



Different covering numbers of compact tree ideals

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Abstract

We investigate the covering numbers of some ideals on ${}^{\omega}2$ associated with tree forcings. We prove that the covering of the Sacks ideal remains small in the Silver and uniform Sacks model, respectively, and that the coverings of the uniform Sacks ideal and the Mycielski ideal, \mathfrak{C}_2 , remain small in the Sacks model.

Keywords Covering numbers · Tree ideals · Mycielski ideals · Uniform trees · Sacks forcing · Silver forcing

Mathematics Subject Classification 03E17 · 03E35

1 Introduction

Let \mathbb{T} be a compact tree-forcing, i.e. a forcing whose conditions are subtrees of ${}^{<\omega}2$. Associate with \mathbb{T} the σ -ideal

$$I^0 := \{A \subseteq {}^{\omega}2 : \forall p \in \mathbb{T} \exists q \in \mathbb{T} (q \leq p \wedge [q] \cap A = \emptyset)\}. \quad (1)$$

In several cases it has been shown that in the forcing extension obtained by a countable support iteration of such a \mathbb{T} of length ω_2 , the covering number of I^0 is equal to the continuum. We will continue some of Brendle's work. In his unpublished habilitation thesis, he proves the following theorem:

Theorem 1.1 ([1, Sec. 7.1 Theorem 12.]) *In the model obtained by adding iteratively ω_2 Miller reals with countable support to a model of CH, we have*

- (1) $\text{cov}(m^0) = \omega_2$ but
- (2) $\text{cov}(r^0) = \text{cov}(I^0) = \text{cov}(s^0) = \text{cov}(v^0) = \omega_1$.

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Here, $m^0, r^0, l^0, s^0,$ and v^0 are the respective ideals associated via (1) with Miller, Mathias, Laver, Sacks, and Silver forcing. Without a proof he mentions the following result concerning iterations of Laver forcing:

Theorem 1.2 ([1]) *In the Laver model, $\text{cov}(m^0) < \text{cov}(l^0)$.*

In contrast to Brendle, who worked on iterations of non-compact tree forcings, we will focus on the case of compact tree ideals. We will extend the scope of our investigation to the covering numbers of the Mycielski ideals that are closely related with Silver forcing and uniform Sacks forcing.

Definition 1.3 We write \mathbb{S} for **Sacks forcing**, i.e. the forcing whose conditions are perfect subtrees of ${}^{<\omega}2$ ordered by inclusion.

A **uniform tree** is a perfect tree u such that

$$\exists a \in [\omega]^\omega \forall \sigma \in u(\sigma \in \text{split}(u) \iff |\sigma| \in a). \tag{*}$$

Let \mathbb{U} be the set of all uniform trees. We call this notion of forcing **uniform Sacks forcing**. We call the unique set a from (*) the **split levels** of u and write a^u for this set.

A uniform tree v is called a **Silver tree**, if there exists a function $f : \omega \setminus a^v \rightarrow 2$ such that $\sigma \upharpoonright (\omega \setminus a^v) \subseteq f$ for every $\sigma \in v$. We write \mathbb{V} for the set of all Silver trees and call it **Silver forcing**.

We write v^0, u^0, s^0 for the ideals defined in (1) for $\mathbb{V}, \mathbb{U},$ and \mathbb{S} , respectively.

Definition 1.4 For an ideal \mathcal{I} on ${}^\omega 2$ we call

$$\text{cov}(\mathcal{I}) := \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} = {}^\omega 2 \right\}$$

the **covering number** of \mathcal{I} .

We will show that consistently, $\text{cov}(j^0) < \text{cov}(i^0)$ for different $i^0, j^0 \in \{v^0, u^0, s^0\}$. In most cases we are able to separate the covering numbers of another type of ideals, which are also closely related to tree forcings:

Definition 1.5 We call the ideals

$$\mathfrak{P}_2 := \left\{ A \subseteq {}^\omega 2 : \forall b \in [\omega]^\omega (A \upharpoonright b \neq b \upharpoonright 2) \right\}$$

and

$$\mathfrak{C}_2 := \left\{ A \subseteq {}^\omega 2 : \forall b \in [\omega]^\omega \exists u \in \mathbb{U} (a^u = \omega \setminus b \wedge [u] \cap A = \emptyset) \right\}$$

Mycielski ideals. Note that

$$\mathfrak{P}_2 = \left\{ A \subseteq {}^\omega 2 : \forall b \in [\omega]^\omega \exists v \in \mathbb{V} (a^v = \omega \setminus b \wedge [v] \cap A = \emptyset) \right\}.$$

Note further that it is not necessary to quantify over all $b \in [\omega]^\omega$. It suffices to consider a dense set $D \subseteq [\omega]^\omega$.

The inclusions $\mathfrak{C}_2 \subseteq u^0$ and $\mathfrak{P}_2 \subseteq v^0$ can be proved in ZFC (see [2, Lemma 4.1]). Trivially, $\mathfrak{P}_2 \subseteq \mathfrak{C}_2$. Thus, one obtains the ZFC-inequalities

$$\text{cov}(u^0) \leq \text{cov}(\mathfrak{C}_2), \quad \text{cov}(v^0) \leq \text{cov}(\mathfrak{P}_2), \quad \text{and} \quad \text{cov}(\mathfrak{C}_2) \leq \text{cov}(\mathfrak{P}_2).$$

It is common knowledge that $\text{cov}(v^0) \leq \mathfrak{r}$, where \mathfrak{r} is the smallest cardinality of a reaping family (cf. [3, Lemma 3 in 3.2]). Since both \mathbb{S} and \mathbb{U} are P -point preserving (see [3, Lemma 1 in 3.2] for the proof that \mathbb{U} preserves P -points), \mathfrak{r} , and hence $\text{cov}(v^0)$, remains small in the Sacks model and the uniform Sacks model.

A standard Löwenheim–Skolem argument shows that $\text{cov}(t^0) = \aleph_2$ in the model obtained by a countable support iteration of $\mathbb{T} \in \{\mathbb{V}, \mathbb{U}, \mathbb{S}\}$ of length ω_2 over a model for $\text{ZFC} + \text{CH}$ (see e.g. [4, Theorem 1.2] for the proof in the case of $\mathbb{T} = \mathbb{S}$). Together with Brendle’s inequality, this yields the (respective) relative consistency of $\text{cov}(v^0) < \text{cov}(s^0)$ and $\text{cov}(v^0) < \text{cov}(u^0)$. Since $\text{cov}(u^0) \leq \text{cov}(\mathfrak{C}_2) \leq \text{cov}(\mathfrak{P}_2)$ is provable in ZFC, $\text{cov}(u^0) = \text{cov}(\mathfrak{C}_2) = \text{cov}(\mathfrak{P}_2) = \aleph_2$ in the uniform Sacks model. Thus, we also obtain the relative consistency of $\text{cov}(v^0) < \text{cov}(\mathfrak{P}_2) = \text{cov}(\mathfrak{C}_2)$.

Finally, the second author has recently shown in [5] that in the Silver model

$$\text{cov}(u^0) \stackrel{\text{(ZFC)}}{\leq} \text{cov}(\mathfrak{C}_2) < \text{cov}(v^0) \stackrel{\text{(ZFC)}}{\leq} \text{cov}(\mathfrak{P}_2).$$

We obtain the following table, summarizing all known results and the results proved in this paper:

$<$	$\text{cov}(v^0)$	$\text{cov}(u^0)$	$\text{cov}(s^0)$	$\text{cov}(\mathfrak{P}_2)$	$\text{cov}(\mathfrak{C}_2)$
$\text{cov}(v^0)$	$=$	[3]	[3]	[3]	[3]
$\text{cov}(u^0)$	[5] + [2]	$=$	4.13	[5]	???
$\text{cov}(s^0)$	3.10	3.10	$=$	[5]	3.10
$\text{cov}(\mathfrak{P}_2)$	ZFC $\vdash \geq$, [2, 4.1]	ZFC $\vdash \geq$, [2]	???	$=$	ZFC $\vdash \geq$, [2]
$\text{cov}(\mathfrak{C}_2)$	[5]	ZFC $\vdash \geq$, [2, 4.1]	4.12	[5]	$=$

Except for two pairs of ideals, all coverings have been separated. The value of $\text{cov}(\mathfrak{P}_2)$ in the Sacks model remains an open question. The value of $\text{cov}(\mathfrak{C}_2)$ can be calculated in all our models; however, it is always the same as $\text{cov}(u^0)$.

In each case, we will construct a family of ideal sets along the iteration. Some care must be made to assure that only \aleph_1 of these sets are required. This is ensured by some form of continuous reading of names that allows us to capture the reals in the final model with some trees in the ground model in such a way that names of the same *type* are captured by the same tree and there are only \aleph_1 many types (see Definitions 3.5 and 4.9 to understand what we mean by types).

Our method is somewhat similar to the one used by Brendle. He, too, constructed the covering sets along the iteration. However, due to the fact that he forced with non-compact tree forcings, he had to use the Laver property of the respective forcing to

capture the reals. Our construction does not rely on the Laver property and we obtain much more control over the reals that are added.

2 Some notation and preliminary results

In this section we will introduce some concepts and notation that will be useful throughout the paper.

Definition 2.1 Let $T \subseteq {}^{<\omega}2$ be a perfect tree. We write $\text{split}(T)$ for the set of split nodes of T . For $n < \omega$, we write $\text{split}_n(T)$ for the set of split nodes that properly contain exactly n split nodes. We call the unique element of $\text{split}_0(T)$ the **stem** of T and write $\text{stem}(T)$ for this element. The **canonical enumeration** of $\text{split}(T)$ is the enumeration $\langle \sigma_s : s \in {}^{<\omega}2 \rangle$ that satisfies $\sigma_\emptyset := \text{stem}(T)$ and, if $\sigma_s \in \text{split}_n(T)$ is defined for some $s \in {}^n2$ and $j < 2$, then $\sigma_{s \frown j}$ is the unique element of $\text{split}_{n+1}(T)$ that extends $\sigma_s \frown j$. For $\sigma \in \text{split}(T)$, we put

$$T_\sigma := \{ \tau \in T : \sigma \subseteq \tau \vee \tau \subseteq \sigma \}.$$

For $s \in {}^{<\omega}2$, put

$$T * s := T_{\sigma_s} = \{ \tau \in T : \sigma_s \subseteq \tau \vee \tau \subseteq \sigma_s \}.$$

Let $p, q \in \mathbb{S}$ and $n < \omega$, put

$$p \leq_n q \iff (p \leq q \wedge \text{split}_n(p) = \text{split}_n(q)).$$

A **fusion sequence** is a sequence $\langle p_n : n < \omega \rangle$ of perfect trees such that $p_{n+1} \leq_n p_n$ for every $n < \omega$. It is easy to see that $p_\omega := \bigcap_{n < \omega} p_n$ is a perfect tree, that satisfies $p_\omega \leq_n p_n$ for every $n < \omega$. Furthermore, if the p_n are all uniform (or Silver), then p_ω is uniform (or Silver). We call p_ω the **fusion limit** of $\langle p_n : n < \omega \rangle$.

For a forcing notion \mathbb{P} and $\alpha \in \mathbf{Ord}$ we will write \mathbb{P}_α for the countable support iteration of \mathbb{P} of length α over some ground model V for $\text{ZFC} + \text{CH}$.

Definition 2.2 Let $\mathbb{P} \in \{ \mathbb{V}, \mathbb{U}, \mathbb{S} \}$ and $\alpha \in \mathbf{Ord}$. A triple (p, F, η) such that $p \in \mathbb{P}_\alpha$, $F \in [\text{supp}(p)]^{<\aleph_0}$, and $\eta : F \rightarrow \omega$ is called a **fusion condition**. Given $p' \in \mathbb{P}_\alpha$ we define

$$p' \leq_{F, \eta} p \iff (p' \leq p \wedge \forall \beta \in F(p' \upharpoonright \beta \Vdash p'(\beta) \leq_{\eta(\beta)} p(\beta))).$$

If (p', F', η') is an other fusion condition, we say that $(p', F', \eta') \leq^1 (p, F, \eta)$, if the following assertions are satisfied:

- (a) $p' \leq_{F, \eta} p$;
- (b) $F \subseteq F'$ and $|F' \setminus F| \leq 1$;
- (c) if $F' \setminus F = \{ \beta \}$, then $\eta \upharpoonright F = \eta' \upharpoonright F$ and $\eta'(\beta) = 0$;

(d) if $F' = F$, there exists precisely one $\beta \in F$ such that $\eta \upharpoonright (F \setminus \{\beta\}) = \eta' \upharpoonright (F \setminus \{\beta\})$ and $\eta'(\beta) = \eta(\beta) + 1$.

If $(p', F', \eta') \leq^1 (p, F, \eta)$ and β is the unique element from (c) or (d), then we call (p', F', η') a β -**extension** of (p, F, η) . A sequence $\langle (p_n, F_n, \eta_n) : n < \omega \rangle$ of fusion conditions is called a **fusion sequence** if

- (a) $|F_0| = 1$;
- (b) $(p_{n+1}, F_{n+1}, \eta_{n+1}) \leq^1 (p_n, F_n, \eta_n)$;
- (c) $\bigcup_{n < \omega} F_n = \bigcup_{n < \omega} \text{supp}(p_n)$;
- (d) $0 \in \bigcup_{n < \omega} F_n$;
- (e) For all $\beta \in \text{supp}(p_n)$ there exists $m < \omega$ such that $\eta_m(\beta) \geq n$.

For every $1 \leq n < \omega$, we call the unique $\beta \in F_n$ from (c) or (d) of the definition of \leq^1 the **n -th active coordinate**. The 0-th active coordinate is the unique element of F_0 . Clearly a fusion sequence uniquely determines the sequence $\langle \beta_n : n < \omega \rangle$ of its active coordinates, and $\langle (p_n, \beta_n) : n < \omega \rangle$ determines the corresponding fusion sequence.

Put $F_\omega := \bigcup_{n < \omega} F_n$ and define $p_\omega \in \mathbb{P}_\alpha$ such that for every $\beta \in \alpha \setminus F_\omega$, $p_\omega \upharpoonright \beta \Vdash_\beta p_\omega(\beta) = \mathbf{1}_\mathbb{P}$ and for every $\beta \in F_\omega$,

$$p_\omega \upharpoonright \beta \Vdash_\beta p_\omega(\beta) = \bigcap_{n < \omega} p_n(\beta).$$

We call p_ω the **fusion limit** of $\langle (p_n, F_n, \eta_n) : n < \omega \rangle$.

Let (p, F, η) be a fusion condition and $\sigma \in \prod_{\beta \in F}^{(\eta(\beta)+1)2}$. We define $p * \sigma \in \mathbb{P}_\alpha$ such that

$$(p * \sigma) \upharpoonright \beta \Vdash (p * \sigma)(\beta) = \begin{cases} p(\beta) * \sigma(\beta) & \beta \in F \\ p(\beta) & \beta \notin F. \end{cases}$$

We will fix some notation: If $\sigma : F \rightarrow {}^{<\omega}2$ for some $F \subseteq \omega_2$, $\beta < \omega_2$ and $j < 2$, we define $\sigma^{(\beta,j)} : F \cup \{\beta\} \rightarrow {}^{<\omega}2$ with $\sigma^{(\beta,j)} \upharpoonright (F \setminus \{\beta\}) = \sigma \upharpoonright (F \setminus \{\beta\})$ and

$$\sigma^{(\beta,j)}(\beta) := \begin{cases} \sigma(\beta) \frown j & \beta \in F \\ \langle j \rangle & \beta \notin F. \end{cases}$$

For $f : \omega \rightarrow \omega_2$ we define $\sigma_s^f : f[|s|] \rightarrow {}^{<\omega}2$ for every $s \in {}^{<\omega}2$ by recursion on s :

$$\begin{aligned} \sigma_\emptyset^f &:= \emptyset \\ \sigma_{s \frown j}^f &:= (\sigma_s^f)^{(\langle |s| \rangle, j)}. \end{aligned} \tag{2}$$

Lemma 2.3 *Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}, \mathbb{S}\}$ and $\alpha \leq \omega_2$. Let $p \in \mathbb{P}_\alpha$ and $f : \omega \rightarrow \text{supp}(p)$ such that $f^{-1}[\{\beta\}]$ is infinite for every $\beta \in \text{supp}(p)$. Let $\langle \sigma_s^f : s \in {}^{<\omega}2 \rangle$ be as in (2). For every $s \in {}^{<\omega}2$, put $a_s := p * \sigma_s^f$.*

- (1) For every $s \in {}^{<\omega}2$ and $|s| \leq n < \omega$, $\langle a_t : s \subseteq t \in {}^n2 \rangle$ is a maximal antichain below a_s .
- (2) For every $s \in {}^{<\omega}2$, $\beta \in \text{supp}(p)$, and $p' \leq a_s$ there exists $t \in {}^{<\omega}2$ such that $f(|t|) = \beta$, $s \subseteq t$, and p' is compatible with both $a_{t \smallfrown 0}$ and $a_{t \smallfrown 1}$.

Proof (1). It suffices to consider $n = |s| + 1$. Clearly, $a_{s \smallfrown 0}$ and $a_{s \smallfrown 1}$ are incompatible. Let $p \leq a_s$ and $\beta := f(|s|)$. Since $p \leq a_s$,

$$p \restriction \beta \Vdash p(\beta) \leq a_s(\beta).$$

Since

$$p \restriction \beta \Vdash \{a_{s \smallfrown 0}(\beta), a_{s \smallfrown 1}(\beta)\} \text{ is a maximal antichain below } a_s(\beta),$$

there exist a \mathbb{P}_β -name \dot{r} , $j < 2$, and $q \leq p \restriction \beta$ such that

$$q \Vdash \dot{r} \leq p(\beta), a_{s \smallfrown j}(\beta).$$

Then $q \widehat{\smallfrown} \dot{r} \widehat{\smallfrown} (p \restriction [\beta + 1, \alpha]) \leq p, a_{s \smallfrown j}$.

(2). We can assume without loss of generality that $p' \restriction \beta$ decides $\text{stem}(p'(\beta))$ and that $a_s \restriction (f(|s|))$ decides $\text{stem}(a_s(f(|s|)))$ for every $s \in {}^{<\omega}2$.

Construct recursively t_0, t_1, \dots and p_0, p_1, \dots with the following properties:

- (a) $p_n \leq a_{t_n}$;
- (b) $p_{n+1} \leq p_n$;
- (c) $t_{n+1} = t_n \widehat{\smallfrown} j$ for some $j < 2$;
- (d) $p_n \restriction \beta \Vdash \text{stem}(a_{t_n}(\beta)) \subseteq \text{stem}(p_n(\beta)) = \text{stem}(p'(\beta))$.

Note that Property (d) guarantees that the construction will finish after finitely many steps.

Put $p_0 = p'$ and $t_0 = s$. Clearly, Properties (a), (b), and (c) are satisfied. For Property (d), note that

$$p' \restriction \beta \Vdash p'(\beta) \leq a_s(\beta),$$

and hence,

$$p' \restriction \beta \Vdash \text{stem}(a_s(\beta)) \subseteq \text{stem}(p'(\beta)).$$

Now, assume that p_n and t_n have been constructed for some $n < \omega$.

Since $\{a_{t_n \smallfrown 0}, a_{t_n \smallfrown 1}\}$ is a maximal antichain below a_{t_n} , there exists $j < 2$ such that p_n is compatible with $a_{t_n \smallfrown j}$. If $\beta_n := f(|t_n|) \neq \beta$, choose any such j and put $t_{n+1} := t_n \widehat{\smallfrown} j$. Define p_{n+1} such that $p_{n+1} \restriction (\alpha \setminus \{\beta_n\}) = p_n \restriction (\alpha \setminus \{\beta_n\})$ and

$$p_{n+1} \restriction \beta_n \Vdash p_{n+1}(\beta_n) = p_n(\beta_n) \cap a_{t_{n+1}}(\beta_n).$$

Since

$$p_n \upharpoonright \beta_n \Vdash (p_n(\beta_n) \leq a_{t_n}(\beta_n) \wedge (p_n(\beta_n) \parallel a_{t_{n+1}}(\beta_n))),$$

it follows that p_{n+1} satisfies Properties (a) and (b). Since $p_n \upharpoonright \beta$ decides both $\text{stem}(a_{t_n}(\beta))$ and $\text{stem}(p_n(\beta))$ and $p_n \upharpoonright \beta \Vdash \text{stem}(a_{t_{n+1}}(\beta)) = \text{stem}(a_{t_n}(\beta))$, Property (d) is satisfied in this case.

If $f(|t_n|) = \beta$, distinguish two cases. If

$$p_n \upharpoonright \beta \Vdash \text{stem}(a_{t_n}(\beta)) \subsetneq \text{stem}(p'(\beta)).$$

Since $p_n \upharpoonright \beta$ decides both $\text{stem}(a_{t_n}(\beta))$ and $\text{stem}(p'(\beta))$, there is a unique $j < 2$ such that $\text{stem}(a_{t_n}(\beta)) \cap j \subseteq \text{stem}(p'(\beta))$. Put $t_{n+1} := t_n \hat{\ } j$ and define p_{n+1} as in the previous case. It is clear that all properties are satisfied. If

$$p_n \upharpoonright \beta \Vdash \text{stem}(a_{t_n}(\beta)) = \text{stem}(p'(\beta)),$$

the construction is finished. It is easy to see that p' is compatible with both $a_{t_n \hat{\ } 0}$ and $a_{t_n \hat{\ } 1}$. □

Building on the proof of the previous lemma we obtain the following result, which one works, if one considers iterations of uniform or Silver trees.

Lemma 2.4 *Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}\}$ and $\alpha \leq \omega_2$. Let $p \in \mathbb{P}_\alpha$ and $f : \omega \rightarrow \text{supp}(p)$ such that $f^{-1}[\{\beta\}]$ is infinite for every $\beta \in \text{supp}(p)$. Let $\langle \sigma_s^f : s \in {}^{<\omega}2 \rangle$ be as in (2) and put $a_s := p * \sigma_s^f$ for every $s \in {}^{<\omega}2$. Let $s \in {}^{<\omega}2$ with $f(|s|) = 0$ and $p' \leq a_s$ such that p' is compatible with both $a_{s \hat{\ } 0}$ and $a_{s \hat{\ } 1}$. Then there exist $t_0, t_1 \in {}^{<\omega}2$ such that $s \hat{\ } j \subseteq t_j$, $|t_0| = |t_1|$, $f(|t_0|) = 0$ and p' is compatible with $a_{t_j \hat{\ } 0}$ and $a_{t_j \hat{\ } 1}$ for $j < 2$.*

Proof The only thing to prove is that t_0, t_1 can be found such that $|t_0| = |t_1|$. We can assume without loss of generality that $p' \leq a_s$. Since $p'(0)$ and $a_s(0)$ are uniform trees with $p'(0) \leq a_s(0)$ such that $p'(0)$ is compatible with both $a_{s \hat{\ } 0}(0)$ and $a_{s \hat{\ } 1}(0)$, we know that $\text{stem}(p'(0)) = \text{stem}(a_s(0))$. Furthermore, there exists $m < \omega$ such that $\text{split}_1(p'(0)) \subseteq \text{split}_m(a_s(0))$. In the proof of the previous lemma with $\beta = 0$, the length of t_0 and t_1 is completely determined by m . In particular, $|t_0| = |t_1|$. □

Definition 2.5 Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}, \mathbb{S}\}$. For $p \in \mathbb{P}_\alpha$, $f : \omega \rightarrow \text{supp}(p)$ such that $f^{-1}[\{\beta\}]$ is infinite for every $\beta \in \text{supp}(p)$, and $\langle \sigma_s^f : s \in {}^{<\omega}2 \rangle$ as in (2), we call $\langle a_s : s \in {}^{<\omega}2 \rangle$ with $a_s = p * \sigma_s^f$ the **antichain refinement** associated with (p, f) . We write $\bar{A}(p, f)$ for this family.

3 cov(s^0) in $V^{\mathbb{U}\omega_2}$ and $V^{\mathbb{V}\omega_2}$

In this section we will show that $\text{cov}(s^0)$ remains small in the Silver and the uniform Sacks model. To obtain a family of s^0 -sets (in the extension models) that covers the

reals, we will construct a family of dense subsets $D \subseteq \mathbb{S}$. Note that sets of the form

$$\mathcal{X}(D) := {}^\omega 2 \setminus \bigcup \{[q] : q \in D\},$$

where $D \subseteq \mathbb{S}$ is pre-dense form a base of s^0 .

The following definition yields a convenient way to describe the continuous reading of names that we need in this section and the next.

Definition 3.1 Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}, \mathbb{S}\}$ and $\alpha < \omega_2$. An \mathbb{P}_α -**coding triple** is a triple (\dot{x}, p, f) , where \dot{x} is a \mathbb{P}_α -name, $p \in \mathbb{P}_\alpha$, and $f : \omega \rightarrow \text{supp}(p)$ such that

- (a) $p \Vdash_{\mathbb{P}_\alpha} \dot{x} \in {}^\omega 2 \setminus \bigcup_{\beta < \alpha} V^{\mathbb{P}^\beta}$;
- (b) $f^{-1}[\{\beta\}]$ is infinite for every $\beta \in \text{supp}(p)$;
- (c) for every $s \in {}^{<\omega} 2$, $(p * \sigma_s^f) \upharpoonright f(|s|)$ decides $\text{stem}((p * \sigma_s^f)(f(|s|)))$;
- (d) the associated antichain refinement $\bar{A}(p, f) = \langle a_s : s \in {}^{<\omega} 2 \rangle$ satisfies

$$\forall s, t \in {}^{<\omega} 2 (s \perp t \implies \dot{x}[a_s] \perp \dot{x}[a_t]).$$

For a \mathbb{P}_α -coding triple (\dot{x}, p, f) with $\bar{A}(p, f) = \langle a_s : s \in {}^{<\omega} 2 \rangle$ define

$$\bar{T}(\dot{x}, p, f) = \langle t_s : s \in {}^{<\omega} 2 \rangle$$

such that t_s is the longest common initial segment of $\dot{x}[a_{s \smallfrown 0}]$ and $\dot{x}[a_{s \smallfrown 1}]$ for every $s \in {}^{<\omega} 2$. Furthermore put

$$a(\dot{x}, p, f) := \begin{cases} f^{-1}[\{\beta\}] & \beta = \max(\text{supp}(p)) \\ \emptyset & \text{supp}(p) \text{ has no maximal element.} \end{cases}$$

A **splitting family** is a strictly increasing family $\bar{T} = \langle t_s : s \in {}^{<\omega} 2 \rangle$ such that for every $s \in {}^{<\omega} 2$, $t_{s \smallfrown 0}$ and $t_{s \smallfrown 1}$ are incompatible and t_s is their longest common initial segment. It is clear that if T is the tree generated by such a family \bar{T} , then T is perfect, \bar{T} enumerates $\text{split}(T)$ and $\bar{T} \upharpoonright^n 2 : {}^n 2 \rightarrow \text{split}_n(T)$ is a bijection for every $n < \omega$.

The following proposition is a reformulation of continuous reading of names for the respective iterations. For \mathbb{V} , a stronger version of this proposition is proved by the second author in [5, Theorem 3.1], for \mathbb{U} the proposition was proved by the authors in [2, Section 7] and for \mathbb{S} a stronger version of the proposition can be found below (see Proposition 4.1).

Proposition 3.2 Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}, \mathbb{S}\}$ and $\alpha \leq \omega_2$. Let \dot{x} be a \mathbb{P}_α -name and $p' \in \mathbb{P}_\alpha$ such that $p' \Vdash_\alpha \dot{x} \in {}^\omega 2 \setminus \bigcup_{\beta < \alpha} V^{\mathbb{P}^\beta}$. Then there exist $p \leq p'$ and $f : \omega \rightarrow \text{supp}(p)$ such that (\dot{x}, p, f) is an \mathbb{P}_α -coding triple.

Fix $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}\}$. Let Ω be the set of infinite, co-infinite subsets of ω . For every splitting family $\bar{T} : {}^{<\omega} 2 \rightarrow {}^{<\omega} 2$ and every $a \in \Omega \cup \{\emptyset, \omega\}$ in the ground model we will construct a family $\langle \dot{D}_\gamma(\bar{T}, a) : \gamma < \omega_2 \rangle$ such that $\dot{D}_\gamma(\bar{T}, a)$ is a $\mathbb{P}_{\gamma+1}$ -name for

an open-dense subset of $\mathbb{S}^{V^{\mathbb{P}_{\gamma+1}}}$ such that for every \mathbb{P}_α -coding triple (\dot{x}, p, f) (for some $\alpha < \omega_2$) with $\bar{T} = \bar{T}(\dot{x}, p, f)$ and $a = a(\dot{x}, p, f)$ we have

$$p \Vdash_{\mathbb{P}_{\omega_2}} \dot{x} \in \mathcal{X}(D_\gamma(\bar{T}, a)).$$

Clearly, there are only \aleph_1 -many of these pairs. This ensures that $\text{cov}(s^0) = \aleph_1$ in $V^{\mathbb{V}_{\omega_2}}$ and $V^{\mathbb{U}_{\omega_2}}$.

Remark 3.3 In the next section we will have to invoke a notion of isomorphism for coding triples (see Definition 4.9). There we will see that reals that come from isomorphic coding triples are captured by the same ideal set. The fact that there are only \aleph_1 -many isomorphism types shows that the respective covering remains small. In this situation, however, it suffices to consider $\bar{T}(\dot{x}, p, f)$ and $a(\dot{x}, p, f)$. Both are invariant with respect to the notion of isomorphism.

Before we define the coding systems, we need one more definition.

Definition 3.4 Given a tree $T \subseteq {}^{<\omega}\omega$, we write $[T]$ for the set of infinite branches of T , i.e.

$$[T] = \{x \in {}^\omega\omega : \forall n < \omega (x \upharpoonright n \in T)\}.$$

Let $\bar{T} = \langle t_s : s \in {}^{<\omega}2 \rangle$ be a splitting family. Let T be the perfect tree generated by \bar{T} . For every $x \in [T]$ there exists a unique real $c_x \in {}^\omega 2$ such that

$$x = \bigcup \{t_{c_x \upharpoonright n} : n < \omega\}.$$

Let $a \subseteq \omega$ and $g : \text{dom}(g) \rightarrow 2$ with $a \subseteq \text{dom}(g) \subseteq \omega$. We call x (\bar{T} -)incompatible with g on a , if $c_x \upharpoonright a \neq g \upharpoonright a$. We say that a perfect subtree S of T is incompatible with g on a , if every $x \in [S]$ is incompatible with g on a . We say that $S \leq_S T$ is incompatible with $A \subseteq {}^\omega 2$ on a , if S is incompatible with every $g \in A$ on a .

Definition 3.5 Let $\bar{T} = \langle t_s : s \in {}^{<\omega}2 \rangle$ be a splitting family that generates the perfect tree T . Let $a \in \Omega \cup \{\emptyset, \omega\}$ and, assuming $a \neq \emptyset$, let $\varphi : a \rightarrow \omega$ be the order isomorphism. Let $\gamma < \omega_2$ and let x_γ be a $\mathbb{P}_{\gamma+1}$ -name for the generic real that is adjoined in the γ -th step of the iteration. Let $\dot{D}_\gamma = \dot{D}_\gamma(\bar{T}, a)$ be a $\mathbb{P}_{\gamma+1}$ -name with the following properties:

- (a) $\Vdash_{\gamma+1} \dot{D}_\gamma$ is an open-dense subset of \mathbb{S} ;
- (b) $\Vdash_{\gamma+1} \forall S \in \dot{D}_\gamma (S \not\subseteq T \implies \text{stem}(S) \notin T)$;
- (c) $\Vdash_{\gamma+1} \forall S \in \dot{D}_\gamma (S \subseteq T \implies (\text{split}(S) \subseteq \bigcup_{n \in a} \text{split}_n(T) \vee \text{split}(S) \subseteq \bigcup_{n \in \omega \setminus a} \text{split}_n(T)))$;
- (d) $\Vdash_{\gamma+1} \forall S \in \dot{D}_\gamma (S \subseteq T \implies \forall n < \omega (|\text{split}(S) \cap \text{split}_n(T)| \leq 1))$;
- (e) $\Vdash_{\gamma+1} \forall S \in \dot{D}_\gamma (\text{split}(S) \subseteq \bigcup_{n \in \omega \setminus a} \text{split}_n(T) \implies S \text{ is } \bar{T}\text{-incompatible with } \omega_2 \cap V^{\mathbb{P}_\gamma} \text{ on } \omega \setminus a)$;
- (f) $\Vdash_{\gamma+1} \forall S \in \dot{D}_\gamma (\text{split}(S) \subseteq \bigcup_{n \in a} \text{split}_n(T) \implies S \text{ is } \bar{T}\text{-incompatible with } x_\gamma \circ \varphi \text{ on } a)$;

(g) $\Vdash_{\gamma+1} \forall S \in \dot{D}_\gamma (S \subseteq T \implies [S] \cap V^{\mathbb{P}_\gamma} = \emptyset)$.

We quickly deal with the fact that the $\dot{D}_\gamma(\bar{T}, a)$'s from the previous definition exist: Since we are only concerned with names for successor steps of the iteration, the analysis can be reduced to a single step of \mathbb{P} . Since the sets of all trees with the respective properties are clearly open, it suffices to prove that they are dense. ZFC proves that trees with properties (b), (c), and (d) are open-dense. The remaining properties are dealt with in the following lemma.

Lemma 3.6 *Let $a \in [\omega]^\omega$, let $\bar{T} = \langle t_s : s \in {}^{<\omega}2 \rangle$ be a splitting family and let T be the tree generated by \bar{T} . Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}, \mathbb{S}\}$, $\mathbb{Q} \in \{\mathbb{U}, \mathbb{S}\}$ and let \dot{S} be a \mathbb{P} -name such that*

$$\Vdash_{\mathbb{P}} \dot{S} \in \mathbb{Q} \wedge \dot{S} \subseteq T \wedge \text{split}(\dot{S}) \subseteq \bigcup_{n \in a} \text{split}_n(T).$$

Then there exists a \mathbb{P} -name \dot{R} such that

$$\Vdash_{\mathbb{P}} \dot{R} \leq_{\mathbb{Q}} \dot{S} \wedge (\dot{R} \text{ is } \bar{T} \text{ -incompatible with } {}^\omega 2 \cap V \text{ on } a).$$

Proof Let \dot{I} be a \mathbb{P} -name such that $\Vdash \dot{I} \in \mathbb{S}$ and

$$\Vdash \text{split}(\dot{I}) = \{t_s : s \in \text{split}(\dot{I})\}.$$

We consider the case $\mathbb{Q} = \mathbb{S}$ first, and deal with $\mathbb{Q} = \mathbb{U}$ in a remark afterwards. It suffices to find a \mathbb{P} -name \dot{J} such that

$$\Vdash_{\mathbb{P}} \dot{J} \leq_{\mathbb{S}} \dot{I} \wedge \forall x \in [\dot{J}](x \upharpoonright a \notin V).$$

Let $p \in \mathbb{P}$. The proof of the following claim is a simple fusion below p and is left to the reader.

Claim 3.6.1 *There exists $q \leq p$ and a family $\langle S_t : t \in {}^{<\omega}2 \rangle$ with the following properties:*

- (a) $S_t \subseteq {}^{<\omega}2$;
- (b) $q * t \Vdash S_t \subseteq \text{split}_{2 \upharpoonright |t|}(\dot{I})$;
- (c) for every $t \in {}^{<\omega}2$, $j < 2$, $\sigma \in S_t$ there exist $\sigma_0, \sigma_1 \in S_{t \smallfrown j}$ such that $\sigma \smallfrown j \subseteq \sigma_0, \sigma_1$ and $\sigma_0 \perp \sigma_1$;
- (d) $|S_t| = 2^{|t|}$.

Let q and $\langle S_t : t \in {}^{<\omega}2 \rangle$ be as in Claim 3.6.1.

Claim 3.6.2 *For every $y \in {}^a 2$ and every $t \in {}^{<\omega}2$,*

$$|\{\sigma \in S_t : \sigma \upharpoonright a \subseteq y\}| \leq 1.$$

Proof By induction on $|t|$. For $t = \emptyset$, this is easy, because $|S_\emptyset| = 2^0 = 1$. Assume that the assertion is true for t . Let $j < 2$. If there is no $\sigma \in S_t$ such that $\sigma \upharpoonright a \subseteq y$, there is no such element in $S_{t \frown j}$, since every element of $S_{t \frown j}$ extends an element from S_t . So, let $\sigma \in S_t$ be the unique element with $\sigma \upharpoonright a \subseteq y$. Since $q * t \Vdash \sigma \in \text{split}(\dot{S})$, $|\sigma| \in a$. Let $\sigma_0, \sigma_1 \in S_{t \frown j}$ be the two elements extending $\sigma \frown j$. Clearly, no other element of $S_{t \frown j}$ can have the desired property. If $j = 1 - y(|\sigma|)$, neither of the two extension is compatible with y . If $j = y(|\sigma|)$, there can exist $k < 2$ such that $\sigma_k \upharpoonright a \subseteq y$. Since $\sigma_0 \perp \sigma_1$ and the split nodes of \dot{S} have lengths in a , $\sigma_0 \upharpoonright a \neq \sigma_1 \upharpoonright a$, so there is at most one such k . □

Let \dot{J} be the \mathbb{P} -name generated by $\bigcup \{S_t : t \subseteq \dot{g}\}$.

Claim 3.6.3 $q \Vdash \forall x \in [\dot{J}](x \upharpoonright a \notin V)$.

Proof Let $y \in {}^{<\omega}2$ and $q' \leq q$. Let $t \in {}^{<\omega}2$ be maximal with $q' \leq q * t$. By Claim 2, there exists at most one $\sigma \in S_t$ such that $\sigma \upharpoonright s \subseteq y$. Since $|\sigma| \in a$ for every $\sigma \in S_t$, there exists $j < 2$ such that for every $\sigma \in S_t$, $(\sigma \frown j) \upharpoonright a \not\subseteq y$. Let $q'' := q' \cap (q * (t \frown j))$ and let \dot{x} be a \mathbb{P} -name with $q'' \Vdash \dot{x} \in [\dot{J}]$. Then there exists a unique $\sigma \in S_{t \frown j}$ such that $q'' \Vdash \sigma \subseteq \dot{x}$. But $\sigma \upharpoonright a \not\subseteq y$, and hence, $q'' \Vdash \dot{x} \upharpoonright a \neq y$. □

This concludes the proof for $\mathbb{Q} = \mathbb{S}$. In case $\mathbb{Q} = \mathbb{U}$, let $\langle \dot{\sigma}_s : s \in {}^{<\omega}2 \rangle$ be a \mathbb{P} -name for the canonical enumeration of $\text{split}(\dot{I})$. Note that there exists a \mathbb{P} -name \dot{u} for a uniform tree such that

$$\Vdash_{\mathbb{P}} \dot{J} = \{ \sigma_s : s \in \text{split}(\dot{u}) \}.$$

In particular, if $\dot{S} = \{t_s : s \in \text{split}(\dot{I})\}$ is a uniform subtree of T , then so is $\dot{R} = \{t_s : s \in \text{split}(\dot{J})\}$. □

The following lemmas show that for all coding triples (\dot{x}, p, f) , p forces that \dot{x} is not in a branch set of one of the open-dense sets defined in Definition 3.5. We will deal with the consequences of Property (d) first. This is the only part of the proof that utilizes the fact that we deal with iterations of uniform trees.

Lemma 3.7 Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}\}$ and $\alpha \leq \omega_2$. Let (\dot{x}, p, f) be an \mathbb{P}_α -coding triple, $\bar{T}(\dot{x}, p, f) = \langle t_s : s \in {}^{<\omega}2 \rangle$, and T the tree generated by \bar{T} . Let $S \in \mathbb{S}$ such that

$$\forall n < \omega (|\text{split}(S) \cap \text{split}_n(T)| \leq 1).$$

Then $p \Vdash \dot{x} \notin [S]$.

Proof Let $\bar{A}(p, f) = \langle a_s : s \in {}^{<\omega}2 \rangle$ be the antichain refinement associated with (\dot{x}, p, f) . If $\text{stem}(S) \notin T$, we are done. So, assume that $\text{stem}(S) \in T$. Then $\text{stem}(S) = t_s$ for some $s \in {}^{<\omega}2$. Let $p' \leq p$. If p' is incompatible with a_s , $p' \Vdash t_s \not\subseteq \dot{x}$, and hence, $p' \Vdash \dot{x} \notin [S]$. So, assume that p' is compatible with a_s . Without loss of generality, $p' \leq a_s$. By Lemma 2.3(2), there exists $s \subseteq s' \in {}^{<\omega}2$ such that $f(\upharpoonright_{s'}) = 0$ and p' is compatible with both $a_{s' \frown 0}$ and $a_{s' \frown 1}$. By Lemma 2.4 there exist $s_0, s_1 \in {}^{<\omega}2$ such that $s' \frown j \subseteq s_j$, $|s_0| = |s_1|$, $f(\upharpoonright_{s_j}) = 0$, and p' is compatible with $a_{s_j \frown 0}$ and $a_{s_j \frown 1}$ ($j < 2$). By choice of S , there exists $j < 2$ such that $t_{s_j} \notin \text{split}(S)$. Hence, there exists

$k < 2$ such that $t_{s_j} \widehat{k} \notin S$. Let $\hat{s} \in |s_j|+1 \cdot 2$ such that $t_{s_j} \widehat{k} \subseteq t_{\hat{s}}$. Then, $a_{\hat{s}} \Vdash t_{\hat{s}} \subseteq \dot{x}$, and thus, $a_{\hat{s}} \Vdash \dot{x} \notin [S]$. By choice of s_j , $a_{\hat{s}}$ is compatible with p' . This concludes the proof of the lemma. \square

The proof of following lemma is a straightforward application of the definition of \bar{T} -compatibility and is therefore left to the reader.

Lemma 3.8 *Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}\}$, $\alpha \leq \omega_2$, and let (\dot{x}, p, f) be an \mathbb{P}_α -coding triple, $\bar{T} = \bar{T}(\dot{x}, p, f)$, and $a = a(\dot{x}, p, f)$. Assume that $a \neq \emptyset$, i.e. $\text{supp}(p)$ has a maximal element β , $\alpha = \beta + 1$ and $a = f^{-1}[\{\beta\}]$. Let $\dot{\phi}$ be a \mathbb{P}_β -name such that*

$$p \restriction \beta \Vdash \dot{\phi} : a \rightarrow a^{p(\beta)} \text{ is an order isomorphism.}$$

Let x_β a \mathbb{P}_α -name for the generic real adjoined in the β -th step of the iteration. Let \dot{S} be an \mathbb{P}_α -name for an element of \mathbb{S} and let $p' \leq p$ such that either

$$p' \Vdash_{\mathbb{P}_\alpha} \dot{S} \text{ is incompatible with } x_\beta \circ \dot{\phi} \text{ on } a$$

or

$$p' \Vdash_{\mathbb{P}_\alpha} \dot{S} \text{ is incompatible with } {}^\omega 2 \cap V \text{ on } \omega \setminus a.$$

Then $p' \Vdash \dot{x} \notin [\dot{S}]$.

The following lemma is the crucial part of the proof that $\text{cov}(s^0)$ remains small in $V^{\mathbb{V}_{\omega_2}}$ and $V^{\mathbb{U}_{\omega_2}}$.

Lemma 3.9 *Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}\}$, $\alpha < \omega_2$, (\dot{x}, p, f) a \mathbb{P}_α -coding triple, $\bar{T} = \bar{T}(\dot{x}, p, f) = \langle t_s : s \in {}^{<\omega} 2 \rangle$, and $a = a(\dot{x}, p, f)$. Let $\gamma < \omega_2$, then*

$$p \Vdash_{\omega_2} \dot{x} \in \mathcal{X}(\dot{D}_\gamma(\bar{T}, a)).$$

Proof Let T be the tree generated by \bar{T} . Put $\dot{D}_\gamma := \dot{D}_\gamma(\bar{T}, a)$. For every $\beta < \omega_2$ let x_β be a $\mathbb{P}_{\beta+1}$ -name for the generic real adjoined in the β -th step of the iteration.

Let \dot{S} be a $\mathbb{P}_{\gamma+1}$ -name with $p \Vdash_{\omega_2} \dot{S} \in \dot{D}_\gamma$. Let $p' \leq p$. We can assume that one of the following statements is true. Either, $p' \Vdash \dot{S} \subseteq T$ or $p' \Vdash \text{stem}(\dot{S}) \notin T$. In the latter case, $p' \Vdash \dot{x} \notin [\dot{S}]$. So, assume that $p' \Vdash \dot{S} \subseteq T$. Now, distinguish three cases:

Case 1. $\gamma + 1 < \alpha$. Let $\beta = \min(\text{supp}(p) \setminus (\gamma + 1))$. Let $G_\beta \subseteq \mathbb{P}_\beta$ be a V -generic filter with $p \restriction \beta \in G_\beta$. Let $b := f^{-1}[\text{supp}(p) \cap (\gamma + 1)]$. For every $n \in b$, there exists $k_n < \omega$ such that n is the k_n -th element of $f^{-1}[\{f(n)\}]$. Let

$$\bar{T}_\beta = \langle t_s : s \in {}^{<\omega} 2, \forall n \in |s| \cap b (s(n) = x_{f(n)}(k_n)) \rangle.$$

Since $\beta < \alpha$, \dot{x} has not been decided in $V[G_\beta]$, and hence, \bar{T}_β generates a perfect tree T_β with

$$V[G_\beta] \models (p \restriction [\beta, \omega_2]) \Vdash \dot{x} \in [T_\beta].$$

By Property (d) and the fact that T_β is a subtree of T , we know that

$$\forall n < \omega (|\text{split}(\dot{S}[G_\beta]) \cap \text{split}_n(T_\beta)| \leq 1).$$

We can apply Lemma 3.7 to this situation to obtain, $p \Vdash [\beta, \omega_2] \Vdash \dot{x} \notin \dot{q}[G_\beta]$. In particular, $p \Vdash \dot{x} \notin [\dot{q}]$.

Case 2. $\gamma + 1 = \alpha$. Then γ is the maximal element of $\text{supp}(p)$ and $a = f^{-1}[\{\gamma\}]$. By Property (c),

$$\Vdash_{\omega_2} \text{split}(\dot{S}) \subseteq \bigcup_{n \in a} \text{split}_n(T) \vee \text{split}(\dot{S}) \subseteq \bigcup_{n \in \omega \setminus a} \text{split}_n(T).$$

Without loss of generality, we can assume that $p' \Vdash_{\omega_2} \text{split}(\dot{S}) \subseteq \bigcup_{n \in a} \text{split}_n(T)$ or $p' \Vdash_{\omega_2} \text{split}(\dot{S}) \subseteq \bigcup_{n \in \omega \setminus a} \text{split}_n(T)$.

First, if $p' \Vdash_{\omega_2} \text{split}(\dot{S}) \subseteq \bigcup_{n \in a} \text{split}_n(T)$, then, by Property (f),

$$p' \Vdash_{\gamma+1} \dot{S} \text{ is incompatible with } x_\gamma \circ \dot{\varphi} \text{ on } a,$$

where $\dot{\varphi}$ is a \mathbb{P}_γ -name such that $p \restriction \gamma$ forces that $\dot{\varphi}$ is the order isomorphism $a \rightarrow a^{p(\gamma)}$. By Lemma 3.8, $p' \Vdash \dot{x} \notin [\dot{S}]$. So, assume $p' \Vdash_{\omega_2} \text{split}(\dot{S}) \subseteq \bigcup_{n \in \omega \setminus a} \text{split}_n(T)$. By Property (e),

$$p' \Vdash \dot{S} \text{ is incompatible with } \omega_2 \cap V^{\mathbb{P}_\gamma} \text{ on } \omega \setminus a.$$

Then, Lemma 3.8 implies $p' \Vdash \dot{x} \notin [\dot{S}]$. It follows that $p \Vdash \dot{x} \notin [\dot{S}]$.

Case 3. $\gamma + 1 > \alpha$. By Property (g), $\Vdash_{\omega_2} [\dot{S}] \cap V^{\mathbb{P}_\gamma} = \emptyset$. Since $p \Vdash \dot{x} \in V^{\mathbb{P}_\gamma}$, this implies $p \Vdash \dot{x} \notin [\dot{S}]$. □

The following theorem is the main result of this section. It shows that $\text{cov}(s^0)$ remains small in $V^{\mathbb{V}_{\omega_2}}$ and $V^{\mathbb{U}_{\omega_2}}$. The fact that both forcings increase the covering number of their respective ideals, is a Löwenheim–Skolem argument which can be found in [4, Theorem 1.2] or [2, Claim 5.5.1].

Theorem 3.10 *Let $\mathbb{P} \in \{\mathbb{V}, \mathbb{U}\}$ and let j^0 be the associated ideal. Let \mathbb{P}_{ω_2} be the countable support iteration of \mathbb{P} over a model V for ZFC + CH. For every V -generic filter $G \subseteq \mathbb{P}_{\omega_2}$,*

$$V[G] \models \text{cov}(s^0) = \aleph_1 < \aleph_2 = 2^{\aleph_0} = \text{cov}(j^0).$$

Proof For every splitting family $\bar{T} = \langle t_s : s \in {}^{<\omega}2 \rangle$ and $a \in \Omega \cup \{\emptyset, \omega\}$ in the ground model we have the family

$$\langle \dot{D}_\gamma(\bar{T}, a) : \gamma < \omega_2 \rangle$$

as in Definition 3.5. In $V[G]$, put

$$D(\bar{T}, a) := \bigcup_{\gamma < \omega_2} (\dot{D}_\gamma(\bar{T}, a))[G].$$

Since every perfect tree in $V[G]$ is added by some initial segment of the iteration, $D(\bar{T}, a)$ is a dense subset of \mathbb{S} (in $V[G]$).

For every real x in $V[G]$, there exist $\alpha < \omega_2$ and a \mathbb{P}_α -coding triple (\dot{x}, p, f) with $p \in G$ and $\dot{x}[G] = x$. Let $\bar{T} = \bar{T}(\dot{x}, p, f)$ and $a = a(\dot{x}, p, f)$. Then

$$p \Vdash \dot{x} \in \bigcap_{\gamma < \omega_2} \mathcal{X}(\dot{D}_\gamma(\bar{T}, a)),$$

by Lemma 3.9. In particular, $x = \dot{x}[G] \in \mathcal{X}(D(\bar{T}, a))$. □

4 cov(u^0) and cov(\mathcal{C}_2) in $V^{\mathbb{S}_{\omega_2}}$

The proof that $\text{cov}(\mathcal{C}_2)$ remains small in $V^{\mathbb{S}_{\omega_2}}$ is very similar to the first author’s proof of this fact in $V^{\mathbb{V}_{\omega_2}}$. We refer the reader to [5] for more motivation on the definition of our coding systems below.

We start with a version of continuous reading of names for Sacks forcing. The analogue theorem for Silver forcing, which can be found in [5, Theorem 3.1], is a little weaker ($f(n) \leq f(|s|)$ has to be replaced with $f(n) < f(|s|)$). This stronger version can easily be proved using the same methods. Sacks forcing allows for this stronger version, since it admits to more liberty in the construction of fusion sequences.

Proposition 4.1 *Let $\alpha \leq \omega_2$ and let \dot{x} be a \mathbb{S}_α -name and $p' \in \mathbb{S}_\alpha$ such that $p' \Vdash \dot{x} \in {}^\omega 2 \setminus \bigcup_{\beta < \alpha} V^{\mathbb{S}_\beta}$. Then there exists an \mathbb{S}_α -coding triple (\dot{x}, p, f) such that $p \leq p'$ and $\bar{T}(\dot{x}, p, f) = \langle t_s : s \in {}^{<\omega} 2 \rangle$ satisfies the following assertions:*

- (a) $\forall s, s' \in {}^{<\omega} 2 (|s| < |s'| \implies |t_s| < |t_{s'}|)$;
- (b) For all $s, s' \in {}^{<\omega} 2$ with $|s| = |s'|$ we have

$$\{n < |s| : s(n) \neq s'(n) \wedge f(n) \leq f(|s|)\} \neq \emptyset \implies |t_s| \neq |t_{s'}|.$$

The proof of the proposition is a fusion. It requires some lemmas:

Lemma 4.2 *Let $\alpha \leq \omega_2$ and let \dot{x} be an \mathbb{S}_α -name and (p, F, η) an \mathbb{S}_α -fusion condition such that $p \Vdash \dot{x} \in {}^\omega 2$. For every $N < \omega$ there exists $q \leq_{F, \eta}$ such that (q, F, η) weakly decides $\dot{x} \upharpoonright N$.*

Proof Left to the reader. □

Remark 4.3 Let \dot{x} be an \mathbb{S}_α -name, let $p \in \mathbb{S}_\alpha$ such that $p \Vdash \dot{x} \in {}^\omega \omega \setminus \bigcup_{\beta < \alpha} V^{\mathbb{S}_\beta}$. Let $\beta < \alpha$. Put $F = \{\beta\}$ and $\eta = \{(\beta, 0)\}$. Then there exists $q \leq_{F, \eta}$ such that (q, F, η) splits \dot{x} at β .

Lemma 4.4 *Let $\alpha \leq \omega_2$ and let \dot{x} be an \mathbb{S}_α -name, let (p, F, η) be an \mathbb{S}_α -fusion condition such that $p \Vdash \dot{x} \in {}^\omega \omega \setminus \bigcup_{\beta < \alpha} V^{\mathbb{S}_\beta}$. Let $\beta \in \text{supp}(p)$ such that $\beta < \gamma$ for every $\gamma \in F$ and $N < \omega$. Then there exists $q \leq_{F, \eta}$ p such that the β -extension (q, F', η') of (p, F, η) splits \dot{x} at β and for every $\sigma \in \prod_{\gamma \in F} ({}^{\eta(\gamma)+1} 2)$,*

$$\left| \dot{x}[q * \sigma^{(\beta, 0)}] \wedge \dot{x}[q * \sigma^{(\beta, 1)}] \right| > N.$$

Proof We prove the claim by induction on $|F|$. For $F = \emptyset$, let $p' \leq p$ such that p' decides $\dot{x} \upharpoonright (N + 1)$. Now, find $q \leq_{F', \eta'} p'$ such that (q, F', η') splits \dot{x} at β .

Now, let $F \neq \emptyset$. We can assume without loss of generality, that (p, F, η) weakly decides $\dot{x} \upharpoonright (N + 1)$. Let δ be the minimal element of F and let $\langle s_l : l < d \rangle$ enumerate $(\eta^{(\delta+1)})2$. Put $q_0 := p$. Let $l < d$ and assume that $q_l \leq_{F', \eta'} p$ has been constructed such that for every $j < l$, $(q_l * \{(\delta, s_j)\}, F' \setminus \{\delta\}, \eta')$ splits \dot{x} at β . By the induction hypothesis, there exists $r_l \leq_{F' \setminus \{\delta\}, \eta'} q_l * \{(\delta, s_l)\}$ such $(r_l, F' \setminus \{\delta\}, \eta')$ splits \dot{x} at β . Define $q_{l+1} \leq_{F', \eta'} q_l$ as follows:

- $q_{l+1} \upharpoonright \delta = r_l \upharpoonright \delta$;
- $q_{l+1} \upharpoonright \delta \Vdash (q_{l+1} * \{(\delta, s_j)\})(\delta) = (q_l * \{(\delta, s_j)\})(\delta)$ for every $j \neq l$;
- $q_{l+1} \upharpoonright \delta \Vdash (q_{l+1} * \{(\delta, s_l)\})(\delta) = (r_l)(\delta)$;
- $(q_{l+1} * \{(\delta, s_l)\}) \upharpoonright (\delta + 1) \Vdash q_{l+1} \upharpoonright (\delta, \alpha) = r_l \upharpoonright (\delta, \alpha)$;
- $(q_{l+1} * \{(\delta, s_j)\}) \upharpoonright (\delta + 1) \Vdash q_{l+1} \upharpoonright (\delta, \alpha) = q_l \upharpoonright (\delta, \alpha)$ for every $j \neq l$.

Finally, $q = q_d$ is the desired element. □

Lemma 4.5 *Let $\alpha \leq \omega_2$ and let \dot{x} be an \mathbb{S}_α -name, let (p, F, η) be an \mathbb{S}_α -fusion condition such that $p \Vdash \dot{x} \in {}^\omega \omega \setminus \bigcup_{\beta < \alpha} V^{\mathbb{S}^\beta}$. Let $\beta = \min(F)$ and $N < \omega$. Then there exists $q \leq_{F, \eta} p$ such that the β -extension (q, F, η') of (p, F, η) splits \dot{x} at β and*

- (a) for every $\sigma \in \prod_{\gamma \in F} (\eta^{(\gamma+1)})2$, $|\dot{x}[q * \sigma^{(\beta, 0)}] \wedge \dot{x}[q * \sigma^{(\beta, 1)}]| > N$;
- (b) for every $\sigma_0, \sigma_1 \in \prod_{\gamma \in F} (\eta^{(\gamma+1)})2$ with $\sigma_0(\beta) \neq \sigma_1(\beta)$,

$$\left| \dot{x}[q * \sigma_0^{(\beta, 0)}] \wedge \dot{x}[q * \sigma_0^{(\beta, 1)}] \right| \neq \left| \dot{x}[q * \sigma_1^{(\beta, 0)}] \wedge \dot{x}[q * \sigma_1^{(\beta, 1)}] \right|.$$

Proof We can assume without loss of generality that (p, F, η) weakly decides $\dot{x} \upharpoonright (N + 1)$. Let $\langle s_l : l < d \rangle$ enumerate $(\eta^{(\beta+1)})2$. Put $q_0 := p$. Let $l < d$ and assume that $q_l \leq_{F, \eta} p$ has been constructed such that for every $j_0 < j_1 < l$, $\sigma_0, \sigma_1 \in \prod_{\gamma \in F} (\eta^{(\gamma+1)})2$ with $\sigma_0(\beta) = s_{j_0}$ and $\sigma_1(\beta) = s_{j_1}$

$$\left| \dot{x}[q_l * \sigma_0^{(\beta, 0)}] \wedge \dot{x}[q_l * \sigma_0^{(\beta, 1)}] \right| \neq \left| \dot{x}[q_l * \sigma_1^{(\beta, 0)}] \wedge \dot{x}[q_l * \sigma_1^{(\beta, 1)}] \right|.$$

By the previous lemma, there exists $r_l \leq_{F \setminus \{\beta\}, \eta} q_l * \{(\beta, s_l)\}$ such that, for every $\sigma \in \prod_{\gamma \in F \setminus \{\beta\}} (\eta^{(\gamma+1)})2$, $|\dot{x}[r_l * \sigma^{(\beta, 0)}] \wedge \dot{x}[r_l * \sigma^{(\beta, 1)}]|$ is longer than any $|\dot{x}[q_l * \hat{\sigma}^{(\beta, 0)}] \wedge \dot{x}[q_l * \hat{\sigma}^{(\beta, 1)}]|$ for $\hat{\sigma} \in \prod_{\gamma \in F} (\eta^{(\gamma+1)})2$ with $\hat{\sigma}(\beta) = s_j$ for some $j < l$. Define $q_{l+1} \leq_{F, \eta} q_l$ as follows:

- $q_{l+1} \upharpoonright \beta = r_l \upharpoonright \beta$,
- $q_{l+1} \upharpoonright \beta \Vdash (q_{l+1} * \{(\beta, s_j)\})(\beta) = (q_l * \{(\beta, s_j)\})(\beta)$ for every $j \neq l$;
- $q_{l+1} \upharpoonright \beta \Vdash (q_{l+1} * \{(\beta, s_l)\})(\beta) = (r_l)(\beta)$;
- $(q_{l+1} * \{(\beta, s_j)\}) \upharpoonright (\beta + 1) \Vdash q_{l+1} \upharpoonright (\beta, \alpha) = r_l \upharpoonright (\beta, \alpha)$ for every $j \leq l$;
- $(q_{l+1} * \{(\beta, s_j)\}) \upharpoonright (\beta + 1) \Vdash q_{l+1} \upharpoonright (\beta, \alpha) = q_l \upharpoonright (\beta, \alpha)$ for every $j > l$.

Finally $q = q_d$ is the desired element. □

Lemma 4.6 *Let $\alpha \leq \omega_2$ and let \dot{x} be an \mathbb{S}_α -name and (p, F, η) and an \mathbb{S}_α -fusion condition such that $p \Vdash \dot{x} \in {}^\omega\omega \setminus \bigcup_{\beta < \alpha} V^{\mathbb{S}^\beta}$. Let $\beta < \alpha$ and $N < \omega$. Then there exists $q \leq_{F,\eta} p$ such that the β -extension (q, F', η') of (p, F, η) splits \dot{x} at β and for every $\sigma_0, \sigma_1 \in \prod_{\gamma \in F} {}^{(\eta(\gamma)+1)}2$ with*

- (a) *for every $\sigma \in \prod_{\gamma \in F} {}^{(\eta(\gamma)+1)}2$, $|\dot{x}[q * \sigma^{(\beta,0)}] \wedge \dot{x}[q * \sigma^{(\beta,1)}]| > N$;*
- (b) *for every $\sigma_0, \sigma_1 \in \prod_{\gamma \in F} {}^{(\eta(\gamma)+1)}2$ with $\sigma_0 \upharpoonright (\beta + 1) \neq \sigma_1 \upharpoonright (\beta + 1)$,*

$$|\dot{x}[q * \sigma_0^{(\beta,0)}] \wedge \dot{x}[q * \sigma_0^{(\beta,1)}]| \neq |\dot{x}[q * \sigma_1^{(\beta,0)}] \wedge \dot{x}[q * \sigma_1^{(\beta,1)}]|.$$

Proof We can assume without loss of generality, that (p, F, η) weakly decides $\dot{x} \upharpoonright (N + 1)$. Let $\langle \delta_0, \dots, \delta_n \rangle$ increasingly enumerate $\{\delta \in F \cup \{\beta\} : \delta \leq \beta\}$. We prove the claim by induction on n . For $n = 0$ this one of the previous lemmas. So, assume $n > 0$. Let $\langle s_j : j < d \rangle$ enumerate ${}^{(\eta(\delta_0)+1)}2$. Put $q_0 := p$. Let $l < d$ and assume that $q_l \leq_{F,\eta} p$ has been constructed such that for every $\sigma_0, \sigma_1 \in \prod_{\gamma \in F} {}^{(\eta(\gamma)+1)}2$ such that $\sigma_0(\delta_0) = s_{j_0}$ and $\sigma_1(\delta_0) = s_{j_1}$ for some $j_0, j_1 < l$ and $\sigma_0 \upharpoonright (\beta + 1) \neq \sigma_1 \upharpoonright (\beta + 1)$, then

$$|\dot{x}[q * \sigma_0^{(\beta,0)}] \wedge \dot{x}[q * \sigma_0^{(\beta,1)}]| \neq |\dot{x}[q * \sigma_1^{(\beta,0)}] \wedge \dot{x}[q * \sigma_1^{(\beta,1)}]|.$$

By the induction hypothesis, there exists $r_l \leq_{F \setminus \{\delta_0\}, \eta} q_l * \{(\delta_0, s_l)\}$ such that for every $\sigma \in \prod_{\gamma \in F \setminus \{\delta_0\}} {}^{(\eta(\gamma)+1)}2$, $|\dot{x}[r_l * \sigma^{(\beta,0)}] \wedge \dot{x}[r_l * \sigma^{(\beta,1)}]|$ is longer than any $|\dot{x}[q_l * \hat{\sigma}^{(\beta,0)}] \wedge \dot{x}[q_l * \hat{\sigma}^{(\beta,1)}]|$ for $\hat{\sigma} \in \prod_{\gamma \in F} {}^{(\eta(\gamma)+1)}2$ with $\hat{\sigma}(\delta_0) = s_j$ for some $j < l$. Define $q_{l+1} \leq_{F,\eta} q_l$ as follows:

- $q_{l+1} \upharpoonright \delta_0 = r_l \upharpoonright \delta_0$,
- $q_{l+1} \upharpoonright \delta_0 \Vdash (q_{l+1} * \{(\delta_0, s_j)\})(\delta_0) = (q_l * \{(\delta_0, s_j)\})(\delta_0)$ for every $j \neq l$;
- $q_{l+1} \upharpoonright \delta_0 \Vdash (q_{l+1} * \{(\delta_0, s_l)\})(\delta_0) = (r_l)(\delta_0)$;
- $(q_{l+1} * \{(\delta_0, s_l)\}) \upharpoonright (\delta_0 + 1) \Vdash q_{l+1} \upharpoonright (\delta_0, \alpha) = r_l \upharpoonright (\delta_0, \alpha)$;
- $(q_{l+1} * \{(\beta, s_j)\}) \upharpoonright (\beta + 1) \Vdash q_{l+1} \upharpoonright (\beta, \alpha) = q_l \upharpoonright (\beta, \alpha)$ for every $j \neq l$.

Finally $q = q_d$ is the desired element. □

Proof of Proposition 4.1 Using the previous lemmas one may construct a fusion sequence $\langle (p_n, F_n, \eta_n) : n < \omega \rangle$ below p' with active coordinates $\langle \beta_n : n < \omega \rangle$ and $\langle \sigma_s : s \in {}^{<\omega}2 \rangle$ as in (2) with the following properties:

- (α) (p_n, F_n, η_n) splits \dot{x} at β_n ;
- (β) For every $s \in {}^n2, s' \in {}^{n+1}2$,

$$|\dot{x}[p_n * \sigma_{s \smallfrown 0}] \wedge \dot{x}[p_n * \sigma_{s \smallfrown 1}]| < |\dot{x}[p_{n+1} * \sigma_{s' \smallfrown 0}] \wedge \dot{x}[p_{n+1} * \sigma_{s' \smallfrown 1}]|;$$

- (γ) For every $s, s' \in {}^{<\omega}2$ with $|s| = |s'| =: n$ and $\sigma_s \upharpoonright (\beta_n + 1) \neq \sigma_{s'} \upharpoonright (\beta_n + 1)$,

$$|\dot{x}[p_n * \sigma_{s \smallfrown 0}] \wedge \dot{x}[p_n * \sigma_{s \smallfrown 1}]| \neq |\dot{x}[p_n * \sigma_{s' \smallfrown 0}] \wedge \dot{x}[p_n * \sigma_{s' \smallfrown 1}]|.$$

Let p_ω be the fusion limit and $f = \langle \beta_n : n < \omega \rangle$. Then (\dot{x}, p_ω, f) is an \mathbb{S}_α -coding triple. Furthermore,

$$t_s = \dot{x}[p_\omega * \sigma_{s \smallfrown 0}] \wedge \dot{x}[p_\omega * \sigma_{s \smallfrown 1}] = \dot{x}[p_{|s|} * \sigma_{s \smallfrown 0}] \wedge \dot{x}[p_{|s|} * \sigma_{s \smallfrown 1}],$$

and hence, Properties (a) and (b) of the assertion follow from Properties (β) and (γ) of the fusion. \square

The following lemma is pivotal in the construction of our coding systems below. It has the same role in this proof as Lemma 3.7 in the proof of the previous section.

Lemma 4.7 *Let $\alpha \leq \omega_2$ and let (\dot{x}, p, f) be an \mathbb{S}_α -coding triple such that the associated splitting family $\bar{T}(\dot{x}, p, f) = \langle t_s : s \in {}^{<\omega}2 \rangle$ satisfies the assertions of Proposition 4.1. Let $s, s_0, s_1 \in {}^{<\omega}2$ such that $f(|s|) = f(|s_0|) = f(|s_1|)$, $|t_{s_0}| = |t_{s_1}|$, and t_{s_0}, t_{s_1} extend t_s . Then $s_0(|s|) = s_1(|s|)$.*

Proof Since t_{s_0} and t_{s_1} both extend t_s , s_0 and s_1 both extend s . Since $|t_{s_0}| = |t_{s_1}|$, Proposition 4.1(1) implies $|s_0| = |s_1|$. Then, Proposition 4.1(2) implies $\{n < |s_0| : s_0(n) \neq s_1(n) \wedge f(n) \leq f(|s_0|)\} = \emptyset$. Since $f(|s|) = f(|s_0|)$, $s_0(|s|) = s_1(|s|)$. \square

Before we come to the definition of the coding systems, we need one final lemma. Its proof is the same as for the analogous lemma for Silver forcing.

Lemma 4.8 (cf. [5, Lemma 5.1]) *Let \dot{b} be a \mathbb{S} -name and $p \in \mathbb{S}$ with $p \Vdash \dot{b} \in [\omega]^\omega \setminus V$. Then there exist \mathbb{S} -names \dot{c}, \dot{y} and $q \leq p$ such that*

$$q \Vdash \dot{c} \in [\dot{b}]^\omega \wedge \dot{y} \in \dot{c}^2 \setminus \{x \mid \dot{c} : x \in {}^\omega 2 \cap V\}.$$

Definition 4.9 Let (\dot{x}, p, f) and (\dot{y}, q, g) be two coding triples. We say that (\dot{x}, p, f) and (\dot{y}, q, g) are **isomorphic**, if there is an order isomorphism $\pi : \text{supp}(p) \rightarrow \text{supp}(q)$ that satisfies $g = \pi \circ f$ and $\bar{T}(\dot{x}, p, f) = \bar{T}(\dot{y}, q, g)$.

It is easy to see that there are only \aleph_1 isomorphism types of coding triples.

Definition 4.10 Let $\alpha \leq \omega_2$ and let (\dot{x}, p, f) be an \mathbb{S}_α -coding triple with associated splitting family $\bar{T}(\dot{x}, p, f) = \langle t_s : s \in {}^{<\omega}2 \rangle$. For every $\beta \in \text{supp}(p)$ put

$$L(\beta) := \{t_s \mid f(|s|) = \beta\}.$$

Let $\Omega(\dot{x}, p, f)$ be the set of all $b \in [\omega]^\omega$ that satisfy one of the following assertions:

- (a) There exists some $\beta \in \text{supp}(p)$ such that $b \subseteq L(\beta)$;
- (b) $b \subseteq \bigcup \{L(\beta) : \beta \in \text{supp}(p)\}$ and $|b \cap L(\beta)| \leq 1$ for every $\beta \in \text{supp}(p)$;
- (c) $b \cap \bigcup \{L(\beta) : \beta \in \text{supp}(p)\} = \emptyset$.

Clearly, $\Omega(\dot{x}, p, f)$ is a dense subset of $[\omega]^\omega$. We come to the definition of the coding systems. These will be extended at successor stages of the iteration. Let $\gamma < \omega_2$ we work in $V^{\mathbb{S}_{\gamma+1}}$. Let

$$\Omega_\gamma(\dot{x}, p, f) = \left\{ b \in \Omega(\dot{x}, p, f) : \neg \exists b_0 \in \Omega(\dot{x}, p, f) \cap V^{\mathbb{S}_\gamma} (b_0 \subseteq b) \right\}.$$

For every $b \in \Omega_\gamma(\dot{x}, p, f)$ we define $u^b \in \mathbb{U}$ depending on b :

- (0) For all cases:
 - (a) $\emptyset \in u^b$;
 - (b) if $\sigma \in u^b$ and $|\sigma| \notin b$, then $\sigma \frown 0, \sigma \frown 1 \in u^b$;
 - (c) if $\sigma \in u^b \setminus \text{split}(T)$ and $|\sigma| \in b$, then let $\sigma \frown j \in u^b$ and $\sigma \frown (1 - j) \notin u^b$, where $j < 2$ is minimal such that $\sigma \frown j \notin T$.
- (1) $b \subseteq L(\beta)$ for some $\beta \in \text{supp}(p)$.
 Split b into three infinite sets b_0, b_1, b_2 . Use Lemma 4.8 to obtain a $y \in {}^{b_2}2$ such that for every $x \in \omega_2 \cap V^{\mathbb{S}_\gamma}, x \upharpoonright b_2 \neq y$. Define u^b recursively.
 - (a) If $\sigma \in u^b, |\sigma| \in b_0, \sigma \in \text{split}(T)$, let n be the minimal element of $b_0 \setminus (|\sigma| + 1)$. If $S := \text{split}(T) \cap T(n) \cap T_\sigma \neq \emptyset$, there exists $j < 2$ such that every element of S extends $\sigma \frown j$ (see Lemma 4.7). Let $\sigma \frown (1 - j) \in u^b$ and $\sigma \frown j \notin u^b$. If S is empty, let $\sigma \frown 0 \in u^b$ and $\sigma \frown 1 \notin u^b$.
 - (b) If $\sigma \in u^b, |\sigma| \in b_1$, and $\sigma \in \text{split}(T)$, then $\sigma = t_s$ for some $s \in {}^{<\omega_2}$ with $f(|s|) = \beta$. Let $k < \omega$ such that $|s|$ is the k -th element of $f^{-1}[\{\beta\}]$. Let $\sigma \frown (1 - t_{s \frown g_\gamma(k)}(|\sigma|)) \in u^b$ and $\sigma \frown (t_{s \frown g_\gamma(k)}(|\sigma|)) \notin u^b$.
 - (c) For $\sigma \in u^b$ with $|\sigma| \in b_2$ and $\sigma \in \text{split}(T)$, i.e. $\sigma = t_s$ for some s , let $\sigma \frown (t_{s \frown \gamma(|\sigma|)}(|\sigma|)) \in u^b$ and $\sigma \frown (1 - t_{s \frown \gamma(|\sigma|)}(|\sigma|)) \notin u^b$.
- (2) $b \subseteq \bigcup \{L(\beta) : \beta \in \text{supp}(p)\}$ and $|b \cap L(\beta)| \leq 1$ for every $\beta \in \text{supp}(p)$. For $b' \subseteq b$ put

$$Y(b') := \{\beta \in \text{supp}(p) : L(\beta) \cap b' \neq \emptyset\}.$$

Let δ be the minimal ordinal such that $Y(b) \setminus \delta$ is finite. Let b_0, b_1 be two disjoint infinite subsets of b such that $\text{sup}(Y(b_0)) = \text{sup}(Y(b_1)) = \delta$ and $\delta \notin Y(b_0), Y(b_1)$. Use Lemma 4.8 to obtain a $y \in {}^{b_1}2$ such that for every $x \in \omega_2 \cap V^{\mathbb{S}_\gamma}, x \upharpoonright b_1 \neq y$. Define u^b recursively:

- (a) if $\sigma \in u^b, |\sigma| \in b_0$, and $\sigma \in \text{split}(T)$, let $\sigma \frown 0 \in u^b$ and $\sigma \frown 1 \notin u^b$;
 - (b) For $\sigma \in u^b$ with $|\sigma| \in b_1$ and $\sigma \in \text{split}(T)$, i.e. $\sigma = t_s$ for some s , let $\sigma \frown (t_{s \frown \gamma(|\sigma|)}(|\sigma|)) \in u^b$ and $\sigma \frown (1 - t_{s \frown \gamma(|\sigma|)}(|\sigma|)) \notin u^b$.
- (3) If $b \cap \bigcup \{L(\beta) : \beta \in \text{supp}(p)\} = \emptyset$, u^b has been completely defined by Case (0).

Proposition 4.11 *Let $\alpha, \alpha' \leq \omega_2$ and let (\dot{x}, p, f) be an \mathbb{S}_α -coding triple. Let $\gamma < \omega_2$ and $b \in \Omega_\gamma(\dot{x}, p, f)$. Let u^b be as in Definition 4.10. Then for every $\mathbb{S}_{\alpha'}$ -coding triple $(\dot{y}, q, g) \cong (\dot{x}, p, f), q \Vdash_{\omega_2} \dot{y} \notin [u^b]$.*

Proof Let $\pi : \text{supp}(p) \rightarrow \text{supp}(q)$ be the order isomorphism. Let $q' \leq q$.

Case 1. $b \subseteq L(\beta)$ for some $\beta \in \text{supp}(p)$. Let $b = b_0 \dot{\cup} b_1 \dot{\cup} b_2$ as in Definition 4.10. Now, distinguish three cases:

Case 1.1. $\pi(\beta) > \gamma$. Without loss of generality, q' decides $\dot{y} \upharpoonright l$ as σ for some $l \in b_0$. If $\sigma \notin \text{split}(T)$, there exists $j < 2$ such that $q' \Vdash \dot{y}(l) = j$ and $\sigma \frown j \notin u^b$, by Property (0)(c). So, assume that σ is a split node of T . Let n be the minimal element

of $b_0 \setminus (|\sigma| + 1)$. By Property (1)(a), there are no split nodes of T in $T_\sigma(n) \cap u^b$. By Property (0)(c), $q' \Vdash \dot{y} \notin [u^b]$.

Case 1.2. $\pi(\beta) = \gamma$. We can assume without loss of generality that q' decides $\dot{y} \upharpoonright l$ as σ for some $l \in b_1$. Again, one may assume that $\sigma \in \text{split}(T)$. Then $\sigma = t_s$ for some $s \in {}^{<\omega}2$. Let $k < \omega$ such that $|s|$ is the k -th element of $f^{-1}[\{\beta\}]$. Then $|s|$ is the k -th element of $g^{-1}[\{\gamma\}]$. In particular,

$$q' \Vdash \dot{y} \upharpoonright (l + 1) = \sigma \frown (t_{s \frown_{g_\gamma(k)}}(|\sigma|)).$$

By Property (1)(b), $\sigma \frown (t_{s \frown_{g_\gamma(k)}}(|\sigma|)) \notin u^b$, and hence, $q' \Vdash \dot{y} \notin [u^b]$.

Case 1.3. $\pi(\beta) < \gamma$. Let $G_\gamma \subseteq \mathbb{S}_\gamma$ be a V -generic filter with $q' \upharpoonright \gamma \in G_\gamma$. In $V[G_\gamma]$, define $F : L(\beta) \rightarrow 2$ as follows: For $l \in L(\beta)$, there exists $s \in {}^{<\omega}2$ with $f(|s|) = \beta$ and $|t_s| = l$. There exists a unique $k < \omega$ such that $|s|$ is the k -th element of $f^{-1}[\{\beta\}]$. By Proposition 4.1(1), k only depends on l and not on s . Define $F(l) := g_{\pi(\beta)}(k)$. Note that

$$(q' \Vdash \dot{y} \upharpoonright l = t_s) \implies (q' \Vdash \dot{y} \upharpoonright (l + 1) \subseteq t_{s \frown_{g_{\pi(\beta)}(k)}}).$$

By choice of y , $y \neq F \upharpoonright b_2$. Hence, there exists $l \in b_2$ such that $F(l) \neq y(l)$. We can assume that q' decides $\dot{y} \upharpoonright l$ as $\sigma \in {}^l 2$. The interesting case is, when $\sigma \in \text{split}(T)$, i.e. $\sigma = t_s$ for some s . Let k be as in the definition of F , then, by Property (1)(c),

$$q' \Vdash \dot{y} \upharpoonright (l + 1) \notin u^b.$$

Case 2. Assume that $b \subseteq \bigcup \{L(\beta) : \beta \in \text{supp}(p)\}$ and $|L(\beta) \cap b| \leq 1$ for every $\beta \in \text{supp}(b)$. Let b_0, b_1 be the partition of b as in Definition 4.10(2).

Case 2.1. $\text{sup}(\pi[Y(b_0)]) > \gamma$. Let $\beta \in Y(b_0)$ such that $\pi(\beta) > \gamma$. Using Lemma 2.3, find $\tilde{s} \in {}^{<\omega}2$ such that $f(|\tilde{s}|) = \beta$ and q' is compatible with $q * \sigma_{\tilde{s} \frown 0}^s$ and $q * \sigma_{\tilde{s} \frown 1}^s$. Since $\{\delta \in Y(b_0) : \pi(\delta) > \gamma\}$ is infinite, there exists $s, s_0, s_1 \in {}^{<\omega}2$ such that $\tilde{s} \frown j \subseteq s_j$, $|s| = |s_0| = |s_1|$, $|t_s| \in b_0 \cap L(f(|s|))$, and $\beta \leq f(|s|)$. By construction,

$$\{n < |s| : s_0(n) \neq s_1(n) \wedge \pi(f(n)) \in [\gamma, \pi(f(|s|))]\} \neq \emptyset.$$

By Proposition 4.1(b), there is at most one $j < 2$ such that $|t_{s_j}| = l$. Hence, one may choose $j < 2$ and $q'' \leq q$ such that $|t_{s_j}| \neq l$. We can assume that q'' decides $\dot{y} \upharpoonright l$, say as σ . By Proposition 4.1(a), $\sigma \notin \text{split}(T)$. By Property (0)(c), $q'' \Vdash \dot{y} \notin [u^b]$.

Case 2.2. Assume that $\text{sup}(\pi[Y(b_0)]) \leq \gamma$. The restricted tree T_γ in the model $V[G_\gamma]$ already decides \dot{y} on coordinates associated with $\beta \in \text{supp}(q) \cap \gamma$. A similar construction as in Case 1.3. yields the claim.

Case 3. If $b \cap \bigcup \{L(\beta) : \beta \in \text{supp}(p)\} = \emptyset$, no split node of T has length in b . Property (0) yields $q' \Vdash \dot{y} \notin [u^b]$.

This concludes the proof that $\{q' \leq q : q' \Vdash \dot{y} \notin [u^b]\}$ is dense below q . It follows that $q \Vdash \dot{y} \notin [u^b]$. □

Theorem 4.12 $V^{\mathbb{S}_{\omega_2}} \models \text{cov}(\mathcal{C}_2) = \aleph_1$.

Since $\text{cov}(u^0) \leq \text{cov}(\mathfrak{C}_2)$ is provable in ZFC, one obtains the following corollary:

Corollary 4.13 $V^{\mathbb{S}_{\omega_2}} \models \text{cov}(u^0) = \aleph_1$.

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