

Colorings of Randomly Augmented Graphs and a
Maker-Strategy for the k -Edge-Connectivity Game

Dissertation
zur Erlangung des Doktorgrades der
Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von
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Kiel, 2024

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Datum der mündlichen Prüfung : 03. September 2024

Danksagungen

An dieser Stelle möchte ich einigen Personen danken, ohne die ich diese Arbeit nicht in dieser Form hätte fertigstellen können.

An erster Stelle bedanke ich mich bei Prof. Dr. Srivastav für die Betreuung und Beratung während meiner Forschungsarbeit.

Weiterhin bedanke ich mich bei allen Mitgliedern der Arbeitsgruppe *Diskrete Optimierung* der Christian-Albrechts-Universität zu Kiel, deren hilfreiche Anmerkungen mir sehr geholfen haben und die mir in unseren Gesprächen in den Oberseminaren ermöglichten, gute Ideen zu finden und schlechte Ideen als solche zu erkennen und zu verwerfen.

Darüber hinaus bedanke ich mich bei meinen Eltern und meiner Familie, deren emotionale und finanzielle Unterstützung es mir ermöglicht hat, stets mit vollem Einsatz zu forschen.

Gillian Krüger danke ich für ihre unermüdlichen Ermutigungen und ihr stets offenes Ohr.

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Summary

In this thesis we study the behavior of two graph properties on randomly augmented graphs as well as multiple Maker-Breaker games. We prove tight upper bounds for the chromatic number and the choice number of randomly augmented graphs and give an upper bound for the number of turns it takes Maker to win several Maker-Breaker games.

In Chapter 2 we give slightly improved probability bounds for the values of the chromatic number and the choice number for the $\mathcal{G}_{n,p}$ random graph model using techniques used by Bollobás [Bol88] and Kahn [Alo93] respectively, combining them with a result by Krivelevich et al. [KSvVW03], as this allows for improved results in Chapter 3 and Chapter 4. We also calculate an explicit bound for the number of turns Maker needs to win the biased k -edge-connectivity game, using well-known strategies, as such results did not give explicit upper bounds for the number of turns but rather focused on identifying certain games as Maker wins.

In Chapter 3 we show that for randomly augmented graphs $\text{pert}_{H,p}$ the chromatic number $\chi(\text{pert}_{H,p})$ for each $\varepsilon > 0$ is bounded from above by

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad a.a.s. \quad (1)$$

where $b = 1/(1-p)$ as long as $p \in (0, 1)$ is constant and $\chi(H) \in o(n)$. We also show that inequality 1 holds, if $p = p(n) = n^{-\theta}$ for some $\theta \in (0, 1/3)$ and $\chi(H) \leq n^{1-3\theta}$. Since the proof is non-constructive we also give slightly less powerful constructive proofs.

In Chapter 4 we show that the upper bound established in Chapter 3 for the chromatic number of a randomly augmented graph also holds for the choice number of the randomly augmented graph, i.e. for each $\varepsilon > 0$

$$\chi_l(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad a.a.s. \quad (2)$$

where $b = 1/(1-p)$ as long as $p \in (0, 1)$ is constant and $\chi(H) \leq n^{-\theta}$ for some $\theta \in (0, 1/2)$.

In Chapter 5 we prove that Maker can win the biased Maker-Breaker k -Hamilton-Cycle game in at most $kn + o(n)$ rounds. We use this result to show that Maker can win the k -edge-connectivity game in at most $\frac{k}{2}n + o(n)$ turns and that Maker can win the k -factor game in at most $\left\lceil \frac{k}{2} \right\rceil n + o(n)$ for each $k \in \mathbb{N}_{\geq 2}$. In the case of the k -edge-connectivity game this bound on the number of turns is optimal up to the $o(n)$ term. The bound in the k -factor game is optimal for even k , while for odd k a lower bound of $\frac{k}{2}n$ could be possible. For each of these three games the bound was improved by a factor of at least 8. We achieve these results by building an expander-like structure on a set of $\theta \left(\frac{n}{\sqrt{\log(n)}} \right)$ vertices and families of differently colored paths which will form the basis for the Hamilton cycles we build, before we start closing the Hamilton cycles. This approach is a generalization of an approach introduced by Brüstle et al. [BCN⁺23] that showed that Maker can win the biased Hamiltonicity game in at most $n + o(n)$ moves.

Zusammenfassung

In dieser Arbeit untersuchen wir das Verhalten von zwei Eigenschaften zufällig augmentierter Graphen sowie mehrere *Maker-Breaker* Spiele. Wir beweisen scharfe obere Schranken für die chromatische Zahl und die listenchromatische Zahl zufällig augmentierter Graphen und geben verschiedene obere Schranken für die Anzahl der Züge, in denen Maker gewisse *biased Maker-Breaker* Spiele gewinnen kann.

In Kapitel 2 geben wir leicht verbesserte Wahrscheinlichkeitsschranken für die Werte der chromatischen Zahl und der listenchromatischen Zahl des $\mathcal{G}_{n,p}$ Zufallsgraphen. Hierfür verwenden wir Techniken von Bollobás [Bol88] bzw. Kahn [Alo93] und kombinieren diese mit einem Resultat von Krivelevich et al. [KSvVW03], da dies eine Verbesserung unserer Ergebnisse in den Kapiteln 3 und 4 ermöglicht. Des Weiteren berechnen wir explizit die beste bisher bekannte obere Schranke für die Anzahl der Züge, die *Maker* benötigt, um das k -Kantenzusammenhangsspiel zu gewinnen. Die existierenden Ergebnisse ließen diese Information oft aus.

In Kapitel 3 zeigen wir, dass die chromatische Zahl von zufällig augmentierten Graphen $\chi(\text{pert}_{H,p})$ für jedes $\varepsilon > 0$ nach oben durch

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad a.a.s. \quad (3)$$

beschränkt ist, wobei $b = 1/(1 - p)$, solange $p \in (0, 1)$ konstant und $\chi(H) \in o(n)$ ist. Wir zeigen außerdem, dass Ungleichung 3 auch gilt, wenn $p = p(n) = n^{-\theta}$ für ein $\theta \in (0, 1/3)$ und $\chi(H) \leq n^{1-3\theta}$ gilt. Da dieser Beweis nicht konstruktiv ist, geben wir ein schwächeres konstruktives Resultat an.

In Kapitel 4 zeigen wir, dass die obere Schranke für $\chi(\text{pert}_{H,p})$ aus Kapitel 3 auch für die listenchromatische Zahl von zufällig augmentierten Graphen gilt. So gilt für alle $\varepsilon > 0$

$$\chi_l(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad a.a.s. \quad (4)$$

wobei $b = 1/(1 - p)$, solange $p \in (0, 1)$ konstant und $\chi(H) \leq n^{-\theta}$ für ein $\theta \in (0, 1)$ ist.

In Kapitel 5 beweisen wir, dass *Maker* für jedes $k \geq 2$ das *biased Maker-Breaker* k -Hamiltonkreisspiel in höchstens $kn + o(n)$ Zügen gewinnen kann. Wir verwenden dieses Resultat, um zu zeigen, dass Maker das k -Kantenzusammenhangsspiel in höchstens $\frac{k}{2}n + o(n)$ Zügen gewinnen kann und dass Maker das k -Faktorspiel in höchstens $\left\lceil \frac{k}{2} \right\rceil n + o(n)$ Zügen gewinnen kann. In dem Fall des k -Kantenzusammenhangspiels stellt dies bis auf den $o(n)$ Term die optimale obere Schranke dar. Die Schranke im k -Faktorspiel ist für gerade k scharf, während für ungerade k unter Umständen eine untere Schranke von $\frac{k}{2}n$ möglich wäre. In jedem dieser Spiele wurde die obere Schranke um einen Faktor größer als 8 gesenkt. Wir beweisen diese Resultate, indem wir eine *expanderähnliche Struktur* auf $\theta \left(\frac{n}{\sqrt{\log(n)}} \right)$ Knoten und verschiedenfarbige Pfade, die die Grundlage der Hamiltonkreise bilden, konstruieren. Dieser Ansatz verallgemeinert ein Resultat von Brüstle et al., das besagt, dass Maker das Hamiltonkreisspiel in höchstens $n + o(n)$ Zügen gewinnen kann.

Chapter 1

Introduction

1.1 On the model of Randomly Augmented Graphs

The most famous random graph model is the $\mathcal{G}_{n,p}$ model. Let $p : \mathbb{N} \rightarrow (0, 1)$ and let $n \in \mathbb{N}$. We say that $G = (V, E)$ is a $\mathcal{G}_{n,p}$ -graph ($G \sim \mathcal{G}_{n,p}$) if $|V| = n$ and each potential edge $e \in \binom{V}{2}$ is chosen independently with probability $p(n)$. In this thesis we will concentrate on two main cases.

1. p is constant or
2. p is some monotonously decreasing function, often in the shape $p(n) = n^{-\gamma}$ for some $\gamma > 0$.

We further assume that $V = [n]$. The $\mathcal{G}_{n,p}$ model was first introduced by Gilbert [Gil59] in 1959 and is highly related to the $\mathcal{G}_{n,M}$ random graph model introduced by Erdős and Rényi [ER59] where a random subset of size $M \leq n$ is chosen. Both of these models have been studied extensively. For a deeper analysis of the $\mathcal{G}_{n,p}$ model we refer the reader to the books of Bollobás [Bol85] and Frieze and Karoński [FK16].

In Chapter 3 and Chapter 4 of this thesis we study a generalization of the $\mathcal{G}_{n,p}$ random graph model. Consider a deterministic graph $H = ([n], E_H)$ and $p : \mathbb{N} \rightarrow (0, 1)$ an arbitrary function. We define the randomly augmented graph $\text{pert}_{H,p} := H \cup \mathcal{G}_{n,p}$ as the union of the so-called host-graph H and a random $\mathcal{G}_{n,p}$ -graph. Note that if H is the empty graph, this model yields the random graph $\mathcal{G}_{n,p}$.

In their foundational paper [BFM03] Bohman et al. introduced a hybrid graph model, where the host-graph has at least linear (in n) minimum degree and the random graph is a $\mathcal{G}_{n,M}$ graph. In the current literature both of the random graph models described above have been studied as augmenting graphs. However, since in the existing literature the term *perturbed graph* is already heavily connected to the restriction on the minimum degree, see for example [CHMP20], [BMPP20] or [BPSS23], and the host graphs we are interested in are graphs H with bounded chromatic number $\chi(H)$, we decided to use the term *augmented graph* in this thesis.

1.2 On the Chromatic Number of Random Graphs

For a graph $G = (V, E)$ the chromatic number $\chi(G)$ of G is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices are of the same color. Since the number of colors in this chapter is not bounded by a constant number we will not use real colors $\{\text{red}, \text{blue}, \dots\}$ but use the first few natural numbers as colors. This is standard procedure [Bol98]. That is, for a $k \in \mathbb{N}$ the function $\varphi : V \rightarrow [k]$ is called a k (vertex) coloring of G . φ is called a proper k -coloring of G if $\{v, w\} \in E \implies \varphi(v) \neq \varphi(w)$.

$$\chi(G) := \min \{k : \exists (\varphi : V \rightarrow [k]) : \{v, w\} \in E \implies \varphi(v) \neq \varphi(w)\}.$$

Note that often times a proper vertex k -coloring $\varphi : V \rightarrow [k]$ will be referred to as just a coloring of G , if the usage is clear from the context. Let $\varphi : V \rightarrow [k]$ be a proper coloring of G then, for each $j \in [k]$ $\varphi^{-1}(j)$ is called the j -color class of G with respect to φ . Note that by definition each color class is an independent set in G .

The chromatic number of the $\mathcal{G}_{n,p}$ model has been of interest for a long time. It was first studied by Erdős in [ER59] and its asymptotic behavior was first described by Bollobás [Bol88] who showed that for each $\varepsilon > 0$ $\chi(\mathcal{G}_{n,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2 \log(n)}$, if p is constant, or $p = p(n) \geq n^{-\theta}$, for some $\theta \in (0, 1/3)$ and $b = 1/(1 - p)$.

In this thesis we give a generalization for the result of Bollobás by showing that for each $\varepsilon > 0$

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))}$$

as long as p is constant. If $p = p(n) = n^{-\theta}$, for some $\theta \in (0, 1/3)$, we can show the same upper bound as long as $\chi(H) \leq n^{1-3\theta}$. It is easy to see that this indeed is a generalization of the famous Bollobás result by plugging in the empty graph which has chromatic number 1.

We achieve this result by considering an optimal coloring of H and dividing its color classes into large color classes which have at least slightly below average size and small color classes which are even smaller. Building upon this we calculate the expected value of the chromatic numbers of the graphs induced by these color classes and finally apply McDiarmid's bounded differences inequality [McD89].

We also show that for certain host graphs, for example the complete $\chi(H)$ -partite graph, this upper bound is tight.

Furthermore, since the approach described above is non-constructive, we study an algorithm that colors the randomly augmented graph $\text{pert}_{H,p}$ by coloring the (purely random) graphs induced by the color classes of an optimal coloring of H . This yields the same upper bound as our non-constructive approach, but fails for host graphs of chromatic number at least $n/\log(n)^{1/2}$.

1.3 On the Choice Number of Random Graphs

Let $G = (V, E)$ be a graph and $U := (U_v)_{v \in V}$ be a family of sets. Then $\varphi : V \rightarrow \bigcup_{v \in V} U_v$ is called a list-coloring of G with respect to U iff for all $v \in V$ there is $\varphi(v) \in U_v$ and for all $\{v, w\} \in E$ there is $\varphi(v) \neq \varphi(w)$. The list-chromatic number or choice number of G , denoted by $\chi_l(G)$ is defined as the minimum $k \in \mathbb{N}$ such that for each family $U := (U_v)_{v \in V}$ of sets with $|U_v| = k$ there exists a list coloring of G with respect to U . It should be noted that if $k \geq n$, we can color G iteratively with one of the remaining colors. We further note that, finding a proper k -coloring of G in the context of Chapter 3 is equivalent to finding a list coloring of G with respect to $([k])_{v \in V}$. Thus $\chi(G) \leq \chi_l(G) \leq n$.

In contrast to the chromatic number of $\mathcal{G}_{n,p}$ the analysis of the choice number of $\mathcal{G}_{n,p}$ proved to be more problematic. In 1979 Erdős et al. [ERT79] failed to give a better bound than $\chi_l(\mathcal{G}_{n,1/2}) \leq n/2$. In 1992 Alon [Alo92] succeeded in proving that indeed $\chi_l(\mathcal{G}_{n,1/2}) \in o(n)$. Finally Kahn managed to give a proof that for all $\varepsilon > 0$ $\chi_l(\mathcal{G}_{n,1/2}) \leq (1 + \varepsilon) \frac{n \log(2)}{2 \log(n)}$ in private communication with Alon [Alo93]. The proof of Kahn is easily generalized for the case where $p \in (0, 1)$ is an arbitrary constant.

In this thesis we give a generalization of the result of Kahn mentioned above. Concretely we prove that for all $\varepsilon > 0$

$$\chi_l(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$$

a.a.s. where $b = 1/(1 - p)$, $p \in (0, 1)$ constant and $\chi(H) \leq n^\gamma$ for some $\gamma < 1/2$.

As the result for the chromatic number, we can easily see that this result is a true generalization of Kahn's result by choosing the empty graph on n vertices as a host graph.

Consider a randomly augmented graph $\text{pert}_{H,p}$ and a $\mathcal{U} := (U_v)_{v \in V}$ family of sets that satisfy the prerequisites stated above. We achieve this result by considering an optimal coloring of the host graph H , categorizing the color classes as "large" or "small" as in the chapter about the chromatic number and randomly assigning a "large" color class to each color in $U = \bigcup_{v \in V} U_v$. Then we show that there exists a coloring for the random graph induced by each large color class such that enough colors remain to color each v in small color classes.

Furthermore we remark that the bound stated above is tight, because $\chi(\text{pert}_{H,p}) \leq \chi_l(G \text{ pert}_{H,p})$.

1.4 On Maker-Breaker Games

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph where \mathcal{V} is finite. Let further $b \in \mathbb{N}_{\geq 1}$. In the $(1 : b)$ Maker-Breaker \mathcal{H} -game two players, Maker (the first player) and Breaker (the second player) alternately claim 1 (Maker) and b (Breaker) elements of \mathcal{V} . Maker wins if she succeeds in claiming all elements of some $E \in \mathcal{E}$. Breaker wins if he can prevent this, i.e. Breaker wins if, at the end of the game, he claimed at least one element of each $E \in \mathcal{E}$. We call \mathcal{V} the *board*, \mathcal{E} the set of *winning sets* and b the *bias* of the game.

Remark 1.1 Since the game is played with perfect information, the result of the game with perfect play is known at the beginning of the game. If Maker wins with optimal play, we call the game a *Maker win*, otherwise we call the game a *Breaker win*.

Remark 1.2 Maker-Breaker-games are bias-monotone in the following sense: Let $b_1, b_2 \in \mathbb{N}_{\geq 1}$ with $b_1 < b_2$ and let \mathcal{H} be a finite hyper-graph. If the $(1 : b_2)$ Maker-Breaker \mathcal{H} -game is a Maker win, so is the $(1 : b_1)$ Maker-Breaker \mathcal{H} -game. Thus, the *threshold bias* $b^* := \max\{b \in \mathbb{N}_{\geq 1} \mid \text{the } (1 : b)\text{-game is a Maker win}\}$ is well-defined for each Maker-Breaker game.

Remark 1.3 For each game studied in this thesis, the board will be the edge set of the complete graph $K_n = ([n], \binom{[n]}{2})$ for some $n \in \mathbb{N}$. In addition, the games will be named after the graph property that Maker wants her graph to have. For example: If for some $k \in \mathbb{N}$ we mention the k -edge-connectivity game, we have

$$\mathcal{V} = \binom{[n]}{2} \text{ and } \mathcal{E} = \left\{ E \subseteq \binom{[n]}{2} : ([n], E) \text{ is } k\text{-edge-connected} \right\}.$$

For further information on Maker-Breaker games and its generalization positional games we refer the reader to the book of Hefetz et al. [HKSS14].

In this thesis we give an upper bound for the number of turns Maker needs to win the $(1 : b)$ biased k -edge-connectivity game, the k -factor-game and the k -edge-disjoint Hamilton Cycle game for each $k \geq 2$ a.a.s. We show that Maker can win the k -edge-disjoint Hamilton Cycle game in at most $kn + o(n)$ turns and the k -edge-connectivity game in at most $\frac{kn}{2} + o(n)$ turns. This constitutes the minimum number of turns Maker needs to achieve a win in these games up to the $o(n)$ term. For the k -factor game we prove that Maker can win in at most $\left\lceil \frac{k}{2} \right\rceil n + o(n)$ turns. This is optimal up to the $o(n)$ term for even k . The former upper bound for these games were constructed using an approach of Krivelevich [Kri10], which assures a Maker win in at most $(9k + 3)n + o(n)$ in the k -edge-disjoint Hamilton Cycle game. Since we could not locate an explicit proof for these bounds we give one in section 2.3.

We achieve these new upper bounds by generalizing an approach found by Brüstle et al. [BCN⁺23] building an expander-like structure on a vertex set of size $C \frac{n}{\log(n)^{1/2}}$ where $C > 0$ is a constant depending on the game and k . Furthermore we construct multiple sets of paths that will form the majority of the Hamilton cycles. The Hamilton cycles are finally constructed using a technique similar to Pósa rotation [P676]. This yields k edge-disjoint Hamilton cycles, which implies the existence of a $2k$ -factor and a $(2k - 1)$ -factor in Maker's graph. Additionally, Maker's graph is $2k$ -edge-connected. To achieve the bound for the k -edge-connectivity game for even k we construct $(k - 1)/2$ edge-disjoint Hamilton cycles and an expander-like structure on a set $S \subseteq V$ of $C \frac{n}{\log(n)^{1/2}}$ vertices. Then we use that in the construction described above no vertex has a high distance from the expander-like structure. We then connect pairs of vertices such that each of these vertices receives an additional path into S .

Chapter 2

Material and Methods

The aim of this chapter is to give an insight into existing results that this thesis relies on. In section 2.1 we give slightly stronger results for the concentration of the chromatic number of the $\mathcal{G}_{n,p}$ random graph, relying on the same techniques used by Bollobás in [Bol88]. In section 2.2 we improve on a result of Kahn and Alon [AKS99] in a similar fashion. In section 2.3 we give an explicit winning strategy for Maker to win the biased k -edge-disjoint Hamilton-Cycle game. This result is often mentioned, but since we could not locate a thorough proof we give one in this chapter.

2.1 On the Chromatic Number of Random Graphs

In this section we will prove upper bounds of $\chi(\mathcal{G}_{n,p})$ that hold a.a.s. The techniques used are those of Bollobás [Bol88]. However, since there have been improvements to the results Bollobás based his proofs on, we will prove improved versions.

To tackle this problem we will split p into multiple ranges to control the vertices that are not included in large independent sets.

The following results and proofs are concerned with the chromatic number of $\mathcal{G}_{n,p}$. Variations of these results are known, but a new result by Krivelevich, Sudakov, van Vu and Wormald [KSvVW03] allows a slight strengthening of the probability bounds with the same techniques.

We start by reminding the reader of the following simple result.

Remark 2.1 Let $p = p(n) \in (0, 1)$, $b = b(n) = \frac{1}{1-p(n)}$ and $\log_b(n) = \frac{\log(n)}{\log(b)}$.

$$\text{If } p = p(n) \rightarrow 0 \text{ then } \log(b) \sim p. \quad (2.1)$$

This can be shown by applying L'Hopital's rule to the term $\frac{x}{\log((1-x)^{-1})}$ for $x \rightarrow 0$.

First we prove the following result which was stated but not proved in [KSvVW03]. Please note that proofs of variations of this result are often left as exercises to the reader. For the sake of completeness the proof shall be included here.

Lemma 2.2 *For each $n \in \mathbb{N}$ and $p = n^{-\theta}$ for some $\theta \in (0, 1)$ let*

$$k_0(n) := \max \left\{ k \in \mathbb{N} : \binom{n}{k} (1-p)^{\binom{k}{2}} > n^4 \right\}.$$

Then $k_0 \sim 2 \log_b(np)$ and more precisely $2 \log_b(np) - 4 \log_b(\log(np)) \leq k_0 \leq 2 \log_b(np)$.

Proof. Let $k_1 = 2 \log_b(np)$. Then

$$\begin{aligned}
 \binom{n}{k_1} (1-p)^{\binom{k_1}{2}} &\leq \left(\frac{ne}{k_1(1-p)^{\frac{1}{2}}} (1-p)^{\frac{k_1}{2}} \right)^{k_1} \\
 &= \left(\frac{n \log(b)e}{2 \log(np)(1-p)^{\frac{1}{2}}} \cdot b^{\frac{-2 \log_b(np)}{2}} \right)^{k_1} \\
 &= \left(\frac{e}{2 \log(np)(1-p)^{\frac{1}{2}}} \cdot \frac{n \log(b)}{np} \right)^{k_1} \\
 &= \left(\frac{(1+o(1))e}{2 \log(np)(1-p)^{\frac{1}{2}}} \right)^{k_1} = o(1). \quad (\text{by Remark 2.1})
 \end{aligned}$$

Let further $k_2 = 2 \log_b(np) - 4 \log_b(\log(np))$. Then

$$\begin{aligned}
 &\binom{n}{k_2} (1-p)^{\binom{k_2}{2}} \\
 &\geq \left(\frac{n}{k_2(1-p)^{\frac{1}{2}}} (1-p)^{\frac{k_2}{2}} \right)^{k_2} \\
 &= \left(\frac{n \log(b)}{2 \log(np)(1-p)^{\frac{1}{2}}} \cdot b^{\frac{-2 \log_b(np) + 4 \log_b(\log(np))}{2}} \right)^{k_2} \\
 &= \left(\frac{1}{2 \log(np)(1-p)^{\frac{1}{2}}} \cdot \frac{n \log(b)}{np} \cdot (\log(np))^2 \right)^{k_2} \\
 &= \left(\frac{(1+o(1)) \log(np)}{2(1-p)^{\frac{1}{2}}} \right)^{k_2} \quad (\text{by Remark 2.1}) \\
 &\geq c^{k_2} \log(np)^{k_2} \quad (\text{for some constant } c > 0) \\
 &= \exp(k_2(\log(c) + \log(\log(np)))) \\
 &\geq \exp(\log_b(np)(\log(c) + \log(\log(np)))) \quad (\text{since } k_2 \geq \log_b(n)) \\
 &= (np)^{\log(b)^{-1}(\log(c) + \log(\log(np)))} \quad (\text{by Remark 2.1}) \\
 &> n^4. \quad \square
 \end{aligned}$$

Thus, we have $k_2 \leq k_0 \leq k_1$ and k_1, k_2 are both asymptotically equal to $2 \log_b(np)$.

It was shown in [KSvVW03] that the independence number of $\mathcal{G}_{n,p}$ is at least k_0 asymptotically almost surely:

Theorem 2.3 *Let $n^{-\frac{2}{5}+\delta} \ll p(n) \leq 1 - \delta$ for some $\delta > 0$. Then for each $\varepsilon > 0$ there exists $n = n(\varepsilon) \in \mathbb{N}$ so that for all $n \geq n(\varepsilon)$ there are constants $c', c > 0$ such that*

$$\mathbb{P}(\alpha(\mathcal{G}_{n,p}) < k_0) \leq \exp\left(-\frac{cn^2}{k_0(n)^4 p}\right) \leq \exp\left(-\frac{c'n^2 p^3}{\log(n)^4}\right).$$

This has been an improvement over the probability bound of the well-known result of Matula [Mat70], which is the foundation of Bollobás' paper.

The following Lemma 2.4 and Theorem 2.5 will be proved with arguments from [Bol88] and [FK16] combining Algorithm 2.1.1 and Theorem 2.3. The difference to the former two results is that they used Matula's result on the independence number of $\mathcal{G}_{n,p}$.

The following result constitutes a minor improvement of the result of Bollobás [Bol88] for the upper bound of the chromatic number of $\mathcal{G}_{n,p}$ for constant p .

Lemma 2.4 *Let $\delta > 0$ and $(1 - \delta) \geq p \gg n^{-\theta}$ for each constant $\theta \in (0, \frac{1}{3})$. Then the probability that there exists a vertex set $V_1 \subseteq V$ set of size $\nu(n) = \frac{n}{\log_b(np)^2}$ such that V_1 does not contain an independent set of size $k_0(\nu) = 2 \log_b(np) - 4 \log_b(\log_b(np))$ is super exponentially small, more exactly*

$$\mathbb{P}(\exists S \subseteq [n] : |S| = \nu \wedge \alpha(\mathcal{G}_{n,p}[S]) < k_0(\nu)) \leq \exp\left(-\frac{cn^2}{\log(n)^8}\right).$$

Proof. Note that with Equation 2.1 we have

$$k_0(\nu) \leq 2 \log_b(np) = 2 \frac{\log(np)}{\log(b)} \quad (2.2)$$

By Theorem 2.3, for sufficiently large $n \in \mathbb{N}$, there exists a constant $c > 0$ such that

$$\begin{aligned} & \mathbb{P}(\exists S \subseteq [n] : |S| = \nu \wedge \alpha(\mathcal{G}_{n,p}[S]) < k_0(\nu)) \\ & \leq \binom{n}{\nu} \cdot \exp\left(-\frac{c\nu^2}{k_0(\nu)^4 p}\right) \quad (\text{union bound and Theorem 2.3}) \\ & \leq n^\nu \cdot \exp\left(-\frac{c\nu^2 \log(b)^4}{\log(n)^4 p}\right) \\ & = \exp\left(-\frac{c_1 \nu^2}{\log(n)^4} + \log(n) \nu\right) \quad (\text{with some } c_1 > 0 \text{ using (2.2)}) \\ & = \exp\left(-\nu \left(\frac{c_1 \nu}{\log(n)^4} - \log(n)\right)\right) \\ & = \exp\left(-\frac{c_1 \nu^2}{\log(n)^4} (1 - o(1))\right) \quad \left(\text{since } \log(n) = o\left(\frac{c_1 \nu}{\log(n)^4}\right)\right) \\ & = \exp\left(-\frac{c_1 n^2}{\log(n)^4 \log_b(np)^4} (1 - o(1))\right) \quad \left(\text{since } \nu = \frac{n}{\log_b(np)}\right) \\ & \leq \exp\left(-\frac{c_1 n^2 \log(b)^4}{\log(n)^8} (1 - o(1))\right) \quad (\text{using (Remark 2.1)}) \\ & \leq \exp\left(-\frac{c_2 n^2}{\log(n)^8}\right). \quad (\text{with some constant } c_2 > 0) \end{aligned}$$

□

Bollobás [Bol88] used the following algorithm to calculate an upper bound for the chromatic number of $\mathcal{G}_{n,p}$, where $0 < p < 1$ is constant.

Algorithm 2.1.1: Coloring for constant $0 < p < 1$

Input: A random graph $\mathcal{G}_{n,p}$
Output: A coloring of the vertices of $\mathcal{G}_{n,p}$

- 1 **while** At least $\frac{n}{\log_b(np)^2}$ vertices remain uncolored and if there is an uncolored independent set I of size $2\log_b(np) - 4\log_b(\log_b(np))$ **do**
 - 2 | Color I with a new color;
 - 3 **end**
 - 4 Color all uncolored vertices with their own color;
-

With a small change to the arguments of Bollobás [Bol88] one can show the following theorem.

Theorem 2.5 *Let $\delta > 0$ and $(1 - \delta) \geq p \gg n^{-\theta}$ for each constant $\theta \in (0, \frac{1}{3})$. For each $\varepsilon > 0$ there exists $n = n(\varepsilon) \in \mathbb{N}$ so that for all $n > n(\varepsilon)$ and a suitable constant $c > 0$ such that the probability that Algorithm 2.1.1 fails to find a proper coloring of $\mathcal{G}_{n,p}$ with at most $(1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}$ colors is at most $\exp\left(-\frac{cn^2 p^7}{\log(n)^8}\right)$. Thus*

$$\mathbb{P}\left(\chi(\mathcal{G}_{n,p}) \geq (1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}\right) \leq \exp\left(-\frac{cn^2 p^7}{\log(n)^8}\right).$$

This improves on the result of Bollobás [Bol88], where it was only shown that for arbitrarily small $\varepsilon > 0$ and some constant $c > 0$

$$\mathbb{P}\left(\chi(\mathcal{G}_{n,p}) \geq (1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}\right) \leq \exp\left(-cn^{2-\varepsilon} p^8\right). \quad (2.3)$$

Proof of Theorem 2.5. We construct a coloring using Algorithm 2.1.1. This algorithm yields a proper coloring. By Lemma 2.4 we can assume that every vertex set $V_1 \subseteq V$ set of size $\nu(n) = \frac{n}{\log_b(np)^2}$ contains an independent subset of size $k_0(\nu) = 2\log_b(np) - 4\log_b(\log_b(np))$ with high probability.

Let $\varepsilon > 0$. Algorithm 2.1.1 colors each of the independent sets of size $k_0(\nu)$ with a new color, so $\left\lceil \frac{n-\nu}{k_0(\nu)} \right\rceil \leq (1 + \varepsilon/2) \frac{n \log(b)}{2 \log(np)}$ colors are required, where the inequality holds for all $n \geq n_1(\varepsilon)$ for some $n_1 \in \mathbb{N}$. Coloring each vertex in the remaining uncolored set with its own color requires at most $\nu = \frac{n}{\log_b(np)^2} \leq \varepsilon/2 \cdot \frac{n \log(b)}{2 \log(np)}$ colors, where the inequality is true for all $n \geq n_2(\varepsilon)$ for some $n_2(\varepsilon) \in \mathbb{N}$. So at most $\frac{n-\nu}{k_0(\nu)} + \nu \leq (1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}$ colors are used.

Since the Algorithm 2.1.1 yields a proper coloring of $\mathcal{G}_{n,p}$, $\chi(\mathcal{G}_{n,p}) > (1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}$ implies that Algorithm 2.1.1 fails to deliver a coloring with at most $(1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}$ colors.

By the argumentation above this is only the case if there exists $V_1 \subseteq V$ of size $\nu(n) = \frac{n}{\log_b(np)^2}$ that does not contain an independent subset of size $k_0(\nu) = 2\log_b(np) - 4\log_b(\log_b(np))$

Thus, by Lemma 2.4, for all $n \geq n_0 := \max\{n_1, n_2\}$

$$\mathbb{P}\left(\chi(\mathcal{G}_{n,p}) \geq (1 + \varepsilon) \frac{n \log(b)}{2 \log(np)}\right) \leq \mathbb{P}(\exists S \subseteq [n] : |S| = \nu \wedge \alpha(\mathcal{G}_{n,p}[S]) < k_0(\nu))$$

$$\leq \exp\left(-\frac{cn^2p^7}{\log(n)^8}\right).$$

□

We consider the following implication of Theorem 2.5.

Corollary 2.6 *Let $p \in (0, 1)$ be constant. For any $\varepsilon > 0$ and sufficiently large $n = n(\varepsilon) \in \mathbb{N}$ we have*

$$\mathbb{E}(\chi(\mathcal{G}_{n,p})) \leq (1 + \varepsilon) \frac{n \log(b)}{2 \log(n)}.$$

Proof of Corollary 2.6. Let $\varepsilon > 0$ and $k = (1 + \frac{\varepsilon}{2}) \frac{n \log(b)}{2 \log(n)}$.

$$\begin{aligned} \mathbb{E}(\chi(\mathcal{G}_{n,p})) &= \sum_{i=1}^n i \cdot \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i) \\ &\leq \sum_{i=1}^k i \cdot \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i) + (n - k) \mathbb{P}(\chi(\mathcal{G}_{n,p}) > k) \\ &\leq k \sum_{i=1}^k \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i) + n \exp\left(-\frac{cn^2p^7}{\log(n)^8}\right) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \frac{n \log(b)}{2 \log(np)} + n \exp\left(-\frac{cn^2p^7}{\log(n)^8}\right) \\ &\quad \text{(By Theorem 2.5 with } \varepsilon' = \varepsilon/2) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \frac{n \log(b)}{2 \log(np)} + \frac{\varepsilon \cdot n \log(b)}{2 \cdot 2 \log(n)} \\ &= (1 + \varepsilon) \frac{n \log(b)}{2 \log(n)}. \end{aligned} \tag{2.4}$$

The inequality in (2.4) holds, since $p(n) > n^{-\frac{1}{7}}$ and thus

$$n \exp\left(-\frac{cn^2p^7}{\log(n)^8}\right) \leq n \exp\left(-\frac{cn}{\log(n)^8}\right) \leq \frac{\varepsilon \cdot n \log(b)}{2 \cdot 2 \log(n)}$$

□

As seen in the proof of Theorem 2.5, if p is constant, in the final step of Algorithm 2.1.1 it is possible to color the remaining vertices with a new color for each such vertex. This strategy is not successful in calculating a tight bound in case of $p = p(n) \sim n^{-\theta}$. Here the number of remaining vertices is too large to be handled in such a way. To be able to analyze colorings of $\mathcal{G}_{n,p}$ for $p = p(n) \sim n^{-\theta}$ where $\theta \in (0, \frac{1}{3})$ we need a coloring algorithm for the small color classes. As Bollobás in [Bol88] we use the well-known greedy color algorithm.

Algorithm 2.1.2: Greedy Color

Input: A random graph $\mathcal{G}_{n,p}$

Output: A coloring of the vertices of $\mathcal{G}_{n,p}$

```

1 while There is a set of uncolored vertices do
2   | Choose a maximally independent set  $I$  of uncolored vertices;
3   | Color  $I$  with a new color;
4 end
    
```

We also give the following minor improvement of the result for the greedy color algorithm.

Theorem 2.7 *Let $n^{-\tau} \leq p \leq n^{-\theta}$ for some arbitrary, but fixed $\tau, \theta \in (0, \frac{1}{3})$. Further let $\chi_g(\mathcal{G}_{n,p})$ be the number of colors used by Greedy Color. Then*

$$\mathbb{P}\left(\chi_g(\mathcal{G}_{n,p}) \geq (1 + \varepsilon) \frac{n \log(b)}{\log(np)}\right) \leq \exp\left(-(1 + o(1))n^3\right).$$

Proof of Theorem 2.7. We will follow the pattern of the proof in [FK16, Theorem 7.9] given for constant p with slight modifications. Suppose that in some iteration of Algorithm 2.1.2 there are at least $n_0 = \frac{np}{(\log_b(n))^2}$ vertices uncolored. Let U be the set of uncolored vertices. Let $\kappa = \frac{4}{\theta}$ and $k_1 = \log_b(np) - \kappa \log_b(\log_b(n))$. With the trivial estimate $k_1 \leq \log_b(np) \leq \log_b(np)$ we get

$$\begin{aligned}
 & \log(k_1) + k_1 \log(n) + 2k_1 - \log_b(n)^{\kappa-2} \\
 & \leq \log(\log_b(np)) + \log_b(np) \log(n) + 2 \log_b(np) - \log_b(n)^{\kappa-2} \\
 & \leq \log_b(n)(\log(n) + 2 - o(1)) - \log_b(n)^{\kappa-2} \\
 & = -\log_b(n)(\log_b(n)^{\kappa-3} - \log(n) - 2 + o(1)) \\
 & = -\log_b(n)(\log(n)^{\kappa-3} \log(b)^{-\kappa+3} - \log(n) - 2 + o(1)) \\
 & = -\log_b(n)(\log(n)^{\kappa-3} (1 \pm o(1))^{-\kappa+3} p^{-\kappa+3} - \log(n) - 2 + o(1)) \\
 & \hspace{15em} (\text{using } \log(b) = (1 \pm o(1))p) \\
 & = -\log_b(n)(\log(n)^{\kappa-3} (1 \pm o(1)) p^{-\kappa+3} - \log(n) - 2 + o(1)) \\
 & \hspace{15em} (\text{using that } \kappa \text{ is constant and thus } (1 \pm o(1))^{-\kappa+3} = (1 \pm o(1))) \\
 & = -(1 \pm o(1)) \log_b(n) (\log(n)^{\kappa-3} p^{-\kappa+3} - \log(n) - 2 + o(1)) \\
 & = -(1 \pm o(1)) \log_b(n) p^{-\kappa+3} (\log(n)^{\kappa-3} - (\log(n) + 2 + o(1)) p^{\kappa-3}) \\
 & \leq -(1 \pm o(1)) \log_b(n) \log(b)^{-\kappa+3} \cdot \underbrace{(\log(n)^{\kappa-3} - (\log(n) + 2) n^{-\tau(\kappa-3)})}_{=o(1)} \\
 & \hspace{15em} (\text{again with } \log(b) = (1 \pm o(1))p \text{ and } p \geq n^{-\tau}) \\
 & = -(1 \pm o(1)) \log_b(n) \log(b)^{-\kappa+3} \cdot \log(n)^{\kappa-3} (1 - o(1)) \\
 & \leq -(1 \pm o(1)) \log_b(n)^{\kappa-2}.
 \end{aligned}$$

Now let U be a set of cardinality n_0 , then

$$\begin{aligned}
 & \mathbb{P}(\exists S : |S| \leq k_1, S \text{ is a maximally independent set in } U) \\
 & \leq \sum_{S \subseteq U; |S| \leq k_1} \mathbb{P}(S \text{ is a maximally independent set in } U)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{t=1}^{k_1} \sum_{S \subseteq U; |S|=t} \mathbb{P}(S \text{ is a maximally independent set in } U) \\
 &= \sum_{t=1}^{k_1} \sum_{S \subseteq U; |S|=t} \mathbb{P}(S \text{ is an independent set in } U) \\
 &\quad \cdot \mathbb{P}(\text{for each } u \in (U \setminus S) \text{ there exists } s \in S : \{u, s\} \in E(\mathcal{G}_{n_0, p})) \\
 &= \sum_{t=1}^{k_1} \sum_{S \subseteq U; |S|=t} \mathbb{P}(S \text{ is an independent set in } U) \\
 &\quad \cdot \prod_{u \in U \setminus S} \mathbb{P}(\text{there exists } s \in S : \{u, s\} \in E(\mathcal{G}_{n_0, p})) \\
 &= \sum_{t=1}^{k_1} \sum_{S \subseteq U; |S|=t} \mathbb{P}(S \text{ is an independent set in } U) \\
 &\quad \cdot \prod_{u \in U \setminus S} (1 - \mathbb{P}(\text{for all } s \in S : \{u, s\} \notin E(\mathcal{G}_{n_0, p}))) \\
 &= \sum_{t=1}^{k_1} \sum_{S \subseteq U; |S|=t} \mathbb{P}(S \text{ is an independent set in } U) \\
 &\quad \cdot \prod_{u \in U \setminus S} (1 - (1 - p)^t) \\
 &= \sum_{t=1}^{k_1} \sum_{S \subseteq U; |S|=t} (1 - p)^{\binom{t}{2}} (1 - (1 - p)^t)^{n_0 - t} \\
 &\hspace{25em} (\text{since } |U \setminus S| = n_0 - t) \\
 &= \sum_{t=1}^{k_1} \binom{n}{t} (1 - p)^{\binom{t}{2}} (1 - (1 - p)^t)^{n_0 - t} \\
 &\leq \sum_{t=1}^{k_1} \left(\frac{ne}{t} (1 - p)^{\frac{t-1}{2}} \right)^t \exp(-(n_0 - t)(1 - p)^t) \quad (\text{since } 1 - x \leq e^{-x} \text{ for all } x) \\
 &= \sum_{t=1}^{k_1} \underbrace{\frac{(1 - p)^{\binom{t}{2}}}{t^t}}_{\leq 1} n^t \exp(t) \exp(t(1 - p)^t) \exp(-n_0(1 - p)^t) \\
 &\leq \sum_{t=1}^{k_1} n^t \exp(t + t(1 - p)^t) \exp(-n_0(1 - p)^t) \\
 &= \sum_{t=1}^{k_1} \left(n \exp(1 + (1 - p)^t) \right)^t \exp(-n_0 \underbrace{(1 - p)^t}_{=b^{-1}}) \\
 &\leq k_1 \left(n \exp(\underbrace{1 + (1 - p)^{k_1}}_{\leq 2}) \right)^{k_1} \exp(-n_0 b^{-k_1}) \quad (\text{since } b > 1) \\
 &= k_1 (ne^2)^{k_1} \exp\left(-\frac{np}{\log_b(n)^2} \exp(-\log(b)(\log_b(np) - \kappa \log_b(\log_b(n))))\right)
 \end{aligned}$$

$$\begin{aligned}
 &= k_1 (ne^2)^{k_1} \exp \left(-\frac{np}{\log_b(n)^2} \exp \left(-\log(b) \frac{(\log(np) - \kappa \log(\log_b(n)))}{\log(b)} \right) \right) \\
 &= k_1 (ne^2)^{k_1} \exp \left(-\frac{np}{\log_b(n)^2} \exp \left(-(\log(np) + \kappa \log(\log_b(n))) \right) \right) \\
 &= k_1 (ne^2)^{k_1} \exp \left(-\frac{np}{\log_b(n)^2} \frac{\log_b(n)^\kappa}{np} \right) \\
 &= k_1 (ne^2)^{k_1} \exp \left(-\log_b(n)^{\kappa-2} \right) \\
 &= \exp \left(\underbrace{\log(k_1) + k_1 \log(n) + 2k_1}_{o(\log_b(n)^{\kappa-2})} - \log_b(n)^{\kappa-2} \right) \\
 &= \exp \left(-(1 - o(1)) \log_b(n)^{\kappa-2} \right) \quad (\text{as shown above}) \\
 &\leq \exp \left(-(1 \pm o(1)) \log(n)^{\kappa-2} p^{-(\kappa-2)} \right) \quad (\text{with } \log(b) = (1 \pm o(1))p) \\
 &\leq \exp \left(-(1 \pm o(1)) n^{\theta(\kappa-2)} \right) \quad (\text{since } p \leq n^{-\theta} \text{ and } \log(n)^{\kappa-2} \geq 1) \\
 &= \exp \left(-(1 \pm o(1)) n^{\theta(\frac{4}{\theta}-2)} \right) \quad (\text{as } \kappa = \frac{4}{\theta}) \\
 &= \exp \left(-(1 \pm o(1)) n^{4-2\theta} \right) \\
 &\leq \exp \left(-(1 \pm o(1)) n^3 \right). \quad (\text{since } \theta \leq \frac{1}{3})
 \end{aligned}$$

Thus the probability that in every set of at least n_0 vertices every maximally independent set is of size at least k_1 is at least $1 - \exp(-cn^3)$. So in each step before the number of uncolored vertices drops below n_0 , at least k_1 vertices are colored. Therefore, the probability that more than $(1 + \varepsilon) \frac{n}{\log_b(n)} \geq \frac{n}{k_1} + n_0$ colors are used is at most $\exp(-(1 + o(1))n^3)$. \square

We now consider colorings of $\mathcal{G}_{n,p}$ using an asymptotically optimal amount of colors for the case $p(n) \leq n^{-\theta}$. The following algorithm can be used to obtain a coloring which is an $1 + \varepsilon$ approximation of an optimal coloring with a slightly better probability bound for $p(n) \sim n^{-\theta}$ as in [Bol88]. This will be used later.

Furthermore we use the following algorithm of Bollobás [Bol88].

Algorithm 2.1.3:

Input: A random graph $\mathcal{G}_{n,p}$

Output: A coloring of the vertices of $\mathcal{G}_{n,p}$

- 1 **while** At least $\frac{n}{\log(np)}$ vertices remain uncolored and if there is an uncolored independent set I of size $2 \log_b(np) - 4 \log_b(\log_b(np))$ **do**
 - 2 | Color I with a new color;
 - 3 **end**
 - 4 color the remaining graph using Algorithm 2.1.2;
-

Lemma 2.8 *Let $n^{-\tau} \leq p \leq n^{-\theta}$ for some arbitrary, but fixed $\tau, \theta \in (0, \frac{1}{3})$. Then the probability that there exists a vertex set $V_1 \subseteq V$ of size $\nu(n) = \frac{n}{\log(np)}$ such that V_1 does not contain an independent set of size $k_0(\nu) \sim 2 \log_b(np) \sim 2 \log_b(np) -$*

$4 \log_b(\log(np))$ is bounded from above by

$$\mathbb{P}(\exists S \subseteq [n] : |S| = \nu \wedge \alpha(\mathcal{G}_{n,p}[S]) < k_0(\nu)) \leq \exp\left(-\left(\frac{c_2 n^2 p^3}{\log(n)^6}\right)\right).$$

Proof. Let $\nu = \frac{n}{\log(np)}$. We have $k_0(\nu) \sim 2 \log_b(np) \sim 2 \log_b(np) - 4 \log_b(\log(np))$. Now,

$$\begin{aligned} & \mathbb{P}(\exists S \subseteq [n] : |S| = \nu \wedge \alpha(\mathcal{G}_{n,p}[S]) < k_0(\nu)) \\ & \leq \binom{n}{\nu} \exp\left(-\frac{c_1 \nu^2 p^3}{\log(\nu)^4}\right) \quad (\text{by Theorem 2.3 for some constant } c_1 > 0) \\ & \leq \exp\left(-\nu \frac{c_1 \nu p^3}{\log(\nu)^4} + \nu \log(n)\right) \\ & = \exp\left(-\frac{n}{\log(np)} \frac{c_1 n p^3}{(\log(n) - \log(\log(np)))^4} + \frac{n}{\log(np)} \log(n)\right) \\ & \leq \exp\left(-\left(\frac{c_2 n^2 p^3}{\log(n)^6}\right)\right), \end{aligned} \tag{a.3}$$

and the last inequality holds for a constant $c_2 > 0$ and sufficiently large n , because $\log(np) \leq \log(nn^{-\theta}) \leq (1 - \theta) \log(n)$. \square

We also give the following minor improvement of a result of Bollobás using Algorithm 2.1.3 and Lemma 2.8.

Theorem 2.9 *Let $\theta \in (0, \frac{1}{3})$, $\delta > 0$ and $n^{-\frac{1}{3}+\delta} \leq p(n) \leq n^{-\theta}$. For sufficiently large $n \in \mathbb{N}$ there is a constant $c > 0$ such that*

$$\mathbb{P}\left(\chi(\mathcal{G}_{n,p}) \geq (1 + \varepsilon) \frac{np}{2 \log(np)}\right) \leq \exp\left(-\left(\frac{cn^2 p^3}{\log(n)^6}\right)\right).$$

Furthermore the probability that Algorithm 2.1.3 fails to construct a coloring with at most $(1 + \varepsilon) \frac{np}{2 \log(np)}$ colors is at most $\exp\left(-\left(\frac{cn^2 p^3}{\log(n)^6}\right)\right)$.

Proof of Theorem 2.9. By Lemma 2.8 the probability that Algorithm 2.1.3 fails to find an independent set of size $k_1 := 2 \log_b(np) - \log_b(\log_b(np))$ as long as at least $\nu = \frac{n}{\log(n)}$ vertices remain is at most $\exp\left(-\left(\frac{c_2 n^2 p^3}{\log(n)^6}\right)\right)$. Hence the number of colors used in the while-loop of Algorithm 2.1.3 is at most $\frac{n}{k_1} \leq (1 + \frac{\varepsilon}{2}) \frac{n \log(b)}{\log(np)}$. Furthermore, since the remaining ν vertices induce a $\mathcal{G}_{\nu,p}$, the probability that Algorithm 2.1.2 uses more than

$$\begin{aligned} & (1 + \varepsilon) \frac{\nu \log(b)}{\log(\nu)} \\ & = (1 + \varepsilon) \frac{\nu \log(b)}{\log(n) - \log \log(np)} \\ & = (1 + \varepsilon) \frac{\nu \log(b)}{(1 - o(1)) \log(n)} \\ & = \frac{1 + \varepsilon}{1 - o(1)} \cdot \frac{n \log(b)}{\log(np) \log(n)} \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1 + \varepsilon}{1 - o(1)} \cdot \frac{n \log(b)}{\log(np)^2} \\
 &= \frac{\varepsilon n \log(b)}{2 \log(np)} \cdot \underbrace{\frac{2}{\varepsilon (1 - o(1)) \log(np)}}_{< 1 \text{ for sufficiently large } n} \\
 &\leq \frac{\varepsilon n \log(b)}{2 \log(np)}
 \end{aligned}$$

colors to color the remaining ν vertices can be estimated by Theorem 2.7 to be at most

$$\exp\left(-(1 + o(1))\nu^3\right) = \exp\left(-\frac{(1 + o(1))n^3}{\log(np)^3}\right) \leq \exp\left(-\left(\frac{(1 + o(1))n^3}{\log(n)^3}\right)\right). \quad (\text{a.4})$$

Thus the probability that Algorithm 2.1.3 uses more than $(1 + \varepsilon)\frac{n \log(b)}{2 \log(np)}$ colors is at most

$$\begin{aligned}
 &\mathbb{P}\left(\text{Algorithm 2.1.3 uses more than } (1 + \varepsilon)\frac{n \log(b)}{2 \log(np)} \text{ colors}\right) \\
 &\leq \mathbb{P}(\text{The while-loop of Algorithm 2.1.3 colors less than } \nu \text{ vertices}) \\
 &\quad + \mathbb{P}\left(\text{Algorithm 2.1.2 uses more than } \frac{\varepsilon n \log(b)}{2 \log(np)} \text{ colors}\right) \\
 &\leq \exp\left(-\left(\frac{c_2 n^2 p^3}{\log(n)^6}\right)\right) + \exp\left(-\left(\frac{(1 + o(1))n^3}{\log(n)^3}\right)\right) \\
 &\hspace{15em} (\text{using Lemma 2.8 and a.4}) \\
 &\leq \exp\left(-\left(\frac{c_3 n^2 p^3}{\log(n)^6}\right)\right),
 \end{aligned}$$

for some constant $c_3 > 0$. The last inequality holds due to the fact that $\exp\left(-\left(\frac{(1 + o(1))n^3}{\log(n)^3}\right)\right) \in o\left(\exp\left(-\left(\frac{c_2 n^2 p^3}{\log(n)^6}\right)\right)\right)$. □

Now we are able to give an upper bound for the expected value of $\chi(\mathcal{G}_{n,p})$.

Corollary 2.10 *Let $\theta \in (0, \frac{1}{3})$, $\delta > 0$ and $n^{-\frac{1}{3} + \delta} \leq p(n) \leq n^{-\theta}$. Let $\varepsilon > 0$. For sufficiently large $n \in \mathbb{N}$ we have*

$$\mathbb{E}(\chi(\mathcal{G}_{n,p})) \leq (1 + \varepsilon) \frac{np}{2 \log(np)}.$$

Proof. Let $\varepsilon > 0$ and $k = (1 + \frac{\varepsilon}{2}) \frac{np}{2 \log(np)}$. Let further n be sufficiently large. Then,

$$\begin{aligned}
 \mathbb{E}(\chi(\mathcal{G}_{n,p})) &= \sum_{i=1}^n i \cdot \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i) \\
 &= \sum_{i=1}^k i \cdot \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i) + \sum_{i=k+1}^n i \cdot \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{i=1}^k i \cdot \mathbb{P}(\chi(\mathcal{G}_{n,p}) = i) + n\mathbb{P}(\chi(\mathcal{G}_{n,p}) > k) \\
 &\leq (1 + \frac{\varepsilon}{2}) \frac{np}{2\log(np)} + n \exp\left(-\left(\frac{cn^2p^3}{\log(n)^5}\right)\right) \\
 &\hspace{15em} \text{(by Theorem 2.9 with } \varepsilon' := \varepsilon/2) \\
 &= (1 + \varepsilon) \frac{np}{2\log(np)}.
 \end{aligned}$$

For the last inequality we use

$$n \exp\left(-\left(\frac{cn^2p^3}{\log(n)^5}\right)\right) \leq \frac{np}{2\log(np)}.$$

This holds for sufficiently large n , because

$$\lim_{n \rightarrow \infty} n \exp\left(-\left(\frac{cn^2p^3}{\log(n)^5}\right)\right) = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{np}{2\log(np)} = \infty.$$

□

In addition to these results that establish bounds on probabilities, we require the following basic result for the chromatic number of the union of any two graphs to analyze the chromatic number of $\text{pert}_{H,p}$ graphs.

Lemma 2.11 (Divide and Color) *Let $G := (V, E_G)$ and $H := (V, E_H)$ be graphs. Further let $\mathcal{C} \subseteq \mathcal{P}(V)$ be a partition of V such that every $S \in \mathcal{C}$ is an independent set in G . Then*

$$\chi(G \cup H) \leq \sum_{S \in \mathcal{C}} \chi(H[S]).$$

If G is a complete k -partite graph and \mathcal{C} is a k -partition of the vertices of G as above, then

$$\chi(G \cup H) = \sum_{S \in \mathcal{C}} \chi(H[S]).$$

The proof is kind of folklore. For readers' convenience we state it here briefly.

Proof. For each $S \in \mathcal{C}$ color the graph with $\chi(H[S])$ colors that are not yet used. This is a proper coloring of $G \cup H$ since no edges in G exist between vertices of the same color, as \mathcal{C} is a set of independent sets. Furthermore, by construction there are no edges in H between vertices of the same color, since no color is used for vertices contained in different members of \mathcal{C} . This proves the inequality. Let G be a complete k -partite graph and \mathcal{C} is a k -partition of the vertices of V as above, the coloring is even optimal, since the $H[S]$ are already optimally colored and for each pair of members of \mathcal{C} , every edge between them already exists. Thus, every other proper coloring of the graph uses at least as many colors as the coloring constructed here. □

Furthermore, we require the following variation of the union bound which is a basic result of stochastic.

Lemma 2.12 *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $k \in \mathbb{N}$. For each $i \in [k]$ let X_i be a random variable and A_i in \mathbb{R} . Then*

$$\mathbb{P} \left(\sum_{i=1}^k X_i \leq \sum_{i=1}^k A_i \right) \leq \sum_{i=1}^k \mathbb{P}(X_i \leq A_i).$$

Proof. Let $\tilde{\Omega} := \{\omega \in \Omega : \sum_{i=1}^k X_i(\omega) > \sum_{i=1}^k A_i\}$ and for each $i \in [k]$ let $\Omega_i := \{\omega \in \Omega : X_i(\omega) > A_i\}$. If for some $\omega \in \Omega$ there is $X_i(\omega) > A_i$ for all $i \in [k]$, then $\sum_{i=1}^k X_i > \sum_{i=1}^k A_i$. So, $\tilde{\Omega} \supseteq \bigcap_{i \in [k]} \Omega_i$ and $\tilde{\Omega}^c \subseteq (\bigcap_{i \in [k]} \Omega_i)^c = \bigcup_{i \in [k]} \Omega_i^c$. This implies

$$\mathbb{P} \left(\sum_{i=1}^k X_i \leq \sum_{i=1}^k A_i \right) = \mathbb{P}(\tilde{\Omega}^c) \leq \mathbb{P} \left(\bigcup_{i \in [k]} \Omega_i^c \right) \leq \sum_{i=1}^k \mathbb{P}(\Omega_i^c) = \sum_{i=1}^k \mathbb{P}(X_i \leq A_i).$$

□

2.2 On the Choice Number of Random Graphs

The proof that the following result holds was first published by Alon [Alo93], who attributes it to private communication from Kahn.

Lemma 2.13 (Kahn, Alon) *Let $G = (V, E)$ be a graph and $b > 1$ be a constant, such that every set of at least $\frac{n}{\log(n)^2}$ vertices contains an independent set of size at least $(1 - \varepsilon)2 \log_b n$. Then $\chi_l(G) \leq (1 + \varepsilon) \frac{n}{2 \log_b(n)}$.*

They then combined this result with the probability result of Bollobás. We will give the slightly stronger result relying on Lemma 2.4. This implies

Theorem 2.14 (Variation of Kahn and Alon) *Let $0 < p < 1$ be constant and $\varepsilon > 0$, then there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$*

$$\mathbb{P} \left(\chi_l(\mathcal{G}_{n,p}) > (1 + \varepsilon) \frac{n}{2 \log_b(n)} \right) \leq \exp \left(\frac{cn^2}{\log(n)^8} \right).$$

Proof. By Lemma 2.4 every set of at least $n/\log(n)^2$ vertices contains an independent set of size at least $(1 - \varepsilon)2 \log_b(n)$. With probability at least $1 - \exp(-cn^2/\log(n)^8)$. Thus by Lemma 2.13 the probability that the choice number of $\mathcal{G}_{H,p}$ exceeds $(1 + \varepsilon) \frac{n}{2 \log_b(n)}$ is at most

$$\mathbb{P} \left(\chi_l(\mathcal{G}_{n,p}) > (1 + \varepsilon) \frac{n}{2 \log_b(n)} \right) \leq \exp \left(\frac{cn^2}{\log(n)^8} \right).$$

□

2.3 Winning the k -edge-Connectivity Game using Krivelevich's techniques

In this section we will prove the following result that is mentioned but not proven by Krivelevich [Kri10]. It is left as an exercise in [HKSS14, Exercise 6.7.2]. It is the currently best-known estimate for the number of rounds Maker needs to win the $(1 : b)$ Maker-Breaker game for k edge-disjoint Hamilton cycles if $b \leq (1 - \varepsilon)n$. Since I could not locate a proof, I will give one here.

Theorem 2.15 (Remark of Krivelevich) *Let $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that for every $n \in N_{\geq n_0}$ Maker has a strategy to construct k edge disjoint Hamilton cycles in the $(1 : b)$ game played on the complete graph on n vertices in at most $\min(13n, (9k + 3)n) + o(n)$ rounds for every $b \leq (1 - \varepsilon)n$.*

In each step Maker will use the following algorithm to construct a graph of minimum degree $\min(8k + 3, 12) =: \kappa$ in κn turns with positive probability. Note that the minimum can be avoided if $k \geq 2$ which is the case we are most interested in. This algorithm is due to Gebauer and Szabo [GS09].

Algorithm 2.3.1: Algorithms for minimum degree game

Input: A set $B \subseteq E$ claimed by Breaker and a set $M \subseteq E$ claimed by Maker

Output: An edge $e \in E \setminus (B \cup M)$

- 1 Choose a vertex $v \in V$ with $\deg_M(v) < \kappa$ that maximizes $\text{dang}v := \deg_B(v) - 2b \deg_M(v)$;
 - 2 Choose a vertex $w \in V \setminus \{v\}$ such that $\{v, w\} \notin (B \cup M)$ is unclaimed uniformly at random.
-

To prove Theorem 2.15 the following Lemma will be used. It is first stated by Krivelevich [Kri10] but was proved with techniques of Gebauer and Szabo [GS09]. It analyzes Algorithm 2.3.1.

Lemma 2.16 (Krivelevich) *For every $\varepsilon > 0$ there exists a fixed $\delta_\varepsilon > 0$ such that by playing according to Algorithm 2.3.1, for all $v \in V$ Maker chooses the κ -th edge incident in v at a time when $d_B(v) \leq (1 - \delta_\varepsilon)n$ in the $(1 : (1 - \varepsilon)\frac{n}{\log(n)})$ game.*

Further it can be shown that Maker constructed a $(\delta_\varepsilon n, 2k)$ -expander with positive probability after κ rounds by following Algorithm 2.3.1, where $\delta_\varepsilon \in (0, 1)$ is the same as in Lemma 2.16.

Theorem 2.17 *For every $\varepsilon > 0$ Maker constructs a $(\delta_\varepsilon n, 2k)$ -expander in at most κn rounds by following Algorithm 2.3.1 in each step in the $(1 : (1 - \varepsilon)\frac{n}{\log(n)})$ game with positive probability.*

Proof. Suppose that Makers graph is not a $(\delta_\varepsilon n, 2k)$ expander after Maker played each round by Algorithm 2.3.1 until each vertex had Maker degree κ . Then there exists a subset $K \subseteq VS$ of size $|A| = i \leq \delta_\varepsilon n$ such that $N_M(K) \subseteq L$ for some $L \subseteq V \setminus K$ with $|L| = 2si - 1$. Since the minimum degree in is Maker's graph is κ , we can assume that $i \geq 5$.

Excursion on $i \geq 5$:

Assume that $|K| \leq 4$. Since every vertex in K has a minimum degree of κ and at most 3 of its neighbours are in K , we can assume that each element of K has at least $\kappa - 3$ neighbours in $V \setminus K$. In the worst case all vertices in K have all neighbours outside of A in common. We see that $|N_M(K)| \geq \kappa - 3$. Since M is supposed to be a $(\delta_\varepsilon n, 2k)$ -expander, we need to ensure that $|N_M(K)| \geq 2k|K|$. This holds because

$$|N_M(K)| \geq \kappa - 3 = (8k + 3) - 3 \geq 8k = 2k|K|.$$

Furthermore there are at least $\frac{\kappa i}{2}$ edges of Maker incident in K . Now one of two cases holds.

Case 1: At least $\frac{\kappa i}{4}$ edges were chosen from K and went into $K \cup L$.

Case 2: At least $\frac{\kappa i}{4}$ edges were chosen from L and went into K .

If at some point during the game a choice is made from a vertex $v \in K$, the Breaker degree in v is at most $(1 - \delta_\varepsilon)n$ and the Maker degree is at most $\kappa - 1$. Therefore there were at least $\delta_\varepsilon n - \kappa + 1$ unclaimed edges incident to v . Thus the probability that Maker chose an edge such that the second endpoint belongs to $K \cup L$ is at most

$$\frac{|K \cup L| - 1}{\delta_\varepsilon n - \kappa + 1} = \frac{2ki - 1 + i - 1}{\delta_\varepsilon n - \kappa + 1} = \frac{(2k + 1)i - 2}{\delta_\varepsilon n - \kappa + 1}$$

regardless of the history of the game.

Thus the probability that **Case 1** holds, which is the case if such a choice has been made at least $\frac{\kappa i}{4}$ times. This is at most

$$\left(\frac{(2k + 1)i - 2}{\delta_\varepsilon n - \kappa + 1} \right)^{\frac{\kappa i}{4}}.$$

For a single choice made from $v \in L$ the probability that the other endpoint is in A is at most $\frac{|K|}{\delta_\varepsilon n - \kappa + 1}$ since as shown above at least $\delta_\varepsilon n - \kappa + 1$ unclaimed edges were incident in v when the edge was chosen. Since at most $\kappa|L|$ choices are made from L , the probability that at least $\frac{\kappa i}{4}$ of them end up in K and thus **Case 2** holds is at most

$$\left(\kappa|L| \right) \left(\frac{i}{\delta_\varepsilon n - \kappa + 1} \right)^{\frac{\kappa i}{4}}.$$

Thus the probability that at least one of the **Cases** holds for a specific $K \subseteq V$ with $|K| = i$ is at most

$$\begin{aligned} & \left(\frac{(2k + 1)i - 2}{\delta_\varepsilon n - \kappa + 1} \right)^{\frac{\kappa i}{4}} + \left(\kappa|L| \right) \left(\frac{i}{\delta_\varepsilon n - \kappa + 1} \right)^{\frac{\kappa i}{4}} \\ & \leq \left(\frac{(2k + 1)i2}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} + \left(\frac{\kappa 2si}{4} \right) \left(\frac{i}{\delta_\varepsilon n - \kappa + 1} \right)^{\frac{\kappa i}{4}} \\ & \leq \left(\frac{(2k + 1)i2}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} + \left(\frac{\kappa 2si4e}{\kappa i} \right)^{\frac{\kappa i}{4}} \left(\frac{2i}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{2(k+1)i2}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} + (8ek)^{\frac{\kappa i}{4}} \left(\frac{2i}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} \\
&\leq \left(\frac{4(k+1)i}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} + \left(\frac{8ek2i}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} \\
&\leq \left(\frac{4(k+1)i}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} + \left(\frac{16eki}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} \\
&\leq 2 \left(\frac{16eki}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} = \left(\frac{32eki}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}}.
\end{aligned}$$

Thus the probability that M is not a $(\delta_\varepsilon n, 2k)$ -expander is at most

$$\begin{aligned}
&\sum_{i=5}^{\delta_\varepsilon n} \binom{n}{i} \binom{n-i}{2i-1} \left(\frac{32eki}{\delta_\varepsilon n} \right)^{\frac{\kappa i}{4}} \\
&\leq \sum_{i=5}^{\delta_\varepsilon n} \left(\frac{en}{i} \left(\frac{n}{2i} \right)^2 \left(\frac{32eki}{\delta_\varepsilon n} \right)^{\frac{\kappa}{4}} \right)^i \\
&= \sum_{i=5}^{\delta_\varepsilon n} \left(\underbrace{\frac{(32ek)^{\frac{\kappa}{4}}}{4e^3}}_{\text{constant}} \frac{1}{\delta_\varepsilon^{\frac{\kappa}{4}}} \left(\frac{i}{n} \right)^{\frac{\kappa}{4}-3} \right)^i.
\end{aligned}$$

Since the δ_ε can be chosen arbitrarily small, we choose it in a way such that $\frac{(32ek)^{\frac{\kappa}{4}}}{4e^3} \delta_\varepsilon^{-\kappa/4} =: c < 1$. Denote the i -th term of the above sum by $g(i)$.

Then for $5 \leq i \leq \sqrt{n}$

$$g(i) \leq \left(c \left(\frac{\sqrt{n}}{n} \right)^{\frac{\kappa}{4}-3} \right)^5 \leq \left(\frac{1}{\sqrt{n}} \right)^5 = o\left(\frac{1}{n} \right)$$

since $\frac{\kappa}{4} - 3 \geq 1$ and for $\sqrt{n} \leq i \leq \delta_\varepsilon n$ there is

$$g(i) \leq c^{\sqrt{n}} = o\left(\frac{1}{n} \right).$$

Thus, the probability that Maker fails to construct a $(\delta_\varepsilon n, 2k)$ -expander is approaching 0 and therefore the probability that Maker constructs a $(\delta_\varepsilon n, 2k)$ -expander is positive. \square

We now give a variation of the algorithm that Krivelevich used to prove that Maker can win the $\left(1 : (1-\varepsilon)\frac{n}{\log(n)}\right)$ Maker-Breaker Hamiltonian Cycle game to prove Theorem 2.15.

Algorithm 2.3.2: Krivelevich's Maker-algorithm for k edge-disjoint Hamilton Cycles

- 1 Build a $(\delta_\epsilon, 2k)$ -expander using Algorithm 2.3.1;
 - 2 If M is disconnected (without the protected edges) connect the components of M ;
 - 3 Add boosters until M is Hamiltonian ;
 - 4 If M contains a Hamiltonian cycle H , protect the edges of H ;
 - 5 If k Hamiltonian Cycles have been protected, end the game;
 - 6 Else go to step 3.
-

Outline of the proof of Theorem 2.15. We use a similar approach as Krivelevich in [Kri10] and thus will use some of his lemmata. First we calculate an upper bound for the number of rounds Maker needs to finish Algorithm 2.3.2. We proceed in three steps.

Step 1 : Construct a $(\delta_\epsilon n, 2k)$ -expander.

By Lemma 2.16 Maker succeeds in this task using at most κn turns.

Step 2 : Connect the components of Maker's graph.

Since this step is carried out, at most $k - 1$ Hamiltonian cycles have been protected in M . Since they can no longer be used to construct Hamiltonian cycles, we can assume that they have been removed from M . Since this assumption removes at most $2(k - 1)$ edges in each vertex, M is still a $(\delta_\epsilon n, 2)$ expander graph. By a lemma of Krivelevich [HKSS14, Lemma 6.3.2] each component of a $(\delta_\epsilon n, 2)$ expander graph is of size at least $3\delta_\epsilon n$. So at most $3\delta_\epsilon n$ turns are needed to connect an unconnected Maker graph. Since this has to be repeated each time a Hamiltonian Cycle is protected and at most k Hamilton cycles are being protected over the course of the game, at most $3k\delta_\epsilon^{-1}$ turns are carried out in this step.

Step 3 : We add boosters until a Hamiltonian cycle is constructed. Since each booster either increases the length of a longest path in a connected non-Hamiltonian path, or makes it Hamiltonian, at most n additional moves are carried out in this step for each Hamiltonian Cycle. Thus at most kn turns are carried out in this step.

In conclusion, assuming $k \geq 2$ Maker makes at most

$$\kappa n + 3k\delta_\epsilon^{-1} + kn = (9k + 3)n + o(n)$$

turns to finish Algorithm 2.3.2.

As a second step we prove that Maker can always make the move that Algorithm 2.3.2 prescribes. We note that since Maker made at most a linear number of moves and Breaker takes b edges every turn, Breaker claimed at most $|B| \leq c \frac{n^2}{\log(n)}$ for some $c > 0$.

As stated above, Maker succeeds in Step 1 with positive probability.

In Step 2 Maker wants to connect two components C and D of a $(\delta_\epsilon n, 2)$ -expander. As stated above each of these components contains at least $3\delta n$ vertices. Thus, at least $9\delta_\epsilon^2 n^2$ edges exist between each pair of components. Since Breaker claimed at most $c \frac{n^2}{\log(n)}$ edges and Maker protected at most $(k - 1)n$ edges, Maker will always find an unclaimed edge that connects C and D .

In Step 3 Maker wants to add a booster to a connected, non-Hamiltonian $(\delta_\epsilon n, 2)$. By [HKSS14, Corollary 6.3.6] there exist at least $\frac{\delta_\epsilon^2 n^2}{2}$ boosters. Since Breaker

claimed at most $c \frac{n^2}{\log(n)}$ edges and Maker protected at most $(k-1)n$ edges, Maker will always find an unclaimed booster.

This shows that Maker can build k edge-disjoint Hamiltonian Cycles in at most $(9k+3)n + o(n)$ turns. \square

Chapter 3

On the Chromatic Number of Randomly Augmented Graphs

3.1 Introduction

In this chapter we give an upper bound for the chromatic number of randomly augmented graphs where we restrict ourselves to host-graphs of bounded chromatic number.

We remind the reader of the following notation. For any function $p : \mathbb{N} \rightarrow [0, 1]$ and any $n \in \mathbb{N}$ let $\mathcal{G}_{n,p}$ denote the random graph model with vertex set $[n] := \mathbb{N}_{\leq n}$, in which there is $\mathbb{P}(\{v, w\} \in E(\mathcal{G}_{n,p})) = p$ independently for each $\{v, w\} \in \binom{[n]}{2}$. We further define $b := b(n) = \frac{1}{1-p(n)}$.

3.1.1 Previous work

For the theory of the chromatic number in $\mathcal{G}_{n,p}$ we refer to the pioneering papers of Erdős and Rényi [ER59], Bollobás [Bol88], Shamir and Spencer [SS87], Łuczak [Łu91] and more recently the non-concentration result of Heckel and Riordan [HR21], as well as the books of Bollobás [Bol85] and Frieze and Karoński [FK16].

The fact that the chromatic number of $\mathcal{G}_{n,p}$ for large p , i.e. $c \geq p = p(n) \geq n^{-\theta}$ for some $\theta \in (0, 1/3)$ and some $c \in (0, 1)$, is equal to $(1 + o(1))^{\frac{\log(b)n}{2\log(n)}}$ a.a.s. was first proved by Bollobás in [Bol88]. Łuczak [Łu91] proved that the same asymptotic bounds hold for $\chi(\mathcal{G}_{n,p})$, for all $p = p(n) > d/n$ with some $d \in \mathbb{R}$. In [SS87] Shamir and Spencer showed that the chromatic number of $\mathcal{G}_{n,p}$ for each p is concentrated around its expected value within a large deviation of essentially \sqrt{n} . Recently, even a long open question about the non-concentration of the chromatic number of $\mathcal{G}_{n,1/2}$ has been answered by Heckel and Riordan [HR21].

In extension of the random graph model $\mathcal{G}_{n,p}$ the interaction between random graphs and deterministic graphs has been studied. One way to analyze the interactions between a random graph and a deterministic graph is to analyze the resilience of the random graph with respect to certain invariants, asking what kind of graph can be added to the $\mathcal{G}_{n,p}$ without changing the asymptotics of the invariant under consideration. Alon and Sudakov [AS10] studied this question for the chromatic number of deterministic graphs of bounded maximum degree and edge number. The

concept of the resilience of a graph invariant is not restricted to random graphs, but can be applied to deterministic graphs as well.

Another way of analyzing the interaction between a deterministic graph and a random graph is the model of perturbed or augmented graphs. In this model a deterministic host graph H is augmented (or perturbed) by a $\mathcal{G}_{n,p}$, so the graph $H \cup \mathcal{G}_{n,p}$ is studied. Generally, whether we speak of a "perturbed" or "augmented" graph depends on the probability p used for $\mathcal{G}_{n,p}$, and we may speak about perturbed graphs for smaller p and of augmented graphs for larger p . This model was first introduced by Bohman et al. in [BFM03] considering the question of Hamiltonicity, and was further studied for example by Dudek et al. in [DRRS20] and Das et al. in [DMT19].

3.1.2 Our results

In this paper we study the chromatic number of a randomly augmented graph $\text{pert}_{H,p}$, where H is a deterministic host graph with chromatic number $o(n)$, augmented by a $\mathcal{G}_{n,p}$ for $n^{-\frac{1}{3}+\delta} \leq p \leq (1-\delta)$ for any fixed $\delta > 0$.

In section 3.2, we first will show that for each $\varepsilon > 0$

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad a.a.s. \quad (\text{N1})$$

for a constant p and host graphs of chromatic number $o(n)$.

This is done by estimating the expected value of the chromatic number of the random graphs induced by the large color classes of an optimal coloring of H and adding them up. Additionally, we estimate the number of vertices in small color classes to be negligible. Using this upper bound on the expected value of the chromatic number and combining it with McDiarmid's bounded differences inequality we get our main result.

We further prove (N1) for augmented graphs for which $p = p(n) = n^{-\theta}$ with $\theta \in (0, \frac{1}{3})$ for some $\delta > 0$. Here we use a similar technique as for constant p to color the random graphs induced by the large color classes of an optimal coloring of H . However, the expected chromatic number of $\mathcal{G}_{n,p}$ is only known to be smaller than $\frac{n \log(b)}{2 \log(n)}$ for $p(n) \geq n^{-\theta}$ with $\theta \in (0, \frac{1}{3})$ for some $\delta > 0$. Since $p(n) = n^{-\theta}$ is monotonously decreasing, for induced subgraphs of size k the probability $p(n)$ does not necessarily satisfy the condition $p(n) \geq k^{-\frac{1}{3}}$ which would be needed to achieve the upper bound or the expected chromatic number. Thus the upper bound is only proved for host graphs with $\chi(H) \leq n^{1-3\theta}$. For the subgraphs induced by small color classes of an optimal coloring of H we use a different approach since the number of vertices they contain is not negligible compared to the desired upper bound. We will color each of these small random graphs with the greedy algorithm. Again the proof will be finalized by C. McDiarmid's bounded differences inequality.

Since in both cases the bounded differences inequality was used the proofs are non-constructive. To give the reader an impression of which coloring strategies can be used to construct a proper coloring of the same asymptotics, we give constructive proofs of (N1) with some additional restrictions on the chromatic number of the host graph H in section 3.4. The construction relies on a slightly improved version

of the coloring of $\mathcal{G}_{n,p}$ proposed by Bollobás [Bol88]. The upper bound for the chromatic number of the host graph for this construction to work is $\chi(H) \leq \frac{n}{\log(n)^{-\frac{1}{2}}}$ if $p \in (0, 1)$ is constant. However in the case $p(n) = n^{-\theta}$ we are able to reach (N1) with no additional restrictions on $\chi(H)$.

In section 3.5 we discuss the results and pose a few open problems not tackled in this paper.

3.2 Bounding the chromatic number of $\text{pert}_{H,p}$

To give an upper bound for the chromatic number of $\text{pert}_{H,p}$. We will combine Lemma 2.11 with the following fact.

Remark 3.1 Let H be a deterministic graph and $p : \mathbb{N} \rightarrow [0, 1]$. Let $S \subseteq V$ be a color class of a proper coloring of H . Then H does not contain edges with both endpoints in S . Thus the only edges in $\text{pert}_{H,p}[S]$ are random edges and since the edges of $\mathcal{G}_{n,p}$ are chosen independently at random we can identify $\text{pert}_{H,p}[S]$ with $\mathcal{G}_{|S|,p}$.

3.2.1 The Case $p \in (0, 1)$ is constant

In this subsection we consider all p , where $p \in (0, 1)$ is a constant.

First, we bound the number of vertices of the host graph in small color classes. Here the adjective small shall refer to color classes that are significantly smaller than the average color class, while the adjective large will be used to describe color classes which are not small.

Lemma 3.2 *Let $H = ([n], E)$ be a graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{R}$ with $\beta(n) \rightarrow \infty$ as $n \rightarrow \infty$. Furthermore let \mathcal{C} be the set of all color classes of an optimal coloring of H . For $\varepsilon > 0$ set $g(n) = \beta(n)^{(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}}$ and define $\mathcal{L}^g = \{S \in \mathcal{C} \mid |S| \geq g(n)\}$ and $\mathcal{S}^g = \mathcal{C} \setminus \mathcal{L}^g$. For sufficiently large $n = n(\varepsilon)$ we have*

$$\left| \bigcup_{S \in \mathcal{S}^g} S \right| \leq \frac{\varepsilon}{2} \cdot \frac{\log(b)n}{2 \log(\beta(n))}.$$

for every b where $b = \frac{1}{1-p}$ for some constant $p \in (0, 1)$.

Proof. Since the sets $S \in \mathcal{S}^g$ are pairwise disjoint and $(1 + \varepsilon)^{-\frac{1}{2}} < 1$, we get the following estimates

$$\begin{aligned} \left| \bigcup_{S \in \mathcal{S}^g} S \right| &= \sum_{S \in \mathcal{S}^g} |S| \\ &\leq \chi(H) g(n) \\ &= \frac{n}{\beta(n)} \cdot \beta(n)^{(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}} \\ &= n \cdot \beta(n)^{(1+\frac{\varepsilon}{2})^{-\frac{1}{2}} - 1} \end{aligned} \quad \text{(Note } (1 + \frac{\varepsilon}{2})^{-\frac{1}{2}} - 1 < 0 \text{)}$$

$$\leq \frac{\varepsilon}{2} \cdot \frac{\log(b)n}{2\log(\beta(n))}.$$

The last inequality can be proved as follows. Let $x := \beta(n)$, $c := \frac{\varepsilon \log(b)}{4}$ and $a := \left(1 + \frac{\varepsilon}{2}\right)^{-\frac{1}{2}} - 1$. Then for sufficiently large x , we have $x^a \leq \frac{c}{\log(x)}$. \square

Now we are ready to give an upper bound for the expected value of the chromatic number of an augmented graph.

Theorem 3.3 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that $\beta(n) \rightarrow \infty$ for $n \rightarrow \infty$. Let further $p \in (0, 1)$ be constant and $b := \frac{1}{1-p}$. Then for each $\varepsilon > 0$ there is $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$*

$$\mathbb{E} \left(\chi \left(\text{pert}_{H,p} \right) \right) \leq (1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}.$$

Proof. Let \mathcal{C} denote the set of all color classes of an optimal coloring of H and let $\varepsilon > 0$. Additionally, we define for $g(n) = \beta(n)^{(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}}$ the set of large color classes $\mathcal{L}^g = \{S \in \mathcal{C} \mid |S| \geq g(n)\}$ and the set of small color classes $\mathcal{S}^g = \mathcal{C} \setminus \mathcal{L}^g$ as in Lemma 3.2. Note that a set $S \in \mathcal{S}^g \cup \mathcal{L}^g$ is an independent set in H , because S is a color class of an optimal (proper) coloring of H . So, the only edges in $\text{pert}_{H,p}[S]$ are the edges of $\mathcal{G}_{n,p}$. We can thus identify $\text{pert}_{H,p}[S]$ with $\mathcal{G}_{|S|,p}$ according to Remark 3.1. Furthermore, since p is constant the assumptions of Corollary 2.6 are fulfilled for sufficiently large n . Now

$$\begin{aligned} \mathbb{E} \left(\chi \left(\text{pert}_{H,p} \right) \right) &\leq \sum_{S \in \mathcal{C}} \mathbb{E} \left(\chi \left(\text{pert}_{H,p}[S] \right) \right) \\ &= \sum_{S \in \mathcal{L}^g} \mathbb{E} \left(\chi \left(\text{pert}_{H,p}[S] \right) \right) + \sum_{S \in \mathcal{S}^g} \mathbb{E} \left(\chi \left(\text{pert}_{H,p}[S] \right) \right) \\ &= \sum_{S \in \mathcal{L}^g} \mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p} \right) \right) + \sum_{S \in \mathcal{S}^g} \underbrace{\mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p} \right) \right)}_{\leq |S|} \\ &\leq \sum_{S \in \mathcal{L}^g} \mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p} \right) \right) + \sum_{S \in \mathcal{S}^g} |S| \\ &\leq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2\log(|S|)} + \sum_{S \in \mathcal{S}^g} |S| \\ &\quad \text{(for sufficiently large } n \text{ by Corollary 2.6)} \\ &\leq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2\log(|S|)} + \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{2\log(\beta(n))} \quad \text{(Lemma 3.2)} \\ &\leq \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \cdot \frac{\log(b)}{2} \cdot \sum_{S \in \mathcal{L}^g} \frac{|S|}{\log(g(n))} + \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{2\log(\beta(n))} \\ &\leq \left(1 + \frac{\varepsilon}{2} \right) \cdot \frac{\log(b)}{2\log(\beta(n))} \underbrace{\sum_{S \in \mathcal{L}^g} |S|}_{\leq n} + \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{2\log(\beta(n))} \end{aligned}$$

$$\begin{aligned} &\leq (1 + \varepsilon) \cdot \frac{n \log(b)}{2 \log(\beta(n))} \\ &= (1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}. \end{aligned}$$

□

We invoke the bounded differences inequality of C. McDiarmid [McD89]:

Theorem 3.4 (McDiarmid's inequality) *Let X_1, \dots, X_n be independent random variables with X_j taking values in some set A_j . Suppose that some measurable function $f : \prod_{j=1}^n A_j \rightarrow \mathbb{R}$ satisfies*

$$|f(x) - f(x')| \leq c_j$$

whenever the vectors $x, x' \in \prod_{j=1}^n A_j$ differ only in the j -th component. Let Y be the (real valued) random variable $Y := f(X_1, \dots, X_n)$. Then for any $t > 0$

$$\mathbb{P}(|Y - \mathbb{E}(Y)| \geq t) \leq 2 \exp\left(\frac{-2t^2}{\sum_{j=1}^n c_j^2}\right).$$

Remark 3.5 In order to apply McDiarmid's inequality to the chromatic number of $\text{pert}_{H,p}$, we first set $X_j := \{\{i, j\} \in E(\text{pert}_{H,p}) \mid i < j\}$. Since the X_j form a partition of the edges of $\text{pert}_{H,p}$, the random variable $\chi(\text{pert}_{H,p})$ depends solely on the values of the X_j . We set

$$Y = f(X_1, \dots, X_n) = \chi(\text{pert}_{H,p})$$

the X_j being mutually independent random variables. Furthermore, for each j , if X and \hat{X} only differ in the j -th component, we have

$$|f(X) - f(\hat{X})| \leq 1,$$

since one additional color will always be enough to color the vertex j , if necessary. Thus, McDiarmid's inequality is applicable with $\sum_{j=1}^n c_j^2 = n$ and we the following upper bound for the chromatic number of $\text{pert}_{H,p}$.

Theorem 3.6 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that $\beta(n) \rightarrow \infty$ for $n \rightarrow \infty$. Let further be $p \in (0, 1)$ and $b = \frac{1}{1-p}$. Then we have for each $\varepsilon > 0$ and sufficiently large n*

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad a.a.s.$$

Proof. Let $\varepsilon > 0$ and $\lambda = \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$. Now by Theorem 3.3 and McDiarmid's inequality we get

$$\begin{aligned} &\mathbb{P}\left(\chi(\text{pert}_{H,p}) \geq (1 + \varepsilon) \cdot \lambda\right) \\ &= \mathbb{P}\left(\chi(\text{pert}_{H,p}) - \left(1 + \frac{\varepsilon}{2}\right) \cdot \lambda \geq \frac{\varepsilon}{2} \cdot \lambda\right) \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbb{P} \left(\chi \left(\text{pert}_{H,p} \right) - \mathbb{E} \left(\chi \left(\text{pert}_{H,p} \right) \right) \geq \frac{\varepsilon}{2} \cdot \lambda \right) && \text{(Theorem 3.3)} \\
 &\leq 2 \exp \left(-\frac{2}{n} \cdot \left(\frac{\varepsilon \lambda}{2} \right)^2 \right) && \text{(McDiarmid's inequality)} \\
 &\leq 2 \exp \left(-\frac{2}{n} \cdot \left(\frac{\varepsilon n \log(b)}{4(\log(n) - \log(\chi(H)))} \right)^2 \right) \\
 &= 2 \exp \left(-\frac{\varepsilon^2 \log(b)^2 n}{8(\log(n) - \log(\chi(H)))^2} \right) = o(1).
 \end{aligned}$$

□

3.2.2 The Case $p(n) = n^{-\theta}$ for some $\theta \in (0, \frac{1}{3})$

We consider now graphs from $\text{pert}_{H,p}$ with non-constant and small p . When trying to apply the same strategy as for constant p , it turns out that for small color classes it is not enough to merely count the number of vertices as in Lemma 3.2. The reason is that the number of vertices in small components can be of a higher order than the desired bound for the chromatic number of $\text{pert}_{H,p}$. Instead we will use another implication of a result of Bollobás [Bol88].

This implies the following result which bounds the number of colors needed to color the small color classes of H .

Lemma 3.7 *Let $\varepsilon > 0$ and $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{Q}$. Let further $g(n) = \frac{\beta(n)}{\log(n)}$ and let \mathcal{C} the set of all color classes of an optimal coloring of H . Define $\mathcal{L}^g = \{S \in \mathcal{C} \mid |S| \geq g(n)\}$ and $\mathcal{S}^g = \mathcal{C} \setminus \mathcal{L}^g$. Let further $n = n(\varepsilon)$ sufficiently large and $p(n) = n^{-\theta}$, where $\theta \in (0, \frac{1}{3})$ and $b = \frac{1}{1-p}$. If $\beta(n) \geq n^{3\theta}$, then we have for $n \geq n(\varepsilon)$ for some $n(\varepsilon) \in \mathbb{N}$,*

$$\mathbb{E} \left(\chi \left(\text{pert}_{H,p} \left[\bigcup_{C \in \mathcal{S}^g} C \right] \right) \right) \leq \frac{\varepsilon n p}{2(\log(np) - \log(\chi(H)))}.$$

Proof. According to Remark 3.1, for each $C \in \mathcal{S}^g$ we can identify $\text{pert}_{H,p(n)}[C]$ with $\mathcal{G}_{|C|,p(n)}$. So by invoking Corollary 2.10 with $\varepsilon = \frac{1}{3}$ we get

$$\begin{aligned}
 \mathbb{E} \left(\chi \left(\text{pert}_{H,p(n)}[C] \right) \right) &= \mathbb{E} \left(\chi \left(\mathcal{G}_{|C|,p(n)} \right) \right) \\
 &\leq \mathbb{E} \left(\chi \left(\mathcal{G}_{|g(n)|,p(n)} \right) \right) && \text{(since } |C| \leq g(n)) \\
 &\leq \left(1 + \frac{1}{3} \right) \frac{g(n)p(n)}{2 \log(g(n)p(n))} && \text{(Corollary 2.10)} \\
 &= \frac{2}{3} \frac{g(n)p(n)}{\log(g(n)p(n))} \\
 &= \frac{2}{3} \frac{\beta(n)p(n)}{\log(n)(\log(\beta(n)p(n)) - \log(\log(n)))} \\
 &\leq \frac{2}{3} \frac{\beta(n)p(n)}{\log(n)(1 - o(1)) \log(\beta(n)p(n))} && \text{(since } \beta(n)p(n) \geq n^{2\theta})
 \end{aligned}$$

$$\leq \frac{\beta(n)p(n)}{\log(n) \log(\beta(n)p(n))}.$$

Thus, since $|\mathcal{S}^g| \leq \chi(H) \leq \frac{n}{\beta(n)}$ we have

$$\begin{aligned} \mathbb{E} \left(\chi \left(\text{pert}_{H,p} [\cup_{C \in \mathcal{S}^g} C] \right) \right) &\leq \sum_{C \in \mathcal{S}^g} \mathbb{E} \left(\chi \left(\text{pert}_{H,p} [C] \right) \right) && \text{(Lemma 2.11)} \\ &\leq \frac{n}{\beta(n)} \frac{\beta(n)p(n)}{\log(n) \log(\beta(n)p)} && (|\mathcal{S}^g| \leq \frac{n}{\beta(n)}) \\ &= \frac{np(n)}{\log(n) \log(\beta(n)p)} \\ &\leq \frac{\varepsilon np}{2 \log(\beta(n)p)} && \text{(for } n \text{ large enough)} \\ &= \frac{\varepsilon np}{2(\log(np) - \log(\chi(H)))}. \end{aligned}$$

□

Next, we bound the expectation of $\chi(\text{pert}_{H,p})$ from above using Lemma 3.7.

Theorem 3.8 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{Q}$. Let further $\varepsilon > 0$ and $p(n) = n^{-\theta}$, where $\theta \in (0, \frac{1}{3})$ and $b = \frac{1}{1-p}$. If $\beta(n) \geq n^{3(\theta+\delta)}$ for some constant $\delta > 0$, then there exists $n(\varepsilon) \in \mathbb{N}$ so that for all $n \geq n(\varepsilon)$ we have*

$$\mathbb{E} \left(\chi \left(\text{pert}_{H,p} \right) \right) \leq (1 + \varepsilon) \cdot \frac{np}{2(\log(np) - \log(\chi(H)))}.$$

Proof. Let \mathcal{C} denote the set of all color classes of an optimal coloring of H and let $\varepsilon > 0$. Additionally we define for $g(n) = \frac{\beta(n)}{\log(n)}$ the set of large color classes $\mathcal{L}^g = \{S \in \mathcal{C} \mid |S| \geq g(n)\}$ and the set of small color classes $\mathcal{S}^g = \mathcal{C} \setminus \mathcal{L}^g$. For a large color class $S \in \mathcal{L}^g$ of an optimal coloring of H there are no edges of H within S . So we can identify $\text{pert}_{H,p(n)}[S]$ with $\mathcal{G}_{|S|,p(n)}$, by Remark 3.1. Since $|S| \geq g(n) = \frac{\beta(n)}{\log(n)} \geq n^{3(\theta+\frac{\delta}{2})}$ for sufficiently large n , we have

$$p(n) = n^{-\theta} \geq |S|^{-\frac{\theta}{3(\theta+\delta/2)}} = |S|^{-\frac{1}{3} + \frac{\delta}{6(\theta+\delta/2)}},$$

and we may invoke Corollary 2.10 with $\varepsilon' = \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} - 1 > 0$ and get

$$\mathbb{E} \left(\chi \left(\text{pert}_{H,p} [S] \right) \right) = \mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p(n)} \right) \right) \leq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \frac{p|S|}{2 \log(|S|p)}. \quad (3.1)$$

For sufficiently large n , $\log(\log(n)) \in o((\log(\beta(n))))$. Thus

$$\left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \log(g(n)) = \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} (\log(\beta(n)) - \underbrace{\log(\log(n))}_{=o(\log(\beta(n)))}) \geq \log(\beta(n)). \quad (3.2)$$

Now

$$\begin{aligned}
 \mathbb{E} \left(\chi \left(\text{pert}_{H,p} \right) \right) &\leq \sum_{S \in \mathcal{C}} \mathbb{E} \left(\chi \left(\text{pert}_{H,p} \right) [S] \right) && (\text{Lemma 2.11}) \\
 &= \sum_{S \in \mathcal{L}^g} \mathbb{E} \left(\chi \left(\text{pert}_{H,p} [S] \right) \right) + \sum_{C \in \mathcal{S}^g} \mathbb{E} \left(\chi \left(\text{pert}_{H,p} [S] \right) \right) \\
 &= \sum_{S \in \mathcal{L}^g} \mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p} \right) \right) + \sum_{C \in \mathcal{S}^g} \mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p} \right) \right) && (\text{Remark 3.1}) \\
 &\leq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \frac{p |S|}{2 \log(|S|p)} + \sum_{C \in \mathcal{S}^g} \mathbb{E} \left(\chi \left(\mathcal{G}_{|S|,p} \right) \right) && (\text{with (3.1)}) \\
 &\leq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \frac{p |S|}{2 \log(|S|p)} + \frac{\varepsilon}{2} \cdot \frac{np}{2 \log(\beta(n)p)} \\
 &\hspace{15em} (\text{Lemma 3.7 for } n \geq n(\varepsilon) \text{ for some } n(\varepsilon) \in \mathbb{N}) \\
 &\leq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \frac{p |S|}{2 \log(g(n)p)} + \frac{\varepsilon}{2} \cdot \frac{np}{2 \log(\beta(n)p)} && (|S| \geq g(n)) \\
 &= \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \frac{p \sum_{S \in \mathcal{L}^g} |S|}{2 \log(g(n)p)} + \frac{\varepsilon}{2} \cdot \frac{np}{2 \log(\beta(n)p)} \\
 &\leq \left(1 + \frac{\varepsilon}{2} \right)^{\frac{1}{2}} \frac{pn}{2 \log(g(n)p)} + \frac{\varepsilon}{2} \cdot \frac{np}{2 \log(\beta(n)p)} \\
 &\leq \left(1 + \frac{\varepsilon}{2} \right) \frac{pn}{2 \log(\beta(n)p)} + \frac{\varepsilon}{2} \cdot \frac{np}{2 \log(\beta(n)p)} && (\text{with (3.2)}) \\
 &= (1 + \varepsilon) \frac{pn}{2(\log(np) - \log(\chi(H)))}
 \end{aligned}$$

□

The main result of this section is the following theorem, where we use McDiarmid's bounded differences inequality (Theorem 3.4) to bound the probability that $\chi(\text{pert}_{H,p})$ differs from its expectation by a large amount.

Theorem 3.9 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{Q}$. Let further $p(n) \geq n^{-\theta}$, where $\theta \in (0, \frac{1}{3})$ and $b = \frac{1}{1-p}$. If $\beta(n) \geq n^{3\theta}$, then*

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \cdot \frac{np}{2(\log(np) - \log(\chi(H)))} \quad a.a.s.$$

Proof. Let $\varepsilon > 0$ and $\lambda = \frac{\varepsilon n \log(b)}{4(\log(np) - \log(\chi(H)))}$. We have

$$\begin{aligned}
 &\mathbb{P} \left(\chi(\text{pert}_{H,p}) \geq (1 + \varepsilon) \frac{np}{2(\log(np) - \log(\chi(H)))} \right) \\
 &= \mathbb{P} \left(\chi(\text{pert}_{H,p}) \geq \left(1 + \frac{\varepsilon}{2} \right) \frac{np}{2(\log(np) - \log(\chi(H)))} + \lambda \right) \\
 &\leq \mathbb{P} \left(\chi(\text{pert}_{H,p}) \geq \mathbb{E} \left(\chi(\text{pert}_{H,p}) \right) + \lambda \right) && (\text{Theorem 3.8})
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2 \exp\left(-\frac{\lambda^2}{2n}\right) && \text{(McDiarmid's inequality, Theorem 3.4)} \\
 &\leq 2 \exp\left(-\left(\frac{\varepsilon^2 np^2}{32(\log(n) - \log(\chi(H)))^2}\right)\right) && \text{(by Remark 2.1 and } \log(np) \leq \log(n)) \\
 &\leq \exp\left(-n^{\frac{1}{4}}\right),
 \end{aligned}$$

and the last inequality holds, because $p \geq n^{-\theta}$, $\theta \in (0, \frac{1}{3})$ implies $np^2 \geq n^{1/3}$. \square

3.3 Tightness of upper bounds for certain graphs

We proved an upper bound for the chromatic number of an augmented graph, depending only on p and the chromatic number of the host graph. A natural question is, of course, whether our bounds are tight. Indeed, the next theorem shows that there exist host graphs for which the asymptotics for the chromatic number is equal to the asymptotics of the upper bound in Theorem 3.6. We will prove this by considering the independence number of a certain class of host graphs. These host graphs are characterized by a "low" number of "large" independent sets. As an example one can consider the complete $\chi(H)$ -partite graphs.

Lemma 3.10 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta = \beta(n)$ with $\beta(n) \rightarrow \infty$ as $n \rightarrow \infty$. Further let $k = \left\lceil \frac{2(\log(n) - \log(\chi(H)))}{\log(b)} \right\rceil$ and $n_{H,k}$ the number of independent sets of size k in H .*

If $n_{H,k} \in o\left(\left(\frac{\chi(H)}{n}\right)^{-k+1}\right)$, then $\alpha(\text{pert}_{H,p}) \leq \frac{2(\log(n) - \log(\chi(H)))}{\log(b)}$ a.a.s.

Proof. Let X_k be the random variable that counts the number of independent sets of size k in $\text{pert}_{H,p}$. Now

$$\begin{aligned}
 \mathbb{P}(\alpha(\text{pert}_{H,p}) \geq k) &= \mathbb{P}(X_k \geq 1) \\
 &\leq \mathbb{E}(X_k) && \text{(Markov's inequality)} \\
 &= n_{H,k}(1-p)^{\binom{k}{2}} \\
 &= n_{H,k} \cdot \exp\left(-\frac{\log(b)}{2}k(k-1)\right) && (b = (1-p)^{-1}) \\
 &\leq n_{H,k} \cdot \exp\left(-\frac{\log(b)}{2} \frac{2(\log(n) - \log(\chi(H)))}{\log(b)}(k-1)\right) \\
 &= n_{H,k} \cdot \left(\frac{\chi(H)}{n}\right)^{-k+1} \\
 &= o(1),
 \end{aligned}$$

with our assumption on $n_{H,k}$. \square

Together with the well-known inequality $\chi(G) \geq \frac{n}{\alpha(G)}$ Lemma 3.10 implies Theorem 3.11:

Theorem 3.11 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta = \beta(n)$ with $\beta(n) \rightarrow \infty$ as $n \rightarrow \infty$. Further let $k = \left\lceil \frac{2(\log(n) - \log(\chi(H)))}{\log(b)} \right\rceil$ and $n_{H,k}$ the number of independent k sets, in H .*

If $n_{H,k} \in o\left(\left(\frac{\chi(H)}{n}\right)^{-k+1}\right)$, then for each $\varepsilon > 0$ there is a $n(\varepsilon) \in \mathbb{N}$, so that for all $n \geq n(\varepsilon)$ we have

1. $\alpha(\text{pert}_{H,p}) \leq \frac{2(\log(n) - \log(\chi(H)))}{\log(b)} \text{ a.a.s.}$
2. $\chi(\text{pert}_{H,p}) \geq (1 - \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \text{ a.a.s.}$

3.4 Coloring Augmented Graphs with Host Graphs of small Chromatic Number

The proofs of Theorem 3.6 and Theorem 3.9 rely on McDiarmid's bounded differences inequality and are not constructive. In this section we will give algorithms based on algorithms of Bollobás [Bol88] (in the following Algorithm 2.1.1 and Algorithm 2.1.3). They find a coloring of $\text{pert}_{H,p}$ with at most $(1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors a.a.s. We restrict ourselves to host graphs H with $\chi(H) \leq \frac{n}{\log(n)^\gamma}$ for some constant $\gamma \geq \frac{1}{2}$ in the case that p is constant and host graphs with $\chi(H) \leq n^{\frac{1-3\theta-2\delta}{2}}$ for some arbitrary small constant $\delta > 0$ if $p(n) = n^{-\theta}$. In preparation for the rest of the section we define the sets of large and small color classes analogously to section 3.2. The Algorithms 3.4.1, 3.4.2, 2.1.1 and 2.1.3 have exponential time complexity.

Definition 3.12 Let $H = ([n], E)$ be a deterministic graph. Let f be a coloring of H and \mathcal{C} the set of all color classes with respect to f . Let further $g : \mathbb{N} \rightarrow \mathbb{R}$ be a function. We define

$$\mathcal{S}^g := \{S \in \mathcal{C} \mid |S| < g(n)\} \text{ and } \mathcal{L}^g := \{S \in \mathcal{C} \mid |S| \geq g(n)\}.$$

\mathcal{S}^g is the set of small and \mathcal{L}^g is the set of large color classes with respect to f .

We will use Algorithm 2.1.1 to color the large color classes of H while Theorem 2.5 allows us to bound the probability that this approach fails to yield a coloring with a small number of colors. We use the following algorithm to construct a proper coloring of $\text{pert}_{H,p}$.

Algorithm 3.4.1: Coloring $\text{pert}_{H,p}$

Input: A deterministic graph H with $\chi(H) \leq \frac{n}{\beta(n)}$ where $\beta(n) \geq \log(n)^\gamma$ for some constant $\gamma > \frac{1}{2}$, an augmented graph $G = \text{pert}_{H,p}$ for some constant $p \in (0, 1)$ and $\varepsilon > 0$ with $\varepsilon < 2(2\gamma - 1)$.

Output: A proper coloring of G using less than $(1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors.

- 1 Construct an optimal coloring χ' of H ;
 - 2 In each color class S of χ' with $|S| \geq \beta(n)^{(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}} =: g(n)$ construct a proper coloring using Algorithm 2.1.1;
 - 3 Color each vertex that is still uncolored with its own color;
-

Theorem 3.13 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) \leq \frac{n}{\log(n)^\gamma}$ for some constant $\gamma > \frac{1}{2}$ and let $\beta(n) = \frac{n}{\chi(H)}$. Let further $p \in (0, 1)$ be constant and $b = \frac{1}{1-p}$. Then Algorithm 3.4.1 a.a.s. constructs a proper coloring for $\text{pert}_{H,p}$ using at most $(1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors.*

Proof. Since this algorithm constructs a proper coloring, the only thing left to prove is that it uses at most $(1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors a.a.s.. We will show that for coloring the large color classes of H at most $(1 + \frac{\varepsilon}{2}) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors are used a.a.s.. Since by Lemma 3.2 there are at most $\frac{\varepsilon}{2} \cdot \frac{\log(b)n}{2\log(n)}$ vertices of $\text{pert}_{H,p}$ left uncolored, the total number of colors used is at most as stated in the theorem.

For each $S \subseteq [n]$, let $\xi(\text{pert}_{H,p}[S])$ be the number of colors that were used by Algorithm 3.4.1 to color the vertices of S . First we recall some preliminary facts: each $S \in \mathcal{L}^g$ is an independent set in H thus $\text{pert}_{H,p}[S]$ can be identified with $\mathcal{G}_{|S|,p}$ (see Remark 3.1). According to Theorem 2.5, Algorithm 2.1.1 uses more than $(1 + \frac{\varepsilon}{2}) \frac{\log(b)n}{2\log(n)}$ colors to color $\mathcal{G}_{n,p}$ with probability at most $\exp\left(-\frac{cn^2}{\log(n)^8}\right)$ for some constant $c > 0$. Note that the probability p is constant by the assumption of the theorem and thus the term cp^7 from Theorem 2.5 is constant here. Choose $\varepsilon' > 0$ such that $1 + \varepsilon' = \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}}$. Combining these facts we get for each color class $S \in \mathcal{L}^g$

$$\begin{aligned} & \mathbb{P}\left(\xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \frac{\log(b)|S|}{2\log(|S|)}\right) \\ &= \mathbb{P}\left(\xi(\mathcal{G}_{|S|,p}) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \frac{\log(b)|S|}{2\log(|S|)}\right) \\ &= \mathbb{P}\left(\xi(\mathcal{G}_{|S|,p}) \geq (1 + \varepsilon') \frac{\log(b)|S|}{2\log(|S|)}\right) \\ &\leq \exp\left(-\frac{c|S|^2}{\log(|S|)^8}\right) \end{aligned} \tag{3.3}$$

for some constant $c > 0$.

We will use

$$\chi(H) = \frac{n}{\beta(n)} \leq \frac{n}{\log(n)^{\frac{1}{2}}} \text{ and thus } \log(\beta(n)) \geq \log\left(\log(n)^{\frac{1}{2}}\right) = \frac{1}{2} \log(\log(n)). \tag{3.4}$$

And continue

$$\begin{aligned} & \mathbb{P}\left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{\log(b)n}{2(\log(n) - \log(\chi(H)))}\right) \\ &= \mathbb{P}\left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{\log(b)n}{2\log(\beta(n))}\right) \\ &\leq \mathbb{P}\left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) \sum_{S \in \mathcal{L}^g} |S|}{2\log(g(n))}\right) \quad (\text{since } \sum_{S \in \mathcal{L}^g} |S| \leq n) \\ &= \mathbb{P}\left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b)|S|}{2\log(g(n))}\right) \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2 \log(|S|)} \right) && (\text{since } g(n) \geq |S|) \\
&\leq \sum_{S \in \mathcal{L}^g} \mathbb{P} \left(\xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2 \log(|S|)} \right) && (\text{union bound}) \\
&\leq \sum_{S \in \mathcal{L}^g} \exp \left(-\frac{c |S|^2}{\log(|S|)^8} \right) && (\text{by (3.3)}) \\
&\leq \sum_{S \in \mathcal{L}^g} \exp \left(-\frac{c g(n)^2}{\log(g(n))^8} \right) && (\text{since } \frac{x^2}{\log(x)^8} \text{ is an increasing function}) \\
&\leq \chi(H) \cdot \exp \left(-\frac{c \cdot g(n)^2}{\log(g(n))^8} \right) && (\text{since } |\mathcal{L}^g| \leq \chi(H)) \\
&\leq \frac{n}{\beta(n)} \cdot \exp \left(-\frac{c \cdot g(n)^2}{\log(g(n))^8} \right) \\
&= \exp \left(\log(n) - \log(\beta(n)) - \frac{c g(n)^2}{\log(g(n))^8} \right) \\
&= \exp \left(\log(n) - \log(\beta(n)) - \frac{c' \beta(n)^{2(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}}}{\log(\beta(n))^8} \right) \\
&\hspace{15em} (\text{for some } c' > 0, \text{ since } g(n) = \beta(n)^{(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}}) \\
&\leq \exp \left(\log(n) - \gamma \log(\log(n)) - \frac{c'' \log(n)^{\gamma 2(1+\frac{\varepsilon}{2})^{-\frac{1}{2}}}}{\log(\log(n))^8} \right) && (\text{for some } c'' > 0 \text{ by 3.4}) \\
&= o(1). && (\text{since } \gamma \geq \frac{1}{2})
\end{aligned}$$

□

We proceed to color an augmented graph, where $p(n) = n^{-\theta}$ for some $\theta \in (0, \frac{1}{3})$. We need the greedy algorithm Algorithm 2.1.2 to color the small color classes.

Based on these results we use the following algorithm.

Algorithm 3.4.2: A coloring of $\text{pert}_{H,p}$

Input: A constant $\theta \in (0, \frac{1}{3})$, an edge probability p with $p = p(n) = n^{-\theta}$, a deterministic graph H with $\chi(H) \leq \frac{n}{\beta(n)}$ where $\beta(n) \geq n^{\frac{3\theta}{2} + \delta}$ for some $\delta > 0$ arbitrary, but fixed and a perturbed graph $G = \text{pert}_{H,p}$. Further an $\varepsilon > 0$ arbitrary small but fixed.

Output: A proper coloring of G using less than $(1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors.

- 1 Construct an optimal coloring χ' of H ;
 - 2 In each color class S of χ' with $|S| \geq \frac{\beta(n)}{\log(n)} =: g(n)$ construct a proper coloring using Algorithm 2.1.3;
 - 3 Color each remaining color class of H with the greedy algorithm Algorithm 2.1.2;
-

Theorem 3.14 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = \frac{n}{\beta(n)}$ for some function $\beta : \mathbb{N} \rightarrow \mathbb{Q}$ such that $\beta(n) \rightarrow \infty$ for $n \rightarrow \infty$. Let further be*

$p = p(n) = n^{-\theta}$ for some constant $\theta \in (0, \frac{1}{3})$, let $b = \frac{1}{1-p}$ and $\varepsilon > 0$. If $\beta(n) \geq n^{3\theta+\varepsilon}$, Algorithm 3.4.2 a.a.s. constructs a proper coloring of $\text{pert}_{H,p}$ with at most $(1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors.

Proof. Let $\xi(\text{pert}_{H,p})$ be the number of colors used by Algorithm 3.4.2. Since this algorithm yields a proper coloring, it is left to prove that

$$\xi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad \text{a.a.s.}$$

With $g = g(n) = \frac{\beta(n)}{\log(n)}$ we consider \mathcal{S}^g as well as \mathcal{L}^g . We split our proof into two parts. First we prove that all "small" color classes are colored using at most $\frac{\varepsilon}{2} \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors. Next we will prove that all "large" color classes are colored with at most $(1 + \frac{\varepsilon}{2}) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors.

Claim 1: $\xi(\text{pert}_{H,p}[\bigcup_{S \in \mathcal{S}^g} S]) < \frac{\varepsilon}{2} \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ a.a.s.

Proof of Claim 1: Let $\chi_g(\mathcal{G}_{n,p})$ be the number of colors that are used when coloring $\mathcal{G}_{n,p}$ with the greedy Algorithm 2.1.2. Since for each $S \in \mathcal{S}^g$ the graph $\text{pert}_{H,p}[S]$ has the same distribution as the random graph $\mathcal{G}_{|S|,p(n)}$ by Remark 3.1. We may consider the random graph $\mathcal{G}_{g(n),p(n)}$ in which each set of $|S|$ vertices induces a $\mathcal{G}_{|S|,p(n)}$. Thus, the greedy algorithm uses at least as many colors to color $\mathcal{G}_{g(n),p(n)}$ as for $\text{pert}_{H,p}[S]$. Since Algorithm 3.4.2 uses the greedy Algorithm 2.1.2 to color S , we get $\xi(\text{pert}_{H,p}[S]) = \chi_g(\mathcal{G}_{|S|,p(n)})$ where $\chi_g(\mathcal{G}_{|S|,p(n)})$ is the number of colors used by Algorithm 2.1.2 to color $\mathcal{G}_{|S|,p(n)}$.

Thus for all $q \in \mathbb{R}$ we have,

$$\mathbb{P}(\xi(\text{pert}_{H,p(n)}[S]) \geq q) = \mathbb{P}(\chi_g(\mathcal{G}_{|S|,p(n)}) \geq q) \leq \mathbb{P}(\chi_g(\mathcal{G}_{g(n),p(n)}) \geq q). \quad (3.5)$$

Now

$$\begin{aligned} & \mathbb{P} \left(\xi \left(\text{pert}_{H,p} \left[\bigcup_{S \in \mathcal{S}^g} S \right] \right) \geq \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{\underbrace{2(\log(n) - \log(\chi(H)))}_{=\log(\beta(n))}} \right) \\ & \leq \mathbb{P} \left(\sum_{S \in \mathcal{S}^g} \xi(\text{pert}_{H,p}[S]) \geq \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{2 \log(\beta(n))} \right) \\ & \leq \mathbb{P} \left(\sum_{S \in \mathcal{S}^g} \xi(\text{pert}_{H,p}[S]) \geq \frac{\varepsilon}{2} \cdot \frac{\sum_{S \in \mathcal{S}^g} \beta(n) \log(b)}{2 \log(\beta(n))} \right) \quad (n \geq \sum_{S \in \mathcal{S}^g} \beta(n)) \\ & \leq \sum_{S \in \mathcal{S}^g} \mathbb{P} \left(\xi(\text{pert}_{H,p}[S]) \geq \frac{\varepsilon}{2} \cdot \frac{\beta(n) \log(b)}{2 \log(\beta(n))} \right) \quad (\text{by Lemma 2.12}) \\ & \leq \sum_{S \in \mathcal{S}^g} \mathbb{P} \left(\chi_g(\mathcal{G}_{g(n),p}) \geq \frac{\varepsilon}{2} \cdot \frac{\beta(n) \log(b)}{2 \log(\beta(n))} \right) \quad (\text{by inequality 3.5}) \\ & \leq \sum_{S \in \mathcal{S}^g} \mathbb{P} \left(\chi_g(\mathcal{G}_{g(n),p}) \geq (1 + \varepsilon) \cdot \frac{g(n) \log(b)}{2 \log(g(n))} \right) \quad (g(n) = \frac{\beta(n)}{\log(n)}) \\ & \leq n \exp(-cg(n)^3) \leq \exp(-cn^2) = o(1). \quad (\text{by Theorem 2.7}) \end{aligned}$$

Now we show that the large color classes are colored with not too many colors.

Claim 2: $\xi(\text{pert}_{H,p}[\bigcup_{S \in \mathcal{L}^g} S]) < \left(1 + \frac{\varepsilon}{2}\right) \frac{n \log(b)}{2(\log(np) - \log(\chi(H)))}$ a.a.s.

Proof of Claim 2: Note that for all $n \geq n(\varepsilon)$ for some $n(\varepsilon) \in \mathbb{N}$, we have

$$(1 + \varepsilon)^{\frac{1}{2}} \log(g(n)p) \geq \log(\beta(n)p) \quad (3.6)$$

We have

$$\begin{aligned} & \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{\log(b)n}{2(\log(np) - \log(\chi(H)))} \right) \\ &= \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{\log(b)n}{2 \log(\beta(n)p)} \right) \quad (\beta(n) = n/\chi(H)) \\ &\leq \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{\log(b) \sum_{S \in \mathcal{L}^g} |S|}{2 \log(\beta(n)p)} \right) \quad (n \geq \sum_{S \in \mathcal{L}^g} |S|) \\ &\leq \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) \sum_{S \in \mathcal{L}^g} |S|}{2 \log(g(n)p)} \right) \quad (\text{by inequality 3.6}) \\ &\leq \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \sum_{S \in \mathcal{L}^g} \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2 \log(|S|p)} \right) \quad (|S| \geq g(n)) \\ &\leq \sum_{S \in \mathcal{L}^g} \mathbb{P} \left(\xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2 \log(|S|p)} \right) \quad (\text{by Lemma 2.12}) \end{aligned}$$

Note that there is some $\delta \in (0, \frac{1}{3})$ such that for all $S \in \mathcal{L}^g$

$$p(n) \geq |S|^{-\frac{1}{3} + \delta}. \quad (3.7)$$

This can be proved as follows

$$p(n) = n^{-\theta} = (n^{3\theta})^{-\frac{1}{3\theta}} \geq \beta(n)^{-\frac{1}{3} + \delta} \geq |S|^{-\frac{1}{3} + \delta}. \quad (3.8)$$

By Remark 3.1 each large color class $S \in \mathcal{L}^g$ of H induces a random graph $\text{pert}_{H,p}[S] = \mathcal{G}_{|S|,p}$. Thus Algorithm 2.1.3 can be used to color each color class in \mathcal{L}^g . Thus we can apply Theorem 2.9 with $\varepsilon' > 0$, $1 + \varepsilon' = \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}}$,

$$\begin{aligned} & \mathbb{P} \left(\xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2 \log(|S|p)} \right) \\ &= \mathbb{P} \left(\xi(\mathcal{G}_{|S|,p}) \geq \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2}} \cdot \frac{\log(b) |S|}{2 \log(|S|p)} \right) \\ &= \mathbb{P} \left(\xi(\mathcal{G}_{|S|,p}) \geq (1 + \varepsilon') \cdot \frac{\log(b) |S|}{2 \log(|S|p)} \right) \\ &\leq \exp \left(-\frac{c |S|^2 p^3}{\log(|S|)^6} \right) \quad (\text{for some } c > 0 \text{ by Theorem 2.9}) \end{aligned}$$

$$\leq \exp \left(-\frac{cg(n)^2 p^3}{\log(g(n))^6} \right). \quad (\text{since } \frac{x^2}{\log(x)^6} \text{ is monotonously increasing})$$

By our definition $\chi(H) = \frac{n}{\beta(n)}$, so trivially $|\mathcal{L}^g| \leq \frac{n}{\beta(n)}$. Thus

$$\begin{aligned} & \mathbb{P} \left(\sum_{S \in \mathcal{L}^g} \xi(\text{pert}_{H,p}[S]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{\log(b)n}{2(\log(np) - \log(\chi(H)))} \right) \\ & \leq \frac{n}{\beta(n)} \exp \left(-\frac{cg(n)^2 p^3}{\log(g(n))^6} \right) \\ & = \exp \left(\log(n) - \log(\beta(n)) - \frac{cg(n)^2 p^3}{\log(g(n))^6} \right) \\ & = \exp \left(\log(n) - \log(\beta(n)) - \frac{c\beta(n)^2 p^3}{\log(n)^2 \log(g(n))^6} \right) \quad (g(n) = \frac{\beta(n)}{\log(n)}) \\ & \leq \exp \left(\log(n) - \log(\beta(n)) - \frac{c\beta(n)^2 p^3}{\log(n)^8} \right) \quad (\log(n) \geq \log(g(n))) \\ & \leq \exp \left(\log(n) - \log(\beta(n)) - \frac{cn^{6\theta+2\varepsilon} p^3}{\log(n)^8} \right) \quad (\beta(n) \geq n^{3\theta}) \\ & = \exp \left(\log(n) - \log(\beta(n)) - \frac{cn^{6\theta+2\varepsilon-3\theta}}{\log(n)^8} \right) \quad (p(n) = n^{-\theta}) \\ & = \exp \left(\log(n) - \log(\beta(n)) - \frac{cn^{3\theta+2\varepsilon}}{\log(n)^8} \right) \\ & = o(1). \end{aligned}$$

We combine claim 1 and 2 conclude the proof:

$$\begin{aligned} & \mathbb{P} \left(\xi(\text{pert}_{H,p}) \geq (1 + \varepsilon) \frac{n \log(n)}{2(\log(np) - \log(\chi(H)))} \right) \\ & \leq \mathbb{P} \left(\xi \left(\text{pert}_{H,p} \left[\bigcup_{S \in \mathcal{L}^g} S \right] \right) < \left(1 + \frac{\varepsilon}{2}\right) \frac{n \log(b)}{2(\log(np) - \log(\chi(H)))} \right) \\ & \quad + \mathbb{P} \left(\xi \left(\text{pert}_{H,p} \left[\bigcup_{S \in \mathcal{S}^g} S \right] \right) < \frac{\varepsilon}{2} \frac{n \log(b)}{2(\log(np) - \log(\chi(H)))} \right) \\ & = o(1). \end{aligned}$$

□

3.5 Conclusion

In this chapter we first proved that for certain combinations of host graphs H with bounded chromatic number, and constant edge probabilities p or edge probabilities $p = p(n) \geq n^{-\frac{1}{3}+\delta}$ for some $\delta > 0$, the chromatic number of the augmented graph $\text{pert}_{H,p}$ is *a.a.s.* bounded from above by

$$\frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$$

(Theorem 3.6 and Theorem 3.9). Furthermore, it was shown that this upper bound is asymptotically tight for host graphs that possess only few independent sets of a certain size (Theorem 3.11).

In addition, the bound is asymptotically tight for host graphs H with finite chromatic number, as in this case for each $\varepsilon > 0$.

$$\chi(\text{pert}_{H,p}) \geq \chi(\mathcal{G}_{n,p}) \geq \left(1 - \frac{\varepsilon}{2}\right) \frac{n \log(b)}{2 \log(n)} \geq (1 - \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}.$$

However, the upper bound cannot be tight for each host graph. Consider for example a host graph chosen uniformly at random from all graphs with n vertices. This is equivalent to considering H to be a $\mathcal{G}_{n, \frac{1}{2}}$ random graph. Thus $\chi(H) = (1 + o(1)) \frac{n \log(2)}{2 \log(n)}$ *a.a.s.* and H satisfies the conditions of Theorem 3.6. So, the upper bound given by Theorem 3.6 yields *a.a.s.*

$$\chi(\text{pert}_{H,p}) \leq (1 + \varepsilon) \frac{n \log(b)}{2 \log(\log(n))}. \quad (3.9)$$

In this case $\text{pert}_{H,p}$ can be considered as a random graph $\mathcal{G}_{n, \hat{p}}$ where $\hat{p} = \frac{1}{2}(p + 1)$, thus with $\hat{b} = \frac{1}{1-\hat{p}}$ we have *a.a.s.*

$$\chi(\text{pert}_{H,p}) = \chi(\mathcal{G}_{n, \hat{p}}) \leq (1 + o(1)) \frac{n \log(\hat{b})}{2 \log(n)}. \quad (3.10)$$

The upper bound on $\chi(\text{pert}_{H,p})$ in 3.9 is smaller than the upper bound in 3.10 by a factor of $c \frac{\log(n)}{\log(\log(n))}$. We ask the following natural question:

Question 1: Since now an asymptotically tight upper bound for the chromatic number of augmented graphs is known, it would be interesting whether there is a better lower bound than the trivial bound

$$\chi(\text{pert}_{H,p}) \geq \max(\chi(H), \chi(\mathcal{G}_{n,p})) = \max\left(\chi(H), \frac{n \log(b)}{2 \log(n)}\right)?$$

The chromatic number of $\mathcal{G}_{n,p}$ is known for a much wider range of p . Furthermore, no result was given for the case $\chi(H) = \Theta(n)$. We may ask

Question 2: Since an upper bound for the chromatic number of $\text{pert}_{H,p}$ for the cases $p(n) \geq n^{\frac{1}{3}+\delta}$ and $\chi(H) = o(n)$ has been established in this paper, can it be extended to include the case $p(n) = o(1)$ and host graphs with $\chi(H) = \Theta(n)$?

The strategy of divide and conquer that was applied in this paper was shown to be useful in our context. However, it appears to be rather ill-suited to study most other classical graph invariants such as the independence number. Thus, we may ask

Question 3: Which graph invariants of randomly augmented graphs are well-suited to be studied by a divide and conquer strategy?

Chapter 4

On the Choice Number of Randomly Augmented Graphs

4.1 Introduction

Consider a graph $G = (V, E)$ and a family of sets $\mathcal{U} := (U_v)_{v \in V}$. We call a function $\varphi : V \rightarrow \bigcup_{v \in V} U_v$ a proper coloring of G with respect to \mathcal{U} if $\varphi(v) \in U_v$ for all $v \in V$ and for each $\{v, w\} \in E$ we have $\varphi(v) \neq \varphi(w)$. The choice number of the graph G is the minimum number $\hat{\varphi}$ such that for each family of sets $\mathcal{U} := (U_v)_{v \in V}$ with $|U_v| = \hat{\varphi}$ for all $v \in V$ there exists a proper coloring of G with respect to \mathcal{U} . The choice number of the random graph model $\mathcal{G}_{n,p}$ was first studied by Erdős, Rubin and Taylor in 1979 [ERT79] who gave an upper bound of $n/2$.

The aim in this chapter is to give an upper bound for the choice number of the randomly augmented graph $\text{pert}_{H,p}$ building upon the results of Alon [Alo93] analogously to the way Chapter 3 builds upon the results of Bollobás [Bol88] and Spencer and Shamir [SS87]. The problem with this approach is that there exists no analogon of Lemma 2.11, the divide and color lemma, for the choice number. This can be seen by the following standard and well-known example given for example by Alon in [Alo93].

Example 4.1 *Let $V_1 := \{1, 3, 5\}$, $V_2 := \{2, 4, 6\}$, $V = V_1 \cup V_2$ and $E = \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$, then $G := (V, E)$ is the complete bipartite graph $K_{3,3}$ and thus $\chi(G) = 2$, with color classes V_1 and V_2 . Each of these color classes is independent in G and thus for $i \in \{1, 2\}$ there is $\chi_l(G[V_i]) = \chi(G[V_i]) = 1$. However, if we set $U_1 = U_2 = \{\text{red}, \text{green}\}$, $U_3 = U_4 = \{\text{green}, \text{blue}\}$ and $U_5 = U_6 = \{\text{red}, \text{blue}\}$, there exists no list coloring of G with respect to $(U_i)_{i \in [6]}$. This can be seen quickly by trying to color 1 blue which implies that there is no color left in U_3 to color 3 and trying to color 1 green implies that there is no color left in U_5 to color 5.*

4.1.1 Previous Results

In comparison to the chromatic number of $\mathcal{G}_{n,p}$ random graphs, determining upper bounds for the choice number of $\mathcal{G}_{n,p}$ proved to be significantly harder. The first upper bound for the choice number of $\mathcal{G}_{n,p}$ was given by Erdős et al. [ERT79] who proved that $\chi_l(\mathcal{G}_{n,1/2}) \leq n/2$ a.a.s. Alon [Alo92] managed to improve this bound

by proving $\chi(\mathcal{G}_{n,1/2}) = o(n)$ a.a.s. The currently known best upper bound is due to Kahn, who proved $\chi_l(\mathcal{G}_{n,1/2}) \leq (1 + \varepsilon) \frac{n \log(2)}{2 \log(n)}$ a.a.s. in private communication with Alon [Alo93].

For non-constant p the currently known best upper bound was established by Krivelevich [Kri00], who proved that for all $p : \mathbb{N} \rightarrow (0, 1)$ that satisfy $n^{-1/4+\varepsilon} \leq p(n) \leq 3/4$ for some $0 < \varepsilon < 1/4$ we have $\chi_l(\mathcal{G}_{n,p(n)}) = (1 + o(1)) = \frac{n \log(b)}{2 \log(n)}$ where $b = 1/(1 - p(n))$.

4.1.2 Our Result

The fact described above precludes us from using the approach in Chapter 3. However, by generalizing the approach of Alon [Alo93] we are able to give the following upper bound:

Theorem 4.1 *Let H be a deterministic graph with $\chi(H) = n^\gamma$ where $\gamma < \frac{1}{2}$. Let further $0 < p < 1$ be a constant and $b = \frac{1}{1-p}$ then for each $\varepsilon > 0$ a.a.s.*

$$\chi_l(\text{pert}_{H,p}) \leq (1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}.$$

We prove this by considering a family of sets $\mathcal{U} = (U_v)_{v \in V}$ where $|U_v| \geq (1 + \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ and an optimal coloring of the host graph H with color classes \mathcal{C} . Using the probabilistic method, we show that there exists a function \hat{f} mapping $U = \bigcup_{v \in V} U_v$ to \mathcal{C} such that each for each v in a large color class of \mathcal{C} at least $(1 + \varepsilon/2) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}$ colors of U_v are mapped to the color class of v . Large color classes are all color classes that are of at least almost average size. Then we use the union bound to show that each of the random graphs induced by the large color classes has a coloring using only the colors assigned to it by \hat{f} . For vertices in the small color classes we show that there has to exist a coloring using only colors that are not used in the large color classes by using Hall's Marriage Theorem. The proof is concluded by showing that combining these colorings for all color classes yields a proper coloring.

4.2 Bounding the Choice Number of Randomly Augmented Graphs

We start defining a setting of random variables with respect to a host graph H .

Definition 4.2 Given a graph $H = (V, E)$ with $\chi(H) = n^\gamma$ for some $\gamma < 1/2$, a family of sets $\mathcal{U} := (U_v)_{v \in V}$ and an optimal coloring χ_H of H we set

1. $U := U(\mathcal{U}) = \bigcup_{v \in V} U_v$ the set of all colors that appear in some set of \mathcal{U} ,
2. $\mathcal{C} = \{V_1, \dots, V_{\chi(H)}\}$ be the color classes of χ_H ,
3. $Y_1, \dots, Y_{|U|}$ independently distributed random variables on \mathcal{C} , with probabilities $\mathbb{P}(Y_j = V_i) = \frac{|V_i|}{n}$ for all $j \in [|U|]$ and $i \in [\chi(H)]$,

4. $f := f_{H, \chi_H} : U \rightarrow \mathcal{C} : c \mapsto Y_c$ a random function,
5. for each $v \in V_i$, $X_v = |f^{-1}(i) \cap S(v)|$ is the number of colors in the list of v that have been assigned to be used in the color class V_i of H that contains v .

Remark 4.3 Let $H = (V, E)$ be a graph with $\chi(H) = n^\gamma$ for some $\gamma < 1/2$, $\mathcal{U} := (U_v)_{v \in V}$ a family of sets and χ_H an optimal coloring of H . Then for each $v \in V$ there exist independent Bernoulli random variables $Z_1^v, \dots, Z_{|U_v|}^v$ such that $X_v = \sum_{c \in U_v} Z_c^v$.

This can be seen by choosing Z_c^v as the indicator variable for the event $\{f(c) = Y_i\}$ for all $c \in U_v$ where V_i is the color class of χ_H that contains v .

We will apply McDiarmid's inequality (Theorem 3.4) to X_v , $v \in V$.

As in Chapter 3, let $g(n) = \left(\frac{n}{\chi(H)}\right)^{(1+\varepsilon)^{-1}} = n^{(1-\gamma)(1+\varepsilon)^{-1}}$ and

$$\mathcal{L} = \{V_i \in \mathcal{C} \mid |V_i| \geq g(n)\}$$

be the set of large color classes of H .

For technical reasons we further assume $0 < \varepsilon < 2(1 - \gamma) - 1$.

Lemma 4.4 *Each vertex in a color class in \mathcal{L} is assigned at least $\left(1 + \frac{\varepsilon}{2}\right) \frac{|V_i| \log(b)}{2 \log(|V_i|)}$ colors by f a.a.s.*

Proof. Let $V_i \in \mathcal{L}^g$ be a large color class and f the function defined in Definition 4.2 and let $v \in V_i$. We have

$$\mathbb{E}(X_v) = (1 + \varepsilon) \frac{|V_i| \log(b)}{2 (\log(n) - \log(\chi(H)))}. \quad (4.1)$$

This is easily proved

$$\begin{aligned} \mathbb{E}(X_v) &= \sum_{c \in U_v} \mathbb{E}(Z_c^v) = |U_v| \mathbb{E}(Z_c^v) \\ &= (1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))} \mathbb{E}(Z_c^v) \\ &= (1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))} \mathbb{P}(Z_c^v = 1) \\ &= (1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))} \mathbb{P}(Y_c = V_i) \\ &= (1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))} \frac{|V_i|}{n} \\ &= (1 + \varepsilon) \frac{|V_i| \log(b)}{2 (\log(n) - \log(\chi(H)))}. \end{aligned}$$

The random variable X_v satisfies the assumptions of McDiarmid's inequality. In fact by Remark 4.3, X_v is the sum of $(1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))}$ Bernoulli random variables. Thus X_v is Lipschitz with constant 1 in each of its $(1 + \varepsilon) \frac{n \log(b)}{2 (\log(n) - \log(\chi(H)))}$ components. McDiarmid's inequality implies that for each $t > 0$

$$\mathbb{P}(X_v - \mathbb{E}(X_v) \leq t) \leq \exp \left(-2t^2 \frac{2 (\log(n) - \log(\chi(H)))}{(1 + \varepsilon) n \log(b)} \right) \quad (4.2)$$

Thus,

$$\begin{aligned}
 & \mathbb{P} \left(X_v \leq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|V_i| \log(b)}{2 \cdot \log(|V_i|)} \right) \\
 &= \mathbb{P} \left(X_v \leq \mathbb{E}(X_v) - \frac{\varepsilon}{2} \cdot \frac{|V_i| \log(b)}{2 \cdot \log(|V_i|)} \right) && \text{(by equation 4.1)} \\
 &\leq \mathbb{P} \left(X_v - \mathbb{E}(X_v) \leq -\frac{\varepsilon |V_i| \log(b)}{4 \cdot \log(|V_i|)} \right) \\
 &\leq \mathbb{P} \left(X_v - \mathbb{E}(X_v) \leq -\frac{\varepsilon g(n) \log(b)}{4 \cdot \log(g(n))} \right) \\
 &\hspace{15em} \text{(since } x/\log(x) \text{ is monotonously increasing)} \\
 &\leq \exp \left(-2 \cdot \frac{\varepsilon^2 g(n)^2 \log(b)^2}{16 \log(g(n))^2} \cdot \frac{2(\log(n) - \log(\chi(H)))}{(1 + \varepsilon)n \log(b)} \right) && \text{(by (4.2))} \\
 &= \exp \left(-\frac{\varepsilon^2 \log(b)}{4(1 + \varepsilon)} \cdot \frac{g(n)^2}{\log(g(n))^2} \cdot \frac{(\log(n) - \log(\chi(H)))}{n} \right) \\
 &= \exp \left(-\frac{\varepsilon^2 \log(b)}{4(1 + \varepsilon)} \cdot \frac{n^{2(1-\gamma)(1+\varepsilon)^{-1}}}{(1 + \varepsilon)^{-2}(\log(n) - \log(\chi(H)))^2} \cdot \frac{(\log(n) - \log(\chi(H)))}{n} \right) \\
 &= \exp \left(-\frac{\varepsilon^2 \log(b)(1 + \varepsilon)}{4} \cdot \frac{n^{2(1-\gamma)(1+\varepsilon)^{-1}-1}}{(\log(n) - \log(\chi(H)))} \right) \\
 &= \exp \left(-\frac{\varepsilon^2 \log(b)(1 + \varepsilon)}{4(1 - \gamma)} \cdot \frac{n^{2(1-\gamma)(1+\varepsilon)^{-1}-1}}{\log(n)} \right) && \text{(as } \chi(H) = n^\gamma) \\
 &= \exp \left(-c_1 \cdot \frac{n^{2(1-\gamma)(1+\varepsilon)^{-1}-1}}{\log(n)} \right) && \text{(for some constant } c_1 > 0) \\
 &\leq \exp(-c_1 n^{c_2})
 \end{aligned}$$

for some constant $c_2 > 0$, since $\gamma < \frac{1}{2}$ and $\varepsilon < 2(1 - \gamma) - 1$.

Thus

$$\begin{aligned}
 & \mathbb{P} \left(\exists V_i \in \mathcal{L}^g : \exists v \in V_i \wedge X_v \leq \left(1 + \frac{\varepsilon}{2}\right) \frac{|V_i| \log(b)}{2 \log(|V_i|)} \right) \\
 &\leq \sum_{V_i \in \mathcal{L}^g} \sum_{v \in V_i} \mathbb{P} \left(X_v \leq \left(1 + \frac{\varepsilon}{2}\right) \frac{|V_i| \log(b)}{2 \log(|V_i|)} \right) && \text{(Union Bound)} \\
 &\leq n \exp(-c_1 n^{c_2}) \\
 &= o(1).
 \end{aligned}$$

So, with positive probability, to every vertex that belongs to a large color class $V_i \in \mathcal{L}^g$ of H at least $\left(1 + \frac{\varepsilon}{2}\right) \frac{|V_i| \log(b)}{2 \log(|V_i|)}$ colors are assigned. \square

Proof of Theorem 4.1. By Lemma 4.4, with positive probability to each vertex in a large color class $V_i \in \mathcal{L}^g$ of H at least $\left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|V_i| \log(b)}{(\log(n) - \log(\chi(H)))}$ colors are assigned.

By the probabilistic method there exists a function $\hat{f} : U \rightarrow \mathcal{C}$ such that for each $V_i \in \mathcal{L}^g$ and each $v \in V_i$ we have $|\hat{f}^{-1}(V_i) \cap U_v| \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|V_i| \log(b)}{(\log(n) - \log(\chi(H)))}$ and $\hat{f}^{-1}(V_i) \cap \hat{f}^{-1}(V_j) = \emptyset$ for all $i \neq j$.

Since each V_i induces a $\mathcal{G}_{|V_i|,p}$ and $|V_i| \geq g(n)$, the probability that V_i does not have a list coloring using only the colors assigned to it by \hat{f} is at most $\exp\left(-\frac{cg(n)^2}{\log(g(n)^8)}\right)$ by Theorem 2.14. This implies that the probability that there exists a $V_i \in \mathcal{L}^g$ that does not have a list coloring using only the colors randomly attributed to it by f , is at most

$$\begin{aligned}
 & \mathbb{P}\left(\exists V_i \in \mathcal{L}^g : \chi_l(\text{pert}_{H,p}[V_i]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|V_i| \log(b)}{(\log(n) - \log(\chi(H)))}\right) \\
 & \hspace{25em} \text{(by the union bound)} \\
 & \leq \sum_{V_i \in \mathcal{L}^g} \mathbb{P}\left(\chi_l(\text{pert}_{H,p}[V_i]) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|V_i| \log(b)}{(\log(n) - \log(\chi(H)))}\right) \\
 & = \sum_{V_i \in \mathcal{L}^g} \mathbb{P}\left(\chi_l(\mathcal{G}_{|V_i|,p}) \geq \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|V_i| \log(b)}{(\log(n) - \log(\chi(H)))}\right) \\
 & \leq \sum_{V_i \in \mathcal{L}^g} \exp\left(-\frac{cg(n)^2}{\log(g(n)^8)}\right) \hspace{10em} \text{(by Theorem 2.14)} \\
 & \leq \chi(H) \exp\left(-\frac{cg(n)^2}{\log(g(n)^8)}\right) \hspace{10em} \text{(since } |\mathcal{L}^g| \leq \chi(H)) \\
 & \leq \exp\left(\log(n) - \frac{n^{2(1-\gamma)(1+\varepsilon)^{-1}}}{8 \log(n)}\right) = o(1).
 \end{aligned}$$

As shown above, for each $V_i \in \mathcal{L}^g$ there exists a coloring $\varphi_i : V_i \rightarrow U$ of $\text{pert}_{H,p}[V_i]$ using only colors assigned to V_i by \hat{f} a.a.s. We combine these colorings by setting $\varphi_{\mathcal{L}^g}(v) = \varphi_{V_i}(v)$ for each $v \in V_i$. Since \hat{f} maps each color to exactly one color class, no two neighbors can have the same colors with respect to $\varphi_{\mathcal{L}^g}$. This implies that $\varphi_{\mathcal{L}^g}$ is a proper coloring of the large color classes which uses at most $\left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))}$.

After having colored the large color classes, for each $v \in V$ there are at least $\frac{\varepsilon}{2} \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))}$ colors in U_v that have not yet been used. We consider the bipartite graph $G_b = (A \cup B)$ with the vertices of $\text{pert}_{H,p}$ that are in small color classes of \mathcal{C} on one side, i.e. $A = \{v \in V : \exists V_i \in \mathcal{S}^v \subseteq V_i\}$ and the colors that are not used by $\varphi_{\mathcal{L}^g}$ on the other side, i.e. $B = \{c \in U \wedge c \notin \text{Im}(\varphi_{\mathcal{L}^g})\}$. For every vertex $v \in A$ and color $c \in B$ we add the edge $\{v, c\}$ iff $c \in U_v$. As shown in Lemma 3.2 at most $\frac{\varepsilon}{2} \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))}$ vertices remain uncolored. This implies

$$|A| \leq \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))}$$

Furthermore, for each $v \in A$ we have

$$|N_{G_b}(v)| \geq |U_v| - |\text{Im}(\varphi_{\mathcal{L}^g})| \geq \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))}$$

because $\varphi_{\mathcal{L}^g}$ used at most $\left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))}$ colors. So, for every $\emptyset \neq A' \subseteq A$ and $v \in A'$ we have

$$|N_{G_b}(A')| \geq |N_{G_b}(v)| \geq \frac{\varepsilon}{2} \cdot \frac{n \log(b)}{(\log(n) - \log(\chi(H)))} \geq |A| \geq |A'|$$

Thus by the use of Hall's Marriage Theorem there exists a Matching M that covers A . The matching M induces a list coloring $\varphi_{\mathcal{S}^g}$ with respect to \mathcal{U} for the vertices in small color classes using only the remaining colors in the following sense $\varphi_{\mathcal{S}^g}(v) = c$ iff $\{v, c\} \in M$.

Since $\varphi_{\mathcal{S}^g}$ and $\varphi_{\mathcal{L}^g}$ use disjunct sets of colors, we define the following proper list coloring of $\text{pert}_{H,p}$ with respect to \mathcal{U} .

$$\varphi : V \rightarrow U, v \mapsto \begin{cases} \varphi_{\mathcal{L}^g}(v), & \exists V_i \in \mathcal{L}^g : v \in V_i \\ \varphi_{\mathcal{S}^g}(v), & \exists V_i \in \mathcal{S}^g : v \in V_i \end{cases}$$

□

4.3 Remarks on tightness

After proving an upper bound for $\chi_l(\text{pert}_{H,p})$ for the cases in which $p \in (0, 1)$ is constant and $\chi(H) < n^\gamma$ for some $\gamma < 1/2$, we will investigate whether for some combinations of H and p the bound in Theorem 4.1 is tight. Our first remark considers host graphs of small chromatic number.

Lemma 4.5 *Let H be a deterministic graph such that $\chi(H)$ is bounded by a constant. Then for all constant $p \in (0, 1)$ and $\varepsilon > 0$ we have*

$$\chi_l(\text{pert}_{H,p}) \geq \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \text{ a.a.s.}$$

Proof. Finding a proper k -coloring of a graph G is equivalent to finding a proper list-coloring of G with respect to the family $\mathcal{U} = \{[k]\}_{v \in V}$, so $\chi(G) \leq \chi_l(G)$. The well-known result $\chi(G) \leq n/\alpha(G)$ combined with a result of Bollobás and Erdős [BE76] that says $\alpha(\mathcal{G}_{n,p}) \leq 2 \log(b)^{-1} (1 - \varepsilon)^{-1/2} \log(n)$ a.a.s., we get

$$\sqrt{(1 - \varepsilon)} \frac{n \log(b)}{2 \log(n)} \leq \frac{n}{\alpha(\mathcal{G}_{n,p})} \leq \chi(\mathcal{G}_{n,p}) \leq \chi_l(\mathcal{G}_{n,p}) \text{ a.a.s.}$$

By assumption of the lemma $\chi(H)$ is bounded by a constant, so $\log(\chi(H)) \in o(\log(n))$ and there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$

$$(1 - \varepsilon)^{-1/2} (\log(n) - \log(\chi(H))) \geq \log(n).$$

This implies for all $n \geq n_\varepsilon$

$$\begin{aligned} (1 - \varepsilon) \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} &\leq \sqrt{(1 - \varepsilon)} \frac{n \log(b)}{2(1 - \varepsilon)^{-1/2} (\log(n) - \log(\chi(H)))} \\ &\leq \sqrt{(1 - \varepsilon)} \frac{n \log(b)}{2 \log(n)} \leq \chi_l(\text{pert}_{H,p}). \end{aligned}$$

□

The following result shows that the bound in Theorem 4.1 is indeed tight in the following sense:

Theorem 4.6 *Let $H = ([n], E)$ be a deterministic graph with $\chi(H) = n^\gamma$ for some $\gamma < 1/2$. Further let $k = \left\lceil \frac{2(\log(n) - \log(\chi(H)))}{\log(b)} \right\rceil$ and $n_{H,k}$ the number of independent k sets, in H .*

If $n_{H,k} \in o\left(\left(\frac{\chi(H)}{n}\right)^{-k+1}\right)$, then for each $\varepsilon > 0$ there is a $n_\varepsilon \in \mathbb{N}$, so that for all $n \geq n_\varepsilon$ we have

$$\chi_l(\text{pert}_{H,p}) \geq (1 - \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad \text{a.a.s.}$$

Proof. Since $\chi(H) = n^\gamma$ the prerequisites of Theorem 3.11 are satisfied. This implies

$$\chi_l(\text{pert}_{H,p}) \geq \chi(\text{pert}_{H,p}) \geq (1 - \varepsilon) \cdot \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))} \quad \text{a.a.s.}$$

□

4.4 Conclusion

In this chapter we proved that for host graphs H with chromatic number $\chi(H) \leq n^\gamma$, for some $\gamma < 1/2$ and $p \in (0, 1)$ constant, the chromatic number of the randomly augmented graph $\text{pert}_{H,p}$ is a.a.s. bounded from above by

$$\chi_l(\text{pert}_{H,p}) \leq \frac{n \log(b)}{2(\log(n) - \log(\chi(H)))}. \quad (4.3)$$

This is the same upper bound that was proved for the chromatic number in Chapter 3. Furthermore, we proved that this bound is tight for certain host graphs that contain only few independent sets of a certain size. All of these results show that $\chi_l(\text{pert}_{H,p})$ in these cases behaves in the same way as $\chi(\text{pert}_{H,p})$.

The following natural questions arise:

Question 1 : Does the upper bound in 4.3 hold for all randomly augmented graphs $\text{pert}_{H,p}$ as long as $p \in (0, 1)$ is constant and H satisfies $\chi(H) \in o(n)$?

Question 2 : For which combinations of host graphs H and probability functions $p : \mathbb{N} \rightarrow (0, 1)$ can a similar bound be shown?

Question 3 : Can the technique used in this chapter be used to show similar bounds for related invariants? We suggest the game chromatic number and the quantum chromatic number as natural candidates.

Chapter 5

An Algorithm to win the biased k -Edge-Connectivity and k -Factor Games fast

5.1 Introduction

5.1.1 Previous work

The biased Maker-Breaker-Connectivity game was first introduced by Chvátal and Erdős [CE78]. They proved the following upper bound for its critical bias.

Theorem 5.1 (Chvátal and Erdős, [CE78]) *Let $\varepsilon > 0$, then there exists $n(\varepsilon) \in \mathbb{N}$, such that for each $n \geq n(\varepsilon)$ and $b > (1+\varepsilon)\frac{n}{\log(n)}$, Breaker wins the $(1-b)$ connectivity game.*

This is achieved by isolating a single vertex $v \in V$, i.e. assuring that at the end of the game its Maker degree is zero and its Breaker degree is $n-1$. Since Hamiltonicity implies connectivity, this also implies that Breaker wins the Hamiltonicity game, and most other games that require Maker to build a global structure, with the same parameters. Krivelevich [Kri10] proved the following theorem for the biased Hamiltonian cycle game determining the asymptotically exact critical bias for this game to be $(1 + o(1))\frac{n}{\log(n)}$.

Theorem 5.2 (Krivelevich 2010, [Kri10]) *There exists $n_0 \in \mathbb{N}$ such that for every $n \in N_{\geq n_0}$ Maker has a strategy to win the $(1 : b)$ Hamiltonian cycle game played on the complete graph on n vertices in at most $14n$ rounds for every $b \leq \left(1 - \frac{30}{\log^{1/4}(n)}\right)n$.*

This result is achieved in three steps. In the first step an expander graph is created by Maker. In the second step the connected components of the expander graph are connected by Maker and a connected expander is achieved. In the last step, the Hamiltonian cycle is completed by adding boosters to Maker's graph.

Krivelevich also remarks that this strategy can be generalized. We remind the reader of Theorem 2.15.

Theorem 5.3 (Theorem 2.15) *Let $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}_{\geq n_0}$ Maker has a strategy to construct k edge disjoint Hamiltonian cycles in the $(1 : b)$ game played on the complete graph on n vertices in at most $\min(13n, (8k + 1)n)$ rounds for every $b \leq (1 - \varepsilon)n$.*

We now consider a graph $G = (V, E)$ in which E is the pairwise disjoint union of k Hamiltonian cycles. Then each vertex $v \in V$ has degree exactly $2k$. Thus G is a $2k$ -factor. Furthermore, for two distinct vertices $v, w \in V$ there exist $2k$ edge disjoint paths connecting v, w . This is true because each Hamiltonian cycle induces two paths from v to w . The techniques of Krivelevich yield a lower bound for the critical bias in the $(1 : b)$ $2k$ -edge-connectivity game and in the $(1 : b)$ $2k$ -factor game and show that both games can be won in a linear number of rounds.

Since the critical bias is known asymptotically, it is further of interest how many rounds Maker has to play to claim a winning graph. We note the following:

Remark 5.4 (Lower Bound) A Hamiltonian graph has to contain a Hamiltonian cycle, which contains exactly n edges. Since Maker can only add one edge per round, Maker cannot win in fewer than n rounds.

For $k \geq 2$, if G is k -edge-connected, its minimum degree has to be at least k . Otherwise it would be possible to destroy its connectivity by removing every edge incident in a vertex v with $\deg(v) < k$. Since every edge connects exactly to vertices, the minimum number of edges needed is $\lceil \frac{k}{2}n \rceil$. Since Maker can add only one edge per round, Maker cannot win in fewer than $\lceil \frac{k}{2}n \rceil$ rounds.

For the case $k = 1$ Brüstle et al. [BCN⁺23] showed the following improvement of Krivelevich's result.

Theorem 5.5 (Brüstle et al. 2023, [BCN⁺23]) *There exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that for any $b < \frac{n}{\log(n)} - \frac{Cn}{\log(n)^{3/2}}$ and $n \geq n_0$ Maker wins the $(1 : b)$ Hamilton cycle game on the complete graph on n vertices in at most $n + \frac{Cn}{\sqrt{\log(n)}}$ rounds.*

5.1.2 Our results

Remark 5.6 Since an edge disjoint union of k Hamiltonian cycles consists of kn edges, Maker needs at least kn rounds to win this $(1 : b)$ game. Thus for the case $k = 1$, Theorem 5.5 states essentially the fastest possible round-complexity, up to the $\frac{Cn}{\sqrt{\log(n)}} = o(n)$ term. Analogously, our main theorem gives the fastest round-complexity, up to an $o(n)$ term in the k -edge-connectivity game and the $2k$ -factor game:

Theorem 5.7 (Main Theorem) *Let $k \in \mathbb{N}$ be arbitrary but fixed. There exists a constant $C_k > 0$ and $n_0 \in \mathbb{N}$ such that for any $b < \frac{n}{\log(n)} - \frac{C_k n}{\log(n)^{3/2}}$ and $n \geq n_0$ Maker has a strategy to construct k edge disjoint Hamiltonian cycles in the $(1 : b)$ game on the complete graph on n vertices in at most $kn + \frac{C_k n}{\sqrt{\log(n)}}$ rounds.*

The main theorem immediately implies a few additional results:

Theorem 5.8 *Let $k \in \mathbb{N}$ be arbitrary but fixed. There exists a constant $C_k > 0$ and $n_0 \in \mathbb{N}$ such that for any $b < \frac{n}{\log(n)} - \frac{C_k n}{\log(n)^{3/2}}$ and $n \geq n_0$ Maker has a strategy to win $(1 : b)$ k -factor-game on the complete graph on n vertices in at most $\left\lceil \frac{k}{2} \right\rceil n + \frac{C_k n}{\sqrt{\log(n)}}$ rounds.*

Proof of Theorem 5.8. If k is even, Maker builds $\frac{k}{2}$ Hamiltonian cycles with the algorithm used in the proof of the main theorem Theorem 5.7 and these cycles form a k -factor. If k is odd, Maker builds $\frac{k+1}{2} = \left\lceil \frac{k}{2} \right\rceil$ Hamiltonian cycles with the algorithm used in the proof of the main Theorem and then chooses one of the Hamiltonian cycles to remove every second vertex in it. By this deletion process, we have created a k -factor. \square

Theorem 5.9 *Let $k \in \mathbb{N}_{\geq 2}$ be arbitrary but fixed. There exists a constant $C_k > 0$ and $n_0 \in \mathbb{N}$ such that for any $b < \frac{n}{\log(n)} - \frac{C_k n}{\log(n)^{3/2}}$ and $n \geq n_0$ Maker has a strategy to win the $(1 : b)$ k -edge-connectivity-game on the complete graph on n vertices in at most $\frac{k}{2} + \frac{C_k n}{\sqrt{\log(n)}}$ rounds.*

5.1.3 Methods and Innovations

In this chapter we use techniques that are similar to those used in a paper of Brüstle et al. [BCN⁺23] that gives an algorithm to win the $(1 : b)$ Hamiltonian cycle game in asymptotically optimal time. Since, in contrast to Brüstle et al., we are interested in building multiple edge-disjoint Hamiltonian cycles, we have to build the foundation for these cycles simultaneously. Where Brüstle et al. built a single set of paths that will be joined to form the single Hamiltonian cycle, we build k edge disjoint sets of paths of different colors simultaneously.

As in [BCN⁺23] we use an expander-like subset of V to facilitate rotations similar to those introduced by Pósa [Pó76]. However, since we want to construct $k \geq 1$ edge-disjoint Hamilton cycles, we cannot use these edges in multiple Hamilton cycles. Thus we increase the required minimum degree of these vertices to $34k$ where Brüstle et al. only required a minimum degree of 10.

This increase in the minimum degree also ensures that the expander-like subset remains connected and keeps expander-like properties even if all k Hamiltonian cycles were removed at the end of the game.

In addition to these modifications we present an innovative strategy that ensures k -connectivity based on the structure that arises at the end of the algorithm *Hamilton-Cycles*.

5.2 Describing the Algorithm *Hamilton-Cycles*

In the algorithm *Hamilton-Cycles* we encounter cases in which one vertex is determined first and Maker is supposed to add an edge to this vertex. Since some of our proofs rely on the fact that a certain amount of random edges were added to each vertex v of a certain subset of V , we want to "ignore" the edges which are incident in v but were chosen from some $w \in V \setminus \{v\}$. To achieve this we consider some edges to be directed edges. Whenever we call for Maker to claim the directed edge

(v, w) , Maker chooses the edge $\{v, w\}$ and denotes that the directed edge (v, w) was supposed to be taken. However, this is only a bookkeeping measure. This approach leads to the following problem:

Let $v, w \in V$. What is Maker supposed to do if the algorithm *Hamilton-Cycles* calls for the addition of the directed edge (v, w) , but $\{v, w\}$ was already claimed as the directed edge (w, v) ? This is easily remedied. Whenever this is the case Maker updates her bookkeeping by including the directed edge (w, v) . Since Maker did not have to claim an edge (as $\{w, v\} = \{v, w\}$ had already been chosen), Maker takes another turn according to the algorithm *Hamilton-Cycles* immediately with an updated out-degree of v .

We define the degree in Breakers graph of some $v \in V$ as $d_B(v)$. We call a vertex $v \in V$ troublesome if $d_B(v) \geq \frac{n}{\sqrt{\log(n)}}$. We further define the number of out-edges Maker added to v after it became troublesome as $d_M^*(v)$ and the number of out-edges Maker added to v before it became troublesome as $d_M'(v)$. Further we define in each step for every $v \in V$, $\text{dang}(v) := d_B(v) - bd_M^*(v)$. These notations are consistent with [BCN⁺23].

In the first stage we create a set S_i of size $\mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ and a set \mathcal{P}_i of families of paths in $V \setminus S_i$. We also make sure that the paths are not too long i.e. each such path has a length of at most $\log(n)$. The relevance of bounding the length will become apparent in Stage 3, when we want to connect the vertices of the paths to ensure connectivity. For clarity we color all edges that are picked with a color between 0 and k , where edges colored with a color i , $1 \leq i \leq k$ will be used to form the i -th Hamiltonian cycle, while the edges colored with 0 are all-purpose edges not yet committed to one of the Hamiltonian cycles.

We call $v, w \in V \setminus S_i$ j -neighbors, if they are joined by an edge of color j . For each round i , color $j \in [k]$ and $v \in V \setminus S_i$, we define $[v]_i^{(j)}$ to be the path of color j that contains v .

At the beginning we choose $S_0 \subseteq V$ arbitrary with $S_0 = \left\lceil \frac{1000(4k+3)^2\sqrt{k+2n}}{\sqrt{\log(n)}} \right\rceil$, and for each $j \in [k]$ we define $\mathcal{P}_0^{(j)}$ to be the set of all singleton paths in $V \setminus S_0$, where a singleton path consists of one vertex only.

5.2.1 Stage 1: Creating a Graph consisting of an Expander and few paths

If, after round i of Breaker a vertex $v \in V \setminus S_i$ has become troublesome, it is added to S_{i+1} and for each j , we delete the path $[v]_{i+1}^{(j)} = P_1 \circ v \circ P_1$ that contains v , from $\mathcal{P}_i^{(j)}$. Furthermore, all Maker-edges incident in v are recolored with $k+1$. Thereafter the paths P_1 and P_2 are added to $\mathcal{P}_i^{(j)}$.

Two paths $P, P' \in \mathcal{P}_i^{(j)}$ of length less than $\log(n)$ are called (u, v) -available in $\mathcal{P}_i^{(j)}$ if u is an endpoint of P , v is an endpoint of P' , $\{u, v\}$ is still unclaimed and no edges of color $k+1$ are incident in u or v . This implies that until this point there was at most one edge of color j incident in u and at most one edge of color j incident in v .

Case 1: Every troublesome vertex x satisfies $d_M^*(x) \geq 34k$.

Case 1.1: There exists a non-troublesome vertex $v \in S_{i-1}$ with $d'(v) < 34k$, then

- choose $w^* \in \{w \in S_0 : (v, w) \notin B \cup M\}$ uniformly at random, add (v, w^*) to M and color it with 0.
- Set $S_i = S_{i-1}$ and $\mathcal{P}_i = \mathcal{P}_{i-1}$.

Case 1.2: Every non-troublesome vertex v in S_{i-1} has a degree of $d'_M(v) \geq 34k$. We have two cases:

Case 1.2.1: There are two elements P and P' in some $\mathcal{P}_{i-1}^{(j)}$ that are (u, v) -available. Then:

- Choose P and P' from $\mathcal{P}_{i-1}^{(j)}$ which are (u, v) available such that $|\mathcal{P}_{i-1}^{(j)}|$ is minimal with respect to j .
- Add $\{u, v\}$ to M and color it with j .
- Set $S_i = S_{i-1}$ and $\mathcal{P}_i^{(j)} = \mathcal{P}_{i-1}^{(j)} - P - P' + P^*$ where P^* is the path $P^* = P \circ (u, v) \circ P'$.

Case 1.2.2: Else end stage 1, move to stage 2, and set $i_0^* = i - 1$.

Case 2: There is a troublesome vertex $v \in S_{i-1}$ such that $d_M^*(v) < 34k$. Select v to be a troublesome vertex $v \in S_{i-1}$ such that $d_M^* < 34k$, having maximum danger (with arbitrarily broken ties). Then we have two cases:

Case 2.1: If there is no $w \in V - S_{i-1}$ such that (v, w) is unclaimed, set $i_0^* = i$ and the game ends.

Case 2.1: Otherwise

- choose such a w and claim (v, w) and color it with 0;
- set $S_i = S_{i-1} \cup \{w\}$ and for each $j \in [k]$ set \mathcal{P}_i^j to be the set obtained from \mathcal{P}_{i-1}^j by deleting the path P that contained w and adding the components of $P - w$ and recolor all j -edges incident in v with $k + 1$.

5.2.2 Stage 2: Closing k Hamilton Cycles

Stage 2 will be repeated until Maker has created k edge disjoint Hamiltonian cycles. To start each iteration set $l \in [k]$ such that $l - 1$ is the number of monochromatic Hamiltonian cycles already constructed. We call an edge of color $j < l$ protected. These edges will no longer be recolored by our algorithm.

For notational ease let \mathcal{E}_i be the set of endpoints of paths in $\mathcal{P}_i^{(l)}$ and $S'_i := S_i \cup \mathcal{E}_i$. Let further $M_i := \{e \in M : c(e) \in \{0, l\}\}$. Now let $P_{i_l}^{(l)}$ be the longest path in M_l that only uses edges in $M_{i_l}^*$ and covers at least one vertex of S'_{i_l} . Maker builds a sequence of paths $P_{i_l}^{(l)}, P_{i_l+1}^{(l)}, \dots$ such that $V(P_{i_l}^{(l)}) \subseteq V(P_{i_l+1}^{(l)})$ and $P_{i_l+1}^{(l)}$ is the longest path that uses only edges in M_l .

Case 1: Every troublesome vertex x has a degree of $d_M^*(x) \geq 34k$.

Case 1.1: There is a non-troublesome vertex $v \in S_{i-1}$ with $d'_M(v) < 34k$. Then

- choose $w^* \in \{w \in S_0 : (v, w) \notin B \cup M\}$ uniformly at random, add (v, w^*) to M and set $c(\{v, w\}) = 0$;
- set $S_i = S_{i-1}$ and $\mathcal{P}_i = \mathcal{P}_{i-1}$.

Case 1.2: Every non-troublesome vertex v in S_{i-1} has a degree of $d'_M(v) \geq 34k$. Then

Case 1.2.1: There is some endpoint $v \in \mathcal{E}_i$ of an l -colored path, which satisfies $d'_M(v) < 34k$. Then

- choose $w^* \in \{w \in S_0 : (v, w) \notin B \cup M\}$ uniformly at random, add (v, w^*) to M and set $c(\{v, w\}) = j$.
- set $S_i = S_{i-1}$ and $\mathcal{P}_i = \mathcal{P}_{i-1}$.

Case 1.2.2:(Prolongation of P_i using rotations)

Every $v \in \mathcal{E}_i$ satisfies $d'_M(v) \geq 34k$.

Case 1.2.2.1: There is no cycle of M_l with vertex set $V(P_i)$ but there exists a path of M_i with vertex set $V(P_i^{(l)})$ and endpoints u, v in S'_i such that $\{u, v\}$ is unclaimed,

- add (u, v) to M and set $c(\{u, v\}) = j$;
- set $S_i = S_{i-1}$ and $\mathcal{P}_i^{(l)} = \mathcal{P}_{i-1}^{(l)}$.

Case 1.2.2.2: Else

- if there is a Hamilton cycle $H \subseteq M_l$,
 - for each edge e on H set $c(e) = l$, protect H , set $l \leftarrow l + 1$, and $i_l^* = i - 1$.
 - Now if $l = k + 1$ end the game. Otherwise repeat stage 2 with the updated values.
- Otherwise end the game reporting failure.

Case 2: There is a troublesome vertex $v \in S_{i-1}$ such that $d_M^*(v) < 34k$. Select v to be a troublesome vertex $v \in S_{i-1}$ such that $d_{M_l}^* < 34k$, having maximum danger (with arbitrarily broken ties). Then

Case 2.1: There is no $w \in V - S_{i-1}$ such that (v, w) is not in $B \cup M$, then set $i_1^* = i$ and end the game.

Case 2.2: Otherwise

- choose such a w , add (v, w) to M and set $c(\{v, w\}) = 0$;
- set $S_i = S_{i-1} \cup \{w\}$ and for each $j \in [k]$ set \mathcal{P}_i^j to be the set obtained from \mathcal{P}_{i-1}^j by deleting the path P that contained w and adding the components of $P - w$.

Remark 5.10 Let $j \in [k]$. Note that paths in $\mathbb{P}^{(j)}$ only grow if case 1.2.1 applies and two available paths are connected. Since paths are only available if both their lengths are at most $\log(n)$. Thus in each step each element of $\mathbb{P}^{(j)}$ are of length at most $\log(n)$.

5.3 Bounding the Number of Rounds played

In this section we show that the game ends in $kn + o(n)$ moves.

We set t to be the number of vertices that become troublesome over the course of the game and f the number of turns played in total.

Lemma 5.11 *The total number of turns carried out in the cases 2 of either stage is at most $34 \cdot k \cdot t$.*

Proof. In these cases, edges are added to a troublesome vertex v with $d_M^*(v) < 34k$. We add one edge per turn. Thus, the number of turns spent on each troublesome vertex is at most $34k$. Since there are t troublesome vertices, the number of rounds spent in these cases is at most $34kt$. \square

Lemma 5.12 *The number of turns carried out in cases 1.1 of either stage is at most $34k \cdot |S_f|$.*

Proof. In case 1.1 of either stage every troublesome vertex v has Maker degree $d_M^*(v) = 34k$. Note that for each $i \leq f$, $S_i \subseteq S_f$. For each non-troublesome vertex v we increase $d_M'(v)$ to $34k$, adding one edge per turn. In the worst case all vertices in S_f are non-troublesome and their degrees d_M' have to be increased in these steps. Thus the number of turns is at most $34k \cdot |S_f|$. \square

Lemma 5.13 *The number of turns carried out in case 1.2 of Stage 1 is at most $k \cdot n$.*

Proof. An edge is only added in this case if there are two vertices u, v with which are (u, v) -available in $\mathcal{P}_i^{(j)}$. Thus, their degree with respect to j -colored edges is at most one, hence they are not in the same j -colored component during turn i . Thus, for each $j \in [k]$, $\mathcal{P}_i^{(j)}$ induces a graph of maximum degree 2 that has no cycles, so this subgraph is a forest. Furthermore, the edge is only added if neither u nor v have a neighbor in S_i , since the paths $[v]_j^{(i)}$ and $[u]_j^{(i)}$ are (u, v) -available. Thus, neither u nor v have been the neighbor of a vertex that became troublesome. Therefore, there are at most n edges added to some $\mathcal{P}_i^{(j)}$ and thus, at most kn edges are added to any element of \mathcal{P}_i over the course of the game. Since $\mathcal{P}_i^{(j)}$ induces a forest, at most n j -colored edges have been added in case 1.2 of Stage 1. \square

Lemma 5.14 *The number of turns carried out in case 1.2.1 of Stage 2 is at most*

$$34k|S_f| + \sum_{j \in [k]} 68k|\mathcal{P}_f^{(j)}|$$

Proof. In case 1.2.1 we choose an endpoint of an l -colored path, say v , and l which is the active color in step i . This vertex satisfies $d_M'(v) < 34k$. So for each of these vertices at most $34k$ rounds are carried out until its Maker degree reaches $34k$. Throughout Stage 2, we only modify $\mathcal{P}_{i-1}^{(l)}$ by moving a vertex v on one of its paths P into S_i and replacing P with the components of $P - v$. This can happen either between rounds, if v becomes troublesome, or in case 2.2 of either stage, if v is chosen as a neighbor of a troublesome vertex. Hence, if there exists an $i > i_0^*$ such that a vertex v is the endpoint of some path in $\mathcal{P}_i^{(j)}$ for some $j \in [k]$ and $i \geq i_0^*$ it

ends in S_f or is an endpoint of some path in $\mathcal{P}_f^{(j)}$. Each path in $\mathcal{P}_f^{(j)}$ has at most 2 endpoints, and thus the upper bound for the number of turns carried out in Case 1.2.1 of Stage 2 is

$$34k|S_f| + \sum_{j \in [k]} 34k \cdot 2 \cdot |\mathcal{P}_f^{(j)}|.$$

□

Lemma 5.15 *For each $j \in [k]$ the number of turns carried out in case 1.2.2.1 of Stage 2 is at most the number of i for which $|P_i| > |P_{i-1}|$. This is at most $|S_f| + 3|\mathcal{P}_f^{(j)}|$. Thus at most $k|S_f| + 3\sum_{j \in [k]} |\mathcal{P}_f^{(j)}|$ turns are carried out in this case.*

Proof. In case 1.2.2.1 of Stage 2 all troublesome vertices u satisfy $d_M^*(u) \geq 34k$. Further, case 1.2 tells us that all non-troublesome elements v of S_{i-1} satisfy $d_M'(w) \geq 34k$ and all endpoints of l -colored paths w satisfy $d_M'(w) \geq 34k$. At the end of Case 1.2.2.1 S_i is updated to be S_{i-1} , so we consider S_i henceforth.

Furthermore, since case 1.2.2.1 applies there exists no cycle on $V(P_{i-1})$. This implies that 1.2.2.1 of Stage 2 was not carried out in some round $k < i$ while $V(P_k) = V(P_{i-1})$, otherwise an edge creating a cycle on $V(P_k)$ would have been added. This means that Maker has succeeded in increasing the length of a longest path in M_l that contains $V(P_k)$. So, there is an i with $|P_i| \geq |P_{i-1}|$. Consider such an i .

Now, consider an interior vertex v of some element P of $\mathcal{P}_f^{(j)}$. Then two of its neighbors are joined to it by j -colored edges. These were chosen during case 1.2.1 of Stage 1. Since v is an interior vertex of P , no edges of color 0 were added to it. Thus, v has degree 2 in M_i and has no neighbors in S_f .

So, if it is in P_i , its neighbors also have to be in P_i . Otherwise v would be an endpoint of P_i and we could increase P_i by adding the second P -neighbor of v . This contradicts the maximality of P_i . Thus, the iteration that adds one interior vertex of P to P_i adds all interior vertices of P to $P_i^{(j)}$, by repetition of this argument. We require at most $|S_f|$ steps to add all vertices of S_f , at most $|\mathcal{P}_f^{(j)}|$ edges to add all internal vertices of elements of $\mathcal{P}_f^{(j)}$ and at most $2|\mathcal{P}_f^{(j)}|$ edges to add all endpoints of elements of $\mathcal{P}_f^{(j)}$. Thus, the number of turns in which $|P_i| > |P_{i-1}|$ while $j \in [k]$ is the active color is at most $|S_f| + |\mathcal{P}_f^{(j)}| + 2|\mathcal{P}_f^{(j)}|$.

The lemma follows by summing over all $j \in [k]$. □

Lemma 5.16 *Let t be the total number of troublesome vertices. The duration of the game, i.e. the number of rounds played is at most*

$$k \cdot n + 69k \cdot |S_f| + 71k \sum_{j \in [k]} |\mathcal{P}_f^{(j)}| + 34k \cdot t \leq (104k + 71k^2) \cdot n.$$

Proof. By the lemmata above the number of moves played is at most

$$\begin{aligned} & 34k \cdot t + 34k \cdot |S_f| + k \cdot n + 34k \cdot |S_f| + \sum_{j \in [k]} 68k|\mathcal{P}_f^{(j)}| + k|S_f| + 3 \sum_{j \in [k]} |\mathcal{P}_f^{(j)}| \\ & \leq k \cdot n + 69k \cdot |S_f| + 71k \sum_{j \in [k]} |\mathcal{P}_f^{(j)}| + 34kt \end{aligned}$$

$$\leq (104k + 71k^2) \cdot n.$$

The last inequality holds because $|S_f|$, t and $|\mathcal{P}_f^{(j)}|$ are all at most n . \square

Lemma 5.17 *The number t of troublesome vertices is at most*

$$t \leq 2(104k + 71k^2) \frac{n}{\sqrt{\log(n)}}.$$

Proof. Since the game lasts for at most $(104k + 71k^2)n$ turns,

$$|B| \leq (104k + 71k^2)n \cdot b \leq \frac{(104k + 71k^2)n^2}{\log(n)}.$$

Furthermore, each troublesome vertex has Breaker degree at least $\frac{n}{\sqrt{\log(n)}}$. In addition each Breaker edge is incident in at most two troublesome vertices. Thus, $t \leq 2|B| \frac{\sqrt{\log(n)}}{n} \leq 2(104k + 71k^2) \frac{n}{\sqrt{\log(n)}}$. \square

Lemma 5.18 $|S_f| < \frac{1000(4k+3)^2\sqrt{k+2n}}{\sqrt{\log(n)}} + 35kt \leq 70.000k^3 \frac{n}{\sqrt{\log(n)}}.$

Proof. Every $v \in (S_f \setminus S_0)$ is either troublesome or one of the first $34k$ out-neighbors chosen from a troublesome vertex and added in case 2 of Stage 1 or Stage 2. So, $|S_f| < |S_0| + 34kt < \frac{1000(4k+3)^2\sqrt{k+2n}}{\sqrt{\log(n)}} + 35kt$. The second inequality is implied by Lemma 5.17. \square

Lemma 5.19 *If Stage 1 ends with case 1.2.2, then for each $j \in [2k]$*

$$|\mathcal{P}_f^{(j)}| < |\mathcal{P}_{i_0^*}^{(j)}| + 35k \cdot t < \frac{2n}{\sqrt{\log(n)}} + 105k \cdot t \leq 210.000k^3 \frac{n}{\sqrt{\log(n)}}.$$

Proof. At the start of Stage 2 there are exactly $|\mathcal{P}_{i_0^*}^{(j)}|$ j -colored paths. The elements of $\mathcal{P}_f^{(j)}$ arise from elements of $|\mathcal{P}_{i_0^*}^{(j)}|$ by either being elements of $|\mathcal{P}_{i_0^*}^{(j)}|$, or by moving troublesome vertices or neighbors of troublesome vertices from $\mathcal{P}_{i-1}^{(j)}$ to $\mathcal{P}_i^{(j)}$ to the set S_i in some step i . Note that the elements of $\mathcal{P}_{i-1}^{(j)}$ were generated in the same way from elements of $\mathcal{P}_{i_0^*}^{(j)}$. This may increase the cardinality.

In the second stage there are only two possibilities in which there is an increase of $|\mathcal{P}_i^{(j)}|$ over $|\mathcal{P}_{i-1}^{(j)}|$. The first possibility is that a troublesome vertex is moved into some S_i between rounds. The second possibility is that the new neighbor say v of a troublesome vertex is moved into some S_i in case 2.2. Since v was moved into S_i in that step, it had at most two j -neighbors. So, the number of paths in $\mathcal{P}_i^{(j)}$ is increased by at most one. This is the case, because the path $[v]_{i-1}^{(j)} = P_1 \circ v \circ P_2$ was replaced with its two components P_1 and P_2 . Since case 2 occurs at most $34k$ times for each troublesome vertex, we have

$$|\mathcal{P}_f^{(j)}| < |\mathcal{P}_{i_0^*}^{(j)}| + t + 34k \cdot t < |\mathcal{P}_{i_0^*}^{(j)}| + 35k \cdot t.$$

with t being the number of troublesome vertices.

For fixed $j \in [k]$ the elements of $\mathcal{P}_{i_0^*}^{(j)}$ are disjoint paths. Furthermore, since $\mathcal{P}_{i_0^*}^{(j)}$ induces a partition of the vertices of $V \setminus S_{i_0^*}$, a vertex of $V \setminus S_{i_0^*}$ cannot be the endpoint of more than one path. This implies that the number of endpoints of j -colored paths is an upper bound for the number of j -colored paths. We continue to bound the number of endpoints of j -colored paths.

If case 1.2.2 was carried out, the prerequisites of case 1.2.1 were not satisfied. That means i_0^* was chosen as an index such that for all $j \in [k]$ there were no more j -colored paths in $\mathcal{P}_{i_0^*}^{(j)}$ that were available. Now consider a fixed vertex $v \in V \setminus S_{i_0^*}$ and $j \in [k]$. Assume that v has j -degree at most one, is not incident in any $(k+1)$ -edge, and v is an endpoint of a j -colored path of size $|V([v]_{i_0^*}^{(j)})| < \log(n)$. Let w be an endpoint of an arbitrary j -colored path $P \in \mathcal{P}_{i_0^*}^{(j)}$. We know that P and $[v]_j^{(i_0^*)}$ are not (v, w) -available, otherwise they would have been connected, because case 1.2.1 would have applied. Since v was not connected to w , w has to satisfy at least one of the following four properties, because otherwise the paths $[v]_{i_0^*}^{(j)}$ and $[w]_{i_0^*}^{(j)}$ would have been (v, w) -available.

Property 1: $|[w]_{i_0^*}^{(j)}| \geq \log(n)$. This means that the j -colored path containing w is too long.

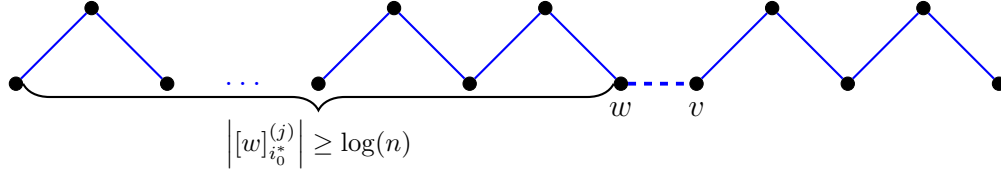


Figure 5.1: The unclaimed edge $\{v, w\}$ will not be claimed by Maker, since the blue path containing w is longer than $\log(n)$.

Property 2: Maker claimed the edge $\{v, w\}$ and its color is $j' \in [k]$.

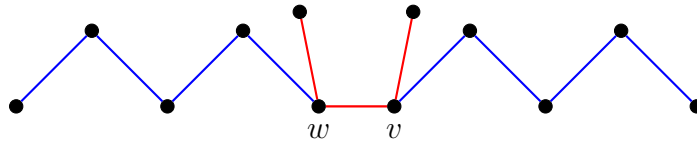


Figure 5.2: Maker cannot claim the edge $\{v, w\}$ for a blue path, because she already chose to use it in a red path.

Property 3: Maker claimed an edge $\{\hat{v}, w\}$ and its color is $k+1$. This means that in some step i prior to step i_0^* , a \hat{v} was moved into $S_i \subseteq S_{i_0^*}$.

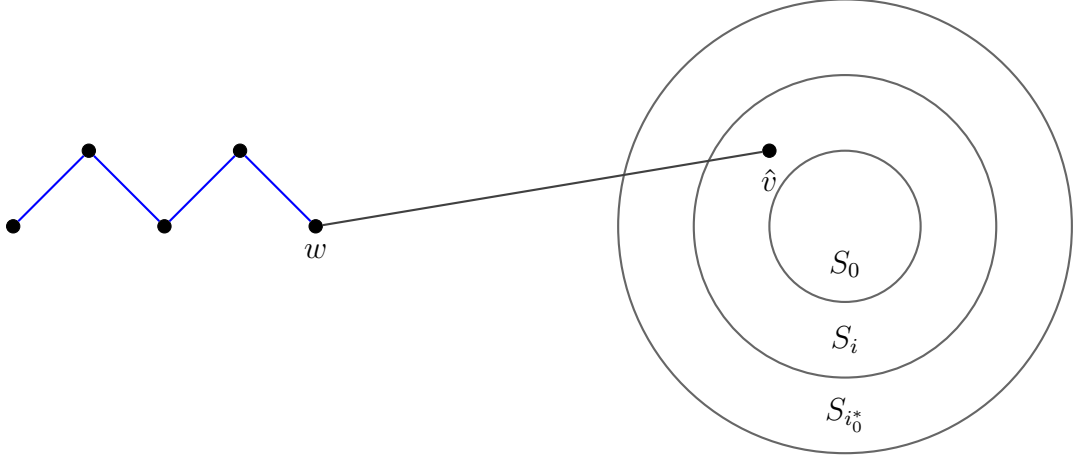


Figure 5.3: A $(k + 1)$ -colored edge is incident in w , joining it to $\hat{v} \in S_i \subseteq S_{i_0}$. This makes w unavailable for further path elongation.

Property 4: Breaker claimed the edge $\{v, w\}$.

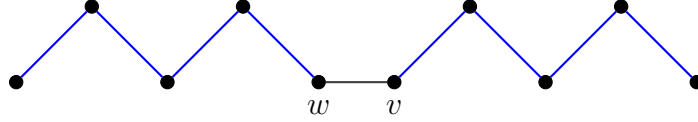


Figure 5.4: Maker cannot claim the edge $\{v, w\}$ as it was already claimed by Breaker.

First, we consider Property 1. Since there are at most n vertices in $V \setminus S_{i_0}^*$, there can be at most $\frac{n}{\log(n)}$ j -colored-paths of size $\log(n)$. Each of them has two endpoints. Thus, at most $\frac{2n}{\log(n)} \leq \frac{n}{2\sqrt{\log(n)}}$ vertices satisfy Property 1.

Property 2: Since there are no troublesome vertices in $V \setminus S_{i_0}^*$, During Stage 1 Maker chooses at most two edges incident in v for each color $j' \in [k]$. Thus at most $2k < \frac{n}{2\sqrt{\log(n)}}$ vertices satisfy Property 2.

Property 3: Maker only colors edges with color $k + 1$ if a troublesome vertex w is moved from a path into some S_i . This can happen between rounds if w becomes troublesome, or in case 2.1. Thus, each vertex that satisfies Property 2 either has a neighbor that is troublesome, or it has a neighbor that is the neighbor of a troublesome vertex. Since each vertex that is moved from a path into some S_i had at most two j -neighbors, at that point, each troublesome vertex induces at most $2(d_M^*(v) + 1)t = 70kt$ vertices that satisfy Property 3.

Property 4: Since v is not troublesome, by definition we have $\deg_B(v) \leq \frac{n}{\sqrt{\log(n)}}$. This implies that the number of vertices that satisfy Property 4 is at most $\frac{n}{\sqrt{\log(n)}}$.

Together these imply that

$$|\mathcal{P}_{i_0}^{(j)}| \leq 70kt + \frac{n}{\sqrt{\log(n)}} + \frac{n}{2\sqrt{\log(n)}} + \frac{n}{2\sqrt{\log(n)}} = 70kt + \frac{2n}{\sqrt{\log(n)}}.$$

□

Lemma 5.20 *If Stage 1 ends in Case 1.2.2, then the game ends after $kn + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ rounds and $t \leq \frac{(2k+1)n}{\sqrt{\log(n)}}$.*

Proof. By Lemma 5.16 the game ends after at most

$$k \cdot n + 69k \cdot |S_f| + 71k \sum_{j \in [k]} |\mathcal{P}_f^{(j)}| + 34k \cdot t$$

turns. By Lemma 5.18 $|S_f| = \mathcal{O}\left(\frac{k^3 n}{\sqrt{\log(n)}}\right) = \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$, by Lemma 5.19 $|\mathcal{P}_f^{(j)}| = \mathcal{O}\left(\frac{k^3 n}{\sqrt{\log(n)}}\right) = \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ for every $j \in [k]$ and $t = \mathcal{O}\left(\frac{k^2 n}{\sqrt{\log(n)}}\right) = \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ by Lemma 5.17. Thus, the number of rounds played is at most $k \cdot n + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right) \leq (k + \frac{1}{2})n$. This implies Breaker claimed at most $|B| < (k + \frac{1}{2}) \frac{n^2}{\log(n)}$ edges. In the proof of Lemma 5.17 we observed that $t \leq \frac{2|B|\sqrt{\log(n)}}{n}$. Thus,

$$t \leq \frac{2|B|\sqrt{\log(n)}}{n} \leq \frac{(2k+1)n}{\sqrt{\log(n)}}.$$

□

Corollary 5.21 *The following inequalities hold:*

1. $|S_f| < \frac{(1000(4k+3)^2 \sqrt{k+2} + 2k+1)n}{\sqrt{\log(n)}} < \frac{(1001(4k+3)^2 \sqrt{k+2})n}{\sqrt{\log(n)}}.$
2. For each $j \in [k]$, we have $|\mathcal{P}_f^{(j)}| < \frac{(2+105k(2k+1))n}{\sqrt{\log(n)}} < \frac{500k^2 n}{\sqrt{\log(n)}}.$

This is proved by inserting $t \leq \frac{(2k+1)n}{\sqrt{\log(n)}}$ in the first inequality of Lemma 5.18, and in the second inequality of Lemma 5.19, respectively.

5.4 Verifying Maker's Strategy

Lemma 5.22 *The constant $C > 0$ in the bias in the theorem of Brüstle et al., Theorem 5.5, can be chosen such that throughout the game every troublesome vertex with $d_M^*(v) < 34k$ satisfies $d_B(v) < n - \frac{(4k+3)^3 n + 2k+1}{\sqrt{\log(n)}}.$*

Proof. See Brüstle et al. Lemma 4.7, where it is made clear that the C_t can be chosen as large as necessary instead of their value of 5000. Alternatively, one can reprove this result, but the proof would be the same as in Brüstle et al. up to the last sentence. □

We need results that are analoga of some technical lemmata of [BCN⁺23] with the exception that up to $2k$ edges of color 0 can be deleted in each vertex. This represents that we cannot use edges that were already protected by belonging to an "earlier" Hamiltonian cycle.

This requires that Maker keeps track of the edge that are already earmarked for a specific Hamilton cycle by building the $\mathcal{P}^{(j)}$ as edge disjoint. This approach further requires that the S_i do not lose their connectedness and expander-like properties after the edges of at most $2k$ edges are "protected" in each vertex. This will be achieved by increasing the Maker out-degree of vertices in S_i .

Assume that $l \in [k]$ is the active color during turn $i \geq i_0^*$. Recall that S'_i is the union of S_i and all endpoints of paths in $\mathcal{P}_i^{(l)}$. Further $S_0 \subseteq V$ was chosen at the beginning of the algorithm *Hamilton-Cycles* arbitrary with

$$|S_0| = \left\lceil \frac{1000(4k+3)^2\sqrt{k+2n}}{\sqrt{\log(n)}} \right\rceil \geq \frac{2004n}{\sqrt{\log(n)}} \quad (5.1)$$

For each $Z \subseteq S'_i$ define

$$N'(Z) := \{v \in S_i \setminus Z : \exists u \in Z \text{ s.t., the edge } (u, v) \text{ was chosen by Maker and } c((u, v)) = 0\}.$$

Lemma 5.23 *Consider $i \geq i^*$ such that Case 1.2.2 applies. Then with probability $1 - o(1)$ for every set $S \subseteq S_i$ that contains no troublesome vertices and that satisfies $|S| \leq \frac{|S_i|}{2}$, we have*

$$|N'(S)| > \min \left(\frac{|S_i|}{2} - |Z|, \frac{2k+1}{30k-1} |Z| \right),$$

if at most $2k$ out-edges have been removed from each vertex in S .

Proof. Let $Z \subseteq S_i$ with $|Z| \leq \frac{|S_i|}{2}$. Note that by Corollary 5.21, and the chosen cardinality of S_0 ,

$$\frac{|S_i|}{2} \leq \frac{|S_f|}{2} \leq \frac{1001|S_0|}{2000}. \quad (5.2)$$

Case 1 : $|Z| \geq \frac{|S_0|}{4}$.

Consider a vertex $v \in S$ and an arbitrary set of $32k$ 0-colored out-edges incident in v . Note that these edges have been chosen in case 1.1 of either stage. This choice was made uniformly at random between all vertices $w \in S_0$, such that (v, w) was unclaimed. Let further $B \subseteq S_0 \setminus |Z|$ with $|B| = \left\lfloor \frac{|S_i|}{2} \right\rfloor - |Z|$. Let $A_{v,i} \subseteq S_0$ be the set of all vertices w such that $\{v, w\}$ was unclaimed by either player in turn i . Since v is not troublesome, at most $\frac{n}{\sqrt{\log(n)}}$ Breaker edges are incident in v . Furthermore, at most $34k \leq \frac{n}{\sqrt{\log(n)}}$ Maker-edges are incident in v . This implies $|A_{v,i}| \geq |S_0| - \frac{2n}{\sqrt{\log(n)}}$.

Thus, the probability that each edge chosen from v (in some step in which Case 1.1 applied) ends in $B \cup Z$, is at most

$$\left(\frac{|B \cup Z|}{|A|} \right)^{32k} \leq \left(\frac{|B| + |Z|}{|S_0| - \frac{2n}{\sqrt{\log(n)}}} \right)^{32k} \leq \left(\frac{|S_i|}{2(|S_0| - \frac{2n}{\sqrt{\log(n)}})} \right)^{32k}$$

$$\begin{aligned}
 (5.2) \quad & \left(\frac{1001|S_0|}{4000(|S_0| - \frac{2n}{\sqrt{\log(n)}})} \right)^{32k} \leq \left(\frac{1001|S_0|1002}{4000|S_0|1001} \right)^{32k} \\
 & \leq \left(\frac{501}{2000} \right)^{32k}
 \end{aligned} \tag{5.3}$$

The second last inequality holds because by inequality (5.1), $|S_0| \geq \frac{2004n}{\sqrt{\log(n)}}$, and thus $|S_0| - \frac{2n}{\sqrt{\log(n)}} \geq \frac{1001|S_0|}{1002}$.

The probability that there exists a set of $2k$ out-edges such that the endpoints of all remaining $32k$ out-edges from v are all in $B \cup Z$ is at most

$$\begin{aligned}
 \binom{34k}{2k} \left(\frac{501}{2000} \right)^{32k} & \leq \left(\frac{34ke}{2k} \right)^{2k} \left(\frac{501}{2000} \right)^{32k} \quad (\text{with 5.3}) \\
 & \leq (17e)^{2k} \left(\frac{501}{2000} \right)^{32k} \\
 & = (17e)^{\frac{32k}{16}} \left(\frac{501}{2000} \right)^{32k} \\
 & \leq \left(\frac{3}{2} \right)^{32k} \left(\frac{501}{2000} \right)^{32k} \\
 & = \left(\frac{1503}{4000} \right)^{32k}
 \end{aligned}$$

Thus, the probability that the statement above is true for all vertices in Z is at most

$$\left(\frac{1503}{4000} \right)^{32k|Z|},$$

and the probability that there exist Z and B with the properties mentioned above is at most

$$\begin{aligned}
 & \binom{|S_i|}{|Z|} \binom{|S_0|}{|B|} \left(\frac{1503}{4000} \right)^{32k|Z|} \\
 & \leq \left(\frac{|S_i|e}{|Z|} \right)^{|Z|} \left(\frac{|S_0|e}{|B|} \right)^{|B|} \left(\frac{1503}{4000} \right)^{32k|Z|} \\
 & \leq \left(\frac{|S_i|e}{|Z|} \right)^{|Z|} \left(\frac{2|S_0|e}{|S_0|} \right)^{\frac{|S_0|}{2}} \left(\frac{1503}{4000} \right)^{32k|Z|} \\
 & \leq \left(\frac{|S_i|e}{|Z|} \right)^{|Z|} (2e)^{2|Z|} \left(\frac{1503}{4000} \right)^{32k|Z|} \quad (\text{since } |Z| \geq \frac{|S_0|}{4}) \\
 & \leq (2e)^{|Z|} (2e)^{2|Z|} \left(\frac{1503}{4000} \right)^{32k|Z|} \\
 & \leq (2e)^{3|Z|} \left(\frac{1503}{4000} \right)^{32k|Z|} \\
 & \leq \left(\frac{5 \cdot 1503}{4 \cdot 4000} \right)^{32k|Z|}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\frac{1}{2}\right)^{|Z|} && (k \geq 1) \\
 &< 2^{-\frac{n}{\sqrt{\log(n)}}} && (\text{since } |Z| \leq \frac{|S_0|}{4} > \frac{n}{\sqrt{\log(n)}}) \\
 &= o\left(\frac{n}{\sqrt{\log(n)}}\right).
 \end{aligned}$$

Case 2 : $|Z| \leq \frac{|S_0|}{4}$.

Consider a vertex $v \in S$ and an arbitrary set of $32k$ 0-colored out-edges incident in v . Note that these edges have been chosen in case 1.1 of either stage. This choice was made uniformly at random between all vertices $w \in S_0$, such that (v, w) was unclaimed. Let further $B \subseteq S_0 \setminus Z$ with $|B| = \left\lfloor \frac{2k+1}{30k-1} |Z| \right\rfloor$. Let $A_{v,i} \subseteq S_0$ be the set of all vertices w such that $\{v, w\}$ was unclaimed by either player in turn i . Since v is not troublesome, at most $\frac{n}{\sqrt{\log(n)}}$ Breaker edges are incident in v . Furthermore, at most $34k \leq \frac{n}{\sqrt{\log(n)}}$ Maker-edges are incident in v . This implies $|A_{v,i}| \geq |S_0| - \frac{2n}{\sqrt{\log(n)}}$.

$$\begin{aligned}
 \left(\frac{|B \cup Z|}{|A|}\right)^{32k} &\leq \left(\frac{|B| + |Z|}{|S_0| - \frac{2n}{\sqrt{\log(n)}}}\right)^{32k} \\
 &\leq \left(\frac{32k}{30k-1} \cdot \frac{|Z|}{(|S_0| - \frac{2n}{\sqrt{\log(n)}})}\right)^{32k} \\
 &\leq \left(\frac{32}{29} \cdot \frac{1000|Z|}{999|S_0|}\right)^{32k}.
 \end{aligned}$$

The last inequality holds, because by inequality (5.1) $|S_0| \geq \frac{2000n}{\sqrt{\log(n)}}$ and thus

$$|S_0| - \frac{2n}{\sqrt{\log(n)}} \geq \frac{999}{1000} |S_0|. \quad (5.4)$$

Therefore, the probability that there exists a set of $2k$ out-edges such that the endpoints of all remaining $32k$ out-edges from v all end in $B \cup Z$ is at most

$$\begin{aligned}
 \binom{34k}{2k} \left(\frac{32}{29} \cdot \frac{1000|Z|}{999|S_0|}\right)^{32k} &\leq \left(\frac{34ke}{2k}\right)^{2k} \left(\frac{32}{29} \cdot \frac{1000|Z|}{999|S_0|}\right)^{32k} \\
 &= (17e)^{2k} \cdot \left(\frac{32}{29} \cdot \frac{1000|Z|}{999|S_0|}\right)^{32k} \\
 &= \left((17e)^{\frac{1}{16}} \cdot \frac{32}{29} \cdot \frac{1000|Z|}{999|S_0|}\right)^{32k} \\
 &< \left(\frac{3|Z|}{2|S_0|}\right)^{32k}.
 \end{aligned}$$

So, the probability that the statement above is true for all vertices in Z is at most

$$\left(\frac{3|Z|}{2|S_0|}\right)^{32k|Z|},$$

and the probability that there exist Z and B with the properties mentioned above is at most

$$\begin{aligned} & \binom{|S_i|}{|Z|} \binom{|S_0|}{|B|} \left(\frac{3|Z|}{2|S_0|}\right)^{32k|Z|} \\ & \leq \left(\frac{|S_i|e}{|Z|}\right)^{|Z|} \left(\frac{|S_0|e}{|B|}\right)^{|B|} \left(\frac{3|Z|}{2|S_0|}\right)^{32k|Z|} \\ & \leq \left(\frac{1001|S_0|e}{1000|Z|}\right)^{|Z|} \left(\frac{|S_0|e(30k-1)}{(2k+1)|Z|}\right)^{\frac{2k+1}{30k-1}|Z|} \left(\frac{3|Z|}{2|S_0|}\right)^{32k|Z|} \quad (\text{by definition of } B) \\ & \leq \left(\frac{1001e|S_0|}{1000|Z|}\right)^{|Z|} \left(\frac{e(30k-1)}{2k+1}\right)^{\frac{2k+1}{30k}} \left(\frac{|S_0|}{|Z|}\right)^{\frac{2k+1}{30k-1}} \left(\frac{3}{2}\right)^{32k} \left(\frac{|Z|}{|S_0|}\right)^{32k|Z|} \\ & \leq \left(\frac{1001e|S_0|}{1000|Z|}\right)^{|Z|} (15e)^{\frac{2k+1}{30k}} \left(\frac{|S_0|}{|Z|}\right)^{\frac{2k+1}{30k-1}} \left(\frac{3}{2}\right)^{32k} \left(\frac{|Z|}{|S_0|}\right)^{32k|Z|} \\ & = \left(\frac{1001e}{1000}\right)^{|Z|} (15e)^{\frac{2k+1}{30k}} \left(\frac{3}{2}\right)^{32k} \left(\frac{|S_0|}{|Z|}\right)^{\frac{2k+1}{30k-1}} \frac{|S_0|}{|Z|} \left(\frac{|Z|}{|S_0|}\right)^{32k|Z|} \\ & = \left(\frac{1001e}{1000}\right)^{|Z|} (15e)^{\frac{2k+1}{30k}} \left(\frac{3}{2}\right)^{32k} \left(\frac{|Z|}{|S_0|}\right)^{-\frac{2k+1}{30k-1}} \left(\frac{|Z|}{|S_0|}\right)^{-1} \left(\frac{|Z|}{|S_0|}\right)^{32k|Z|} \\ & \leq \left(\frac{1001e}{1000}\right)^{|Z|} (15e)^{\frac{2k+1}{30k}} \left(\frac{3}{2}\right)^{32k} \left(\frac{|Z|}{|S_0|}\right)^{30k|Z|}. \end{aligned}$$

Case 2.1 : $|Z| \geq \sqrt{n}$, then

$$\begin{aligned} & \left(\frac{1001e}{1000}\right)^{|Z|} (15e)^{\frac{2k+1}{30k}} \left(\frac{3}{2}\right)^{32k} \left(\frac{|Z|}{|S_0|}\right)^{30k|Z|} \\ & \leq \left(\frac{1001e}{1000}\right)^{|Z|} (15e)^{\frac{2k+1}{30k}} \left(\frac{3}{2}\right)^{32k} \left(\frac{1}{4}\right)^{30k|Z|} \quad (\text{since } |Z| \leq \frac{|S_0|}{4}) \\ & \leq \left(\underbrace{\frac{1001e}{1000} 15e \left(\frac{3^{32}}{2^{32}4^{30}}\right)^k}_{\leq \frac{1}{2}}\right)^{|Z|} \\ & \leq 2^{-|Z|} \leq 2^{-\sqrt{n}} = o\left(\frac{n}{\log(n)}\right). \end{aligned}$$

Case 2.2 : $|Z| \leq \sqrt{n}$. Recall that by definition $|S_0| \geq \frac{n}{\sqrt{\log(n)}}$, thus

$$\begin{aligned} & \left(\frac{1001e}{1000} (15e)^{\frac{2k+1}{30k}} \left(\frac{3}{2} \right)^{32k} \left(\frac{|Z|}{|S_0|} \right)^{30k} \right)^{|Z|} \\ & \leq \left(c \frac{\sqrt{n} \sqrt{\log(n)}}{n} \right)^{30k|Z|} \quad (\text{for some constant } c > 0) \\ & \leq \left(c \frac{\sqrt{\log(n)}}{\sqrt{n}} \right)^{30k|Z|} = o \left(\frac{n}{\sqrt{\log(n)}} \right). \end{aligned}$$

Thus, the proof is completed by summing over all of these probabilities.

For $s \leq \lfloor \frac{|S_i|}{2} \rfloor$ define A_s as the event that there exists a set $Z \subseteq S_i$ of size k that contains no troublesome vertices and that satisfies $N'(Z) \leq \frac{|S_i|}{2} - |Z|$. We further define B_s as the event that there exists a set $Z \subseteq S_i$ of size s that contains no troublesome vertices and that satisfies $N'(Z) \leq \frac{2k+1}{30k-1}|Z|$. Now we have

$$\begin{aligned} \sum_{s=1}^{\frac{|S_i|}{2}} \mathbb{P}(A_s \cap B_s) &= \sum_{s=1}^{\lceil \sqrt{n} \rceil} \mathbb{P}(A_s \cap B_s) + \sum_{s=\lceil \sqrt{n} \rceil}^{\frac{|S_i|}{4}} \mathbb{P}(A_s \cap B_s) + \sum_{s=\frac{|S_i|}{4}}^{\frac{|S_i|}{2}} \mathbb{P}(A_s \cap B_s) \\ &\leq \sum_{s=1}^{\lceil \sqrt{n} \rceil} \mathbb{P}(B_s) + \sum_{s=\lceil \sqrt{n} \rceil}^{\frac{|S_i|}{4}} \mathbb{P}(B_s) + \sum_{s=\frac{|S_i|}{4}}^{\frac{|S_i|}{2}} \mathbb{P}(A_s) \\ &\leq \sum_{s=1}^{\lceil \sqrt{n} \rceil} o \left(\frac{n}{\sqrt{\log(n)}} \right) + \sum_{s=\lceil \sqrt{n} \rceil}^{\frac{|S_i|}{4}} o \left(\frac{n}{\sqrt{\log(n)}} \right) + \sum_{s=\frac{|S_i|}{4}}^{\frac{|S_i|}{2}} o \left(\frac{n}{\sqrt{\log(n)}} \right) \\ &\leq \frac{|S_f|}{2} \cdot o \left(\frac{n}{\sqrt{\log(n)}} \right) \quad (\text{since } |S_i| \leq |S_f|) \\ &\leq \frac{1001(4k+3)^3 \sqrt{k+2}n}{\sqrt{\log(n)}} \cdot o \left(\frac{n}{\sqrt{\log(n)}} \right) \quad (\text{by Corollary 5.21}) \\ &= o(1). \end{aligned}$$

Thus with probability $1 - o(1)$, each set $Z \subseteq S_i$ with $|Z| \leq \frac{|S_i|}{2}$ satisfies $|N'(Z)| > \frac{|S_i|}{2} - |Z|$, or $|N'(Z)| \leq \frac{2k+1}{30k-1}|Z|$ and the result is proved. \square

Lemma 5.24 *With probability $1 - o(1)$, for every $i \geq i^*$ in which case 1.2.2 applies and for every $Z \subseteq S_i$ which contains no troublesome vertex and for which $|Z| \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} \leq \frac{|S_0|}{1.000(4k+3)^2}$, we have*

$$N'(Z) > (4k+2)|Z|.$$

Proof. Let $Z \subseteq S_i$ with $|Z| \leq \frac{|S_0|}{1.000(4k+3)^2}$ such that Z does not contain any troublesome vertices. Let further $B \subseteq S_0 \setminus Z$ with $|B| = (4k+2)|Z|$. Note that every

out-neighbor of such a non-troublesome vertex $v \in Z$ was chosen during case 1.1 during either stage. The neighbors of non-troublesome vertices are chosen uniformly at random from all vertices in S_0 that are not yet joined to v . Let $A \subseteq S_0$ be the set of vertices w such that the edge $\{v, w\}$ was not claimed by neither Breaker nor Maker. Since v is non-troublesome Breaker has chosen at most $\frac{n}{\sqrt{\log(n)}}$ edges that are incident in v . Furthermore, Maker has chosen at most $34k \leq \frac{n}{\sqrt{\log(n)}}$ edges incident in v . This implies $|A| \geq |S_0| - \frac{2n}{\sqrt{\log(n)}}$

Now the probability that from a fixed vertex v that all its $34k$ out-neighbors are in $Z \cup B$ is at most

$$\left(\frac{|B \cup Z|}{|A|} \right)^{34k} \leq \left(\frac{|B| + |Z|}{|S_0| - \frac{2n}{\sqrt{\log(n)}}} \right)^{34k} = \left(\frac{(4k+3)|Z|}{|S_0| - \frac{2n}{\sqrt{\log(n)}}} \right)^{34k} \stackrel{(5.4)}{\leq} \left(\frac{(4k+3)|Z|1000}{|S_0|999} \right)^{34k}$$

So, the probability that every out-neighbor of every vertex in Z end up in $Z \cup B$ is at most

$$\left(\frac{(4k+3)|Z|1000}{|S_0|999} \right)^{34k|Z|}.$$

Thus, the probability that there exist Z and B as above is at most

$$\begin{aligned} & \binom{|S_i|}{|Z|} \binom{|S_0|}{|B|} \left(\frac{(4k+3)|Z|1000}{|S_0|999} \right)^{34k|Z|} \\ & \leq \left(\frac{|S_i|e}{|Z|} \right)^{|Z|} \left(\frac{|S_0|e}{|B|} \right)^{|B|} \left(\frac{(4k+3)|Z|1000}{|S_0|999} \right)^{34k|Z|} \\ & \leq \left(\frac{1001}{1000} \frac{|S_0|e}{|Z|} \right)^{|Z|} \left(\frac{|S_0|e}{|B|} \right)^{|B|} \left(\frac{(4k+3)|Z|1000}{|S_0|999} \right)^{34k|Z|} \\ & = \left(\frac{1001}{1000} \frac{|S_0|e}{|Z|} \right)^{|Z|} \left(\frac{|S_0|e}{(4k+2)|Z|} \right)^{(4k+2)|Z|} \left(\frac{(4k+3)|Z|1000}{|S_0|999} \right)^{34k|Z|} \\ & = \left(\frac{1001e}{1000} \left(\frac{e}{(4k+2)} \right)^{(4k+2)} \left(\frac{(4k+3)1000}{999} \right)^{34k} \left(\frac{|Z|}{|S_0|} \right)^{27k} \right)^{|Z|}. \end{aligned}$$

Now assume that $|Z| \geq \sqrt{n}$. Then we have

$$\begin{aligned} & \left(\frac{1001e}{1000} \left(\frac{e}{(4k+2)} \right)^{(4k+2)} \left(\frac{(4k+3)1000}{999} \right)^{34k} \left(\frac{|Z|}{|S_0|} \right)^{27k} \right)^{|Z|} \\ & \leq \left(\frac{1001e}{1000} \left(\frac{e}{(4k+2)} \right)^{(4k+2)} \left(\frac{(4k+3)1000}{999} \right)^{34k} \left(\frac{1}{1.000(4k+3)^2} \right)^{27k} \right)^{|Z|} \\ & \leq \left(\frac{1001}{1000} e^{(4k+3)} \left(\frac{1000}{999} \right)^{34k} \left(\frac{1}{1.000} \right)^{27k} \left(\frac{1}{(4k+3)^{20k}(4k+2)^{(4k+2)}} \right) \right)^{|Z|} \\ & \leq 2^{-\sqrt{n}} = o\left(\frac{n}{\sqrt{\log(n)}} \right). \end{aligned}$$

On the other hand, if we assume that $|Z| < \sqrt{n}$, then the probability that there exist Z and B as above is at most

$$\begin{aligned} & \left(\frac{1001e}{1000} \left(\frac{e}{(4k+2)} \right)^{(4k+2)} \left(\frac{(4k+3)1000}{999} \right)^{34k} \left(\frac{|Z|}{|S_0|} \right)^{27k} \right)^{|Z|} \\ & \leq \left(c \frac{\sqrt{\log(n)}}{\sqrt{n}} \right)^{27k} \quad (\text{for some } c > 0) \\ & = o \left(\frac{n}{\sqrt{\log(n)}} \right). \end{aligned}$$

For $s \in [k]$ let A_s be the event that there exist sets Z and B as above s with $|Z| = s$. Then the probability that there exists a sets Z with $|Z| \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$ and $N'(Z) \leq (4k+2)|Z|$ is at most

$$\begin{aligned} & \mathbb{P} \left(\bigcup_{s \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}} A_s \right) \leq \sum_{s \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}} \mathbb{P}(A_s) \quad (\text{by the union bound}) \\ & = \sum_{s=1}^{\lceil \sqrt{n} \rceil} \mathbb{P}(A_s) + \sum_{s=\lceil \sqrt{n} \rceil+1}^{\lfloor \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} \rfloor} \mathbb{P}(A_s) \\ & \leq \sum_{s=1}^{\lceil \sqrt{n} \rceil} o \left(\frac{n}{\sqrt{\log(n)}} \right) + \sum_{s=\lceil \sqrt{n} \rceil+1}^{\lfloor \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} \rfloor} o \left(\frac{n}{\sqrt{\log(n)}} \right) \\ & \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} \cdot o \left(\frac{n}{\sqrt{\log(n)}} \right) = o(1). \end{aligned}$$

This proves the lemma. \square

The next result allows us to ensure that for each $i \geq i^*$, if case 1.2.2 applies, the graph induced by S_i is connected, even if up to $2k$ edges have been removed from each vertex. The $2k$ edges that are removed in these steps represent the edges that form the k Hamiltonian cycles that might already have been protected and can thus no longer be used to form other Hamilton cycles.

Lemma 5.25 *For every $i \geq i^*$ in which case 1.2.2 applies and for each $Z \subseteq S_i$, with $|Z| \leq \frac{|S_i|}{2}$ the set $|N'(Z)|$ is non empty with probability $1 - o(1)$ if not more than $2k$ edges have been removed from each vertex. This implies S_i is connected if not more than $2k$ edges have been removed from each vertex and thus S_i is $(2k+1)$ -edge-connected with probability $1 - o(1)$.*

Furthermore, if $|Z| \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$, then $|N'(Z)| > 2k|Z|$ with probability $1 - o(1)$.

Proof. Let $T \subset Z$ be the set of troublesome vertices in Z and $S := Z \setminus T$. Let v be a troublesome vertex. Since case 1.2.2 applies, $34k$ edges that are incident in v were added after v became troublesome. All of these edges were added in a step in which case 2.1 of either stage applied. Whenever case 2.1 of either stage applies in a round \hat{i} , a troublesome vertex is connected to a vertex $w \in V \setminus S_i$. In addition the vertex w is moved into S_{i+1} . Thus w cannot be joined to another troublesome vertex during a round in which case 2.1 applies. This implies that $|N'(T)|$ contains at least $34k|T|$ vertices.

$$|Z| + |N'(Z)| \geq |T| + |N'(T)| \geq 32k \cdot |T|.$$

Now assume $|Z| \leq |S_i|/2$. So $|S| \leq |Z| \leq |S_i|/2$.

Case 1 : $|T| > \frac{2k+1}{32k}|Z|$. This means we have a "large" number of troublesome vertices in Z .

Then $|Z| + |N'(Z)| \geq 32k|T| > (2k+1)|Z|$. This proves the second statement of the lemma. The first statement can be obtained by subtracting $|Z|$ from both sides of the inequality, which yields $|N'(Z)| > 2k|Z| > 0$. Thus $N'(Z) \neq \emptyset$.

Case 2 : $|T| \leq \frac{2k+1}{32k}|Z|$. We first have

$$|S| = |Z| - |T| \geq |Z| - \frac{2k+1}{32k}|Z| = \frac{30k-1}{32k}|Z|. \quad (5.5)$$

Then, since S does not contain any troublesome vertices and $|S| \leq |S_i|/2$ as observed above, by Lemma 5.23 with probability $1 - o(1)$ we get

$$|N'(S)| > \frac{2k+1}{30k-1}|S| \text{ or } |N'(S)| > \frac{|S_i|}{2} - |S|. \quad (5.6)$$

using the assumption $|Z| \leq |S_i|/2$.

This implies, using inequality (5.5),

$$|Z| + |N'(Z)| \geq |S| + |N'(S)| > \frac{2k+1}{30k-1}|S| + |S| \geq \frac{32k}{30k-1}|S| \geq |Z|.$$

or

$$|Z| + |N'(Z)| \geq |N'(S)| + |S| > \frac{|S_i|}{2} \geq |Z|.$$

By subtracting $|Z|$ from the left-hand-sides and the right-hand-sides both inequalities imply $|N'(Z)| > 0$ and thus $N'(Z) \neq \emptyset$.

Furthermore, let us assume $|Z| \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$. Then $|S| \leq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$, and since S does not contain troublesome vertices, we can apply Lemma 5.24. So, with probability $1 - o(1)$ we have $|N'(S)| > (4k+1)|S|$. Thus, with probability $1 - o(1)$ we have

$$\begin{aligned} |Z| + |N'(Z)| &\geq |S| + |N'(S)| \\ &> |S| + (4k+1)|S| \end{aligned} \quad (\text{by Lemma 5.24})$$

$$\begin{aligned}
 &\geq (4k+2)|S| \\
 &\geq (4k+2)\frac{30k-1}{32k}|Z| \quad (\text{with (5.5)}) \\
 &\geq (2k+1)|Z|.
 \end{aligned}$$

By subtracting $|Z|$ from both sides of the inequality we get

$$|N'(Z)| > 2k|Z|.$$

□

Theorem 5.26 *Maker can always make a choice as required by the strategy. Furthermore, the game does not end in Case 2 of any stage. Thus stage 1 ends in Case 1.2.2.*

Proof. We first show that Maker can make the required choice in Case 1.1 of either stage. In Case 1.1 of either stage Maker is supposed to connect a non-troublesome vertex in S_i to a vertex in S_0 . Recall that $|S_0| = \left\lceil \frac{1000(4k+3)^2\sqrt{(k+2)n}}{\sqrt{\log(n)}} \right\rceil$. Now if v is a non-troublesome vertex with $d'_M(v) < 34k$, then we know that Maker chose at most $34k$ 0-colored out-edges from v , and at most $2k(k+1)$ -colored edges incident in v . Thus, there are at most $36k$ Maker out-edges incident in v . Since k is constant, clearly $36k < \frac{n}{\sqrt{\log(n)}}$. Furthermore, since v is non-troublesome, by definition Breaker chose at most $\frac{n}{\sqrt{\log(n)}}$ edges incident in v . So,

$$d'_M(v) + d_B(v) \leq 36k + \frac{n}{\sqrt{\log(n)}} < \frac{2n}{\sqrt{\log(n)}}.$$

Hence in both stages Maker can make the desired choice in Case 1.1, since at least

$$|S_0| - \frac{2n}{\sqrt{\log(n)}}((4k+3)^2\sqrt{(k+2)} - 2) - \frac{n}{\sqrt{\log(n)}}$$

vertices in S_0 are not incident in Maker or Breaker edges and available for Maker's connections.

In case 1.2.1 of Stage 2, Maker is required to connect an endpoint v of a path in $\mathcal{P}_i^{(l)}$ with to a vertex in S_0 . Note that v is not in S_i and thus non-troublesome. Therefore, by the same arguments as were used for the case 1.1, there are at least $((4k+3)^2\sqrt{(k+2)} - 2)\frac{n}{\sqrt{\log(b)}}$ vertices in S_0 that are not yet adjacent to v .

To handle case 2 in each of the stages we combine Corollary 5.21 with Lemma 5.22. Maker has to connect v to some vertex in $V \setminus S_{i-1}$. If case 2 applies in any step i , there exists a troublesome vertex v with $d_{M_i}^*(v) < 34k$. By Lemma 5.22, $d_B(v) < n - \frac{n(4k+3)^3+2k+1}{\sqrt{\log(n)}}$. Furthermore, Maker claimed at most $2k$ edges incident in v that were chosen in Case 1.2.1 of stage 1. They were $(k+1)$ -colored in step i . In addition, Maker has chosen at most $34k$ 0-colored edges from v before v became troublesome and at most $34k - 1$ 0-colored edges from v after v became troublesome. Thus, the

amount of Maker out-edges incident in v is at most $70k - 1 < 70k$. Since for all $i \leq f$, $S_i \subseteq S_f$, Corollary 5.21 part 1 yields

$$|S_i| \leq |S_f| < \frac{((4k+3)^2 \sqrt{(k+2)1001})n}{\sqrt{\log(n)}}.$$

So let $A_v \subseteq V \setminus S_{i-1}$ be the set of vertices that can be chosen to be connected to v by Maker in step i . Then

$$\begin{aligned} |A_v| &\geq n - |S_i| - d_B(v) - 70k \\ &\geq n - \frac{((4k+3)^2 \sqrt{(k+2)1001})n}{\sqrt{\log(n)}} - \left(n - \frac{n1000(4k+3)^3 + 2k+1}{\sqrt{\log(n)}}\right) - 70k \\ &\geq \frac{n((4k+3)^3 1000 - ((4k+3)^2 \sqrt{(k+2)1001}))}{\sqrt{\log(n)}} - 70k > 0, \end{aligned}$$

as k is constant.

Thus Maker can always choose an edge in Case 2.1 and the game does not end in Case 2.2. of either step. \square

The idea of using rotations to build Hamiltonian cycles was first introduced by Posa [Pó76]. His technique uses the following fact. Let $P = v_0 v_1 \dots v_x$ be a path of length $x \geq 4$ and $y \leq x - 2$ such that $\{v_x, v_y\} \in E$. Then $P + \{v_x, v_y\} - \{v_y, v_{y+1}\}$ is a path of length x and endpoints v_{y+1} and v . However, the type of rotations that are helpful in our context is limited.

Definition 5.27 Let $P = vv_1 \dots v_x$ be a path with both endpoints in S'_i , and $y \leq x - 2$ such that $\{v_x, v_y\} \in M_l$ with $v_x \in e$. Then we call $P' = P + \{v_x, v_y\} - \{v_y, v_{y+1}\}$ a limited v - M_l -rotation from P . We call a path P'' constructible from P by repeated limited v - M_l -rotations from P , if there exists a series of limited v - M_l -rotations P, \dots, P'' , such that the end-result is P'' .

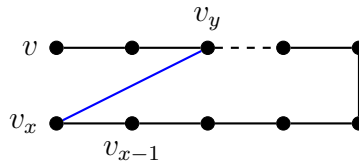


Figure 5.5: Here we depict a limited v - M_l -rotation with the same notation as in Definition 5.27, obtaining P' from P .

This technique allows us to construct a family of paths with the same vertex set and length as $P_i^{(l)}$. By showing that this family is large we will prove that Maker will always be able to close a cycle on the vertices of $P_i^{(l)}$. We begin by showing that at least one of these paths has both of its endpoints in S'_i .

Lemma 5.28 *For every $l \geq i^*$ in which Maker carries out Case 1.2.2, $P_i^{(l)}$ can be chosen to have both endpoints in S'_l .*

Proof. Note that $P_{i-1}^{(l)}$ was defined as the longest path in M_l such that $V(P_{i-1}^{(l)})$ contains at least one vertex of S_{i-1}^* . The path $P_i^{(l)}$ was then constructed recursively by selecting the longest path in M_l that contains all vertices of $P_{i-1}^{(l)}$.

Choose P to be a fixed path on $V(P_i^{(l)})$ that has the most endpoints in S'_i . Assume that an endpoint w of P is in $V \setminus S'_i$. We will deduce a contradiction by constructing a path from P by limited v - M_l -rotations that replaces the endpoint w with an endpoint $x \in S'_i$.

Since $w \in V \setminus S'_i$, it is an interior vertex of some path in $\mathcal{P}_i^{(l)}$, it has degree 2 in M_l . Furthermore, both its M_l neighbors are in P , since otherwise P could be extended by adding the neighbor \hat{w} not yet in P . So one of the following two cases must be true.

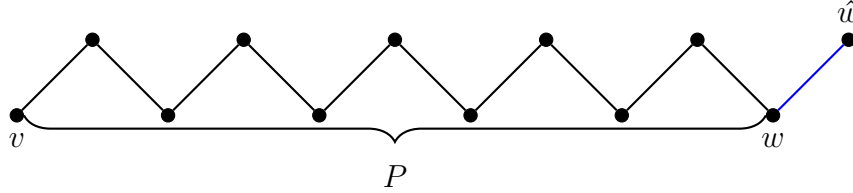


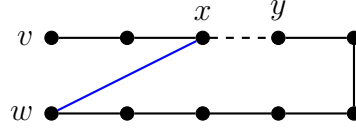
Figure 5.6: The length of P would be increased if it contained \hat{w} and the l -colored (here in blue) edge $\{w, \hat{w}\}$.

Case 1 : w is adjacent to the other endpoint v in M_l .

Then P extends to a circle C via the edge $\{v, w\}$. Now one can construct a path of the same length as P by deleting an arbitrary edge of C . Since by $i_{l-1}^* \leq i$, $V(P_i^{(l)})$ contains $V(P_{i-1}^{(l)})$, it contains an element y of $S_{i-1}^* \subseteq S_i \subseteq S'_i$ and its neighbor $z \in S'_i$. Now $P' = P - \{v, w\} + \{y, z\}$ is a path of the same length as P with both endpoints in S'_i . This contradicts the choice of P as the path with vertex set $V(P_i^{(l)})$ and a maximum number of endpoints in S'_i .

Case 2: w is adjacent to an interior vertex y of P by an M_l -edge that is not in P .

Thus, x has degree at least three in M_l and therefore has to be the endpoint of the element in $\mathcal{P}_i^{(l)}$ that contains w . Now let y be the predecessor of x on P , when starting from w . Consider the path $P' = P - \{x, y\} + \{w, x\}$. P' is of the same length as P , shares one of its endpoints and its other endpoint is y . Since x has only one M_l neighbor in $V \setminus S'_i$, $y \in S'_i$. Therefore, P' has one more endpoint in S'_i , which contradicts the maximality of P .


 Figure 5.7: $y \in S'_i$ replaces $w \notin S'_i$ as an endpoint.

Both cases have lead to a contradiction, so P can be chosen to have both endpoints in S'_i . \square

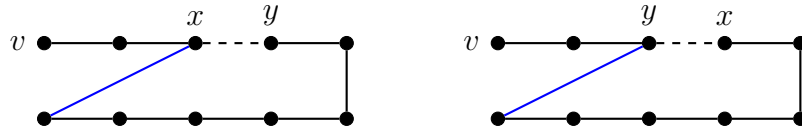
Lemma 5.29 Consider a fixed path P in $V(P_{i+1}^{(l)})$ that has both endpoints v, u in S'_i . Let Z_v be the set of endpoints other than v of paths that are constructible from P by repeated limited v - M_l -rotations. Then $|N'(Z_v)| \leq 2|Z_v|$.

Proof. Note that $u \in Z_v$ and thus $Z_v \neq \emptyset$. We proceed to show that for every w in Z_v and edge $\{w, x\} \in M_l$ such that $x \in S'_i$ there is $x \in Z_v \cup N_P(Z_v)$, where $N_P(Z_v)$ is the set of neighbors of Z_v when only the edges of P are considered. Since P is a path, each vertex $w \in Z_v$ has at most two neighbors in P , which implies $|N'(Z_v)| \leq |N_P(Z_v)| \leq 2|Z_v|$.

Let $w \in Z_v$. If w has no neighbors in S'_i , then $\emptyset = N'(w) \subseteq N_P(Z_v)$ trivially. So, assume there exists $x \in S'_i$ with $\{w, x\} \in M_l$. Consider a shortest sequence of limited v - M_l -rotations from P that results in a path P' with endpoints v and w . We analyze the first rotation in which an edge incident in x is deleted.

Case 1 : Such a rotation exists.

Then the edge $\{x, y\}$ was deleted by the rotation process. Before the rotation no edges incident in x have been removed and the edge $\{x, y\}$ belongs to P . Furthermore, after this rotation either x is left with exactly one edge after the rotation, which makes it an endpoint and thus an element of Z_v , or this rotation left y as an endpoint, which implies $y \in Z_v$ and thus $x \in N_P(Z_v)$, so $x \in nZ_v \cup N_P(Z_v)$ and the claim is proved.


 Figure 5.8: x is an endpoint (left) or a neighbor of an endpoint (right).

Case 2 : No such rotation exists. Then P' is reached without deleting an edge incident in x , and both neighbors of x on P' are neighbors of x on P . If $\{w, x\}$ is on P , the claim is true. Thus, assume that $\{w, x\}$ does not belong to the path P . and thus $\{w, x\}$ does not belong to P' . Since w is an endpoint of P' and P' has maximum length, x must be in P' and the edge $\{w, x\}$ induces a limited v - M_l -rotation. Let $\{x, y\}$ be the edge removed in this rotation, then y is the new endpoint after this rotation. Thus, $y \in Z_v$ and $x \in N_P(Z_v)$. \square

Corollary 5.30 *With probability $1 - o(1)$ for every $l \geq i^*$ in which Case 1.2.2 applies, there are at least $\frac{(k+2)n^2}{2\log(n)}$ pairs of vertices which are the pair of endpoints of a path on $V(P_i^{(l)})$ with probability $1 - o(1)$.*

Proof. Let P be a fixed path on $V(P_i^{(j)})$ with endpoints v, w . By Lemma 5.29 $|N'(Z_v)| < 2|Z_v|$. By Lemma 5.25 the probability that there exists a set $Z \subseteq S_i$ with $|Z| < \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$ and $|N'(Z)| < 2k|Z|$ is $o(1)$. Thus the probability that $|Z_v| \geq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$ is $1 - o(1)$. This argument can be repeated and thus with probability $1 - o(1)$, we have $|Z_u| \geq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}}$ for each $u \in Z_v$.

Now consider $A = \{(x, y) \mid x \in \{v\} \cup Z_v, y \in Z_x\}$. A is the set of pairs of possible endpoints of paths in $V(P_i^{(l)})$. We estimate with probability $1 - o(1)$

$$\begin{aligned}
 |A| &= \sum_{x \in \{v\} \cup Z_v} |\{(x, y) \mid y \in Z_x\}| \geq \sum_{x \in \{v\} \cup Z_v} |Z_x| \\
 &\geq \sum_{x \in \{v\} \cup Z_v} \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} && \text{(with probability } 1 - o(1)) \\
 &> |Z_v| \cdot \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} \\
 &\geq \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} \cdot \frac{\sqrt{(k+2)n}}{\sqrt{\log(n)}} && \text{(again with probability } 1 - o(1)) \\
 &= \frac{(k+2)n^2}{\log(n)}.
 \end{aligned}$$

Each pair in A is counted exactly twice, and thus there exist at least $\frac{(k+2)n^2}{2\log(n)}$ pairs of possible endpoints. \square

Theorem 5.31 *With probability $1 - o(1)$ at the end of stage 2, M contains k edge-disjoint Hamilton cycles.*

Proof. By design the algorithm *Hamilton-Cycles* has to end either in Case 2 of either stage or in Case 1.2.2.2 of stage 2. By Theorem 5.26 with probability $1 - o(1)$ the algorithm does not end in Case 2 of either stage.

Now assume that there exists a turn $i > i_0^*$ in which Case 1.2.2.2 applies and the game ends. We have to consider three cases.

Case 1: There exists a Hamiltonian cycle in M_l .

Since the game ends in this case, we know that this is the k -th Hamiltonian cycle that was formed. Thus, for each $j \in [k]$ there exists a j -colored Hamiltonian cycle and these cycles are pairwise edge-disjoint. Now we have to prove that the second and third case apply with probability $1 - o(1)$.

Case 2: There is no Hamiltonian cycle in M_l but there is an M_l -cycle C with vertex set $V(P_i^{(l)})$.

Recall the following iterative construction. $P_{i_{l-1}^*}$ is the longest M_l path that contains at least one vertex of $S_{i_{l-1}^*}$. Then $P_i^{(l)}$ is chosen to be the longest M_l -path

that contains $V(P_{i-1}^{(l)})$. By its maximality $P_i^{(l)}$ has to contain every vertex in its M_l component.

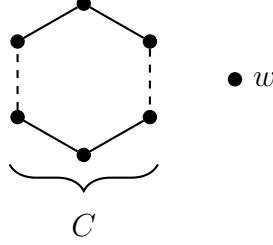


Figure 5.9: By maximality of $P_i^{(l)}$, $w \notin V(P_i^{(l)})$ cannot be connected to a vertex on C .

Since at most $2k$ edges are protected in each vertex, the probability that S_i is M_l -connected is $1 - o(1)$ by Lemma 5.25. Furthermore, by construction $P_i^{(l)} \supseteq P_{i-1}^{(l)}$ contains a vertex of $S_{i-1}^* \subseteq S_i$. Since $P_i^{(l)}$ contains at least one vertex of S_i , P_i and S_i are in the same M_l component. Consider an arbitrary vertex $v \in V \setminus S_i$. Then v is either an endpoint of the path $[v]_i^{(l)}$ or an interior vertex of $[v]_i^{(l)}$. Since case 1.2.2.2 applies, each vertex in S_i' has out-degree at least $34k$ and all these edges have an endpoint in $S_0 \subseteq S_i$. But each endpoint of $[v]_i^{(l)}$ is in S_i' by construction. Therefore each endpoint of $[v]_i^{(l)}$ is in the same M_l -component as S_i . Now, v is in the same M_l -component as its endpoints and thus in the same M_l -component as S_i . Therefore each vertex in $V \setminus S_i$ is in the same M_l -component as $S_0 \subseteq S_i$. This implies that the probability that M_l is connected is $1 - o(1)$. So, the probability that Case 2 applies is $o(1)$.

Case 3: There is no M_l -cycle with vertex set $V(P_i^{(l)})$ and there is no M_l -path with vertex set $V(P_i^{(l)})$ with endpoints u, v such that $\{u, v\}$ is unclaimed.

Since there exists no M_l -path with vertex set $V(P_i^{(l)})$ with endpoints u, v such that $\{u, v\}$ is unclaimed, all these connections have either been taken by Breaker or they have been colored by Maker to build a different Hamiltonian cycle. Since by Lemma 5.20 the game ends in at most $\frac{kn}{2} + o(n)$ rounds, Breaker chose at most $b \cdot \left(\frac{kn}{2} + o(n)\right) \leq (1 - o(1)) \frac{(k+o(1))n^2}{2 \log(n)}$ edges. Furthermore, Maker chooses one edge per round and thus at most $\frac{kn}{2} + o(n) = \frac{o(1)n^2}{\log(n)}$ edges. This implies that the maximum number of claimed vertices is at most $\frac{(k+1)n^2}{2 \log(n)}$. By Corollary 5.30 there are at least $\frac{(k+2)n^2}{2 \log(n)}$ pairs of such endpoints with probability $1 - o(1)$. Thus, the probability that Case 3 applies is $o(1)$. \square

Now we are ready to prove the main theorem (Theorem 5.7).

Proof of Theorem 5.7. Let $k \in \mathbb{N}$ be fixed. We want to prove that Maker can a.a.s. construct a graph that contains k edge disjoint Hamiltonian cycles in at most $k \cdot n + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ turns. We will prove this by showing that this is the case if Maker executes the Stages 1 and 2 of the algorithm *Hamilton-Cycles*.

Assume that Maker executed Step 1 and 2 of the algorithm *Hamilton-Cycles*. Then by Theorem 5.31 Maker's graph contains k edge-disjoint Hamilton cycles with probability $1 - o(1)$.

Furthermore, Lemma 5.20 shows that the game ends after at most $k \cdot n + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ moves if Stage 1 ends in Case 1.2.2.

By Theorem 5.26 this is the case.

This concludes the proof of the main theorem (Theorem 5.7). \square

5.5 Describing the algorithm used to prove k -Edge Connectivity

We proceed to the proof of Theorem 5.9. We cite the theorem of Menger [Men27, Satz β] in a modern version given by Diestel [Die17].

Theorem 5.32 *A graph is k -edge-connected if and only if it contains k edge-disjoint paths between any two vertices.*

Let $k \in \mathbb{N}$. So, we want to give an algorithm that constructs a k -edge-connected graph. We recall that for any two distinct vertices $v, w \in V$ a Hamilton cycle induces two edge-disjoint paths from v to w . We set $k^* = \lfloor \frac{k}{2} \rfloor$. If k is even, we construct k^* edge-disjoint Hamilton cycles with the two steps of the algorithm *Hamilton-Cycles* that were described in section 5.2. Since Maker's graph contains k^* edge-disjoint Hamilton cycles, it contains $2k^* = k$ edge-disjoint paths from v, w for each pair of vertices $\{v, w\} \in \binom{V}{2}$. Thus, by Menger's theorem Maker's graph is k -edge-connected. Maker achieves this in at most $\frac{k}{2} \cdot n + o(n)$ rounds by Theorem 5.7.

If k is odd, an additional stage (Stage 3) is needed. So, let $k \in \mathbb{N}_{\geq 3}$, then $k^* = \lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$. As a first step we construct k^* edge-disjoint Hamilton cycles with the algorithm *Hamilton-Cycles*. We partition the vertices into three sets A_0 , B_0 and C_0 . Let C_0 be the set S'_f delivered by the algorithm *Hamilton-Cycles*. Set further $A_0 = V \setminus C_0$ and $B_0 = \emptyset$. For each step $i > f$, C_i will play the same role as the sets S_i during Stages 1 and 2, i.e. it will contain all troublesome vertices v , as well as $34k^*$ neighbors for each troublesome vertex, all of which were chosen after v became troublesome.

Note that B_0 is initially empty, but at the end of stage 3 B_{f_2} will contain all vertices that start in A_0 and that do not end up in C_{f_2} . By Lemma 5.25, C_0 is $2k^* + 1 = k$ connected. For every $v \in A_i$, $i > 0$ let $[v]_k^{(f)}$ be the path in $\mathcal{P}_f^{(k^*)}$ that contains v .

Since C_0 is k -edge-connected, its vertices will always be in the same component even if we remove at most $k - 1 = 2k^*$ vertices. Thus, it is our aim to increase the connections of vertices not in C_0 by connecting $v \in A_i$ into C_0 with an additional path. Note for each $j \in [k^*]$ there already exist two paths from v into C_0 , namely the sub-paths of $[v]_f^{(j)}$ that start in v and end in either endpoint of $[v]_f^{(j)}$. To achieve this, we want to connect v to a vertex $w \in A_i$ such that $[v]_f^{(k^*)} \neq [w]_f^{(k^*)}$ and $\{v, w\}$ is unclaimed. We connect v to such a w if it exists. Otherwise, if no such w exists, we connect v to an arbitrary vertex of S_0 . This is always possible, because by being

in A_i v is non-troublesome. Furthermore, in non-troublesome vertices a number of edges lower than $|S_0|$ has been claimed.

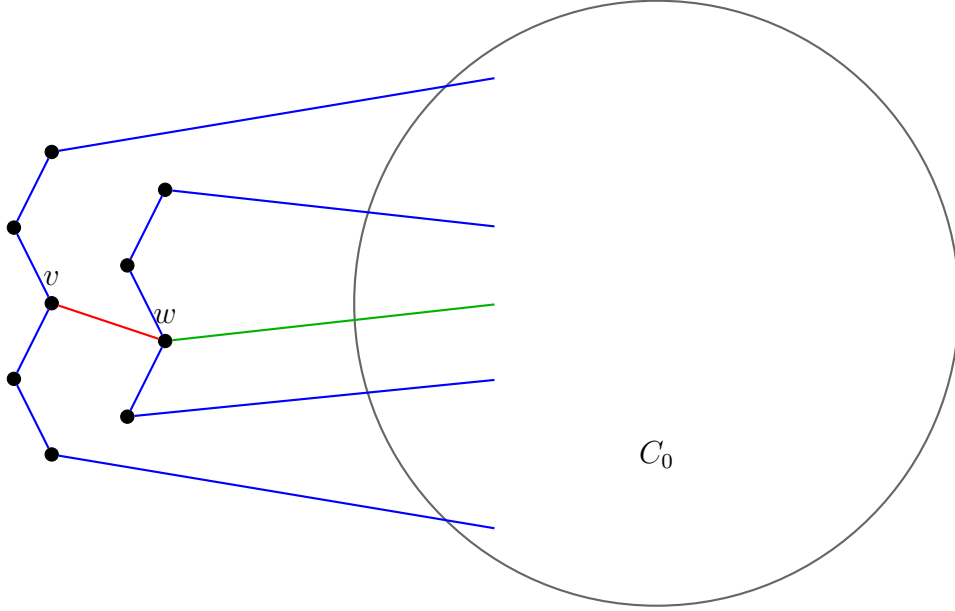


Figure 5.10: Illustration of the basic technique, joining two vertices v and w in A_0 by claiming the edge $\{v, w\}$ depicted in red. If such a pairing is not found, we add an edge connecting $w \in A_0$ to a vertex in S_0 depicted in green.

It is our aim to connect pairs of vertices in A_0 such that they are in different path components of $\mathcal{P}_f^{(k)}$. To keep track of the vertices that have been taken care of in such a way we move both of them into B_i .

5.5.1 Stage 3: Bonus Stage for odd k

We initialize $C_0 = S'_f$, $A_0 = V \setminus C_0$ and $B_0 = \emptyset$. Between each step $i > f$, vertices from A_{i-1} that became troublesome are moved into C_{i-1} .

Case 1: Every troublesome vertex v satisfies $d_M^*(v) \geq 34k$.

Case 1.1: There exists a non-troublesome vertex $v \in C_{i-1}$ with $d'(v) < 34k$. Then

- choose $w^* \in \{w \in S_0 : (v, w) \notin B \cup M\}$ uniformly at random and add (v, w^*) to M . Color $c(v, w^*) = 0$.
- Set $C_i = C_{i-1}$, $A_i = A_{i-1}$ and $B_i = B_{i-1}$.

Case 1.2: Every non-troublesome vertex $v \in C_{i-1}$ satisfies $d'_M(v) \geq 34k$,

Case 1.2.1: If there exists a pair of vertices $v, w \in A_{i-1}$ such that $[v]_f^{(k^*)} \neq [w]_f^{(k^*)}$ and $\{v, w\} \notin M \cup B$, then add $\{v, w\}$ and set $c(\{v, w\}) = k$, $C_i = C_{i-1}$, $A_i = A_{i-1} \setminus \{v, w\}$ and $B_i = B_{i-1} \cup \{v, w\}$.

Case 1.2.2: If there exists a $v \in A_{i-1}$, but no $w \in A_{i-1}$ such that Case 1.2.1 would be satisfied, choose $w' \in S_0$, such that $\{v, w'\} \notin M \cup B$. Then add $\{v, w'\}$, set $c(\{v, w'\}) = k$, $C_i = C_{i-1}$, $A_i = A_{i-1} \setminus \{v\}$ and $B_i = B_{i-1} \cup \{v\}$.

Case 1.2.3: Otherwise $A_{i-1} = \emptyset$ and we end the game.

Case 2: There is a troublesome vertex $w \in C_{i-1}$ such that $d_M^*(v) < 34k$.

Select v to be a troublesome vertex $v \in C_{i-1}$ such that $d_M^*(v) < 34k$, having maximum danger $\text{dang}(v) := d_B(v) - bd_M^*(v)$ (with ties arbitrarily broken). Then

Case 2.1: If there is no $w' \in V - C_{i-1}$ such that (v, w') is not in $B \cup M$, end the game.

Case 2.2: Otherwise,

- choose such a w' and add (v, w') to M . Color (v, w') with color 0, so $c(v, w^*) = 0$.
- Set $C_i = C_{i-1} \cup \{w'\}$ and for each $j \in [k]$ set \mathcal{P}_i^j to be the set obtained from \mathcal{P}_{i-1}^j by deleting the path P that contained w' and adding the components of $P - w'$.

5.6 Analyzing the duration of Stage 3

To analyze the additional stage of the algorithm we recall the following definitions. f is the number of turns carried out in Stage 1 and Stage 2, and t is the number of troublesome vertices at the end of the game. Further, let f_2 be the number of turns played in Stage 1, Stage 2 and Stage 3.

By Theorem 5.7 at most $k^*n + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right) = \frac{(k-1)n}{2}$ turns have been carried out.

We now estimate the number of turns that were carried out in Stage 3.

Lemma 5.33 *The number of turns in which Case 2 of either Stage applies can be estimated to be at most $34 \cdot k^* \cdot t$.*

Proof. The proof is analogous to the proof of Lemma 5.11.

Whenever Case 2 applies in any step of either stage, edges are added to a troublesome vertex v with $d_M^*(v) < 34k^*$. We add one edge per turn, thus, the number of turns spent on each troublesome vertex is $34k^*$. Since there are t troublesome vertices at the end of the algorithm, the number of rounds spent in these cases is at most $34k^*t$. \square

Lemma 5.34 *The number of turns taken in Case 1.1 of either Stage applies can be estimated to be at most $34 \cdot k^* \cdot |C_{f_2}|$.*

Proof. The proof is analogous to the proof of Lemma 5.12.

Whenever Case 1.1 of either stage every troublesome vertex v satisfies $d_M^*(v) = 34k$. Note that for each $i_1 \leq f$, $S_{i_1} \subseteq S_f = C_0$. Furthermore, for each $i_2 \leq f_2$ we have $C_{i_2} \subseteq C_{f_2}$. Thus, for each non-troublesome vertex $v \in C_{f_2}$ we increase $d_M^*(v)$

to $34k$, adding one edge per turn. In the worst case all vertices in C_{f_2} are non-troublesome and have to be increased in these steps. Thus, the number of turns is at most $34k \cdot |C_{f_2}|$. \square

Lemma 5.35 *The number of turns in which Case 1.2.1 of Stage 3 applies, can be estimated to be at most $\frac{|A_0|}{2} \leq \frac{n}{2}$.*

Proof. Since $A_0 \subseteq V$, we have $|A_0| \leq |V| \leq n$. If Case 1.2.1 of Stage 3 applies, there is a pair of vertices $v, w \in A_{i-1} \subseteq A_0$ in different path components of $\mathbb{P}_f^{(k^*)}$.

Since both v, w are then removed from $|A_{i-1}|$, it takes at most $\frac{|A_0|}{2} \leq \frac{n}{2}$ such steps to remove every vertex from A_0 . \square

Lemma 5.36 *The number of turns in which Case 1.2.1 of Stage 3 applies can be estimated to be at most $\frac{2n}{\sqrt{\log(n)}}$.*

Proof. We show that for each $i \leq f_2$ so that in step i Case 1.2.1 of Stage 3 applies, we have $|A_{i-1}| < \frac{2n}{\sqrt{\log(n)}}$. Whenever Case 1.2.1 of Stage 3 applies, Maker reduces the size of A_{i-1} by one. So at most $\frac{2n}{\sqrt{\log(n)}}$ such steps are needed until A_{i-1} is empty.

If Case 1.2.2 of Stage 3 is carried out, Case 1.2.1 of Stage 3 does not apply. Thus, there exists no pair of vertices $v, w \in A_{i-1}$, such that $[v]_f^{(k^*)} \neq [w]_f^{(k^*)}$ and $\{v, w\}$ is unclaimed. Now consider an arbitrary $v \in A_{i-1}$. Since there is no $w \in A_{i-1}$ such that $[v]_f^{(k^*)} \neq [w]_f^{(k^*)}$ and $\{v, w\}$ is unclaimed, each $w \in A_{i-1} \setminus \{v\}$ must fulfill one of the following two properties. Either w is in $[v]_f^{(k^*)}$, or $\{v, w\}$ has been claimed by a player.

Thus

$$|A_i| \leq d_B(v) + d_M(v) + |[v]_f^{(k^*)}|.$$

Observe that $|[v]_f^{(k^*)}| \leq 2 \log(n) < \frac{n}{2\sqrt{\log(n)}}$ by Remark 5.10.

Now we bound the number of $w \in A_{i-1} \setminus \{v\}$ such that $\{v, w\}$ has been claimed by a player. Since v is in A_{i-1} , it is not in C_{i-1} and is non-troublesome, so by definition $d_B(v) < \frac{n}{\sqrt{\log(n)}}$. Furthermore, since v is an interior vertex of $[v]_f^{(j)}$ for any $j \in [k^*]$, exactly $2k^*$ Maker edges incident in v have been added. As k^* is constant and n is assumed to be large enough, $d_M(v) = 2k^* \leq \frac{n}{2\sqrt{\log(n)}}$.

We conclude

$$|A_i| \leq d_B(v) + d_M(v) + |[v]_f^{(k^*)}| \leq \frac{n}{\sqrt{\log(n)}}.$$

\square

We now can give a preliminary upper bound on the number of turns that are carried out during Stages 1, 2 and 3. Note that we count the number of turns carried out in case 2 and case 1.1 double. This will turn out to be not problematic, because the number of steps carried out in these steps is $\mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$.

Lemma 5.37 *The number of turns carried out in Stages 1, 2 and 3 in total is at most*

$$\frac{(2k^* + 1)n}{2} + 34k^* \cdot t + 34k^* \cdot |C_{f_2}| + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right) \leq 70k^* \cdot n.$$

Proof. This result is analogous to the proof of Lemma 5.16. By the lemmata 5.33 through 5.36 and Lemma 5.20 the number of moves played in Stages 1, 2 and 3 combined is at most

$$\begin{aligned} & \underbrace{34k^* \cdot t}_{\text{Lemma 5.33}} + \underbrace{34k^* \cdot |C_{f_2}|}_{\text{Lemma 5.34}} + \underbrace{\frac{n}{2}}_{\text{Lemma 5.35}} + \underbrace{\frac{2n}{\sqrt{\log(n)}}}_{\text{Lemma 5.36}} + \underbrace{2k^*n + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)}_{\text{Lemma 5.20}} \\ &= \frac{(2k^* + 1)n}{2} + \underbrace{34k^* \cdot t + 34k^* \cdot |C_{f_2}|}_{\leq 68k^*n} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right) \\ &\leq 70k^* \cdot n, \end{aligned}$$

using the trivial upper bound n for $|C_{f_2}|$ and t . \square

Lemma 5.38 *The number of troublesome vertices after Stages 1, 2 and 3 are completed is at most*

$$t \leq \frac{140k^*n}{\sqrt{\log(n)}}.$$

Proof. This proof is analogous to the proof of Lemma 5.17. By Lemma 5.37 the game lasts for at most $70k^*n$ turns. Since Breaker claims $b \leq \frac{n}{\log(n)}$ edges in each turn, we have

$$|B| \leq 70 \cdot k^* \cdot n \cdot b \leq \frac{70k^*n^2}{\log(n)}.$$

Furthermore, each troublesome vertex has Breaker degree at least $\frac{n}{\sqrt{\log(n)}}$. In addition each Breaker edge is incident in at most two troublesome vertices. Thus

$$t \leq 2 \cdot |B| \cdot \frac{\sqrt{\log(n)}}{n} \leq 2 \cdot 70 \cdot k^* \cdot \frac{n}{\sqrt{\log(n)}}.$$

\square

Lemma 5.39 *At the end of the game we have*

$$\begin{aligned} |C_{f_2}| &\leq |S_0| + 35k^*t \\ &\leq \frac{1000(4k^* + 3)^2\sqrt{k^* + 2n}}{\sqrt{\log(n)}} + 35k^*t \\ &\leq \frac{1000(4k^* + 3)^2\sqrt{k^* + 2n}}{\sqrt{\log(n)}} + 4900k^* \frac{n}{\sqrt{\log(n)}}. \end{aligned}$$

Proof. This proof is analogous to the proof of Lemma 5.18.

If at the end of Stage 3 a vertex v is in C_{f_2} , it arrived there by one of the following three ways: a) it was in S_0 , b) it became troublesome over the cause of the game (which implies it was added between steps) and c) it is one of the $34k^*$ out-neighbors of a troublesome vertex during case 1.1. of either Stage. Thus for each troublesome vertex at most $35k^*$ vertices end up in C_{f_2} , and at most $35k^*t$ vertices satisfy possibility b) or c). Obviously, there are $|S_0|$ vertices which satisfy possibility a).

The second inequality follows from the definition of S_0 and the third inequality is true by Lemma 5.38. \square

Now that we have given a rough upper bound on the number of rounds played, it is time to refine our result with the new upper bounds for t and $|C_{f_2}|$.

Lemma 5.40 *At most $\frac{kn}{2} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ turns have been carried out in Stages 1, 2 and 3.*

Proof. By Lemma 5.37 the game ends after

$$\frac{(2k^* + 1)n}{2} + 34k^* \cdot t + 34k^* \cdot |C_{f_2}| + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$$

turns. By Lemma 5.38, $t \in \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ and by Lemma 5.39, $|C_{f_2}| \in \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$.

Thus, the number of turns that were carried out by Maker is at most

$$\frac{(2k^* + 1)n}{2} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right) = \frac{k \cdot n}{2} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right).$$

Since the game lasts for at most $\frac{k \cdot n}{2} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ turns and Breaker claims $b \leq \frac{n}{\log(n)}$ edges in each turn,

$$|B| \leq \left(\frac{k \cdot n}{2} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)\right) \cdot b \leq \frac{(k+1)n^2}{2 \log(n)}.$$

Each troublesome vertex has Breaker degree at least $\frac{n}{\sqrt{\log(n)}}$. In addition, each Breaker edge is incident in at most two troublesome vertices. Thus

$$t \leq 2 \cdot |B| \cdot \frac{\sqrt{\log(n)}}{n} \leq 2 \cdot \frac{k+1}{2} \cdot \frac{n}{\sqrt{\log(n)}} \leq (k+1) \cdot \frac{n}{\sqrt{\log(n)}}.$$

\square

Corollary 5.41 *At the end of the game we have*

$$t \leq (k+1) \frac{n}{\sqrt{\log(n)}} \text{ and } |C_{f_2}| \leq |S_0| + 35(k+1)k^* \sqrt{\log(n)}.$$

Proof. Each troublesome vertex has Breaker degree at least $\frac{n}{\sqrt{\log(n)}}$. In addition each Breaker edge is incident in at most two troublesome vertices. Thus

$$t \leq 2 \cdot |B| \cdot \frac{\sqrt{\log(n)}}{n} \leq 2 \cdot \frac{k+1}{2} \frac{n}{\sqrt{\log(n)}} \leq (k+1) \cdot \frac{n}{\sqrt{\log(n)}}.$$

By Lemma 5.39 we have

$$|C_{f_2}| \leq |S_0| + 35k^*t \leq |S_0| + 35(k+1)k^*\sqrt{\log(n)}.$$

□

5.7 Proving the k -Edge-Connectivity of Maker's Graph

We already established that the game ends after at most $\frac{kn}{2} + \mathcal{O}\left(\frac{n}{\sqrt{\log(n)}}\right)$ turns. We now have to show that the Maker graph is k -edge connected after stages 1, 2 and 3 have been carried out. We begin by showing that the algorithm is well-defined, i.e. that the choices that it requires can be made in any given step. Of course, this is true for every step during stages 1 and 2. Thus we will only consider the steps taken during Stage 3.

Lemma 5.42 *Maker can always make the moves required by the algorithm k -edge-connectivity (during Step 3).*

Proof. We consider the cases of Stage 3.

We first observe that Maker can make the move required by the algorithm when Case 1.2.1 applies, since this case applies only when the edge Maker is supposed to choose exists and is unclaimed.

The arguments that Maker can make the required choice in Case 1.1 and Case 1.2.2 are similar and can be handled together. In both of these cases Maker has to connect a non-troublesome vertex v to a vertex $w \in S_0$. Note that v in Case 1.1 is explicitly non-troublesome, while in Case 1.2.2 v is in A_{i-1} . Since all troublesome vertices in step i are in C_i , v is non-troublesome when Case 1.2.2 applies. Thus, $d_B(v) \leq \frac{n}{\sqrt{\log(n)}}$. Furthermore, Case 1.1 requires $d'_M(v) < 34k < \frac{n}{\sqrt{\log(n)}}$ and Case 1.2.2 applies to a vertex of A_{i-1} . So, if Case 1.2.2 applies, v is an interior vertex of $[v]_f^{(j)}$ for each $j \in [k^*]$. These vertices are not incident in Maker edges with an endpoint in $C_{i-1} \supseteq S_0$ by construction. Thus, there exist at least $|S_0| - d_B(v) - d'_M(v) \geq |S_0| - \frac{2n}{\sqrt{\log(n)}}$ vertices $w^* \in S_0$ such that (v, w^*) is unclaimed if Case 1.1 or Case 1.2.2 of Stage 3 apply.

Now we want to show that Maker can make the required choice in Case 2 of Stage 3. So, let $i \leq f_2$ such that Case 2 of Stage 3 applies. Let $v \in C_{i-1}$ be a troublesome vertex with $d'_M(v) \leq 34k^*$ and maximum danger. By Lemma 5.22, $d_B(v) \leq n - \frac{(4k+3)^3 n + 2k+1}{\sqrt{\log(n)}}$. Furthermore, v was non-troublesome when Stage 3 started, otherwise

Stage 2 would not have concluded. The number of edges present in v at the start of Stage 3 that end in $V \setminus C_f$ and were of some color $j \in [k^*]$ when v was moved to C_f is at most $2k^* \leq \frac{n}{\sqrt{\log(n)}}$. In conclusion, there exist at least

$$\begin{aligned}
 & |V \setminus C_{i-1}| - 2k^* - d_B(v) \\
 & \geq |V| - |C_{i-1}| - 2k^* - d_B(v) \\
 & \geq n - |C_{f_2}| - \frac{n}{\sqrt{\log(n)}} - n + \frac{(4k+3)^3 n + 2k + 1}{\sqrt{\log(n)}} \\
 & \geq \frac{(4k+3)^3 n 2k + 1}{\sqrt{\log(n)}} - \frac{n}{\sqrt{\log(n)}} - |C_{f_2}| \quad (\text{by Corollary 5.41})
 \end{aligned}$$

vertices $w^* \in V \setminus C_{i-1}$ such that $\{v, w^*\}$ is unclaimed. \square

Proof of Theorem 5.9. If k is even Maker builds $k^* = \lfloor \frac{k}{2} \rfloor$ Hamilton cycles with the algorithm *Hamilton-Cycles*. By Menger's theorem the constructed graph is k -edge-connected and we are done.

So, let $k \in \mathbb{N}_{\geq 3}$ be odd. Note that $2k^* + 1 = k$. Now, by Lemma 5.25 and since $2k^* + 1 = k$, C_0 is k -edge connected with probability $1 - o(1)$.

We assume for a moment that there exists a set \hat{E} of $k-1$ edges such that $M \setminus \hat{E}$ is not connected. Then there exists $v \in V$ not belonging to the same $(M \setminus \hat{E})$ -component as C_0 .

Since for each color $j \in [k]$ there exist two j -colored paths in M connecting v and C_0 , \hat{E} has to contain two edges of each color from M and both have to be in $[v]_f^{(k^*)}$. Otherwise, there is a path remaining from v to some vertex of C_0 .

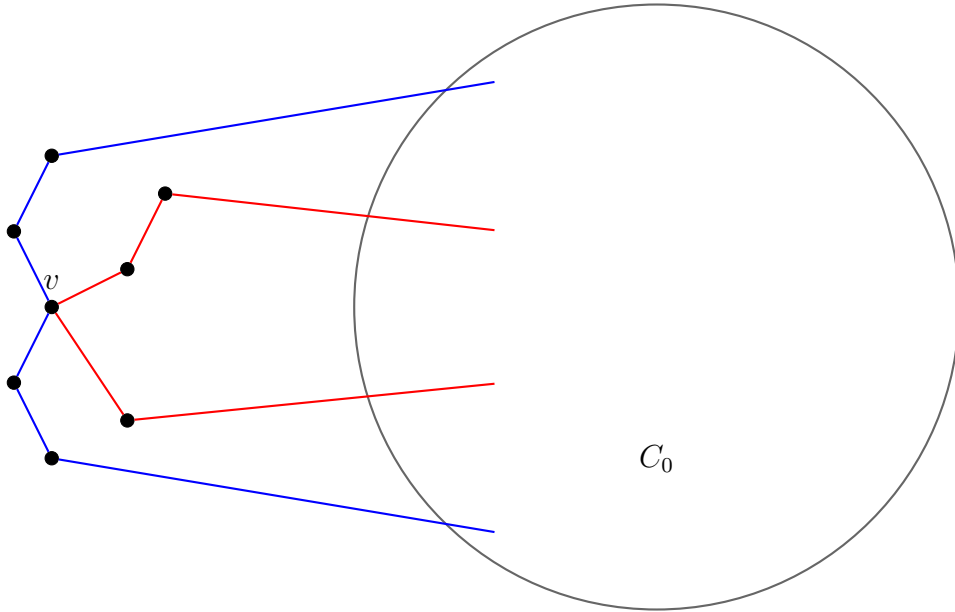


Figure 5.11: An example of a situation where v is an inner vertex of a red-colored path and a blue-colored path i.e. $k^* = 2$, implying $4 = 2k^*$ edges (two red, two blue) have to be removed to disconnect v from C_0 .

If v is in $V \setminus C_0$, it has to be in B_{f_2} , since at the end of the game A_{f_2} is empty. If v ends up to be in B_f , there are two cases.

Case 1: There exists a $w \in S_0 \subseteq C_0$ such that $\{v, w\} \in M$ and $c(\{v, w\}) = k$ and this edge was added in Case 1.2.2 of Stage 3. Since $c(\{v, w\}) = k$, and all edges in \hat{E} are colored with a color in $[k^*]$, $\{v, w\} \notin \hat{E}$. Thus, v is in the same $(M \setminus \hat{E})$ -component as C_0 , which is a contradiction to the assumption that v and C_0 are in different $(M \setminus \hat{E})$ -components.

Case 2: There exists a $w \in B$ with $[v]_f^{(k^*)} \neq [w]_f^{(k^*)}$ such that $\{v, w\} \in M$ and $c(\{v, w\}) = k$. Since \hat{E} contains only two k^* -colored edges, both of which connect vertices in $[v]_f^{(k^*)}$, no edges have been removed from $[w]_f^{(k^*)}$. So, in $M \setminus \hat{E}$, there exists an edge from S_0 to an endpoint u of $[w]_f^{(k^*)}$, which was chosen in Case 1.2.1 of Stage 2. Then there exists a sub-path of $[w]_f^{(k^*)}$ connecting u to w and finally the edge $\{v, w\}$ in $M \setminus \hat{E}$. Because $S_0 \subseteq C_0$, v is in the same $(M \setminus \hat{E})$ -component as C_0 . This is again a contradiction to the assumption that v and C_0 are in different $(M \setminus \hat{E})$ -components.

This concludes the proof. \square

5.8 Conclusion and Open Questions

In this chapter we managed to generalize an approach of Brüstle et al. [BCN⁺23] to show that for each $k \geq 2$ Maker can win the $\left(1 : \frac{n}{\log(n)} - \frac{Cn}{\log(n)^{3/2}}\right)$ Maker-Breaker k -edge-connectivity game in at most $\frac{k}{2}n + o(n)$ turns. This is also the optimal number of rounds up to the $o(n)$ term and improves the formerly known bound by a factor of more than 8. We also used this strategy to show that Maker wins the $\left(1 : \frac{n}{\log(n)} - \frac{Cn}{\log(n)^{3/2}}\right)$ Maker-Breaker k -factor game in $\left\lceil \frac{k}{2} \right\rceil n + o(n)$ rounds for each $k \geq 2$. Since Maker needs at least $\frac{k}{2}n$ rounds to win the k -factor game, the only possibility to improve this result up to the $o(n)$ term is to reduce the number of turns by $\frac{n}{2}$ if k is odd. We pose the following natural questions:

Question 1 : Can the approach described in this chapter be used to show that Maker wins the $\left(1 : \frac{n}{\log(n)} - \frac{Cn}{\log(n)^{3/2}}\right)$ k -vertex-connectivity game in at most $\frac{k}{2}n + o(n)$ turns?

Question 2 : Is it possible to modify the approach in such a way that it yields a Maker win in the $\left(1 : \frac{n}{\log(n)} - \frac{Cn}{\log(n)^{3/2}}\right)$ k -factor game using at most $\frac{k}{2}n + o(n)$ if k is odd?

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Erklärung

Hiermit erkläre ich,

- dass die Abhandlung - abgesehen von der Beratung durch den Betreuer - nach Inhalt und Form die eigene Arbeit ist,
- dass Teile des Kapitels 3 als Preprint unter <https://arxiv.org/abs/2406.14223> veröffentlicht und zur Veröffentlichung in *Random Structures and Algorithms* eingereicht wurden,
- dass die Arbeit darüber hinaus weder ganz noch zum Teil schon einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen hat, veröffentlicht worden ist oder zur Veröffentlichung eingereicht wurde,
- dass die Arbeit unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft entstanden ist,
- dass mir kein akademischer Grad entzogen wurde.

Jan Erhard Geest
Kiel, 2024