

# On the Detection and Combination of Relatively Quasiconvex Subgroups of Relatively Hyperbolic Groups

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# Zusammenfassung

Diese Arbeit untersucht relativ quasikonvexe Untergruppen von relativ hyperbolischen Gruppen. Ihre beiden Forschungsgebiete sind die Erkennung und Kombination von relativ quasikonvexen Untergruppen.

Der erste Hauptsatz dieser Arbeit gibt unter gewissen Voraussetzungen an die periphere Struktur einer relativ hyperbolischen Gruppe  $(G, \mathbb{P})$  einen Semialgorithmus, der relativ quasikonvexe Untergruppen  $H$  von  $G$  erkennt. Das bedeutet, es existiert ein Algorithmus, der, gegeben ein Erzeugendensystem von  $H$ , entweder

- anhält und eine Quasikonvexitätskonstante sowie eine induzierte periphere Struktur von  $(G, \mathbb{P})$  auf  $H$  ausgibt, falls  $H$  relativ quasikonvex ist, oder
- nicht hält, falls  $H$  nicht relativ quasikonvex ist.

Der zweite Hauptsatz dieser Arbeit ist ein Kombinationssatz für relativ quasikonvexe Untergruppen einer torsionsfreien relativ hyperbolischen Gruppe  $(G, \mathbb{P})$ . Um diesen Satz formulieren zu können, wird der Begriff eines  $(G, \mathbb{P})$ -Trägergraphen von Gruppen definiert. Ein solcher Trägergraph von Gruppen ist im Wesentlichen ein beschrifteter Graph von Gruppen mit einer kanonischen Einbettung in  $G$ . Der Kombinationssatz gibt Bedingungen an einen Trägergraphen von Gruppen, die garantieren, dass seine kanonische Einbettung quasiisometrisch ist.

Außerdem wird eine mögliche Anwendung des Kombinationssatzes im Beweis von Endlichkeitsresultaten für Isomorphieklassen von quasikonvexen Untergruppen skizziert. Zusätzlich wird erklärt, warum der Kombinationssatz in der derzeitigen Form nur für torsionsfreie Gruppen anwendbar ist, und eine mögliche Verallgemeinerung auf Gruppen mit Torsion umrissen.

# Abstract

This thesis examines relatively quasiconvex subgroups of relatively hyperbolic groups. Its two areas of research are the detection and combination of relatively quasiconvex subgroups.

The first main theorem of this thesis provides, under certain conditions on the peripheral structure of a relatively hyperbolic group  $(G, \mathbb{P})$ , the existence of a semi-algorithm which detects relatively quasiconvex subgroups  $H$  of  $G$ . More precisely, there is an algorithm which, given a generating set of  $H$ , either

- halts and outputs a quasiconvexity constant and an induced peripheral structure of  $(G, \mathbb{P})$  on  $H$ , if  $H$  is relatively quasiconvex, or
- does not halt, if  $H$  is not relatively quasiconvex.

The second main theorem of this thesis is a combination theorem for relatively quasiconvex subgroups of a torsion-free relatively hyperbolic group  $(G, \mathbb{P})$ . To this end, the concept of a  $(G, \mathbb{P})$ -carrier graph of groups is introduced. Such a carrier graph of groups is essentially a labeled graph of groups with a canonical embedding into  $G$ . The combination theorem then gives conditions on a carrier graph of groups which guarantee that its canonical embedding is quasiisometric.

Additionally, a possible application of the combination theorem in proving finiteness results for the isomorphism classes of quasiconvex subgroups is outlined. Moreover, it is explained why the combination theorem as stated only applies to torsion-free groups and how it might be possible to prove a similar result for groups with torsion.

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# 1 Introduction

In this chapter the topics and goals of this thesis are outlined. The underlying problems and previous work are briefly reviewed and an outline of the thesis is given.

## 1.1 Detecting Relatively Quasiconvex Subgroups

It is well-known that many algorithmic problems are solvable for hyperbolic groups. Examples are the computability of the hyperbolicity constant for a given finite generating set of the group [23], the word problem and the conjugacy problem.

Let  $G$  be a hyperbolic group with finite generating set  $X$  and  $H$  some subgroup of  $G$ .  $H$  is called a *quasiconvex subgroup* of  $G$ , if there is some  $\nu \geq 0$  such that any geodesic  $v$  between two elements of  $H$  in the Cayley graph  $\Gamma(G, X)$  lies within a  $d_X$ -distance of at most  $\nu$  from  $H$ .  $\nu$  is then called a *quasiconvexity constant* for  $H$ .

Quasiconvex subgroups are of interest, because they inherit the geometric structure from the ambient hyperbolic group. They are, for example, hyperbolic groups themselves.

Kapovich [16] provided a semi-algorithm, which given a finite presentation of a hyperbolic group  $G$  and a finite generating set of the subgroup  $H$  stops if and only if  $H$  is a quasiconvex subgroup of  $G$  and if it stops it returns the quasiconvexity constant of  $H$ .

A common generalization of a hyperbolic group is a group that is hyperbolic relative to some finite collection of subgroups. Such a finite collection of subgroups is often referred to as a peripheral structure of the group. (For definitions see Chapter 2.) It turns out that many problems which are solvable for hyperbolic groups have a solution for relatively hyperbolic groups as well, provided the parabolic subgroups are reasonably well-behaved (cf. [9], [12], [7]).

It is also possible to generalize the concept of quasiconvex subgroups to relatively hyperbolic groups, leading to relatively quasiconvex sub-



groups (cf. [15]). As described in Definition 2.2, this definition again hinges on the existence of some constant  $\nu$ , which bounds how far relative geodesics between elements of the subgroup can stray from it. A relatively quasiconvex subgroup is again itself hyperbolic relative to some peripheral structure, which is induced by the peripheral structure of the surrounding group and is called an induced structure. (Thm. & Def. 2.3)

Kharlampovich et al. [18] gave an algorithm which detects relatively quasiconvex subgroups of toral relatively hyperbolic groups with peripherally finite index. Toral relatively hyperbolic groups are torsion-free relatively hyperbolic groups with abelian parabolic subgroups and a relatively quasiconvex subgroup has peripherally finite index, if its induced structure consists of conjugates of finite index subgroups of parabolic groups.

To this end, they show in a more general case that it is possible to find a language  $L$  on a relatively hyperbolic group such that  $L$ -quasiconvex subgroups are relatively quasiconvex and such that relatively quasiconvex subgroups with peripherally finite index are  $L$ -quasiconvex (cf. [18], Thm. 7.5). Using Stallings-graphs they are able to then show that there is an algorithm detecting  $L$ -quasiconvex subgroups.

Generalizing this result, Kim [19] proved that there is an algorithm detecting any finitely generated relatively quasiconvex subgroup of a toral relatively hyperbolic group. The basis of this generalization is a result by Manning and Martínez-Pedrosa [20], which implies that a subgroup  $H$  of a toral relatively hyperbolic group is relatively quasiconvex, if and only if there is a relatively quasiconvex subgroup with peripherally finite index, which is an amalgamated product of  $H$  and its induced structure.

The first main theorem of this paper provides a more general result using independent and more geometric methods of proof.

**Theorem 1.1** *Let  $G$  be a relatively hyperbolic group with peripheral structure  $\mathbb{P} = \{P_1, \dots, P_n\}$ . Let  $X$  be a symmetric finite generating set of  $G$ . Suppose a finite relative presentation of  $(G, \mathbb{P})$  and finite presentations for every  $P_i$  are given. Let  $H$  be a subgroup of  $G$  with finite generating*

set  $Y \subseteq X^*$ .

Suppose further that the following are given:

1. a solution to the membership problem of  $(P_i, O)$  for every finitely generated subgroup  $O \leq P_i$ , and
2. an algorithm, which decides whether a given finitely generated subgroup of some  $P_i$  is finite.

Then there is a semi-algorithm which stops, if and only if  $H$  is a relatively quasiconvex subgroup of  $G$ , and if it stops, it returns

- generating sets of an induced structure  $\mathbb{O}$  on  $H$ , and
- $\lambda \geq 1, c \geq 0$ , such that the inclusion  $(H, d_{Y \cup O}) \rightarrow (G, d_{X \cup P})$  is a  $(\lambda, c)$ -quasiisometric embedding.

It is an easy generalization of a theorem by Osin [22], that the relative quasiconvexity constant  $\nu$  for  $H$  in  $G$  can be calculated from the outputs of this semi-algorithm. (Theorem 2.16)

## 1.2 Combining Relatively Quasiconvex Subgroups

Another nice property of hyperbolic groups is that there are several combination theorems for them, i.e. theorems guaranteeing that a group constructed from hyperbolic groups is itself hyperbolic. The most comprehensive of these combination theorems is a result by Bestvina and Feighn [4].

Gitik [14] proved another combination theorem by using the Ping-Pong-Lemma (cf. [14], p. 1). Their theorem states, that the group generated by two quasiconvex subgroups  $H$  and  $K$  of a hyperbolic group  $G$  is itself quasiconvex and isomorphic to the amalgamated product of  $H$  and  $K$  along their intersection, as long as all short elements of  $H$  and  $K$  lie in this intersection.

Further, there are three results which are of particular interest for this thesis.

Kapovich and Weidmann showed a combination theorem for almost torsion-free hyperbolic groups ([17], Thm. 2.4). A group is called almost torsion-free, if all of its non-trivial elements of finite order have finite centralizer. They then used this combination theorem to prove that every almost torsion-free hyperbolic group, whose subgroups generated by at most  $l \in \mathbb{N}$  elements are quasiconvex, has only a finite number of conjugacy classes of freely indecomposable non-elementary subgroups generated by at most  $l + 1$  elements ([17], Cor. 1.4).

Martínez-Pedrosa ([21], Thms. 1.1 & 1.2) showed results similar to the one by Gitik but for relatively quasiconvex subgroups of relatively hyperbolic groups. Using a version of Lemma 2.19, they show that, under the right conditions, the natural homomorphism into the ambient group of an amalgamated product of two relatively quasiconvex subgroups along their intersection is an injective quasiisometric embedding.

For any group  $G$  and a graph  $\Gamma$  whose edges are labeled by elements of  $G$  there is a natural homomorphism of its fundamental group  $\pi_1(\Gamma)$  into  $G$ . Arzhantseva [2] showed the following:

**Theorem 1.2** *Let  $G$  be a hyperbolic group and  $\Gamma$  a finite graph labeled by elements of  $G$ . There is some constant  $M$  which only depends on  $G$  and the rank of  $\Gamma$ , such that at least one of the following is true:*

- *The natural homomorphism of  $\pi_1(\Gamma)$  into  $G$  is an injective quasi-isometric embedding. (Hence, its image is free and quasiconvex.)*
- *$\Gamma$  can be transformed into a new labeled graph by deleting and adding segments, such that the new graph represents the same subgroup of  $G$  and contains a loop of length at most  $M$  in  $\Gamma$ .*

Since an amalgamated product is a special case of a fundamental group of a graph of groups, the results by Martínez-Pedrosa and Arzhantseva raise the question, whether an analogue result can be found for more general graphs of groups. To give a common framework encompassing both the situations examined by Martínez-Pedrosa and Arzhantseva, the concept of a  $G$ -carrier graph of groups (Def. 4.1) is introduced in this

thesis, which can be thought of as a labeled graph of groups. A carrier graph of groups has a canonical homomorphism of its fundamental group into  $G$ .

This thesis provides a combination theorem for torsion-free relatively hyperbolic groups in its second main theorem (Thm. 4.20), using carrier graphs of groups. Since the statement of the main theorem is rather technical and requires many of the definitions introduced in Chapter 4, it is deferred to that chapter.

It is easy to see that the main theorem implies the result by Arzhantseva and it is proved in Section 4.3 that it also implies the results of Martínez-Pedrosa. It is therefore a generalization of both in the torsion-free case.

Moreover, Section 4.4 gives a short discussion on how this new combination theorem might be used to show similar finiteness results to the one by Kapovich and Weidmann, but for relatively quasiconvex subgroups of torsion-free relatively hyperbolic groups.

### 1.3 Structure

Chapter 2 introduces preliminaries necessary for the proof of the two main theorems. Section 2.1 defines the different notions of length that are used. Section 2.2 introduces some known algorithmic results for relatively hyperbolic groups. Section 2.3 gives some estimates on the length of infinite-order elements of relatively hyperbolic groups. Section 2.4 gives for a subgroup of a relatively hyperbolic group a connection between it being relatively quasiconvex and its inclusion being a quasiisometric embedding. Section 2.5 contains one of the main tools in the proof of the combination theorem, a lemma stating that a path which has some long parabolic edges, is quasigeodesic between these edges and which has restricted backtracking is itself quasigeodesic. Section 2.6 defines trivial segments of a graph of groups.

Chapter 3 is dedicated to the proof of the first main theorem. Section 3.1 gives a short overview of distortion functions, which play a crucial

role in the construction of the semi-algorithm in the main theorem. Section 3.2 contains the proof of the main theorem as well as some auxiliary results that are used in the proof. Section 3.3 gives a short discussion of the first main theorem, in particular how its assumptions might be relaxed and a sketch of a more constructive algorithm which might turn out to be more efficient for an actual implementation.

Chapter 4 is dedicated to the proof of the combination theorem Theorem 4.20. Section 4.1 introduces the concept of carrier graphs of groups, first for general groups and then for torsion-free relatively hyperbolic groups. Section 4.2 consists of the proof of the combination theorem. Section 4.3 shows that the combination theorem implies the previous results by Martínez-Pedrosa [21] and Arzhantseva [2]. Lastly, Section 4.4 discusses the combination theorem, in particular one of its possible applications as well as how it might be extended to relatively hyperbolic groups with torsion.

## 2 Preliminaries

This section gives definitions for some fundamental concepts such as relatively hyperbolic groups and relatively quasiconvex subgroups, as well as important results that provide basic tools for the investigation of these groups.

The first important definition is that of a relatively hyperbolic group. The following definition is due to Osin ([22], Definition 2.35).

**Definition 2.1** Let  $G$  be a group. A finite set  $\mathbb{P} = \{P_1, \dots, P_n\}$  of subgroups of  $G$  is called a *peripheral structure* of  $G$ .

Let  $\langle X, P_1, \dots, P_n \mid R \rangle$  be a finite relative presentation of  $(G, \mathbb{P})$ . For each  $i \in \{1, \dots, n\}$  let  $\tilde{P}_i$  be an isomorphic copy of  $P_i$ , such that  $\tilde{P}_1, \dots, \tilde{P}_n, X$  are mutually disjoint. Let  $\mathcal{P} := \bigcup (\tilde{P}_i \setminus \{1\})$ .

$G$  is *hyperbolic relative to*  $\mathbb{P}$ , if the relative Dehn function of this presentation is linear, i.e. if there exists  $C > 0$ , such that any word  $w \in (X \cup \mathcal{P})^*$  of length at most  $l$  representing the trivial element in  $G$  can be written as the product of at most  $C \cdot l$  conjugates of elements of  $R$  in the free product  $F(X) * (*_{i \in \{1, \dots, n\}} \tilde{P}_i)$ .

A natural class of subgroups of a relatively hyperbolic group  $(G, \mathbb{P})$  are the relatively quasiconvex subgroups.

**Definition 2.2 (Osin [22], Def. 4.9)** Let  $(G, \mathbb{P})$  be a relatively hyperbolic group and  $X$  a finite generating system of  $G$ . Let  $\nu \geq 0$ .  $H \leq G$  is called a  $\nu$ -*relatively quasiconvex subgroup* of  $(G, \mathbb{P})$  with respect to  $X$ , if every vertex of a geodesic  $p$  in  $\Gamma(G, X \cup \mathcal{P})$  with endpoints in  $H$  has at most a  $d_X$ -distance of  $\nu$  from  $H$ .

If  $X$  and  $\mathbb{P}$  are clear from context,  $H$  is simply called a  $\nu$ -relatively quasiconvex subgroup of  $G$ . Moreover,  $H$  is called a relatively quasiconvex subgroup of  $G$ , if it is a  $\nu$ -relatively quasiconvex subgroup for some  $\nu \geq 0$ .

Osin ([22], Prop. 4.10) shows that the definition of a relatively quasiconvex subgroup does not depend upon the choice of  $X$ . If  $Y$  is another

finite generating set of  $G$  and  $H$  is a  $\nu$ -relatively quasiconvex subgroup of  $G$  with respect to  $X$ , then it is a  $\nu'$ -relatively quasiconvex subgroup of  $G$  with respect to  $Y$  for some  $\nu' \geq 0$ .

Relatively quasiconvex subgroups inherit the geometric structure of  $G$  and are themselves hyperbolic relative to a peripheral structure that is induced by  $\mathbb{P}$ .

**Theorem & Definition 2.3 (Hruska [15], Thm. 9.1)** *Let  $(G, \mathbb{P})$  be relatively hyperbolic and  $H$  a relatively quasiconvex subgroup of  $G$ .*

*Then the subgroups in*

$$\bar{\mathbb{O}} = \{H \cap P^g \mid g \in G, P \in \mathbb{P}, |H \cap P^g| = \infty\}$$

*lie in finitely many conjugacy classes in  $H$ . For a set of representatives  $\mathbb{O}$  of these conjugacy classes,  $(H, \mathbb{O})$  is relatively hyperbolic. The peripheral structure  $\mathbb{O}$  is called an induced structure of  $(G, \mathbb{P})$  on  $H$ .*

*Moreover, the inclusion  $(H, d_{Y \cup \mathbb{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding for any finite relative generating set  $Y$  of  $(H, \mathbb{O})$ .*

## 2.1 Notions of Length

The following definition gives two different notions of length for paths in the relative Cayley graph of a relatively hyperbolic group.

**Definition 2.4** Let  $(G, \mathbb{P})$  be a relatively hyperbolic group with finite generating set  $X$ . For any path  $s = (e_1, \dots, e_k)$  in  $\text{Cay}(G, X \cup \mathcal{P})$ ,  $\text{len}(s) := k$  is called the *length* of  $s$ .

The  $X$ -length of  $s$  is defined as

$$\text{len}_X(s) := \sum_{i=1}^k d_X(\alpha(e_i), \omega(e_i)).$$

If  $s$  is a piecewise geodesic, i.e. there is a tuple  $\mathfrak{s} = (s_1, \dots, s_l)$  of geodesics in  $\text{Cay}(G, X \cup \mathcal{P})$  with  $s = s_1 \dots s_l$ , the *piecewise  $X$ -length* of

$s$  is defined as

$$\text{len}_X^p(\mathfrak{s}) := \sum_{i=1}^l d_X((s_i)_-, (s_i)_+).$$

If  $\mathfrak{s}$  is clear from the context, write  $\text{len}_X^p(s)$  for  $\text{len}_X^p(\mathfrak{s})$ .

Moreover, there are also natural notions of lengths for words and group elements. For some set  $X$  and some word  $w \in X^*$  over  $X$ ,  $|w|$  will denote the number of letters in  $w$ .

For some group  $G$  with generating set  $X$  and some element  $g \in G$ ,  $|g|_X$  denotes the word length of  $g$  with respect to  $X$ , i.e. the length of some shortest word in  $X^*$  representing  $g$ .

In a common abuse of notation, a word  $w \in X^*$  will often be identified with the element which it represents. So, while  $|w|$  denotes the length of  $w$ ,  $|w|_X$  denotes the word length with respect to  $X$  of the element represented by  $w$ .

## 2.2 Algorithms for Relatively Hyperbolic Groups

The following algorithmic result by Dahmani [9] gives conditions for when it is possible to compute an upper bound for the relative Dehn function as well as a hyperbolicity constant for the Cayley graph  $\Gamma(G, X \cup \mathcal{P})$ .

**Theorem 2.5 (Dahmani [9])** *Let  $G$  be a group with some peripheral structure  $\mathbb{P} = \{P_1, \dots, P_n\}$  consisting of finitely presented subgroups. Suppose each  $P_i$  has solvable word problem. Let  $\langle X, P_1, \dots, P_n \mid R \rangle$  be a finite relative presentation of  $G$ .*

*Then there is an semi-algorithm that stops if and only if the relative presentation of  $G$  satisfies a linear isoperimetric inequality. If it stops, it provides:*

1.  $K > 0$  such that  $\text{Area}(w) < K|w|$  for all relators  $w$ .
2.  $\delta \geq 0$  such that  $\Gamma(G, X \cup \mathcal{P})$  is  $\delta$ -hyperbolic.

An important restriction of this theorem is that all peripheral subgroups must have a solvable word problem. Moreover, Dahmani gives a



concrete algorithm as in the theorem under the condition that solutions to the word problems of all  $P_i$  are given.

The following terminology was introduced by Osin [22]:

**Definition 2.6** A subpath of a path  $p$  in the Cayley graph  $\Gamma(G, X \cup \mathcal{P})$  is called a  $P_i$ -component of  $p$ , if it is a maximal non-trivial subpath, which is labeled by letters in  $\tilde{P}_i$ .

Every vertex of  $p$  which is not an inner vertex of some  $P_i$ -component, i.e. not a vertex of a  $P_i$ -component other than its endpoints, is called a *phase vertex* of  $p$ .

A path is *locally minimal* if every  $P_i$ -component has length 1.

Two  $P_i$ -components are said to be *connected* if there is an edge labeled by an element of  $P_i$  connecting them. A  $P_i$ -component of a path  $p$  is *isolated*, if it is not connected to some other  $P_i$ -component of  $p$ .

A path  $p$  is *without backtracking* if all its components are isolated.

Note that Osin introduces different but analogue terminology for words in  $(X \cup \mathcal{P})^*$ . It can be interpreted as the result of identifying a word  $w \in (X \cup \mathcal{P})^*$  with the path in the relative Cayley graph  $\Gamma(G, X \cup \mathcal{P})$  starting at 1 and labeled by  $w$ .

For the sake of simplicity this distinction will be ignored, with the understanding that the given terminology applies through this identification in the obvious way.

In addition, by this identification a word  $w \in (X \cup \mathcal{P})^*$  is said to be quasigeodesic, if the corresponding path in  $\Gamma(G, X \cup \mathcal{P})$  is quasigeodesic.

Osin proved that relatively hyperbolic groups fulfill the BCP-property introduced by Farb [12].

**Lemma 2.7 (Osin [22], Theorem 3.23)** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group and  $X$  a finite generating set of  $G$ . Let  $\lambda \geq 1$ ,  $C \geq 0$ . Then there exists a constant  $\varepsilon = \varepsilon(G, \mathbb{P}, X, \lambda, C)$  (or simply  $\varepsilon(\lambda, C)$  if  $G$ ,  $\mathbb{P}$  and  $X$  are clear from the context) such that the following hold for any two  $(\lambda, C)$ -quasigeodesics  $p$  and  $q$  in  $\text{Cay}(G, X \cup \mathcal{P})$  with the same initial and terminal vertex:*

1. The sets of phase vertices of  $p$  and  $q$  are contained in the closed  $\varepsilon$ -neighborhoods (with respect to  $d_X$ ) of each other.
2. Suppose that  $s$  is a  $P_i$ -component of  $p$  such that  $d_X(s_-, s_+) > \varepsilon$ . Then there exists a  $P_i$ -component  $t$  of  $q$  which is connected to  $s$ .
3. Suppose that  $s$  and  $t$  are connected  $P_i$ -components of  $p$  and  $q$ , respectively. Then

$$\max\{d_X(s_-, t_-), d_X(s_+, t_+)\} \leq \varepsilon.$$

While Osin does not give an explicit algorithm to compute  $\varepsilon$ , it is easy to see that his arguments can be rewritten to derive such an algorithm. Osin uses in his proof a linear upper bound on the relative Dehn function. To construct this algorithm it is therefore necessary to calculate such an upper bound. By Theorem 2.5, it suffices to assume that every  $P_i$  is finitely presented and has solvable word problem to be able to compute a linear upper bound for the relative Dehn function. This yields the following theorem.

**Theorem 2.8** *Let  $G$  be a group with symmetric finite generating set  $X$ . Let  $G$  be hyperbolic relative to some peripheral structure  $\mathbb{P} = \{P_1, \dots, P_n\}$  of finitely presented subgroups. Let  $G$  be finitely presented relative  $\mathbb{P}$ . Suppose the word problem in each  $P_i$  is solvable. Let  $\lambda \geq 1$ ,  $C \geq 0$ .*

*Then there is an algorithm which calculates  $\varepsilon = \varepsilon(\lambda, C) > 0$  such that for any locally minimal  $(\lambda, C)$ -quasi-geodesics  $p, q$  with the same endpoints and without backtracking in  $\Gamma(G, X \cup \mathcal{P})$  the following is true:*

- i) *Every vertex of  $p$  has a  $d_X$ -distance of at most  $\varepsilon$  from a vertex of  $q$ .*
- ii) *Each  $P_i$ -component  $s$  of  $p$  with  $d_X(s_-, s_+) > \varepsilon$  is connected to some  $P_i$ -component  $t$  of  $q$ .*
- iii) *For connected  $P_i$ -components  $s$  of  $p$  and  $t$  of  $q$  the following holds:*

$$d_X(s_-, t_-), d_X(s_+, t_+) < \varepsilon$$

## 2.3 Elements of Infinite Order in Relatively Hyperbolic Groups

This section contains several results on the lengths of powers of elements of infinite order in a relatively hyperbolic group.

An important quantity in these results is the translation length as defined by Osin.

**Definition 2.9** (cf. Osin [22], Def. 4.23) Let  $(G, \mathbb{P})$  be a relatively hyperbolic group and  $X$  a symmetric finite generating set of  $G$ . Let  $g \in G$ . The *translation length*  $\tau(g)$  of  $g$  is defined by:

$$\tau(g) := \liminf_{n \rightarrow \infty} \frac{|g^n|_{X \cup \mathcal{P}}}{n}.$$

**Remark 2.10** *The limit inferior in the above definition is in fact a limit. The argument is the same as for the non-relative case which was proven by Gersten and Short ([13], Lemma 6.3).*

*Note further that  $\tau(hgh^{-1}) = \tau(g)$  for all  $g, h \in G$ .*

The following lemma follows from basic methods of hyperbolic geometry as well as the fact, that there is a lower bound on the translation lengths of hyperbolic elements in a relatively hyperbolic group ([22], Thm. 4.25). Recall that the axis of an element acting hyperbolically on a hyperbolic space is the union of all geodesics between its two fixed points in the ideal boundary at infinity.

**Lemma 2.11** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group,  $X$  a symmetric finite generating set of  $G$ . Then there exists  $D_1 = D_1(G, \mathbb{P}, X)$  such that the following holds:*

*Let  $h \in G$  be hyperbolic of infinite order with axis  $Y$  in  $\text{Cay}(G, X \cup \mathcal{P})$ . Then for every  $z \in \mathbb{Z} \setminus \{0\}$ :*

$$|z|\tau(h) + 2d_{X \cup \mathcal{P}}(1, Y) - D_1 \leq |h^z|_{X \cup \mathcal{P}} \leq |z|\tau(h) + 2d_{X \cup \mathcal{P}}(1, Y) + D_1$$

**Proof** Since the axis of a power of a hyperbolic element is the same as its axis and  $\tau(h^z) = |z|\tau(h)$  for all  $z \in \mathbb{Z}$ , it suffices to show the claim for  $z = 1$ .

Let  $\delta \geq 0$  such that  $\text{Cay}(G, X \cup \mathcal{P})$  is  $\delta$ -hyperbolic. Let  $h \in G$  be hyperbolic of infinite order with axis  $Y$  in  $\text{Cay}(G, X \cup \mathcal{P})$ . Let  $g \in VY$  be of minimal  $X \cup \mathcal{P}$ -length.

Since  $g \in VY$ ,  $d_{X \cup \mathcal{P}}(g, hg) \leq \tau(h) + C$  for some constant  $C$  independent of  $g$  and  $h$ . Hence, it has to be shown that the lengths of the (piecewise) geodesic paths  $[1, h]$  and  $[1, g] \cup [g, hg] \cup [hg, h]$  are equal up to some additive constant independent of  $h$ .

If  $\tau(h)$  is at least  $100\delta$ , the claim follows from the minimality of  $g$  and the  $2\delta$ -thinness of the quadrilateral  $[1, g] \cup [g, hg] \cup [hg, h] \cup [h, 1]$ .

Thus, it can be assumed that  $\tau(h) \leq 100\delta$ . Let  $x \in [1, g]$  with  $d_{X \cup \mathcal{P}}(x, g) = d_{X \cup \mathcal{P}}(Y, [1, h])$ . Then  $d_{X \cup \mathcal{P}}(x, hx) \leq \eta$  for some constant  $\eta = \eta(\delta)$ .

Applying the fact that detours in a hyperbolic space are exponentially long to the path

$$[g, x] \cup [x, hx] \cup [hx, h^2x] \cup \dots \cup [h^{N-1}x, h^Nx] \cup [h^Nx, h^Ng]$$

for an appropriate  $N \in \mathbb{N}$  yields that  $d_{X \cup \mathcal{P}}(Y, [1, h])$  must be bounded in terms of  $\delta$  and  $\tau(h)$ , where the bound is monotonically decreasing in  $\tau(h)$ .

By Osin ([22], Thm. 4.25), there is some  $d = d(G, \mathbb{P}, X) > 0$  such that  $\tau(h) \geq d$ . Hence, there is an upper bound on  $d_{X \cup \mathcal{P}}(Y, [1, h])$  independent of  $h$ .

Since  $\tau(h)$  was assumed to be at most  $100\delta$ , this implies the desired approximation.  $\square$

**Lemma 2.12** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group,  $X$  a symmetric finite generating set of  $G$ . Then there exists  $D_2 = D_2(G, \mathbb{P}, X)$  such that the following holds:*

Let  $h \in gP_i g^{-1}$  be a parabolic element of infinite order. Then:

$$2d_{X \cup \mathcal{P}}(1, gP_i) - D_2 \leq |h|_{X \cup \mathcal{P}} \leq 2d_{X \cup \mathcal{P}}(1, gP_i) + 1$$

**Proof** Let  $\delta \geq 0$  such that the Cayley graph  $\text{Cay}(G, X \cup \mathcal{P})$  is  $\delta$ -hyperbolic. Let  $\varepsilon = \varepsilon(2, 12\delta)$  be the constant from Lemma 2.7. Let  $h = gpg^{-1}$ , where  $p \in P_i$  and  $g$  is of minimal  $X \cup \mathcal{P}$ -length in  $gP_i$ .

Suppose first that  $|p|_X \geq \varepsilon$ . Let  $w \in (X \cup \mathcal{P})^*$  be a geodesic word representing  $g$ ,  $t$  be a path in  $\text{Cay}(G, X \cup \mathcal{P})$  labeled by  $wpw^{-1}$  and  $e$  be the edge of  $t$  with label  $p$ .

Show that any subpath  $s$  of  $t$  which is  $(2, 12\delta)$ -quasigeodesic is in fact geodesic.

Since  $g$  is of minimal  $X \cup \mathcal{P}$ -length in  $gP_i$ ,  $e$  is an isolated  $P_i$ -component of  $t$ .  $s$  is without backtracking, since  $w$  is geodesic,  $s$  is a  $(2, 12\delta)$ -quasigeodesic, and  $|p|_X \geq \varepsilon$ . If  $s$  does not contain  $e$ , it is a subpath of a geodesic and therefore itself geodesic. Hence, assume  $s$  contains  $e$ . Let  $s'$  be a geodesic from  $s_-$  to  $s_+$ . Since  $s$  is a  $(2, 12\delta)$ -quasigeodesic and since  $|p|_X \geq \varepsilon$ ,  $s'$  must have a  $P_i$ -component connected to  $e$ . It follows from the minimality of  $g$ , that  $s$  itself must be a geodesic.

It now follows that every subpath of  $t$  of length at most  $12\delta$  is geodesic. Hence,  $t$  is a  $12\delta$ -local geodesic, which in turn implies that it is a  $(2, 2\delta)$ -quasigeodesic ([6], CH. III. H 1.13). Therefore,  $t$  must itself be a geodesic. Thus

$$|gpg^{-1}|_{X \cup \mathcal{P}} = 2|g|_{X \cup \mathcal{P}} + 1 = 2d_{X \cup \mathcal{P}}(1, gP_i) + 1.$$

Now suppose  $|p|_X < \varepsilon$ . Since  $X$  is finite, there are only finitely many such  $p$ . The statement now follows analogously to the proof of Lemma 2.11 by considering a quadrilateral with vertices  $1, g, gp, gpg^{-1}$  in  $\text{Cay}(G, X \cup \mathcal{P})$  and noting that  $d_{X \cup \mathcal{P}}(gP_i, [1, gpg^{-1}])$  is bounded from above by some constant only depending on  $G, \mathbb{P}$  and  $X$ .  $\square$

The following lemma shows, that the length of the product of an element which is essentially orthogonal to a relatively quasiconvex subgroup

with an element of this subgroup is the sum of the lengths of the two elements, up to an additive constant.

**Lemma 2.13** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group,  $X$  a finite generating set of  $G$ . Let  $\delta \geq 0$  such that  $\text{Cay}(G, X \cup \mathcal{P})$  is  $\delta$ -hyperbolic and  $H$  a  $\nu$ -relatively quasiconvex subgroup of  $G$ . Let  $g \in G$  such that  $g$  has minimal  $X \cup \mathcal{P}$ -length in  $gH$  and let  $h \in H$ . Then:*

$$|gh|_{X \cup \mathcal{P}} \geq |g|_{X \cup \mathcal{P}} + |h|_{X \cup \mathcal{P}} - 8\delta - 2\nu$$

The non-relative version of Lemma 2.13 was proven by Arzhantseva ([1], Lemma 9). The proof uses only the hyperbolicity of the Cayley graph and the fact that a quasiconvex subgroup of a hyperbolic group is a quasiconvex subset of the Cayley graph. Hence, it immediately generalizes to the relatively hyperbolic case. Note that they use a slightly different but equivalent definition of hyperbolicity, which is why the constants also differ slightly.

Combining all the previous lemmas of this section yields the following.

**Lemma 2.14** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group,  $X$  a finite generating set of  $G$  and  $H$  a  $\nu$ -relatively quasiconvex subgroup of  $G$ . Then there exists a  $D = D(G, \mathbb{P}, X, \nu)$  such that the following holds:*

*Let  $h \in H$  be of infinite order and  $g \in G$  of minimal  $X \cup \mathcal{P}$ -length in  $gH$ . Then*

$$|ghg^{-1}|_{X \cup \mathcal{P}} \geq |h|_{X \cup \mathcal{P}} + 2|g|_{X \cup \mathcal{P}} - D$$

**Proof** Let  $D_1 = D_1(G, \mathbb{P}, X)$  and  $D_2 = D_2(G, \mathbb{P}, X)$  be the constants from Lemma 2.11 and Lemma 2.12. Let  $\delta \geq 0$  such that  $\text{Cay}(G, X \cup \mathcal{P})$  is  $\delta$ -hyperbolic. Let  $h \in H$  be of infinite order and  $g$  of minimal  $X \cup \mathcal{P}$ -length in  $gH$ .

Suppose  $h$  is hyperbolic, let  $Y$  be the axis of  $h$  in  $\text{Cay}(G, X \cup \mathcal{P})$  and  $x \in VY$  such that  $|gx|_{X \cup \mathcal{P}} = d_{X \cup \mathcal{P}}(1, gx) = d_{X \cup \mathcal{P}}(1, gY)$ . Since quadrilaterals in  $\text{Cay}(G, X \cup \mathcal{P})$  are  $2\delta$ -thin, and  $H$  is  $\nu$ -relatively quasiconvex, it follows that there is some  $k \in H$  with  $d_{X \cup \mathcal{P}}(k, x) \leq 2\delta + \nu$ .

Lemma 2.13 implies:

$$\begin{aligned}
d_{X \cup \mathcal{P}}(1, gY) &= |gx|_{X \cup \mathcal{P}} \\
&\geq |gk|_{X \cup \mathcal{P}} - 2\delta - \nu \\
&\geq |g|_{X \cup \mathcal{P}} + |k|_{X \cup \mathcal{P}} - 10\delta - 3\nu \\
&\geq |g|_{X \cup \mathcal{P}} + |x|_{X \cup \mathcal{P}} - 12\delta - 4\nu \\
&\geq |g|_{X \cup \mathcal{P}} + d_{X \cup \mathcal{P}}(1, Y) - 12\delta - 4\nu
\end{aligned}$$

Since  $gY$  is the axis of  $ghg^{-1}$ , it follows by Lemma 2.11:

$$\begin{aligned}
|ghg^{-1}|_{X \cup \mathcal{P}} &\geq \tau(ghg^{-1}) + 2d_{X \cup \mathcal{P}}(1, gY) - D_1 \\
&\geq \tau(h) + 2|g|_{X \cup \mathcal{P}} + 2d_{X \cup \mathcal{P}}(1, Y) - 24\delta - 8\nu - D_1 \\
&\geq |h|_{X \cup \mathcal{P}} + 2|g|_{X \cup \mathcal{P}} - 24\delta - 8\nu - 2D_1
\end{aligned}$$

Now suppose that  $h$  is parabolic, let  $h = xpx^{-1}$  where  $p \in P_i$  and  $|gx|_{X \cup \mathcal{P}} = d_{X \cup \mathcal{P}}(1, gxP_i)$ . Let  $w \in X^*$  be a word representing  $x$  and for  $n \in \mathbb{N}$  let  $s_n$  be a path in  $\text{Cay}(G, X \cup \mathcal{P})$  starting in 1 and labeled by  $wp^n w^{-1}$ . Since  $h \in H \cap xP_i x^{-1}$  is of infinite order, it follows from Lemma 2.7 that the  $P_i$ -component of  $s_n$  labeled by  $p^n$  is connected to a  $P_i$ -component of any geodesic from 1 to  $h^n$  for large  $n \in \mathbb{N}$ . Hence, there must be some  $k \in H$  with  $d_{X \cup \mathcal{P}}(k, x) \leq \nu + 1$ .

Lemma 2.13 implies:

$$\begin{aligned}
d_{X \cup \mathcal{P}}(1, gxP_i) &= |gx|_{X \cup \mathcal{P}} \\
&\geq |gk|_{X \cup \mathcal{P}} - \nu - 1 \\
&\geq |g|_{X \cup \mathcal{P}} + |k|_{X \cup \mathcal{P}} - 8\delta - 3\nu - 1 \\
&\geq |g|_{X \cup \mathcal{P}} + |x|_{X \cup \mathcal{P}} - 8\delta - 4\nu - 2 \\
&\geq |g|_{X \cup \mathcal{P}} + d_{X \cup \mathcal{P}}(1, xP_i) - 8\delta - 4\nu - 2
\end{aligned}$$

It follows by Lemma 2.11:

$$|ghg^{-1}|_{X \cup \mathcal{P}} = |(gx)p(gx)^{-1}|_{X \cup \mathcal{P}}$$

$$\begin{aligned}
&\geq 2d_{X \cup \mathcal{P}}(1, gxP_i) - D_2 \\
&\geq 2|g|_{X \cup \mathcal{P}} + 2d_{X \cup \mathcal{P}}(1, xP_i) - 16\delta - 8\nu - 4 - D_2 \\
&\geq |h|_{X \cup \mathcal{P}} + 2|g|_{X \cup \mathcal{P}} - 16\delta - 8\nu - 5 - D_2
\end{aligned}$$

Hence  $D := \max\{24\delta + 8\nu + 2D_1, 16\delta + 8\nu + 5 + D_2\}$  is as required.  $\square$

## 2.4 Inclusion of Relatively Quasiconvex Subgroups

Now let  $(G, \mathbb{P})$  be a relatively hyperbolic group with symmetric finite generating set  $X$ , let  $Y \subseteq X^*$  be a generating set of some subgroup  $H \leq G$  and let  $\mathbb{O} = \{O_1, \dots, O_m\}$  where each  $O_j$  is a subgroup of  $H \cap P_{i_j}^{g_j}$  with  $1 \leq i_j \leq n$  and  $g_j \in X^*$ . Then there is a canonical way of mapping each word in  $(Y \cup \mathbb{O})^*$  to some word in  $(X \cup \mathcal{P})^*$  representing the same group element.

**Definition 2.15** The *canonical map*  $\iota: (Y \cup \mathbb{O})^* \rightarrow (X \cup \mathcal{P})^*$  is defined as the canonical extension of the map  $Y \cup \mathbb{O} \rightarrow (X \cup \mathcal{P})^*$  that maps each element of  $Y \subseteq (X \cup \mathcal{P})^*$  to itself and that maps every  $o \in O_j$  to the unique word  $g_j^{-1}pg_j \in (X \cup \mathcal{P})^*$  representing  $o$  with  $p \in P_{i_j}$ .

A finitely generated subgroup of a hyperbolic group is quasiconvex, if and only if the inclusion of the subgroup is a quasiisometric embedding. Theorem & Definition 2.3 generalizes the forward implication by stating that for a relatively quasiconvex subgroup  $H$  of  $G$  with induced structure  $\mathbb{O}$  the inclusion  $(H, d_{Y \cup \mathbb{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding.

The following theorem and its proof are straightforward generalizations of a theorem by Osin ([22], Thm. 4.13) and can be viewed as generalizations of the converse implication in the above statement.

**Theorem 2.16** *Let  $G$  be a group with symmetric finite generating set  $X$  and hyperbolic relative to a collection  $\mathbb{P} = \{P_1, \dots, P_n\}$  of subgroups of  $G$ . Let  $H \leq G$  with finite generating set  $Y \subseteq X^*$ .*

*Let  $\mathbb{O} = \{O_1, \dots, O_m\}$  such that for each  $1 \leq j \leq m$  there is some  $1 \leq i_j \leq n$  and  $g_j \in X^*$  with  $O_j \subseteq H \cap P_{i_j}^{g_j}$ .*



Suppose the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(\lambda, c)$ -quasiisometric embedding.

Then  $H$  is relatively quasiconvex in  $G$  and it is possible to compute a constant  $\nu$  as in Definition 2.2.

**Proof** Let

$$\mu := \max(\{|y| \mid y \in Y\} \cup \{2|g_j| + 1 \mid 1 \leq j \leq m\}).$$

Let  $h \in H$  and  $V = z_1 \dots z_l \in (Y \cup \mathcal{O})^*$  be a minimal word representing  $h$ , i.e.  $l = |h|_{Y \cup \mathcal{O}}$ .

Define  $U := \iota(V)$ , and let  $U_0$  be a subword of  $U$ , i.e.

$$U_0 = A\iota(z_r) \dots \iota(z_{r+s})B$$

with  $|A|, |B| \leq \mu$ .

Since every subword of  $V$  is geodesic, the following holds:

$$\begin{aligned} |U_0| &\leq 2\mu + \mu(s+1) \\ &= 2\mu + \mu|z_r \dots z_{r+s}|_{Y \cup \mathcal{O}} \\ &\leq 2\mu + \mu(\lambda|z_r \dots z_{r+s}|_{X \cup \mathcal{P}} + \lambda c) \\ &\leq 2\mu + \mu(\lambda(|U_0|_{X \cup \mathcal{P}} + 2\mu) + \lambda c) \end{aligned}$$

Therefore, the path  $p$  in  $\Gamma(G, X \cup \mathcal{P})$  from 1 to  $h$ , which is labeled by  $U$ , is a  $(\mu\lambda, 2\mu + 2\mu^2\lambda + \mu\lambda c)$ -quasigeodesic.

Assume  $p$  has connected  $P_i$ -components. Then there is some subword  $p_1wp_2$  of  $U$  for some  $w \in (X \cup \mathcal{P})^*$  and  $p_1, p_2 \in P_i$ , which represents an element  $p_3 \in P_i$ . Replace the subpath of  $p$  labeled by  $p_1wp_2$  with the single edge connecting its endpoints and labeled by  $p_3$ . This new path is still a  $(\mu\lambda, 2\mu + 2\mu^2\lambda + \mu\lambda c)$ -quasigeodesic and its vertex set is a subset of the vertex set of  $p$ .

Repeating this process eventually yields a  $(\mu\lambda, 2\mu + 2\mu^2\lambda + \mu\lambda c)$ -quasigeodesic  $\bar{p}$  without backtracking, whose vertex set is a subset of the vertex set of  $p$ .

For any geodesic  $q$  in  $\Gamma(G, X \cup \mathcal{P})$  from 1 to  $h$  and any vertex  $v$  of  $q$  it follows with  $\varepsilon = \varepsilon(\mu\lambda, 2\mu + 2\mu^2\lambda + \mu\lambda c)$  as in the conclusion of Theorem 2.8, that there exists a vertex  $u$  of  $\bar{p}$  such that:

$$d_X(u, v) \leq \varepsilon.$$

Since  $d_X(u, H) \leq \mu$  for any vertex  $u$  of  $p$  and therefore in particular for any vertex of  $\bar{p}$ , it follows that  $d_X(v, H) \leq \varepsilon + \mu$ . Hence,  $H$  is relatively quasiconvex as in Definition 2.2 with  $\nu := \varepsilon + \mu$ .  $\square$

**Remark 2.17** *The inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding by Theorem & Definition 2.3, if  $\mathcal{O}$  is some induced structure on the relatively quasiconvex subgroup  $H$  of  $G$ .*

*It is however not the case that any structure  $\mathcal{O}$  as in Theorem 2.16 for which this inclusion is a quasiisometric embedding has to be an induced structure. In fact, let  $G$  be a group with finite generating set  $X$  and hyperbolic relative to  $\mathbb{P} = \{P_1, \dots, P_n\}$ , and  $\mathcal{O} := \{O_1, \dots, O_n\}$ , where each  $O_i$  is a finite index subgroup of  $P_i$  and at least one  $O_i$  is a proper subgroup of  $P_i$ . Then  $G$  is not hyperbolic relative  $\mathcal{O}$ , since  $\mathcal{O}$  is not an almost malnormal collection (cf. Osin [22], Prop. 2.36), but the inclusion  $(G, d_{X \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding.*

Utilizing the canonical map allows to show the following consequence of Theorem & Definition 2.3.

**Lemma & Definition 2.18** *Let  $G$  be a group which is hyperbolic relative  $\mathbb{P} = \{P_1, \dots, P_k\}$  and  $X$  a symmetric finite generating set of  $G$ . Let  $M \geq 0$ .*

*Then there exists  $\nu$  such that for every relatively quasiconvex subgroup  $H$  of  $G$ , which is generated by elements of  $X$ -length at most  $M$ , the following hold:*

1.  *$H$  is  $\nu$ -relatively quasiconvex.*
2. *There exist  $l \in \mathbb{N}$ , and  $g_i \in G$  and  $n_i \in \{1, \dots, k\}$  for  $i \in \{1, \dots, l\}$  such that:*

- (a)  $|g_i|_X \leq \nu$  for all  $i \in \{1, \dots, l\}$ , and
- (b)  $\{H \cap P_{n_i}^{g_i} \mid i \in \{1, \dots, l\}\}$  is an induced structure of  $(G, \mathbb{P})$  on  $H$ .
- (c) Let  $s$  be a geodesic in  $\text{Cay}(G, X \cup \mathcal{P})$  with  $\text{lab}(s) \in H$ . Let  $e$  be a  $P_n$ -edge of  $s$  with  $\text{len}_X(e) \geq \nu$  and  $s = s_1 e s_2$ .  
Then there exists some  $i \in \{1, \dots, l\}$  with  $n = n_i$ , and a path  $s' = s'_1 \bar{s} s'_2$  from  $s_-$  to  $s_+$  with the following properties:
- i.  $s'_1$  and  $s'_2$  are geodesics with  $\text{lab}(s'_1), \text{lab}(s'_2) \in H$ , and
  - ii.  $\bar{s} = r_1 e' r_2$ , where  $r_1$  and  $r_2$  are geodesics with  $\text{lab}(r_1)^{-1} = \text{lab}(r_2) = g_i$ , and  $e'$  is a  $P_n$ -edge connected to  $e$  with

$$d_X(\alpha(e), \alpha(e')), d_X(\omega(e), \omega(e')) \leq \nu.$$

In particular,  $\text{lab}(\bar{s}) = g_i^{-1} \text{lab}(e') g_i \in H \cap P_{n_i}^{g_i}$ .

Let  $\nu(G, \mathbb{P}, X, M)$  be defined as the smallest such  $\nu$ .

**Proof** Since there are only finitely many elements of  $G$  of  $X$ -length at most  $M$ , it suffices to show that for every relatively quasiconvex subgroup  $H \leq G$ , which is generated by elements of  $X$ -length at most  $M$ , there is some  $\nu_H$  fulfilling (a)-(c).

Let  $\bar{\nu}_H \geq 0$  such that  $H$  is  $\bar{\nu}_H$ -relatively quasiconvex and such that there is some induced structure  $\mathbb{O} := \{H \cap P_{n_i}^{g_i} \mid i \in \{1, \dots, l\}\}$  of  $(G, \mathbb{P})$  on  $H$  with  $|g_i|_X \leq \bar{\nu}_H$  for all  $i \in \{1, \dots, l\}$ . Let  $\mu := \max(M, 2\bar{\nu}_H + 1)$ .

Let  $Y$  be a finite generating set of  $H$  such that every element of  $Y$  has  $X$ -length at most  $M$ . By Theorem & Definition 2.3, the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(\lambda, c)$ -quasiisometric embedding for some  $\lambda \geq 1$  and  $c \geq 0$ .

Now let  $s$  be a geodesic in  $\text{Cay}(G, X \cup \mathcal{P})$  with  $\text{lab}(s) \in H$ . Let  $e$  be a  $P_n$ -edge of  $s$  with  $\text{len}_X(e) \geq \varepsilon := \varepsilon(\mu\lambda, 2\mu + 2\mu^2\lambda + \mu\lambda c)$  and  $s = s_1 e s_2$ .

Let  $w = w_1 \dots w_m \in (Y \cup \mathcal{O})^*$  be a shortest word representing  $\text{lab}(s)$ . Let  $t$  be the path in  $\text{Cay}(G, X \cup \mathcal{P})$  starting in  $s_-$  and labeled by  $\iota(w)$ . Since every element of  $Y \cup \mathcal{O}$  is mapped by  $\iota$  to a word of length at

most  $\mu$ , it follows analogously to the proof of Theorem 2.16 that  $t$  is a  $(\mu\lambda, 2\mu + 2\mu^2\lambda + \mu\lambda c)$ -quasigeodesic. Hence, by Lemma 2.7,  $t$  has a  $P_n$ -edge  $e'$  which is connected to  $e$  and whose endpoints have  $X$ -distance at most  $\varepsilon$  from the endpoints of  $e$ .

Since  $t$  is labeled by  $\iota(w)$ , there must be a subpath  $\bar{s} = r_1 e' r_2$  of  $t$  as required. Moreover, for geodesics  $s'_1$  and  $s'_2$  from  $s_-$  to  $\bar{s}_-$  and from  $\bar{s}_+$  to  $s_+$ , respectively,  $\text{lab}(s'_1), \text{lab}(s'_2) \in H$ .

The claim now follows for  $\nu_H := \max(\bar{\nu}_H, \varepsilon)$ .  $\square$

## 2.5 Paths with Long Parabolic Edges

The lemma proven in this section is one of the main tools for the proof of Theorem 4.20. A similar result has been given by Martínez-Pedrosa ([21], Prop. 3.1). It states, that a path  $s_0 r_1 s_1 \dots s_{k-1} r_k s_k$  in  $\text{Cay}(G, X \cup \mathcal{P})$ , where the  $r_j$  are long mutually not connected parabolic components and where the  $s_j$  are geodesics, is a quasigeodesic.

The main differences between their result and the following lemma is the fact that the long parabolic components  $r_j$  are replaced by long parabolic edges  $p_j$  and that the  $s_j$  are only required to be quasigeodesics. On the other hand, these relaxations necessitate a stronger restriction on the backtracking which can occur.

**Lemma 2.19** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group with symmetric finite generating set  $X$ . For any  $C, K \geq 1$ ,  $\varepsilon_1 \geq 0$  let  $\varepsilon := \varepsilon(G, \mathbb{P}, X, 1, K)$  be the constant from Lemma 2.7 and let  $\varepsilon_2 := 2^K(K\varepsilon + \varepsilon_1)$ . Then the following holds:*

*Let  $s = s_0 p_1 s_1 \dots s_{k-1} p_k s_k$  be a path of length at most  $K$  in the Cayley graph  $\text{Cay}(G, X \cup \mathcal{P})$ , such that the following conditions are satisfied:*

- 1. The  $p_j$  are  $\mathbb{P}$ -edges of  $s$  with  $\text{len}_X(p_j) > \varepsilon_2$ ,*
- 2. every  $s_j$  is a  $(C, C)$ -quasigeodesic, and*
- 3. if  $ps'p'$  is a subpath of  $s$  with*

*(a)  $p$  and  $p'$  are  $P_i$ -paths,*

(b)  $\text{len}_X(p), \text{len}_X(p') \geq \varepsilon_1$ , and

(c)  $\text{len}_X(s') \leq \varepsilon_2$

then  $\text{lab}(s') \notin P_i$ .

Then  $s$  is a  $(C^2 + 3C, C^2 + 3C)$ -quasigeodesic.

**Proof** Let  $s = s_0 p_1 s_1 \dots s_{k-1} p_k s_k$  be a path of length at most  $K$  in  $\text{Cay}(G, X \cup \mathcal{P})$  fulfilling 1.-3. Since 1.-3. are inherited by subpaths, it suffices to show that

$$|\text{lab}(s)|_{X \cup \mathcal{P}} \geq \frac{\text{len}(s)}{C^2 + 3C} - (C^2 + 3C).$$

It can further be assumed w.l.o.g. that the  $p_j$  are precisely the  $\mathbb{P}$ -edges of  $s$  with  $\text{len}_X(p_j) > \varepsilon_2$ .

Show the following auxiliary lemma:

**Lemma 2.20** *There is some  $N \in \mathbb{N}_0$  with  $N \leq \text{len}(s) - 1 \leq K - 1$  and paths  $s^i = s_1^i p_1^i s_2^i \dots s_{k-1}^i p_k^i s_k^i$  from  $s_-$  to  $s_+$  for  $i \in \{0, \dots, N\}$  such that  $\text{len}(s^i) \leq \text{len}(s) - i$  for  $i \in \{0, \dots, N\}$ ,  $s^N$  is without backtracking, and for each  $s^i$  the following hold:*

1. *Every  $p_j^i$  is a  $\mathbb{P}$ -edge connected to  $p_j$  with*

$$\text{len}_X(p_j^i) > (2^K - 2^i + 1)(K\varepsilon + \varepsilon_1),$$

2. *every  $s_j^i$  is a  $(C, C)$ -quasigeodesic, and*

3. *if  $ps'p'$  is a subpath of  $s^i$  with*

(a)  *$p$  and  $p'$  are  $P_l$ -paths,*

(b)  $\text{len}_X(p), \text{len}_X(p') \geq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon$ , and

(c)  $\text{len}_X(s') \leq (2^K - 2^{i+1} + 2)(K\varepsilon + \varepsilon_1)$

then  $\text{lab}(s') \notin P_l$ .

**Proof** First let  $s^0 := s$ . It is clear, that  $s^0$  fulfills 1.-3. Now suppose that  $s^0, \dots, s^i$  are given and satisfy 1.-3. If  $s^i$  is without backtracking then the claim follows with  $N = i$ .

Thus, assume that  $s^i$  has backtracking. Show that there is a path  $s^{i+1}$  satisfying 1.-3. with  $\text{len}(s^{i+1}) \leq \text{len}(s) - (i + 1)$ .

Let  $us'u'$  be a subpath of  $s^i$ , such that  $u$  and  $u'$  are connected  $P_l$ -edges and  $s'$  is without backtracking and has no  $P_l$ -edges connected to  $u$  (and therefore also none connected to  $u'$ ).

Let  $s^{i+1}$  be the path obtained from  $s^i$  by replacing  $us'u'$  with a single  $P_l$ -edge  $u_i$  connecting its endpoints. Clearly,

$$\text{len}(s^{i+1}) \leq \text{len}(s^i) - 1 \leq \text{len}(s) - (i + 1).$$

As  $s'$  has length less than  $K$  it is a  $(1, K)$ -quasigeodesic. As  $s'$  is moreover without backtracking and none of its  $P_l$ -edges are connected to  $u$  or  $u'$ , it follows from Lemma 2.7 that every  $\mathbb{P}$ -edge of  $s'$  has  $X$ -length at most  $\varepsilon$ . This implies that  $s'$  contains no  $p_j^i$  and that

$$\text{len}_X(s') \leq \text{len}(s')\varepsilon \leq K\varepsilon \leq (2^K - 2^{i+1} + 2)(K\varepsilon + \varepsilon_1).$$

Since  $s^i$  satisfies 3., this implies that

$$\text{len}_X(u) \leq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon$$

or

$$\text{len}_X(u') \leq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon.$$

Hence, there is at most one  $j \in \{1, \dots, k\}$ , such that  $u = p_j^i$  or  $u' = p_j^i$ . If neither is the case, then  $s^{i+1}$  clearly satisfies 1. and 2. with  $p_j^{i+1} := p_j^i$  for all  $j$ .

Hence, assume w.l.o.g  $u = p_j^i$  and therefore

$$\text{len}_X(u') \leq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon.$$

Thus,  $p_j^i$  is connected to  $u_i$  and

$$\begin{aligned}\text{len}_X(u_i) &\geq \text{len}_X(u) - \text{len}_X(s') - \text{len}_X(u') \\ &> (2^K - 2^i + 1)(K\varepsilon + \varepsilon_1) - K\varepsilon - (2^i(K\varepsilon + \varepsilon_1) - K\varepsilon) \\ &= (2^K - 2^{i+1} + 1)(K\varepsilon + \varepsilon_1).\end{aligned}$$

It follows that  $s^{i+1}$  satisfies 1. and 2. with  $q_j^{i+1} := u_i$  and  $q_{j'}^{i+1} := q_{j'}^i$  for every  $j' \neq j$ .

Now let  $ps''p'$  be a subpath of  $s^{i+1}$  fulfilling 3.(a)-(c). If  $u_i$  is not an edge of  $ps''p'$ , it follows from  $s^i$  satisfying 3. that  $\text{lab}(s'') \notin P_l$ .

Suppose  $u_i$  is an edge of  $s''$ , i.e.  $s'' = v_1 u_i v_2$ . Recall that

$$\text{len}_X(u') \leq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon.$$

Thus:

$$\begin{aligned}\text{len}_X(v_1 u_i v_2) &= \text{len}_X(v_1) + \text{len}_X(u) + \text{len}_X(s') + \text{len}_X(u') + \text{len}_X(v_2) \\ &\leq \text{len}_X(v_1) + \text{len}_X(u_i) + 2\text{len}_X(s') + 2\text{len}_X(u') + \text{len}_X(v_2) \\ &= \text{len}_X(s'') + 2(\text{len}_X(s') + \text{len}_X(u')) \\ &\leq (2^K - 2^{i+2} + 2)(K\varepsilon + \varepsilon_1) + 2(K\varepsilon + 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon) \\ &= (2^K - 2^{i+2} + 2)(K\varepsilon + \varepsilon_1) + 2^{i+1}(K\varepsilon + \varepsilon_1) \\ &= (2^K - 2^{i+1} + 2)(K\varepsilon + \varepsilon_1)\end{aligned}$$

Since  $s^i$  satisfies 3. it follows that  $\text{lab}(s'') = \text{lab}(v_1 u_i v_2) \notin P_l$ .

Finally suppose that  $p = u_i$  or  $p' = u_i$ . W.l.o.g. assume  $p = u_i$  and therefore  $\text{len}_X(u_i) \geq 2^{i+1}(K\varepsilon + \varepsilon_1) - K\varepsilon$ . It follows that

$$\begin{aligned}\text{len}_X(u) + \text{len}_X(u') &\geq \text{len}_X(u_i) - \text{len}_X(s') \\ &\geq 2^{i+1}(K\varepsilon + \varepsilon_1) - K\varepsilon - K\varepsilon \\ &= 2(2^i(K\varepsilon + \varepsilon_1) - K\varepsilon).\end{aligned}$$

Hence,  $\text{len}_X(u) \geq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon$  or  $\text{len}_X(u') \geq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon$ . If  $\text{len}_X(u') \geq 2^i(K\varepsilon + \varepsilon_1) - K\varepsilon$ , let  $s''' := s''$ , otherwise let  $s''' := s'u's''$ .

In either case,  $s'''$  is of length at most  $(2^K - 2^{i+1} + 2)(K\varepsilon + \varepsilon_1)$ . Since  $s^i$  satisfies 3.,  $\text{lab}(s''') \notin P_l$ , i.e.  $p = u_i$  and  $p'$  are not connected. Hence,  $\text{lab}(s'') \notin P_l$ .  $\square$

Now let  $r$  be a geodesic from  $s_-$  to  $s_+$ . Since  $s^N$  is a  $(1, K)$ -quasigeodesic without backtracking and

$$\text{len}_X(p_j^N) > (2^K - 2^N + 1)(K\varepsilon + \varepsilon_1) \geq \varepsilon,$$

it follows from Lemma 2.7, that every  $p_j^N$  and therefore every  $p_j$  must be connected to a  $P_{i_j}$ -edge  $q_j$  of  $r$ . Since every  $p_j^N$  is an isolated  $\mathbb{P}$ -edge of  $s^N$ , the  $q_j$  must be pairwise distinct. It is an easy consequence of Lemma 2.7 that  $q_1, \dots, q_k$  appear in order in  $r$ , i.e. that  $r$  is of the form  $r = r_0 q_1 r_1 \dots q_k r_k$ . This implies that for each  $j \in \{0, \dots, k\}$

$$d_{X \cup \mathcal{P}}(\alpha(r_j), \alpha(s_j)), d_{X \cup \mathcal{P}}(\omega(r_j), \omega(s_j)) \leq 1.$$

If  $k \geq \frac{\text{len}(s)}{C^2 + 3C} - (C^2 + 3C)$ , then

$$|\text{lab}(s)|_{X \cup \mathcal{P}} = \text{len}(r) \geq k \geq \frac{\text{len}(s)}{C^2 + 3C} - (C^2 + 3C).$$

If on the other hand  $k \leq \frac{\text{len}(s)}{C^2 + 3C} - (C^2 + 3C)$ , it follows that

$$\begin{aligned} |\text{lab}(s)|_{X \cup \mathcal{P}} &= \text{len}(r) = \sum_{j=0}^k \text{len}(r_j) + k \\ &\geq \sum_{j=0}^k |\text{lab}(s_j)|_{X \cup \mathcal{P}} - k \\ &\geq \sum_{j=0}^k \left( \frac{1}{C} \text{len}(s_j) - C \right) - k \\ &= \frac{1}{C} \left( k + \sum_{j=0}^k \text{len}(s_j) \right) - \frac{k}{C} - (k+1)C - k \\ &= \frac{1}{C} \text{len}(s) - \frac{k}{C} - (k+1)C - k \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{C} \text{len}(s) - \left( C + k \left( C + 1 + \frac{1}{C} \right) \right) \\
&\geq \frac{1}{C} \text{len}(s) - \left( C + \left( \frac{\text{len}(s)}{C^2 + 3C} - (C^2 + 3C) \right) (C + 2) \right) \\
&= \left( \frac{1}{C} - \frac{1}{C^2 + 3C} (C + 2) \right) \text{len}(s) - (C - (C^2 + 3C)(C + 2)) \\
&\geq \frac{\text{len}(s)}{C^2 + 3C} - (C^2 + 3C),
\end{aligned}$$

which proves the claim.  $\square$

## 2.6 Trivial Segments of Graphs of Groups

Graphs of groups were introduced by Bass [3]. The following definition provides the terminology used in this thesis.

**Definition 2.21** A *graph of groups*  $\mathbb{A}$  is a graph  $A$  together with

1. groups  $A_x$  for  $x \in VA \cup EA$  such that  $A_{e^{-1}} = A_e$  for  $e \in EA$ , and
2. *boundary monomorphisms*  $\alpha_e: A_e \rightarrow A_{\alpha(e)}$  for  $e \in EA$ .

Define  $\omega_e := \alpha_{e^{-1}}: A_e \rightarrow A_{\omega(e)}$ .

An  $\mathbb{A}$ -*path*  $t$  from  $t_- \in VA$  to  $t_+ \in VA$  is a tuple  $(a_0, e_1, a_1, \dots, e_k, a_k)$ , where  $(e_1, \dots, e_k)$  is a path in  $A$  from  $t_-$  to  $t_+$ ,  $a_0 \in A_v$ , and  $a_l \in A_{\omega(e_l)}$  for  $l \in \{1, \dots, k\}$ .

For any  $i \leq j \in \{0, \dots, k\}$  and  $\bar{a}_l \in \{1, a_l\}$  for  $l \in \{i, j\}$ , the  $\mathbb{A}$ -path  $s = (\bar{a}_i, e_{i+1}, a_{i+1}, \dots, a_{j-1}, e_j, \bar{a}_j)$  is called an  $\mathbb{A}$ -*subpath* of  $t$ .

Two  $\mathbb{A}$ -paths are called *equivalent*, if one can be transformed into the other by a series of the following operations or their inverses:

1. Replace an  $\mathbb{A}$ -subpath of the form  $(a, e, \omega_e(a'), e^{-1}, a'')$  by  $(a\alpha_e(a')a'')$ .
2. Replace an  $\mathbb{A}$ -subpath of the form  $(a\alpha_e(a'), e, a'')$  by  $(a, e, \omega_e(a')a'')$ .

For  $\mathbb{A}$ -paths  $t = (a_0, e_1, a_1, \dots, e_k, a_k)$  and  $t' = (a'_k, e_{k+1}, a_{k+1}, \dots, e_l, a_l)$  with  $t_+ = t'_-$  the  $\mathbb{A}$ -path

$$tt' := (a_0, e_1, \dots, e_k, a_k a'_k, e_{k+1}, \dots, e_l, a_l)$$

is called the *concatenation of  $t$  and  $t'$* .

For  $v_0 \in VA$  the set  $\pi_1(\mathbb{A}, v_0)$  of the equivalence classes of closed  $\mathbb{A}$ -paths in  $v_0$  together with the operation induced by concatenation is a group called the *fundamental group of  $\mathbb{A}$  at the base point  $v_0$* .

The fundamental groups of  $\mathbb{A}$  at different base points are isomorphic.

In the statement of the combination theorem (Thm. 4.20) the notion of a trivial segment of an  $\mathbb{A}$ -path plays a crucial role.

**Definition 2.22** Let  $\mathbb{A}$  be a graph of groups. A non-degenerate, simple path  $s = (e_1, \dots, e_k)$  in  $A$  is called a *trivial segment* of  $\mathbb{A}$ , if

1.  $\text{val}(\omega(e_l)) = 2$  for all  $l \in \{1, \dots, k-1\}$ , and
2.  $A_{\omega(e_l)} = 1$  for all  $l \in \{1, \dots, k-1\}$ .

Note that  $s$  is a (maximal) trivial segment, if and only if  $s^{-1}$  is a (maximal) trivial segment. Denote the number of pairs  $\{s, s^{-1}\}$  of maximal trivial segments by  $N(\mathbb{A})$ .

Let  $t = (a_0, e_1, \dots, e_k, a_k)$  be an  $\mathbb{A}$ -path and let

$$s = (1, e_i, a_i \dots, a_{j-1}, e_j, 1)$$

be an  $\mathbb{A}$ -subpath of  $t$ . Then  $s$  is called a *trivial segment* of  $t$ , if  $(e_i, \dots, e_j)$  is a trivial segment of  $\mathbb{A}$ . Note that  $a_l = 1$  for  $l \in \{i, \dots, j-1\}$ .

A simple argument shows that there is the following upper bound on  $N(\mathbb{A})$ .

**Lemma 2.23** Let  $\mathbb{A}$  be a minimal graph of groups. Let  $\mathbb{A}_1, \dots, \mathbb{A}_n$  be the components of  $\mathbb{A}$  which arise from deleting all its edges and vertices with trivial edge or vertex group. Then

$$N(\mathbb{A}) \leq 3b_1(A) + 2n,$$

where  $b_1(A)$  denotes the first Betti number of  $A$ .

**Proof** Let  $\bar{A}$  be the graph obtained from  $A$  by replacing every maximal trivial segment of  $\mathbb{A}$  with a single edge and every  $A_i$  with a single vertex.

A simple induction shows, that for any graph  $\Gamma$  the number of its edges (counting both orientations) is bounded from above by  $6b_1(\Gamma) + 4k$ , where  $k$  is the number of its vertices of valence at most 2.

If  $A$  consists of a single loop edge then  $N(\mathbb{A}) \leq 1$  and  $b_1(A) = 1$ , hence the claim is true. Otherwise, every vertex of  $\bar{A}$  of valence 1 or 2 corresponds to some  $\mathbb{A}_i$  and since  $N(\mathbb{A})$  is at most half the number of edges of  $\bar{A}$ , it follows:

$$N(\mathbb{A}) \leq \frac{1}{2}|E\bar{A}| \leq 3b_1(\bar{A}) + 2|\{v \in V\bar{A} \mid \text{val}(v) \leq 2\}| \leq 3b_1(A) + 2n$$

□

### 3 Detecting Relatively Quasiconvex Subgroups

This chapter begins with a brief overview of distortion of subgroups and how it relates to the solvability of the membership problem. Afterwards the first main theorem is proven. This chapter concludes with a short discussion of its results.

#### 3.1 Distortion of Subgroups

Distortion functions are an important tool in proving the first main theorem. This section contains their definition and basic results which are used in Section 3.2.

**Definition 3.1** Let  $G$  be a group and  $H \leq G$  a subgroup. Let  $X$  be a finite generating set of  $G$  and  $Y$  a finite subset of  $G$  such that  $H \leq \langle Y \rangle$ . Then

$$\text{Dist}_{(G,X)}^{(H,Y)}: \mathbb{N} \rightarrow \mathbb{N}, n \mapsto \max\{|h|_Y \mid h \in H, |h|_X \leq n\}$$

is the *distortion function of  $H$  in  $G$  with respect to  $Y$  and  $X$* .

This definition somewhat generalizes the usual definition of distortion functions. It does not require the set  $Y$  to be a generating set of the subgroup  $H$ . Instead it allows for  $Y$  to be a generating set of some subgroup of  $G$  containing  $H$ . In this sense it measures the distortion between two induced word metrics on  $H$ .

The following two lemmas are straightforward consequences of the definition:

**Lemma 3.2** *Let  $G$  be a group and  $H \leq G$ . Let  $X$  be a finite generating set of  $G$  and  $Y$  a finite subset of  $G$  such that  $H \leq \langle Y \rangle$ . Then the following holds:*

1. *Let  $Y'$  be a finite subset of  $G$  such that  $H \leq \langle Y' \rangle \leq \langle Y \rangle$ . Then:*

$$\text{Dist}_{(G,X)}^{(H,Y)} \leq \max\{|y'|_Y \mid y' \in Y'\} \cdot \text{Dist}_{(G,X)}^{(H,Y')}$$

2. Let  $g \in X^*$ . Then for all  $n \in \mathbb{N}$ :

$$\text{Dist}_{(G,X)}^{(H^g, Y^g)}(n) \leq \text{Dist}_{(G,X)}^{(H, Y)}(n + 2|g|)$$

**Lemma 3.3** *Let  $G$  be a group and  $H_2 \leq H_1 \leq G$  subgroups. Let  $X, Y_1, Y_2$  be finite generating sets of  $G, H_1$  and  $H_2$  respectively. Then:*

$$\text{Dist}_{(G,X)}^{(H_2, Y_2)} \leq \text{Dist}_{(H_1, Y_1)}^{(H_2, Y_2)} \circ \text{Dist}_{(G,X)}^{(H_1, Y_1)}$$

**Proof** Let  $n \in \mathbb{N}$  and  $h \in H_2$  with  $|h|_X \leq n$ . Since distortion functions are monotonous, the following holds:

$$\begin{aligned} |h|_{Y_2} &\leq \text{Dist}_{(H_1, Y_1)}^{(H_2, Y_2)}(|h|_{Y_1}) \\ &\leq \text{Dist}_{(H_1, Y_1)}^{(H_2, Y_2)}\left(\text{Dist}_{(G,X)}^{(H_1, Y_1)}(|h|_X)\right) \\ &\leq \text{Dist}_{(H_1, Y_1)}^{(H_2, Y_2)} \circ \text{Dist}_{(G,X)}^{(H_1, Y_1)}(n) \end{aligned}$$

□

It is well-known (cf. [10]) that maximal parabolic subgroups of a finitely generated relatively hyperbolic group are undistorted, i.e. have a linear distortion function. In fact, a result by Osin ([22], Lemma 5.4) states that, up to some computable constants, the distortion function of a peripheral subgroup is bounded in terms of the relative Dehn function. This implies:

**Theorem 3.4** *Let  $G$  be a group which is finitely presented relative to some peripheral structure  $\mathbb{P} = \{P_1, \dots, P_n\}$ . Let  $X$  and  $X_i$  be finite generating sets for  $G$  and each  $P_i$  respectively.*

*Suppose the relative Dehn function of  $(G, \mathbb{P})$  is computable.*

*Then  $\text{Dist}_{(G,X)}^{(P_i, X_i)}$  is computable.*

The final theorem in this section gives a connection between the computation of a distortion function and the solution to the corresponding membership problem. (cf. Farb [11], Proposition 2.1)

**Theorem 3.5** *Let  $G$  be a group and  $H \leq G$ . Let  $X$  be a finite generating set of  $G$  and  $Y \subseteq X^*$  finite, such that  $H \leq \langle Y \rangle$ . Suppose a solution to the word problem of  $G$  is given. Then*

1. *given a solution to the membership problem of  $(G, H)$  it is possible to compute  $\text{Dist}_{(G, X)}^{(H, Y)}$ , and*
2. *given an algorithm that computes  $\text{Dist}_{(G, X)}^{(H, Y)}$  it is possible to construct a solution to the membership problem of  $(G, H)$ .*

**Proof** Suppose a solution to the membership problem of  $(G, H)$  is given. Let  $n \in \mathbb{N}$ .

Using the solution of the membership problem of  $(G, H)$ , it is possible to compute the set

$$W := \{w \in X^* \mid |w| \leq n, w \in H\},$$

which contains representatives of all elements  $h \in H$  with  $|h|_X \leq n$ .

With the solution to the word problem in  $G$  it is now possible to determine the  $Y$ -lengths of all elements which are represented by a word in  $W$ .

The maximum of all these lengths is precisely  $\text{Dist}_{(G, X)}^{(H, Y)}(n)$ .

Suppose now an algorithm computing  $\text{Dist}_{(G, X)}^{(H, Y)}$  is given. Let  $w \in X^*$ .

By definition of  $\text{Dist}_{(G, X)}^{(H, Y)}$ , the element represented by  $w$  lies in  $H$ , if and only if there is a word  $w' \in Y^*$  representing this element with  $|w'| \leq \text{Dist}_{(G, X)}^{(H, Y)}(|w|)$ .

Since a solution to the word problem in  $G$  is given, it is possible to decide for all  $w' \in Y^*$  with  $|w'| \leq \text{Dist}_{(G, X)}^{(H, Y)}(|w|)$ , whether  $\iota(w')w^{-1}$  represents the trivial element in  $G$ .  $\square$

## 3.2 Detection Algorithm

In this section the following is always assumed:

$G$  is a group with symmetric finite generating set  $X$  and  $G$  is relatively hyperbolic with respect to a collection of subgroups  $\mathbb{P} = \{P_1, \dots, P_n\}$ .

There is a finite relative presentation  $\langle X, P_1, \dots, P_n \mid R \rangle$  of  $(G, \mathbb{P})$ . Each  $P_i$  has a finite presentation  $\langle X_i \mid R_i \rangle$ , where  $X_i$  is given as a subset of  $X^*$ .  $H$  is a subgroup of  $G$  with finite generating set  $Y \subseteq X^*$ .

The goal of this section is to prove Theorem 1.1 by giving an explicit algorithm. The core idea of this algorithm will be to successively check for all possible peripheral structures  $\mathbb{O}$  of  $H$  as in Theorem 2.16, whether the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding. If such a structure  $\mathbb{O}$  exists, then by Theorem 2.16,  $H$  is relatively quasiconvex in  $G$ .

However, as mentioned in Remark 2.17, this structure  $\mathbb{O}$  does not always need to be an induced structure. It turns out that to verify both that the above inclusion is a quasiisometric embedding as well as that  $\mathbb{O}$  is an induced structure it is helpful to check for the existence of elements as described in the following lemma. Their existence is an obstruction to  $\mathbb{O}$  being an induced structure.

**Lemma 3.6** *Let  $\mathbb{O} = \{O_1, \dots, O_m\}$  be a peripheral structure of  $H$  consisting of subgroups  $O_j \subseteq H \cap P_{i_j}^{g_j}$  for some  $1 \leq i_j \leq n$  and  $g_j \in X^*$ .*

*Suppose for  $j, k \in \{1, \dots, m\}$  with  $i := i_j = i_k$  there is  $h \in H \cap g_j^{-1}P_i g_k$  with  $h \notin O_j$  if  $j = k$ .*

*Then*

- $O_j$  or  $O_k$  is not maximal parabolic, or
- $j \neq k$  and  $O_j$  and  $O_k$  are conjugated in  $H$ .

**Proof** If  $j = k$ , this is obvious. In this case,  $h \notin O_j$  and therefore  $O_j$  is a proper subgroup of the parabolic group  $\langle O_j, h \rangle \subseteq H \cap P_i^{g_j}$ , i.e.  $O_j$  is not maximal parabolic.

If on the other hand  $j \neq k$  and  $h = g_j^{-1}pg_k$  for some  $p \in P_i$ , then:

$$O_j^h \subseteq (H \cap P_i^{g_j})^h = H^h \cap P_i^{pg_k} = H \cap P_i^{g_k}$$

This implies that  $\langle O_j^h, O_k \rangle$  is a subgroup of  $H \cap P_i^{g_k}$ . So either  $O_j$  or  $O_k$  are not maximal parabolic or  $O_j^h = O_k$ .  $\square$

To verify that the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding it is sufficient to verify that all geodesic words  $w \in (Y \cup \mathcal{O})^*$  are mapped to  $(\lambda, c)$ -quasigeodesic words  $\iota(w) \in (X \cup \mathcal{P})^*$  under the canonical map  $\iota$ , where  $\lambda$  and  $c$  are independent of  $w$ . Since  $(Y \cup \mathcal{O})^*$  contains words of arbitrary length over an infinite alphabet, it is not immediately clear how to verify this property algorithmically.

The reduction to words of bounded length is easy due to the well-known property of hyperbolic spaces that for appropriate constants, local quasigeodesics are quasigeodesics. Hence, it suffices to show for appropriate constants  $L$ ,  $\lambda$  and  $c$  that geodesic words of length at most  $L$  are mapped to  $(\lambda, c)$ -quasigeodesic words.

The problem of the infinite alphabet can be handled by an application of Theorem 2.8. It implies, that for any two quasigeodesic words in  $(X \cup \mathcal{P})^*$  representing the same element and without backtracking every long  $P_i$ -component of one of the quasigeodesics has to be connected to a  $P_i$ -component of the other one. This means that every sufficiently long  $P_i$ -component of the image  $\iota(w)$  of some word  $w \in (Y \cup \mathcal{O})^*$  is connected to a  $P_i$ -component of any geodesic word in  $(X \cup \mathcal{P})^*$  representing the same element, provided that  $\iota(w)$  is without backtracking. It is therefore enough to verify that all geodesic words in  $(Y \cup \mathcal{O})^*$  with short  $O_j$ -components are mapped to uniformly quasigeodesic words.

This is made more precise and proved in Lemma 3.8, but the proof requires a way to ensure that the canonical map  $\iota$  maps geodesic words to words without backtracking. It turns out that this backtracking can only occur, if there are short words as described in Lemma 3.6.

**Lemma 3.7** *Let  $\mathbb{O} = \{O_1, \dots, O_m\}$  be a peripheral structure of  $H$  consisting of subgroups  $O_j \subseteq H \cap P_{i_j}^{g_j}$  for some  $1 \leq i_j \leq n$  and  $g_j \in X^*$ .*

*Let  $L, \lambda \geq 1, c \geq 0$  and*

$$\mu := \max(\{|y| \mid y \in Y\} \cup \{2|g_j| + 1 \mid 1 \leq j \leq m\}).$$



Let  $\varepsilon = \varepsilon(\lambda, c + \mu)$  be as in the conclusion of Theorem 2.8 and

$$D = \max_{1 \leq j \leq m} \left( \text{Dist}_{(G, X)}^{(O_j, Y)}(\varepsilon + 2|g_j|) \right).$$

Suppose that

1. for any geodesic word  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$ , for which  $\iota(w)$  is without backtracking,  $\iota(w) \in (X \cup \mathcal{P})^*$  is a  $(\lambda, c)$ -quasigeodesic word, and
2. there is some geodesic word  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$  such that  $\iota(w)$  has backtracking.

Then there is  $h \in H \cap g_j^{-1}P_i g_k$  with  $i = i_j = i_k$ ,  $h \notin O_j$  if  $j = k$ , and  $|h|_Y \leq D\lambda(\mu + c)$

**Proof** Let  $w \in (Y \cup \mathcal{O})^*$  be a geodesic word with  $|w| \leq L$ . Suppose the path  $v$  in  $\Gamma(G, X \cup \mathcal{P})$  starting at 1 and labeled by  $\iota(w)$  has backtracking. Then there is a subword  $o_1 h o_2$  of  $w$  with  $h \in (Y \cup \mathcal{O})^*$ ,  $o_1 \in O_j$ ,  $o_2 \in O_k$  and  $i := i_j = i_k$  such that the  $P_i$ -components  $p_1$  and  $p_2$  of the subword

$$\iota(o_1 h o_2) = g_j^{-1} p_1 g_j \iota(h) g_k^{-1} p_2 g_k$$

of  $\iota(w)$  are connected and that  $\iota(h)$  is without backtracking and has no  $P_i$ -component connected to  $p_1$  and  $p_2$ . If  $j = k$ , then  $h \notin O_j$  since  $w$  is geodesic and therefore without backtracking.

Since  $p_1$  and  $p_2$  are connected,  $g_j h g_k^{-1}$  represents some element  $p \in P_i$ . As a subword of  $w$ ,  $h$  is geodesic with  $|h| \leq L$  and since  $\iota(h)$  is without backtracking, it is  $(\lambda, c)$ -quasigeodesic by assumption 1. Hence:

$$|h|_{Y \cup \mathcal{O}} \leq |\iota(h)| \leq \lambda|h|_{X \cup \mathcal{P}} + \lambda c = \lambda|g_j^{-1} p g_k|_{X \cup \mathcal{P}} + \lambda c \leq \lambda(\mu + c)$$

Now  $p^{-1} g_j \iota(h) g_k^{-1}$  is a  $(\lambda, c + \mu)$ -quasigeodesic without backtracking and representing the trivial element. The choice of  $\varepsilon$  implies that all of its  $\mathbb{P}$ -components must have a  $d_X$ -length of at most  $\varepsilon$ . Since every  $O_j$ -component  $o$  of  $h$  is mapped under  $\iota$  to  $g_j^{-1} p g_j$  for some  $P_{i_j}$ -component

$p$  of  $\iota(h)$ , it follows that  $|o|_X \leq \varepsilon + 2|g_j|$ . The choice of  $D$  then implies  $|o|_Y \leq D$ , which in turn implies

$$|h|_Y \leq D\lambda(\mu + c). \quad \square$$

It is now possible to prove the following lemma, which states that under the right conditions it suffices to check whether short geodesic words with short  $\mathcal{O}$ -components in  $(Y \cup \mathcal{O})^*$  are mapped to quasigeodesic words in  $(X \cup \mathcal{P})^*$  to verify that the same is true for short geodesic words with arbitrarily large  $\mathcal{O}$ -components.

**Lemma 3.8** *Let  $\mathcal{O} = \{O_1, \dots, O_m\}$  be a peripheral structure of  $H$  consisting of subgroups  $O_j \subseteq H \cap P_{i_j}^{g_j}$  for some  $1 \leq i_j \leq n$  and  $g_j \in X^*$ .*

*Let  $\mu := \max(\{|y| \mid y \in Y\} \cup \{2|g_j| + 1 \mid 1 \leq j \leq m\})$ ,  $N \geq 2(\mu + 1)^2$  and  $L \geq 1$ . Let  $\varepsilon = \varepsilon(N, N + \mu)$  be as in the conclusion of Theorem 2.8,  $C := \sqrt{\frac{N}{2}} - \mu$  and*

$$D = \max_{1 \leq j \leq m} \left( \text{Dist}_{(G, X)}^{(O_j, Y)}(\varepsilon + 2|g_j|) \right).$$

*Suppose that*

- 1. for every geodesic word  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$  and  $|o|_Y \leq D$  for every letter  $o \in \mathcal{O}$  of  $w$  the word  $\iota(w)$  is a  $(C, C)$ -quasigeodesic word in  $(X \cup \mathcal{P})^*$ , and*
- 2. there is no  $h \in H \cap g_j^{-1}P_i g_k$  with  $i = i_j = i_k$ ,  $h \notin O_j$  if  $j = k$ , and  $|h|_Y \leq DN(\mu + N)$ .*

*Then for every geodesic  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$  the word  $\iota(w)$  is a  $(N, N)$ -quasigeodesic word in  $(X \cup \mathcal{P})^*$ .*

**Proof** By Lemma 3.7 and assumption 2., it suffices to show the claim for  $w \in (Y \cup \mathcal{O})^*$  geodesic with  $|w| \leq L$  such that the path  $v'$  in  $\Gamma(G, X \cup \mathcal{P})$  starting at 1 and labeled by  $\iota(w)$  is without backtracking.

If  $|o|_Y \leq D$  for every letter  $o \in \mathcal{O}$  of  $w$ , then  $\iota(w)$  is a  $(C, C)$ -quasigeodesic by assumption 1. and the claim follows as  $C \leq N$ .

Thus, assume there is some letter  $o \in O_j$  of  $w$  with  $|o|_Y \geq D$ . Hence,  $o = g_j^{-1} p g_j$  for some  $p \in P_{i_j}$ . It follows that

$$\text{Dist}_{(G,X)}^{(O_j,Y)}(|o|_X) \geq |o|_Y \geq D \geq \text{Dist}_{(G,X)}^{(O_j,Y)}(\varepsilon + 2|g_j|).$$

The monotony of the distortion function implies that

$$|p|_X \geq |o|_X - 2|g_j| \geq \varepsilon.$$

Since  $\iota(w)$  is without backtracking, it follows from Theorem 2.8 that a geodesic  $v$  in  $\Gamma(G, X \cup \mathcal{P})$  from 1 to  $w$  must have a  $P_{i_j}$ -component connected to the  $P_{i_j}$ -component  $p$  of  $\iota(w)$ .

If  $w$  has at least  $\frac{|\iota(w)|}{N} - (N - 2\mu)$  of such long  $\mathbb{O}$ -components, i.e.  $\mathbb{O}$ -components of  $d_X$ -length at least  $\varepsilon + 2|g_j|$ , then  $v$  must have at least the same number of  $\mathbb{P}$ -components and therefore:

$$|w|_{X \cup \mathcal{P}} = |v| \geq \frac{|\iota(w)|}{N} - (N - 2\mu)$$

Suppose now that  $w$  has  $k \leq \frac{|\iota(w)|}{N} - (N - 2\mu)$  long  $\mathbb{O}$ -components. Then  $w$  can be written as  $w_1 o_1 w_2 o_2 \dots o_k w_{k+1}$ , where the  $o_l$  are the long  $\mathbb{O}$ -components of  $w$  and  $w_1, \dots, w_{k+1} \in (Y \cup \mathcal{O})^*$  are the (possibly empty) subwords of  $w$  between them. Each  $o_l$  is mapped under  $\iota$  to some word  $g_{j_l}^{-1} p_l g_{j_l}$ , where  $p_l \in P_{i_{j_l}}$  is a  $\mathbb{P}$ -component of  $\iota(w)$ . The geodesic  $v$  must then have  $\mathbb{P}$ -components  $q_1, \dots, q_k$ , where each  $q_l$  is connected to the  $\mathbb{P}$ -component  $p_l$  of  $\iota(w)$ . Using Theorem 2.8 and the choice of  $\varepsilon$ , it is easy to show that  $q_1, \dots, q_k$  appear in  $v$  in order, i.e. that  $v$  is of the form  $v_1 q_1 v_2 q_2 \dots q_k v_{k+1}$ . This implies that for each  $l \in \{1, \dots, k+1\}$

$$|v_l|_{X \cup \mathcal{P}} \geq |w_l|_{X \cup \mathcal{P}} - 2(\max\{|g_j| \mid 1 \leq j \leq m\} + 1).$$

Since the  $w_l$  are shorter than  $w$  and contain no long  $\mathbb{O}$ -components, their images  $\iota(w_l)$  are  $(C, C)$ -quasigeodesic by assumption 1. It follows that:

$$\begin{aligned}
|w|_{X \cup \mathcal{P}} &= |v| = \sum_{l=1}^{k+1} |v_l|_{X \cup \mathcal{P}} + k \\
&\geq \sum_{l=1}^{k+1} |w_l|_{X \cup \mathcal{P}} - 2k(\max\{|g_j| \mid 1 \leq j \leq m\} + 1) + k \\
&\geq \sum_{l=1}^{k+1} \left( \frac{1}{C} |\iota(w_l)| - C \right) - k\mu \\
&= \frac{1}{C} \left( k\mu + \sum_{l=1}^{k+1} |\iota(w_l)| \right) - \frac{k\mu}{C} - (k+1)C - k\mu \\
&\geq \frac{1}{C} |\iota(w)| - \frac{k\mu}{C} - (k+1)C - k\mu \\
&= \frac{1}{C} |\iota(w)| - \left( C + k \left( C + \left( 1 + \frac{1}{C} \right) \mu \right) \right) \\
&\geq \frac{1}{C} |\iota(w)| - \left( C + \left( \frac{|\iota(w)|}{N} - (N - 2\mu) \right) (C + 2\mu) \right) \\
&= \left( \frac{1}{C} - \frac{1}{N} (C + 2\mu) \right) |\iota(w)| - (C - (N - 2\mu)(C + 2\mu))
\end{aligned}$$

Since  $C \leq N - 2\mu$  and  $2C(C + \mu) \leq N$ , it follows again that:

$$|w|_{X \cup \mathcal{P}} \geq \frac{|\iota(w)|}{N} - (N - 2\mu)$$

Now let  $w'$  be a non-trivial subword of  $\iota(w)$ . Let  $s$  be minimal, such that  $w'$  is a subword of  $\iota(w_r \dots w_{r+s})$  for some  $r \in \{1, \dots, k+1\}$ . It follows that:

$$|w'|_{X \cup \mathcal{P}} \geq |\iota(w_r \dots w_{r+s})|_{X \cup \mathcal{P}} - 2\mu \geq \frac{|\iota(w_r \dots w_{r+s})|}{N} - N \geq \frac{|w'|}{N} - N$$

Hence,  $\iota(w)$  is  $(N, N)$ -quasigeodesic.  $\square$

The rest of this section is dedicated to the proof of the main theorem:

**Theorem 1.1** *Suppose that for all  $i \in \{1, \dots, n\}$  the following are given:*

1. *a solution to the membership problem of  $(P_i, O)$  for every finitely*

generated subgroup  $O \leq P_i$  and

2. an algorithm, which decides whether a given finitely generated subgroup of some  $P_i$  is finite.

Then there is a semi-algorithm which stops, if and only if  $H$  is a relatively quasiconvex subgroup of  $G$ , and if it stops, it returns

- generating sets of an induced structure  $\mathbb{O}$  on  $H$ , and
- $\lambda \geq 1, c \geq 0$ , such that the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(\lambda, c)$ -quasiisometric embedding.

We will first prove an auxiliary lemma:

**Lemma 3.9** *In the situation of Theorem 1.1, the following operations can be executed in finite time:*

- (O1) For  $g \in G$ , finitely generated  $O \leq H \cap P_i^g$  and  $n \in \mathbb{N}$  compute  $\text{Dist}_{(G,X)}^{(O,Y)}(n)$ .
- (O2) For  $w \in X^*$  and finitely generated  $O \leq H \cap P_i^g$  decide, whether  $w$  represents an element in  $O$ .
- (O3) For  $C \geq 1$  and  $w \in (X \cup \mathcal{P})^*$  decide, whether  $w$  is a  $(C, C)$ -quasigeodesic word.

**Proof** By assumption 1., a solution to the word problem in each  $P_i$  is given. By Zhang ([24], Thm. 4.1.12) it is therefore possible to construct a solution to the word problem in  $G$ .

- (O1) Let  $g \in G$ ,  $O \leq H \cap P_i^g$  with finite generating set  $S \subseteq Y^*$  and  $k \in \mathbb{N}$ .

By Theorem 2.5, an upper bound for the relative Dehn function of  $(G, \mathbb{P})$  can be computed. It follows from Theorem 3.4, that it is possible to compute  $\text{Dist}_{(G,X)}^{(P_i, X_i)}$ . Since a solution to the membership

problem of  $(P_i, O^{g^{-1}})$  is given, Theorem 3.5 implies that it is possible to construct an algorithm that computes  $\text{Dist}_{(P_i, X_i)}^{(O^{g^{-1}}, S^{g^{-1}})}$ . Let  $L := \max\{|s|_Y \mid s \in S\}$ . By Lemma 3.2 and Lemma 3.3:

$$\begin{aligned} \text{Dist}_{(G, X)}^{(O, Y)}(k) &\leq L \cdot \text{Dist}_{(G, X)}^{(O, S)}(k) \\ &\leq L \cdot \text{Dist}_{(G, X)}^{(O^{g^{-1}}, S^{g^{-1}})}(k + 2|g|) \\ &\leq L \cdot \left( \text{Dist}_{(P_i, X_i)}^{(O^{g^{-1}}, S^{g^{-1}})} \circ \text{Dist}_{(G, X)}^{(P_i, X_i)} \right)(k + 2|g|) \end{aligned}$$

Therefore, an upper bound for  $\text{Dist}_{(G, X)}^{(O, Y)}(k)$  can be computed. With this upper bound it is further possible to compute  $\text{Dist}_{(G, X)}^{(O, Y)}(k)$  itself.

(O2) As explained above, it is possible to construct a solution to the word problem in  $G$ . Since  $\text{Dist}_{(G, X)}^{(O, Y)}$  is computable, it is by Theorem 3.5 possible to construct a solution to the membership problem for  $(G, O)$ .

(O3) It suffices to show that it is possible for any  $w \in (X \cup \mathcal{P})^*$  to compute  $|w|_{X \cup \mathcal{P}}$ .

Since  $\text{Dist}_{(G, X)}^{(P_i, X_i)}$  can be computed for all  $i \in \{1, \dots, n\}$ , the membership problem of  $(G, P_i)$  is solvable by Theorem 3.5. Hence, it is possible to construct a word  $w' \in (X \cup \mathcal{P})^*$  representing the same element as  $w$  which is without backtracking.

Let  $\varepsilon(|w'|, |w'|)$  be as in the conclusion of Theorem 2.8 and let  $v \in (X \cup \mathcal{P})^*$  be a geodesic word representing the same element as  $w$ . Since  $v$  and  $w'$  are obviously  $(|w'|, |w'|)$ -quasigeodesic, it follows from Theorem 2.8 that the  $|\cdot|_X$ -length of the  $\mathbb{P}$ -components of  $v$  is bounded by:

$$\max\{|p|_X \mid p \text{ is a } \mathbb{P}\text{-component of } w'\} + 2\varepsilon$$

Since solutions to the membership problem for  $(G, P_i)$  and the word

problem for  $P_i$  are given, it is possible to find a shortest word in  $(X \cup \mathcal{P})^*$  with  $\mathbb{P}$ -components of bounded length as above, which represents the same element as  $w$ . The length of this word is then  $|w|_{X \cup \mathcal{P}}$ .  $\square$

**Proof (of Theorem 1.1)** The algorithm is as follows:

Use the algorithm of Theorem 2.5 to compute a hyperbolicity constant  $\delta$  for the Cayley graph  $\Gamma(G, X \cup \mathcal{P})$ .

Begin enumerating all  $m \in \mathbb{N}$  as well as  $g_j \in G$ ,  $i_j \in \{1, \dots, n\}$  and finite sets  $Y_j \subseteq H \cap P_{i_j}^{g_j}$  for all  $j \in \{1, \dots, m\}$ . This is possible, since (O2) provides a solution to the membership problem of  $(G, P_i)$  for all  $i \in \{1, \dots, n\}$ , and hence the set of all triples  $(h, g, i) \in H \times G \times \{1, \dots, n\}$  with  $ghg^{-1} \in P_i$  can be computed.

While enumerating these sets, run the following partial algorithm for each new set in parallel:

- Step 1:

Let  $\mu := \max(\{|y| \mid y \in Y\} \cup \{2|g_j| + 1 \mid 1 \leq j \leq m\})$  and  $N := \lceil 2(\mu + 1)^2 \rceil$ . Continue with step 2.

- Step 2:

There are computable constants  $L > 0$  and  $\lambda \geq 1, c \geq 0$  only depending on  $\delta$  and  $N$ , such that every  $(L, N, N)$ -local-quasigeodesic in  $\Gamma(G, X \cup \mathcal{P})$  is a  $(\lambda/\mu, c)$ -quasigeodesic. (cf. Coornaert et al. [8], Ch. 3, Thm. 1.4)

Let  $C := \sqrt{\frac{N}{2}} - \mu$  and  $\varepsilon = \max\{\varepsilon(\lambda/\mu, c + \mu), \varepsilon(N, N + \mu)\}$  as in the conclusion of Theorem 2.8. Using (O1), find an upper bound

$$D \geq \max_{1 \leq j \leq m} \left( \text{Dist}_{(G, X)}^{(O_j, Y)}(\varepsilon + 2|g_j|) \right)$$

Continue with step 3.

- Step 3:

Utilizing (O2), check for all  $j, k \in \{1, \dots, m\}$  with  $i := i_j = i_k$  and for all  $h \in Y^*$  with  $|h| \leq \max\{DN(\mu + N), D(\lambda/\mu)(\mu + c)\}$ , whether  $g_j h g_k^{-1} \in P_i$  and  $h \notin O_j$  if  $j = k$ .

If no such  $h$  is found, continue with step 4, else terminate this instance of the partial algorithm.

- Step 4:

Check with (O3) for every geodesic word  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$  and  $|o|_Y \leq D$  for every letter  $o \in \mathcal{O}$  of  $w$ , whether  $\iota(w)$  is a  $(C, C)$ -quasigeodesic word in  $(X \cup \mathcal{P})^*$ .

If there is a word  $w$  such that  $\iota(w)$  is not  $(C, C)$ -quasigeodesic, increase  $N$  by one and repeat from step 2, else continue with step 5.

- Step 5:

Using the algorithm from assumption 2., check whether  $O_j$  is finite for every  $j \in \{1, \dots, m\}$ . Eliminate all finite peripheral subgroups from  $\mathbb{O}$ .

Terminate the whole algorithm and return the constructed peripheral structure  $\mathbb{O}$  of  $H$  as well as  $\lambda$  and  $c$ .

First, assume this algorithm terminates. The elimination of all finite peripheral subgroups in step 5 does not change that, as determined in step 4, all geodesic words  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$  and short  $\mathbb{O}$ -components are mapped to  $(C, C)$ -quasigeodesics under  $\iota$ . So every geodesic word  $w \in (Y \cup \mathcal{O})^*$  with  $|w| \leq L$  is according to Lemma 3.8 mapped to a  $(N, N)$ -quasigeodesic word in  $(X \cup \mathcal{P})^*$ .

Therefore, every geodesic word  $w \in (Y \cup \mathcal{O})^*$  is mapped under  $\iota$  to a  $(L, N, N)$ -locally-quasigeodesic word. Because of the choice of  $L$  and  $N$ , the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(\lambda, c)$ -quasiisometric embedding. In particular,  $H$  is relatively quasiconvex in  $(G, \mathbb{P})$  by Theorem 2.16.



It remains to show that  $\mathbb{O}$  is an induced structure. Let  $i \in \{1, \dots, n\}$  and  $g \in X^*$  such that  $H \cap P_i^g$  is infinite. If  $h \in H \cap P_i^g$ , then:

$$|h|_{Y \cup \mathcal{O}} \leq \lambda |h|_{X \cup \mathcal{P}} + \lambda c \leq \lambda(2|g| + 1) + \lambda c$$

Since  $H \cap P_i^g$  is infinite but also contained in a ball of finite radius in  $\Gamma(H, Y \cup \mathcal{O})$ , there has to be some geodesic word  $w = w_1 o w_2 \in (Y \cup \mathcal{O})^*$ , representing an element  $g^{-1} p g \in H \cap P_i^g$ , which contains an  $O_j$ -component  $o = g_j^{-1} p' g_j$  of  $|\cdot|_Y$ -length at least  $D$ . The  $P_{i_j}$ -component  $p'$  of  $\iota(w) = \iota(w_1) g_j^{-1} p' g_j \iota(w_2)$  therefore has  $|\cdot|_X$ -length at least  $\varepsilon$ .

By Theorem 2.8,  $i = i_j$  and the  $P_i$ -components  $p$  of  $g^{-1} p g \in (X \cup \mathcal{P})^*$  and  $p'$  of  $\iota(w)$  are connected. This means that  $p'' := g_j w_1^{-1} g^{-1} \in P_i$  and therefore:

$$O_j^{w_1^{-1}} \subseteq (H \cap P_i^{g_j})^{w_1^{-1}} = H^{w_1^{-1}} \cap P_i^{p'' g} = H \cap P_i^g$$

It now suffices to show that  $O_j = H \cap P_i^{g_j}$  for all  $1 \leq j \leq m$ .

Since geodesic words of arbitrary length are mapped to  $(\lambda/\mu, c)$ -quasigeodesic words under  $\iota$  and since there is no  $h \in H \cap P_i^{g_j} \setminus O_j$  with  $|h|_Y \leq D_\mu^\lambda(\mu + c)$ ,  $\iota$  must map all geodesic words to words without backtracking by Lemma 3.7. This implies that there can be no  $h \in H \cap P_i^{g_j} \setminus O_j$  at all. Therefore  $H \cap P_i^{g_j} = O_j$ .

Conversely it is easy to see, that the algorithm will terminate if  $H$  is relatively quasiconvex in  $G$ :

Since  $(G, \mathbb{P})$  is relatively hyperbolic, the algorithm of Dahmani from Theorem 2.5 will terminate. Since all subgroups in the induced structure of a finitely generated relatively quasiconvex subgroup are finitely generated (Osin [22], Proposition 2.29), it is also clear that enumerating the sets on which to run the partial algorithm will eventually produce an induced structure, if  $H$  is relatively quasiconvex. For an induced structure the inclusion  $(H, d_{Y \cup \mathcal{O}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding (cf. Hruska [15], Thm. 10.1), hence the partial algorithm will terminate for such a structure.  $\square$

### 3.3 Discussion

This section first discusses the necessity of the main theorem's assumptions and to what extent it might be possible to relax or even omit them. Secondly, a more constructive approach to the algorithm is sketched and evaluated.

The given algorithm relies strongly on the solutions to the membership problems provided by assumption 1. In particular step 3 of the partial algorithm depends on these solutions, since it searches for parabolic elements of  $H$ , which are not accounted for by the peripheral structure. A meaningful relaxation of these conditions, if at all possible, would therefore at least require a fundamental change of the algorithm.

In contrast, it is possible to prove a version of the main theorem, which does not require the decidability of the finiteness of finitely generated parabolic subgroups. In this version of the algorithm the elimination of the finite peripheral subgroups in step 5 is dropped. The consequence of this modification is that the returned peripheral structure will consist of an induced structure together with a collection of finite groups.

It is also easy to see that assumption 2. is a necessary condition to eliminate the finite subgroups from the induced structure:

Let  $G$  be a group with solvable word problem and  $H$  a finitely generated subgroup of  $G$ .  $H$  is relatively quasiconvex in  $(G, \{G\})$  and the induced structure of  $(G, \{G\})$  on  $H$  is  $\{H\}$  if  $H$  is infinite and  $\emptyset$  if  $H$  is finite. Therefore any algorithm determining the induced structure of  $H$  also determines whether  $H$  is finite.

The given algorithm relies on the ability to verify, whether a given peripheral structure of  $H$  is an induced structure and then systematically checks every possible peripheral structure until it finds an induced one. It is also possible to take a more constructive approach to finding an induced structure. Instead of terminating the partial algorithm in step 3 if an  $h$  fulfilling the conditions is found, it would be possible to modify the peripheral structure according to Lemma 3.6, i.e. adding  $h$  to  $Y_j$  if  $j = k$  and replacing  $Y_j$  and  $Y_k$  by  $Y_j^h \cup Y_k$  if  $j \neq k$ , and then repeat from

step 2.

Through this process the peripheral subgroups will be replaced by pairwise non-conjugate maximal parabolic subgroups eventually. However, this new peripheral structure does not have to include an induced structure, since it is not clear at the beginning of this process, that for every infinite maximal parabolic subgroup of  $H$  there is a peripheral subgroup in  $\mathbb{O}$  which can in  $H$  be conjugated into this maximal parabolic subgroup.

A more constructive approach would therefore require the addition of new groups to the peripheral structure, as well as the modification of peripheral groups as described above. Without the ability to verify that a given peripheral structure is not an induced structure this still requires running parallel partial algorithms. Hence it is in general not clear which approach is the more efficient for a concrete implementation of the algorithm.

The algorithm given in the proof above was chosen because it requires less careful bookkeeping than the constructive approach and is therefore more comprehensible.

## 4 Combining Relatively Quasiconvex Subgroups

### 4.1 Carrier Graphs

This section introduces the concept of a carrier graph of groups and applies it to the setting of a torsion-free relatively hyperbolic group.

For a hyperbolic manifold  $M$ , a carrier graph of  $M$  is a graph  $X$  together with a  $\pi_1$ -surjective map  $X \rightarrow M$ . Carrier graphs can be used to study the fundamental groups of hyperbolic manifolds. For example, Biringer and Souto [5] used them to study the ranks of the fundamental groups of mapping tori of closed orientable surfaces.

The following definition of a carrier graph of groups is an algebraic version of the concept of a carrier graph.

**Definition 4.1** Let  $G$  be a group. A  $G$ -carrier graph of groups is a triple

$$\mathcal{A} = (\mathbb{A}, (g_e)_{e \in EA}, E_0)$$

such that the following hold:

1.  $\mathbb{A}$  is a graph of groups such that  $A_x \leq G$  for all  $x \in EA \cup VA$ .
2.  $g_e \in G$  for any  $e \in EA$  such that  $g_{e^{-1}} = g_e^{-1}$  for all  $e \in EA$ .
3.  $E_0$  is an orientation of  $A$ , i.e.  $EA = E_0 \sqcup E_0^{-1}$ .
4. For any  $e \in E_0$  the following hold:
  - (a)  $A_e$  is a subgroup of  $A_{\alpha(e)}$  and  $\alpha_e: A_e \rightarrow A_{\alpha(e)}$  is the inclusion map.
  - (b) The map  $\omega_e: A_e \rightarrow A_{\omega(e)}$  is given by  $\omega_e(a) = g_e^{-1}ag_e$  for all  $a \in A_e$ .

The additional information introduced by the  $g_e$  yields a canonical homomorphism of the fundamental group of  $\mathbb{A}$  into  $G$ .

**Lemma & Definition 4.2** *Let  $G$  be a group and  $\mathcal{A} = (\mathbb{A}, (g_e)_{e \in EA}, E_0)$  be a  $G$ -carrier graph of groups. For any  $\mathbb{A}$ -path  $t = (a_0, e_1, a_1, \dots, e_k, a_k)$  let*

$$\nu_{\mathcal{A}}(t) := a_0 g_{e_1} a_1 \cdot \dots \cdot g_{e_k} a_k.$$

*This induces a well-defined group homomorphism*

$$\nu_{\mathcal{A}}: \pi_1(\mathbb{A}, v_0) \rightarrow G, [t] \mapsto \nu_{\mathcal{A}}([t]) := \nu_{\mathcal{A}}(t).$$

$\mathcal{A}$  is then called a  $G$ -carrier graph of groups of  $H := \text{Im } \nu_{\mathcal{A}}$ .

*If  $G$  is clear from the context then  $\mathcal{A}$  will simply be called a carrier graph of groups or even just a carrier of  $H$ .*

When working with a carrier graph  $\mathcal{A}$  it is sometimes helpful to consider a subdivision of  $\mathcal{A}$ .

**Definition 4.3** Let  $G$  be a group and  $\mathcal{A} = (\mathbb{A}, (g_e)_{e \in EA}, E_0)$  a  $G$ -carrier graph of groups.

Let  $e \in E_0$ . A *subdivision of  $\mathcal{A}$  along  $e$*  is a  $G$ -carrier graph of groups  $\mathcal{A}' = (\mathbb{A}', (g_e)_{e \in EA'}, E'_0)$  with the following properties:

1.  $VA' = VA \sqcup \{v\}$ ,
2.  $EA' = EA \sqcup \{e'^{\pm 1}, e''^{\pm 1}\} \setminus \{e^{\pm 1}\}$ ,  $E'_0 = E_0 \sqcup \{e', e''\} \setminus \{e\}$ ,
3.  $\alpha(e') = \alpha(e)$ ,  $\omega(e') = \alpha(e'') = v$ ,  $\omega(e'') = \omega(e)$ ,
4.  $g_{e'}, g_{e''} \in G$  with  $g_e = g_{e'} g_{e''}$  and  $|g_e|_{X \cup \mathcal{P}} = |g_{e'}|_{X \cup \mathcal{P}} + |g_{e''}|_{X \cup \mathcal{P}}$ ,
5.  $A'_{e'} = A_e$ ,  $A_v = A_{e''} = g_{e'}^{-1} A_e g_{e'}$ , and
6.  $\alpha_{e'}(a') = \alpha_e(a')$ ,  $\omega_{e'}(a') = g_{e'}^{-1} a' g_{e'}$ ,  $\alpha_{e''}(a'') = a''$ ,  $\omega_{e''}(a'') = g_{e''}^{-1} a'' g_{e''}$   
for all  $a' \in A'_{e'}$ ,  $a'' \in A'_{e''}$ .

It is obvious that such a subdivision exists for any decomposition of  $g_e$  as described in 4. Moreover, for any  $v_0 \in VA$  there exists a canonical isomorphism  $\phi: \pi_1(\mathbb{A}, v_0) \rightarrow \pi_1(\mathbb{A}', v_0)$  such that  $\nu_{\mathcal{A}} = \nu_{\mathcal{A}'} \circ \phi$ .

A tree  $Y$  of diameter at most 2 is called a star. A vertex  $v \in VY$  of a star  $Y$  is called central if any vertex of  $Y$  has a distance of at most one from  $v$ . Note that the central vertex of a star is unique, unless  $Y$  consists of a single edge.

In the case of subgroups of torsion-free relatively hyperbolic groups we are interested in a restricted class of carrier graphs of groups.

**Definition 4.4** Let  $G$  be a torsion-free group that is hyperbolic relative to a peripheral structure  $\mathbb{P} = \{P_1, \dots, P_n\}$ . A  $(G, \mathbb{P})$ -carrier graph of groups is a pair

$$(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$$

such that the following is satisfied:

1.  $\mathcal{A} = (\mathbb{A}, (g_e)_{e \in EA}, E_0)$  is a  $G$ -carrier graph of groups.
2. For each  $i \in \{1, \dots, k\}$  there exists  $m_i \in \{1, \dots, n\}$  such that  $\mathcal{C}_i$  is a  $P_{m_i}$ -carrier star of groups with central vertex  $c_i$ . Moreover all boundary monomorphisms of  $\mathcal{C}_i$  are isomorphisms.
3.  $(\mathcal{C}_i)_{1 \leq i \leq k}$  is a disjoint family of  $G$ -sub-carrier graphs of groups of  $\mathcal{A}$ , such that every edge of  $\mathcal{A}$  incident to  $c_i$  is in  $EC_i$ . We call the vertices and edges of the  $\mathcal{C}_i$  *peripheral*.
4. If  $A_e \neq 1$  then  $e$  is peripheral or  $e$  is incident to a unique peripheral vertex.
5.  $\alpha(e)$  is peripheral for any  $e \in E_0$  with  $A_e \neq 1$ . If  $e \in E_0 \cap EC_i$ , then  $\alpha(e) = c_i$ .

Non-peripheral edges of  $\mathbb{A}$  with non-trivial edge group, and non-peripheral vertices of  $\mathbb{A}$  with non-trivial vertex group are called *essential*.

The following two definitions give some more useful terminology in the context of  $(G, \mathbb{P})$ -carrier graphs of groups.

**Definition 4.5** Let  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  be a  $(G, \mathbb{P})$ -carrier graph of groups. Let  $p = (e_1, \dots, e_k)$  be a path in  $A$  and let  $v_0 := \alpha(e_1)$ ,  $v_m := \omega(e_m)$  for all  $m \in \{1, \dots, k\}$ .

$p$  is called

1. *almost-simple*, if  $e_i \neq e_j^{\pm 1}$  for all  $i \neq j \in \{1, \dots, k\}$ , and if  $v_i = v_j$  for  $i \neq j \in \{1, \dots, k\}$  implies that  $v_i$  is a central peripheral vertex, and
2. an *almost-circuit*, if it is a closed almost-simple path.

Let  $t$  be an  $\mathbb{A}$ -path.  $t$  is called *almost-simple* or an  $\mathbb{A}$ -*almost-circuit*, if its underlying path is almost-simple or an almost-circuit, respectively.

For any  $\mathbb{A}$ -path  $t$ , an  $\mathbb{A}$ -subpath  $s$  of  $t$  is called a *full  $\mathbb{A}$ -subpath* of  $t$ , if any maximal peripheral  $\mathbb{A}$ -subpath of  $t$  is either contained in  $s$  or has trivial overlap with  $s$ .

**Definition 4.6** Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group.

1. Let  $\mathcal{A}, \mathcal{B}$  be  $(G, \mathbb{P})$ -carrier graphs without trivial edge groups. Then  $\mathcal{A}$  is *equivalent* to  $\mathcal{B}$ , if there is a graph isomorphism  $f: A \rightarrow B$  such that
  - (a)  $B_{f(x)} = A_x$  for all non-peripheral  $x \in VA \cup EA$ , and
  - (b)  $g_{f(e)} = g_e$  for all non-peripheral  $e \in EA$ .
2. Let  $\mathcal{A}, \mathcal{B}$  be  $(G, \mathbb{P})$ -carrier graphs and let  $\mathcal{A}_1, \dots, \mathcal{A}_k$  and  $\mathcal{B}_1, \dots, \mathcal{B}_l$  be the components of  $\mathcal{A}$  and  $\mathcal{B}$  respectively, which arise from deleting all edges and vertices with trivial edge or vertex group.

Then  $\mathcal{A}$  and  $\mathcal{B}$  are *equivalent*, if

- (a)  $k = l$ ,
- (b) there is a bijection  $f: \{1, \dots, k\} \rightarrow \{1, \dots, k\}$  such that  $\mathcal{A}_i$  is equivalent to  $\mathcal{B}_{f(i)}$  for all  $i \in \{1, \dots, k\}$ , and
- (c)  $b_1(A) = b_1(B)$ .

**Remark 4.7** If  $\mathbb{A}$  is minimal, Lemma 2.23 gives an upper bound for the number  $N(\mathbb{A})$  of maximal trivial segments of  $\mathbb{A}$ , which only depends on the equivalence class of  $\mathcal{A}$ .

The canonical homomorphism  $\nu_{\mathcal{A}}$  from Lemma & Definition 4.2 maps every  $\mathbb{A}$ -path  $t$  in a  $(G, \mathbb{P})$ -carrier graph of groups  $\mathcal{A}$  to some element  $\nu_{\mathcal{A}}(t) \in G$ . It is however possible to retain more information on  $t$  by considering associated paths in  $\text{Cay}(G, X \cup \mathcal{P})$ .

**Definition 4.8** Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with symmetric finite generating set  $X$  and  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  a  $(G, \mathbb{P})$ -carrier graph of groups.

Let  $t = (a_0, e_1, a_1, \dots, e_k, a_k)$  be an  $\mathbb{A}$ -path,  $v_0 := \alpha(e_1)$ , and further  $v_m := \omega(e_m)$  for  $m \in \{1, \dots, k\}$ . Let  $s = s_0 \dots s_{2k}$  be a path in  $\text{Cay}(G, X \cup \mathcal{P})$  such that the following hold:

1.  $s_{2j}$  is a geodesic whose label represents  $a_j$  for  $0 \leq j \leq k$ .
2.  $s_{2j-1}$  is a geodesic whose label represents  $g_{e_j}$  for  $1 \leq j \leq k$ .

Obtain a new path  $\bar{s}$  from  $s$  as follows: For every maximal  $\mathbb{C}_i$ -subpath of  $t$  replace the corresponding subpath  $s'$  of  $s$  by the edge connecting its endpoints and labeled by the element of  $P_{m_i}$  that is represented by the label of  $s'$ .

$\bar{s}$  is called a *realization of  $t$  in  $\text{Cay}(G, X \cup \mathcal{P})$*  or just a *realization of  $t$* .

Suppose there is some  $M \geq 0$  such that every essential vertex group is generated by elements of  $X$ -length at most  $M$ . Let  $\nu := \nu(G, \mathbb{P}, X, M)$  as in Lemma & Definition 2.18.

An edge  $e$  of  $\bar{s}$  which is labeled by some parabolic element is called a *peripheral edge of  $\bar{s}$* , if it was added in the above way, or if it is an edge of some  $s_{2i}$  with  $\text{len}_X(e) \geq \nu$ .

Since, by this construction, any subpath  $s$  of a realization of some  $\mathbb{A}$ -path has a canonical decomposition into geodesics, the piecewise  $X$ -length  $\text{len}_X^p(s)$  will always be taken with regard to this canonical decomposition.



It is immediate that the lengths  $\text{len}(s)$  and  $\text{len}_X^p(s)$  of a realization  $s$  of some  $\mathbb{A}$ -path  $t$  only depend on  $t$ . This observation gives rise to two metrics on  $\pi_1(\mathbb{A}, v_0)$ :

**Lemma & Definition 4.9** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with symmetric finite generating set  $X$  and  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  a  $(G, \mathbb{P})$ -carrier graph of groups.*

1. *For each  $\mathbb{A}$ -path  $t$  and some (any) realization  $s$  of  $t$  define the  $X \cup \mathcal{P}$ -length of  $t$  as  $|t|_{X \cup \mathcal{P}}^{\mathcal{A}} := \text{len}(s)$ . Further define for each equivalency class  $g$  of  $\mathbb{A}$ -paths*

$$|g|_{X \cup \mathcal{P}}^{\mathcal{A}} := \min \{ |t|_{X \cup \mathcal{P}}^{\mathcal{A}} \mid [t] = g \}.$$

*Then*

$$d_{X \cup \mathcal{P}}^{\mathcal{A}}: \pi_1(\mathbb{A}, v_0)^2 \rightarrow \mathbb{N}, (g, h) \mapsto |g^{-1}h|_{X \cup \mathcal{P}}^{\mathcal{A}}$$

*defines a metric and is called the induced metric of  $X \cup \mathcal{P}$  on  $\pi_1(\mathbb{A}, v_0)$ .*

2. *For each  $\mathbb{A}$ -path  $t$  and some (any) realization  $s$  of  $t$  define the  $X$ -length of  $t$  as  $|t|_X^{\mathcal{A}} := \text{len}_X^p(s)$ . Further define for each equivalency class  $g$  of  $\mathbb{A}$ -paths*

$$|g|_X^{\mathcal{A}} := \min \{ |t|_X^{\mathcal{A}} \mid [t] = g \}.$$

*Then*

$$d_X^{\mathcal{A}}: \pi_1(\mathbb{A}, v_0)^2 \rightarrow \mathbb{N}, (g, h) \mapsto |g^{-1}h|_X^{\mathcal{A}}$$

*defines a metric and is called the induced metric of  $X$  on  $\pi_1(\mathbb{A}, v_0)$ .*

*An  $\mathbb{A}$ -path  $t$  is called of minimal  $X \cup \mathcal{P}$ -length, if  $|t|_{X \cup \mathcal{P}}^{\mathcal{A}} = |[t]|_{X \cup \mathcal{P}}^{\mathcal{A}}$ . Any element  $g \in \pi_1(\mathbb{A}, v_0)$  contains a reduced  $\mathbb{A}$ -path of minimal  $X \cup \mathcal{P}$ -length.*

*If  $t$  is an  $\mathbb{A}$ -path and  $s$  a full  $\mathbb{A}$ -subpath of  $t$ , then  $|s|_{X \cup \mathcal{P}}^{\mathcal{A}} \leq |t|_{X \cup \mathcal{P}}^{\mathcal{A}}$  and  $|s|_X^{\mathcal{A}} \leq |t|_X^{\mathcal{A}}$ .*

**Remark 4.10** For the last claim of Lemma & Definition 4.9 it is essential, that  $s$  is a full  $\mathbb{A}$ -subpath of  $t$ . The claim follows from the fact that any full  $\mathbb{A}$ -subpath of  $t$  has a realization which is a subpath of a realization of  $t$ .

The same is not true for an  $\mathbb{A}$ -subpath  $s$  of  $t$  which is not full. In fact, it is in general not possible to give any upper bound on  $|s|_X^{\mathbb{A}}$  in terms of  $|t|_X^{\mathbb{A}}$  if  $s$  is not required to be a full  $\mathbb{A}$ -subpath of  $t$ .

The next two definitions introduce an important subclass of  $(G, \mathbb{P})$ -carrier graphs of groups.

**Definition 4.11** Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with symmetric finite generating set  $X$  and let  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  be a  $(G, \mathbb{P})$ -carrier graph of groups. Let  $M \geq 0$  such that every essential vertex group is generated by elements of  $X$ -length at most  $M$ , and let  $\nu := \nu(G, \mathbb{P}, X, M)$ .

Let  $t$  be an  $\mathbb{A}$ -path.  $t$  is said to have *peripheral  $\mathbb{P}$ -components*, if the following is true for any realization  $s$  of  $t$ :

Let  $p$  be a  $\mathbb{P}$ -component of  $s$  with  $\text{len}_X(p) \geq 3\nu$ . Then  $p = p_1 e p_2$  where  $e$  is a peripheral edge of  $s$  and

$$\text{len}_X(p_1), \text{len}_X(p_2) \leq \nu.$$

**Definition 4.12** Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with finite symmetric generating set  $X$  and  $\mathcal{A}$  a  $(G, \mathbb{P})$ -carrier graph of groups. For every non-peripheral vertex  $v \in VA$  let

$$E_v := \{e \in EA \mid \omega(e) = v, A_e \neq 1\}$$

and  $\mathbb{O}_v := (\omega_e(A_e))_{e \in E_v}$ . Let  $M \geq 0$  and  $\nu := \nu(G, \mathbb{P}, X, M)$ .

$(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  is called  *$M$ -normal*, if:

1. For any essential vertex  $v \in VA$

(a)  $A_v$  is generated by elements of  $X$ -length at most  $M$ ,

- (b)  $A_v$  is relatively quasiconvex in  $(G, \mathbb{P})$  (and therefore  $\nu$ -relatively quasiconvex by Lemma & Definition 2.18), and
- (c) the elements  $g_e$  for essential edges  $e \in EA$  with  $\omega(e) = v$  constitute a tuple as in Lemma 2.18 2. and  $\mathbb{O}_v$  is the corresponding induced structure on  $\mathbb{A}_v$ .

2. Every  $\mathbb{A}$ -path is equivalent to an  $\mathbb{A}$ -path of minimal  $X \cup \mathcal{P}$ -length which has peripheral  $\mathbb{P}$ -components.

The last condition in the definition of an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph of groups  $\mathbb{A}$  essentially guarantees that all (long) parabolic parts of minimal  $\mathbb{A}$ -paths are contained in the peripheral stars.

**Lemma 4.13** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with symmetric finite generating set  $X$  and  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph of groups. Let  $\nu := \nu(G, \mathbb{P}, X, M)$ . Let  $v \in VA$  be essential,  $a \in A_v$ , and  $s$  a realization of the (degenerate)  $\mathbb{A}$ -path  $(a)$ , i.e. a geodesic whose label represents  $a$ .*

*Suppose  $s = s_1 p s_2$  where  $p$  is a peripheral edge of  $s$ , i.e. a  $P_m$ -edge with  $\text{len}(p)_X \geq \nu$ .*

*Then there are an essential edge  $e \in EA$  with  $\alpha(e) = v$  and  $A_e \subseteq P_m$ , an  $\mathbb{A}$ -path  $t' = (a', e, 1)$ , and a realization  $s'$  of  $t'$  starting in  $s_-$ , such that*

1.  $\text{lab}(s') \in \text{lab}(s_1)P_m$ , and
2.  $|\text{len}_X^p(s') - \text{len}_X^p(s_1)| \leq 3\nu$ .

**Proof** Since  $\mathbb{A}$  is  $M$ -normal, it follows from Lemma 2.18 that there is some essential edge  $e \in EA$  with  $\alpha(e) = v$  and a path  $s'' = s'_1 r_1 p' r_2 s'_2$  from  $s_-$  to  $s_+$  with

1.  $s'_1$  and  $s'_2$  are geodesics with  $\text{lab}(s'_1), \text{lab}(s'_2) \in A_v$ ,
2.  $r_1$  and  $r_2$  are geodesics with  $\text{lab}(r_1) = \text{lab}(r_2)^{-1} = g_e$ , and
3.  $p'$  is a  $P_m$ -edge connected to  $p$  with

$$d_X(\alpha(p), \alpha(p')), d_X(\omega(p), \omega(p')) \leq \nu.$$

Let  $t' := (\text{lab}(s'_1), e, 1)$ . Then it is clear, that  $s' := s'_1 r_1$  is a realization of  $t'$  starting in  $s_-$ . Since  $p$  and  $p'$  are connected, it follows that

$$\text{lab}(s') = \text{lab}(s'_1 r_1) \in \text{lab}(s_1) P_m,$$

and further:

$$\begin{aligned} |\text{len}_X^p(s') - \text{len}_X^p(s_1)| &= |d_X((s'_1)_-, (s'_1)_+) + d_X((r_1)_-, (r_1)_+) - d_X((s_1)_-, (s_1)_+)| \\ &\leq d_X((s'_1)_+, (s_1)_+) + d_X((r_1)_-, (r_1)_+) \\ &\leq d_X((r_1)_+, (s_1)_+) + 2d_X((r_1)_-, (r_1)_+) \\ &= d_X(\alpha(p'), \alpha(p)) + 2|g_e|_X \\ &\leq 3\nu. \end{aligned}$$

□

The previous lemma gives some justification for the definition of a peripheral edge in Definition 4.8. Indeed, let  $t$  be an  $\mathbb{A}$ -path with some essential vertex element  $a$ ,  $s$  a realization of  $t$  and  $p$  a peripheral edge in the geodesic subpath of  $s$  associated with  $a$ . Then it follows from the previous lemma, that  $t$  can be modified into an  $\mathbb{A}$ -path  $t'$  with  $\nu_{\mathcal{A}}(t') = \nu_{\mathcal{A}}(t)$  by replacing the  $\mathbb{A}$ -subpath  $(a)$  of  $t$  with some  $\mathbb{A}$ -path  $(a_1, e, a', e^{-1}, a_2)$  for an essential edge  $e$ , such that for some realization  $s'$  of  $t'$  with the same endpoints as  $s$ ,  $p$  is connected to the peripheral edge  $p'$  of  $s'$  corresponding to  $a'$  and the  $X$ -distance of the endpoints of  $p$  and  $p'$  is at most  $3\nu$ .

The second auxiliary lemma below states that any subdivision of a realization of some  $\mathbb{A}$ -path  $t$  yields a corresponding subdivision of  $t$  in some subdivision of  $\mathcal{A}$ .

**Lemma 4.14** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with symmetric finite generating set  $X$  and  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph of groups. Let  $\nu := \nu(G, \mathbb{P}, X, M)$ .*

*Let  $t$  be an  $\mathbb{A}$ -path and let  $s = s_1 s_2$  be a realization of  $t$ .*

*Then there is a subdivision  $\mathcal{A}'$  of  $\mathcal{A}$  along at most one non-peripheral,*

non-essential edge and an  $\mathbb{A}'$ -path  $t'$  starting in  $t_-$ , which has a realization  $s'$  with  $s'_- = s_-$  and  $d_X(s'_+, (s_1)_+) \leq \nu$ .

**Proof** If  $s_1$  is itself the realization of some  $\mathbb{A}$ -subpath  $t'$  of  $t$ , there is nothing to show. Hence, one of the following cases occurs:

1. There is some subpath  $r_1 r_2 r_3$  of  $s$  such that  $s_1 = r_1 r_2$  and  $r_2 r_3$  is a geodesic representing some element  $a \in A_v$  for an essential vertex  $v$  of  $t$ . Then there must be some  $g \in G$  with  $d_X(g, (s_1)_+) \leq \nu$  and  $g^{-1}(r_2)_- \in A_v$ , since  $A_v$  is  $\nu$ -relatively quasiconvex. Let  $r'_2$  be a geodesic from  $(r_2)_-$  to  $g$  and  $s' := r_1 r'_2$ . Since  $r_1$  is a realization of some  $\mathbb{A}$ -subpath of  $t$  starting in  $t_-$  and ending in  $v$ , and  $r'_2$  is a geodesic representing an element of  $A_v$ , it is clear, that  $s'$  is a realization of some  $\mathbb{A}$ -path  $t'$  starting in  $t_-$ .
2. There is some subpath  $r_1 r_2 r_3$  of  $s$  such that  $s_1 = r_1 r_2$  and  $r_2 r_3$  is a geodesic representing some element  $g_e$  for a non-peripheral edge  $e$  of  $t$ .

If  $e$  is non-essential, let  $\mathcal{A}'$  be the subdivision of  $\mathcal{A}$  along  $e$ , such that for the added edge  $e'$  with  $\alpha(e') = \alpha(e)$ ,  $g_{e'} = \text{lab}(r_2)$ . Then  $s' := s_1$  is a realization of some  $\mathbb{A}'$ -path  $t'$  starting in  $t_-$  and ending in  $\omega(e')$ .

If on the other hand  $e$  is essential, let  $s' := r_1$ .  $s'$  is a realization of some  $\mathbb{A}$ -subpath of  $t$  and

$$d_X(s'_+, (s_1)_+) = d_X((r_2)_-, (r_2)_+) \leq |g_e|_X \leq \nu. \quad \square$$

The rest of this section shows that any word metric on an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph  $\mathcal{A}$ , which arises from a finite generating set and the set of peripheral vertex groups, is quasiisometric to the induced metric on  $\mathcal{A}$ .

**Definition 4.15** Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group and  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  a  $(G, \mathbb{P})$ -carrier graph of groups.

For each  $i \in \{1, \dots, k\}$  with  $A_{c_i} \neq \{1\}$  let  $O_i$  be a subgroup of  $\pi_1(\mathbb{A}, v_0)$  that corresponds to the vertex group  $A_{c_i}$ .

The set  $\mathbb{O}_{\mathcal{A}}$  of these subgroups of  $\pi_1(\mathbb{A}, v_0)$  is called a *peripheral structure* of  $\mathcal{A}$ .

**Remark 4.16** *Since every vertex group of a graph of groups corresponds to a uniquely determined conjugacy class of subgroups in its fundamental group, the elements of  $\mathbb{O}_{\mathcal{A}}$  are uniquely determined up to conjugacy in  $\pi_1(\mathbb{A}, v_0)$ .*

**Lemma 4.17** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group with symmetric finite generating set  $X$ . Let  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  be an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph such that  $|t|_{X \cup \mathcal{P}}^{\mathcal{A}} > 0$  for all non-degenerate  $\mathbb{A}$ -almost-circuits  $t$ .*

*Let  $\mathbb{O}_{\mathcal{A}}$  be a peripheral structure of  $\mathcal{A}$ . Let  $Y$  be a finite generating set of  $\pi_1(\mathbb{A}, v_0)$  relative  $\mathbb{O}_{\mathcal{A}}$ .*

*Then  $\text{id}: (\pi_1(\mathbb{A}, v_0), d_{Y \cup \mathcal{O}_{\mathcal{A}}}) \rightarrow (\pi_1(\mathbb{A}, v_0), d_{X \cup \mathcal{P}}^{\mathcal{A}})$  is a quasiisometry.*

**Proof** It follows from standard arguments that the validity of the claim does not depend on the particular choice of the finite generating set  $Y$  and the peripheral structure  $\mathbb{O}_{\mathcal{A}}$ . Hence, one may choose  $\mathbb{O}_{\mathcal{A}}$  and  $Y$  as follows:

Let  $T$  be a spanning tree of  $\mathcal{A}$  and  $\gamma_v$  the unique reduced  $\mathbb{A}$ -path with trivial vertex elements from  $v_0$  to  $v$  in  $T$ . Let  $\mathbb{O}_{\mathcal{A}}$  be the set of non-trivial subgroups  $[\gamma_{c_i} A_{c_i} \gamma_{c_i}^{-1}]$  of  $\pi_1(\mathbb{A}, v_0)$  for  $i \in \{1, \dots, k\}$ .

Let  $E$  be the set of essential vertices of  $\mathcal{A}$ . Since  $\mathcal{A}$  is  $M$ -normal,  $A_v$  is a relatively quasiconvex subgroup of  $(G, \mathbb{P})$  with induced structure  $\mathbb{O}_v$  for every  $v \in E$ . Moreover, there is a finite generating set  $X_v$  of  $A_v$  for every  $v \in E$ . Finally,  $\text{id}: (A_v, d_{X_v \cup \mathcal{O}_v}) \rightarrow (A_v, d_{X \cup \mathcal{P}})$  is a quasiisometry for any  $v \in E$ .

Now let

$$\iota_v: A_v \rightarrow \pi_1(\mathbb{A}, v_0), a \mapsto [\gamma_v a \gamma_v^{-1}]$$

for any  $v \in E$ , let  $y_e := [\gamma_{\alpha(e)}(1, e, 1)\gamma_{\omega(e)}^{-1}]$  for any  $e \in EA \setminus ET$ , and let

$$Y := \{y_e \mid e \in EA \setminus ET\} \cup \bigcup_{v \in E} \iota_v(X_v).$$

Since  $Y$  is finite and  $|o|_{X \cup \mathcal{P}}^A \leq 2|\gamma_{e_i}|_{X \cup \mathcal{P}}^A + 1$  for any  $o \in \mathbb{O}_i$ , there is a  $C_1 \geq 1$  such that  $|z|_{X \cup \mathcal{P}}^A \leq C_1$  for all  $z \in Y \cup \mathcal{O}_A$ .

It is easy to see that  $\iota_v$  is a quasiisometric embedding with respect to  $d_{X_v \cup \mathcal{O}_v}$  and  $d_{Y \cup \mathcal{O}_A}$  for any  $v \in E$ . Now let  $C_2 \geq 1$  such that  $\iota_v$  is a  $(C_2, C_2)$ -quasiisometric embedding with respect to  $d_{X \cup \mathcal{P}}$  and  $d_{Y \cup \mathcal{O}_A}$  for any  $v \in E$ .

Also, since  $A$  is a finite graph and  $|t|_{X \cup \mathcal{P}}^A > 0$  for all non-degenerate  $\mathbb{A}$ -almost-circuits  $t$ , there is some  $C_3 \geq 1$  such that the number of non-peripheral edges of any reduced  $\mathbb{A}$ -path  $t$  is bounded by  $C_3|t|_{X \cup \mathcal{P}}^A$ .

Now let  $g \in \pi_1(\mathbb{A}, v_0)$ . By the choice of  $C_1$ , it is  $|g|_{X \cup \mathcal{P}}^A \leq C_1|g|_{Y \cup \mathcal{O}_A}$ .

Let  $t := t_0 s_1 t_1 \dots t_{l-1} s_l t_l$  be a reduced  $\mathbb{A}$ -path of minimal  $X \cup \mathcal{P}$ -length representing  $g$ , where the  $s_j$  are the maximal peripheral  $\mathbb{A}$ -subpaths of  $t$ . Let  $t_j := (a_0^j, e_1^j, a_1^j, \dots, a_{l_j-1}^j, e_{l_j}^j, a_{l_j}^j)$  for all  $j \in \{0, \dots, l\}$ .

Define the following:

1. For any  $j \in \{0, \dots, l\}$  and  $m \in \{0, \dots, l_j\}$  let  $v \in VA$  such that  $a_m^j \in A_v$ . If  $a_m^j = 1$ , let  $w_{a_m^j}$  be the empty word. Otherwise  $v$  is an essential vertex and there is a word over  $Y \cup \mathcal{O}_A$  representing  $\iota_v(a_m^j)$  of length at most  $C_2|a_m^j|_{X \cup \mathcal{P}} + C_2$ . Let  $w_{a_m^j}$  be such a word.
2. Let  $j \in \{0, \dots, l\}$  and  $m \in \{1, \dots, l_j\}$ . If  $e_m^j \in ET$  let  $w_{e_m^j}$  be the empty word, otherwise let  $w_{e_m^j} := y_{e_m^j}$ .
3. For any  $j \in \{0, \dots, l\}$  let

$$w_{t_j} := w_{a_0^j} w_{e_1^j} w_{a_1^j} \dots w_{a_{l_j-1}^j} w_{e_{l_j}^j} w_{a_{l_j}^j}.$$

4. Let  $j \in \{1, \dots, l\}$ . Since  $T$  contains all peripheral stars,  $\gamma_{\alpha(s_j)} s_j \gamma_{\omega(s_j)}^{-1}$  represents some element  $o_j \in \mathcal{O}_A$ . Let  $w_{s_j} := o_j$ .
5. Let  $w_t := w_{t_0} w_{s_1} w_{t_1} \dots w_{t_{l-1}} w_{s_l} w_{t_l}$ .

By definition,  $w_t \in (Y \cup \mathcal{O}_{\mathcal{A}})^*$  represents  $g$ . Moreover,

$$\sum_{j=0}^l \sum_{m=0}^{l_j} |w_{a_m^j}| + \sum_{j=1}^l |w_{s_j}| \leq C_2 |t|_{X \cup \mathcal{P}}^{\mathcal{A}} + C_2.$$

Since the number of non-peripheral edges is bounded by  $C_3 |t|_{X \cup \mathcal{P}}^{\mathcal{A}}$ , it follows that

$$|g|_{X \cup \mathcal{O}_{\mathcal{A}}} \leq |w_t| \leq (C_2 + C_3) |t|_{X \cup \mathcal{P}}^{\mathcal{A}} + C_2 = (C_2 + C_3) |g|_{X \cup \mathcal{P}}^{\mathcal{A}} + C_2.$$

Hence,  $\text{id}: (\pi_1(\mathbb{A}, v_0), d_{Y \cup \mathcal{O}_{\mathcal{A}}}) \rightarrow (\pi_1(\mathbb{A}, v_0), d_{X \cup \mathcal{P}}^{\mathcal{A}})$  is a quasiisometry.  $\square$

**Remark 4.18** *It is clear, that there is no upper bound on the quasiisometry constants in Lemma 4.17, which is independent of the choice of  $\mathbb{O}_{\mathcal{A}}$  and  $Y$ . But even for the  $\mathbb{O}_{\mathcal{A}}$  and  $Y$  constructed in the proof it is impossible to give such an upper bound without additional information about  $\mathcal{A}$ .*

## 4.2 Combination Theorem

The goal of this section is to prove the second main theorem of this thesis. The proof of this combination theorem is based on the following Proposition. Its main purpose is to apply Lemma 2.19 in the context of carrier graphs of groups.

**Proposition 4.19** *Let  $G$  be a group, which is torsion-free, hyperbolic relative to  $\mathbb{P} = \{P_1, \dots, P_n\}$  and has a symmetric finite generating set  $X$ . Let  $C \geq 1$  and  $M \geq 0$ .*

*There are constants  $C' = C'(G, \mathbb{P}, X, C)$  and  $L' = L'(G, \mathbb{P}, X, M, C)$ , such that one of the following holds for every  $M$ -normal  $(G, \mathbb{P})$ -carrier graph of groups  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$ :*

1. *There is a reduced  $\mathbb{A}$ -path  $t$  with  $|t|_X^{\mathcal{A}} \leq L'$ ,  $t_- \in VC_i$ ,  $t_+ \in VC_j$  and  $m_i = m_j$ , which is not entirely contained in  $C_i$  and whose label  $\nu_{\mathcal{A}}(t)$  represents an element of  $P_{m_i}$ .*



2. There is a subdivision  $\mathcal{A}'$  of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges and a reduced  $\mathbb{A}'$ -path  $t$  with  $|t|_{\mathcal{A}'}^{\mathcal{A}'} \leq L'$ , such that

$$|\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}} < \frac{|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C} - C.$$

3.  $\nu_{\mathcal{A}}: (\pi_1(\mathbb{A}, v_0), d_{X \cup \mathcal{P}}^{\mathcal{A}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(C', C')$ -quasiisometric embedding. Moreover, for any peripheral structure  $\mathbb{O}_{\mathcal{A}}$  of  $\mathcal{A}$ ,  $\nu_{\mathcal{A}}(\mathbb{O}_{\mathcal{A}})$  is an induced structure of  $(G, \mathbb{P})$  on  $\text{Im } \nu_{\mathcal{A}}$ .

**Proof** Let  $\nu := \nu(G, \mathbb{P}, X, M)$  be the constant from Lemma & Definition 2.18. Let  $C_1 := C + 4\nu$ ,  $C_2 := C_1^2 + 3C_1$  and  $C', K \geq 1$  such that any  $(K, C_2, C_2)$ -local-quasigeodesic in  $\text{Cay}(G, X \cup \mathcal{P})$  is a  $(C', C')$ -quasigeodesic. Let  $\varepsilon_1 := 3\nu$ ,  $\varepsilon := \varepsilon(G, \mathbb{P}, X, 1, K)$  as in Lemma 2.7, and let  $\varepsilon_2 := 2^K(K\varepsilon + \varepsilon_1)$ . Let  $L' := \varepsilon_2 K + 8\nu$ .

Suppose that 1. and 2. are false and show that this implies 3.

Let  $g \in \pi_1(\mathbb{A}, v_0)$ ,  $t$  some reduced  $\mathbb{A}$ -path of minimal  $X \cup \mathcal{P}$ -length with peripheral  $\mathbb{P}$ -components representing  $g$ , and  $s$  a realization of  $t$ . It suffices to show that  $s$  is a  $(K, C_2, C_2)$ -local-quasigeodesic, since

$$|g|_{X \cup \mathcal{P}}^{\mathcal{A}} = |t|_{X \cup \mathcal{P}}^{\mathcal{A}} = \text{len}(s)$$

and

$$|\nu_{\mathcal{A}}(g)|_{X \cup \mathcal{P}} = d_{X \cup \mathcal{P}}(s_-, s_+).$$

Write  $s = s_0 p_1 s_1 \dots s_{l-1} p_l s_l$ , where the  $p_m$  are the peripheral edges of  $s$  with  $\text{len}_X(p_m) > \varepsilon_2$ . Let  $i_m$  be such that  $p_m$  is a  $P_{i_m}$ -edge.

Any subpath of  $s$  of length at most  $K$  fulfills Lemma 2.19 1. by definition and it remains to show that it also fulfills Lemma 2.19 2. and 3., where the constant  $C$  in the statement of Lemma 2.19 is  $C_1$ .

It suffices to show that every  $s_j$  is a  $(K, C_1, C_1)$ -local-quasigeodesic to show that every subpath of  $s$  of length at most  $K$  fulfills Lemma 2.19 2.

Let  $\bar{s}$  be a subpath of  $s_j$  with  $\text{len}(\bar{s}) \leq K$ . Since  $t$  has peripheral  $\mathbb{P}$ -components and the  $p_m$  are all peripheral edges of  $s$  with  $X$ -length greater than  $\varepsilon_2$ , all  $\mathbb{P}$ -edges of  $s_j$  and therefore of  $\bar{s}$  have  $X$ -length at

most  $\varepsilon_2 \geq 3\nu$ . It follows that

$$\text{len}_X^p(\bar{s}) \leq \text{len}_X(\bar{s}) \leq \varepsilon_2 \text{len}(\bar{s}) \leq \varepsilon_2 K.$$

If  $\bar{s}$  is a subpath of some geodesic subpath of  $s$  corresponding to some essential vertex element of  $t$  or to some  $g_e$  for an edge  $e$  of  $t$ ,  $\bar{s}$  is itself a geodesic and there is nothing to show. Assuming that this is not the case, Lemma 4.14 can be applied twice to obtain a subdivision  $\mathcal{A}'$  of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges and a path  $s'$  with  $d_X(s'_-, \bar{s}_-), d_X(s'_+, \bar{s}_+) \leq \nu$ , which is a realization of some  $\mathbb{A}'$ -path  $t'$ .

It follows that

$$\text{len}(\bar{s}) \leq \text{len}(s') + 2\nu,$$

$$d_{X \cup \mathcal{P}}(s'_-, s'_+) \leq d_{X \cup \mathcal{P}}(\bar{s}_-, \bar{s}_+) + 2\nu,$$

and

$$|t'|_{\mathcal{X}}^{\mathcal{A}'} = \text{len}_X^p(s') \leq \text{len}_X^p(\bar{s}) + 2\nu \leq \varepsilon_2 K + 2\nu \leq L'.$$

Note that

$$|\nu_{\mathcal{A}'}(t')|_{X \cup \mathcal{P}} \geq \frac{|t'|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C} - C$$

as 2. was assumed to be false. It follows that

$$\begin{aligned} \frac{\text{len}(\bar{s})}{C_1} - C_1 &\leq \frac{\text{len}(s') + 2\nu}{C_1} - C_1 \\ &\leq \frac{|t'|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C} - C - 2\nu \\ &\leq |\nu_{\mathcal{A}'}(t')|_{X \cup \mathcal{P}} - 2\nu \\ &= d_{X \cup \mathcal{P}}(s'_-, s'_+) - 2\nu \\ &\leq d_{X \cup \mathcal{P}}(\bar{s}_-, \bar{s}_+). \end{aligned}$$

Therefore,  $s_j$  is a  $(K, C_1, C_1)$ -local-quasigeodesic and any subpath of  $s$  of length at most  $K$  fulfills Lemma 2.19 2.

Suppose  $s$  does not fulfill Lemma 2.19 3. Thus, there exists a subpath  $ps'p'$  of  $s$  with

1.  $p$  and  $p'$  are  $P_i$ -paths of  $s$ ,

2.  $\text{len}_X(p), \text{len}_X(p') \geq \varepsilon_1 = 3\nu$ , and

3.  $\text{len}_X(s') \leq \varepsilon_2$

where  $s'$  is labeled by an element of  $P_i$ . In particular,

$$\text{len}_X^p(s') \leq \text{len}_X(s') \leq \varepsilon_2.$$

$t$  has peripheral  $\mathbb{P}$ -components, since it is of minimal  $X \cup \mathcal{P}$ -length and  $\mathcal{A}$  is  $M$ -normal. Since  $\text{len}_X(p), \text{len}_X(p') \geq 3\nu$ ,  $p$  and  $p'$  must be of the form  $q_1 q q_2$  and  $q'_1 q' q'_2$ , where  $q_1, q_2, q'_1$  and  $q'_2$  are  $P_i$ -paths of  $X$ -length at most  $\nu$  and  $q$  and  $q'$  are peripheral  $P_i$ -edges with  $\text{len}_X(q), \text{len}_X(q') \geq \nu$ . Moreover,  $p$  and  $p'$  can not be part of the same  $P_i$ -component of  $s$ , hence  $s'$  is not a  $P_i$ -path.

By applying Lemma 4.13 to  $q_2 s' q'_1$  at most twice, obtain a path  $s''$ , which is a realization of some reduced  $\mathbb{A}$ -path  $t'$  with  $t'_-$  and  $t'_+$  peripheral, which is labeled by an element of  $P_i$ , and which fulfills

$$\text{len}_X^p(s'') \leq \text{len}_X^p(q_2 s' q'_1) + 6\nu.$$

$t'$  is not entirely contained in some peripheral star, since  $s'$  was not a  $P_i$ -path, and further

$$|t'|_X = \text{len}_X^p(s'') \leq \text{len}_X^p(q_2 s' q'_1) + 6\nu \leq \varepsilon_2 + 8\nu \leq L'.$$

Hence,  $t'$  fulfills 1., a contradiction.

Therefore,  $\nu_{\mathcal{A}}: (\pi_1(\mathbb{A}, v_0), d_{X \cup \mathcal{P}}^{\mathcal{A}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(C', C')$ -quasiisometric embedding.

Let  $\mathbb{O}_{\mathcal{A}}$  be a peripheral structure of  $\mathcal{A}$ .  $|g^n|_{X \cup \mathcal{P}}^{\mathcal{A}}$  is unbounded for any non-elliptical element  $g \in \pi_1(\mathbb{A}, v_0)$ . Since  $\nu_{\mathcal{A}}$  is a quasiisometric embedding, this also implies that  $|\nu_{\mathcal{A}}(g)^n|_{X \cup \mathcal{P}}$  is unbounded. Thus,  $\nu_{\mathcal{A}}(g)$  is not parabolic. It is now easy to see, that every element of  $\pi_1(\mathbb{A}, v_0)$ , which is mapped to a parabolic element under  $\nu_{\mathcal{A}}$  must be a parabolic element of some vertex group of  $\mathbb{A}$ . Since  $\mathcal{A}$  is  $M$ -normal, it must then be conjugate into some element of  $\mathbb{O}_{\mathcal{A}}$ .

Now let  $O, O' \in \mathbb{O}_{\mathcal{A}}$ ,  $o \in O$ ,  $o' \in O'$  and  $g \in \pi_1(\mathbb{A}, v_0)$  such that  $\nu_{\mathcal{A}}(gog^{-1}o')$  is parabolic. By the above,  $gog^{-1}o'$  must be a parabolic element of some vertex group of  $\mathbb{A}$ . Since  $\mathcal{A}$  is  $M$ -normal, it follows that  $O = O'$ . Thus  $\nu_{\mathcal{A}}(\mathbb{O}_{\mathcal{A}})$  is an induced structure of  $(G, \mathbb{P})$  on  $\text{Im } \nu_{\mathcal{A}}$ .  $\square$

It is now possible to prove the following combination theorem.

**Theorem 4.20** *Let  $G$  be a group, which is torsion-free, hyperbolic relative to  $\mathbb{P} = \{P_1, \dots, P_n\}$  and has finite symmetric generating set  $X$ . Let  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  be an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph.*

*There are  $C = C(G, \mathbb{P}, X, M, N(\mathbb{A}))$  and  $L = L(G, \mathbb{P}, X, M, N(\mathbb{A}))$  such that the following holds:*

$\nu_{\mathcal{A}}: (\pi_1(\mathbb{A}, v_0), d_{X \cup \mathcal{P}}^{\mathcal{A}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(C, C)$ -quasiisometric embedding and for any peripheral structure  $\mathbb{O}_{\mathcal{A}}$  of  $\mathcal{A}$  is  $\nu_{\mathcal{A}}(\mathbb{O}_{\mathcal{A}})$  an induced structure of  $(G, \mathbb{P})$  on  $\text{Im } \nu_{\mathcal{A}}$ , unless one of the following holds:

- (A1) (a) *There is some  $\mathbb{A}$ -path  $t$  with  $|t|_X^{\mathcal{A}} \leq L$ ,  $t_- \in VC_i$ ,  $t_+ \in VC_j$ ,  $m_i = m_j$  and  $\nu_{\mathcal{A}}(t) \in P_{m_i}$ , which passes through some trivial segment of  $\mathbb{A}$  exactly once.*
- (b) *There is some  $\mathbb{A}$ -path  $t$  with  $|t|_X^{\mathcal{A}} \leq L$ ,  $t_- \in VC_i$ ,  $t_+$  essential, and  $A_{t_+} \cap \nu_{\mathcal{A}}(t)^{-1}P_{m_i}\nu_{\mathcal{A}}(t) \neq 1$ , which passes through some trivial segment of  $\mathbb{A}$  exactly once.*
- (A2) *There are essential vertices  $v, v' \in VA$  and some non-degenerate, almost-simple  $\mathbb{A}$ -path  $t$  starting in  $v$  and ending in  $v'$  with  $|t|_X^{\mathcal{A}} \leq L$ .*
- (A3) *There is an  $\mathbb{A}$ -almost-circuit  $t$  with  $|t|_X^{\mathcal{A}} \leq L$  which has a trivial segment.*
- (A4) (a) *There is some essential edge  $e$  with  $\alpha(e) \in VC_i$  and some  $p \in A_{\alpha(e)} \setminus \alpha_e(A_e)$  with*

$$|p|_X \leq L.$$

- (b) *There is some almost-simple  $\mathbb{A}$ -path  $t = (a_0, e_1, a_1, \dots, e_k, a_k)$  with  $|t|_X^{\mathcal{A}} \leq L$ ,  $t_- \in VC_i$ ,  $A_{e_1} = 1$  and  $t_+$  essential, and some*

$p \in A_{t_-} \setminus \{1\}$  such that

$$|p|_X \leq L.$$

(A5) *There is some almost-simple  $\mathbb{A}$ -path  $t$  with  $|t|_X^{\mathbb{A}} \leq L$ ,  $t_- \in VC_i$ ,  $t_+ \in VC_j$  and passing through some trivial segment of  $\mathbb{A}$ , and elements  $p \in A_{t_-} \setminus \{1\}, p' \in A_{t_+} \setminus \{1\}$ , such that*

$$|p|_X, |p'|_X \leq L.$$

(A6) *There is some subdivision  $\mathcal{A}'$  of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges, an  $\mathbb{A}'$ -path  $t$  with  $|t|_X^{\mathbb{A}'} \leq L$ , some trivial segment  $t'$  of  $t$  and some  $a \in G$  with*

(a)  *$t$  passes only once through  $t'$  or  $t'^{-1}$ ,*

(b)  *$a \in A_{t_+}$ , if  $t_+$  is non-peripheral,*

(c)  *$a \in P_{m_j}$ , if  $t_+ \in VC_j$ , and*

(d)  *$|\nu_{\mathcal{A}'}(t)a|_{X \cup \mathcal{P}} < |t'|_{X \cup \mathcal{P}}^{\mathbb{A}'}$ .*

It follows from Theorem 4.20 and Lemma 4.17 that for any  $M$ -normal  $(G, \mathbb{P})$ -carrier graph  $\mathcal{A}$  which does not satisfy any of the conditions (A1)-(A6),  $\nu_{\mathcal{A}}: (\pi_1(\mathbb{A}, v_0), d_{Y \cup \mathcal{O}_{\mathcal{A}}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a quasiisometric embedding for any peripheral structure  $\mathbb{O}_{\mathcal{A}}$  of  $\mathcal{A}$  and finite generating set  $Y$  of  $\pi_1(\mathbb{A}, v_0)$  relative  $\mathbb{O}_{\mathcal{A}}$ .

Although Theorem 4.20 yields that  $\nu_{\mathcal{A}}: (\pi_1(\mathbb{A}, v_0), d_{X \cup \mathcal{P}}^{\mathbb{A}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(C, C)$ -quasiisometric embedding, it follows from Remark 4.18 that without further assumptions on  $\mathcal{A}$ , it is generally not possible to give a  $\bar{C}$  such that  $\nu_{\mathcal{A}}: (\pi_1(\mathbb{A}, v_0), d_{Y \cup \mathcal{O}_{\mathcal{A}}}) \rightarrow (G, d_{X \cup \mathcal{P}})$  is a  $(\bar{C}, \bar{C})$ -quasiisometric embedding.

For the rest of this section, assume that  $\mathcal{A}$  is as in Theorem 4.20, i.e. an  $M$ -normal  $(G, \mathbb{P})$ -carrier graph. Let  $\nu = \nu(G, \mathbb{P}, X, M)$ . The following lemmas are used in the proof of Theorem 4.20 below.

The first lemma shows that, under some additional conditions,  $\mathbb{A}$ -paths as in Proposition 4.19 2. fulfill (A6).

**Lemma 4.21** *Let  $\delta \in (0, 1]$ ,*

$$C \geq \max \left\{ 4, \sqrt{\frac{4}{\delta}}, \frac{N(\mathbb{A}) + 1}{\delta} \right\},$$

*$\mathcal{A}'$  some subdivision of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges, and  $t = s_1 t' s_2$  a reduced  $\mathbb{A}'$ -path such that*

- 1.  $t'$  is almost-simple and there is no edge pair which is crossed by  $t'$  and by  $s_1$  or  $s_2$ ,*
- 2. any vertex element of  $t'$  from one of its essential vertices is trivial and no edge of  $t'$  is essential,*
- 3.  $|t'|_{X \cup \mathcal{P}}^{\mathcal{A}'} \geq \delta |t|_{X \cup \mathcal{P}}^{\mathcal{A}'}$ , and*
- 4.  $|\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}} < \frac{|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C} - C$ .*

*Then  $t'$  contains a trivial segment  $s$  of length greater than  $|\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}}$ . Thus,  $t$  travels  $s$  once and  $s^{-1}$  not at all.*

**Proof** By assumptions 1. and 2.,  $t'$  is of the form

$$t' = p_0 t_1 p_1 \dots t_N p_N,$$

where

1. every  $t_l$  is a maximal trivial segment of  $t'$ ,
2. every  $p_l$  is either trivial or a peripheral  $\mathbb{A}'$ -subpath of  $t$  with non-trivial edge groups.

Since  $t'$  is almost-simple and therefore reduced,  $t_l$  is a maximal trivial segment of  $\mathbb{A}$  for  $l \in \{2, \dots, N-1\}$ . It further follows, that  $\mathbb{A}'$  has at least  $N-1$  pairwise distinct maximal trivial segments. Since  $N(\mathbb{A}') = N(\mathbb{A})$ , this implies  $N \leq N(\mathbb{A}) + 1 \leq \delta C$ .

Let

$$L_t := \sum_{l=1}^N |t_l|_{X \cup \mathcal{P}}^{\mathcal{A}'}$$

and let  $L_p$  be the number of the non-trivial  $p_l$ . Thus,  $L_p \leq N + 1$ .

It follows that

$$\begin{aligned}
|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} &\leq |s_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t'|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} \\
&= L_t + L_p + |s_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} \\
&\leq L_t + L_p + |t|_{X \cup \mathcal{P}}^{\mathcal{A}'} - |t'|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2 \\
&\leq L_t + N + (1 - \delta)|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 3.
\end{aligned}$$

By assumption 4.,  $|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} \geq C^2$  and hence

$$L_t + N \geq \delta|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} - 3 \geq \delta C^2 - 3 \geq \delta \left( \frac{4}{\delta} \right) - 3 = 1$$

Since  $N = 0$  implies  $L_t = 0$ , this means that  $N \neq 0$ . Now, as  $\delta C \geq N$  and  $C \geq 4$ , it follows that

$$\frac{L_t}{N} \geq \frac{\delta|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{N} - \frac{3}{N} - 1 \geq \frac{\delta|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{\delta C} - 4 \geq \frac{|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C} - C > |\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}}.$$

Since  $L_t/N$  is the average  $|\cdot|_{X \cup \mathcal{P}}^{\mathcal{A}'}$ -length of the  $t_l$ , there has to be some  $l \in \{1, \dots, N\}$  such that  $|t_l|_{X \cup \mathcal{P}}^{\mathcal{A}'} > |\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}}$ .  $\square$

The following are two more auxiliary lemmas for the proof of Theorem 4.20.

**Lemma 4.22** *Let  $L \geq 0$  and suppose that none of (A2)-(A5) hold for this  $L$ . Let  $\mathcal{A}'$  be some subdivision of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges. Let  $t$  be a reduced  $\mathbb{A}'$ -path with  $|t|_X^{\mathcal{A}'} \leq L$ . Then one of the following holds:*

1.  *$t$  is almost-simple and has at most one essential vertex, or*
2. *the path  $r$  underlying  $t$  is of the form  $r_1 r_2 (r_2)^{-1} r_3$  with  $r_2$  non-degenerate, such that the following hold:*

- (a)  *$r_1 r_2$ ,  $r_1 r_3$  and  $(r_2)^{-1} r_3$  are almost-simple, and*
- (b) *all vertices of  $r$  other than  $(r_2)_+$  are non-essential.*

**Proof** Let  $t = (a_0, e_1, \dots, e_k, a_k)$  be a reduced  $\mathbb{A}'$ -path with  $|t|_X^{A'} \leq L$ ,  $v_0 := \alpha(e_1)$ ,  $v_i := \omega(e_i)$  for  $i \in \{1, \dots, k\}$ . Suppose that  $t$  is not as in 1. or 2. Show that this implies that one of (A2)-(A5) must hold.

It is easy to see by considering a shortest  $\mathbb{A}$ -subpath with the same property, that if  $t$  is not as in 1. or 2., then one of the following must be true:

1. There are  $l_1 < l_2 \in \{0, \dots, k\}$  such that

$$t' := (1, e_{l_1+1}, a_{l_1+1}, \dots, a_{l_2-1}, e_{l_2}, 1)$$

is almost-simple,  $v_l$  is non-essential for  $l \in \{l_1 + 1, \dots, l_2 - 1\}$ , and for each  $i \in \{1, 2\}$  one of the following holds:

- (a)  $v_{l_i}$  is essential, or
- (b)  $l_i \notin \{0, k\}$  and  $e_{l_i} = e_{l_i+1}^{-1}$ .

2. There are  $l_1 < l_2 \in \{0, \dots, k\}$  with

- (a)  $v_{l_1} = v_{l_2}$ ,
- (b)  $e_{l_1} = e_{l_2+1}^{-1}$  if  $v_{l_1} = v_{l_2}$  is a central peripheral vertex, and
- (c)  $(e_{l_1+1}, \dots, e_{l_2})$  is either an almost-circuit with at most one essential vertex or as described in 2. above with  $r_1$  and  $r_3$  non-degenerate.

Show that in both cases one of (A2)-(A5) must hold.

1. Since  $t$  is reduced,  $e_{l_i} = e_{l_i+1}^{-1}$  implies  $a_{l_i} \notin \omega_{e_{l_i}}(A_{e_{l_i}})$  for  $i \in \{1, 2\}$ . Thus  $v_{l_1}, v_{l_2} \in VA$ , and  $t'$  can be assumed to be an  $\mathbb{A}$ -path rather than an  $\mathbb{A}'$ -path. It follows, that  $e_{l_1}, e_{l_1+1}, e_{l_2}$  and  $e_{l_2+1}$  are non-peripheral. Hence,  $t'$  and  $(a_{l_i})$  are full  $\mathbb{A}'$ -subpaths of  $t$  and therefore  $|t'|_X^A \leq |t|_X^{A'} \leq L$  and  $|a_{l_i}|_X \leq |t|_X^{A'} \leq L$  for  $i \in \{1, 2\}$ .

If  $v_{l_1}$  and  $v_{l_2}$  are both essential,  $t'$  fulfills (A2).



If neither  $v_{l_1}$  nor  $v_{l_2}$  is essential, then both must be peripheral since  $a_{l_1}, a_{l_2} \neq 1$ . Since  $e_{l_1+1}$  is non-peripheral and since  $v_{l_1+1}$  is non-essential, this implies that  $A_{e_{l_1+1}} = 1$ . Hence  $a_{l_1}, a_{l_2}$  and  $t'$  fulfill (A5) with  $p = a_{l_1}$  and  $p' = a_{l_2}$ .

In the remaining case that exactly one of  $v_{l_1}$  and  $v_{l_2}$  is essential it follows analogously that  $a_{l_1}$  and  $t'$  or  $a_{l_2}$  and  $t'^{-1}$  fulfill (A4)a., if  $l_2 = l_1 + 1$  and  $A_{e_{l_2}} \neq 1$ , and (A4)b., otherwise.

2. If  $v_{l_1} = v_{l_2}$  is not a central peripheral vertex, there exists some full  $\mathbb{A}'$ -subpath  $t'$  of  $t$  with underlying path  $(e_{l_1+1}, \dots, e_{l_2})$ .

If  $v_{l_1} = v_{l_2}$  is a central peripheral vertex there is no full  $\mathbb{A}'$ -subpath of  $t$  with underlying path  $(e_{l_1+1}, \dots, e_{l_2})$ . Instead, consider the following  $\mathbb{A}'$ -path

$$t' := (1, e_{l_1+2}, a_{l_1+2}, \dots, e_{l_2}, a_{l_2} g_{e_{l_1}}^{-1} a_{l_2+1} a_{l_1-1} g_{e_{l_1}} a_{l_1}, e_{l_1+1}, a_{l_1+1}),$$

which is equivalent to a cyclic permutation of the full  $\mathbb{A}'$ -subpath of  $t$  with underlying path  $(e_{l_1}, \dots, e_{l_2+1})$ .

In both cases,  $t'$  is an  $\mathbb{A}'$ -almost-circuit with at most one essential vertex or as described in 2. above. Moreover, it follows in both cases that  $|t'|_{X \cup \mathcal{P}}^{\mathbb{A}'} \leq |t|_{X \cup \mathcal{P}}^{\mathbb{A}'}$ .

Shorten  $t'$  by the following procedure:

Let  $s = (b_1, f_1, \dots, f_{k'}, b_{k'})$  an  $\mathcal{A}'$ -path and  $l \in \{1, \dots, k' - 1\}$  with  $f_l = f_{l+1}^{-1}$ . Let

$$s' := (b_1, f'_1, \dots, f_{l-1}, b_{l-1} b_{l+1}, f_{l+2}, \dots, f_{k'}, b_{k'}),$$

if  $f_l$  is non-peripheral, and

$$s' := (b_1, f_1, \dots, f_{l-1}, b_{l-1} g_{f_l} b_l g_{f_l}^{-1} b_{l+1}, f_{l+2}, \dots, f_{k'}, b_{k'}),$$

if  $f_l$  is peripheral. It follows that  $|s'|_X^{\mathbb{A}'} \leq |s|_X^{\mathbb{A}'}$ .

Repeating this process eventually yields an  $\mathbb{A}'$ -almost-circuit  $t''$

with  $|t''|_X^{A'} \leq |t|_X^{A'} \leq L$  and at most one essential vertex. Since  $t''$  is an  $\mathbb{A}'$ -almost-circuit,  $t''$  can be assumed as an  $\mathbb{A}$ -path rather than an  $\mathbb{A}'$ -path after a cyclic permutation.

If  $t''$  has an essential vertex, then some cyclic permutation of  $t''$  fulfills (A2). Otherwise,  $t''$  must contain an edge with trivial edge group and fulfill (A3).  $\square$

**Lemma 4.23** *Let  $L \geq 0$  and  $\mathcal{A}'$  be some subdivision of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges. Let  $t_i = (a_0^i, e_1, a_1^i, \dots, a_{l-1}^i, e_l, 1)$  be almost-simple  $\mathbb{A}'$ -paths for  $i \in \{1, 2\}$ ,  $v_0 := \alpha(e_1)$ ,  $v_m := \omega(e_m)$  for  $m \in \{1, \dots, l\}$  and  $a \in A_{v_l} \setminus \omega_{e_l}(A_{e_l})$ . If*

1.  $|t_1(a)t_2^{-1}|_X^{A'} \leq L$ , and
2.  $[t_2^{-1}t_1] \notin [\omega_{e_l}(A_{e_l})] := \{[(a)] \mid a \in \omega_{e_l}(A_{e_l})\}$ ,

*then one of (A2), (A4) or (A5) must be true for the given  $L$ .*

**Proof** Suppose there is some  $m \in \{0, \dots, l-1\}$  such that  $v_m$  is essential. The same argument as in case 1. in the proof of Lemma 4.22 yields that  $(1, e_{m+1}, a_{m+1}^1, \dots, a_{l-1}^1, e_l, 1)$ ,  $a_m$  and  $a$  fulfill (A2) or (A4). Therefore assume that  $v_m$  is non-essential for all  $m \in \{0, \dots, l-1\}$ .

Replace  $t_i$  by an equivalent  $\mathbb{A}'$ -path such that  $a_{m-1}^i = 1$  for every  $m \in \{1, \dots, l\}$  with  $e_m$  peripheral.

Since  $[t_2^{-1}t_1] \notin [\omega_{e_l}(A_{e_l})]$ , there is some  $m \in \{0, \dots, l-1\}$  such that

$$p := (a_m^2)^{-1}a_m^1 \notin \alpha_{e_{m+1}}(A_{e_{m+1}}).$$

Choose  $m$  minimal with this property. By the previous replacement of the  $t_i$ ,  $e_{m+1}$  is non-peripheral. Hence,  $s_1 := (a_0^1, e_1, \dots, e_m, a_m^1)$  and  $s_2 := (a_0^2, e_1, \dots, e_m, a_m^2)$  are full  $\mathbb{A}'$ -subpaths of  $t_1$  and  $t_2$ , respectively.

Thus, it follows that

$$\begin{aligned} |p|_X &= |(a_m^2)^{-1}a_m^1|_X \\ &= |\nu_{\mathcal{A}'}(s_2)^{-1}\nu_{\mathcal{A}'}(s_1)|_X \end{aligned}$$

$$\begin{aligned}
&\leq |s_1|_X^{A'} + |s_2|_X^{A'} \\
&\leq |t_1(a)t_2^{-1}|_X^{A'} \\
&\leq L.
\end{aligned}$$

Note that  $e_l$  is non-peripheral, since  $a \in A_{v_l} \setminus \omega_{e_l}(A_{e_l})$ . Therefore  $s := (1, e_{m+1}, a_{m+1}^1, \dots, a_{l-1}^1, e_l, 1)$  is a full  $\mathbb{A}'$ -subpath of  $t_1$ . Since  $v_m$  and  $v_l$  have non-trivial vertex groups,  $s$  can be assumed as an  $\mathbb{A}$ -path rather than an  $\mathbb{A}'$ -path. Furthermore,  $(a)$  is a full  $\mathbb{A}'$ -subpath of  $t_1(a)t_2^{-1}$  and it follows that

$$|a|_X, |s|_X^A \leq |t_1(a)t_2^{-1}|_X^{A'} \leq L.$$

If  $v_l$  is essential and  $l = m + 1$ , then  $s$  and  $p$  fulfill (A4)a. If  $v_l$  is essential and  $l > m + 1$  then  $s$  must have an edge with trivial edge group and  $s$  and  $p$  fulfill (A4)b. Otherwise,  $v_l$  is peripheral and  $A_{e_l} = 1$ , since  $a \in A_{v_l} \setminus \omega_{e_l}(A_{e_l})$ . Hence  $s$ ,  $p$  and  $p' := a$  fulfill (A5).  $\square$

**Proof (of Theorem 4.20)** Choose  $C_0$  as in Lemma 4.21 for  $\delta = 1$ . Let  $D = D(G, \mathbb{P}, X, \nu)$  and  $D_2 := D_2(G, \mathbb{P}, X)$  be the constants from Lemma 2.14 and Lemma 2.12, respectively, and

$$C_1 := 2C_0 + \max\{D, D_2 + 1\}.$$

Choose  $C_2$  according to Lemma 4.21 with  $\delta = (3C_1 + 2)^{-1}$  and with

$$C_2 \geq 4\nu, 3C_1 + 2, C_1 + 6.$$

Let  $C := C'(G, \mathbb{P}, X, C_2)$  and  $L := L'(G, \mathbb{P}, X, M, C_2)$  be the constants from Proposition 4.19.

Suppose now that assumption 1. of Proposition 4.19 is true, i.e. there is some reduced  $\mathbb{A}$ -path  $t = (a_0, e_1, \dots, e_k, a_k)$  with  $|t|_X^A \leq L$ ,  $t_- \in VC_i$ ,  $t_+ \in VC_j$  and  $m_i = m_j$ , which is not entirely contained in  $C_i$  and whose label  $\nu_{\mathcal{A}}(t)$  represents an element of  $P_{m_i}$ . Assume  $t$  to be minimal in the sense that no proper  $\mathbb{A}$ -subpath of  $t$  has the same properties. In

particular  $e_1 \notin EC_i$ ,  $e_k \notin EC_j$  and  $a_0 = a_k = 1$ .

Let  $v_0 := \alpha(e_1)$  and  $v_m := \omega(e_m)$  for  $m \in \{1, \dots, k\}$ . By Lemma 4.22, it can be assumed that one of the following cases occurs:

**Case 1:**  $t$  is almost-simple and has at most one essential vertex.

Show that  $t$  has an edge with trivial edge group and hence fulfills (A1)a.

By the structure of  $(G, \mathbb{P})$ -carrier graphs and since at most one vertex of  $t$  is essential, it is easy to see that  $t$  must contain an edge with trivial edge group or else be of length  $k = 2$  with  $v_1$ ,  $e_1$  and  $e_2$  essential. If  $t$  is of this form, it follows that

$$\omega_{e_1}(A_{e_1}) = g_{e_1}^{-1} A_{e_1} g_{e_1} \subseteq g_{e_1}^{-1} P_{m_i} g_{e_1}$$

and, since  $g_{e_1} a_1 g_{e_2} = \nu_{\mathcal{A}}(t) \in P_{m_1}$ ,

$$\begin{aligned} a_1 \alpha_{e_2}(A_{e_2}) a_1^{-1} &= a_1 g_{e_2} A_{e_2} g_{e_2}^{-1} a_1^{-1} \\ &\subseteq a_1 g_{e_2} P_{m_i} g_{e_2}^{-1} a_1^{-1} \\ &= a_1 g_{e_2} t^{-1} P_{m_i} t g_{e_2}^{-1} a_1^{-1} \\ &= g_{e_1}^{-1} P_{m_i} g_{e_1} \end{aligned}$$

This is a contradiction to  $\omega_{e_1}(A_{e_1})$  and  $\alpha_{e_2}(A_{e_2})$  being part of the induced structure  $\mathbb{O}_{v_1}$  of  $G$  on  $A_{v_1}$ , which is guaranteed by  $\mathcal{A}$  being  $M$ -normal.

**Case 2a:** The underlying path of  $t$  is of the form  $rr^{-1}$  for an almost-simple path  $r$  where all vertices of  $rr^{-1}$  other than  $r_+$  are non-essential.

This implies  $k = 2l$  for some  $l$  and  $e_m = e_{k-m+1}^{-1}$  for  $m \in \{1, \dots, l\}$ . Let  $t_1 := (a_0, e_1, \dots, a_{l-1}, e_l, 1)$  and  $t_2 := (a_k^{-1}, e_1, \dots, a_{l+1}^{-1}, e_l, 1)$ . Then  $t = t_1(a_l)t_2^{-1}$  and  $t_1$  and  $t_2$  are almost-simple.

By Lemma 4.23, it can be assumed that  $a' := \nu_{\mathcal{A}}(t_2^{-1}t_1) \in \omega_{e_l}(A_{e_l})$ . Now  $a_l a' \notin \omega_{e_l}(A_{e_l})$  since  $a_l \notin \omega_{e_l}(A_{e_l})$  and moreover:

$$a_l a' = a_l \nu_{\mathcal{A}}(t_2^{-1}t_1) = \nu_{\mathcal{A}}(t_1^{-1}t_1(a_l)t_2^{-1}t_1) = \nu_{\mathcal{A}}(t_1^{-1}t t_1) \in \nu_{\mathcal{A}}(t_1)^{-1} P_{m_i} \nu_{\mathcal{A}}(t_1)$$

If  $v_l$  is peripheral and therefore  $a_l a'$  parabolic, it follows from the

malnormality of  $\mathbb{P}$ , that  $a_l a', \nu_{\mathcal{A}}(t_1) \in P_{m_i}$ . As in the first case, this implies that  $t_1$  fulfills (A1)a. Hence, assume that  $v_l$  is essential.

If  $t_1$  does not have an edge with trivial edge group, it is of length 1 and hence

$$a_l a' \notin \omega_{e_l}(A_{e_l}) = g_{e_l}^{-1} A_{e_l} g_{e_l} = \nu_{\mathcal{A}}(t_1)^{-1} P_{m_i} \nu_{\mathcal{A}}(t_1) \cap A_{v_l},$$

a contradiction. Thus,  $t$  fulfills (A1)b.

**Case 2b:** The underlying path of  $t$  is of the form  $r_1 r_2 (r_2)^{-1} r_3$  as described in Lemma 4.22 2. with  $r_1 r_3$  non-degenerate.

This implies that  $r_1$  travels  $e_1$  or that  $r_3$  travels  $e_k$ . W.l.o.g. assume that  $e_1$  is part of  $r_1$ . In particular, since  $r_2$  is non-degenerate,  $v_1$  cannot be essential. Since  $e_1$  was assumed to be non-peripheral, this implies  $A_{e_1} = 1$  and thus  $t$  fulfills (A1)a.

Now assume Proposition 4.19 2. is true. Let  $\mathcal{A}'$  be a subdivision of  $\mathcal{A}$  along at most two non-peripheral, non-essential edges and  $t = (a_0, e_1, \dots, e_k, a_k)$  a reduced  $\mathbb{A}'$ -path with  $|t|_{\mathcal{X}}^{\mathcal{A}'} \leq L$ , such that

$$|\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}} < \frac{|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_2} - C_2.$$

Let  $v_0 := \alpha(e_1)$  and  $v_l := \omega(e_l)$  for all  $l \in \{1, \dots, k\}$ . By Lemma 4.22, it can be assumed that one of the following cases occurs:

**Case 1:**  $t$  is almost-simple and has at most one essential vertex.

If  $t$  has no essential vertex, it follows from  $C_2 \geq C_0$  and Lemma 4.21 with  $t' := t$  that  $t$  fulfills (A6).

Hence, let there be some  $l \in \{0, \dots, k\}$  such that  $v_l$  is essential. Let  $s = s_l(a_l) s_{l+1}$  where  $s_i = (1, e_i, 1)$  if  $A_{e_i} \neq 1$ , and else  $s_i = (1)$ . It is

$$|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq |a_l|_{X \cup \mathcal{P}} + 2\nu \leq |\nu_{\mathcal{A}'}(s)|_{X \cup \mathcal{P}} + 4\nu.$$

Let  $t = t_1 s t_2$ . It follows:

$$\frac{|t|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_2} - C_2 > |\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}}$$

$$\begin{aligned}
&\geq |\nu_{\mathcal{A}'}(s)|_{X \cup \mathcal{P}} - (|\nu_{\mathcal{A}'}(t_1)|_{X \cup \mathcal{P}} + |\nu_{\mathcal{A}'}(t_2)|_{X \cup \mathcal{P}}) \\
&\geq |s|_{X \cup \mathcal{P}}^{\mathcal{A}'} - 4\nu - (|t_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2|_{X \cup \mathcal{P}}^{\mathcal{A}'}) \\
&= |s|_{X \cup \mathcal{P}}^{\mathcal{A}'} - 4\nu - (|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} - |s|_{X \cup \mathcal{P}}^{\mathcal{A}'}) \\
&= 2|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} - 4\nu - |t|_{X \cup \mathcal{P}}^{\mathcal{A}'}
\end{aligned}$$

Since  $4\nu, 3 \leq C_2$ , this implies

$$|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq \frac{1}{2} \left( 1 + \frac{1}{C_2} \right) |t|_{X \cup \mathcal{P}}^{\mathcal{A}'} - \frac{1}{2}(C_2 - 4\nu) \leq \frac{2}{3} |t|_{X \cup \mathcal{P}}^{\mathcal{A}'}.$$

Thus,  $|t_i|_{X \cup \mathcal{P}}^{\mathcal{A}'} \geq \frac{1}{6} |t|_{X \cup \mathcal{P}}^{\mathcal{A}'}$  for some  $i \in \{1, 2\}$ . With  $\frac{1}{6} \geq \frac{1}{3C_1+2}$  and  $t' := t_i$ , it follows from Lemma 4.21 that  $t$  fulfills (A6).

**Case 2:** The underlying path of  $t$  is of the form  $r_1 r_2 (r_2)^{-1} r_3$  as described in Lemma 4.22 2.

Let  $s$  be the maximal full  $\mathbb{A}'$ -subpath of  $t$  such that the underlying path of  $s$  is a subpath of  $r_2 (r_2)^{-1}$ . Let  $s'$  be the minimal full  $\mathbb{A}'$ -subpath of  $t$  such that  $r_2 (r_2)^{-1}$  is a subpath of the underlying path of  $s'$ . Since  $(r_2)_-$  is non-essential, this implies that

$$|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq |s|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2.$$

Let  $s_1$  and  $s_2$  be the full  $\mathbb{A}'$ -subpaths of  $t$  such that  $t = s_1 s' s_2$ .

It follows from Lemma 4.21, that  $t$  fulfills (A6), if

$$|s_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} \geq \frac{2}{3C_1 + 2} |t|_{X \cup \mathcal{P}}^{\mathcal{A}'},$$

as  $\delta = (3C_1 + 2)^{-1}$ . If on the other hand

$$|s_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq \frac{2}{3C_1 + 2} |t|_{X \cup \mathcal{P}}^{\mathcal{A}'},$$

then

$$|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} = |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} + \frac{2}{3C_1 + 2} |t|_{X \cup \mathcal{P}}^{\mathcal{A}'}$$

and therefore

$$|t|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq \left(1 - \frac{2}{3C_1 + 2}\right)^{-1} |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} \leq \left(1 + \frac{2}{3C_1}\right) (|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2).$$

It follows that

$$\begin{aligned} |\nu_{\mathcal{A}'}(s)|_{X \cup \mathcal{P}} &\leq |\nu_{\mathcal{A}'}(s')|_{X \cup \mathcal{P}} + 2 \\ &\leq |\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}} + |s_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |s_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2 \\ &= |\nu_{\mathcal{A}'}(t)|_{X \cup \mathcal{P}} + |t|_{X \cup \mathcal{P}}^{\mathcal{A}'} - |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2 \\ &< \left(1 + \frac{1}{C_2}\right) |t|_{X \cup \mathcal{P}}^{\mathcal{A}'} - |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} - C_2 + 2 \\ &\leq \left(1 + \frac{1}{C_2}\right) \left(1 + \frac{2}{3C_1}\right) (|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2) - |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} - (C_2 - 2) \\ &\leq \left(1 + \frac{1}{3C_1 + 2}\right) \left(1 + \frac{2}{3C_1}\right) (|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2) - |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} - (C_1 + 4) \\ &= \left(1 + \frac{1}{C_1}\right) (|s|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 2) - |s'|_{X \cup \mathcal{P}}^{\mathcal{A}'} - (C_1 + 4) \\ &= \frac{|s|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_1} - \left(C_1 + 4 - 2\left(1 + \frac{1}{C_1}\right)\right) \\ &\leq \frac{|s|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_1} - C_1. \end{aligned}$$

Let  $l \in \{0, \dots, k\}$  with  $v_l = (r_2)_+$ . Let  $t_1$  and  $t_2$  be  $\mathbb{A}'$ -paths whose underlying path is a subpath of  $r_2$  such that  $s = t_1(a_l)t_2^{-1}$ . By Lemma 4.23, assume that  $\nu_{\mathcal{A}'}(t_2^{-1}t_1) \in A_{v_l}$ .

**Case 2a:**  $v_l$  is essential.

Let  $g \in \nu_{\mathcal{A}'}(t_1)A_{v_l} = \nu_{\mathcal{A}'}(t_2)A_{v_l}$  of minimal  $X \cup \mathcal{P}$ -length in  $gA_{v_l}$  and  $h_i \in A_{v_l}$  such that  $g = \nu_{\mathcal{A}'}(t_i)h_i$  for  $i \in \{1, 2\}$ . Since  $A_{v_l}$  is  $\nu$ -relatively quasiconvex in  $(G, \mathbb{P})$ , Lemma 2.14 implies:

$$\begin{aligned} \frac{|s|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_1} - C_1 &> |\nu_{\mathcal{A}'}(s)|_{X \cup \mathcal{P}} \\ &= |\nu_{\mathcal{A}'}(t_1)a_l\nu_{\mathcal{A}'}(t_2)^{-1}|_{X \cup \mathcal{P}} \\ &= |gh_1^{-1}a_lh_2g^{-1}|_{X \cup \mathcal{P}} \end{aligned}$$

$$\geq |h_1^{-1}a_l h_2|_{X \cup \mathcal{P}} + 2|g|_{X \cup \mathcal{P}} - D$$

Moreover:

$$\begin{aligned} |s|_{X \cup \mathcal{P}}^{\mathcal{A}'} &= |t_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |a_l|_{X \cup \mathcal{P}} + |t_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} \\ &\leq |h_1^{-1}a_l h_2|_{X \cup \mathcal{P}} + |t_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |h_1|_{X \cup \mathcal{P}} + |t_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |h_2|_{X \cup \mathcal{P}} \\ &= |h_1^{-1}a_l h_2|_{X \cup \mathcal{P}} + |t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'} \end{aligned}$$

It follows:

$$\begin{aligned} 2|g|_{X \cup \mathcal{P}} &< \left( \frac{1}{C_1} - 1 \right) |h_1^{-1}a_l h_2|_{X \cup \mathcal{P}} + \frac{|t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_1} - C_1 + D \\ &\leq \frac{|t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_0} - 2C_0 \end{aligned}$$

Hence, for some  $i \in \{1, 2\}$ :

$$|\nu_{\mathcal{A}'}(t_i(h_i))|_{X \cup \mathcal{P}} = |g|_{X \cup \mathcal{P}} < \frac{|t_i(h_i)|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_0} - C_0$$

Thus, by Lemma 4.21,  $t_i$  contains some trivial segment of length greater than  $|\nu_{\mathcal{A}'}(t_i)h_i|_{X \cup \mathcal{P}}$ , which  $t_i$  only passes through once. Since  $t_i$  is a full  $\mathbb{A}'$ -subpath of  $t$ ,  $|t_i|_X^{\mathcal{A}'} \leq |t|_X^{\mathcal{A}'} \leq L$  and  $t_i$  fulfills (A6).

**Case 2b:**  $v_l \in VC_j$  is peripheral.

Let  $g \in \nu_{\mathcal{A}'}(t_1)P_{m_j} = \nu_{\mathcal{A}'}(t_2)P_{m_j}$  of minimal  $X \cup \mathcal{P}$ -length in  $gP_{m_j}$  and  $h_i \in P_{m_j}$ , such that  $g = \nu_{\mathcal{A}'}(t_i)h_i$  for  $i \in \{1, 2\}$ . Lemma 2.12 implies:

$$\begin{aligned} \frac{|s|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_1} - C_1 &> |\nu_{\mathcal{A}'}(s)|_{X \cup \mathcal{P}} \\ &= |\nu_{\mathcal{A}'}(t_1)a_l \nu_{\mathcal{A}'}(t_2)^{-1}|_{X \cup \mathcal{P}} \\ &= |gh_1^{-1}a_l h_2 g^{-1}|_{X \cup \mathcal{P}} \\ &\geq 2|g|_{X \cup \mathcal{P}} - D_2 \end{aligned}$$



Moreover, since  $e_l$  is non-peripheral:

$$\begin{aligned} |s|_{X \cup \mathcal{P}}^{\mathcal{A}'} &= |t_1|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 1 \\ &\leq |t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 1 \end{aligned}$$

It follows:

$$\begin{aligned} 2|g|_{X \cup \mathcal{P}} &< \frac{|t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + 1}{C_1} - C_1 + D_2 \\ &\leq \frac{|t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'} - C_1 + D_2 + 1}{C_1} \\ &\leq \frac{|t_1(h_1)|_{X \cup \mathcal{P}}^{\mathcal{A}'} + |t_2(h_2)|_{X \cup \mathcal{P}}^{\mathcal{A}'} - 2C_0}{C_0} \end{aligned}$$

Hence, for some  $i \in \{1, 2\}$ :

$$|\nu_{\mathcal{A}'}(t_i(h_i))|_{X \cup \mathcal{P}} = |g|_{X \cup \mathcal{P}} < \frac{|t_i(h_i)|_{X \cup \mathcal{P}}^{\mathcal{A}'}}{C_0} - C_0$$

Thus, by Lemma 4.21,  $t_i$  contains some trivial segment of length greater than  $|\nu_{\mathcal{A}'}(t_i)h_i|_{X \cup \mathcal{P}}$ , which  $t_i$  only passes through once. Since  $t_i$  is a full  $\mathbb{A}'$ -subpath of  $t$ ,  $|t_i|_X^{\mathcal{A}'} \leq |t|_X^{\mathcal{A}'} \leq L$  and  $t_i$  fulfills (A6).  $\square$

### 4.3 Recovery of Previous Results

This section shows that the combination theorem is indeed a generalization of previous results, at least in the torsion-free case. It is easy to see that Theorem 4.20 implies the result by Arzhantseva [2] about finitely generated subgroups of hyperbolic groups, since only (A3) and (A6) can occur (cf. Thm.1.2).

The results by Martínez-Pedrosa ([21], Thms. 1.1 & 1.2) are recovered from the combination theorem in the following two corollaries.

**Corollary 4.24** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group and  $X$  a symmetric finite generating set of  $G$ . Let  $H$  be a finitely generated relatively quasiconvex subgroup of  $(G, \mathbb{P})$ . Then there is some  $K_1 = K_1(G, \mathbb{P}, X, H)$  such that the following holds:*

Let  $R \subseteq P$  for some  $P \in \mathbb{P}$  such that  $O := H \cap P \subseteq R$  and such that  $|g|_X \geq K_1$  for every  $g \in R \setminus O$ .

Then the canonical homomorphism  $H *_O R \rightarrow G$  is injective and its image  $\langle H \cup R \rangle$  is relatively quasiconvex in  $(G, \mathbb{P})$ . Moreover, every parabolic subgroup of  $\langle H \cup R \rangle$  is in  $\langle H \cup R \rangle$  conjugate to some parabolic subgroup of  $H$  or some subgroup of  $R$ .

**Proof** Let  $M \geq 0$  such that  $H$  is generated by elements of  $X$ -length at most  $M$ . Let  $\nu := \nu(G, \mathbb{P}, X, M)$ , as well as  $L := L(G, \mathbb{P}, X, M, 0)$ ,  $C := C(G, \mathbb{P}, X, M, 0)$  and  $K_1 := \max\{L, C^2 + C\}$ . W.l.o.g. assume  $R \neq 1$ . Define a  $(G, \mathbb{P})$ -carrier graph  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  as follows:

Let  $\mathbb{O}$  be an induced structure of  $(G, \mathbb{P})$  on  $H$  with  $O \in \mathbb{O}$  if  $O \neq 1$ . Let  $O_1 := O$ ,  $g_1 := 1$  and  $\mathbb{O} \setminus \{O\} = \{O_2, \dots, O_l\}$  with  $O_i = H \cap P_i^{g_i}$  and  $|g_i|_X \leq \nu$  for all  $i \in \{2, \dots, l\}$ .

Start constructing  $\mathcal{A}$  with a segment of length 2 with edges  $e_1$  and  $f_1$ , and vertices  $v_0 = \omega(f_1)$ ,  $v_1 = \alpha(f_1) = \omega(e_1)$  and  $c_1 = \alpha(e_1)$ . Let  $A_{v_0} = H$ ,  $A_{v_1} = A_{c_1} = A_{e_1} = R$ ,  $A_{f_1} = O$  and  $g_{e_1} = g_{f_1} = 1$ . Let  $\mathcal{C}_1$  be the sub-carrier graph of groups with the single edge  $e_1$ .

For each  $i \in \{2, \dots, l\}$  let  $\mathcal{C}_i$  consist of a single edge  $e_i$  with fundamental group  $O_i^{g_i^{-1}}$  and  $g_{e_i} = 1$ . Connect these  $\mathcal{C}_i$  to  $v_0$  by a single edge  $f_i$  with edge group  $O_i^{g_i^{-1}}$  and  $g_{f_i} = g_i$  such that  $\mathbb{O}_{v_0} = \mathbb{O}$ .

It is clear that the  $(G, \mathbb{P})$ -carrier graph of groups  $(\mathcal{A}, (\mathcal{C}_i, c_i)_{1 \leq i \leq l})$  which results from this construction is  $M$ -normal. It is further clear that (A1) can not be fulfilled by  $\mathcal{A}$ , since  $\mathbb{O}$  is an induced structure. Since  $\mathcal{A}$  is a tree, there are no non-trivial  $\mathbb{A}$ -almost circuits, i.e.  $\mathcal{A}$  does not fulfill (A3). Lastly, it follows from  $|g|_X \geq K_1 \geq L$  for all  $g \in R \setminus O$  that  $\mathcal{A}$  can fulfill none of the other statements of Theorem 4.20.

Hence,  $\nu_{\mathcal{A}}$  is a  $(C, C)$ -quasiisometric embedding and thus  $\langle H, R \rangle$  is relatively quasiconvex in  $(G, \mathbb{P})$  by Theorem 2.16. Further, every parabolic subgroup of  $\text{Im}(\nu_{\mathcal{A}}) = \langle H \cup R \rangle$  is conjugate in  $\langle H \cup R \rangle$  into  $R$  or  $O_i$  for some  $i \in \{2, \dots, l\}$ . Since  $A_{v_0} = H$  is mapped injectively and  $K_1 \geq C^2 + C$ , this implies that  $\nu_{\mathcal{A}}$  is injective.  $\square$

**Corollary 4.25** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group and  $X$  a symmetric finite generating set of  $G$ . Let  $H_1$  and  $H_2$  be finitely generated relatively quasiconvex subgroups of  $(G, \mathbb{P})$ . Then there is some  $K_2 = K_2(G, \mathbb{P}, X, H_1, H_2)$  such that the following holds:*

*Let  $P \in \mathbb{P}$ ,  $R := H_1 \cap P = H_2 \cap P$  and let  $h \in G$  such that  $hRh^{-1} = R$  and  $|g|_X \geq K_2$  for every  $g \in RhR$ .*

*Then the canonical homomorphism  $H_1 *_R hH_2h^{-1} \rightarrow G$  is injective and its image  $\langle H_1 \cup hH_2h^{-1} \rangle$  is relatively quasiconvex in  $(G, \mathbb{P})$ . Moreover, every parabolic subgroup of  $\langle H_1 \cup hH_2h^{-1} \rangle$  is in  $\langle H_1 \cup hH_2h^{-1} \rangle$  conjugate to some parabolic subgroup of  $H_1$  or  $hH_2h^{-1}$ .*

**Proof** Let  $M \geq 0$  such that  $H_1$  and  $H_2$  are generated by elements of  $X$ -length at most  $M$ . Let  $\nu := \nu(G, \mathbb{P}, X, M)$ ,  $L := L(G, \mathbb{P}, X, M, 0)$ ,  $C := C(G, \mathbb{P}, X, M, 0)$  and  $K_2 := \max\{L, C^2 + C\}$ . Define a  $(G, \mathbb{P})$ -carrier graph  $(\mathcal{A}, ((\mathcal{C}_i, c_i))_{1 \leq i \leq k})$  as follows:

Let  $\mathbb{O}^j$  be an induced structure of  $(G, \mathbb{P})$  on  $H_j$  for  $j \in \{1, 2\}$  with  $R \in \mathbb{O}^j$  if  $R \neq 1$ . Let  $O_1^j := R$ , and  $\mathbb{O}^j \setminus \{R\} = \{O_2^j, \dots, O_{l_j}^j\}$  with  $O_i^j = H \cap (P_i^j)^{g_i^j}$  and  $|g_i^j|_X \leq \nu$  for all  $j \in \{1, 2\}$  and  $i \in \{2, \dots, l_j\}$ . Let  $k := l_1 + l_2 - 1$ .

Start constructing  $\mathcal{A}$  with a segment of length 4 with edges  $e_1^j$  and  $f_1^j$  for  $j \in \{1, 2\}$ , and vertices  $v_0^j = \omega(f_1^j)$ ,  $v_1^j = \alpha(f_1^j) = \omega(e_1^j)$  and  $c_1 = \alpha(e_1^j)$  for  $j \in \{1, 2\}$ . Let  $A_{v_0^j} = H_j$ ,  $A_{v_1^j} = A_{c_1} = A_{e_1^j} = A_{f_1^j} = R$ ,  $g_{e_1^j} = g_{f_1^j} = 1$  and  $g_{f_1^j} = h$  for  $j \in \{1, 2\}$ . Let  $\mathcal{C}_1$  be the sub-carrier graph of groups consisting of the edges  $e_1^1$  and  $e_1^2$ .

As in the proof of Corollary 4.24, add stars  $\mathcal{C}_i$  for  $i \in \{2, \dots, k\}$  to  $v_0^1$  and  $v_0^2$ , such that  $\mathbb{O}_{v_0^j} = \mathbb{O}^j$  for  $j \in \{1, 2\}$ .

Since  $hRh^{-1} = R$ ,  $(\mathcal{A}, (\mathcal{C}_i, c_i)_{1 \leq i \leq k})$  is indeed a  $(G, \mathbb{P})$ -carrier graph of groups. Moreover, it is clear that  $\mathcal{A}$  is  $M$ -normal. It is further clear that (A1) can not be fulfilled by  $\mathcal{A}$ , since  $\mathbb{O}^1$  and  $\mathbb{O}^2$  are induced structures and since  $|g|_X \geq K_2 \geq L$  for all  $g \in RhR$ . Since  $\mathcal{A}$  is a tree, there are further no non-trivial  $\mathbb{A}$ -almost-circuits, i.e.  $\mathcal{A}$  does not fulfill (A3). Lastly, it follows from  $|g|_X \geq K_2 \geq L$  for all  $g \in RhR$  that  $\mathcal{A}$  can fulfill none of the other statements of Theorem 4.20.

Hence,  $\nu_{\mathcal{A}}$  is a  $(C, C)$ -quasiisometric embedding and thus  $\langle H_1 \cup hH_2h^{-1} \rangle$  is relatively quasiconvex in  $(G, \mathbb{P})$  by Theorem 2.16. Further, every parabolic subgroup of  $\langle H_1 \cup hH_2h^{-1} \rangle$  is conjugate in  $\langle H_1 \cup hH_2h^{-1} \rangle$  into  $O_i^1$  for some  $i \in \{1, \dots, l_1\}$  or  $hO_i^2h^{-1}$  for some  $i \in \{1, \dots, l_2\}$ . Since  $A_{v_0^1} = H_1$  is mapped injectively and  $K_2 \geq C^2 + C$ , this implies that  $\nu_{\mathcal{A}}$  is injective.  $\square$

## 4.4 Discussion

This section discusses one of the possible applications of the combination theorem as well as a possible generalization to relatively hyperbolic groups with torsion.

As described in the introduction, Kapovich and Weidmann [17] proved a combination theorem and derived from it a finiteness result about the number of conjugacy classes of quasiconvex subgroups of an almost torsion-free hyperbolic group.

It seems possible to derive a similar finiteness result from the combination theorem of this thesis, leading to the following conjecture:

**Conjecture 4.26** *Let  $(G, \mathbb{P})$  be a torsion-free relatively hyperbolic group, such that every  $P \in \mathbb{P}$  is noetherian and such that every finitely generated subgroup of  $G$  is relatively quasiconvex. Let  $l \in \mathbb{N}$ .*

*Then there is a finite number of equivalence classes of  $(G, \mathbb{P})$ -carrier graphs of groups, such that for every  $H \leq G$  which is generated by at most  $l$  elements there is some carrier graph  $\mathcal{A}$  of  $H$  in one of these equivalence classes with  $\nu_{\mathcal{A}}$  an injective quasiisometric embedding.*

The following is a possible approach to derive this conjecture from Theorem 4.20.

For any subgroup  $H \leq G$  generated by elements  $h_1, \dots, h_l$  there is a  $(G, \mathbb{P})$ -carrier graph corresponding to the rose with  $l$  petals which are labeled by  $h_1, \dots, h_l$ . This carrier graph can further be made  $M$ -normal by designating appropriate peripheral stars. All these carrier graphs for such subgroups  $H$  belong to the same equivalence class  $E_0$ .

Now define a number of foldings for  $M$ -normal carrier graphs  $\mathcal{A}$ , which in some sense repair the obstructions (A1)-(A6) to  $\nu_{\mathcal{A}}$  being a quasiisometric embedding which are stated in Theorem 4.20. In particular, the carrier graph resulting from a folding should be  $M'$ -normal and represent the same subgroup of  $G$  as the original one.

It then remains to show, by considering a complexity which is strictly decreasing under every folding, that performing the foldings in the correct order transforms every carrier graph in  $E_0$  into a carrier graph in one of a finite number of equivalence classes. This makes use of the fact that every  $P \in \mathbb{P}$  is noetherian and that the constant  $L$  defined in Theorem 4.20 does only depend on the equivalence class of the carrier graph of groups.

The proof of Theorem 4.20 uses the assumption of  $G$  being torsion-free only once. The proof considers a reduced  $\mathbb{A}$ -path  $t$  of the form  $t = r(a)r^{-1}$  where  $r$  is almost-simple, such that a realization  $s$  of  $t$  is not a quasigeodesic. The proof then concludes, that  $r(a')$  is not a quasigeodesic for some particular constants and some  $a' \in A_{r+}$ .

The same is not necessarily true, if  $G$  is not torsion-free. Indeed, if  $a$  is a finite-order element of  $G$  with infinite centralizer, there might be arbitrarily long  $r$  such that  $\nu_{\mathcal{A}}(t) = a$ . A seventh condition would have to be added to Theorem 4.20 to reflect this fact.

Moreover, to address this new obstruction in the folding process described above would require a folding which takes a finite subgroup of some vertex group and pushes it out along an edge path starting in that vertex. Hence, a useful generalization of a  $(G, \mathbb{P})$ -carrier graph of groups for groups with torsion would have to allow some finite edge groups. This in turn would have to be taken into account in the proof of the analogue of Theorem 4.20 for groups with torsion.

A question for further research is therefore to find a workable definition for these carrier graphs of groups in the case with torsion and to prove the analogue of Theorem 4.20 for them.

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## Erklärung

Hiermit erkläre ich, dass die Abhandlung - abgesehen von der Beratung durch meinen Betreuer - nach Inhalt und Form eine eigenständige und nur mit den angegebenen Hilfsmitteln verfasste Arbeit ist, sie unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft entstanden ist, und mir kein akademischer Grad entzogen wurde.

Die Arbeit hat weder ganz noch zum Teil schon einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Ein Teil der Arbeit ist bereits in dem folgenden Vorabdruck auf dem Dokumentenserver arXiv erschienen:

Carstensen, Thomas. "Detecting Relatively Quasiconvex Subgroups and Their Induced Peripheral Structure." arXiv preprint, arXiv:2203.02357 (2022). (<https://doi.org/10.48550/arXiv.2203.02357>)

Kiel, den \_\_\_\_\_