use $\text{mod } A$ to denote the category of finitely

generated right A -modules. For two homomorphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, their composition is written as gf . $\text{gl.dim}A$ denotes the global dimension of A . All subcategories are assumed to be full and closed under taking direct summands. Given an A -module M , $\mathbf{add}M$ means the subcategory of A -modules which are direct summands of direct sums of M , $\mathbf{fac}M$ means the images of $\mathbf{add}M$, $\mathbf{Gen}M$ means the full subcategory of modules that are generated by M , $\mathbf{Cogen}M$ means the full subcategory of modules that are cogenerated by M , M^\perp means the subcategory of A -modules $M^\perp = \{N \in \text{mod } A \mid \text{Ext}_A^i(M, N) = 0 \text{ for } i \geq 1\}$, $\text{rad}M$ means the radical of M , $\text{soc}M$ means the socle of M , $\text{pd}_A M$ means the projective dimension of M , $\text{id}_A M$ means the injective dimension of M . D is the dual functor $D := \text{Hom}_k(-, k) : \text{mod } A \rightarrow \text{mod } A^{op}$, $\text{Tr}M$ means the transpose of M , and $\tau M = D \text{Tr} M$ denotes the Auslander-Reiten translation, $|M|$ denotes the number of non-isomorphic indecomposable summands of M . Let $\{P(i) \mid i = 1, 2, \dots, n\}$ be a complete set of pairwise non-isomorphic indecomposable projective A -modules and $P[i] = \bigoplus_{j \neq i} P(j)$ (indecomposable injective modules $I(i)$ and simple modules $S(i)$ are denoted similarly).

1 Preliminaries

We recall some definitions and basic results which will be used occasionally in the rest of the thesis, most of them are well known.

1.1 Torsion pairs

Torsion pairs (or torsion theory), which is a useful tool when considering some special subcategories, will be used throughout this thesis. So we introduce it firstly, definitions and unmentioned proofs of this section follow [4, VI.1].

Definition 1.1.1. [4, VI. Definition 1.1] *A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod } A$ is called a **torsion pair** if the following conditions are satisfied:*

1. $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$.
2. $\text{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.
3. $\text{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ of $\text{mod } A$ is called a **splitting torsion pair** if each indecomposable A -module lies either in \mathcal{T} or in \mathcal{F} .

When given a torsion pair $(\mathcal{T}, \mathcal{F})$, \mathcal{T} is called the **torsion class** and objects in \mathcal{T} are called **torsion objects**; \mathcal{F} is called the **torsion-free class** and objects in \mathcal{F} are called **torsion-free objects**.

Proposition 1.1.2. [4, VI. Proposition 1.4]

1. *Let \mathcal{T} be a full subcategory of $\text{mod } A$. The following conditions are equivalent:*
 - a) \mathcal{T} is the torsion class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod } A$.
 - b) \mathcal{T} is closed under images, direct sums, and extensions.
2. *Let \mathcal{F} be a full subcategory of $\text{mod } A$. The following conditions are equivalent:*
 - a) \mathcal{F} is the torsion-free class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod } A$.
 - b) \mathcal{F} is closed under submodules, direct products, and extensions.

Proposition 1.1.3. [4, VI. Proposition 1.5] *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\text{mod } A$ and M be an A -module. There exists a short exact sequence*

$$0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$$

with $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. This sequence is unique in the sense that, if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact with $M' \in \mathcal{T}, M'' \in \mathcal{F}$, then the two sequences are isomorphic.

A short exact sequence as in the Proposition 1.1.3 will be called the **canonical sequence** for M .

1.2 Tilting theory

For convenience, we will give the definitions and some important conclusions of tilting theory that will be frequently used in the rest. For more details about tilting theory, the reader is referred to [11], [28] or [3]. The original proof of the results in this section can be found in [11] or [28], but we will follow [4, VI].

Definition 1.2.1. [10, Definition 1.3] *An A -module T is called **classic tilting** provided:*

1. $\text{pd}_A T \leq 1$;
2. $\text{Ext}_A^1(T, T) = 0$;
3. *There exists an exact sequence: $0 \rightarrow A_A \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ with $T_i \in \text{add}T$ for $i = 0, 1$.*

The next definition of tilting modules is a generalization of the classic one, it is easy to see that the classic case is a special case of the generalized one.

Definition 1.2.2. [40] *An A -module T is called **tilting** provided:*

1. *The projective dimension of T is finite;*
2. $\text{Ext}_A^i(T, T) = 0$ for all $i \geq 1$;
3. *There exists an exact sequence: $0 \rightarrow A_A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_r \rightarrow 0$ with $T_i \in \text{add}T$ for $0 \leq i \leq r$.*

In fact, there are also definitions of tilting modules for arbitrary rings, tilting modules of infinite projective dimensions and even tilting objects of abelian categories or some other categories. But since we will not consider them here, we will not give the definitions here. The interested reader is referred to [49], [50], [18], [17].

Modules satisfying the first and second conditions of Definition 1.2.1 are called **classic partial tilting**. And modules that are direct summands of tilting modules are called **partial tilting**. We will see later, due to Bongartz's Lemma, modules satisfying the first two conditions of Definition 1.2.1 are direct summand of some classic tilting modules, i.e., classic partial tilting modules are partial tilting. But for the general case, this is not true, i.e., modules satisfying the first two conditions of Definition 1.2.2 might not be direct summand of any tilting module. And we will say more about this later.

Given an A -module T , consider the full subcategories of $\text{mod-}A$:

$$\mathcal{T}(T) = \{M \in \text{mod-}A \mid \text{Ext}_A^1(T, M) = 0\}$$

and

$$\mathcal{F}(T) = \{N \in \text{mod-}A \mid \text{Hom}_A(T, N) = 0\}.$$

Let $B = \text{End}_A(T)$, we obtain two full subcategories of $\text{mod-}B$:

$$\mathcal{X}(T) = \{X \in \text{mod-}B \mid X \otimes_B T = 0\}$$

and

$$\mathcal{Y}(T) = \{Y \in \text{mod-}B \mid \text{Tor}_1^B(Y, T) = 0\}.$$

The following lemma shows that for each given classic partial tilting module, we can obtain two torsion pairs.

Lemma 1.2.3. [4, VI. Lemma 2.3] *Let T be a classic partial tilting module. Then*

1. $\text{Gen}T$ is a torsion class, and the corresponding torsion-free class is $\mathcal{F}(T) = \{M \mid \text{Hom}_A(T, M) = 0\}$;
2. $\mathcal{T}(T) = \{M \mid \text{Ext}_A^1(T, M) = 0\}$ is a torsion class, and the corresponding torsion-free class is $\text{Cogen}\tau T$.

Lemma 1.2.4. [4, VI. Theorem 2.5] *Let T_A be a classic partial tilting module. The following statements are equivalent:*

1. T_A is a classic tilting module;
2. $\text{Gen}T = \mathcal{T}(T)$;
3. $\text{Cogen}\tau T = \mathcal{F}(T)$;
4. For every module $M \in \mathcal{T}(T)$, there exists a short exact sequence $0 \rightarrow L \rightarrow T_0 \rightarrow M \rightarrow 0$ with $T_0 \in \text{add}T$ and $L \in \mathcal{T}(T)$;
5. For every module $N \in \mathcal{F}(T)$, there exists a short exact sequence $0 \rightarrow N \rightarrow T_1 \rightarrow K \rightarrow 0$ with $T_1 \in \text{add}\tau T$ and $K \in \mathcal{F}(T)$.

Remark. According to Lemma 1.2.4 and Lemma 1.2.3, we got to know: When T is classic tilting, $(\mathcal{T}(T), \mathcal{F}(T))$ are torsion pairs in $\text{mod-}A$. Since ${}_B T_A$ is also a classic tilting module as left B -module ([4, VI. Lemma 3.3]), the dual conclusions imply $(\mathcal{X}(T), \mathcal{Y}(T))$ is also a torsion pair in $\text{mod-}B$.

The following lemma, known as the Bongartz's Lemma [10, Lemma 2.1], asserts that a classic partial tilting module can be completed to be a classic tilting module.

Lemma 1.2.5. [4, VI. Lemma 2.4] *Let T_A be a classic partial tilting module. There exists an A -module E such that $T \oplus E$ is a classic tilting module.*

The following conclusion can be obtained by slightly modifying the proof of Lemma 1.2.5 given in [4]. For the reader's convenience, we will write down the proof.

Corollary 1.2.6. *Let X, Y be two A -modules such that $\dim_k \text{Ext}_A^1(X, Y) = r$. Then we can construct two exact sequences: $0 \rightarrow Y \rightarrow E \rightarrow X^r \rightarrow 0$ and $0 \rightarrow Y^r \rightarrow F \rightarrow X \rightarrow 0$ such that the connecting homomorphisms $\delta : \text{Hom}_A(X, X^r) \rightarrow \text{Ext}_A^1(X, Y)$, $\beta : \text{Hom}_A(Y^r, Y) \rightarrow \text{Ext}_A^1(X, Y)$ are surjective.*

Proof. Let e_1, e_2, \dots, e_r be a basis of the k -vector space $\text{Ext}_A^1(X, Y)$. Represent each e_i by a short exact sequence $0 \rightarrow Y \xrightarrow{f_i} E_i \xrightarrow{g_i} X \rightarrow 0$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y^r & \xrightarrow{f} & \bigoplus_{i=1}^r E_i & \xrightarrow{g} & X^r \longrightarrow 0 \\ & & \downarrow l & & \downarrow u & & \downarrow 1 \\ 0 & \longrightarrow & Y & \xrightarrow{v} & E & \xrightarrow{w} & X^r \longrightarrow 0 \end{array}$$

where $f = \begin{pmatrix} f_1 & 0 \\ & \ddots \\ 0 & f_r \end{pmatrix}$, $g = \begin{pmatrix} g_1 & 0 \\ & \ddots \\ 0 & g_r \end{pmatrix}$, $l = (1, \dots, 1)$ is the codiagonal homomorphism and we mark the lower exact sequence with (\star) . The lower exact sequence is obtained by taking the push-out of f and l , and we denote it by e when considered as an element of $\text{Ext}_A^1(X^r, Y)$. Let $u_i : X \rightarrow X^r$ be the inclusion homomorphism in the i th coordinate. We claim that $e_i = \text{Ext}_A^1(u_i, Y)e$ for each i with $1 \leq i \leq r$. Indeed, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & X & \longrightarrow & 0 \\ & & \downarrow u_i'' & & \downarrow u_i' & & \downarrow u_i & & \\ 0 & \longrightarrow & Y^r & \xrightarrow{f} & \bigoplus_{i=1}^r E_i & \xrightarrow{g} & X^r & \longrightarrow & 0 \\ & & \downarrow l & & \downarrow u & & \downarrow 1 & & \\ 0 & \longrightarrow & Y & \xrightarrow{v} & E & \xrightarrow{w} & X^r & \longrightarrow & 0 \end{array}$$

where u_i' , u_i'' denote the respective inclusion homomorphisms in the i th coordinate. Because $lu_i'' = 1_Y$, we deduce a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & X & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow uu_i' & & \downarrow u_i & & \\ 0 & \longrightarrow & Y & \xrightarrow{v} & E & \xrightarrow{w} & X^r & \longrightarrow & 0 \end{array}$$

hence our claim. Which implies the connecting homomorphism $\delta : \text{Hom}_A(X, X^r) \rightarrow \text{Ext}_A^1(X, Y)$ is surjective. Similarly, consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y^r & \xrightarrow{f} & \bigoplus_{i=1}^r E_i & \xrightarrow{g} & X^r & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow k & & \\ 0 & \longrightarrow & Y^r & \xrightarrow{m} & F & \xrightarrow{n} & X & \longrightarrow & 0 \end{array}$$

where $f = \begin{pmatrix} f_1 & 0 \\ & \ddots \\ 0 & f_r \end{pmatrix}$, $g = \begin{pmatrix} g_1 & 0 \\ & \ddots \\ 0 & g_r \end{pmatrix}$, k is the diagonal homomorphism and the lower exact sequence is obtained by taking the pull-back of g and k . Dual statements show the exact sequence $0 \longrightarrow Y^r \xrightarrow{m} F \xrightarrow{n} X \longrightarrow 0$ is the desired exact sequence. \square

For the rest parts of this section, we will state the well known Tilting Theorem and some of its consequences.

Theorem 1.2.7 (Tilting Theorem). [4, VI. Theorem 3.8] *Let A be an algebra, T_A be a classic tilting module, $B = \text{End}_A(T)$, and $(\mathcal{T}(T), \mathcal{F}(T))$, $(\mathcal{X}(T), \mathcal{Y}(T))$ be the induced torsion pairs in $\text{mod } A$ and $\text{mod } B$, respectively. Then T has the following properties:*

1. *The functors $\text{Hom}_A(T, -)$ and $- \otimes_B T$ induce a quasi-inverse equivalence between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$;*
2. *The functors $\text{Ext}_A^1(T, -)$ and $\text{Tor}_1^B(-, T)$ induce a quasi-inverse equivalence between $\mathcal{F}(T)$ and $\mathcal{X}(T)$.*

The following are some consequences of the Tilting Theorem.

Corollary 1.2.8. [4, VI. Theorem 3.9] *Let M be an arbitrary A -module. Then*

1. $\mathrm{Tor}_1^B(\mathrm{Hom}_A(T, M), T) = 0$;
2. $\mathrm{Ext}_A^1(T, M) \otimes_B T = 0$; and
3. *the canonical sequence of M in $(\mathcal{T}(T), \mathcal{F}(T))$ is*

$$0 \longrightarrow \mathrm{Hom}_A(T, M) \otimes_B T \longrightarrow M \longrightarrow \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M), T) \longrightarrow 0.$$

Let X be an arbitrary B -module. Then

1. $\mathrm{Hom}_A(T, \mathrm{Tor}_1^B(X, T)) = 0$;
2. $\mathrm{Ext}_A^1(T, X \otimes_B T) = 0$; and
3. *the canonical sequence of M in $(\mathcal{X}(T), \mathcal{Y}(T))$ is*

$$0 \longrightarrow \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(X, T)) \longrightarrow X \longrightarrow \mathrm{Hom}_A(T, X \otimes_B T) \longrightarrow 0.$$

Lemma 1.2.9. [4, VI. Lemma 4.1] *Let A be an algebra, T_A be a classic tilting module, and $B = \mathrm{End}_A(T)$. If $M \in \mathcal{T}(T)$, then $\mathrm{pd}_B \mathrm{Hom}_A(T, M) \leq \mathrm{pd}_A M$. If $N \in \mathcal{F}(T)$, then $\mathrm{id}_B \mathrm{Ext}_A^1(T, N) \leq \mathrm{id}_A N$.*

Theorem 1.2.10. [4, VI. Theorem 4.2] *Let A be an algebra, T_A be a classic tilting module, and $B = \mathrm{End}_A(T)$. Then $|\mathrm{gl.dim} A - \mathrm{gl.dim} B| \leq 1$.*

1.3 Traces and rejects

In Chapter 2, we will use traces and rejects to construct tilting modules, so we will give some basic properties about them in this section, for more details, the reader is referred to [2, 8].

Let \mathcal{U} be a class of A -modules, M be an A -module. The trace of \mathcal{U} in M and the reject of \mathcal{U} in M are defined by

$$\mathrm{Tr}_M(\mathcal{U}) = \sum \{ \mathrm{Im} h \mid h : U \rightarrow M \text{ for some } U \in \mathcal{U} \}$$

and

$$\mathrm{Rej}_M(\mathcal{U}) = \bigcap \{ \ker h \mid h : M \rightarrow U \text{ for some } U \in \mathcal{U} \},$$

respectively. In particular, when $\mathcal{U} = \{U\}$, we just write $\mathrm{Tr}_M U$ and $\mathrm{Rej}_M U$.

Proposition 1.3.1. [2, 8.12 and 8.13] *Let \mathcal{U} be a class of A -modules, M be an A -module. Then*

1. $\mathrm{Tr}_M(\mathcal{U})$ is generated by \mathcal{U} , and M is generated by \mathcal{U} if and only if $\mathrm{Tr}_M(\mathcal{U}) = M$;
2. $M/\mathrm{Rej}_M(\mathcal{U})$ is cogenerated by \mathcal{U} , and M is cogenerated by \mathcal{U} if and only if $\mathrm{Rej}_M(\mathcal{U}) = 0$;

Proposition 1.3.2. [2, 8.18] *If $(M_\alpha)_{\alpha \in I}$ is an indexed set of modules, then*

$$\mathrm{Tr}_{\bigoplus_{\alpha \in I} M_\alpha}(\mathcal{U}) = \bigoplus_{\alpha \in I} \mathrm{Tr}_{M_\alpha}(U)$$

and

$$\mathrm{Rej}_{\bigoplus_{\alpha \in I} M_\alpha}(\mathcal{U}) = \bigoplus_{\alpha \in I} \mathrm{Rej}_{M_\alpha}(U).$$

1.4 The Auslander-Reiten formulas and the almost split sequences

Given an A -module M , a pair (P, f) is called a **projective cover** of M if P is a projective module and $f : P \rightarrow M$ is an epimorphism such that any $g : P \rightarrow P$ satisfies $f = fg$ is an automorphism. A short exact sequence: $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_2} M \longrightarrow 0$ is called a **minimal projective presentation** of M , provided (P_0, p_2) and (P_1, p_1) are projective covers of M and $\ker p_2$, respectively. Take a minimal projective resolution of M : $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_2} M \longrightarrow 0$ and then apply the functor $\text{Hom}_A(-, A)$ to the resolution. We define $\text{Tr}(M) := \text{Coker Hom}_A(p_1, A)$, $\tau M := \text{Hom}_k(\text{Tr}(M), k)$, i.e., $\tau M = D \text{Tr } M$ and $\tau^{-1}M := \text{Tr}(\text{Hom}_k(M, k))$, i.e., $\tau^{-1}M = \text{Tr } DM$.

The following conclusion which is also well known as the Auslander-Reiten formulas.

Theorem 1.4.1. [4, IV. Theorem 2.13], [6] *Let A be a k -algebra and M, N be two A -modules in $\text{mod } A$. Then there exist isomorphisms*

$$\text{Ext}_A^1(M, N) \cong D \underline{\text{Hom}}_A(\tau^{-1}N, M) \cong D \overline{\text{Hom}}_A(N, \tau M)$$

that are functorial in both variables.

For two A -modules M, N , $\underline{\text{Hom}}_A(M, N) := \text{Hom}(M, N)/\mathcal{P}(M, N)$ and $\overline{\text{Hom}}_A(M, N) := \text{Hom}(M, N)/\mathcal{I}(M, N)$. Where $\mathcal{P}(M, N)$ and $\mathcal{I}(M, N)$ denote the morphisms $f : M \rightarrow N$ that factor through projective and injective modules, respectively.

Corollary 1.4.2. [4, IV. Corollary 2.14] *Let A be a k -algebra and M, N be two modules in $\text{mod } A$.*

1. *If $\text{pd}_A M \leq 1$ and N is arbitrary, then there exists a k -linear isomorphism*

$$\text{Ext}_A^1(M, N) \cong D \text{Hom}_A(N, \tau M).$$

2. *If $\text{id}_A N \leq 1$ and M is arbitrary, then there exists a k -linear isomorphism*

$$\text{Ext}_A^1(M, N) \cong D \text{Hom}_A(\tau^{-1}N, M).$$

Corollary 1.4.3. [4, IV. Corollary 2.15] *Let A be a k -algebra and M, N be two modules in $\text{mod } A$.*

1. *If $\text{pd}_A M \leq 1$ and $\text{id}_A N \leq 1$, then there exists a k -linear isomorphism*

$$\text{Hom}_A(N, \tau M) \cong \text{Hom}_A(\tau^{-1}N, M).$$

2. *If $\text{pd}_A M \leq 1$, $\text{id}_A \tau N \leq 1$ and N is indecomposable nonprojective, then there exists a k -linear isomorphism*

$$\text{Hom}_A(\tau N, \tau M) \cong \text{Hom}_A(N, M).$$

3. *If $\text{pd}_A \tau^{-1}M \leq 1$, $\text{id}_A N \leq 1$ and M is indecomposable noninjective, then there exists a k -linear isomorphism*

$$\text{Hom}_A(\tau^{-1}N, \tau^{-1}M) \cong \text{Hom}_A(N, M).$$

Recall that a homomorphism $f : B \rightarrow C$ is called a **split epimorphism**, if $id_C : C \rightarrow C$, the identity morphism of C , factors through f .

Definition 1.4.4. [8, pp137] A morphism $f : B \rightarrow C$ is called **minimal right almost split** if

1. it is not a split epimorphism;
2. any morphism $X \rightarrow C$ which is not a split epimorphism factors through f ;
3. any $g : B \rightarrow B$ such that $f = fg$ is an automorphism.

Definition 1.4.5. [8, V. Proposition 1.14] An exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called an **almost split sequence**, if g is minimal right almost split.

The following theorem states the existence of almost split sequences.

Theorem 1.4.6. [8, Theorem1.15]

1. If C is an indecomposable nonprojective module, then there is an almost split sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.
2. If A is an indecomposable noninjective module, then there is an almost split sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

1.5 BB-tilting modules

As we have mentioned in the introduction, APR-tilting modules were historically first studied and play an important role in the research of hereditary algebras. And BB-tilting modules was introduced as a generalization of APR-tilting modules.

A tilting module T_A is called **BB-tilting**, provided T_A is a classic tilting module of the form $T = P[i] \oplus \tau^{-1}S$ for some simple but non-injective S , and is called **APR-tilting**, if moreover, S is projective. In that case, we say $B = \text{End}_A(T)$ is a **BB-tilted algebra** (respectively, **APR-tilted**).

Proposition 1.5.1. [7, Proposition 1.13] Let A be a basic hereditary artin algebra, $T = P[i] \oplus \tau^{-1}S$ be an APR-tilting module. Then $B = \text{End}_A(T)$ is hereditary and $\text{Ext}_A^1(T, S)$ is a simple injective B -module.

Part of the statements in the following Lemma are probably well known, but since we could not find a suitable reference, we give a proof here.

Lemma 1.5.2. Suppose A is a basic finite-dimensional algebra, $T = P[i] \oplus \tau^{-1}S$ is a BB-tilting A -module, $B = \text{End}_A(T)$. Then the following statements hold:

1. S is the unique indecomposable module in $\mathcal{F}(T)$, $\text{Ext}_A^1(T, S)$ is the unique indecomposable module in $\mathcal{X}(T)$, i.e., $\mathcal{F}(T) = \text{add}S$, $\mathcal{X}(T) = \text{add} \text{Ext}_A^1(T, S)$.
2. Assume A is hereditary, then B is hereditary if and only if T is APR-tilting. Moreover, $\text{gl.dim}B = 2$ if and only if T is not APR-tilting. And in this case, $S' = \text{Ext}_A^1(T, S)$ is the unique indecomposable B -module of projective dimension 2.

Proof. 1. For any $X \in \mathcal{F}(T)$, we have $\text{Hom}_A(P[i] \oplus \tau^{-1}S, X) = 0$. Hence $\text{Hom}_A(P[i], X) = 0$ and S is the unique indecomposable A -module in $\mathcal{F}(T)$. Then according to the Tilting Theorem 1.2.7, $\text{Ext}_A^1(T, S)$ is the unique indecomposable module in $\mathcal{X}(T)$.

2. If T is APR-tilting, then according to Proposition 1.5.1 we know B is hereditary. If T is not APR-tilting, we claim $\text{pd}_B \text{Ext}_A^1(T, S) = 2$. According to Theorem 1.2.10, $\text{gl.dim} B \leq 2$. By general theory, all projective B -modules are of the form $\text{Hom}_A(T, T')$ with $T' \in \mathbf{add}(T)$, and are obviously contained in $\mathcal{Y}(T)$. Hence $\text{Ext}_A^1(T, S) \in \mathcal{X}(T)$ is not projective. If $\text{pd}_B \text{Ext}_A^1(T, S) = 1$, we consider its minimal projective resolution:

$$0 \rightarrow \text{Hom}_A(T, T') \rightarrow \text{Hom}_A(T, T'') \rightarrow \text{Ext}_A^1(T, S) \rightarrow 0(\star)$$

with $T', T'' \in \mathbf{add}T$. Applying the functor $-\otimes T$ to it, Corollary 1.2.8 yields a short exact sequence of A -modules

$$0 \rightarrow S \rightarrow T' \rightarrow T'' \rightarrow 0.$$

According to Lemma 2.1.3, $\text{Hom}_B(\text{Hom}_A(T, P[i]), \text{Ext}_A^1(T, S)) \cong \text{Ext}_A^1(P[i], S) = 0$. Since $B = \text{Hom}_A(T, P[i] \oplus \tau^{-1}S)$, and the Tilting Theorem implies $\text{Ext}_A^1(T, S)$ is indecomposable, then we know $\text{Ext}_A^1(T, S)$ is simple. The minimal projective resolution (\star) implies $T'' = \tau^{-1}S$, moreover, the minimal projective resolution shows $\tau^{-1}S$ cannot be a direct summand of T' . Then $T' \in \mathbf{add}P[i]$, and hence is projective. Since A is hereditary and S is a submodule of T' , [4, VII. Theorem 1.4] implies S is projective. Contradicts with our assumption that T is not APR-tilting, so we have $\text{pd}_B(\text{Ext}_A^1(T, S)) = 2$ and $\text{gl.dim} B = 2$. According to Lemma 1.2.9, $\text{pd}_B Y \leq 1$ for all $Y \in \mathcal{Y}(T)$. By [28, Theorem 6.3], $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting. Thus, we know $S' = \text{Ext}_A^1(T, S)$ is the unique indecomposable B -module of projective dimension 2. \square

Lemma 1.5.3. *Suppose A is basic hereditary, $T = P[i] \oplus \tau^{-1}S$ is a BB-tilting A -module. Let $S' =: \text{Ext}_B^1(T, S)$, then $S' \cong \text{Ext}_A^1(T, P(S))$, $\tau_B S' \cong \text{Hom}_A(T, I(S))$ where $P(S)$ (respective, $I(S)$) is the projective cover (respective, injective envelope of S).*

Proof. We have shown in the proof of 1.5.2 that $S' =: \text{Ext}_A^1(T, S)$ is simple. Obviously, $S' \in \mathcal{X}(T)$ and hence can not be projective. Consider the almost split sequence terminating with S'

$$0 \rightarrow \tau_B S' \rightarrow E \rightarrow S' \rightarrow 0.$$

According to Lemma 1.5.2, we know $\tau_B S' \notin \mathcal{X}(T)$. Since A is assumed to be hereditary, [28, Theorem 6.3] implies $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting. Then [4, VI. Proposition 5.2] indicates that the preceding almost split sequence is of the form:

$$0 \rightarrow \text{Hom}_A(T, I) \rightarrow \text{Hom}_A(T, I/\text{soc}I) \oplus \text{Ext}_A^1(T, \text{rad}P) \rightarrow \text{Ext}_A^1(T, P) \rightarrow 0.$$

Where P is an indecomposable projective module not lying in $\mathbf{add}T$ and I is the indecomposable injective module such that $P/\text{rad}P \cong \text{soc}I$. Since $T = P[i] \oplus \tau^{-1}S$, $P(S)$ is the unique indecomposable A -module which does not belong to $\mathbf{add}T$. Thus, $S' \cong \text{Ext}_A^1(T, P(S))$, $\tau_B S' \cong \text{Hom}_A(T, I(S))$. \square

2 Constructions of tilting modules of tilted algebras

Obviously, for any given algebra A , the regular module A_A is a tilting module. There are also some other well known tilting modules, like APR-tilting modules, BB-tilting modules, which we have introduced before. But even till now, there is no general way to construct tilting modules. In most cases, we construct tilting modules from a given one, by replacing one of its indecomposable summands ([16], [29]), which is also called mutations. As has been mentioned, in this chapter, we will introduce some ways to construct new tilting modules from some given one. First, we will give some lemmas that will be frequently referred to in the sequel.

2.1 Consequences of the Tilting Theorem

Lemma 2.1.1. [4, VI. Lemma 3.2] *Let T_A be a classic tilting module, $B = \text{End}_A(T)$, $M, N \in \mathcal{T}(T)$. Then we have functorial isomorphisms:*

1. $\text{Hom}_A(M, N) \cong \text{Hom}_B(\text{Hom}_A(T, M), \text{Hom}_A(T, N));$
2. $\text{Ext}_A^1(M, N) \cong \text{Ext}_B^1(\text{Hom}_A(T, M), \text{Hom}_A(T, N)).$

The following lemma is a special case of a dual version of Lemma 2.1.1. For the reader's convenience, we write down the proof.

Lemma 2.1.2. *Let Λ be hereditary, T_Λ be tilting, $B = \text{End}_\Lambda(T)$, $M, N \in \mathcal{F}(T)$. Then we have functorial isomorphisms:*

1. $\text{Hom}_\Lambda(M, N) \cong \text{Hom}_B(\text{Ext}_\Lambda^1(T, M), \text{Ext}_\Lambda^1(T, N));$
2. $\text{Ext}_\Lambda^1(M, N) \cong \text{Ext}_B^1(\text{Ext}_\Lambda^1(T, M), \text{Ext}_\Lambda^1(T, N)).$

Proof. The statement (1) is a direct consequence of the Tilting Theorem 1.2.7, so it suffices to show (2). Since $N \in \mathcal{F}(T)$, Lemma 1.2.4 (5) provides an exact sequence

$$0 \rightarrow N \rightarrow T' \rightarrow N' \rightarrow 0(\dagger)$$

such that $T' \in \text{add} \tau T$, $N' \in \mathcal{F}(T)$. Since $N' \in \mathcal{F}(T)$ and $\text{pd}_\Lambda T \leq 1$, applying the functor $\text{Hom}_\Lambda(T, -)$ to (\dagger) , we obtain an exact sequence of B -modules

$$0 \rightarrow \text{Ext}_\Lambda^1(T, N) \rightarrow \text{Ext}_\Lambda^1(T, T') \rightarrow \text{Ext}_\Lambda^1(T, N') \rightarrow 0.(\dagger\dagger)$$

Since $\text{Ext}_\Lambda^1(T, T')$ is injective as B -module ([4, VI Proposition 5.8]), applying the functor $\text{Hom}_B(\text{Ext}_\Lambda^1(T, M), -)$ to $(\dagger\dagger)$ yields an exact sequence

$$0 \rightarrow \text{Hom}_B(\text{Ext}_\Lambda^1(T, M), \text{Ext}_\Lambda^1(T, N)) \rightarrow \text{Hom}_B(\text{Ext}_\Lambda^1(T, M), \text{Ext}_\Lambda^1(T, T'))$$

$$\rightarrow \text{Hom}_B(\text{Ext}_\Lambda^1(T, M), \text{Ext}_\Lambda^1(T, N')) \rightarrow \text{Ext}_B^1(\text{Ext}_\Lambda^1(T, M), \text{Ext}_\Lambda^1(T, N)) \rightarrow 0.$$

Corollary 1.4.2 implies $\text{Ext}_\Lambda^1(M, T') \cong D \text{Hom}_\Lambda(\tau^{-1}T', M) = 0$. Applying the functor $\text{Hom}_\Lambda(M, -)$ to (†) we thus obtain

$$0 \rightarrow \text{Hom}_\Lambda(M, N) \rightarrow \text{Hom}_\Lambda(M, T') \rightarrow \text{Hom}_\Lambda(M, N') \rightarrow \text{Ext}_\Lambda^1(M, N) \rightarrow 0.$$

Comparing it with the preceding exact sequence, the result follows the functorial isomorphism of (a). \square

Lemma 2.1.3. [4, VI Lemma 5.5] *Let T_A be a classic tilting module, $B = \text{End}_A(T)$. If $M \in \mathcal{T}(T)$ and $N \in \mathcal{F}(T)$, then, for $j \geq 1$, there is an isomorphism*

$$\text{Ext}_A^j(M, N) \cong \text{Ext}_B^{j-1}(\text{Hom}_A(T, M), \text{Ext}_A^1(T, N)).$$

The following lemma is dual to Lemma 2.1.3, we write down the proof for convenience.

Lemma 2.1.4. *Let T_A be a classic tilting module, $B = \text{End}_A(T)$. If $M \in \mathcal{T}(T)$ and $N \in \mathcal{F}(T)$, then, for $j \geq 0$, there is an isomorphism*

$$\text{Ext}_B^{j+1}(\text{Ext}_A^1(T, N), \text{Hom}_A(T, M)) \cong \text{Ext}_A^j(N, M).$$

Proof. Consider the short exact sequence of B -modules

$$0 \rightarrow K' \rightarrow P' \rightarrow \text{Ext}_A^1(T, N) \rightarrow 0(*),$$

such that $P' \rightarrow \text{Ext}_A^1(T, N)$ is the projective cover of $\text{Ext}_A^1(T, N)$. Since projective B -modules are torsion free and torsion free modules are closed under submodules, P' and K' belong to $\mathcal{Y}(T)$. According to the Tilting Theorem 1.2.7, there exists $M' \in \mathcal{T}(T)$ such that $K' \cong \text{Hom}_A(T, M')$, and since P' is projective $P' \cong \text{Hom}_A(T, \bar{T})$ for some $\bar{T} \in \text{add}T$. Hence the exact sequence(*) can be rewritten as

$$0 \rightarrow \text{Hom}_A(T, M') \rightarrow \text{Hom}_A(T, \bar{T}) \rightarrow \text{Ext}_A^1(T, N) \rightarrow 0(**).$$

Applying the functor $- \otimes T$ to (**), Corollary 1.2.8 yields an exact sequence of A -modules

$$0 \rightarrow \text{Tor}_1^B(\text{Ext}_A^1(T, N), T) \rightarrow \text{Hom}_A(T, M') \otimes_B T \rightarrow \text{Hom}_A(T, \bar{T}) \otimes_B T \rightarrow 0.$$

According to the Tilting Theorem 1.2.7, the preceding exact sequence is isomorphic to

$$0 \rightarrow N \rightarrow M' \rightarrow \bar{T} \rightarrow 0(***)$$

Since $\text{Ext}_A^1(T, N) \in \mathcal{X}(T)$ and $\text{Hom}_A(T, M) \in \mathcal{Y}(T)$, $\text{Hom}_B(\text{Ext}_A^1(T, N), \text{Hom}_A(T, M)) = 0$. Then since $\text{Hom}_A(T, \bar{T})$ is a projective B -module, applying the functor $\text{Hom}_B(-, \text{Hom}_A(T, M))$ to (**), we obtain an exact sequence

$$0 \rightarrow \text{Hom}_B(\text{Hom}(T, \bar{T}), \text{Hom}_A(T, M)) \rightarrow \text{Hom}_B(\text{Hom}_A(T, M'), \text{Hom}_A(T, M)) \rightarrow \text{Ext}_B^1(\text{Ext}_A^1(T, N), \text{Hom}_A(T, M)) \rightarrow 0(****)$$

and isomorphisms $\text{Ext}_B^i(\text{Hom}_A(T, M'), \text{Hom}_A(T, M)) \cong \text{Ext}_B^{i+1}(\text{Ext}_A^1(T, N), \text{Hom}_A(T, M))(\ddagger)$ for $i \geq 1$. Applying the functor $\text{Hom}_A(-, M)$ to (****), we obtain a short exact sequence

$$0 \rightarrow \text{Hom}_A(\bar{T}, M) \rightarrow \text{Hom}_A(M', M) \rightarrow \text{Hom}_A(N, M) \rightarrow 0(*****)$$

and isomorphisms $\text{Ext}_A^i(M', M) \cong \text{Ext}_A^i(N, M)(\ddagger\ddagger)$ for $i \geq 1$. Lemma 2.1.1, applied to $\bar{T}, M, M' \in \mathcal{T}(T)$, shows that the first two terms of the sequences (****) and (*****) are isomorphic. As the isomorphisms are compatible with the sequences, we obtain $\text{Ext}_B^1(\text{Ext}_A^1(T, N), \text{Hom}_A(T, M)) \cong \text{Hom}_A(N, M)$ and also $\text{Ext}_B^{i+1}(\text{Ext}_A^1(T, N), \text{Hom}_A(T, M)) \stackrel{(\ddagger)}{\cong} \text{Ext}_B^i(\text{Hom}_A(T, M'), \text{Hom}_A(T, M)) \stackrel{(\ddagger\ddagger)}{\cong} \text{Ext}_A^i(M', M) \stackrel{(\ddagger\ddagger)}{\cong} \text{Ext}_A^i(N, M)$ for $i \geq 1$. \square

2.2 Perfect exceptional modules and tilting modules

Recall that an A -module M is called **self-orthogonal**, if M satisfies the condition (2) of Definition 1.2.2; it is called **exceptional**, if M satisfies the conditions (1) and (2) of Definition 1.2.2; it is called **perfect exceptional**, if, in addition, M has n non-isomorphic indecomposable summands. In this section, we will give a characterization of tilting modules, that reduces the construction of tilting modules to that of perfect exceptional modules.

Lemma 2.2.1. [25, III. Corollary 6.2] *Let $M = \bigoplus_{i=1}^r M_i^{n_i}$, with M_i be indecomposable and $M_i \not\cong M_j$ for $i \neq j$, be a classic partial tilting module. Then the following two conditions are equivalent*

1. M is classic tilting.
2. r equals the number of isomorphism classes of simple A -modules.

Note that classic partial tilting modules are just modules satisfying the first two conditions of Definition 1.2.1. Hence according to Lemma 2.2.1, for the classic case (tilting modules with projective dimension no more than 1), being tilting is the same as being perfect exceptional. As a generalization of the classic tilting modules, it is natural to ask whether this holds in general (tilting modules of finite projective dimension)? Unfortunately, this is still an open problem. We will see in Proposition 2.2.7, for some special types of algebras, the answer is affirmative. In fact, it is mentioned in the introduction of [46] that this follows from the proof of the main result of [27]. As we can see in [27], the authors used some tools of derived categories and derived functors. By contrast, our proof just necessitates some basic knowledge about representation theory and is somewhat shorter and easier (cf. [27]).

Remark. Lemma 2.2.1 implies that for a given algebra A , the number of non-isomorphic indecomposable summands of classic tilting A -modules is constant.

For an algebra A , $D^b(A)$ denotes the bounded derived category of A , $D^b(A)$ and $D^b(B)$ are called triangle equivalent when they are equivalent as triangulated categories. For more details about derived categories and triangulated categories we refer to [25], [52].

Theorem 2.2.2. [24, Theorem 1.6] *Let M be an A -module such that $\text{Ext}_A^i(M, M) = 0$ for $i > 0$ and suppose there exists an exact sequence $0 \rightarrow A \rightarrow M^0 \rightarrow M^1 \cdots \rightarrow M^s \rightarrow 0$ with $M^i \in \text{add}M$ for $0 \leq i \leq s$. Let $B = \text{End}_A(M)$ and suppose that the global dimension of B is finite. Then there exists a triangle equivalence $F : D^b(A) \rightarrow D^b(B)$.*

In fact, it was proved in [15], the condition in Theorem 2.2.2 that the global dimension of B being finite, can be dropped.

Lemma 2.2.3. [25, III. Proposition 1.5] *Let A, B be basic finite-dimensional k -algebras and assume that A has finite global dimension. If $F : D^b(A) \rightarrow D^b(B)$ is a triangle-equivalence, A and B have the same number of simple modules up to isomorphism.*

Remark 2.2.4. *Let A be a finite dimensional algebra of finite global dimension, T_A be a tilting A -module and $B = \text{End}_A(T)$. It was proved in [25], B has finite global dimension. Then according to Theorem 2.2.2 and Lemma 2.2.3, we know all tilting A -modules and B -modules have the same number of non-isomorphic indecomposable summands. Moreover, then we know if T is a tilting module and X is exceptional such that $\text{Ext}_A^i(T, X) = \text{Ext}_A^i(X, T) = 0$ for $i \geq 1$, then $T \oplus X$ is tilting and $\text{add}T = \text{add}(T \oplus X)$.*

Let us introduce something about the perpendicular categories firstly. Recall that an A -module M is called **basic**, if any indecomposable summands of M has multiplicity one. Let $Z = \bigoplus_{i=1}^s Z_i \in \text{mod } A$ be a basic classic partial tilting module with s pairwise non-isomorphic indecomposable summands, the **right perpendicular subcategory** \mathbf{Z}^{perp} is defined to be $\mathbf{Z}^{\text{perp}} = \{M \in \text{mod } A \mid \text{Hom}_A(Z, M) = \text{Ext}_A^1(Z, M) = 0\}$. Then according to [23] or [33, Theorem 2.1], we have the following theorem.

Theorem 2.2.5. *The perpendicular category \mathbf{Z}^{perp} contains a projective generator Q , such that $\mathbf{Z}^{\text{perp}} \cong \text{mod } \Gamma$, where $\Gamma = \text{End}_A(Q)$ is a hereditary finite dimensional k -algebra, and the number of non-isomorphic simple Γ -modules equals $n - s$. Moreover, $\text{Ext}_A^i(M, N) \cong \text{Ext}_\Gamma^i(M, N)$ for all $M, N \in \mathbf{Z}^{\text{perp}}$ and $i \geq 0$.*

The following lemma states the relationship between exceptional modules and tilting modules and is often used to prove an exceptional module is tilting.

Lemma 2.2.6. [26, Proposition 3.6] *The following statements are equivalent for an exceptional A -module N :*

1. N is tilting;
2. $\mathbf{N}^\perp \subset \text{fac}N$.

Proposition 2.2.7. *Let Λ be hereditary, T_0 be a tilting Λ -module, $B = \text{End}_\Lambda(T_0)$. Then if T is a perfect exceptional B -module, T is tilting.*

Proof. According to [28, Theorem 6.3], the torsion pair $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in $\text{mod } B$. So without loss of generality, we assume $T = Y' \oplus X'$ such that $Y' \in \mathcal{Y}(T_0)$, $X' \in \mathcal{X}(T_0)$. Then the Tilting Theorem 1.2.7 provides $Y \in \mathcal{T}(T_0)$ and $X \in \mathcal{F}(T_0)$ such that $Y' \cong \text{Hom}_\Lambda(T_0, Y)$, $X' \cong \text{Ext}_\Lambda^1(T_0, X)$. In view of Theorem 2.2.5, there is an algebra Γ such that $\text{mod } \Gamma = \mathbf{X}^{\text{perp}}$, we claim that Y is a tilting Γ -module. By Lemma 2.1.4, $\text{Ext}_\Lambda^i(X, Y) \cong \text{Ext}_B^{i+1}(\text{Ext}_\Lambda^1(T_0, X), \text{Hom}_\Lambda(T_0, Y))$ for $i = 0, 1$. Then since T is self-orthogonal, $\text{Ext}_\Lambda^i(X, Y) = 0$ (\mathbb{D}) for $i = 0, 1$, i.e., Y in \mathbf{X}^{perp} . Then Theorem 2.2.5 implies Y is perfect exceptional Γ -module and hence a tilting Γ -module (Lemma 2.2.1).

According to Lemma 2.2.6, to show T is tilting, it suffices to verify $Q \in \text{fac}T$ for every indecomposable B -module Q which belongs to \mathbf{T}^\perp . Without loss of generality, let $Q \in \mathbf{T}^\perp$ be an indecomposable B -module. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in $\text{mod } B$, Q is either in $\mathcal{X}(T_0)$ or in $\mathcal{Y}(T_0)$.

First, we assume $Q \in \mathcal{Y}(T_0)$. According to the Tilting Theorem 1.2.7, there exists $L \in \mathcal{T}(T_0)$ such that $Q \cong \text{Hom}_\Lambda(T_0, L)$. By Lemma 2.1.1, $\text{Ext}_\Lambda^1(Y, L) \cong \text{Ext}_B^1(\text{Hom}_\Lambda(T_0, Y), \text{Hom}_\Lambda(T_0, L))$. As $Q \in \mathbf{T}^\perp$, we thus have $\text{Ext}_\Lambda^1(Y, L) = 0$. Again, $Q \in \mathbf{T}^\perp$ implies $\text{Ext}_B^i(\text{Ext}_\Lambda^1(T_0, X), \text{Hom}_\Lambda(T_0, L)) = 0$ for $i = 1, 2$. Then Lemma 2.1.4 shows that $L \in \mathbf{X}^{\text{perp}} = \text{mod } \Gamma$. Since Y is a tilting Γ -module, Theorem 2.2.5 implies $\text{Ext}_\Gamma^1(Y, L) \cong \text{Ext}_\Lambda^1(Y, L) = 0$. By the same token, Lemma 1.2.4 provides an exact sequence:

$$0 \rightarrow Z \rightarrow \bar{Y} \rightarrow L \rightarrow 0(*)$$

in $\text{mod } \Gamma$ and also in $\text{mod } \Lambda$ such that $\bar{Y} \in \text{add}Y$ and $Z \in \mathcal{T}(Y) = \text{Gen}Y$ (in $\text{mod } \Gamma$ and hence in $\text{mod } \Lambda$). Since $Y \in \mathcal{T}(T_0)$ and the torsion class is closed under taking factor modules, $Z \in \mathcal{T}(T_0)$. Applying the functor $\text{Hom}_\Lambda(T_0, -)$ to $(*)$, we thus obtain $Q \cong \text{Hom}_\Lambda(T_0, L) \in \text{fac Hom}_\Lambda(T_0, Y) \subset \text{fac}T$.

Next, we assume $Q \in \mathcal{X}(T_0)$. According to Theorem 1.2.7, there exists $L \in \mathcal{F}(T_0)$ such that $Q \cong \text{Ext}_\Lambda^1(T_0, L)$. By Lemma 2.1.2, $\text{Ext}_\Lambda^1(X, L) \cong \text{Ext}_B^1(\text{Ext}_\Lambda^1(T_0, X), \text{Ext}_\Lambda^1(T_0, L))$, so $Q \in \Gamma^\perp$ implies $\text{Ext}_\Lambda^1(X, L) = 0$ (②). Since X is classic partial tilting, Lemma 1.2.3 implies $(\text{Gen}X, \mathcal{F}(X))$ is a torsion pair in $\text{mod } \Lambda$. Consider the canonical sequence of L associated to the torsion pair $(\text{Gen}X, \mathcal{F}(X))$:

$$0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0 (**).$$

Identity ② implies $\text{Ext}_\Lambda^1(X, L'') = 0$, and since $L'' \in \mathcal{F}(X)$, $L'' \in \mathcal{X}^{\text{perp}}$.

If $L \in \mathcal{T}(Y)$, then $L'' \in \mathcal{T}(Y)$, so that Theorem 2.2.5 yields $\text{Ext}_\Gamma^1(Y, L'') \cong \text{Ext}_\Lambda^1(Y, L'') = 0$. Since Y is a tilting Γ -module, Lemma 1.2.4 implies that L'' is generated by Y (in $\text{mod } \Gamma$ and hence in $\text{mod } \Lambda$), hence in $\mathcal{T}(T_0)$, i.e., $\text{Ext}_\Lambda^1(T_0, L'') = 0$. Since $L' \in \text{Gen}X$, applying the functor $\text{Hom}_\Lambda(T_0, -)$ to (**), we obtain $Q \cong \text{Ext}_\Lambda^1(T_0, L) \in \text{fac Ext}_\Lambda^1(T_0, L') \subset \text{fac Ext}_\Lambda^1(T_0, X) \subset \text{fac } T$.

If $L \notin \mathcal{T}(Y)$, we have $\dim_k \text{Ext}_\Lambda^1(Y, L) =: s \neq 0$. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \xrightarrow{i} & F_1 & & \\
& & \downarrow h & \swarrow u & \downarrow h' & & \\
0 & \longrightarrow & L & \xrightarrow{f} & F & \xrightarrow{g} & Y^s \longrightarrow 0 \\
& & \downarrow k & & \downarrow l & & \\
0 & \longrightarrow & L_2 & \longrightarrow & F_2 & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

where the existence of the middle row is ensured by Corollary 1.2.6 such that the connecting morphism $\delta : \text{Hom}_\Lambda(Y, Y^s) \rightarrow \text{Ext}_\Lambda^1(Y, L)$ is surjective. The two columns of short exact sequences are the canonical sequences of L and F associated to the torsion pair $(\text{Gen}X, \mathcal{F}(X))$, respectively. The existence of i is due to the universal properties of canonical sequences of torsion pairs and is obviously a monomorphism. In view of $F_1 \in \text{Gen}X$ and ①, we have $\text{Hom}_\Lambda(F_1, Y^s) = 0$, whence $gh' = 0$. So there exists $u : F_1 \rightarrow L$ such that $h' = fu$. Since h' is a monomorphism, u is a monomorphism. By definition, $L_1 = \sum \text{Im} \beta \{ \beta : X \rightarrow L \}$, and $F_1 \in \text{Gen}X$ yields $\dim_k F_1 \leq \dim_k L_1$. Hence i is an isomorphism. Owing to ① and ②, an application of functor $\text{Hom}_\Lambda(X, -)$ to the middle row gives $\text{Ext}_\Lambda^1(X, F) = 0$ and hence $\text{Ext}_\Lambda^1(X, F_2) = 0$. By the construction of the second column, $F_2 \in \mathcal{F}(X)$, i.e., $\text{Hom}_\Lambda(X, F_2) = 0$, so $F_2 \in \mathcal{X}^{\text{perp}}$. By the construction of the middle row, we have $\text{Ext}_\Lambda^1(Y, F) = 0$, so $\text{Ext}_\Lambda^1(Y, F_2) = 0$. Hence Theorem 2.2.5 yields $\text{Ext}_\Gamma^1(Y, F_2) \cong \text{Ext}_\Lambda^1(Y, F_2) = 0$. Since Y is tilting in $\mathcal{X}^{\text{perp}}$, Lemma 1.2.4 implies F_2 is generated by Y (in $\text{mod } \Gamma$ and hence in $\text{mod } \Lambda$), hence in $\mathcal{T}(T_0)$. Consequently, $\text{Ext}_\Lambda^1(T_0, h')$ is an epimorphism. Since $fh = h'i$, $\text{Ext}_\Lambda^1(T_0, fh)$ is also an epimorphism. Applying the functor $\text{Hom}_\Lambda(T_0, -)$ to the first column and the middle row we obtain the following diagram with exact rows:

$$\begin{array}{ccccccc}
\text{Hom}_\Lambda(T_0, F) & \longrightarrow & \text{Hom}_\Lambda(T_0, Y^s) & \xrightarrow{\alpha} & \text{Ext}_\Lambda^1(T_0, L) & \xrightarrow{\text{Ext}_\Lambda^1(T_0, f)} & \text{Ext}_\Lambda^1(T_0, F) \\
& & & & \uparrow \text{id} & & \\
\text{Hom}_\Lambda(T_0, L_2) & \longrightarrow & \text{Ext}_\Lambda^1(T_0, L_1) & \xrightarrow{\text{Ext}_\Lambda^1(T_0, h)} & \text{Ext}_\Lambda^1(T_0, L) & \longrightarrow & \text{Ext}_\Lambda^1(T_0, L_2).
\end{array}$$

For any $\eta \in \text{Ext}_\Lambda^1(T_0, L)$, there exists $\xi \in \text{Ext}_\Lambda^1(T_0, L_1)$ such that $\text{Ext}_\Lambda^1(T_0, f)(\eta) = \text{Ext}_\Lambda^1(T_0, fh)(\xi) = \text{Ext}_\Lambda^1(T_0, f)(\text{Ext}_\Lambda^1(T_0, h)(\xi))$, i.e., $\text{Ext}_\Lambda^1(T_0, f)(\eta - \text{Ext}_\Lambda^1(T_0, h)(\xi)) = 0$. So there exists $v \in \text{Hom}_\Lambda(T_0, Y^s)$ such that $\alpha(v) = \eta - \text{Ext}_\Lambda^1(T_0, h)(\xi)$, i.e., $\eta = \alpha(v) + \text{Ext}_\Lambda^1(T_0, h)(\xi)$. Consequently, the homomorphism $(\alpha, \text{Ext}_\Lambda^1(T_0, h)) : \text{Hom}_\Lambda(T_0, Y^s) \oplus \text{Ext}_\Lambda^1(T_0, L_1) \rightarrow \text{Ext}_\Lambda^1(T_0, L)$ is an epimorphism, i.e., $Q \cong \text{Ext}_\Lambda^1(T_0, L) \in \text{fac}(\text{Hom}_\Lambda(T_0, Y) \oplus \text{Ext}_\Lambda^1(T_0, L_1))$. But since $L_1 \in \text{Gen}X$, this entails $Q \in \text{fac}T$. This completes the proof. \square

An algebra B is called a **tilted algebra** of type Λ , when $B = \text{End}_\Lambda(T)$ and T is a tilting module over a hereditary algebra Λ . Then with Proposition 2.2.7 in hand, we know to construct tilting modules of tilted algebras: We just need to construct perfect exceptional modules.

2.3 Constructions of tilting modules

General assumption: From now on, we let Λ be hereditary, T_0 be a tilting Λ -module, $B = \text{End}_\Lambda(T_0)$. Let $T = X_0 \oplus Y_0$ be a basic tilting Λ -module such that $X_0 \in \mathcal{F}(T_0)$, $Y_0 \in \mathcal{T}(T_0)$. We will refer to this set-up by saying that $(\Lambda, T_0, B, X_0, Y_0)$ fulfills the general assumption.

Consider the following short exact sequence

$$0 \longrightarrow \text{Tr}_{Y_0} X_0 \xrightarrow{l} Y_0 \xrightarrow{p} Y_0 / \text{Tr}_{Y_0} X_0 \longrightarrow 0 (a)$$

such that l is the embedding and $(p, Y_0 / \text{Tr}_{Y_0} X_0)$ is the cokernel. According to Proposition 1.3.1, we know $\text{Tr}_{Y_0} X_0 \in \text{Gen}X_0$. Hence the right exactness of $\text{Ext}_\Lambda^1(X_0, -)$ and the self-orthogonality of $X_0 \oplus Y_0$ imply $\text{Ext}_\Lambda^1(X_0, \text{Tr}_{Y_0} X_0) = 0$. Applying the functor $\text{Hom}_\Lambda(X_0, -)$ to (a), we obtain an exact sequence

$$\text{Hom}_\Lambda(X_0, \text{Tr}_{Y_0} X_0) \rightarrow \text{Hom}_\Lambda(X_0, Y_0) \rightarrow \text{Hom}_\Lambda(X_0, Y_0 / \text{Tr}_{Y_0} X_0) \rightarrow \text{Ext}_\Lambda^1(X_0, \text{Tr}_{Y_0} X_0) = 0$$

By the definition of traces, $\text{Hom}_\Lambda(X_0, l) : \text{Hom}_\Lambda(X_0, \text{Tr}_{Y_0} X_0) \rightarrow \text{Hom}_\Lambda(X_0, Y_0)$ is surjective, hence $\text{Hom}_\Lambda(X_0, Y_0 / \text{Tr}_{Y_0} X_0) = 0$ (③).

The following lemma is quite useful when considering homomorphisms between indecomposable summands of self-orthogonal modules.

Lemma 2.3.1. [28, Lemma 4.1] *Suppose A is hereditary, if T_1 and T_2 are two indecomposable A -module such that $\text{Ext}_A^1(T_2, T_1) = 0$, then any nonzero homomorphism from T_1 to T_2 is either a monomorphism or an epimorphism.*

Lemma 2.3.2. *Suppose that $(\Lambda, T_0, B, X_0, Y_0)$ fulfills the general assumption. Then $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, Y_0 / \text{Tr}_{Y_0} X_0), \text{Hom}_\Lambda(T_0, Y_0 / \text{Tr}_{Y_0} X_0)) = 0$.*

Proof. Since $Y_0 \in \mathcal{T}(T_0)$ and $\mathcal{T}(T_0)$ is closed under factor modules, we have $Y_0 / \text{Tr}_{Y_0} X_0 \in \mathcal{T}(T_0)$. By Lemma 2.1.1, we have $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, Y_0 / \text{Tr}_{Y_0} X_0), \text{Hom}_\Lambda(T_0, Y_0 / \text{Tr}_{Y_0} X_0)) \cong \text{Ext}_\Lambda^1(Y_0 / \text{Tr}_{Y_0} X_0, Y_0 / \text{Tr}_{Y_0} X_0)$, so we just need to show $\text{Ext}_\Lambda^1(Y_0 / \text{Tr}_{Y_0} X_0, Y_0 / \text{Tr}_{Y_0} X_0) = 0$. Applying the functor $\text{Hom}_\Lambda(-, Y_0 / \text{Tr}_{Y_0} X_0)$ to (a) yields an exact sequence

$$\text{Hom}_\Lambda(\text{Tr}_{Y_0} X_0, Y_0 / \text{Tr}_{Y_0} X_0) \rightarrow \text{Ext}_\Lambda^1(Y_0 / \text{Tr}_{Y_0} X_0, Y_0 / \text{Tr}_{Y_0} X_0) \rightarrow \text{Ext}_\Lambda^1(Y_0, Y_0 / \text{Tr}_{Y_0} X_0) (b).$$

In view of $\text{Tr}_{Y_0} X_0 \in \text{Gen}X_0$ and ③, the left exactness of $\text{Hom}_\Lambda(-, Y_0 / \text{Tr}_{Y_0} X_0)$ forces $\text{Hom}_\Lambda(\text{Tr}_{Y_0} X_0, Y_0 / \text{Tr}_{Y_0} X_0) = 0$ (④). Since Y_0 is self-orthogonal, an application of the functor $\text{Hom}_\Lambda(Y_0, -)$ to (a) yields $\text{Ext}_\Lambda^1(Y_0, Y_0 / \text{Tr}_{Y_0} X_0) = 0$. Thus, we have $\text{Ext}_\Lambda^1(Y_0 / \text{Tr}_{Y_0} X_0, Y_0 / \text{Tr}_{Y_0} X_0) = 0$ (⑤). \square

Lemma 2.3.3. *Let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be a short exact sequence of A -modules. Then the following statements hold:*

1. *If $X \neq 0$ and for each $\varphi : X \rightarrow X$, there exists $\phi : Y \rightarrow Y$ such that $\phi f = f\varphi$, then Y being indecomposable implies X is indecomposable.*
2. *If $Z \neq 0$ and for each $\psi : Z \rightarrow Z$ there exists $\phi : Y \rightarrow Y$ such that $g\phi = \psi g$, then Y being indecomposable implies Z is indecomposable.*
3. *If $\text{Hom}_A(Y, X) = \text{Ext}_A^1(Y, X) = 0$ and for each $\phi : Y \rightarrow Y$ there exists $\psi : Z \rightarrow Z$ such that $\psi g = g\phi$, then Z being indecomposable implies Y is indecomposable.*

Proof. 1. Let $\varphi \in \text{End}_A(X)$. By assumption there exists $\phi : Y \rightarrow Y$ such that $\phi f = f\varphi$, whence $\phi^i f = f\varphi^i$ for any $i \geq 1$. If ϕ is nilpotent, then there exists $i \geq 1$ such that $f\varphi^i = 0$, so φ is nilpotent. If ϕ is invertible, due to the Snake Lemma φ is injective, and hence invertible. We conclude that X is indecomposable.

2. the proof is similar to (a).

3. For any $\phi \in \text{End}_A(Y)$, by assumption there exists $\psi \in \text{End}_A(Z)$ such that $\psi g = g\phi$, so $\psi^i g = g\phi^i$ for any $i \geq 1$. If ψ is nilpotent, then there exists $i \geq 1$ such that $g\phi^i = 0$, hence there exists $h : Y \rightarrow X$ such that $\phi^i = fh$. However, $\text{Hom}_A(Y, X) = 0$, so ϕ is nilpotent. If ψ is invertible, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow \phi & & \downarrow \psi & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
& & \downarrow \beta & & \downarrow \epsilon & & \downarrow \psi^{-1} & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0.
\end{array}$$

Where the existence of ϵ is ensured by $\text{Ext}_A^1(Y, X) = 0$. Then $g = g\epsilon\phi$, i.e., $g(\epsilon\phi - id_Y) = 0$, so there exists $h' : Y \rightarrow X$ such that $\epsilon\phi - id_Y = fh'$. Since $\text{Hom}_A(Y, X) = 0$, we obtain $\epsilon\phi = id_Y$. Thus, ϕ is injective and hence invertible. Consequently, Y is indecomposable. \square

Lemma 2.3.4. *Assume that $(\Lambda, T_0, B, T, X_0, Y_0)$ fulfills the general assumption. Then $Y_0/\text{Tr}_{Y_0}X_0$ is basic.*

Proof. Suppose $Y_0 = \bigoplus_{i=1}^t Y_i$ is the decomposition of Y_0 with indecomposable summands, for each i we can consider the following exact sequence:

$$0 \longrightarrow \text{Tr}_{Y_i}X_0 \xrightarrow{l_i} Y_i \xrightarrow{p_i} Y_i/\text{Tr}_{Y_i}X_0 \longrightarrow 0. (a_i)$$

According to Proposition 1.3.1, we know that $\text{Tr}_{Y_0}X_0 = \bigoplus_{i=1}^t \text{Tr}_{Y_i}X_0$, and $Y_0/\text{Tr}_{Y_0}X_0 = \bigoplus_{i=1}^t Y_i/\text{Tr}_{Y_i}X_0$. Since $T = Y_0 \oplus X_0$ is tilting and $\text{Tr}_{Y_0}X_0 \in \text{Gen}X_0$, by right exactness of $\text{Ext}_\Lambda^1(Y_0, -)$, we obtain $\text{Ext}_\Lambda^1(Y_0, \text{Tr}_{Y_0}X_0) = 0$ (©) and hence $\text{Ext}_\Lambda^1(Y_i, \text{Tr}_{Y_i}X_0) = 0$. An application of the functor $\text{Hom}_\Lambda(Y_i, -)$ to (a_i) now shows that the homomorphism $\text{Hom}_\Lambda(Y_i, p_i) : \text{Hom}_\Lambda(Y_i, Y_i) \rightarrow \text{Hom}_\Lambda(Y_i, Y_i/\text{Tr}_{Y_i}X_0)$ is an epimorphism. So for each $\phi : Y_i/\text{Tr}_{Y_i}X_0 \rightarrow$

$Y_i/Tr_{Y_i}X_0$ there exists $\psi : Y_i \rightarrow Y_i$ such that $\phi p_i = p_i \psi$. By Lemma 2.3.3 (b), we know $Y_i/Tr_{Y_i}X_0$ is indecomposable or zero.

Suppose that there exists an isomorphism $\nu : Y_i/Tr_{Y_i}X_0 \rightarrow Y_j/Tr_{Y_j}X_0$ with $Y_i/Tr_{Y_i}X_0 \neq 0, Y_j/Tr_{Y_j}X_0 \neq 0$ and $i \neq j$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Tr_{Y_i}X_0 & \longrightarrow & Y_i & \xrightarrow{p_i} & Y_i/Tr_{Y_i}X_0 & \longrightarrow & 0 & (1) \\ & & \downarrow h & & \downarrow k & & \downarrow \nu & & & \\ 0 & \longrightarrow & Tr_{Y_j}X_0 & \longrightarrow & Y_j & \xrightarrow{p_j} & Y_j/Tr_{Y_j}X_0 & \longrightarrow & 0, & (2) \end{array}$$

where the existence of k is ensured by ⑥. According to Lemma 2.3.1, k is either a monomorphism or an epimorphism.

If k is a monomorphism, consider the following commutative diagram obtained by using the Snake Lemma and taking the cokernels of h and k :

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_i & \longrightarrow & Y_j & \longrightarrow & E & \longrightarrow & 0 & (3) \\ & & \uparrow & & \uparrow & & \uparrow & & & \\ 0 & \longrightarrow & Tr_{Y_i}X_0 & \longrightarrow & Tr_{Y_j}X_0 & \longrightarrow & E & \longrightarrow & 0. & (4) \end{array}$$

Applying $\text{Hom}_\Lambda(-, Y_i/Tr_{Y_i}X_0)$ to (4), we obtain an exact sequence

$$\text{Hom}_\Lambda(Tr_{Y_i}X_0, Y_i/Tr_{Y_i}X_0) \rightarrow \text{Ext}_\Lambda^1(E, Y_i/Tr_{Y_i}X_0) \rightarrow \text{Ext}_\Lambda^1(Tr_{Y_j}X_0, Y_i/Tr_{Y_i}X_0).$$

In view of the exact sequence (b) in Lemma 2.3.2 and ⑤, we have $\text{Ext}_\Lambda^1(Tr_{Y_0}X_0, Y_0/Tr_{Y_0}X_0) = 0$. Then ④ implies $\text{Ext}_\Lambda^1(E, Y_i/Tr_{Y_i}X_0) = 0$ (⑦). Applying the functor $\text{Hom}_\Lambda(-, Y_i/Tr_{Y_i}X_0)$ to (1), ④ yields an isomorphism $\text{Hom}_\Lambda(Y_i/Tr_{Y_i}X_0, Y_i/Tr_{Y_i}X_0) \cong \text{Hom}_\Lambda(Y_i, Y_i/Tr_{Y_i}X_0)$. Thanks to Lemma 2.3.1 and ⑤, $\dim_k \text{Hom}_\Lambda(Y_i, Y_i/Tr_{Y_i}X_0) = \dim_k \text{Hom}_\Lambda(Y_i/Tr_{Y_i}X_0, Y_i/Tr_{Y_i}X_0) = 1$. Applying the functor $\text{Hom}_\Lambda(Y_i, -)$ to (1) while observing ⑥, we obtain an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(Y_i, Tr_{Y_i}X_0) \rightarrow \text{Hom}_\Lambda(Y_i, Y_i) \rightarrow \text{Hom}_\Lambda(Y_i, Y_i/Tr_{Y_i}X_0) \rightarrow 0.$$

Obviously, Y_i is self-orthogonal, by Lemma 2.3.1, $\dim_k \text{Hom}_\Lambda(Y_i, Y_i) = 1$. Then $\dim_k \text{Hom}_\Lambda(Y_i, Y_i/Tr_{Y_i}X_0) = 1$ implies $\dim_k \text{Hom}_\Lambda(Y_i, Tr_{Y_i}X_0) = 0$, and hence $\text{Hom}_\Lambda(Y_i, Tr_{Y_i}X_0) = 0$ (⑧). Applying the functor $\text{Hom}_\Lambda(-, Tr_{Y_i}X_0)$ to (3), we obtain an exact sequence:

$$\text{Hom}_\Lambda(Y_i, Tr_{Y_i}X_0) \rightarrow \text{Ext}_\Lambda^1(E, Tr_{Y_i}X_0) \rightarrow \text{Ext}_\Lambda^1(Y_j, Tr_{Y_i}X_0).$$

Then ⑥ implies $\text{Ext}_\Lambda^1(E, Tr_{Y_i}X_0) = 0$. Upon applying the functor $\text{Hom}_\Lambda(E, -)$ to (1), ⑦ forces $\text{Ext}_\Lambda^1(E, Y_i) = 0$ which implies (3) is split. But since Y_j is indecomposable and $Y_i \neq 0$, we arrive at $E = 0$. So $Y_i \cong Y_j$, a contradiction.

If k is an epimorphism, consider the following commutative diagram obtained by using the Snake Lemma and taking the kernels of h and k

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & Y_i & \longrightarrow & Y_j & \longrightarrow & 0 & (5) \\ & & \uparrow & & \uparrow & & \uparrow & & & \\ 0 & \longrightarrow & F & \longrightarrow & Tr_{Y_i}X_0 & \longrightarrow & Tr_{Y_j}X_0 & \longrightarrow & 0. & (6) \end{array}$$

Applying the functor $\text{Hom}_\Lambda(Y_j, -)$ to (6), yields an exact sequence

$$\text{Hom}_\Lambda(Y_j, Tr_{Y_j}X_0) \rightarrow \text{Ext}_\Lambda^1(Y_j, F) \rightarrow \text{Ext}_\Lambda^1(Y_j, Tr_{Y_i}X_0).$$

Now ⑧ and ⑥ force $\text{Ext}_\Lambda^1(Y_j, F) = 0$ which implies (5) is split, a contradiction. Hence we obtain the assertion. \square

Now we are in a position to verify Theorem A.

Theorem 2.3.5. *Suppose $(\Lambda, T_0, B, T, X_0, Y_0)$ satisfies the general assumption. If no indecomposable summand of Y_0 is generated by X_0 , then $\text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0) \oplus \text{Ext}_\Lambda^1(T_0, X_0)$ is a tilting B -module.*

Proof. By Proposition 1.2.9, we have $\text{pd}_B \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0) \leq \text{pd}_\Lambda Y_0/\text{Tr}_{Y_0}X_0 \leq 1$ and $\text{id}_B \text{Ext}_\Lambda^1(T_0, X_0) \leq \text{id}_\Lambda X_0 \leq 1$, so that $\text{Ext}_B^2(\text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0), \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0)) = \text{Ext}_B^2(\text{Ext}_\Lambda^1(T_0, X_0), \text{Ext}_\Lambda^1(T_0, X_0)) = 0$. Lemma 2.1.3 implies $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0), \text{Ext}_\Lambda^1(T_0, X_0)) \cong \text{Ext}_\Lambda^2(Y_0/\text{Tr}_{Y_0}X_0, X_0) = 0$. Applying the functor $\text{Hom}_\Lambda(X_0, -)$ to the exact sequence (a), we obtain $\text{Ext}_\Lambda^1(X_0, Y_0/\text{Tr}_{Y_0}X_0) = 0$. Then, according to Lemma 2.1.4, we have $\text{Ext}_B^2(\text{Ext}_\Lambda^1(T_0, X_0), \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0)) \cong \text{Ext}_\Lambda^1(X_0, Y_0/\text{Tr}_{Y_0}X_0) = 0$, and $\text{Ext}_B^1(\text{Ext}_\Lambda^1(T_0, X_0), \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0)) \cong \text{Hom}_\Lambda(X_0, Y_0/\text{Tr}_{Y_0}X_0) = 0$. So $\text{Ext}_\Lambda^1(T_0, X_0) \oplus \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0)$ is self-orthogonal and obviously exceptional. Note that Y_i is generated by X_0 whenever $Y_i/\text{Tr}_{Y_i}X_0 = 0$, then since no indecomposable summand of Y_0 is generated by X_0 , we obtain $Y_i/\text{Tr}_{Y_i}X_0 \neq 0$ for $i = 1, 2, \dots, t$. According to Remark 2.2.4, B admits the same number of non-isomorphic simple modules as Λ . Then by the Tilting Theorem 1.2.7 and Lemmas 2.3.2, 2.3.4, we know it is perfect exceptional. By Proposition 2.2.7, we know it is tilting and the conclusion follows. \square

Remark. If we drop the assumption: No indecomposable summand of Y_0 is generated by X_0 , we can only obtain an exceptional module which might not be tilting.

So for each given tilting Λ -module $X_0 \oplus Y_0$ with $X_0 \in \mathcal{F}(T_0)$ and $Y_0 \in \mathcal{T}(T_0)$, if no indecomposable summand of Y_0 is generated by X_0 , we can obtain a tilting B -module $\text{Ext}_\Lambda^1(T_0, X_0) \oplus \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0)$. Similarly, we can also use the rejects to construct tilting modules. As before, we assume that $(\Lambda, T_0, B, X_0, Y_0)$ satisfies the general assumption. Consider the following short exact sequence:

$$0 \longrightarrow \text{Rej}_{X_0}Y_0 \xrightarrow{\pi} X_0 \xrightarrow{q} X_0/\text{Rej}_{X_0}Y_0 \longrightarrow 0 \quad (a')$$

such that π is the embedding and $(q, X_0/\text{Rej}_{X_0}Y_0)$ is the cokernel. Obviously, by the definition of rejects, we have $\text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, Y_0) = 0$ (①). According to Proposition 1.3.1, $X_0/\text{Rej}_{X_0}Y_0$ is cogenerated by Y_0 . Then the left exactness of the functor $\text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, -)$ forces $\text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, X_0/\text{Rej}_{X_0}Y_0) = 0$ (②).

Lemma 2.3.6. *Suppose $(\Lambda, T_0, B, X_0, Y_0)$ satisfies the general assumption. Then $\text{Ext}_B^1(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)) = 0$.*

Proof. Since $X_0 \in \mathcal{F}(T_0)$ and $\mathcal{F}(T_0)$ is closed under taking submodules, $\text{Rej}_{X_0}Y_0 \in \mathcal{F}(T_0)$. Then according to Lemma 2.1.2, we have $\text{Ext}_B^1(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)) \cong \text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, \text{Rej}_{X_0}Y_0)$, so it suffices to show $\text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, \text{Rej}_{X_0}Y_0) = 0$. Since $X_0 \oplus Y_0$ is a tilting Λ -module, $\text{Ext}_\Lambda^1(X_0, X_0) = 0$. Then the right exactness of the functor $\text{Ext}_\Lambda^1(-, X_0)$ forces $\text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, X_0) = 0$ (③). Applying the functor $\text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, -)$ to (a') yields an exact sequence

$$\text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, X_0/\text{Rej}_{X_0}Y_0) \rightarrow \text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, \text{Rej}_{X_0}Y_0) \rightarrow \text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, X_0),$$

then ② and ③ imply $\text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, \text{Rej}_{X_0}Y_0) = 0$. And consequently, our conclusion follows. \square

Lemma 2.3.7. *Suppose $(\Lambda, T_0, B, X_0, Y_0)$ fulfills the general assumption. Then $Rej_{X_0}Y_0$ is basic.*

Proof. Let $X_0 = \bigoplus_{j=1}^{n-t} X_j$ be a decomposition of X_0 into indecomposable modules. Similar to the construction of (a'_j) , for each j , consider the following exact sequence :

$$0 \longrightarrow Rej_{X_j}Y_0 \xrightarrow{\pi_j} X_j \xrightarrow{q_j} X_j/Rej_{X_j}Y_0 \longrightarrow 0. \quad (a'_j)$$

According to Proposition 1.3.2, we know $Rej_{X_0}Y_0 = \bigoplus_{j=1}^{n-t} Rej_{X_j}Y_0$, and hence $X_0/Rej_{X_0}Y_0 = \bigoplus_{j=1}^{n-t} X_j/Rej_{X_j}Y_0$. In view of Proposition 1.3.1, $X_0/Rej_{X_0}Y_0$ is cogenerated by Y_0 . Since $T_0 = X_0 \oplus Y_0$ is tilting, $\text{Ext}_\Lambda^1(Y_0, X_0) = 0$. Then the right exactness of the functor $\text{Ext}_\Lambda^1(-, X_0)$ implies $\text{Ext}_\Lambda^1(X_0/Rej_{X_0}Y_0, X_0) = 0$, and hence $\text{Ext}_\Lambda^1(X_j/Rej_{X_j}Y_0, X_i) = 0$ (④) for $j, i = 0, 1, \dots, n-t$. Applying the functor $\text{Hom}_\Lambda(-, X_j)$ to (a'_j) , we obtain an exact sequence: $\text{Hom}_\Lambda(X_j, X_j) \rightarrow \text{Hom}_\Lambda(Rej_{X_j}Y_0, X_j) \rightarrow 0$, i.e., for each $\phi : Rej_{X_j}Y_0 \rightarrow X_j$, there exists an $\psi : X_j \rightarrow X_j$ such that $\psi\pi_j = \pi_j\phi$. Then according to Lemma 2.3.3 (1), we know $Rej_{X_j}Y_0$ is indecomposable or zero.

Suppose there exists an isomorphism $\alpha : Rej_{X_j}Y_0 \rightarrow Rej_{X_i}Y_0$ with $Rej_{X_j}Y_0 \neq 0, Rej_{X_i}Y_0 \neq 0$ and $i \neq j$. Consider the following commutative diagram with exact rows:

$$0 \longrightarrow Rej_{X_j}Y_0 \xrightarrow{\pi_j} X_j \xrightarrow{q_j} X_j/Rej_{X_j}Y_0 \longrightarrow 0 \quad (1')$$

$$\begin{array}{ccccccc} & & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ & & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Rej_{X_i}Y_0 & \xrightarrow{\pi_i} & X_i & \xrightarrow{q_i} & X_i/Rej_{X_i}Y_0 & \longrightarrow 0, \end{array} \quad (2')$$

where the existence of β is ensured by ④. According to Lemma 2.3.1, β is either an epimorphism or a monomorphism.

If β is a monomorphism. Consider the following commutative diagram with exact rows, which is obtained by using the Snake Lemma and taking the cokernels of β and γ :

$$0 \longrightarrow X_j/Rej_{X_j}Y_0 \longrightarrow X_i/Rej_{X_i}Y_0 \longrightarrow E \longrightarrow 0 \quad (3')$$

$$\begin{array}{ccccccc} & & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & X_j & \longrightarrow & X_i & \longrightarrow & E & \longrightarrow 0. \end{array} \quad (4')$$

Applying the functor $\text{Hom}_\Lambda(Rej_{X_j}Y_0, -)$ to $(1')$, in view of ②, we have an isomorphism $\text{Hom}_\Lambda(Rej_{X_j}Y_0, Rej_{X_j}Y_0) \cong \text{Hom}_\Lambda(Rej_{X_j}Y_0, X_j)$. Since we have proved that $Rej_{X_j}Y_0$ is indecomposable, Lemma 2.3.1 and Lemma 2.3.4 indicate that $\dim_k \text{Hom}_\Lambda(Rej_{X_j}Y_0, X_j) = \dim_k \text{Hom}_\Lambda(Rej_{X_j}Y_0, Rej_{X_j}Y_0) = 1$ (⑤). Applying the functor $\text{Hom}_\Lambda(-, X_j)$ to $(1')$ while observing ④, we obtain an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(X_j/Rej_{X_j}Y_0, X_j) \rightarrow \text{Hom}_\Lambda(X_j, X_j) \rightarrow \text{Hom}_\Lambda(Rej_{X_j}Y_0, X_j) \rightarrow 0.$$

Since X_0 is self-orthogonal, by Lemma 2.3.1, we know $\dim_k \text{Hom}_\Lambda(X_j, X_j) = 1$. Then ⑤ implies $\dim_k \text{Hom}_\Lambda(X_j/Rej_{X_j}Y_0, X_j) = 0$ and hence $\text{Hom}_\Lambda(X_j/Rej_{X_j}Y_0, X_j) = 0$ (⑥). Applying the functor $\text{Hom}_\Lambda(-, X_j)$ to $(3')$, we obtain an exact sequence

$$\text{Hom}_\Lambda(X_j/Rej_{X_j}Y_0, X_j) \rightarrow \text{Ext}_\Lambda^1(E, X_j) \rightarrow \text{Ext}_\Lambda^1(X_i/Rej_{X_i}Y_0, X_j).$$

Then ⑥ and ④ force $\text{Ext}_\Lambda^1(E, X_j) = 0$. Hence (4') is a split sequence which induces a contradiction.

If β is an epimorphism. Consider the following commutative diagram with exact rows, which is obtained by using the Snake Lemma and taking the kernels of β and γ :

$$0 \longrightarrow F \longrightarrow X_j/\text{Rej}_{X_j}Y_0 \longrightarrow X_i/\text{Rej}_{X_i}Y_0 \longrightarrow 0 \quad (5')$$

$$0 \longrightarrow F \longrightarrow X_j \longrightarrow X_i \longrightarrow 0. \quad (6')$$

Applying the functor $\text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, -)$ to (a'), in view of ③, we have $\text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, X_0/\text{Rej}_{X_0}Y_0) = 0$ (⑦). In view of ⑦ and ②, an application of $\text{Hom}_\Lambda(\text{Rej}_{X_i}Y_0, -)$ to (5') implies $\text{Ext}_\Lambda^1(\text{Rej}_{X_i}Y_0, F) = 0$. Applying the functor $\text{Hom}_\Lambda(X_i/\text{Rej}_{X_i}Y_0, -)$ to (6') while observing ④ and ⑥, we have $\text{Ext}_\Lambda^1(X_i/\text{Rej}_{X_i}Y_0, F) = 0$. Thus, (5') is split which induces a contradiction. Now we finish the proof. \square

Theorem 2.3.8. *Suppose $(\Lambda, T_0, B, T, X_0, Y_0)$ satisfies the general assumption. If no indecomposable summand of X_0 is cogenerated by Y_0 , then $\text{Hom}_\Lambda(T_0, Y_0) \oplus \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)$ is a tilting B -module.*

Proof. According to Lemma 1.2.9, $\text{pd}_B \text{Hom}_\Lambda(T_0, Y_0) \leq \text{pd}_\Lambda Y_0 \leq 1$, $\text{id}_B \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0) \leq \text{id}_\Lambda \text{Rej}_{X_0}Y_0 \leq 1$. Then $\text{Ext}_B^2(\text{Hom}_\Lambda(T_0, Y_0), \text{Hom}_\Lambda(T_0, Y_0)) = \text{Ext}_B^2(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)) = \text{Ext}_B^2(\text{Hom}_\Lambda(T_0, Y_0), \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)) = 0$. By Lemma 2.1.3, we know $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, Y_0), \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)) \cong \text{Ext}_\Lambda^2(Y_0, \text{Rej}_{X_0}Y_0) = 0$. And Lemma 2.1.4 implies $\text{Ext}_B^2(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Hom}_\Lambda(T_0, Y_0)) \cong \text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, Y_0)$, $\text{Ext}_B^1(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Hom}_\Lambda(T_0, Y_0)) \cong \text{Hom}_\Lambda(\text{Rej}_{X_0}Y_0, Y_0)$. In view of ①, we have $\text{Ext}_B^1(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Hom}_\Lambda(T_0, Y_0)) = 0$. Since $T_0 = X_0 \oplus Y_0$ is self-orthogonal, an application of the functor $\text{Hom}_\Lambda(-, Y_0)$ to (a') implies $\text{Ext}_\Lambda^1(\text{Rej}_{X_0}Y_0, Y_0) = 0$ and hence Lemma 2.1.4 implies $\text{Ext}_B^2(\text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0), \text{Hom}_\Lambda(T_0, Y_0)) = 0$. So we know $\text{Hom}_\Lambda(T_0, Y_0) \oplus \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)$ is self-orthogonal and obviously exceptional. Notice that $\text{Rej}_{X_j}Y_0 = 0$ if and only if X_j is co-generated by Y_0 , then since no indecomposable summand of X_0 is cogenerated by Y_0 , $\text{Rej}_{X_j}Y_0 \neq 0$ for $j = 1 \cdots n - t$. According to Remark 2.2.4, B admits the same number of non-isomorphic simple modules as Λ . Then by the Tilting Theorem 1.2.7, Lemma 2.3.2 and Lemma 2.3.4, we know it is also perfect exceptional. And Proposition 2.2.7 implies it is tilting. \square

Corollary 2.3.9. *Let Λ be hereditary, T_0 be a tilting Λ -module, $T = X_0 \oplus Y_0$ be a tilting Λ -module such that $Y_0 \in \mathcal{T}(T_0)$, $X_0 \in \mathcal{F}(T_0)$. If no indecomposable summand of Y_0 is generated by X_0 and no indecomposable summand of X_0 is co-generated by Y_0 , then $\text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0) \oplus \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)$ is a tilting B -module.*

Proof. According to Lemma 2.3.6 and Lemma 2.3.2, we know $\text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0) \oplus \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)$ is self-orthogonal. According to Lemmas 2.3.7, Lemma 2.3.4 and Theorem 1.2.7, we know under the assumptions about X_0 and Y_0 , $\text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0}X_0) \oplus \text{Ext}_\Lambda^1(T_0, \text{Rej}_{X_0}Y_0)$ is even perfect exceptional. Thus, our conclusion follows from Proposition 2.2.7. \square

3 Tilting quivers

3.1 Complements of almost complete tilting modules

In this section, we will talk something about complements of partial tilting modules and give the definition of tilting quivers.

Definition 3.1.1. *Let M be a partial tilting A -module, a basic module X is called a **complement** to M , provided: $M \oplus X$ is tilting and $\text{add}M \cap \text{add}X = 0$.*

Remark. According to Lemma 1.2.5, we know that every module satisfies the first two conditions of Definition 1.2.1 can be completed to be a classic tilting module. However, this is not true in general, i.e., the answer to the question: If an A -module M satisfies the first two conditions of Definition 1.2.2, can we find a module E such that $M \oplus E$ is tilting? is Negative. For details you are referred to [46].

Recall that a partial tilting module M is called **almost complete tilting**, if M admits $n - 1$ non-isomorphic indecomposable summands. A module N is called **sincere**, if $\text{Hom}_A(P, N) \neq 0$ for all projective indecomposable A -module P . A module N is called **faithful** if the annihilator $\text{ann}_A(N) = \{a \in A | Na = 0\}$ is $\{0\}$. Obviously, faithful modules are sincere. And it was proved in [36]: For a hereditary algebra A , a self-orthogonal module M is faithful if and only if it is sincere. The following theorem states some conclusions about complements to almost complete tilting modules.

Theorem 3.1.2. [29, Theorem 1] *Let A be a finite-dimensional hereditary algebra, there are at most two non-isomorphic complements to an almost complete tilting module M . Moreover, M has exactly two complements if and only if M is sincere.*

Note that the assumption: A being hereditary cannot be omitted, otherwise, the conclusion may be false (we refer to [16]).

Lemma 3.1.3. [33, Lemma 1.1] *Let $T = \bigoplus_{i=1}^n T_i$ be a tilting A -module, $1 \leq i \leq n$. If $T[i]$ is not faithful, then T_i is the unique (up to isomorphism) complement to $T[i]$. If $T[i]$ is faithful, then T_i is generated or cogenerated by $T[i]$.*

1. *If T_i is generated by $T[i]$, there exists a unique (up to isomorphism) indecomposable complement X not isomorphic to T_i and an exact sequence $0 \longrightarrow X \longrightarrow \overline{T[i]} \longrightarrow T_i \longrightarrow 0$ with $\overline{T[i]} \in \text{add}T[i]$ such that $\text{Ext}_A^i(X, T_i) = 0$ for $i > 0$ and $\text{Ext}_A^i(T_i, X) = 0$ for $i > 1$.*
2. *If T_i is cogenerated by $T[i]$, there exists a unique (up to isomorphism) indecomposable complement Y not isomorphic to T_i and an exact sequence $0 \longrightarrow T_i \longrightarrow \overline{T[i]} \longrightarrow Y \longrightarrow 0$ with $\overline{T[i]} \in \text{add}T[i]$ such that $\text{Ext}_A^i(Y, T_i) = 0$ for $i > 1$ and $\text{Ext}_A^i(T_i, Y) = 0$ for $i > 0$.*

3. If A is hereditary, then the non-vanishing Ext-spaces in (1) and (2) are one dimensional.
4. If A is hereditary, then T_i is never both generated and cogenerated by $T[i]$.

Recall that for an algebra A , \mathcal{T}_A is denoted to be the set of equivalence classes of tilting A -modules, tilting modules T_1, T_2 are supposed to be in the same class provided $\text{add}(T_1) = \text{add}(T_2)$. So when we consider elements of \mathcal{T}_A , we just consider basic tilting modules. The tilting quiver $\overrightarrow{\mathcal{K}}_A$ is defined as follows: the vertices are elements of \mathcal{T}_A , for two basic tilting A -modules, there exists an arrow from T_1 to T_2 if and only if $T_1 = M \oplus X$, $T_2 = M \oplus Y$ with X, Y being indecomposable summands belonging to a non-split exact sequence $0 \rightarrow X \rightarrow \overline{M} \rightarrow Y \rightarrow 0$ with $\overline{M} \in \text{add}M$. For a subset $\mathcal{D} \subset \mathcal{T}_A$, $\overrightarrow{\mathcal{D}}$ denotes the full subquiver of $\overrightarrow{\mathcal{K}}_A$ consisting of elements in \mathcal{D} .

3.2 Successors of the regular module

In this section, we will generalize a well known result about the tilting quivers. It was proved in [33]: Let Λ be a basic hereditary algebra, S_i be a simple Λ -module which is not injective. Then $T_{S_i} := \Lambda_\Lambda[i] \oplus \tau^{-1}S_i$ is tilting, and in this way we obtain all immediate successors of Λ_Λ in $\overrightarrow{\mathcal{K}}_\Lambda$. We will generalize this result to any basic finite dimensional algebra.

Lemma 3.2.1. [48, Ex 8.7] *If $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is a short exact sequence of A -modules and M is projective, then either all three modules are projective or $\text{pd}_A(M_2) = 1 + \text{pd}_A(M_1)$.*

Let M be a basic partial tilting module, the link $\overrightarrow{\text{lk}}(M)$ is defined to be the full subquiver of $\overrightarrow{\mathcal{K}}_A$ with vertices T such that M is a direct summand of T .

Lemma 3.2.2. *Let M be a basic almost complete partial tilting module, X_0 be a complement to M such that $\text{pd}_A X_0 = r$. Then the maximal possible length of the path in $\overrightarrow{\text{lk}}(M)$ terminating with $M \oplus X_0$ is no more than r .*

Proof. Suppose there exists a path of length $r + 1$ in $\text{lk}(M)$ ending with $M \oplus X_0$. According to the definition of tilting quivers, we obtain a long exact sequence

$$0 \longrightarrow X_{r+1} \longrightarrow M_r \xrightarrow{f_r} M_{r-1} \cdots M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} X_0 \longrightarrow 0.$$

Where $M_i \in \text{add}M$ for $0 \leq i \leq r$ and $\ker f_i = X_{i+1}$ for $0 \leq i \leq r$ are complements to M . Then $\text{Ext}_\Lambda^1(X_0, X_1) \cong \text{Ext}_\Lambda^{r+1}(X_0, X_{r+1}) = 0$ and hence X_0 is a direct summand of M , which is a contradiction. \square

Let $M \oplus N$ be a classic tilting module such that $\text{add}M \cap \text{add}N = 0$ and $\text{add}(M \oplus N) = \text{add}(M \oplus E)$, where E is constructed in the Bongartz's Lemma 1.2.5. Then N is called the **Bongartz complement** to M . The following conclusion is a characterization of the Bongartz complements.

Proposition 3.2.3. [47, Proposition 1.2] *Let T_1, T_2, \dots, T_n be pairwise non-isomorphic indecomposables, and suppose that $\bigoplus_{i=1}^n T_i$ is a classic tilting module. Then the following statements are equivalent:*

1. $\bigoplus_{i=r}^n T_i$ is the Bongartz complement to $\bigoplus_{i=1}^r T_i$

2. For $j = r, \dots, n$, there is no surjection from any module in $\mathbf{add}(T_1 \oplus \dots \oplus T_{j-1} \oplus T_{j+1} \oplus \dots \oplus T_n)$ to T_j .

Remark 3.2.4. In our preceding definition of the Bongartz complements, M is classic partial tilting and the tilting module $M \oplus N$ is classic tilting. In some place (like [30]), there is also another definition: Let $M \oplus N$ be a tilting module such that $\mathbf{add}M \cap \mathbf{add}N = 0$ and $M^\perp = (M \oplus N)^\perp$ (recall that $M^\perp = \{Z \in \text{mod } A \mid \text{Ext}_A^i(M, Z) = 0\}$), then N is called the Bongartz complement to M . It is not difficult to see that our definition of the Bongartz complements is just a special case of this definition. Moreover, it was proved in [30], Proposition 3.2.3 is also true for the generalized case.

Lemma 3.2.5. Let $T = P[i] \oplus X$ for some i be a basic tilting module, then there exists an arrow $A \rightarrow T$ in $\overrightarrow{\mathcal{K}}_A$ if and only if $\text{pd}_A X = 1$ and $P[i]$ is the Bongartz complement to X .

Proof. Suppose there exists an arrow: $A \rightarrow T$ in $\overrightarrow{\mathcal{K}}_A$. By the definition of $\overrightarrow{\mathcal{K}}_A$, there exists a non-split exact sequence

$$0 \rightarrow P(i) \rightarrow P \rightarrow X \rightarrow 0$$

such that $P \in \mathbf{add}P[i]$. So X is not projective (otherwise, the exact sequence splits) and $\text{pd}_A X = 1$. If $P[i]$ is not the Bongartz complement to X , then Proposition 3.2.3 induces a contradiction. So we proved the necessary part.

Suppose that $\text{pd}_A X = 1$ and $P[i]$ is the Bongartz complement to X . We claim that X is not the Bongartz complement to $P[i]$. If X is the Bongartz complement to $P[i]$, according to Remark 3.2.4, we have $P[i]^\perp = (P[i] \oplus X)^\perp$. But since $P[i]$ is the Bongartz complement to X , by the same token, $X^\perp = (P[i] \oplus X)^\perp$. Hence $X^\perp = P[i]^\perp$, note that $P[i]$ is projective, this implies X is also projective, a contradiction. So X is not the Bongartz complement to $P[i]$. According to Proposition 3.2.3, we know X is generated by $P[i]$. Then by Lemma 3.1.3, there exists a tilting module $T' = P[i] \oplus Y$ and an arrow $T' \rightarrow T$ in $\overrightarrow{\mathcal{K}}_A$, i.e., there exists a non split exact sequence $0 \rightarrow Y \rightarrow \overline{P} \rightarrow X \rightarrow 0$ with $\overline{P} \in \mathbf{add}P[i]$. Since X is of projective dimension 1, Lemma 3.2.1 forces $\text{pd}_A Y = 0$. Hence $Y = P(i)$ and there exists an arrow $A \rightarrow T$ in $\overrightarrow{\mathcal{K}}_A$. \square

Proposition 3.2.6. Let A be a basic finite dimensional algebra such that $\dim_k \text{End}_A(P(j)) = 1$ for all indecomposable projective A -modules $P(j)$. Then all immediate successors of A in $\overrightarrow{\mathcal{K}}_A$ are BB-tilting modules.

Proof. First, note that if $T = P[i] \oplus \tau^{-1}S$ is a BB-tilting module for some i , then $P[i]$ is the Bongartz complement to $\tau^{-1}S$. Otherwise, Proposition 3.2.3 induces a contradiction. Since T is a BB-tilting module and $\tau^{-1}S$ is not projective, $\text{pd}_A \tau^{-1}S = 1$. Then by Proposition 3.2.5, BB-tilting modules are immediate successors of A .

Suppose $T = P[i] \oplus X$ is an immediate successor of A . Then there exists a non-split exact sequence

$$0 \longrightarrow P(i) \xrightarrow{\epsilon} P \xrightarrow{\mu} X \longrightarrow 0, (\star)$$

with $P \in \mathbf{add}P[i]$. We claim that (\star) is a minimal projective resolution of X . Otherwise, we assume $0 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ is a minimal projective resolution of X . By the Schannel's Lemma ([48, Proposition 3.12]), $P_1 \oplus P \cong P_0 \oplus P(i)$. But since P_0 is a direct summand of P , $P_1 \cong P(i)$, and hence $P \cong P_0$. Applying the functor $\text{Hom}_A(-, A)$ to (\star) yields an exact sequence of A^{op} -modules

$$0 \longrightarrow \text{Hom}_A(X, A) \xrightarrow{\mu^*} \text{Hom}_A(P, A) \xrightarrow{\epsilon^*} \text{Hom}_A(P(i), A) \xrightarrow{\pi} \text{Tr}(X) \cong \text{Ext}_A^1(X, A) \longrightarrow 0 (\star\star)$$

where $\text{Tr}(X)$ is the transpose of X . We claim that $\text{Im}(\epsilon^*) = \text{rad}(\text{Hom}_A(P(i), A))$ as A^{op} -modules. General theory implies $\text{rad}(\text{Hom}_A(P(i), A)) \cong \text{Hom}_A(P(i), J(A))$ as A^{op} -modules. If ϵ^* is surjective, $\text{Tr}(X) = 0$. Then [8, IV. Proposition 1.7] implies X is projective, a contradiction. So $\text{Im}(\epsilon^*) \subseteq \text{rad}(\text{Hom}_A(P(i), A))$ (since $\text{Hom}_A(P(i), A)$ is an indecomposable projective A^{op} -module and $\text{Im}(\epsilon^*) \neq \text{Hom}_A(P(i), A)$). The existence of the non split-sequence (\star) implies $\dim_k \text{Ext}_A^1(X, P(i)) \neq 0$. Applying the functor $\text{Hom}_\Lambda(-, P(i))$ to (\star) , we obtain an exact sequence:

$$\text{Hom}_A(P(i), P(i)) \rightarrow \text{Ext}_A^1(X, P(i)) \rightarrow \text{Ext}_A^1(P, P(i)) = 0.$$

Since $\dim_k \text{End}_A(P(i)) = 1$, this implies $\dim_k \text{Ext}_A^1(X, P(i)) = 1$. Since $T = P[i] \oplus X$ is self-orthogonal, $\dim_k \text{Ext}_A^1(X, A) = \dim_k \text{Ext}_A^1(X, P[i] \oplus P(i)) = \dim_k \text{Ext}_A^1(X, P(i)) = 1$. And since $\text{Hom}_A(P(i), A)$ is indecomposable projective A^{op} -module, $\dim_k \text{Hom}(P(i), J(A)) = \dim_k \text{Hom}_A(P(i), A) - 1$. By the exactness of $(\star\star)$,

$$\dim_k(\text{Im}(\epsilon^*)) = \dim_k(\text{Hom}_A(P(i), A)) - \dim_k \text{Ext}_A^1(X, A).$$

So $\dim_k(\text{Im}(\epsilon^*)) = \dim_k(\text{rad}(\text{Hom}_A(P(i), A)))$, and $\text{Im}(\epsilon^*) = \text{rad}(\text{Hom}_A(P(i), A))$ as A^{op} -modules.

Then $\text{Tr}(X) \cong S'$ for some simple but not projective A^{op} -module S' , and $D \text{Tr}(X) \cong S$ for some simple but not injective A -module S . Since X is indecomposable and $D \text{Tr}(X) \cong S$ is not injective, $X \cong \tau^{-1}S$. \square

Corollary 3.2.7. *Let $B = \text{End}_\Lambda(T)$ be a tilted algebra of type Λ such that T is a basic tilting Λ -module. Then all immediate successors of B in $\overrightarrow{\mathcal{K}}_B$ are BB-tilting modules*

Proof. Let P be an indecomposable projective B -module, then $P = \text{Hom}_\Lambda(T, T_i)$ with T_i being an indecomposable summand of T . According to Lemma 2.1.1, $\text{End}_B(P) \cong \text{End}_\Lambda(T_i)$, and Lemma 2.3.1 implies $\dim_k \text{End}_\Lambda(T_i) = 1$. Hence $\dim_k \text{End}_B P = 1$ and our conclusion follows Proposition 3.2.6. \square

3.3 Constructions of tilting quivers of BB-tilted algebras

Recall that Λ denotes a basic hereditary algebra, and for a given tilting Λ -module T_0 , $\mathcal{T}_{\mathcal{T} \cup \mathcal{F}} \subset \mathcal{T}_\Lambda$ is the subset of \mathcal{T}_Λ consisting of equivalence classes whose representatives has summands only in $\mathcal{T}(T_0) \cup \mathcal{F}(T_0)$.

We say T_0 is **admissible**, if every tilting A -module $T = X_0 \oplus Y_0$ with $X_0 \in \mathcal{T}(T_0)$, $Y_0 \in \mathcal{F}(T_0)$, no indecomposable summand of X_0 is generated by Y_0 . We will see BB-tilting modules are admissible, but also later examples can show there exist admissible tilting modules which are not BB-tilting.

Proposition 3.3.1. *Let $B = \text{End}_\Lambda(T_0)$ such that T_0 is an admissible tilting Λ -module. Then*

$$\Phi : \mathcal{T}_{\mathcal{T} \cup \mathcal{F}} \rightarrow \mathcal{T}_B$$

defined via $T = Y_0 \oplus X_0 \mapsto \text{Hom}_\Lambda(T_0, Y_0/\text{Tr}_{Y_0} X_0) \oplus \text{Ext}_\Lambda^1(T_0, X_0)$, with $Y_0 \in \mathcal{T}(T_0)$, $X_0 \in \mathcal{F}(T_0)$ is a well-defined injective map.

Proof. First, we need to show that Φ is well defined. Let $T = Y_0 \oplus X_0$ be a tilting Λ -module such that $Y_0 \in \mathcal{T}(T_0)$, $X_0 \in \mathcal{F}(T_0)$. Since T_0 is admissible, Theorem 2.3.5 implies $\Phi(T)$ is a tilting B -module. Suppose $T_I = Y_0^{(I)} \oplus X_0^{(I)}$, $T_J = Y_0^{(J)} \oplus X_0^{(J)}$ are two tilting Λ -modules

satisfying $\text{add}(T_I) = \text{add}(T_J)$. It suffices to verify $\text{add}(\Phi(T_I)) = \text{add}(\Phi(T_J))$. By the definition of Φ and the assumptions on T_I, T_J , this is obvious.

Without loss of generality, let $T_1 = Y_1 \oplus X_1 \in \mathcal{T}_{\mathcal{T} \cup \mathcal{F}}$, $T_2 = Y_2 \oplus X_2 \in \mathcal{T}_{\mathcal{T} \cup \mathcal{F}}$ be two basic tilting Λ -modules with $X_1, X_2 \in \mathcal{F}(T_0)$ and $Y_1, Y_2 \in \mathcal{T}(T_0)$, such that $\Phi(T_1) = \Phi(T_2)$. Then by the definition of Φ , we have $\text{Ext}_\Lambda^1(T_0, X_1) = \text{Ext}_\Lambda^1(T_0, X_2)$. While the Tilting Theorem 1.2.7 implies $X_1 = X_2$, so we let $X := X_1 = X_2$. Consider the following two short exact sequences:

$$0 \longrightarrow \text{Tr}_{Y_1} X \longrightarrow Y_1 \longrightarrow Y_1/\text{Tr}_{Y_1} X \longrightarrow 0 \quad (b_1)$$

$$0 \longrightarrow \text{Tr}_{Y_2} X \longrightarrow Y_2 \longrightarrow Y_2/\text{Tr}_{Y_2} X \longrightarrow 0 \quad (b_2).$$

Since $\Phi(T_1) = \Phi(T_2)$, the definition of Φ implies $\text{Hom}_\Lambda(T_0, Y_1/\text{Tr}_{Y_1} X) = \text{Hom}_\Lambda(T_0, Y_2/\text{Tr}_{Y_2} X)$. Since torsion classes are closed under taking factor modules, we have $Y_1/\text{Tr}_{Y_1} X, Y_2/\text{Tr}_{Y_2} X \in \mathcal{T}(T_0)$. Then the Tilting Theorem 1.2.7 forces $Y_1/\text{Tr}_{Y_1} X = Y_2/\text{Tr}_{Y_2} X$. For convenience, let $Y/\text{Tr}_Y X := Y_1/\text{Tr}_{Y_1} X = Y_2/\text{Tr}_{Y_2} X$. Since $\text{Tr}_{Y_i} X$ is generated by X and T_2 is tilting, $\text{Ext}_\Lambda^1(Y_2, \text{Tr}_{Y_1} X) = 0$. Since Y_2 is partial tilting, (b_2) and the right exactness of $\text{Ext}_\Lambda^1(Y_2, -)$ force $\text{Ext}_\Lambda^1(Y_2, Y/\text{Tr}_Y X) = 0$. Then applying the functor $\text{Hom}_\Lambda(Y_2, -)$ to (b_1) , we obtain $\text{Ext}_\Lambda^1(Y_2, Y_1) = 0$. Similarly, we have $\text{Ext}_\Lambda^1(Y_1, Y_2) = 0$. By Remark 2.2.4, we know $X \oplus Y_1 \oplus Y_2$ is tilting. But since $X \oplus Y_1$ and $X \oplus Y_2$ are basic tilting modules, this implies $Y_1 = Y_2$, thus Φ is injective. \square

Lemma 3.3.2. *Suppose $T_0 = P[i] \oplus \tau^{-1}S$ is a BB-tilting module, $B = \text{End}_\Lambda(T_0)$. Then T_0 is admissible.*

Proof. According to the definition of admissible tilting modules and Lemma 1.5.2, to prove that T_0 is admissible, it suffices to show: When $T = S \oplus M$ is a tilting Λ -module with $M \in \mathcal{T}(T_0)$, no indecomposable summand of M is generated by S . Indeed, if there exists some indecomposable summand M_i of M that is generated by S , we obtain an exact sequence

$$0 \rightarrow Z \rightarrow S^r \rightarrow M_i \rightarrow 0.$$

But since S is simple and $\text{Ext}_\Lambda^1(M, S) = 0$, this yields a contradiction. \square

Then according to Proposition 3.3.1, if we take T_0 to be a BB-tilting module and $B = \text{End}_\Lambda(T_0)$, we have a map $\Phi : \mathcal{T}_{\mathcal{T} \cup \mathcal{F}} \rightarrow \mathcal{T}_B$.

Proposition 3.3.3. *When T_0 is a BB-tilting module and $B = \text{End}_\Lambda(T_0)$, the map $\Phi : \mathcal{T}_{\mathcal{T} \cup \mathcal{F}} \rightarrow \mathcal{T}_B$ is a bijection.*

Proof. Without loss of generality, we assume that $T' \in \mathcal{T}_B$ is a basic tilting B -module. Recall that $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in mod B .

If $T' \in \mathcal{Y}(T_0)$, by Theorem 1.2.7, there exists $T \in \mathcal{T}(T_0)$ such that $T' = \text{Hom}_\Lambda(T_0, T)$. It is easy to see that T is tilting and $\Phi(T) = T'$.

If $T' = S' \oplus N'$ with $N' \in \mathcal{Y}(T_0)$, $S' = \text{Ext}_\Lambda^1(T_0, S)$. Theorem 1.2.7 implies there exists $N \in \mathcal{T}(T_0)$ such that $N' = \text{Hom}_\Lambda(T_0, N)$. By Lemma 2.1.4, $\text{Ext}_\Lambda^1(S, N) \cong \text{Ext}_B^2(S', N') = 0$ (❶) and $\text{Hom}_\Lambda(S, N) \cong \text{Ext}_B^1(S', N') = 0$ (❷). By Lemma 2.1.2, $\text{Ext}_\Lambda^1(S, S) \cong \text{Ext}_B^1(S', S') = 0$ (❸) and Lemma 2.1.1 implies $\text{Ext}_\Lambda^1(N, N) \cong \text{Ext}_B^1(N', N') = 0$. So if $\text{Ext}_\Lambda^1(N, S) = 0$, $S \oplus N$ is exceptional and according to the Tilting Theorem 1.2.7, we know it is perfect exceptional and hence tilting. Moreover, it is not difficult to see that $\Phi(S \oplus N) = T'$. Since $T' = S' \oplus N'$ is

basic tilting, let $N = \bigoplus_{i=1}^{n-1} N_i$ be a decomposition of N into indecomposable summands and set $\dim_k \text{Ext}_\Lambda^1(N_i, S) =: r_i$. According to Corollary 1.2.6, consider the following exact sequence

$$0 \rightarrow S^{r_i} \rightarrow M_i \rightarrow N_i \rightarrow 0(c_i)$$

such that the connecting homomorphism $\delta_i : \text{Hom}_\Lambda(S^{r_i}, S) \rightarrow \text{Ext}_\Lambda^1(N_i, S)$ is surjective. Taking the direct sum of (c_i) , we obtain

$$0 \rightarrow S^r \rightarrow M \rightarrow N \rightarrow 0(c)$$

and $M = \bigoplus_{i=1}^{n-1} M_i$. The construction of (c) implies the connecting homomorphism $\delta : \text{Hom}_\Lambda(S^r, S) \rightarrow \text{Ext}_\Lambda^1(N, S)$ is surjective. In view of $\text{Hom}_\Lambda(N, S) = 0$ and **3**, application of the functor $\text{Hom}_\Lambda(-, S)$ to (c) yields the following exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(M, S) \longrightarrow \text{Hom}_\Lambda(S^r, S) \xrightarrow{\delta} \text{Ext}_\Lambda^1(N, S) \longrightarrow \text{Ext}_\Lambda^1(M, S) \longrightarrow 0.$$

Since δ is surjective and $\dim_k \text{Hom}_\Lambda(S^r, S) = \dim_k \text{Ext}_\Lambda^1(N, S) = r$, it follows that δ is an isomorphism and hence $\text{Ext}_\Lambda^1(M, S) = \text{Hom}_\Lambda(M, S) = 0$ (**4**). Since $\text{add} S = \mathcal{F}(T_0)$, $\text{Hom}_\Lambda(M, S) = 0$ implies $M \in \mathcal{T}(T_0)$. Applying the functor $\text{Hom}_\Lambda(T_0, -)$ to (c) yields a short exact sequence:

$$0 \rightarrow \text{Hom}_\Lambda(T_0, M) \rightarrow \text{Hom}_\Lambda(T_0, N) \rightarrow \text{Ext}_\Lambda^1(T_0, S^r) \rightarrow 0(d).$$

Applying the functor $\text{Hom}_B(-, \text{Hom}_\Lambda(T_0, N))$ to (d) , we obtain an exact sequence

$$\text{Ext}_B^1(N', N') \rightarrow \text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, N)) \rightarrow \text{Ext}_B^2(S'^r, N').$$

Since $S' \oplus N'$ is tilting, this implies $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, N)) = 0$. By Lemma 2.1.1, we have $\text{Ext}_\Lambda^1(M, N) \cong \text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, N)) = 0$ (**5**). In view of **1** and **3**, application of the functor $\text{Hom}_\Lambda(S, -)$ to (c) gives $\text{Ext}_\Lambda^1(S, M) = 0$. Applying the functor $\text{Hom}_\Lambda(M, -)$ to (c) , while observing **4** and **5**, we obtain $\text{Ext}_\Lambda^1(M, M) = 0$. Then **4** implies that $S \oplus M$ is exceptional. In view of **2** and **4**, Lemma 2.3.3 (3) implies that each M_i is indecomposable. For any isomorphism $\phi : M_i \rightarrow M_j$, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{r_i} & \xrightarrow{f_i} & M_i & \xrightarrow{g_i} & N_i & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \phi & & \downarrow \psi & & \\ 0 & \longrightarrow & S^{r_j} & \xrightarrow{f_j} & M_j & \xrightarrow{g_j} & N_j & \longrightarrow & 0, \end{array}$$

where the existence of ψ is ensured by **2**. By the Snake Lemma, φ is injective, ψ is surjective and $\ker(\psi) \cong \text{coker}(\varphi)$. Since $\text{coker}(\varphi) \in \text{add} S$, **2** implies ψ is an isomorphism, i.e., $N_i \cong N_j$ and $i = j$. So M is basic and $S \oplus M$ is perfect exceptional and hence tilting. By **2**, (c) is the canonical sequence associated to the torsion pair $(\text{Gen} S, \mathcal{F}(S))$. By the uniqueness of the canonical sequence and the definition of Φ , we have $\Phi(S \oplus M) = T'$. \square

Note that, according to Proposition 1.5.1 and [4, VI. Theorem 5.8], we know when T_0 is APR-tilting, the torsion pair $(\mathcal{T}(T_0), \mathcal{F}(T_0))$ is splitting in mod Λ and hence $\mathcal{T}_{\mathcal{T} \cup \mathcal{F}} = \mathcal{T}_\Lambda$. And in this case, B is a hereditary algebra whose quiver can be obtained by reversing the direction of arrows related to the point associated to S in the quiver of Λ .

Corollary 3.3.4. *Let Λ_1, Λ_2 be the path algebras of two connected acyclic quivers Q_1 and Q_2 , respectively. If Q_1 and Q_2 have the same underlying graph, then Λ_1 and Λ_2 have the same number of non-isomorphic basic tilting modules.*

It was proved in [26]: For a finite dimensional algebra A , \mathcal{T}_A is always countable. Moreover, the number of tilting modules of path algebras of Dynkin diagrams were determined in [21], [43].

Recall the partial order \leq on \mathcal{T}_A is defined via: $T_1 \leq T_2$ if and only if $\mathbb{T}_1^\perp \subset \mathbb{T}_2^\perp$. And any subset of \mathcal{T}_A inherits such a partial order. In [37], the author considered the relationship between $(\mathcal{T}_\Lambda, \leq)$ and (\mathcal{T}_B, \leq) when T_0 is an APR-tilting Λ -module and $B = \text{End}_\Lambda(T_0)$. In this section, we will consider the relationship between $\overrightarrow{\mathcal{K}}_\Lambda$ and $\overrightarrow{\mathcal{K}}_B$ when T_0 is BB-tilting and $B = \text{End}_\Lambda(T_0)$.

Given a partially ordered set (\mathcal{C}, \leq) , we can define a diagram which is called the **Hasse diagram** of the partially ordered set, with vertices are elements of \mathcal{C} , and there exists an arrow from $C_1 \rightarrow C_2$ with $C_1, C_2 \in \mathcal{C}$ if and only if C_1 is a minimal element of $\{D | C_2 \leq D\}$. The following Theorem states the relationship between the tilting quiver $\overrightarrow{\mathcal{K}}_A$ and the Hasse diagram of (\mathcal{T}_A, \leq) .

Theorem 3.3.5. [32, Theorem 2.1] $\overrightarrow{\mathcal{K}}_A$ is the Hasse diagram of (\mathcal{T}_A, \leq) .

Lemma 3.3.6. *Let $0 \longrightarrow X \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} Y \oplus Y' \xrightarrow{(g_1 \ g_2)} Z \longrightarrow 0$ be a non-split short exact sequence. If Z is indecomposable and $\text{Hom}_A(X, Y') = 0$, then $Y' = 0$.*

Proof. Since $\text{Hom}_A(X, Y') = 0$, we have $f_2 = 0$. Since $\ker(g_1 \ g_2) = \text{Im}\begin{pmatrix} f_1 \\ 0 \end{pmatrix} \subset \ker g_1(\text{id}_Y \ 0)$, there exists $h : Z \rightarrow Z$ such that the following diagram with exact rows commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{\begin{pmatrix} f_1 \\ 0 \end{pmatrix}} & Y \oplus Y' & \xrightarrow{(g_1 \ g_2)} & Z \longrightarrow 0 \\ & & \downarrow \text{id}_X & & \downarrow (\text{id}_Y \ 0) & & \downarrow h \\ 0 & \longrightarrow & X & \xrightarrow{f_1} & Y & \xrightarrow{g_1} & Z. \end{array}$$

Since Z is indecomposable, h is either nilpotent or invertible. Since $h(g_1 \ g_2) = g_1(\text{id}_Y \ 0) = (g_1 \ g_2)\begin{pmatrix} \text{id}_Y & 0 \\ 0 & 0 \end{pmatrix}$, $h^i(g_1 \ g_2) = (g_1 \ g_2)\begin{pmatrix} \text{id}_Y & 0 \\ 0 & 0 \end{pmatrix}^i = (g_1 \ 0)$ for $i \geq 1$. If h is nilpotent, then $g_1 = 0$ which implies f_1 is an isomorphism, so that the original sequence splits, a contradiction. If h is invertible, then h is injective and then the Snake Lemma implies $(\text{id}_Y \ 0)$ is injective. Thus, $Y' \subset \ker(\text{id}_Y \ 0) = (0)$, as desired. \square

As has been mentioned in the introduction, $\mathcal{T}_\mathcal{T} \subset \mathcal{T}_\Lambda$ is the subset of \mathcal{T}_Λ consisting of those equivalence classes whose representatives belong to $\mathcal{T}(T_0)$, $\mathcal{T}_\mathcal{F} \subset \mathcal{T}_\Lambda$ is the subset of \mathcal{T}_Λ consisting of those equivalence classes whose representatives belong to $\mathcal{F}(T_0)$. $\mathcal{T}_{\mathcal{T} \cup \mathcal{F}}$ is the subset of \mathcal{T}_Λ consisting of those equivalence classes whose representatives belong to $\mathcal{T}(T_0) \cup \mathcal{F}(T_0)$ and $\mathcal{T}_{\mathcal{T}, \mathcal{F}} \subset \mathcal{T}_\Lambda$ is the subset of \mathcal{T}_Λ consisting of elements in $\mathcal{T}_{\mathcal{T} \cup \mathcal{F}}$ but not in the union of $\mathcal{T}_\mathcal{T}$ and $\mathcal{T}_\mathcal{F}$. Similarly, we have $\mathcal{T}_\mathcal{Y}, \mathcal{T}_\mathcal{X}, \mathcal{T}_{\mathcal{X} \cup \mathcal{Y}}, \mathcal{T}_{\mathcal{X}, \mathcal{Y}} \subset \mathcal{T}_B$.

Note that, when T_0 is BB-tilting, according to Lemma 1.5.2, S is the unique indecomposable module in $\mathcal{F}(T_0)$ and $S' = \text{Ext}_\Lambda^1(T_0, S)$ is the unique indecomposable module in $\mathcal{X}(T_0)$. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting, when T_0 is BB-tilting, $\mathcal{T}_{\mathcal{X}, \mathcal{Y}}$ is just the subset of \mathcal{T}_B consisting of those equivalence classes whose representatives contain S' as a summand. But since $(\mathcal{T}(T_0), \mathcal{F}(T_0))$

may be not splitting, $\mathcal{T}_{\mathcal{T},\mathcal{F}}$ is contained in the subset of \mathcal{T}_Λ consisting of those equivalence classes whose representatives contain S as a summand.

For a quiver Γ , Γ_0 means the set of vertices of Γ . A full subquiver Δ of a quiver Γ is defined to be **convex** in Γ if, for any path $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_t$ in Γ with $x_0, x_t \in \Delta_0$, we have $x_i \in \Delta_0$ for all i such that $0 \leq i \leq t$.

Proposition 3.3.7. *When T_0 is a BB-tilting module, $\overrightarrow{\mathcal{T}_{\mathcal{T},\mathcal{F}}}$ is convex in $\overrightarrow{\mathcal{K}_\Lambda}$, $\overrightarrow{\mathcal{T}_{\mathcal{X},\mathcal{Y}}}$ is convex in $\overrightarrow{\mathcal{K}_B}$.*

Proof. For convenience, in this case we will write \mathcal{T}_S and $\mathcal{T}_{S'}$ instead of $\mathcal{T}_{\mathcal{T},\mathcal{F}}$ and $\mathcal{T}_{\mathcal{X},\mathcal{Y}}$, respectively. Suppose there exists a path $T'_1 \rightarrow \cdots \rightarrow T' \rightarrow \cdots \rightarrow T'_2$ in $\overrightarrow{\mathcal{K}_B}$ such that $T'_1, T'_2 \in \mathcal{T}_{S'}$, i.e., S' is a summand of T'_1 and T'_2 . Hence we can assume T'_1, T'_2 are two basic tilting B -modules of the form: $T'_1 = S' \oplus M'_1, T'_2 = S' \oplus M'_2$. By Theorem 3.3.5, we know that $T'_2 \leq T' \leq T'_1$, i.e., $(S' \oplus M'_2)^\perp \subset (T')^\perp \subset (S' \oplus M'_1)^\perp$, whence $\text{Ext}_B^i(T', S') = \text{Ext}_B^i(S', T') = 0$ for $i \geq 1$. So $S' \oplus T'$ is exceptional, then Remark 2.2.4 implies $\text{add}(T') = \text{add}(S' \oplus T')$. Consequently, S' is a direct summand of T' , so $T' \in \mathcal{T}_{S'}$ and hence $\overrightarrow{\mathcal{T}_{S'}}$ is convex in $\overrightarrow{\mathcal{K}_B}$.

Suppose there exists a path: $T_1 \rightarrow \cdots \rightarrow T \rightarrow \cdots \rightarrow T_n$ in $\overrightarrow{\mathcal{K}_\Lambda}$ such that $T_1, T_n \in \mathcal{T}_S$. We will prove our statement by induction on the length $n - 1$. If $n = 2$, there is nothing to show. We assume our statement to be true for $n = m \geq 2$. Suppose there exists a path: $T_1 \rightarrow T_2 \cdots \rightarrow T \rightarrow \cdots \rightarrow T_{m+1}$ in $\overrightarrow{\mathcal{K}_\Lambda}$ with $T_1, T_{m+1} \in \mathcal{T}_S$. According to the definition of tilting quivers, we may assume $T_1 = M \oplus X, T_2 = M \oplus Y$ are two basic tilting modules with X, Y being indecomposable, and we have a non-split exact sequence

$$0 \longrightarrow X \xrightarrow{f} \overline{M} \xrightarrow{g} Y \longrightarrow 0 (\star)$$

such that $\overline{M} \in \text{add}M$. Since $T_1, T_{m+1} \in \mathcal{T}_S$ and S is the unique indecomposable module in $\mathcal{F}(T_0)$, S is a summand of T_1 and T_{m+1} . By the arguments of the first part of the proof, we know S is a direct summand of T_2 . Again, since $T_1 \in \mathcal{T}_S$ and T_1 is basic, either $X = S$ or S is a summand of M . If $X = S$, the existence of the non-split sequence (\star) implies $\text{Ext}_\Lambda^1(Y, S) \neq 0$. But since S is a direct summand of T_2 and $T_2 = M \oplus Y$, the self-orthogonal property of tilting modules implies $\text{Ext}_\Lambda^1(Y, S) = 0$, a contradiction. Consequently, S is a direct summand of M . By the definition of \mathcal{T}_S and the fact that S is the unique indecomposable module in $\mathcal{F}(T_0)$, $T_1 \in \mathcal{T}_S$ implies S is the unique indecomposable summand of T_1 that does not belong to $\mathcal{T}(T_0)$. Hence $X \in \mathcal{T}(T_0)$, so that $S \in \mathcal{F}(T_0)$ implies $\text{Hom}_\Lambda(X, S) = 0$. Since Y is indecomposable, Lemma 3.3.6 implies S is not a summand of \overline{M} . But $T_1 = M \oplus X \in \mathcal{T}_S$ and S is the unique indecomposable module in $\mathcal{F}(T_0)$, so $\overline{M} \in \mathcal{T}(T_0)$. Since $\mathcal{T}(T_0)$ is closed under taking factors modules, we obtain $Y \in \mathcal{T}(T_0)$. From the definition of \mathcal{T}_S , we now obtain $T_2 \in \mathcal{T}_S$ and the induction hypothesis shows that the path: $T_2 \cdots \rightarrow T \rightarrow \cdots \rightarrow T_{m+1}$ belongs to $\overrightarrow{\mathcal{T}_S}$. Consequently, we have proved our statement. \square

Note that, when $T \in \mathcal{T}(T_0)$ is a tilting Λ -module, the map Φ maps T to $\text{Hom}_\Lambda(T_0, T)$. It is easy to check, when $T' \in \mathcal{Y}(T_0)$ is a tilting B -module, $T' \otimes_B T_0$ is a tilting Λ -module such that $\Phi(T' \otimes_B (T_0)) = T'$. Then by Propositions 3.3.1, 3.3.3, Lemma 1.5.2, when T_0 is BB-tilting, $\Phi_0 = \Phi|_{\mathcal{T}_\mathcal{T}} : \mathcal{T}_\mathcal{T} \rightarrow \mathcal{T}_\mathcal{Y}$ and $\Phi_2 = \Phi|_{\mathcal{T}_{\mathcal{T},\mathcal{F}}} : \mathcal{T}_{\mathcal{T},\mathcal{F}} \rightarrow \mathcal{T}_{\mathcal{X},\mathcal{Y}}$ are bijections. Obviously, Φ_0 is a bijection of partially ordered sets and by [33, Theorem2.7], we know Φ_2 is also a bijection of partially ordered sets.

Theorem 3.3.8. *When T_0 is a BB-tilting module, Φ_0, Φ_2 induce isomorphisms of quivers between $\overrightarrow{\mathcal{T}_\mathcal{T}}$ and $\overrightarrow{\mathcal{T}_\mathcal{Y}}$, $\overrightarrow{\mathcal{T}_{\mathcal{T},\mathcal{F}}}$ and $\overrightarrow{\mathcal{T}_{\mathcal{X},\mathcal{Y}}}$, respectively.*

Proof. The isomorphism between $\overrightarrow{\mathcal{T}}_{\mathcal{T}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{Y}}$ is obvious.

As in the proof of Proposition 3.3.7, we will write $\overrightarrow{\mathcal{T}}_S$ and $\overrightarrow{\mathcal{T}}_{S'}$ instead of $\overrightarrow{\mathcal{T}}_{\mathcal{T},\mathcal{F}}$ and $\overrightarrow{\mathcal{T}}_{\mathcal{X},\mathcal{Y}}$, respectively. Suppose there exists an arrow $T'_1 \rightarrow T'_2$ in $\overrightarrow{\mathcal{T}}_{S'}$ with T'_1, T'_2 being basic tilting. Since $T'_1, T'_2 \in \mathcal{T}_{S'}$, by the definition of tilting quivers and $\mathcal{T}_{S'}$, we assume $T'_1 = S' \oplus C' \oplus C'_1$ and $T'_2 = S' \oplus C' \oplus C'_2$ with C'_1, C'_2 being indecomposable. According to Proposition 3.3.1 and Proposition 3.3.3, Φ_2 is bijective, so we have $T_1 := \Phi_2^{-1}(T'_1), T_2 := \Phi_2^{-1}(T'_2)$. Moreover, according to the proof of Proposition 3.3.3 (exact sequences $(c_i), (c)$ there), we may assume $T_1 = S \oplus X \oplus X_1$ and $T_2 = S \oplus X \oplus X_2$ with X_1, X_2 being indecomposable. Since $T'_2 \leq T'_1$ and Φ_2 is a bijection of partial orders, we have $T_2 \leq T_1$ i.e., $(S \oplus X \oplus X_2)^\perp \subset (S \oplus X \oplus X_1)^\perp$. Consequently, $\text{Ext}_\Lambda^1(X_1, X_2) = 0$, and we claim that $\text{Ext}_\Lambda^1(X_2, X_1) \neq 0$. Otherwise, Remark 2.2.4 implies $\text{add}T_1 = \text{add}(T_1 \oplus X_2) = \text{add}T_2$, which contradicts Φ_2 being injective. Thus $\text{Ext}_\Lambda^1(X_2, X_1) \neq 0$ and [29, Theorem1.1] implies there exists an arrow $T_1 \rightarrow T_2$.

Suppose there exists an arrow $T_1 \rightarrow T_2$ in $\overrightarrow{\mathcal{T}}_S$, let $T'_1 = \Phi_2(T_1), T'_2 = \Phi_2(T_2)$. By Theorem 3.3.5, $T_2 \leq T_1$, and since Φ_2 is a bijection of partially ordered sets, $T'_2 \leq T'_1$. If there exists $T' \in \mathcal{T}_B$ such that $T'_2 \leq T' \leq T'_1$, then by the proof of Proposition 3.3.7, $T' \in \mathcal{T}_{S'}$. Since Φ_2 is a bijection, there exists $T \in \mathcal{T}_S$ such that $T' = \Phi_2(T)$. Since Φ_2 is a bijection of partially ordered sets, $T_2 \leq T \leq T_1$. According to Theorem 3.3.5, this contradicts the minimality of T_1 . So by Theorem 3.3.5, there exists an arrow $T'_1 \rightarrow T'_2$. \square

Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in mod B and when T_0 is BB-tilting, S' is the unique indecomposable B -module in $\mathcal{X}(T_0)$, \mathcal{T}_B is the disjoint union of $\mathcal{T}_{\mathcal{Y}}$ and $\mathcal{T}_{\mathcal{X},\mathcal{Y}}$. Then according to Corollary 3.3.8, when T_0 is BB-tilting, given $\overrightarrow{\mathcal{K}}_\Lambda$ we can obtain all information about $\overrightarrow{\mathcal{K}}_B$ but the arrows between elements of $\mathcal{T}_{\mathcal{Y}}$ and $\mathcal{T}_{\mathcal{X},\mathcal{Y}}$.

Proposition 3.3.9. *Suppose T_0 is a BB-tilting Λ -module. Then*

1. *There are no arrows from $\mathcal{T}_{\mathcal{X},\mathcal{Y}}$ to $\mathcal{T}_{\mathcal{Y}}$ in $\overrightarrow{\mathcal{K}}_B$.*
2. *There are no arrows from $\mathcal{T}_{\mathcal{T}}$ to $\mathcal{T}_{\mathcal{T},\mathcal{F}}$ in $\overrightarrow{\mathcal{K}}_\Lambda$.*

Proof. 1. Let $T'_1 \in \mathcal{T}_{\mathcal{Y}}, T'_2 \in \mathcal{T}_{\mathcal{X},\mathcal{Y}}$ be two basic tilting B -modules. Suppose there exists an arrow from T'_2 to T'_1 , then we can assume $T'_2 = N' \oplus S', T'_1 = N' \oplus Y' \in \mathcal{Y}(T_0)$. By the definition of tilting quivers, there exists a non-split exact sequence $0 \rightarrow S' \rightarrow \overline{N'} \rightarrow Y' \rightarrow 0$ with $\overline{N'} \in \text{add}N'$. If T_0 is APR-tilting, then according to Proposition 1.5.1, S' is injective. Hence the short exact sequence splits, a contradiction. Hence T_0 is not APR-tilting, and Lemma 1.5.2 implies $\text{pd}_B S' = 2$. But since $Y', \overline{N'} \in \mathcal{Y}(T_0)$, Lemma 1.2.9 yields $\text{pd}_B Y', \text{pd}_B \overline{N'} \leq 1$. Then applying the functor $\text{Hom}_B(-, Z')$ to the exact sequence for any B -module Z' , we have $\text{Ext}_B^2(S', Z') = 0$ for any B -module Z' , which implies $\text{pd}_B S' \leq 1$, a contradiction.

2. Let $T_1 \in \mathcal{T}_{\mathcal{T}}, T_2 \in \mathcal{T}_{\mathcal{T},\mathcal{F}}$ be two basic tilting Λ -modules and suppose there exists an arrow from T_1 to T_2 , then we can assume $T_2 = M \oplus S, T_1 = M \oplus Y \in \mathcal{T}(T_0)$. If there exists an arrow from T_1 to T_2 , then there exists a non-split exact sequence $0 \rightarrow Y \rightarrow \overline{M} \rightarrow S \rightarrow 0$ with $\overline{M} \in \text{add}M$. But since $M \in \mathcal{T}(T_0), S \in \mathcal{F}(T_0)$, we have $\text{Hom}_\Lambda(M, S) = 0$. Hence $\text{Hom}_\Lambda(\overline{M}, S) = 0$, a contradiction. \square

Proposition 3.3.10. *Suppose T_0 is a BB-tilting Λ -modules, $T'_1 \in \mathcal{T}_{\mathcal{Y}}, T'_2 \in \overline{\mathcal{T}}_{\mathcal{X},\mathcal{Y}}$ are two basic tilting modules, such that there is an arrow $T'_1 \rightarrow T'_2$ in $\overrightarrow{\mathcal{K}}_B$. Then*

1. Let $T_i = \Phi^{-1}(T'_i)$ for $i = 1, 2$. Then there exists $E_1, E_2, Y \in \mathcal{T}(T_0)$ such that $T_1 = E_1 \oplus Y$, $T_2 = E_2 \oplus S$.

2. There exists an arrow $T_2 \rightarrow T_1$ in $\overrightarrow{\mathcal{K}}_\Lambda$ if and only if $\text{Ext}_\Lambda^1(E_1, E_2) = 0$

Proof. 1. Since $T'_1 \in \mathcal{T}_Y$, $T'_2 \in \mathcal{T}_{X,Y}$ are two basic tilting modules, by the definitions of \mathcal{T}_Y and $\mathcal{T}_{X,Y}$ and the assumption on the existence of the arrow $T'_1 \rightarrow T'_2$, we may assume $T'_1 = E'_1 \oplus Y'$ and $T'_2 = S' \oplus E'_1$ such that $E'_1, Y' \in \mathcal{Y}(T_0)$. Let $E_1 := E'_1 \otimes_B T_0$, $Y := Y' \otimes_B T_0$, then $T_1 = E_1 \oplus Y$ satisfies $\Phi(T_1) = T'_1$. In analogy with the construction of (c) in the proof of Proposition 3.3.3, consider the exact sequence

$$0 \longrightarrow S^r \longrightarrow E_2 \longrightarrow E_1 \longrightarrow 0 (\star).$$

Where $E_1 := E'_1 \otimes_B T_0$ and $r := \dim_k \text{Ext}_\Lambda^1(E_1, S)$, such that the connecting homomorphism $\text{Hom}_\Lambda(S^r, S) \rightarrow \text{Ext}_\Lambda^1(E_1, S)$ is surjective. Then the proof of Proposition 3.3.3 shows $T_2 = E_2 \oplus S$ is as desired.

2. By the definition of tilting quivers, we have a non-split short exact sequence $0 \rightarrow Y' \rightarrow \overline{E'_1} \rightarrow S' \rightarrow 0$ such that $\overline{E'_1} \in \mathbf{add} E'_1$. Applying the functor $- \otimes_B T_0$ to the short exact sequence, Corollary 1.2.8 yields a short exact sequence

$$0 \rightarrow S \rightarrow Y \rightarrow \overline{E_1} \rightarrow 0 (\star\star)$$

such that $\overline{E_1} \in \mathbf{add} E_1$. Since $E_2 \oplus S$ is tilting, $\text{Ext}_\Lambda^1(E_2, E_2) = 0$. Applying the functor $\text{Hom}_\Lambda(E_2, -)$ to (\star) , we have $\text{Ext}_\Lambda^1(E_2, E_1) = 0 (\textcircled{1})$. By the same token, applying the functor $\text{Hom}_\Lambda(E_2, -)$ to $(\star\star)$, we have $\text{Ext}_\Lambda^1(E_2, Y) = 0 (\textcircled{2})$.

If $\text{Ext}_\Lambda^1(E_1, E_2) = 0 (\textcircled{3})$, $\text{Ext}_\Lambda^1(\overline{E_1}, E_2) = 0$. Since $E_2 \oplus S$ is tilting, $\text{Ext}_\Lambda^1(S, E_2) = 0$, so that applying the functor $\text{Hom}_\Lambda(-, E_2)$ to $(\star\star)$, we have $\text{Ext}_\Lambda^1(Y, E_2) = 0 (\textcircled{4})$. Since $E_1 \oplus Y$ is tilting, $\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}$ imply that $E_2 \oplus E_1 \oplus X$ is self-orthogonal. Then Remark 2.2.4 implies $E_2 \in \mathbf{add}(E_1 \oplus Y)$. This means E_2 is a common summand of the two basic tilting modules $E_1 \oplus Y$ and $E_2 \oplus S$. Therefore, [29, Theorem 1.1] provides an arrow between T_1 and T_2 , and by Proposition 3.3.9, we know there exists an arrow from T_2 to T_1 .

If there exists an arrow from T_2 to T_1 , then by the definition of tilting quivers, T_1, T_2 possess a common almost complete tilting module as summand. Since $S \notin \mathbf{add}(E_1 \oplus Y)$, this means E_2 is the common almost complete tilting module, hence $E_2 \in \mathbf{add}(E_1 \oplus Y)$. Since $E_1 \oplus Y$ is tilting, hence is exceptional, we have $\text{Ext}_\Lambda^1(E_1, E_2) = 0$. □

As we have mentioned after Theorem 3.1.2, the condition: A being hereditary is necessary for Theorem 3.1.2. And we can see from the following conclusion.

Proposition 3.3.11. *Suppose that T_0 is a BB-tilting Λ -module, $B = \text{End}_\Lambda(T_0)$. If $M' \in \mathcal{Y}(T_0)$ is a basic almost complete tilting B -module, then M' admits at most three complements.*

Proof. Suppose $M' \oplus Z'$ is a basic tilting B -module. If $Z' \in \mathcal{Y}(T_0)$, let $M \oplus Z \in \mathcal{T}(T_0)$ such that $M' \oplus Z' = \text{Hom}_\Lambda(T_0, M \oplus Z)$. Then according to the Tilting Theorem 1.2.7 and Lemma 2.1.1, we know that $M \oplus Z$ is perfect exceptional if and only if $M' \oplus Z'$ is perfect exceptional. Since $\text{pd}_\Lambda(M \oplus Z) \leq 1$ and $M \oplus Z \in \mathcal{T}(T_0)$, Lemma 1.2.9 implies $\text{pd}_B(M' \oplus Z') \leq 1$. By Proposition 2.2.7, perfect exceptional B modules are tilting (and vice versa). Since Λ is hereditary, according to Theorem 3.1.2, M admits at most two complements belong to $\mathcal{T}(T_0)$.

Then according to the Tilting Theorem 1.2.7, M' admits at most two complements belong to $\mathcal{Y}(T_0)$. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in mod B and S' is the unique indecomposable module in $\mathcal{X}(T_0)$, M' admits at most three complements. \square

Remark 3.3.12. When T_0 is BB-tilting, let $M' \in \mathcal{Y}(T_0)$ be an almost complete tilting B -module, such that $M' \oplus X' \in \mathcal{T}_y$, $M' \oplus S' \in \mathcal{T}_{x,y}$ are two basic tilting B -modules. According to Lemma 3.1.3 and Proposition 3.3.11, there exist either an arrow $M' \oplus X' \rightarrow M' \oplus S'$ or a directed path $M' \oplus X' \rightarrow M' \oplus Y' \rightarrow M' \oplus S'$ in $\overrightarrow{\mathcal{K}}_B$.

Proposition 3.3.13. Suppose T_0 is a BB-tilting Λ -modules, $T_1 \in \mathcal{T}_T$, $T_2 \in \mathcal{T}_{T,\mathcal{F}}$ are two basic tilting modules, such that there is an arrow $T_2 \rightarrow T_1$ in $\overrightarrow{\mathcal{K}}_\Lambda$. Then

1. Let $T'_i = \Phi(T_i)$ for $i = 1, 2$. Then there exists $M', N', Y' \in \mathcal{Y}(T_0)$ such that $T'_1 = M' \oplus Y'$, $T_2 = N' \oplus S'$.

2. There exists an arrow $T'_1 \rightarrow T'_2$ in $\overrightarrow{\mathcal{K}}_B$ if and only if $\text{Ext}_\Lambda^1(N, M) = 0$, where $N, M \in \mathcal{T}(T_0)$ such that $\text{Hom}_\Lambda(T_0, N) = N'$, $\text{Hom}_\Lambda(T_0, M) = M'$.

Proof. 1. Since $T_1 \in \mathcal{T}_T$, $T_2 \in \mathcal{T}_{T,\mathcal{F}}$ are two basic tilting modules, and there exists an arrow $T_2 \rightarrow T_1$ in $\overrightarrow{\mathcal{K}}_\Lambda$, we assume $T_1 = M \oplus Y$, $T_2 = M \oplus S$ such that $M, Y \in \mathcal{T}(T_0)$. Let $M' := \text{Hom}_\Lambda(T_0, M)$, $Y := \text{Hom}_\Lambda(T_0, Y)$, then $T'_1 = M' \oplus Y'$ satisfies $T'_1 = \Phi(T_1)$. By the definition of Φ , $T'_2 = \Phi(T_2) = \text{Hom}_\Lambda(T_0, N) \oplus S'$. Where N is in the following exact sequence

$$0 \longrightarrow S^r \longrightarrow M \longrightarrow N \longrightarrow 0(\star)$$

such that $\dim_k \text{Hom}_\Lambda(S, M) = r$ and $\text{Hom}_\Lambda(S, S^r) \rightarrow \text{Hom}_\Lambda(S, M)$ is surjective. Let $N' := \text{Hom}_\Lambda(T_0, N)$, our statements follow.

2. The existence of the arrow $T_2 \rightarrow T_1$ yields a non-split short exact sequence

$$0 \rightarrow S \rightarrow \overline{M} \rightarrow Y \rightarrow 0(\star\star)$$

with $\overline{M} \in \text{add}M$. Since $M \oplus Y$ is tilting, $\text{Ext}_\Lambda^1(M, M) = \text{Ext}_\Lambda^1(Y, M) = 0$. Applying the functor $\text{Hom}_\Lambda(M, -)$ to (\star) , we have $\text{Ext}_\Lambda^1(M, N) = 0$. Applying the functor $\text{Hom}_\Lambda(Y, -)$ to (\star) , we have $\text{Ext}_\Lambda^1(Y, N) = 0$. Then according to Lemma 2.1.1, we have $\text{Ext}_B^1(M', N') = \text{Ext}_B^1(Y', N') = 0(\bullet)$.

If $\text{Ext}_\Lambda^1(N, M) = 0$, then $\text{Ext}_\Lambda^1(N, \overline{M}) = 0$. Applying the functor $\text{Hom}_\Lambda(N, -)$ to $(\star\star)$, we have $\text{Ext}_\Lambda^1(N, Y) = 0$. According to Lemma 2.1.1, then we have $\text{Ext}_B^1(N', M') = \text{Ext}_B^1(N', Y') = 0(\bullet)$. Since $M' \oplus Y'$ is tilting, N' is partial tilting and $\text{pd}_B N', \text{pd}_B(M' \oplus Y') \leq 1$, \bullet , \bullet imply that $N' \oplus M' \oplus Y'$ is self-orthogonal. Then Remark 2.2.4 implies $N' \in \text{add}(M' \oplus Y')$. But by our assumptions, N' and $M' \oplus Y'$ are basic modules with $n-1$ and n indecomposable summands, respectively. So N' is a summand of $M' \oplus Y'$ and either $N' \cong M'$ or Y' is a summand of N' . If $N' \cong M'$, then $N \cong M$ and (\star) implies $\text{Hom}_\Lambda(S, M) = 0$, a contradiction. So Y' is a summand of N' , and we can write $T'_1 = M' \oplus Y' =: N' \oplus X'$ with $X' \in \text{add}M'(\bullet)$.

If there exists no arrow from T'_1 to T'_2 , then according to Remark 3.3.12, we can find a basic tilting B -module $T' = N' \oplus Z'$ such that $T'_1 \rightarrow T' \rightarrow T'_2$ is a full subquiver of $\overrightarrow{\mathcal{K}}_B$. According to the definition of tilting quivers, there exist two non-split short exact sequences

$$0 \rightarrow X' \rightarrow \overline{N}'_1 \rightarrow Z' \rightarrow 0(\star\star\star)$$

and

$$0 \rightarrow Z' \rightarrow \overline{N'}_2 \rightarrow S' \rightarrow 0(\star\star\star)$$

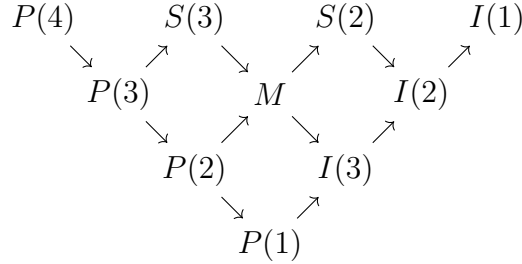
such that $\overline{N'}_1, \overline{N'}_2 \in \mathbf{add}N'$. Since $M \oplus S$ is tilting, $\mathrm{Ext}_\Lambda^1(S, M) = 0$, and Lemma 2.1.4 implies $\mathrm{Ext}_B^2(S', M') = 0$. Then $\textcircled{3}$ implies $\mathrm{Ext}_B^2(S', X') = 0$. Since $S' \oplus N'$ is tilting, $\mathrm{Ext}_B^1(S', N') = 0$. Applying the functor $\mathrm{Hom}_B(S', -)$ to $(\star\star\star)$, we have $\mathrm{Ext}_B^1(S', Z') = 0$ which contradicts $(\star\star\star)$ being non-split. So there exists an arrow from T'_1 to T'_2 .

Suppose there exists an arrow from T'_1 to T'_2 . Since $T'_1 \in \mathcal{T}_y$, N' is the common almost complete tilting module of $T'_1 = N' \oplus S'$ and $T'_2 = M' \oplus Y'$. Since T'_1 is self-orthogonal, $\mathrm{Ext}_B^1(N', M') = 0$. Then by Lemma 2.1.1 we have $\mathrm{Ext}_\Lambda^1(N, M) = 0$, so that our conclusion follows. \square

3.4 Examples

In the end of this chapter, we will illustrate our results by some examples. First of all, we will give an example to show how to construct the tilting quiver in general.

Example 3.4.1. *Let Λ be the path algebra of the quiver: $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, M be the indecomposable module with dimension vector $(0, 1, 1, 0)$ (cf.[4, III. Section 3]). Then the AR quiver of Λ is as follows:*



Let us start with the regular tilting Λ -module $T_{00} := \Lambda_\Lambda$, according to Corollary 3.2.7, all immediate successors of T_{00} in $\overrightarrow{\mathcal{K}}_\Lambda$ are of the form $P[i] \oplus \tau^{-1}S(i)$. Since $S(1)$ is injective, there is no need to consider it. Let us take $T_1 := P[4] \oplus \tau^{-1}S(4) = P[4] \oplus S(3)$ for example. It has four indecomposable summands and because for hereditary algebras, being tilting is the same as being perfect exceptional, we just need to check if it is self-orthogonal. While $\mathrm{Ext}_\Lambda^1(P[4] \oplus S(3), P[4] \oplus S(3)) = \mathrm{Ext}_\Lambda^1(S(3), S(3) \oplus P(2) \oplus P(3))$. By Corollary 1.4.2, this is isomorphic to $D \mathrm{Hom}_\Lambda(\tau^{-1}(S(3) \oplus P(2) \oplus P(3)), S(3)) = D \mathrm{Hom}_\Lambda(S(2) \oplus I(3) \oplus M, S(3))$. According to the AR quiver of Λ (there exist no paths from $S(2), I(3), M$ to $S(3)$), we know $\mathrm{Hom}_\Lambda(S(2) \oplus I(3) \oplus M, S(3)) = 0$. Hence T_1 is self-orthogonal and hence tilting, by the same token, we can check $T_2 = P[3] \oplus \tau^{-1}S(3)$ and $T_3 = P[2] \oplus \tau^{-1}S(2)$ are also tilting. Thus, we obtain all immediate successors of Λ_Λ .

Next, let us consider the immediate successors of $T_1 = P(1) \oplus P(2) \oplus P(3) \oplus S(3)$. It has four almost complete tilting modules as summands: $P(1) \oplus P(2) \oplus P(3)$, $P(1) \oplus P(2) \oplus S(3)$, $P(1) \oplus P(3) \oplus S(3)$ and $P(2) \oplus P(3) \oplus S(3)$. According to Theorem 3.1.2, each of them has at most two complements, and it has two complements if and only if it is faithful.

Since we have obtained the two complements of $P(1) \oplus P(2) \oplus P(3)$: $P(4), S(3)$ already, T_1 doesn't have immediate successors that admit $P(1) \oplus P(2) \oplus P(3)$ as a summand.

Since the dimension vector of $P(1) \oplus P(2) \oplus S(3)$ is $(1, 2, 3, 2)$, we got to know $P(1) \oplus P(2) \oplus S(3)$ is sincere and hence faithful. So T_1 has an immediate successors T_4 such that

$P(1) \oplus P(2) \oplus S(3)$ is a summand of T_4 . By the existence of the almost split sequence:

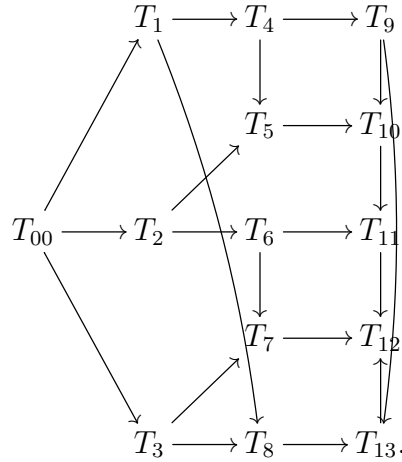
$$0 \longrightarrow P(3) \longrightarrow P(2) \oplus S(3) \longrightarrow M \longrightarrow 0$$

and Lemma 3.1.3, we know $T_4 = P(1) \oplus P(2) \oplus S(3) \oplus M$ is the immediate successor of T_1 corresponding to $P(1) \oplus P(2) \oplus S(3)$.

Since the dimension vector of $P(1) \oplus P(3) \oplus S(3)$ is $(1, 1, 3, 2)$, $P(1) \oplus P(3) \oplus S(3)$ is sincere and hence faithful. So T_1 has an immediate successor such that $P(1) \oplus P(3) \oplus S(3)$ is a direct summand of it. Suppose $P(1) \oplus P(3) \oplus S(3) \oplus X$ is tilting, since $P(1) \oplus P(3) \oplus S(3)$ is self-orthogonal, we have $\text{Ext}_\Lambda^1(P(1) \oplus P(3) \oplus S(3) \oplus X, P(1) \oplus P(3) \oplus S(3) \oplus X) \cong \text{Ext}_\Lambda^1(S(3), X) \oplus \text{Ext}_\Lambda^1(X, P(3)) \oplus \text{Ext}_\Lambda^1(X, S(3)) \oplus \text{Ext}_\Lambda^1(X, X) = 0$. Then Corollary 1.4.2 forces $\text{Hom}_\Lambda(X, P(4)) = \text{Hom}_\Lambda(M, X) = \text{Hom}_\Lambda(S(2), X) = 0$. According to the AR-quiver of Λ , we know X has only two choices: $P(2)$, $I(1)$. Hence $T_8 = P(1) \oplus P(3) \oplus S(3) \oplus I(1)$ is the immediate successor of T_1 corresponding to $P(1) \oplus P(3) \oplus S(3)$.

Since the dimension vector of $P(2) \oplus P(3) \oplus S(3)$ is $(0, 1, 3, 2)$, $P(2) \oplus P(3) \oplus S(3)$ is not sincere and hence not faithful. So T_1 doesn't have immediate successors that admit $P(1) \oplus P(3) \oplus S(3)$ as a summand.

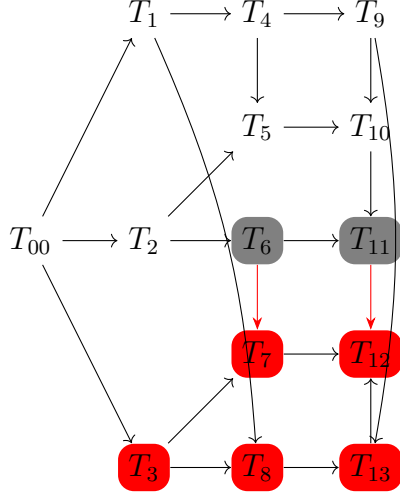
By the same token, we can obtain the tilting quiver $\overrightarrow{\mathcal{K}}_\Lambda$ as follows:



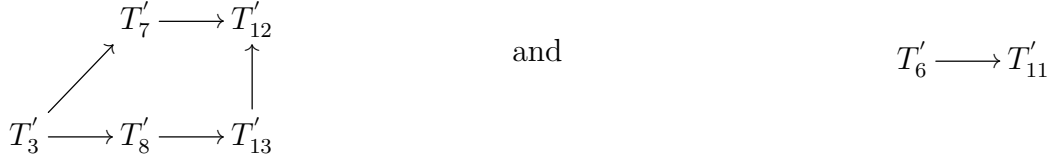
Now we will give an example to show: For some special cases, we can use our results in Chapter 3 to construct the full shape of $\overrightarrow{\mathcal{K}}_B$.

Example 3.4.2. As before, let Λ be the path algebra of the quiver $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, we take T_0 to be $T_0 := P[2] \oplus \tau^{-1}S(2)$, which is BB-tilting but not APR-tilting and $B = \text{End}_\Lambda(T_0)$.

Then $\mathcal{T}_\mathcal{T} = \{T_3, T_7, T_8, T_{12}, T_{13}\}$, $\mathcal{T}_{\mathcal{T}, \mathcal{F}} = \{T_6, T_{11}\}$ which are marked by \bullet and \bullet , respectively. And the arrows between them are marked by red arrows.

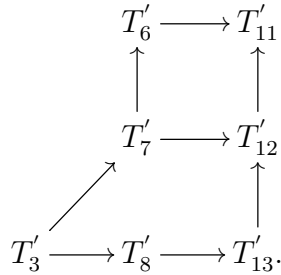


According to Proposition 3.3.8, we know $\overrightarrow{\mathcal{K}}_B$ is the union of



and some more arrows between them.

By computations in the first example, we got to know $T_6 = P(1) \oplus S(2) \oplus P(4) \oplus I(2)$, $T_7 = P(1) \oplus I(1) \oplus P(4) \oplus I(2)$. And according to the definition of Φ , we know $T'_6 = \Phi(T_6) = \text{Hom}_\Lambda(T_0, P(1) \oplus P(4) \oplus I(1)) \oplus \text{Ext}_\Lambda^1(T_0, S(2))$, $T'_7 = \Phi(T_7) = \text{Hom}_\Lambda(T_0, P(1) \oplus I(1) \oplus P(4) \oplus I(2))$. Since there exist arrows from T_6 to T_7 and $\text{Ext}_\Lambda^1(P(1) \oplus P(4) \oplus I(1), P(1) \oplus P(4) \oplus I(2)) = 0$, according to Proposition 3.3.13, we know there exists an arrow from T'_7 to T'_6 in $\overrightarrow{\mathcal{K}}_B$. By the same token, there exists an arrow from T'_{12} to T'_{11} in $\overrightarrow{\mathcal{K}}_B$. By Remark 3.3.12, we know T'_6, T'_{11} have only one predecessor in \mathcal{T}'_y , so the tilting quiver of B should be as :

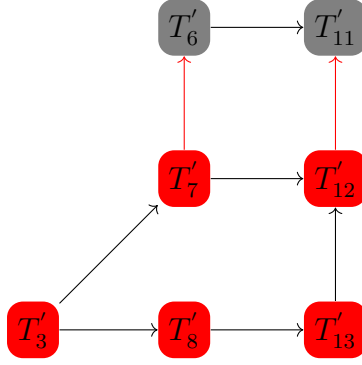


According to [4, VI. Lemma3.10], we know the dimension vectors of indecomposable projective B -modules $P'_1 = \text{Hom}_\Lambda(T_0, I(1))$, $P'_2 = \text{Hom}_\Lambda(T_0, P(1))$, $P'_3 = \text{Hom}_\Lambda(T_0, P(3))$, $P'_4 = \text{Hom}_\Lambda(T_0, P(4))$ are $(1, 1, 0, 0)$, $(0, 1, 1, 1)$, $(0, 0, 1, 1)$, $(0, 0, 0, 1)$. Then we know $B = \text{End}_\Lambda(T_0)$ is given by the path algebra of the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

bound by $\alpha\beta = 0$.

Similar to the computations of $\overrightarrow{\mathcal{K}}_\Lambda$ in Example 3.4.1, we can obtain the tilting quiver $\overrightarrow{\mathcal{K}}_B$ is as:

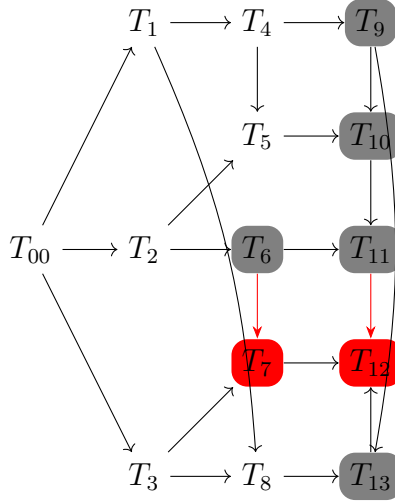


which is the same as the tilting quiver obtained by our results. Elements of \mathcal{T}_y and elements of $\mathcal{T}_{x,y}$ are marked by \bullet and \bullet , respectively. And arrows between them are marked with red arrows.

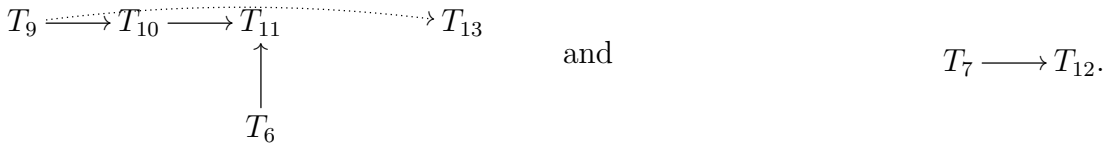
In the following example, we will find an admissible tilting module which is not BB-tilting. And this example shows: Our assumption: B being BB tilted is necessary for Theorem 3.3.8.

Example 3.4.3. Let us take T_0 to be $T_0 = P(1) \oplus P(4) \oplus I(1) \oplus I(2)$, which is not BB-tilting and $B = \text{End}_\Lambda(T_0)$.

Then $\mathcal{T}_T = \{T_7, T_{12}\}$, $\mathcal{T}_{T,\mathcal{F}} = \{T_6, T_9, T_{10}, T_{11}, T_{13}\}$, it's not difficult to check that T_0 is admissible. Let us mark elements of \mathcal{T}_T and $\mathcal{T}_{T,\mathcal{F}}$ by \bullet and \bullet , respectively:



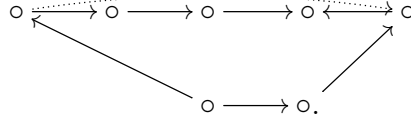
If Theorem 3.3.8 works, the union of the following two graphs should be a subquiver of $\overrightarrow{\mathcal{K}}_B$:



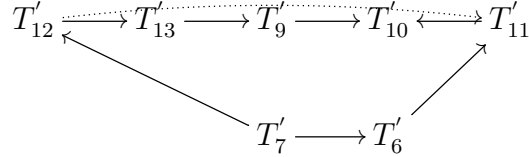
However, by same statements as in Example 3.4.2, we know $B = \text{End}_\Lambda(T_0)$ is given by the path algebra of the quiver

$$\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ \xrightarrow{\gamma} \circ$$

bound by $\beta\gamma = 0$. And similar computations as in Example 3.4.1 show the tilting quiver $\overrightarrow{\mathcal{K}}_B$ is as follows:



For convenience, we will write it in this way:

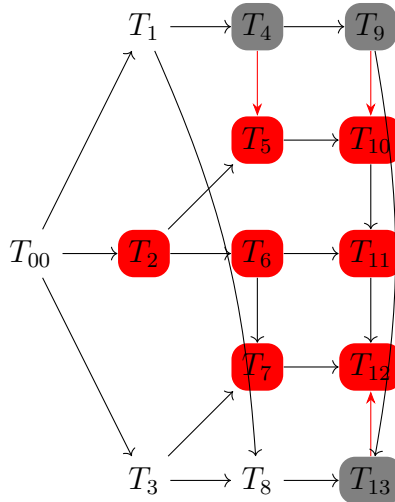


such that $T'_i = \Phi(T_i)$, for $i = 6, 7, 9, 10, 11, 12, 13$. Obviously, Theorem 3.3.8 does not work any more.

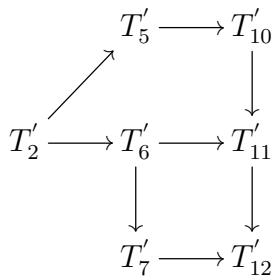
As we can see from Example 3.4.2, in some special cases, we can totally reconstruct the tilting quiver $\overrightarrow{\mathcal{K}}_B$ by using our results. But we need to know in general, even when B is BB-tilted we can only obtain most part of the tilting quiver. And we will end this chapter with the following example.

Example 3.4.4. If we take T_0 to be $T_0 = P[3] \oplus \tau^{-1}S(3) = P(1) \oplus P(2) \oplus P(4) \oplus \tau^{-1}S(3)$, which is BB-tilting but not APR-tilting.

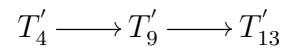
Then $\mathcal{T}_{\mathcal{T}} = \{T_2, T_5, T_6, T_7, T_{10}, T_{11}, T_{12}\}$, $\mathcal{T}_{\mathcal{T}, \mathcal{F}} = \{T_4, T_9, T_{13}\}$ which are marked by \bullet and \bullet , respectively. And the arrows between them are marked by red arrows.



According to Theorem 3.3.8, we know $\overrightarrow{\mathcal{K}}_B$ is the union of

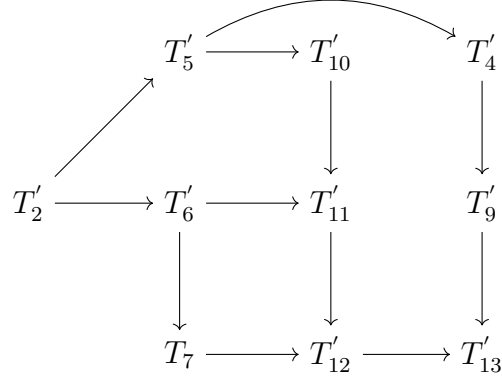


and



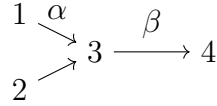
and some more arrows between them. By computations in Example 3.4.1, $T_4 = P(1) \oplus P(2) \oplus$

$M \oplus S(3)$, $T_5 = P(1) \oplus P(2) \oplus M \oplus S(2)$. Then by the definition of Φ , we know $T'_4 = \text{Hom}_\Lambda(T_0, P(1) \oplus P(2) \oplus S(2)) \oplus \text{Ext}_\Lambda^1(T_0, S(3))$, $T'_5 = \text{Hom}_\Lambda(T_0, P(1) \oplus P(2) \oplus M \oplus S(2))$. Since there exists an arrow $T_4 \rightarrow T_5$ in $\overrightarrow{\mathcal{K}}_\Lambda$ and $\text{Ext}_\Lambda^1(P(1) \oplus P(2) \oplus S(2), P(1) \oplus P(2) \oplus M) = 0$, according to Proposition 3.3.13, there exists an arrow $T'_5 \rightarrow T'_4$ in $\overrightarrow{\mathcal{K}}_B$. By the same token, there exists an arrow $T'_{12} \rightarrow T'_{13}$ in $\overrightarrow{\mathcal{K}}_B$. However, $T_9 = P(1) \oplus I(3) \oplus M \oplus S(3)$, $T_{10} = P(1) \oplus I(3) \oplus M \oplus S(2)$. Hence $T'_9 = \text{Hom}_\Lambda(T_0, P(1) \oplus I(2) \oplus S(2)) \oplus \text{Ext}_\Lambda^1(T_0, S(3))$, $T'_{10} = \text{Hom}_\Lambda(T_0, P(1) \oplus I(3) \oplus M \oplus S(2))$. Since $\text{Ext}_\Lambda^1(P(1) \oplus I(2) \oplus S(2), P(1) \oplus I(3) \oplus M) \cong D \text{Hom}_\Lambda(M, M) \neq 0$, Proposition 3.3.13 implies there exists no arrow from T'_{10} to T'_9 . So by our results, we can know

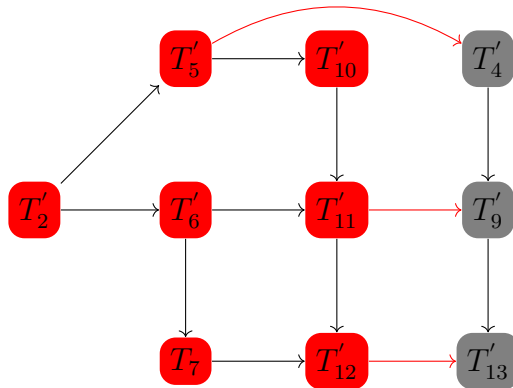


is a subquiver of $\overrightarrow{\mathcal{K}}_B$ and there exists no arrow from T'_{10} to T'_9 .

According to the same statements as in Example 3.4.2, we know $B = \text{End}_\Lambda(T_0)$ is the path algebra of the quiver



bound by $\alpha\beta = 0$. And by the same computations as in Example 3.4.1, we can obtain the tilting quiver $\overrightarrow{\mathcal{K}}_B$ is as:



Elements of \mathcal{T}_T and elements of $\mathcal{T}_{T,\mathcal{F}}$ are marked by \bullet and \bullet , respectively. And arrows between them are marked with red arrows. Then we can see: our results can really obtain "most" parts of $\overrightarrow{\mathcal{K}}_B$ but the arrow $T'_{11} \rightarrow T'_9$.

4 τ tilting theory

Cluster-tilting theory was introduced in [13], which brought a new perspective to this subject. In [13], the authors proved that: Given a finite-dimensional path algebra kQ , for the associated cluster category \mathcal{C}_Q , every almost complete cluster-tilting object has exactly two complements. But as we can see from Theorem 3.1.2, not every almost complete tilting module admits two complements. Considering these facts, in [1], the authors introduced the τ -tilting theory. Which can be considered as a generalization of classic tilting modules and support tilting modules ([34]) from the perspective of mutations.

In analogy with the tilting theory, in [1], the authors also introduced the support τ -tilting quiver $Q(s\tau\text{-tilt}A)$ for a given algebra A and proved some analogous properties. It is quite normal for us to ask: Can we do the same research as in Chapter 3, but in the setting of τ -tilting theory? In fact, this had been considered in [14]. There, the authors compared basic support τ -tilting pairs of A and that of B and the relationship between the support τ -tilting quiver of A and that of B , where (A, T, B) is a **separating and splitting tilting triple**, i.e., T is a classic tilting A -module, $B = \text{End}_A(T)$ such that $(\mathcal{T}(T), \mathcal{F}(T))$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ are splitting in $\text{mod } A$ and $\text{mod } B$, respectively.

In this chapter, we will try to reconstruct the support τ -tilting quiver $Q(s\tau\text{-tilt}B)$ from the support τ -tilting quiver $Q(s\tau\text{-tilt}\Lambda)$, when B is a BB-tilted algebra of type Λ .

4.1 Definitions and basic properties

Recall that $|M|$ denotes the number of isomorphism classes of indecomposable summands of M , in particular, $|A|$ equals the number of isomorphism classes of simple A -modules. Recall that for an idempotent e of A , $\langle e \rangle$ denotes the two side ideal generated by e , i.e., $\langle e \rangle = AeA$.

Definition 4.1.1. [1, Definition 0.1]

1. We call M in $\text{mod } A$ **τ -rigid** if $\text{Hom}_A(M, \tau M) = 0$.
2. We call M in $\text{mod } A$ **τ -tilting** (respectively, **almost complete τ -tilting**) if M is τ -rigid and $|M| = |A|$ (respectively, $|M| = |A| - 1$).
3. We call M in $\text{mod } A$ **support τ -tilting** if there exists an idempotent e of A such that M is a τ -tilting $(A/\langle e \rangle)$ -module.

Definition 4.1.2. [1, Definition 0.3] Let (M, P) be a pair with $M \in \text{mod } A$ and P is projective.

1. We call (M, P) a **τ -rigid pair** if M is τ -rigid and $\text{Hom}_A(P, M) = 0$.
2. We call (M, P) a **support τ -tilting pair** (respectively, **almost complete support τ -tilting pair**) if (M, P) is τ -rigid and $|M| + |P| = |A|$ (respectively, $|M| + |P| = |A| - 1$).

We say (M, P) is **basic** if M and P are basic, and (M, P) is a **direct summand** of (N, Q) if M is a direct summand of N and P is a direct summand of Q .

Proposition 4.1.3. [1, Proposition 2.3] *Let (M, P) be a pair with $M \in \text{mod } A$ and P is projective, e be an idempotent of A such that $\text{add}P = \text{adde}A$.*

1. (M, P) is a τ -rigid (respectively, support τ -tilting, almost complete support τ -tilting) pair for A if and only if M is a τ -rigid (respectively, support τ -tilting, almost complete support τ -tilting) $(A/\langle e \rangle)$ -module.

2. If (M, P) and (M, Q) are support τ -tilting pairs for A , then $\text{add}P = \text{add}Q$. In other words, M determines P and e uniquely.

Remark. Throughout this chapter, sometimes we will say support τ -tilting pairs, but sometimes we just say support τ -tilting modules without mentioning the corresponding projective modules. But the second statement in Proposition 4.1.3 indicates this won't cause any ambiguity.

Proposition 4.1.4. [1, Proposition 1.3] *Any τ -rigid A -module (respectively, τ -rigid pair) M (respectively, (M, P)) satisfies $|M| \leq |A|$ (respectively, $|M| + |P| \leq |A|$).*

The following result is one of the most important results in τ -tilting theory, which is also one of its motivations.

Theorem 4.1.5. [1, Theorem 2.18] *Let A be a finite-dimensional k -algebra. Then any basic almost complete support τ -tilting pair (U, Q) for A is a direct summand of exactly two basic support τ -tilting pairs (T, P) and (T', P') for A .*

4.2 Maps of support τ -tilting modules

In this section, we will discuss the relationship between support τ -tilting Λ -modules and support τ -tilting B -modules, where Λ is basic hereditary and $B = \text{End}_\Lambda(T_0)$ with T_0 being a BB-tilting Λ -module.

For an algebra A , we use τ -rigid A pairs to denote the isomorphism classes of basic τ -rigid A pairs, $s\tau\text{-tilt}A$ to denote the isomorphism classes of basic support τ -tilting A -modules. And for a module M , M_P denotes the projective part of M , i.e., any projective direct summand of M is a direct summand of M_P . The following theorem is one of the main results in [14].

Theorem 4.2.1. [14, Theorem 4.1] *Let (A, T, B) be a separating and splitting tilting triple. Then*

1. *There is a bijection $\phi : \tau\text{-rigid } A \text{ pairs} \rightarrow \tau\text{-rigid } B \text{ pairs}$, given by*

$$(M \oplus N, P) \mapsto (\text{Hom}_A(T, M) \oplus \text{Ext}_A^1(T, \tau_A^{-1}N) \oplus \text{Ext}_A^1(T, P), \text{Hom}_A(T, P) \oplus \text{Hom}_A(T, \tau_A^{-1}N))$$

where $M \in \mathcal{T}(T)$, $N \in \mathcal{F}(T)$, and P is projective. The inverse map $\psi : \tau\text{-rigid } B \text{ pairs} \rightarrow \tau\text{-rigid } A \text{ pairs}$ is given by

$$(X \oplus Y, Q) \mapsto (Y \otimes_B T \oplus \tau_A \text{Tor}_1^B(X, T) \oplus \tau_A(Q \otimes_B T), \text{Tor}_1^B(X, T)_P \oplus (Q \otimes_B T)_P).$$

2. *The restriction of ϕ gives a bijection $s\tau\text{-tilt}A \rightarrow s\tau\text{-tilt}B$.*

According to [4, VI. Theorem 5.6] and Lemma 1.5.2, if we take A to be a hereditary algebra, $T = P[i] \oplus \tau^{-1}S$ to be an APR-tilting A -module, $B = \text{End}_A(T)$, then (A, T, B) satisfies the assumption of Theorem 4.2.1. Note that when T is APR-tilting, $(\mathcal{T}(T), \mathcal{F}(T))$ is splitting and $\mathcal{F}(T) = \text{add}S$. In this case, ϕ restricted on $s\tau\text{-tilt}A$ is defined as:

$$\phi(M, P) = \begin{cases} (\text{Hom}_A(T, M) \oplus \text{Ext}_A^1(T, P), \text{Hom}_A(T, P)) & S \notin \text{add}M \\ (\text{Hom}_A(T, M/S), \text{Hom}_A(T, P) \oplus \text{Hom}_A(T, \tau^{-1}S)) & S \in \text{add}M. \end{cases}$$

According to [4, VI. Theorem 5.6] and Lemma 1.5.2, we know when T is BB-tilting but not APR-tilting, $(\mathcal{T}(T), \mathcal{F}(T))$ is not splitting in $\text{mod } A$ any more. Hence the triple (A, T, B) doesn't satisfy the assumption. And unfortunately, in general, the preceding map doesn't work any more.

For example: If we take A to be the path algebra of the quiver: $3 \rightarrow 2 \rightarrow 1$, $T = P[2] \oplus \tau^{-1}S(2)$, $B = \text{End}_A(T)$. Then obviously, T is BB-tilting but not APR-tilting. Consider the pair $(P(1), P(2) \oplus P(3))$, it is easy to see that it is a support τ -tilting pair. If the map works, according to the definition,

$$\phi(P(1), P(2) \oplus P(3)) = (\text{Hom}_A(T, P(1)) \oplus \text{Ext}_A^1(T, P(2)), \text{Hom}_A(T, P(2)) \oplus \text{Hom}_A(T, P(3))).$$

We will show this is not a support τ -tilting pair. Consider the short exact sequence $0 \rightarrow P(1) \rightarrow P(2) \rightarrow S(2) \rightarrow 0$. Applying the functor $\text{Hom}_A(T, -)$ to it and using the fact that $S(2) \in \mathcal{F}(T)$, we obtain an isomorphism of B -modules: $\text{Hom}_A(T, P(2)) \cong \text{Hom}_A(T, P(1))$. Hence $\text{Hom}_B(\text{Hom}_A(T, P(2)), \text{Hom}_A(T, P(1))) \cong \text{Hom}_B(\text{Hom}_A(T, P(1)), \text{Hom}_A(T, P(1)))$. According to Lemma 2.1.1, this is isomorphic to $\text{Hom}_A(P(1), P(1))$ which is obviously not zero. Then we have $\text{Hom}_B(\text{Hom}_A(T, P(2)), \text{Hom}_A(T, P(1))) \neq 0$ and $\phi(P(1), P(2) \oplus P(3))$ is not a τ -rigid pair and hence not a support τ -tilting pair.

Remark. From now on, we assume that Λ is a basic hereditary algebra, $T_0 = P[i] \oplus \tau^{-1}S$ is a BB-tilting but not APR-tilting Λ -module and $B = \text{End}_\Lambda(T_0)$. And in general, we will use M to denote a Λ -module and M' to denote a B -module.

Then according to Lemma 1.5.2, $\mathcal{F}(T_0) = \text{add}S$. Suppose (M, P) is a basic support τ -tilting pair for Λ , then one of the following cases occurs:

- case (a):** $M \in \mathcal{T}(T_0)$ and $P(S) \notin \text{add}P$, i.e., $P \in \text{add}P[i]$;
- case (b):** $M \in \mathcal{T}(T_0)$ and $P(S) \in \text{add}P$;
- case (c):** $S \in \text{add}M$ and $M/S \in \mathcal{T}(T_0)$;
- case (d):** there exists an indecomposable $Z \in \text{mod } \Lambda$ such that $Z \in \text{add}M$ and $Z \notin \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$.

Remark 4.2.2. Since $\text{Hom}_\Lambda(P(S), S) \neq 0$, $(S, P(S))$ is not a τ -rigid pair and hence cannot be a direct summand of any support τ -tilting pair. So if (M, P) is a basic support τ -tilting pair for Λ of case (c), then $P \in \text{add}P[i]$.

For convenience, we will denote $S' := \text{Ext}_\Lambda^1(T_0, S)$, $P[i]' := \text{Hom}_\Lambda(T_0, P[i])$. Note that Lemma 1.5.2 shows $\mathcal{X}(T_0) = \text{add}S'$ and [10, 1.7] implies $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting. So for a given basic support τ -tilting pair (M', P') for $\text{mod } B$, one of the following cases occurs:

- case (1):** $M' \in \mathcal{Y}(T_0)$ and $\text{Hom}_\Lambda(T_0, \tau^{-1}S) \notin \text{add}P'$, i.e., $P' \in \text{add}P[i]'$;
- case (2):** $S' \in \text{add}M'$;
- case (3):** $M' \in \mathcal{Y}(T_0)$ and $\text{Hom}_\Lambda(T_0, \tau^{-1}S) \in \text{add}P'$.

Remark 4.2.3. *Since $\tau^{-1}S \in \mathcal{T}(T_0)$ and $S \in \mathcal{F}(T_0)$, Lemma 2.1.3 ensures that $\text{Hom}_B(\text{Hom}_\Lambda(T_0, \tau^{-1}S), \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Ext}_\Lambda^1(\tau^{-1}S, S)$. Since Λ is hereditary, Corollary 1.4.2 implies $\text{Ext}_\Lambda^1(\tau^{-1}S, S) \cong D \text{Hom}(S, S) \neq 0$. Thus, $(\text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ is not a τ -rigid pair, hence cannot be a direct summand of any support τ -tilting pair. So if (M', P') is a basic support τ -tilting pair for B of case (2), then $P' \in \text{add}P[i]'$.*

Let $s\tau\text{-tilt}\Lambda_{\mathcal{T}\cup\mathcal{F}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of those isomorphism classes whose representatives belong to $\mathcal{T}(T_0) \cup \mathcal{F}(T_0)$. Hence a basic support τ -tilting module M in $s\tau\text{-tilt}\Lambda_{\mathcal{T}\cup\mathcal{F}}$ if and only if $M \in \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$. Similarly, $s\tau\text{-tilt}B_{\mathcal{X}\cup\mathcal{Y}}$ denotes the subset of $s\tau\text{-tilt}B$ consisting of isomorphism classes whose representatives belong to $\mathcal{X}(T_0) \cup \mathcal{Y}(T_0)$. Then we know $s\tau\text{-tilt}\Lambda_{\mathcal{T}\cup\mathcal{F}}$ is the disjoint union of support τ -tilting modules of cases (a), (b) and (c). Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in mod B , we know $s\tau\text{-tilt}B = s\tau\text{-tilt}B_{\mathcal{X}\cup\mathcal{Y}}$. By slightly modifying the definition of ϕ , we obtain a map F between $s\tau\text{-tilt}\Lambda_{\mathcal{T}\cup\mathcal{F}}$ and $s\tau\text{-tilt}B_{\mathcal{X}\cup\mathcal{Y}}$.

Lemma 4.2.4. [4, III. Proposition 2.11] *Let A be a finite dimensional algebra. For every A -module M and indecomposable projective A -module P_i , there exists a functorial isomorphism of k -vector spaces*

$$\text{Hom}_A(P_i, M) \cong D \text{Hom}_A(M, I_i).$$

Where I_i is the indecomposable injective A -module corresponding to P_i .

Remark 4.2.5. *If $\text{pd}_A M \leq 1$, then Corollary 1.4.2 implies that M is τ -rigid if and only if $\text{Ext}_A^1(M, M) = 0$.*

Proposition 4.2.6. *We can define a map $F : s\tau\text{-tilt}\Lambda_{\mathcal{T}\cup\mathcal{F}} \rightarrow s\tau\text{-tilt}B_{\mathcal{X}\cup\mathcal{Y}}$ as following*

$$F(M, P) = \begin{cases} (\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, P)) & (M, P) \text{ is case (a)} \\ (\text{Hom}_\Lambda(T_0, M) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, P/P(S))) & (M, P) \text{ is case (b)} \\ (\text{Hom}_\Lambda(T_0, M/S), \text{Hom}_\Lambda(T_0, P) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S)) & (M, P) \text{ is case (c)}. \end{cases}$$

Proof. 1. Let (M, P) be a basic support τ -tilting pair for Λ of case (a), i.e., $M \in \mathcal{T}(T_0)$ and $P \in \text{add}P[i]$. Since Λ is hereditary and $M \in \mathcal{T}(T_0)$, Lemma 1.2.9 forces $\text{pd}_B \text{Hom}_\Lambda(T_0, M) \leq \text{pd}_\Lambda M \leq 1$. Then according to Remark 4.2.5, $\text{Hom}_\Lambda(T_0, M)$ is τ -rigid if and only if $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, M)) = 0$. But Lemma 2.1.1 implies $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, M)) \cong \text{Ext}_\Lambda^1(M, M)$. Since Λ is hereditary, then according to Remark 4.2.5, M being τ -rigid implies $\text{Ext}_\Lambda^1(M, M) = 0$. Thus, $\text{Hom}_\Lambda(T_0, M)$ is a τ -rigid B -module. By assumption, $M, P \in \mathcal{T}(T_0)$. Then according to Lemma 2.1.1, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P), \text{Hom}_\Lambda(T_0, M)) \cong \text{Hom}_\Lambda(P, M)$. Since (M, P) is τ -rigid, $\text{Hom}_\Lambda(P, M) = 0$. Thus, we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P), \text{Hom}_\Lambda(T_0, M)) = 0$. And hence $(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, P))$ is a τ -rigid pair. Moreover, the Tilting Theorem 1.2.7 implies it is a basic support τ -tilting pair.

2. Let (M, P) be a basic support τ -tilting pair for Λ of case (b), i.e., $M \in \mathcal{T}(T_0)$ and $P(S) \in \text{add}P$. We propose to show $(\text{Hom}_\Lambda(T_0, M) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, P/P(S)))$ is a support τ -tilting pair. First, we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M) \oplus \text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M) \oplus \tau_B \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \tau_B \text{Hom}_\Lambda(T_0, M)) \oplus \text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \tau_B \text{Ext}_\Lambda^1(T_0, S)) \oplus \text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M)) \oplus \text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Ext}_\Lambda^1(T_0, S))$. By the statements in the first part of the proof, we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \tau_B \text{Hom}_\Lambda(T_0, M)) = 0$. According to Lemma 1.5.3, we have $\tau_B \text{Ext}_\Lambda^1(T_0, S) \cong \text{Hom}_\Lambda(T_0, I(S))$ as B -modules, where $I(S)$ is the injective envelope of S . Hence $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \tau_B \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, I(S)))$. According to Lemma 2.1.1, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, I(S))) \cong \text{Hom}_\Lambda(M,$

$I(S)$), while Lemma 4.2.4 implies $\text{Hom}_\Lambda(M, I(S)) \cong \text{Hom}_\Lambda(P(S), M)$. But by assumption $P(S) \in \text{add}P$, so (M, P) being a τ -rigid pair implies $\text{Hom}_\Lambda(P(S), M) = 0$. Hence $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M), \tau_B \text{Ext}_\Lambda^1(T_0, S)) = 0$. Note that $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in mod B and Lemma 1.5.2 shows that $\mathcal{X}(T_0) = \text{add} \text{Ext}_\Lambda^1(T_0, S)$, then [4, VI. Proposition 5.2] implies $\tau_B \text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M) \in \mathcal{Y}(T_0)$. And according to Proposition 1.1.2, $\text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Ext}_\Lambda^1(T_0, S)) = \text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M)) = 0$. Hence $\text{Hom}_\Lambda(T_0, M) \oplus \text{Ext}_\Lambda^1(T_0, S)$ is τ -rigid. Since $P/P(S) \in \text{add}P[i] \subset \mathcal{T}(T_0)$, Lemma 2.1.1 forces $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P/P(S)), \text{Hom}_\Lambda(T_0, M)) \cong \text{Hom}_\Lambda(P/P(S), M)$. However, since (M, P) is a support τ -tilting pair, we then have $\text{Hom}_\Lambda(P/P(S), M) = 0$. So we obtain $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P/P(S)), \text{Hom}_\Lambda(T_0, M)) = 0$. According to Lemma 2.1.3, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P[i]), \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Ext}_\Lambda^1(P[i], S) = 0$. Hence $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P/P(S)), \text{Ext}_\Lambda^1(T_0, S)) = 0$. Thus, $(\text{Hom}_\Lambda(T_0, M) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, P/P(S)))$ is a τ -rigid pair. Consequently, our assertion follows from the Tilting Theorem 1.2.7.

3. Suppose (M, P) is a basic support τ -tilting pair for Λ of case (c), i.e., $S \in \text{add}M$ and $M/S \in \mathcal{T}(T_0)$. By the statements in the first part of the proof, we know $\text{Hom}_\Lambda(T_0, M/S)$ is τ -rigid. According to Lemma 2.1.1, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P), \text{Hom}_\Lambda(T_0, M/S)) \cong \text{Hom}_\Lambda(P, M/S)$. Then since (M, P) is a support τ -tilting pair, we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P), \text{Hom}_\Lambda(T_0, M/S)) = 0$. By the same token, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, \tau^{-1}S), \text{Hom}_\Lambda(T_0, M/S)) \cong \text{Hom}_\Lambda(\tau^{-1}S, M/S)$. Since Λ is hereditary, Corollary 1.4.3 implies $\text{Hom}_\Lambda(\tau^{-1}S, M/S) \cong \text{Hom}_\Lambda(S, \tau_\Lambda(M/S))$. Since M is τ -rigid and $S \in \text{add}M$, we know $\text{Hom}_\Lambda(S, \tau_\Lambda(M/S)) = 0$. Thus, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, \tau^{-1}S), \text{Hom}_\Lambda(T_0, M/S)) = 0$. And hence we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S), \text{Hom}_\Lambda(T_0, M/S)) = 0$. So $(\text{Hom}_\Lambda(T_0, M/S), \text{Hom}_\Lambda(T_0, P) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ is a τ -rigid pair. Then the Tilting Theorem implies it is a basic support τ -tilting pair. Now we finish the proof. \square

When T_0 is APR-tilting, we have $P(S) = S$ and it is not difficult to see that our map coincides with the map of Theorem 4.2.1.

For convenience, we let $s\tau\text{-tilt}\Lambda_i$ for $i = a, b, c, d$, denote the the subset of $s\tau\text{-tilt}\Lambda$ consisting of isomorphic classes of case (i) for $i = a, b, c, d$. And $s\tau\text{-tilt}B_j$ for $j = 1, 2, 3$, denote the subset of $s\tau\text{-tilt}B$ consisting of isomorphic classes of case (j) for $j = 1, 2, 3$.

Proposition 4.2.7. *The map F is injective. Let F_a be the restriction of F on $s\tau\text{-tilt}\Lambda_a$, then F_a is a bijection between $s\tau\text{-tilt}\Lambda_a$ and $s\tau\text{-tilt}B_1$. Let F_b be the restriction of F on $s\tau\text{-tilt}\Lambda_b$, then F_b is a bijection between $s\tau\text{-tilt}\Lambda_b$ and $s\tau\text{-tilt}B_2$. In particular, if we let $F_{a \cup b}$ be the restriction of F on $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$, then $F_{a \cup b}$ is a bijection between $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$ and $s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2$.*

Proof. Suppose that $(M_1, P_1), (M_2, P_2)$ are two basic support τ -tilting pairs for Λ such that $F(M_1, P_1) = F(M_2, P_2)$, we need to show $(M_1, P_1) = (M_2, P_2)$. Suppose that (M_i, P_i) for $i = 1, 2$ are both of case (j) for $j = a, b, c$. Then by the definition of F and the Tilting Theorem 1.2.7, we know $F(M_1, P_1) = F(M_2, P_2)$ implies $(M_1, P_1) = (M_2, P_2)$.

Note that: If (M, P) is a basic support τ -tilting pair of case (a), our definition of F in Proposition 4.2.6 shows that $F(M, P)$ is of the form: (M'_y, P'_M) for some $M'_y \in \mathcal{Y}(T_0)$ and $P'_M \in \text{add}P[i]'$. If (M, P) is a basic support τ -tilting pair of case (b), our definition of F in Proposition 4.2.6 shows that $F(M, P)$ is of the form: $(\text{Ext}_\Lambda^1(T_0, S) \oplus M'_y, P'_M)$ for some $M'_y \in \mathcal{Y}(T_0)$ and $P'_M \in \text{add}P[i]'$. And if (M, P) is a basic support τ -tilting pair of case (c), our definition of F in Proposition 4.2.6 shows $F(M, P)$ is of the form: $(M'_y, \text{Hom}_\Lambda(T_0, \tau^{-1}S) \oplus P'_M)$ for some $M'_y \in \mathcal{Y}(T_0)$ and $P'_M \in \text{add}P[i]'$. So if (M_1, P_1) is case (a), (M_2, P_2) is case (b), then $(\text{Ext}_\Lambda^1(T_0, S), 0)$ is a direct summand of $F(M_2, P_2)$. But since $\text{Ext}_\Lambda^1(T_0, S) \in \mathcal{X}(T_0)$, it

cannot be a direct summand of $F(M_1, P_1)$. If (M_1, P_1) is case (a), (M_2, P_2) is case (c), then $(0, \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ is a direct summand of $F(M_2, P_2)$. Since $\text{Hom}_\Lambda(T_0, \tau^{-1}S) \notin \mathbf{add}P[i]'$, it cannot be a direct summand of $F(M_1, P_1)$; And if (M_1, P_1) is case (b), (M_2, P_2) is case (c), then $(\text{Ext}_\Lambda^1(T_0, S), 0)$ is a direct summand of $F(M_2, P_2)$. By the same token, we know it cannot be a direct summand of $F(M_1, P_1)$. Thus, F is injective. Moreover, we can see from the definition of F that $\text{Im}F_{a \cup b} \subset s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2$.

First, we show that F_a is surjective. If (M', P') is a support τ -tilting pair of case (1), i.e., $M' \in \mathcal{Y}(T_0), P' \in \mathbf{add}P'[i] \subset \mathcal{Y}(T_0)$. The Tilting Theorem 1.2.7 provides $M \in \mathcal{T}(T_0)$ and general theory provides $P \in \mathbf{add}P[i]$ such that $M' = \text{Hom}_\Lambda(T_0, M), P' = \text{Hom}_\Lambda(T_0, P)$. We claim (M, P) is a basic support τ -tilting pair of Λ and $F(M, P) = (M', P')$. Since Λ is hereditary, Remark 4.2.5 implies M is τ -rigid if and only if $\text{Ext}_\Lambda^1(M, M) = 0$. While according to Lemma 2.1.1, $\text{Ext}_\Lambda^1(M, M) \cong \text{Ext}_B^1(M', M')$. But by Lemma 1.2.9, $\text{pd}_B M' \leq 1$. Then according to Remark 4.2.5, M' being τ -rigid implies $\text{Ext}_B^1(M', M') = 0$. Hence $\text{Ext}_\Lambda^1(M, M) = 0$ and M is τ -rigid. Moreover, according to Lemma 2.1.1, $\text{Hom}_\Lambda(P, M) \cong \text{Hom}_B(P', M')$. Since (M', P') is a support τ -tilting pair, we obtain $\text{Hom}_\Lambda(P, M) = 0$. Hence (M, P) is a τ -rigid pair, and the Tilting Theorem 1.2.7 forces it to be a basic support τ -tilting pair. According to the definition of F , it is easy to see that $(M', P') = F(M, P)$. Moreover, from the definition of case (a), we know (M, P) is case (a).

We next show that F_b is surjective. If (M', P') is a basic support τ -tilting pair of case (2), i.e., $S' \in \mathbf{add}M'$. According to Remark 4.2.3, we know $P' \in \mathbf{add}P'[i]' \subset \mathcal{Y}(T_0)$. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting in mod B and $\mathcal{X}(T_0) = \mathbf{add}S'$, we obtain $M' = M'/S' \oplus S'$ with $M'/S' \in \mathcal{Y}(T_0)$. Then the Tilting Theorem provides $M/S \in \mathcal{T}(T_0)$ such that $M'/S' = \text{Hom}_\Lambda(T_0, M/S)$ and general theory provides $P \in \mathbf{add}P[i] \subset \mathcal{T}(T_0)$ such that $P' = \text{Hom}_\Lambda(T_0, P)$. Moreover, $M' = \text{Hom}_\Lambda(T_0, M/S) \oplus S'$. According to the statements of the preceding part of the proof, we know $\text{Hom}_\Lambda(M/S, \tau_\Lambda(M/S)) = 0$. And Lemma 2.1.1 implies $\text{Hom}_\Lambda(P, M/S) \cong \text{Hom}_B(P', M'/S')$. Since (M', P') is a support τ -tilting pair, we then obtain $\text{Hom}_\Lambda(P, M/S) = 0$. Since $M' = S' \oplus M'/S'$ is τ -rigid, we have $\text{Hom}_B(M'/S', \tau_B S') = 0$. By Lemma 1.5.3, $\tau_B S' \cong \text{Hom}_\Lambda(T_0, I(S))$. So $\text{Hom}_B(M'/S', \text{Hom}_\Lambda(T_0, I(S))) \cong \text{Hom}_B(M'/S', \tau_B S') = 0$. An application of Lemma 2.1.1 provides $\text{Hom}_\Lambda(M/S, I(S)) \cong \text{Hom}_B(M'/S', \text{Hom}_\Lambda(T_0, I(S))) = 0$. But according to Lemma 4.2.4, $\text{Hom}_\Lambda(P(S), M/S) \cong \text{Hom}_\Lambda(M/S, I(S))$. Hence $\text{Hom}_\Lambda(P(S), M/S) = 0$, thus, $(M/S, P \oplus P(S))$ is a τ -rigid pair. Consequently, the Tilting Theorem forces $(M/S, P \oplus P(S))$ to be a basic support τ -tilting pair. According to the definition of F , we know $(M', P') = F(M/S, P \oplus P(S))$. And from the definition of case (b), we know $(M/S, P \oplus P(S))$ is case (b). \square

Note that we have mentioned after Remark 4.2.3: $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ is the disjoint union of $s\tau\text{-tilt}\Lambda_a, s\tau\text{-tilt}\Lambda_b, s\tau\text{-tilt}\Lambda_c$. And $s\tau\text{-tilt}B_{\mathcal{X} \cup \mathcal{Y}}$ is the disjoint union of $s\tau\text{-tilt}B_1, s\tau\text{-tilt}B_2, s\tau\text{-tilt}B_3$. So if F also induces a bijection between $s\tau\text{-tilt}\Lambda_c$ and $s\tau\text{-tilt}B_3$, then F is a bijection between $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ and $s\tau\text{-tilt}B_{\mathcal{X} \cup \mathcal{Y}}$. But unfortunately, in general, this is not true. Before proving this, we will see a proposition firstly.

Proposition 4.2.8. *Let $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ be a basic support τ -tilting pair for B of case (3), and $M, Q \in \mathcal{T}(T_0)$ such that $M' = \text{Hom}_\Lambda(T_0, M), Q' \cong \text{Hom}_\Lambda(T_0, Q)$. Then $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ is contained in $\text{Im}F$ if and only if $(M \oplus S, Q)$ is a support τ -tilting pair for Λ .*

Proof. The sufficient part follows from the definition of F . Suppose that $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S)) = F(\overline{M}, \overline{P})$ for some basic support τ -tilting pair $(\overline{M}, \overline{P})$. By the definition of F , we know $(\overline{M}, \overline{P})$ must be of case (c) and $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S)) = (\text{Hom}_\Lambda(T_0, \overline{M}/S), \text{Hom}_\Lambda(T_0, \overline{P}) \oplus$

$\text{Hom}_\Lambda(T_0, \tau^{-1}S)$). Then the Tilting Theorem 1.2.7 implies $M \cong \overline{M}/S$ and $Q \cong \overline{P}$. Consequently, $(M \oplus S, Q)$ is a support τ -tilting pair. \square

Now we will give an example to show: In general, the map F is not a bijection. Let Λ be the path algebra of the quiver: $3 \rightarrow 2 \rightarrow 1$, $T_0 = P[2] \oplus \tau^{-1}S(2)$ which is BB-tilting but not APR-tilting, $B = \text{End}_\Lambda(T_0)$. Consider the pair:

$$(\text{Hom}_\Lambda(T_0, P(1)), \text{Hom}_\Lambda(T_0, P(3)) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S(2))).$$

Since $\text{Hom}_B(\text{Hom}_\Lambda(T_0, \tau^{-1}S(2)), \text{Hom}_\Lambda(T_0, P(1))) \cong \text{Hom}_\Lambda(\tau^{-1}S(2), P(1)) \cong D \text{Ext}^1(P(1), S(2)) = 0$, it is easy to see that it is a support τ -tilting pair for B . But since $\text{Hom}_\Lambda(P(1), \tau S(2)) \cong \text{Hom}_\Lambda(P(1), P(1)) \neq 0$, $P(1) \oplus S(2)$ is not τ -rigid and hence $(P(1) \oplus S(2), P(3))$ is not a support τ -tilting pair. Then by Proposition 4.2.8, we know it is not in the image of F .

Remark. According to this example and Proposition 4.2.8, we know unlike in Proposition 3.3.3, the numbers of elements in $s\tau\text{-tilt-}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ and $s\tau\text{-tilt-}B_{\mathcal{X} \cup \mathcal{Y}}$ are different in general.

Recall that according to Theorem 4.1.5, given a basic almost complete support τ -tilting pair (M, P) for A , it is a direct summand of exactly two basic support τ -tilting pairs for A . We will call these two support τ -tilting pairs the **completions** of (M, P) . Note that if (M', P') is a basic support τ -tilting pair for B of case (3), then $M' \in \mathcal{Y}(T_0)$ and $P' / \text{Hom}_\Lambda(T_0, \tau^{-1}S) \in \text{add}P[i]$.

Proposition 4.2.9. *Let (M, Q) be a basic almost complete support τ -tilting pair for Λ such that $M \in \mathcal{T}(T_0)$ and $Q \in \text{add}P[i]$. Then $(\text{Hom}_\Lambda(T_0, M), \text{Hom}_\Lambda(T_0, Q) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ is a basic support τ -tilting pair for B such that it is not contained in the image of F if and only if there exists an indecomposable $X \notin \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$ such that $(M \oplus X, Q)$ is one of the completions of (M, Q) and the other completion is either of case (a) or of case (b).*

Proof. For convenience, we let $M' := \text{Hom}_\Lambda(T_0, M)$, $Q' := \text{Hom}_\Lambda(T_0, Q)$. Note that according to Proposition 4.2.8, the support τ -tilting pair $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ doesn't belong to the image of F if and only if $(M \oplus S, Q)$ is not a support τ -tilting pair.

Assume $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ is a basic support τ -tilting pair for B which is not contained in the image of F . Obviously, (M', Q') is a basic almost complete support τ -tilting pair. Then according to Theorem 4.1.5, we know besides $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$, (M', Q') admits one more completion. Statement (2) of Proposition 4.1.3 implies the other completion is of the form $(M' \oplus L', Q')$ for some indecomposable B -module L' . Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting and $\mathcal{X}(T_0) = \text{add}S'$, we know either there exists an indecomposable $Y' \in \mathcal{Y}(T_0)$ such that $(M' \oplus Y', Q')$ is a support τ -tilting pair, or $(M' \oplus S', Q')$ is a support τ -tilting pair. According to Proposition 4.2.7, there exist bijections $s\tau\text{-tilt}\Lambda_a \cong s\tau\text{-tilt}B_1$ and $s\tau\text{-tilt}\Lambda_b \cong s\tau\text{-tilt}B_2$. Moreover, according the proof in Proposition 4.2.7, we know there exists an indecomposable $Y' \in \mathcal{Y}(T_0)$ such that $(M' \oplus Y', Q')$ is a support τ -tilting pair if and only if there exists an indecomposable $Y \in \mathcal{T}(T_0)$ such that $(M \oplus Y, Q)$ is a support τ -tilting pair, $(M' \oplus S', Q')$ is a support τ -tilting pair if and only if $(M, Q \oplus P(S))$ is a support τ -tilting pair.

Hence we know either there exists an indecomposable $Y \in \mathcal{T}(T_0)$ such that $(M \oplus Y, Q)$ is a support τ -tilting pair or $(M, Q \oplus P(S))$ is a support τ -tilting pair, i.e., one of the completions of (M, Q) is either of case (a) or of case (b). But Theorem 4.1.5 implies (M, Q) admits exactly two completions, and according to Proposition 4.2.8, our assumption: $(M', Q' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ being a basic support τ -tilting pair for B which is not contained in the image of F implies $(M \oplus S, Q)$ is not a support τ -tilting pair. So the other completion can only be either of the

form: $(M \oplus X, Q)$ for some indecomposable $X \notin \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$ or of the form: $(M, Q \oplus P)$ for some indecomposable projective $P \in \mathbf{add}P[i]$. But if $(M, Q \oplus P)$ is a support τ -tilting pair with $Q \in \mathbf{add}P[i]$, then Proposition 4.2.7 implies $(\mathrm{Hom}_\Lambda(T_0, M), \mathrm{Hom}_\Lambda(T_0, Q) \oplus \mathrm{Hom}_\Lambda(T_0, P))$ is a basic support τ -tilting pair. But since $(\mathrm{Hom}_\Lambda(T_0, M), \mathrm{Hom}_\Lambda(T_0, Q) \oplus \mathrm{Hom}_\Lambda(T_0, \tau^{-1}S))$ is a basic support τ -tilting pairs, this induces a contradiction with Proposition 4.1.3 (2). Thus, we have proved necessity.

Suppose there exists an indecomposable Λ -module $X \notin \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$ such that $(M \oplus X, Q)$ is one of the completions of (M, Q) and the other completion is either case (a) or (b). As has been mentioned in the first part of the proof, we know (M, Q) admits a completion of case (a) if and only if the almost complete τ -tilting pair (M', Q') admits a completion of case (1), (M, Q) admits a completion of case (b) if and only if the almost complete τ -tilting pair (M', Q') admits a completion of case (2). But Theorem 4.1.5 states (M', Q') admits two completions and recall that we have mentioned after Remark 4.2.3: $s\tau\text{-tilt}B$ is the disjoint union of $s\tau\text{-tilt}B_1, s\tau\text{-tilt}B_2, s\tau\text{-tilt}B_3$. So (M', Q') must admit a completion of case (3), i.e., $(M', Q' \oplus \mathrm{Hom}_\Lambda(T_0, \tau^{-1}S))$ is a support τ -tilting pair. Since we have assumed one of the completions of (M, Q) is $(M \oplus X, Q)$ for some indecomposable Λ -module $X \notin \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$, and the other one is either case (a) or case (b), we know $(M \oplus S, Q)$ can not be a support τ -tilting pair. Then Proposition 4.2.8 implies $(M', Q' \oplus \mathrm{Hom}_\Lambda(T, \tau^{-1}S))$ is not in the image of F . \square

Corollary 4.2.10. *There exists a one to one correspondence between the subset of $s\tau\text{-tilt}\Lambda$ consisting of support τ -tilting pairs of the form $(M \oplus X, Q)$ with $M \in \mathcal{T}(T_0), Q \in \mathbf{add}P[i], X \notin \mathcal{T}(T_0) \cup \mathcal{F}(T_0)$ being indecomposable such that the other completion of (M, Q) is either case (a) or (b) and the subset of $s\tau\text{-tilt}B$ consisting of support τ -tilting pairs of case (3) which are not in the image of F .*

4.3 The support τ -tilting quiver $Q(s\tau\text{-tilt}A)$

In [1], according to Theorem 4.1.5, the authors defined the mutations of support τ -tilting pairs. Two basic support τ -tilting pairs (T, P) and (T', P') for A are said to be **mutations** of each other if there exists a basic almost complete support τ -tilting pair (U, Q) that is a direct summand of (T, P) and (T', P') . In this case, we write $(T', P') = \mu_X(T, P)$ or simply $T' = \mu_X(T)$ if X is a indecomposable A -module satisfying either $T = U \oplus X$ or $P = Q \oplus X$. A partial order \leq is defined on $s\tau\text{-tilt}A$, via $M \leq N$ if and only if $\mathrm{fac}M \subset \mathrm{fac}N$.

Definition 4.3.1. [1, Definition-Proposition 2.28] *Let $T = X \oplus U$ and T' be support τ -tilting A -modules such that $T' = \mu_X(T)$ for some indecomposable A -module X . Then either $T > T'$ or $T < T'$ holds. And we say that T' is a **left mutation** (respectively, **right mutation**) of T , if $T > T'$ (respectively, $T < T'$).*

Definition 4.3.2. [1, Definition 2.29] *We define the support τ -tilting quiver $Q(s\tau\text{-tilt}A)$ of A as follows:*

1. *The set of vertices is $s\tau\text{-tilt}A$*
2. *We draw an arrow from T to U if U is a left mutation of T .*

The following conclusion is analogous to Theorem 3.3.5.

Theorem 4.3.3. [1, Corollary 2.34] *The support τ -tilting quiver $Q(s\tau\text{-tilt}A)$ is the Hasse quiver of the partially ordered set $s\tau\text{-tilt}A$.*

The following lemma will be occasionally used to determine the partial order between two mutations.

Lemma 4.3.4. [1, Lemma 2.25] *Let (T, P) and (U, Q) be support τ -tilting pairs for A . Then the following conditions are equivalent.*

1. $T \geq U$
2. $\text{Hom}_A(U, \tau T) = 0$ and $\text{add}P \subset \text{add}Q$

We have proved in Proposition 4.2.7 that $F_{a \cup b}$ is a bijection, we will see that it is actually an isomorphism of quivers. Note that throughout this section, we suppose Λ is basic hereditary, $T_0 = P[2] \oplus \tau^{-1}S$ is BB-tilting and $B = \text{End}_\Lambda(T_0)$. Recall that $F_{a \cup b}$ is the restriction of F on $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$, and for a subset $\mathcal{C} \subset s\tau\text{-tilt}\Lambda$, $Q(\mathcal{C})$ denotes the full subquiver of $Q(s\tau\text{-tilt}\Lambda)$ consisting of elements in \mathcal{C} .

Proposition 4.3.5. *$F_{a \cup b}$ induces an isomorphism of quivers between $Q(s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b)$ and $Q(s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2)$*

Proof. Let $(M_1, P_1), (M_2, P_2)$ be two basic support τ -tilting pairs for Λ such that they are mutations of each other. Without loss of generality, we assume there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ in $Q(s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b)$. Then according to Lemma 4.3.4, we have $\text{Hom}_\Lambda(M_2, \tau M_1) = 0$ (❶) and $\text{add}P_1 \subset \text{add}P_2$ (❷).

1. Suppose that (M_1, P_1) and (M_2, P_2) are both of case (a). According to the definition of F , we know (M_1, P_1) and (M_2, P_2) being mutations of each other implies $F(M_1, P_1) = (\text{Hom}_\Lambda(T_0, M_1), \text{Hom}_\Lambda(T_0, P_1))$ and $F(M_2, P_2) = (\text{Hom}_\Lambda(T_0, M_2), \text{Hom}_\Lambda(T_0, P_2))$ are mutations of each other. Hence there exists an arrow between $F(M_1, P_1)$ and $F(M_2, P_2)$. Since Λ is hereditary and $M_1 \in \mathcal{T}(T_0)$, Lemma 1.2.9 implies $\text{pd}_B \text{Hom}_\Lambda(T_0, M_1) \leq \text{pd}_\Lambda M_1 \leq 1$. Then by Corollary 1.4.2, we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \tau_B \text{Hom}_\Lambda(T_0, M_1)) \cong D \text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M_1), \text{Hom}_\Lambda(T_0, M_2))$. Since (M_1, P_1) and (M_2, P_2) are of case (a), $M_1, M_2 \in \mathcal{T}(T_0)$. Then according to Lemma 2.1.1, $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M_1), \text{Hom}_\Lambda(T_0, M_2)) \cong \text{Ext}_\Lambda^1(M_1, M_2)$. As Λ is hereditary, Corollary 1.4.2 implies $\text{Ext}_\Lambda^1(M_1, M_2) \cong D \text{Hom}_\Lambda(M_2, \tau M_1)$. Then in view of ❶, we have $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \tau_B \text{Hom}_\Lambda(T_0, M_1)) = 0$. And obviously, identity ❷ implies $\text{add} \text{Hom}_\Lambda(T_0, P_1) \subset \text{add} \text{Hom}_\Lambda(T_0, P_2)$. So $F(M_2) \leq F(M_1)$ and $F(M_2)$ is a left mutation of $F(M_1)$, i.e., there exists an arrow $F(M_1, P_1) \rightarrow F(M_2, P_2)$ in $Q(s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2)$.

2. If (M_1, P_1) and (M_2, P_2) are both of case (b), the existence of an arrow from M_1 to M_2 implies that $F(M_1, P_1) = (\text{Hom}_\Lambda(T_0, M_1) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, P_1/P(S)))$ and $F(M_2, P_2) = (\text{Hom}_\Lambda(T_0, M_2) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, P_2/P(S)))$ are mutations of each other. According to the statements in the foregoing part of the proof, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \tau_B \text{Hom}_\Lambda(T_0, M_1)) = 0$. By Lemma 1.5.3, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \tau_B \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \text{Hom}_\Lambda(T_0, I(S)))$. According to Lemma 2.1.1, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \text{Hom}_\Lambda(T_0, I(S))) \cong \text{Hom}_\Lambda(M_2, I(S))$, while Lemma 4.2.4 implies that $\text{Hom}_\Lambda(M_2, I(S)) \cong \text{Hom}_\Lambda(P(S), M_2)$. As we assume that (M_2, P_2) is a support τ -tilting pair of case (b), we have $\text{Hom}_\Lambda(P(S), M_2) = 0$. Thus, we obtain $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \tau_B \text{Ext}_\Lambda^1(T_0, S)) = 0$. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting, $\mathcal{X}(T_0) = \text{add} \text{Ext}_\Lambda^1(T_0, S)$ and $\text{Ext}_\Lambda^1(T_0, S)$ being injective, [4, VI. Proposition 5.8] forces $\tau_B \text{Hom}_\Lambda(T_0, M_1), \tau_B \text{Ext}_\Lambda^1(T_0, S) \in \mathcal{Y}(T_0)$. Then according to Proposition 1.1.2, $\text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M_1)) = \text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Ext}_\Lambda^1(T_0, S)) = 0$. Hence we obtain $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2) \oplus \text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M_1) \oplus \tau_B \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2), \tau_B \text{Hom}_\Lambda(T_0, M_1)) \oplus \text{Hom}_B(\text{Hom}_\Lambda(T_0,$

$M_2), \tau_B \text{Ext}_\Lambda^1(T_0, S)) \oplus \text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, M_1)) \oplus \text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Ext}_\Lambda^1(T_0, S)) = 0$. Moreover, in view of \bullet , we have $\text{add Hom}_\Lambda(T_0, P_1/P(S)) \subset \text{add Hom}_\Lambda(T_0, P_2/P(S))$. Consequently, Lemma 4.3.4 yields $F(M_2) \leq F(M_1)$. Hence there exists an arrow $F(M_1, P_1) \rightarrow F(M_2, P_2)$ in $Q(s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2)$.

3. Suppose the two pairs belong to different cases. By the definitions of case (a) and case (b) and statement (2) in Lemma 4.3.4, we know: our assumption $(M_2, P_2) \leq (M_1, P_1)$ implies (M_1, P_1) is case (a), so that (M_2, P_2) is case (b). Then we can assume $(M_2, P_2) =: (N, Q \oplus P(S))$ such that $N \in \mathcal{T}(T_0)$, $Q \in \text{add}P[i]$. The existence of an arrow between (M_1, P_1) , (M_2, P_2) and the fact that (M_1, P_1) being case (a) imply: (M_1, P_1) is the other completion of the almost complete support τ -tilting pair (N, Q) . So we assume $(M_1, P_1) =: (N \oplus X, Q)$ such that $X \in \mathcal{T}(T_0)$ is indecomposable. Then $F(M_1, P_1) = (\text{Hom}_\Lambda(T_0, N \oplus X), \text{Hom}_\Lambda(T_0, Q))$, $F(M_2, P_2) = (\text{Hom}_\Lambda(T_0, N) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, Q))$. So $F(M_1, P_1)$ and $F(M_2, P_2)$ are mutations of each other, i.e., there exists an arrow between $F(M_1, P_1)$ and $F(M_2, P_2)$. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting and we have $\mathcal{X}(T_0) = \text{add Ext}_\Lambda^1(T_0, S)$, Proposition 1.1.2 implies $\text{Hom}_B(\text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, N \oplus X)) = 0$. And $\text{Hom}_\Lambda(T_0, N \oplus X)$ being τ -rigid implies $\text{Hom}_B(\text{Hom}_\Lambda(T_0, N), \tau_B \text{Hom}_\Lambda(T_0, N \oplus X)) = 0$. Hence $\text{Hom}_B(\text{Hom}_\Lambda(T_0, N) \oplus \text{Ext}_\Lambda^1(T_0, S), \tau_B \text{Hom}_\Lambda(T_0, N \oplus X)) = 0$. Then according to Lemma 4.3.4, we have $F(M_1, P_1) \geq F(M_2, P_2)$. Consequently, there exists an arrow $F(M_1, P_1) \rightarrow F(M_2, P_2)$ in $Q(s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2)$.

Let $(M'_1, P'_1), (M'_2, P'_2)$ be two basic support τ -tilting pairs for B such that there exists an arrow $(M'_1, P'_1) \rightarrow (M'_2, P'_2)$ in $Q(s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2)$. Since $F_{a \cup b}$ is a bijection, there exist support τ -tilting pairs for Λ : $(M_1, P_1), (M_2, P_2) \in s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$ such that $(M'_1, P'_1) = F(M_1, P_1)$, $(M'_2, P'_2) = F(M_2, P_2)$.

1. If both (M'_1, P'_1) and (M'_2, P'_2) are case (1). According to the proof of Proposition 4.2.7, we know $(M_1, P_1) = (M'_1 \otimes_B T_0, P'_1 \otimes_B T_0)$, $(M_2, P_2) = (M'_2 \otimes_B T_0, P'_2 \otimes_B T_0)$. Since $(M'_1, P'_1), (M'_2, P'_2)$ are mutations, the Tilting Theorem 1.2.7 implies (M_1, P_1) and (M_2, P_2) are mutations. Statements that are dual to those in the first case of the preceding part imply $(M_2, P_2) \leq (M_1, P_1)$. Hence there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ in $Q(s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b)$.

2. If both (M'_1, P'_1) and (M'_2, P'_2) are cases (2). We assume $(M'_1, P'_1) =: (S' \oplus N'_1, P'_1)$, $(M'_2, P'_2) =: (S' \oplus N'_2, P'_2)$ for some $N'_1, N'_2 \in \mathcal{Y}(T_0)$ and $P'_1, P'_2 \in \text{add}P[i]'$. By the proof of Proposition 4.2.7, $(M_1, P_1) = (N'_1 \otimes_B T_0, P(S) \oplus P'_1 \otimes_B T_0)$, $(M_2, P_2) = (N'_2 \otimes_B T_0, P(S) \oplus P'_2 \otimes_B T_0)$. Statements that are dual to those in the second case of the preceding part show $(M_2, P_2) \leq (M_1, P_1)$ and $(M_2, P_2), (M_1, P_1)$ are mutations of each other. Hence there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ in $Q(s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b)$.

3. If the pairs belong to different case, we may assume the one of case (2) is $(N' \oplus S', P')$ for some $N' \in \mathcal{Y}(T_0)$, $P' \in \text{add Hom}_\Lambda(T_0, P[i])$. Since the other one is case (1) and by the definitions of case (1) and case (2), we know $(M'_1, P'_1), (M'_2, P'_2)$ are the two completions of the almost complete support τ -tilting pair (N', P') . So the one of case (1) is either of the form $(N', P' \oplus Q')$ for some indecomposable projective $Q' \in \text{add}P[i]$ or of the form $(N' \oplus X', P')$ for some indecomposable $X' \in \mathcal{Y}(T_0)$. Note that according to Lemma 2.1.3, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, P[i]), \text{Ext}_\Lambda^1(T_0, S)) \cong \text{Ext}_\Lambda^1(P[i], S) = 0$. Hence S' is simple and the projective cover of S' is $\text{Hom}_\Lambda(T_0, \tau^{-1}S)$. And if $(N', P' \oplus Q')$ is a support τ -tilting pair, then $N' \oplus S'$ being τ -rigid and P', Q' belong to $\text{add}P[i]$ imply $(N' \oplus S', P' \oplus Q')$ is a τ -rigid pair. Obviously, it is basic, then this contradicts with Proposition 4.1.4. Hence one of (M'_1, P'_1)

and (M'_2, P'_2) is of the form $(N' \oplus S', P')$ and the other one is of the form $(N' \oplus X', P')$. Since $(N' \oplus S', P')$ and $(N' \oplus X', P')$ are mutations, according to Definition 4.3.1, either $(N' \oplus S', P') \leq (N' \oplus X', P')$ or $(N' \oplus X', P') \leq (N' \oplus S', P')$. Then Lemma 4.3.4 implies either $\text{Hom}_B(N' \oplus S', \tau_B N' \oplus \tau_B X') = 0$ or $\text{Hom}_B(N' \oplus X', \tau_B N' \oplus \tau_B S') = 0$. Since $N' \oplus S'$ and $N' \oplus X'$ are τ -rigid, $\text{Hom}_B(N' \oplus S', \tau_B N') = \text{Hom}_B(N', \tau_B S') = \text{Hom}_B(N' \oplus X', \tau_B N') = \text{Hom}_B(N', \tau_B X') = 0$ (❸). Then either $\text{Hom}_B(S', \tau_B X') = 0$ or $\text{Hom}_B(X', \tau_B S') = 0$. If $\text{Hom}_B(X', \tau_B S') = 0$, in view of ❸, we have $\text{Hom}_B(N' \oplus X', \tau_B S') = 0$. According to Lemma 1.5.3, $\tau_B S' \cong \text{Hom}_\Lambda(T_0, I(S))$, then $\text{Hom}_B(N' \oplus X', \text{Hom}_\Lambda(T_0, I(S))) = 0$. The Tilting Theorem 1.2.7 provides $N, X \in \mathcal{T}(T_0)$ such that $N' = \text{Hom}_\Lambda(T_0, N)$, $X' = \text{Hom}_\Lambda(T_0, X)$. According to Lemma 2.1.1, we obtain $\text{Hom}_\Lambda(N \oplus X, I(S)) \cong \text{Hom}_B(N' \oplus X', \text{Hom}_\Lambda(T_0, I(S))) = 0$. Then an application of Lemma 4.2.4 implies $\text{Hom}_\Lambda(P(S), N \oplus X) \cong \text{Hom}_\Lambda(N \oplus X, I(S)) = 0$. But according to the proof of Proposition 4.2.7, we know $(N' \oplus X', P')$ being a support τ -tilting pair implies $(N \oplus X, P)$ is a support τ -tilting pair, where $P \in \text{add}P[i]$ such that $P' = \text{Hom}_\Lambda(T_0, P)$. Then $\text{Hom}_\Lambda(P(S), N \oplus X) = 0$ implies $(N \oplus X, P \oplus P(S))$ is a τ -rigid pair. And we can see it is basic, but this yields a contradiction with Proposition 4.1.4. Hence $\text{Hom}_B(X', \tau_B S') \neq 0$ and $\text{Hom}_B(S', \tau_B X') = 0$. Consequently, $(N' \oplus S', P') \leq (N' \oplus X', P')$ and thus, $(M'_1, P'_1) = (N' \oplus X', P')$, $(M'_2, P'_2) = (N' \oplus S', P')$. By the proof of Proposition 4.2.7, we know $(M_1, P_1) = (N \oplus X, P)$, $(M_2, P_2) = (N, P(S) \oplus P)$. Obviously, they are mutations of each other. And Lemma 4.3.4 implies $(M_1, P_1) \leq (M_2, P_2)$, i.e., there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ in $Q(s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b)$. Now we finish the proof. \square

According to Proposition 4.2.7, we know F is injective. So if we let F_c denote the restriction of F on $s\tau\text{-tilt}\Lambda_c$, then $F_c : s\tau\text{-tilt}\Lambda_c \rightarrow \text{Im}F_c$ is bijective. Moreover, we will see this induces an isomorphism of quivers.

Proposition 4.3.6. *F induces an isomorphism between the quivers $Q(s\tau\text{-tilt}\Lambda_c)$ and $Q(\text{Im}F_c)$.*

Proof. Let $(M_1, P_1), (M_2, P_2)$ be two basic support τ -tilting pairs for Λ of case (c) such that there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ in $Q(s\tau\text{-tilt}\Lambda)$. According to the definition of F , we know $F(M_1, P_1) = (\text{Hom}_\Lambda(T_0, M_1/S), \text{Hom}_\Lambda(T_0, P_1) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ and $F(M_2, P_2) = (\text{Hom}_\Lambda(T_0, M_2/S), \text{Hom}_\Lambda(T_0, P_2) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$. Since (M_1, P_1) and (M_2, P_2) are mutations, by the Tilting Theorem 1.2.7, $F(M_1, P_1)$ and $F(M_2, P_2)$ are mutations of each other. Since $M_1/S \in \mathcal{T}(T_0)$, Lemma 1.2.9 implies $\text{pd}_B \text{Hom}_\Lambda(T_0, M_1/S) \leq \text{pd}_\Lambda M_1/S \leq 1$. Then according to Corollary 1.4.2, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2/S), \tau_B \text{Hom}_\Lambda(T_0, M_1/S)) \cong D \text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M_1/S), \text{Hom}_\Lambda(T_0, M_2/S))$. Moreover, Lemma 2.1.1 provides an isomorphism $\text{Ext}_B^1(\text{Hom}_\Lambda(T_0, M_1/S), \text{Hom}_\Lambda(T_0, M_2/S)) \cong \text{Ext}_\Lambda^1(M_1/S, M_2/S)$, while Corollary 1.4.2 yields $\text{Ext}_\Lambda^1(M_1/S, M_2/S) \cong D \text{Hom}_\Lambda(M_2/S, \tau_\Lambda(M_1/S))$. Since there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$, an application of Lemma 4.3.4 implies $\text{Hom}_\Lambda(M_2/S, \tau_\Lambda(M_1/S)) = 0$. Thus, $\text{Hom}_B(\text{Hom}_\Lambda(T_0, M_2/S), \tau_B \text{Hom}_\Lambda(T_0, M_1/S)) = 0$. Obviously, $\text{add}P_1 \subset \text{add}P_2$ implies $\text{add} \text{Hom}_\Lambda(T_0, P_1) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S) \subset \text{add} \text{Hom}_\Lambda(T_0, P_2) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S)$. Then according to Lemma 4.3.4, $F(M_2, P_2) \leq F(M_1, P_1)$. Consequently, there exists an arrow $F(M_1, P_1) \rightarrow F(M_2, P_2)$ in $Q(\text{Im}F_c)$.

Conversely, let $(M'_1, P'_1), (M'_2, P'_2)$ be two basic support τ -tilting pairs for B contained in $\text{Im}F_c$, such that there exists an arrow $(M'_1, P'_1) \rightarrow (M'_2, P'_2)$. Without loss of generality, we assume $(M'_1, P'_1) =: (N'_1, Q'_1 \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$, $(M'_2, P'_2) =: (N'_2, Q'_2 \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ such that $N'_1, N'_2 \in \mathcal{Y}(T_0)$, $Q'_1, Q'_2 \in \text{add}P[i']$ and let $(M_1, P_1) := F^{-1}(M'_1, P'_1)$, $(M_2, P_2) := F^{-1}(M'_2, P'_2)$. According to the Definition of F , we know $(M_1, P_1) = (N_1 \oplus S, Q_1)$, $(M_2, P_2) =$

$(N_2 \oplus S, Q_2)$ where $N_i \in \mathcal{T}(T_0)$ such that $N'_i = \text{Hom}_\Lambda(T_0, N_i)$, and $Q_i \in \text{add}P[i]$ such that $Q'_i = \text{Hom}_\Lambda(T_0, Q_i)$ for $i = 1, 2$. Then since (M'_1, P'_1) and (M'_2, P'_2) are mutations, according to the Tilting Theorem 1.2.7, we know (M_1, P_1) and (M_2, P_2) are mutations. The existence of the arrow $(M'_1, P'_1) \rightarrow (M'_2, P'_2)$ implies $M'_2 \leq M'_1$. As before, we can show $\text{Hom}_\Lambda(M_2, \tau_\Lambda M_1) = 0$. Thus, Lemma 4.3.4 implies $(M_2, P_2) \leq (M_1, P_1)$ and hence there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$, i.e., there exists an arrow $F^{-1}(M_1, P_1) \rightarrow F^{-1}(M_2, P_2)$. \square

The next proposition shows that F will reverse the directions of arrows between elements of $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$ and elements of $s\tau\text{-tilt}\Lambda_c$. We will show in Proposition 4.3.8 below that the inverse is also true.

Proposition 4.3.7. *Let $(M_1, P_1), (M_2, P_2)$ be two basic support τ -tilting pairs for Λ such that one of them is an element of $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$, the other one is an element of $s\tau\text{-tilt}\Lambda_c$. If there exists an arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ in $Q(s\tau\text{-tilt}\Lambda)$, then there exists an arrow $F(M_2, P_2) \rightarrow F(M_1, P_1)$ in $Q(s\tau\text{-tilt}B)$.*

Proof. 1. If one of them is case (b) and the other one is case (c). The existence of the arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ implies $(M_2, P_2) \leq (M_1, P_1)$, then Lemma 4.3.4 and the definitions of case (b) and case (c) force (M_1, P_1) to be case (c), (M_2, P_2) to be case (b). We assume $(M_1, P_1) =: (N \oplus S, Q)$ for some $N \in \mathcal{T}(T_0)$, $Q \in \text{add}P[i]$. Then the existence of the arrow $(M_1, P_1) \rightarrow (M_2, P_2)$ implies that $(M_1, P_1), (M_2, P_2)$ are the two completions of the almost complete support τ -tilting pair (N, Q) . Since (M_2, P_2) is case (b), $(M_2, P_2) = (N, Q \oplus P(S))$. Then $F(M_1, P_1) = (\text{Hom}_\Lambda(T_0, N), \text{Hom}_\Lambda(T_0, Q) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$, $F(M_2, P_2) = (\text{Hom}_\Lambda(T_0, N) \oplus \text{Ext}_\Lambda^1(T_0, S), \text{Hom}_\Lambda(T_0, Q))$. Obviously, they are mutations of each other and $\text{fac Hom}_\Lambda(T_0, N) \subset \text{fac Hom}_\Lambda(T_0, N) \oplus \text{Ext}_\Lambda^1(T_0, S)$, i.e., $F(M_1, P_1) \leq F(M_2, P_2)$. Hence there exists an arrow $F(M_2, P_2) \rightarrow F(M_1, P_1)$ in $Q(s\tau\text{-tilt}B)$.

2. If one of them is case (a) and the other one is case (c). Without loss of generality, we assume the one of case (c) is of the form $(N \oplus S, Q)$ for some $N \in \mathcal{T}(T_0)$. Then the one of case (a) is either of the form $(N \oplus X, Q)$ for some indecomposable $X \in \mathcal{T}(T_0)$, or of the form $(N, Q \oplus P)$ for some indecomposable projective $P \in \text{add}P[i]$. But note that the projective cover of the simple module S is $P(S)$. If $(N, Q \oplus P)$ is a support τ -tilting pair with $P \in \text{add}P[i]$, then $(N \oplus S, Q)$ being a support τ -tilting pair implies $(N \oplus S, Q \oplus P)$ is a τ -rigid pair. Obviously, it is basic and hence has $n + 1$ non-isomorphic indecomposable summands. This contradicts with Proposition 4.1.4. Hence we know $(M_1, P_1), (M_2, P_2)$ are of the form $(N \oplus X, Q)$ and $(N \oplus S, Q)$. According to Definition 4.3.1 either $N \oplus X \leq N \oplus S$ or $N \oplus S \leq N \oplus X$, i.e., either $\text{fac}(N \oplus X) \subset \text{fac}(N \oplus S)$ or $\text{fac}(N \oplus S) \subset \text{fac}(N \oplus X)$. Since $N, X \in \mathcal{T}(T_0)$ and Lemma 1.5.2 implies $\mathcal{F}(T_0) = \text{add}S$, if $S \in \text{fac}(N \oplus X)$, we obtain $S \in \mathcal{T}(T_0)$, a contradiction. Thus, $N \oplus X \leq N \oplus S$ and we have $(M_1, P_1) = (N \oplus S, Q)$, $(M_2, P_2) = (N \oplus X, Q)$. Then $F(M_1, P_1) = (\text{Hom}_\Lambda(T_0, N), \text{Hom}_\Lambda(T_0, Q) \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ and $F(M_2, P_2) = (\text{Hom}_\Lambda(T_0, N \oplus X), \text{Hom}_\Lambda(T_0, Q))$. Obviously, they are mutations and $F(M_1, P_1) \leq F(M_2, P_2)$. Consequently, there exists an arrow $F(M_2, P_2) \rightarrow F(M_1, P_1)$ in $Q(s\tau\text{-tilt}B)$. \square

Note that we have proved in Proposition 4.2.7, F induces a bijection between $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$ and $s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2$. Now we will show that the inverse of Proposition 4.3.7.

Proposition 4.3.8. *Let $(M'_1, P'_1), (M'_2, P'_2)$ be two basic support τ -tilting pairs for B such that one of them is an element of $s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2$, the other one is an element of $\text{Im}(F_c)$. Let $(M_1, P_1), (M_2, P_2) \in s\tau\text{-tilt}\Lambda$ be the two basic support τ -tilting pairs for Λ such that $F(M_i, P_i) =$*

(M'_i, P'_i) for $i = 1, 2$. Then if there exists an arrow $(M'_1, P'_1) \rightarrow (M'_2, P'_2)$ in $Q(s\tau\text{-tilt}B)$, there exists an arrow $(M_2, P_2) \rightarrow (M_1, P_1)$ in $Q(s\tau\text{-tilt}\Lambda)$.

Proof. According to the definition of F , without loss of generality, we assume the basic support τ -tilting pair in ImF_c is of the form $(N', P' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$ for some $N' \in \mathcal{Y}(T_0)$, $P' \in \text{add}P[i]'$. Then the corresponding pre-image of it is of the form: $(N \oplus S, P)$ for some $N \in \mathcal{T}(T_0)$, $P \in \text{add}P$, such that $N' = \text{Hom}_\Lambda(T_0, N)$, $P' = \text{Hom}_\Lambda(T_0, P)$.

Suppose the other basic support τ -tilting pair for B is case (1). By the definitions of case (1) and case (3) and the existence of the arrow $(M'_1, P'_1) \rightarrow (M'_2, P'_2)$, we know the support τ -tilting pair of case (1) is of the form: $(N' \oplus Y', P')$ for some indecomposable $Y' \in \mathcal{Y}(T_0)$. Obviously, $\text{fac}(N') \subset \text{fac}(N' \oplus Y')$, i.e., $(N', P' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S)) \leq (N' \oplus Y', P')$. Hence $(M'_1, P'_1) = (N' \oplus Y', P')$, $(M'_2, P'_2) = (N', P' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$. Then $(M_2, P_2) = (N \oplus S, P)$, $(M_1, P_1) = (N \oplus Y, P)$ where $Y \in \mathcal{T}(T_0)$ such that $Y' = \text{Hom}_\Lambda(T_0, Y)$. Obviously, they are mutations of each other. According to Definition 4.3.1, either $(M_1, P_1) \leq (M_2, P_2)$ or $(M_2, P_2) \leq (M_1, P_1)$, i.e., either $\text{fac}M_1 \subset \text{fac}M_2$ or $\text{fac}M_2 \subset \text{fac}M_1$. But if $\text{fac}M_2 \subset \text{fac}M_1$, $S \in \text{fac}(N \oplus Y) \subset \mathcal{T}(T_0)$, contradicts with $S \in \mathcal{F}(T_0)$. Thus, $(M_1, P_1) \leq (M_2, P_2)$ and there exists an arrow $(M_2, P_2) \rightarrow (M_1, P_1)$.

Suppose the other basic support τ -tilting pair for B is case (2). By the definitions of case (2) and case (3) and the existence of the arrow $(M'_1, P'_1) \rightarrow (M'_2, P'_2)$, we assume the support τ -tilting pair of case (2) is of the form: $(N' \oplus S', P')$. Since $M'_2 \leq M'_1$, we have $(M'_1, P'_1) = (N' \oplus S', P')$, $(M'_2, P'_2) = (N', P' \oplus \text{Hom}_\Lambda(T_0, \tau^{-1}S))$. Then $(M_1, P_1) = (N, P \oplus P(S))$, $(M_2, P_2) = (N \oplus S, P)$. Obviously, they are mutations of each other and $\text{fac}(N) \subset \text{fac}(N \oplus S)$, i.e., $(M_1, P_1) \leq (M_2, P_2)$. According to the definition of support τ -tilting quivers, there exists an arrow $(M_2, P_2) \rightarrow (M_1, P_1)$. \square

Let $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of isomorphism classes whose representatives belong to \mathcal{T} , $s\tau\text{-tilt}\Lambda_{\mathcal{F}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of isomorphism classes whose representatives belong to \mathcal{F} , $s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$ denote the subset of $s\tau\text{-tilt}\Lambda$ consisting of elements in $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ but not in the union of $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ and $s\tau\text{-tilt}\Lambda_{\mathcal{F}}$.

It is not difficult to see that $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ is the disjoint union of $s\tau\text{-tilt}\Lambda_a$ and $s\tau\text{-tilt}\Lambda_b$, i.e., $s\tau\text{-tilt}\Lambda_{\mathcal{T}} = s\tau\text{-tilt}\Lambda_a \uplus s\tau\text{-tilt}\Lambda_b$. And $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}}$ is the disjoint union of $s\tau\text{-tilt}\Lambda_a$, $s\tau\text{-tilt}\Lambda_b$ and $s\tau\text{-tilt}\Lambda_c$, i.e., $s\tau\text{-tilt}\Lambda_{\mathcal{T} \cup \mathcal{F}} = s\tau\text{-tilt}\Lambda_a \uplus s\tau\text{-tilt}\Lambda_b \uplus s\tau\text{-tilt}\Lambda_c$. But note that $s\tau\text{-tilt}\Lambda_{\mathcal{F}} = \{(S, P[i])\}$, hence $s\tau\text{-tilt}\Lambda_c = s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{F}}$. Then according to Propositions 4.3.5, 4.3.6, 4.3.7 and 4.3.8, we are ready to state Theorem C.

Theorem 4.3.9. F induces two quiver embeddings $Q(s\tau\text{-tilt}\Lambda_{\mathcal{T}}) \hookrightarrow Q(s\tau\text{-tilt}B)$ and $Q(s\tau\text{-tilt}\Lambda_{\mathcal{F}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}) \hookrightarrow Q(s\tau\text{-tilt}B)$. Moreover, F reverses the directions of arrows between elements of $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ and elements of $s\tau\text{-tilt}\Lambda_{\mathcal{F}} \uplus s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$.

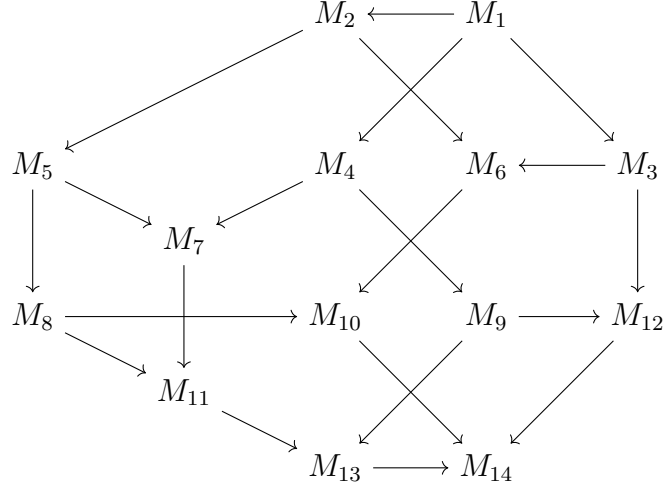
Remark. Since $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting, we know $s\tau\text{-tilt}B$ is just $s\tau\text{-tilt}B_{\mathcal{X} \cup \mathcal{Y}}$ and is equal to the disjoint union of $s\tau\text{-tilt}B_1$, $s\tau\text{-tilt}B_2$ and $s\tau\text{-tilt}B_3$, i.e., $s\tau\text{-tilt}B = s\tau\text{-tilt}B_{\mathcal{X} \cup \mathcal{Y}} = s\tau\text{-tilt}B_1 \uplus s\tau\text{-tilt}B_2 \uplus s\tau\text{-tilt}B_3$. According to Proposition 4.2.7, we know there exists a bijection between $s\tau\text{-tilt}B_1 \uplus s\tau\text{-tilt}B_2$ and $s\tau\text{-tilt}\Lambda_a \uplus s\tau\text{-tilt}\Lambda_b$. Moreover, by Proposition 4.3.5, we have $Q(s\tau\text{-tilt}B_1 \uplus s\tau\text{-tilt}B_2) \cong Q(s\tau\text{-tilt}\Lambda_a \uplus s\tau\text{-tilt}\Lambda_b)$. Obviously, F_c is a bijection between $s\tau\text{-tilt}\Lambda_c$ and ImF_c . And according to Proposition 4.3.6, $Q(ImF_c) \cong Q(s\tau\text{-tilt}\Lambda_c)$. According to Corollary 4.2.10, there exists a one to one correspondence between elements of $s\tau\text{-tilt}B_3/ImF_c$ and elements in $s\tau\text{-tilt}\Lambda_d$ that are neighbors of elements in $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$ in $Q(s\tau\text{-tilt}\Lambda)$. Moreover, Propositions 4.3.7 and 4.3.8 state the relationship between arrows between elements of $s\tau\text{-tilt}B_1 \cup s\tau\text{-tilt}B_2$ and that of ImF_c and arrows between elements of $s\tau\text{-tilt}\Lambda_a \cup s\tau\text{-tilt}\Lambda_b$ and

that of $s\tau\text{-tilt}\Lambda_c$. So given $Q(s\tau\text{-tilt}\Lambda)$, we can obtain all information about $Q(s\tau\text{-tilt}B)$ but arrows that are related to vertices in $s\tau\text{-tilt}B_3/ImF_c$. But unfortunately, we cannot say more about the arrows related to this part.

We will use an example to show the reconstructions of the support τ -tilting quivers to end this chapter.

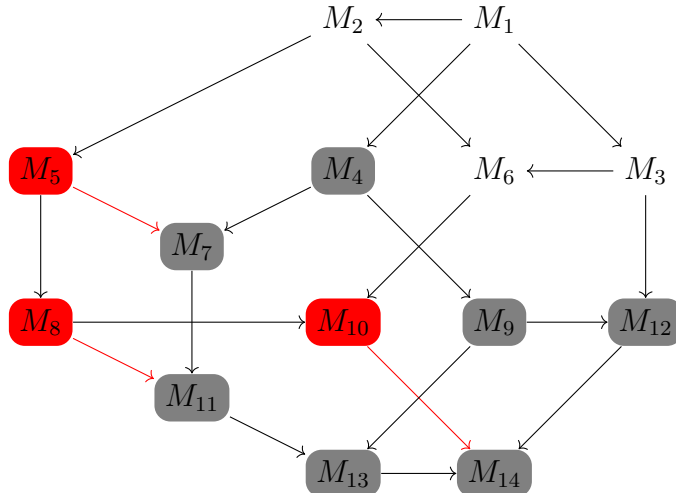
Example 4.3.10. Let Λ be the path algebra of the quiver $3 \rightarrow 2 \rightarrow 1$, $T = P[2] \oplus \tau^{-1}S(2)$ where T is BB -tilting but not APR -tilting.

First, let us see the support τ -tilting quiver of Λ . The following quiver is given in [14]:

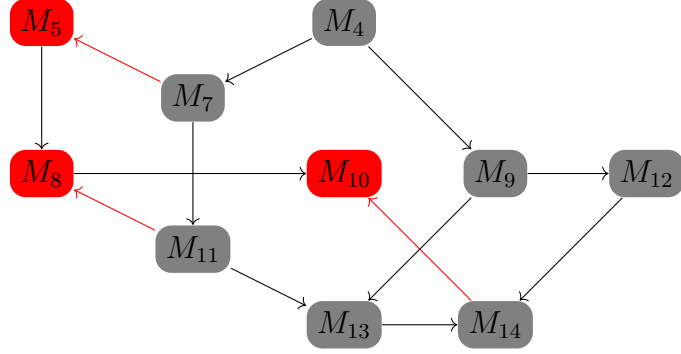


Where $M_1 = (\Lambda_\Lambda, 0)$, $M_2 = (P(2) \oplus P(3) \oplus S(2), 0)$, $M_3 = (P(1) \oplus P(2), P(3))$, $M_4 = (P(1) \oplus P(3) \oplus S(3), 0)$, $M_5 = (P(3) \oplus S(2) \oplus I(2), 0)$, $M_6 = (P(2) \oplus S(2), P(3))$, $M_7 = (P(3) \oplus I(2) \oplus S(3), 0)$, $M_8 = (S(2) \oplus I(2), P(1))$, $M_9 = (P(1) \oplus S(3), P(2))$, $M_{10} = (S(2), P(1) \oplus P(3))$, $M_{11} = (I(2) \oplus S(3), P(1))$, $M_{12} = (P(1), P(2) \oplus P(3))$, $M_{13} = (S(3), P(1) \oplus P(2))$, $M_{14} = (0, \Lambda_\Lambda)$.

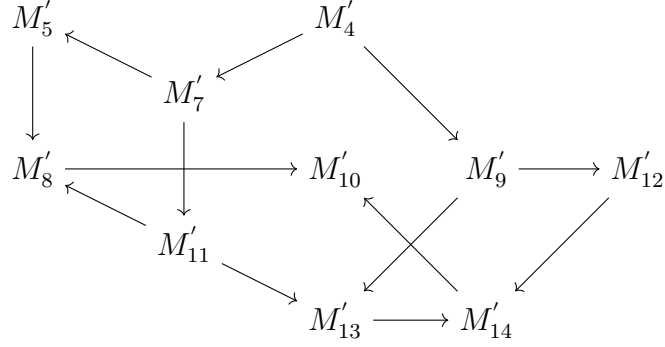
Next, let us try to construct the support τ -tilting quiver of B by using our results in this chapter. Since $\mathcal{T}(T) = \text{add}(S(3) \oplus P(3) \oplus P(1) \oplus I(2))$, $\mathcal{F}(T) = \text{add}S(2)$, we got to know $s\tau\text{-tilt}\Lambda_a = \{M_4, M_7, M_{11}\}$, $s\tau\text{-tilt}\Lambda_b = \{M_9, M_{12}, M_{13}, M_{14}\}$, $s\tau\text{-tilt}\Lambda_c = \{M_5, M_8, M_{10}\}$, $s\tau\text{-tilt}\Lambda_d = \{M_1, M_2, M_3, M_6\}$, $s\tau\text{-tilt}\Lambda_{\mathcal{T}} = \{M_4, M_7, M_{11}, M_9, M_{12}, M_{13}, M_{14}\}$. For convenience, we mark out elements of $s\tau\text{-tilt}\Lambda_{\mathcal{F}} \cup s\tau\text{-tilt}\Lambda_{\mathcal{T}, \mathcal{F}}$ and those of $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$ with \bullet and \bullet , respectively. And arrows between them are marked with red arrows.



According to Theorem 4.3.9, we know $Q(s\tau\text{-tilt}\Lambda_{\mathcal{T}})$ and $Q(s\tau\text{-tilt}\Lambda_{\mathcal{F}} \cup s\tau\text{-tilt}\Lambda_{\mathcal{T},\mathcal{F}})$ are full subquivers of $Q(s\tau\text{-tilt}B)$ and F reverses the directions of arrows between them, so we obtain a subquiver of $Q(s\tau\text{-tilt}B)$ as:

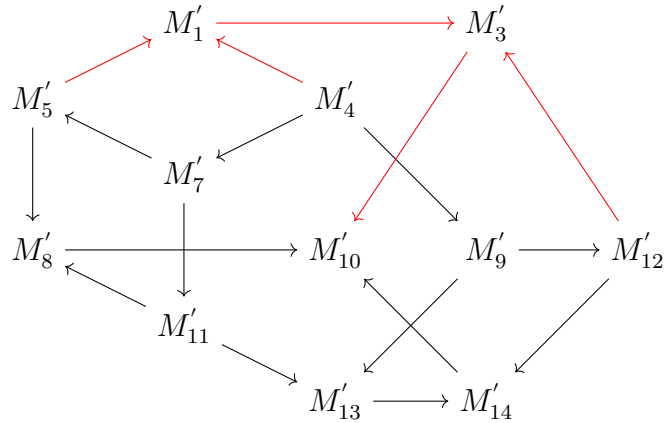


According to the marked support τ -tilting quiver $Q(s\tau\text{-tilt}\Lambda)$, we know M_1, M_3 are the only two elements of $s\tau\text{-tilt}\Lambda_d$ that are neighbours of elements in $s\tau\text{-tilt}\Lambda_{\mathcal{T}}$. Then according to Corollary 4.2.10, we know there are two elements M'_1, M'_3 in $s\tau\text{-tilt}B/ImF$ that are corresponding to M_1 and M_3 . So the support τ -tilting quiver of B is consisting of the quiver



and two more vertices M'_1, M'_3 some more arrows between them.

By the same means as in Example 3.4.2, we know $B = \text{End}_{\Lambda}(T_0)$ is the path algebra of the quiver $3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1$ bound by $\alpha\beta = 0$. Then the support τ -tilting quiver of B is as:



Where $M'_1 = (P(2) \oplus P(3), P(1))$, $M'_3 = (P(3), P(1) \oplus P(2))$, $M'_4 = (B_B, 0)$, $M'_7 = (P(1) \oplus P(2) \oplus S(2), 0)$, $M'_5 = (P(2) \oplus S(2), P(1))$, $M'_{11} = (P(1) \oplus S(2), P(3))$, $M'_9 = (P(1) \oplus P(3) \oplus I(1), 0)$, $M'_{13} = (P(1) \oplus I(1), P(3))$, $M'_{12} = (P(3) \oplus I(1), P(2))$, $M'_8 = (S_2, P(1) \oplus P(3))$, $M'_{14} = (I(1), P(2) \oplus P(3))$, $M'_{10} = (0, B_B)$. And the red arrows are those arrows that cannot be detected by our results.

5 Stable equivalences of tilted algebras

In the representation theory of finite-dimensional algebras, there are three important equivalences: Morita equivalences, derived equivalences and stable equivalences. The first two types of equivalences are defined by functors induced by bimodules or bicomplexes ([41], [2], [44]). But contrast, it is still unknown, whether stable equivalences can be defined via "biobjects". However, it was proved in [45]: If two self-injective algebras are derived equivalent, then they are stably equivalent and the functors are induced by bi-modules. Considering these facts, in [12], the author defined a special type of stable equivalence which is called stable equivalences of Morita type.

It was proved in [38]: For two self-injective algebras A and B without simple projective modules, if there exists an exact functor F which induces a stable equivalence between A and B , and if F maps simple A -modules to simple B -modules, then F is a Morita equivalence. In [39], the author generalized this result to arbitrary finite dimensional algebras.

As we have mentioned in the introduction, in this chapter, we will just propose a question. Throughout this chapter, all algebras are assumed to be finite-dimensional k -algebras with k being an algebraically closed field and we assume all algebras have no simple summands. For an algebra A , the stable category $\underline{\text{mod}}\text{-}A$ of A is defined as follows: objects are the same as those of $\text{mod } A$, and the morphisms between object X and Y are given by $\underline{\text{Hom}}_A(X, Y) := \text{Hom}_A(X, Y)/\mathcal{P}(X, Y)$, where $\mathcal{P}(X, Y)$ denotes the subset of $\text{Hom}_A(X, Y)$ consisting of those homomorphisms from X to Y that factor through a projective A -module. In fact, we can also define the stable category $\overline{\text{mod}} A$ whose objects are the same as those of $\text{mod } A$, and the morphisms between object X and Y are given by $\overline{\text{Hom}}_A(X, Y) := \text{Hom}_A(X, Y)/\mathcal{I}(X, Y)$, with $\mathcal{I}(X, Y)$ denotes the subset of $\text{Hom}_A(X, Y)$ consisting of those homomorphisms from X to Y that factor through an injective A -module. But it is well known: For an algebra A , the two categories $\overline{\text{mod}} A$ and $\underline{\text{mod}} A$ are equivalent. So in general, we will use $\underline{\text{mod}} A$ to represent the stable category.

Recall that two algebras A and B are said to be **stably equivalent** if there exists an equivalence $F : \underline{\text{mod}} A \rightarrow \underline{\text{mod}} B$ of the stable categories.

Theorem 5.1. [5, Theorem 2.1] *If Λ and Λ' are hereditary algebras, then Λ and Λ' are stably equivalent if and only if there are semisimple algebras S and S' such that $\text{mod } (\Lambda \times S)$ and $\text{mod } (\Lambda' \times S')$ are equivalent.*

Proposition 5.2. *Let A be an APR tilted algebra of type Λ_1 , B be an APR tilted algebra of type Λ_2 with Λ_1, Λ_2 being basic connected hereditary. If A and B are stably equivalent, then A and B are Morita equivalent. Moreover, the underlying graphs of the quivers for Λ_1, Λ_2 are the same.*

Proof. Without loss of generality, let $A = \text{End}_{\Lambda_1}(T_1)$, $B = \text{End}_{\Lambda_2}(T_2)$ such that T_1 is an APR-tilting Λ_1 -module, T_2 is an APR-tilting Λ_2 -module. According to Lemma 1.5.2, both A and B are hereditary algebras. Then by Theorem 5.1, being stably equivalent implies A and B are

Morita equivalent. According to Theorem 2.2.2, $D^b(A) \cong D^b(\Lambda_1)$, $D^b(B) \cong D^b(\Lambda_2)$. Obviously, $D^b(A) \cong D^b(B)$, hence $D^b(\Lambda_1) \cong D^b(\Lambda_2)$, then our statements follow from [25, I. Corollary 5.7]. \square

Remark. As a special case of Proposition 5.2, we have: If A, B are two APR tilted algebras of type Λ , then A and B being stably equivalent implies they are Morita equivalent.

Now we are ready to propose our question.

Question 5.3. *Let A, B be two tilted algebras of some basic connected hereditary algebras Λ and Λ' , respectively. If A and B are stably equivalent, does this imply A and B are Morita equivalent?*

As a special case of Question 5.3, we can consider the following question firstly.

Question 5.4. *Let A, B be two tilted algebras of the same basic connected hereditary algebra Λ , if A and B are stably equivalent, does this imply A and B are Morita equivalent?*

As we have mentioned: In [12], the author defined a special type of stable equivalences. We will check first, if the answer to our question is positive for the special type of stable equivalences.

Definition 5.5. *Two algebras A and B are said to be **stably equivalent of Morita type** if there exists an A - B -bimodule ${}_A M_B$ and a B - A -bimodule ${}_B N_A$ such that*

1. M and N are projective as left and right modules, respectively, and
2. $M \otimes_B N \cong A \oplus P$ as A - A -bimodules for some projective A - A -bimodule P , and $N \otimes_A M \cong B \oplus Q$ as B - B -bimodules for some projective B - B -bimodule Q .

For each A -module X , there exists a unique decomposition $X = X_1 \oplus X'$ such that X_1 has no non-zero projective summands and X' is projective.

Lemma 5.6. [39, Lemma 2.3] *If $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is exact in mod A , then there exists an exact sequence*

$$0 \longrightarrow X \xrightarrow{f_1} Y_0 \xrightarrow{g_1} Z_1 \longrightarrow 0,$$

with $Z = Z_1 \oplus Z'$, $Y = Y_0 \oplus Z'$ and $f = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$, $g = \begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix}$, and g_3 is an isomorphism.

The following results are the main results in [39].

Proposition 5.7. [39, Proposition 3.1] *Let A, B be two finite dimensionl k -algebras with no semisimple summands. And let $\{S_1, \dots, S_n\}$ be a complete set of pairwise non-isomorphic simple A -modules and $\{P_1, \dots, P_n\}$ be the corresponding projective covers. If two bimodules ${}_A M_B$ and ${}_B N_A$ induce a stable equivalence of Morita type between A and B , and if $- \otimes_A M$ maps any simple A -module to a simple B -module, then we have the following:*

1. $\{S_i \otimes_A M | i = 1, \dots, n\}$ is a complete set of pairwise non-isomorphic simple B -modules;
2. $\{P_i \otimes_A M | i = 1, \dots, n\}$ is a complete set of pairwise non-isomorphic indecomposable projective B -modules.

Theorem 5.8. [39, Theorem 1.1] *Let A and B be finite dimensional k -algebras (where k is a splitting field for A and B) with no semisimple summands. If two bimodules ${}_A M_B$ and ${}_B N_A$ induce a stable equivalence of Morita type between A and B , and if $- \otimes_A M$ maps any simple A -module to a simple B -module, then $- \otimes_A M$ is a Morita equivalence.*

Proposition 5.9. *Let A, B be two finite dimensional k -algebras without simple summands, $\{S_1, S_2 \cdots S_n\}$ be a complete set of pairwise non-isomorphic simple A -modules and $\{P_1, P_2 \cdots P_n\}$ be the corresponding projective covers. If ${}_A M_B, {}_B N_A$ induce a stable equivalence of Morita type between A and B , then the following statements are equivalent:*

1. $\{S_i \otimes_A M | i = 1, \cdots, n\}$ is a complete set of pairwise non-isomorphic simple B -modules;
2. $\{P_i \otimes_A M | i = 1, \cdots, n\}$ is a complete set of pairwise non-isomorphic indecomposable projective B -modules, and there exists a B -modules isomorphism $J(A)M \cong MJ(B)$ with $J(A), J(B)$ being the Jacobson radicals of A and B , respectively.

Proof. Suppose $\{S_i \otimes_A M | i = 1, \cdots, n\}$ is a complete set of pairwise non-isomorphic simple B -modules. According to Proposition 5.7, $\{P_i \otimes_A M | i = 1, \cdots, n\}$ is a complete set of pairwise non-isomorphic indecomposable projective B -modules. Consider the following exact sequence

$$0 \longrightarrow \text{rad}P_i \longrightarrow P_i \longrightarrow S_i \longrightarrow 0.$$

Since ${}_A M$ is projective, ${}_A M$ is flat. Then we have an exact sequence of B -modules:

$$0 \longrightarrow \text{rad}P_i \otimes_A M \longrightarrow P_i \otimes_A M \longrightarrow S_i \otimes_A M \longrightarrow 0.$$

Since $S_i \otimes_A M$ is simple and $P_i \otimes_A M$ is indecomposable projective, $P_i \otimes_A M \rightarrow S_i \otimes_A M$ is the projective cover of $S_i \otimes_A M$. Hence $\text{rad}P_i \otimes_A M \cong \text{rad}(P_i \otimes_A M)$ (as B -modules). Since M is projective as left A -module, hence is flat as left A -module, general theory in conjunction with [48, Corollary 3.59] provide the following isomorphisms of B -modules:

$$J(A)M \cong J(A) \otimes_A M \cong (\oplus_{i=1}^n n_i \text{rad}P_i) \otimes_A M \cong \oplus_{i=1}^n n_i (\text{rad}P_i \otimes_A M).$$

As well as, we have the following isomorphisms of B -modules:

$$\begin{aligned} MJ(B) &\cong \text{rad}(M) \cong \text{rad}(A \otimes_A M) \cong \text{rad}(\oplus_{i=1}^n (n_i P_i) \otimes_A M) \\ &\cong \text{rad}(\oplus_{i=1}^n n_i P_i \otimes_A M) \cong \oplus_{i=1}^n n_i \text{rad}(P_i \otimes_A M). \end{aligned}$$

Since $\text{rad}P_i \otimes_A M \cong \text{rad}(P_i \otimes_A M)$ for $i = 1, \cdots, n$, we have a B -modules isomorphism $J(A)M \cong MJ(B)$.

Suppose $\{P_i \otimes_A M | i = 1, \cdots, n\}$ is a complete set of pairwise non-isomorphic projective B -modules and $J(A)M \cong MJ(B)$ is an isomorphism of B -modules. First, we assume S is a simple projective A -module. Since A has no simple summand, S is not injective, then [8, V.Proposition 2.6] implies there exists an almost split sequence

$$0 \longrightarrow S \longrightarrow F \longrightarrow \tau^{-1}S \longrightarrow 0(*)$$

with F being projective. Since ${}_A M$ is projective as A -module, applying the functor $- \otimes_A M$ to $(*)$ yields an exact sequence

$$0 \longrightarrow S \otimes_A M \longrightarrow F \otimes_A M \longrightarrow \tau^{-1}S \otimes_A M \longrightarrow 0(**)$$

in mod B . Lemma 5.6 provides a short exact sequence

$$0 \longrightarrow S \otimes_A M \longrightarrow (F \otimes_A M)_0 \longrightarrow (\tau^{-1}S \otimes_A M)_1 \longrightarrow 0 (**).$$

Since $\tau^{-1}S$ is not projective and M, N induce a stable equivalence between A and B , we know $(\tau^{-1}S \otimes_A M)_1$ is indecomposable non-projective. And by assumption, we have $S \otimes_A M$ is indecomposable projective. Hence $(F \otimes_A M)_0 \rightarrow (\tau^{-1}S \otimes_A M)_1$ is a projective cover (Schanuel's Lemma). Consider the almost split sequence terminating in $(\tau^{-1}S \otimes_A M)_1$

$$0 \longrightarrow X \longrightarrow E \longrightarrow (\tau^{-1}S \otimes_A M)_1 \longrightarrow 0 (**).$$

Since $E = E_1 \oplus E' \rightarrow (\tau^{-1}S \otimes_A M)_1$ is minimal right almost split, Proposition [8, X.Proposition 1.3] implies $(E_1 \otimes_A N)_1 \oplus P' \rightarrow ((\tau^{-1}S \otimes_A M)_1 \otimes_B N)_1$ is minimal right almost split with P' being projective, but the stable equivalence forces $((\tau^{-1}S \otimes_A M)_1 \otimes_B N)_1 \cong \tau^{-1}S$. Then F being projective implies E_1 is projective. Hence $E_1 = 0$ and E is projective. [8, V.Theorem 3.3] implies $E \rightarrow (\tau^{-1}S \otimes_A M)_1$ is a projective cover. By the uniqueness (up to isomorphism) of projective covers, we know $S \otimes_A M \cong X$. Then according to [8, V.Theorem 3.3], we know $S \otimes_A M$ is simple.

If S is not projective, let $S := S_i$ for some i . Consider the short exact sequence

$$0 \longrightarrow \text{rad}P_i \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

such that $P_i \rightarrow S_i$ is a projective cover. The exactness of $- \otimes_A M$ yields a short exact sequence of B -modules

$$0 \longrightarrow \text{rad}P_i \otimes_A M \longrightarrow P_i \otimes_A M \longrightarrow S_i \otimes_A M \longrightarrow 0.$$

Since $P_i \otimes_A M$ is indecomposable, $P_i \otimes_A M \rightarrow S_i \otimes_A M$ is a projective cover and hence $\text{rad}P_i \otimes_A M \subset \text{rad}(P_i \otimes_A M)$ for $1 \leq i \leq n$. By the first part of the proof, we have B -modules isomorphisms: $J(A)M \cong \bigoplus_{i=1}^n n_i (\text{rad}P_i \otimes_A M)$ and $MJ(B) \cong \bigoplus_{i=1}^n n_i \text{rad}(P_i \otimes_A M)$. Then the assumption $MJ(B) \cong J(A)M$ forces $\text{rad}P_i \otimes_A M \cong \text{rad}(P_i \otimes_A M)$ as B -modules for $i = 1, \dots, n$. So $S_i \otimes_A M$ is simple for $i = 1, \dots, n$, and then by Proposition 5.7, $\{S_i \otimes_A M | i = 1, \dots, n\}$ is a complete set of pairwise non-isomorphic simple B -modules. \square

Corollary 5.10. *Let A and B be finite dimensional k -algebras without simple summands. If two bimodules ${}_A M_B$ and ${}_B N_A$ induce a stable equivalence of Morita type between A and B , such that $\{P_i \otimes_A M | i = 1, \dots, n\}$ is a complete set of pairwise non-isomorphic indecomposable projective B -modules, and there exists a B -modules isomorphism $J(A)M \cong MJ(B)$ with $J(A), J(B)$ being the Jacobson radicals of A and B , respectively. Then $- \otimes_A M$ is a Morita equivalence.*

Proof. According to Proposition 5.9, if ${}_A M_B$ and ${}_B N_A$ induce a stable equivalence of Morita type between A and B , such that $\{P_i \otimes_A M | i = 1, \dots, n\}$ is a complete set of pairwise non-isomorphic indecomposable projective B -modules, and there exists a B -modules isomorphism $J(A)M \cong MJ(B)$ with $J(A), J(B)$ being the Jacobson radicals of A and B , respectively. Then $\{S_i \otimes_A M | i = 1, \dots, n\}$ is a complete set of pairwise non-isomorphic simple B -modules. Consequently, according to Theorem 5.8, we know $- \otimes_A M$ is a Morita equivalence. \square

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht. Mir wurde kein akademischer Grad entzogen.

Kiel, den 15. August 2022

(Hongwei Peng)